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Some topological proofs and extensions of Gruško's theorem

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Abstract

Homomorphisms $h: F \to \ast_\alpha G_\alpha$ are studied, where F is a free group and $\ast_\alpha G_\alpha$ is the free product.

In Part I, the conditions imposed on the homomorphism h relate to certain commutators in F. The method of proof is to obtain a topological realization of h, interpreting F as the fundamental group of a surface with boundary and $\ast_\alpha G_\alpha$ as the fundamental group of a wedge of complexes. Standard general position arguments and reductions are made; the theorems are obtained from the well-known classification of surfaces.

In Part II, it is assumed that h is an epimorphism. Gruško's Theorem is obtained. Again the method of proof is topological, based on general position arguments, a direct limit construction, and a combinatorial argument.

Besides Gruško's Theorem, several related facts are proved. One of these is Kneser's Conjecture; if M is a closed 3-manifold, then any free factorization of $\pi_1(M)$ is mirrored by a factorization of M into a sum of manifolds. Another is a result on free products with amalgamation; if $(G * H)_A = Q$ has rank n, then there are n elements of $G \cup H$, whose consequence is Q.

A new algebraic proof of Gruško's Theorem is given. This is more or less a translation of the topological proof.

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Introduction

In [**4**], I. Gruško proved that it is possible, given n generators of a free product of groups, to find n other generators, related to the first n in a specific way, such that the new generators lie in the free factors of the free product. What he specifically proved is that if F is free of finite rank and $\varphi: F \to G*H$ is a homomorphism onto a free product, then F has a free decomposition, $F = F_G * F_H$, such that $\varphi(F_G) \subset G$, $\varphi(F_H) \subset H$.

Later, but independently, F. W. Levi [**9**] proved this statement assuming that the rank of F is 2. B. H. Neumann [**10**] subsequently proved just what Gruško proved.

The questions whether F need have finite rank and whether the image need have only a finite number of free factors was solved by Wagner [**17**], whose theorem states: If F is free and $\varphi: F \to \ast_\alpha G_\alpha$ is onto, then there is a free decomposition of F, $\ast_\alpha F_\alpha$, such that $\varphi(F_\alpha) \subset G_\alpha$.

The proofs of Gruško, Neumann, and Wagner make use of the cancellation techniques developed by Kuroš in his proof of the Subgroup Theorem for free products. The major part of this thesis (part II) presents several other proofs of the Gruško-Wagner Theorem.

Topology is the technique of these proofs. Topological methods had been previously used in the theory of free products; the Baer-Levi proof of the Subgroup Theorem [**1**] is one of the most beautiful examples of this. The method used here is a distant relative of Van Kampen's proof of his theorem on the fundamental group of a union of spaces. The first step in any proof here is to construct a "topological realization" of a homomorphism; then apply techniques of "general position"; then, by some sort of combinatorial argument, sometimes coupled with a transfinite induction, simplify the topological realization as much as possible; finally, read off the conclusion of the theorem by applying Van Kampen's Theorem to the simplified situation.

Part I applies this method to the study of commutators in free products; here one makes use of the classification of orientable 2-manifolds according to number of boundary curves, genus, and Euler characteristic.

In Part II, proofs of the Gruško-Wagner Theorem and related theorems are given. Section A studies maps of a 3-manifold into a wedge of spaces; we obtain Gruško's Theorem and the topological result known as Kneser's Conjecture. Section B proves Wagner's version of Gruško's Theorem by using CW-complexes and a limiting process. In Section C this proof is translated into pure algebra, using the notion of "oidgroup", which is somewhat more convenient to use than the related concept of "groupoid". Section D discusses what can be said about free products with amalgamation; it proves a Gruško-like theorem and gives a counterexample to a conjectured theorem.

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O. Preliminaries

O.1. Groups will be written multiplicatively; their identity elements will be given the name "1". The symbols $G_1 * G_2$, $\ast_\alpha G_\alpha$ will denote the free product of groups, as defined, for example, in [**8**], v. 2, p. 11. The symbols $(G_1 * G_2)_A$, $(\ast_\alpha G_\alpha)_A$ will denote the free product with amalgamated subgroup A, as defined in [**8**], v. 2, p. 29. Recall that each element x in a free product $\ast_\alpha G_\alpha$ is described by a unique reduced word:

$$x = a_1 \cdots a_n, \quad a_i \neq 1, \quad a_i \in G_{\alpha_i}, \quad \alpha_i \neq \alpha_{i+1}.$$

And each element x of a free product with amalgamation $(\ast_\alpha G_\alpha)_A$ is either an element of A or else has an expression as a *reduced word*,

$$x = g_1 \cdots g_n, \quad g_i \in G_{\alpha_i} - A, \quad \alpha_i \neq \alpha_{i+1},$$

the number n being unique.

The terms *monomorphism*, *epimorphism*, and *isomorphism* will mean homomorphism which is, respectively, 1-1 into, onto, and 1-1 onto.

The *direct limit* of a system of groups and homomorphisms $\{G_\kappa, p_\lambda^\kappa\}$, indexed on a directed set $\{\kappa\}$, such that $p_\lambda^\kappa : G_\kappa \to G_\lambda$ is defined if $\kappa < \lambda$ and such that $p_\mu^\lambda p_\lambda^\kappa = p_\mu^\kappa$ if $\kappa < \lambda < \mu$, is a group G such that there exist homomorphisms $p^\kappa : G_\kappa \to G$ for all κ, and such that for $\kappa < \lambda$, $p^\lambda p_\lambda^\kappa = p^\kappa$, satisfying the two conditions:

(a) $G = \bigcup_\kappa p^\kappa(G_\kappa)$.

(b) If $p^\kappa(g) = 1$, there exists $\lambda > \kappa$ such that $p_\lambda^\kappa(g) = 1$.

For such a group G we shall use the notation $\varinjlim G_\kappa = G$. The direct limit exists and is unique up to isomorphism.

When G is a group and $\{x_1, \ldots, x_n\}$ is some subset of G, $\langle x_1, \ldots, x_n \rangle$ will denote the smallest normal subgroup of G containing $\{x_1, \ldots, x_n\}$. Hence $G/\langle x_1, \ldots, x_n \rangle$ is the group obtained from G by adding the relations $x_1 = 1, \ldots, x_n = 1$.

O.2. By a manifold will always be meant a finite simplicial complex, or the geometric realization of such a complex, satisfying the condition that the star of each vertex should be a cell of a certain fixed dimension. A subcomplex L of a simplicial complex K is said to be *complete*, if whenever a set of vertices of L span a simplex in K, they span a simplex in L; K' denotes the barycentric subdivision of K; if L is an arbitrary subcomplex of K, then L' is a complete subcomplex of K'. For results on the topology of manifolds, in particular, on the topology of 2-dimensional manifolds, refer to [**15**].

If M is a manifold, $\operatorname{Bd} M$ will denote its boundary and $\operatorname{Int} M = M - \operatorname{Bd} M$. The *double* of a manifold M is the union of $M \times 0$ and $M \times 1$ with each pair of points $x \times 0$, $x \times 1$ identified, for $x \in \operatorname{Bd} M$.

O.3. A complex will usually not be a simplicial complex, but a member of the more general class of CW-complexes, which are studied in [**18**]. If K is a complex, K^n will denote the n-skeleton of K. If $\{K_\alpha\}$ is a set of complexes, with base points p_α, the wedge of $\{K_\alpha\}$, denoted $\bigvee_\alpha K_\alpha$, will be the disjoint union of $\{K_\alpha\}$ with the points $\{p_\alpha\}$ identified.

From time to time, application will be made of Van Kampen's Theorem, a proof of which can be found in [**16**]; a proof in a more modern vein can be found in Olum [**11**].

If G is any group, an aspherical complex $K(G)$ can be constructed, for which $\pi_1(K(G)) \approx G$. If A is another group and $\varphi : A \to G$ is a homomorphism, a cellular map $f : K(A) \to K(G)$ can be found such that the diagram:

$$\begin{array}{ccc} \pi_1(K(A)) & \approx & A \\ \downarrow f_* & & \downarrow \varphi \\ \pi_1(K(G)) & \approx & G \end{array}$$

is commutative. We shall say that f induces the homomorphism φ. By replacing $K(G)$ by the mapping cylinder of f, and $K(A)$ by a neighborhood of $K(A)$ in this mapping cylinder, it is possible to find a complex K and open subset U so that the inclusion $U \subset K$ induces φ. Suppose that $\varphi_\alpha : A \to G_\alpha$ are monomorphisms, and

that $f_\alpha : K(A) \to K(G_\alpha)$ induces φ_α. The union of these mapping cylinders of f_α, identified along $K(A)$ is a complex Q whose fundamental group, by Van Kampen's Theorem, is isomorphic to $(\ast_\alpha G_\alpha)_A$; if U is a neighborhood of $K(A)$, the inclusion $U \subset Q$ induces the inclusion $A \subset (\ast_\alpha G_\alpha)_A$.

The symbol " ∎" will signify the termination of a proof.

O.4. We shall need for reference the following corollary to a Lemma of Pontrjagin [**14**], Lemma 2. I denotes the simplicial complex consisting of two vertices and one 1-simplex; $\frac{1}{2}$ denotes its midpoint.

O.41. Lemma: *If K is a simplicial complex and $f : K \to I$ is a simplicial map, then $f^{-1}(\frac{1}{2}) = L$ is a subcomplex of some subdivision of K; $f^{-1}(\operatorname{Int} I)$ is the Cartesian product of $f^{-1}(\frac{1}{2})$ with $\operatorname{Int} I$. If K is an n-manifold, then L is an $(n-1)$-manifold and $\operatorname{Bd} L \subset \operatorname{Bd} K$.*

O.42. An $(n-1)$-submanifold L of the n-manifold K is said to be *two-sided*, if there is a neighborhood N of L in K such that $N = L \times I$, in such a way that L corresponds to $L \times \frac{1}{2}$. (This is apparently a stronger condition than is required in the definition of "zweiseitig" of Seifert-Threlfall [**15**], pp. 273–274.)

O.43. Lemma: *Let K be an n-manifold, let P be a subcomplex of K, and let V be the open star of P in K. Then there exists an $(n-1)$-manifold L embedded in a two-sided manner in K, such that $\operatorname{Bd} L \subset \operatorname{Bd} K$, $L \subset V$, and $K - L = A \cup B$, where A and B are disjoint open sets, and $P \subset A \subset V$.*

Proof: Let C denote the union of those vertices of K' contained in P', and let D denote the union of those vertices of K' not contained in P'. There is a simplicial map $f : K' \to I$, defined by $f(C) = 0$, $f(D) = 1$. Let $L = f^{-1}(\frac{1}{2})$, $A = f^{-1}([0, \frac{1}{2}))$, $B = f^{-1}((\frac{1}{2}, 1])$. The verification that L, A, and B now satisfy the conclusions of the lemma is trivial. ∎

O.44. Lemma: *Let X be a Hausdorff space, X_α and Y open subsets of X such*

that

$$X = \bigcup_\alpha X_\alpha, \quad X_\alpha \cap X_\beta = Y \text{ for } \alpha \neq \beta,$$

and such that each set $X_\alpha - Y$ *is closed. Let* $f: M \to X$ *be continuous, where* M *is an* n*-dimensional manifold. Then there exists a two-sided* $(n-1)$*-manifold* L *in* M*, such that* $\operatorname{Bd} L \subset \operatorname{Bd} M$*, such that* $M - L = \bigcup_\alpha A_\alpha$*, where the* A_α *are pairwise disjoint open sets such that* $f(A_\alpha) \subset X_\alpha$*. If each component* Q *of* $\operatorname{Bd} M$ *is such that there is an* α *such that* $f(Q) \subset X_\alpha$*,* L *can be chosen without boundary.*

Proof: We note that all but a finite number of the sets $f^{-1}(X_\alpha - Y)$ are empty, since M is compact. Let ρ be a metric on M. There is a positive ε such that, for each α,

$$\rho[f^{-1}(X_\alpha - Y), M - f^{-1}(X_\alpha)] > \varepsilon \text{ and such that } \rho(Q, M - f^{-1}(X_\alpha)) > \varepsilon,$$

where Q is any component of $\operatorname{Bd} M$ such that $f(Q) \subset X_\alpha$. Triangulate M with mesh less than $\varepsilon/4$. Let $\operatorname{St}(V)$ for $V \subset M$ be the union of all those open simplexes whose closures meet V. Let P denote the closure of $\operatorname{St}(f^{-1}(X - Y) \cup Q^*)$, where Q^* is the union of all boundary components Q of M such that there exists α with $f(Q) \subset X_\alpha$. By Lemma O.43, there is a two-sided $(n-1)$-manifold L embedded in M such that $M - L = A \cup B$, where A and B are disjoint open sets, and $f^{-1}(X - Y) \subset P \subset A \subset \operatorname{St} P$; hence $B \subset f^{-1}(Y)$. It follows from the choice of ε that each component of $\operatorname{St} P$ is contained in some $f^{-1}(X_\alpha)$. Hence it is possible to write $A = \bigcup_\alpha \widetilde{A}_\alpha$, where the $\widetilde{A}_\alpha$ are disjoint open sets and $f(\widetilde{A}_\alpha) \subset X_\alpha$. Define $A_1 = B \cup \widetilde{A}_1$, $A_\alpha = \widetilde{A}_\alpha$ for $\alpha \neq 1$. It is clear that L and A_α satisfy the lemma. ∎

O.5. Limits of complexes. We return our attention to CW-complexes.

Let $\{K_\alpha\}$ be a collection of complexes indexed on a directed set $\{\alpha, <\}$, such that for $\alpha < \beta$, K_α is a closed subcomplex of K_β. Then on the set $\bigcup_\alpha K_\alpha$ a topology is defined by requiring that a subset C be closed if and only if $C \cap K_\alpha$ is closed for each α. Clearly $\bigcup_\alpha K_\alpha$ is again a CW-complex.

O.51. Lemma: *Choose any β and any $x \in K_\beta$. Then for each positive integer i, $\pi_i(\bigcup_\alpha K_\alpha, x) = \varinjlim_{\alpha>\beta} \pi_i(K_\alpha, x)$ where $\varinjlim$ denotes the direct limit of the groups $\pi_i(K_\alpha, x)$ and the homomorphisms $\pi_i(K_\alpha, x) \to \pi_i(K_\gamma, x)$ induced by the inclusion $K_\alpha \subset K_\gamma$.*

Proof: (a) Each element of $\pi_i(\bigcup_\alpha K_\alpha, x)$ is the image of some element of $\pi_i(K_\alpha, x)$ for some α. An element of $\pi_i(\bigcup_\alpha K_\alpha, x)$ is represented by a map $f: S^i \to \bigcup_\alpha K_\alpha$, where S^i is the i-sphere; $f(S^i)$, being compact, is contained in a finite subcomplex of $\bigcup_\alpha K_\alpha$, hence in some K_α.

(b) Let $j: \pi_i(K_\alpha, x) \to \pi_i(\bigcup_\alpha K_\alpha, x)$ be induced by inclusion; if $\mu \in \pi_i(K_\alpha, x)$, and $j\mu$ is the trivial element of $\pi_i(\bigcup_\alpha K_\alpha, x)$, then there is a map $g: D^{i+1} \to \bigcup_\alpha K_\alpha$ such that $g|S^i$ represents μ, where D^{i+1} is the $(i+1)$-cell. $g(D^{i+1})$ being compact lies in some K_γ, and if $h: \pi_i(K_\alpha, x) \to \pi_i(K_\gamma, x)$ is induced by inclusion, then $h\mu$ is trivial. ∎

O.52. Corollary: *If each component of each K_α is contractible, then each component of $\bigcup_\alpha K_\alpha$ is contractible.*

For by Lemma 0.51, if Q is a component of $\bigcup_\alpha K_\alpha$, $\pi_i(Q) = 0$ for all i. Hence Q is contractible.

O.53. Lemma: *If $\{K_\alpha\}$, $\{L_\alpha\}$ are two such directed systems of cell complexes and $f_\alpha: K_\alpha \to L_\alpha$ is continuous, and if for $\alpha < \beta$, $f_\alpha = f_\beta|K_\alpha$, then the unique extension of the maps f_α to $f: \bigcup_\alpha K_\alpha \to \bigcup_\alpha L_\alpha$ is continuous.*

Proof: f is continuous on the closure of each cell. Hence by [**18**], f is continuous. ∎

O.54. Addendum: *If f_α is an embedding of K_α as a closed subcomplex of L_α, then f is an embedding of $\bigcup_\alpha K_\alpha$ as a closed subcomplex of $\bigcup_\alpha L_\alpha$.*

This is obvious.

I. Commutators in free products with amalgamation

I.1. Let G be a group. For an element $x \in G$, the *genus* of x, $\mathfrak{G}_G(x)$, will be the least positive integer n such that there exist $2n$ elements of G, $u_1, v_1, u_2, v_2, \ldots, u_n, v_n$, such that (where $[u,v] = uvu^{-1}v^{-1}$):

$$x = [u_1, v_1][u_2, v_2]\cdots[u_n, v_n].$$

By convention, define $\mathfrak{G}_G(1) = 0$; for x not in the commutator subgroup of G, define $\mathfrak{G}_G(x) = \infty$. It is clear that

$$\mathfrak{G}_G(xy) \leq \mathfrak{G}_G(x) + \mathfrak{G}_G(y).$$

$$\mathfrak{G}_G(x^{-1}) = \mathfrak{G}_G(x).$$

$$\mathfrak{G}_G(yxy^{-1}) = \mathfrak{G}_G(x).$$

If $A \subset G$, then $\mathfrak{G}_G(x) \leq \mathfrak{G}_A(x)$.

$\mathfrak{G}_G(x) < \infty$ if and only if x is in the commutator subgroup of G.

A topological interpretation of the genus of an element of G is as follows. Let K be a complex whose fundamental group is isomorphic to G; let $x \in \pi_1(K)$; then $\mathfrak{G}_{\pi_1(K)}(x) \leq n$ if and only if there is a map $\varphi: T_n \to K$, (T_n = orientable, compact surface of genus n with one boundary component), such that $\varphi|\mathrm{Bd}\, T_n$ is a representative of x.

We use this interpretation to prove a theorem of Griffiths [**3**].

I.11. Theorem: *Let $H = G_1 * G_2 * \cdots * G_k$. Let $x_1 \in G_1$, $x_2 \in G_2$, ..., $x_k \in G_k$, have genera γ_1, γ_2, ..., γ_k, respectively: $\gamma_i = \mathfrak{G}_{G_i}(x_i)$. Let $y = x_1 x_2 \cdots x_k \in H$. Then $\mathfrak{G}_H(y) = \gamma_1 + \gamma_2 + \cdots + \gamma_k$.*

Proof: We have

$$\mathfrak{G}_H(y) \leq \mathfrak{G}_H(x_1) + \cdots + \mathfrak{G}_H(x_k) \leq \mathfrak{G}_{G_1}(x_1) + \cdots + \mathfrak{G}_{G_k}(x_k) = \gamma_1 + \cdots + \gamma_k.$$

The opposite inequality must therefore be proved.

Let $K_1, \ldots, K_k$ be complexes with base points $p_1, \ldots, p_k$ such that

$$\pi_1(K_1) \approx G_1, \quad \ldots, \quad \pi_1(K_k) \approx G_k.$$

Let L be the space obtained from the disjoint union of $K_1, \ldots, K_k$ by identifying the points $p_1, \ldots, p_k$ to the point p; i.e., $L = \bigvee_i K_i$. we may suppose that p has a contractible neighborhood $N \subset L$. Then $\pi_1(L) \approx G_1 * \cdots * G_k$ according to Van Kampen's Theorem.

Define $\widehat{K}_i = K \cup N$ and suppose that $\mathfrak{G}_H(y) = n$. Then, if T is the compact, orientable surface of genus n with k boundary components, $J_1, \ldots, J_k$, there is a map $\varphi: T \to L$ such that $\varphi | J_i$ represents x_i, and $\varphi(J_i) \subset \widehat{K}_i$.

One can find, by Lemma O.44, a finite number of disjoint simple closed curves $C_1, \ldots, C_q$ in $T - \operatorname{Bd} T$, such that for each j, $\varphi(C_j) \subset N$; and such that for each component Q_ℓ of $T - \bigcup_{j=1}^{q} C_j$, $\varphi(\overline{Q}_\ell)$ is contained in some set $\widehat{K}_i$ depending on Q_ℓ. In particular, let Q_i be the component of $T - \bigcup_{j=1}^{q} C_j$ containing J_i; then $\varphi(\overline{Q}_i) \subset \widehat{K}_i$.

Now, by the elementary topology of surfaces, we have

$$\sum_{i=1}^{k} \operatorname{Genus}(\overline{Q}_i) \leq \sum_{\ell} \operatorname{Genus}(Q_\ell) \leq \operatorname{Genus}(T) = n.$$

But on the other hand

$$\gamma_i = \mathfrak{G}_{G_i}(x_i) \leq \operatorname{Genus}(\overline{Q}_i)$$

for the following reason.

If $\widetilde{Q}_i = \overline{Q}_i \cup$ (Disks attached to each boundary component of $\overline{Q}_i$ except J_i), then there is an extension $\widetilde{\varphi}: \widetilde{Q}_i \to \widehat{K}_i$ of $\varphi | \overline{Q}_i$, since all the boundary components of

$\overline{Q}_i$ except J_i are mapped by φ into the contractible set N; Genus $(\widetilde{Q}_i)$ = Genus $(\overline{Q}_i)$. Hence $\mathfrak{G}_{\pi_1(\widehat{K}_i)}(x_i) \leq$ Genus $(\overline{Q}_i)$; and obviously

$$\mathfrak{G}_{G_i}(x_i) = \mathfrak{G}_{\pi_1(K_i)}(x_i) = \mathfrak{G}_{\pi_1(\widehat{K}_i)}(x_i).$$

Thus, $\gamma_1 + \cdots + \gamma_k \leq n$, which was to be proved. ∎

I.2. Now let G be a group and A a subgroup of G. For $x \in G$, we define the *(Euler) characteristic of x in G with respect to A*, as the maximum of the integers $1-2n-k$, for which there exist $u_1, v_1, u_2, v_2, \ldots, u_n, v_n$; $w_1, \ldots, w_k \in G$ and $a_1, \ldots, a_k \in A$ such that

$$x = [u_1, v_1] \cdots [u_n, v_n] w_1 a w_1^{-1} \cdots w_k a_k w_k^{-1}.$$

We denote this number by $\chi(x; G, A)$.

Let $[G, G]$ denote the commutator subgroup of G, and $\langle A \rangle$ denote the smallest normal subgroup of G containing A; then for $x \notin [G, G]\langle A \rangle$, define $\chi(x) = -\infty$. Clearly:

$$\chi(x) \leq 1.$$

$$\chi(x) = 1 \text{ if and only if } x = 1.$$

$$\chi(x) > -\infty \text{ if and only if } x \in [G, G]\langle A \rangle.$$

$$\chi(xy) \geq \chi(x) + \chi(y) - 1.$$

$$\text{If } G \subset H, \quad \chi(x; G, A) \geq \chi(x; H, A).$$

$$\text{If } A \subset B, \quad \chi(x; G, A) \leq \chi(x; G, B).$$

The characteristic has the following topological interpretation:

Let K be a complex with fundamental group $\approx G$, $L \subset K$ a subset with fundamental group A, such that the inclusion $L \subset K$ induces the inclusion $A \subset G$. Let $x \in G$ be represented by a loop γ in K; then $\chi(x; G, A) \geq n$ if and only if there is a map of an orientable surface T_n of Euler characteristic n into K, $f: T_n \to K$, such that α is a boundary component of T_n, $f|\alpha = \gamma$ and $f(\mathrm{Bd}\, T_n - \alpha) \subset L$.

We now obtain a theorem similar to that of Griffiths.

I.21. Theorem: *Let* $H = (G_1 * \cdots * G_p)_A$ *and suppose* $x_i \in G_i$ *has characteristic* $\chi(x_i; G_i, A) = \gamma_i$. *Let* $y = x_1 x_2 \cdots x_p$. *Then*

$$\chi(y; H, A) \leq \left(\sum_{i=1}^{p} \gamma_i \right) + p - 1.$$

Remark: This theorem does not give the exact value of $\chi(y; H, A)$, but only bounds:

$$\left(\sum_{i=1}^{p} \gamma_i \right) - p + 1 \leq \chi(y; H, A) \leq \left(\sum_{i=1}^{p} \gamma_i \right) + p - 1.$$

Proof: We suppose $G_i \approx \pi_1(K_i)$ and that L is a connected open subset of each K_i for which $\pi_1(L) \approx A$ and that the inclusion $L \subset K$ induces the inclusion $A \subset G$. Let M be the union of the K_i with the copies of L identified; by Van Kampen's Theorem, $\pi_1(M) \approx H$. We suppose as before that $\chi(y; H, A) = n$. Then there is a map $f : T_n \to M$ (where T_n = orientable surface of characteristic n with boundary curves α, β_1, ..., β_k) such that $f|\alpha$ represents y, and $f(\beta_i) \subset L$. Now since $f|\alpha$ represents $y = x_1 x_2 \cdots x_p$, it is possible to add a disk with p holes to T_n (the outer boundary of the disk with holes maps to α and the inner p boundary components, $\{\alpha_j\}$, map to curves representing the x_j), obtaining T with $\chi(T) = n - p + 1$, $\operatorname{Bd} T = \alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_k$; and to extend f to $\varphi : T \to M$ such that $\varphi|\alpha_j$ represents x_j and $\varphi(\beta_i) \subset L$ and $\varphi(\alpha_j) \subset K_j$.

Now as before, we find curves $J_1, \ldots, J_q$ on $T - \operatorname{Bd} T$ such that

$$J_1 \cup \cdots \cup J_q \subset \varphi^{-1}(L),$$

and each component Q_ℓ of $T - (J_1 \cup \cdots \cup J_q)$ is contained in some $\varphi^{-1}(K_i)$. In particular, Q_j = component containing α_j, $Q_j \subset \varphi^{-1}(K_j)$. We may also assume that no Q_ℓ is a disk; for if Q_ℓ is a disk we can redefine φ on Q_ℓ so that $\varphi(Q_\ell) \subset L$, $\pi_1(L) \to \pi_1(K_i)$ being a monomorphism; and then we can delete a curve $J_k = \operatorname{Bd} Q_\ell$ from the set of J curves; continuing in this way, we reduce the number of J curves until for each ℓ, we have $\chi(Q_\ell) \leq 0$.

By the elementary topology of surfaces, we know:

$$\sum_{i=1}^{p} \gamma_i \geq \sum_{j=1}^{p} \chi(Q_j) \geq \sum \chi(Q_\ell) = \chi(T) = n - p + 1.$$

Thus $n \leq (\sum_{i=1}^{p} \gamma_i) + p - 1$, which was to be proved. ∎

I.3. The next theorem, For $A = 1$, $n = 1$, was due to Kuroš [**7**]. It was proved again by Levi [**9**], who utilized in his proof a weak form of Gruško's Theorem.

I.31. Theorem: *Let* $H = (\ast_\alpha G_\alpha)_A$. *Let* $u_1, \ldots, u_n, v_1, \ldots, v_n$ *be elements of* H *such that*

$$h = [u_1, v_1] \cdots [u_n, v_n] \in G_1,$$

and such that h *cannot be expressed in the form*

$$h = [x_1, y_1] \cdots [x_q, y_q] z_1 a_1 z_1^{-1} \cdots z_r a_r z_r^{-1}$$

for $x_i, y_i, z_i \in H$, $a_i \in A$, $q < n$, $r \leq 2(n-q)$. *Then* u_i *and* $v_i \in G_1$ *for* $i = 1, \ldots, n$.

Proof: Let T be an orientable 2-manifold of genus n with one boundary curve γ, and let $K(H)$ be a geometric realization of H in the sense of the preceding theorem. Then there is a map $f: T \to K(H)$ taking γ into a loop in $K(G_1)$ which represents h, and taking a canonical set of curves into representatives of the elements $u_1, v_1, \ldots, u_n, v_n$. In the neighborhood of $f^{-1}(K(A))$ are curves $c_1, \ldots, c_k$ such that for each component Q_ℓ of $T - \bigcup_{i=1}^{k} C_i$ there is an α such that $f(Q_\ell) \subset K(G_\alpha)$. As above, we can simplify f so that for each ℓ, $\chi(Q_\ell) \leq 0$. Let Q be the component containing γ. $\operatorname{Genus} Q = q \leq \operatorname{Genus} T = n$. $\chi(Q) \geq \chi(T) = 1 - 2n$, since $\sum_\ell \chi(Q_\ell) = \chi(T)$, and $\chi(Q_\ell) \leq 0$ for all ℓ. Since $\chi(Q) = 2 - 2q -$ (number of boundary curves of Q), we have (number of boundary curves of Q) $\leq 2(n-q) + 1$; that is, (number of boundary curves of Q, not counting γ) $\leq 2(n-q)$. From the hypothesis on h, which $f|\gamma$ represents, it is seen that $n = q$, and hence $\gamma = \operatorname{Bd} Q$; hence $Q = T$. Therefore $h(T) \subset K(G_1)$, and thus the elements $u_1, v_1, \ldots, u_n, v_n$ which are represented by loops in $f(T)$, are in G_1. ∎

II. Four variations on a theorem of I. Gruško

II.A. 3-manifolds and Kneser's Conjecture

Before presenting a proof of the complete form of Gruško's Theorem (concerning infinitely generated free products), a proof of the finite version will be given, which involves 3-dimensional manifolds. A proof of the so-called Kneser's Conjecture will be obtained; this result is in a certain sense a very strong, topological form of Gruško's Theorem, and uses much more delicate topological results than are needed for the algebraic theorem.

Many of the results of this Section are suggested in Kneser [**6**].

II.A.1. Let M be a closed 3-manifold. Let $K \vee L$ be a wedge of two complexes at a point p; let N be a contractible neighborhood of p. Note that if C is an n-cell and $\alpha : C \to K \vee L$ is a map sending the boundary of C into N, then α represents an element of the n-th homotopy group of $K \vee L$ with base point p.

Let $f : M \to K \vee L$ be a continuous function, preserving base points. According to Lemma O.44, there is a two-sided 2-manifold T contained in M, such that $f(T) \subset N$ and T contains the base point of M, and such that if the components of $M - T$ are $M_1, \ldots, M_n$, then for each i, either $f(M_i) \subset K \cup N$ or $f(M_i) \subset L \cup N$. Such a manifold T will be called *sane* with respect to f.

II.A.11. Theorem: *Suppose that M is a closed 3-manifold, $K \vee L$ is a wedge of complexes at the point p, N a contractible neighborhood of p. Let $f : M \to K \vee L$ be a map and let $T \subset M$ be a sane manifold with respect to the map f. Suppose furthermore that each component of T is a 2-sphere; and that the homomorphism $f_* : \pi_1(M) \to \pi_1(K \vee L)$ is onto. Then there is a map $g : M \to K \vee L$, homotopic mod base point to f, and a sane manifold $S \subset M$ with respect to g, such that S consists of only one 2-sphere.*

Proof: Let a *tie* be a path P in M, i.e., a map $[0,1] \to M$, such that $P(0) \in T_i$ and $P(1) \in T_j$, where T_i and T_j are distinct components of T and the image of P

intersects T only at these two points. fP then represents an element of $\pi_1(K \vee L)$, which lies in either $\pi_1(K)$ or $\pi_1(L)$; this element will be called f_*P. Since M is a 3-manifold, every tie P is homotopic mod end points to a nonsingular tie, that is, a tie whose image is a simple, semilinear arc; also, every tie is homotopic to one whose image does not contain the base point of M, such that the homotopy moves the ends of the tie along the manifold T.

When the tie P is such that $f_*P = 1$, P is said to be a *binding* tie. The significance of binding ties is that they enable one to reduce the number of components of T.

Consider the set of all maps $g: M \to K \vee L$ which are homotopic to f mod basepoint, for which there exist sane 2-manifolds whose components are all spheres. There can be found such a map g and such a 2-manifold T that the number of components of T is minimal. It must be shown that T has only one component.

(a). *In this situation there are no binding ties*. For let P be a binding tie; for concreteness, suppose fP lies in $K \cup N$. According to the remarks above, P can be supposed to be non-singular and disjoint from the base point of M. Since P is a binding tie, there is a homotopy f_t of $f|P$, such that $f_0 = f|P$, $f_tP \subset K \cup N$, and $f_1P \subset N$. By the Homotopy Extension Theorem, f_t can be extended to a homotopy of f, such that $f_t(x) = f(x)$, for x not in a small neighborhood of P; in particular f_t preserves base point. f_t is of such a nature that T is sane with respect to F_t for all t. Now let U be a tubular neighborhood of P such that $f_1(U) \subset N$. $\operatorname{Bd} U - T$ consists of two 2-cells and an annulus A. Let $T' = (T - U) \cup A$. It follows that T' is sane with respect to f_1, and that the number of components of T' is one less than the number of components of T. This proves assertion (a).

(b) *If the number of components of T is more than one, then there exists a binding tie*. There certainly exists a tie P such that $P(0) =$ base point of M. If $f_*P = 1$, the assertion follows. Suppose $f_*P \neq 1$. Since $f_*: \pi_1(M) \to \pi_1(K \vee L)$ is onto, there is a loop based at the base point of M, call it L, such that $f_*L = f_*P$. L can be smoothed out so that it becomes the product of ties and loops which

intersect T only at their end points; thus $L = R_1 R_2 \cdots R_n$; the number n will be called the length of L. We suppose that the length of L is minimal compatible with the condition that $f_* L = f_* P$. If $f_* R_i = 1$, either R_i is a tie, in which case assertion (b) follows, or R_i is a loop; but if R_i is a loop, then $L^* = R_1 \cdots R_{i-1} R_{i+1} \cdots R_n$, is a loop of less length satisfying $f_*(L^*) = f_*(P)$. Hence for no i is $f_* R_i = 1$. Furthermore, the group elements $f_* R_i$ and $f_* R_{i+1}$ must lie in distinct free factors $\pi_1(K)$ and $\pi_1(L)$, otherwise it would follow that $R_i R_{i+1}$ is itself deformable into a loop or tie, whose endpoints alone intersect T. And this too would diminish the length of L. Thus, the expression $f_* L = f_* R_1 \, f_* R_2 \cdots f_* R_n$, is a reduced word in the free product $\pi_1(K \vee L)$; since $f_* L$ has length 1, it follows that $n = 1$. $L^{-1} P$ can now be deformed into a binding tie. This proves assertion (b).

The theorem follows immediately. ∎

We now prove

II.A.12. Gruško's Theorem for finitely generated groups: *If F is a finitely generated free group, and $\varphi : F \to A * B$ is a homomorphism onto the free product of A and B, then there exists a free factorization, $F = F_A * F_B$, such that $\varphi(F_A) \subset A$ and $\varphi(F_B) \subset B$.*

Proof: Let F have rank n. Let G be the wedge of n circles, embedded in the 3-sphere; and let M be the double of a tubular neighborhood of G. M is called the 3-sphere with n handles. Let K and L be complexes with fundamental groups isomorphic to A and B respectively. F may be identified with the fundamental group of G and with the fundamental group of M. There is then a continuous map $f : G \to K \vee L$ which induces φ. There is an extension (also called "f") of f to M, since there is a retraction r of M onto G; this retraction r may be constructed so that the inverse image of each point of G is simply connected. It is now possible to find a sane 2-manifold with respect to f, which consists entirely of 2-spheres. According to the preceding theorem, there is a map $g : M \to K \vee L$, homotopic to f mod base point, and a connected sane 2-sphere S with respect to g. Now S must separate M

into two parts, M_A and M_B, such that $g(M_A) \subset K \cup N$ and $g(M_B) \subset L \cup N$. By Van Kampen's Theorem, $F = \pi_1(M) = \pi_1(M_A) * \pi_1(M_B)$. Since $g_* = f_* = \varphi$, and by the above remarks, we have $\varphi(\pi_1(M_A)) \subset A$ and $\varphi(\pi_1(M_B)) \subset B$. The theorem is now proved, letting $F_A = \pi_1(M_A)$, $F_B = \pi_1(M_B)$. ∎

II.A.2. Kneser's Conjecture

In [**6**], H. Kneser announced a theorem to the effect that any factorization of the fundamental group of a 3-manifold could be realized by a factorization of the 3-manifold into the sum of manifolds. His proof was not altogether convincing. Recently, the work of C. D. Papakyriakopoulos [**12**], [**13**] has led the way to the solution of many of the topological problems involving 3-manifolds. J. H. C. Whitehead [**19**] has given a partial proof of Kneser's Conjecture.

Here is described a proof of a strong form of Kneser's Conjecture.

II.A.21. Lemma: *Let X be a space, X_1, X_2 open subsets, $X = X_1 \cup X_2$. Let $A = X_1 \cap X_2$. Suppose that for each component A_i of A the homomorphisms $\pi_1(A_i, a_i) \to \pi_1(X_1, a_i)$, $\pi_1(A_i, a_i) \to \pi_1(X_2, a_i)$ have kernel 1. Then the homomorphisms $\pi_1(A_i) \to \pi_1(X)$ have kernel 1.*

Proof: Let ξ be a loop in A_i which is contractible in X. There is a map $f: D \to X$, where D is a 2-cell, $J = \operatorname{Bd} D$, such that $f|J$ is the loop ξ. So $f(J) \subset A_i$. Now by Lemma O.44, there are curves $C_i \subset D - J$ such that each component Q_j of $D - C$ lies in either $f^{-1}(X_1)$ or $f^{-1}(X_2)$. Suppose that f has been chosen so that $f|J = \xi$ and so that the number n of curves C_i is minimal. Then $n = 0$; for suppose $n > 0$; let C_0 be an innermost curve on D, so that C_0 bounds a disk Δ in D and $f(\Delta) \subset X_1$ or $f(\Delta) \subset X_2$; suppose, e.g., $f(\Delta) \subset X_1$. This means that $f|C_0$ represents an element of $\pi_1(A_i)$ which injects into 1 in $\pi_1(X_1)$. By hypothesis, $f|C_0$ represents 1 in $\pi_1(A_i)$. Hence $f|\Delta$ can be changed so as to map Δ into A_i and the curve C_j can be removed.

Thus $n = 0$ and $f(D) \subset X_1$ or $f(D) \subset X_2$. By a repetition of the above argument, $f|J$ represents 1 in $\pi_1(A_i)$. ∎

Let M be a manifold. If M is orientable, define $\mathfrak{O}(M) = \pi_1(M)$. If M is nonorientable, then there is an orientable 2-sheeted covering M' which corresponds to a subgroup of index 2 in $\pi_1(M)$; this subgroup will be called $\mathfrak{O}(M)$ in this case. Intuitively, $\mathfrak{O}(M)$ is the subgroup of $\pi_1(M)$, each element of which is represented by an orientation-preserving loop.

II.A.22. Theorem: *Suppose M is a closed 3-manifold. Let $\varphi\colon \pi_1(M) \to A*B$ be an epimorphism onto a free product. Further let $\operatorname{Ker}\varphi \subset \mathfrak{O}(M)$; and let it be true that no subgroup of $\operatorname{Ker}\varphi$ is isomorphic to the fundamental group of a closed orientable 2-manifold of positive genus. Then there exists in M a 2-sphere S which contains the base point m_0 of M and which splits M into two pieces M_1 and M_2, so that the following commutative diagram exists (i.e., the two vertical outside homomorphisms exist).*

$$\begin{array}{ccccc} \pi_1(M_1) & \to & \pi_1(M) & \leftarrow & \pi_1(M_2) \\ \downarrow & & \downarrow\varphi & & \downarrow \\ A & \to & A*B & \leftarrow & B \end{array}$$

Proof: (1) construct complexes K and L such that $\pi_1(K) \approx A$, $\pi_1(L) \approx B$, and $\pi_2(K) = \pi_2(L) = 0$. Then let $K \vee L$ be the wedge of K and L, and let N be a contractible neighborhood of the joining basepoint of $K \vee L$.

(2) Now construct a map $f\colon M \to K \vee L$ which "induces" the homomorphism φ. I.e., so that the diagram

$$\begin{array}{ccc} & & \pi_1(M) \\ & \swarrow f_* & \downarrow\varphi \\ \pi_1(K \vee L) & \approx & A*B \end{array}$$

is commutative. This may be done as follows: Triangulate M. Map a maximal tree of M^1 into the base point of $K \vee L$. The remaining 1-simplexes on M define generators of $\pi_1(M)$; say σ defines $g \in \pi_1(M)$. $\varphi(g) \in A*B$ is represented by a loop in $K \vee L$; map σ along this loop. Each 2-simplex δ of M^2 now gives a relation in $\pi_1(M)$; from the fact φ is a homomorphism we can extend $f|M^1$ to Δ. Each 3-simplex e of M^3, gives under $f|\operatorname{Bd} e$ an element of $\pi_2(K \vee L) = 0$. Hence $f|M^2$ can be extended to $M^3 = M$.

(3) Now, according to Lemma O.44, there is a 2-manifold T in M (perhaps not connected), such that:

(i) $f(T) \subset N$.

(ii) $M - T = Q_1 \cup Q_2$, disjoint open sets, and $f(Q_1) \subset K \cup N$, $f(Q_2) \subset L \cup N$.

(iii) A neighborhood of T has the form $T \times I$.

From condition (i) and the hypothesis that $\operatorname{Ker}\varphi \subset \mathfrak{O}(M)$, it follows that in the covering space $\widetilde{M}$ of M corresponding to $\operatorname{Ker}\varphi$, lying above T are a number of disjoint copies of T; and $\widetilde{M}$ is orientable. From this and condition (iii) it follows that each component of T is orientable. Let the components of T be $T_0, T_1, \ldots, T_n$, and suppose $m_0 \in T_0$.

(4) We now simplify the picture so that each component of T is a sphere. We first go to the covering $\widetilde{M}$, $p\colon \widetilde{M} \to M$. If not each component of T is a sphere, then there are components of $p^{-1}(T)$, $\widetilde{T}_i$, for which the inclusion homomorphism $\pi_1(\widetilde{T}_i) \to \pi_1(\widetilde{M})$ is not a monomorphism (this follows from the hypothesis that no subgroup of $\operatorname{Ker}\varphi \approx \pi_1(\widetilde{M})$ is isomorphic to $\pi_1(\widetilde{T}_i)$).

Let $\widetilde{Q}_1 = p^{-1}(Q_1)$, $\widetilde{Q}_2 = p^{-1}(Q_2)$. From Lemma II.A.21, it follows that there is a component $\widetilde{T}_i$ of $p^{-1}(T)$, such that either $\pi_1(\widetilde{T}_i) \to \widetilde{Q}_1$ or $\pi_1(\widetilde{T}_i) \to \widetilde{Q}_2$ is not a monomorphism. By the Loop Theorem [**12**], there is a simple closed curve $C \subset \widetilde{T}_i$ which is not contractible on $\widetilde{T}_i$, which is contractible in $\widetilde{Q}_1$ or $\widetilde{Q}_2$, say $\widetilde{Q}_1$. Since $p|\widetilde{T}_i$ is a homeomorphism, on $T_i = p(\widetilde{T}_i)$ there is a simple closed curve C' not contractible on T_i but contractible in Q_1. Apply Dehn's Lemma [**13**] to this situation; there is a disk $D \subset \overline{Q}_1$ such that $D \cap T = \operatorname{Bd} D = C'$.

By the Homotopy Extension Theorem, there is a homotopy f_t, such that $f_0 = f$, f_t differs from f only on a small neighborhood of D, and $f_1(D) \subset N$ (this is possible since $f(D)$ represents an element of $\pi_2(K \vee L) = 0$). Then we can split T_i along D; that is, let U be a nice 3-cell neighborhood of D which intersects T_i in an annulus, and whose boundary minus T_i consists of an annulus and two 2-cells D_1 and D_2. Make the change

$$T_i \to T_i' = (T_i - U) \cup D_1 \cup D_2.$$

This change, since C' is not contractible on T_i, decreases the number

$$\sum_{i=0}^{n} (\text{Genus}\, T_i)^2;$$

and conditions (i), (ii), and (iii) of (3) are still satisfied. This shows that if we consider all maps $f: M \to K \vee L$ inducing the homomorphism φ, and all 2-manifolds $T \subset M$, satisfying the conditions of (3), and among these pick a map f and a manifold T such that the number $\sum_{i=0}^{n} (\text{Genus}\, T_i)^2$, where T_i are the components of T, is minimal, then this number must be 0, and hence each T_i must be a sphere.

(5) Now we apply Theorem II.A.11. The hypotheses of that theorem are now satisfied, since each component of T is a sphere, and since the homomorphism $f_*: \pi_1(M) \to \pi_1(K \vee L)$ is onto. Hence there exists a map f which induces φ, and a manifold $T \subset M$ satisfying the conditions (3), such that T consists of one 2-sphere.

The conclusions of this theorem are now clear. ∎

As a corollary we obtain Kneser's Conjecture:

II.A.3. Theorem: *Let M be a closed 3-manifold and let $\varphi: \pi_1(M) \approx A * B$. Then there is a 2-sphere S tamely embedded in M, containing the base point of M, such that S splits M into manifolds M_1, M_2 such that $\pi_1(M_1)$ corresponds to A, $\pi_1(M_2)$ corresponds to B under φ.*

For since φ is onto and $\text{Ker}\, \varphi = 1$, the hypotheses of Theorem II.A.22 are satisfied, and the conclusion follows.

II.B. Cell-complexes

Here we prove the complete version of Gruško's Theorem; that is, we do not restrict ourselves to finitely generated groups. The first proof of this form of the theorem was due to Wagner [**17**].

II.B.1. Theorem: *Let F be a free group, $\ast_\alpha G_\alpha$ a free product, $\varphi: F \to \ast_\alpha G_\alpha$ a homomorphism onto. Then there is a free decomposition of F, $\ast_\alpha F_\alpha = F$, such that $\varphi(F_\alpha) \subset G_\alpha$.*

Proof: We may make the following topological interpretation: F is the fundamental group of a complex L; G_α is the fundamental group of a complex K_α with base point p_α. The fundamental group of the complex $\bigvee_\alpha K_\alpha$, obtained from the disjoint union of the K_α by identifying the points p_α to one point p, is $\ast_\alpha G_\alpha$. φ is the homomorphism induced by a map $f: L \to \bigvee_\alpha K_\alpha$. Now there exist such topologiccal interpretations for which f is a cellular map, and for which $f^{-1}(p)$ is a subcomplex of L, *each of whose components is contractible*; such a topological interpretation will be called *tractable*. The existence of a tractable topological interpretation is seen as follows: We can take L to be a 1-dimensional complex; we can assume f to be a cellular map; and by subdividing L, we obtain a complex L' such that $f^{-1}(p)$ is a 0-dimensional subcomplex of L'.

We define ties, binding ties, etc., similar to the method of the preceding Section. Namely, a *tie* is a path $P: [0,1] \to L$ such that (1) there exists an α such that fP lies in K_α, and (2) $P(0)$ and $P(1)$ lie in different components of $f^{-1}(p)$. The *value*, f_*P, of P is the element of G_α which the loop fP represents. A *binding tie* is a tie P for which $f_*P = 1$.

Now suppose that P is a binding tie, and that fP lies in K_α. We can suppose that P is a cellular map. Suppose that the ends of P lie in C_1 and C_2, distinct components of $f^{-1}(p)$. Let Q denote the square $\{(x_1, x_2) \mid 0 \le x_1 \le 1,\ 0 \le x_2 \le 1\}$; identify $[0,1]$ with the points of Q for which $x_2 = 0$. Then since P is a binding tie, the fundamental group of K_α is not changed if a copy of Q is attached to K_α by

the map fP on $[0,1]$ and by mapping $\{(0,x_2),\ (x_1,1),\ (1,x_2)\} \to p$. The resulting space $K'_\alpha = K_\alpha \cup Q$ is a complex since the attaching map is cellular. Now attach another copy of Q, Q_1, to L by the map P on $[0,1]$, leaving the remainder of the boundary as a free 1-cell T'. $L' = L \cup Q$ is also a complex. And finally the map f can be extended to a map $f' : L \to K'_\alpha \vee \bigvee_{\beta \neq \alpha} K_\beta$ by mapping $Q_1 \to Q$ by the identity map. This is still a tractable topological interpretation of φ, for clearly f'_* induces the homomorphism φ, and each component of $(f')^{-1}(p)$ is either a component of $f^{-1}(p)$, hence contractible, or else the set $C_1 \cup C_2 \cup T'$, which is contractible.

We now construct a tractable topological interpretation, $g : L \to \bigvee_\alpha K_\alpha$, of φ for which there are no binding ties, by transfinite induction, as follows.

Let $f_0 : L_0 \to \bigvee_\alpha K_{0,\alpha}$ be a tractable topological interpretation of φ.

Let ρ be an ordinal number and suppose there have been defined tractable topological interpretations of φ, $f_\sigma : L_\sigma \to \bigvee_\alpha K_{\sigma,\alpha}$ for all $\sigma < \rho$, satisfying the conditions:

For $\sigma < \tau$, L_σ is a closed subcomplex of L_τ;

$K_{\sigma,\alpha}$ is a closed subcomplex of $K_{\tau,\alpha}$;

and $f_\sigma = f_\tau | L_\sigma$.

We define f_ρ as follows:

If ρ is a limit ordinal, let

$$L_\rho = \bigcup_{\sigma<\rho} L_\sigma, \quad K_{\rho,\alpha} = \bigcup_{\sigma<\rho} K_{\sigma,\alpha}.$$

$f_\rho : L_\rho \to \bigvee_\alpha K_{\rho,\alpha}$ is the unique extension of $\{f_\sigma\}$ to L_ρ. It follows from Lemmas O.51, 52, 53, 54, that f_ρ is a tractable topological interpretation of φ.

If ρ is not a limit ordinal, then $\rho = \chi + 1$. If in f_χ there exists a binding tie, we make the construction described above which binds together two components of $f_\chi^{-1}(p)$; we call this f_ρ. If in f_χ there is no binding tie, then f_χ is the tractable topological interpretation of φ for which we seek.

This process cannot continue indefinitely. In fact, let $\aleph$ be the number of

components of $f_0^{-1}(p)$. Then there is an ordinal χ, for which $\overline{\chi} \leq \aleph$ and for which in f_χ there is no binding tie. Let $g = f_\chi$.

We now use the hypothesis that φ is onto, to prove $g^{-1}(p)$ is connected. If $g^{-1}(p)$ is not connected, there is a tie P which begins at the base point of L. By the construction of g, $g_*(P) \neq 1$. Since φ is onto there is a loop Q in L such that $g_*Q = g_*P$. Write Q as a product of ties and loops R_i such that $g(R_i) \subset K_{\alpha_i}$,

$$Q = R_1 R_2 \cdots R_n.$$

The length of Q is the number n. Let Q be a loop of minimum length such that $g_*(Q) = g_*(P)$. If $g_*(R_i) = 1$, then R_i is not a tie; hence the loop

$$Q' = R_1 \cdots R_{i-1} R_{i+1} \cdots R_n$$

has smaller length than Q and $g_*(Q') = g_*(Q)$. Hence for no i is $g_*(R_i) = 1$. Similarly $g_*(R_i)$ and $g_*(R_{i+1})$ lie in distinct free factors G_{α_i} and $G_{\alpha_{i+1}}$. Thus $g_*(Q) = g_*(R_1) \cdots g_*(R_n)$ is a reduced word in $\ast_\alpha G_\alpha$; since $g_*(Q) = g_*(P)$ has length 1, $n = 1$. But then $Q^{-1}P$ is a tie with value $g_*(Q^{-1}P) = 1$. This contradiction shows that $g^{-1}(p)$ is connected.

Now $L = \bigcup_\alpha g^{-1}(K_\alpha)$, and $g^{-1}(K_\alpha) \cap g^{-1}(K_\beta) = g^{-1}(p)$ for $\alpha \neq \beta$. Since $g^{-1}(p)$ is contractible, it follows from Van Kampen's Theorem that

$$F = \pi_1(L) = \ast_\alpha \pi_1(g^{-1}(K_\alpha)).$$

Let $F_\alpha = \pi_1(g^{-1}(K_\alpha))$. Then $F = \ast_\alpha F_\alpha$, and $\varphi(F_\alpha) \subset G_\alpha$. ∎

II.C. Oidgroups

In trying to translate these proofs of Gruško's Theorem into purely algebraic and combinatorial arguments, it seemed natural to me to use the notion of Brandt groupoid (see [**2**], or [**5**], pp. 132–133, for the definition of groupoid). After exploring this method for a while, I found that it would be necessary to develop a theory for groupoids analogous to that of free products of groups. Rather than do this, I sought to utilize the already developed theory of free products.

I decided therefore to use a modification of the notion of groupoid, which I call "oidgroup".

Definition: An *oidgroup* $\Gamma = \{G, C, I, \delta\}$ is a group G, a sub*set* $C \subset G$, and a set I (disjoint from G), with a function $\delta : C \to I$, satisfying the following axioms:

1. C generates G.
2. $1 \notin C$.
3. $C = C^{-1}$.
4. For $a, b \in C$, and $ab \neq 1$, it is true that $ab \in C$ if and only if $\delta(a) = \delta(b^{-1})$.
5. For $\varepsilon, \eta \in I$, $\varepsilon \neq \eta$, there exists $a \in C$, such that $\varepsilon = \delta(a)$, $\eta = \delta(a^{-1})$.
6. For any sequence $a_1, \ldots, a_n \in C$, if $a_i a_{i+1} \notin C \cup \{1\}$, $1 \leq i \leq n-1$, then $a_1 a_2 \cdots a_n \neq 1$.

Axiom 5 will be called the *axiom of connectedness*. If Γ does not, or perhaps does not, satisfy axiom 5, Γ shall be called a *non-connected oidgroup*.

Example of an oidgroup: Let K be a connected complex and let Q be the groupoid consisting of homotopy classes (mod end points) of paths in K going between vertices. Let $K^* = K$ with all vertices identified to a point, and let $h : Q \to \pi_1(K^*)$ be the induced map. Define $G = \pi_1(K^*)$, $C = h(Q) - \{1\}$, $I = \{\text{vertices of } K\}$; let $\delta : C \to I$ be defined thus: If $a \in C$, let $q \in Q$ be such that $h(q) = a$; $\delta(a) =$ endpoint of q on the right. (Elements of Q have two end-

points, one on the left and one on the right.) It can be checked that $\Gamma = \{G, C, I, \delta\}$ is an oidgroup.

II.C.1. Proposition: *If* $\delta(a^{-1}) \neq \delta(b)$, $ab \neq 1$. *Also, if* $a, b, ab \in C$, *then* $\delta(ab) = \delta(b)$.

Proof: The first statement is equivalent to the tautology: If $a^{-1} = b$, then $\delta(a^{-1}) = \delta(b)$. To prove the second statement, we have $(ab)b^{-1} = a \in C$, hence by axiom 4, $\delta(ab) = \delta((b^{-1})^{-1}) = \delta(b)$. ∎

Definition: Let G be an oidgroup, and let $\eta \in I$. The *fundamental group of* Γ *at the point* η, denoted by $\pi(\Gamma, \eta)$, is the following subset of G:

$$\{1\} \cup \{a \in C \mid \delta(a) = \delta(a^{-1}) = \eta\}.$$

II.C.2. Proposition: $\pi(\Gamma, \eta)$ *is a group and* $\pi(\Gamma, \eta) \approx \pi(\Gamma, \varepsilon)$, *where the isomorphism is given by an inner automorphism of* G.

Proof: That $\pi(\Gamma, \eta)$ is a group follows from axiom 3 and Proposition II.C.1. The isomorphism $\pi(\Gamma, \varepsilon) \approx \pi(\Gamma, \eta)$ is given by the inner automorphism $x \to axa^{-1}$, where a is given by axiom 5. ∎

Definition: A *tree* at $\eta \in I$ is a function $t: I \to G$, (we shall use the notation $t(\varepsilon) = t_\varepsilon$) such that for $\varepsilon \neq \eta$, $t_\varepsilon \in C$ and $\delta(t_\varepsilon) = \varepsilon$, $\delta(t_\varepsilon^{-1}) = \eta$; and $t_\eta = 1$.

A tree in an oidgroup is the analogue of a based maximal tree in a complex. The existence of trees follows from axiom 5.

II.C.3. Let t be a tree at η and let F be a free group with a basis $\{\tau_\varepsilon\}$, $\varepsilon \in I - \{\eta\}$. Define

$$H = \pi(\Gamma, \eta) * F$$

and $h: H \to G$ by defining $h|\pi(\Gamma, \eta)$ to be the inclusion map and $h(\tau_\varepsilon) = t_\varepsilon$, for $\varepsilon \in I - \{\eta\}$. Define $\tau_\eta = 1$; then $h(\tau_\eta) = t_\eta = 1$.

We define on H the structure of an oidgroup: $\Lambda = \{H, C_H, I, \delta_H\}$. The set I is the same as occurs in Γ.

$$C_H = \{\tau_\varepsilon^{-1}\gamma\tau_\vartheta \mid \varepsilon, \vartheta \in \pi(\Gamma, \eta);\ \gamma \neq 1, \text{ or } \varepsilon \neq \vartheta\}.$$
$$\delta_H(\tau_\varepsilon^{-1}\gamma\tau_\vartheta) = \vartheta.$$

It must be shown that Λ satisfies the six axioms for an oidgroup:

1. C_H generates H. For each element $\gamma \in \pi(\Gamma, \eta)$, the element $\gamma = \tau_\eta^{-1}\gamma\tau_\eta \in C_H$. For each basis element τ_ε of F, $\varepsilon \neq \eta$, we have $\tau_\varepsilon = \tau_\eta^{-1}1\tau_\varepsilon \in C_H$.

2. $1 \notin C_H$. This is clear from the definition of C_H and the free product structure of H.

3. $C_H = C_H^{-1}$. If $a = \tau_\varepsilon^{-1}\gamma\tau_\vartheta \in C_H$, then $a^{-1} = \tau_\vartheta^{-1}\gamma^{-1}\tau_\varepsilon \in C_H$.

4. If $a, b \in C_H$, $ab \neq 1$, then $ab \in C_H$ if and only if $\delta_H(a) = \delta_H(b^{-1})$. Let $a = \tau_\varepsilon^{-1}\gamma_a\tau_{\delta_H(a)}$, $b = \tau_{\delta_H(b^{-1})}^{-1}\gamma_b\tau_\vartheta$. If $\delta_H(a) = \delta_H(b^{-1})$, then $ab = \tau_\varepsilon^{-1}\gamma_a\gamma_b\tau_\vartheta$; since $ab \neq 1$, it must be that $\gamma_a\gamma_b \neq 1$, or $\varepsilon \neq \vartheta$; thus $ab \in C_H$. But if $\delta_H(a) \neq \delta_H(b^{-1})$, then $ab = \tau_\varepsilon^{-1}\gamma_a\tau_{\delta_H(a)}\tau_{\delta_H(b^{-1})}^{-1}\gamma_b\tau_\vartheta$; from the free product structure of H, and the definition of C_H, it follows that $ab \notin C_H$.

5. For $\varepsilon, \vartheta \in I$, $\varepsilon \neq \vartheta$, there exists $a \in C_H$, such that $\varepsilon = \delta_H(a)$, $\vartheta = \delta_H(a^{-1})$. Let $a = \tau_\vartheta^{-1}1\tau_\varepsilon$.

6. For $a_1, \ldots, a_n \in C_H$, $a_i a_{i+1} \notin C_H \cup \{1\}$, then $a_1 a_2 \cdots a_n \neq 1$. $a_i = \tau_{\varepsilon_i}^{-1}\gamma_i\tau_{\vartheta_i}$, and $\vartheta_i \neq \varepsilon_{i+1}$. Hence $a_1 a_2 \cdots a_n$ has length $\geq n$ in the free product structure of H, and so is not 1.

Definition: Let $\Gamma_1 = \{G_1, C_1, I_1, \delta_1\}$, $\Gamma_2 = \{G_2, C_2, I_2, \delta_2\}$ be two oidgroups. An *oidmorphism* $f\colon\Gamma_1 \to \Gamma_2$ is defined by a homomorphism $f_G\colon G_1 \to G_2$ and a function $f_I\colon I_1 \to I_2$, such that (i) $f_G(C_1) \subset C_2 \cup \{1\}$, (ii) $f_I\delta_1(a) = \delta_2 f_G(a)$ for all $a \in C_1$ for which $f_G(a) \neq 1$.

II.C.31. Proposition: *Suppose the oidmorphism $f\colon\Gamma_1 \to \Gamma_2$ is such that f_I is* 1-1. *Then the kernel of f_G is the consequence in G_1 of a subset of C_1.*

Proof: Let x be an element in the kernel of f_G. x has a representation in the form

$$x = a_1 a_2 \cdots a_n, \quad a_i \in C_1, \quad a_i a_{i+1} \notin C_1 \cup \{1\};$$

thus $\delta_1(a_i) \neq \delta_1(a_{i+1}^{-1})$. Since f_I is 1-1, assuming $f_G(a_i) \neq 1 \neq f_G(a_{i+1})$, we have $\delta_2(f_G(a_i)) \neq \delta_2(f_G(a_{i+1})^{-1})$. Let $b_i = f_G(a_i)$. We have $b_i \in C_2 \cup \{1\}$, $\delta_2(b_i) \neq \delta_2(b_{i+1}^{-1})$, and $b_1 b_2 \cdots b_n = 1$. From axiom 6, it follows that some $b_i = 1$. The proposition now follows by induction on n. ∎

II.C.4. Proposition (Structure Theorem): *$h: H \to G$ and the identity $\iota: I \to I$ define an isomorphism of the oidgroups $\Lambda \approx \Gamma$.*

(Λ is the oidgroup defined by means of Γ on page 23.)

Proof: It will first be shown that h is onto. According to axiom 1, it is enough to show that for $a \in C$ there is some element x of H such that $h(x) = a$. Let $\varepsilon = \delta(a)$, $\vartheta = \delta(a^{-1})$. Then $a = t_\vartheta^{-1}(t_\vartheta a t_\varepsilon^{-1})t_\varepsilon$. Now $t_\vartheta a t_\varepsilon^{-1} \in \pi(\Gamma, \eta)$, hence $a = h(\tau_\vartheta^{-1}(t_\vartheta a t_\varepsilon^{-1})\tau_\varepsilon)$.

Now it will be shown that in fact the homomorphism $h: H \to G$ and the identity $\iota: I \to I$ define an oidmorphism $h: \Lambda \to \Gamma$. We must show that $h(C_H) \subset C \cup \{1\}$; $h(\tau_\varepsilon^{-1}\gamma\tau_\vartheta) = t_\varepsilon^{-1}\gamma t_\vartheta$; from the definitions of t_ε, t_ϑ, $\pi(\Gamma, \eta)$, it follows that $t_\varepsilon^{-1}\gamma t_\vartheta \in C$.

Since ι is 1-1, from the above Proposition II.C.31, it follows that if $\operatorname{Ker} h$ is not trivial, there is an element of C_H, say $\tau_\varepsilon^{-1}\gamma\tau_\vartheta$, such that $h(\tau_\varepsilon^{-1}\gamma\tau_\vartheta) = 1$; but $h(\tau_\varepsilon^{-1}\gamma\tau_\vartheta) = t_\varepsilon^{-1}\gamma t_\vartheta \neq 1$; if $\varepsilon = \vartheta$ and $\gamma \neq 1$, then $t_\varepsilon^{-1}\gamma t_\varepsilon \neq 1$. Thus, h is a monomorphism. ∎

Definition: An *oid free product* $K = \ast_\alpha G_\alpha$ is a free product of groups, which belongs to an oidgroup $\Gamma = \{K, C, I, \delta\}$, such that $C \cap G_\alpha$ generates G_α for each α.

For each α, let $\Gamma_\alpha = \{G_\alpha, C \cap G_\alpha, \delta(C \cap G_\alpha), \delta | C \cap G_\alpha\}$. Then clearly Γ_α is a non-connected oidgroup.

Definition: A *tie* ξ in an oid free product K as above, is an element of $C \cap G_\alpha$ for some α, such that $\delta(\xi) \neq \delta(\xi^{-1})$.

Definition: An *oidmap* f of the oid free product $K = \ast_\alpha G_\alpha$ into the oid free product $L = \ast_\alpha H_\alpha$, is an oidmorphism f of oidgroups, such that $f(G_\alpha) \subset H_\alpha$.

II.C.5. Proposition: *Let ξ be a tie in the oid free product K, and let $K' = K/\langle\xi\rangle$. Then K' can be given a structure of oid free product in a certain natural manner, so that the natural homomorphism $h: K \to K'$ is an oidmap, and so that h_* is an isomorphism $\pi(K) \to \pi(K')$.*

Proof: Suppose $\xi \in G_1$; then K' has the form: $(\ast_{\alpha\neq 1} G_\alpha) * G_1'$, where $G_1' = G_1/\langle\xi\rangle$. Define $C' = h(C) - \{1\}$. Let $\eta = \delta(\xi)$, $\varepsilon = \delta(\xi^{-1})$; let $I' = I - \{\varepsilon\}$. Define δ' as follows:

Let $x' \in C'$; pick $x \in C$ such that $h(x) = x'$. Define $\delta'(x') = \delta(x)$ if $\delta(x) \neq \varepsilon$; if $\delta(x) = \varepsilon$, $\delta'(x') = \eta$. It must be shown that δ' is well-defined.

Let t be a tree in K at η, such that $t_\varepsilon = \xi$. According to the Structure Theorem, $K = \pi(K,\eta) * F$, where F has as basis $\{\tau_\vartheta\}$, $\vartheta \in I - \{\eta\}$, and $C = \{\tau_\rho^{-1}\gamma\tau_\vartheta\}$, $\gamma \in \pi(K,\eta)$, $\tau_\rho = t_\rho$, $\delta(\tau_\rho^{-1}\gamma\tau_\vartheta) = \vartheta$. Then $K' = \pi(K,\eta) * F'$, where F' has as basis $\{\tau_\vartheta\}$, $\vartheta \in I - \{\eta,\varepsilon\}$, and $C' = \{\tau_\rho^{-1}\gamma\tau_\vartheta\}$, where $\rho, \vartheta \neq \varepsilon$. Let $x' = \tau_\rho^{-1}\gamma\tau_\vartheta$; now there are two cases: $\vartheta \neq \eta$ and $\vartheta = \eta$; in the first case, any $x \in C$ such that $h(x) = x'$, has the form $x = q^{-1}\gamma\tau_\vartheta$, where q is either τ_ρ, or if $\rho = \eta$, q may be τ_ε. Thus $\delta(x) = \vartheta$; $\delta'(x') = \vartheta$. In the second case, such an x has the form $x = q^{-1}\gamma\tau_\eta$ or $x = q^{-1}\gamma\tau_\varepsilon$ in which case $\delta(x) = \eta$ or ε, so that $\delta'(x') = \eta$. Then this shows that δ' is well-defined, and that the oidgroup structure of K' is described by the free product $\pi(K,\eta) * F'$ and the basis $\{\tau_\vartheta\}$, $\vartheta \in I - \{\varepsilon,\eta\}$ of F'. It follows that $\pi(K',\eta) = \pi(K,\eta)$ for this description of K'; hence h_* is an isomorphism $\pi(K) \approx \pi(K')$. ∎

An oidmap $f: K_1 \to K_2$ will be said to *preserve base point*, if in I_1 and I_2 are chosen elements η_1, η_2 respectively, such that $f_I(\eta_1) = \eta_2$. In this case we have a homomorphism $f_*: \pi(K_1,\eta_1) \to \pi(K_2,\eta_2)$.

II.C.6. *A directed system of oid free products and oidmaps* $\{K_\kappa; p_\lambda^\kappa\}$ is a set of oid free products K_κ indexed on a directed set $\{\kappa,\lambda\}$, with oidmaps $p_\lambda^\kappa: K_\kappa \to K_\lambda$,

$\kappa < \lambda$, preserving base point (the base point of K_κ will be η_κ), such that if $\kappa < \lambda < \mu$, $p^\lambda_\mu p^\kappa_\lambda = p^\kappa_\mu$. We now define $\varinjlim K_\kappa = K$ as follows:

$$K_\kappa = \{G_\kappa = \mathop{\ast}_{\alpha} G_{\kappa,\alpha}, C_\kappa, I_\kappa, \delta_\kappa\}.$$

Let $G = \varinjlim G_\kappa$ defined by the homomorphisms $(p^\kappa_\lambda)_G$. It is easy to see that $G = \ast_\alpha \varinjlim_\kappa G_{\kappa,\alpha}$.

Let $I = \varinjlim I_\kappa$ under the functions $(p^\kappa_\lambda)_I$.

Let $p^\kappa_G : G_\kappa \to G$ be the direct limit homomorphism, and similarly $p^\kappa_I : I_\kappa \to I$ the direct limit function.

Define $C = (\bigcup_\kappa p^\kappa_G(C_\kappa)) - \{1\}$.

If $a \in C$, let a_κ be some element of some C_κ such that $p^\kappa_G(a_\kappa) = a$; define $\delta(a) = p^\kappa_I \delta_\kappa(a_\kappa)$. It is easy to see that δ is well-defined.

Then $K = \varinjlim K_\kappa = \{G, C, I, \delta\}$.

II.C.61. Proposition: $\varinjlim K_\kappa = K$ *is an oid free product, and*

$$\pi(K, \eta) = \varinjlim \pi(K_\kappa, \eta_\kappa).$$

Furthermore if L is an oid free product and $f_\kappa : K_\kappa \to L$ are oidmaps such that

$$\begin{array}{ccc} K_\kappa & \xrightarrow{p^\kappa_\lambda} & K_\lambda \\ & {\scriptstyle f_\kappa}\searrow \quad \swarrow{\scriptstyle f_\lambda} & \\ & L & \end{array}$$

is commutative whenever $\kappa < \lambda$, then there is a unique oidmap $f : K \to L$, such that

$$\begin{array}{ccc} K_\kappa & \xrightarrow{p^\kappa} & K \\ & {\scriptstyle f_\kappa}\searrow \quad \swarrow{\scriptstyle f} & \\ & L & \end{array}$$

is commutative for all κ.

Proof: To show K is an oid free product, we have some axioms to verify. In particular, $C \cap G_\alpha$ should generate G_α.

$$C \cap G_\alpha = \bigcup_\kappa p_G^\kappa(C_\kappa \cap G_{\kappa,\alpha}) - \{1\};$$

since $C_\kappa \cap G_{\kappa,\alpha}$ generates $G_{\kappa,\alpha}$ and since $G_\alpha = \varinjlim G_{\kappa,\alpha}$, it follows that $C \cap G_\alpha$ generates G_α. The six axioms for an oidgroup must now be verified. Of these, 1,2, and 3 are immediate.

4. Suppose $\delta(a) = \delta(b^{-1})$, $a, b \in C$. Then there exist κ, and $a_\kappa, b_\kappa \in C_\kappa$ with $\delta_\kappa(a_\kappa) = \delta_\kappa(b_\kappa^{-1})$, $p^\kappa(a_\kappa) = a$, $p^\kappa(b_\kappa) = b$. Then $a_\kappa b_\kappa \in C_\kappa \cup \{1\}$; hence $p^\kappa(a_\kappa b_\kappa) = ab \in C \cup \{1\}$. Conversely suppose $a, b, ab \in C$; there is a κ such that $a_\kappa, b_\kappa, (ab)_\kappa \in C_\kappa$, $p^\kappa(a_\kappa) = a$, $p^\kappa(b_\kappa) = b$, $p^\kappa((ab)_\kappa) = ab$; since $a \cdot b = ab$, there is $\lambda > \kappa$ such that $p_\lambda^\kappa(a_\kappa) \cdot p_\lambda^\kappa(b_\kappa) = p_\lambda^\kappa((ab)_\kappa)$. Let $a_\lambda = p_\lambda^\kappa(a_\kappa)$, $b_\lambda = p_\lambda^\kappa(b_\kappa)$. Then a_λ, b_λ and $a_\lambda b_\lambda \in C_\lambda$; therefore $\delta_\lambda(a_\lambda) = \delta_\lambda(b_\lambda^{-1})$, hence $\delta(a) = \delta(b^{-1})$.

5. Let $\varepsilon, \eta \in I$, $\varepsilon \neq \eta$. There exists κ such that $\varepsilon_\kappa, \eta_\kappa \in I_\kappa$, $p^\kappa(\varepsilon_\kappa) = \varepsilon$, $p^\kappa(\eta_\kappa) = \eta$; then there is $a_\kappa \in C_\kappa$ such that $\varepsilon_\kappa = \delta_\kappa(a_\kappa)$, $\eta_\kappa = \delta_\kappa(a_\kappa^{-1})$. Let $a = p^\kappa(a_\kappa)$; then $\delta(a) = \varepsilon$, $\delta(a^{-1}) = \eta$.

6. Let $a_1, \ldots, a_n \in C$, $a_i a_{i+1} \notin C \cup \{1\}$. There is κ such that a_i has a representative $a_{\kappa i} \in C_\kappa$. To obtain a contradiction, assume $a_1 a_2 \cdots a_n = 1$. Then there is $\lambda > \kappa$, such that if $a_{\lambda i} = p_\lambda^\kappa(a_{\kappa i})$, $a_{\lambda 1} a_{\lambda 2} \cdots a_{\lambda n} = 1$; so there must be i such that $a_{\lambda i} a_{\lambda, i+1} \in C_\lambda \cup \{1\}$. But this would imply $a_i a_{i+1} \in C \cup \{1\}$, contrary to supposition.

The first part of the proposition is now proved.

For the proof that $\varinjlim \pi(K_\kappa, \eta_\kappa) = \pi(K, \eta)$, we must prove two things:

(a) $\pi(K, \eta) = \bigcup_\kappa p^\kappa(\pi(K_\kappa, \eta_\kappa))$ and

(b) any relation among the elements of $\pi(K, \eta)$ occurs among some representative elements of $\pi(K_\kappa, \eta_\kappa)$.

(a) Let $\gamma \in \pi(K, \eta)$; i.e., either $\gamma = 1$, or $\gamma \in C$, $\delta(\gamma) = \delta(\gamma^{-1}) = \eta$. To prove (a) we may assume $\gamma \in C$; there is κ and $\gamma_\kappa \in C_\kappa$ such that $p^\kappa(\gamma_\kappa) = \gamma$; then there

is $\lambda > \kappa$, such that if $\gamma_\lambda = p^\kappa_\lambda \gamma_\kappa$, $\delta(\gamma_\lambda) = \delta(\gamma_\lambda^{-1}) = \eta_\lambda$. Hence $\gamma_\lambda \in \pi(K_\lambda, \eta_\lambda)$, $p^\lambda \gamma_\lambda = \gamma$.

(b) Suppose $\gamma_1, \ldots, \gamma_n \in \pi(K, \eta)$, and $\gamma_1 \cdots \gamma_n = 1$. According to (a) we find λ, and $\gamma_{\lambda i} \in \pi(K_\lambda, \eta_\lambda)$ such that $p^\lambda \gamma_{\lambda i} = \gamma_i$; since $p^\lambda(\gamma_{\lambda 1}\gamma_{\lambda 2} \cdots \gamma_{\lambda n}) = 1$, there is $\mu > \lambda$ such that if $\gamma_{\mu i} = p^\lambda_\mu \gamma_{\lambda i}$, then $\gamma_{\mu 1}\gamma_{\mu 2} \cdots \gamma_{\mu n} = 1$, and $p^\mu(\gamma_{\mu i}) = \gamma_i$.

This proves the second part of the proposition.

The last part about the uniqueness and existence of f follows from the similar theorems for the existence of direct limits of homomorphisms (to obtain $f_G : G_K \to G_L$) and functions (to obtain $f_I : I_K \to I_L$). ∎

Definition: Let $\varphi : G \to \ast_\alpha H_\alpha$ be a homomorphism of groups. We identify $H = \ast_\alpha H_\alpha$ with the oid free product where I_H contains one point, and $C_H = H - \{1\}$. φ is said to be a *free homomorphism* (with respect to the given free product decomposition of H), if there exists an oid free product K and an oidmap $f : K \to H$ and an isomorphism $G \approx \pi(K, \eta)$ such that the diagram

$$\begin{array}{ccc} G & \approx & \pi(K,\eta) \\ & \varphi \searrow \quad \swarrow f_* & \\ & H & \end{array}$$

is commutative. (Note: $\pi(H) = H$).

II.C.7. Theorem: *Let $\varphi : G \to \ast_\alpha H_\alpha = H$ be a free homomorphism. Then there exists an oid free product Λ and an oidmap $g : \Lambda \to H$, such that G may be identified with $\pi(\Lambda, \eta)$ and φ with g_*, such that if ξ is any tie of Λ, $g(\xi) \neq 1$.*

Proof: We shall construct Λ by a transfinite induction. According to the definition of free homomorphism, there is an oid free product K_0, oidmap $f_0 : K_0 \to H$; and $G \approx \pi(K_0, \eta_0)$ is such a way that $\varphi = f_{0*}$.

Suppose ρ is an ordinal number and that for all $\sigma, \tau, \upsilon < \rho$ there have been defined oid free products K_σ, oidmaps $f_\sigma : K_\sigma \to H$, and oidmaps $p^\sigma_\tau : K_\sigma \to K_\tau$,

for $\sigma < \tau$, and isomorphisms: $G \approx \pi(K_\sigma, \eta_\sigma)$ such that the following diagrams are commutative for $\sigma < \tau < \upsilon$;

$$\begin{array}{ccc} & \pi(K_\sigma,\eta_\sigma) & \\ \approx\nearrow & & \\ G & \downarrow p_\tau^\sigma & \\ \approx\searrow & & \\ & \pi(K_\tau,\eta_\tau) & \end{array} \qquad \begin{array}{ccc} & K_\tau & \\ p_\tau^\sigma\nearrow & & \searrow p_\upsilon^\tau \\ & p_\upsilon^\sigma & \\ K_\sigma & \longrightarrow & K_\upsilon \end{array} \qquad \begin{array}{ccc} K_\sigma & \overset{p_\tau^\sigma}{\longrightarrow} & K_\tau \\ & f_\sigma\searrow \quad \swarrow f_\tau & \\ & H & \end{array}$$

and such that $p_{\tau,I}^\sigma : I_\sigma \to I_\tau$ is onto.

If ρ is a limit ordinal number, define $K_\rho = \varinjlim_{\sigma<\rho} \{K_\sigma, p_\tau^\sigma\}$. Define f_ρ as the limit oidmap; and p_ρ^σ as the oidmap defined by the direct limit. The isomorphism $G \approx \pi(K_\rho, \eta_\rho)$ is given by:

$$G \approx \pi(K_\sigma, \eta_\sigma) \overset{p_\rho^\sigma}{\longrightarrow} \pi(K_\rho, \eta_\rho).$$

From the preceding proposition II.C.61, it follows that this is an isomorphism, and that all the diagrams are commutative.

Or suppose ρ is not a limit ordinal, i.e., $\rho = \chi + 1$. If K_χ contains no tie ξ with $f_\chi(\xi) = 1$, define $\Lambda = K_\chi$. If on the other hand K_χ contains some tie ξ with $f_\chi(\xi) = 1$, define $K_\rho = K_{\chi+1} = K_\chi/\langle\xi\rangle$. The fact that $f_\chi(\xi) = 1$ implies there is an oidmap $f_\rho : K_\rho \to H$ such that the diagram

$$\begin{array}{ccc} K_\chi & \overset{p_\rho^\chi}{\longrightarrow} & K_\rho \\ & f_\chi\searrow \quad \swarrow f_\rho & \\ & H & \end{array}$$

is commutative; p_ρ^χ denotes the quotient map; note that $p_{\rho,I}^\chi : I_\chi \to I_\rho$ is onto and collapses two points to one. For $\sigma < \chi$, define p_ρ^σ to be the composition $K_\sigma \overset{p_\chi^\sigma}{\longrightarrow} K_\chi \overset{p_\rho^\chi}{\longrightarrow} K_\rho$. According to Proposition II.C.5, $p_\rho^\chi : \pi(K_\chi, \eta_\chi) \to \pi(K_\rho, \eta_\rho)$ is an isomorphism. Define $G \approx \pi(K_\rho, \eta_\rho)$ as the composition

$$G \approx \pi(K_\chi, \eta_\chi) \overset{p_\rho^\chi}{\longrightarrow} \pi(K_\rho, \eta_\rho).$$

This procedure of transfinite induction cannot proceed indefinitely since no function $p_\tau^\sigma : I_\sigma \to I_\tau$ is 1-1. Hence there is some χ, with Cardinal $\chi \leq$ Cardinal I_0, for which in K_χ there is no tie ξ with $f_\chi(\xi) = 1$. Thus has $\Lambda = K_\chi$ been constructed. ∎

II.C.81. Lemma: *Let $\Gamma = \{G, C, I, \delta\}$ be a non-connected oidgroup. Suppose $g \in G$. Then there is a unique expression for g as:*

$$g = a_1 \cdots a_n, \qquad a_i \in C, \qquad a_i a_{i+1} \notin C \cup \{1\}. \tag{1}$$

(Unique Factorization Theorem.)

Proof: From axiom 1 it is possible to represent g as a product of elements of C. Collecting these elements together into the largest blocks which lie in C, we obtain an expression (1).

To prove uniqueness, suppose $g = b_1 \cdots b_m$, $b_j \in C$, $b_j b_{j+1} \notin C \cup \{1\}$. It will be shown that this is the same as the expression (1) by induction on n.

If $n = 0$, then $g = 1$. Axiom 6 assures that $m = 0$.

Assume $n > 0$. Axiom 6 implies $m > 0$. Then $1 = gg^{-1} = a_1 \cdots a_n b_m^{-1} \cdots b_1^{-1}$. Axioms 2 and 6 imply that $a_n b_m^{-1} \in C \cup \{1\}$; while axioms 4 and 6 show that $a_n b_m^{-1} \notin C$ since axiom 4 implies $a_{n-1} a_n b_m^{-1} \notin C$ and $a_n b_m^{-1} b_{m-1}^{-1} \notin C$, if $a_n b_m^{-1} \in C$. Thus $a_n b_m^{-1} = 1$; $a_n = b_m$. Now let $g_1 = a_1 \cdots a_{n-1} = b_1 \cdots b_{m-1}$. The induction hypothesis shows that $n - 1 = m - 1$ and that $a_i = b_i$ for $i \leq n - 1$. ∎

II.C.82. Lemma: *Let K be an oid free product: $G_K = \ast_\alpha G_{K,\alpha}$. Suppose $a \in C_K$ and that the reduced word of a in this free product is $a = a_1 a_2 \cdots a_n$, $a_i \in G_{K,\alpha_i} - \{1\}$, $\alpha_i \neq \alpha_{i+1}$. Then $a_i \in C_K$.*

Proof: Let $a_i = b_{i1} b_{i2} \cdots b_{in_i}$, where each $b_{ij} \in G_{K,\alpha_i} \cap C_K$, and $b_{ij} b_{i,j+1} \notin C_K \cup \{1\}$. Each a_i has such an expression.

Then $a = b_{1,1} \cdots b_{1,n_1} b_{2,1} \cdots b_{2,n_2} \cdots b_{n,1} \cdots b_{n,n_n}$, and there is no cancellation of adjacent terms here (i.e., each segment of this expression is a non-trivial element

of G_K). Collect the terms of this expression into pieces belonging to C_K:

$$a = c_1 c_2 \cdots c_p, \quad c_j \in C_K, \quad c_j c_{j+1} \notin C_K \cup \{1\}.$$

If, for some i, $n_i > 1$, it follows that $p > 1$. But this contradicts Lemma II.C.81, since $a \in C_K$. ∎

Definition: If Λ is an oid free product, $H = \ast_\alpha H_\alpha$, and $g: \Lambda \to H$ is an oidmap such that if ξ is a tie of Λ, $g(\xi) \neq 1$, g will be said to be *irreducible*.

II.C.9. Theorem: *Let $g: \Lambda \to H = \ast_\alpha H_\alpha$ be an oidmap, such that $g_*: \pi(\Lambda, \eta) \to H$ is onto. And let g be irreducible. Then in I_Λ there is just one point.*

Proof: Suppose ε is a point distinct from η. There exists $a \in C_\Lambda$ with $\delta(a) = \varepsilon$, $\delta(a^{-1}) = \eta$, and of these choose one which has minimal length with respect to the free product decomposition $\ast_\alpha G_{\Lambda,\alpha} = \Lambda$. Then I say that the length of a is 1; for suppose $a = a_1 a_2 \cdots a_n$, $a_i \in G_{\Lambda,\alpha_i} - \{1\}$, $\alpha_i \neq \alpha_{i+1}$. From Lemma II.C.82, it follows that $a_i \in C_\Lambda$. Then either a_1 is a tie, in which case $a = a_1$, or a_1 is a loop, in which case $a_2 \cdots a_n$ has less length than a.

Hence a is a tie, $\delta(a) = \varepsilon \neq \delta(a^{-1}) = \eta$. By the hypothesis that g is irreducible, $g(a) \neq 1$.

Let $a \in G_{\Lambda,\alpha}$; then $g(a) \in H_\alpha - \{1\}$. By the hypothesis that g_* is onto, there exists $\gamma \in \pi(\Lambda, \eta)$ for which $g(a) = g(\gamma)$. Consider the set of all those $\gamma \in \pi(\Lambda, \eta)$, $g(\gamma) = g(a)$, and choose one for which γ has the minimal length in terms of $\ast_\alpha G_{\Lambda,\alpha} = G_\Lambda$. Then I say that $\gamma \in G_{\Lambda,\alpha}$; this will be shown if we show that γ has length 1, since $g(\gamma) \in H_\alpha - \{1\}$.

Let $\gamma = c_1 c_2 \cdots c_n$, $c_i \in G_{\Lambda,\alpha_i} - \{1\}$, $\alpha_i \neq \alpha_{i+1}$. Then $g(c_i) \in H_{\alpha_i} - \{1\}$. For suppose $g(c_i) = 1$; then from the irreducibility of g, c_i cannot be a tie; hence c_i is a loop and $\gamma' = c_1 \cdots c_{i-1} c_{i+1} \cdots c_n \in \pi(\Lambda, \eta)$, $g(\gamma') = g(\gamma) = g(a)$, and γ' has less length than γ.

Hence $g(\gamma) = g(c_1) \cdots g(c_n) = g(a)$ is a reduced word in $\ast_\alpha H_\alpha$; hence $n = 1$.

But then $\gamma^{-1}a$ is a tie in Λ and $g(\gamma^{-1}a) = g(\gamma)^{-1}g(a) = 1$. This is the contradiction which proves the theorem. ∎

II.C.10. Theorem: *If G is a free group, then any homomorphism $\varphi: G \to \ast_\alpha H_\alpha$ is free.*

Proof: Let $\{x_i\}$ be a basis of G. And let $\varphi(x_i) = h_{i1}h_{i2}\cdots h_{in_i}$, $h_{ij} \in H_{\alpha_{ij}}$, $n_i \geq 1$ (h_{ij} may be 1). Let K be the free group on a basis $\{y_{ij} \mid j = 1, \ldots, n_i\}$. Let $K_\alpha =$ subgroup of K generated by those y_{ij} for which $\alpha_{ij} = \alpha$. Then $K = \ast_\alpha K_\alpha$. Define $g: K \to \ast_\alpha H_\alpha$ by $g(y_{ij}) = h_{ij}$; clearly $g(K_\alpha) \subset H_\alpha$.

$$\text{Define } I = \{(i,j) \mid j = 2, \ldots, n_i\} \cup \{\eta\}.$$

$$\text{Let } \delta(y_{ij}) = (i, j+1) \text{ if } j < n_i.$$

$$\delta(y_{in_i}) = \eta.$$

$$\delta(y_{ij}^{-1}) = (i,j) \text{ if } i < j.$$

$$\delta(y_{i1}^{-1}) = \eta.$$

Define C and $\delta: C \to I$ as follows:

An element a of C is of the form $a = u_1u_2\cdots u_n$, where u_k is either $y_{(ij)k}$ or $y_{(ij)k}^{-1}$, where this expression of a is a reduced word in the basis $\{y_{ij}\}$ of K, and $n > 0$, and for $k = 1, \ldots, n-1$, $\delta(u_k) = \delta(u_{k+1}^{-1})$.

Define $\delta(a) = \delta(u_n)$.

It is clear by checking the axioms that with this structure, K is an oid free product, and $g: K \to \ast_\alpha H_\alpha$ is an oidmap.

Define $m: G \to \pi(K, \eta)$ as follows:

$$m(x_i) = y_{i1}y_{i2}\cdots y_{in_i}.$$

The diagram

$$\begin{array}{ccc} G & \xrightarrow{m} & \pi(K,\eta) \\ & \varphi \searrow \quad \swarrow g & \\ & H & \end{array}$$

is clearly commutative.

To prove the theorem, it suffices to show that m is an isomorphism. The fact that $\operatorname{Ker} m = 1$, is obvious; in fact, if ℓ is the length of $q \in G$ in terms of the basis $\{x_i\}$, the length of $m(q)$ in terms of the basis $\{y_{ij}\}$ is not less than ℓ. To show that m is onto, let $\gamma \in C$ be such that $\delta(\gamma) = \delta(\gamma^{-1}) = \eta$. We may write $\gamma = a_1 a_2 \cdots a_n$ where $a_k \in C$, $\delta(a_k) = \delta(a_k^{-1}) = \eta$; and if a_k is written as a reduced word in the basis $\{y_{ij}\}$, no proper segment of a_k is a loop at η. Then it is clear from the definitions of C, δ, and m, that $a_k = m(x_{i_k}^{\varepsilon_k})$, where $\varepsilon_k = \pm 1$. Thus m is onto.

The theorem is proved. ∎

As a corollary we have Gruško's Theorem.

II.C.11. Theorem: *If F is a free group, $\varphi: F \to \ast_\alpha H_\alpha$ is an epimorphism, then there is a free product decomposition $F = \ast_\alpha F_\alpha$ such that $\varphi(F_\alpha) \subset H_\alpha$.*

Proof: By the preceding theorem, φ is free; by Theorem II.C.7, we may find an oid free product Λ and an oidmap $g: \Lambda \to H$ such that $F \approx \pi(\Lambda)$ and such that $g_* = \varphi$ and such that g is irreducible. By Theorem II.C.9, since φ is onto, I_Λ has just one point; thus $G_\Lambda = \pi(\Lambda, \eta) = F$. Since Λ is an oid free product and g an oidmap, $G_\Lambda = \ast_a G_{\Lambda,\alpha}$, $g(G_{\Lambda,\alpha}) \subset H_\alpha$. So by defining $F_\alpha = G_{\Lambda,\alpha}$, the theorem is proved. ∎

II.D. Free Products with Amalgamation

We begin this discussion with a counterexample.

II.D.1. Theorem: *Let F be the free group of rank 2. F is a non-trivial free product with amalgamation, $F = (G*H)_A$, where G, H, and A are free groups of rank 2; and the subgroup $H \subset F$ contains no basis element of F.*

Proof: Let F have a basis $\{x, y\}$; $G = \{g_1, g_2\}$; $H = \{h_1, h_2\}$, $A = \{a_1, a_2\}$. Define monomorphisms $m_1 : A \to G$, $m_2 : A \to H$ as follows:

$$m_1(a_1) = g_1^2.$$
$$m_1(a_2) = g_2^2.$$
$$m_2(a_1) = h_1.$$
$$m_2(a_2) = h_2 h_1 h_2 h_1^{-1} h_2^{-1}.$$

The fact that m_1 and m_2 are monomorphisms follows from the facts that $m_i(a_1)$ and $m_i(a_2)$ do not commute, and hence are basis elements of the subgroup of G or H which they generate. The image of m_i is clearly a proper subgroup of G or H. Hence we can construct the group $(G*H)_A$. Now define a homomorphism $\varphi : (G*H) \to F$ thus:

$$\varphi(g_1) = x; \quad \varphi(g_2) = y^2 x^2 y x^{-2} y^{-2}.$$
$$\varphi(h_1) = x^2; \quad \varphi(h_2) = y^2.$$

Notice that $\varphi m_1(a_1) = \varphi m_2(a_1) = x^2$, and $\varphi m_1(a_2) = \varphi m_2(a_2) = y^2 x^2 y^2 x^{-2} y^{-2}$; hence φ is well-defined. Now define a homomorphism $\psi : F \to (G*H)_A$ thus:

$$\psi(x) = g_1.$$
$$\psi(y) = h_1^{-1} h_2^{-1} g_2 h_2 h_1.$$

It is easy to verify that $\varphi\psi$ is the identity on F and that $\psi\varphi$ is the identity on $(G*H)_A$. Hence φ is an isomorphism. The group $\varphi(H)$ lies in the subgroup of F generated by squares and hence contains no basis element of F. ∎

This example shows that in the statement of Gruško's Theorem, it is impossible to replace the free product by a free product with amalgamation. It is possible, however, to prove the following result, whih may be considered a weak form of Gruško's Theorem.

II.D.2. Theorem: *Let $H = (\ast_\alpha G_\alpha)_A$ be a free product with amalgamation; let F be a free group, and $\varphi: F \to H$ an epimorphism. Then there is a basis $\{x_\beta\}$ of F and an α such that $\varphi(x_1) \in G_\alpha$.*

Proof: We proceed by analogy to the proof of Gruško's Theorem which uses cell-complexes. We construct a complex X, which is the union of its open connected subspaces, X_α, such that for $\alpha \neq \gamma$, $X_\alpha \cap X_\gamma = K$, a connected set, such that the injection homomorphism $\pi_1(K) \to \pi_1(X_\alpha)$ induces the inclusion $A \subset G_\alpha$; thus $\pi_1(K) \approx A$ and $\pi_1(X_\alpha) \approx G_\alpha$. F is interpreted as the fundamental group of a complex Q, $f: Q \to X$ induces the homomorphism φ. And f is a tractable interpretation, in the sense that in $f^{-1}(K)$ there is a subcomplex L, each component of which is contractible, and such that each component of $Q - L$ lies in some $f^{-1}(X_\alpha)$. We can modify f so that in each component of L there can be chosen a base point which is mapped by f into the base point of X (which is in K). Then a *tie* will be a path going from the base point of one component of L to the base point of another component of L, which lies entirely within some $f^{-1}(X_\alpha)$. A *binding tie* will be a tie whose value in H lies in A. As before, a binding tie can be used to change f so as to bind together two components of L; and as before, a transfinite induction can be performed to demonstrate the existence of such a Q and f for which there is no binding tie.

In this case, it is asserted that some component of $Q-L$ is not simply connected. Suppose to the contrary. Then L is certainly not connected; otherwise, F would be trivial. Hence there is a tie P from q_0 = base point of Q, to a base point of some other component of L, and $f_*(P) \notin A$. Since φ is onto, there is a loop R based at q_0 such that $f_*(R) = f_*(P)$. R can be represented as a product of ties and loops, as in the previous proof, $R = U_1 \cdots U_n$. Now, if U_i is a loop,

$f_*(U_i) = 1$, since we assume each component of $Q - L$ is simply connected. Hence we reduce R as before by deleting the loop U_i; we also reduce R by amalgamating two consecutive factors which lie in the same $f^{-1}(X_\alpha)$. Finally after making these reductions, $f_*(R) = f_*(U_1)\cdots f_*(U_n)$, becomes a reduced word in H in the sense that $f_*(U_i) \notin A$ and $f_*(U_i)$ and $f_*(U_{i+1})$ are in distinct factors G_{α_i} and $G_{\alpha_{i+1}}$. It is well known that the length of such a representation is unique; hence, since $f_*(R) = f_*(P) \in G_\alpha$, $n = 1$. But then R is a loop in some $f^{-1}(X_\alpha)$, and since every component of $Q - L$ is supposed simply connected, it follows that $f_*(R) = 1$; hence $f_*(P) = 1 \in A$, contrary to the assumption that there are no binding ties. This contradiction shows that some component C of $Q - L$ is not simply connected; now $\pi_1(C)$ is a free factor of $\pi_1(Q)$, and $f_*(\pi_1(C))$ is conjugate to a subgroup of G_α; since f_* is onto, there is some element $y \in \pi_1(X)$, such that $y\pi_1(C)y^{-1}$ is mapped by f_* into G_α. We now choose a basis of H, $\{z_\beta\}$, such that $z_1 \in \pi_1(C)$. Then the basis $\{yz_\beta y^{-1}\}$, satisfies the conclusion of the theorem. ∎

The last theorem, which is to follow, even more closely resembles Gruško's Theorem, in that it singles out a collection of elements of the free group which are almost a basis.

II.D.3. Lemma: *Let $(G*H)_A$ be a free product with amalgamation, and let $h:(G*H)_A \to Q$ be onto, such that the kernel of h is the consequence of elements $\gamma_\alpha \in G$. Let $G' = h(G)$, $H' = h(H)$, $A' = h(A)$, and let $(G'*H')_{A'}$ be the free product with amalgamation defined by the inclusions $A' \subset G'$, $A' \subset H'$. Then $Q = (G'*H')_{A'}$.*

Proof: We have a homomorphism $\varphi:(G'*H')_{A'} \to Q$ defined by $\varphi|G' =$ identity, $\varphi|H' =$ identity.

Define a homomorphism $\psi:Q \to (G'*H')_{A'}$ as follows. Let

$$\overline{\psi}:(G*H)_A \to (G'*H')_{A'}$$

be defined as $\overline{\psi}|G = h|G$, $\overline{\psi}|H = h|H$. Then if $q \in Q$, define $\psi(q) =$ the element of $\overline{\psi}(h^{-1}(q))$. To show that ψ is well-defined, we must show that $\overline{\psi}(h^{-1}(q))$ has

exactly one element. $\overline{\psi}(h^{-1}(q))$ is not empty, since h is onto. Now it must be shown that if $p \in (G*H)_A$ is such that $h(p) = 1$, then $\overline{\psi}(p) = 1$; since such a p is a consequence of $\{\gamma_\alpha\}$, it is sufficient to show that $\overline{\psi}(\gamma_\alpha) = 1$; but this is obvious. Hence ψ is a homomorphism. Clearly $\varphi\psi$ is the identity on Q and $\psi\varphi$ is the identity on $(G'*H')_{A'}$; hence φ is an isomorphism. ∎

II.D.4. Theorem: *Let F be a free group of rank n, and let $(G*H)_A$ be a free product with amalgamation, and let $f: F \to (G*H)_A$ be onto. Then there are n elements of F, $x_1, \ldots, x_n$, such that $F_k = F$ modulo the consequence of $\{x_1, \ldots, x_k\}$, is free of rank $n-k$, and such that for each i, $f(x_i) \in G$ or $f(x_i) \in H$.*

Proof: By induction on n. This is clearly true for $n = 1$.

By the previous Theorem II.D.2, there is a basis element x_1 of F such that $f(x_1) \in G$ or $\in H$; suppose $f(x_1) \in G$. Let $Q = (G*H)_A$ modulo the consequence of $f(x_1)$; let $h: (G*H)_A \to Q$ be the natural homomorphism. By the lemma, $Q = (G'*H')_{A'}$ where $G' = h(G)$, $H' = h(H)$, $A' = h(A)$. Let $F' = F/\langle x_1\rangle$; define a homomorphism $f': F' \to Q = (G'*H')_{A'}$ by $f'(u\langle x_1\rangle) = hf(u)$; this is well-defined since $hf(\langle x_1\rangle) = 1$. Clearly f' is onto, and F' has rank $n-1$. By the induction hypothesis, there exist $y_2, \ldots, y_n \in F'$, $f'(y_i) \in G'$ or $\in H'$, and $F'/\langle y_2, \ldots, y_k\rangle$ is free of rank $n-k$. Let $z_2, \ldots, z_n$ be representatives of $y_1, \ldots, y_n$ in F. Then it follows that $f(z_i) \in G\langle f(x_1)\rangle$ or $\in H\langle f(x_1)\rangle$. Suppose, e.g.,

$$f(z_i) = g \prod u_j f(x_1)^{\varepsilon_j} u_j^{-1}, \qquad g \in G.$$

Let $v_j \in F$ such that $f(v_j) = u_j$; then let $w_i = \prod v_j x_1^{\varepsilon_j} v_j^{-1}$. it follows that $f(z_i w_i^{-1}) = g \in G$. So there can be found $w_i \in \langle x_1\rangle$ such that $f(z_i w_i^{-1}) \in G$ or H. Let

$$x_2 = z_2 w_2^{-1}, \ldots, x_n = z_n w_n^{-1}.$$

Then $x_1, \ldots, x_n$ satisfy the conclusions of the theorem. ∎

Remark: Because of the counterexample, it is possible that $\{x_1, \ldots, x_n\}$ may not be a basis of F. In fact in the counterexample, we may find x_1, x_2 to be the elements: x, $y^2x^2yx^{-2}y^{-2}$, which are both mapped by ψ into G.

I do not know whether this theorem in some form remains true if F has infinite rank.

E N D

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Epilogue

I typed this thesis in Princeton New Jersey in 1959. I have rendered it into TeX in November 1996, in Berkeley California; the wording and order has remained unchanged; an occasional formula has been displayed rather than having it in the middle of a line; a very few misprints have been corrected (and many new ones have been introduced); one picture (on page 9) has been rendered in words rather than being drawn by computer. Here are references to some of my own papers which are related to this thesis:

[1] 'A topological proof of Grushko's theorem on free products', Math. Z. 90 (1965), 1–8.

[2] 'Foldings of G-trees', *Arboreal Group Theory*, MSRI Publ. 19, Springer (1991), 355–368.

[3] 'A remark about the description of free product of groups', Proc. Cambridge Philos. Soc. 62 (1966), 129–134.

[4] *Group theory and three-dimensional manifolds*, Yale Math. Monographs 4, Yale U. Press (1971).

[5] 'Sur un generalization del notion de producto libere amalgamate de gruppos', Magnus preprint 93-06-16B.

The basic matters in II.B and II.D were included in [1], and redone much later in the language of foldings of trees in [2]. The theory of "oidgroups" was exposited under the name "partial group" in [3]. This is somewhat related to the notion of "pregroup" which is included in [4]; the original version of this [5], in Interlingua, originally written in 1968, can be located in the Magnus archive `http://zebra.sci.ccny.cuny.edu/web/html/1993.html`.