The Embedding of Homotopy Types into Manifolds

John R. Stallings
(stall@math.berkeley.edu)
University of California, Berkeley

Keywords: embedding, simple homotopy type, general position

MSC2000: 57Q10, 57Q35, 57Q65

0. Introduction

The theorem is:

If K is a polyhedron of dimension k, M a manifold of dimension m, and $f : K \to M$ a map which is $(2k - m + 1)$-connected, then there is a procedure which, when $k \leq m - 3$, yields a k-dimensional subpolyhedron $K_1 \subset M$ and a simple homotopy equivalence $K \to K_1$ for which the composition $K \to K_1 \subset M$ is homotopic to $f : K \to M$.

A notion of "equivalence" of such "homotopic embeddings" is discussed; this is briefly related to regular neighborhoods and quasi-isotopies. This is strongly related to Mazur's [9] theory of "neighborhoods".

The proof uses a "general position" argument to ameliorate singularities and a simple construction which must be iterated several times.

Our result is related to Haefliger's [5] and Hudson's [6] theorems on embedding differentiable and polyhedral manifolds. By trying to embed only up to homotopy, we simplify matters.

The theories of Shapiro [12] and Wu [17] about obstructions to the embedding of polyhedra suggest that our result is a special case of an obstruction theory, whose explicit character we can only faintly surmise.

A conjecture of Peterson [11] was the starting point of this investigation. We prove this conjecture (with its misprint corrected) as Corollary **4.2**.

1. Polyhedra and general position

We consider *polyhedra* (that is, finite unions of convex hulls of finite subsets of a real vector space) and *polyhedral maps* (that is, functions whose graphs are polyhedra). We know, among other things, that poly-

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hedra have simplicial triangulations and that continuous functions can be approximated by polyhedral maps.

If $f : K \to L$ is a polyhedral map and $\Sigma \subset K$, we shall say that Σ *contains the singularities of* f, if f sends $K - \Sigma$ in a 1-1 way onto $f(K) - f(\Sigma)$.

A polyhedral map $K \to L$ is *non-degenerate* if, for some triangulation of K, every simplex is mapped 1-1 onto its image.

The *dimension*, dim K, of a polyhedron K or of an open subset of a polyhedron, has an obvious meaning. We find it convenient to understand that dim $K < 0$ means $K = \emptyset$, the empty set.

A polyhedron M is said to be *like an* m*-manifold*, if the following porposition is true for every K, L, and f.

$$\mathcal{GP}(K^k, L^{(\ell)}, f, M^{(m)}, \Sigma; g, \Sigma_1) :$$

Let K *be a subpolyhedron of* L*; let* $f : L \to M$ *be a map such that* $f|K$ *is non-degenerate; let* $\Sigma \subset K$ *contain the singularities of* $f|K$*. Suppose* dim $L \leq m$*.*

Then f *can be approximated by a map* $g : L \to M$ *such that* $g|K = f|K$*; and there exists* $\Sigma_1 \subset L$ *which contains the singularities of* g*, such that*

$$\text{dim } (\Sigma_1 - \Sigma) \leq \text{Max } (2\ell - m, k + \ell - m)$$

where $k = \text{dim } K$, $\ell = \text{dim } (L - K)$.

If $M = \mathbf{R}^m$, this proposition can be proved by triangulating L so that K is a full subcomplex (i.e., determined by its vertices) and so that f is linear on each simplex; g is obtained by perturbing the images of the vertices of $L - K$, so that they are in general position with respect to each other and to the images of K-vertices.

If M is locally polyhedrally like $\mathbf{R}^m$ (i.e., a PL m-manifold), the proposition can be deduced by a not completely trivial argument based on the fact that $\mathbf{R}^m$ is like an m-manifold. It is also true that PL m-manifolds with boundary are like m-manifolds. For details see [18].

Clearly there are polyhedra which are like m-manifolds but are not PL manifolds, such as $P \times M$ where P is arbitrary and M is a PL m-manifold; there are, no doubt, more interesting examples.

2. Cell-complexes and simple homotopy equivalence

We shall need to consider identification spaces. It is not a natural thing to consider them in the category of polyhedra; but it is natural to consider them in the category of finite cell-complexes (maps in this

category are to be *cellular*, i.e., for all n to send n-skeleton into n-skeleton). In order to get from a polyhedron to a cell-complex, we triangulate. To go the other way, we need the proposition:

Proposition 2.1. *If C is a finite cell-complex with a subcomplex K which is a polyhedron, then there is a polyhedron L containing K and a simple homotopy equivalence $C \approx L$ which is the identity on K, such that*

$$\dim (C - K) = \dim (L - K).$$

We construct L a cell at a time, starting with the lowest dimensional cells of $C - K$. Thus the problem is reduced to the case that $C - K$ consists of a single n-cell. Approximate the attaching map of the boundary, S^{n-1}, of the n-cell D^n, by a polyhedral map. Triangulate K and D^n so that the attaching map is simplicial and so that S^{n-1} is a full subcomplex of D^n. Make minimal identifications on the barycentric subdivision of D^n so that the resulting identification space of $K \cup D^n$ has a triangulation; the result is L. L is obtained by adding some fringe which collapses to K, plus an honest n-cell.

Simple homotopy equivalence is discussed in [10, 16]. One calls certain maps of cell-complexes, simple homotopy equivalences; with regard to polyhedra, the notion is independent of which (linear) triangulations we choose.

Given a map $f : C_1 \to C_2$, we form the *mapping cylinder*, Map (f) as the cell-complex obtained from $C_1 \times [0, 1]$ and C_2 by identifying $(a, 1)$ with $f(a)$ for all $a \in C_1$. There is a *standard retraction* $r :$ Map $(f) \to C_2$ defined by $r(a, t) = f(a)$ for $a \in C_1$; there is also an inclusion $C_2 \subset$ Map (f). The following are well-known facts:

2.2. *Given $f : C_1 \to C_2$, both the inclusion $C_2 \to$ Map (f) and the standard retraction* Map $(f) \to C_2$ *are simple homotopy equivalences.*

2.3. *If C_3 contains as a subcomplex* Map (f), *where $f : C_1 \to C_2$, and if E is the identification space of C_3 obtained by setting $b = r(b)$ for $b \in$ Map (f), where r is the standard retraction, then the identification map $C_3 \to E$ is a simple homotopy equivalence.*

3. The elementary construction

A map $f : K \to M$ is said to be *p-connected* if the relative homotopy groups $\pi_1(\text{Map } (f), K \times 0)$ are trivial for $i \leq p$. This is equivalent to the condition that f induces an isomorphism $\pi_i(K) \to \pi_i(M)$ for $i < p$ and a surjection $\pi_p(K) \to \pi_p(M)$.

Theorem 3.1. *$(K, s, M, f; L, g)$: Let $f : K \to M$ be a polyhedral map which is $(s+1)$-connected; let $\Sigma \subset K$ contain the singularities of f, where $\dim \Sigma \leq s$. Then there exists a polyhedron L which contains $f(K)$, such that the composition*

$$K \to f(K) \subset L$$

is a simple homotopy equivalence, and such that

$$\dim\ (L - f(K)) \leq s + 2;$$

and there exists a map $g : L \to M$ which extends the inclusion $f(K) \subset M$.

Proof. Consider $f' : \Sigma \to f(\Sigma)$, the restriction of f. Then

$$\text{Map } (f') \subset \text{Map } (f)$$

with

$$\Sigma \times 0 \subset K \times 0.$$

And

$$\dim\ (\text{Map } (f')) \leq s + 1.$$

Since $(\text{Map } (f), K \times 0)$ is $(s+1)$-connected, there is a deformation of Map (f') into $K \times 0$, leaving $\Sigma \times 0$ fixed. That is, there is a map

$$h : [0,1] \times \text{Map } (f') \to \text{Map } (f),$$

such that

$h(0, z) = z$
$h(1, z) \in K \times 0$, for $z \in \text{Map } (f')$.
$h(t, \sigma) = \sigma$, for $\sigma \in \Sigma \times 0$, $0 \leq t \leq 1$.

Define K_1 to be

$$\{K \times 0\} \cup \{[0,1] \times \text{Map } (f')\},$$

identifying $h(1, z)$ with $(1, z)$. K_1 is just the mapping cylinder of $h(1, \ )$, and so the inclusion

$$K \times 0 \subset K_1$$

is a simple homotopy equivalence.

Define K_2 to be obtained from K_1 by identifying (t, σ) to σ for all $0 \leq t \leq 1$ and $\sigma \in \Sigma \times 0$. Since $[0,1] \times (\Sigma \times 0)$ is a mapping cylinder (of the identity map on $\Sigma \times 0$), and this identification is obtained from the standard retraction, it follows that the identification map

$$K_1 \to K_2$$

is a simple homotopy equivalence.

There is a map $h^* : K_2 \to \text{Map } (f)$, induced from h and the identity map on $K \times 0$, since our identifications have been designed with this in mind.

Define r to be the standard retraction Map $(f) \to M$ and f' to be the standard retraction Map $(f') \to f(\Sigma)$.

Define K_3 to be obtained from K_2 by identifying $(0, z)$ to $(0, r'(z))$ for $z \in \text{Map } (f')$. The resulting identification map

$$K_2 \to K_3$$

is a simple homotopy equivalence, since in the earlier identifications the subcomplex $0 \times \text{Map } (f')$ has not been affected.

The map $r \circ h^* : K_2 \to M$ respects the identifications forming K_3, and so defines a map

$$\varphi : K_3 \to M.$$

The subset of K_3 which was derived from

$$(K \times 0) \cup ([0, 1] \times \Sigma \times 0) \cup (0 \times \text{Map } (f'))$$

under our identifications can be equated with K, in which Σ has been identified to $f(\Sigma)$; this is nothing but $f(K)$. On this subset φ is the inclusion of $f(K)$ into M. The composite simple homotopy equivalence

$$K \to K \times 0 \to K_1 \to K_2 \to K_3$$

can be factored through $K \to f(K)$ and $f(K) \subset K_3$.

K_3 is a cell-complex containing the polyhedron $f(K)$. By **2.1**, we can replace it by a polyhedron L of the same simple homotopy type with

$$\dim (L - f(K)) = \dim (K_3 - f(K)).$$

φ is replaced by a polyhedral map

$$g : L \to M$$

We note that dim $(K_3 - f(K))$ is the dimension of $I \times \text{Map } (f')$, which is dim $(\Sigma) + 2$, which is $\leq s + 2$.

This completes the proof of **3.1**.

4. The embedding process.

Theorem 4.1. $(K^k, M^{(m)}, f; K_1)$: *If K is a polyhedron of dimension k, if M is like an m-manifold, and if $f : K \to M$ is a map which is*

$(2k-m+1)$-connected, then there is a procedure which, when $k \leq m-3$, yields a k-dimensional subpolyhedron $K_1 \subset M$ and a simple homotopy equivalence $h : K \to K_1$ for which the composite $K \to K_1 \subset M$ is homotopic to $K \to M$.

Proof.

STEP 1. Apply

$$\mathcal{GP}(\emptyset, K^{(k)}, f, M^{(m)}, \emptyset; f_1, \Sigma).$$

f_1 approximates f, and $\Sigma \subset K$ contains the singularities of f_1, and $\dim \Sigma \leq 2k - m$.

Set $K_1 = K$.

Set $h : K \to K_1$ to be the identity on K.

Set $s = 2k - m$.

STEP 2. Apply

$$\mathbf{3.1}(K_1, s, M, f_1; L, g).$$

This yields a map $g : L \to M$, where L contains $f_1(K_1)$, and where the map, which will be called h_1,

$$K_1 \to f_1(K_1) \subset L$$

is a simple homotopy equivalence. Furthermore, g is the embedding on $f_1(K_1)$ into M, and

$$\dim\ (L - f_1(K_1)) \leq s + 2.$$

STEP 3. Apply

$$\mathcal{GP}(f_1(K_1)^k, L^{(s+2)}, g, M^{(m)}, \emptyset; g_1, \Sigma_1).$$

Then g_1 approximates g and is the embedding on $f_1(K_1)$, and Σ_1 contains the singularities of g_1, where

$$\dim \Sigma_1 \leq \text{Max}\ (k + s + 2 - m, 2s + 4 - m).$$

STEP 4. If Σ_1 is not empty,

Set $s = \dim \Sigma_1$.

Set h to be the composition

$$K \overset{h}{\to} K_1 \overset{h_1}{\to} L.$$

Set K_1 to be L.

Set f_1 to be g_1.

And return to Step 2.

STEP 5. If Σ_1 is empty,

Set h to be the composite

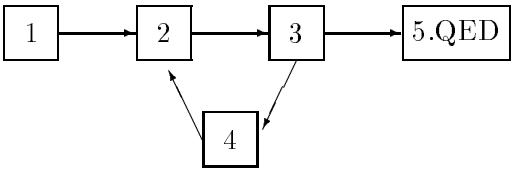
$$K \stackrel{h}{\to} K_1 \stackrel{h_1}{\to} L.$$

Set K_1 to be L.

Set the inclusion of K_1 into M to be the map g_1.

QED.

This procedure can be summarized by a flow diagram [4]:



We must determine to what situations this procedure can be applied and what conditions assure that eventually we can escape from the loop in the diagram to get the desired final result at Step 5.

We want to have dim $K_1 \leq k$ always. This is true to start with; in Step 4 we change K_1 to L. The dimension of L, constructed in Step 2, is $\leq$ Max $(k, s+2)$; hence it is necessary to have

$$(*) \qquad s+2 \leq k.$$

To start with, $s = 2k-m$, and so to make a beginning we must have

$$s+2 = (2k-m)+2 \leq k, \text{ or}$$

$$(**) \qquad k \leq m-2.$$

Step 4 changes s into dim Σ_1, which by Step 3 is $\leq$ Max $(k+s+2-m, 2s+4-m)$. Assuming (**) and that we have got condition (*), both

$$k+s+2-m \leq s, \text{ and}$$

$$2s+4-m \leq s.$$

Hence, under assumption (**), Step 4 never increases s, and condition (*) is always satisfied.

In Step 2, to apply **3.1** we need the hypothesis that f_1 is $(s+1)$-connected. To start with, f_1 is homotopic to f, and the homotopy properties of f_1 never change. We assume, in the hypothesis of the theorem, that f is $(2k-m+1)$-connected; the earlier discussion showed that, assuming (**), we always have $s \leq 2k-m$; hence Step 2 can always be done if $k \leq m-2$.

Our procedure is thus applicable whenever $k \leq m-2$. But we may never escape from the loop in the diagram; the changes occurring in Steps 2 and 3 may not decrease the dimension of the singularities.

If we assume

$$(***) \qquad k \leq m-3,$$

then in Step 3 we have

$$\dim \Sigma_1 \leq \text{Max } (k+s+2-m, 2s+4-m).$$

We start with $s = 2k-m \leq k-3$, and never increase s, and so:

$$k+s+2-m \leq s-1$$

$$2s+4-m \leq s-2.$$

Hence dim $\Sigma_1 \leq s-1$. This becomes the new s in step 4. The dimension of Σ_1 decreases each time the loop is traversed. Finally, when dim $\Sigma_1 < 0$, we proceed to Step 5 and escape with the desired result.

The theorem is proved.

Corollary 4.2. *A k-connected, p-dimensional polyhedron K is the same simple homotopy type as a subpolyhedron of R^{2p-k}.*

Proof. If $k \geq p$, K is contractible and so has the homotopy type of a point.

If $k = p-1$, then K has the homotopy type of a bouquet of p-spheres, which can be embedded in R^{p+1}.

If $k = p-2$, then K is the homotopy type of the $(p-2)$-fold suspension of a 2-complex; a 2-complex can be embedded up to homotopy in $\mathbf{R}^4$ [3]. Hence its $(p-2)$-fold suspension embeds in $\mathbf{R}^{p+2}$.

If $k \leq p-3$, apply

$$\mathbf{4.1} \ (K^p, \mathbf{R}^{2p-k}, f; K_1).$$

This can be done since $p \leq 2p-k-3$, and K is k-connected, and $\mathbf{R}^{2p-k}$ is $(k+1)$-connected; hence any map $f : K \to \mathbf{R}^{2p-k}$ is k-connected. K_1 is the answer.

Theorem 4.3. $(K^k, M^{(m)}, f, s; K_1)$*: Let K be a polyhedron of dimension k, let M be like an m-manifold. Let $f : K \to M$ be a polyhedral map, with $\Sigma \subset K$ containing the singularities of f, and* dim $\Sigma \leq s$. *Suppose f is $(s+1)$-connected. Then there is a procedure which, when $k \leq m-3$ and $s \leq m-5$, yields a k-dimensional subpolyhedron $K_1 \subset M$, with $f(K) \subset K_1$ and the composite*

$$K \to f(K) \subset K_1$$

a simple homotopy equivalence.

Proof. This is really just a sharper form of **4.1**, wherein we omit Step 1 in the proof. We start by setting $K_1 = K$; set

$$h : K \to K_1 \text{ to be the identity on } K.$$

Now proceed directly to Step 2 in the procedure used for **4.1**.

The key point is the computation of dim Σ_1 in Step 3.

$$\dim \Sigma_1 \leq \text{Max } (k + s + 2 - m, 2s + r - m).$$

If $k \leq m-3$, then $k+s+2-m \leq s-1$. If $s \leq m-5$, then $2s+4-m \leq s-1$. Hence dim $\Sigma_1 \leq s - 1$. Thus our looping terminates, and we have a result K_1 embedded into M. After going through Step 4 or Step 5, from that point on the map h factors into

$$K \to f(K) \subset K_1.$$

This completes the proof of **4.3**.

5. Equivalence of embeddings.

If K_0 and K_1 are subpolyhedra of M, and if $\varphi : K_0 \times 0 \to K_1 \times 1$ is a specific simple homotopy equivalence, then we say K_0 is *equivalent* to K_1, if there is a polyhedron $L \subset M \times [0, 1]$, such that

$$L \cap M \times 0 = K_0 \times 0$$

$$L \cap M \times 1 = K_1 \times 1$$

and such that the inclusions

$$K_0 \times 0 \subset L \text{ and } K_1 \times 1 \subset L$$

are simple homotopy equivalences, with the composite

$$K_0 \times 0 \xrightarrow{\varphi} K_1 \times 1 \subset L$$

homotopic to the inclusion $K_o \times 0 \subset L$. We call L an *equivalence* between K_0 and K_1. In what follows, the simple homotopy equivalence φ will, up to homotopy, sometimes be obvious from the context.

Theorem 5.1. $(M^{(m)}, K_0, K_1^k, f; L)$*: Let* M *be a polyhedron such that* $M \times [0, 1]$ *is like an* $(m+1)$*-manifold. Let* K_0 *and* K_1 *be* k*-dimensional*

subpolyhedra of M, with a simple homotopy equivalence $f : K_0 \to K_1$ between them, such that the composite

$$K_0 \xrightarrow{f} K_1 \subset M$$

is homotopic to the inclusion $K_0 \subset M$. Suppose the pair (M, K_0) is $(2k - m + 2)$-connected, and that $k \leq m - 3$.

Then K_0 and K_1 are equivalent by an equivalence $L \subset M \times [0, 1]$, where dim $L = k + 1$.

Proof. Let L_1 be the mapping cylinder of f, suitably polyhedralized by **2.1**. The homotopy between $f : K_0 \to K_1 \subset M$ and $K_0 \subset M$, defines a map of L_1 into M which is the embedding on K_1 and on $K_0 \times 0$. We thus get a map $h : L_1 \to M \times [0, 1]$, which is the identity on $K_0 \times 0$ and maps K_1 isomorphically onto $K_1 \times 1$. Apply:

$$\mathcal{GP}((K_0 \times 0) \cup K_1^k, L_1^{(k+1)}, h, M \times [0, 1]^{(m+1)}, \emptyset; g, \Sigma).$$

The resultant g maps $K_0 \times 0$ and K_1 just as h did. $\Sigma \subset L$ contains the singularities of g, where

$$\dim \Sigma \leq 2(k + 1) - (m + 1) = 2k - m + 1.$$

The connectivity of g is the connectivity of h, which is the connectivity of (M, K_0), which is $2k - m + 2$. Furthermore, the inclusions $K_0 \to K_0 \times 0 \subset L_1$ and $K_1 \subset L_1$ are simple homotopy equivalences consistent with f.

Apply:

$$\mathbf{4.3}.(L_1^{k+1}, M \times [0, 1]^{(m+1)}, g, 2k - m + 1; L_2).$$

We can do this because $k+1 \leq (m+1)-3$ and $2k-m+1 \leq (m+1)-5$. The resultant L_2 contains the image of g, in particular $K_0 \times 0$ and $K_1 \times 1$; the desired L is obtained from L_2 by pushing L_2 into $M \times (0, 1)$ except on these subsets.

An omitted check of the necessary facts completes the proof of **5.1**.

We can summarize **4.1** and **5.1** as follows:

Comment 5.2. *Let M be like an m-manifold, $M \times [0, 1]$ like an $(m+1)$-manifold (this is the case in particular when M is a PL m-manifold). Let K be a k-dimensional polyhedron, $k \leq m - 3$; let $f : K \to M$ be a map.*

If f is $(2k - m + 1)$-connected, then f can be replaced by an embedding of something, K_1, of the same simple homotopy type as K.

If f is $(2k - m + 2)$-connected, then homotopic embeddings K_0, K_1 of things of the same simple homotopy type as K are equivalent.

If we were to assume the s-cobordism theorem for PL manifolds [2, 7, 9, 8, 13, 15], then, using the fact that the equivalence between K_0 and K_1 has "codimension" ≥ 3, we could then prove that there is a quasi-isotopy of M (i.e., a polyhedral equivalence $M \times [0,1] \to M \times [0,1]$ taking $M \times 0$ to $M \times 0$ and $M \times 1$ to $M \times 1$) which takes a regular neighborhood of K_0 onto a regular neighborhood of K_1, in case M is a PL manifold of dimension at least 6.

Let us sketch the proof. We assume:

S-COBORDISM. $(M_0^m, M_1, Q, f; g)$: *Let Q be a PL $(m+1)$-manifold with PL m-manifolds M_0 and M_1 contained in the boundary of Q (denoted ∂Q). Let f be a polyhedral equivalence of the closure of $\partial Q - (M_0 \cup M_1)$ with $(\partial M_0) \times [0,1]$, in such a way that ∂M_0 corresponds to $\partial M_0 \times 0$ by the identity map and ∂M_1 corresponds to $\partial M_0 \times 1$. Suppose that the inclusions $M_0 \subset Q$ and $M_1 \subset Q$ are simple homotopy equivalences, and that $m \geq 5$.*

Then there exists a polyhedral equivalence g of Q with $M_0 \times [0,1]$, extending f, taking M_0 to $M_0 \times 0$ by the identity map and M_1 to $M_0 \times 1$.

This is what we want to prove:

NBHD-ISOT. $(K_0^k, K_1^k, L^{k+1}, M^m; g)$: *Let K_0 and K_1 be k-dimensional polyhedra in a PL m-manifold M without boundary, which are equivalent by a polyhedron $L \subset M \times [0,1]$, where* dim $L = k+1$. *If $k \leq m-3$ and $m \geq 6$, then there is a polyhedral equivalence $g : M \times [0,1] \to M \times [0,1]$, which is the identity on $M \times 0$, which maps $M \times 1$ to $M \times 1$, and which takes $N_1 \times 1$ onto $N_0 \times 1$, where N_0 and N_1 are regular neighborhoods of K_0 and K_1.*

Proof. Triangulate $N \times [0,1]$ with L a subcomplex. Let U denote the closed star of L in the second barycentrifc subdivision. Define N_0 and N_1 by $N_0 \times 0 = U \cap (M \times 0)$; $N_1 \times 1 = U \cap (M \times 1)$. Let V denote the closure of $\partial U - (N_0 \times 0 \cup N_1 \times 1)$. Because dim $L \leq$ dim $(M \times [0,1]) - 3$, and dim $K_1 \leq$ dim $(M) - 3$, a simple application of $\mathcal{GP}$ will prove that $\pi_1(K_0) \approx \pi_1(L) \approx \pi_1(K_1)$, and

$$\pi_1(K_0) \approx \pi_1(N_0) \approx \pi_1(\partial N_0)$$

$$\pi_1(L) \approx \pi_1(U) \approx \pi_1(V)$$

$$\pi_1(K_1) \approx \pi_1(N_1) \approx \pi_1(\partial N_1)$$

A non-complicated algebraic argument, using universal covering spaces and other techniques of simple homotopy theory, will then show that since

$$K_0 \times 0 \subset L \quad \text{and} \quad K_1 \times 1 \subset L$$

are simple homotopy equivalences, the same is true of

$$\partial N_0 \times 0 \subset V \quad \text{and} \quad \partial N_1 \times 1 \subset V.$$

Apply

$$\text{S-COBORDISM}((\partial N_0 \times 0)^{m-1}, \partial N_1 \times 1, V, \emptyset; g_1).$$

(Here $\emptyset$ denotes the empty function.) The result is a polyhedral equivalence g_1 of V with $\partial N_0 \times [0,1]$. Next, apply

$$\text{S-COBORDISM}((N_0 \times 0)^m, N_1 \times 1, U, g_1; g_2).$$

The result is a polyhedral equivalence g_2 of U with $N_0 \times [0,1]$ which extends g_1.

Finally, let W be the closure of $M \times [0,1] - U$, with $P_0 = W \cap (M \times 0)$ and $P_1 = W \cap (M \times 1)$. Because of the codimension 3 condition we have isomorphisms

$$\pi_1(M \times [0,1]) \approx \pi_1(P_0) \approx \pi_1(W) \approx \pi_1(P_1).$$

Furthermore, the complement of W, *viz.* U, has just been shown to offer no problems with simple homotopy type. Hence

$$P_0 \subset W \quad \text{and} \quad P_1 \subset W$$

are simple homotopy equivalences. Apply

$$\text{S-COBORDISM}(P_0^m, P_1, W, g_1; g_3).$$

Then g_3 is a polyhedral equivalence of W with $P_0 \times [0,1]$ which extends g_1 on V.

The desired map g is just the union of g_2 and g_3; this takes $W \cup U$ onto $(P_0 \times [0,1]) \cup (N_0 \times [0,1])$, agreeing with g_1 which takes

$$V \quad \text{onto} \quad \partial N_0 \times [0,1].$$

This completes the proof of NBHD-ISOT, assuming S-COBORDISM.

6. Afterword

This paper was dated November, 1965; Princeton University.

Comment 6.1. In a later version, dated 1971, I explained the "subroutine" notation thus:

This paper was originally written in 1965 when the author was obsessed with electronic computers. This may explain a certain peculiarity

of style, which in turn led to an emotional block against submitting it for publication. But now that more and more mathematicians have gone through a similar phase, the style may not be distasteful.

In geometric topology especially, the reference to theorems by "Subroutine" notation seems to me quite valid. The title of a theorem, for instance,

$$\mathcal{GP}(K^k, L^{(\ell)}, F, M^{(m)}, \Sigma; g, \Sigma_1)$$

contains various "symbolic parameters", such as in this case, K, k, L, ℓ, F, M, m, Σ, g, Σ_1, whose relationships form the theorem's assertion. When this theorem is referred to in a later discussion, various "actual parameters" are inserted; e.g.,

$$\mathcal{GP}(\emptyset, K^{(k)}, f, M^{(m)}, \emptyset; f_1, \Sigma)$$

This is understood to mean that in the context of the discussion, we consider the variant of $\mathcal{GP}$ in which K^k is replaced by the empty set, L by K, ℓ by k, F by f, etc.

It is customary in mathematical discussion to say simply, "...Now apply general position...". Unless the subject is intuitive and familiar, this manner of explanation is difficult to follow. Since our general position statement is involved and contains many variables, it may well take an extraordinary intellectual effort to determine just how general position is to be applied. The Subroutine notation, on the other hand, requires no intellectual effort at all, at least in theory, if there are no typographical mishaps.

Comment 6.2. The idea of something which is "like an m-manifold", in that maps into it behave as maps into an m-manifold with respect to general position, has been gone into further by E. Akin [1] and myself [14, Chapter V], so that a similar result is true for these $ND(m)$ polyhedra. An $ND(m)$ polyhedron is a polyhedron P satisfying one of the following equivalent three descriptive propositions:

(1) In a triangulation of P, the link of each simplex σ is $(m - (\dim\sigma) - 2)$-connected.

(2) The link of each point of P is an $(m-2)$-connected $ND(m-1)$ polyhedron. An $ND(0)$ polyhedron is any polyhedron; a (-1)-connected space is any *non-empty* space.

(3) If K is any m-dimensional polyhedron, $f : K \to P$ any map, then f can be approximated arbitrarily closely by a non-degenerate polyhedral map. A *non-degenerate* map is one for which, for every point of P, the preimage consists of only a finite set.

A polyhedral m-manifold, with or without boundary, is $ND(m)$. The cartesian product of $ND(r)$ and $ND(s)$ is $ND(r+s)$. The k-skeleton of any triangulation of an $ND(m)$ polyhedron, where $k \leq m$, is $ND(k)$.

Comment 6.3. Let me give a non-Subroutine idea of how to prove **4.2**; this is directed to non-machines who have some intuitive idea of general position.

We try to embed a k-connected p-dimensional polyhedron K up to homotopy in $\mathbf{R}^n$, and we are assuming that $k \leq p-3$ and $n = 2p-k$. A generic map $f : K \to \mathbf{R}^n$, by general position, will have singularities of dimension at most $2p-n$, which happens to equal k. Thus, these singularities can be contracted in K, across tracks of dimension $k+1$ and this enables us to add to $f(K)$ some extra cells of dimension at most $k+2$ to get a polyhedron K' of the same (simple) homotopy type as K. We then extend the embedding of $f(K)$ in $\mathbf{R}^n$ to a generic map $f' : K' \to \mathbf{R}^n$; this extension will have singularities that involve intersections of $(k+2)$-dimensional cells (the ones we added on) with $(k+2)$-cells (other added on cells), and with p-cells (the cells of $f(K)$) in $\mathbf{R}^n$; and so, by general position will have singularities of dimension at most the maximum of

$$2(k+2)-n = 3k+4-2p = k+2k+(4-2p) \leq k+2(p-3)+4-2p = k-2$$

and

$$(k+2)+p-n = k+2+p-(2p-k) = k-1+(k-p+3) \leq k-1.$$

We started with singularities of dimension k, and now we have singularities of dimension at most $k-1$. After a while, continuing this process, the singularities have negative dimension, and *voilà!* we have embedded K up to simple homotopy type in $\mathbf{R}^n$.

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