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STABLE MODULE THEORY

by

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Introduction. The notions of torsion and torsion freeness have played a very important role in module theory--particularly in the study of modules over integral domains. Furthermore, the use of homological techniques in this connection has been well-established. It is the aim of this paper to extend these techniques and to show that this extension leads naturally to several new concepts (e.g. k-torsion freeness and Gorenstein dimension) which are useful in the classification of modules and rings.

We shall assume henceforth that all rings are noetherian and all modules finitely generated. It is well-known that, when R is a domain (commutative) and M an R-module, then M is torsion free if and only if the canonical map $M \to M^{**}$ is an injection; here $M^* = \mathrm{Hom}_R(M,R)$. Furthermore, if $t(M)$ is the kernel of this map, then $t(M)$ is the torsion submodule of M, $M/t(M)$ is torsion free, and $M \to M/t(M)$ is universal with respect to maps of M into torsion free R-modules. It is therefore of some interest to have a description of the kernel (and cokernel) of the map $M \to M^{**}$.

We now place ourselves in the following setting. R will denote a not necessarily commutative ring with unit, noetherian on both sides. All modules under consideration will be assumed finitely generated, and R will always be considered a two-sided R-module. For an R-module M we can find an exact sequence $P_1 \to P_0 \to M \to 0$ where P_0 and P_1 are finitely generated projectives. Dualizing, we obtain an exact sequence $0 \to M^* \to P_0^* \to P_1^*$. Let us denote by $D(M)$ the cokernel of $P_0^* \to P_1^*$. In [1] it is shown that there is an exact sequence:

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(0.1) $\quad 0 \to \mathrm{Ext}^1(D(M),R) \to M \to M^{**} \to \mathrm{Ext}^2(D(M),R) \to 0$

where $M \to M^{**}$ is the usual map of a module into its double dual. Furthermore, it is shown in (2.6) below that the functors $\mathrm{Ext}^i(D(M), \ )$ for $i > 0$ depend only on M and not on the particular presentation $P_1 \to P_0 \to M \to 0$. We therefore may make the following definition.

Definition: The R-module M is said to be k-torsion free (k is an integer ≥ 1) when $\mathrm{Ext}^i(D(M),R) = 0$ for $1 \leq i \leq k$.

In the case where R is a domain, we see from the sequence (0.1) that 1-torsion freeness is equivalent to the usual notion of torsion freeness. In the general case, we also see from (0.1) that M is 2-torsion free if and only if M is reflexive (i.e. $M \to M^{**}$ is an isomorphism). Finally, it is easily shown (2.17) that M is k-torsion free if and only if there exists an exact sequence $0 \to M \to P_{k-1} \to \cdots \to P_0 \to N \to 0$ such that P_i is projective for $0 \leq i \leq k-1$ and the dualized sequence: $0 \to N^* \to P_0^* \to \cdots \to P_{k-1}^* \to M^* \to 0$ is also exact.

This last description of k-torsion freeness leads to the related notion of syzygies. If $\cdots \to P_{j+1} \to P_j \to \cdots \to P_1 \to P_0 \to N \to 0$ is a projective resolution of N, then we shall denote by $\Omega^j(N)$ the cokernel of $P_{j+1} \to P_j$. The module $\Omega^j N$ is called a j^{th} syzygy of N. Since $\mathrm{Ext}^i(\Omega^j(N), \ ) \approx \mathrm{Ext}^{i+j}(N, \ )$, for $i > 0$, we see that the functors $\mathrm{Ext}^i(\Omega^j(N), \ )$ depend only on N when $i > 0$. From the criterion for k-torsion freeness of the preceding paragraph we note that every k-torsion free module is a k-syzygy.

If one denotes by $D_k(M)$ the module $\Omega^k D(M)$, then it is shown in the Appendix that $D_0^2(M) \approx M$, $D_1^2(M) \approx M/t(M)$ and $D_2^2(M) \approx M^{**}$ where $D_i^2(M) = D_i(D_i(M))$ (Actually, these isomorphisms hold only up to the addition of a finitely

generated projective module). Furthermore, it is shown in Chapter 2 (and, alternatively, in the Appendix) that there is a natural map: $M \to D_k^2(M)$ for each integer $k \geq 0$. For $k = 0, 1, 2$ these maps are induced, respectively, by the identity: $M \to M$, the natural map $M \to M/t(M)$, and the natural map: $M \to M^{**}$.

Definition: The module M is said to be k-reflexive when $M \to D_k^2(M)$ induces an isomorphism $\mathrm{Ext}^1(D_k^2(M), \ ) \xrightarrow{\sim} \mathrm{Ext}^1(M, \ )$.

It is easily seen that every module is 0-reflexive, while M is 1-reflexive if and only if M is torsion free (or 1-torsion free). In the commutative case, M is 2-reflexive if and only if M is reflexive in the usual sense (see Appendix). When R is commutative, the condition M is k-torsion free, M is k-reflexive. M is a k-syzygy for all $k \geq 0$, are equivalent if and only if R is a Gorenstein ring (Corollary (4.22)).

In trying to find criteria for $\Omega^i M$ to be i-torsion free we naturally encounter the notion of grade [9]. We shall write grade $M = n$ when $\mathrm{Ext}^i(M,R) = 0$ for $0 \leq i < n$ and $\mathrm{Ext}^n(M,R) \neq 0$; grade $M = \infty$ will mean that $\mathrm{Ext}^i(M,R) = 0$ for all $i \geq 0$. In Proposition (2.26) it is shown that $\Omega^i M$ is i-torsion free for $1 \leq i \leq k$ if and only if grade $\mathrm{Ext}^i(M,R) \geq i - 1$ for each $1 \leq i \leq k$.

The grade of the modules $\mathrm{Ext}^i(M,R)$ is bound up with a number of other properties of M. The fairly strong condition: grade $\mathrm{Ext}^i(M,R) \geq i$ is related to the existence of certain modules of finite projective dimension, described below. Furthermore, when R is commutative and M has finite projective dimension (or finite "Gorenstein" dimension, as we shall see), then the torsion freeness properties of M are completely determined by the grade of the $\mathrm{Ext}^i(M,R)$. When R is commutative, this grade is seen to depend only on the support of $\mathrm{Ext}^i(M,R)$.

We now give a brief description of some of the principal results to be found in various sections of this paper.

In Chapter one we establish the rather general functorial techniques to be applied in the proofs of subsequent module-theoretic results. Let us consider half-exact functors F from Γ to Δ, where Γ and Δ are abelian categories, Γ having enough projectives and injectives. Let $\underline{F} = \operatorname{coker}(L_0F \to F)$ where $L_0F \to F$ is the usual map of the zeroth left derived of F into F. We say that F is projectively stable when the epimorphism $F \to \underline{F}$ is an isomorphism. This condition is equivalent to the condition that F vanish on every projective object. Dually, we let $\overline{F} = \ker(F \to R^0F)$, call F injectively stable when $\overline{F} \tilde{\to} F$ and observe that this happens if and only if F vanishes on injectives. Our fundamental tools are the following exact sequences (1.15):

$$\dots \to S_{i+2}S^1F \to L_iF \to S_iF \to S_{i+1}S^1F \to \dots$$

$$\dots \to L_0F \to F \to S_1S^1F \to 0$$

and:

$$0 \to S^1S_1F \to F \to R^0F \to \dots$$

$$\dots \to R^{i-1}F \to S^{i+1}S_1F \to S^iF \to R^iF \to \dots$$

Here, S_iF(resp. S^iF) denotes the i^{th} left (resp. right) satellite of F. We see immediately that $\underline{F} = S_1S^1F$ and $\overline{F} = S^1S_1F$.

Using these sequences we define transformations $F \to S_iS^iF$ and

$S^iS_iF \to F$ for $i \geq 0$. In Theorem (1.25) it is shown that $S^iS_iF \to F$ is an isomorphism for $0 \leq i \leq n$ when S_iF is injectively stable for $0 \leq i < n$. Theorem (1.26) gives a dual version of this result. Other parts of these theorems lead to the following rather interesting result. Let $n - \underline{\operatorname{Hex}}(\Gamma,\Delta)$ denote the category of half exact functors $F: \Gamma \to \Delta$ such that S^iF is projectively stable for $0 \leq i < n$. Dually, $n - \overline{\operatorname{Hex}}(\Gamma,\Delta)$ denotes the category of F such that S_iF is injectively stable for $0 \leq i < n$. In (1.25) we see that $S_n(n - \overline{\operatorname{Hex}}(\Gamma,\Delta)) \subset n - \underline{\operatorname{Hex}}(\Gamma,\Delta)$, and in (1.26) that $S^n(n - \underline{\operatorname{Hex}}(\Gamma,\Delta)) \subset n - \overline{\operatorname{Hex}}(\Gamma,\Delta)$. Furthermore, we have seen that $F \approx S_nS^nF$ (resp. $S^nS_nF \approx F$) when F is in $n - \underline{\operatorname{Hex}}(\Gamma,\Delta)$ (resp. $n - \overline{\operatorname{Hex}}(\Gamma,\Delta)$). Taking these isomorphisms as identifications, we conclude that $n - \underline{\operatorname{Hex}}(\Gamma,\Delta)$ and $n - \overline{\operatorname{Hex}}(\Gamma,\Delta)$ are isomorphic categories via the functors S_n and S^n.

After establishing a number of technical results about stability of satellites, we turn to a consideration of the relation between the natural transformations of the functor F to the functor G, and those of S^iF into S^iG. Denoting these by (F,G) and (S^iF, S^iG) respectively, we observe that S^i induces a map:

$$(\underline{F},\underline{G}) \to (S^iF, S^iG)$$

for $i > 0$. In Corollary (1.34) it is shown that S^iF projectively stable for $1 \leq i < n$ implies that this map is an isomorphism for $1 \leq i < n$ and all half exact functors G. When $i = 2$ the converse holds as well (1.33).

If Δ = the category of abelian groups, then, for each object A of Γ we have a functor $\operatorname{Hom}(A, _): \Gamma \to \Delta$, which we denote $(A, _)$. Its projective stabilization is written $(\underline{A}, _)$. One sees easily that $(\underline{A},\underline{B}) \approx ((\underline{B}, _), (\underline{A}, _))$ (natural transformations of $(\underline{B}, _)$ into $(\underline{A}, _)$)

while this last is isomorphic to $(\mathrm{Ext}^1(B,\ ),\ \mathrm{Ext}^1(A,\ ))$. We also have an exact sequence:

$$0 \to K \to (A,B) \to (\underline{A},\underline{B}) \to 0$$

where $K = \{f \in (A,B) \mid f$ factors through a projective$\}$. When $F = (B,\ )$, $G = (A,\ )$, the map $(\underline{F},\underline{G}) \to (S^iF,\ S^iG)$ becomes $(\underline{A},\underline{B}) \to (\mathrm{Ext}^i(B,\ ),\ \mathrm{Ext}^i(A,\ ))$. Applying the general theorem mentioned above shows that this map is an isomorphism for $1 \leq i < n$ and all B if $\mathrm{Ext}^i(A,P) = 0$ for $1 \leq i < n$ and each projective P; when $i = 2$ the converse holds as well. This is also a result of Hilton-Rees [6].

Finally, to provide a suitable setting for some of the preceding matters we introduce the (projectively) stabilized category $\underline{\Gamma}$. Here, the objects are those of Γ, but the morphisms between two objects A and B are the elements of $(\underline{A},\underline{B})$. In (1.44) it is shown that A and B are isomorphic in $\underline{\Gamma}$ (written $\underline{A} \approx \underline{B}$) if and only if $A + P \approx B + Q$ in Γ, for suitable projectives P and Q.

In Chapter two we deal with finitely generated modules over an arbitrary ring R with unit, noetherian on both sides. We introduce the functors D and Ω^i already discussed, and record some of their elementary properties in (2.6) and (2.7). Our fundamental tools of calculation are the following exact sequence of functors (Theorem (2.8)) whose existence comes from results in [1]: $(k \geq 0)$

(a) $0 \to \mathrm{Ext}^1_R(D\Omega^k M,\ ) \to \mathrm{Tor}^R_k(M,\ ) \to (\mathrm{Ext}^k_R(M,R),\ ) \to \mathrm{Ext}^2_R(D\Omega^k M,\ )$

(b) $\mathrm{Tor}^R_2(D\Omega^k M,\ ) \to (\mathrm{Ext}^k_R(M,R) \otimes_R\ ) \to \mathrm{Ext}^k_R(M,\ ) \to \mathrm{Tor}^R_1(D\Omega^k M,\ ) \to 0$

(M is a finitely generated left or right R-module, while R is always considered as a two-sided R-module).

When ${}_R\mathcal{M}$ is the category of finitely generated R-modules, then the isomorphism class of a module $\underline{M}$ in ${}_R\underline{\mathcal{M}}$ (the "projective equivalence class" of M) is determined by the functor $(\underline{M},_)$. Since the torsion freeness and reflexivity properties of M depend only on the projective equivalence class of M, we shall "represent" M by the functor $(\underline{M},_)$. An easy calculation, using the above exact sequences, gives the following: $(k,j > 0)$

(a) $$S_kS^j(\underline{M},_) \approx \mathrm{Tor}_k(D\Omega^{j-1}M,\ )$$

(b) $$S^kS_j(\underline{M},_) \approx \mathrm{Ext}^k(D\Omega^jDM,\ )$$

Recalling that $D_k = \Omega^kD$, and letting $J_k = D\Omega^k$, we also find that:

(c) $$S_kS^k(\underline{M},_) \approx (\underline{J^2_{k-1}(M)},_\ )$$

(d) $$\underline{S^kS_k(\underline{M},_)} \approx (\underline{D^2_k(M)},\ _\ )$$

Letting $F = (\underline{M},_)$ the usual maps $F \to S_kS^kF$ and $S^kS_kF \to F$ defined in Chapter one give us maps $J^2_{k-1}M \to M$ and $M \to D^2_kM$. The latter is the map already discussed in relation k-reflexivity (see Appendix for an alternate construction). To apply Theorems (1.25) and (1.26), describing when these maps are isomorphisms, we need the key observation that M is k-torsion free if and only if $\mathrm{Tor}_i(D(M),\ ) \approx S_{i-1}(\underline{M},_)$ is injectively stable for $1 \leq i \leq n$. Theorem (2.17) now asserts the equivalence of the following statements (for $n \geq 1$ and integer, and M finitely generated):

(1) M is n-torsion free.

(2) M is n-reflexive and $D_n(M)$ is n-torsion free.

(3) M is an n-syzygy and $D^2_n(M)$ is n-torsion free.

In Chapter One it was shown (Proposition 1.21) that S^kF is a direct summand of $S^kS_kS^kF$ in a natural way. If we let $F = (\underline{M},_)$, this is reflected in Proposition (2.20) where we see that $D_k(M)$ is a direct summand (up to projective equivalence) of $D_k^3(M)$. Thinking of D_k as a generalized "dualization" functor, we observe the analogy with the well-known fact that M^* is a direct summand of M^{***}.

In §4 we discuss the relation between grade and k-torsion freeness. We observe first that, for a module E, grade $E \geq n$ if and only if $S_j(E \otimes_R \ )$ is injectively stable for $0 \leq j < n$. Now when $E = \mathrm{Ext}^{i+1}(M,R)$, the functor $(E \otimes_R \ )$ is simply $L_0(\mathrm{Ext}^{i+1}(M, \ ))$. In Proposition (2.21) it is shown that Ω^iM is i-torsion free if and only if $S_j\mathrm{Ext}^{i+1}(M, \ )$ is injectively stable for $0 \leq j < i+1$. The relation between the stability properties of F and of L_0F is described in Proposition (2.25). Putting these pieces together, we obtain Proposition (2.26) which asserts that Ω^iM is i-torsion free for $1 \leq i \leq n$ if and only if grade $\mathrm{Ext}^{i+1}(M,R) \geq i$ for $0 \leq i < n$.

In §5 we discuss the question of when an element of $(\underline{A},\underline{B})$ is representable by a monomorphism: $A \to B + P$ for some projective P (equivalently, when a given natural transformation $\mathrm{Ext}^1(B, \ ) \to \mathrm{Ext}^1(A, \ )$ is induced by a monomorphism $A \to B + P$). A necessary and sufficient condition is that the cokernel of the induced map $(\underline{B},__) \to (\underline{A},__)$ be injectively stable. The special cases of the maps $J_k^2(M) \to J_{k-1}^2(M)$ and $J_k^2(M) \to M$ are dealt with in Theorem (2.31).

In §6 we use Theorem (2.31) and the techniques of the previous section to prove two of our main results: the Spherical Filtration and Approximation Theorems. To state the first of these, let us say that a module S is n-spherical of type E (E is an R-module) when pd $S \leq n$, $\mathrm{Ext}^i(S,R) = 0$ for $0 < i < n$, and $\mathrm{Ext}^n(S,R) \approx E$. Suppose now that M is a finitely generated R-module with the property that grade $\mathrm{Ext}^i(M,R) \geq i$ for $1 \leq i \leq k$.

The Spherical Filtration Theorem gives the existence of a finitely generated projective P and a sequence of submodules:

$$M_k \subset M_{k-1} \subset \ldots \subset M_1 \subset M_0 = M + P$$

such that $M_i \approx J_i^2(M)$ (up to projective equivalence) and M_{i-1}/M_i is i-spherical of type $\mathrm{Ext}^i(M,R)$.

These sequences seem to be related to filtrations of M derived and studied by J.-E. Roos (Comptes Rendues, 26 fevrier 1962) and R. M. Fossum (Math. Scand. - to appear). The interested reader is referred to these papers.

Under the same hypotheses, namely that grade $\mathrm{Ext}^i(M,R) \geq i$ for $1 \leq i \leq k$, we can find a module $H_k(M)$ and a map $M \to H_k(M)$ with the properties:

(a) pd $H_k(M) \leq k$

(b) $\mathrm{Ext}^i(H_k(M),R) \xrightarrow{\sim} \mathrm{Ext}^i(M,R)$ for $1 \leq i \leq k$. If $M \to N$ is given where pd $N \leq k$, then there is a unique map $\underline{H_k(M)} \to \underline{N}$ such that $\underline{M} \to \underline{N}$ is the composition $\underline{M} \to \underline{H_k(M)} \to \underline{N}$. The existence of $H_k(M)$ is the content of the Approximation Theorem.

We close Chapter two with a strengthening of previous results about the maps $(\underline{A},\underline{B}) \to (\mathrm{Ext}^i(B, \ ), \mathrm{Ext}^i(A, \ ))$.

Chapter three is a short one, devoted to the definition of Gorenstein dimension for half exact functors. The most important application of this definition is to the functor $(\underline{M},_)$, whose Gorenstein dimension is called the Gorenstein dimension of the module M. The following conditions on M are equivalent (Proposition 3.8):

(a) G-dim $M = 0$

(b) $\mathrm{Ext}^i(M,R) = \mathrm{Ext}^i(D(M),R) = 0$ for $i > 0$

(c) $M \xrightarrow{\sim} M^{**}$ and $\mathrm{Ext}^i(M,R) = \mathrm{Ext}^i(M^*,R) = 0$ for $i > 0$

(d) M and $D(M)$ are k-torsion free for all $k \geq 0$.

It turns out that G-dim $M \leq n$ when there is an exact sequence: $0 \to L_n \to \dots \to L_i \to L_0 \to M \to 0$ where G-dim $L_i = 0$ for $0 \leq i \leq n$. When there is no shorter such resolution, G-dim $M = n$. If it is known that G-dim $M < \infty$ (and $M \neq 0$), then G-dim $M = \sup\{t \mid \mathrm{Ext}^t(M,R) \neq 0\}$. From this it follows that G-dim $M \leq$ pd M, and that these agree when the latter is finite.

Finally, Chapter four is devoted to the study of modules (always finitely generated) over commutative noetherian rings--a setting in which the notion of Gorenstein dimension is particularly useful.

Suppose first that $(R,\underline{m})$ is a local ring. When G-dim $M = n < \infty$, then n is also the smallest integer ≥ 0 such that $\mathrm{Tor}_i(M,A) = 0$ for $i > n$ and all modules A of finite projective dimension (Theorem 4.13). Furthermore, we have the important formula (4.13):

$$(\text{G-dim } M) + (\text{depth } M) = (\text{depth } R)$$

(Here, depth M means the length of a maximal M-regular sequence contained in the maximal ideal $\underline{m}$ of R: this number is also known as the "codimension" of M). This generalizes a similar formula for modules of finite projective dimension.

Using this last formula and the techniques of localization, we arrive at the key property of modules of finite G-dimension over an arbitrary (not necessarily local) ring R. Namely: if G-dim $M < \infty$ then grade $\mathrm{Ext}^i(M,R) \geq i$ for all $i > 0$ (Corollary 4.17). As a consequence of previous results, we see that $\Omega^i M$ is always i-torsion free for $i \geq 0$ when M has finite Gorenstein dimension.

In §4 we discuss Gorenstein Rings (rings which, locally, have finite injective dimension over themselves). Our main result, Theorem (4.20), asserts that a local ring R is Gorenstein if and only if every finitely generated R-module has finite Gorenstein dimension. More precisely, we see in Proposition (4.21) that the following conditions are equivalent for a commutative ring R and an integer $k \geq 1$:

(a) $\Omega^{k+1}M$ is $(k+1)$-torsion free for all R-modules M.

(b) $\Omega^j M$ is j-torsion free for all M and $1 \leq j \leq k+1$

(c) For a prime $\underline{p}$, depth $R_{\underline{p}} < k$ implies $R_{\underline{p}}$ is a Gorenstein ring.

(d) grade $\mathrm{Ext}^i(M,R) \geq i$ for all M and $1 \leq i \leq k$.

Consequently, we see that R is a Gorenstein ring if and only if the following properties are equivalent for any R-module M and integer $k \geq 0$:

(i) M is k-torsion free (ii) M is k-reflexive, and (iii) M is a k-syzygy.

In §6 we discuss locally free sheaves (vector bundles) at isolated singularities. More precisely, let R be a local ring which is regular at every non-maximal prime. Let $Y = \mathrm{Spec}\, R - \{m\}$, $X = \mathrm{Spec}\, R$, and suppose depth $R = k \geq 3$. Let $\mathcal{O}_Y$ be the sheaf of local rings induced on Y by the usual sheaf $\mathcal{O}_X$ on Spec R. Any coherent sheaf $\mathcal{F}$ of $\mathcal{O}_Y$-modules is the restriction to Y of a sheaf of the form $\tilde{M}$ on X, where M is an R-module (for details, see [7]). Consider the map $M \to D^2_{k-1}(M)$ described above. It is easily seen that $D^2_{k-1}(M)$ is free at every non-maximal prime, so determines a locally free sheaf $\mathcal{L}_{k-1}(\mathcal{F})$ on Y. Since $D^2_{k-1}(M)$ is defined up to the addition of a projective (hence free) module, $\mathcal{L}_{k-1}(\mathcal{F})$ is defined up to the addition of a free sheaf of $\mathcal{O}_Y$-modules. The map

$\mathcal{J} \to \mathcal{L}_{k-1}(\mathcal{J})$ induced by $M \to D^2_{k-1}(M)$ satisfies the following universal mapping property. Suppose $\mathcal{J} \to \mathcal{L}$ is a map of coherent $\mathcal{O}_Y$-modules such that $\mathcal{L}$ comes from an R-module N which is a (k-1)-syzygy. Then there is a map $\mathcal{L}_{k-1}(\mathcal{J}) \to \mathcal{L}$, unique up to the addition of a map factoring through a free $\mathcal{O}_Y$-module, such that the maps $(\mathcal{J} \to \mathcal{L})$ and $(\mathcal{J} \to \mathcal{L}_{k-1}(\mathcal{J}) \to \mathcal{L})$ differ (at most) by a map factoring through a free $\mathcal{O}_Y$-module. Thus $\mathcal{J} \to \mathcal{L}_{k-1}(\mathcal{J})$ is "stably" universal with respect to maps of $\mathcal{J}$ into sheaves coming from (k-1)-syzygies.

An important property of modules M of finite Gorenstein dimension is that their torsion freeness properties are completely determined by their $\mathrm{Ext}^i(M,R)$ (in fact, by the supports of the $\mathrm{Ext}^i(M,R)$). In Theorem (4.25) the following properties for a module M of finite G-dimension, and integer $k \geq 0$, are seen to be equivalent.

(1) M is k-torsion free.

(2) M is a k-syzygy.

(3) grade $\mathrm{Ext}^i(M,R) \geq i + k$ for all $i > 0$.

(4) For any prime $\underline{p}$ every $R_{\underline{p}}$ regular sequence of length $\leq k$ is also $M_{\underline{p}}$ regular.

An insight into the structure of modules of finite G-dimension is given by the following result. When G-dim $M = n < \infty$, we can find a projective P and an exact sequence $0 \to M_n \to P + M \to H_n(M) \to 0$ where M_n is given by the Spherical Filtration Theorem and $H_n(M)$ by the Approximation Theorem. It is easily seen that G-dim $M_n = 0$ and pd $H_n(M) = n$. Therefore we see that, up to the addition of a projective, every module of finite G-dimension is the extension of a module of zero G-dimension by a module of finite projective dimension (4.26 and 4.27). Furthermore, the k-torsion freeness of M (G-dim $M = n < \infty$) for any $k \geq 0$ is equivalent to the k-torsion freeness of the modules in any of the following groups:

(a) The modules M_i, $0 \leq i \leq n$ given by the Spherical Filtration Theorem.

(b) The i-spherical modules of type $\mathrm{Ext}^i(M,R)$ for $1 \leq i \leq n$

(c) The module $H_n(M)$ given by the Approximation Theorem.

In §7 we consider perfect modules. The R-module M is called perfect when grade M = G-dim $M < \infty$. Our main result is that when R is a local ring, $x_1,\ldots,x_t$ an R-regular sequence, and M an $R/(x_1,\ldots,x_t)$-module, then M is a perfect R-module of G-dimension t if and only if M has zero G-dimension as an $R/(x_1,\ldots,x_t)$-module (a result of C. Peskine and L. Szpiro). As an application, we compute $\mathrm{Tor}^R_i(M,N)$ when M is a perfect R-module of G-dimension t, $(x_1,\ldots,x_t)$ is an R-regular sequence in the annihilator of M (such always exists), and N is an $R/(x_1,\ldots,x_t)$-module of finite projective dimension. We see in Proposition (4.36) that:

$$\mathrm{Tor}^R_i(M,N) = \binom{t}{i} \cdot (M \otimes_{R/(x_1,\ldots,x_t)} N)$$

for $0 \leq i \leq t$ (and zero for $i > t$). By $\binom{t}{i} \cdot M$ is meant $\binom{t}{i}$ direct sum copies of M. A similar result is proved for $\mathrm{Ext}^i_R(M,N)$ in (4.37).

Chapter 1: Satellites and Stable Functors

§1: Stability

In this Section we define stable functors and list some of their basic properties. These ideas will be central to the remainder of the paper.

Let Γ be an abelian category with enough projectives and injectives, and let Δ be an arbitrary abelian category. We shall denote by $\mathrm{Hom}(\Gamma,\Delta)$ the abelian category of all covariant additive functors from Γ to Δ. (Technically, Γ should be small; in all applications described here, Γ may, indeed, be so chosen).

(1.1) We recall that we can construct a functor $L_0\colon \mathrm{Hom}(\Gamma,\Delta) \to \mathrm{Hom}(\Gamma,\Delta)$ --the "zero$^{th.}$ left derived"--together with a natural transformation: $L_0 \to$ Identity (so we have for any F a map: $L_0(F) \to F$) which satisfies:

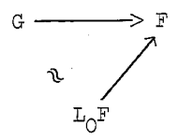
(a) For all F in $\mathrm{Hom}(\Gamma,\Delta)$, L_0F is right exact.

(b) If G in $\mathrm{Hom}(\Gamma,\Delta)$ is right exact, then the map $L_0F \to F$ induces an isomorphism: $(G,L_0F) \xrightarrow{\sim} (G,F)$.

(We shall denote the group of maps from A to B by (A,B) for A and B objects of a fixed abelian category)

(c) The map $L_0F \to F$ is the identity on the full subcategory of projective objects of Γ.

(d) (Uniqueness of L_0F): If G is a right exact functor in $\mathrm{Hom}(\Gamma,\Delta)$ and there is a map $G \to F$ which is an isomorphism on projectives, then there exists a unique isomorphism $G \approx L_0F$ such that the following diagram commutes:



For details and proofs, see [4, ch. V §5].

(1.2) Definition: For each functor F in $\mathrm{Hom}(\Gamma,\Delta)$ we define $\underline{F}$, the projective stabilization of F, to be the cokernel of $L_0F \to F$. We thus have the exact sequence of functors $L_0F \to F \to \underline{F} \to 0$. We say that F is projectively stable if the map $F \to \underline{F}$ is an isomorphism.

(1.3) PROPOSITION: For any F in $\mathrm{Hom}(\Gamma,\Delta)$ the following statements are equivalent:

(a) F is projectively stable.

(b) $F(P) = 0$ for all projectives P.

(c) $L_0F = 0$.

(d) $(G,F) = 0$ for all right exact functors G in $\mathrm{Hom}(\Gamma,\Delta)$.

(e) $(G,F) = 0$ for all functors G in $\mathrm{Hom}(\Gamma,\Delta)$ which preserve epimorphisms.

Proof: If F is projectively stable then $F \to \underline{F}$ is an isomorphism. Let P be any projective. Then the exact sequence $L_0F \to F \xrightarrow{\sim} \underline{F} \to 0$ gives the exact sequence $L_0F(P) \to F(P) \xrightarrow{\sim} \underline{F}(P) \to 0$. Since $L_0F \to F$ is an isomorphism on projectives, we conclude that $L_0F(P) = F(P) = 0$. Therefore (a) implies (b). To prove that (b) implies (c) note that the zero functor is right exact and agrees with F on projectives when F vanishes on projectives. By the uniqueness of L_0F (property (1.1d) we have $L_0F = 0$. Now suppose $L_0F = 0$; we prove (c) implies (d). Property (1.1)b tells us that when G is right exact, $(G,L_0F) \approx (G,F)$. We therefore conclude that $(G,F) = 0$. Now suppose (d) holds. Let G be a functor preserving epimorphisms. Claim: the sequence $L_0G \to G \to 0$ is exact. In fact, let A be any object and let $P \to A \to 0$ be exact with P projective (we assumed that Γ has enough projectives). We have the exact and commutative diagram:

$$\begin{array}{ccccc} L_0G(P) & \to & L_0G(A) & \to & 0 \\ \wr & & \downarrow & & \\ G(P) & \to & G(A) & \to & 0 \end{array}$$

Therefore $L_0G(A) \to G(A)$ is an epimorphism and the claim is proved. From $L_0G \to G \to 0$ we get the exact sequence: $0 \to (G,F) \to (L_0G,F)$. Since L_0G is right exact, our hypothesis (d) gives $(L_0G,F) = 0$; we conclude that $(G,F) = 0$ as well. Finally, we prove (e) implies (a). Since L_0F is right exact it preserves epimorphisms. By assumption $(L_0F,F) = 0$. Therefore $L_0F \to F$ is the zero map, and the exact sequence $L_0F \to F \to \underline{F} \to 0$ gives $F \xrightarrow{\sim} \underline{F}$.

We shall denote by $\underline{\mathrm{Hom}}(\Gamma,\Delta)$ the full subcategory of $\mathrm{Hom}(\Gamma,\Delta)$ consisting of the projectively stable functors. It is easily seen that given an exact sequence $0 \to F_1 \to F_2 \to F_3 \to 0$ in $\mathrm{Hom}(\Gamma,\Delta)$ then F_2 is in $\underline{\mathrm{Hom}}(\Gamma,\Delta)$ if and only if both F_1 and F_3 are in $\underline{\mathrm{Hom}}(\Gamma,\Delta)$. In other words, $\underline{\mathrm{Hom}}(\Gamma,\Delta)$ is a dense subcategory of $\mathrm{Hom}(\Gamma,\Delta)$.

(1.4) PROPOSITION: For each F in $\mathrm{Hom}(\Gamma,\Delta)$ we have:

(a) $\underline{F}$ is projectively stable.

(b) For each projectively stable G in $\mathrm{Hom}(\Gamma,\Delta)$ the map $F \to \underline{F}$ induces an isomorphism: $(\underline{F},G) \xrightarrow{\sim} (F,G)$. Thus, the functor $\mathrm{Hom}(\Gamma,\Delta) \to \underline{\mathrm{Hom}}(\Gamma,\Delta)$ given by $F \rightsquigarrow \underline{F}$ is a left adjoint to the inclusion functor $\underline{\mathrm{Hom}}(\Gamma,\Delta) \to \mathrm{Hom}(\Gamma,\Delta)$.

Proof: To prove (a) note that $L_0F(P) \to F(P) \to \underline{F}(P) \to 0$ is exact. When P is any projective, $L_0F(P) \xrightarrow{\sim} F(P)$, so $\underline{F}(P) = 0$. By Proposition (1.3) a functor is projectively stable if and only if it vanishes on projectives; therefore (a) holds.

Now suppose that G is projectively stable. Since $F \to \underline{F} \to 0$ is exact, we have that $(\underline{F},G) \to (F,G)$ is one-one. To prove it is surjective,

choose any map $F \to G$ and consider the diagram:

$$\begin{array}{ccccccc} L_0F & \to & F & \to & \underline{F} & \to & 0 \\ & & \downarrow & & & & \\ & & G & & & & \end{array}$$

The composition $L_0F \to F \to G$ is an element of (L_0F,G). By Proposition (1.3) the fact that G is projectively stable and L_0F is right exact gives $(L_0F,G) = 0$. Thus the composition $L_0F \to F \to G$ is zero. Since $\underline{F}$ is a cokernel, there is a (unique) map $\underline{F} \to G$ which leaves the diagram commutative. Therefore $(\underline{F},G) \to (F,G)$ is surjective, hence an isomorphism.

We now record the dual version of everything which we have done so far.

(1.5) There is a functor $R^0\colon \mathrm{Hom}(\Gamma,\Delta) \to \mathrm{Hom}(\Gamma,\Delta)$--called the "zeroth right derived"--together with a natural transformation: Identity $\to R^0$ (so a map $F \to R^0F$ for any F) having the following properties:

(a) R^0F is left exact.

(b) If G is left exact then $F \to R^0F$ induces an isomorphism $(R^0F,G) \xrightarrow{\sim} (F,G)$.

(c) The map $F \to R^0F$ is the identity on the full subcategory of injective objects of Γ.

(d) (Uniqueness of R^0F): If G is a left exact functor in $\mathrm{Hom}(\Gamma,\Delta)$ and there is a map $F \to G$ which is an isomorphism on injectives, then there exists a unique isomorphism $R^0F \approx G$ such that the following diagram commutes:

$$\begin{array}{ccc} F & \longrightarrow & G \\ & \searrow & \wr \uparrow \\ & & R^0F \end{array}$$

(1.6) Definition: For each functor F in $\mathrm{Hom}(\Gamma,\Delta)$ we define $\overline{F}$, the injective stabilization of F, to be the kernel of $F \to R^0F$. We then have the exact sequence of functors $0 \to \overline{F} \to F \to R^0F$. We say that F is

injectively stable if the map $\overline{F} \to F$ is an isomorphism.

The proof of the following is entirely dual to the proof of Proposition (1.3).

(1.7) PROPOSITION: For any F in $\mathrm{Hom}(\Gamma,\Delta)$ the following statements are equivalent:

(a) F is injectively stable.

(b) $R^0F = 0$.

(c) $F(I) = 0$ for all injectives I in Γ.

(d) $(F,G) = 0$ for all left exact functors G in $\mathrm{Hom}(\Gamma,\Delta)$.

(e) $(F,G) = 0$ for all functors G in $\mathrm{Hom}(\Gamma,\Delta)$ which preserve monomorphisms.

We shall denote the full subcategory of injectively stable functors in $\mathrm{Hom}(\Gamma,\Delta)$ by $\overline{\mathrm{Hom}}(\Gamma,\Delta)$.

(1.8) PROPOSITION: For each F in $\mathrm{Hom}(\Gamma,\Delta)$ we have:

(a) $\overline{F}$ is injectively stable.

(b) If G is injectively stable the map $(G,\overline{F}) \to (G,F)$ induced by $\overline{F} \to F$ is an isomorphism. Thus, the functor $\mathrm{Hom}(\Gamma,\Delta) \to \overline{\mathrm{Hom}}(\Gamma,\Delta)$ given by $F \rightsquigarrow \overline{F}$ is a right adjoint to the inclusion functor: $\overline{\mathrm{Hom}}(\Gamma,\Delta) \to \mathrm{Hom}(\Gamma,\Delta)$.

(1.9) Remark: Suppose that $G \to F$ is a map of functors where G is right exact and F is half exact. A simple diagram chase proves that coker $(G \to F)$ is a half exact functor. As a result, we see that if F is half exact then coker $(L_0F \to F) = \underline{F}$ is also half exact. In a similar fashion $\overline{F} = \ker(F \to R^0F)$ is half-exact when F is.

Recall now that an epimorphism $A \to B$ is said to split if there exists a map $B \to A$ such that $B \to A \to B$ is the identity map of B. Dually, a monomorphism $C \to A$ is said to split if there is a map $A \to C$ such that $C \to A \to C$ is the identity of C. The following lemma will be of use later.

(1.10) LEMMA: Let F be any half exact functor in $\mathrm{Hom}(\Gamma,\Delta)$.

(a) Suppose the epimorphism $F \to \underline{F}$ splits. Then the sequence $0 \to L_0F \to F \to \underline{F} \to 0$ is exact and splits.

(b) Suppose the monomorphism $\overline{F} \to F$ splits. Then the sequence $0 \to \overline{F} \to F \to R^0F \to 0$ is exact and splits.

Proof: Let $K = \ker(F \to \underline{F})$; then the sequence $0 \to K \to F \to \underline{F} \to 0$ is exact and splits. We show that $K \approx L_0F$. If P is any projective then $\underline{F}(P) = 0$ since $\underline{F}$ is always projectively stable. Therefore the map $K(P) \to F(P)$ is an isomorphism. Furthermore K is half-exact since it is a direct summand of the half exact functor F. From the exact and commutative diagram:

$$\begin{array}{ccccccc} L_0F & \to & F & \to & \underline{F} & \to & 0 \\ \downarrow & & \| & & \| & & \\ 0 \to K & \to & F & \to & \underline{F} & \to & 0 \end{array}$$

we conclude that $L_0F \to K$ is an epimorphism. Since L_0F is right exact, it follows easily that K preserves epimorphisms. Since K is also half exact, K is right exact. By the uniqueness of L_0F we conclude that $L_0F \approx K$ and (a) now follows. The proof of (b) is entirely similar.

For the remainder of this paper we shall identify F and $\underline{F}$ if F is projectively stable. We also identify F and $\overline{F}$ if F is injectively stable.

§2: Satellites

Important examples of projectively (resp. injectively) stable functors are provided by taking various satellites of a functor. Let us briefly recall some of the main properties of satellites. For details and proofs

see [4, Ch. III].

(1.11) The first left satellite is an (additive) functor S_1: $\mathrm{Hom}(\Gamma,\Delta) \to \mathrm{Hom}(\Gamma,\Delta)$ together with the following additional structure. For each F in $\mathrm{Hom}(\Gamma,\Delta)$ and each exact sequence $E\colon 0 \to C' \to C \to C'' \to 0$ in Γ we are given a map $(S_1F)(C'') \to F(C')$ which is functorial in both F and E and which satisfies:

(a) If C is projective then $0 \to S_1F(C'') \to F(C') \to F(C)$ is exact (this serves to define $S_1F(C'')$).

(b) The sequence: $S_1F(C') \to S_1F(C) \to S_1F(C'') \to F(C') \to F(C) \to F(C'')$ is of order two and is exact if F is half exact.

(c) S_1F is projectively stable (i.e. S_1F vanishes on projectives).

(1.12) Remark: From (b) we see that if F is half exact, so is S_1F. If, in addition, F is left exact, property (a) gives $S_1F = 0$.

We shall write S_0 = identity on $\mathrm{Hom}(\Gamma,\Delta)$, and define S_n to be $S_1(S_{n-1})$ for $n \geq 1$.

In a dual manner we define the right satellites.

(1.13) The first right satellite is an additive functor S^1: $\mathrm{Hom}(\Gamma,\Delta) \to \mathrm{Hom}(\Gamma,\Delta)$ together with the following structure. For each F in $\mathrm{Hom}(\Gamma,\Delta)$ and exact sequence $E\colon 0 \to C' \to C \to C'' \to 0$ in Γ there is a map $F(C'') \to S^1F(C')$, functorial in F and E, satisfying:

(a) If C is injective then $F(C) \to F(C'') \to S^1F(C') \to 0$ is exact (this serves to define $S^1F(C')$).

(b) The sequence $F(C') \to F(C) \to F(C'') \to S^1F(C') \to S^1F(C) \to S^1F(C'')$ is of order two and is exact if F is half exact.

(c) S^1F is injectively stable (i.e. S^1F vanishes on injectives).

(1.14) Remark: From (b) we see that if F is half exact, so is S^1F. If F is right exact, then (a) shows that $S^1F = 0$. We define: S^0 = identity on $\mathrm{Hom}(\Gamma,\Delta)$ and $S^nF = S^1(S^{n-1}F)$ for $n \geq 1$.

From now on we shall be primarily interested in half exact functors. We shall denote by $\mathrm{Hex}(\Gamma,\Delta)$ the full subcategory of $\mathrm{Hom}(\Gamma,\Delta)$ consisting of the half exact functors. We shall denote by $\underline{\mathrm{Hex}}(\Gamma,\Delta)$ and $\overline{\mathrm{Hex}}(\Gamma,\Delta)$ the full subcategories of half exact projectively stable and half exact injectively stable functors respectively. Our fundamental tool in determining the stability properties of satellites is the following result.

(1.15) PROPOSITION: (Fundamental exact sequences): For any F in $\mathrm{Hex}(\Gamma,\Delta)$ (i.e. F half exact) there are exact sequences:

$$(*) \qquad \ldots \to S_{n+2}S^1F \to L_nF \to S_nF \to S_{n+1}S^1F \to \ldots$$

$$\ldots \to L_1F \to S_1F \to S_2S^1F \to L_0F \to F \to S_1S^1F \to 0$$

$$(**) \qquad 0 \longrightarrow S^1S_1F \to F \to R^0F \to S^2S_1F \to S^1F \to R^1F \to \ldots$$

$$\ldots \to S^{n+1}S_1F \to S^nF \to R^nF \to S^{n+2}S_1F \to \ldots$$

Furthermore, these sequences are functorial in F.

Remark: The construction of these sequences is described in [4, Exercise 3, p. 105]. In the case of (*) one constructs the exact sequence $L_0F \to F \to S_1S^1F \to 0$ (where $L_0F \to F$ is the usual map), then applies S_i to it, obtaining sequences $S_iL_0F \to S_iF \to S_{i+1}S^1F$. These are then spliced together to form a long exact sequence. Finally, one observes that $S_iL_0F \approx L_iF = i^{\text{th}}$ left derived of F. In the case of (**) one starts with $0 \to S^1S_1F \to F \to R^0F$, applies S^i, splices, and notes that $S^iR^0F \approx R^iF = i^{\text{th}}$ right derived of F.

(1.16) COROLLARY: Suppose F is in $\mathrm{Hex}(\Gamma,\Delta)$. Then there are unique isomorphisms $S_1S^1F \approx \underline{F}$ and $S^1S_1F \approx \overline{F}$ such that the following diagrams commute:



Proof: The assertion follows directly from the existence of the exact sequences $L_0F \to F \to S_1S^1F \to 0$ and $0 \to S^1S_1F \to F \to R^0F$.

Now recall that by (1.11)c we know that S_1F is projectively stable for any functor F. In particular S_1F is in $\underline{\mathrm{Hex}}(\Gamma,\Delta)$ when F is in $\overline{\mathrm{Hex}}(\Gamma,\Delta)$. Thus, we may think of S_1 as a functor $S_1\colon \overline{\mathrm{Hex}}(\Gamma,\Delta) \to \underline{\mathrm{Hex}}(\Gamma,\Delta)$. On the other hand, for any F we have S^1F is injectively stable, i.e. S^1F is in $\overline{\mathrm{Hom}}(\Gamma,\Delta)$. In particular, we can think of S^1 as a functor $S^1\colon \underline{\mathrm{Hex}}(\Gamma,\Delta) \to \overline{\mathrm{Hex}}(\Gamma,\Delta)$.

Now for any half exact F, the previous Corollary gives $\underline{F} \approx S_1S^1F$. If F lies in $\underline{\mathrm{Hex}}(\Gamma,\Delta)$, then we have $F = \underline{F} \approx S_1S^1F$. Let us agree to take the isomorphism $\underline{F} \approx S_1S^1F$ as an identification. Then for F in $\underline{\mathrm{Hex}}(\Gamma,\Delta)$, $F = S_1S^1F$. Similarly, if F is in $\overline{\mathrm{Hex}}(\Gamma,\Delta)$ and we take the isomorphism $F = \overline{F} \approx S^1S_1F$ as an identification, then $F = S^1S_1F$. Thus, the compositions $\underline{\mathrm{Hex}}(\Gamma,\Delta) \xrightarrow{S^1} \overline{\mathrm{Hex}}(\Gamma,\Delta) \xrightarrow{S_1} \underline{\mathrm{Hex}}(\Gamma,\Delta)$ and $\overline{\mathrm{Hex}}(\Gamma,\Delta) \xrightarrow{S_1} \underline{\mathrm{Hex}}(\Gamma,\Delta) \xrightarrow{S^1} \overline{\mathrm{Hex}}(\Gamma,\Delta)$ are the respective identities. Consequently we have proved:

(1.17) COROLLARY: The functors $S^1\colon \underline{\mathrm{Hex}}(\Gamma,\Delta) \to \overline{\mathrm{Hex}}(\Gamma,\Delta)$ and $S_1\colon \overline{\mathrm{Hex}}(\Gamma,\Delta) \to \underline{\mathrm{Hex}}(\Gamma,\Delta)$ establish an isomorphism of categories.

(1.18) COROLLARY: Let F and G be in $\mathrm{Hex}(\Gamma,\Delta)$.

(a) $S^1F \approx S^1G$ if and only if $\underline{F} \approx \underline{G}$, while $S_1F \approx S_1G$ if and only if $\overline{F} \approx \overline{G}$.

(b) $F \to \underline{F}$ induces $S^1F \xrightarrow{\sim} S^1\underline{F}$, while $\overline{F} \to F$ induces $S_1\overline{F} \xrightarrow{\sim} S_1F$.

(c) Taking the isomorphisms of (b) as identifications, we have that the maps $(\underline{G},\underline{F}) \xrightarrow{S^1} (S^1G,S^1F)$ and $(\overline{G},\overline{F}) \xrightarrow{S_1} (S_1G,S_1F)$ are isomorphisms.

Proof: Suppose $S^1F \approx S^1G$. Applying S_1 we have $\underline{F} = S_1S^1F \approx S_1S^1G = \underline{G}$. Conversely, suppose $\underline{F} \approx \underline{G}$. Applying S^1 we have $S^1S_1(S^1F) = S^1\underline{F} \approx S^1\underline{G} = S^1S_1(S^1G)$. Now S^1G is in $\overline{\mathrm{Hex}}(\Gamma,\Delta)$, so $S^1S_1(S^1G) = S^1G$. Similarly $S^1S_1(S^1F) = S^1F$. We conclude $S^1F \approx S^1G$. The other part of (a) is proved dually.

Part (b) follows from part (a).

To prove part (c), just note that S_1 and S^1 respectively provide inverses to the maps in question.

§3: Stability of Satellites

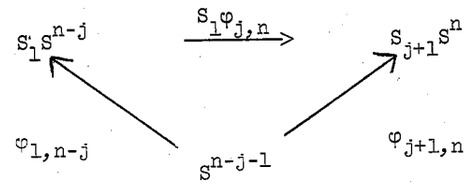
The satellites S_iF $(i > 0)$ are always projectively stable, while the satellites S^iF $(i > 0)$ are always injectively stable. In this Section we shall be concerned with finding conditions under which the S_iF are injectively stable, and the S^iF are projectively stable. One such characterization will involve the construction of maps $F \to S_iS^iF$ and $S^iS_iF \to F$, and will have later applications to reflexivity and torsion-freeness of modules. Another set of criteria will involve the maps $(\underline{G},\underline{F}) \to (S^iG,S^iF)$ and will be useful later in determining when maps $\mathrm{Ext}^i(N, \ ) \to \mathrm{Ext}^i(M, \ )$ come from maps $M \to N$.

We begin by defining natural transformations $S^{n-j} \to S_jS^n$ for $0 \le j \le n$. We proceed by induction on j. When $j = 0$ we have simply the identity map $S^n \to S^n$. Suppose, then, that the map $S^{n-(j-1)} \to S_{j-1}S^n$

has been defined, and that $j \geq 1$. If we apply S_1 to this map we obtain a map $S_1S^{n-j+1} \to S_jS^n$. Let us now write S_1S^{n-j+1} as $S_1S^1(S^{n-j})$. For any half exact functor F we have a map $(S^{n-j}F) \to (\underline{S^{n-j}F}) = S_1S^1(S^{n-j}F)$. Therefore we have a natural transformation $S^{n-j} \to S_1S^1(S^{n-j})$. By composition we obtain our desired map: $S^{n-j} \to S_1S^1S^{n-j} \to S_jS^n$.

If we denote by $\varphi_{j,n}$ the transformation: $S^{n-j} \to S_jS^n$ then we have the following commutative diagram:

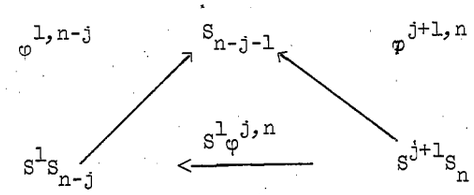
(1.19)



for $0 \leq j < n$.

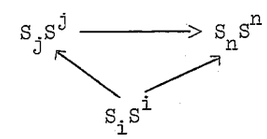
In a dual manner we may construct natural transformations $\varphi^{j,n}: S^jS_n \to S_{n-j}$, for which the following diagram commutes:

(1.20)

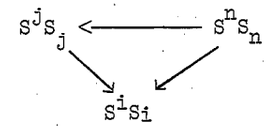


for $0 \leq j < n$.

We now consider the following special case. When $j = 1$ we have a map $\varphi_{1,n}: S^{n-1} \to S_1S^n$. This map is simply $S^{n-1} \to \underline{S^{n-1}} = S_1S^1(S^{n-1})$. If we now apply S_{n-1} we get: $S_{n-1}S^{n-1} \to S_nS^n$. We obtain, by iteration of this construction, a natural transformation $S_iS^i \to S_jS^j$. Furthermore, if $i \leq j \leq n$ it is easily seen that the following diagram commutes:



Dually, of course, we have transformations $S^jS_j \to S^iS_i$ for $i \leq j$, and when $i \leq j \leq n$ we have a commutative diagram:



Before going further, let us recall that in Corollary (1.18) we proved that for any half exact F, the map $F \to \underline{F}$ induces an isomorphism $S^1F \xrightarrow{\sim} S^1\underline{F}$, while the map $\overline{F} \to F$ induces an isomorphism $S_1\overline{F} \xrightarrow{\sim} S_1F$. Henceforth we shall take these isomorphisms as indentifications, and write $S^1F = S^1\underline{F}$, $S_1\overline{F} = S_1F$. Furthermore, we shall write $\underline{F} = S_1S^1F$, $\overline{F} = S^1S_1F$, and $F = \underline{F}$ (resp. $F = \overline{F}$) when F is projectively (resp. injectively) stable.

Let us now go back to the maps $S_iS^i \to S_kS^k$ and $S^kS_k \to S^iS_i$. When $i = 0$ we have, for a functor F, the maps: $F \to S_kS^kF$ and $S^kS_kF \to F$. Applying S^k to the first of these, we obtain $S^kF \to S^kS_kS^kF$. On the other hand, replacing F by the functor S^kF in the second, we have the map $S^kS_kS^kF \to S^kF$. We therefore have $S^kF \to S^kS_kS^kF \to S^kF$. In a similar fashion we have maps $S_kF \to S_kS^kS_kF \to S_kF$. As we shall see later, the following result is a generalization of the fact that M^*, the dual of a module, is a direct summand of M^{***}.

(1.21) PROPOSITION: Let F be a half exact functor and $k > 0$ an integer.

(a) The composition $S^kF \to S^kS_kS^kF \to S^kF$ is the identity. Therefore S^kF is a direct summand of $S^kS_kS^kF$.

(b) The composition $S_kF \to S_kS^kS_kF \to S_kF$ is the identity. Therefore

S_kF is a direct summand of $S_kS^kS_kF$.

Proof: We shall prove only (a) since (b) is proved by a dual demonstration. We proceed by induction on k. When $k = 0$, $S^0F = S^0S_0S^0F = S^0F = F$ and the maps in question are all the identity transformation. Suppose $k = 1$. The map $F \to \underline{F} = S_1S^1F$ induces $S^1F \xrightarrow{\sim} S^1S_1S^1F$ which we have agreed to write $S^1F = S^1S_1S^1F$. On the other hand, S^1F is always injectively stable, so we have $S^1S_1S^1F = \overline{S^1F} = S^1F$. Therefore both maps $S^1F \to S^1S_1S^1F$ and $S^1S_1S^1F \to S^1F$ are the identity (modulo our identification), so their composition: $S^1F \to S^1F$ is also the identity. This takes care of $k = 1$.

Suppose that the proposition holds for $1,\dots,k$. Let us apply S_k to the map $S^kF \to S_1S^1(S^kF)$. We obtain the usual map $S_kS^kF \to S_{k+1}S^{k+1}F$. If we apply S^{k+1} to the following commutative diagram:

$$\begin{array}{ccc} S_kS^kF & \longrightarrow & S_{k+1}S^{k+1}F \\ & \nwarrow \quad \nearrow & \\ & F & \end{array}$$

we obtain the commutative diagram:

$$\begin{array}{ccc} S^{k+1}S_kS^kF & \longrightarrow & S^{k+1}S_{k+1}S^{k+1}F \\ & \nwarrow \quad \nearrow & \\ & S^{k+1}F & \end{array} \tag{1.22}$$

Let us now apply the natural transformation $S^kS_k \to S^0S_0 =$ identity, to the map $S^kF \to S_1S^1(S^kF)$. We obtain a commutative diagram:

$$\begin{array}{ccc} S^kS_k(S^kF) & \longrightarrow & S^kF \\ \downarrow & & \downarrow \\ S^kS_k(S_1S^1S^kF) & \longrightarrow & S_1S^1(S^kF) \end{array}$$

If we now apply S^1 to this diagram, and combine it with diagram (1.22),

we obtain the following commutative diagram:

$$\begin{array}{ccccc} S^{k+1}F & \longrightarrow & S^{k+1}S_kS^kF & \longrightarrow & S^{k+1}F \\ \| & & \downarrow & & \downarrow \\ S^{k+1}F & \longrightarrow & S^{k+1}S_{k+1}S^{k+1}F & \to & S^1S_1S^{k+1}F \end{array}$$

Now the right-hand vertical map was obtained by applying S^1 to the map $S^kF \to \underline{S^kF} = S_1S^1(S^kF)$. The induced isomorphism $S^{k+1}F \xrightarrow{\sim} S^1S_1S^{k+1}F$ we agreed to call the identity. Furthermore, the composite $S^{k+1}S_{k+1}S^{k+1}F \to S^1S_1S^{k+1}F \to S^{k+1}F$ is the usual map $S^{k+1}S_{k+1}S^{k+1}F \to S^{k+1}F$. We now rewrite our diagram as:

$$\begin{array}{ccccc} S^1(S^kF) & \longrightarrow & S^1(S^kS_kS^kF) & \longrightarrow & S^1(S^kF) \\ \| & & \downarrow & & \| \\ S^{k+1}F & \longrightarrow & S^{k+1}S_{k+1}S^{k+1}F & \longrightarrow & S^{k+1}F \end{array}$$

The top row is obtained by applying S^1 to the composition $S^kF \to S^kS_kS^kF \to S^kF$. Since this composition is the identity by induction hypothesis, we see that the composition of the maps in the top row is the identity. It now follows that the composition of the maps in the bottom row is the identity. This completes the induction and the proof.

Let us now return to the general situation and the maps $\varphi_{j,n}$. We shall soon see that $\varphi_{j,n}(F)\colon S^{n-j}(F) \to S_jS^n(F)$ is an isomorphism for all $0 \le j \le n$ if and only if $S_j(F)$ is injectively stable for $0 \le j < n$. In order to phrase this last condition more economically, we establish the following terminology.

(1.23) Definition:

(a) Denote by Stab F the largest integer n such that S^iF is projectively stable for all $0 \le i < n$. If S^iF is projectively stable for

all $i \geq 0$ we write $\underline{\text{Stab}}\ F = \infty$.

(b) Denote by $\overline{\text{Stab}}\ F$ the largest integer n such that S_iF is injectively stable for all $0 \leq i < n$. If S_iF is injectively stable for all $i \geq 0$, we write $\overline{\text{Stab}}\ F = \infty$

(1.24) Examples:

(i) If $i \geq 1$, then $\overline{\text{Stab}}\ S^iF \geq 1$. This is clear since S^1F is always injectively stable.

(ii) If $i \geq 2$, then $\overline{\text{Stab}}\ S^iF \geq 2$. To see this, note that $S^{i-1}F$ is injectively stable, and we have an epimorphism $S^{i-1}F \to S_1S^1(S^{i-1}F)$. It follows that $S_1S^1(S^{i-1}F) = S_1S^iF$ is also injectively stable. By definition, to say that $S_j(S^iF)$ is injectively stable for $j = 0,1$ is to say that $\overline{\text{Stab}}\ S^iF \geq 2$. As we shall see, it is not in general true that $\overline{\text{Stab}}\ S^iF \geq i$.

(iii) Similar arguments give $\underline{\text{Stab}}\ S_iF \geq i$ for $i = 1,2$ and any half-exact F.

We now come to our first important results. The following two theorems, dual to each other, give important criteria for stability, and will have many applications in the sequel.

(1.25) THEOREM: For F in $\text{Hex}(\Gamma,\Delta)$ and an integer $n \geq 0$, the following statements are equivalent:

(a) $\overline{\text{Stab}}\ F \geq n$

(b) The maps $\varphi^{j,k}(F)\colon S^jS_kF \to S_{k-j}F$ are isomorphisms for all $0 \leq j \leq k \leq n$.

(c) The maps $\varphi^{j,n}(F)\colon S^jS_nF \to S_{n-j}F$ are isomorphisms for $0 \leq j \leq n$.

(d) For $0 \leq k \leq n$, $\varphi^{k,k}(F)\colon S^kS_kF \to F$ is an isomorphism, $\underline{\text{Stab}}\ S_kF \geq k$, and, when F is projectively stable, $\underline{\text{Stab}}\ S_kF > k$

(e) $\underline{\text{Stab}}\ S_nF \geq n$ and $\varphi^{n,n}(F)\colon S^nS_nF \to F$ is an isomorphism.

(f) $F \approx S^nG$ for some G in $\text{Hex}(\Gamma,\Delta)$ and $\overline{\text{Stab}}\ S^nS_nF \geq n$.

(1.26) THEOREM: For G in $\text{Hex}(\Gamma,\Delta)$ and an integer $n \geq 0$, the following statements are equivalent:

(a) $\underline{\text{Stab}}\ G \geq n$

(b) The maps $\varphi_{j,k}(G)\colon S^{k-j}G \to S_jS^kG$ are isomorphisms for all $0 \leq j \leq k \leq n$.

(c) The maps $\varphi_{j,n}(G)\colon S^{n-j}G \to S_jS^nG$ are isomorphisms for all $0 \leq j \leq n$.

(d) For $0 \leq k \leq n$, $\varphi_{k,k}(G)\colon G \to S_kS^kG$ is an isomorphism, $\overline{\text{Stab}}\ S^kG \geq k$, and, when G is injectively stable, $\overline{\text{Stab}}\ S^kG > k$.

(e) $\overline{\text{Stab}}\ S^nG \geq n$ and $\varphi_{n,n}(G)\colon G \to S_nS^nG$ is an isomorphism.

(f) $G \approx S_nH$ for some H in $\text{Hex}(\Gamma,\Delta)$, and $\underline{\text{Stab}}\ S_nS^nG \geq n$.

Proof: We prove only (1.25), leaving the dual proof to the reader.

To show (a) implies (b) we proceed by induction on k. When $k = 0$ there is nothing to prove. Suppose the implication holds for $0,\ldots,k-1$. We now prove the implication for k by induction on j. When $j = 0$ there is again nothing to prove. When $j = 1$ we note that $S^1S_1(S_{k-1}F) \to S_{k-1}F$ is an isomorphism since the assumption $\text{Stab}\ F \geq k$ contains the assertion that $S_{k-1}F$ is injectively stable. This map, however, is simply $\varphi^{1,k}(F)\colon S^1S_kF \to S_{k-1}F$. Thus, $j = 1$ is taken care of. To complete our induction, suppose $S^jS_kF \xrightarrow{\sim} S_{k-j}F$ for some $0 < j < k$. Applying S^1 we get $S^{j+1}S_kF \xrightarrow{\sim} S^1S_{k-j}F$ via $S^1\varphi^{j,k}(F)$. Since $j > 0$, we have $k - j < k$. By our induction hypothesis (on k) we have $S^1S_{k-j}F \xrightarrow{\sim} S_{k-(j+1)}F$ via $\varphi^{1,k}(F)$. Now the commutative diagram (1.20) tells us that $\varphi^{j+1,k}(F)$, the map we are interested in, is the composition of isomorphisms: $\varphi^{1,k-j}(F) \circ S^1\varphi^{j,k}(F)$; hence $\varphi^{j+1,k}(F)$ is an isomorphism and we are done.

The implication (b) implies (c) is clear.

We now show that (b) and (c) are equivalent by proving (c) implies (b).