An Extension Theorem for Euler Characteristics of Groups

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Abstract. We give a homological definition of the Euler characteristic $\mu(G)$ of a group G; if N is a normal subgroup of G with quotient group H, and if $\mu(H)$ and $\mu(N)$ are defined, then $\mu(G)$ is defined, and is the product of the other two. Several conjectures and problems are proposed.

Keywords: homological algebra, rational Euler characteristic, group extensions

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0. Introduction and Historical Note

Introduction

Wall [10] introduced the notion of Euler characteristic $\chi(G)$ for certain groups G. It is the topological Euler characteristic of the Eilenberg-MacLane space $K(G,1)$ when this has the homotopy type of a finite complex. If G has finite index n in a group H, then $\chi(H)$ is the rational number $\frac{1}{n}\chi(G)$.

Serre [7] gave a homological definition which generalized Wall's to those groups N containing a subgroup G of finite index for which $\mathbf{Z}$ has a finite free resolution over $\mathbf{Z}G$. Brown [1] considered a more general case that included the situation where $\mathbf{Z}$ has a finite projective resolution over $\mathbf{Z}G$.

In a slightly different direction, Chiswell [3] has defined $\chi(G)$ for those groups G for which $\mathbf{Q}$ has a finite projective resolution over $\mathbf{Q}G$. The present work uses Chiswell's definition. Recall from [8] that $\chi(G;\mathbf{Q})$ is a $\mathbf{Q}$-linear combination of conjugacy classes of G; the analogue of the topological characteristic is the coefficient of the trivial class and is denoted $\mu(G)$ by Chiswell. The major aim of this paper is a proof of the conjecture that if $\chi(N)$ and $\chi(G/N)$ are defined, where N is normal in G, then

$$\mu(G) = \mu(N) \cdot \mu(G/N).$$

What we do is to compute $\chi(G)$ in its entirety, out of $\chi(G/N)$ and the way in which G/N acts (modulo inner automorphisms of N) on a finite projective resolution of $\mathbf{Q}$ over $\mathbf{Q}N$.


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In addition to the multiplicative result on μ sketched above, we describe the homological algebra needed in detail. We show that extending the coefficient field does nothing. We describe the results of Chiswell on amalgamated free products. We include some material on finite-dimensional vector spaces on which G acts. Since this research raised more questions than it settled, some of these are collected at the end of the paper.

Historical Note

Chiswell's work relies on our paper [8] explaining a result of Gottlieb [4] in homological terms. The "trace function," which is the basic idea, was described from another point of view by Hattori [5]. Gottlieb generalized the ideas of Nielsen [6] on the Lefschetz fixed-point theorem. What is needed to get the calculation of the Euler characteristic of an extension, is to generalize the basic ideas of homological algebra (chain-mappings, homotopies, etc.) to include what Nielsen was considering. Wecken [11] did much of this algebra.

1. Homological Algebra.

Definitions.

1.1. Let Λ be a ring with 1. By a *module* we mean a left Λ-module. A *map* means a homomorphism of left Λ-modules. We shall call these maps "I-maps" when "α-maps" (see **1.4**) are simultaneously being discussed.

A *projective* P is a projective module. *Finite projective* means finitely generated projective. We shall write maps on the left of elements on which they operate, and denote by $1_M : M \to M$ the identity map. If P is a (finite) projective, there is a (finitely generated) free module F and maps

$$\pi : F \to P \text{ and } \iota : P \to F$$

such that $\pi\iota = 1_P$. We say that (F, π, ι) *represents* P *as a projective*.

1.2. A *complex* $\mathcal{C}$ is a sequence of modules $\{C_i\}$ and maps $\{\partial_i : C_i \to C_{i-1}\}$ such that $C_i = 0$ for $i < 0$, and $\partial_{i-1}\partial_i = 0$ for all i.

The *homology* $H(\mathcal{C})$ is the sequence of modules

$$\{H_i(\mathcal{C}) = \operatorname{Ker}\partial_i / \operatorname{Im}\partial_{i+1}\}.$$

We write $H(\mathcal{C}) = M$ when $H_0(\mathcal{C}) = M$ and $H_i(\mathcal{C}) = 0$ for $i \neq 0$.

If $H(\mathcal{C}) = 0$, we say $\mathcal{C}$ is an *exact sequence*.

If a map $\varepsilon : C_0 \to M$ is given such that $\varepsilon\partial_1 = 0$, we say $\mathcal{C} \to M$ is a *complex over* M.

If $H(\mathcal{C}) \approx M$ we say $\mathcal{C} \to M$ is an *exact complex* over M.

1.3. If $\mathcal{P}$ is a complex such that each P_i is projective, $\mathcal{P}$ is a *projective complex*. An exact projective complex over M is called a *projective resolution* of M. If each P_i is a finite projective, we call $\mathcal{P}$ *locally finite*. If there is n such that $P_i = 0$ for $i > n$, we call $\mathcal{P}$ a *bounded complex* of *length* $\leq n$ (length $= n$ if $P_n \neq 0$).

If $\mathcal{P} \to M$ is a locally finite, bounded projective resolution, we say $\mathcal{P}$ is an FP-resolution of M. A module M is an *FP-module* if it has an FP-resolution.

1.4. Ring homomorphisms are supposed to preserve 1. Let $\alpha : \Lambda \to \Lambda$ be a ring endomorphism. If A and B are modules and $\varphi : A \to B$ a homomorphism of abelian groups, we call φ an *α-map* if

$$\varphi(\lambda a) = \alpha(\lambda)\varphi(a) \quad \text{, for } \lambda \in \Lambda,\ a \in A.$$

We note that the set of α-maps from A to B, being closed under addition, forms an abelian group $\mathrm{Hom}_\alpha(A, B)$. If $f : A \to B$ is an α-map and $g : B \to C$ is a β-map, then the composition $gf : A \to C$ is a $\beta\alpha$-map. In this paper we shall *never* consider such a composition unless either α or β or both are the identity ring-homomorphism $I : \Lambda \to \Lambda$. In other words, one of f and g, at least, must be an ordinary Λ-homomorphism.

As an aid to keeping things straight, when a diagram of I-maps and α-maps is drawn, as far as possible the I-maps will be represented by horizontal, the α-maps by vertical, arrows.

All direct sums, projective modules, complexes, exact sequences of complexes, are defined in terms of I-maps, and the mapping cone (see **1.8**) of a chain map φ is defined only when φ is an I-map.

1.5. Let $\mathcal{A}$, $\mathcal{B}$ be complexes. A sequence of α-maps $\{\varphi_i : A_i \to B_i\}$ is called a *chain α-map* $\varphi : \mathcal{A} \to \mathcal{B}$ if

$$\partial_{i,B}\varphi_i = \varphi_{i-1}\partial_{i,A} \quad \text{for all } i.$$

If $\mathcal{A} \xrightarrow{\varepsilon} M$, $\mathcal{B} \xrightarrow{\eta} N$ are complexes over M and N, and $f : M \to N$ is an α-map, we say that a chain α-map $\varphi : \mathcal{A} \to \mathcal{B}$ *extends* f when

$$\eta\varphi_0 = f\varepsilon.$$

1.6 Let $\mathcal{A}$, $\mathcal{B}$ be complexes and let $\varphi, \psi : \mathcal{A} \to \mathcal{B}$ be two chain α-maps. An *α-homotopy d between φ and ψ* is a sequence of α-maps

$d_i : A_i \to B_{i+1}$ such that

$$\partial_{i+1,B} d_i + d_{i-1}\partial_{i,A} = \varphi_i - \psi_i \quad \text{, for all } i.$$

φ and ψ are *α-homotopic* if there exists such a homotopy.

1.7. A *short exact sequence of complexes*

$$0 \to \mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \to 0$$

consists of two chain I-maps φ and ψ, such that for all i

$$0 \to A_i \xrightarrow{\varphi_i} B_i \xrightarrow{\psi_i} C_i \to 0$$

is an exact sequence.

1.8. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a chain I-map. The *mapping cone* $\mathcal{M}(\varphi)$ is the chain complex defined thus:

$$M_i = A_{i-1} \oplus B_i$$

$$\partial_{\mathcal{M}}(a,b) = (-\partial_A a, \varphi a + \partial_B b)$$

Note that $\partial^2_{\mathcal{M}}(a,b) = (\partial^2_A, -\varphi\partial_A a + \partial_B\varphi a + \partial^2_B b) = 0$.

Theorems.

1.9. A chain α-map $\varphi : \mathcal{A} \to \mathcal{B}$ induces a sequence of α-maps on homology

$$\varphi_* : H_i(\mathcal{A}) \to H_i(\mathcal{B})$$

in the usual way. If two chain α-maps φ, ψ are α-homotopic, then $\varphi_* = \psi_*$.

1.10. If

$$0 \to \mathcal{A} \xrightarrow{\varphi} \mathcal{B} \xrightarrow{\psi} \mathcal{C} \to 0$$

is a short exact sequence of complexes, there is a sequence of I-maps

$$\partial_* : H_i(\mathcal{C}) \to H_{i-1}(\mathcal{A})$$

and a long exact sequence

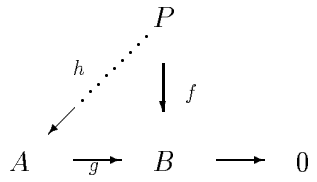
$$\cdots \to H_i(\mathcal{A}) \xrightarrow{\varphi_*} H_i(\mathcal{B}) \xrightarrow{\psi_*} H_i(\mathcal{C}) \xrightarrow{\partial_*} H_{i-1}(\mathcal{A}) \to \cdots$$

1.11. If $\varphi : \mathcal{A} \to \mathcal{B}$ is a chain I-map and $\mathcal{M} = \mathcal{M}(\varphi)$ is the mapping cone, there is a short exact sequence

$$0 \to \mathcal{B} \to \mathcal{M} \to \mathcal{A}' \to 0$$

where $A'_i = A_{i-1}$, $\partial'_i = -\partial_{i-1}$ in $\mathcal{A}'$. The maps $\partial_* : H_{i+1}(\mathcal{A}') \to H_i(\mathcal{B})$ can be identified with $\varphi_* : H_i(\mathcal{A}) \to H_i(\mathcal{B})$.

1.12. In the situation



where P is projective, g is an I-epimorphism, and f is an α-map, there exists an α-map $h : P \to A$ such that $gh = f$.

Proof. Let (F, π, ι) represent P as a retract of a free module F with basis $\{x_i\}$. For each x_i pick $a_i \in A$ such that $f\pi(x_i) = g(a_i)$. Define

$$k(\lambda_i x_i) = \alpha(\lambda_i) a_i$$

—Note the *summation convention* that in a monomial when indices are repeated they are to be summed over, unless the contrary is explicitly stated.—

Then $k : F \to A$ is an α-map; the trivial proof of this uses the fact that α is a ring endomorphism. And $gk = f\pi$. Define $h = k\iota : P \to A$. Then $gh = gk\iota = f\pi\iota = f$.

Now, using **1.12** as the major tool, we can prove (Cf. Cartan-Eilenberg [2], Ch. V.1.):

1.13. Let $\mathcal{P} \to A$ be a projective complex over A, and $\mathcal{C} \to B$ an exact complex over B; let $f : A \to B$ be an α-map. Then f extends to a chain α-map $\varphi : \mathcal{P} \to \mathcal{C}$, and any two such extensions are α-homotopic.

The proof is exactly like the usual one of homological algebra.

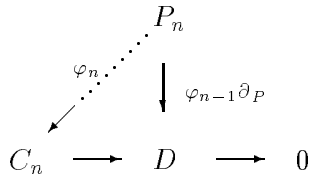
The chain α-map φ is defined bit by bit. Suppose $\varphi_i : P_i \to C_i$ has been defined for $i < n$, commuting with ∂. Then on P_n

$$\partial_C \varphi_{n-1} \partial_P = \varphi_{n-2} \partial_P \partial_P = 0$$

and hence $\mathrm{Im}\,(\varphi_{n-1}\partial_P)$ lies in

$$D = \mathrm{Ker}\,(C_{n-1} \to C_{n-2}) = \mathrm{Im}\,(C_n \to C_{n-1}).$$

So to define φ_n requires solving the diagram



which can be done by **1.12**.

Similarly, the α-homotopy d between two chain α-maps φ and ψ is defined bit by bit. Suppose $d_i : P_i \to C_{i+1}$ has been defined for $i < n-1$ with

$$\partial d + d\partial = \varphi - \psi$$

when possible. As before, let $D = \mathrm{Im}\,(C_n \to C_{n-1})$. On P_{n-1} we have

$$\begin{aligned}
\partial_C(\varphi_{n-1} - \psi_{n-1} - d_{n-2}\partial_P) &= \\
\partial_C(\varphi_{n-1} - \psi_{n-1}) - (\partial_C d_{n-2})\partial_P &= \\
\partial_C(\varphi_{n-1} - \psi_{n-1}) - (\varphi_{n-2} - \psi_{n-2} - d_{n-3}\partial_P)\partial_P &= \\
\partial_C(\varphi_{n-1} - \psi_{n-1}) - (\varphi_{n-2} - \psi_{n-2})\partial_P &= 0.
\end{aligned}$$

Thus the image of $\varphi_{n-1} - \psi_{n-1} - d_{n-2}\partial_P$ lies in D, and to define d_{n-1} requires solving the diagram

$$\begin{array}{ccccc}
 & & P_{n-1} & & \\
 & \overset{d_{n-1}}{\swarrow} & \downarrow{\scriptstyle \varphi_{n-1}-\psi_{n-1}-d_{n-2}\partial_P} & & \\
C_n & \longrightarrow & D & \longrightarrow & 0
\end{array}$$

which is done by **1.12**.

By noticing carefully what happens at the last stage when the complexes are bounded, we can say a sharper result.

1.14. Let $\mathcal{D} \to A$ be a complex over A and $\mathcal{C} \to B$ an exact complex over B; let $f : A \to B$ be an α-map. Suppose both $\mathcal{D}$ and $\mathcal{C}$ are of length $\leq n$:

(a) If $D_0, \ldots, D_{n-1}$ are projective (nothing assumed about D_n), there exists a chain α-map $\varphi : \mathcal{D} \to \mathcal{C}$ extending f.

(b) If $D_0, \ldots, D_{n-2}$ are projective (nothing assumed about D_{n-1} and D_n), and φ,$\psi : \mathcal{D} \to \mathcal{C}$ are two chain α-maps extending f, then there exists an α-homotopy d between φ and ψ.

The following corollary can be proved in a mundane way, but we prefer an elegant trick due to J.H.C. Whitehead [12], that is reminiscent of harmonic functions:

1.15. Let

$$0 \to D_n \to D_{n-1} \to D_{n-2} \to \cdots \to D_0 \to 0$$

be an exact sequence, where $D_0, \ldots, D_{n-2}$ are projective. Then

$$D_0 \oplus D_2 \oplus \cdots = D_{even}$$

is isomorphic to

$$D_1 \oplus D_3 \oplus \cdots = D_{odd}.$$

Proof. Regard $\mathcal{D}$ as a complex over 0. Two chain I-maps, namely 0 and $1_{\mathcal{D}}$, extend $0 \to 0$. By **1.14**(b) there is an I-homotopy d between $1_{\mathcal{D}}$ and 0. In other words, $d\partial + \partial d = 1_{\mathcal{D}}$. Define $\delta = d\partial d$. Compute:

$$\begin{aligned} \delta\partial + \partial\delta &= d(\partial d)\partial + \partial(d\partial)d \\ &= d(1 - d\partial)\partial + \partial(1 - \partial d)d \\ &= d\partial + \partial d \quad \text{, because } \partial^2 = 0 \\ &= 1_{\mathcal{D}} \end{aligned}$$

$$\begin{aligned} \delta^2 &= d(\partial d)d\partial d \\ &= d(1 - d\partial)d\partial d \\ &= d^2\partial d - d^2(\partial d)\partial d \\ &= d^2\partial d - d^2(1 - d\partial)\partial d \\ &= d^2\partial d - d^2\partial d = 0 \end{aligned}$$

Therefore, as an endomorphism of $D_0 \oplus D_1 \oplus D_2 \oplus \cdots$,

$$(\partial + \delta)^2 = \partial^2 + \delta^2 + \partial\delta + \delta\partial = 1_{\mathcal{D}}.$$

Since $\partial + \delta$ interchanges D_{even} and D_{odd} , then $(\partial + \delta)_{even \to odd}$ has inverse $(\partial + \delta)_{odd \to even}$ and so these two are isomorphisms between D_{even} and D_{odd}.

As a corollary of this we have *Schanuel's Lemma*:

1.16. Let $\mathcal{D}$ and $\mathcal{C}$ be two exact complexes over A, both of length $\leq n$. Assume $D_0, \ldots, D_{n-1}, C_0, \ldots, C_{n-1}$ are projective. Then

$$D_n \oplus C_{n-1} \oplus D_{n-2} \oplus C_{n-3} \oplus \cdots$$

is isomorphic to

$$C_n \oplus D_{n-1} \oplus C_{n-2} \oplus D_{n-3} \oplus \cdots$$

Proof. The identity map 1_A extends by **1.14**(a) to a chain I-map $\varphi : \mathcal{D} \to \mathcal{C}$. Let $\mathcal{M} = \mathcal{M}(\varphi)$ be the mapping cone of φ. From **1.11** and **1.10**, since φ_* is an isomorphism on homology, we see $H(\mathcal{M}) = 0$. Therefore we can apply **1.15** to $\mathcal{M}$, noting that

$$M_i = D_{i-1} \oplus C_i$$

so that M_i is projective except for $M_n = D_{n-1} \oplus C_n$ and $M_{n+1} = D_n$. The result of **1.15** is what **1.16** asserts.

1.17. Under the hypotheses of **1.16** if D_n is projective, then C_n is projective. If $D_0, \ldots, D_n, C_0, \ldots, C_{n-1}$ are finitely generated, then C_n is finitely generated.

Here is an application:

1.18. Suppose $A = B \oplus C$ is an FP module. Then B and C are FP modules.

Proof. Let $\mathcal{P} \to A$ be an FP resolution of length n. In particular, P_0 is finitely generated and maps onto A, and so A and hence B and C are finitely generated. Let

$$Q_0 \to B \quad \text{and} \quad R_0 \to C$$

be epimorphisms of finite free modules. We continue by induction.

Suppose, for $i \leq n$, we have exact sequences of finite free modules

$$Q_{i-1} \to \cdots \to Q_0 \to B \to 0$$

$$R_{i-1} \to \cdots \to R_0 \to C \to 0$$

Let U and V be the kernels of $Q_{i-1} \to Q_{i-2}$ and $R_{i-1} \to R_{i-2}$ and let W be the kernel of $P_{i-1} \to P_{i-2}$. We have two exact sequences over $B \oplus C$.

$$0 \to W \to P_{i-1} \to \cdots \to P_0 \to B \oplus C \to 0$$

$$0 \to U \oplus V \to Q_{i-1} \oplus R_{i-1} \to \cdots \to Q_0 \oplus R_0 \to B \oplus C \to 0$$

Now, W is finitely generated, being the image of P_i, and so by **1.17** $U \oplus V$ is finitely generated and the Q- and R- sequences can be extended further. If $i = n$, then $W = P_n$, and so $U \oplus V$ is a finite projective by **1.17**. Thus U and V are finite projectives and we have constructed finite projective resolutions of B and C, each of length $\leq n$.

1.19. If $0 \to A \to B \to C \to 0$ is an exact sequence of I-maps and any two of the modules are FP, then so is the third.

Proof. (a) Let $\mathcal{P} \to A$, $\mathcal{Q} \to B$ be FP resolutions. The I-map $A \to B$ extends to $\varphi : \mathcal{P} \to \mathcal{Q}$. Let $\mathcal{M} = \mathcal{M}(\varphi)$ be the mapping cone. From the exact sequence of homology from

$$0 \to \mathcal{Q} \to \mathcal{M} \to \mathcal{P}' \to 0$$

if follows that $H(\mathcal{M}) \approx C$, so that $\mathcal{M}$ is an FP resolution of C.

(b) Let $\mathcal{P} \to A$ and $\mathcal{R} \to C$ be FP resolutions. We can splice $\mathcal{P} \to A$ and $0 \to A \to B \to C \to 0$ to get an exact complex $\widetilde{\mathcal{P}}$ over C.

$$\underbrace{\cdots \to P_1 \to P_0 \to B}_{\widetilde{\mathcal{P}}} \to C \to 0$$

The identity map of C extends to a chain map $\psi : \mathcal{R} \to \widetilde{\mathcal{P}}$ and the mapping cone $\mathcal{M}(\psi)$ has $H(\mathcal{M}) = 0$. It is therefore an exact sequence, which looks like this:

$$\cdots \to \begin{matrix} R_1 \\ \oplus \\ P_1 \end{matrix} \to \begin{matrix} R_0 \\ \oplus \\ P_0 \end{matrix} \to B \to 0$$

which can be considered an FP resolution of $\mathcal{B}$.

(c) Let $\mathcal{Q} \to B$ and $\mathcal{R} \to C$ be FP resolutions. Let $\varphi : \mathcal{Q} \to \mathcal{R}$ extend $B \to C$. Let $\mathcal{M} = \mathcal{M}(\varphi)$. We compute the homology of $\mathcal{M}$ and find $H_i(\mathcal{M}) = 0$ for $i \neq 1$ and $H_1(\mathcal{M}) \approx A$. Then there is an exact sequence

$$0 \to \operatorname{Ker} \partial_1 \to M_1 \to M_0 \to 0.$$

Since M_0 and M_1 are finite projectives, $\operatorname{Ker} \partial_1$ is a finite projective. There is a long exact sequence

$$\begin{matrix} \cdots \to M_3 \to M_2 \to \operatorname{Ker} \partial_1 \to & H_1 & \to 0 \\ & \wr\!\wr & \\ & A & \end{matrix}$$

which is an FP resolution of A.

2. Trace

Recall that Λ is a ring, $\alpha : \Lambda \to \Lambda$ is a ring endomorphism. This allows us to define α-maps. The set of α-maps $f : M \to N$ forms an abelian group called $\operatorname{Hom}_\alpha(M, N)$.

We are going to define an abelian group $T_\alpha(\Lambda)$, and for every FP module M, a homomorphism of abelian groups, the *α-trace*:

$$\tau_\alpha : \operatorname{Hom}_\alpha(M, M) \to T_\alpha(\Lambda)$$

such that τ_α is *additive on short exact sequences*, in that:

$$\begin{array}{ccccccccc} 0 \to & A & \xrightarrow{i} & B & \xrightarrow{j} & C & \to 0 \\ & \downarrow f & & \downarrow g & & \downarrow h & \\ 0 \to & A & \xrightarrow{i} & B & \xrightarrow{j} & C & \to 0 \end{array}$$

if the above is a commutative diagram with equal exact rows of I-maps and the vertical arrows represent α-maps, then

$$\tau_\alpha(g) = \tau_\alpha(f) + \tau_\alpha(h).$$

(Furthermore, the group $T_\alpha(\Lambda)$ will be the "universal solution" of this problem.)

We now show that a commutativity property of τ_α, though not instantly obvious, follows easily from additivity on short exact sequences.

2.1. If $f : A \to B$ is an α-map and $g : B \to A$ an I-map, and if A and B are FP modules, then

$$\tau_\alpha(fg) = \tau_\alpha(gf).$$

Proof. We create two diagrams

$$\begin{array}{ccccccccc} 0 \to & B & \xrightarrow{i} & A \oplus B & \xrightarrow{j} & A & \to 0 \\ & \downarrow{\scriptstyle fg} & & \downarrow{\scriptstyle \vartheta} & & \downarrow{\scriptstyle 0} & \\ 0 \to & B & \longrightarrow & A \oplus B & \longrightarrow & A & \to 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 \to & B & \xrightarrow{i'} & A \oplus B & \xrightarrow{j'} & A & \to 0 \\ & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \vartheta} & & \downarrow{\scriptstyle gf} & \\ 0 \to & B & \longrightarrow & A \oplus B & \longrightarrow & A & \to 0 \end{array}$$

where

$$\vartheta(a,b) = (0, f(a) + fg(b))$$
$$i(b) = (0,b)\ ,\ j(a,b) = a$$
$$i'(b) = (-g(b), b)\ ,\ j'(a,b) = a + g(b).$$

Since τ_α is a homomorphism on $\mathrm{Hom}_\alpha(X,X)$, we have $\tau_\alpha(0) = 0$. From exact sequence additivity the first diagram gives $\tau_\alpha(fg) = \tau_\alpha(\vartheta)$, and the second diagram gives $\tau_\alpha(\vartheta) = \tau_\alpha(gf)$.

Now, consider Λ itself as a left module. Each α-map $f : \Lambda \to \Lambda$ is completely determined by $f(1)$, and conversely $\lambda \in \Lambda$ determines the α-map r_λ given by

$$r_\lambda(x) = \alpha(x)\lambda.$$

Likewise, $\mu \in \Lambda$ determines the I-map i_μ:

$$i_\mu(x) = x\mu.$$

Then, $r_\lambda(i_\mu(x)) = \alpha(x\mu)\lambda = \alpha(x)\alpha(\mu)\lambda = r_{\alpha(\mu)\lambda}$ and $i_\mu(r_\lambda(x)) = i_\mu(\alpha(x)\lambda) = \alpha(x)\lambda\mu = r_{\lambda\mu}$. Therefore, in order for **2.1** to hold, the least we need is

$$\tau_\alpha(r_{\alpha(\mu)\lambda}) = \tau_\alpha(r_{\lambda\mu}).$$

Our first approximation to $T_\alpha(\Lambda)$ will turn out to be perfect. Hence the formal definition:

2.2. Definition. $T_\alpha(\Lambda)$ is the quotient of the additive group of Λ by the subgroup generated by the elements of the form

$$[\lambda, \mu]_\alpha = \lambda\mu - \alpha(\mu)\lambda.$$

For $\lambda \in \Lambda$, let $\langle\lambda\rangle \in T_\alpha(\Lambda)$ be the element represented by λ. We have

$$\begin{aligned} \langle\lambda + \mu\rangle &= \langle\lambda\rangle + \langle\mu\rangle \\ \langle\lambda\mu\rangle &= \langle\alpha(\mu)\lambda\rangle \end{aligned}$$

Define

$$\tau_\alpha(r_\lambda) = \langle\lambda\rangle.$$

We shall now extend this definition of τ_α to α-endomorphisms of finite free, finite projective, and, finally, FP modules, and prove it is additive on endomorphisms of exact sequences.

Let F be a finite free module with basis $\{x_i\}$ and let $f : F \to F$ be an α-map. Then f is exactly determined by a square matrix (λ_{ij}) over Λ, given by

$$f(x_i) = \lambda_{ij} x_j$$

It easily follows from additivity on exact sequences that if $\lambda_{ij} = 0$ for all $i \leq j$ or for all $i \geq j$, we would have $\tau_\alpha(f) = 0$; and that if $\lambda_{ij} = 0$ except when $i = j$, then

$$\tau_\alpha(f) = \tau_\alpha(r_{\lambda_{ii}}) = \langle\lambda_{ii}\rangle.$$

Hence, since $\tau_\alpha : \mathrm{Hom}_\alpha(F, F) \to T_\alpha(\Lambda)$ is supposed to be a homomorphism, we are forced to have in general: $\tau_\alpha(f) = \langle\lambda_{ii}\rangle$. Apparently this depends on the basis, but in fact it does not (see **2.5** below).

If $M = (\mu_{ij})$ is a square matrix, define

$$\tau_\alpha(M) = \langle\mu_{ii}\rangle.$$

2.3. If $L = (\lambda_{ij})$ is an $n \times k$ matrix and $M = (\mu_{ij})$ is a $k \times n$ matrix, then

$$\tau_\alpha(LM) = \tau_\alpha(\alpha(M)L).$$

Proof. Here $\alpha(M) = (\alpha(\mu_{ij}))$.

$$\begin{aligned} \tau_\alpha(LM) &= \tau_\alpha(\lambda_{ij}\mu_{jk}) \\ &= \langle\lambda_{ij}\mu_{ji}\rangle \\ &= \langle\alpha(\mu_{ji})\lambda_{ij}\rangle \\ &= \tau_\alpha(\alpha(M)L). \end{aligned}$$

2.4. If F_1 is finite free with basis $\{x_i\}$, F_2 finite free with basis $\{y_i\}$, and if $f : F_1 \to F_2$ is an α-map and $g : F_2 \to F_1$ is an I-map, define the matrices $L = (\lambda_{ij})$, $M = (\mu_{ij})$ of f and g by

$$\begin{aligned} f(x_i) &= \lambda_{ij} y_j \\ g(y_i) &= \mu_{ij} x_j \end{aligned}$$

Then the matrix of fg is $\alpha(M)L$ and the matrix of gf is LM.

Proof.

$$\begin{aligned} fg(y_i) &= f(\mu_{ij} x_j) \\ &= \alpha(\mu_{ij}) f(x_j) \text{ , since } f \text{ is an } \alpha\text{-map} \\ &= \alpha(\mu_{ij})\lambda_{jk} y_k \\ \text{And, } gf(x_i) &= g(\lambda_{ij} y_j) \\ &= \lambda_{ij} g(y_j) \text{ , since } g \text{ is an } I\text{-map} \\ &= \lambda_{ij}\mu_{jk} x_k \end{aligned}$$

2.5. If F is a finite free module with bases $\{x_i\}$ and $\{y_j\}$ and $f : F \to F$ is an α-map, then the definition of $\tau_a(f)$ as the trace of the matrix of f is the same for both bases.

Proof. Let

$$\begin{aligned} x_i &= \mu_{ij} y_j \\ y_i &= \nu_{ij} x_j \end{aligned}$$

define two matrices $M = (\mu_{ij})$ and $N = (\nu_{ij})$. For certain bizarre rings M and N need not be square. Let

$$f(x_i) = \lambda_{ij} x_j \quad , \quad L = (\lambda_{ij})$$

Then

$$\begin{aligned} f(y_i) &= f(\nu_{ij} x_j) \\ &= \alpha(\nu_{ij}) f(x_j) \quad \text{, since } f \text{ is an } \alpha\text{-map} \\ &= \alpha(\nu_{ij})\lambda_{jk} x_k \\ &= \alpha(\nu_{ij})\lambda_{jk}\mu_{kl} y_l \end{aligned}$$

Thus the matrix of f with respect to $\{y_i\}$ is $\alpha(N)LM$. Now

$$\begin{aligned} \tau_\alpha(L) &= \tau_\alpha(LM \cdot N) \\ &= \tau_\alpha(\alpha(N) \cdot LM) \quad \text{by } \mathbf{2.3} \end{aligned}$$

This shows independence of basis choice, and allows us to define $\tau_\alpha(f)$ by choosing a basis of F and looking at the trace of the matrix of f with respect to that basis, interpreted as an element of $T_\alpha(\Lambda)$.

2.6. Now let P be a finite projective having a representation (F, π, ι) as a retract of a finite free F. Let $f : P \to P$ be an α-map. We define

$$\tau_\alpha(f) = \tau_\alpha(\iota f \pi)$$

where $\iota f \pi$ already has a defined α-trace, since F is free.

This definition is forced by additivity in the following diagram, where $Q = \operatorname{Ker} \pi$:

$$\begin{array}{ccccccccc} 0 \to & Q & \longrightarrow & F & \xrightarrow{\pi} & P & \to 0 \\ & \downarrow 0 & & \downarrow \iota f \pi & & \downarrow f & \\ 0 \to & Q & \longrightarrow & F & \longrightarrow & P & \to 0 \end{array}$$

2.7. The definition above is independent of the representation of P.

Proof. Let (F', π', ι') be another such representation. Define

$$f_1, f_2 : F \oplus F' \to F \oplus F'$$

by

$$\begin{aligned} f_1(x, y) &= (\iota f \pi(x), 0) \\ f_2(x, y) &= (0, \iota' f \pi'(y)) \end{aligned}$$

Choosing a basis for $F \oplus F'$ consisting of the union of bases for F and F', we conclude

$$(*) \qquad \begin{aligned} \tau_\alpha(f_1) &= \tau_\alpha(\iota f \pi) \\ \tau_\alpha(f_2) &= \tau_\alpha(\iota' f \pi') \end{aligned}$$

Now define $g : F \oplus F' \to F \oplus F'$ by

$$g(x, y) = (x - \iota\pi(x) + \iota\pi'(y), y - \iota'\pi'(y) + \iota'\pi(x))$$

Then g is an I-map, and computation using the fact that $\pi\iota = \pi'\iota' = 1_P$ shows:

$$g^2 = 1_{F \oplus F'} \text{ , so that } g \text{ is its own inverse}$$

$$g f_1 g^{-1} = g f_1 g = f_2.$$

It then follows from **2.3** and **2.4** that

$$\tau_\alpha(f_1) = \tau_\alpha(f_2).$$

Using $(*)$ this completes **2.7**. (What g does is to interchange in $F \oplus F'$ the two summands which are isomorphic to P.)

2.8. If P and Q are two finite projectives, and $f : P \to Q$ is an α-map, $g : Q \to P$ an I-map, then $\tau_\alpha(fg) = \tau_\alpha(gf)$.

Proof. Let (F_1, π_1, ι_1), (F_1, π_2, ι_2) represent P and Q respectively. Then

$$\begin{aligned}
\tau_\alpha(fg) &= \tau_\alpha(\iota_2 f g \pi_2) \\
&= \tau_\alpha(i_2 f \pi_1 \iota_1 g \pi_2) \\
&= \tau_\alpha(\iota_1 g \pi_2 \iota_2 f \pi_1) \text{ by } \mathbf{2.3} \text{ and } \mathbf{2.4} \\
&= \tau_\alpha(\iota_1 g f \pi_1) = \tau_\alpha(gf).
\end{aligned}$$

2.9. Now let A be an FP module with $\mathcal{P} \to A$ an FP resolution. Let $f : A \to A$ be an α-map. By **1.13**, f extends to a chain α-map $\varphi : \mathcal{P} \to \mathcal{P}$. We define

$$\tau_\alpha(f) = \sum (-1)^i \tau_\alpha(\varphi_i)$$

This is forced on us by the additivity of τ_α on exact sequences.

2.10. The above definition is independent of the choice of the chain α-map extending f.

Proof. Let φ and ψ be two such extensions, chain α-maps $\mathcal{P} \to \mathcal{P}$. By **1.13** there is a chain α-homotopy $\{d_i : P_i \to P_{i+1}\}$ such that

$$\varphi_i - \psi_i = \partial_{i+1} d_i + d_{i-1} \partial_i$$

Then

$$\begin{aligned}
&\sum (-1)^i \tau_\alpha(\varphi_i) - \sum (-1)^i \tau_\alpha(\psi_i) \\
= &\sum (-1)^i \tau_\alpha(\varphi_i - \psi_i) \\
= &\sum (-1)^i (\tau_\alpha(\partial_{i+1} d_i) + \tau_\alpha(d_{i-1} \partial_i)) \\
= &\sum (-1)^i (\tau_\alpha(\partial_{i+1} d_i) + \tau_\alpha(\partial_i d_{i-1})) \text{ , by } \mathbf{2.8} \\
= &\ 0.
\end{aligned}$$

2.11. The definition in **2.9** is independent of the choice of FP resolution of A.

Proof. Let $\mathcal{P} \to A$, $\mathcal{Q} \to A$ be two FP resolutions. The α-map $f : A \to A$ extends to a chain α-map $\varphi : \mathcal{P} \to \mathcal{Q}$ by **1.13**, and the I-map

$1_A : A \to A$ extends to a chain I-map $\gamma : \mathcal{Q} \to \mathcal{P}$. The compositions $\gamma\varphi : \mathcal{P} \to \mathcal{P}$ and $\varphi\gamma : \mathcal{Q} \to \mathcal{Q}$ both are chain α-maps extending f. The alternating sums of the traces of $\{\gamma_i\varphi_i\}$ and $\{\varphi_i\gamma_i\}$ are equal by **2.8**.

2.12. τ_α is additive on short exact sequences.

Proof. First consider the diagram:

$$\begin{array}{ccccccccc} 0 \to & P & \longrightarrow & Q & \longrightarrow & R & \to 0 \\ & \downarrow f & & \downarrow g & & \downarrow h & \\ 0 \to & P & \longrightarrow & Q & \longrightarrow & R & \to 0 \end{array}$$

where P, Q, R are finite projectives and f, g, h are α-maps. Regard the rows as a projective resolution of 0, and (f, g, h) and $(0, 0, 0)$ as chain α-maps extending 0. By **2.10**,

$$\tau_\alpha(f) - \tau_\alpha(g) + \tau_\alpha(h) = \tau_\alpha(0) - \tau_\alpha(0) + \tau_\alpha(0) = 0$$

Now consider the general case (Cf. Cartan-Eilenberg [2], V.2.)

$$(*) \qquad \begin{array}{ccccccccc} 0 \to & A & \longrightarrow & B & \longrightarrow & C & \to 0 \\ & \downarrow f & & \downarrow g & & \downarrow h & \\ 0 \to & A & \longrightarrow & B & \longrightarrow & C & \to 0 \end{array}$$

where A, B, C are FP-modules.

By the construction in **1.19** part (b), out of FP resolutions $\mathcal{P} \to A$, $\mathcal{R} \to C$, we obtain an FP resolution $\mathcal{M} \to B$ and a short exact sequence of FP resolutions.

$$\begin{array}{ccccccccc} 0 \to & \mathcal{P} & \to & \mathcal{M} & \to & R & \to & 0 \\ & & \searrow & & \searrow & & \searrow & \\ & 0 & \to & A & \to & B & \to & C & \to 0 \end{array}$$

Now $M_i = P_i \oplus R_i$ and the horizontal maps

$$0 \to P_i \to P_i \oplus R_i \to R_i \to 0$$

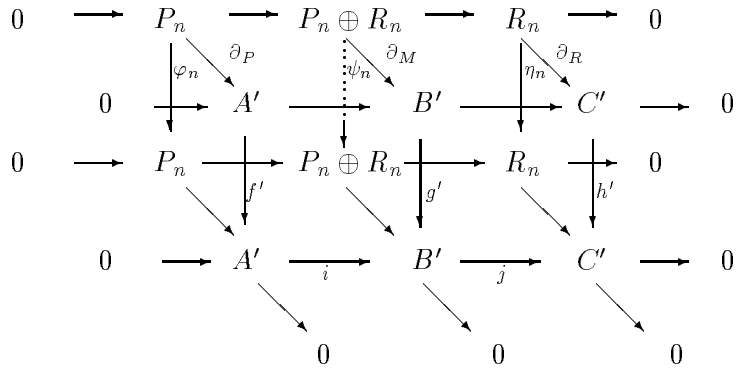
are the obvious ones $p \mapsto (p, 0)$ and $(p, r) \mapsto r$. Let f, g, h be α-maps making $(*)$ a commutative diagram.

We construct chain α-maps $\varphi : \mathcal{P} \to \mathcal{P}$, $\psi : \mathcal{M} \to \mathcal{M}$, $\eta : \mathcal{R} \to \mathcal{R}$ making these diagrams commute (φ and η are constructed extending f and h by **1.13**):

$$(**) \qquad \begin{array}{ccccccccc} 0 \to & P_i & \longrightarrow & P_i \oplus R_i & \longrightarrow & R_i & \to 0 \\ & \downarrow \varphi_i & & \downarrow \psi_i & & \downarrow \eta_i & \\ 0 \to & P_i & \longrightarrow & P_i \oplus R_i & \longrightarrow & R_i & \to 0 \end{array}$$

The construction of ψ_i is inductive; we suppose ψ_i has been constructed for $i < n$. Let A', B', C' be the kernels of ∂_{n-1} on $\mathcal{P}$, $\mathcal{M}$, $\mathcal{R}$. It is easy

to see that φ_{n-1}, ψ_{n-1}, η_{n-1} restrict to α-endomorphisms f', g', h' of these. The situation can be summarized by a commutative diagram



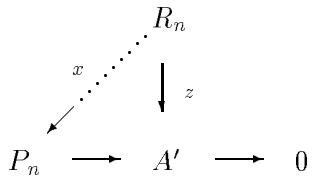
The condition that $(**)$ commutes, for $i = n$, implies that ψ_n has the form:

$$\psi_n(p, r) = (\varphi_n(p) + x(r), \eta_n(r))$$

where $x : R_n \to P_n$ is an α-map. The condition that ψ be a chain α-map, i.e. that $g'\partial_M = \partial_M\psi_n$, says of x that:

$$(***) \qquad i\partial_P(x(r)) = g'\partial_M(0, r) - \partial_M(0, \eta_n(r)).$$

The right side of this equation is an α-map $z : R_n \to B'$. Checking that $jz = 0$, we see that z, in effect, maps R_n into A'. The α-map x is then the solution of this diagram:



The α-map x exists since R_n is projective, satisfying $(***)$. So ψ_n can be defined.

So finally we have chain α-maps φ, ψ, η extending f, g, h, making all the diagrams $(**)$ commutative. From the remark at the start of this proof,

$$\tau_\alpha(\psi_i) = \tau_\alpha(\varphi_i) + \tau_\alpha(\eta_i).$$

Taking the alternating sum over i, we get the desired result:

$$\tau_\alpha(g) = \tau_\alpha(f) + \tau_\alpha(h).$$

2.13. If $\mathcal{C}$ is a bounded complex of FP modules and $\phi : \mathcal{C} \to \mathcal{C}$ is a chain α-map, we define

$$\tau_\alpha(\varphi) = \sum (-1)^i \tau_\alpha(\varphi_i).$$

2.14. (a) If $\mathcal{C}$ is a bounded exact sequence of FP modules and $\varphi : \mathcal{C} \to \mathcal{C}$ is a chain α-map, then $\tau_\alpha(\varphi) = 0$.

(b) If $\mathcal{C}$ is a bounded complex of FP modules and if $H_\iota(\mathcal{C})$ is FP for all i, and if $\varphi : \mathcal{C} \to \mathcal{C}$ is a chain α-map inducing homology α-maps $\varphi_{*,i} : H_i(\mathcal{C}) \to H_i(\mathcal{C})$, then

$$\tau_\alpha(\varphi) = \sum (-1)^i \tau_\alpha(\varphi_{*,i}).$$

Proof. These generalized additivity formulas follows from **2.12** by breaking the problems into short exact sequences which will be of FP modules by **1.19**. For example, in (b), let $Z_i = \operatorname{Ker} \partial_i$, $B_i = \operatorname{Im} \partial_{i+1}$. We have sequences

$$0 \to B_i \to Z_i \to H_i \to 0$$

$$0 \to Z_i \to C_i \to B_{i-1} \to 0$$

with α-endomorphisms on them. One proves inductively that B_{i-1}, Z_i, B_i, etc. are FP and uses aditivity on these sequences.

2.15. Now consider I-maps, traces into the group

$$T(\Lambda) = T_I(\Lambda) = \Lambda / \{\lambda\mu - \mu\lambda\}.$$

If A is an FP module, define

$$\operatorname{Rank}(A) = \tau_I(1_A).$$

When it is necessary to mention the ring Λ, we call this $\operatorname{Rank}_\Lambda(A)$.

Consequences of **2.14** are:

(i) If $\mathcal{P} \to A$ is an FP resolution, then

$$\operatorname{Rank}(A) = \sum (-1)^i \operatorname{Rank}(\mathcal{P}_i).$$

(ii) If $\mathcal{C}$ is a bounded complex of FP modules, and each $H_i(\mathcal{C})$ is FP, then

$$\sum (-1)^i \operatorname{Rank}(C_i) = \sum (-1)^i \operatorname{Rank}(H_i(\mathcal{C})).$$

(iii) If F is a free module with a basis of n elements, then $\operatorname{Rank} F = n\langle 1 \rangle$.

(iv) If P is a finite projective, represented as a retract of a finite free by (F, π, ι) and $\iota\pi : F \to F$ has the matrix (λ_{ij}) with respect to some basis, then

$$\mathrm{Rank}P = \langle \lambda_{ii} \rangle.$$

2.16. Example. If G is a finite group having n elements and Λ is the group ring $\mathbf{Q}G$ where $\mathbf{Q}$ is the rational numbers, then the augmentation $\pi : \mathbf{Q}G \to \mathbf{Q}$ has a splitting $\iota : \mathbf{Q} \to \mathbf{Q}G$ given by $1 \mapsto (g_1 + \cdots + g_n)/n$. So

$$\mathrm{Rank}_{\mathbf{Q}G}(\mathbf{Q}) = \langle (g_1 + \cdots + g_n) \rangle / n.$$

We can be even more explicit, by noting that $T(\mathbf{Q}G)$ can be identified with the free $\mathbf{Q}$-module with basis the set $\{\gamma_1, \ldots, \gamma_k\}$ of conjugacy classes of G. The number of elements in a conjugacy class γ is a_γ, which is the index of the centralizer of $g \in \gamma$ in G; if this centralizer contains b_γ elements, then

$$\frac{a_\gamma}{n} = \frac{1}{b_\gamma}$$

and so

$$\mathrm{Rank}_{\mathbf{Q}G}(\mathbf{Q}) = \sum_1^k \frac{\gamma_i}{b_{\gamma_i}}.$$

2.17. We now discuss change of rings. The group $T(\Lambda) = T_I(\Lambda)$ is a functor of rings, in that if $J : \Lambda \to \Gamma$ is a ring homomorphism, then there is an induced homomorphism $J_\# : T(\Lambda) \to T(\Gamma)$ defined in the obvious way. By means of J we consider Γ a Γ-Λ bimodule; the situation is ${}_\Gamma\Gamma_\Lambda$ in the terminology of Cartan-Eilenberg. There is then a functor

$$\Gamma \otimes_\Lambda \{\cdot\}$$

from Λ-modules to Γ-modules. The functor is exact when Γ is a projective right Λ-module; this happens in particular when R is a commutative ring, A, B are groups, $\Lambda = RA$, $\Gamma = RB$ are the group rings, and J is induced by a monomorphism of groups $A \to B$; in this case RB is a free right RA-module with basis a set of representatives of the cosets bA.

(a) In any case, if P is a (finite) Λ-projective, then $\Gamma \otimes_\Lambda P$ is a (finite) Γ-projective with

$$\mathrm{Rank}_\Gamma(\Gamma \otimes_\Lambda P) = J_\#(\mathrm{Rank}_\Lambda P).$$

(b) If $\Gamma \otimes_\Lambda \{\cdot\}$ is exact and A is an FP Λ-module, then $\Gamma \otimes_\Lambda A$ is an FP Γ-module. If $f : A \to A$ is an I-map, then

$$\Gamma \otimes_\Lambda f : \Gamma \otimes_\Lambda A \to \Gamma \otimes_\Lambda A$$

is an I-map (in the category of Γ-modules). The obvious formula holds:

$$\tau_\Gamma(\Gamma \otimes_\Lambda f) = J_\# \tau_\Lambda(f).$$

In particular,

$$\mathrm{Rank}_\Gamma(\Gamma \otimes_\Lambda A) = J_\# \mathrm{Rank}_\Lambda(A).$$

One proves (b) by letting $\mathcal{P} \to A$ be an FP Λ-resolution, $\varphi : \mathcal{P} \to \mathcal{P}$ a chain map extending f. Then $\Gamma \otimes_\Lambda \mathcal{P} \to \Gamma \otimes_\Lambda A$ is an FP Γ-resolution, and $\Gamma \otimes_\Lambda \varphi$ extends $\Gamma \otimes_\Lambda f$. The matrices for computing the traces of $\Gamma \otimes_\Lambda \varphi_i$ are the images by J of the matrices for computing the traces of φ_i.

Note: We shall write $J^\#$ for $J_\#$ when it seems typographically advisable.

2.18. Suppose $J : \Lambda \to \Gamma$, $\alpha : \Lambda \to \Lambda$, $\beta : \Gamma \to \Gamma$ are ring homomorphisms such that $\beta J = J\alpha$. Let $r : \Gamma \to \Gamma$ be a left I-, right β-map — i.e., a homomorphism of the additive group such that $r(\gamma_1 x \gamma_2) = \gamma_1 r(x) \beta(\gamma_2)$.

In this situation, there is a *natural* construction which for every α-map $f : A \to B$ of Λ-modules, gives an I-map of Γ-modules,

$$r \otimes f : \Gamma \otimes_\Lambda A \to \Gamma \otimes_\Lambda B.$$

[The example to keep in mind is this: $J : RN \to RG$ is the map of group rings induced by an inclusion of N onto a normal subgroup of G. Fix an element $g \in G$. Let $\beta : RG \to RG$ be the inner automorphism $\gamma \mapsto g^{-1}\gamma g$, and let $\alpha : RN \to RN$ be the restriction of β. The map $r : RG \to RG$ will be right-multiplication by g, so that

$$r(\gamma_1 x \gamma_2) = \gamma_1 x \gamma_2 g = \gamma_1 \cdot xg \cdot g^{-1}\gamma_2 g = \gamma_1 \cdot r(x) \cdot \beta(\gamma_2)$$

which shows r to be a left I-, right β-map.]

The word "natural" means this: Suppose the category $[\Lambda, \alpha]$ has as its objects α-maps $f : A \to B$ and as morphisms commutative diagrams

$$\begin{array}{ccc} A_1 & \xrightarrow[g_1]{} & A_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ B_1 & \xrightarrow[g_2]{} & B_2 \end{array}$$

the horizontal maps being I-maps. Then there is a functor $[\Lambda, \alpha] \to [\Gamma, I]$ transforming the object f into $r \otimes f$ and transforming the horizontal maps in a morphism by $\Gamma \otimes_\Lambda \{\cdot\}$.

Proof. We define $r \otimes f : \Gamma \otimes_\Lambda A \to \Gamma \otimes_\Lambda B$ by $(r \otimes f)(\gamma \otimes a) = r(\gamma) \otimes f(a)$. To show this is well-defined, we need, for $\lambda \in \Lambda$,

$$(r \otimes f)(\gamma\lambda \otimes a) = (r \otimes f)(\gamma \otimes \lambda a).$$

This is the proof:

$$\begin{aligned}
(r \otimes f)(\gamma\lambda \otimes a) &= r(\gamma\lambda) \otimes f(a) \\
&= r(\gamma J(\lambda)) \otimes f(a) \text{ , by the def. of } \Gamma \text{ as r. } \Lambda\text{-module} \\
&= r(\gamma)\beta J(\lambda) \otimes f(a) \text{ , since } r \text{ is a right } \beta\text{-map} \\
&= r(\gamma)J\alpha(\lambda) \otimes f(a) \text{ , since } \beta J = J\alpha \\
&= r(\gamma)\alpha(\lambda) \otimes f(a) \text{ , by def. of r. } \Lambda\text{-module str. on } \Gamma \\
&= r(\gamma) \otimes \alpha(\lambda)f(a) \text{ , in } \Gamma \otimes_\Lambda B \\
&= r(\gamma) \otimes f(\lambda a) \text{ , since } f \text{ is a left } \alpha\text{-map} \\
&= (r \otimes f)(\gamma \otimes \lambda a).
\end{aligned}$$

The reason that $r \otimes f$ is a Γ-homomorphism is derived from the fact that r is a left Γ-homomorphism. Naturality follows directly from the definition of $r \otimes f$.

2.19. In the situation of **2.18**, there is an induced homomorphism

$$r_\# : T_\alpha(\Lambda) \to T(\Gamma)$$

defined by

$$r_\#(\langle\lambda\rangle_\alpha) = \langle rJ(\lambda)\rangle.$$

If $\Gamma \otimes_\Lambda \{\cdot\}$ is exact and $f : A \to A$ is an α-map of FP Λ-modules then

$$\tau(r \otimes f) = r_\#(\tau_\alpha(f)).$$

Proof. To show $r_\#$ is well-defined, we will show it maps $[\lambda, \mu]_\alpha$ to some $[\gamma, \delta]$:

$$\begin{aligned}
[\lambda, \mu]_\alpha &= \lambda\mu - \alpha(\mu)\lambda \mapsto rJ(\lambda\mu - \alpha(\mu)\lambda) \\
&= r(J\lambda \cdot J\mu - J\alpha\mu \cdot J\lambda) \\
(*) \qquad &= rJ\lambda \cdot \beta J\mu - J\alpha\mu \cdot rJ\lambda \\
(**) \qquad &= rJ\lambda \cdot J\alpha\mu - J\alpha\mu \cdot rJ\lambda \\
&= [rJ(\lambda), J\alpha(\mu)].
\end{aligned}$$

Note that at $(*)$ we use the fact that r is a left I-, right β-map, and at $(**)$ that $\beta J = J\alpha$.

If $\mathcal{P} \to A$ is an FP Λ-resolution, then since $\Gamma \otimes_\Lambda \{\cdot\}$ is exact, $\Gamma \otimes_\Lambda \mathcal{P} \to \Gamma \otimes_\Lambda A$ is an FP Γ-resolution. If $\varphi : \mathcal{P} \to \mathcal{P}$ is a chain α-map extending f, then $r \otimes \varphi$ extends $r \otimes f$ by **2.18**. Thus the question how traces behave, reduces to maps on projectives, and by the usual argument, to maps on free modules.

Thus, let $f : F \to F$ be an α-map, where F is a finite free Λ-module with basis $\{x_i\}$ and $f(x_i) = \lambda_{ij}x_j$. Then $\Gamma \otimes_\Lambda F$ is free with basis $\{1 \otimes x_i\}$, and

$$\begin{aligned}(r \otimes f)(1 \otimes x_i) &= r(1) \otimes \lambda_{ij}x_j \\ &= r(1)J(\lambda_{ij}) \otimes x_j \\ &= (J(\lambda_{ij})r(1) + [r(1), J(\lambda_{ij})]) \otimes x_j \\ &= (rJ(\lambda_{ij}) + [r(1), J(\lambda_{ij})]) \cdot (1 \otimes x_j)\end{aligned}$$

and so $\tau(r \otimes f) = \langle rJ(\lambda_{ii})\rangle = r_\# \tau_\alpha(f)$.

3. Group rings and finite vector spaces on which G acts.

We shall now specialize drastically to the case when $\Lambda = KG$, the group algebra with coefficients in a field K. This section, except for the remarks in **3.1**, **3.2**, **3.3**, is used in the rest of the paper only in the proof of **4.2**, that if the Euler characteristic of a subgroup of finite index is defined, so is that of the total group. The same result follows from our main theorem on group extensions **6.13**, but this latter uses more complex techniques. Here we use a method commonly used in the elementary theory of representations of finite groups.

A word about possible more general results: Many of our theorems remain valid if $\Lambda = RG$ where R is a commutative ring. In this setting, it may be possible to replace "finite KG-space V" by "FP R-module M on which G acts", "dim V"by "$\mathrm{Rank}_R(M)$", and the condition "the characteristic of K does not divide dim V" by "$\mathrm{Rank}_R(M)$ is a unit in R". We leave this kind of generalization for further investigation since some interesting subtle questions of relative homological algebra are involved.

Another direction of generalization would replace KG by a Hopf algebra with involution. It may be remarked that we implicitly use the diagonal map $KG \to KG \otimes_K KG$ given by $g \mapsto g \otimes g$, and the involution $KG \to KG$ given by $g \mapsto g^{-1}$, frequently. The fact that $T(KG)$ is a coalgebra makes the consideration of coefficients of certain conjugacy classes a natural thing. Most probably, this entire theory can be generalized to some class of Hopf algebras; this could be used in a homological assault on the topological problem (problem 4 in the final section of this paper). It would be an interesting research effort.

3.1. Now we suppose K is a field of characteristic p, the case $p = 0$ being of most interest. KG is the group ring of the group G.

With $\Lambda = KG$, the machinery of sections **1** and **2** is available. We suppose $\alpha : G \to G$ is a fixed group endomorphism, extending to $\alpha : KG \to KG$ in the obvious way.

Let $[G] = [G]_I$ be the set of conjugacy classes of G. If $g \in G$ then $[g]$ or $[g]_G$ will be its conjugacy class.

$[G]_\alpha$ is the set of *α-conjugacy classes*, where this is the definition of α-conjugacy:

$g_1 \sim_\alpha g_2$ iff there is $h \in G$ such that $g_2 = \alpha(h) g_1 h^{-1}$.

$K[G]_\alpha$ is the free K-module with basis $[G]_\alpha$. It is easy to show that the α-trace group $T_\alpha(KG)$ can be identified with $K[G]_\alpha$.

If $\zeta = \sum a_\gamma \gamma \in K[G] = K[G]_I$, where $a_\gamma \in K$, $\gamma \in [G]$, and almost all $a_\gamma = 0$, then ζ_γ is defined to be a_γ. In particular, $\alpha_{[1]}$ is the coefficient of $[1]$.

3.2 "Module" will mean left KG-module. A "finite *KG-module*" is a finitely generated module. A "finite *KG-space*" is a module which is of finite dimension over the field K. If V is a finite KG-space, then $\dim V$ is its dimension over K.

Given a finite KG-space V, the *character* of V is defined. It is the function

$$\mathrm{ch}_V : [G] \to K$$

where $\mathrm{ch}_V(\gamma)$ is defined by picking $g \in \gamma$ and defining $\mathrm{ch}_V(\gamma)$ to be the trace of the K-map

$$v \to gv$$

on V.

Suppose $\zeta \in K[G]$ and $c : [G] \to K$ is a function. Their *convolution* is the element $\zeta \circledast c \in K[G]$ defined thus:

If $\zeta = \sum a_\gamma \gamma$, then

$$\zeta \circledast c = \sum a_\gamma c(\gamma^{-1}) \gamma.$$

Usually, we shall convolve ζ with various characters of finite KG-spaces.

3.3. We shall abbreviate some notation slightly:

$$\mathrm{Rank} A = \mathrm{Rank}_G(A) = \mathrm{Rank}_{KG}(A) \in K[G]$$

is the Rank of the FP KG-module A.

$$\otimes \quad \text{means} \quad \otimes_K .$$

$$\otimes_G \quad \text{means} \quad \otimes_{KG} .$$

If A and B are modules, we make $A \otimes B$ into a module by the *diagonal action*:

$$g(a \otimes b) = (ga) \otimes (gb).$$

A *trivial* KG-module A is one for which $ga = a$ for all $g \in G$. When speaking of K as a module we mean to imply that this is K with the trivial KG-module structure.

If A is any KG-module, we can forget the G action and regard A simply as a K-module. We can then impose on this K-module a trivial KG-module structure; the result is denoted $\overline{A}$. The point of this circumlocution is to allow us to compare, for instance, two modules

$$A \otimes B \quad \text{and} \quad A \otimes \overline{B}$$

where the action of G on $A \otimes B$ is

$$g(a \otimes b) = (ga) \otimes (gb),$$

and the action of G on $A \otimes \overline{B}$ is

$$g(a \otimes b) = (ga) \otimes b.$$

3.5. Let A be a KG-module. Then $KG \otimes A$ is isomorphic to $KG \otimes \overline{A}$; both are free KG-modules with basis $\{1 \otimes x_i\}$ if $\{x_i\}$ is a K-basis of A.

Proof. $f : KG \otimes \overline{A} \to KG \otimes A$ is given by

$$(f(k_i g_i \otimes a) = k_i(g_i \otimes g_i a)$$

Its inverse has the formula:

$$f^{-1}(k_i g_i \otimes a) = k_i(g_i \otimes g_i^{-1} a).$$

One has to note that both f and f^{-1} are maps of KG-modules.

3.6. Let P be a finite projective KG-module and let V be a finite KG-space. Then $P \otimes V$ is a finite projective KG-module and

$$\mathrm{Rank}(P \otimes V) = (\mathrm{Rank} P) \circledast \mathrm{ch}_V.$$

Proof. Let (F, π, ι) represent P as a retract of a finite free module with basis $\{x_1, \ldots, x_k\}$ and let $\{v_1, \ldots, v_n\}$ be a K-basis of V. By **3.5**, $F \otimes V$ is a free module with basis $\{x_i \otimes v_j\}$. Then

$$(F \otimes V, \pi \otimes 1_V, \iota \otimes 1_V)$$

represents $P \otimes V$ as a retract of $F \otimes F$; hence $P \otimes V$ is a finite projective module.

Let $\iota\pi(x_i) = \lambda_{ij} x_i$, so that

$$\mathrm{Rank} P = \langle \lambda_{ii} \rangle \in K[G].$$

Write

$$\lambda_{ij} = \lambda_{ijg}g \quad \text{, summed over } g$$

with $\lambda_{ijg} \in K$. Let the action of G on V be given by $gv_i = b_{ijg}v_j$, $b_{ijg} \in K$.

Then $\mathrm{Rank}(P \otimes V)$ is the trace of the matrix of $(\iota \otimes 1_V)(\pi \otimes 1_V)$ with respect to the basis $\{x_i \otimes v_j\}$ of $F \otimes V$. Then

$$\begin{aligned}(\iota \otimes 1_V)(\pi \otimes 1_V)(x_i \otimes v_j) &= \iota\pi(x_i) \otimes v_j \\ &= (\lambda_{ikg} g x_k) \otimes v_j \\ &= \lambda_{ikg} g(x_k \otimes g^{-1} v_j) \\ &= \lambda_{ikg} g(x_k \otimes b_{jlg^{-1}} v_l) \\ &= (\lambda_{ikg} b_{jlg^{-1}} g)(x_k \otimes v_l)\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{Rank}(P \otimes V) &= \langle \lambda_{iig} b_{jjg^{-1}} g \rangle \\ &= \lambda_{iig} b_{jjg^{-1}} [g] \\ &= (\lambda_{iig}[g]) \circledast \mathrm{ch}_V \\ &= (\mathrm{Rank} P) \circledast \mathrm{ch}_V .\end{aligned}$$

Note that in the above, all expressions are to be summed over repeated indices and over $g \in G$.

As a corollary:

3.7. Looking at the coefficient of [1] in the situation of **3.6**,

$$\mathrm{Rank}(P \otimes V)_{[1]} = \mathrm{Rank}(P)_{[1]} \cdot \dim V.$$

We are now going to define a sort of "transfer" which will go from $A \otimes V$ to A when V is a finite KG-space. It is in fact a kind of trace.

3.8. Let V be a finite KG-space with K-basis $\{v_1, \ldots, v_n\}$, let A and B be modules, and let $f : A \otimes V \to B \otimes V$ be a KG-map. Then

$$f(a \otimes v_i) = f_{ij}(a) \otimes v_j$$

defines functions f_{ij} which are K-maps. Define

$$f_V^* : A \to B \quad \text{by}$$

$$f_V^*(a) = f_{ii}(a).$$

We call f_V^* the *transfer-trace* of f.

3.9. The definition of f_V^* is independent of the basis of V.

Proof. Let $\{w_1, \ldots, w_n\}$ be another basis, which is related to the basis $\{v_1, \ldots, v_n\}$ by matrices $M = (\mu_{ij})$, $M^{-1} = N = (\nu_{ij})$, where

$$v_i = \mu_{ij} w_j \quad \text{and} \quad w_j = \nu_{jk} v_k .$$

Let

$$\begin{aligned} f(a \otimes v_i) &= f_{ij}(a) \otimes v_j \\ f(a \otimes w_i) &= e_{ij}(a) \otimes w_j . \end{aligned}$$

Then

$$\begin{aligned} f(a \otimes w_i) &= f(a \otimes \nu_{ij} v_j) \\ &= \nu_{ij} f_{jk}(a) \otimes v_k \\ &= \nu_{ij} f_{jk}(a) \otimes \mu_{kl} w_l \\ &= \nu_{ij} f_{jk}(a) \mu_{kl} \otimes w_l . \end{aligned}$$

It follows that $e_{il}(a) = \nu_{ij} f_{jk}(a) \mu_{kl}$ and

$$\begin{aligned} e_{ii}(a) &= \mu_{ki} \nu_{ij} f_{jk}(a) \\ &= \delta_{kj} f_{jk}(a) \quad , \delta_{kj} = \text{Kronecker delta} \\ &= f_{ii}(a). \end{aligned}$$

3.10. $f_V^* : A \to B$ is a KG-map.

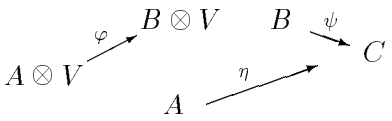
Proof. Knowing that f is a KG-map,

$$\begin{aligned} f(ga \otimes gv_i) &= g f(a \otimes v_i) \\ &= g \cdot (f_{ij}(a) \otimes v_j) \\ &= (g f_{ij}(a)) \otimes (g v_j). \end{aligned}$$

Using the basis $\{v_i\}$, we have $f_V^*(a) = f_{ii}(a)$. Using the basis $\{gv_i\}$, we have

$$f_V^*(ga) = g f_{ii}(a) = g \cdot f_V^*(a).$$

3.11. Let φ, ψ, η be KG-maps:



Suppose $\eta \otimes 1_V = (\psi \otimes 1_V)\varphi : A \otimes V \to C \otimes V$.
Then $\psi \varphi_V^* = (\dim V)\eta$.

Proof. Let $\varphi(a \otimes v_i) = \varphi_{ij}(a) \otimes v_j$. Then

$$\begin{aligned} \eta(a) \otimes v_i &= (\eta \otimes 1_V)(a \otimes v_i) \\ &= (\psi \otimes 1_V)\varphi(a \otimes v_i) \\ &= (\psi \otimes 1_V)(\varphi_{ij}(a) \otimes v_j) \\ &= \psi(\varphi_{ij}(a)) \otimes v_j \end{aligned}$$

Hence $\psi\varphi_{ij} = \delta_{ij}\eta$. So

$$\psi\varphi_V^* = \psi\varphi_{ii} = \delta_{ii}\eta = (\dim\ V)\eta.$$

3.12. Theorem. *Suppose A is a KG-module and V a finite KG-space.*

(i) *If $V \neq 0$ and $A \otimes V$ is finitely generated, then A is finitely generated.*

(ii) *If $A \otimes V$ is projective and the characteristic p of the field K does not divide* dim V*, then A is projective.*

Proof. (i) Obvious.

(ii) Let F be free and $\psi : F \twoheadrightarrow A$ an epimorphism. Since $A \otimes V$ is projective, the map $\psi \otimes 1_V : F \otimes V \twoheadrightarrow A \otimes V$ splits; i.e., there is

$$\varphi : A \otimes V \to F \otimes V$$

such that $(\psi \otimes 1_V)\varphi = 1_{A\otimes V} = 1_A \otimes 1_V$. By an application of **3.11**, with $\eta = 1_A$, we conclude,

$$1_A = \psi\left(\frac{1}{\dim\ V}\varphi_V^*\right)$$

so that $\left(F, \psi, \frac{1}{\dim\ V}\varphi_V^*\right)$ represents A as a retract of F, so A is projective.

3.13. Remarks.

(a) A neater categorical definition of the transfer-trace may be desirable. If F is a finite-dimensional K-vector space, let $V^* = \text{Hom}_K(V, K)$, and note that there are isomorphisms

$$V^* \otimes V \to \text{Hom}_K(V, V)$$

$$\text{Hom}_K(A, B) \otimes \text{Hom}_K(V, V) \to \text{Hom}_K(A \otimes V, B \otimes V).$$

Natural transformations as above exist in any case, and they are isomorphisms when V is finite-dimensional. We have then the K-trace:

$$\text{trace} : \text{Hom}_K(V, V) \approx V^* \otimes V \to K$$

and the map $1_{\mathrm{Hom}\,(A,B)} \otimes \mathrm{trace}$ on:

$$\mathrm{Hom}_K(A\otimes V, B\otimes V) \approx \mathrm{Hom}_K(A,B)\otimes \mathrm{Hom}_K(V,V) \to \mathrm{Hom}_K(A,B).$$

The argument in **3.10** shows (in case everything is a KG-module) that this maps the subgroup $\mathrm{Hom}_G(A\otimes V, B\otimes V)$ of the left side into the subgroup $\mathrm{Hom}_G(A,B)$ of the right side. This is the transfer-trace.

Using properties of trace, such as additivity on short exact sequences, one could find interesting and perhaps useful properties of the transfer-trace. It is an interesting philosophical problem to embed both the α-trace and the transfer-trace into a general categorical setting.

Another problem would be to extend the theory by permitting K to be an arbitrary commutative ring (not necessarily a field). In this case one can immediately extend the definitions and theorems (especially **3.12** (ii)) to cover the case when "finite KG-space" is interpreted to mean "KG-module which is a finite K-projective", but it would be more fascinating to see to what extent one can get away with requiring finite K-spaces to be merely FP K-modules.

(b) The condition $p \nmid \dim V$ is important. For instance, if K has characteristic p and G has order p, then K is not KG-projective, but $K\otimes KG$ is KG-projective; the point is that $V = KG$ has dimension p over K and thus **3.12** (ii) cannot be applied.

3.14. (a) Suppose V is a finite KG-space, not zero, and that V has a locally finite projective resolution. Then K has a locally finite projective resolution.

(b) Suppose V, a finite KG-space, has a bounded projective resolution, and that p, the field characteristic, does not divide $\dim V$. Then K has a bounded projective resolution of the same length as that of V; if V is an FP module, so it K.

Proof. Let $\mathcal{Q} \to K$ be a projective resolution of K. Then $\mathcal{Q}\otimes V \to V$ is a projective resolution of V.

(a) Let $\mathcal{P} \to V$ be a locally finite projective resolution of V. Suppose $Q_0, \ldots, Q_{n-1}$ are finitely generated. Let $A = \mathrm{Ker}\,(Q_{n-1} \to Q_{n-2})$, and $B = \mathrm{Ker}\,(P_{n-1} \to P_{n-2})$. Then B is finitely generated since it is the image of P_n. Apply **1.17** to see that $A\otimes V$ is finitely generated and so by **3.12** (i), A is finitely generated. Thus A is the image of a fintie free module, so that Q_n can be taken finitely generated. This inductive process ultimately makes $\mathcal{Q}$ locally finite.

(b) Let $\mathcal{P} \to V$ be a bounded projective resolution of length n. Let A be the kernel of $Q_{n-1} \to Q_{n-2}$. Then by **1.17**, $A\otimes V$ is projective, and hence by **3.12** (ii), A is projective, and

$$0 \to A \to Q_{n-1} \to \cdots \to Q_0 \to K \to 0$$

is a bounded projective resolution of K, which is locally finite if $\mathcal{P}$ and $\mathcal{Q}$ are.

3.15. Corollary. If there is any FP KG-space V, such that $p \nmid \dim V$, then K is an FP KG-module.

3.16. If K is an FP module, and V is any finite KG-space, then V is an FP module, with

$$\mathrm{Rank}V = (\mathrm{Rank}K) \circledast \mathrm{ch}_V$$

$$(\mathrm{Rank}V)_{[1]} = (\mathrm{Rank}K)_{[1]} \cdot \dim V.$$

Conversely if $p \nmid \dim V$ and V is FP, then K is FP, and

$$(\mathrm{Rank}K)_{[1]} = \frac{1}{\dim V} \cdot (\mathrm{Rank}V)_{[1]}$$

Proof. If $\mathcal{P} \to K$ is an FP resolution, then $\mathcal{P} \otimes F \to V$ is an FP resolution, and

$$\begin{aligned} \mathrm{Rank}V &= \sum (-1)^i \mathrm{Rank}(P_i \otimes V) \\ &= \sum (-1)^i (\mathrm{Rank}P_i) \circledast \mathrm{ch}_V \quad \text{by } \mathbf{3.6} \\ &= (\mathrm{Rank}K) \circledast \mathrm{ch}_V \end{aligned}$$

The rest follows from **3.15** and the fact that $\mathrm{ch}_V(1) = \dim V$.

4. Definition of Euler characteristic

If K is an FP KG-module, we define the *total Euler characteristic* of G to be

$$\chi(G;K) = \mathrm{Rank}_{KG}(K).$$

The *special Euler characteristic* $\mu(G;K)$, is the coefficient of $[1]$, i.e.,

$$\mu(G;K) = \chi(G;K)_{[1]}$$

We say "$\chi(G;K)$ is defined" exactly when "K is an FP KG-module".

4.1. If K has a finite free KG-resolution, then $\chi(G;K)$ is defined and is an integral multiple of $[1]$.

This is because the Rank of a free module is the number of elements in a basis, times $[1]$.

4.2. If $i : H \to G$ is an inclusion of a subgroup of finite index n in G, and p (which is the field-characteristic) does not divide n, and if $\chi(H;K)$ is defined, then $\chi(G;K)$ is defined and

$$\mu(G;K) = \frac{1}{n}\mu(H;K).$$

Proof. Let $\mathcal{P} \to K$ be an FP KH-resolution; then

$$KG \otimes_H \mathcal{P} \to KG \otimes_H K = K(G/H)$$

is an FP KG-resolution. We have

$$\mathrm{Rank}_{KG}(K(G/H)) = i^{\#}\mathrm{Rank}_{KH}(K).$$

Now $K(G/H)$ is a finite KG-space of dimension n. So by **3.16**, K has an FP KG-resolution and the equation involving special characteristics follows.

This shows that the special characteristic (whose definition is due to Chiswell), agrees with Wall's and Serre's "rational Euler characteristic", when these are defined, if we take K to be a field of characteristic 0.

4.3. Example. Let H be an infinite cyclic subgroup, of index 2 in $G = Z_2 * Z_2$. There is a free KH resolution of K given by

$$(*) \qquad \begin{array}{ccccccc} 0 \to & KH & \to & KH & \to K \to 0. \\ & 1 & \to & 1-t & & \end{array}$$

where t is a generator of H. Thus $\chi(H;K) = 0$. If the characteristic of K is not 2, then by **4.2**, K has a finite projective resolution over KG, of length 1, with $\mu(G;K) = 0$.

Explicitly,

$$G = \{t, s : s^2 = 1, (st)^2 = 1\}.$$

We are assured by **3.14** that the kernel A of $KG \to K$ is KG-projective. Let $V = K(G/H)$, a finite KG-space of dimension 2. Then we have

$$0 \to KG \to KG \to V \to 0$$

$$0 \to A \otimes V \to KG \otimes V \to V \to 0$$

where the first is $KG \otimes_H (*)$, and the second is

$$(0 \to A \to KG \to K \to 0) \otimes V$$

By Schanuel's Lemma,

$$(A \otimes V) \oplus KG \approx KG \oplus (KG \otimes V).$$

Therefore $(\mathrm{Rank}A) \circledast \mathrm{ch}_V = 2 \cdot [1]$. We conclude that $(\mathrm{Rank}A)_{[1]} = 1$ and that ch_V has the value 0 on all the other conjugacy classes occurring in $\mathrm{Rank}A$. The only conjugacy classes on which $\mathrm{ch}_V \neq 0$ are $[s]$ and $[st]$. What happens is that there are idempotents e_1 and e_2 in KG, where

$$e_1 = \frac{1}{2}(1 - s) \quad , \quad e_2 = \frac{1}{2}(1 - st)$$

and that

$$A = (KG \cdot e_1) \oplus (KG \cdot e_2)$$

so that

$$\mathrm{Rank}A = \langle e_1 + e_2 \rangle = [1] - \frac{1}{2}[s] - \frac{1}{2}[st].$$

It follows that

$$\begin{aligned} \chi(G) &= \mathrm{Rank}_{KG}(K) \\ &= \mathrm{Rank}(KG) - \mathrm{Rank}(A) \\ &= [1] - ([1] - \frac{1}{2}[s] - \frac{1}{2}[st]) \\ &= \frac{1}{2}[s] + \frac{1}{2}[st] \end{aligned}$$

This result could also be gotten from the theorem of Chiswell:

4.4. If $G = A *_B C$ is an amalgamated free product, and i_A, i_B, i_C are the inclusions of A, B, C into G, and if $\chi(A;K)$, $\chi(B;K)$, $\chi(C;K)$ are defined, then $\chi(G;K)$ is defined, and

$$\chi(A *_B C) = i_A^{\#}\chi(A) + i_C^{\#}\chi(C) - i_B^{\#}\chi(B)$$

$$\mu(A *_B C) = \mu(A) + \mu(C) - \mu(B).$$

Sketch of proof. Let $\mathcal{P} \to K$, $\mathcal{Q} \to K$, $\mathcal{R} \to K$ be FP KA- , KB-, KC-resolutions respectively. Then $\mathcal{P}$ and $\mathcal{R}$ are projective KB-resolutions and there are chain maps over KB:

$$\begin{array}{ccccc} \mathcal{P} & \longleftarrow & \mathcal{Q} & \longrightarrow & \mathcal{R} \\ \downarrow & & \downarrow & & \downarrow \\ K & = & K & = & K \end{array}$$

From this we get FP KG-resolutions

$$\begin{array}{ccccc} KG \otimes_A \mathcal{P} & \overset{\varphi}{\longleftarrow} & KG \otimes_B \mathcal{Q} & \overset{\psi}{\longrightarrow} & KG \otimes_C \mathcal{R} \\ \downarrow & & \downarrow & & \downarrow \\ K(G/A) & \longleftarrow & K(G/B) & \longrightarrow & K(G/C) \end{array}$$

We then have

$$(\varphi, -\psi) : KG \otimes_B \mathcal{Q} \to (KG \otimes_A \mathcal{P}) \oplus (KG \otimes_C \mathcal{R})$$

yielding a mapping cone $\mathcal{M} = \mathcal{M}(\varphi, -\psi)$. The homology fits into an exact sequence whose interesting part is

$$0 \to H_1(\mathcal{M}) \to K(G/B) \to K(G/A) \oplus K(G/C) \to H_0(\mathcal{M}) \to 0.$$

Comparing this with the remark of Swan [9], that, with $G = A *_B C$, there is an exact sequence

$$0 \to K(G/B) \to K(G/A) \oplus K(G/C) \to K \to 0$$

and we see that $H(\mathcal{M}) \approx K$, and $\mathcal{M}$ is an FP KG-resolution of K. We figure out what $\mathrm{Rank}K = \mathrm{Rank}M$ is by **2.17**.

4.5. It is worth noting that the formula for the special Euler characteristics is nicer than that for the total characteristic. The following phenomenon is why:

If $i : A \to C$ is a monomorphism of groups, inducing $i^{\#} : K[A] \to K[C]$, and $\eta \in K[A]$, then $\eta_{[1]} = \iota^{\#}(\eta)_{[1]}$. This is because a monomorphism, although it may coalesce several conjugacy classes to a single one, never sends any non-trivial conjugacy class into [1].

The analogue of **4.4** for HNN-products can now be described. The HNN-product $G = A *_B \varphi$ is defined when B is a subgroup of A and $\varphi : B \to A$ a monomorphism, as the group

$$\{A, t : tbt^{-1} = \varphi(b) \text{ for all } b \in B\}.$$

If $\mathcal{P} \to K$, $\mathcal{Q} \to K$ are FP KA- and KB-resolutions, the inclusion $B \subset A$ and $\varphi : B \to A$ induce two KB-module structures on KB, with respect to which there are two KB-chain maps:

$$\begin{array}{ccccc} \mathcal{P} & \xleftarrow{\vartheta} & \mathcal{Q} & \xrightarrow{\psi} & \mathcal{P} \\ \downarrow & & \downarrow & & \downarrow \\ K & = & K & = & K \end{array}$$

A certain amount of care is necessary when we tensor this picture with KG, because of the two different KB-module structures on KA and hence on $\mathcal{P}$; the reason it can be done is that the two KG-structures on KA are related by an inner automorphism of G. In any case, we get

$$\vartheta - \psi : KG \otimes_B \mathcal{Q} \to KG \otimes_A \mathcal{P}$$

and an FP resolution of K over KG is $\mathcal{M}(\vartheta - \psi)$.

Thus

$$\chi(A *_B \varphi) = i_A^{\#}\chi(A) - i_B^{\#}\chi(B).$$

Another way to get the same result its to establish an isomorphism, where C_1 and C_2 are infinite cyclic groups with generators s and st^{-1}, and B_1, B_2 are two copies of B:

$$(A *_B \varphi) * C_1 = (A * C_1) *_{B_1 * B_2} (B * C_2).$$

The amalgamation is:

$$B = B_1 = B$$

$$s\varphi(B)s^{-1} = B_2 = st^{-1}B(st^{-1})^{-1}.$$

We then apply **4.4** several times, remembering that by **4.3**, $\chi(C_1) = \chi(C_2) = 0$.

The full statement of the theorem is:

4.6. Suppose $\chi(A;K)$ and $\chi(B;K)$ are defined, $G = A *_B \varphi$. If i_B, $j : B \to G$ are the two inclusions, induced by $B \subset A$ and $\varphi : B \to A$, and $\eta \in K[B]$, then $i_B^{\#}(\eta) = j^{\#}(\eta)$, since i_B and j are conjugate in G. Let $i_A : A \subset G$ be the inclusion. Then $\chi(G;K)$ is defined, and:

$$\chi(G;K) = i_A^{\#}\chi(A;K) - i_B^{\#}\chi(B;K)$$

$$\mu(G;K) = \mu(A;K) - \mu(B;K).$$

5. Extension of coefficient field

Here we note that $\chi(G;K) = \chi(G;L)$ when $K \subset L$, and if either is defined, so is the other. Therefore, for all practical purposes, we can restrict attention to the prime field K_p. Here $K_0 =$ the rational numbers and, for $p \neq 0$, K_p is the Galois field of p elements. It is conceivable that $\chi(G;K_0)$ determines $\chi(G;K_p)$, but this appears to be a deep problem.

It is, of course, not generally true that every finite projective over LG comes from one over KG when $K \subset L$; in fact, there are often finite projectives over LG whose Ranks are not contained in $K[G]$, as examples from the theory of representations of finite groups amply demonstrate. This very interesting phenomenon is completely ignored by the Euler characteristic.

5.1. Suppose $K \subset L$, i.e., L is an extension field of K. Identify $K[G]$ with the obvious K-submodule of $L[G]$.

(a) If $\chi(G;K)$ is defined, so is $\chi(G;L)$, and $\chi(G;L) = \chi(G;K)$.

(b) If $\chi(G;L)$ is defined, then $\chi(G;K)$ is defined.

Proof. (a) Let $\mathcal{P} \to K$ be an FP KG-resolution. Then $L \otimes_K \mathcal{P} \to L$ is an FP LG-resolution with the same χ.

(b) Let $\mathcal{P} \to K$ be a projective KG-resolution and $\mathcal{Q} \to L$ an FP LG-resolution. Suppose $P_0, \ldots, P_{n-1}$ are finitely generated, $A = \mathrm{Ker}\,(P_{n-1} \to P_{n-2})$ and $B = \mathrm{Ker}\,(Q_{n-1} \to Q_{n-2})$. Compare

$$L \otimes_K \mathcal{P} \to L \quad \text{and} \quad \mathcal{Q} \to L$$

by the corollary to Schanuel's Lemma, **1.17**. Since B is finitely generated, $L \otimes_K A$ is finitely generated as an LG-module. Let $\{x_i\}$ be a basis, finite or infinite, of L as a K-vector space. Let $\{x_i \otimes a_{ij}\}$ j running over a finite set, be a set of LG-generators of $L \otimes_K A$. Then the finite set $\{a_{ij}\}$ — note that $a_{ij} = 0$ for all but finitely many indices — generates A as a KG-module. For, let A' be the submodule of A generated by $\{a_{ij}\}$. Then

$$L \otimes_K A' \to L \otimes_K A \to L \otimes_K (A/A') \to 0$$

is exact, $L \otimes_K A' \to L \otimes_K A$ is onto, and so $L \otimes_K (A/A') = 0$, which implies, since L is K-free, that $A/A' = 0$. Thus we can assume $\mathcal{P}$ is locally finite.

If the length of $\mathcal{Q}$ is n, then $B = Q_n$ is projective. By **1.17**, $L \otimes_K A$ is a projective LG-module. Since L is K-free, $L \otimes_K A$ is a projective KG-module then, and $A \approx K \otimes_K A$ is a direct sumand and hence a projective KG-module. The resolution $\mathcal{P}$ can be terminated with A.

6. Group extensions

In this section, $N \triangleleft G$ and $H = G/N$, i.e., N is a normal subgroup of G with quotient group H. We fix the field K and let $\chi(A) = \chi(A; K) \in K[A]$ for various cases of groups A. The inclusion $i : N \to G$ and quotient $j : G \to H$ make KG a KN-module and KH a KG-module.

6.1. If K is an FP KN-module, then KH is an FP KG-module, and

$$\mathrm{Rank}_{KG}(KH) = i^{\#}(\mathrm{Rank}_{KN}(K)) = i^{\#}(\chi(N)).$$

Proof. Let $\mathcal{P} \to K$ be an FP KN-resolution. Then $KG \otimes_N \mathcal{P} \to KG \otimes_N K = KH$ is an FP KG-resolution of $\mathrm{Rank} = i^{\#}(\mathrm{Rank}_{KN}(K))$, by **2.17**. Then $\mathrm{Rank}_{KN}(K) = \chi(N)$ by definition.

6.2. If C is any FP KH-module, and K is an FP KN-module, then C is an FP KG-module.

Proof. By **6.1**, every finite free KH-module is an FP KG-module. By **1.18**, every direct summand of a finite free KH-module is an FP KG-module, so that every finite projective KH-module is an FP KG-module. Now, let $\mathcal{R} \to C$ be an FP KH-resolution. Break $\mathcal{R}$ into short exact sequences, and starting with those of highest degree, successively prove using **1.19** that the image of $R_i \to R_{i-1}$ is an FP KG-module. Finally at the end we prove that C is an FP KG-module.

6.3. Corollary (Chiswell). If $\chi(N)$ and $\chi(H)$ are defined, where $G/N = H$, then $\chi(G)$ is defined.

For, that $\chi(N)$ and $\chi(H)$ are defined, means K is an FP KN-module and also an FP KH-module. By **6.2**, K is an FP KG-module, and $\chi(G)$ is defined.

6.4. If $\chi(G)$ is defined, and $H_*(N; K) = K$ in dimension 0, zero in other dimensions, then $\chi(H)$ is defined and $\chi(H) = j^{\#}(\chi(G))$.

Proof. Let $\mathcal{Q} \to K$ be an FP KG-resolution. It is simultaneously a projective KN-resolution. Now $H_*(N; K) = H_*(K \otimes_N \mathcal{Q}) = K$, and so $K \otimes_N \mathcal{Q}$ is an exact complex over K. Since $K \otimes_N KG \approx KH$, it follows that $K \otimes_N \mathcal{Q} \approx KH \otimes_G \mathcal{Q}$ is a projective KH-complex, hence an FP resolution of K over KH. By **2.17**, we have

$$\mathrm{Rank}_{KH}(KH \otimes_G \mathcal{Q}) = j^{\#}\mathrm{Rank}_{KG}(\mathcal{Q}).$$

The above, **6.4**, has some interesting group-theoretic application, as an example in the final section shows.

6.5. We now define a kind of transfer from $K[H]$ to $K[G]$ whenever $\chi(N)$ is defined.

Let K have an FP KN-resolution, so that, by **6.1**, KH has an FP KG-resolution $\mathcal{P} \to KH$. Each $c \in H$ determines

$$r_c : KH \to KH$$

by $r_c(x) = xc$. Conversely, each left (KG- or KH-) endomorphism of KH is right multiplication by a fixed element of H. Now r_c has a trace in $K[G]$ and we define

$$\mathrm{tr}_{GH}(c) = \tau(r_c) \in K[G].$$

Some easily proved facts:

6.6. tr_{GH}, as defined above, is a K-map from KH to $K[G]$. Furthermore,

$$\mathrm{tr}_{GH}(cd) = \mathrm{tr}_{GH}(dc).$$

Thus, tr_{GH} defines a map $K[H] \to K[G]$, also denoted by tr_{GH}.

Proof. Let φ and ψ be KG-endomorphisms $\mathcal{P} \to \mathcal{P}$ that extend r_c and r_d on KH. Then, for $a \in K$, we have $\varphi + a\psi$ extending $r_c + ar_d = r_{c+ad}$ and $\tau(\varphi + a\psi) = \tau(\varphi) + a\tau(\psi)$. This shows tr_{GH} is a K-map.

Likewise, $\varphi\psi$ extends $r_c r_d = r_{cd}$, while $\psi\varphi$ extends r_{cd}. Since

$$\tau(\varphi\psi) = \tau(\psi\varphi),$$

we have $\mathrm{tr}_{GH}(cd) = \mathrm{tr}_{GH}(dc)$; thus tr_{GH} is zero on the kernel of $KH \to K[H]$ and defines

$$\mathrm{tr}_{GH} : K[H] \to K[G].$$

6.7. Example. Let G be the free group of rank 2 with basis $\{x, y\}$. Let N be the subgroup of index 2 with basis

$$\{u = x, v = yxy^{-1}, w = y^2\}$$

and let H be the group of order 2 consisting of $\{1, t\}$. The map $j : G \to H$ takes $x \mapsto 1$, $y \mapsto t$.

An FP KN-resolution of K can be found:

$$0 \to A \to KN \to K \to 0$$

where A, the augmentation ideal, is the free left KN-module with basis $\{1 - u, 1 - v, 1 - w\}$.

By tensoring $KG \otimes_N \{\cdot\}$, we get an FP KG-resolution of KH like this:

$$0 \to B \to KG \to KH \to 0$$

where B, an ideal in KG, is the free left KG-module with basis $\{1 - x, 1 - yxy^{-1}, 1 - y^2\}$.

To compute $\mathrm{tr}_{GH}(1)$, we extend $r_1 = 1_{KH}$ by the identity 1_{KG} and 1_B, for example, and get just the rank of this resolution, which is

$$\mathrm{tr}_{GH}(1) = \mathrm{Rank}_G(KG) - \mathrm{Rank}_G(B) = -2[1],$$

which is, of course, $i^{\#}(\chi(N))$.

To compute $\mathrm{tr}_{GH}(t)$, extend r_t on KH by right multiplication by y, r_y on both KG and B.

Since $r_y(1) = y$, the matrix of r_y on KG, with respect to the basis $\{1\}$ is (y).

On B,

$$r_y(1 - x) = (1 - x)y = y^{-1}(1 - yxy^{-1}) + (-y^{-1} + xy^{-1})(1 - y^2)$$

$$r_y(1 - yxy^{-1}) = y - yx = y(1 - x)$$

$$r_y(1-y^2) = y - y^3 = y(1-y^2)$$

Thus, the matrix of r_y on B, with respect to the basis $\{1-x, 1-yxy^{-1}, 1-y_2\}$ is

$$\begin{pmatrix} 0 & y^{-1} & -y^{-1}+xy^{-1} \\ y & 0 & 0 \\ 0 & 0 & y \end{pmatrix}$$

This shows $\mathrm{tr}_{GH}(t) = [y] - [y] = 0$.

6.8. Remark. As we see, $\mathrm{tr}_{GH}(1)$ is an easy computation in general. But, if $t \neq 1$, the computation of $\mathrm{tr}_{GH}(t)$ is more subtle; in the example it involved the transforming of a simple, diagonal-looking right-multiplication into a matrix of left-multiplications. The major points of this paper (α-maps and such) are useful in dealing with this subtlety.

6.9. Suppose $\chi(N)$ is defined and that U is an FP KH-module, and $f : U \to U$ is a KH-endomorphism. Then f may be considered to be either a KH- or a KG-map, and by **6.2** U is an FP KG-module. There are two possible traces for f, and the relation between them is

$$\tau_{KG}(f) = \mathrm{tr}_{GH}\tau_{KH}(f).$$

Proof. This is the definition of tr_{GH} when $U = KH$.

Suppose **6.9** is true for FP KH-modules U and V. Let $f : U \oplus V \to U \oplus V$ be a KH-map. Then $f(u,v) = (f_{11}(u) + f_{12}(v), f_{21}(u) + f_{22}(v))$ where f_{ij} are KH-maps. This is the sum of four KH-maps. The maps $(f_{12}, 0)$ and $(0, f_{21})$ have trace in either sense zero; for instance

$$\begin{array}{ccccccccc} 0 \to & U & \to & U \oplus V & \to & V & \to 0 \\ & \downarrow 0 & & \downarrow (f_{12},0) & & \downarrow 0 & \\ 0 \to & U & \to & U \oplus V & \to & V & \to 0 \end{array}$$

This shows $\tau(f_{12}, 0) = 0$. finally we have $\tau(f) = \tau(f_{11}) + \tau(f_{22})$, in both senses, and so **6.9** is true for $U \oplus V$. So **6.9** is true for any finite free KH-module.

If U is a projective KH-module, with representation (F, π, ι) as retract of a finite free KH-module F, let $U_1 = \mathrm{Ker}\,\pi : F \to U$. The following diagram, valid over both KG and KH:

$$\begin{array}{ccccccccc} 0 \to & U_1 & \longrightarrow & F & \xrightarrow{\pi} & U & \to 0 \\ & \downarrow 0 & & \downarrow \iota f \pi & & \downarrow f & \\ 0 \to & U_1 & \longrightarrow & F & \longrightarrow & U & \to 0 \end{array}$$

shows that $\tau(f) = \tau(\iota f \pi)$ for both kinds of traces. By **6.9** for F free, we then get

$$\tau_{KG}(f) = \tau_{KG}(\iota f \pi) = \mathrm{tr}_{GH}(\iota f \pi) = \mathrm{tr}_{GH}\tau_{KH}(f).$$

Thus **6.9** is true if U is a finite KH-projective.

In the general case, if $\mathcal{P} \to U$ is an FP KH-resolution, extend $f : U \to U$ to a chain map $\varphi : \mathcal{P} \to \mathcal{P}$ over KH and hence also over KG. Using **2.14**,

$$\begin{aligned} \tau_{KG}(f) &= \tau_{KG}(\varphi) \\ &= \mathrm{tr}_{GH}\tau_{KH}(\varphi) \quad \text{, by } \mathbf{6.9} \text{ for projectives} \\ &= \mathrm{tr}_{GH}\tau_{KH}(f). \end{aligned}$$

6.10. Corollary. If $\chi(N)$ and $\chi(H)$ are defined, then by **6.3** $\chi(G)$ is defined, and by **6.9** applied to $f = 1_K : K \to K$, we have

$$\chi(G) = \mathrm{tr}_{GH}(\chi(H)).$$

To compute $\chi(G)$ more explicitly, and in particular $\mu(G) = \chi(G)_{[1]}$, therefore necessitates saying something about tr_{GH}.

6.11. If $\chi(N)$ is defined, then

$$\mathrm{tr}_{GH}([1]) = i^{\#}(\chi(N)).$$

Proof. Let $\mathcal{P} \to K$ be an FP KN-resolution. Then $KG \otimes_N \mathcal{P} \to KH$ is an FP KG-resolution.

$$r_1 : KH \to KH$$

is the identity, and so extends to the identity on $KG \otimes_N P$ which is $KG \otimes_N (1_{\mathcal{P}})$. Hence

$$\begin{aligned} \mathrm{tr}_{GH}([1]) &= \mathrm{Rank}_{KG}(KG \otimes_N \mathcal{P}) \\ &= i^{\#}(\mathrm{Rank}_{KN}(\mathcal{P})) \qquad \text{by } \mathbf{2.16} \\ &= i^{\#}(\chi(N)). \end{aligned}$$

The *support* of $\eta = \sum a_\gamma \gamma \in K[G]$ is defined as the set of those conjugacy classes γ for which $a_\gamma \neq 0$. The next proposition is crucial, but surprisingly difficult.

6.12. If $\chi(N)$ is defined and $h \in H$, then $\mathrm{tr}_{GH}([h])$ has support contained in those conjugacy classes of G contained in $j^{-1}([h])$.

Proof. The proof gives a fairly explicit computation of $\mathrm{tr}_{GH}([h])$.

Let $g \in G$ with $j(g) = h$. Conjugation by g determines an automorphism $\alpha : N \to N$

$$\alpha(n) = g^{-1}ng \quad \text{for } n \in N.$$

Recall that with respect to α there is an entire theory of α-maps of left KN-modules in sections **1** and **2**.

Let $\mathcal{P} \to K$ be an FP KN-resolution. The identity $K \to K$ is an α-map since N acts trivially on K. The identity on $\mathcal{P}$ will *not* in general be an α-map. However, there is some α-endomorphism $\varphi : \mathcal{P} \to \mathcal{P}$ extending 1_K. This has a trace

$$\tau_{\alpha,N}(\varphi) \in K[N]_\alpha$$

$(*)$ Recall that $[N]_\alpha$ is a set of equivalence classes. $n_1 \sim_\alpha n_2$ if and only if there is $m \in N$ such that

$$n_2 = g^{-1}mgn_1m^{-1}$$

i.e., $n_2 g = (g^{-1}mg)n_1 g(g^{-1}mg)^{-1}$, i.e., $n_2 g$ is conjugate to $n_1 g$, conjugating by an element of N.

Now let $\beta : G \to G$ be conjugation by g.

$$\beta(a) = g^{-1}ag \quad \text{for } a \in G.$$

Then $\beta i = i\alpha$ and we find ourselves in the situation of **2.17** et sequitur. There is a right β-map $r : G \to G$ defined by $r(a) = ag$, because

$$r(ab) = abg = ag \cdot g^{-1}bg = r(a)\beta(b).$$

Therefore,

$$r \otimes \varphi : KG \otimes_N \mathcal{P} \to KG \otimes_N \mathcal{P}$$

is an ordinary map of left KG-modules. Then by **2.19**,

$$(**) \quad \tau_{KG}(r \otimes \varphi) = r_\#(\tau_{\alpha,N}(\varphi)).$$

Recall that $r_\# : K[N]_\alpha \to K[G]$ is the map taking an α-conjugacy class $[n]_\alpha$ to $[r(n)] = [ng]$, the conjugacy class in G in which ng lies. The fact that $r_\#$ is well-defined checks with $(*)$. In particular:

$(***)$ The support of the image of $r_\#$ is contained in those conjugacy classes of G of the form $[ng]$. These are just the conjugacy classes in $j^{-1}([h])$.

Now, $r \otimes \varphi : KG \otimes_N \mathcal{P} \to KG \otimes_N \mathcal{P}$ extends $r \otimes 1_K : KG \otimes_N K \to KG \otimes_N K$, which is right multiplication by g, mod N. So, if we identify $KG \otimes_N K$ with KH, we have:

$$KG \otimes_N \mathcal{P} \to KH$$

is an FP KG-resolution, $r \otimes \varphi$ extends $r_h : KH \to KH$.

Thus, $\mathrm{tr}_{GH}(h) = \tau_{KG}(r \otimes \varphi)$, by definition of tr_{GH}

$$= r_{\#}(\tau_{\alpha,N}(\varphi)) \quad \text{by } (**)$$

has support in $j^{-1}([h])$, by $(***)$.

6.13. Main Theorem. *Let N be a normal subgroup of G with quotient group H. Suppose $\chi(N)$ and $\chi(H)$ are defined. Then:*

(i) *$\chi(G)$ is defined,*

(ii) $\chi(G) = \mathrm{tr}_{GH}\chi(H)$,

(iii) $\mu(G) = \mu(N) \cdot \mu(H)$,

where $\mu(A) = \chi(A)_{[1]}$.

Proof. The first two conclusions are **6.3** and **6.10**, stated here for the skimming reader. As for (iii), let

$$\chi(H) = \sum a_\eta \eta,$$

where $a_\eta \in K$ and η ranges over conjugacy classes in H. By (ii) and the fact tr_{GH} is a K-map,

$$\begin{aligned} \chi(G) &= \sum a_\eta \mathrm{tr}_{GH}(\eta) \\ &= a_{[1]}\mathrm{tr}_{GH}([1]) + \sum_{\eta \neq [1]} a_\eta \mathrm{tr}_{GH}(\eta) \\ &= a_{[1]} \cdot i^{\#}\chi(N) + \beta \end{aligned}$$

since by **6.11** we have $\mathrm{tr}_{GH}([1]) = i^{\#}\chi(N)$. By **6.12** we have that β has support contained in the complement of N.

Thus, the coefficient of [1],

$$\begin{aligned} \chi(G)_{[1]} &= a_{[1]} \cdot \chi(N)_{[1]} \\ &= \chi(H)_{[1]} \cdot \chi(N)_{[1]}. \end{aligned}$$

(Cf. the comment in **4.5**.)

6.14. To complete the picture of the behavior of tr_{GH}:

When $\chi(N)$ is defined, then

$$j^{\#}\mathrm{tr}_{GH} : K[H] \to K[H]$$

is

$$j^{\#}\mathrm{tr}_{GH}(\eta) = \eta \circledast \mathrm{ch}$$

where $\mathrm{ch} = \sum(-1)^i \mathrm{ch}_{H_i(N;K)}$ regarding $H_i(N;K)$ as a module over $G/N = H$.

Sketch of proof. One ordinarily defines the action of H on $H_*(N;K)$ by looking at a projective KG-resolution $\mathcal{Q} \to K$, which can be regarded also as a projective KH-resolution. Then $H_*(N;K) \approx H_*(K \otimes_N \mathcal{Q})$ which has an obvious H-action on it.

But, equivalently, if $h^{-1} \in H$, pick $g \in G$ with $j(g) = h$; define $\alpha : N \to N$ by $n \mapsto g^{-1}ng$. Let $\mathcal{P} \to K$ be an FP KN-resolution and extend 1_K by an α-map $\varphi : \mathcal{P} \to \mathcal{P}$. Tensor with K and get

$$1_K \otimes \varphi : K \otimes_N \mathcal{P} \to K \otimes_N \mathcal{P}.$$

If we had taken $\mathcal{Q}$ instead of $\mathcal{P}$ (where $\mathcal{Q}$ is a KG-resolution) the α-map φ could be taken to be simply left-multiplication by g^{-1}:

$$g^{-1}(ab) = g^{-1}ag \cdot g^{-1}b \quad \text{for } a \in N.$$

Thus, the map induced on homology by $1_K \otimes \varphi$ will be left multiplication by h^{-1}.

By **2.14**, we have:

$$\tau_K(1_K \otimes \varphi) = \tau_K(1_K \otimes \varphi)_* : H_*(N) \to H_*(N).$$

That is, if $\varepsilon : KN \to K$ is the augmentation:

$$\varepsilon^{\#}\tau_\alpha(\varphi) = \tau_K(1_K \otimes \varphi) = \tau_K((1_K \otimes \varphi)_*)$$

and, since $(1_K \otimes \varphi)_*$ is left-multiplication by h^{-1}, this is just $\mathrm{ch}(h^{-1})$.

We compute $\mathrm{tr}_{GH}(h)$ as the KG-trace of $r_g \otimes \varphi$ on $KG \otimes_N \mathcal{P}$, and then $j^{\#}\mathrm{tr}_{GH}(h)$ as the KH-trace of

$$1_{KH} \otimes r_g \otimes \varphi \quad \text{on} \quad KH \otimes_G (KG \otimes_N \mathcal{P})$$

i.e.,

$$r_h \otimes (1_K \otimes \varphi) \quad \text{on} \quad KH \otimes_K (K \otimes_N \mathcal{P})$$

This expression has trace

$$[h] \cdot \tau_K(1_K \otimes \varphi) = [h] \cdot \mathrm{ch}(h^{-1}).$$

This is the claimed formula for $\mathrm{tr}_{GH}([h])$; using linearity it is valid in general.

7. Questions and Conjectures

Let K be a field. Recall that an FP KG-module M is one having a finite projective resolution, and $\chi(G;K)$ is defined exactly when K is an FP KG-module. Now, $\chi(G;K)$ is defined as a finite linear combination of

conjugacy classes, using a trace function, and $\mu(G;K)$ is the coefficient of the trivial class [1].

Conjectures 1. Let $A *_B C$ be an amalgamated free product.

1(a). If $\chi(A *_B C)$ and $\chi(B)$ are defined, then both $\chi(A)$ and $\chi(C)$ are defined.

1(b). If $\chi(A *_B C)$, $\chi(A)$ and $\chi(C)$ are defined, then $\chi(B)$ is defined.

Conjectures 2. Consider a group-extension $N \lhd G$, $G/N = H$.

2(a). If $\chi(G)$ and $\chi(N)$ are defined, then so is $\chi(H)$.

2(b). If $\chi(G)$ and $\chi(N)$ are defined, and $\chi(N) \neq 0$, then $\chi(H)$ is defined.

2(c). If $\chi(G)$ is defined and $H_*(N;K)$ is finitely generated, then $\chi(H)$ is defined.

2(d). If $\chi(G)$ is defined, $H_*(N;K)$ is finitely generated, and the alternating sum of the K-ranks of $H_i(N;K)$ is not divisible by the characteristic of K, then $\chi(H)$ is defined.

Conjectures 3. Let $K = \mathbf{Q}$ the rational numbers.

3(a). If the denominator of $\mu(G)$ contains p^n where p is prime, then G contains a subgroup of order p^n.

3(b). If K_p is an FP K_pG-module (K_p = prime field of characteristic p), then $\mathbf{Q}$ is an FP $\mathbf{Q}G$-module, $\mathrm{Rank}_{\mathbf{Q}G}(\mathbf{Q})$ has no coefficients whose denominators are divisible by p, and the reduction mod p of $\mathrm{Rank}_{\mathbf{Q}G}(\mathbf{Q})$ is $\mathrm{Rank}_{K_pG}(K_p)$.

Background for these conjectures.

1. These were derived from looking at the resolution of K over $K(A *_B C)$ made out of those for A, B, and C. Both conjectures seem highly probable.

2. 2(d) is a generalization of **6.4**; it is the likeliest of these conjectures.

6.4 has a consequence worthy of note:

Theorem. *Suppose $Z_2 * Z_2 \lhd G$ and that $\chi(G)$ is defined; then $\mu(G) = 0$.*

The reason is that $H_*(Z_2 * Z_2; \mathbf{Q}) = \mathbf{Q}$ and so **6.4** is satisfied. We have $\mu(Z_2 * Z_2) = \frac{1}{2} + \frac{1}{2} - 1 = 0$, and so

$$\mu(G) = \mu(Z_2 * Z_2) \cdot \mu(H) = 0,$$

where H is the quotient group, which by **6.4** has an Euler characteristic.

This is peculiarly similar to Gottlieb's theorem which says, essentially, that if Z is an infinite cyclic group, and $Z \lhd G$, and $K(G,1)$ is a

finite complex, then $\chi(K(G,1))$ (i.e., $\mu(G;Q)$) is zero. Gottlieb's result would have a proof similar to the above if **2(a)** or **2(c)** were true. I do not know however if G/Z even has finitely generated rational homology in general.

3. 3(a) is a theorem under K. Brown's definition of rational Euler characteristic. The closest I can come to this is the following:

Theorem. *Suppose $e \in \mathbf{Q}G$ is idempotent, p is a prime in the denominator of the coefficient of 1 in e, and suppose that pe has no coefficient in which p occurs in the denominator. Then G contains an element of order p.*

Proof. pe can be reduced mod p to give $e_1 \in K_pG$ such that $e_1^2 = 0$ and the coefficient of 1 in e_1 is non-zero.

Let $\langle e_1 \rangle = \tau(e_1) \in K_p[G]$ where $[G]$ = conjugacy classes of G.

$$\langle e_1 \rangle = \sum a_g[g].$$

A little thought will show that the p-th power operation in K_pG is well-defined in $K_p[G]$ and that $\langle e_1^p \rangle = \sum a_g[g^p]$. For this to be zero, since $a_1 \neq 0$, there must be some $g \neq 1$ with $a_g \neq 0$, such that $g^p = 1$.

A more bizarre conjecture along the lines of **3(a)**, derived by analogy with finite groups, would be: If a conjugacy class $[g]$ occurs in $\chi(G;\mathbf{Q})$ and its coefficient's denominator is divisible by p^n, p a prime, then the centralizer of g in G contains a subgroup of order p^n.

Conjecture **3(b)** derives from the natural desire to reduce all conceivable Euler characteristics to the rational one; it seems true in simple examples. Either **3(a)** or **3(b)** would imply that if G is torsion-free, then $\mu(G)$ is an integer.

Among other problems that come to mind, here are some:

Problem 1. Given a group G and normal subgroup $N \triangleleft G$, find reasonable conditions on G and G/N that will imply that $\chi(N)$ is defined.

Problem 2(a). Find a group G such that $\mu(G;\mathbf{Q})$ is a rational number with denominator $m > 1$ for which all subgroups of finite index have index prime to m.

Problem 2(b). Find a group G for which $\chi(G;\mathbf{Q})$ is defined, in which there are an infinite number of conjugacy classes of elements of finite order. (Or prove this impossible.)

Problem 2(c). Prove that if G is non-trivial, torsion-free and $\chi(G;\mathbf{Q})$ is defined, then G contains infinitely many conjugacy classes. (This is a question from Wu-Chung Hsiang.)

Problem 3. Delineate the relation between the Chiswell and Brown definitions of rational Euler characteristic.

Problem 4. (For topologists) Define the rational $\chi(K)$ as usual (an integer) for a finite complex K. If B, an infinite complex, is the base space of a fibration whose fibre F and total space E have the homotopy type of finite complexes, and $\chi(F) \neq 0$, define

$$\chi(B) = \frac{\chi(E)}{\chi(F)}.$$

Extend this definition — by homological algebra, by S-fibration theory, by defining rational domination by finite complexes, or by wine, poetry or virtue, as you will — so that (1) χ behaves reasonably with respect to both fibrations and cofibrations, and (2) the characteristic of the loop space of the 2-sphere is defined and equal to $\frac{1}{2}$.

8. Postscript

This paper was written by hand in November 1974, at the University of California, Berkeley. It was then typed and xeroxed, but never published before. This is my retyping of it into TEX, as close to the original as possible. At that time, it was partly supported by NSF contract GP-43843X.

9. Index of terminology

There are several items of possibly non-standard terminology and symbolism that I would like to list here:

The ring involved, or the field involved, or the group involved, are usually not added to the terminology. For instance, a "projective" means a projective Λ-module; a "finite" module is a finitely generated module. **[1.1]**

FP. An FP-module is one which has a projective resolution by finitely generated modules. **[1.3]**

α-map. When $\alpha : R \to S$ is a ring-map, this gives every S-module the structure of an R-module, and the concept of α-map involves this. **[1.4]**

$\mathcal{M}$. The "mapping cone" $\mathcal{M}(\varphi)$ is the chain-complex version of the mapping cylinder with the source taken out. **[1.8]**

$\varphi_{ij}v_j$. The "summation convention" means that any formula with repeated indices in a monomial is to be summed over those indices,

where the range of the indices is understood from context. This means that $\sum$ is omitted in many places. **[1.12]**

τ_α and T_α; $\langle x \rangle$. The "trace" and α-trace are defined for endomorphisms of FP modules, and is denoted by τ_α. It belongs to a group denoted $T_\alpha(\Lambda)$, and when α is the identity map, $T(\Lambda)$; this is a quotient of the additive group of the group ring by certain additive commutators; the element in the quotient corresponding to x is written $\langle x \rangle$. **[2.2, 2.25]**

Rank. The "Rank" of an FP module is the trace of the identity map. **[2.15]**

$f_\#$, $i^\#$. A ring homomorphism $f : R \to S$ induces an abelian group homomorphism of $T(R)$ to $T(S)$ denoted by $f_\#$ or, in some cases, especially when f has a subscript, such as f_A, then this is denoted by $f_A^\#$. **[2.17]**

$[g]$, $[g]_\alpha$. Conjugacy class and α-conjugacy class, in a group understood from context. **[3.1]**

Finite KG-space. K and L are generally fields. A "finite KG-space" is a KG-module which is finite-dimensional as a vector space over K. **[3.1]**

ch_V. The "character" of a finite KG-space V, is ch_V, which is a function on conjugacy classes of G into K. **[3.2]**

$z \circledast c$. The "convolution" $z \circledast c$ is defined for a linear combination z of conjugacy classes with a function c on conjugacy classes. **[3.2]**

$\overline{A}$. The KG-module A in which the G-action has been trivialized. **[3.4]**

f_V^*. The "transfer-trace" f_V^* from $A \to B$, given $f : A \otimes V \to B$, where V is a finite K-space. **[3.8]**

$\chi(G;K)$, $\mu(G;K)$. $\chi(G;K)$ is the Rank of K as KG-module; and $\mu(G;K)$ is the coefficient of the identity conjugacy class in this.

$$\mu(G;K) = \chi(G;K)_{[1]}$$

They are called, respectively, the total and special Euler characteristics. **[4]**

tr_{GH}. This is defined in **[6.5]**, where $i : N \lhd G$ and $H = G/N$; and where $\mathcal{P}$ is an FP KN-resolution of K:

$$\begin{aligned} \mathrm{tr}_{GH}([1]) &= \mathrm{Rank}_{KG}(KG \otimes_N \mathcal{P}) \\ &= i^\#(\mathrm{Rank}_{KN}(\mathcal{P})) \qquad \text{by } \mathbf{[2.16]} \\ &= i^\#(\chi(N)). \end{aligned}$$

r_c. Right multiplication by c. **[2.1]**

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