arXiv:0904.4013v1 [math.AC] 26 Apr 2009

STRONGLY n-GORENSTEIN PROJECTIVE, INJECTIVE AND FLAT MODULES

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Abstract. This paper generalize the idea of the authors in [2]. Namely, we define and study a particular case of modules with Gorenstein projective, injective, and flat dimension less or equal than $n \geq 0$, which we call, respectively, strongly n-Gorenstein projective, injective and flat modules. These three classes of modules give us a new characterization of the first modules, and they are a generalization of the notions of strongly Gorenstein projective, injective, and flat modules respectively.

1. Introduction

Throughout this paper, all rings are commutative with identity element, and all modules are unital.

Let R be a ring, and let M be an R-module. As usual we use $\mathrm{pd}_R(M)$, $\mathrm{id}_R(M)$ and $\mathrm{fd}_R(M)$ to denote, respectively, the classical projective dimension, injective dimension and flat dimension of M. By $\mathrm{gldim}(R)$ and $\mathrm{wdim}(R)$ we denote, respectively, the classical global dimension and weak dimension of R. It is convenient to use "local" to refer to (not necessarily Noetherian) rings with a unique maximal ideal.

For a two-sided Noetherian ring R, Auslander and Bridger [1] introduced the G-dimension, $\mathrm{Gdim}_R(M)$, for every finitely generated R-module M. They proved the inequality $\mathrm{Gdim}_R(M) \leq \mathrm{pd}_R(M)$ with equality $\mathrm{Gdim}_R(M) = \mathrm{pd}_R(M)$ when $\mathrm{pd}_R(M)$ is finite.

Several decades later, Enochs and Jenda [9, 10] defined the notion of Gorenstein projective dimension (G-projective dimension for short), as an extension of G-dimension to modules that are not necessarily finitely generated, and the Gorenstein injective dimension (G-injective dimension for short) as a dual notion of Gorenstein projective dimension. Then, to complete the analogy with the classical homological dimension, Enochs, Jenda and Torrecillas [11] introduced the Gorenstein flat dimension. Some references are [6, 7, 9, 10, 11].

In 2004, Holm [13] generalized several results which are already obtained over Noetherian rings to associative rings.

Recently in [3], the authors started the study of global Gorenstein dimensions of rings, which are called, for a commutative ring R, projective, injective, and weak dimensions of R, denoted by $GPD(R)$, $GID(R)$, and $G.wdim(R)$, respectively, and, respectively, defined as follows:

2000 *Mathematics Subject Classification.* 13D05, 13D02.

Key words and phrases. Strongly n-Gorenstein projective, injective and flat modules, Gorenstein global dimension.

$$\begin{array}{lrcl} 1) & GPD(R) & = & sup\{Gpd_R(M) \mid M\ R-module\} \\ 2) & GID(R) & = & sup\{Gid_R(M) \mid M\ R-module\} \\ 3) & G.wdim(R) & = & sup\{Gfd_R(M) \mid M\ R-module\} \end{array}$$

They proved that, for any ring R, $G.wdim(R) \leq GID(R) = GPD(R)$ ([3, Theorems 2.1 and 2.11]). So, according to the terminology of the classical theory of homological dimensions of rings, the common value of $GPD(R)$ and $GID(R)$ is called Gorenstein global dimension of R, and denoted by $G.gldim(R)$.
They also proved that the Gorenstein global and weak dimensions are refinement of the classical global and weak dimensions of rings. That is : $G.gldim(R) \leq gldim(R)$ and $G.wdim(R) \leq wdim(R)$ with equality if $wdim(R)$ is finite ([3, Propositions 2.12]).

In [2], the authors introduced a particular case of Gorenstein projective, injective, and flat modules, which are defined, respectively, as follows:

Definitions 1.1.

(1) A module M is said to be strongly Gorenstein projective (SG-projective for short), if there exists an exact sequence of projective modules of the form:

$$\mathbf{P} = \cdots \to P \xrightarrow{f} P \xrightarrow{f} P \xrightarrow{f} P \to \cdots$$

such that $M \cong \mathrm{Im}\,(f)$ and such that $Hom(-, Q)$ leaves $\mathbf{P}$ exact whenever Q is a projective module.

The exact sequence $\mathbf{P}$ is called a strongly complete projective resolution and denoted by (P, f).

(2) The strongly Gorenstein injective module is defined dually.

(3) A module M is said to be strongly Gorenstein flat (SG-flat for short), if there exists an exact sequence of flat modules of the form:

$$\mathbf{F} = \cdots \to F \xrightarrow{f} F \xrightarrow{f} F \xrightarrow{f} F \to \cdots$$

such that $M \cong \mathrm{Im}\,(f)$ and such that $I \otimes -$ leaves $\mathbf{F}$ exact whenever I is an injective module. The exact sequence $\mathbf{F}$ is called a strongly complete flat resolution and denoted by (F, f).

The principal role of the strongly Gorenstein projective and injective modules is to give a simple characterization of Gorenstein projective and injective modules, respectively, as follows:

Theorem 1.2 ([2],Theorem 2.7)**.** *A module is Gorenstein projective (resp., injective) if, and only if, it is a direct summand of a strongly Gorenstein projective (resp., injective) module.*

Using [2, Theorem 3.5] together with [13, Theorem 3.7], we have the next result:

Proposition 1.3. *Let R be a coherent ring. A module is Gorenstein flat if, and only if, it is a direct summand of a strongly Gorenstein flat module.*

This result allows us to show that the strongly Gorenstein projective, injective and flat modules have simpler characterizations than their Gorenstein correspondent modules. For instance:

Theorem 1.4 ([2], Propositions 2.9 and 3.6)**.**

(1) *A module M is strongly Gorenstein projective if, and only if, there exists a short exact sequence of modules: $0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$ where P is projective and $Ext(M,Q) = 0$ for any projective module Q.*
(2) *A module M is strongly Gorenstein injective if, and only if, there exists a short exact sequence of modules: $0 \longrightarrow M \longrightarrow I \longrightarrow M \longrightarrow 0$ where I is injective and $Ext(E,M) = 0$ for any injective module E.*
(3) *A module M is strongly Gorenstein flat if, and only if, there exists a short exact sequence of modules: $0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$ where F is flat and $Tor(M,I) = 0$ for any injective module I.*

Along this paper we need the following Lemmas:

Lemma 1.5. *Let $0 \to N \to N' \to N'' \to 0$ be an exact sequence of R-modules. Then:*

(1) *$Gpd_R(N) \leq max\{Gpd_R(N'), Gpd_R(N'') - 1\}$ with equality if $Gpd_R(N') \neq Gpd_R(N'')$.*
(2) *$Gpd_R(N') \leq max\{Gpd_R(N), Gpd_R(N")\}$ with equality if $Gpd_R(N'') \neq Gpd_R(N) + 1$.*
(3) *$Gpd_R(N'') \leq max\{Gpd_R(N'), Gpd_R(N) + 1\}$ with equality if $Gpd_R(N') \neq Gpd_R(N)$.*

Proof. Using [13, Theorems 2.20 and 2.24] the argument is analogous to the one of [5, Corollary 2, p. 135]. □

Dually we have:

Lemma 1.6. *Let $0 \to N \to N' \to N'' \to 0$ be an exact sequence of R-modules. Then:*

(1) *$Gid_R(N) \leq max\{Gid_R(N'), Gid_R(N'') + 1\}$ with equality if $Gid_R(N') \neq Gid_R(N'')$.*
(2) *$Gid_R(N') \leq max\{Gid_R(N), Gid_R(N'')\}$ with equality if $Gid_R(N'') + 1 \neq Gid_R(N)$.*
(3) *$Gid_R(N'') \leq max\{Gid_R(N'), Gid_R(N) - 1\}$ with equality if $Gid_R(N') \neq Gid_R(N)$.*

And using [13, Proposition 3.11] and Lemma 1.6 we get the following Lemma

Lemma 1.7. *Let $0 \to N \to N' \to N'' \to 0$ be an exact sequence of modules over a coherent ring R. Then:*

(1) *$Gfd_R(N) \leq max\{Gfd_R(N'), Gfd_R(N'') - 1\}$ with equality if $Gfd_R(N') \neq Gfd_R(N'')$.*
(2) *$Gfd_R(N') \leq max\{Gfd_R(N), Gfd_R(N'')\}$ with equality if $Gfd_R(N'') \neq Gfd_R(N) + 1$.*
(3) *$Gfd_R(N'') \leq max\{Gfd_R(N'), Gfd_R(N) + 1\}$ with equality if $Gfd_R(N') \neq Gfd_R(N)$.*

In [13], Holm gives a characterization of modules with finite Gorenstein projective, injective and flat modules ([13, Theorems 2.20, 2.22 and 3.14]). In this three characterizations, Holm impose the finitely of this dimensions. Almost by definition one has the inclusion

$$\{M | pd(M) \leq n\} \subseteq \{M | Gpd(M) \leq n\}.$$

The main idea of this paper is to introduce and study an intermediate class of modules called strongly n-Gorenstein projective modules. Similarly, we define the strongly n-Gorenstein injective and flat modules.
The simplicity of these modules manifests in the fact that they has simpler characterizations than their corresponding Gorenstein modules. Moreover, with such modules, we are able to give nice new characterizations of modules with Gorenstein projective, injective and flat dimensions equal to n.

2. Strongly n-Gorenstein projective and injective modules

In this section, we introduce and study strongly n-Gorenstein projective and injective modules which are defined as follows:

Definitions 2.1. Let n be a positive integer.

(1) An R-module M is said to be strongly n-Gorenstein projective, if there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$$

where $pd(P) \leq n$ and $Ext^{n+1}(M, Q) = 0$ whenever Q is projective.

(2) An R-module M is said to be strongly n-Gorenstein injective, if there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow I \longrightarrow M \longrightarrow 0$$

where $id(I) \leq n$ and $Ext^{n+1}(E, M) = 0$ whenever E is injective.

A direct consequence of the above definition is that, the strongly 0-Gorenstein projective modules are just the strongly Gorenstein projective modules (by [2, Proposition 2.9]). Also every module with finite projective dimension less or equal than n is a strongly n-Gorenstein projective module and we have:

Proposition 2.2. *Let n be a positive integer and M be a strongly n-Gorenstein projective module. Then, the following hold:*

(1) If $0 \to N \to P_n \to ... \to P_1 \to M \to 0$ is an exact sequence where all P_i are projective, then N is strongly Gorenstein projective module and consequently $Gpd(M) \leq n$.

(2) Moreover, if $0 \longrightarrow M \longrightarrow P \longrightarrow M \longrightarrow 0$ is a short exact sequence where $pd(P) < \infty$ then $Gpd(M) = pd(P)$ and consequently M is strongly k-Gorenstein projective module with $k := pd(P)$.

Proof. (1) If $n = 0$ the result holds from [2, Proposition 2.9]. Otherwise, since M is strongly n-Gorenstein projective module, there is a short exact sequence

$$0 \to M \to P \to M \to 0$$

where $pd(P) \leq n$. Consider the following n-step projective resolution of M:

$$0 \to N \to P_n \to ... \to P_1 \to M \to 0$$

Hence, there is a module Q such that the following diagram is commutative:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N & \to & P_n & \to ... \to & P_1 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Q & \to & P_n \oplus P_n & \to ... \to & P_1 \oplus P_1 & \to & P & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N & \to & P_n & \to ... \to & P_1 & \to & M & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

Clearly, Q is projective since $pd(P) \leq n$ and for every projective module K, $Ext(N, K) = Ext^{n+1}(M, K) = 0$. Thus, by [2, Proposition 2.9], N is strongly Gorenstein projective module (then, Gorenstein projective). So, $Gpd(M) \leq n$.

(2) From the short exact sequence $0 \longrightarrow M \to P \to M \to 0$ and [13, Proposition 2.27] and Lemma 1.5 and since $Gpd(M)$ is finite by (1) above, we have

$$k := pd(P) = Gpd(P) = max\{Gpd(M), Gpd(M)\} = Gpd(M)$$

Thus, $Gpd(M) = pd(P)$. By [13, Theorem 2.20], $Ext^{k+1}(M, K) = 0$ whenever K is projective. Consequently, M is strongly k-Gorenstein projective module. □

Using [13, Theorem 2.20], a direct consequence of Proposition 3.2, is that every strongly n-Gorenstein projective module is strongly m-Gorenstein module whenever $n \leq m$.

Proposition 2.3.

(1) If $(M_i)_{i\in I}$ is a family of strongly n-Gorenstein projective modules, then $\bigoplus M_i$ is strongly n-Gorenstein projective.

(2) If $(M_i)_{i\in I}$ is a family of strongly n-Gorenstein injective modules, then $\prod M_i$ is strongly n-Gorenstein injective.

Proof. Clear since $pd(\oplus M_i) = sup\{pd(M_i)\}$ and $id(\prod M_i) = sup\{id(M_i)\}$ and also since $Ext^i(\bigoplus M_i, N) \cong \bigoplus Ext^i(M_i, N)$ and $Ext^i(M, \prod N_i) \cong \prod Ext^i(M, N_i)$ for every modules M, N, M_i, N_i and all $i \geq 0$. □

It is clear that, for a positive integer n and an R-module M:

"$pd(M) \leq n$" $\Longrightarrow$ "M *is strongly* $n - Gorenstein\ projective$" $\Longrightarrow$ "$Gpd(M) \leq n$"

The converse is false as the following two examples shows:

Example 2.4. Consider the quasi-Frobenius local ring $R := K[X]/(X^2)$ where K is a field and we denote by $\overline{X}$ the residue class in R of X. Let S be a Noetherian ring such $gldim(S) = n$. Consider a finitely generated S-module M of S such that $pd_S(M) = n$. Set $T = R \times S$ and set $E := (\overline{X}) \times M$. Then:

(1) E is strongly n-Gorenstein projective T-module and $Gpd_T(E) = n$.
(2) However, $pd_T(E) = \infty$.

Proof. (1) Consider the short exact sequence of R-modules:

$$0 \to (\overline{X}) \xrightarrow{\kappa} R \xrightarrow{\phi} (\overline{X}) \to 0$$

where κ is the injection and ϕ is the multiplication by $\overline{X}$. And consider also the short exact sequence of S-module:

$$0 \to M \xrightarrow{\iota} M \oplus M \xrightarrow{\pi} M \to 0$$

where ι and π are respectively the canonical injection and projection. Hence, we have the short exact sequence of $R \times S$-module:

$$(\star) \qquad 0 \to E \to R \times (M \oplus M) \to E \to 0$$

By [15, Lemma 2.5(2)], $pd_T(R \times (M \oplus M)) = pd_S(M \oplus M) = n$. On the other hand, by [4, Theorem 3.1] and [3, Propositions 2.8 and 2.12], we have

$$G.gldim(T) = sup\{G.gldim(R), G.gldim(S)\} = gldim(S) = n < \infty$$

Then, $Gpd_T(E) < \infty$. Therefore, applying Lemma 1.5 to $(\star)$,

$$Gpd_T(E) \leq max\{Gpd_T(R \times (M \oplus M)), Gpd_T(E) - 1\}$$

Thus, $Gpd_T(E) \leq Gpd_T(R \times (M \oplus M))$. Using Lemma 1.5 again to $(\star)$, we have

$$Gpd_T(R \times (M \oplus M)) \leq max\{Gpd_T(E), Gpd_T(E)\} = Gpd_T(E)$$

So, $Gpd_T(E) = Gpd_T(R \times (M \oplus M))$. On the other hand, by [13, Propostion 2.27], $Gpd_T(R \times (M \oplus M)) = pd_T(R \times (M \oplus M)) = n$. Consequently, $Gpd_T(E) = n$ and by $(\star)$ and [13, Theorem 2.20], E is strongly n-Gorenstein projective T-module, as desired.

(2) Using [15, Lemma 2.5(2)], $pd_T(E) = sup\{pd_R(\overline{X}), pd_S(M)\}$. Now, suppose that $pd_R(\overline{X}) < \infty$. Thus, by [3, Proposition 2.8 and Corollary 2.10], $\overline{X}$ is projective and then free since R is local. Absurd, since $\overline{X}^2 = 0$. Consequently, $pd_T(E) = \infty$. □

Example 2.5. Consider the Noetherian local ring $R := K[[X, Y]]/(XY)$ where K is a field, and we denote by $\overline{X}$ the residue class in R of X. Let S be a Noetherian ring such that $gldim(S) = n$. Let M be a finitely generated S-module such that $pd_S(M) = n$. Set $T = R \times S$ and set $E := (\overline{X}) \times M$. Then:

(1) $Gpd_T(E) = n$.
(2) There is no positive integer k for such E is strongly k-Gorenstein T-module.

Proof. (1) By [4, Lemma 3.2] and [13, Theorem 2.27],

$$n = pd_S(M) = Gpd_S(M) = Gp_S(E \otimes_T S) \leq Gpd_T(E)$$

On the other hand, seen [3, Propostions 2.8 and 2.10 and 2.12], the conditions of [4, Lemma 3.3] are satisfied. Hence, we have

$$Gpd_T(E) \leq sup\{Gpd_R(\overline{X}), Gpd_S(M)\} = pd_S(M) = n$$

Consequently, $Gpd_T(E) = n$, as desired.

(2) Suppose the existence of a positive integer k such that E is strongly k-Gorenstein projective T-module. Then, there exist a short exact sequence of T-modules $0 \to E \to P \to E \to 0$ where $pd_T(P) < \infty$. Since R is a projective T-module and since $(\overline{X}) \cong_R E \otimes_T R$ we have a short exact sequence of R-modules

$$0 \to (\overline{X}) \to P \otimes_T R \to (\overline{X}) \to 0$$

Notice that $pd_R(P \otimes_T R) < \infty$ since R is a projective T-module. Using [3, Propositions 2.8 and 2.10], we get that $P \otimes_T R$ is a projective R-module and that $(\overline{X})$ is a Gorenstein projective R-module. So, by [13, Theorem 2.20], $(\overline{X})$ is strongly Gorenstein projective module. Absurd (by [2, Example 2.13(2)]). □

Now we give our main result of this paper.

Theorem 2.6. *Let M be an R-module and n a positive integer. Then, $Gpd_R(M) \leq n$ if, and only if, M is a direct summand of a strongly n-Gorenstein projective module.*

Proof. If $n = 0$ the result holds from [2, Theorem 2.7]. So, assume that $0 < Gpd(M) \leq n$. From, [13, Theorem 2.10], there is an exact sequence of R-module $0 \to K \to G \to M$ where G is Gorenstein projective and $pd(K) \leq n-1$. By definition of Gorenstein projective module there is a short exact sequence

$$0 \to G \to P \to G^0 \to 0$$

where P is projective and G' is Gorenstein projective. Hence, consider the following pushout diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & K & = & K & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 \to & G & \to & P & \to & G^0 & \to 0 \\
 & \downarrow & & \downarrow & & \| & \\
0 \to & M & \to & D & \to & G^0 & \to 0 \\
 & \downarrow & & \downarrow & & & & & \\
 & 0 & & 0 & & & & &
\end{array}$$

From the vertical middle short exact sequence, $pd(D) \leq pd(K) + 1 \leq n$. Now, consider the Gorenstein projective resolution of M:

$$0 \to G_n \to P_n \to ... \to P_1 \to M \to 0$$

where all P_i are projective and G_n is Gorenstein projective. Devise this sequence on short exact sequence as

$$\begin{array}{ccccccccc}
0 \to & G_1 & \to & P_1 & \to & M & \to 0 \\
0 \to & G_2 & \to & P_2 & \to & G_1 & \to 0 \\
 & \vdots & & \vdots & & \vdots & \\
0 \to & G_n & \to & P_n & \to & G_{n-1} & \to 0
\end{array}$$

Clearly, by Lemma 1.5, for all $1 \leq i \leq n$, $Gpd(G_i) \leq n - i \leq n$.
Consider also the following projective resolution of G_n:

$$... \to P_{n+2} \to P_{n+1} \to G_n \to 0$$

and devise this long sequence on short exact sequences as $0 \to G_{i+1} \to P_{i+1} \to G_i \to 0$ for all $i \geq n$. It is clear that for all $i \geq n$, G_i is Gorenstein projective module (by [13, Theorem 2.5]).
On the other hand, since G^0 is Gorenstein projective, there a co-proper right projective resolution of G^0

$$0 \to G^0 \to P^1 \to P^2 \to P^3 \to ...$$

such that for every $i \geq 1$, $G^i = Im(P^i \to P^{i+1})$ is Gorenstein projective. If we devise this sequence on short exact sequence we get $0 \to G^i \to P^{i+1} \to G^{i+1} \to 0$ for all $i \geq 0$.
Briefly, we have

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
0 & \to & G^1 & \to & P^2 & \to & G^2 & \to & 0 \\
0 & \to & G^0 & \to & P^1 & \to & G^1 & \to & 0 \\
0 & \to & M & \to & D & \to & G^0 & \to & 0 \\
0 & \to & G_1 & \to & P_1 & \to & M & \to & 0 \\
0 & \to & G_2 & \to & P_2 & \to & G_1 & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Thus, we have a sum short exact sequence $0 \to N \to Q \to N \to 0$ where $N = \oplus_{i\geq 1} G_i \oplus M \oplus_{i\geq 0} G^i$ and $Q = \oplus_{i\geq 1} P_i \oplus D \oplus_{i\geq 1} P^i$. And clearly $pd(Q) = pd(D) \leq n$ and $Gpd(N) = sup\{Gpd(G_i), Gpd(G^i), Gpd(M)\} \leq n$ (by [13, Proposition 2.19]). Thus, by [13, Theorem 2.20], N is strongly n-Gorenstein projective module and M is a direct summand of N.

The condition "if" follows from [13, Propostion 2.19] and Proposition 2.2. □

Dually, we have:

Theorem 2.7. *Let M be an R-module and n a positive integer. Then, $Gid_R(M) \leq n$ if, and only if, M is a direct summand of a strongly n-Gorenstein injective module.*

Proof. The proof is similar to the one of Theorem 2.6 by replacing the direct sum by the direct product and by using [13, Theorem 2.15] and the dual of [13, Propostion 2.19] and [2, Theorem 2.7]. □

Remark 2.8. From the proof of Theorem 2.6, if $Gpd(M) = n$ then, there exists a strongly n-Gorenstein projective module N such that $Gpd(N) = n$ and M is a direct summand of N.

Proposition 2.9. *For any module M and any positive integer n, the following are equivalent:*

1. *M is strongly n-Gorenstein projective.*
2. *There is an exact sequence $0 \to M \to Q \to M \to 0$ where $pd(Q) \leq n$ and $Ext^i(M, P) = 0$ for every module P with finite projective dimension and all $i > n$.*
3. *There is an exact sequence $0 \to M \to Q \to M \to 0$ where $pd(Q) < \infty$ and $Ext^i(M, P) = 0$ for every projective module P and all $i > n$.*

Proof.
$1 \Rightarrow 2$. By definition of strongly n-Gorenstein projective modules, we have just to prove that for every $i > n$ and all module P with finite projective dimension we have $Ext^i(M, P) = 0$. That is clear from [13, Theorem 2.20] since $Gpd(M) \leq n$ (by Proposition 2.2).
$2 \Rightarrow 3$. Obvious.
$3 \Rightarrow 1$. Since $Ext^i(M, P) = 0$ for every projective module P and all $i > n$, from the

short exact sequence $0 \to M \to Q \to M \to 0$ we have for all $i > n$

$$... \to 0 = Ext^i(M,P) \to Ext^i(Q,P) \to Ext^i(Q,P) = 0 \to ...$$

Thus $Ext^i(Q,P) = 0$. On the other hand, $Gpd(Q) = pd(Q) < \infty$ (by [13, Proposition 2.27]). Then, from [13, Theorem 2.20], $pd(Q) = Gpd(Q) \leq n$. Consequently, M is strongly n-Gorenstein projective. □

Proposition 2.10. *If $G.gldim(R) < \infty$. Then:*

1. *M is strongly n-Gorenstein projective if, and only if, there exists an exact sequence $0 \to M \to Q \to M \to 0$ where $pd(Q) \leq n$.*
2. *M is strongly n-Gorenstein injective if, and only if, there exists an exact sequence $0 \to M \to E \to M \to 0$ where $id(E) \leq n$.*

Proof. (1) The condition "only if" is clear by definition of the strongly n-Gorenstein projective module. So, we claim the "if" condition. Since $G.gldime(R) < \infty$, $Gpd(M) < \infty$. Thus, there is an integer k such that $Ext^i(M,P) = 0$ for all $i > k$ and for all projective module P. Thus, using the long exact sequence

$$... \to Ext^i(Q,P) \to Ext^i(M,P) \to Ext^{i+1}(M,P) \to Ext^{i+1}(Q,P) \to ...$$

we deduce that $Ext^{n+1}(M,P) = Ext^{n+j}(M,P)$ for all $j > 0$ (since $pd(Q) \leq n$). Thus, if $j > k$. we conclude that for every projective module P, $Ext^{n+1}(M,P) = 0$. Consequently, M is strongly n-Gorenstein projective.

(2) The proof is dual to (1). □

Proposition 2.11. *Let M be a strongly n-Gorenstein projective R-module ($n \geq 1$). Then, there is an epimorphism $\varphi : N \twoheadrightarrow M$ where N is strongly Gorenstein projective and $K = Ker(\varphi)$ satisfies $pd(K) = Gpd(M) - 1 \leq n - 1$.*

Proof. Assume that M is strongly n-Gorenstein projective module.

The proof will be similar to the one of [13, Theorem 2.10]. For exactness we give the proof here. Let $0 \to N \to P_n \to ... \to P_1 \to M \to 0$ be an exact sequence where all P_i are projective and N is strongly Gorenstein projective (the existence of this sequence is guaranties by Proposition 2.2). By definition of strongly Gorenstein projective module, there is an exact sequence $0 \to N \to Q \to ... \to Q \to N \to 0$ where Q is projective and such that the functor $Hom(-,P)$ leaves this sequence exact, whenever P is projective. Thus, there exists homomorphisms, $Q \to P_i$ for $i = 1, ..., n$ and $N \to M$, such that the following diagram is commutative.

$$\begin{array}{ccccccccc} 0 \to & N & \to & Q & \to ... \to & Q & \to & N & \to 0 \\ & \| & & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & N & \to & P_n & \to ... \to & P_1 & \to & M & \to 0 \end{array}$$

This diagram gives a chain map between complexes,

$$\begin{array}{ccccccccc} 0 \to & Q & \to ... \to & Q & \to & N & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & P_n & \to ... \to & P_1 & \to & M & \to 0 \end{array}$$

which induces an isomorphism in homology. Its mapping cone is exact, and all the modules in it, except for $P_1 \oplus N$, which is strongly Gorenstein projective, are projective. Hence the kernel K of $\varphi : P_1 \oplus N \twoheadrightarrow M$ satisfies $pd(K) \leq n - 1$, as desired. □

Proposition 2.12.

(A): *Let* $0 \to N \xrightarrow{\alpha} P \xrightarrow{\beta} N' \to 0$ *be an exact sequence of* R*-modules.*

Case1 "P projective and $Gpd(N') = n < \infty$**":**

(1) If N' *is strongly Gorenstein projective then so is* N*.*

(2) if $n \geq 1$ *and* N' *is strongly* n*-Gorenstein projective then* N *is strongly* $(n-1)$*-Gorenstein projective and* $Gpd(N) = n - 1$*.*

Case 2 "*$pd(P) = n < \infty$**":**

If N *is strongly Gorenstein projective module then* N' *is strongly* $(n+1)$*-Gorenstein projective.*

(B): *Let* $0 \to N \xrightarrow{\mu} N' \xrightarrow{\nu} Q \to 0$ *be an exact sequence where* $pd(Q) = n < \infty$*.*

(1) If $n > 0$ *and* N' *is strongly Gorenstein projective then* N *is strongly* $(n-1)$*-Gorenstein projective.*

(2) If Q *is projective then* N *is strongly Gorenstein projective if, and only if,* N' *is strongly Gorenstein projective.*

Proof. **(A)**

Case 1:

(1) Clear.

(2) If N' is strongly n-Gorenstein projective module, there is a short exact sequence $0 \to N' \to Q \to N' \to 0$ where $pd(Q) \leq n$. Since $Gpd(N') = n$ we deduce that $pd(Q) = n$ (by Proposition 2.2). On the other hand, we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N & \to & P & \to & N' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & Q' & \to & P \oplus P & \to & Q & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & N & \to & P & \to & N' & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since P is projective, we get $pd(Q') = n - 1$ and since $Gpd(N') = n$ we deduce that $Gpd(N) = n-1$ (by Lemma 1.5). Thus, N' is strongly $(n-1)$-Gorenstein projective (by [13, Theorem 2.20]).

Case 2:

Since N is strongly Gorenstein projective module there is an exact sequence $0 \to N \xrightarrow{u} Q \xrightarrow{v} N \to 0$ where Q is projective and $Ext(N, K) = 0$ for every module K with finite projective dimension. Thus, since $pd(P) < \infty$, the short sequence

$$0 \to Hom(N, P) \xrightarrow{\circ v} Hom(Q, P) \xrightarrow{\circ u} Hom(N, P) \to 0$$

is exact. Hence, for $\alpha : N \to P$ there is a morphism $\lambda : Q \to P$ such that $\alpha = \lambda \circ u$. Thus, the following diagram is commutative

$$\begin{array}{ccccccccc}
0 & \to & N & \xrightarrow{u} & Q & \xrightarrow{v} & N & \to & 0 \\
 & & \alpha \downarrow & & \phi \downarrow & & \downarrow \alpha & & \\
0 & \to & P & \xrightarrow{i} & P \oplus P & \xrightarrow{j} & P & \to & 0
\end{array}$$

where $\phi : Q \to P \oplus P$ is defined by $\phi(q) = (\lambda(q), \alpha \circ v(q))$ and i and j are respectively the canonical injection and projection. Thus, applying the Snake Lemma, we deduce an exact sequence of the from:

$$0 \to N' \to (P \oplus P)/\phi(Q) \to N' \to 0$$

and clearly $pd(P \oplus P/\phi(Q)) \leq n+1$ and $Gpd(N') \leq n+1$. Thus, by [13, Theorem 2.20], N' is strongly (n+1)-Gorenstein projective, as desired.

(B)

Suppose that N' is strongly Gorenstein projective. Thus, there is an exact sequence $0 \to N' \xrightarrow{u} P \xrightarrow{u} N' \to 0$ where P is projective and $Ext(N, K) = 0$ for every module K with finite projective dimension. Then, similar as in **(A) Case 2**, there is a morphism $\phi : P \to Q \oplus Q$ such that the following diagram is commutative:

$$\begin{array}{ccccccc} 0 \to & N' & \xrightarrow{u} & P & \xrightarrow{v} & N' & \to 0 \\ & \nu \downarrow & & \phi \downarrow & & \downarrow \nu & \\ 0 \to & Q & \xrightarrow{i} & Q \oplus Q & \xrightarrow{j} & Q & \to 0 \end{array}$$

Hence, applying Snake Lemma, we get an exact sequence of the form $0 \to N \to Ker(\phi) \to N \to 0$.

(1) If $n > 0$ then $pd(ker(\phi)) = n-1$ and also $Gpd(N) = n-1$ (by Lemma 1.5). Therefore, N is strongly $(n-1)$-Gorenstein projective (by [13, Theorem 2.20]).
(2) If Q is projective, then $ker(\phi)$ is projective and N is Gorenstein projective. Thus, N is strongly Gorenstein projective. Conversely, if N is strongly Gorenstein projective it is clear that $N' \cong N \oplus P$ is strongly Gorenstein projective, as desired.

□

Dually, we have:

Proposition 2.13.

(A): *Let $0 \to N \xrightarrow{\alpha} I \xrightarrow{\beta} N' \leftrightarrow 0$ be an exact sequence of R-modules.*
Case1 "I injective and $Gid(N) = n < \infty$":
(1) If N is strongly Gorenstein injective then so is N'.
(2) If $n \geq 1$ and N is strongly n-Gorenstein injective then N' is strongly $(n-1)$-Gorenstein injective and $Gid(N') = n-1$.
Case 2 "$id(P) = n < \infty$":
If N' is strongly Gorenstein injective module then N is strongly (n+1)-Gorenstein injective.

(B): *Let $0 \to E \xrightarrow{\mu} N' \xrightarrow{\nu} N \to 0$ be an exact sequence where $id(E) = n < \infty$.*
(1) If $n > 0$ and N' is strongly Gorenstein injective then N is strongly $(n-1)$-Gorenstein injective.
(2) If E is injective then N is strongly Gorenstein injective if, and only if, N' is strongly Gorenstein injective.

Corollary 2.14. *Let R be a ring. The following are equivalent:*

(1) Every Gorenstein projective module is strongly Gorenstein projective.
(2) Every module such $Gpd(M) \leq 1$ is strongly 1-Gorenstein projective.

Proof. Assume that every Gorenstein projective module is strongly Gorenstein projective and consider a module M such that $Gpd(M) \leq 1$. Consider a short exact sequence $0 \to N \to P \to M \to 0$ where P is projective and so N is Gorenstein projective. Hence, by the hypothesis condition, N is strongly Gorenstein projective module. Thus, by Proposition 2.12 (Case 2), M is strongly 1-Gorenstein projective module, as desired.
Conversely, assume that every module such $Gpd(M) \leq 1$ is strongly 1-Gorenstein projective. Let M be a Gorenstein projective module. Thus, by the hypothesis condition M is strongly 1-Gorenstein projective. Then, there is an exact sequence $0 \to M \to Q \to M \to 0$ where $pd(Q) \leq 1$. Since M is Gorenstein projective, so is Q and then it is projective (by [13, Theorem 2.5 and Proposition 2.27]). Consequently, M is strongly Gorenstein projective module. □

Proposition 2.15. *Let R be a ring. The following are equivalent:*

1. *Every module is strongly n-Gorenstein projective.*
2. *Every module is strongly n-Gorenstein injective.*

Proof. We prove only one implication and the other is similar.
Assume that every module is strongly n-Gorenstein projective. Thus $G.gldim(R) \leq n$ (by Proposition 2.2 and the hypothesis condition). Now, consider an arbitrary module M. Clearly $Gid(M) \leq n$ (since $G.gldim(R) \leq n$). Then, for every injective module I, $Ext^{n+1}(I, M) = 0$ ([13, Theorem 2.22]). On the other hand, there is an exact sequence, $0 \to M \to P \to M \to 0$ where $pd(P) \leq n$. By [3, Corollary 2.10], $id(P) \leq n$. Consequently, M is strongly n-Gorenstein injective, as desired. □

Proposition 2.16. *Let R be a ring with finite Gorenstein global dimension and n a positive integer. The following are equivalent:*

1. *$G.gldim(R) \leq n$.*
2. *Every strongly Gorenstein projective module is strongly n-Gorenstein injective module.*
3. *Every strongly Gorenstein injective module is strongly n-Gorenstein projective module.*

Proof. We claim that $G.gldim(R) \leq n$ if, and only if, every strongly Gorenstein projective module is strongly n-Gorenstein injective module. The proof of the other equivalence is analogous. So, suppose that $G.gldim(R) \leq n$ and consider a strongly Gorenstein projective module M. For such module there is an exact sequence $0 \to M \to P \to M \to 0$ where P is projective. From [3, Corollary 2.10], $id(P) \leq n$. Hence, from Proposition 2.10, M is strongly n-Gorenstein injective.
Conversely, suppose that every strongly Gorenstein projective module is strongly n-Gorenstein injective module and let P be a projective module (then strongly Gorenstein projective). By the hypothesis condition, P is strongly n-Gorenstein injective. Thus, there is an exact sequence $0 \to P \to E \to P \to 0$ where $id(E) \leq n$. Hence, $P \oplus P \cong E$. Consequently, $id(P) \leq n$. Then, from [3, Theorem 2.1 and Lemma 2.2], $G.gldim(R) \leq n$, as desired. □

3. Strongly n-Gorenstein flat modules

In this section, we introduce and study the strongly n-Gorenstein flat modules which are defined as follows:

Definition 3.1. An R-module M is said to be strongly n-Gorenstein flat, if there exists a short exact sequence

$$0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$$

where $fd_R(P) \leq n$ and $Tor_R^{n+1}(M, I) = 0$ whenever I is injective.

A direct consequence of the above definition is that, the strongly 0-Gorenstein flat modules are just the strongly Gorenstein flat modules (by [2, Proposition 3.6]). Also every module with finite flat dimension less or equal than n is a strongly n-Gorenstein flat module. Also we have:

Proposition 3.2. *Let n be a positive integer and M be a strongly n-Gorenstein flat R-module. Then, the following hold:*

(1) If $0 \to N \to P_n \to ... \to P_1 \to M \to 0$ is an exact sequence where all P_i are projective, then N is strongly Gorenstein flat module and consequently $Gfd(M) \leq n$.

(2) Moreover, if $0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$ is a short exact sequence, where $fd(F) < \infty$, then $Gfd(M) = fd(F)$ and consequently M is strongly k-Gorenstein flat module with $k := pd(P)$.

Proof. (1) Using an n-step projective resolution of M and [2, Proposition 3.6], the proof is analogous to Proposition 2.2.

(2) Consider an exact short sequence $(\mp)$ $0 \longrightarrow M \longrightarrow F \longrightarrow M \longrightarrow 0$ where $fd(F) < \infty$. We claim $Gfd(M) = fd(F)$.
Consider an n step projective resolution

$$0 \to N \to P_n \to ... \to P_1 \to M \to 0$$

From (1) above N is strongly Gorenstein flat module. Thus, there is an short exact sequence $(\star)$ $0 \to N \to P \to N \to 0$ where P is flat and $Tor(N, I) = 0$ whenever I is injective. Hence, from $(\star)$, for all $i > 0$, $Tor^i(N, I) = 0$. So, we have $Tor^{n+i}(M, I) = Tor^i(N, I) = 0$.
Now, suppose that $fd(F) := k$ and let I be an arbitrary injective. From the short exact sequence $(\mp)$ we have the long exact sequence

$$....Tor^{i+1}(F, I) \to Tor^{i+1}(M, I) \to Tor^i(M, I) \to Tor^i(F, I) \to ...$$

Hence, for all $i > k$, $Tor^i(M, I) = Tor^{i+1}(M, I) = ... = Tor^{i+n}(M, I) = 0$. In particular, $Tor^{k+1}(M, I) = 0$. Consequently, M is strongly k-Gorenstein flat module. Then, from (1) above $Gfd(M) \leq k = fd(F)$.
Conversely, we claim $fd(F) \leq Gfd(M)$. Applying $Hom_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ to the short exact sequence $0 \to M \to F \to M \to 0$ we get the exactness of $0 \to Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \to Hom_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z}) \to Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \to 0$. On the other hand, from [13, Proposition 3.11], $Gid(Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \leq Gfd(M) \leq n$ and by [17, Lemma 3.51 and Theorem 3.52], $id(Hom_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})) = fd(F) < \infty$. Hence, by [13, Theorem 2.22] and the injective counterpart of Proposition 2.9, $Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is strongly n-Gorenstein injective module. So, from the injective counterpart of Proposition 2.2 and by [13, Proposition 3.11],

$$fd(F) = id(Hom_{\mathbb{Z}}(F, \mathbb{Q}/\mathbb{Z})) = Gid(Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \leq Gfd(M)$$

Thus, we have the desired equality.

□

Theorem 3.3. *Let R be a coherent ring, M be an R-module and n be a positive integer. Then, $Gf(M) \leq n$ if, and only if, M is a direct summand of a strongly n-Gorenstein flat module.*

Proof. Using [2, Theorem 3.5], [13, Proposition 3.13 and Theorems 3.14 and 3.23], Lemma 1.7 and Proposition 3.2 the proof of this result is analogous to the one of Theorem 2.6. □

Proposition 3.4. *For a module M and a positive integer n, the following are equivalent:*

1. *M is strongly n-Gorenstein flat.*
2. *There is an exact sequence $0 \to M \to F \to M \to 0$ where $fd(F) \leq n$ and $Tor^i(M, I) = 0$ for every module I with finite injective dimension and all $i > n$.*
3. *There is an exact sequence $0 \to M \to F \to M \to 0$ where $fd(F) < \infty$ and $Tor^i(M, I) = 0$ for every injective module P and all $i > n$.*

Proof. $1 \Rightarrow 2$. Assume that M is strongly n-Gorenstein flat module. Then, there is an exact sequence $0 \to M \to F \to M \to 0$ where $fd(F) \leq n$. On the other hand, if $0 \to N \to P_n \to ... \to P_1 \to M \to 0$ is an n-step projective resolution of M, by Proposition 3.2, N is strongly Gorenstein flat module. Thus, there is a short exact sequence $(\star)$ $0 \to N \to P \to N \to 0$ where P is flat and $Tor(N, I) = 0$ whenever $id(I) < \infty$ (from [2, Proposition 3.6]). Hence, from $(\star)$, for all $i > 0$, $Tor^i(N, I) = 0$. So, we have $Tor^{n+i}(M, I) = Tor^i(N, I) = 0$, as desired.

$2 \Rightarrow 3$. Obvious.

$3 \Rightarrow 1$. As in the proof of Proposition 2.9($3 \Rightarrow 1$), we prove that for every injective module I and all $i > n$ we have $Tor^i(F, I) = 0$. Suppose that $m := fd(F) > n$ and let M be an arbitrary module. Pick a short exact sequence $0 \to M \to I \to I/M \to 0$ where I is injective. So, we have the long exact sequence

$$... \to Tor^{i+1}(F, I) \to Tor^{i+1}(F, I/M) \to Tor^i(F, M) \to Tor^i(F, I) \to ...$$

Thus, for $i > n$ we have $Tor^i(F, M) = Tor^{i+1}(F, I/M)$. Hence, $Tor^m(F, M) = Tor^{m+1}(F, I/M) = 0$. Then, $fd(F) \leq m - 1$. Absurd. Thus, $fd(F) \leq n$. So, it is clear that M is strongly n-Gorenstein flat module, as desired. □

Proposition 3.5. *Let M be a strongly n-Gorenstein flat module over a coherent ring R ($n \geq 1$). Then, there is an epimorphism $\varphi : N \twoheadrightarrow M$ where N is strongly Gorenstein flat and $K = Ker(\varphi)$ satisfies $fd(K) = Gfd(M) - 1 \leq n - 1$.*

Proof. Using [2, Proposition 3.6] and [13, Lemma 3.17], the proof is analogous that in [13, Lemma 3.17] and Proposition 2.11. □

Proposition 3.6. *Let M be an R-module and n a positive integer. Then, following are equivalent:*

1. *M is strongly n-Gorenstein projective and M admits a finite n-presentation.*
2. *M is strongly n-Gorenstein flat module and M admits a finite $n + 1$-presentation.*

Proof. Any way, in this Proposition M admits a finite n-presentation. Thus, we can consider an n-step free resolution $0 \to N \to F_n \to ... \to F_1 \twoheadrightarrow M \to 0$ where

F_i are finitely generated free and N is finitely generated.
(1) If M is strongly n-Gorenstein projective module, then N is a finitely generated strongly Gorenstein projective module. Thus, from [2, Proposition 3.9], N is a finitely presented strongly Gorenstein flat module. Then, M admits a finite $(n+1)$-presentation and , for all injective module I, we have $Tor^{n+1}(M, I) = Tor(N, I) = 0$. On the other hand, there is an exact sequence $0 \to M \to Q \to M \to 0$ where $fd(Q) \leq pd(Q) \leq n$. Consequently, M is strongly n-Gorenstein flat module which admits a finite $n+1$-presentation.
(2) Now, if M is strongly n-Gorenstein flat module which admits a finite $(n+1)$-presentation. Then, N is a finitely presented strongly flat module. Thus, from [2, Proposition 3.9], N is strongly Gorenstein projective module. Hence, for every projective module P, $Ext^{n+1}(M, P) = Ext(N, P) = 0$. On the other hand, there is an exact sequence $0 \to M \to F \to M \to 0$ where $fd(F) \leq n$. But, from this short exact sequence we see that F also admits a finite $(n+1)$-presentation. Thus, $pd(F) = fd(F) \leq n$. Consequently, M is strongly n-Gorenstein projective module, as desired. □

Corollary 3.7. *If R is a coherent ring and M a finitely presented module. Then, M is strongly n-Gorenstein projective if, and only if, M is strongly n-Gorenstein flat.*

Finally, it is clear that for a module M and a positive integer n we have:

$$\text{“}fd(M) \leq n\text{”} \Longrightarrow \text{“}M \text{ is strongly } n - Gorenstein\ flat\text{”} \Longrightarrow \text{“}Gf(M) \leq n\text{”}$$

Also, the converse are false, in general, by the same Examples (2.4 and 2.5) in section 2 since T is Noetherian and E is finitely presented (since E is finitely generated and T is Noetherian) (by Corollary 3.7).

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