Again we proceed by induction on k. When $k = 0$ there is nothing to prove. Suppose the implication holds for $0,\ldots,k-1$; we prove it for k by induction on j. The case $j = 0$ is trivial, so suppose $j = 1$. We have the commutative diagram:

$$\begin{array}{ccc} S^1S_k F & \longrightarrow & S_{k-1}F \\ \| & & \| \\ S^1S_1(S_{k-1}F) & \longrightarrow & S_{k-1}F \\ \wr & & \wr \\ S^1S_1(S^{n-k+1}S_n F) & \longrightarrow & S^{n-k+1}S_n F \end{array}$$

where the vertical maps are isomorphisms by Assumptions (c). Since $k \leq n$, we have $n - k + 1 > 0$, hence $S^{n-k+1}(S_n F)$ is injectively stable. We conclude that the bottom horizontal map is an isomorphism; therefore $S^1S_1F \to S_{k-1}F$ is an isomorphism. This disposes of $j = 1$. Suppose then that we have our result for some $0 < j < k$. By induction hypothesis (on j) we have $S^jS_kF \approx S_{k-j}F$. Applying S^1 gives $S^{j+1}S_kF \approx S^1S_{k-j}F$. Since $k - j < k$, our induction hypothesis on k gives $S^1S_{k-j}F \approx S_{k-j-1}F$. Therefore we have $S^{j+1}S_kF \approx S_{k-(j+1)}F$, completing the proof.

Let us now assume that (b) holds. Then $S^kS_kF \to F$ is clearly an isomorphism for $0 \leq k \leq n$. Let us consider S^jS_kF. For any $j \leq k$ we have, from (b), that $S^jS_kF \approx S_{k-j}F$. When $0 \leq j < k$ we have $k - j > 0$ so $S_{k-j}F$ is projectively stable. Therefore S^jS_kF is projectively stable for all $0 \leq j < k$ which is exactly the conditions $\underline{\text{Stab}}\ S_kF \geq k$. When $j = k$ we have $S^jS_kF \approx F$. When F is projectively stable, so is S^kS_kF; hence $\underline{\text{Stab}}\ S_kF > k$. Thus, condition (d) holds.

That (d) implies (e) is clear.

At this point let us remark that arguments completely dual to the ones just given imply, for Theorem (1.26), that the implications

(a) ===> (b) <===> (c) ===> (d) ===> (e) all hold. In particular, $\underline{\text{Stab}}\ G > n$ implies $\overline{\text{Stab}}\ S^nG \geq n$ for any half exact G.

Returning to the proof of Theorem (1.25), we assume that (e) holds. Then $F \approx S^n(S_nF)$ and $\underline{\text{Stab}}\ (S_nF) \geq n$. Using the remark just made, that Theorem (1.26) (a) ===> (e) holds, we see that $\overline{\text{Stab}}\ S^n(S_nF) \geq n$ (simply replace G by S_nF). Thus, (e) implies (f).

Finally, let us prove (f) implies (a). Suppose $F \approx S^nG$. As we have seen in Proposition (1.21), S^nG is a direct summand of $S^nS_nS^nG$. Therefore F is a direct summand of S^nS_nF and S_iF is a direct summand of $S_i(S^nS_nF)$ for all $i \geq 0$. The assumption $\overline{\text{Stab}}\ S^nS_nF \geq n$ tells us that $S_iS^nS_nF$ vanishes on injectives for $0 \leq i < n$. Consequently S_iF also vanishes on injectives for $0 \leq i < n$. This means that $\overline{\text{Stab}}\ F \geq n$ which completes our proof.

Recall now that it was shown in Corollary (1.17) that $S^1\colon \underline{\text{Hex}}(\Gamma,\Delta) \to \overline{\text{Hex}}(\Gamma,\Delta)$ and $S_1\colon \overline{\text{Hex}}(\Gamma,\Delta) \to \underline{\text{Hex}}(\Gamma,\Delta)$ establish an isomorphism of categories (modulo the identification $F = S_1S^1F$ when F is projectively stable, and $F = S^1S_1F$ when F is injectively stable). To use the preceding two theorems to generalize this, we establish the following terminology:

(1.27) Denote by $n - \underline{\text{Hex}}(\Gamma,\Delta)$ (resp. $n - \overline{\text{Hex}}(\Gamma,\Delta)$) the full subcategory of $\text{Hex}(\Gamma,\Delta)$ determined by those functors F such that $\underline{\text{Stab}}\ F \geq n$ (resp. $\overline{\text{Stab}}\ F \geq n$).

By Theorem (1.25), if F is in $n - \overline{\text{Hex}}(\Gamma,\Delta)$ then S_nF is in $n - \underline{\text{Hex}}(\Gamma,\Delta)$, and the map $S^nS_nF \to F$ is an isomorphism. By Theorem (1.26), if F is in $n - \underline{\text{Hex}}(\Gamma,\Delta)$ then S^nF is in $n - \overline{\text{Hex}}(\Gamma,\Delta)$ and $F \to S_nS^nF$ is an isomorphism. Thus, we can think of S^n and S_n as functors:

S_n: n - $\overline{\text{Hex}}(\Gamma,\Delta)$ -> n - $\underline{\text{Hex}}(\Gamma,\Delta)$

S^n: n - $\underline{\text{Hex}}(\Gamma,\Delta)$ -> n - $\overline{\text{Hex}}(\Gamma,\Delta)$

whose compositions S^nS_n and S_nS^n are isomorphic to the appropriate identity functors. If we agree to take these isomorphisms as identifications then we get:

(1.28) COROLLARY: The categories n - $\overline{\text{Hex}}(\Gamma,\Delta)$ and n - $\underline{\text{Hex}}(\Gamma,\Delta)$ are isomorphic via the functors S^n and S_n.

The next Theorem gives a characterization of half-exact functors F such that $\overline{\text{Stab}}\ S^nF \geq n$. (by (1.26) we know that this holds when $\underline{\text{Stab}}\ F \geq n$). Furthermore, it gives a universal mapping property for the functor S_nS^n restricted to the full subcategory of such F.

(1.29) THEOREM: Suppose F is in Hex(Γ,Δ). Then $\overline{\text{Stab}}\ S^nF \geq n$ if and only if there is a map F -> G such that $\underline{\text{Stab}}\ G \geq n$ and $S^nF \to S^nG$ is an isomorphism. Furthermore, if such a map exists then there is a unique isomorphism $S_nS^nF \to G$ such that the composition $F \to S_nS^nF \to G$ is the given map: F -> G.

Proof: Suppose $\overline{\text{Stab}}\ S^nF \geq n$. By applying Theorem (1.25) to S^nF we have $\underline{\text{Stab}}\ S_n(S^nF) \geq n$ and $S^nS_n(S^nF) \to (S^nF)$ is an isomorphism. Since the composition $S^nF \to S^nS_nS^nF \to S^nF$ is an isomorphism, we conclude that $S^nF \to S^nS_nS^nF$ is an isomorphism. This map, however, is obtained by applying S^n to $F \to S_nS^nF$. Therefore, letting $G = S_nS^nF$, we obtain the desired map.

Conversely, if F -> G induces $S^nF \xrightarrow{\sim} S^nG$ then it induces $S^nS_nS^nF \xrightarrow{\sim} S^nS_nS^nG$. If $\underline{\text{Stab}}\ G \geq n$ then applications of (1.25) and (1.26) give $\overline{\text{Stab}}\ S^nS_nS^nG \geq n$. Since S^nF is a direct summand of $S^nS_nS^nF \approx S^nS_nS^nG$, we conclude that $\overline{\text{Stab}}\ S^nF \geq n$ which is what we wanted.

Suppose now that the conditions just proved equivalent hold. Consider the commutative diagram:

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ S_nS^nF & \longrightarrow & S_nS^nG \end{array}$$

Since Stab $G \geq n$, the map $G \to S_nS^nG$ is an isomorphism. Therefore we may define a map $\alpha: S_nS^nF \xrightarrow{\sim} G$ such that the following commutes:

$$\begin{array}{ccc} F & \longrightarrow & G \\ & \searrow \quad \nearrow_{\alpha} & \\ & S_nS^nF & \end{array}$$

Now for any α rendering this diagram commutative we have the following commutative diagram:

$$\begin{array}{ccccc} F & & \longrightarrow & & G \\ \downarrow & & & & \downarrow \wr \\ S_nS^nF & & \longrightarrow & & S_nS^nG \\ \downarrow \wr & & & & \| \\ S_nS^nS_nS^nF & & \xrightarrow{S_nS^n\alpha} & & S_nS^nG \\ \uparrow \wr & & & & \uparrow \wr \\ S_nS^nF & & \xrightarrow{\alpha} & & G \end{array}$$

Therefore α is uniquely determined by F -> G, and our proof is complete.

We now state the dual result, whose proof is left to the reader.

(1.30) THEOREM: Suppose F is in Hex(Γ,Δ). Then $\underline{\text{Stab}}\ S_nF \geq n$ if and only if there is a map G -> F such that $S_nG \to S_nF$ is an isomorphism and $\overline{\text{Stab}}\ G \geq n$. If such a map exists, then there is a unique isomorphism $G \xrightarrow{\sim} S^nS_nF$ such that G -> F is the composition $G \xrightarrow{\sim} S^nS_nF \to F$.

Now let us turn our attention to a different sort of characterization of

of stability of satellites. Given a map: $\underline{G} \to \underline{F}$ we can apply S^i to it, obtaining a map $S^i\underline{G} \to S^i\underline{F}$. Since $S^i\underline{F} = S^iF$ for $i > 0$, we get a map $S^iG \to S^iF$. Thus, S^i determines a homomorphism: $(\underline{G},\underline{F}) \to (S^iG, S^iF)$ for any $i > 0$. In a similar way we have, for $i > 0$, a homomorphism: $(\overline{G},\overline{F}) \to (S_iG, S_iF)$.

(1.31) LEMMA:

(a) The maps $(\underline{G},\underline{F}) \xrightarrow{S^2} (S^2G, S^2F)$ and $(S^1G, S^1F) \to (S^1G, \underline{S^1F})$ (induced by $S^1F \to \underline{S^1F}$) have isomorphic kernels and cokernels.

(b) The maps $(\overline{G},\overline{F}) \xrightarrow{S_2} (S_2G, S_2F)$ and $(S_1G, S_1F) \to (\overline{S_1G}, S_1F)$ have isomorphic kernels and cokernels.

Proof: We prove (a) only, leaving the dual proof to the reader. As was shown in Corollary (1.18), S^1 defines an isomorphism $(\underline{G},\underline{F}) \xrightarrow{\sim} (S^1G, S^1F)$. Replacing F by S^1F, and G by S^1G, we also get an isomorphism: $(\underline{S^1G}, \underline{S^1F}) \xrightarrow{\sim} (S^2G, S^2F)$. Finally, since $\underline{S^1F}$ is projectively stable, the map $S^1G \to \underline{S^1G}$ induces an isomorphism $(\underline{S^1G}, \underline{S^1F}) \approx (S^1G, \underline{S^1F})$. Putting these maps and isomorphisms together, one sees easily that the following commutative diagram results:

$$\begin{array}{ccc} (\underline{G},\underline{F}) & \longrightarrow & (S^2G, S^2F) \\ & & \wr \uparrow S^1 \\ \wr & & (\underline{S^1G}, \underline{S^1F}) \\ & & \wr \\ (S^1G, S^1F) & \longrightarrow & (S^1G, \underline{S^1F}) \end{array}$$

Therefore the horizontal maps have isomorphic kernels and isomorphic cokernel

(1.32) PROPOSITION: For F in $\mathrm{Hex}(\Gamma,\Delta)$ the following statements are equivalent:

(a) $(\underline{G},\underline{F}) \to (S^2G, S^2F) \to 0$ is exact for all G in $\mathrm{Hex}(\Gamma,\Delta)$.

(b) There exists a G in $\mathrm{Hex}(\Gamma,\Delta)$ such that $S^1G \approx \underline{S^1F}$ and $(\underline{G},\underline{F}) \to (S^2G, S^2F) \to 0$ is exact.

(c) The epimorphism $S^1F \to \underline{S^1F} \to 0$ splits.

Proof: To show that (a) implies (b) we need only demonstrate the existence of G. Since S^1F vanishes on injectives, the exact sequence $S^1F \to \underline{S^1F} \to 0$ shows that $\underline{S^1F}$ does as well. Therefore $\underline{S^1F} = S^1S_1(\underline{S^1F})$ so we may take $G = S_1(\underline{S^1F})$.

Suppose now that (b) holds. Since $(\underline{G},\underline{F}) \to (S^2G, S^2F)$ and $(S^1G, S^1F) \to (S^1G, \underline{S^1F})$ have isomorphic cokernels by Lemma (1.31), we conclude that the latter map is surjective. Since $S^1G \approx \underline{S^1F}$ by assumption, we see that the exact sequence: $(\underline{S^1F}, S^1F) \to (\underline{S^1F}, \underline{S^1F}) \to 0$. Thus, there is a map $\underline{S^1F} \to S^1F$ such that the composition $\underline{S^1F} \to S^1F \to \underline{S^1F}$ is the identity. This, however, is what is menat by $S^1F \to \underline{S^1F} \to 0$ splitting.

Finally, suppose that $S^1F \to \underline{S^1F} \to 0$ splits. As we have seen in Lemma (1.10), this is equivalent to the splitting of the exact sequence $0 \to L_0(S^1F) \to (S^1F) \to \underline{S^1F} \to 0$. Applying the (additive functor $(S^1G, \ )$, we get the sequence

$$0 \to (S^1G, L_0(S^1F)) \to (S^1G, S^1F) \to (S^1G, \underline{S^1F}) \to 0$$

which is exact and splits. Since the cokernel of $(S^1G, S^1F) \to (S^1G, \underline{S^1F})$ is isomorphic to the cokernel of $(\underline{G},\underline{F}) \to (S^2G, S^2F)$, we see that this last map is surjective, and we are done.

(1.33) PROPOSITION: For F in $\mathrm{Hex}(\Gamma,\Delta)$, the following statements are equivalent:

(a) $(\underline{G},\underline{F}) \to (S^2G, S^2F)$ is an isomorphism for all G in $\mathrm{Hex}(\Gamma,\Delta)$.

(b) there exists a G in $\mathrm{Hex}(\Gamma,\Delta)$ such that $S^1G \approx L_0S^1F$ and $(\underline{G},\underline{F}) \to (S^2G, S^2F)$ is a monomorphism.

(c) S^1F is projectively stable.

Proof: To prove (a) implies (b) we must find G. By the previous Proposition $S^1F \to \underline{S^1F} \to 0$ splits. Again, by Lemma (1.10), this shows that $0 \to L_0(S^1F) \to S^1F \to \underline{S^1F} \to 0$ is exact and splits. Since L_0S^1F is then a direct summand of the injectively stable functor S^1F, it follows that L_0S^1F is injectively stable. Therefore $L_0S^1F = S^1S_1(L_0S^1F)$ so we may take $G = S_1L_0S^1F$.

Now suppose (b) holds. Since $S^1G \approx L_0S^1F$, we see that S^1G is right exact. As was pointed out in (1.14), the right satellites of a right exact functor are all zero. Therefore $S^2G = S^1(S^1G) = 0$. Since $0 \to (\underline{G},\underline{F}) \to (S^2G,S^2F)$ is exact we conclude that $(\underline{G},\underline{F}) = 0$. Since $(\underline{G},\underline{F}) \approx (S^1G,S^1F)$, we have $(S^1G,S^1F) = 0$. But $S^1G \approx L_0S^1F$, so $(L_0S^1F,S^1F) = 0$. Since the canonical map $L_0(S^1F) \to S^1F$ is an isomorphism on projectives, we must have S^1F vanishing on projectives. Therefore S^1F is projectively stable.

Finally, suppose S^1F is projectively stable. Then $S^1F = \underline{S^1F}$ so $(S^1G,S^1F) \to (S^1G,\underline{S^1F})$ is an isomorphism. Since this map has the same kernel and cokernel as $(\underline{G},\underline{F}) \to (S^2G,S^2F)$, we conclude that this last map is also an isomorphism. Therefore (c) implies (a) and our proof is complete.

We shall see in Proposition (1.39) below that when Δ is the category of abelian groups, and F is of the form $F = (A, \ )$ where A is an object of Γ, then we need only test condition (a) of the previous proposition on functors of the form $G = (B, \ )$.

(1.34) COROLLARY: Consider the following conditions on the half exact functor F and the integer n:

(a) S^iF is projectively stable for $1 \leq i < n$.

(b) $(\underline{G},\underline{F}) \to (S^iG,S^iF)$ is an isomorphism for $1 \leq i \leq n$ and all G in $\mathrm{Hex}(\Gamma,\Delta)$.

Then for any F, condition (a) implies condition (b). If, for each $1 \leq j < n$ there is a G_j in $\mathrm{Hex}(\Gamma,\Delta)$ such that $S^jG_j \approx L_0S^jF$, then (a) and (b) are each equivalent to:

(c) $(\underline{G_j},\underline{F}) \xrightarrow{\sim} (S^iG_j,S^iF)$ for $1 \leq i \leq j+1$ and $1 \leq j < n$.

Proof: That (a) implies (b) follows, by straight-forward induction on n, from Proposition (1.33).

Suppose the functors G_j exist. Then clearly (b) implies (c). Let us prove (c) implies (a). We proceed by induction on n. When $n = 1$ condition (a) is vacuous so there is nothing to prove. When $n = 2$ we have precisely the condition: (b) implies (c); of Proposition (1.33). Assume our implication holds, then, for $1,2,\ldots,n-1$. By induction hypothesis S^iF is projectively stable for $1 \leq i < n-1$, so we need only show that $S^{n-1}F$ is projectively stable.

Let $G = S^{n-2}G_{n-1}$. By assumption we have $S^1G \approx L_0S^1(S^{n-2}F)$. Consider the following commutative diagram:

$$\begin{array}{ccc}
(\underline{G},\underline{S^{n-2}F}) & \longrightarrow & (S^2G,S^2(S^{n-2}F)) \\
\| & & \| \\
(S^1\underline{G},S^1(\underline{S^{n-2}F})) & & (S^nG_{n-1},S^nF) \\
\| & & \uparrow \\
(S^{n-1}G_{n-1},S^{n-1}F) & \longleftarrow & (\underline{G_{n-1}},\underline{F})
\end{array}$$

Property (c) gives that the maps $(\underline{G_{n-1}},\underline{F}) \to (S^{n-1}G_{n-1},S^{n-1}F)$ and $(\underline{G_{n-1}},\underline{F}) \to (S^nG_{n-1},S^nF)$ are isomorphisms. Therefore $(\underline{G},\underline{S^{n-2}F}) \to (S^2G,S^2(S^{n-2}F))$ is an isomorphism. From Proposition (1.33) we conclude that $S^1(S^{n-2}F) = S^{n-1}F$ is projectively stable, finishing the induction and the proof.

We now pursue a slightly different line of investigation. If F is

half exact we have, for every short exact sequence $0 \to C' \to C \to C'' \to 0$ the usual long exact sequence:

$$F(C') \to F(C) \to F(C'') \to S^1F(C') \to S^1F(C) \to S^1F(C'') \to \cdots$$

$$\cdots \to S^{i-1}F(C'') \to S^iF(C') \to S^iF(C) \to S^iF(C'') \to \cdots .$$

If we apply S_1S^1 to this we obtain a sequence of order two:

($): $\underline{F}(C') \to \underline{F}(C) \to \underline{F}(C'') \to \cdots \to \underline{S^iF}(C') \to \underline{S^iF}(C) \to \underline{S^iF}(C'') \to \cdots$

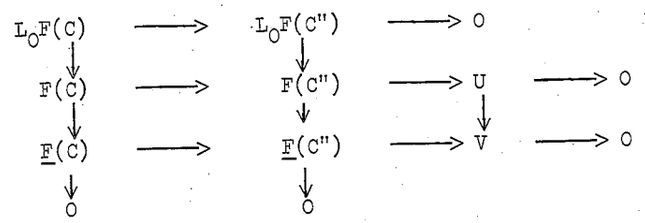
We wish to determine when this is exact for $1 \leq i \leq n$ and all short exact sequences $0 \to C' \to C \to C'' \to 0$.

We note first that if S^iF is projectively stable for $1 \leq i \leq n$ then $S^iF = \underline{S^iF}$ so the sequence ($) reduces to the preceding sequence for $1 \leq i \leq n$, hence is exact. To prove the converse we need the following Lemma.

(1.35) LEMMA: For any F in $\mathrm{Hex}(\Gamma,\Delta)$ and short exact sequence $0 \to C' \to C \to C'' \to 0$ we have:

$\operatorname{coker}(F(C) \to F(C'')) \approx \operatorname{coker}(\underline{F}(C) \to \underline{F}(C''))$.

Proof: Consider the following exact and commutative diagram:



We must prove $U \approx V$. But this is a trivial exercise in diagram chasing.

We now can prove:

(1.36) PROPOSITION: Let F be half exact and $n \geq 0$ be an integer. Then the following statements are equivalent:

(a) S^iF is projectively stable for $1 \leq i \leq n$.

(b) The sequence ($) is exact for $1 \leq i \leq n$ and all short exact sequences $0 \to C' \to C \to C'' \to 0$.

(c) The sequence ($) is exact for $1 \leq i \leq n$ and all short exact sequences $0 \to C' \to C \to C'' \to 0$ where C' is projective and C is injective.

Proof: As we have seen, condition (a) gives $S^iF = \underline{S^iF}$ for $1 \leq i \leq n$ so we need only consider the sequence $\underline{F}(C') \to \underline{F}(C) \to \underline{F}(C'') \to S^1F(C') \to S^1F(C) \to S^1F(C'')$. Since $\underline{F}$ is half exact when F is, the sequence $\underline{F}(C') \to \underline{F}(C) \to \underline{F}(C'')$ is exact. By the previous Lemma, $\operatorname{coker}(\underline{F}(C) \to \underline{F}(C'')) \approx \operatorname{coker}(F(C) \to F(C''))$. This last cokernel is isomorphic to $\ker(S^1F(C') \to S^1F(C))$ since the sequence $F(C') \to F(C) \to F(C'') \to S^1F(C') \to S^1F(C) \to S^1F(C'')$ is exact. Therefore we may splice $\underline{F}(C') \to \underline{F}(C) \to \underline{F}(C'')$ to $S^1F(C') \to S^1F(C) \to S^1F(C'')$, preserving exactness.

Since it is clear that (b) implies (c), we need only prove that (c) implies (a). Let P be any projective. Choose an injective I so that there is an exact sequence $0 \to P \to I \to I/P \to 0$. Let $G = S^{i-1}F$. Since $G(P) \to G(I) \to G(I/P) \to S^1G(P) \to S^1G(I) \to S^1G(I/P)$ is exact, we have $\ker(S^1G(P) \to S^1G(I)) \approx \operatorname{coker}(G(I) \to G(I/P))$. By Lemma (1.35) we have $\operatorname{coker}(G(I) \to G(I/P)) \approx \operatorname{coker}(\underline{G}(I) \to \underline{G}(I/P))$. Finally, if $1 \leq i \leq n$, then condition (c) gives the exactness of:

$\underline{G}(P) \to \underline{G}(I) \to \underline{G}(I/P) \to \underline{S^1G}(P) \to \underline{S^1G}(I) \to \underline{S^1G}(I/P)$ so

$\operatorname{coker}(\underline{G}(I) \to \underline{G}(I/P)) \approx \ker(\underline{S^1G}(P) \to \underline{S^1G}(I))$. Putting these isomorphisms together, we deduce that $\ker(S^1G(P) \to S^1G(I)) \approx \ker(\underline{S^1G}(P) \to \underline{S^1G}(I))$. Denoting these isomorphic kernels by K, we have the exact and commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & S^1G(P) & \longrightarrow & S^1G(I) \\ & & \| & & \downarrow & & \downarrow \\ 0 & \longrightarrow & K & \longrightarrow & \underline{S^1G}(P) & \longrightarrow & \underline{S^1G}(I) \end{array}$$

Since I is injective, $S^1G(I) = 0 = \underline{S^1G}(I)$. Since P is projective, $\underline{S^1G}(P) = 0$. We conclude that $S^1G(P) = 0$, or: $S^iF(P) = 0$. Since P was arbitrary, we conclude that S^iF is projectively stable for $1 \leq i \leq n$.

We conclude this section by using the Proposition just proved to strengthen a previous result. In (1.33) it was shown that S^1F is projectively stable if and only if $(G,F) \to (S^2G, S^2F)$ is an isomorphism for all half exact G. We shall now suppose that Δ is the category of abelian groups and F is of the form $(A, \ )$ for A an object of Γ. We shall show that it suffices to test the map $(G,F) \to (S^2G, S^2F)$ on functors of the form $G = (B, \ )$ in order to ascertain whether S^1F is projectively stable.

We shall need the following well-known result of Yoneda:

(1.37) LEMMA: Let χ be a category and F a set-valued functor. Let A be any object of χ. If Φ is a natural transformation $(A, \ ) \to F$, then $\Phi(1_A) \in F(A)$ and the correspondence $\Phi \rightsquigarrow \Phi(1_A)$ establishes an isomorphism: $((A, \ ),F) \xrightarrow{\sim} F(A)$. Furthermore, this isomorphism is functorial in F and A.

(1.38) COROLLARY: Let Γ be an abelian category with enough projectives. Let $\underline{\mathrm{Hom}}(A, \ )$, for A in Γ, be the projective stabilization of the functor $\mathrm{Hom}(A, \ )$, and let $\mathrm{Hom}(A, \ ) \to \underline{\mathrm{Hom}}(A, \ )$ be the natural map of a functor into its projective stabilization. Then there is a natural isomorphism: $(\underline{\mathrm{Hom}}(B, \ ), \underline{\mathrm{Hom}}(A, \ )) \approx \underline{\mathrm{Hom}}(A,B)$ such that the following diagram commutes:

$$\begin{array}{ccc} & \mathrm{Hom}(A,B) & \\ \swarrow & & \searrow \\ (\underline{\mathrm{Hom}}(B, \ ), \underline{\mathrm{Hom}}(A, \ )) & \approx & \underline{\mathrm{Hom}}(A,B) \end{array}$$

Proof: We note first that the map $\mathrm{Hom}(A,B) \to (\underline{\mathrm{Hom}}(B, \ ), \underline{\mathrm{Hom}}(A, \ ))$ may be described in the following way. For $f \in \mathrm{Hom}(A,B)$ we get a transformation $(\mathrm{Hom}(B, \ ), \mathrm{Hom}(A, \ ))$. Composing with the natural map $\mathrm{Hom}(A, \ ) \to \underline{\mathrm{Hom}}(A, \ )$ shows that f determines an element of $(\mathrm{Hom}(B, \ ), \underline{\mathrm{Hom}}(A, \ ))$. Now we have seen--Proposition (1.4)--for any functors F and G we have $(F, \underline{G}) \approx (\underline{F}, \underline{G})$. Our map is then the composition: $\mathrm{Hom}(A,B) \to (\mathrm{Hom}(B, \ ), \underline{\mathrm{Hom}}(A, \ )) \approx (\underline{\mathrm{Hom}}(B, \ ), \underline{\mathrm{Hom}}(A, \ ))$. Furthermore, Yoneda's result gives $(\mathrm{Hom}(B, \ ), \underline{\mathrm{Hom}}(A, \ )) \approx \underline{\mathrm{Hom}}(A,B)$. We now have the following diagram which we claim is commutative:

$$\begin{array}{ccc} & \mathrm{Hom}(A,B) & \\ \swarrow & & \searrow \\ (\underline{\mathrm{Hom}}(B, \ ), \underline{\mathrm{Hom}}(A, \ )) & \approx & (\mathrm{Hom}(B, \ ), \underline{\mathrm{Hom}}(A, \ )) \approx \underline{\mathrm{Hom}}(A,B) \end{array}$$

In fact, starting with $f \in \mathrm{Hom}(A,B)$ we see that its image in $(\mathrm{Hom}(B, \ ), \underline{\mathrm{Hom}}(A, \ ))$ is the composition $\mathrm{Hom}(B, \ ) \xrightarrow{(f, \ )} \mathrm{Hom}(A, \ ) \to \underline{\mathrm{Hom}}(A, \ )$. The image of this in $\underline{\mathrm{Hom}}(A,B)$ under the Yoneda isomorphism is simply the image of 1_B (identity of B) via the composition: $\mathrm{Hom}(B,B) \to \mathrm{Hom}(A,B) \to \underline{\mathrm{Hom}}(A, \ )(B)$. This, however, is precisely the image of $1_B \circ f = f$ under the map: $\mathrm{Hom}(A,B) \to \underline{\mathrm{Hom}}(A,B)$. Thus the diagram commutes, and our desired isomorphism is the composition of the two isomorphisms appearing in its bottom row.

We now come to our main result. Let Γ be once again an abelian category with enough projectives and injectives, Δ the category of abelian groups,

and A any object in Γ. Let F in $\mathrm{Hom}(\Gamma,\Delta)$ be the functor $\mathrm{Hom}(A, \ )$. Then $\underline{F} = \underline{\mathrm{Hom}}(A, \ )$. Furthermore, it is easily seen that $S^iF = \mathrm{Ext}^i(A, \ )$ for $i \geq 0$--for details see [4, Ch. V, 6.1]. When $i > 0$ we also have $S^iF = S^i\underline{F} = \mathrm{Ext}^i(A, \ )$. If B is another object of Γ and $G = \mathrm{Hom}(B, \ )$, then the maps $(\underline{G},\underline{F}) \to (S^iG, S^iF)$ become the maps $(\underline{\mathrm{Hom}}(B, \ ), \underline{\mathrm{Hom}}(A, \ )) \to (\mathrm{Ext}^i(B, \ ), \mathrm{Ext}^i(A, \ ))$ if $i > 0$. We now prove:

(1.39) PROPOSITION: Let $F = \mathrm{Hom}(A, \ )$ for A in Γ. The following conditions are equivalent:

(a) $\mathrm{Ext}^1(A, \ )$ is projectively stable.

(b) $(\underline{G},\underline{\mathrm{Hom}}(A, \ )) \to (S^2G, \mathrm{Ext}^2(A, \ ))$ is an isomorphism for all half exact functors $G: \Gamma \to$ abelian groups.

(c) $(\underline{\mathrm{Hom}}(B, \ ), \underline{\mathrm{Hom}}(A, \ )) \to (\mathrm{Ext}^2(B, \ ), \mathrm{Ext}^2(A, \ ))$ is an isomorphism for all objects B in Γ.

Proof: That (a) implies (b) has been shown in Proposition (1.33), while (b) clearly implies (c). We now prove that (c) implies (a). We recall that Proposition (1.36) tells us that $S^1F = \mathrm{Ext}^1(A, \ )$ will be projectively stable if, for any short exact sequence $0 \to P \to I \to I/P \to 0$, with P projective and I injective, the following six term sequence is exact:
$\underline{\mathrm{Hom}}(A,P) \to \underline{\mathrm{Hom}}(A,I) \to \underline{\mathrm{Hom}}(A,I/P) \to \underline{\mathrm{Ext}}^1(A,P) \to \underline{\mathrm{Ext}}^1(A,I) \to \underline{\mathrm{Ext}}^1(A,I/P)$. Since $\underline{\mathrm{Hom}}(A, \ )$ and $\underline{\mathrm{Ext}}^1(A, \ )$ are projectively stable, and P is projective, $\underline{\mathrm{Hom}}(A,P) = \underline{\mathrm{Ext}}^1(A,P) = 0$. Furthermore, $\underline{\mathrm{Hom}}(A, \ )$ and $\underline{\mathrm{Ext}}^1(A, \ )$ are half-exact, so we have, in any case, the exact sequences:
$0 \to \underline{\mathrm{Hom}}(A,I) \to \underline{\mathrm{Hom}}(A,I/P)$ and $0 \to \underline{\mathrm{Ext}}^1(A,I) \to \underline{\mathrm{Ext}}^1(A,I/P)$. Thus, to prove the six term sequence is exact, it is only required to show that $\underline{\mathrm{Hom}}(A,I) \to \underline{\mathrm{Hom}}(A,I/P)$ is an isomorphism. By Corollary (1.38) we have the following commutative diagram:

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(A,I) & \approx & (\underline{\mathrm{Hom}}(I, \ ), \underline{\mathrm{Hom}}(A, \ )) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(A,I/P) & \approx & (\underline{\mathrm{Hom}}(I/P, \ ), \underline{\mathrm{Hom}}(A, \ )) \end{array}$$

Therefore it will suffice to prove that the right-hand vertical map is an isomorphism. However, by Assumption (c) the horizontal maps in the following commutative diagram are isomorphisms:

$$\begin{array}{ccc} (\underline{\mathrm{Hom}}(I, \ ), \underline{\mathrm{Hom}}(A, \ )) & \approx & (\mathrm{Ext}^2(I, \ ), \mathrm{Ext}^2(A, \ )) \\ \downarrow & & \downarrow \\ (\mathrm{Hom}(I/P, \ ), \mathrm{Hom}(A, \ )) & \approx & (\mathrm{Ext}^2(I/P, \ ), \mathrm{Ext}^2(A, \ )) \end{array}$$

It now suffices to prove that this right-hand vertical map is an isomorphism. Now the exact sequence $0 \to P \to I \to I/P \to 0$ and the fact that P is projective give $\mathrm{Ext}^2(I/P, \ ) \approx \mathrm{Ext}^2(I, \ )$, so $(\mathrm{Ext}^2(I, \ ), \mathrm{Ext}^2(A, \ )) \xrightarrow{\sim} (\mathrm{Ext}^2(I/P, \ ), \mathrm{Ext}^2(A, \ ))$. Therefore our six term sequence was exact. Since $0 \to P \to I \to I/P \to 0$ was arbitrary, Proposition (1.36) gives $S^1F = \mathrm{Ext}^1(A, \ )$ is projectively stable.

§4: The Stabilization of $\mathrm{Hom}(A, \ )$

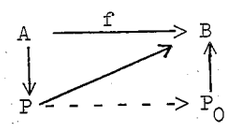
In this Section we take a close look at the functor $\underline{\mathrm{Hom}}(A, \ )$ = the projective stabilization of the functor $\mathrm{Hom}(A, \ )$. To simplify notation we shall write $(\underline{A},\underline{\ })$ for $\underline{\mathrm{Hom}}(A, \ )$. Thus, $(\underline{A},\underline{B})$ will denote $\underline{\mathrm{Hom}}(A, \ )(B)$. Since our first results will be quite general we shall assume, until further notice, only that Γ is an abelian category with enough projectives. All functors under consideration will have values in the category of abelian groups. Unless otherwise stated, objects and maps will be those of Γ.

We have, by definition, an exact sequence of functors:
$I_0(A, \ ) \to (A, \ ) \to (\underline{A},\underline{\ }) \to 0$. Therefore, for any object B we have:

$K = \ker((A,B) \to (\underline{A},\underline{B})) = \mathrm{im}(L_0(A,\ )(B) \to (A,B))$. To compute K, let $P_1 \to P_0 \to B \to 0$ be exact with P_0 and P_1 projective. By definition $L_0(A,\ )(B) = \mathrm{coker}((A,P_1) \to (A,P_0))$ and the transformation $L_0(A,\ ) \to (A,\ )$, applied to B, renders the following diagram commutativ

$$\begin{array}{ccccc} (A,P_1) & \to & (A,P_0) & \to & L_0(A,\ )(B) \to 0 \\ \| & & \| & & \downarrow \\ (A,P_1) & \to & (A,P_0) & \to & (A,B) \end{array}$$

Claim: The element $f \in (A,B)$ lies in K if and only if there is a factorization of f of the form: $f = (A \to P \to B)$ where P is projective. To see this, note that $K = \mathrm{im}(L_0(A,\ )(B) \to (A,B)) = \mathrm{im}((A,P_0) \to (A,B))$. Thus, if $f \in K$ then f factors $A \to P_0 \to B$, and P_0 was chosen projective. Conversely, if f factors: $A \to P \to B$ with P projective, then we have the following commutative diagram:



Since $P_0 \to B$ is an epimorphism and P is projective, the dotted map exists, leaving the diagram commutative. We can now write f as the composition $A \to P \to P_0 \to B$ so $f \in \mathrm{im}((A,P_0) \to (A,B)) = K$ and the claim is proven. We conclude that for each A and B in Γ there is an exact sequence:

$$0 \to \{f \in (A,B) \mid f \text{ factors through a projective}\} \to (A,B) \to (\underline{A},\underline{B}) \to 0.$$

We may therefore identify $(\underline{A},\underline{B})$ with (A,B) modulo the equivalence relati defined by: $f \sim g$ if and only if $f - g$ factors through a projective.

We can give another description of $(\underline{A},\underline{B})$. As was seen in Corollary (1. there is an isomorphism $(\underline{A},\underline{B}) \approx ((\underline{B},__),(\underline{A},__))$ such that the following

commutes:

$$\begin{array}{ccc} & (A,B) & \\ \swarrow & & \searrow \\ ((\underline{B},__),(\underline{A},__)) & \approx & (\underline{A},\underline{B}) \end{array}$$

However, for any functors F and G we know that $(\underline{F},\underline{G}) \approx (S^1F,S^1G)$. Applying this to $F = (B,\ )$ and $G = (A,\ )$ we obtain $((\underline{B},__),(\underline{A},__)) \approx (\mathrm{Ext}^1(B,\ ),\mathrm{Ext}^1(A,\ ))$. We now have the commutative diagram:

$$\begin{array}{ccc} & (A,B) & \\ \swarrow & & \searrow \\ (\mathrm{Ext}^1(B,\ ),\mathrm{Ext}^1(A,\ )) & \approx & (\underline{A},\underline{B}) \end{array}$$

From this and the previous description of $(\underline{A},\underline{B})$ we obtain the well-known result of Eckmann-Hilton [5] that every natural transformation $\mathrm{Ext}^1(B,\ ) \to \mathrm{Ext}^1(A,\ )$ comes from a map: $A \to B$, while a map $A \to B$ induces the zero transformation if and only if it factors through a projective. We summarize our results in the following.

(1.40) THEOREM: Let Γ be an abelian category with enough projectives. For any object A let $(\underline{A},__)$ denote the projective stabilization of the functor $(A,\ )$. Finally, for objects A and B let $K = \{f \in (A,B) \mid f$ factors through a projective$\}$. Then there is an isomorphism $(\underline{A},\underline{B}) \approx (\mathrm{Ext}^1(B,\ ),\mathrm{Ext}^1(A,\ ))$ and an exact and commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & K & \to & (A,B) & \to & (\underline{A},\underline{B}) & \to & 0 \\ & & \| & & \| & & \wr & & \\ 0 & \to & K & \to & (A,B) & \to & (\mathrm{Ext}^1(B,\ ),\mathrm{Ext}^1(A,\ )) & \to & 0 \end{array}$$

Turning now from maps to objects, we note that since Ext is an additive functor, and since $\mathrm{Ext}^1(P,\ ) = 0$ if (and only if) P is projective, we see that $\mathrm{Ext}^1(A + P,\ ) \approx \mathrm{Ext}^1(A,\ )$ if the object P is projective (the

isomorphism being induced by $A \to A + P$). It follows that if A and B are two objects such that, for some projectives P and Q, $A + P \approx B + Q$, then $\mathrm{Ext}^1(A, \ ) \approx \mathrm{Ext}^1(B, \ )$. Here the isomorphism is induced by: $B \to B + Q \approx A + P \to A$. That the converse is also true is a result of Eckmann-Hilton [5] and Hilton-Rees [6] which we record here for future reference:

(1.41) PROPOSITION: For objects A and B of Γ, the following are equivalent:

(a) If $Q \to A \to 0$ is exact with Q projective, then there is a projective P and an isomorphism $B + Q \approx A + P$.

(b) There exist projectives P and Q such that $A + P \approx B + Q$.

(c) There is a natural isomorphism $\mathrm{Ext}^1(A, \ ) \approx \mathrm{Ext}^1(B, \ )$.

(1.42) COROLLARY: If $\mathrm{Ext}^1(A, \ ) \approx \mathrm{Ext}^1(B, \ )$, then $\mathrm{Ext}^i(A, \ ) \approx \mathrm{Ext}^i(B, \ )$ for all $i > 0$.

Proof: By Proposition (1.41), there exist projectives P and Q such that $A + P \approx B + Q$. Therefore we have for $i > 0$: $\mathrm{Ext}^i(A, \ ) \approx \mathrm{Ext}^i(A + P, \ ) \approx \mathrm{Ext}^i(B + Q, \ ) \approx \mathrm{Ext}^i(B, \ )$.

§5: The Stabilized Category $\underline{\Gamma}$

In order to formalize the discussion of the previous section - §4 - it is convenient to construct from the category Γ (abelian, with enough projectives) a new category called the (projective) stabilization of Γ. This category will provide a natural setting for the study of the Ext and Tor functors, and their stability properties.

(1.43) Definition: If Γ is an abelian category with enough projectives, define the category $\underline{\Gamma}$ in the following way:

(a) The objects of $\underline{\Gamma}$ are the objects of Γ.

(b) $\mathrm{Hom}_{\underline{\Gamma}}(A,B) = \underline{\mathrm{Hom}}(A,B) = (\underline{A},\underline{B})$.

$\underline{\Gamma}$ is called the projective stabilization of Γ.

In order to avoid confusion, we shall denote the object A by $\underline{A}$ when considering it as an object of $\underline{\Gamma}$. If $f \in (A,B)$, we shall denote its image in $(\underline{A},\underline{B})$ by $\underline{f}$. This image is an equivalence class of maps in (A,B) which differ from f by a map factoring through a projective (1.40).

Now in any category, the isomorphism class of an object is determined by its maps into other objects, i.e. by the functor $(A, \ )$. It follows that $A \approx B$ in $\underline{\Gamma}$ if and only if $(\underline{A}, _) \approx (\underline{B}, _)$. Furthermore, a straight forward application of the isomorphism $((\underline{A}, _), (\underline{B}, _)) \approx (\mathrm{Ext}^1(A, \ ), \mathrm{Ext}^1(B, \ ))$ shows that $(\underline{A}, _) \approx (\underline{B}, _)$ if and only if $\mathrm{Ext}^1(A, \ ) \approx \mathrm{Ext}^1(B, \ )$. Combining this with Proposition (1.41) we obtain:

(1.44) PROPOSITION: The following are equivalent:

(a) $\underline{A} \approx \underline{B}$ in $\underline{\Gamma}$.

(b) $(\underline{A}, _) \approx (\underline{B}, _)$ as functors.

(c) $\mathrm{Ext}^1(A, \ ) \approx \mathrm{Ext}^1(B, \ )$ as functors.

(d) There exist projectives P and Q in Γ such that $A + P \approx B + Q$.

(3.6) Definition

(a) If f and g are in (A,B), we say that f and g are projectively equivalent if $\underline{f} = \underline{g}$ in $(\underline{A},\underline{B})$.

(b) We say that A and B are projectively equivalent if $\underline{A} \approx \underline{B}$.

Suppose now that AG is the category of abelian groups and $\underline{\Gamma}^*$ denotes the dual category of $\underline{\Gamma}$. Then the functor $\underline{\Gamma}^* \to (\Gamma, AG)$ given by $A \rightsquigarrow \mathrm{Ext}^1(A, \ )$ establishes a natural equivalence between $\underline{\Gamma}^*$ and the full subcategory of (Γ, AG) determined by functors of the form $\mathrm{Ext}^1(A, \ )$ for A in Γ. For a discussion of this representation and related questions see [1] and [2].

Chapter 2: Applications To Noetherian Rings

In this Chapter we shall apply the techniques of Chapter one to the study of finitely generated modules over noetherian rings. Such a module M will be represented by the functor $F = (\underline{M}, _)$ and the stability properties of F will be examined and interpreted. Throughout, R will be a ring with unit, noetherian on both sides. Unless stated otherwise, those R-modules under consideration will be assumed finitely generated.

Let us denote by ${}_R\mathcal{M}$ (resp. $\mathcal{M}_R$) the category of finitely generated left (resp. right) R-modules. In general, our results will be stated for modules in ${}_R\mathcal{M}$; however, if no remark is made to the contrary it will be assumed that a similar result holds for right R-modules. Finally, ${}_R\underline{\mathcal{M}}$ and $\underline{\mathcal{M}}_R$ will denote the (projective) stabilizations of the categories ${}_R\mathcal{M}$ and $\mathcal{M}_R$ respectively.

§1: The Functors D and Ω^i.

In this section we construct a functor D which will be useful in describing the left satellites of $(\underline{M}, _)$. We begin by recalling from [1] that if $P_1 \to P_0 \to M \to 0$ is any exact sequence of finitely generated modules with P_1 and P_0 projective, then there is an exact sequence of functors:

$$(2.1) \qquad 0 \to \mathrm{Ext}^1_R(Z', \ ) \to (M \otimes_R \ ) \xrightarrow{\varphi} (M^*, \ ) \to \mathrm{Ext}^2_R(Z', \ ) \to 0$$

where

(a) $Z' = \mathrm{coker}\,(P_0^* \to P_1^*)$ and

(b) $(M \otimes_R \ ) \xrightarrow{\varphi} (M^*, \ )$ is defined in the following manner. For a

module N, $\varphi(m \otimes n)\,(m^*) = m^*(m) \cdot n$ where $m \in M$, $n \in N$ and $m^* \in M^*$.

Furthermore, this exact sequence is functorial in the exact sequence $P_1 \to P_0 \to M \to 0$. In other words, for every exact commutative diagram of finitely generated modules, with the P_i and Q_i projective:

$$\begin{array}{ccccccc} P_1 & \to & P_0 & \to & M & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ Q_1 & \to & Q_0 & \to & N & \to & 0 \end{array}$$

we obtain an exact and commutative diagram of functors:

$$\begin{array}{ccccccccc} 0 \to \mathrm{Ext}^1(Z', \ ) & \to & (M \otimes_R \ ) & \to & (M^*, \ ) & \to & \mathrm{Ext}^2(Z', \ ) & \to & 0 \\ \downarrow & & \downarrow (f \otimes_R \ ) & & \downarrow (f^*, \ ) & & \downarrow & & \\ 0 \to \mathrm{Ext}^1(Z'', \ ) & \to & (N \otimes_R \ ) & \to & (N^*, \ ) & \to & \mathrm{Ext}^2(Z'', \ ) & \to & 0 \end{array}$$

where $Z'' = \mathrm{coker}\,(Q_0^* \to Q_1^*)$.

From this diagram it is clear that the map $\mathrm{Ext}^1(Z', \ ) \to \mathrm{Ext}^1(Z'', \ )$ is uniquely determined by f. This map is also obtained by applying Ext to the map $Z'' \to Z'$ induced by the following diagram:

$$\begin{array}{ccccccc} P_0^* & \to & P_1^* & \to & Z' & \to & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ Q_0^* & \to & Q_1^* & \to & Z'' & \to & 0 \end{array}$$

We now show that $\mathrm{Ext}^1(Z', \ ) \to \mathrm{Ext}^1(Z'', \ )$ depends only on the image $\underline{f}$ of f in $(\underline{M}, \underline{N})$ (Recall that $(\underline{M}, \underline{N}) = \underline{\mathrm{Hom}}_R(M, \ )(N)$). Suppose that

f factors through a projective--i.e. $\underline{f} = 0$. It is easily seen that f then factors through any projective mapping onto N. Since N is finitely generated, we may assume that f factors through a projective P which is finitely generated. It follows that $\mathrm{Ext}^1(Z', \ ) \to \mathrm{Ext}^1(Z'', \ )$ factors through the kernel of $(P \otimes_R \ ) \to (P^*, \ )$. Now it is well known that a finitely generated module P is projective if and only if $(P \otimes_R \ ) \to (P^*, \ )$ is an isomorphism. By our choice of P, we conclude that $\mathrm{Ext}^1(Z', \ ) \to \mathrm{Ext}^1(Z'', \ )$ is zero. Since all functors involved here are additive, we conclude that if f and g lie in (M,N) and if $f-g$ factors through a projective, then f and g induce the same map $\mathrm{Ext}^1(Z', \ ) \to \mathrm{Ext}^1(Z'', \ )$. In other words, we have:

(2.2) LEMMA: The map $\mathrm{Ext}^1(Z', \ ) \to \mathrm{Ext}^1(Z'', \ )$ induced by $f \in (M,N)$ depends only on $\underline{f} \in (\underline{M},\underline{N})$.

(2.3) COROLLARY: If $\underline{M} \approx \underline{N}$ in ${}_R\underline{\mathcal{M}}$, then $\underline{Z}' \approx \underline{Z}''$ in $\underline{\mathcal{M}}_R$.

Proof: By Proposition (1.44), A and B are isomorphic in ${}_R\underline{\mathcal{M}}$ (resp. $\underline{\mathcal{M}}_R$) if and only if $\mathrm{Ext}^1(A, \ ) \approx \mathrm{Ext}^1(B, \ )$. If $\underline{M} \approx \underline{N}$ then we can find maps $\alpha: M \to N$ and $\beta: N \to M$ such that $\underline{\beta} \cdot \underline{\alpha} = \underline{1}_M$ and $\underline{\alpha} \cdot \underline{\beta} = \underline{1}_N$. If $\mathrm{Ext}^1(Z', \ ) \to \mathrm{Ext}^1(Z'', \ )$ and $\mathrm{Ext}^1(Z'', \ ) \to \mathrm{Ext}^1(Z', \ )$ are induced by α and β respectively, then Lemma (2.2) tells us that the compositions $\mathrm{Ext}^1(Z', \ ) \to \mathrm{Ext}^1(Z'', \ ) \to \mathrm{Ext}^1(Z', \ )$ and $\mathrm{Ext}^1(Z'', \ ) \to \mathrm{Ext}^1(Z', \ ) \to \mathrm{Ext}^1(Z'', \ )$ are induced by 1_M and 1_N respectively. Therefore $\mathrm{Ext}^1(Z', \ ) \approx \mathrm{Ext}^1(Z'', \ )$ so $\underline{Z}' \approx \underline{Z}''$.

As we have just seen, if $P_1 \to P_0 \to M \to 0$ is an exact sequence of finitely generated (left) R-modules, with P_1 and P_0 projective, then $\mathrm{coker}\,(P_0^* \to P_1^*)$ is a (right) R-module whose projective equivalence class depends only on the projective equivalence class of M. We now construct a functor $D: {}_R\underline{\mathcal{M}} \to \underline{\mathcal{M}}_R$ which formalizes this correspondence between M and Z'.

We begin by choosing once and for all, for each R-module M (not necessarily finitely generated) an epimorphism $P(M) \to M \to 0$ such that:

(a) $P(M)$ is projective and $P(M) \to M$ is the identity if M is itself projective.

(b) If M is finitely generated, so is $P(M)$.

(2.4) Definition: $\Omega^0(M) = M$

$\Omega^1(M) = \ker(P(M) \to M)$

$\Omega^n(M) = \Omega^1(\Omega^{n-1}M)$.

The module $\Omega^i M$ is called the i^{th} syzygy of M (sometimes the i^{th} suspension of M). We also denote by $P_i(M)$ the projective: $P(\Omega^i M)$.

In effect, we have supplied each module M with a projective resolution $\ldots \to P_i \to \ldots \to P_1 \to P_0 \to M \to 0$ (where $P_i = P_i(M)$) made up from short exact sequences:

$$0 \to \Omega^i M \to P_{i-1} \to \Omega^{i-1}M \to 0.$$

If M is finitely generated, so are all the modules P_i and $\Omega^i M$. Unless otherwise indicated, a projective resolution for a module will mean this one.

(2.5) Definition: For M in $\mathcal{M}_R$, (i.e. finitely generated) define $D(M)$ to be $\mathrm{coker}(P_0(M)^* \to P_1(M)^*)$. If $\underline{f} \in (\underline{M},\underline{N})$ then f induces a map $\mathrm{Ext}^1(D(M), \ ) \to \mathrm{Ext}^1(D(N), \ )$, hence a map $\underline{D(N)} \to \underline{D(M)}$ which we shall denote $D(\underline{f})$.

Remarks: That $D(\underline{f}) \in (\underline{D(N)}, \underline{D(M)})$ depends only on $\underline{f}$ is the content of Lemma (2.2). The Corollary to this same Lemma asserts that the isomorphism class of $\underline{D(M)}$ in $\underline{\mathcal{M}}_R$ depends only on the isomorphism class of

$\underline{M}$ in ${}_R\underline{\mathcal{M}}$. This entire construction can be carried out for modules in $\mathcal{M}_R$. For such a module M, $D(M)$ is in ${}_R\underline{\mathcal{M}}$. We shall use the same letter D for both constructions since it will always be clear (or unimportant) whether left or right modules are involved.

(2.6) PROPOSITION: D is a functor ${}_R\underline{\mathcal{M}} \to \mathcal{M}_R$ and $\mathcal{M}_R \to \mathcal{M}_R$ with the following properties:

(a) For every finitely generated module M, there is an exact sequence of functors:

$$0 \to \mathrm{Ext}^1(D\underline{M}, \ ) \to (M \otimes_R \ ) \to (M^*, \ ) \to \mathrm{Ext}^2(D\underline{M}, \ ) \to 0$$

(b) $D(\underline{P}) = 0$ if and only if P is projective.

(c) Suppose $P \to Q \to M \to 0$ is an exact sequence of finitely generated R-modules with P and Q projective. If $C = \mathrm{coker}(Q^* \to P^*)$ then there is a unique isomorphism $\underline{C} \approx \underline{D(M)}$ such that the following commutes:

$$\begin{array}{ccc} \mathrm{Ext}^1(C, \ ) & \to & (M \otimes_R \ ) \\ \wr & & \| \\ \mathrm{Ext}^1(D\underline{M}, \ ) & \to & (M \otimes_R \ ) \end{array}$$

(d) There is a natural isomorphism between $D \circ D$ and the identity functor of ${}_R\underline{\mathcal{M}}$ (resp. $\mathcal{M}_R$).

Proof: We have already noted property (a). Part (b) follows from (a) and the fact, already mentioned, that $(M \otimes_R \ ) \to (M^*, \ )$, for finitely generated M, is an isomorphism if and only if M is projective. Part (c) also follows directly from part (a) and the diagram:

$$\begin{array}{ccccc} 0 \to \mathrm{Ext}^1(D\underline{M}, \ ) & \to & (M \otimes_R \ ) & \to & (M^*, \ ) \\ \wr & & \| & & \| \\ 0 \to \mathrm{Ext}^1(C, \ ) & \to & (M \otimes_R \ ) & \to & (M^*, \ ). \end{array}$$

To prove (d) recall that if P is a finitely generated projective, so is P^*, and $P \to P^{**}$ is an isomorphism. Starting with the exact sequence $P_1(M) \to P_0(M) \to M \to 0$ we obtain the exact sequence $P_0(M)^* \to P_1(M)^* \to D(\underline{M}) \to 0$. From part (c) already proved, we have that $\mathrm{coker}\,(P_1(M)^{**} \to P_0(M)^{**})$ is projectively equivalent to $D(D\underline{M})$. But $\mathrm{coker}\,(P_1(M)^{**} \to P_0(M)^{**}) \approx \mathrm{coker}\,(P_1(M) \to P_0(M)) \approx M$. Thus $\underline{M} \approx \underline{D(DM)}$ defines our isomorphism: $D \circ D \approx$ identity.

Remark: As a consequence of this proposition we see that as far as those properties of $D(\underline{M})$ which depend only on its projective equivalence class are concerned, the particular choice of projective resolutions for modules has no effect. Indeed, another choice for such resolutions would yield a functor D' isomorphic to D, the isomorphism being given by part (c) above.

We can also make the Ω^i into functors. Here we need only assume that we are working in an abelian category Γ in which every object has been assigned an epimorphism $P(A) \to A$, where $P(A)$ is projective. Then Ω^i is the i^{th} iterate of Ω^1, where $\Omega^1(A) = \ker\,(P(A) \to A)$. From the exact sequence $0 \to \Omega^1 A \to P(A) \to A \to 0$ we see that $\mathrm{Ext}^1(\Omega^1 A, \ ) \approx \mathrm{Ext}^2(A, \ )$. By induction it follows that $\mathrm{Ext}^1(\Omega^i A, \ ) \approx \mathrm{Ext}^{i+1}(A, \ )$. Thus, $\Omega^i A$ is determined up to projective equivalence by the projective equivalence class of A. To make Ω^i into a functor on $\underline{\Gamma}$, we must define $\Omega^i(\underline{f})$ for $\underline{f} \in (\underline{A},\underline{B})$. Now $\underline{f} \in (\underline{A},\underline{B})$ determines a transformation: $\mathrm{Ext}^{i+1}(\underline{f}, \ ): \mathrm{Ext}^{i+1}(B, \ ) \to \mathrm{Ext}^{i+1}(A, \ )$, hence a transformation: $\mathrm{Ext}^1(\Omega^i B, \ ) \to \mathrm{Ext}^1(\Omega^i A, \ )$. Since $(\mathrm{Ext}^1(\Omega^i B, \ ), \mathrm{Ext}^1(\Omega^i A, \ )) \approx (\underline{\Omega^i A},\underline{\Omega^i B})$, we define $\Omega^i(\underline{f})$ to be the element of $(\underline{\Omega^i A},\underline{\Omega^i B})$ corresponding to this last transformation. It is easily seen that if $f \in (M,N)$ represents $\underline{f} \in (\underline{M},\underline{N})$, then the map $\Omega^i(\underline{f})$ is represented by any map $\Omega^i M \to \Omega^i N$ coming from a commutative diagram:

$$\begin{array}{ccccccccc} 0 \to & \Omega^i M & \to & P_{i-1}(M) & \to \cdots \to & P_0(M) & \to & M & \to 0 \\ & \downarrow & & & & \downarrow & & \downarrow f & \\ 0 \to & \Omega^i N & \to & P_{i-1}(N) & \to \cdots \to & P_0(N) & \to & N & \to 0 \end{array}$$

Summarizing this discussion we have:

(2.7) PROPOSITION: For each $i \geq 0$ there is a functor $\Omega^i: \underline{\Gamma} \to \underline{\Gamma}$ with the properties:

(a) For any object A, $\mathrm{Ext}^1(\Omega^i(\underline{A}), \ ) \approx \mathrm{Ext}^{i+1}(A, \ )$

(b) If $\underline{f}: \underline{M} \to \underline{N}$, then there is a commutative diagram:

$$\begin{array}{ccc} & \mathrm{Ext}^{i+1}(\underline{f}, \ ) & \\ \mathrm{Ext}^{i+1}(N, \ ) & \longrightarrow & \mathrm{Ext}^{i+1}(M, \ ) \\ \wr & \mathrm{Ext}^1(\Omega^i \underline{f}, \ ) & \wr \\ \mathrm{Ext}^1(\Omega^i \underline{N}, \ ) & \longrightarrow & \mathrm{Ext}^1(\Omega^i \underline{M}, \ ) \end{array}$$

(c) $\Omega^i(\underline{A})$ is determined up to $\underline{\Gamma}$-isomorphism by property (a).

§2: Calculation of Satellites.

We now turn to the calculation of the satellites of Tor and Ext. Of central importance will be the following:

(2.8) THEOREM: Let M be a finitely generated R-module and $k \geq 0$ an integer. Then there are exact sequences of functors:

(a) $0 \to \mathrm{Ext}^1_R(D\Omega^k M, \ ) \to \mathrm{Tor}^R_k(M, \ ) \to (\mathrm{Ext}^k_R(M,R), \ ) \to \mathrm{Ext}^2_R(D\Omega^k M, \ )$

(b) $\mathrm{Tor}^R_2(D\Omega^k M, \ ) \to (\mathrm{Ext}^k_R(M,R) \otimes_R \ ) \to \mathrm{Ext}^k_R(M, \ ) \to \mathrm{Tor}^R_1(D\Omega^k M, \ ) \to 0$

Proof: These statements follow, respectively, from [1, Prop. 5.8c] and [1, Theorem 7.5]. In the case of (a) we note that Z'_{i-1} of [1] is actually $D\Omega^i M$, while in the case of (b) a careful reading of the discussion following [1, Lemma 7.4] shows that $N = D\Omega^k M$ when $F = \mathrm{Ext}^k(M, \ )$.

We can immediately obtain some results about satellites from the exact sequences (a) and (b). If we apply (a) to an arbitrary injective I, and note that $\mathrm{Ext}^1(D\Omega^k M, I) = \mathrm{Ext}^2(D\Omega^k M, I) = 0$, we see that the map $\mathrm{Tor}_k(M, \ ) \to (\mathrm{Ext}^k(M,R), \ )$ is an isomorphism. Thus $\mathrm{Tor}_k(M, \ ) \to (\mathrm{Ext}^k(M,R), \ )$ is an isomorphism on injectives, while the functor $(\mathrm{Ext}^k(M,R), \ )$ is clearly left exact. It follows from the uniqueness of R^0 (1.5d) that we have an isomorphism of functors $(\mathrm{Ext}^k(M,R), \ ) \approx R^0 \mathrm{Tor}_k(M, \ )$, and a commutative diagram:

$$\begin{array}{ccccccc} 0 \to & \overline{\mathrm{Tor}_k(M, \ )} & \to \mathrm{Tor}_k(M, \ ) & \to R^0 \mathrm{Tor}_k(M, \ ) & & & \\ & \| & & \wr & & & \\ 0 \to & \mathrm{Ext}^1(D\Omega^k M, \ ) & \to \mathrm{Tor}_k(M, \ ) & \to (\mathrm{Ext}^k(M,R), \ ). & & & \end{array}$$

We conclude that $\mathrm{Ext}^1(D\Omega^k M, \ )$ is the injective stabilization of $\mathrm{Tor}_k(M, \ )$.

By applying sequence (b) to a projective P, and noting that $\mathrm{Tor}_1(D\Omega^k M, P) = \mathrm{Tor}_2(D\Omega^k M, P) = 0$, we obtain isomorphisms and a commutative diagram:

$$\begin{array}{ccccccc} (\mathrm{Ext}^k(M,R) \otimes_R \ ) & \to \mathrm{Ext}^k(M, \ ) & \to \mathrm{Tor}_1(D\Omega^k M, \ ) & \to 0 & & & \\ \wr & \| & \wr & & & & \\ L_0 \mathrm{Ext}^k(M, \ ) & \to \mathrm{Ext}^k(M, \ ) & \to \underline{\mathrm{Ext}^k(M, \ )} & \to 0 & & & \end{array}$$

Reiterating these observations we have:

(2.9) COROLLARY: If M is in ${}_R\mathcal{M}$ then $D\Omega^k M$ is in $\mathcal{M}_R$ and we have $(k \geq 0)$:

(a) $\mathrm{Ext}^1(D\Omega^k M, \ ) \approx \overline{\mathrm{Tor}_k(M, \ )} = S^1 S_1 \mathrm{Tor}_k(M, \ )$, all as functors: $\mathcal{M}_R \to \mathcal{M}_R$.

(b) $\mathrm{Tor}_1(D\Omega^k M, \ ) \approx \underline{\mathrm{Ext}^k(M, \ )} = S_1 S^1 \mathrm{Ext}^k(M, \ )$, all as functors: ${}_R\mathcal{M} \to {}_R\mathcal{M}$.

(2.10) COROLLARY: For any finitely generated M and integer $k \geq 0$, the following conditions are equivalent:

(a) $\mathrm{Ext}^k(M,R) = 0$

(b) $\mathrm{Ext}^k(M, \ )$ is projectively stable

(c) $\mathrm{Tor}_k(M, \ )$ is injectively stable

Proof: If $\mathrm{Ext}^k(M,R) = 0$ then the second sequence of Theorem (2.8) gives: $\mathrm{Tor}_1(D\Omega^k M, \ ) \approx \mathrm{Ext}^k(M, \ )$, so $\mathrm{Ext}^k(M, \ )$ is projectively stable. Hence (a) implies (b).

On the other hand, if $\mathrm{Ext}^k(M, \ )$ is projectively stable, $\mathrm{Ext}^k(M,R) = 0$ (since R is projective). From (2.8), sequence (a), we have $\mathrm{Tor}_k(M, \ ) \approx \mathrm{Ext}^1(D\Omega^k M, \ )$. Therefore $\mathrm{Tor}_k(M, \ )$ is injectively stable.

Finally, suppose $\mathrm{Tor}_k(M,I) = 0$ for each injective module I. From (2.8), sequence (a), we see that $(\mathrm{Ext}^k(M,R),I) = 0$ for each injective I. Since any module — in particular $\mathrm{Ext}^k(M,R)$ — can be embedded in some injective, we conclude that $\mathrm{Ext}^k(M,R) = 0$, completing the proof.

Remark: The equivalence of (a) and (b) may by proved independently of (2.8). Clearly (b) implies (a). To see the converse, recall that when M is finitely generated, $(M, \ )$, hence its derived functors $\mathrm{Ext}^i(M, \ )$, commute with direct limits. In particular, $\mathrm{Ext}^i(M, \ )$ commutes with arbitrary direct sums. Clearly (a) implies (b) since any projective is a direct summand of a direct sum of copies of R.

Before proceeding, let us recall the following facts already observed about satellites:

(1) $S_i(M \otimes_R \ ) \approx \mathrm{Tor}_i(M, \ )$
(2) $S^i(M, \ ) \approx \mathrm{Ext}^i(M, \ )$
} [4, Theorem 6.1]

(3) $S^i F = S^i \underline{F}$ for $i > 0$
(4) $S_i F = S_i \overline{F}$ for $i > 0$
} Corollary (1.18)

We can now prove the following.

(2.11) PROPOSITION: Let M be finitely generated. Then

(a) For $k, \ j > 0$ we have $S_k S^j(\underline{M}, _) \approx \mathrm{Tor}_k(D\Omega^{j-1} M, \ )$

(b) For $k, \ j \geq 0$ we have $S^k S_j(\underline{M}, _) \approx S^k \mathrm{Tor}_{j+1}(DM, \ )$

When $k > 0$, $S^k \mathrm{Tor}_{j+1}(DM, \ ) \approx \mathrm{Ext}^k(D\Omega^j DM, \ )$.

Proof: $S_k S^j(\underline{M}, _) = S_{k-1} S_1 S^1 S^{j-1}(\underline{M}, _) \approx S_{k-1}(S_1 S^1 \mathrm{Ext}^{j-1}(M, \ )) = S_{k-1} \underline{\mathrm{Ext}^{j-1}(M, \ )} \approx S_{k-1} \mathrm{Tor}_1(D\Omega^{j-1} M, \ ) \approx \mathrm{Tor}_k(D\Omega^{j-1} M, \ )$. This proves (a).

To prove (b) we first note that since $(\underline{M}, _)$ is projectively stable, $(\underline{M}, _) = S_1 S^1(\underline{M}, _)$. By part (a) already proved we have $S_1 S^1(\underline{M}, _) \approx \mathrm{Tor}_1(DM, \ )$. Therefore $S^k S_j(\underline{M}, _) \approx S^k S_j \mathrm{Tor}_1(DM, \ ) \approx S^k \mathrm{Tor}_{j+1}(DM, \ )$. This proves the first assertion of (b). Now suppose $k > 0$. Then we can write $S^k \mathrm{Tor}_{j+1}(DM, \ )$ as $S^{k-1} S^1 S_1 \mathrm{Tor}_j(DM, \ ) \approx S^{k-1} \overline{\mathrm{Tor}_j(DM, \ )} \approx S^{k-1} \mathrm{Ext}^1(D\Omega^j DM, \ ) \approx \mathrm{Ext}^k(D\Omega^j DM, \ )$. This completes the proof.

To simplify our notation in future calculations we shall introduce the following functors ($k \geq 0$ is an integer):

(2.12) Definition: $D_k = \Omega^k D$ and $D_k^2 = D_k \circ D_k$

$J_k = D\Omega^k$ and $J_k^2 = J_k \circ J_k$

(2.13) LEMMA:

(a) $S_k S^k(\underline{M}, _) \approx (\underline{J_{k-1}^2(M)}, _)$

(b) $S^k S_k(\underline{M},_) \approx (D_k^2(M),_)$

Proof: The long exact sequence for Tor applied to: $0 \to \Omega^1 N \to P \to N \to 0$ gives $\mathrm{Tor}_1(\Omega^1 N, \ ) \approx \mathrm{Tor}_2(N, \ )$. Induction now gives $\mathrm{Tor}_1(\Omega^i N, \ ) \approx \mathrm{Tor}_{i+1}(N, \ )$. Proposition (2.11a) gives $(\underline{N},_) \approx \mathrm{Tor}_1(DN, \ )$. By the same proposition we have $S_k S^k(\underline{M},_) \approx \mathrm{Tor}_k(D\Omega^{k-1}M, \ ) \approx \mathrm{Tor}_1(\Omega^{k-1}D\Omega^{k-1}M, \ )$. Since $\underline{DDN} \approx \underline{N}$ we have (letting $N = \Omega^{k-1}D\Omega^{k-1}M$): $\mathrm{Tor}_1(\Omega^{k-1}D\Omega^{k-1}M, \ ) \approx \mathrm{Tor}_1(DD\Omega^{k-1}D\Omega^{k-1}M, \ ) \approx (\underline{D\Omega^{k-1}D\Omega^{k-1}M},_) = (\underline{J_{k-1}^2 M},_)$. This proves (a).

To demonstrate (b), we compute: $\underline{S^k S_k(\underline{M},_)} = S_1 S^1 S^k S_k(\underline{M},_) = S_1 S^{k+1} S_k(\underline{M},_) \approx S_1 \mathrm{Ext}^{k+1}(D\Omega^k DM, \ ) \approx S_1 \mathrm{Ext}^1(\Omega^k D\Omega^k DM, \ ) \approx S_1 S^1(\Omega^k D\Omega^k DM, \ ) = (\underline{\Omega^k D\Omega^k DM},_) = (\underline{D_k^2(M)},_)$. This completes the proof.

Now the usual map $S^k S_k F \to F$ defined in Chapter 1, §3 can be applied to $F = (\underline{M},_)$, giving a map $S^k S_k(\underline{M},_) \to (\underline{M},_)$. Applying $S_1 S^1$ to this gives, using the Lemma just proved, the following:

$$\begin{array}{ccc} \underline{S^k S_k(\underline{M},_)} & \to & (\underline{M},_) \\ \wr & & \| \\ \underline{(D_k^2(M),_)} & \to & (\underline{M},_) \end{array}$$

The transformation $\underline{(D_k^2(M),_)} \to (\underline{M},_)$ now gives a map: $\underline{M} \to \underline{D_k^2(M)}$.

In an analogous manner, the transformation $(\underline{M},_) \to S_k S^k(\underline{M},_)$ determin a map $\underline{J_{k-1}^2(M)} \to \underline{M}$.

For completeness we state the following:

(2.14) PROPOSITION: The map $\underline{M} \to \underline{D_k^2(M)}$ just described is induced by the map $M \to \Omega^k D\Omega^k DM$, defined via resolutions, which is described in the Appendix.

The proof of this Proposition, though cumbersome, is essentially straightforward and will be omitted. The demonstration is achieved by reducing (via induction) to the case $k = 1$, and then unravelling the map $\mathrm{Ext}^1(D\Omega DM, \ ) \to \mathrm{Tor}_1(DM, \ )$, as described in [1,4.12].

§3: Applications to Torsion Freeness and Reflexivity.

We now return to some of the other concepts mentioned in the Introduction.

(2.15) Definition: Let M be in ${}_R\mathcal{M}$.

(a) M is said to be k-reflexive when $\underline{M} \to \underline{D_k^2(M)}$ is an isomorphism in ${}_R\underline{\mathcal{M}}$.

(b) M is said to be k-torsion free when $\mathrm{Ext}^i(DM,R) = 0$ for $1 \leq i \leq k$.

(c) M is said to be a k-syzygy when $\underline{M} \approx \underline{\Omega^k N}$ for some N in ${}_R\mathcal{M}$.

We shall now let F be the functor $(\underline{M},_)$ and interpret some of the results already proved about stable functors in this special case. First, however, we need a Lemma.

(2.16) LEMMA: The following are equivalent for M in ${}_R\mathcal{M}$:

(a) M is n-torsion free.

(b) $\mathrm{Tor}_i(DM, \ )$ is injectively stable for $1 \leq i \leq n$.

(c) $S_i(\underline{M},_)$ is injectively stable for $0 \leq i \leq n-1$.

(d) $\overline{\mathrm{Stab}}\,(\underline{M},_) \geq n$.

(e) $\overline{\mathrm{Stab}}\,\mathrm{Ext}^1(M, \ ) \geq n+1$.

Proof: By corollary (2.10) we have $\mathrm{Tor}_i(M, \ )$ is injectively stable if and only if $\mathrm{Ext}^i(M,R) = 0$. Replacing M by DM we see that (a) is equivalent to (b). Now Proposition (2.11b) with $k = j = 0$ gives $(\underline{M},_) \approx \mathrm{Tor}_1(DM, \ )$. Consequently $S_i(\underline{M},_) \approx S_i \mathrm{Tor}_1(DM, \ ) \approx \mathrm{Tor}_{i+1}(DM, \ )$. Therefore $S_i(\underline{M},_)$ is injectively stable if and only if $\mathrm{Tor}_{i+1}(DM, \ )$ is. We conclude that (c) and (b) are equivalent. Statements (d) and (c) are

equivalent by definition. Finally, to prove that (d) and (e) are equivalent we observe that $S_i(\underline{M},_) = S_i(S_1S^1(\underline{M},_)) = S_{i+1}S^1(\underline{M},_) \approx S_{i+1}\mathrm{Ext}^1(M,\ )$ when $i > 0$. We now get our result by definition of $\overline{\mathrm{Stab}}$ and the fact that $\mathrm{Ext}^1(M,\ )$ is always injectively stable.

We now come to the first important application of our results on stability of functors. We shall make strong use of Theorem (1.25), so the reader is advised to look back at that result.

(2.17) THEOREM: For $n \geq 1$ and M in ${}_R\mathcal{M}$ the following statements are equivalent:

(a) M is n-torsion free.

(b) There exists an exact sequence $0 \to M \to F_1 \to \dots \to F_n$ with the F_i finitely generated free modules, such that the sequence $F_n^* \to \dots \to F_1^* \to M^* \to 0$ is exact.

(c) M is k-reflexive and D_kM is k-torsion free for $0 \leq k \leq n$.

(d) M is n-reflexive and D_nM is n-torsion free.

(e) M is an n-syzygy and $D_n^2(M)$ is n-torsion free.

Proof: To show that (a) implies (b) we proceed by induction on n. Suppose M is 1-torsion free. From the exact sequence $0 \to \mathrm{Ext}^1(DM,R) \to M \to M^{**} \to \mathrm{Ext}^2(DM,R) \to 0$ (sequence (2.1) applied to R) we see that $M \to M^{**}$ is an injection. Let $G \to M^* \to 0$ be exact with G a finitely generated free module. Then G^* is a finitely generated free module and $G \to G^{**}$ is an isomorphism. Letting $F_1 = G^*$, we have an exact sequence $F_1^* \to M^* \to 0$. Dualizing, we have a monomorphism $0 \to M^{**} \to F_1$. Composing with $M \to M^{**}$ we obtain the exact sequence $0 \to M \to F_1$. By construction $F_1^* \to M^* \to 0$ is exact, so the case $n = 1$ is proven. When M is 2-torsion free we have $M \approx M^{**}$. In a manner similar to the preceding, construct $F_2^* \to F_1^* \to M^* \to 0$ exact with F_1 and F_2 free. Dualizing and

replacing M^{**} by M, we obtain the desired exact sequence $0 \to M \to F_1 \to F_2$. Now suppose the implication holds for $1,\dots,n-1$ (with $n > 2$) and suppose that M is n-torsion free. Then M is 2-torsion free so we get the sequence $0 \to M \to F_1 \to F_2$ as above. Let C be defined by the exact sequence $F_1 \to F_2 \to C \to 0$. Then $\underline{D(C)} \approx \underline{\mathrm{coker}(F_2^* \to F_1^*)} = \underline{M}^*$. However, dualizing the exact sequence $P_1 \to P_0 \to M \to 0$ gives $0 \to M^* \to P_0^* \to P_1^* \to D(M) \to 0$ so $\underline{M}^* \approx \underline{\Omega^2 D(M)}$. When $i > 0$ these calculations give:

$$\mathrm{Ext}^i(DC,\ ) \approx \mathrm{Ext}^i(M^*,\ ) \approx \mathrm{Ext}^i(\Omega^2 DM,\ ) \approx \mathrm{Ext}^{i+2}(DM,\ ).$$

Since M is n-torsion free, $\mathrm{Ext}^{i+2}(DM,R) = 0$ for $0 \leq i \leq n-2$. Hence $\mathrm{Ext}^i(DC,R) = 0$ for $1 \leq i \leq n-2$, which means that C is n-2 torsion free. By induction hypothesis we have an exact sequence $0 \to C \to F_3 \to \dots \to F_n$ with the required properties. Splicing this to $0 \to M \to F_1 \to F_2 \to C \to 0$ gives our result.

To prove that (b) implies (a), splice the sequences $0 \to M^* \to P_0^* \to P_1^* \to DM \to 0$ and $F_n^* \to \dots \to F_1^* \to M^* \to 0$ together, obtaining the exact sequence $F_n^* \to \dots \to F_1^* \to P_0^* \to P_1^* \to DM \to 0$. Since the dual of this sequence is exact, we have $\mathrm{Ext}^i(DM,R) = 0$ for $1 \leq i \leq n$. This means that M is n-torsion free.

Now suppose that (a) holds; we shall deduce (c). By Lemma (2.16) we know that (a) implies $\overline{\mathrm{Stab}}(\underline{M},_) \geq n$. By Theorem (1.25) (with $F = (\underline{M},_)$) we know that this last condition is equivalent to the conjunction of the conditions: $S^kS_k(\underline{M},_) \approx (\underline{M},_)$ and $\underline{\mathrm{Stab}}\ S_k(\underline{M},_) > k$ (since $F = (\underline{M},_)$ is projectively stable) for $0 \leq k \leq n$. We conclude that $\underline{S^kS_k(M,_)} \approx (\underline{M},_)$ so $(\underline{D_k^2M},_) \approx (\underline{M},_)$. This implies that $\underline{M} \to \underline{D_k^2M}$ is an isomorphism for $0 \leq k \leq n$; i.e. M is k-reflexive for $0 \leq k \leq n$. The condition

$\underline{\text{Stab}}\ S_k(\underline{M},_) > k$ means that $S^iS_k(\underline{M},_)$ is projectively stable for $0 \leq i \leq k$. The calculation of Proposition (2.11), however, gives $S^iS_k(\underline{M},_) \approx \text{Ext}^i(D\Omega^kDM, \ ) = \text{Ext}^i(DD_k(M), \ )$. To say that this functor is projectively stable for $1 \leq i \leq k$ is to say that $D_k(M)$ is k-torsion free. Thus (a) implies (c).

It is clear that (c) implies (d).

Suppose (d) holds. Since M is n-reflexive, $\underline{M} \approx \underline{D_n^2(M)} = \underline{\Omega^n(D\Omega^nDM)}$; therefore M is an n-syzygy. Since $D_n(M)$ is n-torsion free, the fact that (a) implies (d) which we have just proved allows us to conclude that $D_n(D_nM)$ is n-torsion free. Thus, (d) implies (e).

Finally, we prove that (e) implies (a). Let $F = \text{Ext}^1(M, \ )$. Straight forward calculations of the type used to prove (2.11) give $S^{n+1}S_{n+1}F \approx \text{Ext}^1(D_n^2(M), \ )$. Lemma (2.16) tells us that $D_n^2(M)$ is n-torsion free if and only if $\overline{\text{Stab}}\ \text{Ext}^1(D_n^2(M), \ ) \geq n+1$. Thus, assumption (e) yields $\overline{\text{Stab}}\ S^{n+1}S_{n+1}F \geq n+1$. Furthermore, $\underline{M} \approx \underline{\Omega^nN}$ for some N, so $F = \text{Ext}^{n+1}(N, \ ) = S^{n+1}(N, \ )$. By Theorem (1.25) we may now conclude that $\overline{\text{Stab}}\ F \geq n+1$. By Lemma (2.16) again, we know that $\overline{\text{Stab}}\ \text{Ext}^1(M, \ ) \geq n+1$ is equivalent to M being n-torsion free. Thus (a) holds and we are done.

Recall now that for any module M, we have that M* (single dual) is a direct summand of M*** (triple dual). We show that on analogous results holds when the dualization functor $(\ )^*$ is replaced by $D_k(\ )$ (resp: J_k) and the phrase "direct summand" is suitably interpreted in terms of projecti equivalence. To obtain this result we shall apply a general fact about functors, already proved.

By Proposition (1.21) the composition:

$$S^{k+1}F \to S^{k+1}S_{k+1}S^{k+1}F \to S^{k+1}F$$

is the identity, for any half-exact functor F. Applying S_1 to this gives that $S_1S^{k+1}F \to S_1S^{k+1}S_{k+1}S^{k+1}F \to S_1S^{k+1}F$ is the identity. Now let $F = (\underline{M},_)$. A simple calculation of the type made previously shows that $S_1S^{k+1}F = (\underline{\Omega^kM},_)$ and $S_1S^{k+1}S_{k+1}S^{k+1}F = (\underline{\Omega^kD\Omega^kD\Omega^kM},_)$. We conclude that the composition: $(\underline{\Omega^kM},_) \to (\underline{\Omega^kD\Omega^kD\Omega^kM},_) \to (\underline{\Omega^kM},_)$ is the identity. Hence we have maps:

$$\underline{\Omega^kM} \to \underline{\Omega^kD\Omega^kD\Omega^kM} \to \underline{\Omega^kM} \tag{2.18}$$

whose composition is the identity. If we replace M by DM, and recall that $\underline{D_kM} = \underline{\Omega^kDM}$, we see that the composition $\underline{D_kM} \to \underline{D_k^3M} \to \underline{D_kM}$ is the identity. On the other hand, if we apply D to (2.18) and recall that $J_kM = D\Omega^kM$, we see that the composition $\underline{J_kM} \to \underline{J_k^3M} \to \underline{J_kM}$ is the identity. It follows that, in the category $\mathcal{M}_R$, $\underline{D_kM}$ is a direct summand of $\underline{D_k^3M}$ and $\underline{J_kM}$ is a direct summand of $\underline{J_k^3M}$. Furthermore, it is easily verified that the various maps involved are induced by the maps $\underline{M} \to \underline{D_k^2M}$ and $\underline{J_k^2M} \to \underline{M}$ already described.

It remains to determine what being a direct summand in $\mathcal{M}_R$ means from the module theoretic standpoint. The answer is given by the following.

(2.19) LEMMA: Let Γ be an abelian category with enough projectives. Suppose the composition $\underline{A} \to \underline{B} \to \underline{A}$ is the identity of $\underline{A}$ in $\underline{\Gamma}$. If P is any projective for which there is an epimorphism $P \to A$, then A is a direct summand of $B + P$ in Γ.

Proof: Applying Ext^1 we see that $\text{Ext}^1(A, \ ) \to \text{Ext}^1(B, \ ) \to \text{Ext}^1(A, \ )$ is the identity. Therefore $\text{Ext}^1(A, \ ) \to \text{Ext}^1(B, \ )$ is a monomorphism. Our result now follows directly from [6, Theorem 2.4].

(2.20) PROPOSITION: Let M be in ${}_R\mathcal{M}$.

(a) $\underline{D_k M} \to \underline{D_k^3 M} \to \underline{D_k M}$ is the identity and $D_k M$ is a direct summand of $(D_k^3 M) + P$ for any projective P mapping onto $D_k M$.

(b) $\underline{J_k M} \to \underline{J_k^3 M} \to \underline{J_k M}$ is the identity and $J_k M$ is a direct summand of $(J_k^3 M) + Q$ for any projective Q mapping onto $J_k M$.

Let us return now to the study of torsion freeness. We have seen that every k-torsion free module is a k-syzygy. We now ask about the converse: when is $\Omega^k M$ k-torsion free, and if it is, when does this imply that $\Omega^i M$ i-torsion free for $1 \leq i \leq k$? The answers will be provided below.

(2.21) PROPOSITION: For M in ${}_R\mathcal{M}$ and $k \geq 1$ an integer, the following are equivalent.

(a) $\Omega^k M$ is k-torsion free

(b) $\overline{\text{Stab}}\ \text{Ext}^{k+1}(M, \ ) \geq k+1$

(c) There exists a map $\underline{N} \to \underline{M}$ such that:

(i) $\text{Ext}^i(N,R) = 0$ for $1 \leq i \leq k$

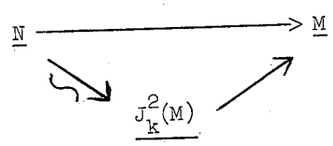
(ii) $\text{Ext}^{k+1}(M, \ ) \Rightarrow \text{Ext}^{k+1}(N, \ )$ (so $\underline{\Omega^k M} \approx \underline{\Omega^k N}$)

(d) There is an exact sequence $0 \to B \to C \to M \to 0$ in ${}_R\mathcal{M}$ such that

(i) $\text{pd}_R B < k$

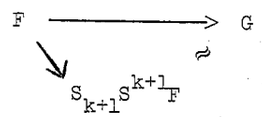
(ii) $B^* \to \text{Ext}^1(M,R) \to 0$ is exact and $\text{Ext}^i(B,R) \Rightarrow \text{Ext}^{i+1}(M,R)$ for $1 \leq i < k$.

Furthermore, when condition (c) holds there is a unique ${}_R\underline{\mathcal{M}}$ isomorphism $\underline{N} \approx \underline{J_k^2 M}$ such that the following commutes:



Proof: By Lemma (2.16) we have that $\Omega^k M$ is k-torsion free if and only if $\overline{\text{Stab}}\ \text{Ext}^1(\Omega^k M, \ ) \geq k+1$. Since $\text{Ext}^1(\Omega^k M, \ ) \approx \text{Ext}^{k+1}(M, \ )$, we see that (a) and (b) are equivalent.

Now let $F = (\underline{M}, _)$. Then $\text{Ext}^{k+1}(M, \ ) \approx S^{k+1}F$ and we want to know when $\overline{\text{Stab}}\ S^{k+1}F \geq k+1$. By Theorem (1.29) this happens if and only if $\underline{\text{Stab}}\ S_{k+1}(S^{k+1}F) \geq k+1$ and $S^{k+1}F \to S^{k+1}(S_{k+1}S^{k+1}F)$ is an isomorphism. As we have seen, $S_{k+1}S^{k+1}(\underline{M}, _) \approx (\underline{J_k^2(M)}, _)$. The condition $\underline{\text{Stab}}\ S_{k+1}S^{k+1}F \geq k+1$ is therefore equivalent to $\text{Ext}^i(J_k^2(M),R) = 0$ for $1 \leq i \leq k$, while the second condition becomes $\text{Ext}^{k+1}(M, \ ) \Rightarrow \text{Ext}^{k+1}(J_k^2(M), \ )$. Thus, conditions (b) and (c) are equivalent. Now if $\underline{N} \to \underline{M}$ is any other map satisfying condition (c), let $G = (\underline{N}, _)$. Then $\underline{\text{Stab}}\ G \geq k+1$ since $\text{Ext}^i(N,R) = 0$ for $1 \leq i \leq k$ (see Corollary (2.10)). Also $S^{k+1}F \to S^{k+1}G$ is an isomorphism. By the uniqueness assertion of Theorem (1.29) there is a unique isomorphism $G \approx S_{k+1}S^{k+1}F$ such that the diagram



commutes. Thus $(\underline{N}, _) \approx (\underline{J_k^2 M}, _)$, so $\underline{N} \approx \underline{J_k^2 M}$.

We now show that (c) implies (d). Let $N \to M$ represent the map $\underline{N} \to \underline{M}$. Let $P \to M$ be an epimorphism with P projective. By addition we obtain a map $N + P \to M$ which is evidently surjective. Let $C = N + P$ and $B = \ker (N + P \to M)$. Then $0 \to B \to C \to M \to 0$ is exact. By assumption, $\text{Ext}^i(C,R) = \text{Ext}^i(N,R) = 0$ for $1 \leq i \leq k$. The long exact sequence for $\text{Ext}(\ ,R)$ gives $B^* \to \text{Ext}^1(M,R) \to 0$ exact and $\text{Ext}^i(B,R) \Rightarrow \text{Ext}^{i+1}(M,R)$ for $1 \leq i < k$. Since $\text{Ext}^{k+1}(M, \ ) \approx \text{Ext}^{k+1}(C, \ )$, the same sequence gives $\text{Ext}^k(B,R) = 0$. Now apply Ext to $0 \to B \to C \to M \to 0$, obtaining the

exact sequence of functors:

$\ldots \to \mathrm{Ext}^{k+1}(M, \ ) \to \mathrm{Ext}^{k+1}(C, \ ) \to \mathrm{Ext}^{k+1}(B, \ ) \to \mathrm{Ext}^{k+2}(M, \ ) \to \mathrm{Ext}^{k+2}(C, \ )$

Since $\mathrm{Ext}^{k+1}(M, \ ) \simeq \mathrm{Ext}^{k+1}(C, \ )$, we have $\mathrm{Ext}^{k+j}(M, \ ) \simeq \mathrm{Ext}^{k+j}(C, \ )$ for $j > 0$ by Corollary (1.42). Therefore $\mathrm{Ext}^{k+1}(B, \ ) = 0$, and so $\mathrm{pd}_R B \leq k$. On the other hand, since $\mathrm{pd}\, B < \infty$, we have

$\mathrm{pd}_R B = \mathrm{Sup}\{t \mid \mathrm{Ext}^t(B,P) \neq 0$ for some projective $P\}$. Since $\mathrm{Ext}^k(B,R) = 0$ we have that $\mathrm{Ext}^k(B, \ )$ is projectively stable. Therefore $\mathrm{pd}_R B < k$ and we have deduced (d).

Finally, suppose (d) holds. We shall prove (a). The hypothesis $\mathrm{pd}\, B <$ gives $\mathrm{Ext}^j(B, \ ) = 0$ for $j \geq k$ so the long exact sequence for Ext yiel $\mathrm{Ext}^{k+1}(C, \ ) \approx \mathrm{Ext}^{k+1}(M, \ )$. We conclude that $\underline{\Omega^k M} \approx \underline{\Omega^k C}$. It will therefore suffice to prove that $\Omega^k C$ is k-torsion free. Now part (ii) of (d) togethe with the long exact sequence for $\mathrm{Ext}(\ ,R)$ gives $\mathrm{Ext}^i(C,R) = 0$ for $1 \leq i \leq k$. Thus, $\underline{\mathrm{Stab}}\ (\underline{C},_) \geq k+1$. By Theorem (1.26) we have $\underline{\mathrm{Stab}}\ G \geq n$ implies $\overline{\mathrm{Stab}}\ S^n G \geq n$. Therefore $\overline{\mathrm{Stab}}\ S^{k+1}(\underline{C},_) \geq k+1$. Since $S^{k+1}(\underline{C},_) = \mathrm{Ext}^1(\Omega^k C, \ )$, we conclude that $\Omega^k C$, hence $\Omega^k M$, is k-torsion free. Our proof is now complete.

§4: Grade and Torsion Freeness.

We now find a criterion for $\Omega^i M$ to be i-torsion free for $1 \leq i \leq k$. order to state this we recall the notion of depth.

(2.21) Definition: If M and N are left (resp. right) R-modules, then M-depth of N is the smallest non-negative integer t such that $\mathrm{Ext}^t(M,N) \neq 0$. If no such exists (i.e. $\mathrm{Ext}^i(M,N) = 0$ for all $i \geq 0$) th M-depth $N = \infty$.

The M-depth of R will be written: $\mathrm{grade}_R M$, (or simply grade M).

(2.22) LEMMA:

(a) If E is finitely generated, then grade $E \geq n$ if and only if $\overline{\mathrm{Stab}}\ (E \otimes_R \ ) \geq n$.

(b) If M is finitely generated, then grade $\mathrm{Ext}^i(M,R) \geq n$ if and only if $\overline{\mathrm{Stab}}\ L_0 \mathrm{Ext}^i(M, \ ) \geq n$.

Proof: We first prove (a). To say that $\overline{\mathrm{Stab}}\ (E \otimes_R \ ) \geq n$ is to say that $S_i(E \otimes_R \ )$ is injectively stable for $0 \leq i < n$. Since $S_i(E \otimes_R \ ) \approx \mathrm{Tor}_i(E, \ )$, we see that $\overline{\mathrm{Stab}}\ (E \otimes_R \ ) \geq n$ if and only if $\mathrm{Tor}_i(E, \ )$ is injectively stable for $0 \leq i < n$. Now in Corollary (2.10) it was shown that $\mathrm{Tor}_i(E, \ )$ injectively stable is equivalent to $\mathrm{Ext}^i(E,R) = 0$. Thus, $\overline{\mathrm{Stab}}\ (E \otimes_R \ ) \geq n$ if and only if $\mathrm{Ext}^i(E,R) = 0$ for $0 \leq i < n$. This last condition, by Definition (2.21), means $\mathrm{grade}_R E \geq n$.

To prove (b) we recall that when M is finitely generated, so is $\mathrm{Ext}^i(M,R)$, and we have the exact sequence:

$$\mathrm{Tor}_2(D\Omega^i M, \ ) \to (\mathrm{Ext}^i(M,R) \otimes_R \ ) \to \mathrm{Ext}^i(M, \ ) \to \mathrm{Tor}_1(D\Omega^i M, \ ) \to 0 .$$

As we have seen, this gives $L_0 \mathrm{Ext}^i(M, \ ) \approx (\mathrm{Ext}^i(M,R) \otimes_R \ )$. Thus, $\overline{\mathrm{Stab}}\ L_0 \mathrm{Ext}^i(M, \ ) \geq n$ if and only if $\overline{\mathrm{Stab}}\ (\mathrm{Ext}^i(M,R) \otimes_R \ ) \geq n$. By part (a) just proved, this means grade $\mathrm{Ext}^i(M,R) \geq n$, so we are done.

We have seen in Proposition (2.21) that $\Omega^i M$ is i-torsion free if and only if $\overline{\mathrm{Stab}}\ \mathrm{Ext}^{i+1}(M, \ ) \geq i+1$. If we can relate $\overline{\mathrm{Stab}}\ \mathrm{Ext}^{i+1}(M, \ )$ to $\overline{\mathrm{Stab}}\ L_0 \mathrm{Ext}^{i+1}(M, \ )$ then we can relate the torsion freeness of $\Omega^i M$ to the grade of $\mathrm{Ext}^i(M,R)$, by means of the Lemma just proved. The following results will describe the connection between $\overline{\mathrm{Stab}}\ S^i F$ and $\overline{\mathrm{Stab}}\ L_0 S^i F$ for an arbitrary half exact functor F. They make use of the fundamental sequence (*) of (1.15).

(2.23) LEMMA: Suppose $\overline{\mathrm{Stab}}\ L_0 G \geq k$. Then the map $S_i G \to S_{i+1} S^1 G$ is an

isomorphism on injectives for $0 \leq i \leq k-1$, and is an epimorphism for $i = k$. Thus, if $0 \leq t \leq k+1$ and $\overline{\text{Stab}}\ G \geq t$, then $\overline{\text{Stab}}\ S^1G \geq t+1$. If $0 \leq t \leq$ then $\overline{\text{Stab}}\ G \geq t$ if and only if $\overline{\text{Stab}}\ S^1G \geq t+1$.

Proof: The assumption $\overline{\text{Stab}}\ L_0G \geq k$ means that S_iL_0G, which equals L_iG (ith left derived), is injectively stable. The conclusion of the Lemma may now be deduced from the following section of fundamental sequence (1.15)

$$\ldots \to L_iF \to S_iF \to S_{i+1}S^1F \to L_{i-1}F \to \ldots$$

The following Lemma is proved in a similar fashion.

(2.24) LEMMA: Suppose $\overline{\text{Stab}}\ S^1G \geq k$. Then $L_iG \to S_iG$ is an isomorphism on injectives for $1 \leq i \leq k-3$, and is an epimorphism for $i = k-2$. Thus, for $1 \leq t \leq k-1$ we have that $\overline{\text{Stab}}\ L_0G \geq t$ implies $\overline{\text{Stab}}\ G \geq t$. When $1 \leq t \leq k-2$, we have $\overline{\text{Stab}}\ L_0G \geq t$ if and only if $\overline{\text{Stab}}\ G \geq t$.

Our main result along these lines is the following:

(2.25) PROPOSITION: For any half exact functor F and integer $n \geq 1$ th following statements are equivalent.

(a) $\overline{\text{Stab}}\ S^jF \geq j$ for $1 \leq j \leq n+2$.

(b) $\overline{\text{Stab}}\ L_0S^{i+1}F \geq i$ for $1 \leq i \leq n$.

Proof: We prove that (a) implies (b), using induction on n. Suppose $n = 1$ and $\overline{\text{Stab}}\ S^jF \geq j$ for $1 \leq j \leq 3$. In particular, we have $\overline{\text{Stab}}\ S^1(S^2F) \geq 3$. If we let $G = S^2F$ and $k = 3$ in Lemma (2.24) we concl that $\overline{\text{Stab}}\ L_0S^2F \geq 1$ if and only if $\overline{\text{Stab}}\ S^2F \geq 1$. The condition $\overline{\text{Stab}}\ S^2F \geq 1$ holds for any functor, since S^2F is always injectively sta Thus $\overline{\text{Stab}}\ L_0S^2F \geq 1$ and so $n = 1$ is disposed of. Now assume $n > 1$ a the implication holds for $1,\ldots,n-1$. Suppose $\overline{\text{Stab}}\ S^jF \geq j$ for $1 \leq j \leq n+2$. In particular, we have $\overline{\text{Stab}}\ S^1(S^{n+1}F) \geq n+2$. If we let $k = n+2$ and $G = S^{n+1}F$ in Lemma (2.24), we conclude that $\overline{\text{Stab}}\ L_0S^{n+1}F \geq$ if and only if $\overline{\text{Stab}}\ S^{n+1}F \geq n$. But by assumption $\overline{\text{Stab}}\ S^{n+1}F \geq n+1 > n$

we conclude $\overline{\text{Stab}}\ L_0S^{n+1}F \geq n$. By induction hypothesis $\overline{\text{Stab}}\ L_0S^{i+1}F \geq i$ for $1 \leq i \leq n$, so our argument is complete.

To prove that (b) implies (a) we again proceed by induction on n. Let $n = 1$ and assume $\overline{\text{Stab}}\ L_0S^2F \geq 1$. When $G = S^2F$ and $k = 1$ in Lemma (2.23) we conclude that $\overline{\text{Stab}}\ S^2F \geq 2$ implies $\overline{\text{Stab}}\ S^3F \geq 3$. As we have seen in example (1.24), $\overline{\text{Stab}}\ S^2F \geq 2$ for any functor F, so we conclude $\overline{\text{Stab}}\ S^3F \geq 3$. Thus, the case $n = 1$ is proved. Now suppose (b) implies (a) for $1,\ldots,n-1$ and $n > 1$. Assume $\overline{\text{Stab}}\ L_0S^{i+1}F \geq i$ for $1 \leq i \leq n$. By virtue of our induction hypothesis we need only show $\overline{\text{Stab}}\ S^{n+2}F \geq n+2$. Let $G = S^{n+1}F$ and $k = n$ in Lemma (2.23). We conclude that $\overline{\text{Stab}}\ S^{n+1}F \geq n+1$ implies $\overline{\text{Stab}}\ S^{n+2}F \geq n+2$. By induction hypothesis the inequality $\overline{\text{Stab}}\ S^{n+1}F \geq n+1$ holds, so we are done.

We can now state and prove a criterion for Ω^iM be i-torsion free for $1 \leq i \leq k$.

(2.26) PROPOSITION: Let M be finitely generated. Then Ω^iM is i-torsion free for $1 \leq i \leq k$ if and only if grade $\text{Ext}^{i+1}(M,R) \geq i$ for $1 \leq i \leq k-1$.

Proof: As we have seen, Ω^iM is i-torsion free if and only if $\overline{\text{Stab}}\ \text{Ext}^{i+1}(M, \ ) \geq i+1$. Therefore Ω^iM is i-torsion free for $1 \leq i \leq k$ if and only if $\overline{\text{Stab}}\ S^iF \geq i$ for $1 \leq i \leq k+1$, where $F = (M, _)$. By the preceding Proposition this is equivalent to $\overline{\text{Stab}}\ L_0S^{i+1}F \geq i$ for $1 \leq i \leq k-1$. By Lemma (2.22b) this last condition is equivalent to grade $\text{Ext}^{i+1}(M,R) \geq i$ for $1 \leq i \leq k-1$.

(2.27) COROLLARY: Suppose R has the property that grade $\text{Ext}^i(B,R) \geq i$ for all $i > 0$ and every left and right R-module B of finite projective dimension. Then for any M we have Ω^kM is k-torsion free if and only if Ω^iM is i-torsion free for all $1 \leq i \leq k$.

Proof: Clearly $\Omega^i M$ being i-torsion free for $1 \leq i \leq k$ implies $\Omega^k M$ is k-torsion free. Let us suppose, then, that $\Omega^k M$ is k-torsion free. By Proposition (2.21) there is an exact sequence $0 \to B \to C \to M \to 0$ with $pd_R B < k$ and $Ext^i(B,R) \approx Ext^{i+1}(M,R)$ for $1 \leq i \leq k-1$. When R has the stated property, grade $Ext^i(B,R) \geq i$ for all $i > 0$, so grade $Ext^{i+1}(M,R) \geq i$ for $1 \leq i \leq k-1$. By the previous Proposition we conclude $\Omega^i M$ is i-torsion free for $1 \leq i \leq k$.

Remark: We shall see in the next section that for any commutative Noetherian ring R we have grade $Ext^i_R(B,R) \geq i$ for any R-module B of finite projective dimension. For these rings, then, we have $\Omega^k M$ is k-torsion free if and only if $\Omega^i M$ is i-torsion free for $1 \leq i \leq k$ (for any M).

We now reverse our outlook and ask: When does $\Omega^n M$ being (n+j)-torsion free imply that M is j-torsion free? As usual we shall state the answer in terms of modules, but view the problem in a more general setting to provide a proof.

(2.28) PROPOSITION: Suppose $n,j \geq 0$ and grade $Ext^i(M,R) \geq i+j$ for all $1 \leq i \leq n$. Then the following are equivalent (M is, of course, assumed finitely generated):

(a) M is j-torsion free.

(b) $\Omega^n M$ is (n+j)-torsion free.

We immediately translate this into the language of functors. Let $F = (\underline{M}, _)$. Then $S^i F \approx Ext^i(M, \ )$ and $L_0 S^i F \approx (Ext^i(M,R) \otimes_R \ )$. As we have seen, M is j-torsion free if and only if $\overline{Stab}\ S^1 F \geq j+1$, while $\Omega^n M$ is (n+j)-torsion free if and only if $\overline{Stab}\ S^{n+1} F \geq n+j+1$. Finally, the condition grade $Ext^i(M,R) \geq i+j$ we have seen to be equivalent to $\overline{Stab}\ L_0 S^i F \geq i+j$. Reformulating Proposition (2.28) in these terms, we obtain

the following result about half exact functors.

(2.29) PROPOSITION: Let F be half exact, n and j be non-negative integers, and suppose $\overline{Stab}\ L_0 S^i F \geq i+j$ for $0 \leq i \leq n$. Then the following are equivalent:

(a) $\overline{Stab}\ S^1 F \geq j+1$.

(b) $\overline{Stab}\ S^{n+1} F \geq n+j+1$.

Proof: We prove that (a) implies (b) by induction on n. When $n = 0$, conditions (a) and (b) are clearly the same. When $n = 1$ we are given $\overline{Stab}\ L_0 S^1 F \geq 1+j$ and $\overline{Stab}\ S^1 F \geq j+1$. Let $G = S^1 F$ and $k = j+1$ in Lemma (2.23). Then $\overline{Stab}\ S^1 F \geq j+1$ implies $\overline{Stab}\ S^2 F \geq j+2$ so the case $n = 1$ holds. Now suppose (a) implies (b) for $0,1,\ldots,n-1$ where $n > 1$. Assume that (a) holds. By induction hypothesis we have $\overline{Stab}\ S^n F \geq n+j$. We now apply the case $n = 1$ already proved to the functor $G = S^{n-1} F$. Since $Stab\ L_0 S^1 G \geq (n+j-1) + 1$ and $\overline{Stab}\ S^1 G \geq (n+j-1) + 1$, we conclude that $\overline{Stab}\ S^2 G \geq (n+j-1) + 2$. Since $S^2 G = S^{n+1} F$, we see that $\overline{Stab}\ S^{n+1} F \geq n+j+1$, so our induction argument is complete.

We prove (b) implies (a) by induction on n. When $n = 0$ it is clear that (a) and (b) are equivalent. Suppose that (b) implies (a) for $0,\ldots,n$, and suppose $n \geq 0$. If we let $G = S^n F$ then condition (b) for n+1 becomes: $\overline{Stab}\ S^1 G \geq n+j+1$. Taking $k = n+j+1$ in Lemma (2.24) we see that the assumption $\overline{Stab}\ L_0 G \geq n+j$ gives $\overline{Stab}\ G \geq n+j$. Thus, $\overline{Stab}\ S^n F \geq n+j$. By induction hypothesis $\overline{Stab}\ S^n F \geq n+j$ implies that $\overline{Stab}\ S^1 F \geq j+1$, so we are done.

§5: Representation by Monomorphisms.

Let us recall the following argument used to prove that (c) implies (d) in Proposition (2.21). We had a map $\underline{f}: \underline{N} \to \underline{M}$ in the projectively

stabilized category ${}_R\underline{\mathcal{M}}$. This map was represented by $f: N \to M$ in ${}_R\underline{\mathcal{M}}$. Choosing $P \to M$ to be any epimorphism with P projective, we obtained the induced epimorphism $P + N \to M$. Furthermore, the following diagram evidently commutes:

$$\begin{array}{ccc} P + N & \longrightarrow & M \\ \uparrow & & \| \\ N & \xrightarrow{f} & M \end{array}$$

where $N \to P+N$ is the canonical map of a direct summand. Passing to ${}_R\underline{\mathcal{M}}$, and noting that $\underline{N} \to \underline{P+N}$ is an isomorphism in this category, we obtain the following commutative diagram:

$$\begin{array}{ccc} \underline{P + N} & \longrightarrow & \underline{M} \\ \wr & & \| \\ \underline{N} & \xrightarrow{\underline{f}} & \underline{M} \end{array}$$

In this sense, then, we represented the map $\underline{f}$ in ${}_R\underline{\mathcal{M}}$ by an epimorphism $P+N \to M$ in ${}_R\mathcal{M}$. Dually, we may ask when $\underline{f}$ may be represented by a monomorphism in ${}_R\mathcal{M}$. The answer, which will have applications to the maps $\underline{J_k^2(M)} \to \underline{M}$, is provided by the following result.

(2.30) PROPOSITION: Let $\underline{f}: \underline{N} \to \underline{M}$ be a morphism in ${}_R\underline{\mathcal{M}}$. The following statements are equivalent.

(a) There is a projective P and a monomorphism $N \to M+P$ in ${}_R\mathcal{M}$ such that the following commutes in ${}_R\underline{\mathcal{M}}$:

$$\begin{array}{ccc} \underline{N} & \xrightarrow{\underline{f}} & \underline{M} \\ \| & & \wr\downarrow \\ \underline{N} & \longrightarrow & \underline{M + P} \end{array}$$

where $\underline{M} \xrightarrow{\sim} \underline{M+P}$ is induced by the direct summand map $M \to M+P$. Furthermore, $N \to M+P$ may be chosen so that the sequence $(M+P)^* \to N^* \to 0$ is exact.

(b) coker $((M, \ ) \to (N, \ ))$ is an injectively stable functor.

Proof: Suppose (a) holds. From the commutative diagram, we see that coker $((\underline{M}, _) \to (\underline{N}, _)) \approx$ coker $((\underline{M+P}, _) \to (\underline{N}, _))$. Therefore it suffices to prove that coker $((\underline{M+P}, _) \to (\underline{N}, _))$ is injectively stable in order to prove (b). Consider the following exact and commutative diagram:

$$\begin{array}{ccccccc} (M + P, \ ) & \longrightarrow & (N, \ ) & \longrightarrow & C & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ (\underline{M + P}, _) & \longrightarrow & (\underline{N}, _) & \longrightarrow & C' & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ 0 & & 0 & & & & \end{array}$$

It is easily seen that if C vanishes on injectives then $C' = \text{coker}\ ((\underline{M+P}, _) \to (\underline{N}, _))$ does also. Thus, it suffices to prove that C is injectively stable. Let W be the cokernel of the monomorphism $N \to M+P$. From the exact sequence $0 \to N \to M+P \to W \to 0$ we get the exact sequence: $(M+P, \ ) \to (N, \ ) \to \text{Ext}^1(W, \ ) \to \text{Ext}^1(M+P, \ )$. Since $C = \text{coker}\ ((M+P, \ ) \to (N, \ ))$, we see that $C \approx \ker (\text{Ext}^1(W, \ ) \to \text{Ext}^1(M+P, \ ))$. Thus C, being a subfunctor of $\text{Ext}^1(W, \ )$, is injectively stable. We conclude that C, hence cokernel $((M, \ ) \to (N, \ ))$, is injectively stable. Thus (a) implies (b).

Now suppose that (b) holds. Let $F \to N^* \to 0$ be exact in ${}_R\mathcal{M}$ with F free. If $P = F^*$, then $P^* \to N^* \to 0$ is exact. Dualizing we have a map: $N^{**} \to P^{**} = P$. By composition with $N \to N^{**}$ we obtain a map $h: N \to P$ with the property that $h^*: P^* \to N^*$ is surjective. Now let $f \in (N,M)$ represent $\underline{f} \in (\underline{N},\underline{M})$. Define the map $N \to M+P$ to be (f,h). It is easily

seen that the following commutes in $_R\underline{\mathcal{M}}$:

$$\begin{array}{ccc} \underline{N} & \xrightarrow{\underline{f}} & \underline{M} \\ \| & & \downarrow \\ \underline{N} & \longrightarrow & \underline{M+P} \end{array}$$

Furthermore $((M+P)^* \to N^*) = (M^*+P^* \to N^*)$ is surjective since $h^*: P^* \to$ is surjective. Therefore it remains only to prove that $N \to M+P$ is a monomorphism. From the above commutative diagram we obtain the following co

$$\begin{array}{ccc} (\underline{M},_) & \longrightarrow & (\underline{N},_) \\ \wr & & \| \\ (\underline{M+P},_) & \longrightarrow & (\underline{N},_) \end{array}$$

By assumption the top horizontal map has an injectively stable cokernel. Therefore $C = \operatorname{coker}((\underline{M+P}),_) \to ((\underline{N},_))$ is injectively stable as well. We now claim that $L_0((M+P,)) \to L_0((N,))$ is surjective. In fact, the exact sequence $\operatorname{Tor}_2(DU,) \to (U^* \otimes_R) \to (U,) \to \operatorname{Tor}_1(DU,) \to 0$ shows that $L_0(U,) \approx (U^* \otimes_R)$. Therefore we have:

$$L_0(M+P,) \approx ((M+P)^* \otimes_R)$$

$$L_0(N,) \approx (N^* \otimes_R)$$

Since $(M+P)^* \to N^* \to 0$ is exact, we see that $L_0(M+P,) \to L_0(N,) \to 0$ is exact, which establishes our claim. Now consider the exact and commutat diagram:

$$\begin{array}{ccccccc} L_0(M+P,) & \longrightarrow & L_0(N,) & \longrightarrow & 0 & & \\ \downarrow & & \downarrow & & & & \\ (M+P,) & \longrightarrow & (N,) & \longrightarrow & E & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ (\underline{M+P},_) & \longrightarrow & (\underline{N},_) & \longrightarrow & C & \longrightarrow & 0 \\ \downarrow & & \downarrow & & & & \\ 0 & & 0 & & & & \end{array}$$

Easy chasing gives $E \approx C$. As we have seen, C is injectively stable; therefore E vanishes on injectives. Choose an exact $0 \to N \to I$ with I injectives. Since $E(I) = 0$ we have that $(M+P,I) \to (N,I) \to 0$ is exact. Thus, there is a map $M+P \to I$ such that $N \to I$ is the composite $N \to M+P \to I$. Since $N \to I$ is a monomorphism, $N \to M+P$ must be a monomorphism. This completes the proof.

We now give an important application of this Proposition. Recall that $J_k(M) = D\Omega^k M$ and that we have the commutative diagram:

$$\begin{array}{ccc} (\underline{M},_) & \longrightarrow & S_{k+1}S^{k+1}(\underline{M},_) \\ \| & & \wr \\ (\underline{M},_) & \longrightarrow & (\underline{J_k^2(M)},_) \end{array}$$

We shall consider the maps $\underline{J_k^2(M)} \to \underline{M}$ and ask when $\operatorname{coker}((\underline{M},_) \to (\underline{J_k^2(M)},_))$ is injectively stable for $1 \le k \le n$. Since $(\underline{J_k^2(M)},_) \approx S_{k+1}S^{k+1}(\underline{M},_)$ we turn to the following general result.

(2.31) THEOREM: For an integer $n \ge 1$ and half exact functor F, the following statements are equivalent.

(a) $\overline{\text{Stab}}\, L_0 S^i F \ge i$ for $1 \le i \le n$.

(b) For $1 \leq i \leq n$ we have coker $(F \to S_{i+1}S^{i+1}F)$ is injectively stable, and $\overline{\text{Stab}}\ L_0S^iF \geq i\text{-}1$.

(c) For $1 \leq i \leq n$ we have coker $(S_iS^iF \to S_{i+1}S^{i+1}F)$ is injectively stable, and $\overline{\text{Stab}}\ L_0S^iF \geq i\text{-}1$.

Proof: First we prove (a) implies (b). Clearly $\overline{\text{Stab}}\ L_0S^iF \geq i$ implies $\overline{\text{Stab}}\ L_0S^iF \geq i\text{-}1$ (for $1 \leq i \leq n$). We now proceed by induction on n. If $n = 1$ the fundamental sequence (1.15*) applied to the functor S^1F gives the exact sequence $S_1S^1F \to S_2S^2F \to L_0S^1F$. Since L_0S^1F was assumed to be injectively stable, it follows that coker $(S_1S^1F \to S_2S^2F)$ is injectively stable. Since $F \to S_1S^1F \to 0$ is exact, we have the exact and commutative diagram:

$$\begin{array}{ccccc} F & \longrightarrow & S_1S^1F & \longrightarrow & 0 \\ \| & & \downarrow & & \\ F & \longrightarrow & S_2S^2F & & \end{array}$$

Diagram chasing gives coker $(F \to S_2S^2F) \approx$ coker $(S_1S^1F \to S_2S^2F)$. Therefore coker $(F \to S_2S^2F)$ is injectively stable, giving our result for $n = 1$.

Suppose now that $n > 1$ and the implication holds for all smaller integers. Consider the exact and commutative diagram:

$$\begin{array}{ccccccc} F & \longrightarrow & S_nS^nF & \longrightarrow & U_1 & \longrightarrow & 0 \\ \| & & \downarrow & & & & \\ F & \longrightarrow & S_{n+1}S^{n+1}F & \longrightarrow & U_2 & \longrightarrow & 0 \\ & & \downarrow & & & & \\ & & L_{n-1}S^nF & & & & \end{array}$$

where the exact middle column comes from the fundamental sequence (*) applied to the functor S^nF. From the usual properties of cokernels we deduce the existence of maps $U_1 \to U_2$ and $U_2 \to L_{n-1}S^nF$ which commute with the others in the diagram. Straightforward diagram chasing gives the exactness of: $U_1 \to U_2 \to L_{n-1}S^nF$. Now U_1 is injectively stable by induction hypothesis, while $L_{n-1}S^nF \approx S_{n-1}(L_0S^nF)$ is injectively stable since $\overline{\text{Stab}}\ L_0S^nF \geq n$. Therefore $U_2 =$ coker $(F \to S_{n+1}S^{n+1}F)$ is injectively stable, completing our induction argument.

To prove that (b) implies (c), consider the exact and commutative diagram:

$$\begin{array}{ccccccc} F & \longrightarrow & S_iS^iF & & & & \\ \| & & \downarrow & & & & \\ F & \longrightarrow & S_{i+1}S^{i+1}F & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & & & \\ & & C' & & & & \\ & & \downarrow & & & & \\ & & 0 & & & & \end{array}$$

By the usual properties of cokernels there is a map $C \to C'$, which is easily seen to be surjective. By assumption (b), C vanishes on injectives. Therefore C' does as well. Thus, (b) implies (c).

Now suppose that (c) holds; we prove (a). By assumption $\overline{\text{Stab}}\ L_0S^iF \geq i\text{-}1$ for $1 \leq i \leq n$, so we need only show that $S_{i-1}L_0S^iF$ is injectively stable for $1 \leq i \leq n$ in order to prove (a). If we apply the fundamental sequence (*) to the functor S^iF we obtain the exact sequence:

$$S_iS^iF \to S_{i+1}S^{i+1}F \to S_{i-1}(L_0S^iF) \to S_{i-1}S^iF$$

Proposition (2.25) tells us that the condition $\overline{\text{Stab}}\ L_0S^iF \geq i-1$ for $1 \leq i \leq n$ is equivalent to $\overline{\text{Stab}}\ S^iF \geq i$ for $1 \leq i \leq n+1$. Therefore $S_{i-1}S^iF$ is certainly injectively stable for $1 \leq i \leq n$. Let $C_i = \text{coker}\ (S_iS^iF \to S_{i+1}S^{i+1}F)$; then C_i is injectively stable by assumption when $1 \leq i \leq n$. From the above exact sequence we see that the following is exact:

$$0 \to C_i \to S_{i-1}(L_0S^iF) \to S_{i-1}S^iF$$

Since C_i and $S_{i-1}S^iF$ both vanish on injectives for $1 \leq i \leq n$, we conclude that $S_{i-1}L_0S^iF$ is injectively stable, and we are done.

We now apply Theorem (2.31) just proved to the functor $F = (\underline{M},_)$. Condition (a) of the Theorem becomes $\overline{\text{Stab}}\ L_0\ \text{Ext}^i(M,\ ) \geq i$ for $1 \leq i \leq n$. By Lemma (2.22) this is equivalent to grade $\text{Ext}^i(M,R) \geq i$ for $1 \leq i \leq n$. Now consider condition (b). Since $S_{i+1}S^{i+1}F \approx (\underline{J_i^2(M)},_)$, the injective stability of coker $(F \to S_{i+1}S^{i+1}F)$ is clearly equivalent to the injective stability of coker $((\underline{M},_) \to (\underline{J_i^2(M)},_))$. Now this last condition is equivalent, by Proposition (2.30), to the existence of a projective P_i and a monomorphism $J_i^2(M) \to M+P_i$ such that the following commutes in $_R\underline{\mathcal{M}}$:

$$\begin{array}{ccc} \underline{J_i^2(M)} & \longrightarrow & \underline{M+P} \\ \| & & \uparrow \wr \\ \underline{J_i^2(M)} & \longrightarrow & \underline{M} \end{array}$$

and $(M+P_i)^* \to (J_i^2(M))^* \to 0$ is exact. The remaining part of condition (b), that $\overline{\text{Stab}}\ L_0S^iF \geq i-1$ for $1 \leq i \leq n$, is clearly equivalent to grade $\text{Ext}^i(M,R) \geq i-1$ for $1 \leq i \leq n$. This last condition we know, by Proposition (2.26), to be equivalent to Ω^iM being i-torsion free for $1 \leq i$

The parts of condition (c) of Theorem (2.31) may be interpreted in a similar fashion for $F = (\underline{M},_)$. We therefore have the following special case of (2.31):

(2.32) COROLLARY: For M in $_R\mathcal{M}$ and $n \geq 1$ the following are equivalent:

(a) grade $\text{Ext}^i(M,R) \geq i$ for $1 \leq i \leq n$.

(b) For $1 \leq i \leq n$ we have Ω^iM is i-torsion free and there is a projective P_i and a monomorphism $J_i^2(M) \to M+P_i$ such that the following commutes in $_R\underline{\mathcal{M}}$:

$$\begin{array}{ccc} \underline{J_i^2(M)} & \longrightarrow & \underline{M+P} \\ \| & & \uparrow \wr \\ \underline{J_i^2(M)} & \longrightarrow & \underline{M} \end{array}$$

(c) For $1 \leq i \leq n$ we have Ω^iM is i-torsion free and there is a projective Q_i and a monomorphism $J_i^2(M) \to M+Q_i$ such that the following commutes in $_R\underline{\mathcal{M}}$:

$$\begin{array}{ccc} \underline{J_i^2(M)} & \longrightarrow & \underline{J_{i-1}^2(M)+Q_1} \\ \| & & \uparrow \wr \\ \underline{J_i^2(M)} & \longrightarrow & \underline{J_{i-1}^2(M)} \end{array}$$

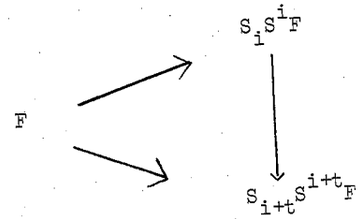
Furthermore, when these equivalent conditions hold, the maps $J_i^2(M) \to M+P_i$ and $J_i^2(M) \to J_{i-1}^2(M)+Q_i$ may be chosen so that $(M+P^i)^* \to (J_i^2(M))^* \to 0$ and $(J_{i-1}^2(M)+Q_i)^* \to (J_i^2(M))^* \to 0$ are exact for $1 \leq i \leq n$.

§6: Spherical Filtrations and the Approximation Theorem.

Recall that for an arbitrary functor F we have a sequence of transformations:

$$F \to S_1S^1F \to S_2S^2F \to \dots \to S_{k+1}S^{k+1}F \to \dots$$

such that the following commutes:



Now suppose $F = (\underline{M},_)$ for a finitely generated R-module M. Then $S_{k+1}S^{k+1}F \approx (\underline{J_k^2M},_)$ and the above sequence of functors may be replaced by a sequence of modules:

$$\dots \to \underline{J_k^2M} \to \underline{J_{k-1}^2M} \to \dots \to \underline{J_1^2M} \to \underline{J_0^2M} = \underline{M}$$

Let us assume, until further notice, that M has the property that grade $\mathrm{Ext}^i(M,R) \geq i$ for $1 \leq i \leq n$. By Corollary (2.32) of the previous section we can find a projective Q_{n-1} and a monomorphism $J_n^2M \to J_{n-1}^2M+Q_{n-1}$ such that $(J_{n-1}^2M+Q_{n-1})^* \to (J_n^2M)^* \to 0$ is exact and the following commutes:

$$\begin{array}{ccc} \underline{J_n^2M} & \longrightarrow & \underline{J_{n-1}^2M + Q_{n-1}} \\ \| & & \wr \\ \underline{J_n^2M} & \longrightarrow & \underline{J_{n-1}^2M} \end{array}$$

Using the same Corollary again, we can find a projective Q_{n-2} and a monomorphism: $J_{n-1}^2M \to J_{n-2}+Q_{n-2}$ such that $(J_{n-2}^2+Q_{n-2})^* \to (J_{n-1}^2M)^* \to 0$ is exact and a diagram similar to the one above commutes. By adding the identity map $Q_{n-1} \to Q_{n-1}$, we obtain monomorphism: $J_{n-1}^2M+Q_{n-1} \to J_{n-2}^2+Q_{n-2}+Q_{n-1}$. Now let $P_n = 0$, $P_{n-1} = Q_{n-1}+P_n = Q_{n-1}$, and $P_{n-2} = Q_{n-2}+P_{n-1} = Q_{n-2}+Q_{n-1}$. Then we have a sequence of monomorphism:

$$J_n^2M + P_n \to J_{n-1}^2M + P_{n-1} \to J_{n-2}^2M + P_{n-2}$$

We now proceed inductively. Assuming $J_k^2M + P_k \to J_{k-1}^2M + P_{k-1}$ has been constructed, apply Corollary (2.32) to obtain $J_{k-1}^2M \to J_{k-2}^2M + Q_{k-2}$. This induces $J_{k-1}^2M + P_{k-1} \to J_{k-2}^2M + Q_{k-2} + P_{k-1}$. Let $P_{k-2} = Q_{k-2} + P_{k-1}$. Then we obtain $J_k^2M + P_k \to J_{k-1}^2M + P_{k-1} \to J_{k-2}^2M + P_{k-2}$. Furthermore $(J_{k-2}^2M + P_{k-2})^* \to (J_{k-1}^2M + P_{k-1})^* \to 0$ is exact.

Thus, when grade $\mathrm{Ext}^i(M,R) \geq i$ for $1 \leq i \leq n$, we obtain a sequence of monomorphisms:

$$J_n^2M + P_n \to J_{n-1}^2M + P_{n-1} \to \dots \to J_1^2M + P_1 \to M + P_0$$

These monomorphisms have the property that their duals are surjective and the following diagram commutes in $R\underline{\mathcal{M}}$.

(2.33)

$$\begin{array}{ccc} \underline{M} & \approx & \underline{M + P_0} \\ \uparrow & & \uparrow \\ \underline{J_k^2M} & \approx & \underline{J_k^2M + P_k} \\ \downarrow & & \downarrow \\ \underline{J_{k-1}^2M} & \approx & \underline{J_{k-1}^2M + P_{k-1}} \end{array}$$

(Here the horizontal maps are induced by direct summand morphisms, while the right-hand vertical maps are induced by the sequence of monomorphisms).

(2.34) LEMMA (Assumptions and notation as above):

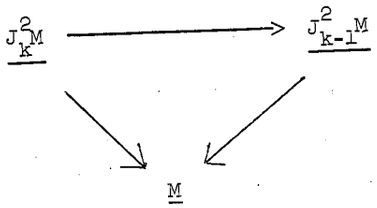
(a) For each $1 \leq k \leq n$ the monomorphism $J_k^2M + P_k \to M + P_0$ induces isomorphisms $Ext^{k+t}(M + P_0, \ ) \xrightarrow{\sim} Ext^{k+t}(J_k^2M + P_k, \ )$ for all $t > 0$.

(b) For each $1 \leq k \leq n$ the monomorphism $J_k^2M + P_k \to J_{k-1}^2M + P_{k-1}$ induces isomorphisms: $Ext^{k+t}(J_{k-1}^2M + P_{k-1}, \ ) \xrightarrow{\sim} Ext^{k+t}(J_k^2M + P_k, \ )$ for all $t > 0$.

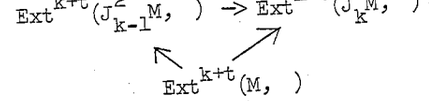
(c) $Ext^i(J_k^2M + P_k, R) \approx Ext^i(J_k^2M, R) = 0$ for $1 \leq i \leq k$ and $1 \leq k \leq n$.

Proof: Since grade $Ext^i(M,R) \geq i > i-1$ for $1 \leq i \leq n$, we have that Ω^iM is i-torsion free for $1 \leq i \leq n$. By Proposition (2.21) we know that when Ω^kM is k-torsion free, the map $J_k^2M \to \underline{M}$ induces an isomorphism $Ext^{k+1}(M, \ ) \xrightarrow{\sim} Ext^{k+1}(J_k^2M, \ )$. It follows that $\Omega^kJ_k^2M \to \underline{\Omega^kM}$ is an isomorphism. Therefore $Ext^{k+t}(M, \ ) \xrightarrow{\sim} Ext^{k+t}(J_k^2M, \ )$ for all $t > 0$. Applying Ext^{k+t} to the top square of diagram (2.33) above, and using this isomorphism, gives part (a) of the Lemma.

To prove part (b), consider the following commutative diagram:



Applying Ext^{k+t} yields:



When $1 \leq k \leq n$, part (a) just proved shows that the diagonal maps are isomorphisms. Therefore $Ext^{k+t}(J_{k-1}^2M, \ ) \xrightarrow{\sim} Ext^{k+t}(J_k^2M, \ )$ for all $t > 0$. Using this fact, we obtain our result by applying Ext^{k+t} to the bottom square of diagram (2.33).

Finally, part (c) follows directly from Proposition (2.21).

In order to facilitate the statement of our next result, we introduce the following terminology:

(2.35) Definition: Given $E \neq 0$ in $\mathcal{M}_R$ and an integer $k > 0$, we say that a module M in ${}_R\mathcal{M}$ is k-spherical of type E when M satisfies:

(a) $pd_RM = k$

(b) $Ext^i(M,R) = 0$ for $1 \leq i < k$

$Ext^k(M,R) \approx E$

We say that M is k-spherical of type 0 when M is projective.

(2.36) Remark: We shall see in Proposition (2.45) below that if M and N are both k-spherical of type E then $\underline{M} \approx \underline{N}$ in ${}_R\mathcal{M}$.

(2.37) THEOREM (Spherical Filtration): Suppose M is finitely generated and grade $Ext^i(M,R) \geq i$ for $1 \leq i \leq n$. Then there exists a projective P and a sequence of submodules:

$$M_n \subset M_{n-1} \subset \ldots \subset M_1 \subset M_0 = M + P$$

such that:

(a) $\underline{M_k} \approx \underline{J_k^2M}$ for $0 \leq k \leq n$.

(b) M_{k-1}/M_k is k-spherical of type $Ext^k(M,R)$ for $1 \leq k \leq n$.

Proof: As we have seen, there is, under our hypotheses, a sequence of monomorphisms: $J_n^2M + P_n \to J_{n-1}^2M + P_{n-1} \to \ldots \to J_1^2M + P_1 \to M + P_0$. For each $1 \leq k \leq n$ consider S_kM defined by the exact sequence:

(2.38) $$0 \to J_k^2 M + P_k \to J_{k-1}^2 M + P_{k-1} \to S_k M \to 0$$

By Lemma (2.34) the map: $J_k^2 M+P_k \to J_{k-1}^2 M+P_{k-1}$ induces isomorphisms $\mathrm{Ext}^{k+t}(J_{k-1}^2 M+P_{k-1}, \ ) \xrightarrow{\sim} \mathrm{Ext}^{k+t}(J_k^2 M+P_k, \ )$ for all $t > 0$. This fact, together with the long exact sequence for Ext determined by (2.38) shows that $\mathrm{Ext}^{k+t}(S_k M, \ ) = 0$. Therefore $\mathrm{pd}_R S_k M \leq k+1$.

Now apply $\mathrm{Ext}(\ , R)$ to (2.38). One obtains exact sequences:

$$(J_{k-1}^2 M+P_{k-1})^* \to (J_k^2 M+P_k)^* \to \mathrm{Ext}^1(S_k M, R)$$
$$\to \mathrm{Ext}^1(J_{k-1}^2 M+P_{k-1}, R) \to \dots \to \mathrm{Ext}^i(J_k^2 M+P_k, R) \to \mathrm{Ext}^{i+1}(S_k M, R)$$
$$\to \mathrm{Ext}^{i+1}(J_{k-1}^2 M+P_{k-1}, R) \to \mathrm{Ext}^{i+1}(J_k^2 M+P_k, R) \to \dots$$

By construction, $(J_{k-1}^2 M+P_{k-1})^* \to (J_k M+P_k)^*$ is surjective, while $\mathrm{Ext}^i(J_k^2 M+P_k, R) = 0$ for $1 \leq i \leq k$ and $1 \leq k \leq n$, by Lemma (2.34). Furthermore we have $\mathrm{Ext}^{k+t}(J_{k-1}^2 M+P_{k-1}, R) \xrightarrow{\sim} \mathrm{Ext}^{k+t}(J_k^2 M+P_k, R)$ for all $t > 0$. We conclude from the last exact sequences that $\mathrm{Ext}^i(S_k M, R) = 0$ for $1 \leq i < k$ and for $i > k$. Also we get $\mathrm{Ext}^k(S_k M, R) \xrightarrow{\sim} \mathrm{Ext}^k(J_{k-1}^2 M+P_{k-1}, R) \approx \mathrm{Ext}^k(M, R)$ (this last isomorphism coming from Lemma (2.34a)). Since $\mathrm{pd}\, S_k M < \infty$ we have $\mathrm{pd}\, S_k M = k$ if $\mathrm{Ext}^k(M,R) \neq 0$ and $\mathrm{pd}\, S_k M = 0$ if $\mathrm{Ext}^k(M,R) = 0$ for $1 \leq k \leq n$.

If we let $M_k = \mathrm{im}(J_k^2 M+P_k \to J_{k-1}^2 M+P_{k-1})$ for $1 \leq k \leq n$, $M_0 = M+P_0$, then we have inclusions: $M_n \subset M_{n-1} \subset \dots \subset M_1 \subset M_0 = M+P_0$. It is easily seen that $M_{k-1}/M_k \approx S_k M$ for $1 \leq k \leq n$ so we are done.

We now turn to the following question: Given $E_1, \dots, E_n$, when does there exist a module M, $\mathrm{pd}\, M \leq n$, such that $\mathrm{Ext}^i(M,R) \approx E_i$ for $1 \leq i \leq n$? As one would expect, the answer involves a depth condition on the E_i.

(2.39) LEMMA: Suppose $E \neq 0$ is in $\mathcal{M}_R$ and grade $E \geq k$. Then there is a k-spherical module M in ${}_R\mathcal{M}$ of type E.

Proof: Let $P_k \to P_{k-1} \to \dots \to P_1 \to P_0 \to E \to 0$ be part of a resolution of E by finitely generated projectives. Since grade $E \geq k$ we have $\mathrm{Ext}^i(E,R) = 0$ for $0 \leq i \leq k-1$. Therefore $0 \to P_0^* \to P_1^* \to \dots \to P_{k-1}^* \to P_k^*$ is an exact sequence of projective modules in ${}_R\mathcal{M}$. Let $M = \mathrm{coker}\,(P_{k-1}^* \to P_k^*)$; an easy calculation verifies that M is k-spherical of type E.

(2.40) COROLLARY: If grade $E_k \geq k$ for $1 \leq k \leq n$ then there is a module M such that $\mathrm{pd}\, M \leq n$ (equality if and only if $E_n \neq 0$) and $\mathrm{Ext}^k(M,R) \approx E_k$ for $1 \leq k \leq n$.

Proof: By the preceding Lemma we can find S_k which is k-spherical of type E_k for $1 \leq k \leq n$. Let $M = \sum_{k=1}^{n} S_k$; then M satisfies our requirements.

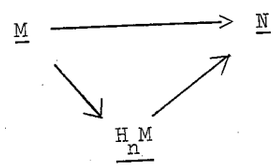
Now suppose that the modules $E_1, \dots, E_n$ are of the form: $E_i \approx \mathrm{Ext}^i(M,R)$ for $1 \leq i \leq n$, where M does not necessarily have finite projective dimension. Our next result shows that under suitable depth conditions we can "approximate" M by a module of projective dimension $\leq n$. Furthermore, this approximation satisfies a universal mapping property.

(2.41) THEOREM (Approximation Theorem): Let M be finitely generated with grade $\mathrm{Ext}^i(M,R) \geq i$ for $1 \leq i \leq n$. Then there is a finitely generated module $H_n M$, projective P, and exact sequence $0 \to J_n^2 M \to M+P \to H_n M \to 0$ with the following properties:

(a) $\mathrm{pd}\, H_n M \leq n$, with equality if and only if $\mathrm{Ext}^n(M,R) \neq 0$

(b) The map $M+P \to H_n M$ induces isomorphisms: $\mathrm{Ext}^i(H_n M, R) \xrightarrow{\sim} \mathrm{Ext}^i(M,R)$ for $1 \leq i \leq n$.

(c) If $\underline{M} \to \underline{N}$ is any map with $\mathrm{pd}\, N \leq n$ then there is a unique ${}_R\mathcal{M}$-morphism $\underline{H_n M} \to \underline{N}$ such that the diagram:



commutes in $_R\mathcal{M}$. It follows that H_nM is unique up to $_R\mathcal{M}$-isomorphism.

Proof: As we have seen, our hypotheses imply the existence of a monomorphism $J_n^2M \to M+P_0$ such that $(M+P_0)^* \to (J_n^2M)^*$ is surjective and $Ext^{n+t}(M+P_0, \ ) \to Ext^{n+t}(J_n^2M, \ )$ is an isomorphism for all $t > 0$.

Let $H_nM = \operatorname{coker}(J_n^2M \to M+P_0)^\sim$ Applying Ext to the exact sequence $0 \to J_n^2M \to M+P_0 \to H_nM \to 0$ we obtain the exact sequence of functors:

$Ext^{n+1}(M+P_0, \ ) \to Ext^{n+1}(J_n^2M, \ ) \to Ext^{n+2}(H_nM, \ ) \to Ext^{n+2}(M+P_0, \ ) \to Ext^{n+2}(J_n^2M, \ )$. We conclude that $Ext^{n+2}(H_nM, \ ) = 0$ so pd. $H_nM \leq n+1$. From the above short exact sequence we obtain also the following exact sequences:

$(M+P_0)^* \to (J_n^2M)^* \to Ext^1(H_nM,R) \to Ext^1(M+P_0,R) \to Ext^1(J_n^2M,R) \to \cdots$

$\cdots \to Ext^i(J_n^2M,R) \to Ext^{i+1}(H_nM,R) \to Ext^{i+1}(M+P_0,R) \to Ext^{i+1}(J_n^2M,R) \to \cdots$

Since $Ext^i(J_n^2M,R) = 0$ for $1 \leq i \leq n$ by Lemma (2.34) and $Ext^{i+t}(M+P_0,R) \to Ext^{i+t}(J_n^2M,R)$ for all $t > 0$, we conclude from these exact sequences that $Ext^i(H_nM,R) \to Ext^i(M+P_0,R)$ for $1 \leq i \leq n$, and $Ext^i(H_nM,R) = 0$ for $i > n$. This proves (b). Since pd $H_nM < \infty$, we have pd. $H_nM \leq n$ which gives (a). Part (c) now follows from parts (a) and (b) and the following Lemma.

(2.42) LEMMA: Let $f: A \to B$ be a map of finitely generated modules which induces $Ext^i(B,R) \to Ext^i(A,R)$ for $1 \leq i \leq n$. If pd. $C = t \leq n$, then f induces:

(1) $Ext^j(B,C) \to Ext^j(A,C)$ for $1 \leq j \leq n-t$.

(2) $(\underline{B},\underline{C}) \to (\underline{A},\underline{C})$.

Proof: To show (1) we proceed by induction on t. We first note that since A and B are finitely generated, $Ext^i(B,R) \to Ext^i(A,R)$ is equivalent to $Ext^i(B,P) \to Ext^i(A,P)$ for any projective P. Thus, when $t = 0$ we get our result (1). Now suppose (1) holds for modules of projective dimension $< t$, and suppose pd. $C = t > 0$. Let $0 \to K \to P \to C \to 0$ be exact with P projective. Then pd. $K = t-1 < t$, and we have the following exact commutative diagram:

$$\begin{array}{ccccccccc} Ext^j(B,K) & \to & Ext^j(B,P) & \to & Ext^j(B,C) & \to & Ext^{j+1}(B,K) & \to & Ext^{j+1}(B,P) \\ \downarrow\wr & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \downarrow\wr \\ Ext^j(A,K) & \to & Ext^j(A,P) & \to & Ext^j(A,C) & \to & Ext^{j+1}(A,K) & \to & Ext^{j+1}(A,P) \end{array}$$

The indicated isomorphisms exist by induction hypothesis (noting that $1 \leq j \leq n-t$ implies $1 \leq j+1 \leq n-(t-1)$). Thus, the middle map is an isomorphism by the five Lemma. This completes the induction and the proof of (1).

To prove (2), let $0 \to K \to P \to C \to 0$ be exact with P projective. We obtain a commutative diagram whose rows are exact by Lemma (1.35):

$$\begin{array}{ccccccc} (\underline{B},\underline{P}) & \to & (\underline{B},\underline{C}) & \to & Ext^1(B,K) & \to & Ext^1(B,P) \\ \downarrow & & \downarrow & & \downarrow\wr & & \downarrow\wr \\ (\underline{A},\underline{P}) & \to & (\underline{A},\underline{C}) & \to & Ext^1(A,K) & \to & Ext^1(A,P) \end{array}$$

Here the isomorphisms exist by part (1) already proved. Since P is projective, $(\underline{B},\underline{P}) = (\underline{A},\underline{P}) = 0$. Therefore the map $(\underline{B},\underline{C}) \to (\underline{A},\underline{C})$ is an isomorphism. This completes the proof of the Lemma.

To see how part (c) of Theorem (2.41) follows, let $A = M$ and $B = H_nM$.

By part (2) of the Lemma, $(\underline{H_n M},\underline{N}) \twoheadrightarrow (\underline{M},\underline{N})$ for any N with $\operatorname{pd} N \leq n$. This is the statement of part (c), so our proof of Theorem (2.41) is complete.

§7: Lifting Maps of Syzygies.

We conclude this Chapter with a discussion of the map $(\underline{M},\underline{N}) \to (\underline{\Omega^i M},\underline{\Omega^i N})$, given by $\underline{f} \rightsquigarrow \underline{\Omega^i f}$. We shall extend Proposition (1.39) and deduce the uniqueness--up to ${}_R\underline{\mathcal{M}}$-isomorphism--of a k-spherical module of given type.

Recall from Chapter 1 that for any two half exact functors G and F we have a map $(\underline{G},\underline{F}) \to (S^iG,S^iF)$ induced by S^i If we fix F, then Corollary (1.34) gives a sufficient condition for this map to be an isomorphism for $1 \leq i \leq n+1$ and all half exact G; namely: that S^iF be projectively stable for $1 \leq i \leq n$. When $F = (\underline{M},_)$ where M is in ${}_R\mathcal{M}$, then $S^iF \approx \operatorname{Ext}^i(M,\ )$ and S^iF is projectively stable if and only if $\operatorname{Ext}^i(M,R) = 0$. If $G = (\underline{N},_)$ then the map $(\underline{G},\underline{F}) \to (S^{i+1}G,S^{i+1}F)$ becomes: $((\underline{N},_)\ ,\ (\underline{M},_)) \to (\operatorname{Ext}^{i+1}(N,\ )\ ,\ \operatorname{Ext}^{i+1}(M,\ ))$. Now $\operatorname{Ext}^{i+1}(M,\ ) \approx \operatorname{Ext}^1(\Omega^iM,\ )$ and we have seen that $(\operatorname{Ext}^1(\Omega^iN,\ )\ ,\ \operatorname{Ext}^1(\Omega^iM,\ )) \approx (\underline{\Omega^iM},\underline{\Omega^iN})$. Furthermore, $((\underline{N},_)\ ,\ (\underline{M},_)) \approx (\underline{M},\underline{N})$. It is easily seen that we have a commutative diagram

$$\begin{array}{ccc} ((\underline{N},_)\ ,\ (\underline{M},_)) & \xrightarrow{S^{i+1}} & (\operatorname{Ext}^{i+1}(N,\ )\ ,\ \operatorname{Ext}^{i+1}(M,\ )) \\ \wr & & \wr \\ (\underline{M},\underline{N}) & \xrightarrow{\Omega^i} & (\underline{\Omega^iM},\underline{\Omega^iN}) \end{array}$$

As a direct result of our sufficient condition for the top map to be an isomorphism, we deduce the following.

(2.43) PROPOSITION: If $\operatorname{Ext}^i(M,R) = 0$ for $1 \leq i \leq n$ then the maps $(\underline{M},\underline{N}) \to (\underline{\Omega^iM},\underline{\Omega^iN})$, induced by Ω^i, are isomorphisms for $1 \leq i \leq n$ and any R-module N.

To apply this to spherical modules we will need the following result:

(2.44) LEMMA: If $\operatorname{pd}. M = n$ and M is finitely generated, then $\operatorname{Ext}^n(M,\ ) \approx (\operatorname{Ext}^n(M,R) \otimes_R\ )$.

Proof: Since $\operatorname{pd}. M = n$ we see that Ω^nM, hence $D\Omega^nM$, is projective. Therefore $\operatorname{Ext}^1(D\Omega^nM,\ ) = \operatorname{Ext}^2(D\Omega^nM,\ ) = 0$ and we get our result from the exact sequence (2.8):

$$0 \to \operatorname{Ext}^1(D\Omega^nM,\ ) \to (\operatorname{Ext}^n(M,R) \otimes_R\ ) \to \operatorname{Ext}^n(M,\ ) \to \operatorname{Ext}^2(D\Omega^nM,\ )$$

Now suppose that S and T are both k-spherical modules of type E. Then $\operatorname{pd}. S = \operatorname{pd}. T = k$ and $\operatorname{Ext}^k(S,R) \approx \operatorname{Ext}^k(T,R) \approx E$. By the Lemma we have $\operatorname{Ext}^k(S,\ ) \approx (E \otimes_R\ ) \approx \operatorname{Ext}^k(T,\ )$. Therefore $\underline{\Omega^{k-1}S} \approx \underline{\Omega^{k-1}T}$. Since S is k-spherical we have: $\operatorname{Ext}^i(S,R) = 0$ for $1 \leq i \leq k-1$. By Proposition (2.43) we conclude that $(\underline{S},\underline{N}) \approx (\underline{\Omega^{k-1}S},\underline{\Omega^{k-1}N})$ for any R-module N. Similarly we have $(\underline{T},\underline{N}) \approx (\underline{\Omega^{k-1}T},\underline{\Omega^{k-1}N})$. Since $\underline{\Omega^{k-1}S} \approx \underline{\Omega^{k-1}T}$, it follows that $(\underline{S},\underline{N}) \approx (\underline{T},\underline{N})$ for any R-module N. In other words: $(\underline{S},_) \approx (\underline{T},_)$. This, of course, is equivalent to $\underline{S} \approx \underline{T}$. We have proved:

(2.45) PROPOSITION: Any two k-spherical modules of the same type are projectively equivalent (i.e. isomorphic in ${}_R\underline{\mathcal{M}}$).

When the $\operatorname{Ext}^i(M,R)$ have the right grade we can prove the following converse to Proposition (2.43) with the help of the Approximation Theorem and Lemma (2.44).

(2.46) PROPOSITION: Suppose M is finitely generated and $\operatorname{grade} \operatorname{Ext}^i(M,R) \geq i$ for $1 \leq i \leq n$. Then the following conditions are equivalent:

(a) $\operatorname{Ext}^i(M,R) = 0$ for $1 \leq i \leq n$.

(b) $(\underline{M},\underline{N}) \to (\underline{\Omega^iM},\underline{\Omega^iN})$ is an isomorphism for $1 \leq i \leq n$ and all R-modules N.

(c) $(\underline{M},\underline{N}) \to (\underline{\Omega^i M},\underline{\Omega^i N})$ is an isomorphism for $1 \leq i \leq n$ and every R-module N with pd. $N \leq n$.

Proof: That (a) implies (b) is Proposition (2.43), while (b) clearly implies (c). We need only prove that (c) implies (a). Let $F = (\underline{M},_)$; we must show that S^iF is projectively stable for $1 \leq i \leq n$. By Corollary (1.34) this will be true if we can find functors G_j for $1 \leq j \leq n$ such that

(1) $S^jG_j \approx L_0S^jF$

(2) $(\underline{G_j},\underline{F}) \to (S^iG_j,S^iF)$ is an isomorphism for $1 \leq i \leq n+1$ and $1 \leq j \leq n$.

As we have observed before, $L_0S^jF \approx L_0 \operatorname{Ext}^j(M,\ ) \approx (\operatorname{Ext}^j(M,R)\otimes_R\ )$.

Suppose that $1 \leq j \leq n$. Since grade $\operatorname{Ext}^i(M,R) \geq i$ for $1 \leq i \leq j$, the Approximation Theorem (2.41) gives the existence of a module H_jM such that pd. $H_jM \leq j$ and $\operatorname{Ext}^i(H_jM,R) \approx \operatorname{Ext}^i(M,R)$ for $1 \leq i \leq j$.

Let $G_j = (\underline{H_jM},_)$ if $\operatorname{Ext}^j(M,R) \neq 0$, and let $G_j = 0$ if $\operatorname{Ext}^j(M,R) = 0$. When $\operatorname{Ext}^j(M,R) = 0$ we have $L_0S^jF = (\operatorname{Ext}^j(M,R)\otimes_R\ ) = 0$ so conditions (1) and (2) are satisfied. When $\operatorname{Ext}^j(M,R) \neq 0$ we have pd. $H_j(M) = j$. By Lemma (2.44) $S^jG_j \approx \operatorname{Ext}^j(H_jM,\ ) \approx (\operatorname{Ext}^j(H_jM,R)\otimes_R\ ) \approx (\operatorname{Ext}^j(M,R)\otimes_R\ ) \approx L_0S^jF$. Thus, condition (1) is satisfied. Since pd. $H_jM = j \leq n$, our hypothesis (c) gives $(\underline{M},\underline{H_jM}) \Rightarrow (\underline{\Omega^iM},\underline{\Omega^iH_jM})$ for $1 \leq i \leq n$. From the commutative diagram:

$$\begin{array}{ccc}
(\underline{G_j},\underline{F}) & \longrightarrow & (S^{i+1}G_j,S^{i+1}F) \\
\wr & & \wr \\
((\underline{H_jM},_)\ ,\ (\underline{M},_)) & \longrightarrow & (\operatorname{Ext}^{i+1}(H_jM,\ )\ ,\ \operatorname{Ext}^{i+1}(M,\ )) \\
\wr & & \wr \\
(\underline{M},\underline{H_jM}) & \xrightarrow{\sim} & (\underline{\Omega^iM},\underline{\Omega^iN})
\end{array}$$

we see that $(\underline{G_j},\underline{F}) \Rightarrow (S^{i+1}G_j,S^{i+1}F)$ for $0 \leq i \leq n$. Therefore the G_j satisfy (2). Corollary (1.34) therefore applies and we are done.

Chapter 3: Gorenstein Dimension

In this Chapter we define the Gorenstein dimension of a half-exact functor F--written: G-dim F. When M is an R-module, we define G-dim M to be G-dim F where $F = (M, _)$. As we shall see, in this Chapter and the next, modules of finite Gorenstein dimension have many pleasing properties with respect to torsion freeness and reflexivity. As we shall see, modules of finite projective dimension and modules over Gorenstein rings are all examples of modules having finite G-dimension.

We deal with functors first. Suppose that Γ and Δ are abelian categories where Γ has enough projectives and injectives.

(3.1) Definition: Let $F:\Gamma \to \Delta$ be a half-exact functor.

(a) G-dim F = 0 when $\overline{\text{Stab}}\ F = \infty = \underline{\text{Stab}}\ F$.

(b) G-dim F = k for $k \geq 1$ when G-dim$(S^k F) \neq 0$ but G-dim$(S^{k+1}F) = 0$.

(3.2) Remark: When $k \geq 1$ we see that G-dim F = k if and only if $k = \min\{t \mid \text{G-dim}(S^{t+1}F) = 0\}$. If k = 0 this is no longer true. For a counter-example, let $F = (P, \ )$ where P is a non-zero projective. Then $(P,P) \neq 0$ so F is not projectively stable. Thus, G-dim F > 0. On the other hand, $S^1F = \text{Ext}^1(P, \ ) = 0$ so $0 = \min\{t \mid \text{G-dim } S^{t+1}F = 0\}$. It is clear from this example, however, that when F is projectively stable we do get G-dim $F = \min\{t \mid \text{G-dim } S^{t+1}F = 0\}$.

(3.3) Remark: Technically, what we have defined should be called the right Gorenstein dimension of F, and should be denoted: $\overline{\text{G-dim}}\ F$. To define $\underline{\text{G-dim}}\ F$, we should replace S^k(resp. S^{k+1}) by S_k(resp. S_{k+1}) in part (b) of the definition, obtaining:

(b): $\underline{\text{G-dim}}\ F = k$ if G-dim$(S_k F) = 0$ but G-dim$(S_{k+1}F) \neq 0$.

Statements about the right G-dimension of F when $F = (M, _)$ can be translated into statements about M, while statements about the left G-dim are

related to DM. Because of the duality between M and DM we have suppressed the notion of $\underline{\text{G-dim}}$ in favor of $\overline{\text{G-dim}}$. For the sake of notation we have written this last as simply: G-dim.

(3.4) LEMMA: If G-dim F = 0 then G-dim S^1F = G-dim S_1F = 0.

Proof: Since the property "G-dim F = 0" is symmetrical in left and right satellites, we shall only prove that G-dim $S^1F = 0$, leaving the dual proof to the reader.

First, we have $\underline{\text{Stab}}(S^1F) = \infty$ since $S^i(S^1F) = S^{i+1}F$ which is projectively stable since $\underline{\text{Stab}}\ F = \infty$.

On the other hand, F is projectively stable, so $F = S_1S^1F$. Since $\overline{\text{Stab}}\ F = \infty$, $S_jF = S_{j+1}(S^1F)$ is injectively stable for all $j \geq 0$. Furthermore, S^1F is always injectively stable. Thus, $S_j(S^1F)$ is injectively stable for all $j \geq 0$, so $\overline{\text{Stab}}\ S^1F = \infty$. By definition of G-dimension, G-dim $S^1F = 0$ so we are done.

Where we are dealing with a functor known to have finite G-dimension, this dimension depends only on the stability of its right satellites, as is shown by the following.

(3.5) PROPOSITION: Let $k > 0$ be an integer and suppose that G-dim $F < \infty$. Then G-dim $F \leq k$ if and only if S^iF is projectively stable for all $i > k$.

Proof: Suppose G-dim $F \leq k$. Then G-dim$(S^{k+1}F) = 0$. By repeated application of Lemma (4.4) we see that G-dim$(S^iF) = 0$ for $i > k$. Thus, S^iF is projectively stable for all $i > k$.

Conversely, suppose that S^iF is projectively stable for all $i > k$. To prove that G-dim $F \leq k$ we must show that G-dim $S^{k+1}F = 0$; i.e. that $\overline{\text{Stab}}\ S^{k+1}F = \infty = \underline{\text{Stab}}\ S^{k+1}F$. Now $S^i(S^{k+1}F) = S^{k+1+i}F$ and is projectively stable when $i \geq 0$ by assumption (since $k+1+i > k$). Thus, $\underline{\text{Stab}}\ S^{k+1}F = \infty$.

It remains to show that $\overline{\text{Stab}}\ S^{k+1}F = \infty$. Let $H = S^kF$. By assumption

$S^iH = S^{k+i}F$ vanishes on projectives for $i > 0$. By Proposition (1.3) this is equivalent to $L_0(S^iH) = 0$ for all $i > 0$. This being the case, we have, trivially: $\overline{\text{Stab}}\ L_0S^iH = \infty \geq j + i$ for all $i > 0$, $j \geq 0$. Now when this condition holds, Proposition (2.29) tells us that the following statements are equivalent (for any $j \geq 0$):

(a) $\overline{\text{Stab}}\ S^1H \geq j + 1$

(b) $\overline{\text{Stab}}\ S^{n+1}H \geq n + j + 1$

where n is any integer ≥ 0. In particular, when n = G-dim F we have $S^{n+1}H = S^{n+k+1}F$ which has G-dimension zero by repeated application of Lemma (3.4). It follows that $\overline{\text{Stab}}\ S^{k+1}F \geq j + 1$ for any $j \geq 0$. Thus $\overline{\text{Stab}}\ S^{k+1}F = \infty$ and our proof is complete.

(3.6) COROLLARY: Let G-dim $F < \infty$.

(a) If S^iF is not projectively stable for some $i \geq 0$, then G-dim F = sup{t|S^tF is not projectively stable}.

(b) If S^iF is projectively stable for all $i \geq 0$, then G-dim F = 0.

Proof: Part (a) follows directly from the preceding Proposition. To prove (b), note that the argument of Proposition (3.5) shows that G-dim $S^1F = 0$. By Lemma (3.4), G-dim $S_1(S^1F) = 0$. Since F is projectively stable, $S_1S^1F = F$ so G-dim F = 0.

With the case Γ = category of R-modules in mind, we make the following definition:

(3.7) Definition: For M an object of Γ we define G-dim M (= Gorenstein dimension of M) to be G-dim F where $F = (\underline{M},_)$.

Remarks: We note immediately that if $\underline{M} \approx \underline{N}$ then $(\underline{M},_) \approx (\underline{N},_)$ so G-dim M = G-dim N. In particular, G-dim P = 0 if P is projective. If we now write $(\underline{M},_)$ for F in Corollary (3.6) and note that $(\underline{M},_)$ is projectively stable, we see that if G-dim $M < \infty$, then G-dim M = sup{t|$\text{Ext}^t(M,\ )$

is not projectively stable}.

Now, let us suppose that Γ is the category of finitely generated left (resp. right) R-modules, where R is noetherian on both sides. Since $\text{Ext}^i(M,\ )$ is projectively stable if and only if $\text{Ext}^i(M,R) = 0$, we see that if G-dim $M < \infty$ then:

$$\text{G-dim } M = \sup\{t|\text{Ext}^t(M,R) \neq 0\} .$$

We now turn our attention to modules of zero G-dimension. To say that G-dim M = 0 is to say that $S^i(\underline{M},_) \approx \text{Ext}^i(M,\ )$ is projectively stable for all $i \geq 0$ and that $S_i(\underline{M},_) \approx \text{Tor}_{i+1}(DM,\ )$ is injectively stable for all $i \geq 0$. The first condition is just simply: $\text{Ext}^i(M,R) = 0$ for all $i > 0$. As we have seen in Corollary (2.10), the second condition is equivalent to: $\text{Ext}^i(DM,R) = 0$ for all $i > 0$.

We can write these conditions in another form. From the exact sequence:

$$0 \to \text{Ext}^1(DM,R) \to M \to M^{**} \to \text{Ext}^2(DM,R) \to 0$$

we see that M is reflexive if and only if $\text{Ext}^1(DM,R) = \text{Ext}^2(DM,R) = 0$. From the exact sequence $0 \to M^* \to P_0^* \to P_1^* \to DM \to 0$ we see that $\text{Ext}^i(DM,R) = 0$ if and only if $\text{Ext}^{i-2}(M^*,R) = 0$ (for $i > 2$). Thus, $\text{Ext}^i(DM,R) = 0$ for all $i > 0$ if and only if M is reflexive and $\text{Ext}^i(M^*,R) = 0$ for all $i > 0$. In summary, we have

(3.8) PROPOSITION: For M a finitely generated R-module the following statements are equivalent:

(a) G-dim M = 0.

(b) $\text{Ext}^i(M,R) = \text{Ext}^i(DM,R) = 0$ for all $i > 0$.

(c) M is reflexive and $\text{Ext}^i(M,R) = \text{Ext}^i(M^*,R) = 0$ for all $i > 0$.

(d) M and DM are both i-torsion free for all $i \geq 0$. (Condition (d) is equivalent to (b) by definition of i-torsion freeness and the fact that $\underline{DD(M)} \approx \underline{M}$).

To proceed further, we need the following Lemmas.

(3.9) LEMMA: Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence of finitely generated left R-modules. Then there exist finitely generated right R-modules D_1, D_2, D_3 and an exact sequence $0 \to M_3^* \to M_2^* \to M_1^* \to D_3 \to D_2 \to D_1 \to 0$ such that:

(a) There exist isomorphisms $\underline{D_i} \approx \underline{D(M_i)}$, $1 \leq i \leq 3$, and a commutative diagram:

$$\begin{array}{ccccc} \underline{D_3} & \longrightarrow & \underline{D_2} & \longrightarrow & \underline{D_1} \\ \wr & & \wr & & \wr \\ \underline{D(M_3)} & \longrightarrow & \underline{D(M_2)} & \longrightarrow & \underline{D(M_1)} \end{array}$$

(b) $0 \to D_1^* \to D_2^* \to D_3^* \to 0$ is exact.

Proof: We can find finitely generated projectives P_i, Q_i ($i = 1, 2, 3$) and maps so that the following diagram is exact and commutative:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q_1 & \longrightarrow & Q_2 & \longrightarrow & Q_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_1 & \longrightarrow & P_2 & \longrightarrow & P_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

We note that the top two rows split. Define: $D_i = \operatorname{coker}(P_i^* \to Q_i^*)$ for $i = 1,2,3$. The uniqueness properties of the functor D, described in Proposition (2.6), give us assertion (a). If we dualize the first diagram we obtain the exact commutative diagram:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & M_3^* & \longrightarrow & M_2^* & \longrightarrow & M_1^* & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_3^* & \longrightarrow & P_2^* & \longrightarrow & P_1^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Q_3^* & \longrightarrow & Q_2^* & \longrightarrow & Q_1^* & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & D_3 & \longrightarrow & D_2 & \longrightarrow & D_1 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

It is now a standard result of diagram chasing (the so-called "snake diagram") that we have an exact sequence $0 \to M_3^* \to M_2^* \to M_1^* \to D_3 \to D_2 \to D_1 \to 0$. If we now dualize once more, we obtain the following exact and commutative diagram (recalling that finitely generated projectives are reflexive):

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & D_1^* & \longrightarrow & D_2^* & \longrightarrow & D_3^* & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Q_1 & \longrightarrow & Q_2 & \longrightarrow & Q_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & P_1 & \longrightarrow & P_2 & \longrightarrow & P_3 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Applying the "snake diagram" again, we see that the sequence $0 \to D_1^* \to D_2^* \to D_3^* \to M_1 \to M_2 \to M_3 \to 0$ is exact. Furthermore, since $M_1 \to M_2$ is a monomorphism, we conclude that $0 \to D_1^* \to D_2^* \to D_3^* \to 0$ is exact. This proves (b) and we are done.

(3.10) LEMMA: Suppose $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact with the M_i finitely generated. Suppose also that G-dim $M_3 = 0$. Then G-dim $M_2 = 0$ if and only if G-dim $M_1 = 0$.

Proof: Since G-dim $M_3 = 0$, we know from Proposition (3.8) that $\operatorname{Ext}^i(M_3,R) = \operatorname{Ext}^i(DM_3,R) = 0$ for all $i > 0$. In particular, $\operatorname{Ext}^1(M_3,R) = 0$. Dualizing our short exact sequence then gives the exact sequence $0 \to M_3^* \to M_2^* \to M_1^* \to 0$. Now by Lemma (3.9) there exists an exact sequence: $0 \to M_3^* \to M_2^* \to M_1^* \to D_3 \to D_2 \to D_1 \to 0$ where $\underline{D_i} \approx D(M_i)$ for $i = 1,2,3$. Since $M_2^* \to M_1^*$ is surjective, we deduce the exactness of: $0 \to D_3 \to D_2 \to D_1 \to 0$. We now have two long exact sequences, obtained from $0 \to M_1 \to M_2 \to M_3 \to 0$ and $0 \to D_3 \to D_2 \to D_1 \to 0$ respectively:

$$\dots \to \operatorname{Ext}^i(M_3,R) \to \operatorname{Ext}^i(M_2,R) \to \operatorname{Ext}^i(M_1,R) \to \operatorname{Ext}^{i+1}(M_3,R) \to \dots$$

and:

$$0 \to D_1^* \to D_2^* \to D_3^* \to \operatorname{Ext}^1(D_1,R) \to \operatorname{Ext}^1(D_2,R) \to \operatorname{Ext}^1(D_3,R) \to \dots$$

$$\dots \to \operatorname{Ext}^{i-1}(D_3,R) \to \operatorname{Ext}^i(D_1,R) \to \operatorname{Ext}^i(D_2,R) \to \operatorname{Ext}^i(D_3,R) \to \dots$$

Since $\operatorname{Ext}^i(M_3,R) = 0$ for all $i > 0$, we deduce from the first of these sequences that $\operatorname{Ext}^i(M_2,R) \xrightarrow{\sim} \operatorname{Ext}^i(M_1,R)$ for all $i > 0$.

From Lemma (3.9) we have that $D_2^* \to D_3^*$ is surjective. Furthermore, $\operatorname{Ext}^i(D_3,R) = \operatorname{Ext}^i(D(M_3),R) = 0$ for all $i > 0$ since $\underline{D_3} \approx \underline{D(M_3)}$ and G-dim $M_3 = 0$. Thus, the second long exact sequence gives $\operatorname{Ext}^i(D_1,R) \xrightarrow{\sim} \operatorname{Ext}^i(D_2,R)$ for all $i > 0$. Since $\underline{D_1} \approx \underline{D(M_1)}$ and $\underline{D_2} \approx \underline{D(M_2)}$ we have, in summary:

$$\operatorname{Ext}^i(M_1,R) \approx \operatorname{Ext}^i(M_2,R)$$

$$\operatorname{Ext}^i(D(M_1),R) \approx \operatorname{Ext}^i(D(M_2),R)$$

for all $i > 0$. By the characterization of G-dim = 0 contained in Proposition (3.8), we see that G-dim $M_1 = 0$ if and only if G-dim $M_2 = 0$.

Suppose now that $\mathcal{L}$ denotes the full subcategory of ${}_R\mathcal{M}$ determined by the modules of zero G-dimension. Then $\mathcal{L}$ has the following properties:

(3.11) (a) Suppose $0 \to A \to B \to C \to 0$ is exact and C lies in $\mathcal{L}$. Then A is in $\mathcal{L}$ if and only if B is in $\mathcal{L}$.

(b) If $0 \to A \to B \to C \to 0$ splits, then B is in $\mathcal{L}$ if and only if both A and C are in $\mathcal{L}$.

(c) If P is projective, then P is in $\mathcal{L}$.

Property (a) is a restatement of Lemma (3.10). Now if $B \approx A + C$ then $(\underline{B},_) \approx (\underline{A},_) + (\underline{C},_)$. Since satellites commute with taking direct sums of functors, we see that G-dim$(\underline{B},_) = 0$ if and only if G-dim$(\underline{A},_)$ = G-dim$(\underline{C},_) = 0$. This proves (b). Finally, if P is projective, so is $D(P)$ and $\operatorname{Ext}^i(P,R) = \operatorname{Ext}^i(D(P),R) = 0$ for all $i > 0$. By Proposition (3.8), G-dim $P = 0$.

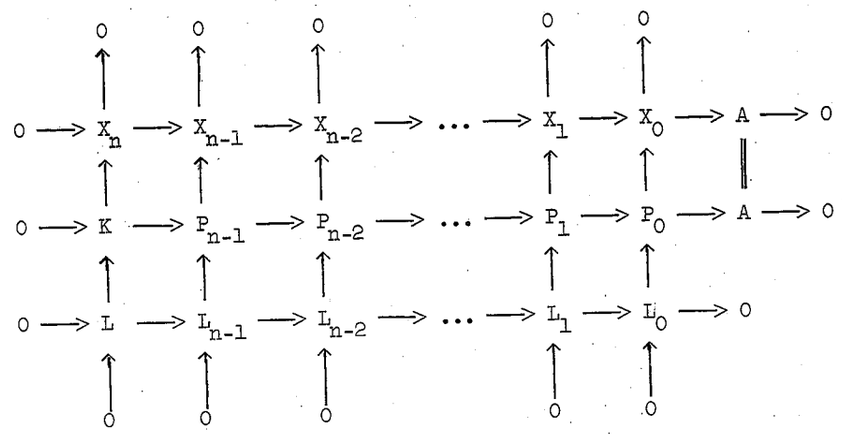
We now prove a general Lemma about subcategories having properties (a), (b) and (c) listed above. Such categories are sometimes called resolving subcategories.

(3.12) LEMMA: Suppose Γ is an abelian category with enough projectives, and Λ is a full subcategory of Γ having properties (a), (b) and (c) of (3.11). Then Λ has the following property:

If $0 \to X_n \to X_{n-1} \to \dots \to X_0 \to A \to 0$ and $0 \to Y_n \to Y_{n-1} \to \dots \to Y_0 \to A \to 0$ are exact in Γ with the X_i and Y_i in Λ for $0 \leq i \leq n-1$, then X_n is in Λ if and only if Y_n is in Λ.

(This tells us that any two "minimal" resolutions, by objects of Λ of an object in Γ, have the same length.)

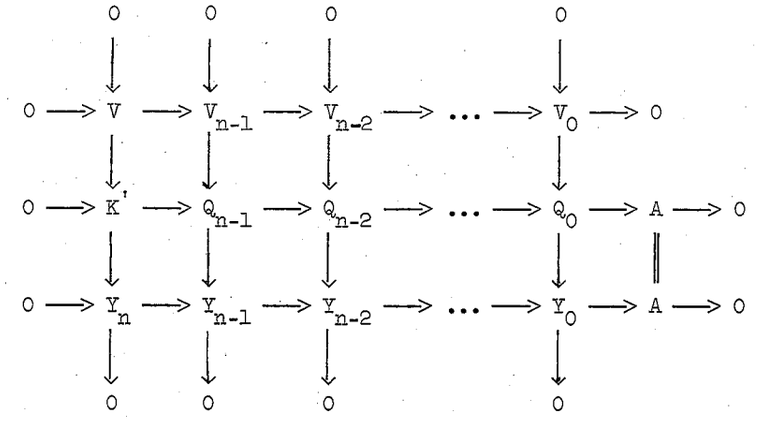
Proof: We sketch a proof, leaving the details to the reader. It clearly suffices to prove that if X_n is in Λ then Y_n is in Λ. We can construct a commutative exact diagram:



where the P_i $(0 \leq i \leq n-1)$ are projective--hence in Λ by property (c). Since the X_i, for $1 \leq i \leq n$ are in Λ, property (a) applied to the exact columns gives that L_i is in Λ for $1 \leq i \leq n-1$. Repeated application of (a) to the bottom row now shows that L is in Λ as well. Since X_n is in Λ, (a) applied to the first column shows that K is in Λ.

Now suppose that $0 \to K' \to Q_{n-1} \to \cdots \to Q_0 \to A \to 0$ is any exact sequence with the Q_i projective. Then $\underline{K'} \approx \underline{K}$ (from the above diagram) since $\mathrm{Ext}^1(K', \ ) \approx \mathrm{Ext}^{n+1}(A, \ ) \approx \mathrm{Ext}^1(K, \ )$. Thus, $K' + Q \approx K + P$ for suitable projectives P and Q. We have shown that K is in Λ. By (c), P is in Λ hence $K + P \approx K' + Q$ is in Λ by (b). Again by (b), we have K' is in Λ.

We now construct an exact and commutative diagram:



where the Q_i are all projective. Then the Q_i are all in Λ. By assumption, the Y_i are in Λ for $1 \leq i \leq n-1$: therefore, property (a) applied to the columns of the diagram assures that the V_i are in Λ for $1 \leq i \leq n-1$. Property (a) invoked repeatedly now shows that all kernels and cokernels in the exact sequence $0 \to V \to V_{n-1} \to \cdots \to V_0 \to 0$ belong to Λ. In particular, V is in Λ and $U = \mathrm{coker}(V \to V_{n-1})$ is in Λ. Consider the following exact and commutative diagram:

