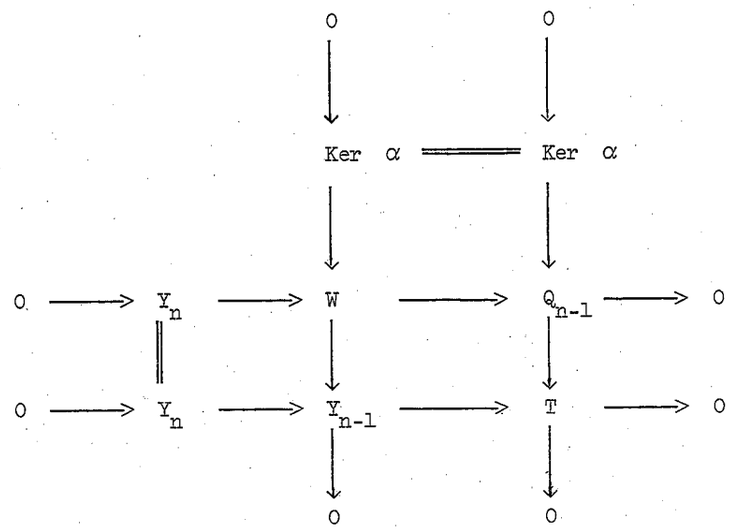
Defining α to be the composition $Q_{n-1} \to S \to T$, elementary diagram chasing gives an exact sequence $0 \to K' \to \operatorname{Ker}\alpha \to U \to 0$. Since U and K' are in Λ, so is $\operatorname{Ker}\alpha$. Now let W be the pull-back of the maps $(Y_{n-1} \to T)$ and $(Q_{n-1} \xrightarrow{\alpha} T)$. Then we have an exact and commutative diagram:



Since Y_{n-1} and $\operatorname{Ker}\alpha$ are in Λ, we see that W is in Λ. Since Q_{n-1} is projective, the middle row splits and Y_n is a direct summand of W. By property (b) we conclude that Y_n is in Λ, completing the proof.

Let us return to the category of R-modules. As we observed prior to (3.11), the full subcategory $\mathcal{L}$ of modules of zero G-dimension satisfies the hypotheses of Lemma (3.12). An application of this fact is contained in the following important characterization of G-dimension.

(3.13) THEOREM: For an integer $k \geq 0$ and a finitely generated module M, the following statements are equivalent:

(a) G-dim $M \leq k$.

(b) G-dim $\Omega^k M = 0$.

(c) There is an exact sequence of finitely generated modules:

$0 \to X_k \to \dots \to X_0 \to M \to 0$ where G-dim $X_i = 0$ for each $0 \leq i \leq k$.

Proof: First, we show the equivalence of (a) and (b). By definition, G-dim $M \leq k$ if and only if G-dim $\operatorname{Ext}^{k+1}(M, \ ) = 0$. Suppose (a) holds; then G-dim $\operatorname{Ext}^1(\Omega^k M, \ ) = 0$. Now G-dim $F = 0$ implies G-dim $S_1 F = 0$ by Lemma (3.4). It follows that G-dim $S_1\operatorname{Ext}^1(\Omega^k M, \ ) = 0$. Since $S_1\operatorname{Ext}^1(\Omega^k M, \ ) \approx S_1 S^1(\Omega^k M, \ ) = (\underline{\Omega^k M}, _)$ we have G-dim$(\underline{\Omega^k M}, _) = 0$. Therefore, G-dim $\Omega^k M = 0$ and hence (b) holds. Conversely, suppose (b) holds. Then G-dim$(\underline{\Omega^k M}, _) = 0$. By Lemma (3.4) we see that

$0 =$ G-dim$(\underline{\Omega^k M}, _) =$ G-dim $S^1(\underline{\Omega^k M}, _) =$ G-dim $\operatorname{Ext}^1(\Omega^k M, \ ) =$ G-dim $\operatorname{Ext}^{k+1}(M, \ )$.

Therefore, G-dim $M \leq k$ and so (a) holds. Thus, (a) and (b) are equivalent.

To prove that (b) implies (c) we just observe that the sequence $0 \to \Omega^k M \to P_{k-1} \to \dots \to P_0 \to M \to 0$, with the P_i the usual projectives, satisfies our requirement (since projectives have G-dimension zero).

Finally, we show that (c) implies (b). Consider the sequence $0 \to \Omega^k M \to P_{k-1} \to \dots \to P_0 \to M \to 0$. Since the P_i are projectives, they have zero G-dimension. By assumption there exists an exact sequence $0 \to X_k \to X_{k-1} \to \dots \to X_0 \to M \to 0$ where the X_i have zero G-dimension. By Lemma (3.12) we conclude that G-dim $\Omega^k M = 0$, concluding our proof.

(3.14) COROLLARY: When M is finitely generated, G-dim $M \leq$ pd. M.

Proof: If M has projective dimension k, then there is an exact sequence $0 \to P_k \to P_{k-1} \to \dots \to P_0 \to M \to 0$ where the P_i are finitely generated projectives. Since G-dim $P_i = 0$ for $0 \leq i \leq k$, we apply the previous Theorem to conclude that G-dim $M \leq k$.

(3.15) COROLLARY: Suppose M is finitely generated.

(a) If G-dim $M < \infty$ then G-dim $M =$ the smallest integer $k \geq 0$ such that there exists an exact sequence

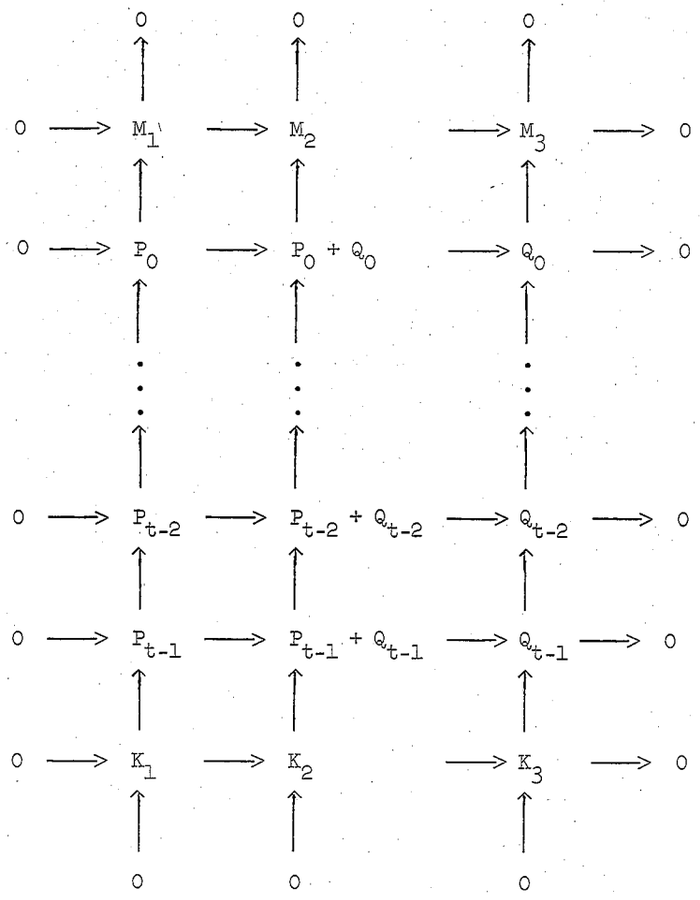
$0 \to X_k \to \dots \to X_0 \to M \to 0$ with G-dim $X_i = 0$ for $0 \le i \le k$.

(b) If G-dim $M \neq 0$ and $0 \to N \to X \to M \to 0$ is exact with G-dim $X = 0$, then (G-dim N) = (G-dim M) - 1.

The proof of this Corollary (using Theorem (3.13)) is analogous to the proof of similar statements for projective dimension.

(3.16) COROLLARY: Suppose $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact with M_i finitely generated ($i = 1, 2, 3$) and with G-dim $M_3 = n < \infty$. If $t \ge n$, then G-dim $M_2 \le t$ if and only if G-dim $M_1 \le t$.

Proof: Consider the exact and commutative diagram:



where the P_i and Q_i are projective ($0 \le i \le t - 1$). Then $K_i \approx \Omega^t(M_i)$ so G-dim K_i = G-dim $\Omega^t(M_i)$ for $i = 1, 2, 3$. Since G-dim $M_3 = n$, and $n \le t$ we have G-dim $\Omega^t M_3$ = G-dim $K_3 = 0$ (by Theorem 3.13). Since $0 \to K_1 \to K_2 \to K_3 \to 0$ is exact, we have G-dim $K_2 = 0$ if and only if G-dim $K_1 = 0$. Thus, G-dim $\Omega^t M_2 = 0$ if and only if G-dim $\Omega^t M_1 = 0$. By Theorem (3.13) this is equivalent to saying G-dim $M_2 \le t$ if and only if G-dim $M_1 \le t$, so our result is proved.

We conclude this Chapter with a result which will be considerably strengthened in the next Chapter where our rings will be commutative.

(3.17) PROPOSITION: Suppose M is finitely generated and G-dim $M = n < \infty$. If grade $\mathrm{Ext}^i(M,R) \ge i + k$ for all $1 \le i \le n$ then M is k-torsion free.

Proof: Since grade $\mathrm{Ext}^i(M,R) \ge i + k$ for $1 \le i \le n$, Proposition (2.28) tells us that M is k-torsion free if and only if $\Omega^n M$ is $(n + k)$-torsion free. Since G-dim $M \le n$ we have G-dim $\Omega^n M = 0$ by Theorem (3.13). By Proposition (3.8) we must have the $\Omega^n M$ is j-torsion free for all j--in particular, for $j = n + k$. Therefore, M is k-torsion free and we are done.

Chapter Four: Applications to Commutative Rings

In this chapter we shall strengthen many of our previous results--particularly for modules of finite Gorenstein dimension. This will be possible for two important reasons. First, we shall have at our disposal the techniques of localization. Second, we shall be able to describe the previously defined notion of depth in terms of so-called regular sequences. This will enable us to argue inductively on the depth of a ring or module.

In §4 we shall relate Gorenstein dimension with the notion of a Gorenstein ring [3].

For the remainder of this paper, all rings will be assumed commutative and noetherian and, unless stated to the contrary, all modules will be finitely generated.

§1. Depth in Commutative Rings

If M and N are R-modules, recall that the M-depth of N is the smallest non-negative integer i such that $\operatorname{Ext}^i(M,N) \neq 0$. (If $\operatorname{Ext}^i(M,N) = 0$ for all i, then M-depth of N is ∞).

If $\underline{a}$ is an ideal of R, we shall say that a sequence $(x_1,\ldots,x_s)$ of elements of R is an N-regular sequence in $\underline{a}$ (N is an R-module) if (a) each x_i lies in $\underline{a}$; (b) x_i is regular (non-zero divisor) in $N/(x_1,\ldots,x_{i-1})N$ for $i = 1,\ldots,s$ (when $i = 1$ then x_1 is regular in N); and (c) $(x_1,\ldots,x_s)N \neq N$. If these conditions hold, then $(x_1,\ldots,x_s)$ will be called a <u>maximal</u> N-regular sequence in $\underline{a}$ if, for any $y \in R$, $(x_1,\ldots,x_s, y)$ is no longer an N-regular sequence in $\underline{a}$.

Proofs of the following properties of depth may be found in [9] and [3].

(4.1) Any N-regular sequence in $\underline{a}$ can be completed to a maximal such sequence. If $\underline{a} \subset \operatorname{rad}(R)$, then any permutation of an N-regular sequence in $\underline{a}$ is again such a sequence.

(4.2) Let M be an R-module and $\underline{a} = \operatorname{Ann} M$ (the annihilator of M). If $\underline{a}N \neq N$ then M-depth $N < \infty$ and M-depth N = the length of any maximal N-regular sequence contained in $\underline{a}$ or the radical of $\underline{a}$. Furthermore, if $(x_1,\ldots,x_t)$ is any N-regular sequence in $\underline{a}$, then we have:

(a) $\operatorname{Ext}^i_R(M,N) = 0$ for $i < t$

(b) $\operatorname{Ext}^i_R(M,N) \approx \operatorname{Ext}^{i-t}_S(M,N/(x_1,\ldots,x_t)N)$ for $i \geq t$ where $S = R/(x_1,\ldots,x_t)R$.

(4.3) Let $\operatorname{Supp}(M) = \{\underline{p} \in \operatorname{Spec} R \mid M_{\underline{p}} \neq 0\}$. If M-depth $N < \infty$ then:

M-depth N = Min $\{R/\underline{p}$-depth $N \mid \underline{p} \in \operatorname{Supp} M\}$

= Min $\{R/\underline{p}$-depth $N \mid \underline{p}$ is minimal in $\operatorname{Supp} M\}$.

(4.4) <u>Definition</u>: Let R be a local ring with maximal ideal $\underline{m}$. Then $\operatorname{depth}_R M$ for an R-module M will denote the $R/\underline{m}$-depth of M (this is sometimes called the codimension of M).

The following Lemma will be quite useful to us in later sections.

(4.5) LEMMA: If M-depth $N < \infty$ then:

M-depth$_R N$ = Min $\{\operatorname{depth}_{R_{\underline{p}}} N_{\underline{p}} \mid \underline{p} \in \operatorname{Supp} M\}$.

<u>Proof</u>: Suppose $\underline{p} \in \operatorname{Supp} M$; then $\underline{p} \supset \operatorname{Ann} M$. If $(x_1,\ldots,x_t)$ is a maximal N-regular sequence in $\operatorname{Ann} M$ then $(x_1,\ldots,x_t) \subset \underline{p}R_{\underline{p}}$ and $(x_1,\ldots,x_t)$ is $N_{\underline{p}}$-regular. Therefore $\operatorname{depth}_{R_{\underline{p}}} N_{\underline{p}} \geq t$ and we have shown the inequality:

M-depth $N \leq$ Min $\{\operatorname{depth}_{R_{\underline{p}}} N_{\underline{p}} \mid \underline{p} \in \operatorname{Supp} M\}$.

Now let $\underline{p}$ be arbitrary in $\operatorname{Supp} M$ and suppose $R/\underline{p}$-depth $M = t$. We shall produce a prime $\underline{q} \supset \underline{p}$ (hence in $\operatorname{Supp} M$) such that $\operatorname{depth}_{R_{\underline{q}}} N_{\underline{q}} = t$. Let $(x_1,\ldots,x_t)$ be a maximal N-regular sequence in $\underline{p}$. Then every element of $\underline{p}$ is a zero-divisor for $N/(x_1,\ldots,x_t)N$. Since the

set of zero-divisors for a module is the union of its associated primes, $\underline{p}$ is contained in this union for $N/(x_1,\ldots,x_t)N$. Since there are finitely many primes associated to a finitely generated module, and since an ideal contained in a finite union of primes is contained in one of them, we see that $\underline{p}$ is contained in some prime $\underline{q}$ associated to $N/(x_1,\ldots,x_t)N$. Now $\underline{q}R_{\underline{q}}$ is an associated prime of $N_{\underline{q}}/(x_1,\ldots,x_t)N_{\underline{q}}$ so the elements of $\underline{q}R_{\underline{q}}$ are all zero divisors for this module. Hence $(x_1,\ldots,x_t)$ is a maximal $N_{\underline{q}}$-regular sequence for $\underline{q}R_{\underline{q}}$, so depth $_{R_{\underline{q}}}N_{\underline{q}} = t$.

Thus, we have shown that for any prime $\underline{p} \in$ Supp M,

$R/\underline{p}$-depth N = depth $_{R_{\underline{q}}}N_{\underline{q}}$ for some $\underline{q}$ in Supp M. Therefore:

$$M\text{-depth } N \geq \text{Min } \{\text{depth}_{R_{\underline{p}}} N_{\underline{p}} \mid \underline{p} \in \text{Supp } M\}$$

Since we have already proved the reverse inequality, we are done.

(4.6) COROLLARY: If M is a non-zero R-module, then $\text{grade}_R M < \infty$ and

$$\text{grade}_R M = \text{Min } \{\text{depth } R_{\underline{p}} \mid \underline{p} \in \text{Supp } M\}$$

(recall that $\text{grade}_R M = M\text{-depth } R$).

Proof: By virtue of Lemma (4.5) we need only show that $\text{grade}_R M < \infty$. However, if $\underline{a}$ = annihilator of M, then $\underline{a}R = \underline{a} \neq R$ (since $M \neq 0$); therefore $\text{grade}_R M = M\text{-depth } R < \infty$ by (4.2).

§2: Gorenstein Dimension in Commutative Rings

We now turn to the study of R-modules of finite G-dimension. Our first goal will be to prove for such modules that (when R is local):

$$(\text{G-dim } M) + (\text{depth}_R M) = \text{depth}_R R$$

To do this, we must first see how G-dimension behaves with respect to regular sequences.

The following is a standard result:

(4.7) LEMMA: If $x \in R$ is both M and R regular, then

(a) $\text{Tor}_i^R(M, R/(x)) = 0$ for all $i > 0$.

(b) $\text{Ext}_R^i(M, R/(x)) \approx \text{Ext}_{R/(x)}^i(M/(x)M, R/(x))$ for all $i \geq 0$.

Proof: Part (a) is obtained by tensoring the exact sequence: $0 \to R \xrightarrow{x} R \to R/(x) \to 0$ by M. Part (b) follows from part (a) via a standard result on change of rings [4, p. 118, Prop. 4.1.3].

Since we shall often be working with quotients and localizations of R, we shall use the following notation when confusion may arise. If S is any commutative ring, $\mathcal{M}(S)$ will denote the category of finitely generated S-modules, and G(S)-dim M will denote the Gorenstein dimension of the S-module M. We shall let $\mathcal{L}(S)$ denote the full subcategory of modules of zero G(S)-dimension. Finally, $D_S(\ )$ will denote the functor D defined on the category $\mathcal{M}(S)$.

(4.8) LEMMA: If x is R-regular, then $D_{R/(x)}(M/(x)M) \approx D_R(M)/(x)\, D_R(M)$, the isomorphism occurring in $\mathcal{M}(R/(x))$.

Proof: Let $P_1 \to P_0 \to M \to 0$ be exact with P_0, P_1 the usual projectives. Tensoring by $R/(x)$ gives the exact sequence:

$$P_1/(x)P_1 \to P_0/(x)P_0 \to M/(x)M \to 0$$

Furthermore, it is well-known that $P/(x)P$ is $R/(x)$-projective when P is R-projective. Therefore we can use this sequence to compute $D_{R/(x)}(M/(x)M)$:

$$D_{R/(x)}(M/(x)M) \approx \text{coker}(\text{Hom}_{R/(x)}(P_0/(x)P_0, R/(x)) \to \text{Hom}_{R/(x)}(P_1/(x)P_1, R/(x))$$

Since each P_i is projective and x is R-regular, we see that x is P_i-regular also. From Lemma (4.7) we conclude that

$\text{Hom}_{R/(x)}(P_i/(x)P_i, R/(x)) \approx \text{Hom}_R(P_i, R/(x))$ for $i = 0, 1$. Therefore:

$$D_{R/(x)}(M/(x)M) \approx \text{coker}(\text{Hom}_R(P_0, R/x) \to \text{Hom}_R(P_1, R/(x)))$$

But now we have the exact and commutative diagram:

$$\begin{array}{ccccccc}
 & 0 & & 0 & & & \\
 & \downarrow & & \downarrow & & & \\
\mathrm{Hom}_R(P_0,R) & \longrightarrow & \mathrm{Hom}_R(P_1,R) & \longrightarrow & D_R(M) & \longrightarrow & 0 \\
\downarrow x & & \downarrow x & & \downarrow x & & \\
\mathrm{Hom}_R(P_0,R) & \longrightarrow & \mathrm{Hom}_R(P_1,R) & \longrightarrow & D_R(M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Hom}_R(P_0,R/(x)) & \longrightarrow & \mathrm{Hom}_R(P_1,R/(x)) & \longrightarrow & D_R(M)/(x)D_R(M) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Therefore coker $(\mathrm{Hom}_R(P_0,R/(x)) \to \mathrm{Hom}_R(P_1,R/(x))) \approx D_R(M)/(x)D_R(M)$ hence $\underline{D_{R/(x)}(M/(x)M)} \approx \underline{D_R(M)/(x)D_R(M)}$, concluding the proof.

(4.9) LEMMA: The full subcategory $\mathcal{L}(R)$ of R-modules having zero G-dimension has the following properties:

(a) L is in $\mathcal{L}(R)$ if and only if $D_R(L)$ is in $\mathcal{L}(R)$.

(b) If L is in $\mathcal{L}(R)$ then L^* is in $\mathcal{L}(R)$.

(c) If $(x_1,\ldots,x_t)$ is an R-regular sequence and L is in $\mathcal{L}(R)$, then $(x_1,\ldots,x_t)$ is L-regular and $L/(x_1,\ldots,x_t)L$ is in $\mathcal{L}(R/(x_1,\ldots,x_t))$.

Proof: To prove (a) we recall that G-dim $L = 0$ if and only if $\mathrm{Ext}^i(L,R) = \mathrm{Ext}^i(D(L),R) = 0$ for all $i > 0$. Since $\underline{D(D(L))} \approx \underline{L}$ we see that $\mathrm{Ext}^i(D(D(L)),R) \approx \mathrm{Ext}^i(L,R)$ for $i > 0$. Therefore G-dim $L = 0$ if and only if $\mathrm{Ext}^i(D(D(L)),R) = \mathrm{Ext}^i(D(L),R) = 0$ for all $i > 0$; that is, if and only if G-dim $D(L) = 0$. This proves (a).

Now suppose L is in $\mathcal{L}(R)$. From the usual presentation: $P_1 \to P_0 \to L \to 0$ we get the exact sequence $0 \to L^* \to P_0^* \to P_1^* \to D(L) \to 0$. By part (a) we have that $D(L)$ is in $\mathcal{L}(R)$. Since P_0 and P_1 are finitely generated projectives, so are P_0^* and P_1^*, hence they too are in $\mathcal{L}(R)$. Now when $0 \to A \to B \to C \to 0$ is exact with C in $\mathcal{L}(R)$, then A is in $\mathcal{L}(R)$ if and only if B is in $\mathcal{L}(R)$--this by Lemma (3.10). Applying this twice to $0 \to L^* \to P_0^* \to P_1^* \to D(L) \to 0$ we conclude that L^* is in $\mathcal{L}(R)$, proving (b).

We now prove (c) by induction on t. When $t = 1$ let $x = x_1$. Then x is R-regular. Since G(R)-dim $L = 0$, we know that L is 1-torsion free (since $\mathrm{Ext}^1(D(L),R) = 0$), hence a submodule of a free module. It follows that x is L-regular. Now the exact sequence $0 \to R \xrightarrow{x} R \to R/(x) \to 0$ gives rise to exact sequences $\mathrm{Ext}^i_R(L,R) \to \mathrm{Ext}^i_R(L,R/(x)) \to \mathrm{Ext}^{i+1}_R(L,R)$. Since G-dim $L = 0$, we have $\mathrm{Ext}^i_R(L,R) = 0$ for all $i > 0$. Therefore $\mathrm{Ext}^i_R(L,R/(x)) = 0$ for all $i > 0$. Now x is both R and L-regular, so Lemma (4.7) tells us that $\mathrm{Ext}^i_R(L,R/(x) \approx \mathrm{Ext}^i_{R/(x)}(L/(x)L,R/(x))$, so $\mathrm{Ext}^i_{R/(x)}(L/(x)L,R/(x)) = 0$ for all $i > 0$. By part (a), already proved, $D_R(L)$ is in $\mathcal{L}(R)$, so the same reasoning just used gives $\mathrm{Ext}^i_{R/(x)}(D_R(L)/(x)D_R(L),R/(x)) = 0$ for all $i > 0$. But $\underline{D_R(L)/(x)D_R(L)} \approx \underline{D_{R/(x)}(L/(x)L)}$ by Lemma (4.8). We therefore have: $0 = \mathrm{Ext}^i_{R/(x)}(D_{R/(x)}(L/(x)L),R/(x)) = \mathrm{Ext}^i_{R/(x)}(L/(x)L,R/(x))$ $i > 0$. We conclude that $L/(x)$ is in $\mathcal{L}(R/x)$, completing the case $t = 1$.

Now suppose that $t > 1$ and (c) holds for all positive integers less than t. Then $L/(x_1,\ldots,x_{t-1})L$ is in $G(R/(x_1,\ldots,x_{t-1})$. Since x_t is $R/(x_1,\ldots,x_{t-1})$-regular by assumption, the case $t = 1$ already proved shows that x_t is $(L/(x_1,\ldots,x_{t-1})L)$ regular; therefore $(x_1,\ldots,x_t)$ is an L-regular sequence. Also, by the case $t = 1$, we conclude that $L/(x_1,\ldots,x_{t-1},x_t)L$ is in $\mathcal{L}(R/(x_1,\ldots,x_t))$. This completes the induction and the proof.

(4.10) LEMMA: If G(R)-dim $M < \infty$ and x is both R and M regular, then

$G(R/(x))$-dim $M/(x)$ $M < \infty$.

Proof: When $G(R)$-dim $M = 0$ we have $G(R/(x))$-dim $M/(x)M = 0$ by the previous Lemma. We proceed by induction on $G(R)$-dim M, assuming the statement true for $0 \leq$ G-dim $M < n$. When G-dim $M = n$, choose an exact sequence $0 \to K \to P \to M \to 0$ with P projective. By Corollary (3.15) we must have $G(R)$-dim $K = n - 1 < n$. By induction hypothesis we have $G(R/(x))$-dim $K/(x)K = t < \infty$. This means that we have an exact sequence $0 \to X_t \to \dots \to X_0 \to K/(x)K \to 0$ where $G(R/(x))$-dim $X_i = 0$ for $0 \leq i \leq t$.

Since x is R and M-regular, $\mathrm{Tor}_1(M,R/(x)) = 0$ by Lemma (4.7); therefore $0 \to K/(x)K \to P/(x)P \to M/(x)M \to 0$ is exact. Since P is R-projective, $P/(x)P$ is $R/(x)$-projective, hence has $G(R/(x))$-dimension zero. Splicing, we have an exact sequence:

$$0 \to X_t \to \dots \to X_0 \to P/(x)P \to M/(x)M \to 0$$

where all modules except $M/(x)M$ have $G(R/(x))$-dimension equal to zero. Therefore $G(R/(x))$-dim $M/(x)M \leq t + 1 < \infty$ and we are done.

When R is a ring which has no R-regular sequences (e.g. R is local of dimension zero) then we can give a fairly simple criterion for a module to have zero (or finite) G-dimension. We observe first that if $E^* = 0$ for an R-module E, then $\mathrm{Hom}_R(E,R) = 0$ so grade $E > 0$. Since grade E for $E \neq 0$ is always finite (by (4.6)) we see that the annihilator of E contains an R-regular sequence of length at least 1. This being impossible by assumption on R, we must have $E = 0$. Thus, for a ring of the type under consideration, $E^* = 0$ implies $E = 0$. We now can prove:

(4.11) PROPOSITION: Suppose every non-zero-divisor of R is a unit. For an R-module M the following statements are then equivalent:

(a) G-dim $M = 0$

(b) G-dim $M < \infty$

(c) There exists an exact sequence:

$$\dots \to F_i \to \dots \to F_1 \to F_0 \to F_{-1} \to F_{-2} \to \dots \to F_{-i} \to \dots$$

of free modules such that $M = \mathrm{coker}(F_1 \to F_0)$ and $M_t = \mathrm{coker}(F_{t+1} \to F_t)$ is reflexive for each $-\infty < t < \infty$.

Proof: We show the equivalence of (a) and (b) first. Since (a) clearly implies (b), suppose that (b) holds, so that G-dim $M \leq n < \infty$ for some positive integer n. We shall prove (a) by induction on n. Suppose first that G-dim $M = n \leq 1$. Then we have an exact sequence $0 \to L_1 \to L_0 \to M \to 0$ where G-dim $L_i = 0$ for $i = 0,1$. Dualizing this sequence, we obtain the exact sequence: $0 \to M^* \to L_0^* \to L_1^* \to \mathrm{Ext}^1(M,R) \to 0$. Dualizing again, we see that $0 \to (\mathrm{Ext}^1(M,R))^* \to L_1^{**} \to L_0^{**}$ is exact. Since modules of finite G-dimension are reflexive (see Proposition (3.8)), we have $L_i^{**} \approx L_i$ for $i = 0,1$. Therefore $(\mathrm{Ext}^1(M,R))^* = 0$. By remarks made above, we conclude that $\mathrm{Ext}^1(M,R) = 0$. Now G-dim $M = \max\{t \mid \mathrm{Ext}^t(M,R) \neq 0\} \leq 1$ (see remarks after (3.7)). We conclude that G-dim $M = 0$. Now suppose that G-dim $M \leq n - 1$ implies G-dim $M = 0$. If G-dim $M \leq n$, consider the exact sequence $0 \to K \to P \to M \to 0$ where G-dim $P = 0$ (say P is projective). Then G-dim $K \leq n - 1$ so G-dim $K = 0$ by induction hypothesis. We then have G-dim $M \leq 1$, so G-dim $M = 0$ by the case $n = 1$ already proved.

Thus, G-dim $M < \infty$ if and only if G-dim $M = 0$.

We now show that (a) and (c) are equivalent. Suppose first that G-dim $M = 0$. Let $\dots \to F_i \to \dots \to F_1 \to F_0 \to M \to 0$ be a free resolution of M. When $M_t = \mathrm{coker}(F_{t+1} \to F_t)$ we have exact sequences $0 \to M_{t+1} \to F_t \to M_t \to 0$. If G-dim $M_t = 0$ then G-dim $M_{t+1} = 0$ since F_t (being free) has G-dimension zero (see (3.11)). However, $M_0 = M$ has

G-dimension zero; therefore G-dim $M_t = 0$ for all $t \geq 0$, so M_t is reflexive for all $t \geq 0$. Now since G-dim $M = 0$, we have that M is k torsion free for all k. By Theorem (2.17) we can find an exact sequence $0 \to M \to F_{-1} \to F_{-2} \to \dots \to F_{-i} \to \dots$ where the $F_t (t < 0)$ are free modules. If $M_t = \operatorname{coker}(F_{t+1} \to F_t)$ then we have an exact sequence:

$$0 \to M \to F_{-1} \to \dots \to F_{t+1} \to F_t \to M_t \to 0.$$

Since G-dim $M = 0$, we have G-dim $M_t \leq t < \infty$. Therefore G-dim $M_t = 0$, so M_t is reflexive. Thus, the sequence

$$\dots \to F_i \to \dots \to F_1 \to F_0 \to F_{-1} \to \dots \to F_{-i} \to \dots$$

obtained by splicing satisfies the requirements of (c).

Finally, suppose (c) holds. From the exact sequence $0 \to M_{t+1} \to F_t \to M_t \to 0$ we get the exact sequence:

$$0 \to M_t^* \to F_t^* \to M_{t+1}^* \to \operatorname{Ext}^1(M_t,R) \to 0.$$

This gives the exactness of $0 \to (\operatorname{Ext}^1(M_t,R))^* \to M_{t+1}^{**} \to F_t^{**}$. Since F_t is free, F_t is reflexive, while M_{t+1} is reflexive by assumption. Therefor $(\operatorname{Ext}^1(M_t,R))^* = 0$, so $\operatorname{Ext}^1(M_t,R) = 0$ (this for each $-\infty < t < \infty$). Furthermore, $\underline{M_{t+s}} \approx \underline{\Omega^s M_t}$ so $\operatorname{Ext}^i(M_t,R) \approx \operatorname{Ext}^1(M_{t+i-1},R) = 0$. Thus $\operatorname{Ext}^i(M_t,R) = 0$ for each t and all $i > 0$; in particular, this holds for $M_0 = M$. Now suppose that $k > 0$ is an arbitrary integer. We have an exact sequence $0 \to M \to F_{-1} \to \dots \to F_{-k} \to M_{-k} \to 0$. Since $\operatorname{Ext}^i(M_{-k},R) = 0$ for all $i > 0$, we see that $F_{-k}^* \to F_{-(k-1)}^* \to \dots F_{-1}^* \to M^* \to 0$ is exact. The existence of a sequence $0 \to M \to F_{-1} \to \dots \to F_{-k}$ with this property (and the F_t free) is equivalent, by Theorem (2.17), to M being k-torsion free. Since k was arbitrary, we have that M is k-torsion free for all k; that is, $\operatorname{Ext}^i(D(M),R) = 0$ for all $i > 0$. Since $\operatorname{Ext}^i(M,R) = 0$ for all $i > 0$ as

well, we have G-dim $M = 0$, concluding the proof.

We now turn our attention to local rings. When R is local with maximal ideal $\underline{m}$, and M is an R-module, let us recall that depth M means $R/\underline{m}$-depth M. This is the length of a maximal M-regular sequence in $\underline{m}$, and is always finite when $M \neq 0$ (since $\underline{m}M \neq M$ by Nakayama's Lemma).

(4.12) PROPOSITION: Let $(R,\underline{m})$ be a local ring and $M \neq 0$ an R-module of finite G-dimension. The following statements are equivalent:

(a) G-dim $M = 0$

(b) $\operatorname{Tor}_i^R(M,A) = 0$ for all $i > 0$ and every R-module A such that pd $A < \infty$.

(c) Every R-regular sequence in $\underline{m}$ is M-regular

(d) depth $M \geq$ depth R

(e) depth M = depth R

(f) $\operatorname{Ext}^i(M,R) = 0$ for all $i > 0$.

(g) $\operatorname{Ext}^i(M,A) = 0$ for all $i > 0$ and every R-module A such that pd $A < \infty$.

Proof: Assume G-dim $M = 0$; we shall prove (b). Suppose pd $A = n$. Since G-dim $M = 0$, we have that M is n-torsion free (for all n), so M is an n-syzygy; i.e. $\underline{M} \approx \underline{\Omega^n K}$ for some K (see Theorem (2.17)). Since Tor_i vanishes on projectives for $i > 0$ we have: $\operatorname{Tor}_i(M,A) \approx \operatorname{Tor}_i(\Omega^n K,A) \approx \operatorname{Tor}_{i+n}(K,A) = 0$ (since $i > 0$ and hence $n + i > n =$ pd A).

Now suppose (b) holds, and let $(x_1,\dots,x_t)$ be an R-regular sequence in $\underline{m}$. The Koszul complex $R_{x:1,\dots,t}$ provides a finite projective resolution for $R/(x_1,\dots,x_t)R$. If we tensor this resolution by M, we obtain the Koszul complex $M_{x:1,\dots,t}$ whose homology modules are clearly the $\operatorname{Tor}_i(M,R/(x_1,\dots,x_t)R)$. Since pd $R/(x_1,\dots,x_t)R < \infty$ these are all zero by assumption. Thus, $M_{x:1,\dots,t}$ is acyclic. It is a

standard result that this acyclicity is equivalent to $(x_1,\dots,x_t)$ being M-regular. Thus, (b) implies (c).

It is clear from the definitions that (c) implies (d).

We now deduce (e) from (d) by induction on depth R. Suppose depth R = 0. Then every element of $\underline{m}$ is a zero divisor so $\underline{m}$ is contained in the union of the associated primes of R. Since $\underline{m}$ is contained in a finite union of primes, and is maximal, $\underline{m}$ itself is associated. We therefore have an injection: $0 \to R/\underline{m} \to R$. Applying the functor $(M^*, \ )$ we obtain an exact sequence: $0 \to (M^*, R/\underline{m}) \to M^{**}$. Since depth R = 0 and G-dim $M < \infty$, Proposition (4.11) gives G-dim M = 0. Therefore $M \approx M^{**}$. Now the module $(M^*, R/\underline{m})$ cannot be zero. If it were, then the annihilator of M^* would have to contain an $R/\underline{m}$-regular element which would then be a unit. Consequently we would have $M^* = 0$ which gives $M \approx M^{**} = 0$, contrary to assumption. Thus, $M \approx M^{**}$ contains the non-zero submodule $(M^*, R/\underline{m})$ which is clearly annihilated by $\underline{m}$. It follows that $\underline{m}$ cannot contain an M-regular element, so depth M = depth R = 0.

Now suppose depth $R \geq 1$ and that the implication holds for rings of smaller depth. By assumption, depth $M \geq$ depth $R \geq 1$. We can find an element in $\underline{m}$ which is both M and R regular. If not, $\underline{m}$ would be contained in the union of the associated primes of M and of R. In this case, $\underline{m}$ would be associated to either M or R which is not true since M and R each have depth ≥ 1. Therefore let $x \in \underline{m}$ be M and R-regular. Since G(R)-dim $M < \infty$, Lemma (4.10) gives G(R/(x))-dim $M/(x)M < \infty$. By induction assumption, depth $M/(x)M$ = depth $R/(x)$ = (depth R) - 1. Since depth $M/(x)M$ = (depth M) - 1, we conclude that depth M = depth R, completing the induction.

Now suppose (e) holds. We prove (f) by induction, again on depth R. When

depth R = 0, Lemma (4.11) gives G-dim M = 0, so $\operatorname{Ext}^i(M,R) = 0$ for all $i > 0$. Thus, (f) follows in this case. Now suppose depth $R \geq 1$ and the implication holds for rings of smaller depth. As above, we can find an x which is both R and M regular. Since depth M = depth R we have depth $M/(x)M$ = depth $R/(x)$ = (depth R) - 1. Since G(R/(x))-dim $M/(x)M < \infty$, our induction hypothesis gives $\operatorname{Ext}^i_{R/(x)}(M/(x)M, R/(x)) = 0$ for all $i > 0$. Since x is M and R-regular we also have, from Lemma (4.7), that

$$\operatorname{Ext}^i_R(M,R/(x)) \approx \operatorname{Ext}^i_{R/(x)}(M/(x)M,R/(x)) = 0$$

for all $i > 0$. Now the exact sequence

$$0 \to R \xrightarrow{x} R \to R/(x) \to 0$$

yields the exact sequence: $\operatorname{Ext}^i_R(M,R) \xrightarrow{x} \operatorname{Ext}^i_R(M,R) \to \operatorname{Ext}^i_R(M,R/(x))$ whose last term is zero for $i > 0$. We conclude that $\operatorname{Ext}^i_R(M,R) = (x)\operatorname{Ext}^i(M,R)$. By Nakayama's Lemma we obtain our result: $\operatorname{Ext}^i(M,R) = 0$ for $i > 0$.

Now suppose (f) holds, so that $\operatorname{Ext}^i(M,R) = 0$ for all $i > 0$. We proceed by induction on pd A. Since $\operatorname{Ext}^i(M, \ )$ is projectively stable for $i > 0$, $\operatorname{Ext}^i(M,A) = 0$ when pd A = 0 and $i > 0$. Now suppose pd $A > 0$ and (f) holds for modules of smaller projective dimension. Let $0 \to K \to P \to A \to 0$ be exact with P projective. Then pd K = (pd A) - 1 so $\operatorname{Ext}^i(M,K) = \operatorname{Ext}^i(M,P) = 0$ for all $i > 0$ by induction hypothesis. The exact sequence $\operatorname{Ext}^i(M,P) \to \operatorname{Ext}^i(M,A) \to \operatorname{Ext}^{i+1}(M,K)$ gives $\operatorname{Ext}^i(M,A) = 0$ for all $i > 0$ completing the induction. Thus, (f) implies (g).

Finally, when (g) holds we have $\operatorname{Ext}^i(M,R) = 0$ for all $i > 0$. Since G-dim $M < \infty$ we have

$$\text{G-dim } M = \max\{t \mid \operatorname{Ext}^t(M,R) \neq 0\} = 0;$$

thus, (g) implies (a). This completes our proof.

We now come to the main result of this section, relating depth and G-dimension. Note that part (b) generalizes the well-known result for

modules of finite projective dimension.

(4.13) THEOREM: Let $(R, \underline{m})$ be a local ring and M an R-module with G-dim $M < \infty$.

(a) The following numbers are equal:

(i) G-dim M

(ii) The smallest integer $t \geq 0$ such that $\operatorname{Tor}_i^R(M,A) = 0$ for all $i > t$ and all modules A with pd $A < \infty$.

(iii) The smallest integer $t \geq 0$ such that $\operatorname{Ext}^i_R(M,A) = 0$ for all $i > t$ and all modules A with pd $A < \infty$.

(iv) The smallest integer $t \geq 0$ such that $\operatorname{Ext}^i_R(M,R) = 0$ for all $i > t$.

(b) (G-dim M) + (depth M) = depth R.

Proof: We shall prove only that the integer described in (ii) is equal to G-dim M. The remaining equalities of part (a) are proved in an analogous manner, using Proposition (4.12), and the details will be left to the reader.

Suppose then that k is the smallest non-negative integer such that $\operatorname{Tor}_i^R(M,A) = 0$ for all $i > k$ and all A of finite projective dimension. Consider $\Omega^k M$. Since G-dim $M < \infty$ we have G-dim $\Omega^k M \leq$ G-dim $M < \infty$. If pd $A < \infty$ then $\operatorname{Tor}_j^R(\Omega^k M, A) \approx \operatorname{Tor}^R_{k+j}(M,A) = 0$ when $j > 0$ (so $j + k > k$). By Proposition (4.12) we conclude that G-dim $\Omega^k M = 0$; therefore G-dim $M \leq k$. On the other hand, if G-dim $M = n < k$, then G-dim $\Omega^n M = 0$. If $i > n$ we have $\operatorname{Tor}_i^R(M,A) \approx \operatorname{Tor}^R_{i-n}(\Omega^n M, A) = 0$ by Proposition (4.12) again. This, however, contradicts the minimality of k, so we must have G-dim $M = k$.

As was mentioned, the other parts of (a) are proved similarly, so we turn our attention to part (b). We shall proceed by induction on G-dim M. When G-dim $M = 0$ we have already proved in Proposition (4.12) that depth M = depth R. Therefore suppose G-dim $M = n > 0$ and the assertion holds

for modules of smaller G-dimension. Consider the exact sequence

$$0 \to K \to P \to M \to 0$$

where P is projective. It is well-known and easily proved that depth K = (depth M) + 1. Furthermore, G-dim $K = n - 1$, so our induction hypothesis gives $n-1$ = G-dim K = (depth R)-(depth K) = (depth R)-(depth M)-1. We therefore obtain (since n = G-dim M): (depth M) + (G-dim M) = depth R. This completes our induction and the proof of the theorem.

§3: Localization

We now turn to the behavior of G-dimension under localization. We first recall that when S is a multiplicative subset of a commutative ring R, and the R-module M is finitely presented, then $S^{-1}(\operatorname{Ext}^i_R(M,N)) \approx \operatorname{Ext}^i_{S^{-1}R}(S^{-1}M, S^{-1}N)$. Since all rings under consideration are noetherian and all modules finitely generated, we have, in particular, that $(\operatorname{Ext}^i_R(M,R))_{\underline{p}} \approx \operatorname{Ext}^i_{R_{\underline{p}}}(M_{\underline{p}}, R_{\underline{p}})$ where $\underline{p}$ is a prime ideal. Furthermore, it is easily seen that $(D_R(M))_{\underline{p}} \approx D_{R_{\underline{p}}}(M_{\underline{p}})$, the isomorphism taking place in $\mathcal{M}(R_{\underline{p}})$. Now G(R)-dim $M = 0$ if and only if $\operatorname{Ext}^i(M,R) = \operatorname{Ext}^i(D(M),R) = 0$ for all $i > 0$. Since a module is zero if and only if each of its localizations at prime (resp. maximal) ideals is zero, we have the following:

(4.14) LEMMA: G(R)-dim $M = 0$ if and only if $G(R_{\underline{p}})$-dim $M_{\underline{p}} = 0$ for each prime (resp. maximal) ideal $\underline{p}$.

Since localization preserves projectives and exactness, we see that $(\Omega^n M)_{\underline{p}} \approx (\Omega^n M_{\underline{p}})$. Therefore, applying Lemma (4.14) to $\Omega^n M$ gives the following.

(4.15) COROLLARY: G(R)-dim $M \leq n$ if and only if $G(R_{\underline{p}})$-dim $M_{\underline{p}} \leq n$ for each prime (resp. maximal) ideal $\underline{p}$.

As we have seen in the preceding Chapter, the assumption grade $\mathrm{Ext}^i(M,R) \geq i$ (or $i-1$) is very intimately related to questions of torsion freeness and the existence of certain modules of finite projective dimension. In the case of modules of finite G-dimension we have a very satisfactory situation.

(4.16) PROPOSITION: Suppose $G(R_{\underline{p}})\text{-dim } M_{\underline{p}} < \infty$ for each prime ideal $\underline{p}$ of R. Then grade $\mathrm{Ext}^i(M,R) \geq i$ for all $i > 0$.

Proof: As we have seen in Corollary (4.6):

$$\text{grade } E = \min \{\text{depth } R_{\underline{p}} \mid \underline{p} \in \text{Supp } E\} .$$

Suppose $\underline{p} \in \text{Supp } \mathrm{Ext}^i(M,R)$ for some $i > 0$. We must show that depth $R_{\underline{p}} \geq i$. Since $(\mathrm{Ext}^i_R(M,R))_{\underline{p}} \neq 0$, we have $\mathrm{Ext}^i_{R_{\underline{p}}}(M_{\underline{p}}, R_{\underline{p}}) \neq 0$. Therefore $G(R_{\underline{p}})\text{-dim } M_{\underline{p}} \geq i$. By Theorem (4.13) we have: depth $R_{\underline{p}} = (\text{depth } M_{\underline{p}}) + (G(R_{\underline{p}})\text{-dim } M_{\underline{p}}) \geq G(R_{\underline{p}})\text{-dim } M \geq i$. Therefore depth $R_{\underline{p}} \geq i$ and we are done.

(4.17) COROLLARY: If M has finite projective (or Gorenstein) dimension, then grade $\mathrm{Ext}^i(M,R) \geq i$ for all $i > 0$.

Recall that in Corollary (2.27) it was shown that if R is such that grade $\mathrm{Ext}^i(A,R) \geq i$ for $1 \leq i \leq k$ and for each module A of finite projective dimension, then for any R-module M, $\Omega^k M$ being k-torsion free implies $\Omega^i M$ is i-torsion free, $1 \leq i \leq k$. Consequently we have:

(4.18) COROLLARY: If $\Omega^k M$ is k-torsion free then $\Omega^i M$ is i-torsion free for all $1 \leq i \leq k$ (R being assumed commutative).

Now let us go back to Theorem (2.41), the Approximation Theroem. This result states that if grade $\mathrm{Ext}^i(M,R) \geq i$ for $1 \leq i \leq k$ then there is a module $H_k(M)$ with pd $H_k(M) \leq k$, and a map $\underline{M} \to \underline{H_k(M)}$ which is universal

with respect to maps of $\underline{M}$ into modules of projective dimension $\leq k$. Let us then define the following full subcategories of $\mathcal{M}(R)$:

$\mathcal{J}_k(R)$ = category of modules M with grade $\mathrm{Ext}^i(M,R) \geq i$ for $1 \leq i \leq k$.

$\mathcal{L}_k(R)$ = category of modules M with pd $M \leq k$.

Denote by $\underline{\mathcal{J}_k(R)}$ and $\underline{\mathcal{L}_k(R)}$ the corresponding full subcategories of $\underline{\mathcal{M}(R)}$. By virtue of Corollary (4.17) we have inclusion functors:

$$\mathcal{L}_k(R) \to \mathcal{J}_k(R) \quad \text{and} \quad \underline{\mathcal{L}_k(R)} \to \underline{\mathcal{J}_k(R)}$$

We now make H_k into a functor $\underline{\mathcal{J}_k(R)} \to \underline{\mathcal{L}_k(R)}$. Clearly $\underline{H_k(M)}$ is in $\mathcal{L}_k(R)$. Suppose $\underline{f}: \underline{M} \to \underline{N}$ is a morphism in $\underline{\mathcal{J}_k(R)}$. Then the universality of $\underline{M} \to \underline{H_k(M)}$ gives a unique morphism $H_k(\underline{f}): \underline{H_k(M)} \to \underline{H_k(N)}$ such that the following commutes in $\underline{\mathcal{M}(R)}$:

$$\begin{array}{ccc} \underline{M} & \longrightarrow & \underline{H_k(M)} \\ \underline{f}\downarrow & & \downarrow H_k(\underline{f}) \\ \underline{N} & \longrightarrow & \underline{H_k(N)} \end{array}$$

Thus, H_k defines a functor $\mathcal{J}_k(R) \to \mathcal{L}_k(R)$. Furthermore, if $\underline{M}$ is in $\underline{\mathcal{J}_k(R)}$ and $\underline{A}$ is in $\mathcal{L}_k(R)$ we have an isomorphism:

$$(\underline{M},\underline{A}) \approx (\underline{H_k(M)},\underline{A})$$

(since $\underline{M} \to \underline{A}$ factors uniquely through $\underline{H_k(M)}$ by the universality of $\underline{M} \to \underline{H_k(M)}$). This may be stated in the following way.

(4.19) PROPOSITION: $H_k: \underline{\mathcal{J}_k(R)} \to \underline{\mathcal{L}_k(R)}$ is a left adjoint to the inclusion functor $\underline{\mathcal{L}_k(R)} \longrightarrow \underline{\mathcal{J}_k(R)}$.

§4: Gorenstein Rings

Let us recall that a local ring R is called a Gorenstein ring when $\text{inj. dim}_R R < \infty$. An arbitrary commutative ring R is called a Gorenstein

ring when $R_{\underline{p}}$ is a Gorenstein ring for each prime ideal $\underline{p}$. We now justify some of our terminology by proving the following result.

(4.20) THEOREM: The local ring R is a Gorenstein ring if and only if every finitely generated R-module has finite Gorenstein dimension.

Proof: First let us suppose R is Gorenstein, with inj. $\dim_R R = n < \infty$. If M is finitely generated, we show that G-dim $\leq n$. Consider $\Omega^n M$. Since inj. $\dim_R R = n$ we have $\operatorname{Ext}^i(\Omega^n M, R) \approx \operatorname{Ext}^{n+i}(M,R) = 0$ for all $i > 0$. Now consider an exact sequence $P_{n+1} \to P_n \to \dots \to P_1 \to P_0 \to \Omega^n M \to 0$ where the P_i are finitely generated projectives. Since $\operatorname{Ext}^i(\Omega^n M, R) = 0$ for all $i > 0$ we have, upon dualizing, the following exact sequence:

$0 \to (\Omega^n M)^* \to P_0^* \to P_1^* \to \dots \to P_n^* \to P_{n+1}^*$. Let $C = \operatorname{coker}(P_n^* \to P_{n+1}^*)$. Since $D(\Omega^n M) \approx \operatorname{coker}(P_0^* \to P_1^*)$ (at least up to projective equivalence) we have an exact sequence

$$0 \to D(\Omega^n M) \to P_2^* \to \dots \to P_n^* \to P_{n+1}^* \to C \to 0.$$

Therefore $\underline{D(\Omega^n M)} \approx \underline{\Omega^n C}$ and we have

$$\operatorname{Ext}^i(D(\Omega^n M), R) \to \operatorname{Ext}^{n+i}(C,R) = 0$$

for all $i > 0$ (by our assumption on R). We conclude that G-dim $\Omega^n M = 0$ so G-dim $M \leq n < \infty$.

Conversely, suppose G-dim $M < \infty$ for every finitely generated R-module M. Let $n = \operatorname{depth} R < \infty$ (since R is local). Then no proper ideal of R contains an R-regular sequence of length $> n$. Since G-dim $M < \infty$ we have grade $\operatorname{Ext}^i(M,R) \geq i$ for all $i > 0$ by Proposition (4.16). When $i > n$ this would mean that the annihilator of $\operatorname{Ext}^i(M,R)$ contains an R-regular sequence of length $i > n$. This being impossible we must have $\operatorname{Ext}^i(M,R) = 0$ for $i > n$ and every finitely generated R-module M. Since injective dimension can be tested by cyclic (hence finitely generated) modules, we

conclude that inj. $\dim_R R \leq n < \infty$ and our proof is complete.

Remark: When inj. $\dim_R R < \infty$ then inj. $\dim_R R = \operatorname{depth} R$. We have already proved inj. dim. $R \leq \operatorname{depth} R$ (see last line of above proof). If $(x_1, \dots, x_n)$ is a maximal R-regular sequence (so $n = \operatorname{depth} R$) then $R/(x_1, \dots, x_n)$R-depth $R = n$. Therefore $\operatorname{Ext}^n(R/(x_1, \dots, x_n)R, R) \neq 0$ and we get our equality. For an alternate proof of this fact, see [3].

We now record a series of conditions relating k-syzygies, and k-torsion free modules.

(4.21) PROPOSITION: For R commutative and $k \geq 1$ an integer, the following statements are equivalent.

(a) $\Omega^{k+1} M$ is $(k+1)$-torsion free for all R-modules M.

(b) $\Omega^j M$ is j-torsion free for all M and $1 \leq j \leq k+1$

(c) For a prime ideal $\underline{p}$, depth $R_{\underline{p}} < k$ implies $R_{\underline{p}}$ is a Gorenstein ring.

(d) grade $\operatorname{Ext}^i(M,R) \geq i$ for all M and $1 \leq i \leq k$.

Proof: We have seen in Corollary (4.18) that (a) and (b) are equivalent for commutative rings. Now suppose that (b) holds. In Proposition (2.26) it was shown that (b) is equivalent to the condition:

grade $\operatorname{Ext}^i(M,R) \geq i - 1$ for $1 \leq i \leq k+1$ and all M. Therefore for any $\underline{p}$ we have either $(\operatorname{Ext}^{k+1}(M,R))_{\underline{p}} = 0$ or $\underline{p}$ lies in $\operatorname{Supp}(\operatorname{Ext}^{k+1}(M,R))$, in which case depth $R_{\underline{p}} \geq k$. If depth $R_{\underline{p}} < k$ we must therefore have $(\operatorname{Ext}_R^{k+1}(M,R))_{\underline{p}} \approx \operatorname{Ext}_{R_{\underline{p}}}^{k+1}(M_{\underline{p}}, R_{\underline{p}}) = 0$ for all M. Since every $R_{\underline{p}}$-module is of the form $M_{\underline{p}}$ for some R-module M, we conclude that inj. $\dim_{R_{\underline{p}}} R_{\underline{p}} \leq k$ so $R_{\underline{p}}$ is Gorenstein. Thus, (b) implies (c).

Now suppose (c) holds. Choose any prime $\underline{p} \in \operatorname{Supp}(\operatorname{Ext}^i M, R))$ where $1 \leq i \leq k$. To prove (d) we must show that depth $R_{\underline{p}} \geq i$. If depth $R_{\underline{p}} \geq k$

then we are done (since $k \geq i$). If depth $R_{\underline{p}} < k$ then $R_{\underline{p}}$ is a Gorenstein ring so grade$_{R_{\underline{p}}} \operatorname{Ext}^i_{R_{\underline{p}}}(M_{\underline{p}}, R_{\underline{p}}) \geq i$. Since $\operatorname{Ext}^i_{R_{\underline{p}}}(M_{\underline{p}}, R_{\underline{p}}) \neq 0$ by assumption, we must have depth $R_{\underline{p}} \geq i$ in this case as well.

To complete the proof, we show that (d) implies (b). Let M be any R-module. It suffices to show that grade $\operatorname{Ext}^{k+1}(M,R) \geq k$ since (d) gives grade $\operatorname{Ext}^{i+1}(M,R) \geq i$ for $0 \leq i < k$. Now $\operatorname{Ext}^{k+1}(M,R) \approx \operatorname{Ext}^k(\Omega^1 M, R)$. By assumption (d), grade $\operatorname{Ext}^k(\Omega^1 M, R) \geq k$ so we are done.

(4.22) COROLLARY: The commutative ring R is a Gorenstein ring if and only if, for each integer $k \geq 1$, the following conditions are equivalent for any R-module M:

(i) M is k-torsion free.

(ii) M is k-reflexive.

(iii) M is a k-syzygy.

Proof: We always have that (i) implies (ii) and (ii) implies (iii). If (iii) implies (i) then $\Omega^k N$ is k-torsion free for all N and all $k \geq 1$. Therefore $R_{\underline{p}}$ is a Gorenstein ring for all $\underline{p}$ since (b) and (c) of the previous proposition are equivalent. Therefore R is a Gorenstein ring. Conversely when $R_{\underline{p}}$ is a Gorenstein ring for each $\underline{p}$, then the equivalence of (b) and (c) tells us that (iii) implies (i), so we are done.

§5: Application to Vector Bundles

Let R be a local ring with depth $R \geq 2$. Let $X = \operatorname{Spec} R$, $Y = \operatorname{Spec} R - \{\underline{m}\}$ where $\underline{m}$ is the maximal ideal of R. It is well-known that the category of K (= $R/\underline{m}$)-vector bundles on Y is isomorphic to the full subcategory of R-modules determined by those M satisfying:

(a) depth $M \geq 2$

(b) $M_{\underline{p}}$ is free for all $\underline{p} \neq \underline{m}$.

This isomorphism of categories is given by $\mathcal{E} \rightsquigarrow \Gamma(\mathcal{E})$ where $\Gamma(\mathcal{E})$ denotes the global sections of the bundle $i_*\mathcal{E}$ $i: Y \longrightarrow X$. That $\Gamma(\mathcal{E})$ is an R-module follows from the hypothesis depth $R \geq 2$. For details and proofs, see [7]. That depth $\Gamma(\mathcal{E}) \geq 2$ follows from the fact that $\Gamma(\mathcal{E})$ is reflexive (since $\mathcal{E}$ is) and the following general result:

(4.23) LEMMA: Suppose $M = \Omega^k N$. Then every R-regular sequence of length $\leq k$ is also M-regular.

Proof: Let $x_1, \ldots, x_t$ be R-regular with $t \leq k$. To prove this sequence is M-regular it suffices to prove that $\operatorname{Tor}_i(M, R/(x_1, \ldots, x_t)R) = 0$ for $1 \leq i \leq t$. However, we have

$\operatorname{Tor}_i(M, R/(x_1, \ldots, x_t)R) \approx \operatorname{Tor}_i(\Omega^k N, R/(x_1, \ldots, x_t)R)) \approx \operatorname{Tor}_{i+k}(N, R/(x_1, \ldots, x_t)R)$.

Since $t \leq k$ we see that (when $i > 0$) $i + k > t = \operatorname{pd} R/(x_1, \ldots, x_t)R$; therefore $\operatorname{Tor}_i(M, R/(x_1, \ldots, x_t)R) = 0$ for all $i > 0$.

Suppose now that R is local and has the following properties:

(i) depth $R = k \geq 3$

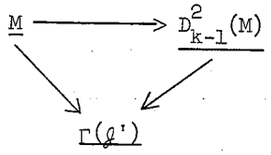
(ii) $R_{\underline{p}}$ is a regular local ring when $\underline{p} \neq \underline{m}$.

(Condition (ii) holds when R is the local ring at an isolated singularity).

If M is any R-module, consider the map: $\underline{M} \to \underline{D^2_{k-1}(M)}$.

Claim: (a) $D^2_{k-1}(M) = \Gamma(\mathcal{E})$ where $\mathcal{E}$ is a vector bundle on Y.

(b) Given $M \to (\mathcal{E}')$ where $\mathcal{E}'$ is a vector bundle on Y and $\Gamma(\mathcal{E}')$ is a (k-1)-syzygy, there is a unique map: $\underline{D^2_{k-1}(M)} \to \Gamma(\mathcal{E}')$ such that the following commutes:



Proof: To show (a) we must verify that depth $D^2_{k-1}(M) \geq 2$ and $(D^2_{k-1}(M))_{\underline{p}}$ is free for $\underline{p} \neq \underline{m}$. Since depth $R = k \geq 3$, we have $k - 1 \geq 2$. Therefore $D^2_{k-1}(M) = \Omega^{k-1}D\Omega^{k-1}D(M)$ is at least a 2-syzygy. By Lemma (4.23) we conclude that depth $D^2_{k-1}(M) \geq 2$. Furthermore, if $\underline{p} \neq \underline{m}$ then $R_{\underline{p}}$ is regular by assumption (ii), so dim $R_{\underline{p}}$ = depth $R_{\underline{p}} \leq k - 1$. Since $(D^2_{k-1}(M))_{\underline{p}}$ is a (k-1)-syzygy over a regular local ring of dimension less than k, we must have $(D^2_{k-1}(M))_{\underline{p}}$ projective--hence free. This proves (a).

To prove (b) we must construct a map: $\underline{D^2_{k-1}(M)} \to \underline{\Gamma(\mathcal{G}')}$. We are given a map $\underline{M} \to \Gamma(\mathcal{G}')$ so we may apply D^2_{k-1}, obtaining: $\underline{D^2_{k-1}(M)} \to \underline{D^2_{k-1}(\Gamma(\mathcal{G}'))}$. We shall prove that $\Gamma(\mathcal{G}')$ is (k-1)-reflexive, so that $\underline{D^2_{k-1}(\Gamma(\mathcal{G}'))} \approx \underline{\Gamma(\mathcal{G}')}$. Here we shall use the fact that $\Gamma(\mathcal{G}')$ was assumed to be a (k-1)-syzygy. Now $R_{\underline{p}}$ is a regular local ring, hence a Gorenstein ring, for $\underline{p} \neq \underline{m}$. Thus, if depth $R_{\underline{p}} < k$ then $R_{\underline{p}}$ is Gorenstein. As we have seen in Proposition (4.21), this last condition on R is equivalent to the condition that $\Omega^j M$ be j-torsion free for $1 \leq j \leq k + 1$ and each R-module M. Therefore $\Gamma(\mathcal{G}')$, being a k-1 syzygy, is (k-1)-torsion free. Since (k-1)-torsion freeness implies (k-1)-reflexivity (see Theorem (2.17)), we see that $\Gamma(\mathcal{G}')$ is (k-1)-reflexive. Thus, the map $\underline{M} \to \underline{\Gamma(\mathcal{G}')}$ gives us a map $\underline{D^2_{k-1}(M)} \to \underline{\Gamma(\mathcal{G}')}$. It is easily seen that the required diagram commutes and that this uniquely determines our map. This proves the claim.

We can phrase this result in another fashion. Let $\underline{\mathcal{G}_{k-1}(Y)}$ denote the category of bundles on Y whose global sections are (k-1)-syzygies. We now form the category $\underline{\mathcal{G}_{k-1}(Y)}$ in a way analogous to the construction of the projective stabilization of a category. The objects of $\underline{\mathcal{G}_{k-1}(Y)}$ are the objects of $\mathcal{G}_{k-1}(Y)$ while the morphisms are equivalence classes of bundle

maps: two such maps being equivalent when their difference factors through a trivial (i.e. product) bundle. This notation is reasonable since a bundle $\mathcal{G}$ is trivial if and only if $\Gamma(\mathcal{G})$ is free, while $f: \mathcal{G} \to \mathcal{G}'$ is equivalent to zero if and only if $\underline{\Gamma(f)}: \underline{\Gamma(\mathcal{G})} \to \underline{\Gamma(\mathcal{G}')}$ (in $\underline{\mathcal{M}(R)}$) is zero. We now have:

(4.24) PROPOSITION: The global section functor $\Gamma: \underline{\mathcal{G}_{k-1}(Y)} \to \underline{\mathcal{M}(R)}$ has a left adjoint--namely $\underline{M} \to \underline{D^2_{k-1}(M)}$

§6: Modules of finite G-dimension

The next result shows that the torsion freeness of M is directly related to grade $\mathrm{Ext}^i(M,R)$ for $i > 0$.

(4.25) THEOREM: Suppose for every prime ideal $\underline{p}$ of R we have that $G(R_{\underline{p}})\text{-dim } M_{\underline{p}} < \infty$. For any integer $k \geq 1$ the following statements are then equivalent:

(a) M is k-torsion free

(b) There is an exact sequence $0 \to M \to P_{k-1} \to \cdots \to P_1 \to P_0$ where the P_i are projective

(c) There is an exact sequence $0 \to M \to X_{k-1} \to \cdots \to X_1 \to X_0$ where the X_i have G-dimension zero.

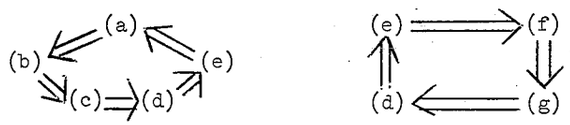
(d) $\mathrm{grade}_R \mathrm{Ext}^i(M,R) \geq i + k$ for all $i > 0$.

(e) $\mathrm{grade}_{R_{\underline{p}}} (\mathrm{Ext}^i(M,R))_{\underline{p}} \geq i + k$ for all $i > 0$ and each prime $\underline{p}$.

(f) For each prime ideal $\underline{p}$ every $R_{\underline{p}}$-regular sequence of length $\leq k$ is $M_{\underline{p}}$-regular

(g) For all $1 \leq i \leq k$ and each prime ideal $\underline{p}$, depth $R_{\underline{p}} \geq i$ implies that depth $M_{\underline{p}} \geq i$.

Proof: We shall use the following scheme of implications:



We have already shown that (a) implies (b) in Proposition (2.17), while (b) implies (c) since projectives have zero G-dimension. To show that (c) implies (d), let $N = \operatorname{coker}(X_1 \to X_0)$. Since $\operatorname{Ext}^i(X_j,R) = 0$ for each j and all $i > 0$, we see that $\operatorname{Ext}^i(M,R) \approx \operatorname{Ext}^{i+k}(N,R)$. Furthermore, $G(R_{\underline{p}})\text{-dim } N_{\underline{p}} \leq (G(R_{\underline{p}})\text{-dim } M_{\underline{p}}) + k < \infty$, for each $\underline{p}$. Therefore grade $\operatorname{Ext}^i(M,R) = $ grade $\operatorname{Ext}^{i+k}(N,R) \geq i + k$ by Proposition (4.16) - this for all $i > 0$. Thus, (d) holds. That (d) implies (e) is clear. Now suppose (e) holds. Proposition (3.17) tells us that $M_{\underline{p}}$ is k-torsion free (over $R_{\underline{p}}$) for each $\underline{p}$. Therefore $(\operatorname{Ext}^i(D(M),R))_{\underline{p}} = 0$ for $1 \leq i \leq k$ and all $\underline{p}$. We conclude that $\operatorname{Ext}^i(D(M),R) = 0$ for $1 \leq i \leq k$ so M is k-torsion free and (a) holds.

Now assume that (e) holds. Since (e) implies (a), we see that $M_{\underline{p}}$ is k-torsion free, hence $M_{\underline{p}}$ is a k-syzygy for each $\underline{p}$. By Lemma (4.23) this means that every $R_{\underline{p}}$-regular sequence of length $\leq k$ is $M_{\underline{p}}$-regular. Thus statement (f) holds. That (f) implies (g) is obvious. We complete our proof by showing that (g) implies (d) (since we already know that (d) implies (e)).

Choose $\underline{p} \in \operatorname{Supp}(\operatorname{Ext}^i(M,R))$ for any $i > 0$. We must show that depth $R_{\underline{p}} \geq i + k$. Since $\operatorname{Ext}^i_{R_{\underline{p}}}(M_{\underline{p}},R_{\underline{p}}) \neq 0$, we must have $G(R_{\underline{p}})\text{-dim } M_{\underline{p}} \geq i$. Now if depth $R_{\underline{p}} = t \leq k$ our assumption gives depth $M_{\underline{p}} \geq t$. Since $M_{\underline{p}}$ has finite $G(R_{\underline{p}})$-dimension over $R_{\underline{p}}$, we have the formula: depth $R_{\underline{p}} = t = (G(R_{\underline{p}})\text{-dim } M_{\underline{p}}) + (\text{depth } M_{\underline{p}}) \geq i + t$. This can only happen when $i = 0$, contradicting $i > 0$. Therefore depth $R_{\underline{p}} > k$ and our assumption (g) gives depth $M_{\underline{p}} \geq k$. By the above formula we have: depth $R_{\underline{p}} \geq i + k$, completing our proof.

Remark: From the exact sequence $0 \to M^* \to P_0^* \to P_1^* \to D(M) \to 0$ defining $D(M)$, we have observed that $\operatorname{Ext}^i(M^*,R) \approx \operatorname{Ext}^{i+2}(D(M),R)$ for $i > 0$. Furthermore, M is reflexive if and only if $\operatorname{Ext}^1(D(M),R) = \operatorname{Ext}^2(D(M),R) = 0$. Therefore, if M is reflexive and $k > 2$, we see that M is k-torsion free if and only if $\operatorname{Ext}^i(M^*,R) = 0$ for $1 \leq i \leq k - 2$. Combining this fact with the equivalence of (a) and (b) of the theorem above, we see that for a reflexive module M of finite G-dimension, M is a k-syzygy ($k > 2$) if and only if $\operatorname{Ext}^i(M^*,R) = 0$ for $1 \leq i \leq k - 2$. In the case where R is a Gorenstein ring (so all modules have finite G-dimension) we obtain Theorem 8.2 of [3].

Looking back at those results which require the condition grade $\operatorname{Ext}^i(M,R) \geq i$, we reexamine the Spherical Filtration Theorem (2.37). We shall apply this result to a module of finite G-dimension. Recall that if N and E are R-modules, $E \neq 0$, then N is said to be j-spherical of type E ($j > 0$) if pd $N = j$, $\operatorname{Ext}^j(N,R) \approx E$ and $\operatorname{Ext}^i(N,R) = 0$ for $1 \leq i \leq j - 1$. When $E = 0$, the spherical modules of type E are the projectives.

(4.26) PROPOSITION: Suppose G-dim $M = k < \infty$. Then there is a projective P and a filtration:

$$M_k \subset M_{k-1} \subset \cdots \subset M_1 \subset M_0 = M + P$$

such that:

(a) $\underline{\Omega^j M_j} \approx \underline{\Omega^j M}$ and $\operatorname{Ext}^i(M_j,R) = 0$ for $1 \leq i \leq j$, and $0 \leq j \leq k$.

(b) M_{j-1}/M_j is j-spherical of type $\operatorname{Ext}^j(M,R)$, $1 \leq j \leq k$.

(c) G-dim $M_j = k$ for $1 \leq j \leq k - 1$.

G-dim $M_k = 0$.

(d) If $H = \operatorname{coker}(M_k \longrightarrow M + P)$ then pd $H = k$ and $\operatorname{Ext}^i(H,R) \approx \operatorname{Ext}^i(M,R)$ for all $i > 0$.

Proof: The existence of the filtration satisfying (a) and (b) is obtained from Theorem (2.37) - since grade $\mathrm{Ext}^i(M,R) \geq i$ for all $i > 0$. If $1 \leq j \leq k-1$, the fact that $\underline{\Omega^j M_j} \approx \underline{\Omega^j M}$ gives G-dim M_j = G-dim M = k. Now consider M_k; we have the exact sequence $0 \to M_k \to M_{k-1} \to M_{k-1}/M_k \to 0$. Since M_{k-1}/M_k is k-spherical of type $\mathrm{Ext}^k(M,R) \neq 0$, its G-dimension is precisely k. As we have seen, G-dim $M_{k-1} = k$ as well. By Corollary (3.16), G-dim $M_k \leq$ G-dim $M_{k-1} = k$. However, $\mathrm{Ext}^i(M_k,R) = 0$ for $1 \leq i \leq k$, so G-dim $M_k = 0$. This proves (c).

To prove (d), we just observe that $\mathrm{coker}(M_k \to M + P)$ is $H_k(M)$ - the module of projective dimension $\leq k$ associated to M by the Approximation Theorem (2.41). This theorem gives us $\mathrm{Ext}^i(M,R) \approx \mathrm{Ext}^i(H_j(M),R)$ for $1 \leq i \leq k$. Since G-dim $M = k$, $\mathrm{Ext}^k(M,R) \neq 0$ so pd $H_k(M) = k$ and $\mathrm{Ext}^i(M,R) = \mathrm{Ext}^i(H_k(M),R) = 0$ for all $i > k$. This completes our proof.

(4.27) COROLLARY: Every module of finite G-dimension is projectively equivalent to an extension of a module of zero G-dimension by a module of finite projective dimension.

Proof: We have the exact sequence $0 \to M_k \to M + P \to H_k(M) \to 0$ where $\underline{M + P} \approx \underline{M}$.

Suppose now that we focus our attention on a module M of G-dimension k, and its associated filtration:

$$M_k \subset \ldots \subset M_0 = M + P.$$

Let $S_j(M) = M_{j-1}/M_j$. Then S_j is j-spherical of type $\mathrm{Ext}^j(M,R)$ and, by Proposition (2.45), is determined up to projective equivalence by this property. Our next result relates the torsion freeness properties of M with those of the M_i, the $S_j(M)$, and of $H_k(M)$.

(4.28) PROPOSITION: For any $n \geq 0$ the following statements are equivalent:

(a) M is n-torsion free

(b) Each M_j, $j = 0,\ldots,k$, is n-torsion free.

(c) Each $S_j(M)$, $j = 1,\ldots,k$, is n-torsion free.

(d) $H_k(M) = (M+P)/M_k$ is n-torsion free.

Proof: We note that all of the modules M, M_j, $S_j(M)$, and $H_k(M)$ have G-dimension $\leq k$. Therefore Theorem (4.25) shows that their torsion freeness properties are determined by the grades of the $\mathrm{Ext}^i(M,R)$, $\mathrm{Ext}^i(M_j,R)$, $\mathrm{Ext}^i(S_j(M),R)$ and $\mathrm{Ext}^i(H_k(M),R)$ respectively $(i > 0)$. Now each $\mathrm{Ext}^i(M,R)$ is isomorphic to $\mathrm{Ext}^i(M_j,R)$ for at least one value of j. Conversely, each $\mathrm{Ext}^i(M_j,R)$ is either zero (so has infinite grade) or is isomorphic to $\mathrm{Ext}^i(M,R)$. Thus, with the aid of Theorem (4.25) we see that (a) and (b) are equivalent. The remainder of the proposition is proved similarly.

Thus, the torsion freeness properties of M are determined by (and determine) those of the associated sphericals $S_j(M)$. Now the relation between grade and torsion freeness for sphericals is relatively straightforward. Suppose S is spherical with pd $S = n \geq 2$. Let $0 \to P_n \to \ldots \to P_0 \to S \to 0$ be a projective resolution. Since $\mathrm{Ext}^i(S,R) = 0$ for $1 \leq i \leq n-1$ we have the exact diagram:

$$0 \to S^* \to P_0^* \to P_1^* \longrightarrow P_2^* \to \ldots \to P_n^* \to \mathrm{Ext}^n(S,R) \to 0$$

$$\begin{array}{ccccc} & & & \searrow & \nearrow \\ & & & D(S) & \\ & & \nearrow & & \searrow \\ & 0 & & & 0 \end{array}$$

We deduce that: $\underline{D(S)} \approx \underline{\Omega^{n-1}\mathrm{Ext}^n(S,R)}$

$$\underline{S^*} \approx \underline{\Omega^{n+1}\mathrm{Ext}^n(S,R)}$$

We can now rewrite the exact sequence $0 \to \mathrm{Ext}^1(D(S),R) \to S \to S^{**} \to \mathrm{Ext}^2(D(S),R) \to 0$ as

$$0 \to \mathrm{Ext}^n(\mathrm{Ext}^n(S,R),R) \to S \to S^{**} \to \mathrm{Ext}^{n+1}(\mathrm{Ext}^n(S,R),R) \to 0$$

and record the following:

(a) S is 1-torsion free if and only if $\mathrm{Ext}^n(\mathrm{Ext}^n(S,R),R) = 0$

(b) S is 2-torsion free if and only if $\mathrm{Ext}^n(\mathrm{Ext}^n(S,R),R) = \mathrm{Ext}^{n+1}(\mathrm{Ext}^n(S,R),R) = 0$

Note that since $\mathrm{pd}\, S < \infty$, $\mathrm{Ext}^i(\mathrm{Ext}^n(S,R),R) = 0$ for $0 \leq i < n$. Therefore $\mathrm{Ext}^n(\mathrm{Ext}^n(S,R),R) = 0$ is equivalent to $\mathrm{grade}\, \mathrm{Ext}^n(S,R) \geq n + 1$. If, in addition, $\mathrm{Ext}^{n+1}(\mathrm{Ext}^n(S,R),R) = 0$, then $\mathrm{grade}\, \mathrm{Ext}^n(S,R) \geq n + 2$. Consequently, statements (a) and (b) are consistent with Theorem (4.25), and the preceding exact sequence serves to explain (at least for sphericals) why $\mathrm{grade}\, \mathrm{Ext}^i(M,R)$ is related to 1 and 2-torsion freeness.

Now let us examine k-torsion freeness for S when $k > 2$. Since $\mathrm{pd}\, S < \infty$, being k-torsion free is equivalent to being a k-syzygy. To see when S is such, we consider a projective resolution of S^*:

$$\ldots \to Q_i \to \ldots \to Q_1 \to Q_0 \to S^* \to 0$$

Dualizing, and supposing that S is 2-torsion free (so reflexive), we obtain a complex:

$$0 \to S \to Q_0^* \to Q_1^* \to \ldots \to Q_i^* \to \ldots$$

It is easily seen that S will be a k-syzygy when this complex is exact at Q_i^* for $0 \leq i \leq k - 2$. This is clearly equivalent to: $\mathrm{Ext}^i(S^*,R) = 0$ for $1 \leq i \leq k - 2$. Since $\underline{S^*} \approx \underline{\Omega^{n+1}\mathrm{Ext}^n(S,R)}$, we can write this condition as $\mathrm{Ext}^{n+i+1}(\mathrm{Ext}^n(S,R),R) = 0$ for $1 \leq i \leq k - 2$. Since S is 2-torsion free, $\mathrm{Ext}^{n+i+1}(\mathrm{Ext}^n(S,R),R) = 0$ for $i = -1, 0$ as well. Thus, the sequence:

$$0 \to S \to Q_0^* \to Q_1^* \to \ldots \to Q_{k-1}^*$$

is exact (and S is a k-syzygy) if (and only if)

$$\mathrm{Grade}\, \mathrm{Ext}^n(S,R) \geq n + k.$$

§7: Perfect Modules

We now have two fairly extensive classes of modules of finite Gorenstein dimension: the modules of finite projective dimension over an arbitrary Noetherian ring, and the modules over any Gorenstein ring. The following series of results gives us means to construct new modules of finite G-dimension from these.

In Lemma (4.9 c) we proved that if x is both R and M regular then $G(R)\text{-dim}\, M = 0$ implies $G(R/(x))\text{-dim}\, M/(x)M = k$. We now establish the converse when R is local.

(4.29) LEMMA: Let R be local and M be an R-module. If x is both R and M regular then $G(R/(x))\text{-dim}\, M/(x)M = 0$ implies $G(R)\text{-dim}\, M = 0$.

Proof: Since x is both M and R regular, $\mathrm{Ext}^i_R(M,R/(x)) \approx \mathrm{Ext}^i_{R/(x)}(M/(x)M, R/(x)) = 0$ (for all $i > 0$) when $G(R/(x))\text{-dim}\, M/(x)M = 0$. From the exact sequence $0 \to R \xrightarrow{x} R \to R/(x) \to 0$ we get for each i an exact sequence:

$$\mathrm{Ext}^i_R(M,R) \xrightarrow{x} \mathrm{Ext}^i_R(M,R) \to \mathrm{Ext}^i_R(M,R/(x)).$$

Since $\mathrm{Ext}^i_R(M,R/(x)) = 0$ for all $i > 0$, we may use Nakayama's lemma to conclude that $\mathrm{Ext}^i_R(M,R) = 0$ for all $i > 0$. It remains to show the same for $D(M)$.

Claim: x is $D(M)$ regular. To prove this we must show that $\mathrm{Tor}^R_1(D(M),R/(x)) = 0$. As we have seen in Corollary (2.9), $\mathrm{Tor}^R_1(D(M), __) \approx (\underline{M}, __)$. Furthermore, we have from $0 \to R \xrightarrow{x} R \to R/(x) \to 0$ the exact sequence:

$$(\underline{M},\underline{R}) \to (\underline{M},\underline{R}) \to (\underline{M},\underline{R/(x)}) \to \mathrm{Ext}^1(M,R) \quad \text{(see Lemma (1.35))}.$$

Since R is projective, $(\underline{M},\underline{R}) = 0$; also, we have shown $\mathrm{Ext}^1(M,R) = 0$. Therefore $\mathrm{Tor}_1^R(M,R/(x) \approx (\underline{M},\underline{R/(x)}) = 0$ and our claim is proved.

As we have seen, G-dim $L = 0$ if and only if G-dim $D(L) = 0$ (by Lemma (4.9)). Therefore $G(R/(x))$-dim $D_{R/(x)}(M/(x)M) = 0$. Since $\underline{D_{R/(x)}(M/(x)M)} \approx \underline{D_R(M)/(x)D_R(M)}$ we see that $G(R/(x))$-dim$[D_R(M)/(x)D_R(M)] = 0$. Since x is both R and $D_R(M)$ regular, $D_R(M)$ satisfies the same hypothesis as M so the same argument as for M shows: $\mathrm{Ext}_R^i(D(M),R) = 0$ for all $i > 0$. Thus, $G(R)$-dim $M = 0$ and we are done.

(4.30) COROLLARY: Let R be local, M an R-module, and let x be R and M regular. Then $G(R)$-dim $M = k$ if and only if $G(R/(x))$-dim $M/)x)M=k$.

Proof: Suppose $G(R/(x))$-dim $M/(x)M = k$. We proceed by induction on k. When $k = 0$ we get our result from the previous lemma. Now suppose $k > 0$ and the result true for $k - 1$. Let $0 \to N \to P \to M \to 0$ be an exact sequence of R-modules with P projective. Since x is R and M regular, $\mathrm{Tor}_1(M,R/(x)) = 0$ so $0 \to N/(x)N \to P/(x)P \to M/(x)M \to 0$ is an exact sequence of $R/(x)$-modules. Since $G(R/(x))$-dim $P/(x)P = 0$ and $G(R/(x))$-dim $M/(x)M = k \neq 0$, we see that $G(R/(x))$-dim $N/(x)N = k-1$ (note that x is N regular since x is R-regular and $N \subset$ the projective P). Now $G(R)$-dim $M \neq 0$ since, if it were, $G(R/(x))$-dim $M/(x)M$ would also be zero-contrary to assumption. Therefore $G(R)$-dim $M = (G(R)$-dim $N) + 1 = k$ and we are done.

An easy induction argument now gives:

(4.31) COROLLARY: Let R be local and M an R-module. Suppose $(x_1,\ldots,x_t)$ is an R and M-regular sequence. Then we have $G(R)$-dim $M = G(R/(x_1,\ldots,x_t))$-dim $M/(x_1,\ldots,x_t)M$.

We also need the following.

(4.32) LEMMA: Let R be local and let x be an R-regular element contained in the annihilator of M. If $G(R/(x))$-dim $M = k$ then $G(R)$-dim $M = k + 1$.

Proof: Tensoring the exact sequence $0 \to R \xrightarrow{x} R \to R/(x) \to 0$ by M gives $\mathrm{Tor}_1(M,R/(x)) = \ker(M \xrightarrow{x} M) = M$ since x annihilates M. Now let $0 \to K \to P \to M \to 0$ be an exact sequence of R-modules with P projective. Since $\mathrm{Tor}_1(M,R/(x)) = M$, tensoring this sequence by $R/(x)$ gives an exact sequence of $R/(x)$-modules: $0 \to M \to K/(x)K \to P/(x)P \to M \to 0$ where $P/(x)P$ is $R/(x)$-projective. If $L = \ker(P/(x)P \to M)$, then we have short exact sequences (of $R/(x)$-modules):

$$0 \to L \to P/(x)P \to M \to 0$$

$$0 \to M \to K/(x)K \to L \to 0$$

Since $G(R/(x))$-dim $M = k$ and $G(R/(x))$-dim $P/(x)P = 0$, we have $G(R/(x))$-dim $L \leq k - 1$. By Corollary (3.16) we get, from the second sequence, that $G(R/(x))$-dim $K/(x)K \leq \max\{G(R/(x))$-dim M, $G(R/(x))$-dim $L\} = k$. On the other hand, if $G(R/(x))$-dim $K/(x)K \leq k - 1$, the same Corollary gives $G(R/(x))$-dim $M \leq k - 1$, which is false. Therefore $G(R/(x))$-dim $K/(x)K = k$. Since K is contained in the projective P, x is K-regular (x being R-regular). By the previous Lemma, $G(R)$-dim $K = k$, so $G(R)$-dim $M = k + 1$ (note that M cannot have zero $G(R)$-dimension since x annihilates M).

(4.33) COROLLARY: Let R be a local ring and M an R-module. If $(x_1,\ldots,x_t)$ is an R-sequence contained in the annihilator of M, then $G(R/(x_1,\ldots,x_t))$-dim $M/(x_1,\ldots,x_t)M = k$ implies that $G(R)$-dim $M = k + t$.

Proof: This result follows from Lemma (4.32) by straightforward induction

on t.

We now extend the notion of perfect module from the case of finite projective dimension to the case of finite Gorenstein dimension.

(4.34) Definition: The R-module M is perfect if grade M = G-dim $M < \infty$. Our main result about perfect modules is the following.

(4.35) PROPOSITION: (C. Peskine, L. Szpiro): Let R be a local ring and $(x_1,\dots,x_t)$ an R-regular sequence. Let M be an $R/(x_1,\dots,x_t)$-module (hence an R-module as well). Then M is a perfect R-module with G(R)-dim $M = t$ if and only if $G(R/(x_1,\dots,x_t))$-dim $M = 0$. When this is the case, we have $\mathrm{Ext}^t_R(\mathrm{Ext}^t_R(M,R),R) \approx M$.

Proof: Suppose first that $G(R/(x_1,\dots,x_t))$-dim $M = 0$. By Corollary (4.33) above, we see that G(R)-dim $M = t$. Since $(x_1,\dots,x_t)$ is R-regular and annihilates M, we have $\mathrm{grade}_R(M) \geq t$. If $\mathrm{grade}_R(M) > t$ then we would have $\mathrm{Ext}^t_R(M,R) = 0$ which is impossible since G(R)-dim $M = t$. Therefore M is perfect with G(R)-dim $M = t$.

Suppose, conversely, that M is a perfect R-module of dimension t. Consider the exact sequence:

$$0 \to \Omega^t M \to P_{t-1} \to \dots \to P_0 \to M \to 0$$

where the P_i are R-projective, and G(R)-dim $\Omega^t M = 0$. Since grade $M = t$, $\mathrm{Ext}^i(M,R) = 0$ for $0 \leq i < t$. Therefore the sequence:

$$0 \to P_0^* \to P_1^* \to \dots \to P_{t-1}^* \to (\Omega^t(M))^* \to C \to 0$$

is exact, where $C = \mathrm{coker}(P_{t-1}^* \to (\Omega^t M)^*)$. Dualizing the exact sequence $0 \to \Omega^t M \to P_{t-1} \to \Omega^{t-1} M \to 0$ we get the exact sequence: $P_{t-1}^* \to (\Omega^t M)^* \to \mathrm{Ext}^1(\Omega^{t-1}M,R) \to 0$. We know that $\mathrm{Ext}^1(\Omega^{t-1}M,R) \approx \mathrm{Ext}^t(M,R)$, so $C \approx \mathrm{Ext}^t(M,R)$. Thus, from the exact sequence: $0 \to P_0^* \to \dots \to (\Omega^t M)^* \to \mathrm{Ext}^t(M,R) \to 0$ we see that G(R)-dim $\mathrm{Ext}^t(M,R) \leq t$.

Since the diagram

$$\begin{array}{ccccccccc} 0 & \to & (\Omega^t M)^{**} & \to & P_{t-1}^{**} & \to \dots \to & P_0^{**} & \to M \to & 0 \\ & & \wr & & \wr & & \wr & \| & \\ 0 & \to & \Omega^t M & \to & P_{t-1} & \to \dots \to & P_0 & \to M \to & 0 \end{array}$$

is exact and commutative, we see that $\mathrm{Ext}^i(\mathrm{Ext}^t(M,R),R) = 0$ for $0 \leq i < t$, and $\mathrm{Ext}^t(\mathrm{Ext}^t(M,R),R) \approx M$. Thus, $\mathrm{Ext}^t(M,R)$ is also a perfect module of dimension t. Furthermore, since the x_i annihilate M, they also annihilate $\mathrm{Ext}^t(M,R)$. We conclude that $\mathrm{Ext}^t(M,R)$ satisfies the same hypothesis as M.

Now let us consider M and $\mathrm{Ext}^t(M,R)$ as modules over $R/(x_1,\dots,x_t)$. By Rees' result (4.2) we have:

$$\mathrm{Ext}^i_{R/(x_1,\dots,x_t)}(M,R/(x_1,\dots,x_t)) \approx \mathrm{Ext}^{i+t}_R(M,R).$$

Since G(R)-dim $M = t$, we conclude that $\mathrm{Ext}^i_{R/(x_1,\dots,x_t)}(M,R/x_1,\dots,x_t) = 0$ for all $i > 0$, and $\mathrm{Hom}_{R/(x_1,\dots,x_t)}(M,R/(x_1,\dots,x_t) \approx \mathrm{Ext}^t_R(M,R)$.

Let E^i denote the functor $\mathrm{Ext}^i_{R/(x_1,\dots,x_t)}(\ ,R/(x_1,\dots,x_t))$. Then we have shown that $E^i(M) = 0$ for all $i > 0$ and $E^0(M) \approx \mathrm{Ext}^t_R(M,R)$. Since we have observed that $\mathrm{Ext}^t(M,R)$ satisfies the same hypothesis as M, we have

$$E^i(\mathrm{Ext}^t(M,R)) = E^i(E^0(M)) = 0$$

for all $i > 0$, and

$$E^0(\mathrm{Ext}^t(M,R) = E^0(E^0(M)) \approx M.$$

A tedious calculation shows that this last isomorphism is induced by $M \to E^0(E^0(M))$: the map of M into its $R/(x_1,\dots,x_0)$-double dual. Thus, M is $R/(x_1,\dots,x_t)$ reflexive. Now Proposition (3.8), characterizing

modules of zero G-dimension, tells us that $G(R/(x_1,\ldots,x_t))\text{-dim } M = 0$ if and only if M is $R/(x_1,\ldots,x_t)$-reflexive and $E^i(M) = E^i(E^0(M)) = 0$ for all $i < 0$. Since this has been verified, we are done.

§8: A Computation

Let us introduce the following notation. If n is a positive integer and M a module, $n \cdot M$ will denote the direct sum: $M + M + \ldots + M$ (n factors). If $n = \binom{t}{i}$ (binomial coefficient), then $\binom{t}{i} \cdot M$ will be the direct sum of M with itself $\binom{t}{i}$ times.

(4.36) PROPOSITION:

Let R be a local ring and M a perfect R-module with $G(R)\text{-dim } M = t$ (= grade M). Suppose $(x_1,\ldots,x_t)$ is an R-regular sequence in the annihilator of M, and N is an $R/(x_1,\ldots,x_t)$-module with $\mathrm{pd}_{R/(x_1,\ldots,x_t)} N < \infty$. Then we have for $0 \le i \le t$:

$$\mathrm{Tor}_i^R(M,N) \approx \binom{t}{i} \cdot (M \otimes_{R/(x_1,\ldots,x_t)} N)$$

while $\mathrm{Tor}_i^R(M,N) = 0$ for $i > t$.

Proof:

Since $(x_1,\ldots,x_t)$ is an R-regular sequence, the Koszul complex $R_{x:1,\ldots,t}$ provides a free resolution of $R/(x_1,\ldots,x_t)$:
$0 \to X_t \to \ldots \to X_0 \to R/(x_1,\ldots,x_t) \to 0$. Here, $X_i = \Lambda^i(t \cdot R) \approx \binom{t}{i} \cdot R$. If we tensor this complex by M, we obtain a complex whose homology is the $\{\mathrm{Tor}_i^R(M,R/(x_1,\ldots,x_t))\}$. However, since the $\{x_i\}$ annihilate M, the boundary operator in this new complex is zero. Therefore we have: $\mathrm{Tor}_i^R(M,R/(x_1,\ldots,x_t)) \approx [\binom{t}{i} \cdot R] \otimes M \approx \binom{t}{i} \cdot M$ for $0 \le i \le t$, and 0 for $i > t$.

Now there exists a spectral sequence ([4], XVI, §5):

$$\mathrm{Tor}_p^{R/(x_1,\ldots,x_t)}(\mathrm{Tor}_q^R(M,R/(x_1,\ldots,x_t)),N) \Longrightarrow \mathrm{Tor}_n^R(M,N)$$

Since M is perfect with $G(R)\text{-dim } M = t$, we see that $G(R)/(x_1,\ldots,x_t))\text{-dim } M = 0$ by Proposition (4.35) of the previous section. Since $\mathrm{Tor}_q^R(M,R/(x_1,\ldots,x_t)) \approx \binom{t}{i} \cdot M$, we see that $G(R/(x_1,\ldots,x_t))\text{-dim } \mathrm{Tor}_q^R(M,R/(x_1,\ldots,x_t)) = 0$. Now if S is a ring, $G(S)\text{-dim } M = 0$ and $\mathrm{pd}_S N < \infty$, then $\mathrm{Tor}_i^S(M,N) = 0$ for all $i > 0$ by Proposition (4.12). We conclude that

$$\mathrm{Tor}_p^{R/(x_1,\ldots,x_t)}(\mathrm{Tor}_q(M,R/(x_1,\ldots,x_t)),N) = 0$$

for all $p > 0$. Thus, the spectral sequence collapses to give isomorphisms

$$\mathrm{Tor}_n^R(M,N) \approx \mathrm{Tor}_0^{R/(x_1,\ldots,x_t)}(\binom{t}{i} \cdot M,N)$$

$$= \binom{t}{i}(M \otimes_{R/(x_1,\ldots,x_t)} N).$$

(4.37) PROPOSITION: Under the same hypothesis as (4.36) we have:

$$\mathrm{Ext}_R^i(M,N) \approx \binom{t}{i} \cdot \mathrm{Hom}_{R/(x_1,\ldots,x_t)}(M,N)$$

for $0 \le i \le t$, and $\mathrm{Ext}_R^i(M,N) = 0$ for $i > t$.

Proof: The proof is entirely similar to that of (4.36), except that the spectral sequence ([4], XVI, §5):

$$\mathrm{Ext}_{R/(x_1,\ldots,x_t)}^p(\mathrm{Tor}_q^R(M,R/(x_1,\ldots,x_t)),N) \Longrightarrow \mathrm{Ext}_R^n(M,N)$$

is used, together with the fact that $G(S)\text{-dim } M = 0$ and $\mathrm{pd}_S N < \infty$ imply $\mathrm{Ext}_S^i(M,N) = 0$ for all $i > 0$ (Proposition (4.12)).

Bibliography

[1]: M. Auslander: Coherent Functors, Proceedings of the Conference on Categorical Algebra, La Jolla 1965 (Springer-Verlag) pp. 189-231.

[2]: M. Auslander: Comments on the Functor Ext (To appear).

[3]: H. Bass: On the Ubiquity of Gorenstein Rings, Math. Zeitschr. Vol. 82 (1963) pp. 8-28.

[4]: H. Cartan and S. Eilenberg: Homological Algebra, Princeton University Press, 1956.

[5]: B. Eckmann and P. Hilton: Homotopy Groups of Maps and Exact Sequences, Comment. Math. Helv. 34 (1960) pp. 271-304.

[6]: P. Hilton and D. Rees: Natural Maps of Extension Functors and a Theorem of R. G. Swan, Proc. Camb. Phil. Soc. Vol. 57 (1961) pp. 489-502.

[7]: G. Horrocks: Vector Bundles on the Punctured Spectrum of a Local Ring, Proc. London Math. Soc. (3) 14 (1964) pp. 689-713.

[8]: F. Oort.

[9]: D. Rees: The Grade of an Ideal or Module, Proc. Camb. Phil. Soc. 53 (1957), 28-42.

Appendix: The Maps $M \to D_k^2(M)$

We take a closer look here at the modules $D_k^2(M)$ and the maps $M \to D_k^2(M)$. As we have seen, when $k = 0$, $D_0^2(M) = DD(M)$ is projectively equivalent to M, the isomorphism being induced by the indentity map (so every module is 0-reflexive). To compute $D_1^2(M)$, let $P_1 \to P_0 \to M \to 0$ be exact with the P_i projective. Dualizing, we obtain the exact sequence: $0 \to M^* \to P_0^* \to P_1^* \to DM \to 0$. If $K = \ker(P_1^* \to DM)$, then there is a unique isomorphism $\mathrm{Ext}^1(K, \ ) \approx \mathrm{Ext}^1(\Omega^1 DM, \ )$ such that the following commutes:

$$\begin{array}{ccc} \mathrm{Ext}^1(K, \ ) & & \mathrm{Ext}^2(DM, \ ) \\ & & \| \\ \mathrm{Ext}^1(\Omega^1 DM, \ ) & & \mathrm{Ext}^2(DM, \ ) \end{array}$$

Henceforth, all such isomorphisms will be taken as identifications. We therefore have an exact sequence $0 \to M^* \to P_0^* \to \Omega^1 DM \to 0$. If we now choose $Q \to M^* \to 0$ exact, with Q projective, we have the presentation: $Q \to P_0^* \to \Omega^1 DM \to 0$. Dualizing then gives the exact sequence: $0 \to (\Omega^1 DM)^* \to P_0 \to Q^* \to D\Omega^1 DM \to 0$. Therefore $\Omega^1(D\Omega^1 DM) = \ker(Q^* \to D\Omega^1 DM) = \mathrm{coker}((\Omega^1 DM)^* \to P_0)$, and the following is exact: $0 \to (\Omega^1 DM)^* \to P_0 \to D_1^2(M) \to 0$. From the exact sequence $Q \to P_0^* \to P_1^*$, we see that the composition $(P_1 \to P_0 \to Q^*)$ is zero; therefore there is a map $P_1 \to (\Omega^1 DM)^*$ which renders the following diagram commutative (the bottom row is exact):

$$\begin{array}{ccccccc} P_1 & \longrightarrow & P_0 & & & & \\ \downarrow & & \| & & & & \\ 0 & \longrightarrow & (\Omega^1 DM)^* & \longrightarrow & P_0 & \longrightarrow & Q^* \end{array}$$

From this, we construct an exact and commutative diagram.

$$\begin{array}{ccccccc}
P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow & & \| & & \downarrow & & \\
(\Omega^1 DM)^* & \longrightarrow & P_0 & \longrightarrow & D_1^2(M) & \longrightarrow & 0 \\
\| & & \| & & \downarrow & & \\
(\Omega^1 DM)^* & \longrightarrow & P_0 & \longrightarrow & M^{**} & &
\end{array}$$

Diagram chasing shows that $M \to D_1^2(M)$ is surjective, $D_1^2(M) \to M^{**}$ is injective, and the composition $(M \to D_1^2(M) \to M^{**})$ is the canonical map $M \to M^{**}$. We conclude that, up to projective equivalence, $D_1^2(M) \approx \mathrm{im}(M \to M^{**}) = M/t(M)$ and $M \to D_1^2(M)$ is induced by $M \to M/t(M)$. In general, $t(M) = \mathrm{Ext}^1(DM,R)$. When R is a (commutative) domain, $t(M)$ is the torsion submodule of M.

Claim: M is 1-torsion free if and only if M is 1-reflexive. In fact, when M is 1-torsion free, $\mathrm{Ext}^1(DM,R) = 0$ so $M \tilde{\to} M/t(M)$ and M is 1-reflexive. To prove the converse, we observe that when $A \subset B$ and B is 1-torsion free, we have the diagram:

$$\begin{array}{ccc}
A & \hookrightarrow & B \\
\downarrow & & \downarrow \text{ injective} \\
A^{**} & \longrightarrow & B^{**}
\end{array}$$

Therefore $A \to A^{**}$ is injective so A is 1-torsion free. Now M^{**}, being a submodule of a projective, is 1-torsion free by this observation; $M/t(M)$, being a submodule of M^{**}, is also 1-torsion free. If M is 1-reflexive, then $P + M \approx M/t(M) + Q$ for suitable projectives P and Q. Therefore M, being a submodule of the 1-torsion free module $M/t(M) + Q$, is itself 1-torsion free.

Now we turn to the case $k = 2$. From the exact sequence

$0 \to M^* \to P_0^* \to P_1^* \to DM \to 0$ we see that $\underline{M^*} \approx \underline{\Omega^2 DM}$; replacing M by $\Omega^2 DM$, we see that $\underline{D_2^2(M)} \approx \underline{(\Omega^2 DM)^*} \approx \underline{M^{**}}$. It can be shown (see 2.14) that the map $\underline{M} \to \underline{D_2^2(M)}$ is induced by $M \to M^{**}$. This being true, it is clear that if M is reflexive (i.e. $M \tilde{\to} M^{**}$) then M is 2-reflexive (i.e. $\underline{M} \tilde{\to} \underline{M^{**}}$). We now look at the question as to when a module which is 2-reflexive is reflexive.

We begin with the following.

PROPOSITION: Let R be a two-sided noetherian ring and M a finitely generated left R-module which is 1-reflexive (i.e. 1-torsion free). Then M is 2-reflexive if and only if the $\mathrm{coker}(M \to M^{**})$ is projective, or what is the same thing, if and only if $\mathrm{Ext}^2(D(M),R)$ is projective.

Proof: Since M is 1-torsion free, we have the exact sequence $0 \to M \to M^{**} \to \mathrm{Ext}^2(D(M),R) \to 0$. Hence, if $\mathrm{Ext}^2(D(M),R)$ is projective then we have that $M \oplus \mathrm{Ext}^2(D(M),R) \cong M^{**}$. This shows that the map $M \to M^{**}$ gives an isomorphism $\underline{M} \tilde{\to} \underline{M^{**}}$ and hence M is 2-reflexive.

Suppose now that M is a 2-reflexive module over the ring R. Since $M + P \approx M^{**} + Q$ for suitable projectives P and Q, we have:

(1) $\mathrm{Ext}^i(M^{**}, \ ) \tilde{\to} \mathrm{Ext}^i(M, \ )$ for all $i > 0$, and

(2) M is 1-torsion free.

From (2) we obtain the short exact sequence:

$$0 \to M \to M^{**} \to \mathrm{Ext}^2(DM,R) \to 0$$

This leads to the exact sequence:

$$\mathrm{Ext}^1(M^{**}, \ ) \tilde{\to} \mathrm{Ext}^1(M, \ ) \to \mathrm{Ext}^2(\mathrm{Ext}^2(DM,R), \ ) \to \mathrm{Ext}^2(M^{**}, \ ) \tilde{\to} \mathrm{Ext}^2(M, \ )$$

Therefore $\mathrm{Ext}^2(\mathrm{Ext}^2(DM,R), \ ) = 0$, so $\mathrm{pd.}\ \mathrm{Ext}^2(DM,R) \leq 1$. However, applying $\mathrm{Ext}(\ ,R)$ to our short exact sequence gives the exact sequence:

$$0 \to (\mathrm{Ext}^2(DM,R))^* \to M^{***} \to M^* \to \mathrm{Ext}^1(\mathrm{Ext}^2(DM,R),R) \to \mathrm{Ext}^1(M^{**},R) \tilde{\to} \mathrm{Ext}^1(M,R)$$

Since $M^* \to M^{***} \to M^*$ is always the identity map of M^*, we see that

$M^{***} \to M^*$ is surjective. We conclude that $\mathrm{Ext}^1(\mathrm{Ext}^2(DM,R),R) = 0$; therefore $\mathrm{Ext}^2(DM,R)$ is projective.

We describe in the following Proposition exactly what it means for a two-sided noetherian ring that every left 2-reflexive module is reflexive.

PROPOSITION: Let R be a two-sided neotherian ring. Then the following statements are equivalent:

(a) Each finitely generated left R-module which is 2-reflexive is reflexive.

(b) If for a finitely generated right R-module N we have that $\mathrm{Ext}^2(N,R)$ is projective, then $\mathrm{Ext}^2(N,R)$ is zero.

Proof: (a) => (b). Let N be a finitely generated right R-module such that $\mathrm{Ext}^2(N,R)$ is projective. If we let M be the left 1-torsion free module $D(N)/_{tD(N)}$ we deduce from the exact sequence

$$0 \to \mathrm{Ext}^1(N,R) \to D(N) \to D(N)^{**} \to \mathrm{Ext}^2(N,R) \to 0$$

the exact sequence

$$0 \to M \to M^{**} \to \mathrm{Ext}^2(N,R) \to 0$$

since $M^{**} = D(N)^{**}$. Since $\mathrm{Ext}^2(N,R)$ is projective, by hypothesis, and M is 1-torsion free, it follows from our previous Proposition that M is 2-reflexive. Therefore by the hypothesis of (a) we know that M is reflexive, i.e. The Map $M \to M^{**}$ is an isomorphism. From this we have our desired conclusion that $\mathrm{Ext}^2(N,R) = 0$.

(b) => (a). Suppose M is 2-reflexive. Then M is clearly 1-torsion free and therefore by our previous Proposition $\mathrm{Ext}^2(D(M),R)$ is projective. Since the hypothesis of (b) implies that $\mathrm{Ext}^2(D(M),R) = 0$ we conclude that M is reflexive.

Now for an arbitrary commutative ring R we have by Theorem (4.1) of [2] that if M is a finitely generated R-module and C is an arbitrary R-module,

then zero is the only projective submodule of $\mathrm{Ext}^1(M,C)$. Combining the last Proposition with this remark, we obtain:

PROPOSITION: If R is a commutative noetherian ring, and M is a finitely generated R-module, then M is reflexive if and only if M is 2-reflexive.

Remark: It would be interesting to know if it is possible for $\mathrm{Ext}^2(N,R)$ to be a non-trivial projective module over a non-commutative two-sided noetherian ring R and N finitely generated.

We now turn our attention to the maps $M \to D_k^2(M)$ where $k > 2$. In Chapter Two we defined these maps as special cases of the morphisms of functors: $S^k S_k F \to F$ (by letting $F = (M, __)$). We now give a module-theoretic description using resolutions. As a "generic" example, we construct $M \to D_5^2(M)$.

Start by choosing $P_1 \to P_0 \to M \to 0$ exact with the P_i projective. Then $0 \to M^* \to P_0^* \to P_1^* \to DM \to 0$ is exact. Choose $Q_2 \to Q_1 \to Q_0 \to M^*$ exact with the Q_i projective, and splice; we obtain the exact sequence:

$$0 \to \Omega^5 DM \to Q_2 \to Q_1 \to Q_0 \to P_0^* \to P_1^* \to DM \to 0 \quad .$$

Now let $Q_4 \to Q_3 \to \Omega^5 DM \to 0$ be a presentation with the Q_i projective. Then $0 \to (\Omega^5 DM)^* \to Q_3^* \to Q_4^* \to D(\Omega^5 DM) \to 0$ is exact. Finally, choose $G_0 \to G_1 \to G_2 \to (\Omega^5 DM)^* \to 0$ exact with the G_i projective. We then have an exact sequence:

$$0 \to \Omega^5 D\Omega^5 DM \to G_0 \to G_1 \to G_2 \to Q_3^* \to Q_4^* \to D\Omega^5 DM \to 0 \quad .$$

Now dualizing the exact sequence $Q_4 \to Q_3 \to Q_2 \to Q_1 \to Q_0 \to M^* \to 0$ gives a complex: $0 \to M^{**} \to Q_0^* \to Q_1^* \to Q_2^* \to Q_3^* \to Q_4^*$. Consider the diagram:

$$\begin{array}{ccccccccccc} 0 & \to & D_5^2(M) & \to & G_0 & \to & G_1 & \to & G_2 & \to & Q_3^* \to Q_4^* \\ & & & & & & & & & & \| \quad \| \\ 0 & \to & M^{**} & \to & Q_0^* & \to & Q_1^* & \to & Q_2^* & \to & Q_3^* \to Q_4^* \end{array}$$

Since the top row is acyclic and Q_i^* is projective for each i, we can find maps $Q_i^* \to G_i$ for $0 \leq i \leq 2$ which leave the diagram commutative. Since $0 \to M^{**} \to Q_0^* \to Q_1^*$ is exact, this induces a map $M^{**} \to D_5^2(M)$. Composing with $M \to M^{**}$ gives our desired map $M \to D_5^2(M)$. As pointed out in Proposition (2.14) this induces our standard map $\underline{M} \to \underline{D_5^2(M)}$.