# Chapter 1

# The Category of Graded Rings

## 1.1 Graded Rings

Unless otherwise stated, all rings are assumed to be associative rings and any ring $R$ has an identity $1 \in R$. If $X$ and $Y$ are nonempty subsets of a ring $R$ then $XY$ denotes the set of all finite sums of elements of the form $xy$ with $x \in X$ and $y \in Y$. The group of multiplication invertible elements of $R$ will be denoted by $U(R)$.

Consider a multiplicatively written group $G$ with identity element $e \in G$. A ring $R$ is **graded of type** $G$ **or** $R$, is $G$-graded, if there is a family $\{R_\sigma, \sigma \in G\}$ of additive subgroups $R_\sigma$ of $R$ such that $R = \oplus_{\sigma \in G} R_\sigma$ and $R_\sigma R_\tau \subset R_{\sigma\tau}$, for every $\sigma, \tau \in G$. For a $G$-graded ring $R$ such that $R_\sigma R_\tau = R_{\sigma\tau}$ for all $\sigma, \tau \in G$, we say that $R$ is **strongly graded** by $G$.

The set $h(R) = \cup_{\sigma \in G} R_\sigma$ is the set of **homogeneous elements** of $R$; a nonzero element $x \in R_\sigma$ is said to be **homogeneous of degree** $\sigma$ and we write : $\deg(x) = \sigma$. An element $r$ of $R$ has a unique decomposition as $r = \sum_{\sigma \in G} r_\sigma$ with $r_\sigma \in R_\sigma$ for all $\sigma \in G$, but the sum being a finite sum i.e. almost all $r_\sigma$ zero. The set $\sup(r) = \{\sigma \in G, r_\sigma \neq 0\}$ is the **support of** $r$ **in** $G$. By $\sup(R) = \{\sigma \in G, R_\sigma \neq 0\}$ we denote the **support of the graded ring** $R$. In case $\sup(R)$ is a finite set we will write $\sup(R) < \infty$ and then $R$ is said to be a $G$**-graded ring of finite support**.

If $X$ is a nontrivial additive subgroup of $R$ then we write $X_\sigma = X \cap R_\sigma$ for $\sigma \in G$. We say that $X$ is graded (or homogeneous) if : $X = \sum_{\sigma \in G} X_\sigma$. In particular, when $X$ is a subring, respectively : a left ideal, a right ideal, an ideal, then we obtain the notions of graded subring, respectively : a graded left ideal, a graded right ideal, graded ideal. In case $I$ is a graded ideal of $R$ then the factor ring $R/I$ is a graded ring with gradation defined by : $(R/I)_\sigma = R_\sigma + I/I$, $R/I = \oplus_{\sigma \in G}(R/I)_\sigma$.

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### 1.1.1 Proposition

Let $R = \oplus_{\sigma\in G} R_\sigma$ be a $G$-graded ring. Then the following assertions hold :

1. $1 \in R_e$ and $R_e$ is a subring of $R$.
2. The inverse $r^{-1}$ of a homogeneous element $r \in U(R)$ is also homogeneous.
3. $R$ is a strongly graded ring if and only if $1 \in R_\sigma R_{\sigma^{-1}}$ for any $\sigma \in G$.

**Proof**

1. Since $R_eR_e \subseteq R_e$, we only have to prove that $1 \in R_e$. Let $1 = \sum r_\sigma$ be the decomposition of 1 with $r_\sigma \in R_\sigma$. Then for any $s_\lambda \in R_\lambda (\lambda \in G)$, we have that $s_\lambda = s_\lambda.1 = \sum_{\sigma\in G} s_\lambda r_\sigma$, and $s_\lambda r_\sigma \in R_{\lambda\sigma}$. Consequently $s_\lambda r_\sigma = 0$ for any $\sigma \neq e$, so we have that $sr_\sigma = 0$ for any $s \in R$. In particular for $s = 1$ we obtain that $r_\sigma = 0$ for any $\sigma \neq e$. Hence $1 = r_e \in R_e$.
2. Assume that $r \in U(R) \cap R_\lambda$. If $r^{-1} = \sum_{\sigma\in G}(r^{-1})_\sigma$ with $(r^{-1})_\sigma \in R_\sigma$, then $1 = rr^{-1} = \sum_{\sigma\in G} r(r^{-1})_\sigma$. Since $1 \in R_e$ and $r(r^{-1})_\sigma \in R_{\lambda\sigma}$, we have that $r(r^{-1})_\sigma = 0$ for any $\sigma \neq \lambda^{-1}$. Since $r \in U(R)$ we get that $(r^{-1})_\sigma \neq 0$ for $\sigma \neq \lambda^{-1}$, therefore $r^{-1} = (r^{-1})_{\lambda^{-1}} \in R_{\lambda^{-1}}$.
3. Suppose that $1 \in R_\sigma R_{\sigma^{-1}}$ for any $\sigma \in G$. Then for $\sigma, \tau \in G$ we have:

$$R_{\sigma\tau} = R_eR_{\sigma\tau} = (R_\sigma R_{\sigma^{-1}})R_{\sigma\tau} = R_\sigma(R_{\sigma^{-1}}R_{\sigma\tau}) \subseteq R_\sigma R_\tau$$

therefore $R_{\sigma\tau} = R_\sigma R_\tau$, which means that $R$ is strongly graded. The converse is clear. □

### 1.1.2 Remark

The previous proposition shows that $R_eR_\sigma = R_\sigma R_e = R_\sigma$, proving that $R_\sigma$ is an $R_e$-bimodule.

If $R$ is a $G$-graded ring, we denote by $U^{\mathrm{gr}}(R) = \cup_{\sigma\in G}(U(R) \cap R_\sigma)$ the set of the invertible homogeneous elements. It follows from Proposition 1.1.1 that $U^{\mathrm{gr}}(R)$ is a subgroup of $U(R)$. Clearly the degree map $\deg : U^{\mathrm{gr}}(R) \to G$ is a group morphism with $\mathrm{Ker}(\deg) = U(R_e)$.

A $G$-graded ring $R$ is called a crossed product if $U(R) \cap R_\sigma \neq \emptyset$ for any $\sigma \in G$, which is equivalent to the map *deg* being surjective. Note that a $G$-crossed product $R = \oplus_{\sigma\in G} R_\sigma$ is a strongly graded ring. Indeed, if $u_\sigma \in U(R) \cap R_\sigma$, then $u_\sigma^{-1} \in R_{\sigma^{-1}}$ (by Proposition 1.1.1), and $1 = u_\sigma u_\sigma^{-1} \in R_\sigma R_{\sigma^{-1}}$.

## 1.2 The Category of Graded Rings

The category of all rings is denoted by RING. If $G$ is a group, the category of $G$-graded rings, denoted by $G$-RING, is obtained by taking the $G$-graded rings for the objects and for the morphisms between $G$-graded rings $R$ and $S$ we take the ring morphisms $\varphi : R \to S$ such that $\varphi(R_\sigma) \subseteq S_\sigma$ for any $\sigma \in G$.

Note that for $G = \{1\}$ we have $G$-RING=RING. If $R$ is a $G$-graded ring, and $X$ is a non-empty subset of $G$, we denote $R_X = \oplus_{x\in X} R_x$. In particular, if $H \leq G$ is a subgroup, $R_H = \oplus_{h\in H} R_h$ is a subring of $R$. In fact $R_H$ is an $H$-graded ring. If $H = \{e\}$, then $R_H = R_e$. Clearly the correspondence $R \mapsto R_H$ defines a functor $(-)_H : G - \mathrm{RING} \to H - \mathrm{RING}$.

### 1.2.1 Proposition

The functor $(-)_H$ has a left adjoint.

**Proof** Let $S \in H - \mathrm{RING}$, $S = \oplus_{h\in H} S_h$. We define a $G$-graded ring $\overline{S}$ as follows: $\overline{S} = S$ as rings, and $\overline{S}_\sigma = S_\sigma$ if $\sigma \in H$, and $\overline{S}_\sigma = 0$ elsewhere. Then the correspondence $S \mapsto \overline{S}$ defines a functor which is a left adjoint of $(-)_H$. □

We note that if $S \in \mathrm{RING} = H - \mathrm{RING}$ for $H = \{1\}$, then the $G$-graded ring $\overline{S}$ is said to have the **trivial $G$-grading**. Let $H \trianglelefteq G$ be a normal subgroup. Then we can consider the factor group $G/H$. If $R \in G - \mathrm{RING}$, then for any class $C \in G/H$ let us consider the set $R_C = \oplus_{x\in C} R_x$. Clearly $R = \oplus_{C\in G/H} R_C$, and $R_C R_{C'} \subseteq R_{CC'}$ for any $C, C' \in G/H$. Therefore $R$ has a natural $G/H$-grading, and we can define a functor $U_{G/H} : G - \mathrm{RING} \to G/H - \mathrm{RING}$, associating to the $G$-graded ring $R$ the same ring with the $G/H$-grading described above. If $H = G$, then $G/G - \mathrm{RING} = \mathrm{RING}$, and the functor $U_{G/G}$ is exactly the forgetful functor $U : G - \mathrm{RING} \to \mathrm{RING}$, which associates to the $G$-graded ring $R$ the underlying ring $R$.

### 1.2.2 Proposition

The functor $U_{G/H} : G - \mathrm{RING} \to G/H - \mathrm{RING}$ has a right adjoint.

**Proof** Let $S \in G/H - \mathrm{RING}$. We consider the group ring $S[G]$, which is a $G$-graded ring with the natural grading $S[G]_g = Sg$ for any $g \in G$. Since $S = \oplus_{C\in G/H} S_C$, we define the subset $A$ of $S[G]$ by $A = \oplus_{C\in G/H} S_C[C]$. If $g \in G$, there exists a unique $C \in G/H$ such that $g \in C$; define $A_g = S_C g$. Clearly the $A_g$'s define a $G$-grading on $A$, in such a way that $A$ becomes a $G$-graded subring of $S[G]$. We have defined a functor $F : G/H - \mathrm{RING} \to G - \mathrm{RING}$, associating to $S$ the $G$-graded ring $A$. This functor is a right adjoint of the

functor $U_{G/H}$. Indeed, if $R \in G-\mathrm{RING}$ and $S \in G/H-\mathrm{RING}$, we define a map

$$\varphi : \mathrm{Hom}_{G/H-\mathrm{RING}}(U_{G/H}(R), S) \to \mathrm{Hom}_{G-\mathrm{RING}}(R, F(S))$$

in the following way: if $u \in \mathrm{Hom}_{G/H-\mathrm{RING}}(U_{G/H}(R), S)$, then $\varphi(u)(r_g) = u(r_g)g$ for any $r_g \in R_g$. Then $\varphi$ is a natural bijection; its inverse is defined by $\varphi^{-1}(v) = \varepsilon \circ i \circ v$ for any $v \in \mathrm{Hom}_{G-\mathrm{RING}}(R, A)$, where $i : A \to S[G]$ is the inclusion map, and $\varepsilon : S[G] \to S$ is the augmentation map, i.e. $\varepsilon(\sum_{g\in G} s_g g) = \sum_{g\in G} s_g$. In case $S$ is a strongly graded ring (resp. a crossed product, then the ring $A$, constructed in the foregoing proof, is also strongly graded (resp. a crossed product). □

Clearly if $H \leq G$ and $R$ is a $G$-strongly graded ring (respectively a crossed product), then $R_H$ is an $H$-strongly graded ring (respectively a crossed product). Moreover, if $H \trianglelefteq G$ is a normal subgroup, then $U_{G/H}(R)$ is a $G/H$-strongly graded ring (respectively a crossed product). □

### 1.2.3 Remark

The category $G$-RING has arbitrary direct products. Indeed, if $(R_i)_{i\in I}$ is a family of $G$-graded rings, then $R = \oplus_{\sigma\in G}(\prod_i (R_i)_\sigma)$ is a $G$-graded ring, which is the product of the family $(R_i)_{i\in I}$ in the category $G$-RING. Note that $R$ is a subring of $\prod_{i\in I} R_i$, the product of the family in the category RING. The ring $R$ is denoted by $\prod^{\mathrm{gr}}_{i\in I} R_i$. If $G$ is finite or $I$ is a finite set, we have $\prod^{\mathrm{gr}}_{i\in I} R_i = \prod_{i\in I} R_i$.

### 1.2.4 Remark

Let $R = \oplus_{\sigma\in G} R_\sigma$ be a $G$-graded ring. We denote by $R^o$ the opposite ring of $R$, i.e. $R^o$ has the same underlying additive group as $R$, and the multiplication defined by $r \circ r' = r'r$ for $r, r' \in R$. The assignment $(R^o)_\sigma = R_{\sigma^{-1}}$ makes $R$ into a $G$-graded ring. The association $R \mapsto R^o$ defines an isomorphism between the categories $G-\mathrm{RING}$ and $G-\mathrm{RING}$.

## 1.3 Examples

### 1.3.1 Example *The polynomial ring*

If $A$ is a ring, then the polynomial ring $R = A[X]$ is a $\mathbb{Z}$-graded ring with the standard grading $R_n = AX^n$ for $0 \leq n$, and $R_n = 0$ for $n < 0$. Clearly $R$ is not strongly graded.

### 1.3.2 Example *The Laurent polynomial ring*

If $A$ is a ring, let $R = A[X, X^{-1}]$ be the ring of Laurent polynomials with the indeterminate $X$. An element of $R$ is of the form $\sum_{i\geq m} a_i X^i$ with $m \in \mathbb{Z}$

and finitely many non-zero $a_i$'s. Then $R$ has the standard $\mathbb{Z}$-grading $R_n = AX^n$, $n \in \mathbb{Z}$. Clearly $R$ is a crossed product.

### 1.3.3 Example *Semitrivial extension*

Let $A$ be a ring and ${}_AM_A$ a bimodule. Assume that $\varphi = [-,-] : M \otimes_A M \to A$ is an $A-A$-bilinear map such that $[m_1, m_2]m_3 = m_1[m_2, m_3]$ for any $m_1, m_2, m_3 \in M$. Then we can define a multiplication on the abelian group $A \times M$ by

$$(a, m)(a', m') = (aa' + [m, m'], am' + ma')$$

which makes $A \times M$ a ring called the semi-trivial extension of $A$ by $M$ and $\varphi$, and is denoted by $A \times_\varphi M$. The ring $R = A \times_\varphi M$ can be regarded as a $\mathbb{Z}_2$-graded ring with $R_0 = A \times \{0\}$ and $R_1 = \{0\} \times M$. We have that $R_1R_1 = Im\varphi \times \{0\}$, so if $\varphi$ is surjective then $R$ is a $\mathbb{Z}_2$-strongly graded ring.

### 1.3.4 Example *The "Morita Ring"*

Let $(A, B, {}_AM_B, {}_BN_A, \varphi, \psi)$ be a Morita context, where $\varphi : M \otimes_B N \to A$ is an $A-A$-bimodule morphism, and $\psi : N \otimes_A M \to B$ is a $B-B$-bimodule morphism such that $\varphi(m \otimes n)m' = m\psi(n \otimes m')$ and $\psi(n \otimes m)n' = n\varphi(m \otimes n')$ for all $m, m' \in M, n, n' \in N$. With this set-up we can form the Morita ring

$$R = \begin{pmatrix} A & M \\ N & B \end{pmatrix}$$

where the multiplication is defined by means of $\varphi$ and $\psi$. Moreover, $R$ is a $\mathbb{Z}$-graded ring with the grading given by:

$$R_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \qquad R_1 = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \qquad R_{-1} = \begin{pmatrix} 0 & 0 \\ N & 0 \end{pmatrix}$$

and $R_i = 0$ for $i \neq -1, 0, 1$.

Since $R_1R_{-1} = \begin{pmatrix} Im\,\varphi & 0 \\ 0 & 0 \end{pmatrix}$ and $R_{-1}R_1 = \begin{pmatrix} 0 & 0 \\ 0 & Im\,\psi \end{pmatrix}$, then $R$ is not strongly graded.

### 1.3.5 Example *The matrix rings*

Let $A$ be a ring, and $R = M_n(A)$ the matrix ring. Let $\{e_{ij} | 1 \leq i, j \leq n\}$ be the set of matrix units, i.e. $e_{ij}$ is the matrix having 1 on the $(i, j)$-position and 0 elsewhere. We have that $e_{ij}e_{kl} = \delta_{jk}e_{il}$ for any $i, j, k, l$, where $\delta_{jk}$ is Kronecker's symbol. For $t \in \mathbb{Z}$ set $R_t = 0$ if $|t| \geq n$, $R_t = \sum_{i=1,n-t} Re_{i,i+t}$ if $0 \leq t < n$, and $R_t = \sum_{i=-t+1,n} Re_{i,i+t}$ if $-n < t < 0$. Clearly $R = \oplus_{t \in \mathbb{Z}} R_t$, and this defines a $\mathbb{Z}$-grading on $R$.

On the other hand we can define various gradings on the matrix ring. We mention an example of a $\mathbb{Z}_2$-grading on $R = M_3(A)$, defined by :

$$R_0 = \begin{pmatrix} A & A & 0 \\ A & A & 0 \\ 0 & 0 & A \end{pmatrix} \quad \text{and} \quad R_1 = \begin{pmatrix} 0 & 0 & A \\ 0 & 0 & A \\ A & A & 0 \end{pmatrix}$$

Since $R_1R_1 = R_0$, $R$ is a strongly $\mathbb{Z}_2$-graded ring; however $R$ is not a crossed product, since there is no invertible element in $R_1$. It is possible to define such "block-gradings" on every $M_n(A)$.

### 1.3.6 Example *The $G \times G$-matrix ring*

Let $G$ be a finite group and let $A$ be an arbitrary ring. We denote by $R = M_G(A)$ the set of all $G \times G$-matrices with entries in $A$. We view such a matrix as a map $\alpha : G \times G \to A$. Then $R$ is a ring with the multiplication defined by:

$$(\alpha\beta)(x, y) = \sum_{z \in G} \alpha(x, z)\beta(z, y)$$

for $\alpha, \beta \in R,\ x, y \in G$. If

$$R_g = \{\alpha \in M_G(R) | \alpha(x, y) = 0 \quad \text{for every} \quad x, y \in G \text{ with } x^{-1}y \neq g\}$$

for $g \in G$, then $R$ is a $G$-graded ring with $g$-homogeneous component $R_g$. Indeed, let $\alpha \in R_g, \beta \in R_{g'}$. Then for every $x, y \in G$ such that $x^{-1}y \neq gg'$, and any $z \in G$, we have either $x^{-1}z \neq g$ or $z^{-1}y \neq g'$, therefore $(\alpha\beta)(x, y) = \sum_{z \in G} \alpha(x, z)\beta(z, y) = 0$, which means that $\alpha\beta \in R_{gg'}$. If for $x, y \in G$ we consider $e_{x,y}$ the matrix having 1 on the $(x, y)$-position, and 0 elsewhere, then $e_{x,y}e_{u,v} = \delta_{y,u}e_{x,v}$. Clearly $e_{x,xg} \in R_g,\ e_{yg,y} \in R_{g^{-1}}$, and $(\sum_{x \in G} e_{x,xg})(\sum_{y \in G} e_{yg,y}) = 1$, hence $R$ is a crossed product.

### 1.3.7 Example *Extensions of fields*

Let $K \subseteq E$ be a field extension, and suppose that $E = K(\alpha)$, where $\alpha$ is algebraic over $K$, and has minimal polynomial of the form $X^n - a,\ a \in K$ (this means that $E$ is a radical extension of $K$). Then the elements $1, \alpha, \alpha^2, ..., \alpha^{n-1}$ form a basis of $E$ over $K$. Hence $E = \oplus_{i=0,n-1} K\alpha^i$, and this yields a $\mathbb{Z}_n$-grading of $E$, with $E_0 = K$. Moreover $E$ is a crossed product with this grading.

As particular examples of the above example we obtain two very interesting ones:

### 1.3.8 Example

Let $k(X)$ be the field of rational fractions with the indeterminate $X$ over the field $k$. Then the conditions in the previous example are satisfied by the

extension $k(X^n) \subseteq k(X)$, where $n \geq 1$ is a natural number. Therefore $k(X)$ may be endowed with a $\mathbb{Z}_n$-grading, which even defines a crossed product.

### 1.3.9 Example

Let $P$ be the set of the positive prime numbers in $\mathbb{Z}$, and let $\{n_p | p \in P\}$ be a set of positive integers. Let $\xi_p = p^{1/n_p}$ and $K_p = Q(\xi_p)$the extension of $Q$ obtained by adjoining $\xi_p$. Then there exists a natural $\mathbb{Z}_{n_p}$-grading on $K_p$, which defines a crossed product. If we denote by $E$ the extension of $Q$ obtained by adjoining all the elements $\xi_p$, $p \in P$, we obtain a field with a $G$-grading, where $G = \oplus_{p \in P} \mathbb{Z}_{n_p}$. The grading is given as follows: if $\sigma \in G$ has the non-zero entries $k_1, k_2, ..., k_n$ at the positions $p_1, p_2, ..., p_n$, then $E_\sigma = Q\xi_{p_1}^{k_1} ... \xi_{p_n}^{k_n}$.

### 1.3.10 Remark

We observe that in the previous examples some fields have a non-trivial $G$-grading, where $G$ is a finite or an infinite torsion group. We prove now that any $\mathbb{Z}$-grading of a field $K$ is trivial. Indeed, if we assume that $K = \oplus_{n \in \mathbb{Z}} K_n$ is a non-trivial grading of $K$, let us pick some $n \neq 0$ and some non-zero $a \in K_n$. Then $a^{-1} \in K_{-n}$, so we may assume that $n > 0$. Since $K$ is a field, $1 - a$ is invertible, so there exists $b \in K$ with $(1 - a)b = 1$. Let $b = b_{n_1} + ... + b_{n_s}$, with $n_1 < ... < n_s$, the decomposition of $b$ as a sum of non-zero homogeneous components. Then $b_{n_1} + ... + b_{n_s} - ab_{n_1} - ... - ab_{n_s} = 1$. Looking at the degrees in the expression on the left hand side, $ab_{n_1}$ (resp. $ab_{n_s}$) is the unique term of the sum of smallest (resp. greatest) degree, and this is a contradiction, since 1 is homogeneous.

### 1.3.11 Example *Tensor product of graded algebras*

Let $A$ be a commutative ring. Then $R = \oplus_{\sigma \in G} R_\sigma$ is called a $G$-graded $A$-algebra if $R$ is a $G$-graded ring and $R$ is an $A$-algebra with $a.1 = 1.a \in R_e$ for any $a \in A$. If $R$ is a $G$-graded algebra, and $S$ is an $A$-algebra, then we may consider the tensor product $R \otimes_A S$, which is an $A$-algebra with multiplication given by $(r \otimes s)(r' \otimes s') = rr' \otimes ss'$ for $r, r' \in R, s, s' \in S$. Since any $R_\sigma$ is an $A - A$-bimodule, we obtain $R \otimes S = \sum_{\sigma \in G} R_\sigma \otimes_A S$, and this yields a $G$-grading of $R \otimes_A S$.

Assume now that $G$ is an abelian group and that $R = \oplus_{\sigma \in G} R_\sigma, S = \oplus_{\sigma \in G} S_\sigma$ are two $G$-graded $A$-algebras.Then the tensor algebra $R \otimes_{\mathbb{Z}} S$ is a $\mathbb{Z}$-algebra having a $G$-grading defined as follows: for any $\sigma \in G$ let $(R \otimes_{\mathbb{Z}} S)_\sigma = \oplus_{xy=\sigma} R_x \otimes_{\mathbb{Z}} S_y$. Clearly $R \otimes_{\mathbb{Z}} S = \oplus_{\sigma \in G} (R \otimes_{\mathbb{Z}} S)_\sigma$, and since $G$ is abelian this defines a $G$-grading. Now $R \otimes_A S$ is the abelian group $R \otimes_{\mathbb{Z}} S/I$, where $I$ is the subgroup of $R \otimes_{\mathbb{Z}} S$ generated by the elements of the form $ra \otimes s - r \otimes as$, with $r \in R, s \in S, a \in A$. Since $I$ is a homogeneous two-sided ideal of $R \otimes_{\mathbb{Z}} S$,

the $A$-algebra $R \otimes_A S$ has the natural $G$-grading defined by: $(R \otimes_A S)_\sigma$ is the abelian subgroup generated by the elements $r_x \otimes s_y$, with $r_x \in R_x, s_y \in S_y$, and $xy = \sigma$.

Assume now that $R = \oplus_{\sigma \in G} R_\sigma$ is a $G$-graded $A$-algebra, and $S = \oplus_{\tau \in H} S_\tau$ is an $H$-graded algebra. The tensor product algebra $R \otimes_A S$ has a natural $G \times H$-grading with $(R \otimes_A S)_{(\sigma,\tau)} = R_\sigma \otimes S_\tau$ for any $(\sigma, \tau) \in G \times H$.

### 1.3.12 Example *Group actions on rings*

Let $R$ be a ring and $\sigma \in \mathrm{Aut}(R)$ a ring automorphism of $R$ of finite order $n$. Denote by $G$ the cyclic group $< \sigma >$, and assume that $n.1$ is invertible in $R$ and that $R$ contains a primitive $n^{\text{th}}$-root of unity, say $\omega$. We consider the character group $\widehat{G} = \mathrm{Hom}(G, < \omega >)$ of $G$, and for any $\gamma \in \widehat{G}$ we consider the set

$$R_\gamma = \{r \in R | g(r) = \gamma(g)r \ \text{ for all } \ g \in G\}$$

If $\chi \in \widehat{G}$ is the identity element, i.e. $\chi(g) = 1$ for any $g \in G$, then $R_\chi = R^G$, the $G$-invariant subring of $R$, $R^G = \{r \in R | g(r) = r \ \text{ for all } \ g \in G\}$.

### 1.3.13 Proposition

With notation as above we have that $R = \oplus_{\gamma \in \widehat{G}} R_\gamma$, and $R_\gamma R_{\gamma'} \subseteq R_{\gamma\gamma'}$ for any $\gamma, \gamma' \in \widehat{G}$, i.e. $R$ is a $\widehat{G}$-graded ring.

**Proof**

If $r \in R_\gamma, r' \in R_{\gamma'}$, then we have that $g(rr') = g(r)g(r') = (\gamma(g)r)(\gamma'(g)r') = \gamma(g)\gamma'(g)rr' = (\gamma\gamma')(g)rr'$ for any $g \in G$, therefore $R_\gamma R_{\gamma'} \subseteq R_{\gamma\gamma'}$. On the other hand $\widehat{G}$ ia a cyclic group generated by $\gamma_0 \in \widehat{G}$, where $\gamma_0(g) = \omega.1$, so $\widehat{G} = \{\chi = \gamma_0^0, \gamma_0, ..., \gamma_0^{n-1}\}$. If $r \in R$, since $n.1$ is invertible in $R$, we can define the elements:

$r_0 = n^{-1}(r + g(r) + ... + g^{n-1}(r))$
$r_1 = n^{-1}(r + \omega^{n-1}g(r) + \omega^{n-2}g^2(r) + ... + \omega g^{n-1}(r))$
.....
$r_{n-1} = n^{-1}(r + \omega g(r) + \omega^2 g^2(r) + ... + \omega^{n-1}g^{n-1}(r))$

Since $1 + \omega + \omega^2 + ... + \omega^{n-1} = 0$, we have that $r_0 + r_1 + ... + r_{n-1} = r$. On the other hand for any $g \in G$ $g(r_0) = n^{-1}(g(r) + g^2(r) + ... + r) = r_0$, therefore $r_0 \in R_\chi$, $g(r_1) = n^{-1}(g(r) + \omega^{n-1}g^2(r) + \omega^{n-2}g^3(r) + ... + \omega r) = \omega r_1 = \gamma_0(g)r_1$, therefore $r_1 \in R_{\gamma_0}$.

In a similar way, we prove that $r_2 \in R_{\gamma_0^2}, ..., r_{n-1} \in R_{\gamma_0^{n-1}}$, and we obtain $R = \sum_{\gamma \in \widehat{G}} R_\gamma$. It remains to show that this sum is direct. Let $r_0 \in R_\chi, r_1 \in R_{\gamma_0}, ..., r_{n-1} \in R_{\gamma_0^{n-1}}$ be such that $r_0 + r_1 + ... + r_{n-1} = 0$. By applying

$g, g^2, ..., g^{n-1}$ we obtain a set of equations :

$$\begin{cases} r_0 + r_1 + \cdots + r_{n-1} = 0 \\ r_0 + \omega r_1 + \cdots + \omega^{n-1} r_{n-1} = 0 \\ r_0 + \omega^2 r_1 + \cdots + \omega^{2(n-1)} r_{n-1} = 0 \\ r_0 + \omega^{n-1} r_1 + \cdots + \omega^{(n-1)(n-1)} r_{n-1} = 0 \end{cases}$$

If we define the matrix $A$ by :

$$A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \omega & \cdots & \omega^{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & \omega^{n-1} & \cdots & \omega^{(n-1)(n-1)} \end{pmatrix}$$

then the foregoing system may be rewritten as :

$$A \begin{pmatrix} r_0 \\ r_1 \\ \cdots \\ r_{n-1} \end{pmatrix} = 0$$

The determinant of $A$ is of Vandermonde type, thus :

$$\det(A) = \prod_{0 \le i < j < n} (\omega^i - \omega^j)$$

We prove that $\det(A)$ is an invertible element of $R$, and for this it is enough to show that $d = \prod_{1 \le i < n}(1 - \omega^i)$ is invertible in $R$.

Let us consider the polynomial $f(X) = 1 + X + X^2 + ... + X^{n-1}$. Then $f(X) = \prod_{1 \le i \le n-1}(X - \omega^i)$, in particular $f(1) = d$. On the other hand $f(1) = n.1$, which is invertible. Hence $\det(A)$ is invertible, and then so is $A$. We obtain that:

$$A^{-1}(A \begin{pmatrix} r_0 \\ r_1 \\ \cdots \\ r_{n-1} \end{pmatrix}) = 0$$

which shows that $r_0 = r_1 = ... = r_{n-1} = 0$. □

### 1.3.14 Remarks

1. It is possible to state a more general result : Assume that $G$ is a finite abelian group acting by automorphisms on the ring $R$. If $R$ is an algebra over $\mathbb{Z}[n^{-1}, \omega]$, where $n$ is the exponent of $G$ and $\omega$ is a primitive $n^{\text{th}}$ root of unity, then $R = \oplus_{\gamma \in \widehat{G}} R_\gamma$, where $\widehat{G} = \text{Hom}(G, < \omega >)$ is the character group of $G$, and $R_\gamma = \{r \in R | g(r) = \gamma(g)r \text{ for all } g \in G\}$.

The proof goes by induction on $s$, where $G = G_1 \times \cdots \times G_s$ and $G_i$ are cylic groups.

Such an example arises from the consideration of a finite abelian group $G$ of automorphisms of the ring $R$, such that the exponent $n$ of $G$ is invertible in $R$. If $\omega$ is a primitive $n^{\text{th}}$ root of unity, we can construct $R' = R \otimes \mathbb{Z}[\omega]$. The ring $R'$ satisfies the conditions in the above remark, and $R'^G = R^G \otimes \mathbb{Z}[\omega]$.

2. Let $R$ be a ring and $G = \{1, g\}$ where $g$ is an automorphism of $R$ such that $g^2 = 1$. Assume that 2 is invertible in $R$. Since $-1 \in R$ is a primitive root of unity of order 2, then $R = R_1 \oplus R_g$, where $R_1 = R^G$ and $R_g = \{r \in R | g(r) = -r\}$, i.e. $R$ is a $G$-graded ring.

   We obtain such an example by taking $R = M_n(k)$, $k$ any field of characteristic not 2, and letting $g \in \mathrm{Aut}(R)$ be the inner automorphism induced by the diagonal matrix

$$U = \begin{pmatrix} -1 & & & & & \\ & \cdots & & & & \\ & & -1 & & & \\ & & & 1 & & \\ & & & & \cdots & \\ & & & & & 1 \end{pmatrix}$$

   with $m$ entries equal to $-1$, and $n-m$ entries equal to 1 (where $1 \leq m < n$). We have $g : M_n(k) \to M_n(k), g(X) = UXU^{-1}$. Then $G =< g >$ has order 2, and $R$ has the grading $R = R_1 \oplus R_g$, where $R_1 = R^G$ is the set of all matrices of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with $A \in M_m(k), B \in M_{n-m}(k)$, and $R_g$ is the set of all matrices of the form $\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$.

## 1.4 Crossed Products

In Section 1.1. a $G$-crossed product has been defined as a $G$-graded ring $R = \oplus_{\sigma \in G} R_\sigma$ such that there exists an invertible element in every homogeneous component $R_\sigma$ of $R$. We now provide a general construction leading to crossed products. We call $(A, G, \sigma, \alpha)$ a *crossed system* if $A$ is a ring, $G$ is a group with unit element written as 1, and $\sigma : G \to \mathrm{Aut}(A), \alpha : G \times G \to U(A)$ are two maps with the following properties for any $x, y, z \in G$ and $a \in A$ :

i) ${}^x({}^y a) = \alpha(x, y)\, {}^{xy}a\alpha(x, y)^{-1}$

ii) $\alpha(x, y)\alpha(xy, z) = {}^x\, \alpha(y, z)\alpha(x, yz)$

iii) $\alpha(x, 1) = \alpha(1, x) = 1$

where we have denoted $\sigma(x)(a)$ by ${}^{x}a$ for $x \in G, a \in A$.

The map $\sigma$ is called a *weak action* of $G$ on $A$, and $\alpha$ is called a $\sigma$-cocycle. Let $\overline{G} = \{\overline{g} | g \in G\}$ be a copy (as a set) of $G$. We denote by $A_\alpha^\sigma[G]$ the free left $A$-module with the basis $\overline{G}$, and we define a multiplication on this set by:

$$(a_1\overline{x})(a_2\overline{y}) = a_1{}^{x}a_2\alpha(x,y)\overline{xy}$$

for $a_1, a_2 \in A,\ x, y \in G$.

### 1.4.1 Proposition

The foregoing operation makes the set $A_\alpha^\sigma[G]$ into a ring. Moreover, this ring is $G$-graded by $(A_\alpha^\sigma[G])_g = A\overline{g}$, and it is a crossed product.

**Proof** In the first part we establish that the multiplication is associative. Indeed, if $a_1, a_2, a_3 \in A$ and $x, y, z \in G$, then:

$(a_1\overline{x})((a_2\overline{y})(a_3\overline{z}))$

$= (a_1\overline{x})(a_2{}^{y}a_3\alpha(y,z)\overline{yz})$

$= a_1{}^{x}(a_2{}^{y}a_3\alpha(y,z))\alpha(x,yz)\overline{xyz}$

$= a_1{}^{x}a_2{}^{x}({}^{y}a_3)\ {}^{x}\alpha(y,z)\alpha(x,yz)\overline{xyz}$

$= a_1{}^{x}a_2\alpha(x,y)\ {}^{xy}a_3\alpha(x,y)^{-1}\ {}^{x}\alpha(y,z)\alpha(x,yz)\overline{xyz}$ (by i)

$= a_1{}^{x}a_2\alpha(x,y)\ {}^{xy}a_3{}^{x}\alpha(xy,z)\overline{xyz}$ (by ii)

$= (a_1{}^{x}a_2\alpha(x,y)\overline{xy})(a_3\overline{z})$

$= ((a_1\overline{x})(a_2\overline{y}))(a_3\overline{z})$

Clearly the element $1_A\overline{e}$ is the identity element of $A_\alpha^\sigma[G]$ (here $1_A$ is the unity of $A$ and $e$ is the identity of $G$). We have by construction that $A_\alpha^\sigma[G] = \oplus_{x\in G}A\overline{x}$, and $(A\overline{x})(A\overline{y}) = A\overline{xy}$, therefore $A_\alpha^\sigma[G]$ is a $G$-graded ring. Since $\overline{x}\overline{x^{-1}} = \alpha(x,x^{-1})\overline{1}$, $\overline{x^{-1}}\overline{x} = \alpha(x^{-1},x)\overline{1}$, and $\alpha(x,x^{-1}), \alpha(x^{-1},x)$ are invertible elements of $A$, we obtain that $\overline{x}$ and $\overline{x^{-1}}$ are invertible elements of $A_\alpha^\sigma[G]$, and this shows that $A_\alpha^\sigma[G]$ is a $G$-crossed product. □

We now consider some particular cases of the above construction.

a. Assume first that $\alpha : G \times G \to U(A)$ is the trivial map, i.e. $\alpha(x,y) = 1$ for all $x, y \in G$. Then ii and iii hold, and i means that $\sigma : G \to \mathrm{Aut}(A)$ is a group morphism. In this case the crossed product $A_\alpha^\sigma[G]$ will be denoted $A *_\sigma G$, and it is called the **skew groupring** associated to $\sigma$. In this case the multiplication is defined by :

$$(a\overline{x})(b\overline{y}) = a\sigma(x)(b)\overline{xy}$$

for $a, b \in A,\ x, y \in G$.

b. Assume that $\sigma : G \to \mathrm{Aut}(A)$ is the trivial map, i.e. $\sigma(g) = 1$ for any $g \in G$. In this case the condition i) implies that $\alpha(x, y) \in U(Z(A))$ for any $x, y \in G$, where $Z(A)$ is the center of the ring $A$. Also the conditions ii. and iii. show that $\alpha$ is a 2-cocycle in the classical sense, i.e. $\alpha \in Z^2(G, U(Z(A)))$. The crossed product $A^\sigma_\alpha[G]$ is in this situation denoted by $A_\alpha[G]$, and it is called the **twisted groupring** associated to the cocycle $\alpha$. The multiplication of the twisted groupring is defined by :

$$(a\overline{x})(b\overline{y}) = ab\alpha(x, y)\overline{xy}$$

Let us consider again a general crossed product. We have seen in Section 1.1 that a $G$-graded ring $R = \oplus_{\sigma \in G} R_\sigma$ is a crossed product if and only if the sequence

$$1 \to U(R_1) \to U^{gr}(R) \to G \to 1 \tag{1}$$

is exact (here $U^{gr} = \bigcup_{g \in G}(R_g \bigcap U(R))$ , the second map is the inclusion, and the third map is the degree map).

### 1.4.2 Proposition

Every $G$-crossed product $R$ is of the form $A^\sigma_\alpha[G]$ for some ring $A$ and some maps $\sigma, \alpha$.

**Proof** Start by putting $A = R_e$. Since $R_g \cap U(R) \neq \emptyset$ for any $g \in G$, we may choose for $g \in G$ some $u_g \in R_g \cap U(R)$. We take $u_e = 1$. Then it is clear that $R_g = R_e u_g = u_g R_e$, and that the set $\{u_g | g \in G\}$ is a basis for $R$ as a left (and right) $R_e$-module. Let us define the maps:

$$\sigma : G \to \mathrm{Aut}(R_e) \text{ by } \sigma(g)(a) = u_g a u_g^{-1} \text{ for } g \in G,\ a \in R_e,$$

and

$$\alpha : G \times G \to U(R_e) \text{ by } \alpha(x, y) = u_x u_y u_{xy}^{-1} \text{ for } x, y \in G$$

We show that $\sigma$ and $\alpha$ satisfy the conditions i.,ii.,iii. Indeed,
$\alpha(x, y)^{xy} a \alpha(x, y)^{-1}$
$= u_x u_y u_{xy}^{-1}(u_{xy} a u_{xy}^{-1}) u_{xy} u_y^{-1} u_x^{-1}$
$= u_x(u_y a u_y^{-1}) u_x^{-1}$
$= {}^x({}^y a)$

therefore i. holds. Next

$\alpha(x, y)\alpha(xy, z) = u_x u_y u_{xy}^{-1} u_{xy} u_z u_{xyz}^{-1} = u_x u_y u_z u_{xyz}^{-1}$

On the other hand:

$$^x\alpha(y,z)\alpha(x,yz) = u_x\alpha(y,z)u_x^{-1}u_xu_{yz}u_{xyz}^{-1}$$
$$= u_xu_yu_zu_{yz}^{-1}u_{yz}u_{xyz}^{-1} = u_xu_yu_zu_{xyz}^{-1}$$

so ii holds, too.

Finally $\alpha(x,e) = \alpha(e,x) = 1$ since $u_e = 1$, therefore iii. holds. Let $a \in R_x$ and $b \in R_y$ be homogeneous elements of $R$. We compute the product $ab$ via the maps $\sigma, \alpha$. We have that $a$ and $b$ can be uniquely expressed as $a = a_1u_x$ and $b = b_1u_y$, with $a_1, b_1 \in R_e$. Then

$$\begin{aligned} ab &= (a_1u_x)(b_1u_y) = a_1(u_xb_1u_x^{-1})u_xu_y \\ &= a_1(u_xb_1u_x^{-1})(u_xu_yu_{xy}^{-1})u_{xy} = a_1{}^xb_1\alpha(x,y)u_{xy} \end{aligned}$$

This entails that the ring $R$ is isomorphic to $(R_e)_\alpha^\sigma[G]$. □

### 1.4.3 Remarks

1. If the exact sequence (1) splits, i.e. there exists a group morphism $\varphi : G \to U^{gr}(R)$ such that $deg \circ \varphi = Id_G$, then $u_x = \varphi(x)$ is an invertible element of $R_x$, for any $x \in G$. In this case $\alpha(x,y) = u_xu_yu_{xy}^{-1} = \varphi(x)\varphi(y)\varphi(xy)^{-1} = 1$, therefore $\varphi$ is a group morphism and $R$ is isomorphic to the skew group ring $R_e *_\varphi G$. Conversely, if $R = A *_\varphi G$ is a skew group ring over $G$ then (1) is a split exact sequence.

2. If $R = \oplus_{\sigma\in G}R_\sigma$ is a $G$-graded ring, we denote by $C_R(R_e)$ the centralizer of $R_e$ in $R$, i.e. $C_R(R_e) = \{r \in R | rr_e = r_er \text{ for all } r_e \in R_e\}$. Assume that for any $x \in G$ we can choose an invertible element $u_x \in R_x$ such that $u_x \in C_R(R_e)$. Then clearly $\sigma(x) = 1$ for any $x \in G$, and $\alpha(x,y) \in u(Z(R_e))$, so $\alpha$ is a 2-cocycle and $R$ is isomorphic to a twisted group ring. Conversely, it is easy to see that a twisted group ring $A_\alpha[G]$ has the property that every homogeneous component contains an invertible element centralizing the homogeneous part of degree $e$.

## 1.5 Exercises

1. Let $R = \oplus_{\sigma\in G}R_\sigma$ be a $G$-graded ring and $(\sigma_i)_{i\in I}$ a set of generators for the group $G$. Then the following assertions are equivalent :

   i) $R$ is a strongly graded ring.

   ii) $R_e = R_{\sigma_i}R_{\sigma_i^{-1}} = R_{\sigma_i^{-1}}R_{\sigma_i}$ for any $i \in I$.

2. Let $R = \oplus_{\sigma\in G}R_\sigma$ be a $G$-graded ring and $H$ a subgroup of $G$. Prove that $U(R_H) = R_H \cap U(R)$.

3. Let $R, S$ be two objects and $\varphi : R \to S$ a morphism in the category $G$-RING. Prove that the following assertions are equivalent :

   i) $\varphi$ is an epimorphism in the category $G$-RING.

   ii) $\varphi$ is an epimorphism in the category RING.

   **Hint:** ii. $\Rightarrow$ i. is obvious.

   i.$\Rightarrow$ ii. Let us consider $T$ another object in the category RING, and $u, v : S \to T$ morphisms in this category such that $u \circ \varphi = v \circ \varphi$. Let $T[G]$ be the group ring associated to $T$ with the natural $G$-grading. We define the maps $\overline{u}, \overline{v} : S \to T[G]$ by $\overline{u}(s_g) = u(s)g$, and $\overline{v}(s_g) = v(s)g$. Then $\overline{u}, \overline{v}$ are morphisms in the category $G$-RING, and $\overline{u} \circ \varphi = \overline{v} \circ \varphi$. We obtain that $\overline{u} = \overline{v}$, hence $u = v$.

4. Let $(R_i)_{i \in I}$ be a family of $G$-graded rings. Prove that :

   i) If $I$ is finite, then the direct product $R = \prod_{i \in I}^{\mathrm{gr}} R_i$ is a strongly graded ring if and only if $R_i$ is a strongly graded ring for every $i \in I$.

   ii) If $I$ is an arbitrary set, then the direct product $R = \prod_{i \in I}^{\mathrm{gr}} R_i$ in the category $G - RING$ is a crossed product if and only if $R_i$ is a crossed product for every $i \in I$.

5. Let $A$ be a commutative ring, $R$ a $G$-graded $A$-algebra, and $S$ an $H$-graded $A$-algebra ($G$ and $H$ are two groups). We consider $R \otimes_A S$ with the natural $G \times H$-grading (see Example 1.3.17). If $R$ and $S$ are strongly graded (resp. crossed products), then $R \otimes_A S$ is a $G \times H$-strongly graded ring (resp. a crossed product).

6. A ring $R$ is called **almost strongly graded by the group** $G$ if there exists a family $(R_\sigma)_{\sigma \in G}$ of additive subgroups of $R$ with $1 \in R_e$, $R = \sum_{\sigma \in G} R_\sigma$, and $R_\sigma R_\tau = R_{\sigma\tau}$ for all $\sigma, \tau \in G$. Prove that :

   i) If $R$ is an almost strongly graded ring and $I$ is a two-sided ideal of $R$, then $R/I$ is also an almost strongly graded ring. In particular if $R = \oplus_{\sigma \in G} R_\sigma$ is a $G$-strongly graded ring and $I$ is a two-sided ideal of $R$, then $R/I$ is an almost strongly graded ring.

   ii) If $S$ is an almost strongly graded ring (by the group $G$), then there exist a strongly graded ring $R$ and an ideal $I$ of $R$ such that $S \simeq R/I$.

   iii) If $R = \sum_{\sigma \in G} R_\sigma$ is almost strongly graded, then $R_\sigma$ is an $R_e$-bimodule for every $\sigma \in G$. Moreover, $R_\sigma$ is a left (and right) finitely generated projective $R_1$-module.

**Hint :** Since $1 \in R_{\sigma^{-1}}R_\sigma$, we can find some $a_i$'s in $R_\sigma$, $b_i$'s in $R_{\sigma^{-1}}$ such that $1 = \sum_{i=1,n} b_i a_i$. Define the maps :

$$u : R_\sigma \to R_e^n, \quad u(r_\sigma) = (r_\sigma b_1, ..., r_\sigma b_n)$$

and

$$v : R_e^n \to R_\sigma, \quad v(\lambda_1, ..., \lambda_n) = \sum_{i=1,n} \lambda_i a_i$$

Then $u$ and $v$ are morphisms of left $R_e$-modules and $v \circ u = Id$, therefore $R_\sigma$ is a direct summand of the left $R_e$-module $R_e^n$.

7. Let $R = \oplus_{\sigma \in G} R_\sigma$ be a $G$-strongly graded ring such that $R$ is a commutative ring. Prove that $G$ is an abelian group.

   **Hint :** Since $R$ is strongly graded, we have that $R_\sigma R_\tau = R_{\sigma\tau} \neq 0$ for any $\sigma, \tau \in G$. Therefore there exist $r_\sigma \in R_\sigma$ and $r_\tau \in R_\tau$ such that $r_\sigma r_\tau \neq 0$. We have that $r_\sigma r_\tau = r_\tau r_\sigma \neq 0$, and this implies $\sigma\tau = \tau\sigma$.

8. Let $R = \oplus_{\sigma \in G} R_\sigma$ be a ring graded by the abelian group $G$. Prove that the center $Z(R)$ of $R$ is a graded subring of $R$.

9. Let $(A, G, \sigma, \alpha)$ be a crossed system (see Section 1.4). If $(A, G, \sigma', \alpha')$ is another crossed system, we say that $(A, G, \sigma, \alpha)$ and $(A, G, \sigma', \alpha')$ are *equivalent* if there exists a map $u : G \to U(A)$ with the properties :

   i) $u(e) = 1$

   ii) $\sigma'(g) = \varphi_{u(g)}\sigma(g)$, where $\varphi_{u(g)} \in \mathrm{Aut}(A)$ is defined by $\varphi_{u(g)}(a) = u(g)au(g)^{-1}$ for any $a \in A, g \in G$

   iii) $\alpha'(x, y) = u(x)^x u(y)\alpha(x, y)u(xy)^{-1}$ for all $x, y \in G$.

   Prove that :

   a. The above defined relation between crossed systems (with the same $A$ and $G$) is an equivalence relation.

   b. The crossed systems $(A, G, \sigma, \alpha)$ and $(A, G, \sigma', \alpha')$ are equivalent if and only if there exists a graded isomorphism $f : A_\alpha^\sigma[G] \to A_{\alpha'}^{\sigma'}[G]$ such that $f(a) = a$ for every $a \in A$.

   c. If $Z(A)$ is the center of the ring $A$ and $(A, G, \sigma, \alpha)$ is a crossed system, prove that the map $\sigma$ defines a group morphism $G \to \mathrm{Aut}(U(Z(A)))$, i.e. $U(Z(A))$ is a $G$-module.

10. Let $(A, G, \sigma, \alpha)$ be a crossed system. We denote by $Z^2(G, U(Z(A)))$ the set of all functions $\beta : G \times G \to U(Z(A))$ satisfying the following conditions :

    i) $\beta(x, y)\beta(xy, z) = {}^x\beta(y, z)\alpha(x, yz)$ for every $x, y, z \in G$ (recall that by ${}^x a$ we mean $\sigma(x)(a)$).

ii) $\beta(x,e) = \beta(e,x) = 1$.

Prove that :

a. $Z^2(G, U(Z(A)))$ is an abelian group with the product defined by $(\beta\beta')(x,y) = \beta(x,y)\beta'(x,y)$ for every $\beta, \beta' \in Z^2(G, U(Z(A)))$. The elements of this group are called 2-cocycles with respect to the action of $G$ on $U(Z(A))$ defined by $\sigma$.

b. If $\beta \in Z^2(G, U(Z(A)))$ then $(A, G, \sigma, \beta\alpha)$ is also a crossed system.

c. If for every map $t : G \to U(Z(A))$ with $t(e) = 1$ we define the 2-coboundary $\delta t$ by $(\delta t)(x,y) = t(x)^x t(y) t(xy)^{-1}$, and we denote by $B^2(G, U(Z(A)))$ the set of 2-coboundaries, prove that $B^2(G, U(Z(A)))$ is a subgroup of $Z^2(G, U(Z(A)))$. Prove also that for $\beta \in Z^2(G, U(Z(A)))$ the crossed systems $(A, G, \sigma, \alpha)$ and $(A, G, \sigma, \beta\alpha)$ are equivalent if and only if $\beta \in B^2(G, U(Z(A)))$.

d. If $(A, G, \sigma, \alpha)$ and $(A, G, \sigma, \alpha')$ are two crossed systems, prove that there exists $\beta \in Z^2(G, U(Z(A)))$ with $\alpha' = \beta\alpha$.

e. We denote by

$$H^2(G, U(Z(A))) = Z^2(G, U(Z(A)))/B^2(G, U(Z(A)))$$

which is called the second cohomology group of $G$ over $U(Z(A))$. If $\beta \in Z^2(G, U(Z(A)))$ then $\hat{\beta}$ denotes the class of $\beta$ in the factor group $H^2(G, U(Z(A)))$. Prove that the map associating to $\hat{\beta} \in H^2(G, U(Z(A)))$ the equivalence class of the crossed system $(A, G, \sigma, \beta\alpha)$ is bijective (here $\sigma$ is the weak action of $G$ on $A$).

11. Let $(A, G, \sigma, \alpha)$ be a crossed system such that $\alpha(x,y) \in Z(A)$ for every $x, y \in G$. If $H^2(G, U(Z(A))) = \{1\}$, prove that the graded ring $A^\sigma_\alpha[G]$ is isomorphic to a skew group ring.

Let $G$ be a group ring with identity 1. A ring $R$ is said to be **$G$-system** if $R = \sum_{g \in G} R_g$, where $R_g$ are such additive subgroups of $R$ that $R_g R_n \subseteq R_{gh}$ for all $g, h \in G$. If for all $g, h \in G$, $R_g R_h = R_{gh}$, $R$ is called **Clifford system** (or almost strongly graded ring - see exercise 6.)

12. Prove the following assertions :

i) If $I$ is two-sided ideal of $R$ then $R/I$ is also a $G$-system (resp. a Clifford system).

ii) Every a $G$-system (resp. a Clifford system) is an image of a $G$-graded ring (resp. of a $G$-strongly graded ring).

13. (P. Greszczuk, [91]) Prove that if the the $G$-system $R = \sum_{g\in G} R_g$ has unity 1 and the group $G$ is finite then $1 \in R_e$.

    **Hint :** (Proof of author). For any nonempty subset $S$ of $G$ we put $R_s = \sum_{s\in S} R_1$. If $S, T \subseteq G$ are two nonempty sets we have $R_S.R_T \subseteq R_{ST}$. We prove by induction on $|G-S|$ that if $e \in S$ then $1 \in R_S$. If $S = G$, it is clear. Assume that the result is true for subsets of cardinality $> |S|$. Let $x \in G - S$, then $|S \cup \{x\}| > |S|$ and $|x^{-1}S \cup \{e\}| > |S|$. Hence by the induction assumption : $1 \in R_S + R_x$ and $1 \in R_{x^{-1}S} + R_1$. Hence $1 = a + b = c + d$ where $a \in R_S, b \in R_x$, $C \in R_{x^{-1}S}$ and $d \in R_1$. So, $(1 - a)(1 - d) \in R_x R_{x^{-1}S} \subseteq R_S$. Since $(1 - a)(1 - d) = 1 - a - d + ad$ since $e \in S$ we have $R_e \subseteq R_S$ and therefore $ad - a - d \in R_S$. Hence $1 \in R_S$. Thus, since $S = \{e\}$ satisfies the hypothesis it results that $1 \in R_e$. For $n = 1$ it is clear.

14. (P. Greszczuk) Let $R = \sum_{g\in G} R_g$ be a $G$-system with unity 1 ($G$ is a finite group). To prove :

    a. If $I$ is a right (left) ideal of $R_e$, then $IR = R$ (resp. $RI = R$) if and only if $I = R_e$.

    b. An element $x \in R_e$ is right (left) invertible in $R$ if and only if $x$ is right (left) invertible in $R_e$.

    c. $J(R) \cap R_e \subseteq J(R_e)$ where $J(R)$ (resp. $J(R_e)$) is the Jacobson radical of $R$ (resp. $R_e$).

    **Hint :** a. If $R = IR = \sum_{g\in G} IR_g$, $IR = \sum_{g\in G} IR_g$ is a $G$-system. By exercise 13. $1 \in JR_e \subseteq I$. For b. and c. we apply a.

15. Let $R = \oplus_{\sigma\in G} R_\sigma$ be a $G$-graded ring. Prove that the polynomial ring $R[X]$ is a $G$-graded ring with the grading $R[X]_\sigma = R_\sigma[X]$ for any $\sigma \in G$. Moreover if $R$ is a strongly graded ring (respectively a crossed product) then $R[X]$ is strongly graded (respectively crossed product).

## 1.6 Comments and References for Chapter 1

This chapter is of a preliminary nature. It contains the definitions of a $G$-graded ring and $G$-strongly graded ring for an arbitrary group $G$, together with the main examples. In Section 1.2 the category of $G$-graded rings is introduced, i.e. $G$-Ring, and properties of functorial nature are being looked at. As a special class of strongly graded rings, $G$-crossed products are introduced in Section 1.4. Crossed product constructions appear in different areas of Algebra, in particular the crossed product results in the theory of the Brauer group and Galois cohomology are well-known, see also exercise 1.5.10 before. The interested reader may also look at the book by S. Caenepeel, F. Van Oystaeyen, *Brauer Groups and Cohomology of Graded Rings*, M. Dekker, New York. Let us also mention that crossed products also appear in $C^*$-algebra theory, it is a type of construction with a rather general applicability. In this chapter we included some properties of crossed products as an initiation to the vast field of possible applications, we complete this by several exercises in Section 1.5

## Some References

- M.Cohen, L.Rowen, *Group Graded Rings*, [42].
- E. C. Dade, *Group Graded Rings and Modules*, [49].
- C. Năstăsescu, F. Van Oystaeyen, *Graded and Filtered Rings and Modules*, [147].
- C. Năstăsescu, F. Van Oystaeyen, *Graded Ring Theory*, [150].
- C. Năstăsescu, F. Van Oystaeyen, *On Strongly Graded Rings and Crossed Products*, [149].
- D. Passman, *Infinite Crossed Products and Group Graded Rings*, [169].
- D. Passman, *Group Rings, Crossed Products and Galois Theory*, [170].

# Chapter 2

# The Category of Graded Modules

## 2.1 Graded Modules

Throughout this section $R = \oplus_{\sigma\in G} R_\sigma$ is a graded ring of type $G$ for some fixed group $G$. A (left) **$G$-graded $R$-module** (or simple graded module) is a left $R$-module $M$ such that $M = \oplus_{x\in G} M_x$ where every $M_x$ is an additive subgroup of $M$, and for every $\sigma \in G$ and $x \in G$ we have $R_\sigma M_x \subseteq M_{\sigma x}$. Since $R_e M_x \subseteq M_x$ we see that every $M_x$ is an $R_e$-submodule of $M$. The elements of $\cup_{x\in G} M_x$ are called the homogeneous elements of $M$. A nonzero element $m \in M_x$ is said to be homogeneous of degree $x$, and we write $\deg(m) = x$. Every $m \in M$ can be uniquely represented as a sum $m = \sum_{x\in G} m_x$, with $m_x \in M_x$ and finitely many nonzero $m_x$. The nonzero elements $m_x$ in this sum are called the **homogenous components** of $m$. The set $\sup(m) = \{x \in G | m_x \neq 0\}$ is called the **support** of $m$. We also denote by $\sup(M) = \{x \in G | M_x \neq 0\}$ the **support** of the graded module $M$. If $\sup(M)$ is a finite set (we denote by $\sup(M) < \infty$) we say that $M$ is a graded module of finite support.

An $R$-submodule $N$ of $M$ is said to be a **graded submodule** if for every $n \in N$ all its homogeous components are also in $N$, i.e. : $N = \oplus_{\sigma\in G}(N \cap M_\sigma)$. For a graded submodule $N$ of $M$ we may define a quotient- (or factor-) structure on $M/N$ by defining a gradation as follows : $(M/N)_\sigma = M_\sigma + N/N$, for $\sigma \in G$. For an arbitrary submodule $N$ of a graded module $M$ we let $(N)_g$, resp. $(N)^g$, be the largest, resp. smallest, graded submodule of $M$ contained in $N$, resp. containing $N$. It is clear that $(N)_g$ equals the sum of all graded submodules of $M$ contained in $N$, while $(N)^g$ is the intersection of all graded submodules of $M$ containing $N$. We have : $(N)_g \subset N \subset (N)^g$. Of course, when $N$ itself is a graded submodule of $M$ then $(N)_g = N = (N)^g$. The set of $R$-submodules of a given module $M$ is usually denoted by $\mathcal{L}_R(M)$; in case $M$

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is a graded $R$-module we look at the set $\mathcal{L}_R^g(M)$ of all graded $R$-submodules of $M$. It is easily verified that $\mathcal{L}_R(M)$ is a lattice with respecty to the partial ordering given by inclusion and the operations $\cap$ and $\cup$; moreover $\mathcal{L}_R^g(M)$ is a sublattice of $\mathcal{L}_R(M)$.

Note that $(N)_g = \oplus_{\sigma\in G}(N \cap M_\sigma)$ is the submodule of $M$ generated by $N \cap h(M)$; on the other hand $(N)^g$ is the submodule of $M$ generated by the set $\cup_{n\in N}\{n_\sigma, \sigma \in G\}$, where $\{n_\sigma, \sigma \in G\}$ is the set of homogeneous components of $n \in N$.

From these observations it is also clear that an $R$-submodule $N$ of $M$ is a graded $R$-submodule if and only if $N$ has a set of generators consisting of homogeneous elements in $M$. All of the foregoing may be applied to left ideals $L$ of $R$ and two-sided ideals $I$ of $R$, in particular $(I)_g$ and $(I)^g$ are two-sided when $I$ is.

## 2.2 The category of Graded Modules

When the ring $R$ is graded by the group $G$ we consider the **category** $G-R$-**gr**, simply written $R$-gr if no ambiguity can arise, defined as follows. For the objects of $R$-gr we take the graded (left) $R$-modules and for graded $R$-modules $M$ and $N$ we define the morphisms in the graded category as :

$$\mathrm{Hom}_{R-\mathrm{gr}}(M,N) = \{f \in \mathrm{Hom}_R(M,N), f(M_\sigma) \subset N_\sigma, \text{ for all } \sigma \in G\}$$

From the definition it is clear thet $\mathrm{Hom}_{R-\mathrm{gr}}(M,N)$ is an additive subgroup of $\mathrm{Hom}_R(M,N)$.

At this point it is useful though not really essential to have knowledge of a few basic facts in Category Theory; we include a short introduction in Appendix A.

The category $R$-gr has **coproducts** and **products**. Indeed, for a family of graded modules $\{M_i, i \in J\}$ a coproduct $S_J = \oplus_{\sigma\in G} S_\sigma$ may be given by taking $S_\sigma = \oplus_{i\in J}(M_i)_\sigma$ and a product $P_J$ may be obtained by taking $P_\sigma = \prod_{i\in J}(M_i)_\sigma$, so $P_J = \oplus_{\sigma\in G}\prod_{i\in J}(M_i)_\sigma$.

Since for any $f \in \mathrm{Hom}_{R-\mathrm{gr}}(M,N)$ we have a kernel, $\mathrm{Ker}f$, and an image object, $\mathrm{Im}f$, which are in $R$-gr and such that : $M/\mathrm{Ker}f \cong \mathrm{Im}f$ are naturally isomorphic in this category, the category $R$-gr is an abelian category. It also follows that a graded morphism $f$ is a monomorphism, resp. epimorphism, in this category if and only if $f$ is injective, resp. surjective in the set theoretic sense.

In a straightforward way one may verify that $R$-gr satisfies Grothendieck's axioms : $Ab3$, $Ab4$, $Ab3^*$, $Ab^*$ and also $Ab5$.

For $M \in R$-gr and $\sigma \in G$ we define the **$\sigma$-suspension** $M(\sigma)$ of $M$ to be the graded $R$-module obtained from $M$ by putting $M(\sigma)_\tau = M_{\tau\sigma}$ for all $\tau \in G$. This defines a functor $T_\sigma : R\text{-gr} \to R\text{-gr}$ by putting $T_\sigma(M) = M(\sigma)$. The family of functors $\{T_\sigma, \sigma \in G\}$ satisfies :

1. $T_\sigma \circ T_\tau = T_{\sigma\tau}$ for all $\sigma, \tau \in G$
2. $T_\sigma \circ T_{\sigma^{-1}} = T_{\sigma^{-1}} \circ T_\sigma = \mathrm{Id.}$, for all $\sigma \in G$.

In particular it follows that each $T_\sigma$ is an isomorphism in the category $R$-gr. The left $R$-module ${}_RR$ is of course a graded $R$-module, hence it is clear that the family $\{{}_RR(\sigma), \sigma \in G\}$ is a set of generators of the category $R$-gr (see Appendix A, for the definition of a family of generators). Therefore the category $R$-gr is a Grothendieck category (see appendix A). One easily checks that each ${}_RR(\sigma)$ is a projective object in $R$-gr, such objects will be referred to as **graded projective modules**, hence $R$-gr has a projective family of generators. The general theory of Grothendieck categories then implies that $R$-gr has enough injective objects, these are referred to as **graded injective modules** (or **gr-injective modules**). Now $F \in R$-gr is said to be **gr-free** if it has an $R$-basis consisting of homogeneous elements, equivalently $F \cong \oplus_{i\in J} R(\sigma_i)$, where $\{\sigma_i, i \in J\}$ is a family of elements of $G$

Since a graded module $M$ has a set of homogeneous generators it is in an obvious way isomorphic to a quotient of a gr-free object of $R$-gr. Note that any gr-free object in $R$-gr is necessarily a free $R$-module when viewed as an $R$-module by forgetting the gradation. A more detailed treatment of the forgetful functor $U : R\text{-gr} \to R\text{-mod}$, associating to a graded $R$-module $M$ the underlying $R$-module $U(M)$, is given in Section 2.5. If $P$ is gr-projective then it is isomorphic to a direct summand of a gr-free $F$; in fact find a gr-free $F$ mapping to $P$ epimorphically in $R$-gr and use the projectivity of $P$ in $R$-gr. Hence it follows that a gr-projective in $R$-gr is just a graded and projective $R$-module. A similar property will fail for gr-free modules ! Indeed, taking $R = \mathbb{Z} \times \mathbb{Z}$ with trivial gradation and taking for $F$ the $R$-module $R$ endowed with the gradation given by $F_0 = \mathbb{Z} \times \{0\}, F_1 = \{0\} \times \mathbb{Z}$ and $F_i = 0$ for $i \neq 0, 1$, then it is clear that $F$ cannot have a homogeneous basis ! Hence gr-free is a stronger property than "graded plus free".

### 2.2.1 Remark

The category of right $G$-graded $R$-modules, $G$-gr-$R$ (or shortly gr-$R$) may be defined in a similar way. However, if we let $R^{\mathrm{op}}$ be the opposite graded ring with respect to the opposite group $G^{\mathrm{op}}$, then $G$-gr-$R$ is exactly the category $G^{\mathrm{op}}$-$R^{\mathrm{op}}$-gr and so we need not repeat any "right" versions of earlier observations.

For a given $M \in R$-gr we define $G(M) = \{\sigma \in G, M \cong M(\sigma)\}$. It is an easy exercise to establish that $G(M)$ is a subgroup of $G$. The latter subgroup is

called the **stabilizer** (or **inertia** group) of $M$. In case $G(M) = G$ we say that $M$ is $G$-**invariant**. The subgroup $G(M)$ is connected to $\sup(M)$.

### 2.2.2 Proposition

With notation as before, if $\{\sigma_i, i \in I\}$ is a left transversal of $G(M)$ in $G$ then there is a $J \subset I$ such that : $\sup(M) = \cup_{i\in J}\ \sigma_i G(M)$. Note that $J = \emptyset$ may be allowed. Moreover the cardinality of $J$ does not depend on $\{\sigma_i, i \in J\}$.

**Proof** If $\tau \in G(M)$ then $M \cong M(\tau)$ in $R$-gr. Take $J \subset I$ such that $\sigma_i \in \sup(M)$ for any $i \in J$. Then : $M(\tau)_{\sigma_i} \cong M_{\sigma_i} \neq 0$, hence also $M_{\sigma_i\tau} \neq 0$ and $\cup_{i\in J}\sigma_i G(M) = \sup(M)$. Conversely, , if $\sigma \in \sup(M)$ then it follows from $G = \cup_{i\in I}\sigma_i G(M)$ that $\sigma \in \sigma_i G(M)$ for a certain $i \in I$. Thus $\sigma = \sigma_i\tau$ for some $\tau \in G(M)$. Now from $M_\sigma = M_{\sigma_i\tau} = M(\tau)_{\sigma_i} \cong M_{\sigma_i}$ and $M_\sigma \neq 0$ we derive that $\sigma_i \in \sup(M)$ and thus $i \in J$, or $\sup(M) = \cup_{i\in J}\sigma_i G(M)$. The final part of the statement is clear. □

In view of the foregoing proposition we may put $|J| = [\sup(M) : G(M)]$.

### 2.2.3 Proposition

For $M \in R$-gr and any $\sigma \in G$ we obtain : $G(M(\sigma)) = \sigma G(M)\sigma^{-1}$. Also $\sup(M(\sigma)) = \sup(M)\sigma^{-1}$.

**Proof** Look at $\lambda \in G(M(\sigma))$, then $M(\sigma) \cong M(\sigma)(\lambda) = M(\lambda\sigma)$ for every $\sigma \in G$. Therefore $M \cong M(\lambda\sigma)(\sigma^{-1}) = M(\sigma^{-1}\lambda\sigma)$ and so $\sigma^{-1}\lambda\sigma \in G(M)$ or $\lambda \in \sigma G(M)\sigma^{-1}$. This proves the inclusion $G(M(\sigma)) \subset \sigma G(M)\sigma^{-1}$ and the reversed inclusion may be established in formally the same way. The last part of the proposition is clear. □

## 2.3 Elementary Properties of the Category $R$-gr

In this section we focus on some elementary properties of the category of graded $R$-modules for an arbitrary $G$-graded ring.

### 2.3.1 Proposition

Consider $M, N, P$ in $R$-gr with given $R$-linear maps, $f : M \to P$, $h : M \to N$, $g : N \to P$, such that $f = g \circ h$ and $f$ is a morphism in $R$-gr. If $g$, resp. $h$ is a morphism in the category $R$-gr, then there exists a morphism $h' : M \to N$, resp. $g' : N \to P$, in $R$-gr such that $f = g \circ h'$, resp. $f = g' \circ h$.

**Proof** Let us prove the case where $g$ is a morphism in $R$-gr. Pick an homogeneous $m \in M_\sigma$ for some $\sigma \in G$. We decompose $h(m)$ as $h(m) = \sum_{\tau \in G} h(m)_\tau$. The assumption $f = g \circ h$ entails that $f(m) = \sum_{\tau \in G} g(h(m)_\tau)$ with $g(h(m)_\tau) \in P_\tau$. Since $f(m) \in P_\sigma$ we may define the morphism (in $R$-gr) $h'$ by putting $h'(m) = h(m)_\sigma$. That $f = g \circ h'$ follows easily. □

### 2.3.2 Corollary

If $M \in R$-gr is projective (resp. injective), when considered as an ungraded module then $M$ is also projective, resp. injective in the category $R$-gr.

**Proof** Let us prove the statement concerning projectivity, the version for injectivity is dual. Consider an epimorphism $u : N \to N'$ in $R$-gr and any morphism $f : M \to N'$ in $R$-gr. Since $u$ is also surjective as a morphism in $R$-mod, there exists an $R$-linear $g : M \to N$ such that $f = u \circ g$. Applying the foregoing proposition yields the existence of a morphism $g'$ in $R$-gr, $g'M \to N$ such that $f = u \circ g'$ and this establishes the projectivity of $M$ in $R$-gr. □

### 2.3.3 Remark

We have observed in Section 2.2. that a gr-projective $R$-module $P$ is also projective when viewed as an ungraded $R$-module. However, if $Q$ is gr-injective, that is an injective object in the category $R$-gr, then $Q$ need not be injective in $R$-mod when viewed as an $R$-module. Particular cases where a positive solution exists are encountered in Corollary 2.3.2 and Corollary 2.5.2. For now let us just provide the following easy example. Over an arbitrary field $k$, consider the Laurent polynomial ring $R = k[T, T^{-1}]$ with $\mathbb{Z}$-gradation defined by $R_n = \{aT^n, a \in k\}$ for $n \in \mathbb{Z}$. In view of the graded version of Baer's theorem, cf. Corollary 2.4.8, it is easy to check that ${}_RR$ is gr-injective but not injective. A note about the language; we often refer to "the graded version of $X$", e.g. in the foregoing or in the corollary hereafter, to indicate that a result is in some way a graded version of some "well-known" result in module theory or general algebra. Even though, for the logical understanding of this text, knowledge of such results is not assumed (in fact, by restricting to the case of a trivial gradation one does recover a proof for the classical result referred to) it is of course beneficary to have studied some elementary algebra course as we pointed out in the introduction. This will provide more insight and shed light on the origin of some of the problems we encounter and why certain modifications have been made.

For $M \in R$-gr we define the **projective dimension of $M$ in the category $R$-gr** similar to the definition of projective dimension in $R$-mod and we adapt the notation $\mathrm{gr-pdim}_R(M)$ for this (writing $\mathrm{pdim}_R(M)$ for the projective dimension of $M$ viewed as an ungraded module). That this is not really a new invariant follows from the foregoing Corollary 2.3.2 and remark 2.3.3.

### 2.3.4 **Corollary** (A graded version of Maschke's theorem)

Consider a graded submodule $N$ of the graded module $M$. Then $N$ is a graded direct summand of $M$ (i.e. a direct summand as an $R$-gr object) if and only if $N$ is a direct summand of $M$ viewed as ungraded $R$-modules.

**Proof** If $N$ is a direct summand of $M$ as an $R$-module then there is an $R$-linear $f : M \to N$ such that $f \circ i = 1_N$, where $i$ is the canonical inclusion $N \to M$. In view of Proposition 2.3.1. we may find a graded morphism $f' : M \to N$ such that $f' \circ i = 1_N$. But this shows exactly that $i$ splits as a morphism in $R$-gr and thus $M = N \oplus N'$, where $N' = \mathrm{Ker}(f')$, in $R$-gr. The other implication is trivial enough, so the properties in the statement are indeed equivalent. □

Recall that in any category $\underline{\mathcal{C}}$ a subobject $N \subset M$ is said to be an **essential subobject** if for every other non-zero subobject $L \subset M$ we have $L \cap N \neq 0$ (we assume $\underline{\mathcal{C}}$ has a suitable initial object 0). In particular for the category $R$-gr a graded submodule $N$ of $M$ is gr-essential if it is an essential subobject in the above sense for the category $R$-gr, this is obviously equivalent to : for every nonzero homogeneous element $m \in M$ we have $N \cap Rm \neq 0$, in other words there is an $a \in h(R)$ such that $am \neq 0$.

### 2.3.5 Proposition

Let $N \subset M$ in $R$-gr. Then $N$ is gr-essential in $M$ if and only if $N$ is essential in $M$ in $R$-mod. Moreover, in this case we have that for every $m \in M$ there is a homogeneous $a_\tau \in R_\tau$ such that $a_\tau m \in N$ and $a_\tau m \neq 0$.

**Proof** First it is clear that an essential submodule $N$ of $M$ in $R$-mod is certainly gr-essential ($M$ and $N$ as in the statement). Conversely, assume that $N \subset M$ is essential in $R$-gr. Pick $m \neq 0$ in $M$, write $\mathrm{supp}(m) = \{\sigma_1, \ldots, \sigma_n\}, m = m_{\sigma_1} + \ldots + m_{\sigma_n}$ with $0 \neq m_{\sigma_i} \in M_{\sigma_i}$, $i = 1, \ldots, n$. By induction on $n$, we now establish $Rm \cap N \neq 0$, in fact we establish that there is an $a \in h(R)$ such that : $am \in N$, $am \neq 0$. In case $n = 1$ this follows from gr-essentiality of $N$ in $M$. In general, applying the induction hypothesis we select $b \in h(R)$ such that $b(m - m_{\sigma_1}) \neq 0$ in $N$. If $bm_{\sigma_1} = 0$ then $bm = b(m - m_{\sigma_1}) \neq 0$ in $N$ and we are done, so assume that $bm_{\sigma_1} \neq 0$. Take $c \in h(R)$ such that $cbm_{\sigma_1} \neq 0$ in $N$ and look at $cbm = cbm_{\sigma_1} + \ldots + cbm_{\sigma_n}$; since $cb \in h(R)$ the latter is the homogeneous decomposition of $cbm$ and as $cbm_{\sigma_1} \neq 0$ we must have $cbm \neq 0$. Since $cbm \in N$ follows from $b(m_{\sigma_2} + \ldots + m_{\sigma_m}) \in N$ and $cbm_{\sigma_1} \in N$, the claim has been established. □

## 2.4 The functor $\mathrm{HOM}_R(-,-)$

Let $R$ be a $G$-graded ring and $M = \oplus_{x\in G} M_x$ and $N = \oplus_{x\in G} N_x$ two objects from the category $R$-gr. An $R$-linear $f : M \to N$ is said to be a **graded morphism of degree** $\sigma, \sigma \in G$, if $f(M_x) \subseteq N_{x\sigma}$ for all $x \in G$. Graded morphism of degree $\sigma$ build an additive subgroup $\mathrm{HOM}_R(M,N)_\sigma$ of $\mathrm{Hom}_R(M,N)$. The following equalities hold :

$$\mathrm{HOM}_R(M,N)_e = \mathrm{Hom}_{R-\mathrm{gr}}(M,N)$$
$$\mathrm{HOM}_R(M,N)_\sigma = \mathrm{Hom}_{R-\mathrm{gr}}(M,N(\sigma)) = \mathrm{Hom}_{R-\mathrm{gr}}(M(\sigma^{-1}),N)$$

Also, if we put $\mathrm{HOM}_R(M,N) = \sum_{\sigma\in G} \mathrm{HOM}_R(M,N)_\sigma$, then we have $\mathrm{HOM}_R(M,N) = \oplus_{\sigma\in G}\mathrm{HOM}_R(M,N)_\sigma$ so $\mathrm{HOM}_R(M,N)$ is a graded abelian group of type $G$. At places in the literature the Hom is used in stead of HOM. e.g. in [84], [85].

Denote by $G$-gr-$Ab$ the category of graded abelian groups of type $G$. The correspondance $(M,N) \to \mathrm{HOM}_R(M,N)$ defines a left exact functor : $(R-\mathrm{gr})^\circ \times (R-\mathrm{gr}) \to G-\mathrm{gr}-Ab$. If $M,N,P \in R$-gr and $f : M \to N$ and $g : N \to P$ are morphisms of degree $\sigma$, respectively $\tau$, then $g \circ f : M \to P$ is a graded morphism of degree $\sigma\tau$. It follows that for $N = M$, the abelian group $\mathrm{HOM}_R(M,M)$ with the multiplication : $f * g = g \circ f$ where $f, g \in \mathrm{HOM}_R(M,M)$ is a $G$-graded ring. This ring is denoted by $\mathrm{END}_R(M)$. In general, the inclusion of $\mathrm{HOM}_R(M,N)$ in $\mathrm{Hom}_R(M,N)$ is proper as can be seen in the following example.

### 2.4.1 Example

Let $R = \oplus_{n\in\mathbb{Z}} R_n$ be a $\mathbb{Z}$-graded ring such that $R_n \neq 0$ for any $n \in \mathbb{Z}$. Then there exists an element $a = \sum_n a_n$ such that $a_n \in R_n$ and $a_n \neq 0$. Put $M =_R R^{(\mathbb{Z})}$, then $M$ is graded $R$-module. Define $f \in \mathrm{Hom}_R(M,R)$ by putting $f((x_n)_{n\in\mathbb{Z}}) = \sum_{i\in\mathbb{Z}} x_i a_i$, where $(x_n)_n \in M$. If $f \in \mathrm{HOM}_R(M,R)$ then there exist $f_{n_1},\ldots,f_{n_s}$ such that each $f_{n_i}$ is a morphism of degree $n_i$ for $1 \leq i \leq s$. In this case we have $f(M_0) \subseteq \sum_{i=1}^s R_{n_i}$. But $M_0 = R_0^{(\mathbb{Z})}$ and we consider $(x_n)_{n\in\mathbb{Z}} \in M_0$ such that $x_{n_0} = 1$ where $n_0 \neq \{n_1,\ldots,n_s\}$ and $x_n = 0$ for $n \neq n_0$. In this case we have $f((x_n)_{n\in\mathbb{Z}}) = a_{n_0} \notin \oplus_{i=1}^s R_{n_i}$. The connection between $\mathrm{HOM}_R(M,N)$ and $\mathrm{Hom}_R(M,N)$ can best be expressed by topological methods.

For this it is necessary to introduce some considerations concerning the "finite topology". Let $X$ and $Y$ be arbitrary sets and $Y^X$ the set of all mappings from $X$ to $Y$. It is clear that we may view $Y^X$ as the product of the sets $Y_x = Y$, where $x$ ranges over the index set $X$. The **finite topology** of $Y^X$ is obtained by taking the product space in the category of topological spaces, with each $Y$ being regarded as a discrete space. A basis for the open sets of this topology is given by the sets of the form $\{g \in Y^X | g(x_i) = f(x_i), 1 \leq i \leq n\}$, where

$\{x_i | 1 \leq i \leq n\}$ is a finite set of elements of $X$ and $f$ is a fixed element of $Y^X$. Every open set is a union of sets of this form.

If $X$ and $Y$ are abelian groups, and $\mathrm{Hom}_{\mathbb{Z}}(X,Y)$ is the set of all homomorphisms from $X$ to $Y$, then $\mathrm{Hom}_{\mathbb{Z}}(X,Y)$ is a subset of $Y^X$. In fact $\mathrm{Hom}_{\mathbb{Z}}(X,Y)$ is a closed set in $Y^X$. Indeed if $f \in Y^X$ belongs to the closure of $\mathrm{Hom}_{\mathbb{Z}}(X,Y)$ and $x, x' \in X$, there exist $g \in \mathrm{Hom}_{\mathbb{Z}}(X,Y)$ such that $g(x) = f(x), g(x') = f(x')$ and $g(x+x') = f(x+x')$. From this it follows that $f(x+x') = f(x) + f(x')$ thus $f \in \mathrm{Hom}_{\mathbb{Z}}(X,Y)$. $\mathrm{Hom}_{\mathbb{Z}}(X,Y)$ is in fact a topological abelian group for the topology induced by the finite topology.

If $f \in \mathrm{Hom}_{\mathbb{Z}}(X,Y)$, then the sets

$$V(f, x_1, \ldots, x_n) = \{g \in \mathrm{Hom}_{\mathbb{Z}}(X,Y) | g(x_i) = f(x_i), 1 \leq i \leq n\}$$

form a basis for the filter of neighbourhoods of $f$, where $\{x_i | 1 \leq i \leq n\}$ ranges over the finite subsets of $X$. Note that $V(f, x_1, \ldots, x_n) = \cap_{i=1} V(f, x_i)$ and $V(f, x_1, \ldots, x_n) = f + V(0, x_1, \ldots, x_n)$. Moreover $V(0, x_1, \ldots, x_n)$ is a subgroup of $\mathrm{Hom}_{\mathbb{Z}}(X,Y)$.

Assume now that $R$ is a $G$-graded ring, and $M, N \in R$-gr. We have the inclusion :

$$\mathrm{HOM}_R(M,N) \subseteq \mathrm{Hom}_R(M,N) \subseteq \mathrm{Hom}_{\mathbb{Z}}(M,N)$$

It is easy to see (same argument as above) that $\mathrm{Hom}_R(M,N)$ is a closed subset of $\mathrm{Hom}_{\mathbb{Z}}(M,N)$ which is a topological abelian group with respect to the topology induced by the finite topology. If $m \in M$ and $m = m_{x_1} + \ldots m_{x_s}$ where $\{m_{x_i} | 1 \leq i \leq s\}$ is the set of homogeneous components of $m$, then we clearly have the inclusion $V(f, m_{x_1}, \ldots, m_{x_s}) \subseteq V(f,m)$ and therefore the sets $V(f, m_1, \ldots, m_t)$ form a basis for the filter of neighbourhoods of $f$ when $\{m_i | 1 \leq i \leq t\}$ ranges over the finite subsets of $h(M)$.

### 2.4.2 Proposition

For any $\sigma \in G$, $\mathrm{HOM}_R(M,N)_\sigma$ is a closed subset of $\mathrm{Hom}_R(M,N)$ in the finite topology.

**Proof** Assume that $f$ belongs to the closure of $\mathrm{HOM}_R(M,N)$. If $m_x \in M_x$ then there exist $g \in \mathrm{HOM}_R(M,N)_\sigma$ such that $g \in V(f, m_x)$ and $f(m_x) = g(m_x)$. Since $g(M_x) \subseteq N_{x\sigma}$, we have that $f(m_x) = g(m_x) \in N_{x\sigma}$ hence $f(M_x) \subseteq N_{x\sigma}$, for every $x \in G$, and thus $f \in \mathrm{HOM}_R(M,N)_\sigma$.

Select $f \in \mathrm{Hom}_R(M,N)$ and $\sigma \in G$. We define a map $f_\sigma : M \to N$ in the following way. If $m_x \in M_x$ for some $x \in G$, we put $f_\sigma(m_x) = f(m_x)_{x\sigma}$, i.e. $f_\sigma(m_x)$ is the homogeneous component of degree $x\sigma$ of the element $f(m_x) \in N$. Then, if $a_\lambda \in R_\lambda$ is a homogeneous element of $R$ we have $a_\lambda m_x \in$

$M_{\lambda x}$ and so $f_\sigma(a_\lambda m_x) = f(a_\lambda m_x)_{\lambda x\sigma}$. On the other hand, $a_\lambda f_\sigma(m_x) = a_\lambda f(m_x)_{x\sigma}$. But since $a_\lambda f(m_x)_{x\sigma}$ is the homogeneous component of degree $\lambda x\sigma$ of the element $a_\lambda f(m_x) = f(a_\lambda m_x)$, we see that $f_\sigma(a_\lambda m_x) = a_\lambda f_\sigma(m_x)$ thus $f_\sigma$ is $R$-linear. Furthermore, since $f_\sigma(M_x) \subseteq M_{x\sigma}$, it follows that $f_\sigma \in \mathrm{HOM}_R(M,N)_\sigma$. □

Recall that if $(G,+)$ is a topological abelian group, $x \in G$ and $(x_i)_{i\in I}$ a family of elements of $G$, then this family is said to be **summable** to $x$ if, for any neighbourhood $V(x)$ of $x$, there exists a finite subset $J_0$ of $I$ such that $\sum_{i\in J} x_i \in V(x)$ for any finite subset $J$ of $I$ such that $J_0 \subseteq J$. If the family $(x_i)_{i\in I}$ is summable to $x$ then we write $\sum_{i\in I} x_i = x$. We have the following result, establishing the topological relation between $\mathrm{HOM}_R(M,N)$ and $\mathrm{Hom}_R(M,N)$.

### 2.4.3 Theorem

Let $R$ be a $G$-graded ring, $M, N \in R$-gr and $f \in \mathrm{Hom}_R(M,N)$. Then the following assertions hold :

i) The family $(f_\sigma)_{\sigma\in G}$ is summable to $f$ in the finite topology, i.e. $f = \sum_{\sigma\in G} f_\sigma$, where the $f_\sigma$ are uniquely determined by $f$, that is, if $(g_\sigma)_{\sigma\in G}$, with $g_\sigma \in \mathrm{HOM}_R(M,N)_\sigma$ is another family summable to $f$, then $f_\sigma = g_\sigma$ for any $\sigma \in G$.

ii) $\mathrm{Hom}_R(M,N)$ is the completion of $\mathrm{HOM}_R(M,N)$ in the finite topology.

**Proof** It is clear that in the definition of summable family of $\mathrm{Hom}_R(M,N)$ with respect to the finite topology we may restrict to considering neighbourhoods of $f$ of the form $V(f,m)$, where $m$ is a homogneous element of $M$. Assume that $m \in M_x$ for some $x \in G$. By definition of the maps $f_\sigma$ there exist a finite subset $J_o$ of $G$ such that $f(m) = (\sum_{\sigma\in J} f_\sigma)(m)$ and, moreover, for any $\sigma \notin J_0$, $f_\sigma(m) = 0$. Thus, for any finite subset $J$ of $G$ such that $J_0 \subseteq J$ we have $\sum_{\sigma\in J} f_\sigma \in V(f,m)$ and hence the family $(f_\sigma)_{\sigma\in G}$ is summable to $f$. Now, for the uniqueless property, assume that $(g_\sigma)_{\sigma\in G}$ is another family summable to $f$ such that there exist $\sigma_0 \in G$ with $f_{\sigma_0} \neq g_{\sigma_0}$. Then there exist a homogeneous element $m_x \in M_x$ such that $f_{\sigma_0}(m_x) \neq g_{\sigma_0}(m_x)$ and if we consider the neighbourhood $V(f,m_x)$ of $f$, there exists a finite subset $J_0$ of $G$ such that for any subset $J$ of $G$ that contains $J_0$ we have that $\sum_{\sigma\in J} f_\sigma \in V(f,m_x)$ and $\sum_{g\in G} g_\sigma \in V(f,m_x)$. If $\sigma_0 \notin J_0$ we put $J = J_0 \cup \{\sigma_0\}$; so we may assume :$\sigma_0 \in J$. Then we have $(\sum_{\sigma\in J} f_\sigma)(m_x) = (\sum_{\sigma\in J} g_\sigma)(m_x)$ and since $f_\sigma, g_\sigma \in \mathrm{HOM}_R(M,N)_\sigma$ we obtain that $f_\sigma(m_x) = g_\sigma(m_x)$ for any $\sigma \in J$. The latter is a contradiction and it completes the proof of the uniqueness. Finally, in order to prove ii. observe first that, since the finite sum $\sum_{\sigma\in J_0} f_\sigma$ belongs to $\mathrm{HOM}_R(M,N)$, it follows from i. that $V(f,m) \cap \mathrm{HOM}_R(M,N) \neq \emptyset$ and hence that $\mathrm{HOM}_R(M,N)$ is dense in $\mathrm{Hom}_R(M,N)$. Now, it is wellknown that

$N^M$, being a topological product of discrete topological groups, is complete in the finite topology. Thus $\mathrm{Hom}_R(M,N)$ being a closed subgroup of $N^M$, is complete in the induced topology and therefore $\mathrm{Hom}_R(M,N)$ is the completion of the Hansdorff topological group $\mathrm{HOM}_R(M,N)$ endowed with the topology induced by the finite topology of $\mathrm{Hom}_R(M,N)$, that is exactly the finite topology of $\mathrm{HOM}_R(M,N)$. □

### 2.4.4 Corollary

If $M, N \in R$-gr and $M$ is finitely generated then

$$\mathrm{HOM}_R(M,N) = \mathrm{Hom}_R(M,N)$$

**Proof** Since $M$ is finitely generated, $\mathrm{Hom}_R(M,N)$ is a discrete abelian group in the finite topology. Now by Theorem 2.4.3. the assertion follows. □

### 2.4.5 Corollary

Let $M, N \in R$-gr such that both $M$ and $N$ have finite support. Then $\mathrm{HOM}_R(M,N) = \mathrm{Hom}_R(M,N)$.

**Proof** Assume that $\sup(M) = \{\sigma_1, \ldots, \sigma_r\}$ and $\sup(N) = \{\tau_1, \ldots, \tau_s\}$. If $f \in \mathrm{Hom}_R(M,N)$, then by Theorem 2.4.3 we have that $\sum_{\sigma \in G} f_\sigma = f$ in the finite topology. If $\sigma \in G$ and $\sigma \notin \{\sigma_i^{-1}\tau_j | i = 1, \ldots, r; j = 1, \ldots, s\}$ then for any $i, 1 \leq i \leq r$ we have $f_\sigma(M_{\sigma_i}) \subseteq N_{\sigma_i \sigma}$, and since in this case $\sigma_i \sigma \notin \mathrm{supp}(N)$, $f_\sigma(M_{\sigma_i}) = 0$ so $f_\sigma = 0$. Then it is clear that $f = \sum_{i=1}^{r} \sum_{j=1}^{s} f_{\sigma_i^{-1}\tau_j}$ and hence $f \in \mathrm{HOM}_R(M,N)$. □

### 2.4.6 Corollary

Let $R$ be a $G$-graded ring where $G$ is a finite group. If $M, N \in R$-gr then we have

$$\mathrm{HOM}_R(M,N) = \mathrm{Hom}_R(M,N)$$

**Proof** Apply Corollary 2.4.5. □

Since the functor $\mathrm{HOM}_R(-,-)$ is left exact we can define the right derived functor denoted by $\mathrm{EXT}_R^n(-,-)$ where $n \geq 0$ (for the functor $\mathrm{Hom}_R(-,-)$ the right derived functors are denoted usually by $\mathrm{Ext}_R^n(-,-)$).

Let $R$ be a $G$-graded ring and $\mathcal{M} \in R$-gr.

A Noetherian (resp. Artinian) object in $R$-gr will be called gr-Noetherian (resp. gr-Artinian). As in the ungraded case it is very easy to prove that $M$

is gr-Noetherian if and only if every graded submodule is finitely generated, if and only if every ascending chain of graded submodules terminates. Similarly, $M$ is gr-Artinian if and only if every descinding chain of graded submodules terminates.

When the graded left $R$-module ${}_RR$ is gr-Noetherian, resp. gr-Artinian, we say that $R$ is left gr-Noetherian, resp gr-Artinian. Left-right symmetric versions of these definitions may be phrased in a similar way.

### 2.4.7 Corollary

Consider graded $R$-modules $M$ and $N$ and assume that one of the following conditions holds :

i) The ring $R$ is left gr-Noetherian and $M$ is finitely generated.

ii) the group $G$ is a finite group.

Then, for every $n \geq 0$ we have :

$$\mathrm{EXT}^n_R(M,N) = \mathrm{Ext}^n_R(M,N)$$

**Proof** The case where $G$ is a finite group follows as a consequence of Corollary 2.4.6. So let us assume we are in the situation i. The assumptions then allow to construct the free resolution :

$$\to F_2 \to F_1 \to F_0 \to M \to 0$$

where each $F_i$ is a free graded $R$-module of finite rank. As a consequence of Corollary 2.4.4., and using the well-known calculus of right derived functors, the statement follows. □

A graded $R$-module $M$ is said to be gr-injective, resp. gr-projective, if $M$ is injective, resp. projective, as an object of $R$-gr. The property of gr-projectivity is a very well-behaved one, see Corollary 2.3.2. and Remark 2.3.3.

From loc. cit. it is easy to obtain that $M$ is gr-projective if and only if the functor $\mathrm{Hom}_{R-\mathrm{gr}}(M,-)$ is exact, if and only if the functor $\mathrm{HOM}_R(M,-)$ is exact, if and only if $M$ is a direct summand in $R$-gr of a gr-free $R$-module. Remark 2.3.3. already pointed at the more erratic behaviour of the notion "gr-injective".

### 2.4.8 Corollary

The following statements are equivalent for some $Q \in R$-gr :

i) $Q$ is a gr-injective $R$-module

ii) The functors $\mathrm{Hom}_{R-\mathrm{gr}}(-,Q)$ as well as $\mathrm{HOM}_R(-,Q)$ are exact.

iii) *(Graded Version of Baer's theorem)* For every graded left ideal $L$ of $R$ we obtain from the canonical inclusion $i : L \to R$ a surjective morphism :

$$\mathrm{HOM}(i, 1_Q) : \mathrm{HOM}_R(R, Q) \to \mathrm{HOM}_R(L, Q)$$

**Proof** The equivalence i. $\Leftrightarrow$ ii. as well as the implication ii. $\Rightarrow$ iii. are clear. The proof of iii. $\Rightarrow$ ii. is formally similar to the proof given in the ungraded case so we delete it here. □

It is possible and easy to construct a theory of graded bimodules with respect to two graded rings of type $G$ (for the same group; more general situations may be considered too but that is out of the scope of this book). Let us just introduce some basic notions and a version of the hom-tensor relation.

Consider the $G$-graded rings $R$ and $S$. An abelian group $M$ is said to be a **graded $R$-$S$-bimodule** if $M = \oplus_{\sigma\in G} M_\sigma$ is an $R$-$S$-bimodule such that the structure of a left $R$-module makes it a graded $R$-module and the structure of right $S$-module makes it into a graded right $S$-module, i.e. for $\sigma, \tau, \gamma \in G$ we have $R_\sigma M_\tau R_\gamma \subset M_{\sigma\tau\gamma}$.

For $N \in R$-gr we have that $\mathrm{Hom}_R(M, N)$ is a left $S$-module by putting : $(s.f)(m) = f(ms)$ for $s \in S$, $f \in \mathrm{Hom}_R(M, N)$ and $m \in M$. Moreover, if $f : M \to N$ is $R$-linear of degree $\sigma$ and $s \in S_\lambda$ then $s.f : M \to N$ has degree $\lambda\sigma$. Indeed, for $m_\tau \in M_\tau$ we obtain : $(s.f)(m_\tau) = f(m_\tau s) \in N_{\tau\lambda\sigma}$ and $(s.f)(M_\sigma) \subset N_{\tau\lambda\sigma}$, or $s.f$ has degree $\lambda\sigma$. The foregoing establishes that $\mathrm{HOM}_R(M, N)$ is in fact a graded left $S$-module.

Now look at $M \in$gr-$R$ and $N \in R$-gr. We may consider the abelian group $M \otimes_R N$, which may be $G$-graded by putting $(M \otimes_R N)_\sigma$ equal to the additive subgroup generated by all elements $x \otimes y$ with $x \in M_\tau$, $y \in N_\gamma$ such that $\tau\gamma = \sigma$. To see that this is well-defined (the tensor product over a noncommutative ring always is to be handled with some care !) we may start the construction from the abelian group $M \otimes_{\mathbb{Z}} N$ which is $G$-graded by putting :

$$(M \underset{\mathbb{Z}}{\otimes} N)_\sigma = \underset{\substack{\tau,\gamma\in G \\ \tau\gamma=\sigma}}{\oplus} (M_\tau \otimes N_\gamma), \text{ for each } \sigma \in G$$

In this $G$-graded abelian group $M \otimes_{\mathbb{Z}} N$ the additive subgroup $K$ generated by all elements of the form $mr \otimes n - m \otimes rn$, with $m \in M, n \in N, r \in R$, is a graded subgroup of $M \otimes_{\mathbb{Z}} N$. By definition we have that $M \otimes_R N = (M \otimes_{\mathbb{Z}} N)/K$ and the gradation we have defined on $M \otimes_R N$ is just the one induced by the $G$-gradation of the abelian group $M \otimes_{\mathbb{Z}} N$. Considering $\mathbb{Z}$ with trivial $G$-gradation as a graded ring, the object $M \otimes_R N \in \mathbb{Z}$-gr defined is called the **graded tensor product** of $M$ and $N$.

Now if ${}_RN_S$ is a graded $R$-$S$-bimodule then $M \otimes_R N$ inherits the structure of a graded right $S$-module defined by putting : $(m \otimes n)s = m \otimes ns$, for any

$m \in M, n \in N, s \in S$. There is a well-known relation between the functors $\mathrm{Hom}_R(-,-)$ and $-\otimes_R -$, often referred to as the hom-tensor relation, we do have a graded version of such a relation.

### 2.4.9 Proposition

For $M \in$gr-$R$, $P \in$ gr-$S$, $N \in R$-gr-$S$ we obtain a natural graded isomorphism :

$$\mathrm{HOM}_S(M \otimes_R N, P) \cong \mathrm{HOM}_R(M, \mathrm{HOM}_S(N, P)$$

defined as follows : for $m \in M, n \in N, f \in \mathrm{HOM}_S(M \otimes_R N, P) : \varphi(f)(m)(n) = f(m \otimes n)$ In particular we obtain a canonical isomorphism :

$$\mathrm{Hom}_{S-\mathrm{gr}}(M \otimes_R N, P) \cong \mathrm{Hom}_{R-\mathrm{gr}}(M, \mathrm{HOM}_S(N, P))$$

This means that the functor $- \otimes_R N : \mathrm{gr}{-}R \to \mathrm{gr}{-}S$ is left adjoint of the functor $\mathrm{HOM}_S(N, -) : \mathrm{gr}{-}S \to \mathrm{gr}{-}R$.

**Proof** The argument is an absolutely straightforward graded version of the classical (ungraded) argument in the classical situation. This may be found in any textbook on homological algebra or basic algebra, e.g. N. Bourbaki, so we refer to the literature for this. □

Let $R = \oplus_{\sigma \in G} R_\sigma$ and $S = \oplus_{\sigma \in G} S_\sigma$ be two $G$-graded rings and $\varphi : R \to S$ is a graded morphism of rings (i.e. $\varphi(R_\sigma) \subseteq S_\sigma$ for any $\sigma \in G$). We denote by $\varphi_*^{\mathrm{gr}} : S$-gr $\to R$-gr the functor of "restriction of scalars" so if $M \in S$-gr, then $\varphi_*^{\mathrm{gr}}(M) = M$ where $M$ has the following structure as left $R$-module : if $a \in R$ and $m \in M$, then $a * m = \varphi(a).m$. We denote by $S \otimes_R - : R$-gr $\to S$-gr, $M \to S \otimes_R M$ the **functor induced by** $\varphi$ and by

$$\mathrm{HOM}_R({}_RS_R, -) : R{-}\mathrm{gr} \ \to S{-}\mathrm{gr}$$

the **coinduced functor**.

### 2.4.10 Corollary

With notation as above, the functor $S \otimes_R$- is a left adjoint of the functor $\varphi_*^{\mathrm{gr}}$ and the functor $\mathrm{HOM}_R({}_RS_S, -)$ is a right adjoint of the functor $\varphi_*^{\mathrm{gr}}$.

**Proof** We apply Proposition 2.4.9, because $\varphi_*^{\mathrm{gr}} \simeq S_S \otimes_S -$ and also $\varphi_*^{\mathrm{gr}} \simeq \mathrm{HOM}_S({}_SS_R, -)$.

## 2.5 Some Functorial Constructions

Maybe it is surprizing to learn that the forgetful functor $R$-gr $\to R$-mod is in fact useful at places. Perhaps the most obvious reason for this is that we may

construct a right adjoint functor for it. The first part of this section deals with this. In a second part induced and coinduced functors are the topic of study.

To a $G$-graded ring $R$ we have associated several categories, e.g. $R$-mod, $R$-gr, $R_e$-mod,.... The study of relations between these categories is the main topic of "graded module theory". Let us write $U : R\text{-gr} \to R\text{-mod}$ for the forgetful functor (forgetting the graded structure on the $R$-modules). We know, from the basic properties of $R$-gr, that $U$ is an exact functor. We may define a right adjoint $F : R\text{-mod} \to R\text{-gr}$ for the functor $U$, in the following way. To $M \in R$-mod, associate the $G$-graded abelian group ${}^GM = \oplus_{\sigma\in G}{}^\sigma M$ where ${}^\sigma M$ is just a copy of $M$ indexed by $\sigma$, but with $R$-module structure given by : $r.^\tau m = {}^{\sigma\tau}(rm)$ for ${}^\tau m \in {}^\tau M$ and $r \in R_\sigma$, i.e. $r.^\tau m$ is the element $rm$ viewed in the copy ${}^{\sigma\tau}M$ of $M$. It is obvious that the foregoing does define a structure on ${}^GM$ making it into a $G$-graded $R$-module. The latter graded object is denoted by $F(M)$, $F(M) = \oplus_{\sigma\in G}F(M)_\sigma$, where $F(M)_\sigma$ is nothing but ${}^\sigma M$ as an abelian group.

To an $R$-linear map $f : M \to N$ we correspond $F(f) : F(M) \to F(N)$, such that for $x \in M, \sigma \in G$ we have $F(f)({}^\sigma x) = {}^\sigma f(x)$. Obviously, $F(f)$ is a morphism in $R$-gr. By construction $F$ is an exact functor. Observe that $UF(M) = \oplus_{\sigma\in G}{}^\sigma M$, but this is not a direct sum of copies of $M$ as an $R$-module because each ${}^\sigma M$ is not an $R$-submodule of $F(M)$ (it is of course an $R_e$-submodule).

### 2.5.1 Theorem

The functor $F$ is a right adjoint for $U$. In case $G$ is a finite group, then $F$ is also a left adjoint for $U$.

#### Proof

Consider $M \in R$-gr and $N \in R$-mod. Define $\varphi(M,N) : \mathrm{Hom}_R(U(M),N) \to \mathrm{Hom}_{R-\mathrm{gr}}(M,F(N))$, by putting : $\varphi(M,N)(f)(m_\sigma) = {}^\sigma(f(m_\sigma))$ for $m_\sigma \in M_\sigma$ and $f : U(M) \to N$ an $R$-linear map. Clearly $\varphi(M,N)(f)$ is also $R$-linear and in fact even a morphism in the category $R$-gr as it preserves degrees. If $f$ is such that $\varphi(M,N)(f) = 0$ then we must have $\varphi(M,N)(f)(m_\sigma) = 0$ for all $m_\sigma \in M_\sigma$, all $\sigma \in G$, hence ${}^\sigma(f(m_\sigma)) = 0$ and thus $f(m_\sigma) = 0$ for each $m_\sigma \in M$ and each $\sigma \in G$. Consequently $\varphi(M,N)(f) = 0$ entails that $f = 0$ and $\varphi(M,N)$ is an injective map. On the other hand, look at a given $g \in \mathrm{Hom}_{R-\mathrm{gr}}(M,F(N))$. Now define $\alpha_N : F(N) \to N$ by putting : $\alpha_N(({}^\sigma n)_{\sigma\in G}) = \sum_{\sigma\in G}{}^\sigma n$ (since $F(N)$ is a direct sum, the latter sum is defined). Clearly $\alpha_N$ is $R$-linear and if we define $f = \alpha_N \circ g$ we have an $R$-linear map $f : M \to N$. One easily verifies that $\varphi(M,N)(f) = g$ and it follows that $\varphi(M,N)$ is an isomorphism. The system $\{\varphi(M,N), M \in R\text{-mod}, N \in R\text{-gr}\}$ does define a functorial isomorphism.

For the second part, assume that $G$ is a finite group. For $M \in R$-mod and $N \in R$-gr we now define :

$$\psi(M,N) : \mathrm{Hom}_{R-\mathrm{gr}}(F(M),N) \to \mathrm{Hom}_R(M,U(N))$$

in the following way. Define $\alpha_M : M \to F(M), m \mapsto ({}^\sigma m)_{\sigma\in G}$ where ${}^\sigma m = m$ for any $\sigma \in G$. It is clear that for $\lambda_\sigma \in R_\sigma, \alpha_M(\lambda_\sigma m) = ({}^{\sigma x}(\lambda_\sigma m))_{x\in G} = \lambda_\sigma({}^x m)_{x\in G}$. Observe that finiteness of $G$ is used in the construction of $\alpha_M$ because $({}^\sigma m)_{\sigma\in G}$ must be in the direct sum ! Now to $f \in \mathrm{Hom}_{R-\mathrm{gr}}(F(M),N)$ we associate $\psi(M,N)(f) = f \circ \alpha_M$ in $\mathrm{Hom}_R(M,U(N))$. Again $\psi(M,N)$ is injective because whenever $\psi(M,N)(f) = 0$ we have $(f\circ\alpha)(m) = 0$ for all $m \in F(M)_\sigma$, all $\sigma \in G$, thus $f(({}^\sigma m)_{\sigma\in G}) = 0$ and as $f$ preserves degrees this leads to $f(0,\ldots,m,0\ldots,0) = 0$ with $m$ in the $\sigma^{\mathrm{th}}$-position. The latter means $f(F(M)_\sigma) = 0$ for all $\sigma \in G$, or $f = 0$. On the other hand, for a given $g \in \mathrm{Hom}_R(M,N)$ we define the map $f : F(M) \to N$ by $f({}^\sigma m) = g(m)_\sigma$, where ${}^\sigma m \in F(M)_\sigma = {}^\sigma M$. For any $m \in M$ we have $(f\circ\alpha_M)(m) = f(({}^\sigma m)_{\sigma\in G}) = \sum_{\sigma\in G} g(m)_\sigma = g(m)$. Thus $\psi(M,N)(f) = g$ and it follows that $\psi(M,N)$ is also surjective, thus an isomorphism. One verifies that the system $\{\psi(M,N), M \in R\text{-mod}, N \in R\text{-gr}\}$ defines a functorial morphism. □

### 2.5.2 Corollary

Let $R$ be graded by a finite group $G$ and let $Q \in R$-gr. Then $Q$ is gr-injective if and only if $U(Q)$ is injective in $R$-mod.

**Proof** One implication has been proved in Corollary 2.3.2. Assuming that $Q$ is gr-injective and using the exactness of $U$ and the properties of adjoints (see Appendix A), it is easy to see that $U(Q)$ is injective in $R$-mod. □

The reader may already have noticed that often we write $Q$ for $U(Q)$ (we forget to forget !) when there is no danger of confusion; often we will say "$Q$ viewed as an $R$-module".

There is a partial converse to Theorem 2.5.1. :

### 2.5.3 Proposition

If $U : R$-gr $\to R$-mod has a left adjoint then $G$ is finite.

**Proof** When $U$ has a left adjoint then $U$ commutes with arbitrary direct products in the sense that the natural $R$-morphism $f : U(\prod^{\mathrm{gr}}_{i\in J} M_i) \to \prod_{i\in J} U(M_i)$ is an isomorphism for any set $J$, and any family of graded $R$-modules $\{M_i, i \in J\}$, where $\prod\limits_{i\in J}{}^{\mathrm{gr}} M_i = \oplus_{\sigma\in G}(\prod_{i\in J}(M_i)_\sigma)$. Take $J = G$ and $M_i = R(i^{-1})$ for $i \in G$ and $m_i \in M_i, m_i = 1$, hence $m_i$ is of degree $i \in G$. In this case$(m_i)_{i\in J} \in \prod_{i\in J} U(M_i)$ cannot be in Imf unless $G$ is finite, so necessarily $G$ must have been a finite group. □

### 2.5.4 Proposition

For $M \in R$-gr we have : $FU(M) \cong \oplus_{\sigma\in G} M(\sigma)$.

**Proof** Let us consider ${}^x m$ in ${}^x M$ for $m \in M$ and a fixed $x \in G$. Write $m = \sum_{\sigma\in G} m_{x\sigma}$ with $m_{x\sigma} \in M_{x\sigma}$. Define $\alpha : F(U(M)) \to \oplus_{\sigma\in G} M(\sigma)$, $x_m \mapsto (m_{x\sigma})_{\sigma\in G}$ where $m_{x\sigma} \in M(\sigma)_x$; this is allowed because only finitely many of the $\{m_{x\sigma}, \sigma \in G\}$ are nonzero. For $\lambda_\tau \in R_\tau, \tau \in G$, we calculate : $\alpha(\lambda_\tau {}^x m) = \alpha({}^{\tau x}(\lambda_\tau m_{x\sigma})_{\sigma\in G}) = \lambda_\tau (m_{x\sigma})_{\sigma\in G} = \lambda_\tau \alpha({}^x m)$. Consequently, $\alpha$ is $R$-linear. Moreover,

$$\alpha(F(U(M))_x) \subset (\oplus_{\sigma\in G} M(\sigma))_x$$

hence $\alpha$ is in fact a morphism in the category $R$-gr. A standard verification learns that $\alpha$ is also bijective and thus an isomorphism. □

In the remainder of this section we study induced and coinduced functors. Let $R = \oplus_{\sigma\in G} R_\sigma$ be a $G$-graded ring, and $N \in R_e$-mod. We consider the graded $R$-module $M = R \otimes_{R_e} N$, where $M$ has the gradation given by : $M_\sigma = R_\sigma \otimes_{R_e} N$, for $\sigma \in G$. The graded $R$-module $M = \oplus_{\sigma\in G} M_\sigma$ is called the $R$-module **induced** by the $R_e$-module $N$. We denote this module by $\mathrm{Ind}(N)$. It is obvious that the mapping $N \to \mathrm{Ind}(N)$ defines a covariant functor $\mathrm{Ind} : R_e$-mod $\to R$-gr, called the **induced functor**. This functor is right exact. Moreover, if $R$ is a flat right $R_e$ module (i.e. $R_\sigma$ is a flat right $R_e$-module for any $\sigma \in G$), then the functor Ind is exact.

Since $R$ is an $R_e - R$-bimodule, we may consider the left $R$-module $M' = \mathrm{Hom}_{R_e}(R, N)$. If $f \in \mathrm{Hom}_{R_e}(R, N)$ and $a \in R$, the multiplication $af$ is given by $(af)(x) = f(xa), x \in R$. For any $\sigma \in G$, we define the set $M'_\sigma = \{f \in \mathrm{Hom}_{R_e}(R, N) | f(R_{\sigma'}) = 0$ for any $\sigma' \neq \sigma^{-1}\}$. It is obvious that $M'_\sigma$ is a subgroup of $M'$ (in fact $M'_\sigma \simeq \mathrm{Hom}_{R_e}(R_{\sigma^{-1}}, N)$). The sum $M^* = \sum_{\sigma\in G} M'_\sigma$ is a direct sum. Indeed, if $f \in M'_\sigma \cap (\sum_{\tau\neq\sigma} M'_\tau)$, we have that $f \in M'_\sigma$ and $f = \sum_{\tau\neq\sigma} f'_\tau$, $f'_\tau \in M'_\tau$, thus if $x \in R_{\sigma^{-1}}$ we have $f(x) = \sum_{\tau\neq\sigma} f'_\tau(x) = 0$, so $f(R_{\sigma^{-1}}) = 0$. Since $f(R_\tau) = 0$ for any $\tau \neq \sigma^{-1}$, we obtain that $f = 0$. Now we prove that $R_\sigma M^*_\tau \subseteq M^*_{\sigma\tau}$ for any $\sigma, \tau \in G$. Indeed if $a \in R_\sigma$ and $f \in M_\tau$ we have for any $x \in R_\lambda$, where $\lambda \neq (\sigma\tau)^{-1} = \tau^{-1}\sigma^{-1} : (af)(x) = f(xa) = 0$ since $xa \in R_{\lambda\sigma}$ and $\lambda\sigma \neq \tau^{-1}$. Therefore, $af \in M^*_{\sigma\tau}$. Consequently $M^* = \oplus_{\sigma\in G} M'_\sigma$ is an object in the category $R$-gr. This object is called the **coinduced module** for $N$, and is denoted by $\mathrm{Coind}(N)$. It is obvious that the mapping $N \to \mathrm{Coind}(N)$ defines a covariant functor $\mathrm{Coind} : R_e$-mod$\to R$-gr, called the **coinduced functor**. It is obvious that Coind is a left exact functor. Furthermore, if $R$ is a projective left $R_e$-module, then Coind is an exact functor. Now if $\sigma \in G$ is fixed, we can define the functor $(-)_\sigma : R$-gr $\to R_e$-mod, given by $M \to M_\sigma$, where $M = \oplus_{\tau\in G} M_\tau \in R$-gr. It is obvious that $(-)_\sigma$ is an exact functor.

We recall that by $T_\sigma : R\text{-gr} \to R\text{-gr}$ we have denoted the $\sigma$-suspension functor. The main result of this section is the following :

### 2.5.5 Theorem

With notation as above we have :

a. The functor $T_{\sigma^{-1}} \circ \mathrm{Ind}$ is a left adjoint functor of the functor$(-)_\sigma$. Moreover, $(-)_\sigma \circ T_{\sigma^{-1}} \circ \mathrm{Ind} \simeq 1_{R_e-\mathrm{mod}}$

b. The functor $T_{\sigma^{-1}} \circ \mathrm{Coind}$ is a right adjoint functor of the functor $(-)_\sigma$. Moreover, $(-)_\sigma \circ T_{\sigma^{-1}} \circ \mathrm{Coind} \simeq 1_{R_e-\mathrm{mod}}$.

**Proof**

a. Since $(-)_\sigma = (-)_e \circ T_\sigma$ and $T_\sigma$ is an isomorphism of categories with inverse $T_{\sigma^{-1}}$, it is enough to prove a. and b. when $\sigma = e$. So for a. we prove now that Ind is a left adjoint functor of the functor $(-)_e$. For this we define the functorial morphisms

$$\mathrm{Hom}_{R-\mathrm{gr}}(\mathrm{Ind}, -) \underset{\beta}{\overset{\alpha}{\rightleftarrows}} \mathrm{Hom}_{R_e}(-, (-)_e)$$

as follows :
if $N \in R_e$-mod and $M \in R$-gr, then $\alpha(N, M) : \mathrm{Hom}_{R-\mathrm{gr}}(\mathrm{Ind}(N), M) \to \mathrm{Hom}_{R_e}(N, M_e)$ is defined by $\alpha(N, M)(u)(x) = u(1 \otimes x)$ where $u : R \otimes_{R_e} N \to M$ is a morphism in $R$-gr. Clearly $u(1 \otimes x) \in M_e$, since $1 \otimes x \in (R \otimes_{R_e} N)_e = R_e \otimes_{R_e} N \simeq N$.

Now we define $\beta(N, M) : \mathrm{Hom}_{R_e}(N, M_1) \to \mathrm{Hom}_{R-\mathrm{gr}}(\mathrm{Ind}(N), M)$ as follows :
if $v \in \mathrm{Hom}_{R_e}(N, M_e)$, we put $\beta(N, M)(v) : R \otimes_{R_e} M \to M$ defined by $\beta(N, M)(v)(\lambda \otimes x) = \lambda v(x)$. It is clear that $\beta(N, M)(v) \in \mathrm{Hom}_{R-\mathrm{gr}}(\mathrm{Ind}(N), M)$. It is easy to see that $\alpha$ and $\beta$ are functorial morphisms and $\beta(N, M)$ is inverse to $\alpha(N, M)$, hence $\alpha$ and $\beta$ are functorial isomorphisms with $\beta$ inverse to $\alpha$. So Ind is a left adjoint functor of $(-)_e$. The last assertion of a. is obvious. □

b. We define the functorial morphism

$$\mathrm{Hom}_{R_e}((-)_e, -) \underset{\delta}{\overset{\gamma}{\rightleftarrows}} \mathrm{Hom}_{R-\mathrm{gr}}(-, \mathrm{Coind})$$

as follows :
if $N \in R_e$-mod and $M \in R$-gr, then $\gamma(M, N) : \mathrm{Hom}_{R_e}(M_e, N) \to \mathrm{Hom}_{R-\mathrm{gr}}(M, \mathrm{Coind}(N))$ is defined by putting for each $u \in \mathrm{Hom}_{R_e}(M_e,$

$N$) and $m_x \in M_x$,$\gamma(M,N)(u)(m_x) : R \to N$, $\gamma(M,N)(u)(m_x)(a) = u(a_{x^{-1}}m_x)$ where $a = \sum_{g\in G} a_g \in R$, $a_g \in R_g$.

It is clear that $\gamma(M,N)(u)(M_x) \subset \mathrm{Coind}(N)_x$ for any $x \in G$, hence $\gamma(M,N)(u) \in \mathrm{Hom}_{R-\mathrm{gr}}(M, \mathrm{Coind}(N))$ and therefore the map $\gamma(M,N)$ is well-defined.

Conversely, if $v \in \mathrm{Hom}_{R-\mathrm{gr}}(M, \mathrm{Coind}(N))$, we define $\delta(M,N)(v) : M_e \to N$ by $\delta(M,N)(v)(m_e) = v(m_e)(1)$.

If $a \in R_e$ we have $\delta(M,N)(v)(am_e) = v(am_e)(1) = (av(me))(1) = v(m_e)(a) = av(m_e)(1) = a\delta(M,N)(v)(m_e)$ and therefore $\delta(M,N)(v) \in \mathrm{Hom}_{Re}(M_e, N)$ so $\delta(M,N)$ is well defined. Now if $u \in \mathrm{Hom}_{R_e}(M_e, N)$, we have that $(\delta(M,N) \circ \gamma(M,N))(u) = \delta(M,N)(\gamma(M,N)(u))$. If $m_e \in M_e$ then we have $\delta(M,N)(\gamma(M,N)(u))(m_e) = \gamma(M,N)(u)(m_e)(1) = u(1.m_e) = u(m_e)$, $\delta(M,N) \circ \gamma(M,N) = 1_{\mathrm{Hom}_{R_e}(N_e,N)}$. Conversely, if $v \in \mathrm{Hom}_{R-\mathrm{gr}}(M, \mathrm{Coind}(N))$ the we have $(\gamma(M,N)) \circ \delta(M,N))(v) = \gamma(M,N)(\delta(M,N)(v))$. Now if $m_x \in M_x$ and $a \in R$ then we have $\gamma(M,N)(\delta(M,N)(v)(m_x)(a) = \delta(M,N)(v)(a_{x^{-1}}m_x) = v(a^m_{x^{-1}}x)(1) = (a_{x^{-1}}v(m_x))(1) = v(m_x)(1.a_{x^{-1}}) = v(m_x)(a_{x^{-1}}) = v(m_x)(a)$ (because $v(m_x)(a_y) = 0$ for any $y \neq x^{-1}$). Consequently $(\gamma(M,N) \circ \delta(M,N) = 1_{\mathrm{Hom}R-\mathrm{gr}}(M, \mathrm{Coind})$, Finally, the functor Coind is a right adjoint of the functor $(-)_e$. The last assertion of b. is obvious. □

### 2.5.6 Corollary

1. If $N$ is an injective $R_e$-module, then $\mathrm{Coind}(N)$ is gr-injective.
2. If $M = \oplus_{\sigma\in G} M_\sigma$ is gr-injective, and for any $\sigma \in G$, $R_\sigma$ is a flat right $R_e$-module, then $M_\sigma$ is an injective $R_e$-module for each $\sigma \in G$.

**Proof**

1. In view of Theorem 2.5.5., the functor Coind is a right adjoint of the functor $(-)_e$. Since the latter functor is exact, the general theory of adjoint functors (see Appendix A.) implies that $\mathrm{Coind}(N)$ is an injective object in $R$-gr.
2. Again from Theorem 2.5.5. it follows that $(-)_\sigma$ is a right adjoint of the functor $T_{\sigma^{-1}} \circ \mathrm{Ind}$. Since $R$ is a flat $R_e$-module, Ind must be an exact functor and therefore $T_{\sigma^{-1}} \circ \mathrm{Ind}$ is then an exact functor too. Consequenty, $M_\sigma$ is an injective $R_e$-module.

□

### 2.5.7 Corollary

For a $G$-graded ring of finite support $R$, every $R$-module may be embedded into a graded $R$-module.

**Proof** Consider $M \in R$-mod and consider ${}_{R_e}M$ by restriction of scalars with respect to $R_e \to R$. Since $\mathrm{Supp}(R) < \infty$ we have $\mathrm{Coind}(M) = \mathrm{Hom}_{R_e}(R, M)$. On the other hand there is a canonical map :

$$\varphi(M) : M \to \mathrm{Hom}_{R_e}(R, M)$$

given by

$$\varphi(M)(m)(r) = rm, \text{ for } m \in M, r \in R$$

Obviously $\varphi(M)$ is $R$-linear and injective. Since $\mathrm{Hom}_{R_e}(R, M) = \mathrm{Coind}(M)$ and the latter is a graded $R$-module, the assertion follows. □

## 2.6 Some Topics in Torsion Theory on $R$-gr

In commutative algebra, in fact in Ring Theory in general, localization is a useful technique. It has become customary to present the theory of localization in its abstract categorical form mainly because it allows a very unified approach to the concept of localization. In the literature one may find "localization" applied to rings, algebras, modules, groups, topological spaces. Obvious problems dealing with properties of objects that may be preserved under localization may best be dealt with by viewing the localization as a localization in a specific category of those objects with the property under consideration and morphisms preserving that property. There are however still several different, but equivalent ways to introduce the localization theory in a categorical setting e.g. via Serre's localizing subcategories, torsion theories on Grothendieck categories and their generalizations to additive categories, torsion radicals (B. Stenstrom [181]) Gabriel topologies (P. Gabriel [67]),...

Before focusing on localization in $R$-gr we provide a short introduction to the theory of localization in a Grothendieck category, along the way we point out how several of the concepts, mentioned above, do appear in the theory and we give a hint about the interrelations between these. We shall return to general localization theory for graded rings in Chapter 8. The preliminary results we present in this section are necessary to relate the category of $R_e$-modules to a suitable full subcategory of $R$-gr (see Proposition 2.6.3) related to a certain localization. This in turn will be applied to the problem of recognizing strongly graded rings.

Let $\mathcal{A}$ be a Grothendieck category and $\mathcal{C}$ be a full subcategory of A. $\mathcal{C}$ is called a **closed subcategory** of $\mathcal{A}$ if it is closed under subobjects, quotients objects and arbitrary direct sums. Moreover, if $\mathcal{C}$ is also closed under taking extensions, then it is called a **localizing subcategory** of $\mathcal{A}$. A closed subcategory $\mathcal{C}$ of a Grothendieck category $\mathcal{A}$ is also a Grothendieck category. Indeed, if $U \in \mathcal{A}$ is a generator of $\mathcal{A}$, then the set

$$\{U|K|K \in \mathcal{A} \text{ such that } U|K \in \mathcal{C}\}$$

is a family of generators of $\mathcal{C}$, and then the direct sum of this family is a generator of $\mathcal{C}$. For any closed subcategory $\mathcal{C}$ of $\mathcal{A}$ and for any $M \in \mathcal{A}$ we can consider the greatest subobject $t_{\mathcal{C}}(M)$ of $M$ belonging to $\mathcal{C}$. In fact, $t_{\mathcal{C}}(M)$ is the sum of all subobjects of $M$ which belong to $\mathcal{C}$, and it exists because $\mathcal{C}$ is closed under taking quotient objects and arbitrary sums. The mapping $M \to t_{\mathcal{C}}(M)$ defines a left exact functor $t_{\mathcal{C}} : \mathcal{A} \to \mathcal{A}$. If $M = t_{\mathcal{C}}(M)$ i.e. $M \in \mathcal{C}$ we say that $M$ is a **$\mathcal{C}$-torsion** object. If $t_{\mathcal{C}}(M) = 0$, $M$ is called a **C-torsion free object**. Moreover, if $\mathcal{C}$ is a localizing subcategory then for any $M \in \mathcal{M}$, we have $t_{\mathcal{C}}(M/t_{\mathcal{C}}(M)) = 0$ i.e. $M/t_{\mathcal{C}}(M)$ is $\mathcal{C}$-torsion free. In this case $t_{\mathcal{C}}$ is called the **radical** associated to the localizing subcategory $\mathcal{C}$.

If $\mathcal{A}$ is a Grothendieck category and $M \in \mathcal{A}$ an object then we denote by $\sigma_{\mathcal{A}}[M]$ (or shorthly $\sigma[M]$) the full subcategory of all objects of $\mathcal{A}$ which are **subgenerated** by $M$, i.e. which are isomorphic to a subobjects of quotient objects of direct sums of copies of $M$.

### 2.6.1 Proposition

With notation as introduced above :

1. $\sigma[M]$ is a closed subcategory
2. $\sigma[M]$ is the smallest closed subcategory of $\mathcal{A}$, containing $M$

**Proof**

1. Since the direct sum functor is exact, we obtain that $\sigma[M]$ is closed under taking direct sums. Now consider :

$$0 \to Y' \to Y \to Y'' \to 0$$

an exact sequence in $\mathcal{A}$ such that $Y \in \sigma[M]$. By the definition of $\sigma[M]$ it follows immediately that $Y' \in \sigma[M]$. Since $Y \in \sigma[M]$, there exists an epimorphism $f : M^{(J)} \to X$ and a monomorphism $u : Y \to X$. We have $Y'' \simeq Y/Y'$ and $Y/Y' \subset X/Y' = X''$, so $X''$ is a quotient object of $X$, it is also a quotient object of $M^{(j)}$ and $Y'' \in \sigma[M]$. Thus $\sigma[M]$ is a closed subcategory of $\mathcal{A}$.

2. Assume that $\mathcal{C}$ is a closed subcategory of $\mathcal{A}$ and $M \in \mathcal{C}$. Then for $Y$, $f$ and $u$ as above, we have that $M^{(j)} \in \mathcal{C}$, so $X \in \mathcal{C}$, showing that $Y \in \mathcal{C}$. Therefore $\sigma[M] \subseteq \mathcal{C}$. □

Now we consider the case : $\mathcal{A} = R\text{-gr}$, where $R = \oplus_{\lambda \in G} R_\lambda$ is a $G$-graded ring. Fixing $\sigma \in G$. We define the subclass $\mathcal{C}_\sigma$ to be

$$\mathcal{C}_\sigma = \{M \in R\text{−gr} | M = \oplus_{\lambda \in G} M_\lambda, \text{ with } M_\sigma = 0\}$$

A graded left module $M = \oplus_{\lambda\in G} M_\lambda$ is said to be **$\sigma$-faithful** if $R_{\sigma\tau^{-1}}x_\tau \neq 0$ for any nonzero $x_\tau \in M_\tau$. We say that $M$ is **faithful** if it is $\sigma$-faithful for all $\sigma \in G$. The graded ring $R$ is left **$\sigma$-faithful**, respectively **faithful** if ${}_RR$ is. Similarly, the notion **right $\sigma$-faithful** respectively **faithful** may be introduced for the ring $R$. When $R$ is a strongly graded ring, Proposition 1.1.1. yields that $R$ is left and right faithful. Now, if $M \in R$-gr, the functorial isomorphisms $\alpha$ and $\beta$ (see Section 2.5.) define the canonical graded functorial morphism.

1. $\mu(M) : \mathrm{Ind}(M_\sigma(\sigma^{-1}) \to M,\ \mu(M)(\lambda \otimes x) = \lambda x, \lambda \in R,\ x \in M_\sigma$. Analogously, the functorial isomorphisms $\gamma$ and $\delta$ define the canonical graded functorial morphism

2. $v(M) : M \to \mathrm{Coind}(M_\sigma(\sigma^{-1})), v(M)(x_\lambda)(a) = a_{\sigma\lambda^{-1}}x_\lambda$, where $x_\lambda \in M_\lambda, a = \sum_{\tau\in G} a_\tau, a_\gamma \in R_\tau$.

## 2.6.2 Proposition

With notation as above : $\mathrm{Im}v(M)$ is an essential submodule of $\mathrm{Coind}(M_\sigma)(\sigma^{-1})$.

**Proof** Since $M$ may be changed to the $\sigma$-suspension $M(\sigma)$ it is sufficient to deal with the case where $\sigma = e$. Let $f \in \mathrm{Coind}(M_e)_\lambda, f \neq 0$, for some $\lambda \in G$. So we have $f \in \mathrm{Hom}_{R_e}(R, M_e)$ such that $f(R_\tau) = 0$ if $\tau = \lambda^{-1}$. Since $f \neq 0$, there exist $a_{\lambda^{-1}} \in R_{\lambda^{-1}}$ such that $f(a_{\lambda^{-1}}) \neq 0$. We put $x_e = f(a_{\lambda^{-1}}) \in M_e$. We have $v(M)(x_e)(b) = b_e x_e = b_e f(a_{\lambda^{-1}}) = f(b_e a_{\lambda^{-1}})$ where $b = \sum_{\tau\in G} b_\tau$ is an arbitrary element from $R$. On the other hand $(a_{\lambda^{-1}}f)(b) = f(ba_{\lambda^{-1}}) = \sum_{\tau\in G} f(b_\tau a_{\lambda^{-1}}) = f(b_e a_{\lambda^{-1}})$ so we have $a_{\lambda^{-1}}f = v(M)(x_e) \in \mathrm{Im}v(M)$. Since $v(M)(x_e)(1) = x_e \neq 0$, we have $a_{\lambda^{-1}}f \in \mathrm{Im}v(M)$ and $a_{\lambda^{-1}}f \neq 0$. Now by Proposition 2.3.6. it follows that $\mathrm{Im}v(M)$ is an essential submodule in $\mathrm{Coind}(M_e)$. □

## 2.6.3 Proposition

With notation as before :

a. For every $\sigma \in G$, $\mathcal{C}_\sigma$ is a localizing subcategory of $R$-gr which is closed under arbitrary direct products.

b. If $M = \oplus_{\sigma\in G} M_\sigma$, then $M \in \mathcal{C}_\sigma$ if and only if for every $x_\tau \in M_\tau$, $R_{\sigma\tau^{-1}}x_\tau = 0$.

c. If $M = \oplus_{\sigma\in G} M_\sigma$ is a nonzero graded module, then $M$ is $\mathcal{C}_\sigma$-torsion free if and only if $M$ is $\sigma$-faithful if and only if every non-zero graded submodule of $M$ intersects $M_\sigma$ nontrivially.

d. $M$ is faithful if and only if $M$ is $\mathcal{C}_\sigma$-torsionfree for any $\sigma \in G$.

e. If $\mu(M)$ and $v(M)$ are the morphisms of 1. and 2. then $\mathrm{Ker}(\mu(M))$, $\mathrm{Ker}(v(M))$, $\mathrm{coker}(\mu(M))$ and $\mathrm{coker}(v(M))$ are in $\mathcal{C}_\sigma$.

Moreover $\mathrm{Ker}(v(M)) = t_{\mathcal{C}_\sigma}(M)$ and $\mathrm{Im}(\mu(M))$ is the smallest graded submodule $L$ of $M$, such that $M/L \in \mathcal{C}_\sigma$.

f. If $M$ is $\sigma$-faithful then $\mathrm{Ker}(\mu(M)) = t_{\mathcal{C}_\sigma}(\mathrm{Ind}(M_\sigma)(\sigma^{-1})$.

g. $\mathcal{C}_\sigma = 0$ if and only if $R$ is a strongly graded ring.

**Proof**

a. That $\mathcal{C}_\sigma$ is a localizing subcategory of $R$-gr follows from the fact that $(-)_\sigma$ is an exact functor. Now let $(M_i)_{i\in I}$ be a family of objects from $\mathcal{C}_\sigma$. If $M$ is a direct product in $R$-gr of the family $(M_i)_{i\in I}$, then $M = \oplus_{\lambda\in G} M_\lambda$ where $M_\lambda = \prod_{i\in I}(M_i)_\lambda$. Therefore, $M_i \in \mathcal{C}_\sigma$ yields $(M_i)_\sigma = 0$ and therefore $M_\sigma = 0$, so $M \in \mathcal{C}_\sigma$.

b. This is routine.

c. Assume that $M$ is $\mathcal{C}_\sigma$-torsionfree and let $x_\tau \in M_\tau, x_\tau \neq 0$. Then $Rx_\tau$ is also $\mathcal{C}_\sigma$-torsion free and therefore $(Rx_\tau)_\sigma \neq 0$. But $(Rx_\tau)_\sigma = R_{\sigma\tau^{-1}}x_\tau$. Hence $R_{\sigma\tau^{-1}}x_\tau \neq 0$; i.e. $M$ is $\sigma$-faithful. Conversely, assume that $M$ is $\sigma$-faithful. If $t_{\mathcal{C}\sigma}(M) \neq 0$, then there exists $x_\tau \in (t_{\mathcal{C}_\sigma}(M))_\tau$, $x_\tau \neq 0$, for some $\tau \in G$. Thus $R_{\sigma\tau^{-1}}x_\tau \neq 0$. On the other hand $(t_{\mathcal{C}_\sigma}(M))_\sigma = 0$, and from $R_{\sigma\tau^{-1}}x_\tau \subseteq (t_{\mathcal{C}_\sigma}(M))_\sigma$ we obtain that $R_{\sigma\tau^{-1}}x_\tau = 0$, a contradiction. Hence $t_{\mathcal{C}_\sigma}(M) = 0$, i.e. $M$ is $\mathcal{C}_\sigma$-torsion free. That $M$ is $\sigma$-faithful if and only if every nonzero graded submodule of $M$ is intersecting $M_\sigma$ non-trivially, is obvious.

d. This follows directly from c.

e. Since $\mathrm{Ind}(M_\sigma)(\sigma^{-1})_\sigma = \mathrm{Ind}(M_\sigma)_e = R_e \otimes_{R_e} M_\sigma \simeq M_\sigma$, $\mathrm{Coind}(M_\sigma)(\sigma^{-1})_\sigma = \mathrm{Coind}(M_\sigma)_e = \mathrm{Hom}_{R_e}(R_e, M_\sigma) \simeq M_\sigma$, and because the functor $(-)_\sigma$ is exact it follows that $\mathrm{Ker}(\mu(M))$, $\mathrm{ker}(v(M))$, $\mathrm{coker}(\mu(M))$, $\mathrm{coker}(v(M))$ belong to $\mathcal{C}_\sigma$. In particular it follows that $\mathrm{Ker}v(M) \subseteq t_{\mathcal{C}_\sigma}(M)$. Conversely, let $x_\lambda \in t_{\mathcal{C}_\sigma}(M)$. Hence $(Rx_\lambda)_\sigma = 0$ so $R_{\sigma\lambda^{-1}}x_\lambda = 0$. Therefore $\nu(M)(x_\lambda)(a) = a_{\sigma\lambda^{-1}}x_\lambda = 0$ for any $a \in R$. Consequently, $\nu(M)(x_\lambda) = 0$ so $x_\lambda \in \mathrm{Ker}v(M)$ and hence $\mathrm{Ker}v(M) = t_{\mathcal{C}_\sigma}(M)$. On the other hand, note that $\mathrm{Im}(\mu(M)) = RM_\sigma$, and since $(M/RM_\sigma)_\sigma = 0$ we have $M/RM_\sigma \in \mathcal{C}_\sigma$. Now if $L$ is a graded submodule of $M$ such that $M/L \in \mathcal{C}_\sigma$ then $(M/L)_\sigma = 0$ and hence $L_\sigma = M_\sigma$ so $RM_\sigma = RL_\sigma \subseteq L$.

f. Easy enough.

g. Assume tat $\mathcal{C}_\sigma = 0$ and let $M \oplus_{\sigma\in G} M_\sigma$ be a nonzero graded module. So $M_\sigma \neq 0$ and since $(M/RM_\sigma)_\sigma = 0$ it follows that $M = RM_\sigma$ and so

we have $R_\lambda M_\sigma = M_{\lambda\sigma}$ for any $\lambda \in G$. Now, if we replace $M$ by the $\tau$-suspension $M(\tau)$ we obtain $M(\tau) \neq 0$ so we have $R_\lambda(M(\tau))_\sigma = M(\tau)_{\lambda\sigma}$ and thus $R_\lambda M_{\sigma\tau} = M_{\lambda\sigma\tau}$ for any $\lambda, \tau \in G$. Putting $\tau = \sigma^{-1}\theta$ yields $R_\lambda M_\theta = M_{\lambda\theta}$ for $\lambda, \theta \in G$. In particular for $M =_R R$ we obtain that $R_\lambda R_\theta = R_{\lambda\theta}$ for any $\lambda, \theta \in G$ so $R$ is a strongly graded ring. The converse is obvious. □

### 2.6.4 Remark

A strongly graded ring is left and right faithful (see the first part of this section). The following example proves that there are left and right faithful rings which are not strongly graded. Let $K$ be a field and let $R = K[X]$ be the polynomial ring in the indeterminate $X$. $R$ becomes a $\mathbb{Z}_n$-graded ring by putting $R_{\widehat{o}} = K[X^n]$ and $R_{\widehat{k}} = X^k K[X^n]$, for every $k = 1, \ldots, n-1$. $R$ is not strongly graded, as $R_{\widehat{1}} R_{\widehat{n-1}} = X^n K[X^n] \neq R_{\widehat{o}}$. On the other hand $R$ is a faithful ring, being a commutative domain.

Let $M = \oplus_{\sigma \in G} M_\sigma$ be an object from the category $R$-gr. The $t_M : M \to M_e$, $t(m) = m_e$, where $m = \sum_{\sigma \in G} m_\sigma$ is an element from $M$ is called the **trace map**. It is clear that $t_M$ is $R_e$-linear. Also if $a \in R$, $a = \sum_{\sigma \in G} a_\sigma$ then we have $t_M(am) = \sum_{\sigma \in G} a_{\sigma^{-1}} m_\sigma$.

We denote by $\mathrm{rad}(t_M) = \{m \in M | t_M(am) = 0 \text{ for any } a \in R\}$. It is clear that $\mathrm{rad}(t_M)$ is a submodule of $M$. Also if $m = \sum_{\sigma \in G} m_\sigma \in \mathrm{rad}(t_M)$, then we have for $\lambda \in G$ that $R_{\lambda^{-1}} m_\lambda = 0$. Hence $t_M(Rm_\lambda) = R_{\lambda^{-1}} m_\lambda = 0$ and therefore $m_\lambda \in \mathrm{rad}(t_M)$. Hence $\mathrm{rad}(t_M)$ is a graded submodule of $M$, called the **radical of the trace map** $t_M$.

In fact we have that $\mathrm{rad}(t_M) = t_{\mathcal{C}_e}(M)$. In particular it follows that $M$ is $e$-faithful if and only if $\mathrm{rad}(t_M) = 0$. If $M = {}_R R$ we have the trace map $t_R : R \to R_e$ (denoted by $t$). Clearly $t : R \to R_e$ is left and right $R_e$-linear. In this case we may define the left radical, $l.\mathrm{rad}(t)$, and right radical $r.\mathrm{rad}(t)$ of $t$.

So we have $l.\mathrm{rad}(t) = \{a \in R | t(Ra) = 0\}$ and $r.\mathrm{rad}(t) = |\{a \in R | t(aR) = 0\}$. We remark that $t(ab) = \sum_{\sigma \in G} a_{\sigma^{-1}}.b_\sigma$ for any $a, b \in R$.

An object $M \in R$-gr is called $\mathcal{C}_e$-closed if $M$ is $\mathcal{C}_e$-torsionfree and has the following property : for any diagram in $R$-gr

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \xrightarrow{u} & X & \longrightarrow & \mathrm{coker}\ u & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & & & & & \\ & & M & & & & & & \end{array}$$

where coker $u \in \mathcal{C}_e$, there exists a unique morphism $g : X \to M$ such that $g \circ u = f$.

### 2.6.5 Proposition

The following assertions hold :

1. If $N \in R_e$-mod then $\mathrm{Coind}(N)$ is $\mathcal{C}_e$-closed.
2. If $M \in R$-gr is $\mathcal{C}_e$-closed then $M \simeq \mathrm{Coind}(M_e)$.
3. If we denote by $\mathcal{A}$ the full subcategory of all $\mathcal{C}_e$-closed objects of $R$-gr, then the functor $\mathrm{Coind} : R_e - \mathrm{mod} \to \mathcal{A}$ is an equivalence of categories.

**Proof**

1. If $K = t_{\mathcal{C}_e}(\mathrm{Coind}(N))$, then by Theorem 2.5. we have

$$\mathrm{Hom}_{R-\mathrm{gr}}(K, \mathrm{Coind}(N)) \simeq \mathrm{Hom}_{R_e}(K_e, N) = 0$$

since $K_e = 0$. Since $K \subseteq \mathrm{Coind}(N)$ it follows that $K = 0$ so $\mathrm{Coind}(N)$ is $\mathcal{C}_e$-torsionfree. We consider the diagram in $R$-gr :

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X' & \xrightarrow{u} & X & \longrightarrow & \mathrm{coker}\ u & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & & & & & \\ & & \mathrm{Coind}(N) & & & & & & \end{array}$$

where coker $u \in \mathcal{C}_e$. By Theorem 2.5.

$$\mathrm{Hom}_{R-\mathrm{gr}}(X', \mathrm{Coind}(N)) \simeq \mathrm{Hom}_{R_e}(X'_e, N)$$

Since $(\mathrm{coker}\ u)_e = 0$ then $u_e : X'_e \to X_e$ is an isomorphism. Using the same Theorem 2.5. it followsthat there exists $g : X \to \mathrm{Coind}(N)$ such that $g \circ u = f$.

2. We consider the canonical morphism

$$v(M) : M \to \mathrm{Coind}(M_e), v(M)(x_\lambda)(a) = a_{\lambda^{-1}} x_\lambda$$

where $x_\lambda \in M_\lambda, a = \sum_{\sigma \in G} a_\sigma, a_\sigma \in R_\sigma$. Since $\ker v(M) \in \mathcal{C}_e$, and $M$ is $\mathcal{C}_e$-closed, then $\ker v(M) = 0$. Now from the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{v(M)} & \mathrm{Coind}(M_e) & \longrightarrow & \mathrm{coker}(M) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1_M} & & & & & & \\ & & M & & & & & & \end{array}$$

where $\mathrm{coker} v(M) \in \mathcal{C}_e$ implies that there exists a morphism

$$g : \mathrm{Coind}(M_e) \to M$$

such that $g \circ v(M) = 1_M$. Therefore $\mathrm{Im} v(M)$ is direct summand of $\mathrm{Coind}(M_e)$. By Proposition 2.6.2., $\mathrm{Im} v(M)$ is essential in $\mathrm{Coind}(M_e)$. hence $\mathrm{Im} v(M) = \mathrm{Coind}(M_e)$ and therefore $v(M)$ is an isomorphism.

3. If we consider the functor

$$(-)_e : \mathcal{A} \to R_e\text{−mod}, M \to M_e$$

then it is clear that $(-)_e \circ \text{Coind} \simeq 1_{R_e\text{mod}}$ and by assertion 2. it follows also that $\text{Coind} \circ (-)_e \simeq 1_{\mathcal{A}}$. □

### 2.6.6 Remark

Using the notion of quotient category (see P. Gabriel [67]), assertion 3. in the foregoing proposition states that $R_e$-mod is equivalent to the quotient category $R\text{−gr}/\mathcal{C}_e$.

Let $N \in R_e$-mod, if $M = \text{Ind}(N) = R \otimes_{R_e} N$, then the morphism $\nu(M)$ defines the canonical morphism $\eta(N) : \text{Ind}(N) \to \text{Coind}(N), \eta(N)(a \otimes x)(b) = \sum_{g \in G}(b_{g^{-1}}a_g).x$ where $a, b \in R$ and $x \in N$. So $\eta(N)(a \otimes x)(b) = t(ba).x$. It is easy to see that the class of morphisms $\{\eta(N), N \in R_e\text{−mod}\}$ defines a functorial morphism $\eta : \text{Ind} \to \text{Coind}$. Also for any $N \in R_e$-mod, we have $\ker\eta(N) = t_{\mathcal{C}_e}(\text{Ind}(N))$ and $\text{coker}\eta(N) \in \mathcal{C}_e$. Moreover, $\text{Im}\eta(N)$ is an essential subobject of $\text{Coind}(N)$.

### 2.6.7 Proposition

If $R$ is a strongly graded then $\eta : \text{Ind} \to \text{Coind}$ is an functorial isomorphism.

**Proof** From $\mathcal{C}_e = 0$ it follows that $\ker\eta(N) = \text{coker}\eta(N) = 0$ for any $N \in R_e$-mod. Thus $\eta(N)$ is an isomorphism.

### 2.6.8 Remark

Let $G$ be a non-trivial group i.e. $G \neq \{e\}$ and let $R$ be an arbitrary ring. Then $R$ can be considered as a $G$-graded ring with the trivial grading. Obviously, in this case we have $\text{Ind} \simeq \text{Coind}$ but $R$ is not strongly graded.

Thus, in this case, we may ask the following question "If $R$ is a graded ring and the functors Ind and Coind are isomorphic, how close is $R$ to being a strongly graded ring ?"

### 2.6.9 Theorem

Let $R$ be a $G$-graded ring. The following assertions are equivalent :

a. The functors Ind and Coind are isomorphic.

b. The canonical morphism $\eta : \text{Ind} \to \text{Coind}$ is a functorial isomorphism.

c. The map $\eta(R) : R \to \text{Coind}(R_e), \eta(R)(a)(b) = t(ba)$ is an isomorphism and for every $g \in G$, $R_g$ is projective and finitely generated in $R_e$-mod.

d. There exists an isomorphism $\vartheta : R \to \mathrm{Coind}(R_e)$ in $R$-gr that is also a morphism in mod-$R_e$ and for every $g \in G$, $R_g$ is finitely generated and projective in $R_e$-mod

**Proof**

a. $\Rightarrow$ b. Assume that there exists a functorial isomorphism $\varphi : \mathrm{Ind} \to \mathrm{Coind}$. Hence for any $N \in R_e$-mod, $\varphi(N) : \mathrm{Ind}(N) \to \mathrm{Coind}(N)$ is an isomorphism. Since $\mathrm{Coind}(N)$ is $\mathcal{C}_e$-closed, it follows that $\mathrm{Ind}(N)$ is $\mathcal{C}_e$-closed. In particular $\mathrm{Ind}(N)$ is $\mathcal{C}_e$-torsionfree so $\mathrm{Ker}\eta(N) = 0$. From the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ind}(N) & \xrightarrow{\eta(N)} & \mathrm{Coind}(N) & \longrightarrow & \mathrm{coker}\ \eta(N) \\ & & \downarrow{\scriptstyle 1_{\mathrm{Ind}(N)}} & & & & \\ & & \mathrm{Ind}(N) & & & & \end{array}$$

we conclude that there exists a morphism $v : \mathrm{Coind}(N) \to \mathrm{Ind}(N)$ such that $v \circ \eta(N) = 1_{\mathrm{Ind}(N)}$. So, $\mathrm{Im}\eta(N)$ is a direct summand of $\mathrm{Coind}(N)$. Since $\mathrm{Im}\eta(N)$ is essential in $\mathrm{Coind}(N)$, we have that $\mathrm{Im}\eta(N) = \mathrm{Coind}(N)$, so $\eta(N)$ is an isomorphism.

b. $\Rightarrow$ c. It is clear that $\eta(R)$ is an isomorphism. Now Theorem 2.5. and the fact that Ind $\simeq$ Coind imply that the functors Ind and Coind are exact. By the properties of adjoint functors (see Appendix A.) it follows that the functor $(-)_e : R\text{−gr} \to R_e$-mod has the following property : if $P \in R$-gr is a finitely generated projective object then $P_e \in R_e$-mod is a finitely generated projective object in $R_e$-mod. In particular if $P = {}_RR(\sigma)$ then $P_e = R_\sigma$ is a projective and finitely generated $R_e$-module.

c. $\Rightarrow$ d. obvious.

d. $\Rightarrow$ a. If $N \in R_e$-mod, then the fact that $\mathrm{Hom}_{R_e}(R_\sigma, R_e) \otimes_{R_e} N$ is canonical isomorphic to $\mathrm{Hom}_{R_e}(R\sigma, N)$ entails that :

$$\vartheta \otimes 1_N : R \otimes_{R_e} N \to \mathrm{Coind}(R_e) \otimes_{R_e} N \simeq \mathrm{Coind}(N) \text{ for } N \in R_e\text{−mod}$$

defines a functorial isomorphism. □

## 2.6.10 Theorem

Let $R$ be a $G$-graded ring. Assume that Ind $\simeq$ Coind and let $g \in \sup(R)$. Then there exist elements $a_i \in R_g, b_i \in R_{g^{-1}}$, $1 \leq i \leq n$, such that for every $a \in R_g$, $b \in R_{g^{-1}}$ we have

$$a = (\sum_{i=1}^{n} a_i b_i)a, b = b(\sum_{i=1}^{n} a_i b_i)$$

**Proof** From Theorem 2.6.9 we retain that the map

$$\eta(R)_g : R_g \to \mathrm{Hom}_{R_e}(R_{g^{-1}}, R_e)\ \eta(R)_g(r_g)(s_{g^{-1}}) = s_{g^{-1}}r_g$$

is an $R_e$-isomorphism. Since $R_{g^{-1}}$ is finitely generated and projective in $R_e$-mod, the dual basis lemma entails the existence of $b_1, \ldots, b_n \in R_{g^{-1}}$ and $f_1, \ldots, f_n \in \mathrm{Hom}_{R_e}(R_{g^{-1}}, R_e)$, such that for each $b \in R_{g^{-1}}$ we have

$$b = \sum_{i=1}^{n} f_i(b)b_i$$

For every $i = 1, \ldots, n$, there is an $a_i \in R_g$ such that $f_i = \eta(R)_g(a_i)$. Hence $b = \sum_{i=1}^n \eta({}_RR)_g(a_i)(b)b_i = \sum_{i=1}^n ba_ib_i = b(\sum_{i=1}^n a_ib_i)$. Let $c = \sum_{i=1}^n a_ib_i$, then $b = bc$ for every $b \in R_{g^{-1}}$ and thus $R_{g^{-1}}(1-c) = 0$. It follows that $R_{g^{-1}}(1-c)R_g = 0$ so that $\eta(R)_g((1-c)R_g) = 0$. Since $\eta(R)_g$ is injective, we get $(1-c)R_g = 0$ and hence $a = (\sum_{i=1}^n a_ib_i)a$ for every $a \in R_g$. □

## 2.6.11 Corollary

Let $R = \oplus_{g\in G}R_g$ be a $G$-graded ring. Assume that Ind $\simeq$ Coind. If every $R_g$, $g \in \sup(R)$ is faithful as left (or right) $R_e$-module, then $H = \sup(R)$ is a subgroup of $G$ and $R = \oplus_{h\in H}R_h$ is an $H$-strongly graded ring.

**Proof** If $g \in \sup(R)$ then $g^{-1} \in \sup(R)$ and it follows from the theorem above that $R_gR_{g^{-1}} = R_e$. If now $g, h \in \sup(R)$ and $gh \notin \sup(R)$. Then $0 = R_{gh}R_{h^{-1}} \supseteq R_gR_hR_{h^{-1}} = R_g$, contradiction. Hence $H = \sup(R)$ is a subgroup of $G$ and $R = \oplus_{h\in H}R_h$ is an $H$-strongly graded ring.

If $A$ is a ring, we denote by $\Omega_A$ the set of all isomorphism classes of simple objects in $A$-mod i.e. $\Omega_A = \{[S]|S$ is a simple left $A$-module $\}$, and $[S] = \{S' \in A-\mathrm{mod}|S' \simeq S\}$.

We recall that the ring $A$ is called **local** if $A/J(A)$ is a simple artinian ring ($J(A)$ is the Jacobson radical). □

We conclude this section with two useful corollaries.

## 2.6.12 Corollary

For a graded ring $R = \oplus_{\sigma\in G}R_\sigma$ we assume that Ind $\simeq$ Coind. If $|\Omega_{R_e}| = 1$ (in particular if $R_e$ is a local ring), then $H = \sup(R)$ is a subgroup of $G$ and $R = \oplus_{h\in H}R_h$ is an $H$-strongly graded ring.

**Proof** Since $|\Omega_{R_e}| = 1$ every finitely generated and projective module in $R_e$-mod is a generator and hence is faithful. Apply now Corollary 2.6.11. □

### 2.6.13 Corollary

Assume that for a graded ring $R$, we have Ind $\simeq$ Coind. If $R_e$ has only two idempotents 0 and 1 (in particular when $R_e$ ia a domain), then $H = \sup(R)$ is a subgroup of $G$ and $R = \oplus_{n\in H} R_h$ is an $H$-strongly graded ring.

**Proof** By Theorem 2.6.10, if $g \in \sup(R)$, there exist elements $a_i \in R_g$, $b_i \in R_{g^{-1}}$ $(1 \leq i \leq n)$ such that for every $a \in R_g$ we have $a = (\sum_{i=1}^{n} a_i b_i)a$. We put $c = \sum_{i=1}^{n} a_i b_i$. In particular for every $1 \leq r \leq n$, $a_r = ca_r$ so $a_r b_r = ca_r b_r$ and therefore $c^2 = c$. Since $R_g \neq 0$, $c \neq 0$ and hence the hypothesis forces $c = 1$, so $R_g R_{g^{-1}} = R_e$ for any $g \in \sup(R)$. □

## 2.7 The Structure of Simple Objects in $R$-gr

We consider a $G$-graded ring $R = \oplus_{\sigma\in G} R_\sigma$. A nonzero object $\Sigma \in R$-gr is said to be a *gr*-**simple** object if 0 and $\Sigma$ are the only gr-submodules of $\Sigma$. An object $M \in R$-gr is called **gr-semisimple** if $M$ is a direct sum of gr-simple modules.

A gr-submodule $N$ of $M$ is said to be a **gr-maximal** submodule whenever $M/N$ is gr-simple. Clearly $N$ is gr-maximal in $M$ if and only if $N \neq M$ and $N + Rx = M$ for any $x \in h(M), x \notin N$. Observe that a gr-maximal submodule of $M$ need not be a maximal submodule; indeed the zero ideal in $K[X, X^{-1}] = R$, endowed with the natural $\mathbb{Z}$-gradation $R_n = KX^n$ for $n \in \mathbb{Z}$, is gr-maximal (because every homogeneous element is invertible) but not maximal (because not every element is invertible). A more complete answer, elucidating the structure of gr-simple rings is given in Theorem 2.10.10 (deriving from general results in Chapter 4.).

A $G$-graded ring $\Delta$ is a **gr-division ring**, or a **gr-skewfield**, if every nonzero homogeneous element of $\Delta$ is invertible. For example, the ring of Laurent series $K[X, X^{-1}]$, where $K$ is a field, is a gr-skewfield in fact a gr-field. If we write $H = \sup(\Delta) = \{\sigma \in G, \Delta_\sigma \neq 0\}$, then it is easily verified that $H$ is a subgroup of $G$ and $\Delta = \oplus_{\tau\in H} \Delta_\tau$. It follows that as an $H$-graded ring, $\Delta$ is a crossed product. gr-skewfields appear naturally in the study of gr-simple modules because of a graded version of Schur's lemma, contained in 3. of the following Proposition.

### 2.7.1 Proposition

Consider a gr-simple module $\Sigma$ in $R$-gr.

1. For $\sigma \in G$ either $\Sigma_\sigma = 0$ or $\Sigma_\sigma$ is a simple $R_e$-module.
2. If $\Sigma_\sigma \neq 0$. Then $\Sigma \cong (R/I)(\sigma^{-1})$ for some gr-maximal $I$ of $R$.

3. Put $\Delta = \mathrm{END}_R(\Sigma)$, then $\Delta$ is a gr-skewfield such that $\Delta = \oplus_{\sigma \in G(\Sigma)} \Delta_\sigma$ where $G(\Sigma)$ is the stabilizer subgroup of $G$ for $\Sigma$.

4. When $\sigma \in \sup(\Sigma)$, $\Sigma$ is $\sigma$-faithful.

**Proof**

1. In case $\Sigma_\sigma \neq 0$ we can take an $x_\sigma \in \Sigma_\sigma$, $x_\sigma \neq 0$. Then $Rx_\sigma$ is a nonzero graded submodule of $\Sigma$ and therefore $\Sigma = Rx_\sigma$ and $\Sigma_\sigma = R_e x_\sigma$. This establishes that $\Sigma_\sigma$ is a simple $R_e$-module.

2. Again choose $x_\sigma \neq 0$ in $\Sigma_\sigma$. The canonical morphism $f : R \to \Sigma$, $r \mapsto rx_\sigma$, is a nonzero graded map of degree $\sigma$. Thus $f : R \to \Sigma(\sigma)$, obtained by taking the $\sigma$-suspension, is a morphism in $R$-gr. Of course, $\Sigma(\sigma)$ is gr-simple too and hence $f$ must be surjective. It follows that $\Sigma(\sigma) \cong R/I$ where $I = \mathrm{Ker} f \in R$-gr; obviously $I$ is then a gr-maximal left ideal of $R$ such that $\Sigma \cong (R/I)(\sigma^{-1})$.

3. Take $f \neq 0$ in $\Delta_\sigma$, i.e. $f$ is a nonzero morphism in $R$-gr when viewed as a map $\Sigma \to \Sigma(\sigma)$. The latter is a gr-simple module since it is a $\sigma$-suspension of $\Sigma$. Therefore $f$ is necessarily an isomorphism and as such it is invertible in $\Delta$. Finally observe that $\Delta_\sigma \neq 0$ if and only if $\sigma \in G\{\Sigma\}$.

4. The gr-simplicity of $\Sigma$ entails that $t_{\mathcal{C}_\sigma}(\Sigma) = \Sigma$ whenever $t_{\mathcal{C}_\sigma}(\Sigma) \neq 0$, so one would obtain that $\Sigma_\sigma = (t_{\mathcal{C}_\sigma}(\Sigma))_\sigma = 0$ but that contradicts $\sigma \in \sup(\Sigma)$.

   Consequently, we must have $t_{\mathcal{C}_\sigma}(\Sigma) = 0$ and therefore $\Sigma$ is $\sigma$-faithful.

□

We now aim to describe explicitely the structure of gr-simple modules; the torsion-reduced induction fiunctor will play the important part here.

Recall that we have the induction functor :

$$\mathrm{Ind} = R \otimes_{R_e} - : R_e\text{−mod} \to R\text{−gr}$$

where $\mathrm{Ind}(N)$ is graded by putting $\mathrm{Ind}(N)_\sigma = R_\sigma \otimes_{R_e} N$. Let us denote by $t_e$ the radical $t_{\mathcal{C}_e}$ associated to the localizing subcategory $\mathcal{C}_e$. It is not hard to verify that the corresponce $N \to R\overline{\otimes}_{R_e} N$, where $R\overline{\otimes}_{R_e} N = R \otimes_{R_e} M/t_e(R \otimes_{R_e} N)$, defines a functor $R\overline{\otimes}_{R_e} - : R_e$-mod $\to R$-gr.

Note, in the particular case where $R$ is a strongly graded ring, that then $R\overline{\otimes}_{R_e} - = R \otimes_{R_e} -$.

### 2.7.2 Theorem. (Structure theorem for gr-simple modules)

1. If $N \in R_e$-mod is a simple $R_e$-module then $R\overline{\otimes}_{R_e}N$ is gr-simple in $R$-gr.
2. If $\Sigma = \otimes_{\sigma\in G}\Sigma_\sigma$ is gr-simple in $R$-gr such that $\Sigma_\sigma \neq 0$ for at least one $\sigma \in G$, then we have : $\Sigma \cong (R\overline{\otimes}_{R_e}\Sigma_\sigma)(\sigma^{-1})$.

**Proof**

1. Let us write $M = R\overline{\otimes}_{R_e}N$; by definition of the gradation on $M$ we have that $M_e = R_e \otimes_{R_e} N \cong N$ (taking into account that $(t_{\mathcal{C}_e}(R \otimes_{R_e} N))_e = 0$).

   In particular $M \neq 0$ and also we have that $M$ is $e$-faithful (cf. Proposition 2.6.2.). Consequently if $X \neq 0$ is a gr-submodule of $M$ then $X \cap M_e \neq 0$ and it follows that $X_e \neq 0$. Now $M_e \cong N$ is a simple $R_e$-module and thus $X_e = M_e$, or $RX_e = RM_e = M$. However, we must have $RX_e \subset X$ and thus $X = M$ follows. This establishes that $M$ is a gr-simple $R$-module, as desired.

2. Start with a gr-simple $R$-module $\Sigma$ such that $\Sigma_\sigma \neq 0$ for a certain $\sigma \in G$. Up to passing to the gr-simple $\Sigma(\sigma)$ we may assume that $\sigma = e$, i.e. $\Sigma_e \neq 0$. Then, in view of Proposition 2.6.2., $R\overline{\otimes}_{R_e}\Sigma_e$ is a nonzero gr-essential submodule of $\Sigma$. As the latter is gr-simple we must have $\Sigma = R\overline{\otimes}_{R_e}\Sigma_e$. □

For a graded $R$-module $M$ we let $\mathrm{soc}^{\mathrm{gr}}(M)$ be the sum of the gr-simple gr-submodule of $M$; we call $\mathrm{soc}^{\mathrm{gr}}(M)$ the **gr-socle (graded socle)** of $M$. By $\mathrm{soc}(M)$ we refer to the socle of the $R$-module $M$ (i.e. the sum of the simple submodules of $M$).

### 2.7.3 Proposition

With notation as introduced before :

1. $\mathrm{soc}^{\mathrm{gr}}(M)$ equals the intersection of all gr-essential gr-submodules of $M$.
2. We have the inclusion $\mathrm{soc}(M) \subset \mathrm{soc}^{\mathrm{gr}}(M)$.

**Proof**

1. Let $L$ be the intersection of all gr-essential gr-submodules of $M$. It is obvious that $\mathrm{soc}^{\mathrm{gr}}(M) \subset L$. First we establish now that $L$ is gr-semisimple i.e. every gr-submodule $K$ of $L$ is a direct summand in $R$-gr. Consider a gr-submodule $X$ in $M$, maximal with respect to $X \cap K = 0$. Clearly, $K \oplus X$ is then gr-essential in $M$ and so Proposition 2.3.6.

entails that $K \oplus X$ is essential as an $R$-submodule of $M$. It follows that $L \subset K \oplus X$ and from $K \subset L$ it then also follows that $L = K \oplus (L \cap X)$. Hence, $L$ is gr-semisimple as claimed but then $L = \mathrm{soc}^{\mathrm{gr}}(M)$ is clear.

2. It is well-known that $\mathrm{soc}(M)$ is the intersection of all essential submodules of $M$. Since gr-essential implies essential in the ungraded sense, the inclusion $\mathrm{soc}(M) \subset \mathrm{soc}^{\mathrm{gr}}(M)$ follows directly from statement i. □

**Remark** If we consider the Laurent polynomial ring $R = R[T, T^{-1}]$ with $\mathbb{Z}$-gradation $R_n = RT^n, x \in \mathbb{Z}$ where $R$ is a field, it is easy to see that ${}_RR$ is a gr-simple object so $\mathrm{Soc}^{\mathrm{gr}}(R) = R$. Since $R$ is domain then $\mathrm{soc}(R) = 0$.

## 2.7.4 Corollary

In case $R$ is $G$-graded, with finite support, we obtain :

1. For every simple $R$-module, $S$ say, there exists a gr-simple $\Sigma$ in $R$-gr such that $S$ is isomorphic to an $R$-submodule in $\Sigma$.

2. Any $S$ as before is a semisimple $R_e$-module of finite length.

**Proof**

1. We have seen, cf. Corollary 2.5.5., that there exists a graded $R$-module $M$ such that $S$ is isomorphic to a simple $R$-submodule of $M$. Then part ii. of Proposition 2.7.3. finishes the proof.

2. Let $\Sigma$ be the gr-simple in $R$-gr the existence of which is stated in i. Proposition 2.7.1. yields that $\Sigma$ is a semisimple $R_e$-module and therefore $S$ is a semisimple $R_e$-module too. On the other hand, $\Sigma$ is an epimorphic image of ${}_RR(\sigma)$ for some $\sigma \in G$. If $\sup(R) = \{\sigma_1, \ldots, \sigma_n\}$ then $\sup({}_RR(\sigma)) = \{\sigma_1\sigma^{-1}, \ldots, \sigma_n\sigma^{-1}\}$ and therefore $\sup(\Sigma) \subset \{\sigma_1\sigma^{-1}, \ldots, \sigma_n\sigma^{-1}\}$. hence it follows that the $R_e$-length of $\Sigma$ is at most $n$.

□

## 2.7.5 Corollary

If $\sigma \in \sup(\Sigma)$ for a gr-simple $\Sigma$ in $R$-gr then : $\mathrm{End}_{R-\mathrm{gr}}(\Sigma) \cong \mathrm{End}_{R_e}(\Sigma_\sigma)$.

**Proof** We may define a ring morphism $\varphi$ by :

$$\varphi : \mathrm{End}_{R-\mathrm{gr}}(\Sigma) \to \mathrm{End}_{R_e}(\Sigma_\sigma), u \mapsto u|\Sigma_\sigma$$

If $u|\Sigma_\sigma = 0$, then $u(\Sigma) = u(R\Sigma_\sigma) = Ru(\Sigma_\sigma) = 0$. Therefore $\varphi$ is injective. On the other hand, if $f \in \mathrm{End}_{R_e}(\Sigma_e)$ then we define $u = 1\overline{\otimes}f : \Sigma(\sigma) \to \Sigma(\sigma)$. From the fact that the $\sigma$-suspension functor is a category isomorphism,

it follows that $\mathrm{End}_{R-\mathrm{gr}}(\Sigma) = \mathrm{End}_{R-\mathrm{gr}}(\Sigma(\sigma))$. Consequently, $u = 1\overline{\otimes}f \in \mathrm{End}_{R-\mathrm{gr}}(\Sigma)$. But we have $\varphi(u) = f$ as it is the restriction of $1\overline{\otimes}f$ to $\Sigma_\sigma$, hence $\varphi$ is also surjective and thus an isomorphism. □

## 2.8 The Structure of Gr-injective Modules

Let $R$ be a $G$-graded ring and $Q \in R$-gr an injective object. Assume that $Q$ is $\sigma$-faithful, where $\sigma \in G$. With notation as in Section 2.6., we have $t_{\mathcal{C}_\sigma}(Q) = 0$, and therefore the canonical morphism $\nu(Q) : Q \to \mathrm{Coind}(Q_\sigma)(\sigma^{-1})$ is a monomorphism. Again, by Proposition 1.6.1., $\mathrm{Im}\nu(Q)$ is essential in $\mathrm{Coind}(Q_\sigma)(\sigma^{-1})$. Since $Q$ is gr-injective it follows that $\nu(Q)$ is an isomorphism. We now prove that $Q_\sigma$ is an injective $R_e$-module. Let $E(Q_\sigma)$ be the injective envelope of $Q_\sigma$ in $R_e$-mod. Since Coind is a left exact functor, there is a monomorphism $\mathrm{Coind}(Q_\sigma) \subset \mathrm{Coind}(E(Q_\sigma))$ in $R$-gr. Since $\mathrm{Coind}(Q_\sigma) \simeq Q(\sigma)$, $\mathrm{Coind}(Q_\sigma)$ is gr-injective and therefore $\mathrm{Coind}(E(Q_\sigma)) = \mathrm{Coind}(Q_\sigma) \oplus X$ for some $X \in R$-gr. In particular we have $\mathrm{Coind}(E(Q_\sigma))_e = \mathrm{Coind}(Q_\sigma) \oplus X_e$, thus $E(Q_\sigma) = Q_\sigma \oplus X_e$ and $X_e = 0$. Hence $E(Q_\sigma) = Q_\sigma$, and therefore $Q_\sigma$ is an injective $R_e$-module. We have in fact the following result.

### 2.8.1 Proposition

Let $Q = \oplus_{\sigma\in G} Q_\sigma$ be a gr-injective module. If $Q$ is $\sigma$-faithful, then $Q_\sigma$ is an injective $R_e$-module and $Q \simeq \mathrm{Coind}(Q_\sigma)(\sigma^{-1})$.

### 2.8.2 Corollary

Let $M = \oplus_{\sigma\in G} M_\sigma$ be a graded $R$-module. If $M$ is $\sigma$-faithful, then

$$E^g(M) \simeq \mathrm{Coind}(E(M_\sigma))(\sigma^{-1})$$

(recall that $E^g(M)$ denotes the injective envelope of $M$ in $R$-gr).

**Proof** Since $M$ is $\sigma$-faithful and $E^g(M)$ is an essential extension of $M$, $E^g(M)$ is $\sigma$-faithful too. By Proposition 2.8.1, we have that $E^g(M) = \mathrm{Coind}(E^g(M)_\sigma)(\sigma^{-1})$ and $E^g(M)_\sigma$ is an injective $R_e$-module. But since $E^g(M)$ is $\sigma$-faithful, Proposition 2.6.1 entails than $M_\sigma$ is an essential $R_e$-submodule of $E^g(M)_\sigma$, and therefore $E(M_\sigma) = E^g(M)_\sigma$ in $R_e$-mod. □

### 2.8.3 Lemma

Let $M \in R$-gr be a nonzero graded $R$-module such that $\sup(M) < \infty$. Then there exists a $\sigma \in \sup(M)$ and a nonzero graded submodule $M'$ of $M$ such that $M'$ is $\sigma$-faithful.

**Proof** By induction on the cardinality of $\sup(M)$. Noting that the claim holds when $\sup(M)$ consists of one element only, assume that the result holds when $|\sup(M)| < n$ and consider an $M \in R$-gr with $\sup(M)| = n > 1$. Choose $g \in \sup(M)$; if $M$ is $g$-faithful we were lucky and stop the proof, otherwise there exists a nonzero graded submodule $N$ of $M$ such that $N_g = 0$. Then $\sup(N) \underset{\neq}{\subset} \sup(M)$ and the induction hypothesis applied to $N$ yields the result. □

## 2.8.4 Corollary

If $M \in R$-gr is gr-injective and $\sup(M)$ is finite, then there is a graded nonzero submodule $Q$ of $M$ such that $Q$ is gr-injective and $Q$ is $\sigma$-faithful for some $\sigma \in G$. In particular, every injective indecomposable object of finite suport of $R$-gr is $\sigma$-faithful for some $\sigma \in G$.

**Proof** In view of Lemma 2.8.3 there exist $\sigma \in \sup(M)$ and a nonzero graded submodule $N$ of $M$ such that $N$ is $\sigma$-faithful. We put $Q = E^g(M)$ and it is clear that $Q$ is $\sigma$-faithful and $Q \subseteq M$ (in fact it is a direct summand).

□

For $M \in R$-gr we put $\mathcal{F}(M) = \{N, N$ a graded subobject of $M$ such that $N$ is $\sigma$-faithful for some $\sigma \in G\}$.

## 2.8.5 Proposition

If $M \in R$-gr has finite support then there is a finite direct sum of elements of $\mathcal{F}(M)$ which is essential as an $R$-module in $M$.

**Proof** Let $\mathcal{F}(M) = \{N_i | i \in I\}$ and $\mathcal{A} = \{J | J \subseteq I$ such that the sum $\sum_{i \in J} N_i$ is direct $\}$. Inclusion makes $\mathcal{A}$ into an inductively ordered set , so, using Zorn's lemma we may select a maximal element $J$ of $\mathcal{A}$. If $S = \sum_{i \in J} N_i$ is not gr-essential in $M$, then there is a nonzero graded submodule $N$ of $M$ such that $S \cap N = 0$. Lemma 2.8.3 entails the existence of $i \in I$ with $N_i \subseteq N$, hence $J \cup \{i\} \in \mathcal{A}$. But the latter is a contradiction. Therefore $S$ is gr-essential in $M$, hence also essential as an $R$-submodule. Since the direct sum of $\sigma$-faithful graded modules it is $\sigma$-faithful and the result now follows because $\sup(M)$ is finite. □

## 2.8.6 Lemma

Let $Q \in R$-gr be gr-injective of finite support and $\sigma$-faithful for some $\sigma \in G$. Then $Q$ is injective in $R$-mod.

**Proof** By Proposition 2.8.1, $Q_\sigma$ is injective in $R_e$-mod and $Q \simeq \mathrm{Coind}(Q_\sigma)(\sigma^{-1})$. Now we know that $\mathrm{Coind}(Q_\sigma)_= \{f \in \mathrm{Hom}_{R_e}(R, Q_\sigma),\ f(R_\lambda) = 0, \lambda \neq \sigma^{-1}\}$. Since $Q$ has finite support only a finite number of the components $\mathrm{Coind}(Q_g)_x$ is nonzero. Therefore, we arrive at $\mathrm{Coind}(Q_\sigma) = \mathrm{Hom}_{R_e}(R, Q_\sigma)$ which is injective in $R$-mod, hence $Q$ is injective in $R$-mod too. □

The following provides apartial converse to Corollary 2.3.2., under the hypothesis that modules having finite support are being considered. It is also an extension of Corollary 2.5.2.

### 2.8.7 Theorem

If $M \in R$-gr is gr-injective and of finite support, then $M$ is injective as an $R$-module.

**Proof** Proposition 2.8.5 provides us with $M_1 \oplus, \ldots \oplus M_n$ essential in $M$, $M_i \in \mathcal{F}(M_i), i = 1, \ldots, n$. Let $E^g(M_i)$ be the injective hull of $M_i (1 \leq i \leq n)$ in $R$-gr. Clearly : $M = E^g(M) = \oplus_{i=1}^n E^g(M_i)$. On the other hand, $E^g(M_i)$ has finite support for any $1 \leq i \leq n$ and therefore Lemma 2.8.6 entails that $E^g(M_i)$ is an injective module in $R$-mod. Thus $M$ is injective in $R$-mod. □

### 2.8.8 Corollary

Let $M \in R$-gr be gr-injective having finite support. Then there exist $\sigma_1, \ldots, \sigma_n \in \mathrm{supp}(M)$ and injective $R_e$-modules $N_1, \ldots, N_n$ such that $M \simeq \oplus_{i=1}^n \mathrm{Coind}(N_i)(\sigma_i^{-1})$.

**Proof** Directly from the proof of Theorem 2.8.7 and Proposition 2.8.1.

## 2.9 The Graded Jacobson Radical (Graded Version of Hopkins' Theorem)

Let $R$ be a $G$-graded ring. If $M$ is a graded $R$-module we denote by $J^g(M)$ the **graded Jacobson radical** of $M$, that is the intersection of all gr-maximal submodules of $M$ (if $M$ has no gr-maximal submodule then we shall take, by definition, $J^g(M) = M$).

### 2.9.1 Proposition

Let $M$ be a nonzero graded $R$-module.

i) If $M$ is finitely generated then $J^g(M) \neq M$.

ii) $J^g(M) = \cap\{\mathrm{Ker} f | f \in \mathrm{Hom}_{R-\mathrm{gr}}(M, \Sigma), \Sigma \text{ is gr-simple}\} = \cap\{\mathrm{Ker} f \mid f \in \mathrm{HOM}_{R-\mathrm{gr}}(M, \Sigma), \Sigma \text{ is gr-simple}\}$.

iii) If $f \in \mathrm{HOM}_{R-\mathrm{gr}}(M,N)$ then $f(J^g(M)) \subseteq J^g(N)$.

iv) $J^g({}_RR) = \cap\{\mathrm{Ann}_R(\Sigma), \Sigma \text{ is graded simple}\}$.

v) $J^g({}_RR)$ is a (two-sided) graded ideal.

vi) $J^g({}_RR)$ is the largest proper graded ideal $I$ such that any $a \in h(R)$ is invertible, if the class of $a$ in $R/I$ is invertible.

vii) $J^g({}_RR) = J^g(R_R)$.

**Proof** The proofs of the first five statements in the graded and ungraded case are similar, so we shall omit them. Obviously, vi. implies vii., hence we have to prove vi.

To this end, let $\pi : R \to R/J^g({}_RR)$ be the canonical map. Let $a$ be a homogeneous element in $R$ such that its class in $R/J^g({}_RR)$ is invertible. If $Ra \neq R$ then there exists a gr-maximal left ideal $M$ of $R$ containing $Ra$. Since $J^g({}_RR) \subseteq M$ and $\pi(a)$ is invertible it follows that $R = M$, a contradiction. Hence $Ra = R$, i.e. there is $b \in R$ such that $ba = 1$, and we obviously may assume that $b \in h(R)$. Moreover, $\pi(b)$ is the inverse of $\pi(a)$ in $R/J^g({}_RR)$, so there exists $c \in h(R)$ such that $cb = 1 = ba$. Therefore $a = c$, which implies that $a$ is invertible in $R$.

Let us prove that $J^g({}_RR)$ is the largest graded ideal of $R$ having this property. Suppose, if $I$ is a left ideal of $R$, such that any $a \in h(R)$ is invertible if its class in $R/I$ is invertible. Let $p : R \to R/I$ be the canonical projection. If we suppose that $I \not\subseteq J^g({}_RR)$ then there exists a maximal graded left ideal $M$ which does not contain $I$. One gets $I + M = R$, so we may select two homogeneous elements, $a \in I$ and $b \in M$, with $a + b = 1$. As $p(1) = p(b)$, it follows that $b$ is invertible in $R$, hence $M = R$, a contradiction. In conclusion, $I = J^g({}_RR)$. □

We have proved that $J^g(R_R) = J^g({}_RR)$. We denote this graded ideal by $J^g(R)$ and call it the **graded Jacobson radical of** $R$. The following corollary is the graded version of Nakayama's Lemma.

### 2.9.2 Corollary (Graded Version of Nakayama's Lemma)

If $M$ is a finitely generated graded left $R$-module then $J^g(R)M \neq M$.

**Proof** By the second assertion of Proposition 2.9.1, $J^g(R)\Sigma = 0$, for any gr-simple module $\Sigma$. In particular, if we take $N$ to be any proper maximal graded submodule of $M$, we obtain $J^g(R)(M/N) = 0$, so $J^g(R)M \subseteq N \neq M$. □

### 2.9.3 Corollary

If $R$ is a $G$-graded ring then $J^g(R) \cap R_e = J(R_e)$. Moreover, $J^g(R)$ is the largest proper graded ideal of $R$ having this property.

**Proof** If $\Sigma \in R$-gr is a gr-simple module then, by Proposition 2.7.1, $\Sigma$ is a semi-simple $R_e$-module, so $J(R_e)\Sigma = 0$. Thus $J(R_e) \subseteq R_e \cap J^g(R)$. To prove the other inclusion, let $N$ be a simple left $R_e$-module. Then $\Sigma = R\overline{\otimes}N$ is gr-simple and $e$-faithful, therefore $J^g(R)\Sigma = 0$. Since $\Sigma_e \simeq N$ we obtain $(J^g(R) \cap R_e)N = 0$, thus $J^g(R) \bigcap R_e \subseteq J(R_e)$.

Let us prove that any proper graded ideal $I$, with $I \bigcap R_e = J(R_e)$, is contained in $J^g(M)$. Let $a \in h(R)$ be such that the class $\widehat{a}$ in $R/I$ is invertible. There exists a homogeneous element $b$ in $R$, $\widehat{a}\widehat{b} = \widehat{b}\widehat{a} = \widehat{1}$. Therefore, $1 - ab$ and $1 - ba$ belong to $I \cap R_e = J(R_e)$. Then $ab = 1 - (1 - ab)$ and $ba = 1 - (1 - ba)$ are invertible in $R_e$. Let $c$, respectively $d$, be the inverses. We have $a(bc) = 1 = (db)a$, so $a$ is invertible. By Proposition 2.9.1, $a \in J^g(R)$, and this implies that $I \subseteq J^g(R)$. □

### 2.9.4 Corollary

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a $G$-graded ring of finite support. Assume that $n = |\sup(R)|$. Then :

i) $J^g(R) \subseteq J(R)$, where $J(R)$ is the classical Jacobson radical.

ii) If $(J(R))_g$ is the largest graded ideal contained in $J(R)$ (see Section 2.1) then $(J(R))_g = J^g(R)$.

iii) $J(R)^n \subseteq J^g(R)$.

**Proof**

i) Follows from Corollary 2.7.4.

ii) By i. we have $J^g(R) \subseteq (J(R))_g$. Now if $\Sigma$ is a gr-simple $R$-module then $\Sigma$ is finitely generated as an $R$-module and we have $J(R)\Sigma \neq \Sigma$, thus $((J(R))_g\Sigma \neq \Sigma$. Since $(J(R))_g$ is a graded ideal, $(J(R))_g\Sigma$ is a graded submodule of $\Sigma$. But $\Sigma$ is gr-simple, thus $(J(R))_g\Sigma = 0$ and we have $(J(R))_g \subseteq J^g(R)$. Therefore $(J(R))_g = J^g(R)$.

iii) If $\Sigma$ is gr-simple then (see Proposion 2.7.1), $\Sigma$ is a semisimple $R_e$ of finite length. As an object in the category $R$-mod, $\Sigma$ is Noetherian and Artinian, hence $\Sigma$ is an $R$-module of finite length. Clearly $l_{R_e}(\Sigma) \leq n$. Since we have $l_R(\Sigma) \leq n$ hence $J(R)^n\Sigma = 0$ and therefore $J(R)^n \subseteq J^g(R)$. □

We recall that $M \in R$-gr is said to be **left gr-Noetherian** (respectively **left gr-Artinian**) if $M$ satisfies the ascending (respectively descending) chain condition for graded left $R$-submodules. A graded ring $R$ is called **left gr-semisimple** if and only if :

$$(*) \qquad R = L_1 \oplus \ldots \oplus L_n,$$

where $L_i$ are minimal graded left ideals of $R$, $i = 1, \ldots, n$. Obviously, if $R$ is a gr-semisimple ring, then $R_e$ is semisimple and Artinian. It is easy to see, by using the decomposition $(*)$, that a gr-semisimple ring is left gr-Noetherian and gr-Artinian. A $G$-graded ring is called **gr-simple** if it admits a decomposition $(*)$ with $\mathrm{HOM}_R(L_i, L_j) \neq 0$, for $i, j \in \{1, \ldots, n\}$. The last condition is satisfied, of course, if and only if there are $\sigma_{ij} \in G$ such that $L_j \simeq L_i(\sigma_{ij})$.

If $R$ is gr-semisimple, having the decomposition $(*)$, for any $L_i$ we consider the sum of all $L_j$ which are isomorphic with $L_i(\sigma_{ij})$, for a certain $\sigma_{ij} \in G$. This sum is a graded two-sided ideal of $R$, so any gr-semisimple ring is a finite direct product of gr-simple rings. A gr-simple ring $R$ is said to be **gr-uniformly simple** if $R$ has a decomposition $(*)$, where $L_i \simeq L_j$ in $R$-gr, $i, j \in \{1, \ldots, n\}$. In this case the ring $R_e$ is simple Artinian (if $R$ is an arbitrary gr-simple ring then $R_e$ is not in general a simple Artinian ring).

## 2.9.5 Proposition

Let $R$ be a $G$-graded ring. $R$ is left gr-semisimple if and only if $R$ is right-gr-semisimple.

**Proof** If $R$ is left gr-semisimple then $J^g(R) = 0$. By the decomposition $(*)$, there are some orthogonal idempotents $e_i$, $i = 1, \ldots, n$, such that $L_i = Re_i$. Moreover, $e_i \in R_e$ and it is easy to see that $\Delta_i = e_i R e_i$ is a gr-division ring. Let $K_i$ denote the right graded $R$ module $e_i R$. Obviously, $R_R = K_1 \oplus \ldots \oplus K_n$, so to end the proof it suffices to show that each $K_i$ is gr-minimal. To this end we shall prove that $K = eR$ is gr-minimal, for any graded ring $R$ and any homogeneous idempotent $e$ such that $J^g(R) = 0$ and $\Delta = eRe$ is a gr-division ring. Indeed, let $0 \neq J \subseteq K$ be a graded right ideal of $R$. Since $J = eJ$ and $J \neq 0$, $eJ \neq 0$. If $eJe = 0$ then $(eJ)^2) = 0$, hence $eJ \subseteq J^g(R) = 0$, a contradiction. In conclusion, $eJe \neq 0$. On the other hand $(eJe)(eRe) = eJeRe \subseteq eJe$, proving that $eJe$ is a right graded ideal in $\Delta$. By the assumption, $\Delta$ is a gr-division ring, so $eJe = \Delta$. It results $e = e\lambda e \in eJ = J$ $(\lambda \in J)$, therefore $eR = JR \subseteq J$, i.e. $J = eR = K$. □

## 2.9.6 Proposition

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a $G$-graded ring. The following assertions hold :

i) If $R$ is gr-semi-simple then $R$ is left and right $e$-faithful.

ii) Conversely, if $R$ is left $e$-faithful and $R_e$ is semisimple Artinian ring, then $R$ is gr-semi-simple.

**Proof**

i) The above proposition entails that ${}_RR = \oplus_{i=1}^n L_i$ where the $L_i$ are graded minimal ideals $L_i = Re_i$, $e_i$ is homogeneous idempotent. Since $e_i^2 = e_i$, $e_i \in R_e$ so $(L_i)_e = R_e e_i \neq 0$. Hence, any $L_i$ is left $e$-faithful and therefore $R$ is left $e$-faithful. In a similar way it follows that $R$ is right $e$-faithful.

ii) We have $R_e = \oplus_{i=1}^n S_i$ where the $S_i$ are simple left $R_e$-modules. Since $R$ is left $e$-faithful then $R\overline{\otimes}_{R_e} R_e = R \otimes_{R_e} R_e \simeq R$. On the other hand $R\overline{\otimes}_{R_e} R_e \simeq \oplus_{i=1}^n R\overline{\otimes}_{R_e} S_i$ where $R\overline{\otimes}_{R_e} S_i = \Sigma_i$ is gr-simple left module. So ${}_RR$ is a gr-semisimple left module.

The next corollary is the graded version of Hopkins' Theorem.

## 2.9.7 Corollary

If $R$ is left gr-Artinian then it is left gr-Noetherian.

**Proof** We shall adapt the proof of Hopkins' Theorem to the graded case. Because, $R$ is gr-Artinian, $J^g(R)$ is a finite intersection $J^g(R) = M_1 \cap \ldots \cap M_n$ of gr-maximal left ideals. In particular, $R/J^g(R)$ is a gr-semisimple ring. The descending chain

$$J^g(R) \supseteq J^g(R)^2 \supseteq \ldots \supseteq J^g(R)^n \supseteq \ldots$$

must terminate, so there exists $n \in \mathbb{N}$ such that $J^g(R)^n = J^g(R)^{n+1} = \ldots$. If $J^g(R)^n \neq 0$ then there exists an homogeneous $x \in J^g(R)^n$ such that $J^g(R)x \neq 0$. We can choose the element $x \in J^g(R)^n$ such that $Rx$ is minimal with the property that $J^g(R)x \neq 0$. By Nakayama's Lemma we have $J^g(R)x = J^g(R)Rx \neq Rx$. If $a \in J^g(R)$ is a homogeneous element we have $Rax \subseteq J^g(R)x \subseteq Rx$ and $Rax \neq Rx$. Therefore $ax = 0$, which implies $J^g(R)x = 0$, contradiction. One gets $J^g(R)^n = 0$. For each $i = 1, \ldots, n-1$, $J^g(R)^i/J^g(R)^{i+1}$ is annihilated by $J^g(R)$, so it is an $R/J^g(R)$-module. Since the ring $R/J^g(R)$ is gr-semisimple it follows that $J^g(R)^i/J^g(R)^{i+1}$ is a gr-semisimple module. It is also a gr-Artinian module, thus it is left gr-Noetherian, so the corollary is proved. □

A Grothendieck category $\mathcal{A}$ is said to be **semisimple** if every object is semisimple, i.e. a direct sum of simple objects. For any semisimple abelian category, a subobject of an object $M$ is a direct summand of $M$. The converse

is not necessarily true in general but for a locally finitely generated one, i.e. whenever there exists a family $\{U_i, i \in J\}$ of generators for $\mathcal{A}$ such that each $U_i$ is finitely generated, it is true that the property for subobjects of an object to be direct summands does imply the semisimplicity of $\mathcal{A}$.

We are interested in the particular case $\mathcal{A} = R$-gr, then $\mathcal{A}$ has the family of generators $\{{}_RR(\sigma), \sigma \in G\}$ and each ${}_RR(\sigma)$ is of course finitely generated.

### 2.9.8 Proposition

The following statements hold for a $G$-graded ring $R$ :

i) $R$ is left gr-semisimple if and only if $R$-gr is semisimple.

ii) $R$ is gr-simple if and only if $R$-gr is semisimple and there exist a gr-simple object $\Sigma$ such that every gr-simple object of $R$-gr is isomorphic to $\Sigma(\sigma)$ for some $\sigma \in G$.

iii) $R$ is uniformly gr-simple if and only if there exists a gr-simple object $\Sigma$ such that ${}_RR \cong \Sigma^n$ for some $n \geq 1 \in \mathbb{N}$. In that case $\sup(R) = \sup(\Sigma)$ and $\sup(R)$ is a subgroup of $G$.

iv) $R$ is uniformly gr-simple if and only if $R$ is gr-semisimple and $R_e$ is a simple Artinian ring.

**Proof** Both i. and ii. follow from the definitions.

iii. Look at the decomposition $(*)$ and put $\Sigma = L_1$. We have $L_i \cong \Sigma$ for $1 \leq i \leq n$ and thus ${}_RR \cong \Sigma^n$. It is clear enough that $\sup(R) = \sup(\Sigma)$. Since $\Sigma_e \neq 0$, $e \in \sup(\Sigma)$ follows. Assume that $\sigma, \tau \in \sup(\sigma)$. Since $\Sigma_\tau \neq 0$ then $R\Sigma_\tau = \Sigma$ and therefore $\Sigma_\sigma = R_{\sigma\sigma^{-1}}\Sigma_\tau$. Since $\Sigma_{\sigma \neq 0}$ then $R_{\sigma\sigma^{-1}} \neq 0$. On the other hand from the isomorphism ${}_RR \cong \Sigma^n$ it follows that $\Sigma_{\sigma\tau^{-1}} \neq 0$ so $\sigma\tau^{-1} \in \mathrm{supp}(\Sigma)$. Hence $\sup(\Sigma)$ is a subgroup of $G$.

iv. If $R$ is uniformly gr-simple then the statements in iv. do follow trivially. For the converse assume that $R_e$ is simple Artinian; then $R_e \cong S^n$ for some simple $R_e$-module $S$. But $R\overline{\otimes}_{R_e}R_e \cong (R\overline{\otimes}_{R_e}S)^n$ ($R$ is $e$-faithful). If we put $\Sigma = R\overline{\otimes}_{R_e}S$ then $R \cong \Sigma^n$ and it follows from iii. that $R$ is uniformly gr-simple.

## 2.10 Graded Endomorphism Rings and Graded Matrix Rings

For a $G$-graded ring $R$ and an $M \in R$-gr the ring $\mathrm{END}_R(M) = \mathrm{HOM}_R(M, M)$ is a $G$-graded ring with obvious addition and multiplication defined by $g.f =$

$f \circ g$. In this section we provide necessary and sufficient conditions for the graded ring $\mathrm{END}_R(M)$ to be strongly graded or a crossed product

First we have to introduce a few general notions.

Let $\mathcal{A}$ be an abelian category and $M, N \in \mathcal{A}$. We say that $N$ divides $M$ in $\mathcal{A}$ if $N$ is isomorphic to a direct summand of $M$, i.e. there exists $f \in \mathrm{Hom}_{\mathcal{A}}(M, N)$ and $g \in \mathrm{Hom}_{\mathcal{A}}(N, M)$ such that $f \circ g = 1_N$. We say that $N$ weakly divides $M$ in $\mathcal{A}$ if it divides a finite direct sum $M^t$ of copies of $M$. It is clear that $N$ weakly divides $M$ in $\mathcal{A}$ if and only if there exist $f_1, \ldots, f_t \in \mathrm{Hom}_{\mathcal{A}}(M, N)$ and $g_1, \ldots, g_t \in \mathrm{Hom}_{\mathcal{A}}(N, M)$ such that $1_N = f_1 \circ g_1 + \ldots + f_t \circ g_t$.

We say that $M, N \in \mathcal{A}$ are **weakly isomorphic** in $\mathcal{A}$ (we denote this by $M \sim N$) if and only if they weakly divide each other in $\mathcal{A}$. Thus $M \sim N$ if and only if there exist positive integers $n, m$ and objects $M', N' \in \mathcal{A}$ such that $M \oplus M' \simeq N^m$ and $N \oplus N' \simeq M^n$. Clearly $\sim$ is an equivalence relation on the class of objects of $\mathcal{A}$.

Let us consider $\mathcal{A} = R$-gr in particular. An object $M \in R$-gr is said to be **weakly $G$-invariant** if $M \sim M(\sigma)$ in $R$-gr for all $\sigma \in G$.

## 2.10.1 Theorem

Let $M \in R$-gr. Then the $G$-graded ring $\mathrm{END}_R(M)$ is strongly graded if and only if $M$ is weakly $G$-invariant. In particular $R$ is a strongly graded ring if and only if $R$ is weakly $G$-invariant in $R$-gr.

**Proof**
$\mathrm{END}_R(M)$ is strongly graded if and only if $1 \in \mathrm{END}_R(M)_\lambda \mathrm{END}_R(M)_{\lambda^{-1}}$ for any $\lambda \in G$. This condition is equivalent to the fact that there exist $g_1, \ldots, g_n \in \mathrm{END}_R(M)_\lambda$ and $f_1, \ldots, f_n \in \mathrm{END}_R(M)_{\lambda^{-1}}$ such that

$$1_M = \sum_{i=1,n} g_i \cdot f_i = \sum_{i=1,n} f_i \circ g_i \tag{1}$$

But

$$\mathrm{END}_R(M)_\lambda = \mathrm{Hom}_{R-gr}(M(\sigma), M(\sigma\lambda))$$

and

$$\mathrm{END}_R(M)_{\lambda^{-1}} = \mathrm{Hom}_{R-gr}(M(\sigma\lambda), M(\sigma))$$

for any $\sigma \in G$. The equation (1) is equivalent to the fact that $M(\sigma)$ weakly divides $M(\sigma\lambda)$ for any $\sigma, \lambda \in G$. For $\sigma = e$ this means that $M$ weakly divides $M(\lambda)$ for any $\lambda \in G$. Also, for $\lambda = \sigma^{-1}$, it means that $M(\sigma)$ weakly divides $M$ for any $\sigma \in G$. Therefore $M \sim M(\sigma)$ for any $\sigma \in G$. The last part of the theorem follows from the fact that $\mathrm{End}_R(R) \simeq R$ as graded rings. □

An object $M \in R - gr$ with the property that $M \simeq M(\sigma)$ for any $\sigma \in G$ is called a $G$-invariant graded module.

### 2.10.2 Theorem

Let $M \in R$-gr. Then $\mathrm{END}_R(M)$ is a crossed product if and only if $M$ is $G$-invariant. In particular, $R$ is a crossed product if and only if $R \simeq R(\sigma)$ for any $\sigma \in G$.

**Proof** $\mathrm{END}_R(M)$ is a crossed product if and only if $\mathrm{END}_R(M)_\sigma$ contains an invertible element for any $\sigma \in G$. Since $\mathrm{END}_R(M)_\sigma = \mathrm{Hom}_{R-gr}(M, M(\sigma))$, we see that $\mathrm{END}_R(M)$ is a crossed product if and only if $M$ is $G$-invariant. □

For any object $M \in R$-gr we denote by $\overline{M} = \oplus_{\sigma\in G} M(\sigma)$ and $\overline{\overline{M}} = \prod^{\mathrm{gr}}_{\sigma\in G} M(\sigma)$. Clearly $\overline{M}$ and $\overline{\overline{M}}$ are $G$-invariant graded modules, so the rings $\mathrm{END}_R(\overline{M})$ and $\mathrm{END}_R(\overline{\overline{M}})$ are crossed products. The next result describes the structure of these rings more precisely.

### 2.10.3 Theorem

$\mathrm{END}_R(\overline{M})$ (respectively $\mathrm{END}_R(\overline{\overline{M}})$) is a skew groupring over the ring $\mathrm{End}_{R-\mathrm{gr}}(\overline{M})$ (respectively $\mathrm{End}_{R-\mathrm{gr}}(\overline{\overline{M}})$).

**Proof** Let us consider a family $(M^x)_{x\in G}$ of copies of $M$ as an $R$-module indexed by the group $G$. Let $\prod_{x\in G} M^x$ be the direct product of this family in the category $R$-mod, and define for any $g \in G$ a map

$$\overline{g} : \prod_{x\in G} M^x \to \prod_{x\in G} M^x, \ \ \overline{g}((m^x)_{x\in G}) = ((n^x)_{x\in G})$$

where $n^x = m^{gx}$ for any $x \in G$, which is clearly a morphism of $R$-modules. If $g, h \in G$ we have that $\overline{gh} = \overline{h}\overline{g}$. Indeed $(\overline{h} \circ \overline{g})((m^x)_{x\in G}) = \overline{h}((n^x)_{x\in G})$ where $n^x = m^{gx}$. If we put $\overline{h}((n^x)_{x\in G}) = (p^x)_{x\in G}$, then $p^x = n^{hx} = m^{g(hx)} = m^{(gh)x}$ for any $x \in G$. Hence $\overline{gh} = \overline{h}\overline{g}$. In particular $\overline{g}\overline{g^{-1}} = \overline{g^{-1}}\overline{g} = 1$, and $\overline{g}$ is an isomorphism. Then we can define

$$\varphi : G \to U(\mathrm{End}_R(\prod_{x\in G} M^x)), \ \ \varphi(g) = \overline{g}$$

Clearly $\varphi$ is a group morphism and $\overline{g}(\oplus_{x\in G} M^x) = \oplus_{x\in G} M^x$.

We consider the case where $M \in R$-gr and $M^x = M(x)$ for any $x \in G$. Let $\overline{\overline{M}} = \prod^{\mathrm{gr}}_{x\in G} M(x)$ be the direct product of this family in the category $R$-gr. We have $\overline{\overline{M}} = \oplus_{\lambda\in G} \overline{\overline{M}}_\lambda$, where

$$\overline{\overline{M}}_\lambda = \prod_{x\in G} M(x)_\lambda = \prod_{x\in G} M_{\lambda x}$$

If $\overline{\overline{m}} = (m^x)_{x\in G} \in \overline{\overline{M}}_\lambda$, then $m^x \in M_{\lambda x}$, hence $\overline{g}(\overline{\overline{m}}) = (n^x)_{x\in G}$, where $n^x = m^{gx} \in M_{\lambda gx}$. Since

$$\overline{\overline{M}}_{\lambda g} = \prod_{x\in G} M(x)_{\lambda g} = \prod_{x\in G} M_{\lambda gx}$$

we obtain that $\overline{g}(\overline{\overline{M}}_\lambda) \subseteq \overline{\overline{M}}_{\lambda g}$. In particular $\overline{g}(\overline{\overline{M}}) \subseteq \overline{\overline{M}}$, thus $\overline{g}$ is a morphism of degree $g$ when considered as an element of $\mathrm{END}_R(\overline{\overline{M}})$. The construction of the map $\varphi$ yields group morphisms $\varphi' : G \to U^{\mathrm{gr}}(\mathrm{END}_R(\overline{M}))$ and $\varphi'' : G \to U^{\mathrm{gr}}(\mathrm{END}_R(\overline{\overline{M}}))$, now we just apply the results of Chapter I. □

Now we focus our attention on matrix rings with entries in a graded ring, and investigate how these can be made into graded rings themselves. Let $R = \oplus_{\sigma\in G} R_\sigma$ be a $G$-graded ring, $n$ is a positive integer, and $M_n(R)$ the ring of $n \times n$-matrices with entries in $R$. Fix some $\overline{\sigma} = (\sigma_1, \ldots, \sigma_n) \in G^n$. To any $\lambda \in G$ we associate the following additive subgroup of $M_n(R)$

$$M_n(R)_\lambda(\overline{\sigma}) = \begin{pmatrix} R_{\sigma_1\lambda\sigma_1^{-1}} & R_{\sigma_1\lambda\sigma_2^{-1}} & \ldots & R_{\sigma_1\lambda\sigma_n^{-1}} \\ R_{\sigma_2\lambda\sigma_1^{-1}} & R_{\sigma_2\lambda\sigma_2^{-1}} & \ldots & R_{\sigma_2\lambda\sigma_n^{-1}} \\ \ldots & \ldots & \ldots & \ldots \\ R_{\sigma_n\lambda\sigma_1^{-1}} & R_{\sigma_n\lambda\sigma_2^{-1}} & \ldots & R_{\sigma_n\lambda\sigma_n^{-1}} \end{pmatrix}$$

### 2.10.4 Proposition

The family of additive subgroups $\{M_n(R)_\lambda(\overline{\sigma}) |\ \lambda \in G\ \}$ defines a $G$-grading of the ring $M_n(R)$. We will denote this graded ring by $M_n(R)(\overline{\sigma})$.

**Proof** It is easy to see that $M_n(R)_\lambda(\overline{\sigma})M_n(R)_\mu(\overline{\sigma}) \subseteq M_n(R)_{\lambda\mu}(\overline{\sigma})$ for any $\lambda, \mu \in G$. Since $R_{\sigma_i\lambda\sigma_j^{-1}} \cap (\sum_{\mu\neq\lambda} R_{\sigma_i\mu\sigma_j^{-1}}) = 0$ for any $i, j \in \{1, \ldots, n\}$, we see that $M_n(R)_\lambda(\overline{\sigma}) \cap (\sum_{\mu\neq\lambda} M_n(R)_\mu(\overline{\sigma})) = 0$ so $\sum_{\lambda\in G} M_n(R)_\lambda(\overline{\sigma}) = \oplus_{\lambda\in G} M_n(R)_\lambda(\overline{\sigma})$ To finish the proof it is enough to show that $M_n(R) = \sum_{\lambda\in G} M_n(R)_\lambda(\overline{\sigma})$. Any matrix in $M_n(R)$ is a sum of matrices with a homogeneous element on a certain entry, and zero elsewhere, therefore it is enough to show that any such matrix belongs to $\sum_{\lambda\in G} M_n(R)_\lambda(\overline{\sigma})$. Let $A$ be such a matrix, having the element $r \in R_g$ on the $(i, j)$-entry, and zero elsewhere. If we take $\lambda = \sigma_i^{-1} g \sigma_j$, we have that $A \in M_n(R)_\lambda(\overline{\sigma})$, which ends the proof. □

If we take $\sigma_1 = \ldots = \sigma_n = e$, we simply denote the resulting graded ring structure on $M_n(R)(\overline{e})$ by $M_n(R)$, In this case the grading is given by $M_n(R) = \oplus_{\lambda\in G} M_n(R)_\lambda$, where

$$M_n(R)_\lambda = \begin{pmatrix} R_\lambda & \ldots & R_\lambda \\ \ldots & \ldots & \ldots \\ R_\lambda & \ldots & R_\lambda \end{pmatrix}$$

for any $\lambda \in G$.

### 2.10.5 Proposition

Let $M \in R$-gr be gr-free with homogeneous basis $e_1, \ldots, e_n$, say $deg(e_i) = \sigma_i$. Then $\mathrm{END}_R(M) \simeq M_n(R)(\overline{\sigma})$, where $\overline{\sigma} = (\sigma_1, \ldots, \sigma_n)$.

**Proof** If $f \in \mathrm{END}_R(M)_\lambda$, $\lambda \in G$, then $f(M_\sigma) \subseteq M_{\sigma\lambda}$ for any $\sigma \in G$. In particular $f(e_i) \in M_{\sigma_i\lambda}$ for any $i = 1, \ldots, n$. Hence $f(e_i) = \sum_{j=1,n} a_{ij}e_j$ with $a_{ij}$ homogeneous of degree $\sigma_i\lambda\sigma_j^{-1}$. Therefore the matrix $(a_{ij})$ associated to $f$ is in $M_n(R)_\lambda(\overline{\sigma})$. □

### 2.10.6 Remarks

i) If $\overline{\sigma} = (\sigma_1, \ldots, \sigma_n) \in G^n$, put $M = R(\sigma_1^{-1}) \oplus \ldots \oplus R(\sigma_n^{-1})$, where $e_i$ is the element of $M$ with 1 on the $i$-th slot and 0 elsewhere, then $e_i$ is a homogeneous element of degree $\sigma_i$ and $e_1, \ldots, e_n$ is a basis of $M$.

ii) It follows from Propositions 3.5.4 and 3.5.5 that if $M \in R$-gr is gr-free with finite homogeneous basis, then $\mathrm{End}_R(M) = \mathrm{END}_R(M)$.

iii) Let $\overline{\sigma} = (\sigma_1, \ldots, \sigma_n) \in G^n$ and $\varphi \in S_n$ a permutation. Put $\varphi(\overline{\sigma}) = (\sigma_{\varphi(1)}, \ldots, \sigma_{\varphi(n)})$ Then $M_n(R)(\overline{\sigma}) \simeq M_n(R)(\varphi(\overline{\sigma}))$ as $G$-graded rings. Indeed, by the first remark, we have $M_n(R)(\overline{\sigma}) \simeq \mathrm{END}_R(M)$, where $M = R(\sigma_1^{-1}) \oplus \ldots \oplus R(\sigma_n^{-1})$. If we put $\sigma_{\varphi(i)} = \tau_i$ and $N = R(\tau_1^{-1}) \oplus \ldots \oplus R(\tau_n^{-1})$ then clearly $M \simeq N$ in $R$-gr. The assertion follows from the fact that $M_n(R)(\varphi(\overline{\sigma})) \simeq \mathrm{END}_R(N)$ and $\mathrm{END}_R(M) \simeq \mathrm{END}_R(N)$.

iv) Let $\overline{\sigma} = (\sigma_1, \ldots, \sigma_n) \in G^n$ and $\tau \in Z(G)$, the centre of the group $G$. We put $\overline{\sigma}\tau = (\sigma_1\tau, \ldots, \sigma_n\tau)$. Then we have that $M_n(R)(\overline{\sigma}) = M_n(R)(\overline{\sigma\tau})$. Indeed, since $\tau \in Z(G)$, we clearly have that $M_n(R)_\lambda(\overline{\sigma}) = M_n(R)_\lambda(\overline{\sigma\tau})$

□

### 2.10.7 Example

Let $K$ be a field and $G = \mathbf{Z}_2$. Regard $K$ as a trivially $G$-graded ring. From the preceding remark, we obtain the following. If $n = 2$, then $M_2(K)$ has two $G$-gradings. The first one is the trivial grading, i.e. $M_2(K)_{\hat{0}} = M_2(K)$ and $M_2(K)_{\hat{1}} = 0$. The other one is the grading given by $M_2(K)_{\hat{0}} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}$ and $M_2(K)_{\hat{1}} = \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}$. For $n = 3$, on $M_3(K)$ we have two gradings.

The first is the trivial one, $M_3(K)_{\hat{0}} = M_3(K)$ and $M_3(K)_{\hat{1}} = 0$. The second one is given by

$$M_3(K)_{\hat{0}} = \begin{pmatrix} K & K & 0 \\ K & K & 0 \\ 0 & 0 & K \end{pmatrix} \text{ and } M_3(K)_{\hat{1}} = \begin{pmatrix} 0 & 0 & K \\ 0 & 0 & K \\ K & K & 0 \end{pmatrix}$$

□

## 2.10.8 Corollary

If $R$ is a $G$-strongly graded ring and $\overline{\sigma} = (\sigma_1, \ldots, \sigma_n) \in G^n$, then the $G$-graded ring $M_n(R)(\overline{\sigma})$ is strongly graded.

**Proof** Consider the gr-free module $M = R(\sigma_1^{-1}) \oplus \ldots \oplus R(\sigma_n^{-1})$. We know from Theorem [!] that $R$ is strongly graded if and only if $R \sim R(\sigma)$ for any $\sigma \in G$. Since $\sim$ is an equivalence relation, we have that

$$\begin{aligned} M(\sigma) &= R(\sigma_1^{-1})(\sigma) \oplus \ldots \oplus R(\sigma_n^{-1})(\sigma) && (2.1) \\ &= R(\sigma\sigma_1^{-1}) \oplus \ldots \oplus R(\sigma\sigma_n^{-1}) && (2.2) \end{aligned}$$

Since $R$ is strongly graded, $R \sim R(\sigma\sigma_i^{-1})$ and $R \sim R(\sigma_i^{-1})$. By transitivity we obtain that $R(\sigma\sigma_i^{-1}) \sim R(\sigma_i^{-1})$ for any $1 \leq i \leq n$. Clearly :

$$\oplus_{i=1,n} R(\sigma\sigma_i^{-1}) \sim \oplus_{i=1,n} R(\sigma_i^{-1})$$

therefore $M \sim M(\sigma)$ for all $\sigma \in G$. Then by Theorem 1.9.2 and Proposition 1.9.5 it follows $M_n(R)(\overline{\sigma})$ is a strongly graded ring. □

## 2.10.9 Corollary

If $R$ is a $G$-crossed product (respectively a skew groupring over $G$), then for any $\overline{\sigma} = (\sigma_1, \ldots, \sigma_n) \in G^n$, $M_n(R)(\overline{\sigma})$ is a crossed product (respectively a skew groupring over $G$).

**Proof** For any $\lambda \in G$ we have

$$M_n(R)_\lambda(\overline{\sigma}) = \begin{pmatrix} R_{\sigma_1\lambda\sigma_1^{-1}} & R_{\sigma_1\lambda\sigma_2^{-1}} & \ldots & R_{\sigma_1\lambda\sigma_n^{-1}} \\ R_{\sigma_2\lambda\sigma_1^{-1}} & R_{\sigma_2\lambda\sigma_2^{-1}} & \ldots & R_{\sigma_2\lambda\sigma_n^{-1}} \\ \ldots & \ldots & \ldots & \ldots \\ R_{\sigma_n\lambda\sigma_1^{-1}} & R_{\sigma_n\lambda\sigma_2^{-1}} & \ldots & R_{\sigma_n\lambda\sigma_n^{-1}} \end{pmatrix}$$

Since $R$ is a $G$-crossed product, there exists an invertible element $u_{\sigma_i\lambda\sigma_i^{-1}} \in R_{\sigma_i\lambda\sigma_i^{-1}}$. Then the matrix

$$A_\lambda = \begin{pmatrix} u_{\sigma_1\lambda\sigma_1^{-1}} & & \ldots & 0 \\ 0 & u_{\sigma_2\lambda\sigma_2^{-1}} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & u_{\sigma_n\lambda\sigma_n^{-1}} \end{pmatrix}$$

is an invertible element of $M_n(R)_\lambda(\overline{\sigma})$, so $M_n(R)(\overline{\sigma})$ is a $G$-crossed product. Now assume that $R$ is a skew groupring over $G$. Then there is a group morphism $\varphi : G \to U^g(R)$ such that $deg \circ \varphi = 1_G$, i.e. $\varphi(g) \in R_g$ for any $g \in G$. In this case we consider the matrix

$$A_\lambda = \begin{pmatrix} \varphi(\sigma_1\lambda\sigma_1^{-1}) & & \dots & 0 \\ 0 & \varphi(\sigma_2\lambda\sigma_2^{-1}) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \varphi(\sigma_n\lambda\sigma_n^{-1}) \end{pmatrix}$$

Then for any $\lambda, \mu \in G$ we have that $A_\lambda A_\mu = A_{\lambda\mu}$, so the map $\overline{\varphi} : G \to U^g(M_n(R)(\overline{\sigma}))$ defined by $\overline{\varphi}(\lambda) = A_\lambda$ for any $\lambda \in G$, is a group morphism with the property that $deg \circ \overline{\varphi} = 1_G$. Hence $M_n(R)(\overline{\sigma})$ is a skew groupring. □

### 2.10.10 Theorem (Graded version of Wedderburn's Theorem)

Let $R$ be a graded ring of type $G$. Then the following statements are quivalent :

i) $R$ id gr-simple (resp. gr-unifromly simple)

ii) There exists a graded division ring $D$ and a $\overline{\sigma} = (\sigma_1, \dots, \sigma_n) \in G^n$ such that $R \simeq M_n(D)(\overline{\sigma})$ (resp. $R \simeq M_n(D)$).

**Proof** A full proof is given in Chapter IV. In fact, by Corollary 4.5.8, putting $D = \mathrm{End}({}_R\Sigma) = \mathrm{END}({}_R\Sigma)$ and $V = \Sigma_D$, we have $R \cong \mathrm{End}(V_D)$ here $V_D$ is a graded vector space of finite dimension. □

## 2.11 Graded Prime Ideals. The Graded Spectrum

An ideal $P$ of a ring $R$ is prime if $P \neq R$ and if $IJ \subset P$ implies that $I \subset P$ or $J \subset P$, for ideals $I$ and $J$ of $R$. Equivalently $P \neq R$ is prime if for $a, b \in R$, $aRb \subset P$ if and only if $a \in P$ or $b \in P$. The set of prime ideals of $R$ is denoted by Spec$R$; it is called the prime **spectrum** of $R$.

Using Zorn's lemma, we may find for $P \in \mathrm{Spec}R$ a minimal prime ideal $Q \in \mathrm{Spec}R$ such that $P \supset Q$. The **prime radical** $\mathrm{rad}(R)$ of $R$ is defined as : $\mathrm{rad}(R) = \cap\{P, P \in \mathrm{Spec}R\}$. It is clear that $\mathrm{rad}(R)$ is the intersection of all minimal prime ideals of $R$. A basic fact in elementary Ring Theory states that $\mathrm{rad}(R)$ is exactly the set of all **strongly nilpotent** elements in $R$, i.e. those $a \in \mathrm{rad}(R)$ such that for every sequence $a_0, a_1, \dots, a_n$, such that $a_0 = a, a_1 \in a_0Ra_0$, $a_2 \in a_1Ra_1, \dots, a_n \in a_{n-1}Ra_{n-1}$, there is an $n \geq 0$ such

that $a_n = 0$. In particular, every element of $\mathrm{rad}(R)$ is nilpotent. For a simple $R$-module $S$ the annihilator $\mathrm{ann}_R(S)$ is certainly a prime ideal, consequently it follows that $\mathrm{rad}(R) \subset J(R)$, the latter being the Jacobson radical of $R$. If $\mathrm{rad}(R) = 0$ then $R$ is said to be a **semiprime** ring. It is not difficult (see Exercises, Section 2.12) to verify that $R$ is semiprime if and only if $R$ has no nonzero nilpotent ideals.

A proper ideal $I \neq R$ is said to be **semiprime** if the ring $R/I$ is a semiprime ring.

In a $G$-graded ring $R$ a graded ideal $P$ of $R$ is **gr-prime** if $P \neq R$ and for graded ideals $I$ and $J$ of $R$ we have $I \subset P$ or $J \subset P$ only when $IJ \subset P$. A graded ideal $P$ is gr-prime if and only if $aRb \subset P$ with $a, b \in h(R)$ implies $a \in P$ or $b \in P$. The ring $R$ is said to be **gr-prime** when $(0)$ is a gr-prime ideal of $R$. The set of all gr-prime ideals of $R$ is denoted by $\mathrm{Spec}^g(R)$ and it is called the **graded (prime) spectrum** of $R$. Every gr-prime ideal contains a minimal gr-prime ideal. We write $\mathrm{rad}^g(R)$ for $\cap\{P, P \in \mathrm{Spec}^g(R)\}$ and call it the **gr-prime radical** of $R$. Just like in the ungraded case we obtain $\mathrm{rad}^g(R) \subset J^g(R)$, where $J^g(R)$ is the graded Jacobson radical of $R$. The ring $R$ is said to be **gr-semiprime** whenever $\mathrm{rad}^g(R) = 0$. Again, as in the ungraded case we have that $R$ is gr-semiprime if and only if $R$ has no nonzero nilpotent graded ideals.

With these conventions and notation we have :

## 2.11.1 Proposition

1. If $P \in \mathrm{Spec}(R)$ then $(P)_g \in \mathrm{Spec}^g(R)$.
2. If $Q \in \mathrm{Spec}^g(R)$ then there exists a $P \in \mathrm{Spec}(R)$ such that $Q = (P)_g$
3. $\mathrm{rad}^g(R) = (\mathrm{rad}(R))_g$

**Proof**

1. Easy.
2. In view of Zorn's lemma we may choose an ideal $P$ maximal with respect to the property that $Q = (P)_g$. First we establish that $P$ is a prime ideal, therefore look at ideals $I, J$ of $R$ such that $I \underset{\neq}{\supset} P$, $J \underset{\neq}{\supset} P$. By the maximality assumption on $P$ it follows that $(I)_g \underset{\neq}{\supset} Q$ and $(J)_g \underset{\neq}{\supset} Q$ and thus $(I)_g(J)_g \not\subset Q$. Since $(I)_g(J)_g \subset IJ$ it follows that $IJ \not\subset P$ and therefore $P$ is a prime ideal (for arbitrary ideals $I$ and $J$ such that $IJ \subset P$ we may always look at $I+P$ and $J+P$ with $(I+P)(J+P) \subset P$).
3. Follows directly from 1. and 2. □

We include some specific results in case $R$ is $G$-graded of finite support.

### 2.11.2 Lemma (Cohen, Rowen)

Assume that $R = \oplus_{\sigma\in G} R_\sigma$ is a $G$-graded ring of finite support with $n = |\sup(R)|$. Consider $d \geq 1$ in $\mathbb{N}$ and $\lambda_1, \ldots, \lambda_n d \in G$. We define $\alpha_i = \prod_{j=1}^{i} \lambda_j$ and assume that $\alpha_i \in \sup(R)$, $1 \leq i \leq nd$. Then there exists a sequence : $0 \leq j_0 < j_1 < \ldots < j_d \leq nd$, such that :

$$e = \lambda_{j_0+1} \ldots \lambda_{j_1} = \lambda_{j_1+1} \ldots \lambda_{j_2} = \ldots = \lambda_{j_{d-1}+1} \ldots \lambda_{j_d}$$

**Proof** We consider the elements $1, \alpha_1, \ldots, \alpha_{nd}$. Since $|\sup(\mathrm{R})| = \mathrm{n}$ there are least $d+1$ elements $\alpha_i$ that are equal, say $\alpha_{j_0} = \alpha_{j_1} = \ldots = \alpha_{j_d}$ with $1 \leq j_0 < j_1 < \ldots < j_d$. If $0 \leq k \leq d-1$ then from $\lambda_1 \ldots \lambda_{j_k} = (\lambda_1 \ldots \lambda_{j_k})(\lambda_{j_k+1} \ldots \lambda_{j_{k+1}})$, hence $e = \lambda_{j_k+1} \ldots \lambda_{j_{k+1}}$.

### 2.11.3 Proposition. (Cohen and Rowen)

Let $R$ be $G$-graded of finite support, say $n = |\sup(R)|$.

1. If $S \subset R$ is a graded subring such that $S_e = 0$ then $S^n = 0$.
2. If $L \subset R_e$ is a left ideal such that $L^d = 0$ then $(RL)^{\mathrm{nd}} = 0$.

**Proof**

1. Write $\sup(R) = \{x_1, \ldots, x_n\}$. Since $S = \oplus_{i=1}^{n} S_{x_i}$ we obtain $S^n = \sum S_{y_1} \ldots S_{y_n}$ where the $(y_1, \ldots, y_n)$ range over all choices of $n$ elements from $\{x_1, \ldots, x_n\}$. If for $t \leq n$, $y_1 y_2 \ldots y_t \notin \sup(R)$ then we have $S_{y_1} S_{y_2} \ldots S_{y_n} = 0$. On the other hand if now $1, y_1, y_1 y_2, \ldots, y_1 \ldots y_n \in \sup(R)$ then by lemma 2.11.2, for $d = 1$, it follows that there exist $1 \leq r < s \leq n$ such that $y_{r+1} \ldots y_s = 1$. So $S_{y_1} S_{y_2} \ldots S_{y_n} \subset S_{y_1} \ldots S_{y_r} S_{y_{r+1} \ldots y_s} S_{y_{s+1}} S_{y_n} = 0$ since $S_{y_{r+1}} \ldots y_s = S_1 = 0$. Hence $S^n = 0$
2. If we put $S = RL$, $S$ is a left graded ideal of $R$ where $S_e = RL \cap R_e = L$. We get $\lambda_1, \ldots, \lambda_{nd} \in G$. By Lemma 2.11.2, we have $S_{\lambda_1} \ldots S_{\lambda_{nd}} \subset AL^d B = 0$ where $A, B$ are suitable subproducts. Hence $S^{nd} = 0$.

### 2.11.4 Theorem

Let $R$ be a $G$-graded ring of finite support. Assume that $R$ is gr-semiprime. Then

1. $R$ is $e$-faithful
2. If $\sigma \in \sup(R)$ then $\sigma^{-1} \in \sup(R)$
3. $R_e$ is a semiprime ring

**Proof**

1. We denote by $I = t_{\mathcal{C}_e}({}_RR)$; then $I$ is a graded left ideal of $R$, such that $I_e = 0$. Then by Proposition 2.11.3 we have $I^n = 0$. Since $R$ is gr-semiprime we must have $I = 0$ so $R$ is left $e$-faithful. In a similar way, it follows that $R$ is also right $e$-faithful

2. If $R_\sigma \neq 0$, pick $x_\sigma \in R_\sigma, x_\sigma \neq 0$. Since $R$ is left $e$-faithful we have $R_{\sigma^{-1}}x_\sigma \neq 0$. So $R_{\sigma^{-1}} \neq 0$ and therefore $\sigma^{-1} \in \sup(R)$.

3. Follows from Proposition 2.11.3, assertion 2.

□

## 2.11.5 Corollary

Let $R$ be a $G$-graded ring. Then

1. $\mathrm{rad}(R) \cap R_e = \mathrm{rad}^g(R) \cap R_e \subseteq \mathrm{rad}(R_e)$

2. If $\sup(R) < \infty$ then $\mathrm{rad}^g(R) \cap R_e = \mathrm{rad}(R_e)$

**Proof**

1. The equality $\mathrm{rad}(R) \cap R_e = \mathrm{rad}^g(R) \cap R_e$ follows from Proposition 2.11.1 assertion 3. Let $a \in \mathrm{rad}^g(R) \cap R_e$; to prove that $a \in \mathrm{rad}(R_e)$ it suffices to show that $a$ is strongly nilpotent. Indeed let the sequence $a_0, a_1, \ldots, a_n$ of elements from $R_e$ such that $a_0 = a, a_1 \in a_0R_ea_0, \ldots a_n \in a_{n-1}R_ea_{n-1}, \ldots$.

   Clearly $a_1 \in a_0Ra_0, \ldots, a_n \in a_{n-1}Ra_{n-1}, \ldots$. Since $a \in \mathrm{rad}^g(R)$ there exists an $n$ such that $a_n = 0$ so $a \in \mathrm{rad}(R)$.

2. Direct from Theorem 2.11.4. assertion 3. □

Assume now that $R = \oplus_{\sigma \in G} R_\sigma$ is a strongly graded ring. We denote by $\mathrm{Mod}(R_e, R)$ the set of all two-sided $R_e$ submodules of $R$. If $A \in \mathrm{Mod}(R_e, R)$ and $\sigma \in G$ then put $A^\sigma = R_{\sigma^{-1}}AR_\sigma$. It is clear that $A^\sigma \in \mathrm{Mod}(R_e, R)$, we say that $A^\sigma$ is the **$\sigma$-conjugate of** $A$.

## 2.11.6 Lemma

Fixing $\sigma \in G$, the map $\varphi_\sigma : \mathrm{Mod}(R_e, R) \to \mathrm{Mod}(R_e, R)$, $A \to A^\sigma$ is bijective and it preserves inclusion, sums, intersection and products in $R$ of elements of $\mathrm{Mod}(R_e, R)$.

**Proof** For $\sigma, \tau \in G$ we obtain $(A^\sigma)^\tau = A^{\sigma\tau}$ for all $A \in \mathrm{Mod}(R_e, R)$. From this it follows that $\varphi_\sigma$ is bijective with the map $\varphi_{\sigma^{-1}}$ as an inverse. Now, if we consider $(A_i)_{i\in I}$, a family of elements from $\mathrm{Mod}(R_e, R)$, then $\cap_{i\in I} A_i = R_\sigma R_{\sigma^{-1}}(\cap_{i\in I} A_i)R_\sigma R_{\sigma^{-1}} \subseteq R_\sigma(\cap_{i\in I} A_i^\sigma)R_{\sigma^{-1}}$ so $R_{\sigma^{-1}}(\cap_{i\in I} A_i)R_\sigma \subseteq \cap_{i\in I} A_i^\sigma$ hence $(\cap A_i)^\sigma \subseteq \cap_{i\in I} A_i^\sigma$. Therefore $(\cap_{i\in I} A_i^\sigma)^{\sigma^{-1}} \subseteq \cap_{i\in I} A_i$ and $\cap_{i\in I} A^{\sigma_i} \subseteq (\cap A_i)^\sigma$, thus $(\cap A_i)^\sigma = \cap_{i\in I} A_i^\sigma$. In a similar way one checks that $\varphi_\sigma$ preserves sums and poducts. □

We say that $A \in \mathrm{Mod}(R_e, R)$ is $G$-invariant if $A^\sigma = A$ for all $\sigma \in G$. It is straightforward to check that any ideal of $R$ is a $G$-invariant element of $\mathrm{Mod}(R_e, R)$. If $A \subset R_e$ is an ideal then $A \in \mathrm{Mod}(R_e, R)$. Clearly, $A^\sigma$ is also an ideal of $R_e$. If $P$ is a prime ideal of $R_e$, then $P^\sigma$ is also a prime ideal of $R_e$. If $P$ is a $G$-invariant ideal of $R_e$, then $P$ is called **$G$-prime** if and only if $A_1A_2 \subset P$ for $G$-invariant ideals $A_i (i = 1, 2)$ of $R_e$ implies that $A_1 \subset P$ or $A_2 \subset P$.

## 2.11.7 Proposition

Let $R$ be a strongly graded ring of type $G$ and let $I$ be a graded ideal of $R$. Then $I_e$ is an ideal of $R_e$, it is $G$-invariant and $I = RI_e = I_eR$. Moreover

1. The correspondence $I \to I_e$, defines a bijection between the set of graded ideals of $R$ and the set of $G$-invariant ideals of $R_e$.

2. The above correspondence induces a bijection between the set of gr-prime ideals of $R$ and the set of $G$-prime ideals of $R_e$.

**Proof** If $\sigma \in G$, since $R_{\sigma^{-1}} I_e R_\sigma \subset I$ and $R_{\sigma^{-1}} I_e R_\sigma \subset R_e$, then $R_{\sigma^{-1}} I_e R_\sigma \subset I_e$, so $I_e^\sigma \subset I_e$. Since $I_e = (I_e^\sigma)^{\sigma^{-1}} \subset I_e^{\sigma^{-1}}$ we obtain (because $\sigma$ is arbitrary) that $I_e \subset I_e^{\tau^{-1}}$ where $\tau = \sigma^{-1}$. So $I_e \subset I_e^\sigma$ and $I_e = I_e^\sigma$, thus $I_e$ is a $G$-invariant ideal of $R_e$. Since $R$ is strongly graded we have $I = RI_e = I_eR$. Consequently, if $A$ is a $G$-invariant ideal of $R_e$, then $I = RA = AR$ is a graded ideal of $R$. Moreover if $P$ is a gr-prime ideal of $R$, then $P_e$ is a $G$-prime ideal of $R_e$. Indeed, if $A.B \subset P_e$ where $A, B$ are $G$-invariant ideals of $R_e$ then we have $(RA)(BR) \subset RP_eR = P$. Since $RA = AR$ and $BR = RB$ so we have $RA \subset P$ or $BR \subset P$ so $A \subset P_e$ or $B \subset P_e$. Now the statements 1. and 2. follow. □

## 2.11.8 Corollary

If $R$ is a strongly graded ring, then the Jacobson radical of $R_e, J(R_e)$, and the prime radical $\mathrm{rad}(R_e)$ are both $G$-invariant.

**Proof** We have $J(R_e) = R_e \cap J^g(R)$ and $\mathrm{rad}(R_e) = R_e \cap \mathrm{rad}^g(R)$ hence we may apply Proposition 2.11.7 □

### 2.11.9 Corollary

Let $R$ be a $G$-strongly graded ring. If $P$ is a prime ideal of $R$, then $P \cap R_e$ is a $G$-prime ideal of $R_e$. Conversely, if $Q$ is a $G$-prime ideal of $R$, then there exists at least one prime ideal $P$ of $R$ such that $P \cap R_e = Q$.

**Proof** Just apply Proposition 2.11.1 and Proposition 2.11.7. □

### 2.11.10 Corollary

When $R$ is strongly graded by a finite group $G$ then $\mathrm{rad}(R_e)$ is exactly the intersection of all $G$-prime ideals of $R_e$.

**Proof** Apply Corollary 2.11.5.2 and Proposition 2.11.7 □

A graded ring $R$ of type $G$ is **almost strongly graded** $R = \sum_{\sigma \in G} R_\sigma$, where each $R_\sigma$ is an additive subgroup of $R$ such that $1 \in R_e$ and $R_\sigma R_\tau = R_{\sigma\tau}$ for all $\sigma, \tau \in G$. Such rings have also been termed **Clifford systems** in the literature.

Any epimorphic image of a strongly graded ring is an almost strongly graded ring (in fact they may be characterized as such). In particular, an epimorphic image (factor ring) of an almost strongly graded ring is again almost strongly graded. For an almost strongly graded ring $R = \sum_{\sigma \in G} R_\sigma$, every $R_\sigma$ is an invertible $R_e - R_e$-bimodule and in particular $R_\sigma$ is (left as well as right) a projective and finitely generated $R_e$-module.

In the sequel of this section we assume that $R$ is **almost strongly graded by a finite group** $G$. Consider an $R$-module $M$ and $N$ an $R_e$-submodule of $M$, we let $N^* = \cap_{\sigma \in G} R_\sigma N$ denoted the "largest" $R$-submodule of $M$ contained in $N$.

### 2.11.11 Lemma

Let $R$ be almost strongly graded by a finite group $G$ and consider an $R$-module $M$. Then $M$ contains an $R_e$-submodule $N$ which is maximal with respect to the property $N^* = 0$.

**Proof** Let $(N_i)_{i \in I}$ be a chain of $R_e$-submodules of $M$ such that $N_i^* = 0$ for each $i \in I$. If $(\cup_{i \in I} N_i)^* \neq 0$ there is an $x \in (\cup_{i \in I} N_i)^*, x \neq 0$ i.e. $Rx \subset (\cup_{i \in I} N_i)^* \subseteq \cup_{i \in I} N_i$. Hence $R_\sigma x \subset \cup_{i \in I} N_i$ for all $\sigma \in G$. Since $R_\sigma$ is a finitely generated $R_e$-module it follows that for $\sigma \in G$, there exists an $i_0$ such that $R_\sigma x \subset N_{i_0}$. Since $G$ is finite we can assume that $R_\sigma x \subset N_{i_0}$ for any $\sigma \in G$ hence $x \in N_{i_0}^*$ i.e. $N_{i_0}^* \neq 0$, contradiction. Therefore we must have $(\cup_{i \in I} N_i)^* = 0$ and then Zorn's Lemma may be applied in order to yield the existence of an $N$ as desired. □

### 2.11.12 Theorem

Let $R = \sum_{\sigma \in G} R_\sigma$ be an almost strongly graded ring ($G$ is a finite group). If $P$ is a prime ideal of $R$ there exists a prime ideal $Q$ of $R_e$ such that $P \cap R_e = \cap_{\sigma \in G} Q^\sigma$. Moreover, $Q$ is minimal over $P \cap R_e$.

**Proof** Since the ring $R/P$ is also an almost strongly graded ring, we may assume that $P = 0$. We view $R$ as an $R_e - R$-bimodule. Using Lemma 2.11.11 we may choose an $R_e - R$ subbimodule $Y$ maximal with respect to $Y^* = 0$. Hence $\cap_{\sigma \in G} R_\sigma Y = 0$. First we note that $aR_eb \subset Y$ for $a \in R_e, b \in R$, implies $a \in Y$ or $b \in Y$. Suppose that $a, b \notin Y$. By maximality of $Y$ we have $I = (R_eaR + Y)^* \neq 0$ and $J = (R_ebR + Y)^* \neq 0$. Clearly $I$ and $J$ are ideals of $R$. Since $I \neq 0$, $J \neq 0$ and $R$ is prime we have $IJ \neq 0$. On the other hand we obtain : $IJ \subset (R_eaR + Y)J \subset R_eaJ + YJ \subset R_eaJ + Y \subset R_ea(R_ebR + Y) + Y \subset R_eaR_ebR + Y \subset Y$. Hence $0 \neq IJ \subset Y$, and therefore $0 \neq IJ \subset Y^*$ which leads to a contradiction.

Put $Q = Y \cap R_e$. As a consequence of the preceding statement it follows that $Q$ is a prime ideal of $R_e$. From $\cap_{\sigma \in G} Q^\sigma = \cap_{\sigma \in G} R_{\sigma^{-1}} Q R_\sigma \subseteq \cap_{\sigma \in G} R_{\sigma^{-1}} Y R_\sigma \subset \cap_{\sigma \in G} R_{\sigma^{-1}} Y = Y^* = 0$, it then follows that $\cap_{\sigma \in G} Q^\sigma = 0$. For the second statement, consider a prime ideal $Q'$ minimal over $P \cap R_e$. From $Q' \supset \cap_{\sigma \in G} Q^\sigma$ we derive that $Q' \supset \prod_{\sigma \in G} Q^\sigma$ and hence $Q' \supset Q^\sigma$ for a certain $\sigma \in G$. However, $Q^\sigma$ is prime too, so by the minimality assumption on $Q$, we arrive at $Q' = Q^\sigma$. Consequently, $Q = (Q')^{\sigma^{-1}}$. Of course, $Q^\sigma \supset P \cap R_e$ and so it follows that $Q$ is a prime ideal minimal over $P \cap R_e$.

□

### 2.11.13 Corollary

If the almost strongly graded ring $R$ with respect to a finite group $G$ is semiprime, then $R_e$ is a semiprime ring.

### 2.11.14 Corollary

Assume that $R$ is strongly graded by a finite group $G$ and consider a prime ideal $Q$ of $R_e$. There exists a prime ideal $P$ of $R$ such that $P \cap R_e = Q$ if and only if $Q$ is $G$-invariant.

**Proof** Apply Corollary 2.11.9 and Theorem 2.11.12. □

### 2.11.15 Corollary

In the situation of Corollary 2.11.14, there exists a prime ideal $P$ of $R$ such that $Q$ is in the set of minimal primes over $P \cap R_e$.

**Proof** Put $A = \cap_{\sigma\in G} Q^\sigma$. Clearly $A$ is $G$-invariant and a $G$-prime ideal of $R_e$. Now, there exists a prime ideal $P$ of $R$ such that $P \cap R_e = A$. Clearly $Q$ is a minimal prime ideal over $P \cap R_e$.

## 2.12 Exercises

Let $G$ be a multiplicative group with identity element $e$. We recall that a left $G$-set is a non-empty set, say $A$, together with a left action $G \times A \to A$ of $G$ on $A$ given by $(\sigma, x) \to \sigma x$, such that $e.x = x$ and $(\sigma\tau)x = \sigma(\tau x)$ for every $\sigma, \tau \in G, x \in A$. If $A$ and $A'$ are left $G$-sets, then a map $\varphi : A \to A'$ is a morphism of $G$-sets if $\varphi(\sigma x) = \sigma\varphi(x)$ for every $\sigma \in G, x \in A$. We denote by $G$-SET the category of left $G$-sets. Analogously, we define the category of right $G$-sets denoted by SET-$G$.

1. Prove that $G$-SET is a category with coproducts and products.

2. If $A$ is a left $G$-set, we denote by $A^{\mathrm{op}}$ the right $G$-set where $A^{\mathrm{op}} = A$ (as the set) and the right action is defined by $x * \sigma = \sigma^{-1}x$. Prove that the correspondence $A \to A^{\mathrm{op}}$ is an isomorphism between categories $G$-SET and SET-$G$.

3. If $H \leq G$ is a subgroup of $G$, we denote by $G/H$ (resp. $G\backslash H$) the set of all left $H$-coset $\sigma H$, (resp. the set of all right $H$-coset $H\sigma$) with $\sigma \in G$. Prove that $G/H$ is a left $G$-set if we put $\tau(\sigma H) = (\tau\sigma)H$ for any $\tau \in G$. Prove that, if $G$ acts transitively on $A$ ((i.e. for any $x, y \in A$, there exists $\sigma \in G$, such that $y = \sigma x$), then $A$ is isomorphic with a $G$-set of the form $G/H$ for some subgroup $H$ of $G$.

4. Prove that in the category $G$-set every object is the coproduct of $G$-sets of the form $G/H$.

5. Let $R = \oplus_{\sigma\in G} R_\sigma$ be a $G$-graded ring and $A$ a left $G$-set.

   A (left) $A$-graded $R$-module is a left $R$-module $M$ such that $M = \oplus_{x\in A} M_x$, where every $M_x$ is an additive subgroup of $M$, and every $\sigma \in G$, $x \in A$ we have $R_\sigma M_x \subseteq M_{\sigma x}$. We can define the category $(G, A, R)$-gr of $A$-graded $R$-modules as follows : the objects are all $A$-graded $R$-modules; if $M = \oplus_{x\in A} M_x$ and $N = \oplus_{x\in A} N_x$ are $A$-graded left $R$-modules, the morphisms between $M$ and $N$ on the category $(G, A, R)$-gr are the $R$-linear $f : M \to N$ such that $f(M_x) \subseteq N_x$ for every $x \in A$ i.e.

   $$\mathrm{Hom}_{(G,A,R)-\mathrm{gr}}(M, N) = \{f \in \mathrm{Hom}_R(M, N) | f(M_x) \subseteq N_x, \forall x \in A\}$$

   If $H \leq G$ is a subgroup of $G$, we denote by $(G/H, R)$-gr the category $(G, G/H, R)$-gr.

   i) Prove if $A$ is singleton $(G, A, R)$-gr is just $R$-mod.

ii) Prove that $(G, A, R)$-gr is equivalent to a product of categories $\prod_{i\in I}(G/H_i, R)$-gr where $(H_i)_{i\in I}$ is a family of subgroups of $G$.

iii) Prove that $(G, A, R)$-gr is a Grothendieck category with a family of projective generators.

**Hint :** To show that $(G, A, R)$-gr is an abelian category, use the same argument as for the case when $A = G$. But, in order to prove that $(G, A, R)$-gr has a family of projective generators, we procede in the following way : if $R = \oplus_{\sigma\in G} R_\sigma$ is a $G$-graded ring, for each $x \in A$ we define the $x$-suspension $R(x)$ of $R$ to be the object of $(G, A, R)$-gr which coincides with $R$ as an $R$-module, but with the gradation defined by

$$R(x)_y = \oplus\{R_\sigma | \sigma \in G, \sigma x = y\} \text{ for } y \in A$$

The family $\{R(x), x \in A\}$ is a family of projective generators for the category $(G, A, R)$-gr.

6. Prove that if $M \in (G, A, R)$-gr then $M$ is a projective object if and only if $M$ is projective as left $R$-module.

   **Hint :** Apply exercise 5.

7. Let $A$ and $A'$ be $G$-sets and $\varphi : A \to A'$ a morphism of $G$-sets. To $\varphi$, we associate a canonical covariant functor $T_\varphi : (G, A, R)$-gr $\to (G, A', R)$-gr, defined as follows : $T_\varphi(M)$ is the $R$-module $M$ with an $A'$-gradation defined by

   $$M_{x'} = \oplus\{M_x | x \in A, \varphi(x) = x'\} \text{ for } x' \in A'$$

   where we put $M_{x'} = 0$ if $x' \notin \varphi(A)$. If $f \in \mathrm{Hom}_{(G,A,R)-\mathrm{gr}}(M, N)$, we put $T_\varphi(f) = f$. To prove that $T_\varphi$ has a right adjoint $S^\varphi$ and the latter is an exact functor. Moreover, if $\varphi^{-1}(x')$ is a finite set for all $x' \in A'$, then $S^\varphi$ is also a left adjoint for $T_\varphi$.

   **Hint :** The same proof as in section 2.5. (the first part).

8. If $A$ is a finite $G$-set prove that $Q \in (G, A, R)$-gr is an injective $R$-module.

9. Let $G$ be a group and $H \leq G$ a subgroup of $G$. If $R = \oplus_{\sigma\in G} R_\sigma$ is a $G$-graded ring we consider the category $(G/H, R)$-gr. We denote by $T_H : (G/H, R)$-gr $\to R_H$-mod (here $R_H = \oplus_{x\in H} R_x$) the exact functor $T_H(M) = M_H$ where $M = \oplus_{C\in G/H} M_C$ and if $f \in \mathrm{Hom}_{(G/H,R)-\mathrm{gr}}(M, N)$ then $T_H(f) = f/M_H : M_H \to N_H$.

   Prove that $T_H$ has a left adjoint functor $\mathrm{Ind}_H : R_H-\mathrm{mod} \to (G/H, R)$-gr, called the induced functor and also a right adjoint functor $\mathrm{Coind}_H : R_H$-mod $\to (G/H, R)$-gr, called the coinduced functor.

Prove also that when $R$ is a strongly graded ring, $\mathrm{Ind}_H$ is an equivalence of categories having $T_H$ for its inverse.

**Hint :** The construction of $\mathrm{Ind}_H$ is simple; if $N \in R_H$-mod then $\mathrm{Ind}_H(N) = R \otimes_{R_H} N$ with the $G/H$-gradation $\mathrm{Ind}_H(N) = \oplus_{G \in G_H}(\mathrm{Ind}_H(N))_C$ where $(\mathrm{Ind}_H(N))_C = R_C \otimes_{R_H} N$. For the rest we apply the techniques used in Section 2.5.

10. With notation as in exercise 9, put $\mathcal{C}_H = \{M = \oplus_{C \in G/H} M_C \in (G/H, R)$-gr, such that $M_H = 0\}$.

    i) Prove that $\mathcal{C}_H$ is a localizing subcategory of $(G/H, R)$-gr which is also stable under direct products. We denote by $t_H$ the radical associated to $\mathcal{C}_H$, i.e. if $M \in (G/H, R)$-gr, $t_H(M)$ is the sum of all subobjects of $M$ which belong to $\mathcal{C}_H$. Now if $N \in R_H$-mod we have $\mathrm{Ind}_H(N) = R \otimes_{R_H} N$. We denote by $R\overline{\otimes}_{R_H} N = R \otimes_{R_H} N / t_H(R \otimes_{R_H} N)$. So we obtain the functor

    $$R\overline{\otimes}_{R_H} - : R_H\text{−mod} \to (G/H, R)\text{−gr}$$

    ii) Prove the following assertions :

    a. If $N \in R_H$-mod is simple module then $R\overline{\otimes}_{R_H} N$ is a simple object in $(G/H, R)$-gr.

    b. If $\Sigma = \oplus_{C \in G/H} \Sigma_C$ is a simple object in $(G/H, R)$-gr such that $\Sigma_H \neq 0$ then prove that $\Sigma_H$ is simple in $R_H$-mod and in this case

    $$\Sigma \simeq R\overline{\otimes}_{R_H} \Sigma_H$$

    **Hint :** For i. see section 2.6. and for ii. use the same proof as in Section 2.7.

11. Let $R$ be a graded ring and $N \in R$-gr; $N$ is called gr-**flat** if the functor $- \otimes_R N : \mathrm{gr}\text{−}R \to \mathbb{Z}$-gr is exact. Prove that $N$ is gr-flat if and only if $N$ is flat in $R$-mod.

    **Hint :** The implication $\Leftarrow$ is obvious. Conversely if $N$ is gr-flat, as in the non-graded case $N$ is the inductive limit in $R$-gr of gr-free modules. The implication then follows easily.

12. If $M \in R$-gr we write gr-w.dim$_R M$ for the gr-**flat dimension** (defined as the corresponding ungraded concept which is denoted by w.dim$_R M$). Prove that gr.w.dim$_R M$ = w.dim$_R M$.

    **Hint :** Apply exercise 10.

13. Assume that $R = \sum_{\sigma \in G} R_\sigma$ is an almost strongly graded ring over a finite group $G$. Because $R_{\sigma^{-1}} R_\sigma = R_e$ for all $\sigma \in G$, there exist $a_i^\sigma \in R_{\sigma^{-1}}$, $b_i^\sigma \in R_\sigma$ such that for some finite set $T_\sigma$

    (1) $$1 = \sum_{i \in T_\sigma} a_i^\sigma b_i^\sigma$$

Assume also that $M, N \in R$-mod and $f \in \mathrm{Hom}_{R_e}(M, N)$. We define the map $\widetilde{f} : M \to N$ as follows :

$$\widetilde{f}(m) = \sum_{\sigma \in G} \sum_{i \in I_\sigma} a_i^\sigma f(b_i^\sigma m) \tag{2}$$

Then prove

i) $\widetilde{f}$ is an $R$-linear map

ii) Assume that $n = |G| < \infty$. Let $N \subset M$ be a submodule of $M$ such that $N$ is a direct summand of $M$ in $R_e$-mod. If $M$ has no $n$-torsion, prove there exists an $R$-submodule $P$ of $M$ such that $N \oplus P$ is essential in $M$ as an $R_e$-module. Furthermore, if $M = nM$, then $N$ is a direct summand of $M$ as $R$-module.

**Hint :** (Following the proof of Lemma 1. from [122]). We have $f : M \to N$ as $R_e$-modules such that $f(m) = m$ for all $m \in N$. Let $\widetilde{f} : M \to N$ be as in assertion 1. If $x \in N$, then $\widetilde{f}(x) = nx$. We put $P = \mathrm{Ker}\widetilde{f}$ and prove that $N \cap P = 0$ (since $M$ has no $n$-torsion) and $N \oplus P$ is essential in $M$ as an $R_e$-module. The last part of the exercise is clear.

iii) Assume that $N$ is an $R$-submodule of $M$ and $M$ has no $n$-torsion. Prove there exists an $R$-submodule $P \subset M$, such that $N \oplus P$ is essential on $M$ as $R_e$-submodule.

iv) If $M$ is semisimple as a left $R_e$-module and $M$ has $n$-torsion then $M$ is semisimple as $R$-module.

**Hint :** Apply 2.

v) If $R_e$ is a semisimple Artinian ring and $n$ is invertible in $R_e$ then $R$ is a semisimple Artinian ring.

(Compare to Section 3.5. for similar ideas).

14. Assume that $R$ is a ring with identity. Prove :

i) $R$ is semiprime if and only if has no nonzero nilpotent ideals.

ii) Assume that $R$ is a $G$-graded ring. Then $a \in \mathrm{rad}^g(R)$ and $a \in h(R) \Leftrightarrow a$ is **gr-strongly nilpotent element** i.e. for any sequence $a_0 = a$, $a_1 \in a_0 R a_0, \ldots, a_n \in a_{n-1} R a_{n-1}, \ldots$ where $a_0, a_1, \ldots a_n \in h(R)$ there is an $n \geq 0$ such that $a_n = 0$.

iii) $R$ is gr-semiprime if and only if $R$ has no nonzero nilpotent graded ideals.

**Hint :**

i) "⇒" is obvious. "⇐" Let $a \in \mathrm{rad}(R), a \neq 0, I = RaR \neq 0$. Then $I^2 \neq 0$, Since $I^2 = RaRaR$ then $aRa \neq 0 \Rightarrow$ there exists $a_1 \in aRa, a_1 \neq 0$. If we put $J = Ra_1R \neq 0$ then $J^2 \neq 0 \Rightarrow a_1Ra_1 \neq 0 \Rightarrow$ there exists $a_2 \in a_1Ra_1, a_2 \neq 0$. So by induction we obtain the sequence $a_0 = a, a_1 \in a_0R_{a_0} \ldots, a_n, \ldots$ such that $a_n \in a_{n-1}Ra_{n-1}$ for any $n \geq 1$ and $a_n \neq 0 (n \geq 0)$, contradiction.

ii) and iii. are similar to the non-graded case.

15. (Levitzki) If $R$ is a ring with identity and $I$ is a nonzero nil ideal of bounded index, there exists a nonzero ideal $J \subseteq I$ such that $J^2 = 0$.

**Hint** Assume that there exists $n > 1$, such that $x^n = 0$ for any $x \in I$. For element $x \in I$, we can assume that $x^n = 0$ and $x^{n-1} \neq 0$. If we can assume that $x^n = 0$ and $x^{n-1} \neq 0$. If we put $J = Rx^{n-1}$, then $J \neq 0$, $J \subseteq I$ and $J^2 = 0$. Indeed if $a \in R$, then $y = ax^{n-1} \in I$ and $yx = 0$. So $(y + x)^{n-1} = y^{n-1} + xy^{n-2} + \ldots + x^{n-2}y + x^{n-1}$. Hence $xy^{n-2} + \ldots + x^{n-2}y = (y + x)^{n-1} - y^{n-1} - x^{n-1} = (ax^{n-1} + x)^{n-1} + (ax^{n-1})^{n-1} - x^{n-1} = tx^{n-1}$ with $t \in I$. Since $0 = (y + x)^n = xy^{n-1} + x^2y^{n-2} + \ldots + x^{n-2}y^2 + x^{n-1}y$, then $tx^{n-1}y + x^{n-1}y = 0$ and therefore $(1 + t)x^{n-1}y = 0$. Since $t \in I$, $t$ is nilpotent and hence $1 + t$ is invertible. Hence $x^{n-1}y = 0$ and so $x^{n-1}ax^{n-1} = 0$. Consequently $(Rx^{n-1}R)^2 = 0$.

16. Let $R = \oplus_{\sigma \in G} R_\sigma$ be a graded ring of finite support. If $I$ is a nonzero graded ideal of $R$ such that it is a gr-nil ideal of bounded index (i.e. there exists an $n > 1$ such that $a^n = 0$, for anyhomogeneous element $a \in I$) then $R$ contains a nonzero nilpotent graded idal $J \subseteq I$.

**Hint** $I = \oplus_{\sigma \in G} I_\sigma$. If $I_e = 0$ then $I_e^s = 0$, where $s = |\mathrm{supp}(R)|$. If $I_e \neq 0$, by exercise 15. there exists $J_e \subset I_e, J_e \neq 0$ a nonzero nilpotent ideal of $R_e$. But $J = RJ_eR$ is a nilpotent nonzero graded ideal of $R$ and $J \subseteq I$.

The following exercises are related to **small objects** and **steady categories**. The notion of "small object" will also be used later in the text.

17. Let $A$ be an abelian category with arbitrary direct sums (i.e. an AB3 category Appendix) and $M$ an object of $A$. $M$ is called small when the functor $\mathrm{Hom}_A(M, -)$ preserves direct sums. Prove that the following assertions are equivalent :

i) $M$ is small.

ii) The functor $\mathrm{Hom}_A(A, -)$ preserves countable direct sums, i.e. any morphism $f : M \to \oplus_{i \in N} X_i$ factors through a finite subcoproduct $\oplus_{i \in F} X_i$ of $\oplus_{i \in N} X_i$ where $F$ is a finite subset of $N$.

iii) For any ascending chain $M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n \subseteq \ldots$ of proper subobjects of $M$, the direct union $\sum_{i\geq 1} M_i$ is a proper subobject of $M$.

**Hint :** For detail see [181], p. 134).

18. If $M \in R$-mod is countable generated then $M$ is small if and only if $M$ is finitely generated.

**Hint :** We apply assertion iii. of 17.

More generally we will say that $M$ is $X$-small (or small relative to $X$ where $X$ is an object of $A$ such that $\mathrm{Hom}_A(M, -)$ preserves (countable) direct sums of copies of $X$.

19. Let $M \in A$. Prove that $M$ is small $\Leftrightarrow$ $M$ is $X$-small for every object of $A$.

**Hint :** The necessity id obvious. To prove sufficiency, let $A = \oplus_{i\in I\!N} X_i$ and $f : M \to A$ a morphism in $A$. Let $\varepsilon_i : X_i \to A, i \in I\!N$, be the canonical injections. Then we have monomorphism $u = \oplus_{i\in I\!N}\varepsilon_i : A \to A^{(N)}$ and since $M$ is $A$-small, $\mathrm{Im}(u \circ f) \subset A^{(F)}$ for some finite subset $F$ of $I\!N$. This implies $\mathrm{Im}f \subseteq \oplus_{i\in F} X_i$.

20. Let $A$ and $B$ be AB3 categories and $M \in A, N \in B$. Let $F : A \to B$ and $G : B \to A$ be functors such that $F$ is left adjoint of $G$ and $G$ preserves direct sums.Prove the following assertions :

i) $M$ is $GN$-small in $A \Leftrightarrow FM$ is $N$-small in $B$.

ii) If $M$ is small in $A$ then $FM$ is small in $B$.

iii) If furthermore we assume that $G \circ F \simeq \mathrm{Id}_A$, then $M$ is small in $A \Leftrightarrow FM$ is small in $B$

**Hint :**

i) Since $G$ preserves coproducts, we have the canonical isomorphism $\mathrm{Hom}_B(FM, N^{I\!N}) \simeq \mathrm{Hom}_A(M, G(N^{I\!N})) \simeq \mathrm{Hom}_a(M, (G(N))^{(I\!N)})$ from which i. follows.

ii) and iii. follow from i. and exercise 19.

21. Let $R$ be a $G$-graded ring and $M, N \in R$-gr. Prove that the following assertions hold :

i) $M$ is small in $R$-gr if and only if $M$ is small in $R$-mod.

ii) $M$ is $N$-small in $R$-gr if and only if $M$ is $N$-small in $R$-mod.

iii) If $G$ is a finite group and $M \in R$-mod is a small object then $F(M)$ is small in the category $R$-gr. ($F : R$-mod$\to$ $R$-gr is the right adjoint of the forgetful functor $U : R$-gr $\to R$-mod (see Section 2.5).



**Hint :** We apply the foregoing exercise, Theorem 2.5.9 and Proposition 2.5.4.

A Grothendieck category $A$ is called a steady category if every small object of $A$ is finitely generated (every finitely generated object in $A$ is small). Recall that a Grothendieck category is called locally Noetherian when it has a set of Noetherian generators.

22. Prove that every locally Noetherian Grothendieck category is a steady category.

**Hint :** Let $M$ be a small object of $A$ and $X_1 \subseteq X_2 \subseteq \dots X_n \subseteq \dots$ an ascending chain of subobjects of $M$, with $X = \sum_{i\geq 1} X_i$. For any $k \in \mathbb{N}$ let $\pi_k : \prod_{n\geq 1} X/X_n \to X/X_k$ be the canonical projections and $\pi : X \to \prod_{n\geq 1} X/X_n$ the canonical morphism induced by the canonical projections $p_n : X \to X/X_n$. If we set $Y_n = \pi(X_n)$ then $\pi_k(Y_n) = p_k(X_k)$ and hence $\pi_k(Y_n) = 0$ for each $k \geq n$. Therefore we have $Y_n \subseteq \oplus_{n\geq 1}\pi(X_n) = \sum_{n\in\mathbb{N}} Y_n \subseteq \oplus_{n\in\mathbb{N}} X/X_n$. Now for each $n \in \mathbb{N}$ consider $v_n : X/X_n \to E(X/X_n)$ Since $A$ is a locally Noetherian category, $\oplus_{n\in\mathbb{N}} E(X/X_n)$ is an injective object of $A$ and so there exists a morphism $\overline{f} : M \to \oplus_{n\in\mathbb{N}} E(X/X_n)$ satisfying $\overline{f} \circ i = f$ where $i$ is the inclusion of $X$ in $M$. As $M$ is a small object of $A$, $f$ factors through a finite subcoproduct $\oplus_{n=1}^{k} E(X/X_n)$ and so we see that $X/X_n = 0$ for every $n \geq k$, i.e. the chain $(X_n)_{n\geq 1}$ is stationary and $M$ is Noetherian and hence, in particular $M$ is finitely generated.

23. Assume that $R$ is a $G$-graded ring. If $R$ is gr-Noetherian and $M \in R$-gr, then $M$ is small in $R$-gr if and only if $M$ is finitely generated.

**Hint :** The category $R$-gr is locally Noetherian and we can apply the exercise 20.

24. Let $R$ be a $G$-graded ring with $G$ a finite group. Prove that the category $R$-gr is steady if and only if the category $R$-mod is steady.

**Hint :** The implication "$\Longleftarrow$" follows from assertion i. of exercise 19. and "$\Longrightarrow$" follows from assertion iii. of exercise 21.

25. Let $R$ be a $G$-graded ring and $M, N \in R$-gr such that $M$ is $N$-gr-small (i.e. $M$ is $N$ small in the category $R$-gr). Prove that $\mathrm{HOM}_R(M,N) = \mathrm{Hom}_R(M,N)$.

**Hint :** If $\mathrm{Hom}_R(M,N)$ and $\sigma \in G$ we have the morphism $f_\sigma \in \mathrm{HOM}_R(M,N)_\sigma = \mathrm{Hom}_{R-\mathrm{gr}}(M, N(\sigma))$ defined by $f_\sigma(m_\lambda) = f(m_\lambda)_{\lambda\sigma}$ for every $\lambda \in G, m_\lambda \in M_\lambda$. Then $f = \sum_{\sigma\in G} f_\sigma$ i.e. the family $(f_\sigma)_{\sigma\in G}$ is sumable to $f$ in the finite topology (see Section 2.4). We can define a morphism $g \in \mathrm{Hom}_{R-\mathrm{gr}}(M, \oplus_{\sigma\in G} N(\sigma))$ by $g(x) = (f_\sigma(x))_{\sigma\in G}$ for every $x \in M$. Since $M$ is $N$ small in $R$-gr, then there are $\sigma_1, \dots, \sigma_n \in G$ such that $g(M) \subseteq \oplus_{i=1}^{n} N(\sigma_i)$. Therefore $f_\sigma = 0$ for every $\sigma \notin \{\sigma_1, \dots, \sigma_n\}$ and hence $f = \sum_{i=1}^{n} f_{\sigma_i}$, and therefore $f \in \mathrm{HOM}_R(M,N)$.

26. Let $R$ be a $G$-graded ring, where $G$ is an infinite group and let $M \in R\text{-gr}$. Prove that the following statements are equivalent :

    i) $M$ is small in $R$-gr (or $R$-mod).

    ii) $\mathrm{HOM}_R(M, N) = \mathrm{Hom}_R(M, N)$ for every $N \in R\text{-gr}$.

    **Hint :** The implication i. $\Longrightarrow$ ii. follows from exercise 25. For implication ii. $\Rightarrow i.$ the first step is to show that if $g \in \mathrm{Hom}_{R-\mathrm{gr}}(M, \oplus_{\sigma\in G} N(\sigma))$ there are some $\tau_1, \ldots, \tau_n \in G$ such that $\mathrm{Im} g \subseteq \oplus_{i=1}^{n} N(\tau_i)$. Let now $f : M \to \oplus_{i\in \mathbb{N}} X_i$ be a morphism in $R$-gr and let $A = \oplus_{i\in\mathbb{N}} X_i$ and $N = \oplus_{\sigma\in G} A(\sigma)$. Then $N$ is $G$-invariant i.e. $N(\sigma) \simeq N$ in $R$-gr for every $\sigma \in G$. Since $G$ is infinite, we may assume that $\mathbb{N}$ is a subset of $G$. In this case there is a monomorphism in $R$-gr :

$$v : N^{(\mathbb{N})} \to \oplus_{\sigma\in G} N(\sigma)$$

    If $u_1 : A(e) = A \to N$ and $\varepsilon_i : X_i \to A, i \in N$ be the canonical injection we set $u = \oplus_{i\in\mathbb{Z}}(u_1 \circ \varepsilon_i) : A \to N(\mathbb{N})$. We obtain a morphism in $R$-gr, $v \circ u \circ f : M \to \oplus_{\sigma\in G} N(\sigma)$ so $\mathrm{Im}(v \circ u \circ f) \subseteq \oplus_{i=1}^{n} N(\sigma_i)$ for some elements $\sigma_1, \ldots, \sigma_n \in G$. This implies that $\mathrm{im} f \subseteq \oplus_{i\in F} X_i$. where $F = \mathbb{N} \cap \{\sigma_1, \ldots, \sigma_n\}$ and therefore $M$ is small in $R$-gr.

## 2.13 Comments and References for Chapter 2

Focus in Chapter 2 is on the categorial language. For a group $G$ with identity element $e$, a $G$-graded ring $R$ has three module categories associated to it in a natural way : $R$-gr, $R$-mod and $R_e$-mod. The connections between these categories may be studied by considering various pairs of adjoint functors between them. For example, the forgetful functor $U : R\text{-gr} \to R\text{-mod}$ has a right adjoint $F : R\text{-mod} \to R\text{-gr}$. Between $R_e$-mod and $R$-gr we study the functors :

$$(-)_e : R\text{-gr} \to R_e\text{-mod}$$

$$\text{Ind} : R_e\text{-mod} \to R\text{-gr}$$

$$\text{Coind} : R_e\text{-mod} \to R\text{-gr}$$

The construction of these functors is inspired by Representation Groups Theory. The functor Ind, resp. Coind, is a left, resp. right, adjoint of the functor $(-)_e$; we study the relations between these functors in Section 2.5 and in Section 2.8. apply this to describe the structure of gr-injective modules. Up to a minor modification, Ind is applied in Section 2.7 in order to elucidate the structure of the simple objects of $R$-gr. Using a functorial isomorphism between the functors Ind and Coind, we provide a characterization for a graded ring $R$ to be strongly graded in Section 2.6. Moreover in Section 2.6. the methods from Torsion Theory are introduced.

The graded version of the Jacobson radical is as important in the theory of graded rings as its ungraded equivalent is in classical (noncommutative) algebra. For graded rings of finite support there is a stringent relation between the Jacobson radical and its graded version, cf. Corollary 2.9.4; a graded version of Hopkins' theorem (corollary 2.9.7) is obtained.

Section 2.10 deals with some theory about graded endomorphism rings; this leads to graded matrix rings and applications, e.g. to a graded version of Wedederburn's theorem (Theorem 2.10.10). We point out that Section 2.10 may be seen as a motivation for the study of the graded HOM functors in Section 2.4.

For a certain version of the local-global methods of commutative algebra the consideration of prime spectra is important, needless to say that prime and semiprime ideals have been cornerstones of most theories in Ring Theory. In Section 2.11, gr-prime ideals are being studied and the relation between the prime spectrum and the gr-prime spectrum of a graded ring is being studied. Here Theorem 2.11.4 and Theorem 2.11.12 may be viewed as main results. Finally, the exercises in Section 2.12 complete and clarify some of the results in this chapter.

## Some references

A broad area is covered in this chapter, we just mention some papers from the bibliography very related to the contents.

- M. Cohen, S. Montgomery [43], [44]
- M. Cohen, L. Rowen [42]
- E. Dade [49], [51], [52], [53], [55]
- S. Dascalescu, C. Năstăsescu, A. Del Rio, F. Van Oystaeyen [59]
- J.-L. Gomez Pardo, C. Năstăsescu [74], [75]
- J.-L. Gomez Pardo, G. Militaru, C. Năstăsescu, [77]
- M. Lorenz, D. Passman [122]
- C. Menini, C. Năstăsescu [130], [131], [132], [133]
- C. Năstăsescu, S. Raianu, F. Van Oystaeyen [160]
- C. Năstăsescu [140], [141], [143]
- G. Sjöding [179]
- F. Van Oystaeyen [189], [190], [193], [195]

# Chapter 3

# Modules over Strongly Graded Rings

## 3.1 Dade's Theorem

Recall that, for a $G$-graded ring $R$, we have the induction functor : Ind $= R \otimes_{R_e} (-) : R_e\text{-mod} \to R\text{-gr}$, appearing as a special case of the general induction functor in Section 2.5. To $M \in R_e$-mod there coresponds $R \otimes_{R_e} M$ graded by $(R \otimes_{R_e} M)_\sigma = R_\sigma \otimes_{R_e} M$, for $\sigma \in G$.

For any $\sigma \in G$ we define $(-)_\sigma : R\text{-gr} \to R_e\text{-mod}$ to be the functor given by $M \to M_\sigma$ and $f \to f_\sigma$ where for $f : M \to N$ in $R$-gr, we define $f_\sigma$ as $f|M_\sigma$. We have observed that the functor Ind is a left adjoint of the functor $(-)_e$, cf. Theorem 2.5.3., and moreover $(-)_e \circ \text{Ind} \cong \text{Id}_{R_e-\text{mod}}$.

The main result of this section provides us with a characterization of strongly graded rings in terms of equivalences of certain categories.

### 3.1.1 Theorem (E. Dade)

The following statements concerning the $G$-graded ring $R$ are equivalent :

1. $R$ is strongly graded
2. The induction functor Ind : $R_e$-mod $\to$ $R$-gr is an equivalence of categories
3. The functor $(-)_e : R\text{-gr} \to R_e\text{-mod}$ is an equivalence of categories.
4. For any $\sigma \in G$, the functor $(-)_\sigma : R\text{-gr} \to R_e\text{-mod}$ is an equivalence of cetegories.

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5. For $M = \oplus_{\sigma\in G} M_\sigma$ in $R$-gr to say that $M = 0$, is equivalent to $M_e = 0$ or to $M_\sigma = 0$ for some $\sigma \in G$.

**Proof** We have $(-)_\sigma = (-)_e \circ T_\sigma$ where $T_\sigma$ is the $\sigma$-suspension functor, therefore it is clear that equivalence of 3. and 4. follows from the fact that $T_\sigma$ is an isomorphism of categories. The implications 3. $\Rightarrow$ 5, and 4. $\Rightarrow$ 5. are obvious. The equivalence of 2. and 3. follows directly from $(-)_e \circ \mathrm{Ind} \cong \mathrm{Id}_{R_e-\mathrm{mod}}$. The implication 1. $\Rightarrow$ 2. is consequence of Proposition 2.6.2, indeed for $M \in R$-gr we have that the canonical morphism $\mu(M) : R \otimes_{R_e} M_e \to M$, $r \otimes m \mapsto rm$, for $r \in R$, $m \in M_e$, is a functorial isomorphism because $\mathrm{Ker}\mu(M) = \mathrm{Coker}\mu(M) = 0$. Finally, the assertion 5. $\Rightarrow$ 1. is a consequence of proposition 2.6.2. g. □

Recall that an $R'$-bimodule ${}_RM_R$ is said to be an **invertible $R$-bimodule** if there exists an $R$-bimodule ${}_RN_R$ such that $M \otimes_R N \cong R \cong N \otimes_R M$ as $R$-bimodules.

## 3.1.2 Corollary

For a $G$-strongly ring $R$ we have for every $\sigma, \tau$ in $G$, that the canonical morphisms :

$$f_{\sigma\tau} : R_\sigma \otimes_{R_e} R_\tau \to R_{\sigma\tau}, a \otimes b \mapsto ab$$

are isomorphisms of $R_e$-bimodules. In particular, $R_\sigma$ is an invertible $R_e$-bimodule for every $\sigma \in G$.

**Proof** Consider the graded $R$-module $R(\sigma)$, for $\sigma \in G$. Since $R(\tau)_e = R_\tau$, it follows from Theorem 3.1.1., that the canonical morphism $\mu(R(\tau)) : R \otimes_{R_e} R_\tau \to R(\tau)$, defined by $r \otimes b \mapsto rb$ for $r \in R$, $b \in R_\tau$, is an isomorphism in the category $R$-gr. Consequently, the restriction $\mu(R(\tau))_\sigma : R_\sigma \otimes_{R_e} R_\tau \to R(\tau)_\sigma = R_{\sigma\tau}$ must be an isomorphism of $R_e$-bimodules. That $f_{\sigma\tau} = \mu(R(\tau))_\sigma$ needs no explanation and the first part of the statement in the Corollary follows from the foregoing. Furthermore, note that $R_\sigma \otimes_{R_e} R_{\sigma^{-1}} \cong R_e = R_{\sigma^{-1}} \otimes_{R_e} R_\sigma$ as $R_e$-bimodules, hence $R_\sigma$ is an invertible $R_e$-bimodule with "inverse" in fact given by $R_{\sigma^{-1}}$ (we wrote "inverse" because it is not unique; this follows from the fact that we are considering objects up to isomorphism and so the term inverse should more correctly be applied to the isomorphism classes of $R_e$-bimodules). □

## 3.1.3 Corollary

Consider a right $R$-module $M$ and a graded left $R$-module $N$ over the strongly graded ring $R$. The canonical set map :

$$\alpha : M \otimes_{R_e} N_e \to M \otimes_R N, x \otimes_{R_e} y \mapsto x \otimes_R y$$

where $x \in M$, $y \in N_e$, is an additive group isomorphism.

**Proof** Consider the canonical map :

$$\mu(N) : R \otimes_{R_e} N_e \to N, r \otimes y \mapsto ry$$

for $r \in R, y \in N_e$. As a consequence of Theorem 3.1.1, $\mu(N)$ is an isomorphism of additive groups. Using the canonical isomorphism $M \cong M \otimes_R R$, we obtain $\alpha$ as the composition of the following isomorphisms :

$$M \otimes_{R_e} N_e \simeq (M \otimes_R R) \otimes_{R_e} N_e \cong M \otimes_R (R \otimes_{R_e} N_e) = M \otimes_R N$$

and therefore $\alpha$ is an isomorphism. □

Again, let $R$ be strongly graded by $G$. Consider an $R_e$-module $N$. Since $R_\sigma$ is an $R_e$-bimodule it makes sense to define $G\{N\} = \{\sigma \in G, R_\sigma \otimes_{R_e} N \cong N\}$. That $G\{N\}$ is a subgroup of $G$ follows from Corollary 3.1.3. We call $G\{N\}$ the **stabilizer- (or inertia)** subgroup for $N$.

An $N \in R_e$-mod is said to be $G$**-invariant** whenever $G\{N\} = G$.

### 3.1.4 Corollary

With notation as above, put $M = R \otimes_{R_e} N \in R$-gr. If $R$ is strongly graded then the stabilizer group for $M$ in $R$-gr equals the stabilizer group of $N$ in $R_e$-mod.

**Proof** An immediate consequence of Theorem 3.1.1. □

### 3.1.5 Corollary

If $R$ is strongly graded then for every $\sigma \in G$, the functor :

$$R_\sigma \otimes_{R_e} - : R_e\text{−mod} \to R_e\text{−mod}, X \mapsto R_\sigma \otimes_{R_e} X$$

for $X \in R_e$-mod, is an equivalence of categories.

**Proof** An easy consequence of Corollary 3.1.2. □

## 3.2 Graded Rings with $R$-gr Equivalent to $R_e$-mod

If we have an equivalence between categories for $R$-gr and $R_e$-mod, but one that is not necessarily given by the functor Ind, does it follow that $R$ is necessarily strongly graded ? This does not hold in general, but we present some positive results in particular cases.

First some general definitions and notation. Let $\mathcal{A}$ be an abelian category. Denote by $\Omega_{\mathcal{A}}$ the set of isomorphism classes of simple objects of $\mathcal{A}$; for a simple object $S$ of $\mathcal{A}$ we let $[S]$ denote its isomorphism class. In case $\mathcal{A} = A$-mod we write $\Omega_A$ instead of $\Omega_{A-\mathrm{mod}}$.

### 3.2.1 Theorem

Suppose that $F : R\text{-gr} \to R_e\text{-mod}$ is an equivalence of categories. If $\Omega_{R_e}$ is a finite set then $R$ is strongly graded.

**Proof** As a consequence of assertion 5. in Theorem 3.1.1. it will suffice to show that $\mathcal{C}_e = 0$. Assume $\mathcal{C}_e \neq 0$. From Theorem 2.7.2, it follows that there is a bijective correspondence between $\Omega_{R_e}$ and the subset of all $[S]$ in $\Omega_{R-\mathrm{gr}}$ where $S$ is gr-simple and $e$-faithful. The hypothesis about $F$ yields that $\Omega_{R_e}$ and $\Omega_{R-\mathrm{gr}}$ correspond bijectively too, hence in view of the foregoing it follows that every gr-simple in $R$-gr is $e$-faithful. Consequently $\Omega_{\mathcal{C}_e} = \emptyset$. On the other hand, since $\mathcal{C}_e \neq 0$ there must be a nonzero $M$ in $\mathcal{C}_e$ and $M$ is finitely generated. There exists a maximal gr-submodule $N$ of $M$ with $N \neq M$ (Zorn's lemma). We then obtain a gr-simple $\Sigma = M/N$ in $\mathcal{C}_e$ and that is a contradiction. □

The set $\Omega_{R_e}$ will be examined further in Chapter 4, e.g. Theorem 4.2.5 and Section 4.3.

### 3.2.2 Corollary

Let $F : R\text{-gr} \to R_e\text{-mod}$ be an equivalence. If $R_e$ is a semilocal ring, then $R$ is strongly graded (recall that a ring $A$ is semilocal whenever $A/J(A)$ is a semisimple Artinian ring).

**Proof** If $R_e$ is semilocal then $\Omega_{R_e}$ is a finite set. □

The theorem may be slightly extended by allowing an equivalence between $R$-gr and any module category $A$-mod.

### 3.2.3 Corollary

If there exists a ring $A$ such that $|\Omega_A| = 1$ and $R$-gr becomes equivalent to $A$-mod, then $R$ is strongly graded.

**Proof** Since $|\Omega_{R-\mathrm{gr}}| = |\Omega_A|$, the same argument as in the proof of theorem 3.2.1 may be used to derive that either $\mathcal{C}_e = 0$ or $\mathcal{C}_e = R$-gr. In case $\mathcal{C}_e = R$-gr we have $M(\sigma) \in \mathcal{C}_e$ for any nonzero $M \in R$-gr and $\sigma \in G$. Thus $M(\sigma)_e = M_\sigma = 0$ for all $\sigma \in G$, hence $M = 0$ and that is a contradiction. So we must have $\mathcal{C}_e = 0$ and it follows that $R$ is strongly graded. □

To end this paragraph, we provide an example of a graded ring that is not strongly graded but nevertheless the categories $R$-gr and $R_e$-mod being equivalent.

### 3.2.4 Example

Let $k$ be a field and $I$ an infinite set. We consider the direct product $R = k^I$, $R$ is a ring. Since $I$ is infinite $R \simeq R \times R$ as rings. We consider the group $G = \{1, g\}$ where $g^2 = 1$. On $R$ we consider the trivial gradation i.e. $R_1 = R$ and $R_g = 0$. Clearly $R$-gr $\simeq R$-mod $\times R$-mod. Since $R \simeq R \times R$ then $R$-gr is equivalent with the category $R_e$-mod. But it is clear that $R$ is not a strongly graded ring. Of course, the ring $R = k^I$ with $I$ infinite is not a semi-local ring.

## 3.3 Strongly Graded Rings over a Local Ring

The main result of this paragraph is the following .

### 3.3.1 Theorem

Let $R = \oplus_{\lambda \in G} R_\lambda$ be a strongly graded ring. If $R_e$ is a local ring (i.e. $R_e/J(R_e)$ is a simple Artinian ring), then $R$ is a crossed product.

**Proof** We consider the graded ring $R/J^g(R)$, which is also strongly graded. By Corollary 2.9.3 $J^g(R) \cap R_e = J(R_e)$ therefore $(R/J^g(R))_e = R_e/J(R_e)$. Using Proposition 2.9.1 assertion 6. we may suppose $J^g(R) = 0$. In this case $R_e$ is simple and Artinian. Obviously, any strongly graded ring as before is gr-uniformly simple, hence by Corollary 3.1.5 for any $\sigma \in G$, the left $R_e$-module $R_\sigma$ is isomorphic to $R_e$. Let $f : R_e \to R_\sigma$ be such an isomorphism. If $u_\sigma = f(1)$ then $R_\sigma = f(R_e) = R_e u_\sigma$ an $f(a) = au_\sigma, a \in R_e$. Similarly, taking the right $R_e$-module $R_{\sigma^{-1}}$ we find an element $v_\sigma \in R_{\sigma^{-1}}$ such that $v_\sigma R_e = R_{\sigma^{-1}}$. Since $R_{\sigma^{-1}}.R_\sigma = R_e$, there exists $a \in R_e$, such that $1 = v_\sigma a u_\sigma$ so $w_\sigma = v_\sigma a \in R_{\sigma^{-1}}$ is a left inverse of $u_\sigma$. From $u_\sigma w_\sigma u_\sigma = u_\sigma$ we obtain $(u_\sigma w_\sigma - 1)u_\sigma = 0$ and $u_\sigma w_\sigma - 1 \in \mathrm{Ker} f = 0$. Hence $u_\sigma w_\sigma = 1$ and therefore $u_\sigma$ is invertible for all $\sigma$, so it follows also that $R$ is a crossed product.

**Another proof :** Since $R_\sigma \simeq R$ in as left $R_e$-modules ($R$ is strongly graded) we have $R(\sigma) \simeq R$ in $R$-gr so ${}_R R$ is $G$-invariant. By Theorem 2.10.2 it follows that $R$ is a crossed product. □

The foregoing result has applications in the theory of the Brauer group of a commutative ring where the crossed product structure has importance.

### 3.3.2 Example

If $R_e$ is not a local ring then $R$ is not necessarily a crossed product, as the following example shows.

Let $K$ be a field and $R$ be the ring of $3 \times 3$ matrices over $K$. We define a $\mathbb{Z}_2$-grading on $R$ as follows :

$$R_{\widehat{0}} = \begin{pmatrix} K & K & O \\ K & K & O \\ O & O & K \end{pmatrix}; R_{\widehat{1}} = \begin{pmatrix} O & O & K \\ O & O & K \\ K & K & K \end{pmatrix}$$

Clearly, $R_{\widehat{1}}R_{\widehat{1}} = R_{\widehat{0}}$, hence $R$ is a strongly graded ring, which is not a crossed product (there is no invertible element in $R_{\widehat{1}}$). Also $R_{\widehat{0}} \simeq M_2(K) \times K$, $R_{\widehat{0}}$ is not a local ring. We observe that $R$ is a gr-simple ring, but $R_{\widehat{0}}$ is not a simple ring (in fact it is only semi-simple).

## 3.4 Endomorphism $G$-Rings

For rings $S \subset T$ we let the centralizer of $S$ in $T$ be $C_T(S) = \{t \in T, st = ts$ for every $s \in S\}$. Clearly $C_T(S)$ is a subring of $T$ and $S \cap C_T(S) = Z(S)$, the center of $S$.

In particular if $R$ is a $G$-graded ring then $C_R(R_e)$ is a graded subring of $R$.

In general an abelian group $A$ is called a $G$-module for the group $G$ if there is a group morphism $\phi : G \to \mathrm{Aut}(A)$. When $A$ is a ring and $\mathrm{Aut}(A)$ is the group of all ring automorphisms of $A$ then we say that $A$ is a $G$-ring if a $\phi$ as before exists. We denote $\phi(g)(a)$ by $ga$ for $g \in G$ and $a \in A$. An abelian group $A$ is a (left) $G$-module exactly when $A$ is a $\mathbb{Z}[G]$-module, where $\mathbb{Z}[G]$ is the groupring of $G$ over $\mathbb{Z}$.

First we introduce some general terminology and notation. For an arbitrary ring $S$ and $R - S$-bimodules ${}_RM_S$ and ${}_RN_S$ we let $\mathrm{Hom}_{R,S}(M, N)$ denote the group of bimodule morphisms. In case $M = N$, $\mathrm{Hom}_{R,S}(M, M)$ will be denoted by $\mathrm{End}_{R,S}(M)$. In the sequel of this section $R$ will be strongly graded by $G$. An $R - S$-bimodule is also an $R_e - S$-bimodule and we have an inclusion $\mathrm{Hom}_{R,S}(M, N) \hookrightarrow \mathrm{Hom}_{R_e,S}(M.N)$. The fact that $R$ is also an $R_e - R$-bimodule in the obvious way entails that $\mathrm{End}_{R_e,R}(R) \cong C_R(R_e)$.

### 3.4.1 Theorem (Miyashita)

With notation and terminology as above, consider $R-S$-bimodules $M, N$ and $P$, then the following assertions hold :

1. For any $\sigma \in G$ and $f \in \mathrm{Hom}_{R_e,S}(M, N)$ there exists a unique $f^\sigma \in \mathrm{Hom}_{R_e,S}(M, N)$ such that $f^\sigma(\lambda_\sigma x) = \lambda_\sigma f(x)$ for any $x \in M$ and $\lambda_\sigma \in R_\sigma$.

2. The map $(\sigma, f) \mapsto f^\sigma$ is an action of the group $G$ on the additive group $\mathrm{Hom}_{R_e,S}(M,N)$.

3. For any subgroup $H$ of $G$, we have $\mathrm{Hom}_{R_e,S}(M,N)^H = \mathrm{Hom}_{R_H,S}(M,N)$. In particular $\mathrm{Hom}_{R_e,S}(M,N)^G = \mathrm{Hom}_{R,S}(M,N)$.

4. For any $\sigma \in G$, $f \in \mathrm{Hom}_{R_e,S}(M,N)$, and $g \in \mathrm{Hom}_{R_e,S}(N,P)$, we have $(g \circ f)^\sigma = g^\sigma \circ f^\sigma$. In particular, $\mathrm{End}_{R_e,S}(M)$ is a $G$-ring.

**Proof**

1. Since $R_\sigma R_{\sigma^{-1}} = R_e$, there exist elements $a_1, \ldots, a_n \in R_\sigma$ and $b_1, \ldots, b_n \in R_{\sigma^{-1}}$ such that $\sum_{i=1,n} a_i b_i = 1$. If $f^\sigma(\lambda_\sigma x) = \lambda_\sigma f(x)$ for any $x \in M$ and $\lambda_\sigma \in R_\sigma$, then we have

$$f^\sigma(m) = f^\sigma(\sum_{i=1,n} a_i b_i m) = \sum_{i=1,n} a_i f(b_i m)$$

therefore $f^\sigma$ is uniquely determined by $f$ and $\sigma$. Hence we define $f^\sigma$ by $f^\sigma(m) = \sum_{i=1,n} a_i f(b_i m)$. If $\lambda_\sigma \in R_\sigma$, then we have

$$\begin{aligned} f^\sigma(\lambda_\sigma m) &= \sum_{i=1,n} a_i f(b_i \lambda_\sigma m) \\ &= \sum_{i=1,n} a_i b_i \lambda_\sigma f(m) \\ &= \lambda_\sigma f(m) \end{aligned}$$

since $b_i \lambda_\sigma \in R_e$ for any $i = 1, \ldots, n$.

We show now that $f^\sigma \in \mathrm{Hom}_{R_e,S}(M,N)$. Indeed, if $\lambda \in R_e$, then :

$$f^\sigma(\lambda \lambda_\sigma m) = \lambda \lambda_\sigma f(m) = \lambda f^\sigma(\lambda_\sigma m)$$

Since $R_\sigma M = M$, we obtain for $x \in M$, that $x = \sum_{i=1,s} \lambda_\sigma^i m_i$. Hence :

$$\begin{aligned} f^\sigma(\lambda x) &= f^\sigma(\lambda \sum_{i=1,s} \lambda_\sigma^i m_i) \\ &= \lambda \sum_{i=1,s} f^\sigma(\lambda_\sigma^i m_i) \\ &= \lambda f^\sigma(\sum_{i=1,s} \lambda_\sigma^i m_i) \\ &= \lambda f^\sigma(x) \end{aligned}$$

which shows that $f^\sigma$ is a morphism of $R_e$-modules. The fact that $f^\sigma$ is an $S$-morphism is clear.

2. If $\sigma = e$, we clearly have $f^e = 1$. Pick $\sigma, \tau \in G$. If $\lambda_\sigma \in R_\sigma, \lambda_\tau \in R_\tau$, we have that $f^{\sigma\tau}(\lambda_\sigma\lambda_\tau m) = \lambda_\sigma\lambda_\tau f(m)$, and $(f^\tau)^\sigma(\lambda_\sigma\lambda_\tau m) = \lambda_\sigma f^\tau(\lambda_\tau m) = \lambda_\sigma\lambda_\tau f(m)$ Since $R_\sigma R_\tau = R_{\sigma\tau}$, we obtain that $f^{\sigma\tau} = (f^\tau)^\sigma$, i.e. $(\sigma\tau)f = \sigma(\tau f)$, therefore the group $\mathrm{Hom}_{R_e,S}(M,N)$ is a $G$-module.

3. If $f \in \mathrm{Hom}_{R_e,S}(M,N)^H$, we have $f^\sigma = f$ for any $\sigma \in H$. Hence if $\lambda_\sigma \in R_\sigma$ with $\sigma \in H$, then $f(\lambda_\sigma m) = f^\sigma(\lambda_\sigma m) = \lambda_\sigma f(m)$ showing that $f$ is an $R_H$-morphism. The converse is obvious.

4. If $\lambda_\sigma \in R_\sigma$ and $m \in M$, we have that

$$\begin{aligned}(g \circ f)^\sigma(\lambda_\sigma m) &= \lambda_\sigma(g \circ f)(m)\\ &= \lambda_\sigma g(f)(m)\\ &= g^\sigma(\lambda_\sigma f(m))\\ &= g^\sigma(f^\sigma(\lambda_\sigma m))\\ &= (g^\sigma \circ f^\sigma)(\lambda_\sigma m)\end{aligned}$$

so $(g \circ f)^\sigma = g^\sigma \circ f^\sigma$. □

### 3.4.2 Remark

1. The map $f^\sigma$ has the following property : $\lambda_{\sigma^{-1}} f^\sigma(x) = f(\lambda_{\sigma^{-1}} x)$ for all $\lambda_{\sigma^{-1}} \in R_{\sigma^{-1}}$ and $x \in M$, which follows immediatelly from part 1. of the previous theorem. Indeed, if we write $1 = \sum_i a_i b_i$ with $a_i \in R_{\sigma^{-1}}$ and $b_i \in R_\sigma$, then

$$\begin{aligned}f(\lambda_{\sigma^{-1}} x) &= \sum_i a_i b_i f(\lambda_{\sigma^{-1}} x)\\ &= \sum_i a_i f^\sigma(b_i \lambda_{\sigma^{-1}} x)\\ &= \sum_i a_i b_i \lambda_{\sigma^{-1}} f^\sigma(x)\\ &= \lambda_{\sigma^{-1}} f^\sigma(x)\end{aligned}$$

2. If $A$ and $B$ are two $G$-modules, then by Theorem 3.4.1 it follows that the abelian group $\mathrm{Hom}_{\mathbb{Z}}(A,B)$ is a $G$-module. Indeed, if $f \in \mathrm{Hom}_{\mathbf{Z}}(A,B)$ and $\sigma \in G$, then $(\sigma f)(x) = \sigma f(\sigma^{-1}x)$. Thus, we obtain the canonical functor

$$\begin{aligned}\mathrm{Hom}_{R_e}(\cdot,\cdot) : (R-\mathrm{mod})^o \times R-\mathrm{mod} &\to \mathbb{Z}[G]-\mathrm{mod},\\ (M,N) &\mapsto \mathrm{Hom}_{R_e}(M,N), \quad M,N \in R-\mathrm{mod}\end{aligned}$$

### 3.4.3 Corollary

The ring $C_R(R_e)$ is a $G$-ring with the $G$-action defined as follows : if $\sigma \in G$, then $\alpha_\sigma : C_R(R_e) \to C_R(R_e)$, $\alpha_\sigma(z) = \sum_{i=1,n} a_i z b_i$, is an automorphism, where $a_i \in R_{\sigma^{-1}}$ and $b_i \in R_\sigma$ are such that $\sum_{i=1,n} a_i b_i = 1$. Moreover, $C_R(R_e)^G \simeq Z(R)$. In particular $Z(R_e)$ is a $G$-ring: if $c \in Z(R_e)$ and $\sigma \in G$, then $\sigma c \in Z(R_e)$ is the element with the property that $(\sigma c)\lambda_\sigma = \lambda_\sigma c$ for any $\lambda_\sigma \in R_\sigma$.

**Proof** We have that $C_R(R_e) \simeq \mathrm{End}_{R_e,R}(R)$. If $z \in C_R(R_e)$, we define

$$\phi_z \in \mathrm{End}_{R_e,R}(R) \text{ by } \phi_z(\lambda) = z\lambda \text{ for any } \lambda \in R$$

Then

$$\begin{aligned} \phi_z^\sigma(\lambda) &= \sum_{i=1,n} a_i \phi_z(b_i \lambda) \\ &= \sum_{i=1,n} a_i z b_i \lambda \end{aligned}$$

The element of $C_R(R_e)$ associated to $\phi_z^\sigma(\lambda)$ is $\phi_z^\sigma(1) = \sum_{i=1,n} a_i z b_i$. Hence $\sigma z = \sum_{i=1,n} a_i z b_i$, i.e. $\phi_z^\sigma = \sigma z$ is an automorphism. Since $C_R(R_e) \simeq \mathrm{End}_{R_e,R}(R)$, we have that

$$C_R(R_e)^G \simeq \mathrm{End}_{R_e,R}(R)^G \simeq \mathrm{End}_{R,R}(R) \simeq Z(R)$$

Now if $z \in C_R(R_e)$, then $\sigma z \lambda_\sigma = \lambda_\sigma z$. Indeed, we have

$$\begin{aligned} \sigma z \lambda_\sigma &= \sum_{i=1,n} a_i z b_i \lambda_\sigma \\ &= \sum_{i=1,n} a_i b_i \lambda_\sigma z \\ &= \lambda_\sigma z \end{aligned}$$

□

### 3.4.4 Remark

Let ${}_RM_R$ be an $R$-bimodule. If $a \in C_R(R_e)$, then $\phi_a : M \to M, \phi_a(x) = ax$, is an $R_e - R$-bimodule morphism. If $\sigma \in G$, then $\phi_a^\sigma = \phi_{\sigma a}$. Indeed, we have

$$\begin{aligned} \phi_a^\sigma(\lambda_\sigma x) &= \lambda_\sigma \phi_a(x) = \lambda_\sigma a x \\ &= \sigma a \lambda_\sigma x \\ &= \phi_{\sigma a}(\lambda_\sigma x) \end{aligned}$$

therefore since $R_\sigma M = M$ we obtain that $\phi_a^\sigma = \phi_{\sigma a}$. □

If $H$ is a subgroup of $G$ of index $s$, let $(\sigma_i)_{i=1,s}$ be a system of representatives for the left $H$-cosets of $G$. Thus $G = \cup_{i=1,s}\sigma_i H$, a disjoint union. For any $R-S$-bimodules $M$ and $N$, we define the trace map

$$t_H^G : \mathrm{Hom}_{R_H,S}(M,N) \to \mathrm{Hom}_{R,S}(M,N) \text{by } t_H^G(f) = \sum_{i=1,s} f^{\sigma_i} = \sum_{i=1,s} \sigma_i f$$

We will write $t_H$ instead of $t_H^G$ whenever there is no danger of confusion. The definition of $t_H$ does not depend on the choice of the elements $\sigma_1, \ldots, \sigma_s$. Indeed, if $\sigma_i H = \sigma_i' H$ for any $i = 1, \ldots, s$, then we have $\sigma_i = \sigma_i' h_i$ for some $h_i \in H$ for any $i$. Then

$$t_H(f) = \sum_{i=1,s} f^{\sigma_i} = \sum_{i=1,s} f^{\sigma_i' h_i} = \sum_{i=1,s} (f^{h_i})^{\sigma_i'} = \sum_{i=1,s} f^{\sigma_i'}$$

since $f^{h_i} = f$ for any $i = 1, \ldots, s$ and $f \in \mathrm{Hom}_{R_H,S}(M,N)$.

We show now that $t_H(f) \in \mathrm{Hom}_{R,S}(M,N)$. Indeed, if $\sigma \in G$, we have that

$$t_H(f)^\sigma = (\sum_{i=1,s} f^{\sigma_i})^\sigma = \sum_{i=1,s} f^{\sigma\sigma_i}$$

But $\sigma\sigma_i \in \cup_{j=1,s}\sigma_j H$, so

$$\{\ \sigma_i H \mid i = 1, \ldots, s\ \} = \{\ \sigma\sigma_i H \mid i = 1, \ldots, s\ \}$$

showing that $t_H(f)^\sigma = t_H(f)$. In particular, if $M = N$, we have the application

$$t_H : \mathrm{End}_{R^{(H)},S}(M) \to \mathrm{End}_{R,S}(M)$$

and $Im(t_H)$ is an ideal of the ring $End_{R,S}(M)$. Indeed, if $g, h \in End_{R,S}(M)$, then

$$g \circ t_H(f) = g \circ \sum_{i=1,s} f^{\sigma_i} = \sum_{i=1,s} g \circ f^{\sigma_i} = \sum_{i=1,s} g^{\sigma_i} \circ f^{\sigma_i} = \sum_{i=1,s} (g \circ f)^{\sigma_i} = t_H(g \circ f)$$

Analogously we have $l_H(f) \circ h = l_H(f \circ h)$ When $H = \{\ 1\ \}$ and $G$ is finite, we denote $t_{\{1\}}^G$ by $t^G$.

## 3.5 The Maschke Theorem for Strongly Graded Rings

Throughout this section $R = \oplus_{\sigma \in G} R_\sigma$ will be a strongly graded ring, and $H$ a subgroup of $G$ of finite index $s$. We denote by $(\sigma_i)_{i=1,s}$ a system of representatives for the left $H$-cosets of $G$.

### 3.5.1 Theorem

Let $M$ be a left $R$-module having no $s$-torsion, and $N$ an $R$-submodule of $M$. If $N$ is a direct summand of $M$ as an $R_H$-module, then there exists an $R$-submodule $P$ of $M$ such that $N \oplus P$ is essential in $M$ as an $R_H$-submodule. Furthermore, if $sM = M$ (for example if $s$ is invertible in $R$), then $N$ is a direct summand of $M$ as an $R$-module.

**Proof** We know that there exists a morphism $f : M \to N$ of $R_H$-modules such that $f \circ i = 1_N$, where $i : N \to M$ is the inclusion map. Let $\tilde{f} = t_G(f) = \sum_{i=1,s} f^{\sigma_i}$ We calculate :

$$\widetilde{f} \circ i = \sum_{i=1,s} (f^{\sigma_i} \circ i) = \sum_{i=1,s} (f^{\sigma_i} \circ i^{\sigma_i}) = \sum_{i=1,s} (f \circ i)^{\sigma_i} = \sum_{i=1,s} (1_N)^{\sigma_i} = s \cdot 1_N$$

Therefore we have $\tilde{f}(x) = sx$ for any $x \in N$. Put $P = Ker(\tilde{f})$. If $x \in N \cap P$, then $sx = 0$, and since $M$ has no $s$-torsion we obtain that $x = 0$. Thus $N \cap P = 0$. We show now that $N \oplus P$ is essential in $M$ as an $R_H$-module. Let $x \in M$, and $y = \tilde{f}(x) \in N$. Then :

$$\tilde{f}(sx) = s\tilde{f}(x) = sy = \tilde{f}(y)$$

so $\tilde{f}(sx - y) = 0$ and $sx - y \in P$. This shows that $sx \in N \oplus P$. Thus $sM \subseteq N \oplus P$, and $N \oplus P$ is an essential $R_H$-submodule of $M$. The second part is clear. □

### 3.5.2 Corollary

Let $M \in R$-mod such that $M$ is semisimple as an $R_H$-module. If $s$ is invertible in $R$, then $M$ is semisimple as an $R$-module.

**Proof** Let $N$ be an $R$-submodule of $M$. Since $M$ is semisimple as an $R_H$-module, $N$ is a direct summand in $M$ as an $R_H$-submodule. Now Theorem 3.51. entails that $N$ is a direct summand of $M$ as an $R$-submodule. □

### 3.5.3 Corollary

If $R_H$ is a semisimple ring and $s$ is invertible in $R$, then $R$ is a semisimple ring.

The foregoing results generalize the classical result due to Maschke, stating that for a finite group $G$ with $n = |G|$ being invertible in the ring $A$, the groupring $A[G]$ is semisimple whenever $A$ is semisimple.

### 3.5.4 Remarks

1. For $A[G] = R$ with $n = |G| < \infty$, the invertibility of $n$ in $A$ follows from the semisimplicity of $R$. To prove this, look at $\sigma \in G$ and assume $\sigma$ is of order $m$ in $G$, i.e. $\sigma^m = e$. We claim that the left annihilator of $1-\sigma$ in $R$ is exactly $R.(1+\sigma+\ldots+\sigma^{m-1})$ where we have written 1 again for the unit $1_A.e$ of $R$. Indeed from $(\Sigma_{\tau\in G} r_\tau \tau)(1-\sigma) = 0$ it follows that $r_\tau = r_{\tau\sigma^{-1}}$ for all $\tau \in G$, hence $r_\tau = r_{\tau\sigma} = r_{\tau\sigma^2} = \ldots = r_{\tau\sigma^{m-1}}$ (note : $\sigma^{-1} = \sigma^{m-1}$) for all $\tau$ in $G$. Since $m$ divides $n$ we have established that $\sum_{\tau\in G} r_\tau \tau = r'(1+\sigma+\ldots+\sigma^{m-1})$ for some $r' \in R$.

   The semisimplicty of $R$ entails that $R$ is so-called von Neumann regular, cf. [24], hence there exists an $r \in R$ such that $(1-\sigma)r(1-\sigma) = 1-\sigma$, hence $(1-(1-\sigma)r)(1-\sigma) = 0$. The foregoing then yields that $1-(1-\sigma)r = r'(1+\sigma+\ldots+\sigma^{m-1})$ for some $r' \in R$. Applying the augmentation $\varepsilon : A[G] \to A$, $g \mapsto 1$, to the foregoing relation yields : $1 = \varepsilon(r')m$. This proves already that the order of any $\sigma \in G$ is invertible in $A$, hence the exponent of $G$ is invertible in $A$. However $n$ and the exponent of $G$ have the same prime factors, therefore also $n$ is invertible in $A$.

2. Recall that if $R = A[G]$ is left Artinian then $|G| < \infty$.

3. For a strongly graded ring $R$ with $G$ finite, the semisimplicity of $R$ does not necessarily yield that $|G|$ is invertible in $R$. As an example one may consider the well-known $\mathbb{Z}/n\mathbb{Z}$-gradation on the rational function field $k(X)$, where $k$ is any field, given by $k(X)_{\hat{o}} = k(X^n)$ and $k(X)_{\hat{i}} = X^i k(X^n)$. Obviously this gradation is a strong gradation, $k(X)$ is a field thus certainly semisimple Artinian and if $\mathrm{char} k = n \neq 0$, which we are free to consider, $n$ is not invertible in $k$.

   We return to the consideration of a strongly graded ring $R$ with respect to a finite group $G$. For an $R_e$-submodule $N$ in an $R$-submodule $M$, we define $N^* = \cap_{\sigma\in G} R_\sigma N$.

### 3.5.5 Lemma

$N^*$ is an $R$-submodule of $M$. Moreover, if $N$ is an essential $R_e$-submodule of $M$, then so is $N^*$.

**Proof** For any $\lambda \in G$ we have $R_\lambda N^* \subseteq \cap_{\sigma\in G} R_\lambda R_\sigma N = \cap_{\sigma\in G} R_{\lambda\sigma} N = N^*$.

Clearly $N^*$ is the largest $R$-submodule of $M$ contained in $N$.

Assume now that $N$ is an essential $R_e$-submodule of $M$. Let $X$ be a non-zero $R_e$-submodule of $M$. Then $R_{\sigma^{-1}} X \neq 0$ since $R_\sigma R_{\sigma^{-1}} = R_e$. Hence $R_{\sigma^{-1}} X \cap N \neq 0$, and then $R_\sigma(R_{\sigma^{-1}} X \cap N) \neq 0$. Since $R_\sigma(R_{\sigma^{-1}} X \cap N) = R_\sigma N \cap X$, we have that $R_\sigma N \cap X \neq 0$. We have obtained that $R_\sigma N$ is an

essential $R_e$-submodule of $M$. Since $G$ is finite, $N^*$ is also essential as an $R_e$-submodule of $M$. □

### 3.5.6 Corollary

Let $R$ be a strongly $G$-graded ring, where $n = |G|$ is finite. If $M$ is an $R$-module without $n$-torsion and $N$ an $R$-submodule of $M$, then $N$ is essential in $M$ as an $R$-submodule if and only if it is essential in $M$ as an $R_e$-submodule.

**Proof** Clearly if $N$ is $R_e$-essential, it is also $R$-essential. Assume now that $N$ is $R$-essential in $M$. Using Zorn's Lemma, we can find an $R_e$-submodule $L$ of $M$ maximal with the property that $N \cap L = 0$. Then $N \oplus L$ is essential in $M$ as an $R_e$-submodule. Lemma 3.5.5 shows that $(N \oplus L)^* = N^* \oplus L^* = N \oplus L^*$ is essential in $M$ as an $R_e$-submodule. On the other hand $K = N \oplus L^*$ is an $R$-submodule of $M$, and $N \subseteq K \subseteq N \oplus L$, so $K = N \oplus (K \cap L)$. By Theorem 3.5.1 there exists an $R$-submodule $U$ of $K$ such that $N \oplus U$ is essential in $K$ as an $R_e$-submodule. Then $N \oplus U$ is essential in $M$ as an $R_e$-submodule, therefore it is essential in $M$ as an $R$-submodule. This shows that $U = 0$, so $N$ is $R_e$-essential in $M$. □

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a graded ring and $M \in R$-mod. We say that $M$ is **component regular** whenever $0 \neq m \in M$, implies $R_\sigma m \neq 0$ for all $\sigma \in G$. We note that if $R$ is a strongly graded ring, then every $R$-module is component regular.

The following result is another version of the essential Maschke Theorem for graded rings which generalize Corollary 3.5.6.

### 3.5.7 Theorem (D. Quinn [174])

Let $R = \oplus_{\sigma \in R} R_\sigma$ be a graded ring, where $n = |G| < \infty$. Let $N \leq M$ be left $R$-modules, and suppose that $M$ is component regular and has no $n$-torsion. Then there exists $P \subseteq M$, an $R$-submodule, such that $N \cap P = 0$ and $N \oplus P$ is an essential $R_e$-submodule of $M$. In particular, if $N$ is an essential $R$-submodule of $M$, then $N$ is an essential $R_e$-submodule of $M$.

**Proof** By Zorn's Lemma, there exists an $R_e$-submodule $L \subseteq M$ such that $L$ is maximal with respect to the property that $N \cap L = 0$. Thus $N \oplus L$ is an essential $R_e$-submodule of $M$. Since $M$ is component regular, then for any $\sigma \in G, R_\sigma(N \oplus L)$ is also an essential $R_e$-submodule of $M$. Indeed, if $m \in M$, $m \neq 0$, then $R_{\sigma^{-1}} m \neq 0$, thus $R_{\sigma^{-1}} m \cap (N \oplus L) \neq 0$. Thus $R_\sigma(R_{\sigma^{-1}} m \cap (N \oplus L)) \neq 0$, and hence $R_\sigma R_{\sigma^{-1}} m \neq 0$ and $R_\sigma R_{\sigma^{-1}} m \cap R_\sigma(N \oplus L) \neq 0$. Since $R_\sigma R_{\sigma^{-1}} \subseteq R_e$, there exists $a \in R_e$ such that $0 \neq am \in R_\sigma(N \oplus L)$. We denote by $K = \cap_{\sigma \in G} R_\sigma(N \oplus L) = \cap_{\sigma \in G}(N + R_\sigma L)$. Since $G$ is finite, $K$ is obviously an essential $R_e$-submodule of $M$. If $\tau \in G$, we have $R_\tau K \subseteq$

$\cap_{\sigma\in G}R_\tau(N+R_\sigma L)\subseteq\cap_{\sigma\in G}(N+R_{\tau\sigma}L)=K$, hence $K$ is an $R$-submodule of $M$. Since $N\leq K\leq N\oplus R_\sigma L$, we have $K=N\oplus(K\cap R_\sigma L)=N\oplus L^\sigma$, where $L^\sigma=K\cap R_\sigma L$. Let $\pi_\sigma$ be the projection from $K$ to $N$ relative to this decomposition, and let $f:K\to N, f(m)=\sum_{\sigma\in G}\pi_\sigma(m), m\in K$. We prove that $f$ is $R$-linear; let $m\in K$ and write $m=n+l, n\in N, l\in L^\sigma$. Then $\pi_\sigma(m)=n$. If $\lambda\in R_\tau$, we get $\lambda m=\lambda n+\lambda l$, where $\lambda n\in N$ and $\lambda l\in R_\tau L^\sigma\subseteq R_\tau K\cap R_\tau R_\sigma L\subset K\cap R_{\tau\sigma}L=L^{\tau\sigma}$. Thus $\pi_{\tau\sigma}(\lambda m)=\lambda n=\lambda\pi_\sigma(m)$. Summing over all $\sigma\in G$, we obtain $\sum_{\sigma\in G}\pi_{\tau\sigma}(\lambda m)=\lambda\sum_{\sigma\in G}\pi_\sigma(m)$, so $f(\lambda m)=\lambda f(m)$, i.e. $f$ is $R$-linear.

Put $P=\mathrm{Ker}f$, which is an $R$-submodule of $M$. If $m\in N\cap P$, we have $f(m)=0$, and $\pi_\sigma(m)=m$ for any $\sigma\in G$. Hence $f(m)=|G|m$. Since $M$ has no $|G|$-torsion, we obtain $m=0$. Hence $N\cap P=0$. Also, if $x\in K$ and $y=f(x)$, we have $f(y)=ny$, and therefore $f(nx-y)=nf(x)-f(y)=nf(x)-ny=nf(x)-nf(x)=0$. Thus $nx-y\in P$, i.e. $nx\in N\oplus P$. Since $K$ is an essential $R_e$-submodule of $M$, we obtain that $N\oplus P$ is an essential $R_e$-submodule of $M$.

## 3.6 $H$-regular Modules

Throughout this section $R$ is again a strongly graded ring of type $G$ and we consider a subgroup $H$ having finite index $s$ in $G$. We let $\{\sigma_i, i=1,\ldots,s\}$ be a left transversal of $H$ in $G$, i.e. a system of representatives of the left $H$-cosets in $G$.

An $R$-module $M$ is said to be **$H$-regular** if there exists $f\in\mathrm{End}_{R_H}(M)$ such that $t_H^G(f)=1_M$. When $H=\{e\}$ we say that $M$ is regular (instead of $\{e\}$-regular). If $M$ is a left $R$-module, we denote by $\mathrm{Res}_H^G(M)$ the module $M$ considered as an $R_H$-module by restricting the scalars.

### 3.6.1 Proposition

The following assertions hold :

1. If $M\in(G/H,R)$-gr, then $M$ is $H$-regular (the definition of the category $(G/H,R)$-gr and some detail is given in Section 2.12)
2. If $M=\oplus_{i\in I}M_i$, then $M$ is $H$-regular if and only if $M_i$ is $H$-regular for any $i\in I$.
3. If $M$ is $H$-regular, then $M$ is $\sigma H\sigma^{-1}$-regular for any $\sigma\in G$.
4. Let $K<H<G$ be subgroups such that $K$ has finite index in $G$. If $M$ is $K$-regular, then $M$ is $H$-regular.
5. Let $K<H<G$ be subgroups such that $K$ has finite index in $G$. If $M$ is $H$-regular and $\mathrm{Res}_H^G(M)$ is $K$-regular, then $M$ is $K$-regular.

6. Let $M, N \in R$-mod, and $H$ a normal subgroup of $G$. If $M$ or $N$ is $H$-regular, then the abelian group $\mathrm{Hom}_{R_H}(M, N)$ is $H$-regular as a $G$-module.

**Proof**

1. Since $M \in (G/H, R)$-gr, then we have $M = \oplus_{c\in G/H} M_c$. If $m \in M$, we can write $m = \sum_{c\in G/H} m_c$. Then define $f \in \mathrm{Hom}_{R_H}(M, M)$ by $f(m) = m_H$ We have that $t_H^G(f) = \sum_{i=1,s} f^{\sigma_i}$ and $f^{\sigma_i}(m) = \sum_{j=1,n} a_j f(b_j m)$ where $1 = \sum_{j=1,n} a_j b_j, a_j \in R_{\sigma_i}, b_j \in R_{\sigma_i^{-1}}$ We remark that $f(b_j m) = b_j m_{\sigma_i H}$, so $f^{\sigma_i}(m) = \sum_{j=1,n} a_j b_j m_{\sigma_i H} = m_{\sigma_i H}$. Therefore $t_H^G(f)(m) = \sum_{i=1,s} m_{\sigma_i H} = m$ i.e. $t_H^G(f) = 1_M$.

2. Let $M \in R$-mod be $H$-regular and $N$ a direct summand of $M$. There exists $f \in \mathrm{End}_{R_H}(M)$ such that $t_H^G(f) = 1_M$. Since $N$ is a direct summand of $M$, there exists $p \in \mathrm{Hom}_R(M, N)$ such that $p \circ i = 1_N$, where $i : N \to M$ is the inclusion. If we put $g = p \circ f \circ i$, then

$$\begin{aligned} t_H^G(g) &= \sum_{i=1,s} g^{\sigma_i} = \sum_{i=1,s} (p \circ f \circ i)^{\sigma_i} = \sum_{i=1,s} p^{\sigma_i} \circ f^{\sigma_i} \circ i^{\sigma_i} \\ &= \sum_{i=1,s} p \circ f^{\sigma_i} \circ i = p \circ (\sum_{i=1,s} f^{\sigma_i}) \circ i = p \circ i = 1_N \end{aligned}$$

   so $N$ is $H$-regular.

   Assume that $M = \oplus_{i\in I} M_i$ and $M_i$ is $H$-regular for any $i \in I$. Then there exists for any $i \in I$ some $f_i \in \mathrm{End}_{R_H}(M_i)$ such that $t_H^G(f_i) = 1_{M_i}$. If we put $f = \oplus f_i$, then it is easy to see that $t_H^G(f) = 1_M$.

3. Since $M$ is $H$-regular, there exists $f \in \mathrm{End}_{R_H}(M)$ such that $t_H^G(f) = 1_M$. Then $f^\sigma \in \mathrm{End}_{R_{\sigma H \sigma^{-1}}}(M)$. Indeed, for any $\lambda_\sigma \in R_\sigma, \mu \in R_H$ and $\lambda_{\sigma^{-1}} \in R_{\sigma^{-1}}$ we have that

$$f^\sigma(\lambda_\sigma \mu \lambda_{\sigma^{-1}} x) = \lambda_\sigma f(\mu \lambda_{\sigma^{-1}} x) = \lambda_\sigma \mu f(\lambda_{\sigma^{-1}} x) = \lambda_\sigma \mu \lambda_{\sigma^{-1}} f^\sigma(x)$$

   Now the set $\{\ \sigma\sigma_i\sigma^{-1} \mid i = 1, \ldots, s\ \}$ is a set of representatives for the left $\sigma H \sigma^{-1}$-cosets of $G$. Indeed, we have that $G = \cup_{i=1,s} \sigma_i H$. If $g \in G$, then $\sigma^{-1} g \sigma = \sigma_i h$ for some $h \in H$. Thus $g = \sigma\sigma_i h\sigma^{-1} = (\sigma\sigma_i\sigma^{-1})(\sigma h \sigma^{-1})$, so $G = \cup_{i=1,s}(\sigma\sigma_i\sigma^{-1})(\sigma H \sigma^{-1})$. On the other hand, if $(\sigma\sigma_i\sigma^{-1})^{-1}(\sigma_j\sigma^{-1}) \in \sigma H \sigma^{-1}$, we obtain that $\sigma_i^{-1}\sigma_j \in H$, i.e. $\sigma_i = \sigma_j$. Thus we have that

$$t_{\sigma H \sigma^{-1}}^G(f^\sigma) = \sum_{i=1,s} (f^\sigma)^{\sigma\sigma_i\sigma^{-1}} = \left(\sum_{i=1,s} f^{\sigma_i}\right)^\sigma = 1_M^\sigma = 1_M$$

4. There exists $f \in \mathrm{End}i_{R_K}(M)$ such that $t_K^G(f) = 1_M$. Let $(\tau_j)_{j=1,r}$ be a set of representatives for the left $K$-cosets of $H$. Then $(\sigma_i\tau_j)_{i=1,s,\ j=1,r}$ is a set of representatives for the left $K$-cosets of $G$. Thus we have $\sum_{i=1,s}\sum_{j=1,r} f^{\sigma_i\tau_j} = 1_M$ so $\sum_{i=1,s}\sum_{j=1,r}(f^{\tau_j})^{\sigma_i} = 1_M$ If we denote by $g = \sum_{j=1,r} f^{\tau_j}$, then $g \in \mathrm{End}_{R_H}(M)$, and $t_H^G(g) = 1_M$.

5. Notation as in 4. Since $\mathrm{Res}_H^G(M)$ is $K$- regular, there exists $f \in \mathrm{End}_{R_K}(M)$ such that $t_K^H(f) = 1_M$, i.e. $\sum_{j=1,r} f^{\tau_j} = 1_M$. Since $M$ is $K$-regular, there exists $g \in \mathrm{End}_{R_H}(M)$ such that $t_H^G(g) = 1_M$, i.e. $\sum_{i=1,s} g^{\sigma_i} = 1_M$. Let $h = g \circ f$. Then $h \in \mathrm{End}_{R_K}(M)$ and

$$\begin{aligned} t_H^G(h) &= \sum_{i=1,s}\sum_{j=1,r} h^{\sigma_i\tau_j} = \sum_{i=1,s}\sum_{j=1,r} g^{\sigma_i\tau_j} f^{\sigma_i\tau_j} \\ &= \sum_{i=1,s}\sum_{j=1,r} (g^{\tau_j})^{\sigma_i}(f^{\tau_j})^{\sigma_i} = \sum_{i=1,s}\sum_{j=1,r} g^{\sigma_i}(f^{\tau_j})^{\sigma_i} \\ &= \sum_{i=1,s} g^{\sigma_i} \circ (\sum_{j=1,r} f^{\tau_j})^{\sigma_i} = \sum_{i=1,s} g^{\sigma_i} = 1_M \end{aligned}$$

6. Assume that $M$ is $H$-regular. Then there exists $f \in \mathrm{End}_{R_H}(M)$ such $t_H^G(f) = 1_M$. We denote by $A = \mathrm{Hom}_{R_H}(M, N)$ and define $f^* : A \to A$ by $f^*(u) = u \circ f$ for any $u \in \mathrm{Hom}_{R_H}(M, N)$. Then $t_H^G(f^*) = \sum_{i=1,s} \sigma_i f^*$, where

$$\begin{aligned} (\sigma_i f^*)(u) &= \sigma_i f^*(\sigma_i^{-1}u) = \sigma_i f^*(u^{\sigma_i^{-1}}) \\ &= (f^*(u^{\sigma_i^{-1}}))^{\sigma_i} = (u^{\sigma_i^{-1}} \circ f)^{\sigma_i} = u \circ f^{\sigma_i} \end{aligned}$$

Hence

$$t_H^G(f^*)(u) = \sum_{i=1,s} u \circ f^{\sigma_i} = u \circ \sum_{i=1,s} f^{\sigma_i} = u$$

so $t_H^G(f^*) = 1_A$. The situation where $N$ is $H$-regular can be dealt with in a similar way. □

### 3.6.2 Corollary

If $N \in R_H$-mod, then $\mathrm{Ind}_H^G(N)$ (or $\mathrm{Coind}_H^G(N)$) is $H$-regular. Recall that $\mathrm{Ind}_H^G(N) = R \otimes_{R_H} N$, see Section 2.12)

**Proof** Apply 1. of Proposition 3.6.1. □

Let $R = \oplus_{\sigma\in G} R_\sigma$ be a graded ring and $H$ a subgroup of $G$ of finite index $s$. For $M \in R$-mod, we have a surjective morphism of $R$-modules

$$\alpha : R \otimes_{R_H} M \to M,\ \ \alpha(\lambda \otimes m) = \lambda m,\ \ \lambda \in R,\ m \in M$$

and an injective morphism of $R$-modules

$$\beta : M \to \mathrm{Hom}_{R_H}(R, M), \;\; \beta(m)(\lambda) = \lambda m, \;\; \lambda \in R, \; m \in M$$

Then $\alpha$ is a split surjection and $\beta$ is a split injection in $R_H$-mod. Indeed, define $\alpha' : M \to R \otimes_{R_H} M$ by $\alpha'(m) = 1 \otimes m$, and $\beta' : \mathrm{Hom}_{R_H}(R, M) \to M$ by $\beta'(u) = u(1)$. We have $\alpha \circ \alpha' = 1_M$ and $\beta' \circ \beta = 1_M$. Moreover, $\alpha'$ and $\beta'$ are clearly $R_H$-linear.

If $n \in \mathbb{Z}$, we say that $n$ is invertible in $M$ if the morphism $\phi_n : M \to M, \phi_n(x) = nx$ is bijective.

### 3.6.3 Proposition

Let $M \in R$-mod. If $s = [G : H]$ is invertible in $M$, then $M$ is $H$-regular.

**Proof** With notation as above, define $\gamma = s^{-1} t_H^G(\alpha')$. Then $\gamma$ is $R$-linear and

$$\alpha \circ \gamma = s^{-1} \sum_{i=1,s} \alpha \circ \alpha'^{\sigma_i} = s^{-1} \sum_{i=1,s} (\alpha \circ \alpha')^{\sigma_i} = 1_M$$

so $M$ is $H$-regular. □

If $M, N \in R$-mod, we say that $M$ divides $N$ (or $M$ is a component of $N$), and write $M|N$, if $M$ is isomorphic to a direct summand of $N$ in $R$-mod. If $H$ is a subgroup of $G$, then $M$ is said to be $R_H$-projective (respectively $R_H$-injective) if $M$ divides an induced (respectively coinduced) $R$-module $R \otimes_{R_H} L$ for some $R_H$-module $L$. It is obvious that the two concepts coincide in the case where $R$ is a strongly graded ring.

### 3.6.4 Theorem

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a strongly graded ring and $H$ a subgroup of $G$ of finite index $s$. If $M \in R$-mod, then the following assertions are equivalent.

1. $M$ is $H$-regular.
2. $M$ is $R_H$-projective.

**Proof** 2. $\Rightarrow$ 1. follows from Corollary 3.6.2 and assertion 2. of Proposition 3.6.1. In order to prove that 1. $\Rightarrow$ 2., assume that $M$ is $H$-regular, and let $f \in \mathrm{End}_{R_H}(M)$ such that $t_H^G(f) = 1_M$. We consider the map $\alpha : R \otimes_{R_H} M \to M$ defined before, and define $\gamma : M \to R \otimes_{R_H} M, \;\; \gamma(m) = 1 \otimes f(m)$ We have $\alpha \circ \gamma = f$ and $\gamma$ is $R_H$-linear. Then

$$\alpha \circ t_H^G(\gamma) = t_H^G(\alpha \circ \gamma) = t_H^G(f) = 1_M$$

showing that $\alpha$ is a split surjection in $R$-mod. □

### 3.6.5 Corollary

If $M$ is a projective (respectively injective) $R$-module, then $M$ is $H$-regular.

**Proof** If $M$ is projective then $\alpha$ is a split surjection, and if $M$ is injective then $\beta$ is a split injection in $R$-mod, and we can apply Theorem 3.6.4. □

### 3.6.6 Corollary

Let $H$ be a subgroup of $G$ and $N$ an $R_H$-module which is $K$-regular for some subgroup $K$ of $H$. Then $\mathrm{Ind}_H^G(N)$ is a $K$-regular module.

**Proof** Since $N$ is $K$-regular, there exists $L \in R_K$-mod such that $N$ is a component of $R_H \otimes_{R_K} L$. Thus $\mathrm{Ind}_N^G = R \otimes_{R_H} N$ is a component of

$$R \otimes_{R_H} (R_H \otimes_{R_K} L) \simeq R \otimes_{R_K} L$$

and the assertion follows from Theorem 3.6.4. □

Assume now that $G$ is a finite group. If $M \in R$-mod, we denote by $\mathcal{B}(M)$ the set of all subgroups $H$ of $G$ such that $M$ is $H$-regular. $\mathcal{B}(M)$ is not empty since $M$ is clearly $G$-regular. Since $\mathcal{B}(M)$ is a finite set, it has at least one minimal element. We denote by $\mathcal{V}(M)$ the set of all minimal elements of $\mathcal{B}(M)$. By assertion 3) of Proposition 3.6.1, we know that $\mathcal{V}(M)$ is closed under conjugation. An $R$-module $M$ is said to be **indecomposable** if $M$ is not a direct sum of two non-zero submodules. $M$ is said to be **strongly indecomposable** if $\mathrm{End}_R(M)$ is a local ring. Obviously, if $M$ is strongly indecomposable, then $M$ is indecomposable.

Let $n = |G|$, and write $n = p_1^{k_1} \dots p_r^{k_r}$, where $p_1, \dots, p_r$ are distinct primes. If we denote $u = p_1^{k_1}$ and $v = p_2^{k_2} \dots p_r^{k_r}$, we have that $(u, v) = 1$, so there exist integers $k$ and $l$ such that $ku + lv = 1$. Thus $\phi_{ku} + \phi_{lv} = 1_M$, and since $\mathrm{End}_R(M)$ is a local ring, either $\phi_{ku}$ or $\phi_{lv}$ is invertible. If $\phi_{lv}$ is invertible, then $p_2^{k_2}, \dots, p_r^{k_r}$ are invertible on $M$. If $u$ is invertible, then we repeat the procedure with $p_2^{k_2}, \dots, p_r^{k_r}$, until we obtain that at most one of the elements $p_1^{k_1}, \dots, p_r^{k_r}$ is not invertible on $M$. Hence we can write $n = p^k m$, where $p$ is prime, $(m, p) = 1$, and $m$ is invertible on $M$.

### 3.6.7 Proposition

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a strongly graded ring, where $G$ is a finite group of order $n$. If $M \in R$-mod is strongly indecomposable, then any element of $\mathcal{V}(M)$ is a $p$-subgroup of $G$ for some prime divisor $p$ of $n$.

**Proof** Let $n = p^k m$, where $(p, m) = 1$ and $m$ is invertible in $M$. Let $H \in \mathcal{V}(M)$ and $K < H, K$ a Sylow-$p$-subgroup of $H$. Then $K$ is $p$-subgroup of $G$ and $[H : K]$ is prime with $p$. Since $[H : K]$ divides $n = |G|$, then $[H : K]$ divides $m$ and therefore $[H : K]$ is invertible on $M$. Now clearly $[H : K]$ is invertible on $\mathrm{Res}_H^G M$, too. Hence $\mathrm{Res}_H^G(M)$ is $K$-regular. Since $M$ is $H$-regular Proposition 3.6.1 yields that $M$ is $K$-regular. But $H \in \mathcal{V}(M)$ and $K \subseteq H$, so $H = K$. □

## 3.7 Green Theory for Strongly Graded Rings

The finitely generated indecomposable modules for a group ring have a representation theoretic meaning. In the case $R = k[G]$ where $k$ is a field of characteristic $p$ and $G$ a finite group, a characterization of the finitely generated indecomposable modules is part of the work of J. A. Green, [88], [89], that became known as "Green Theory". The idea(s) of proof used by J. A. Green may be extended to the case of certain strongly graded rings. We also made use of E. Dade's paper [55]; for some generalities the reader may also consult G. Karpilowsky [112] or Curtis, Reiner [46].

In this section, $R$ is strongly graded by $G$ such that the following conditions hold :

G.a. $R_e$ is an algebra over a commutative Noetherian complete local ring $A$ with residue field $k$ of nonzero characteristic $p$.

G.b. $G$ is a finite group.

G.c. For every $\sigma \in G$, $R_\sigma$ is a finitely generated $A$-submodule of $R$.

From G.c. it follows that $R$ is a finitely generated $A$-module (in fact a Noetherian $A$-module) hence any finitely generated $R$-module $M$ is a Noetherian $A$-module and moreover $\mathrm{End}_R(M)$ as well as $\mathrm{End}_{R_e}(M)$ are Noetherian $A$-modules.

To start off the theory we need a few lemmas of group theoretic nature dealing mainly with double cosets in $G$.

### 3.7.1 Lemma

Let $G$ be a finite group, and $H, K$ two subgroups of $G$. Let $KgH$ be a double coset of $g \in G$ relative to $H$ and $K$. If $\{h_1 \ldots, h_s\}$ is a set of representatives for the left cosets of $gHg^{-1} \cap K$ in $K$, then $KgH = \cup_{j=1}^s h_j gH$ and $\{h_j gH | j = 1, \ldots, s\}$ are the left cosets of $H$ contained in $KgH$.

**Proof** If $z = xgy \in KgH$, then $x = h_j ghg^{-1}$ so $z = h_j ghg^{-1}gy = h_j ghy$ so $z \in \bigcup_{j=1}^{s} h_j gH$. On the other hand, if $h_j gH \cap h_{j'} gH \neq \emptyset$, then there exists $z = h_j gh_1 = h_{j'} gh_2$, and hence $h_{j'}^{-1} h_j = gh_2 h_1^{-1} g^{-1} \in gHg^{-1} \cap K$, so $h_{j'} = h_j$. Hence $h_j gH = h_{j'} gH$.

### 3.7.2 Lemma

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a strongly graded ring, where $G$ is a finite group. Let $H, K$ be two subgroups of $G$. If $M$ is an $R_H$-module, then we have the isomorphism of $R_K$-modules

$$R_{KgH} \otimes_{R_H} M \simeq R_K \otimes_{R_{gHg^{-1} \cap K}} (R_{gH} \otimes_{R_H} M)$$

**Proof** Let us denote for the sake of simplicity $M^{(g)} = R_{gH} \otimes_{R_H} M$ for each $g \in G$.
We have $R_{KgH} \otimes_{R_H} M \cong \oplus_{j=1}^{s} M^{(h_j g)}$, as $R_e$-module, $R_K \otimes_{R_{gHg^{-1} \cap K}} M^{(g)} = \oplus_{j=1}^{s} R_{h_j (gHg^{-1} \cap K)} \otimes_{R_{gHg^{-1} \cap K}} M^{(g)} \cong \oplus_{j=1}^{s} (R_{h_j} \otimes_{R_e} R_{gHg^{-1} \cap K}) \otimes_{R_{gHg^{-1} \cap K}} M^{(g)} \cong \oplus_{j=1}^{s} (R_{h_j} \otimes_{R_e} R_{gH}) \otimes_{R^{(H)}} M = \oplus_{j=1}^{s} M^{(h_j g)}$, as $R_e$-modules. Now it is easy to see that the above isomorphism is an $R_K$-isomorphism. □

### 3.7.3 Lemma (Mackey formula)

If $\{g_1, g_2, \ldots, g_r\}$ is a set of representatives for the double cosets of $K$ and $H$ in $G$, then

$$R \otimes_{R_H} M = \oplus_{i=1}^{r} \left( R_K \otimes_{R_{K \cap g_i H g_i^{-1}}} (R_{g_i H} \otimes_{R_H} M) \right)$$

as $R_K$-modules.

**Proof** $R \otimes_{R_H} M = \oplus_{i=1}^{r} R_{Kg_i H} \otimes_{R_H} M$ as $R_K$-modules, and apply Lemma 3.7.2. □

### 3.7.4 Lemma

Let $M \in R$-mod be a finitely generated $R$-module. If $M$ is an indecomposable $R$-module, then $M$ is strongly indecomposable.

**Proof** Put : $T = \mathrm{End}_R(M)$. Let $\underline{m}$ be the maximal ideal of $A$, and consider the $\underline{m}$-adic topology on $T : (\underline{m}^i T)_{i \geq 0}$. It is obvious that each $\underline{m}^i T$ is an ideal of $T$. By assertion 8. p. 302 of [150], we have that $T$ is complete in the $\underline{m}$-adic topology. Then every idempotent of $T/\underline{m}T$ may be lifted to $T$ (see Lemma VII.1 of [157], page 312). Since $M$ is an indecomposable $R$-module $T$ has only two idempotent elements, 0 and 1, and hence $T/\underline{m}T$ has only two

idempotent elements, 0 and 1. We show now that $T/J(T)$ is a field, where $J(T)$ denotes the Jacobson radical of $T$. We have $\underline{m}T \subset J(T)$ (see [136], page 299, Property 3.). Since $T/\underline{m}T$ is a finitely generated $(A/\underline{m})$-module, then $T/\underline{m}T$ is an Artinian algebra, and therefore $T/J(T)$ is a semisimple Artinian ring. Since there exists a natural number $k$ such that $J(T)^k \subset \underline{m}T$, every idempotent of $T/J(T)$ may be lifted to $T/\underline{m}T$. Therefore, $T/J(T)$ has only trivial idempotents, (0 and 1), i.e. $T/J(T)$ is a field. Thus, $T$ is a local ring, i.e. $M$ is a strongly indecomposable $R$-module.

### 3.7.5 Corollary

Let $R$ be a strongly graded ring satisfying G.a, G.b, G.c. If $M$ is a finitely generated $R$-module, then $M$ is a finite direct sum $M_1 \oplus M_2 \ldots \oplus M_n$ of indecomposable $R$-modules $M_i$. Moreover, this decomposition is unique up to order and isomorphisms.

**Proof** The fact that $M$ is a finite direct sum of indecomposable modules follows from the fact that $M$ is a left Noetherian $R$-module. Apply Lemma 3.7.4 and the classical Krull-Schmidt theorem.

In Section 3.6, we defined the set $\mathcal{B}(M)$ of all subgroups $H$ of $G$ such that $M$ is $H$-regular, and we denoted by $\mathcal{V}(M)$ the set of minimal elements of $\mathcal{B}(M)$. From now on, all modules will be finitely generated.

### 3.7.6 Theorem

Suppose that $R$ is a strongly graded ring which satisfies conditions G.a., G.b. and G.c.. Let $M \in R$-mod be an indecomposable module. If $H \in \mathcal{V}(M)$ and $K \in \mathcal{B}(M)$, then the following assertions hold :

1. $H$ is a $p$-group, where $p = \mathrm{char}k$, $k$ is the residue class field of $A$ (see condition G.a).
2. There exists $\sigma \in G$ such that $\sigma H \sigma^{-1} \leq K$.
3. The elements of $\mathcal{V}(M)$ form a unique conjugacy class of $p$-subgroups of $G$.

**Proof**

1. Let $n = |G|$. Then we can write $n = p^k m$, where $(p, m) = 1$. But it is easy to see that $m$ is invertible in the ring $A$ i.e. $m$ is invertible in $R$. Now we apply Proposition 3.6.7.
2. Since $M$ is $H$-regular, there exists $N \in R_H$-mod such that $M$ divides $R \otimes R_H N$. Since $M$ is indecomposable, we may assume that $N$ is an indecomposable $R_H$-module. On the other hand we can write $\mathrm{Res}^G_K M =$

$M_1 \oplus \ldots \oplus M_t$, where $M_i \in R_K$-mod and $M_i$ are indecomposable. Using Lemma 3.7.3, each $M_i$ divides $R_K \otimes_{R_{K \cap g_i H g_i^{-1}}} (R_{g_i H} \otimes_{R_H} N)$ for some $g_i$. But $M$ divides $R \otimes_{R_K} M \cong R \otimes_{R_K} M_1 \oplus \ldots \oplus R \oplus_{R_K} M_t$. Since $M$ is indecomposable, there exists $i$ such that $M$ divides $R \otimes_{R_K} M_i$. Thus, $M$ divides

$$R \otimes_{R_K} \left( R_K \otimes_{R_{K \cap g_i H g_i^{-1}}} (R_{g_i H} \otimes_{R_H} N) \right) \cong R \otimes_{R_{K \cap g_i H g_i^{-1}}} (R_{g_i H} \otimes_{R_H} N)$$

Consequently, $M$ is $(K \cap g_i H g_i^{-1})$-regular i.e. $(K \cap g_i H g_i^{-1}) \in \mathcal{B}(M)$. Since $H$ is minimal in $\mathcal{B}(M)$, $g_i H g_i^{-1}$ is minimal in $\mathcal{B}(M)$, and (since $K \cap g_i H g_i^{-1} \subseteq g_i H g_i^{-1}$) we have $K \cap g_i H g_i^{-1} = g_i H g_i^{-1}$, i.e. $g_i H g_i^{-1} \subseteq K$.

3. This follows from 2.

A subgroup $H$ of $G$ which is minimal in $\mathcal{B}(M)$ is called a **vertex** of $M$, and an indecomposable $R_H$-module $N$ such that $M$ divides $R \otimes_{R_H} N$ is called a **source** of $M$.

## 3.7.7 Theorem

Assume that $H$ is a vertex of the indecomposable $R$-module $M$. If $N, N' \in R_H$-mod are two sources of $M$, then there exists $\sigma \in N(H)$ ($N(H)$ is the normalizer of $H$) such that $N' \cong R_{\sigma H} \otimes_{R_H} N$ (as $R_H$-modules).

**Proof** Write $\mathrm{Res}^G_H M = M_1 \oplus \ldots \oplus M_t$, where the $M_i$ are indecomposable $R_H$-modules. On the other hand, $M$ divides $R \otimes_{R_H} N$. If we apply Lemma 3.7.3 for the case $K = H$, we have

$$R \otimes_{R_H} N \cong \oplus_{i=1}^r \left( R_H \otimes_{R_{H \cap g_i H g_i^{-1}}} (R_{g_i H} \otimes_{R_H} N) \right)$$

where $g_1, g_2, \ldots, g_r$ is a set of representatives for the double cosets of $H$ and $H$ in $G$. Then $M_i$ divides $R \otimes_{R_H} N$, and therefore there exists $g_i$ such that $M_i$ divides $R_H \otimes_{R_{H \cap g_i H g_i^{-1}}} (R_{g_i H} \otimes_{R_H} N)$. Since $M$ is a direct summand in $R \otimes_{R_H} M = \oplus_{i=1}^t R \otimes_{R_H} M_i$, and $M$ is indecomposable, $M$ is a direct summand in $R \otimes_{R_H} M_i$ for some $i$. Thus $M$ divides $R \otimes_{R_H} (R_H \otimes_{R_{H \cap g_i H g_i^{-1}}} (R_{g_i H} \otimes^{R_H} \otimes_{R_H} N)) \cong R \otimes_{R_{H \cap g_i H g_i^{-1}}} (R_{g_i H} \otimes R_H N)$. So $M$ is $(H \cap g_i H g_i^{-1})$-regular, and since $H$ is a vertex, we have $H \cap (g_i H g_i^{-1}) = H$, so $H = g_i H g_i^{-1}$. Concequently, $g_i \in N(H)$, and $M_i$ is a direct summand in $R_H \otimes_{R_H} R_{g_i H} \otimes_{R_H} N) \cong R_{g_i H} \otimes_{R_H} N$. Because $R_{g_i H} \otimes R_{g_i^{-1} H} \cong R_H$, then $R_{g_i H} \otimes_{R_H} N$ is also an indecomposable $R_H$-module. Hence $M_i \cong R_{g_i H} \otimes_{R_H} N$.

Analogously, if we replace $N$ by $N'$, there exists $g_j$ such that $M_i \cong R_{g_jH} \otimes_{R_H} N'$. Thus $R_{g_iH} \otimes_{R_H} N \cong R_{g_jH} \otimes_{R_H} N'$, and therefore $N' = R_{\sigma H} \otimes_{R_H} N$, where $\sigma = g_j^{-1} g_i \in N(H)$.

If $K, H$ are subgroups of $G$, we write $K \subset_c H$ (inclusion by conjugation) if there exists $\sigma \in G$ such that $K \subset \sigma H \sigma^{-1}$. If $K = \sigma H \sigma^{-1}$ we write $K =_c H$.

## 3.7.8 Theorem

Let $M$ be an indecomposable $R$-module, $H$ a vertex of $M$ and $K$ a subgroup of $G$ such that $M$ is $R_K$-projective. Let us write $\mathrm{Res}_K^G M = M_1 \oplus \ldots \oplus M_s$, where $M_i$ are indecomposable $R_K$-modules, and for each $i, 1 \leq i \leq s$, let $K_i$ be a vertex of $M_i$. Then :

a. $K_i \subset_c H$ for $1 \leq i \leq s$

b. There exists an $M_i$ such that $M$ is a direct summand in $R \otimes_{R_K} M_i$, and for this $i$ we have :

c. $K_i =_c H$.

**Proof**

a. Let $N \in R_H$-mod be a source of $M$. By Lemma 3.7.3 we have $R \otimes_{R_H} N \cong \oplus_{i=1}^r R_K \otimes_{R^{(K \cap g_i H g_i^{-1})}} (R_{g_iH} \otimes_{R_H} N)$ as $R_K$. Since $M_i$ is a direct summand of $R \otimes_{R_H} N$, there exists a $g_i$ such that $M_i$ is a direct summand of $R_K \otimes_{R_{K \cap g_i H g_i^{-1}}} (R_{g_iH} \otimes_{R_H} \otimes_{R_H} N)$. Therefore $M_i$ is $(K \cap g_i H g_i^{-1})$-regular. Since $K_i$ is a vertex of $M_i$, then $K \cap g_i H g_i^{-1}$ contains $\sigma_i K_i \sigma_i^{-1}$ for some $\sigma_i \in K$. But it is easy to see that $K_i \subset \sigma H \sigma^{-1}$, where $\sigma = \sigma_i^{-1} g_i$.

b. Since $M$ is a direct summand of $R \otimes_{R_K} M \cong \oplus_{i=1}^s R \otimes_{R_K} M_i$, and since $M$ is indecomposable, there exists an $M_i$ such that $M$ divides $R \otimes_{R_K} M_i$. Since $K_i$ is a vertex of $M_i$, $M_i$ is a direct summand in $R_K \otimes_{R_{K_i}} M_i$. Hence $M$ is a direct summand in $R \otimes_{R_K} (R_K \otimes_{R_{K_i}} M)$, and therefore $M$ is $K_i$-regular. Since $H$ is a vertex of $M$, and $K_i \subseteq_c H$, then $K_i =_c H$.

An indecomposable $R$-module $M$ is called **principal** if $M$ is isomorphic to a direct summand of ${}_R R$.

## 3.7.9 Corollary

If $R_e$ is a semisimple Artinian ring ($R = \oplus_{\sigma \in G} R_\sigma$ is strongly graded, and satisfies G.a., G.b. and G.c), $M$ is a principal indecomposable $R$-module, and $K$ is a subgroup of $G$, such that $M$ is $K$-regular, then

$$\mathrm{Res}_K^G M = M_1 \oplus M_2 \oplus \ldots \oplus M_s$$

where the $M_i$ are principal indecomposable $R_K$-modules.

**Proof** Since $M$ is a principal indecomposable $R$-module, then $\{e\}$ is a vertex of $M$. From Theorem 3.7.3. it follows that $K_i = \{e\}$. If $N_i$ is a source of $M_i$, then $N_i$ is an indecomposable $R_e$-modules, and $M_i$ is isomorphic to a direct summand of $R_K \otimes_{R_e} N_i$. But $N_i$ is isomorphic to a minimal left ideal of $R_e$. Thus $M_i$ is isomorphic with a direct summand of $R_K \otimes_{R_e} R_e = R_K$.

We end this section by the following fundamental result :

## 3.7.10 Theorem (J.A.Green)

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a strongly graded ring satisfying conditions G.a, G.b, and G.c. Let $P$ be a $p$-subgroup of $G$ and let $H$ be a subgroup of $G$ containing $N(P)$. Then there is a one-to-one correspondence between the isomorphism classes $[M]$ of finitely generated indecomposable $R$-modules $M$ with vertex $P$, and the isomorphism classes $[N]$ of finitely generated indecomposable $R_H$-modules with vertex $P$. Here $[M]$ corresponds to $[N]$ if and only if $M$ divides $R \otimes_{R_H} N$, which occurs if and only if $N$ divides $\mathrm{Res}^G_H M$.

**Proof**

*Step 1.*
$P$ is not $G$-conjugate to a subgroup of $xPx^{-1} \cap P$ for all $x \in G - H$. Indeed, if $P$ is $G$-conjugate to a subgroup of $xPx^{-1} \cap P$, then there exists $y \in G$, such that $yPy^{-1} \subseteq xPx^{-1} \cap P$. Thus $yPy^{-1} \subseteq P$ and therefore $yPy^{-1} = P$, so $y \in N(P)$. Hence $P \subseteq xPx^{-1}$, so $P = xPx^{-1}$ i.e. $x \in N(P)$, a contradiction.
*Step 2.*
Let $N \in R_H$-mod be finitely generated indecomposable, such that $N$ is $P$-regular. Then $N$ is the only possible component of $\mathrm{Res}^G_H(R \otimes_{R_H} N)$ in the sense that any other indecomposable component which is not isomorphic to $N$ is $(xPx^{-1} \cap H)$-regular for some $x \in G - H$. Indeed, since $R = R_H \oplus R_{G-H}$ as $R_H$-bimodules, there exists $N' \in R_H$-mod such that $\mathrm{Res}^G_H(R \otimes_{R_H} N) = N \oplus N'$. Since $N$ is $P$-regular, there exists an $R_P$-modules $L$ such that $R_H \otimes_{R_P} L = N \oplus N''$ for a suitable $R_H$-module $N''$. Then $R \otimes_{R_P} L = R \otimes_{R_H} (R_H \otimes_{R_P} L) \cong (R \otimes_{R_H} N) \oplus (R \otimes_{R_H} N'') = N \oplus N' \oplus N'' \oplus N'''$ as $R_H$-modules for a suitable $R_H$-module $N'''$. On the other hand, by Lemma 3.7.3. we have $R \otimes_{R_P} L \cong \oplus_{i=1}^r V_i$, where $V_i = R_H \otimes_{R_{H \cap g_i P g_i^{-1}}} (R_{g_i P} \otimes_{R_P} L)$, and $\{g_1 = 1, g_2, \ldots, g_r\}$ is a set of representatives for the double cosets of $H$ and $P$ in $G$. Now $V_1 = R_H \otimes_{R_P} L = N \oplus N''$. Thus $N \oplus N' \oplus N'' \oplus N''' = N \oplus N'' \oplus \oplus_{i=2}^r V_i$, and so we have $N' \oplus N''' \cong \oplus_{i=2}^r V_i$. If we write $N' = N'_1 \oplus \ldots \oplus N'_s$ are indecomposable $R_H$-modules, then we obtain that each $N'_k$ is a component of a suitable $V_i$, hence $N'_k$ is $(H \cap g_k P g_k^{-1})$-regular for some $g_k \notin H$. Now $R \otimes_{R_H} N = N \oplus N'$, and the assertion follows.

*Step 3.*

Let $M \in R$-mod be an indecomposable $R$-module. By Theorem 3.7.3. $\mathrm{Res}_H^G M$ has at least one indecomposable component $N \in R_H$-mod such that $N$ has vertex $P$ and $M$ is a component of $R \otimes_{R_H} N$. By Step 2., $N$ is the unique component of $\mathrm{Res}_H^G(R \otimes_{R_H} N)$ which is not $(xPx^{-1} \cap H)$-regular for all $x \in G - H$. We prove that $N$ is the unique indecomposable component of $\mathrm{Res}_H^G M$, with vertex $P$, such that $N$ is not $(xPx^{-1} \cap H)$-regular for all $x \in G - H$. Indeed, let $N'$ be any indecomposable component of $\mathrm{Res}_H^G M$, with vertex $P$. Thus $N'$ is component of $\mathrm{Res}_H^G(R \otimes_{R_H} N)$. If $N'$ is $(xPx^{-1} \cap H)$-regular for some $x \in G - H$, then since $N'$ has vertex $P$, there exists $y \in H$ such that $yPy^{-1} \subseteq xPx^{-1} \cap H$, and hence $P \subseteq y^{-1}xP(y^{-1}x)^{-1}$, so $P = (y^{-1}x)P(y^{-1}x)^{-1}$. Thus $y^{-1}x \in N(P) \subset H$. Since $y \in H$, $x \notin H$ follows, contradiction. Thus $N'$ is not $(xRx^{-1} \cap H)$-regular for all $x \in G - H$. By Step 2. since $N'$ is also a component of $\mathrm{Res}_H^G(R \otimes_{R_H} N)$ we obtain that $N \cong N'$.

*Step 4.*

We denote by $\mathcal{C}_{R,P}$ the set of all classes $[M]$ of finitely generated indecomposable $R$-modules $M$ with vertex $P$, and by $\mathcal{C}_{R_H,P}$ the set of all classes $[N]$ of finitely generated indecomposable $R_H$-modules with vertex $P$. We define the map $f : \mathcal{C}_{R,P} \to \mathcal{C}_{R_H,P}, f([M]) = [N]$, given by Step 3. Let $N$ be an indecomposable $R_H$-module with vertex $P$. Decompose $R \otimes_{R_H} N = M_1 \oplus \ldots \oplus M_t$, where $M_i$ are indecomposable $R$-modules. Since $N$ has vertex $P$, then it is obvious that $N$ is not $(xPx^{-1} \cap H)$-regular for all $x \in G - H$. Step 2. implies that $N$ is the only indecomposable component of $\mathrm{Res}_H^G(R \otimes_{R_H} N)$ which is not $(xPx^{-1} \cap H)$ regular for all $x \in G - H$. Hence there exists a unique $M \in \{M_1, \ldots, M_t\}$ such that $N$ is a component of $\mathrm{Res}_H^G M$. We prove now that $M$ is unique in $\{M_1, \ldots, M_t\}$ with vertex $P$. Since $N$ is $P$-regular, there exists $N' \in R_P$-mod such that $N$ is a component of $R_H \otimes_{R_P} N'$. Thus $R \otimes_{R_H} N$ is a component of $R \otimes_{R_P} N'$, therefore $R \otimes_{R_H} N$ is $P$-regular. In particular, $M$ is $P$-regular. Let $\underline{Q} \subseteq P$ be a vertex of $M$. If there exists $x \in G - H$ such that $M$ is $(xPx^{-1} \cap P)$-regular, then $\underline{Q}$ is $G$-conjugate to a subgroup of $xPx^{-1} \cap P$, so there exists $y \in G$ such that $y\underline{Q}y^{-1} \subseteq xPx^{-1} \cap P$, hence $\underline{Q} \subseteq P \cap y^{-1}Py \cap y^{-1}xP(y^{-1}x)^{-1}$. But, since $x \notin H, y \notin H$ or $y^{-1}x \notin H$. Therefore $\underline{Q} \subseteq P \cap y^{-1}Py$ or $\underline{Q} \subseteq P \cap (y^{-1}x)P(y^{-1}x)^{-1}$. Consequently there exists $z \notin H$ such that $\underline{Q} \subseteq P \cap z^{-1}Pz \subseteq z^{-1}Pz \cap H$. We consider $X$ to be an indecomposable component of $\mathrm{Res}_H^G M$. By Theorem 3.7.8. there exists an indecomposable $Y \in R_H$-mod with vertex $\underline{Q}$ such that $Y$ is a component of $\mathrm{Res}_H^G M$, and $M$ is a component of $R \otimes_{R_H} Y$. Thus $Y$ is $(z^{-1}Pz \cap H)$-regular and therefore $X$ is $(z^{-1}Pz \cap H)$-regular. In particular, $N$ is $(z^{-1}Pz \cap H)$-regular for some $z \notin H$, a contradiction. Therefore $\underline{Q}$ is not $G$-conjugate to a subgroup of $zPz^{-1} \cap P$ for all $x \in G - H$. Since $N$ is the only indecomposable component of $\mathrm{Res}_H^G M$ which is not $(xPx^{-1} \cap H)$-regular for all $x \in G - H$, we obtain that $P$ is a vertex of $M$. Suppose now that any $W \in \{M_1, \ldots, M_t\}$ has vertex $P$. Then there exists an indecomposable component of $\mathrm{Res}_H^G W$

which is not $(xPx^{-1} \cap H)$-regular for all $x \in G - H$. Hence $W = M$. From the above considerations we obtain $g : \mathcal{C}_{R_H,P} \to \mathcal{C}_{R,P}, g([N]) = [M]$, and $g \circ f$ and $f \circ g$ are identical maps. □

The map $f : \mathcal{C}_{R,P} \to \mathcal{C}_{R^{(H)},P}$ given above is called a **Green correspondence**.

## 3.8 Exercises

The standing assumption here is that $R = \sum_{\sigma \in G} R_\sigma$ is an almost strongly graded ring over a finite group $G$. Because $R_{\sigma^{-1}} R_\sigma = R_e$ for all $\sigma \in G$, then there exists $a_i^\sigma \in R_{\sigma^{-1}}$, $b_i^\sigma \in R_\sigma$ such that

$$1 = \sum_{i \in I_\sigma} a_i^\sigma b_i^\sigma, I_\sigma \text{ is a finite set} \tag{1}$$

Assume also that $M, N \in R$-mod and $f \in \mathrm{Hom}_{R_e}(M, N)$. We define the map $\widetilde{f} : M \to N$ by the equality

$$\widetilde{f}(m) = \sum_{\sigma \in G} \sum_{i \in I_\sigma} a_i^\sigma f(b_i^\sigma m) \tag{2}$$

Then prove the following statements :

1. $\widetilde{f}$ is $R$-linear map
2. Assume that $n = |G| < \infty$. Let $N \subset M$ be a submodule of $N$ such that $N$ is a direct summand of $M$ in $R_e$-mod. If $M$ has no $n$-torsion, prove there exists an $R$-submodule $P$ of $M$ such that $N \oplus P$ is essential in $M$ as an $R_e$-module. Furthermore, if $M = nM$, then $N$ is a direct summand of $M$ as $R$-module.

   **Hint :** (Following the proof of Lemma 1. from [122]). We have $f : M \to N$ as $R_e$-modules such that $f(m) = m$ for all $m \in N$. Let $\widetilde{f} : M \to N$ be as in Exercise 1. If $x \in N$, then $\widetilde{f}(x) = nx$. We put $P = \mathrm{Ker}\widetilde{f}$ and we prove that $N \cap P = 0$ (since $M$ has no $n$-torsion) and $N \oplus P$ is essential in $M$ as an $R_e$-module. The last part of the exercise is clear.
3. Assume that $N$ is an $R$-submodule of $M$ and $M$ has no $n$-torsion. Prove

   i) There exists an $R$-submodule $P \subset M$, such that $N \oplus P$ is essential in $M$ as an $R_e$-submodule

   ii) $N$ is essential in $M$ as $R$-module if and only if $N$ is essential in $M$ as $R_e$-module.

   **Hint :** (Following the proof of Lemma 2 from [122])

   i) Let $L$ be an $R_e$-submodule of $M$ maximal with respect to the property that $N \cap L = 0$. Then $N \oplus L$ is essential in $M$ as $R_e$-module. Let $(N \oplus L)^* = \cap_{\sigma \in G} R_\sigma (N \oplus L) = \cap_{\sigma \in G} (R_\sigma N +$

$R_\sigma L) = \cap_{\sigma\in G}(N + R_\sigma L)$. By Lemma 3.5.5, $(N \oplus L)^*$ is essential in $M$ as $R_e$-module. If $K = (N \oplus L)^*$, then $N \subset K \subset N \oplus L$, so $K = N \oplus (K \cap L)$. By exercise 2. there exists an $R$-submodule $U$ of $K$ such that $N \oplus U$ is essential in $K$ as $R_e$-module. Hence $N \oplus U$ is essential in $M$ as $R_e$-module.

ii) follows directly from i.

4. Let $R = \sum_{\sigma\in G} R_\sigma$ be an almost strongly graded ring with $n = |G| < \infty$ and $M$ a simple left $R$-module. Prove :

   i) The restriction ${}_{R_e}M$ of $M$ over the ring $R_e$ contains a simple submodule $W$ and we have ${}_{R_e}M = \sum_{\sigma\in G} R_\sigma W$

   ii) ${}_{R_e}M$ is semi-simple in $R_e$-mod.

   **Hint :** If $x \in M, x \neq 0$, then $M = Rx = \sum_{\sigma\in G} R_\sigma x$. Since $R_\sigma$ is a finitely generated $R_e$-module (see Section 1.) for any $\sigma \in G$, then $R_\sigma x$ is a finitely generated $R_e$-module for any $\sigma \in G$. So $M$ is a finitely generated $R_e$-module. Then there exists a maximal $R_e$-module $K$ of ${}_{R_e}M$. If we put $K_0 = \cap_{\sigma\in G} R_\sigma K$ then $K_0$ is an $R$-submodule of $M$, hence $K_0 = 0$. From the exact exact sequence

$$0 \longrightarrow M \longrightarrow \oplus_{\sigma\in G} M/R_\sigma K$$

   results that $M$ is a semisimple $R_e$-module (since $M/R_\sigma K$ is also maximal for any $\sigma \in G$). Let $W$ be a simple $R_e$-submodule of $M$. On the other hand since $\sum_{\sigma\in G} R_\sigma W$ is also a nonzero submodule of $M$, we have $M = \sum_{\sigma\in G} R_\sigma W$.

5. Let $R = \sum_{\sigma\in G} R_\sigma$ be an almost strongly graded ring $n = |G| < \infty$. Prove :

   i) If $R$ is a semisimple Artinian ring, then $R_e$ is a semisimple Artinian ring.

   ii) If $n$ is invertible in $R_e$ and $R_e$ is a semisimple Artinian ring, then $R$ is a semisimple Artinian ring

   **Hint :**

   i) Results from exercise 4.

   ii) We can apply exercise 2.

6. Let $R = \oplus_{\sigma\in G} R_\sigma$ be a strongly graded ring, where $n = |G| < \infty$. Prove :

   i) If $R$ is a von Neumann regular ring, then $R_e$ is a von Neumann regular ring (it is not necessary that $n$ is finite). (Recall that the ring $R$ is Von Neumann regular if for any $x \in R$, there exists $y \in R$ such that $x = xyx$).

ii) If $n$ is invertible in $R$ and $R_e$ is a von Neumann regular ring then $R$ is a von Neumann regular ring.

**Hint :**

i) If $a \in R_e$ then $a = aba$ with $b \in R$. Since $a$ is a homogeneous element, then $a = ab_e a$ where $b_e$ is a homogeneous composand of degree $e$ of $b$.

ii) Let $a \in R$ and $I = Ra$. Because $R$ is finitely generated and a projective left $R_e$-module and because $I$ is a finitely generated $R_e$-submodule of $R$, since $R_e$ is von Neumann regular results that $Ra$ is direct sumand of ${}_R R$ as $R_e$-module. By exercise 2 results that $I = Ra$ is direct sumand of ${}_R R$ as $R$-module so $R$ is von Neumann regular (we remark that for ii. we can assume that $R$ is only an almost strongly graded ring).

7. Let $R = \oplus_{\sigma \in G} R_\sigma$ be a strongly graded ring where $n = |G| < \infty$. If $M \in R$-mod we denote by $Z_R(M)$ (resp. $Z_{R_e}(M)$) the singular submodule of $M$ in $R$-mod (resp. in $R_e$-mod) (we recall that $Z_R(M) = \{x \in M | \}\mathrm{Ann}_R(x)$ is an essential left ideal of $R$). Prove :

i) $Z_{R_e}(M) \subseteq Z_R(M)$

ii) If $R$ has no $n$-torsion then $Z_{R_e}(M) = Z_R(M)$.

iii) If $M \in R$-gr and $R$ has no $n$-torsion then $Z_R(M)$ is a graded submodule of $M$ and $Z_R(M) = RZ_{R_e}(M_e)$.

**Hint :**

i) If $x \in Z_{R_e}(M)$, then $J = \mathrm{Ann}_{R_e}(x)$ is an essential left ideal of $R_e$. If we put $I = RJ$, since $R$ is strongly graded then $I$ is an essential left ideal of $R$ clearly $Ix = 0$ and $x \in Z_R(M)$

ii) If $x \in Z_R(M)$ then $I = \mathrm{Ann}_R(x)$ is an essential left ideal of $R$. By Exercise 3. results that $I$ is essential in $R$ as left $R_e$-module. Clearly $J = I \cap R_e$ is essential left ideal of $R_e$ and $Jx = 0$ so $x \in Z_{R_e}(M)$.

iii) Follows from ii.

8. Let $R = \oplus_{\sigma \in G} R_\sigma$ be a strongly graded ring where $n = |G| < \infty$ and $n$ is invertible in $R_e$. Then $R$ is a left hereditary (resp. semi-hereditary) ring, if and only if $R_e$ is left hereditary (resp. semi-hereditary).

An extensive study of tame orders in relation with the strongly graded situation is contained in [138].

**Hint :** Suppose that $R$ is a left hereditary (resp. semi-hereditary) ring. If $I$ is a left ideal (a left finitely generated ideal) of $R_e$, then $RI$ is a left ideal (resp. a left finitely generated ideal) of $R$ and therefore $RI$ is a

projective module. Because $RI$ is a left graded ideal, then $(RI)_e = I$ is a projective $R_e$-module. Conversely, we assume that $R_e$ is a left hereditary (resp. semi-hereditary) ring. Let $K \subseteq R$ be a left ideal (resp. finitely generated) of $R$. There exists a set (resp. a finite set) $I$ such that

$$R^{(I)} \xrightarrow{\varphi} K \hookrightarrow R$$

where $\varphi$ is a surjective $R$-linear. Because $R$ is a finitely generated projective left $R_e$-module, then $K$ is also a projective $R_e$-module and therefore there exits an $R_e$-homomorphism $\psi : K \to R^{(I)}$ such that $\varphi \circ \psi = 1_K$. Using the result of exercise 1. we may consider an $R$-homomorphism $\widetilde{\psi} : K \to R^{(I)}$. But $(\varphi \circ \frac{1}{n}\widetilde{\psi})(x) = x$ for any $x \in K$ so $\varphi \circ \frac{1}{n}\widetilde{\psi} = 1_K$. In conclusion : $K$ is a left projective $R$-module.

In the following exercises $R = \sum_{\sigma \in G} R_\sigma$ is an almost strongly ring over a finite group $G$.

**Notations :**

1. If $M \in R$-mod, then by $\mathcal{L}_R(M)$ (resp. $\mathcal{L}_{R_e}(M)$) we denote the lattice of $R$-submodule of $M$ (resp. $R_e$-submodules of $M$).

2. By $K\text{-dim}_R(M)$ (resp. $K\text{-dim}_{R_e}M$) we denote the Krull dimension of $M$ over the ring $R$ (resp. over the ring $R_e$).

3. By $G\text{-}_R M$ (resp. $G\text{-dim}_{R_e}M$) we denote the Gabriel dimension of $M$ over the ring $R$ (resp. over the ring $R_e$ - see appendix B).

Let $M \in R$-mod and $N$ be an $R_e$-submodule of $M$. We denote by $N^* = \cap_{\sigma \in G} R_\sigma N$ and $N^{**} = \sum_{\sigma \in G} R_\sigma N$. Clearly $N^*$ is the largest $R$-submodule of $M$ contained in $N$ and $N^{**}$ is the smallest $R$-submodule of $M$ such that $N \subseteq N^{**}$.

9. Let $M \in \mathbb{Z}$-mod and $L$ be an $R_e$-submodule of $M$. Then

   i) $\mathcal{L}_{R_e}(M/R_\sigma L) \simeq \mathcal{L}_{R_e}(M/L)$

   ii) $\mathcal{L}_{R_e}(R_\sigma L) \simeq \mathcal{L}_{R_e}(L)$

   **Hint**

   i) If $X/L \in \mathcal{L}_{R_e}(M/L)$ then $X/L \to R_\sigma X/R_\sigma L$ is an isomorphism of lattice. The inverse isomorphism is $Y/R_\sigma L \to R_{\sigma^{-1}} Y/L$ where $Y/R_\sigma L \in \mathcal{L}_{R_e}(M/R_\sigma L)$

   ii) Is clear

   If $M \in R$-mod, we let $\text{rank}_R M$, $\text{rank}_{R_e} M$ denote the Goldie dimension of $M$ over these rings.

10. Let $|G| = n$ and $\text{rank}_R M = m$. Then $\text{rank}_{R_e} M \leq mn$.

    **Hint :** Let $N$ be a $R_e$-submodule of $M$ maximal with the property $N^* = 0$. Let $A_1, \ldots, A_t$ be $R_e$-submodules of $M$ strictly containing $N$ whose sum is direct modulo $N$. Then $A_i^* \neq 0$ for each $1 \leq i \leq t$. If $t > m$ then for some $i$ $(\sum_{j\neq i} A_j^*) \cap A_i^* \neq 0$. Because $\left((\sum_{j\neq i} A_j) \cap A_i\right)^* \supseteq (\sum_{j\neq i} A_j^*) \cap A_i^*$ it follows that $\left((\sum_{j\neq i} A_j) \cap A_i\right)^* \neq 0$ and so $\sum_{j\neq i} A_j \cap A_i \supsetneq N$, a contradiction. Then $\text{rank}_{R_e}(M/N) \leq m$. By Exercise 9. $\text{rank}_{R_e}(M/R_\sigma N) \leq m$ for all $\sigma \in G$. Because $0 = N^* = \cap_{\sigma\in G} R_\sigma N$ we have $0 \to M \to \oplus_{\sigma\in G} M/R_\sigma N$ and so $\text{rank}_{R_e} M \leq mn$.

11. If $R$ has finite Goldie dimension, then $R_e$ has finite Goldie dimension.

    **Hint :** We apply Exercise 10.

12. Let $M \in R$-mod. Then ${}_R M$ is Noetherian in $R$-mod if and only if ${}_{R_e} M$ is Noetherian in $R_e$-mod.

    **Hint :** The inplication " $\Leftarrow$ is clear. So assume that ${}_R M$ is Noetherian. Let $N$ be the $R_e$-submodule of $M$ maximal such that $N^* = 0$. Since $0 = N^* = \cap_{\sigma\in G} R_\sigma N$ then $0 \to M \to \oplus_{\sigma\in G} M/R_\sigma N$. So by exercise 9 it is sufficient to prove that $M/N$ is $R_e$-Noetherian.

    Let $M_1/N \subset M_2/N \subset \ldots \subset M_n/N \subset \ldots$ an ascending chain of non-zero $R_e$-submodules of $M/N$. Then $N \subsetneq M_1$, and therefore $M_1^* \neq 0$. Since $M_1^*$ is a nonzero $R$-submodule of $M$, using the Noetherian induction, we have that $M/M_1^*$ is $R_e$-Noetherian. Since $M_1^* \subset M_1$ then there exists $n$ such that $M_i/M_1^* = M_{i+1}/M_1^*$ for any $i \geq n$ so $M_i = M_{i+1} = \ldots$.

13. Let $M \in R$-mod and $N \subseteq M$ be an $R_e$-submodule of $M$ such that $N^* = 0$. Then

    i) $K\text{-dim}_{R_e} M = K\text{-dim}_{R_e}(M/N)$ if either side exists.

    ii) $G\text{-dim}_{R_e} M = G\text{-dim}_{R_e}(M/N)$ if either side exists.

    **Hint :** Since $0 = N^* = \cap_{\sigma\in G} R_\sigma N$ then there exits in $R_e$-mod the canonical monomorphism $0 \to M \to \oplus_{\sigma\in G} M/R_\sigma N$. Now we can apply exercise 9.

14. Let $M \in R$-mod. Then $K\text{-dim}_R M = K\text{-dim}_{R_e} M$ if either side exists. In particular $M$ is an Artinian $R$-module if and only if $M$ is an Artinian $R_e$-module.

    **Hint** If $K\text{-dim}_{R_e} M$ exists, clearly $K\text{-dim}_R M$ exists and moreover $K\text{-dim}_R M \leq K\text{-dim}_{R_e} M$. Assume that $K\text{-dim}_R M = \alpha$ and by induction on $\alpha$ we prove that $K\text{-dim}_{R_e} M \leq \alpha$. Clearly we can reduce the problem when $M$ is $\alpha$-critical (see Appendix B). Let $N$ be an $R_e$-submodule of

$M$, maximal with the property that $N^* = 0$. If $X$ is $R_e$-submodule of $M$ such that $N \subsetneq X$, then $X^* \neq 0$ and therefore $K\text{-dim}_R M/X^* < \alpha$ and by induction we have $K\text{-dim}_{R_e} M/X^* < \alpha$. Since $X^* \subset X$, we have $K\text{-dim}_{R_e} M/X < \alpha$. Using the Appendix B, yields that $K\text{-dim}_{R_e} M/N \leq \alpha$ so by exercise 13. we have $K\text{-dim}_{R_e} M \leq \alpha$. Conversely if we assume that $K\text{-dim}_R M$ exists then $K\text{-dim}_{R_e} M$ exists, cf. [119] or [157].

15. Let $M \in R$-mod, then $G\text{-dim}_R M = G\text{-dim}_{R_e} M$ if either side exists. In particular $M$ is a semi-Artinian $R_e$-module if and only if ${}_R M$ is semi-Artinian.

    **Hint :** See [20], [21], [141], [157].

## 3.9 Comments and References for Chapter 3

Strongly graded rings have enjoyed a growing interest in connection to the theory of graded rings and generalized crossed products; also it turned out that the property of being strongly graded is exactly providing a tool for introducing "affine" open sets in the study of the noncommutative geometry of a projective noncommutative scheme.

It is unnecessary to repeat most of the material about strongly graded rings already included in the book [150], so we focussed on the study of the category $R$-gr for a strongly graded ring $R$. The first main result is "Dade's Theorem" mentioned in Section 3.1; it provides an equivalence of categories between $R$-gr and $R_e$-mod, defined via the induction functor. In Section 3.2 the foregoing subject is elaborated upon and necessary conditions are given such that a category equivalence between $R$-gr and $R_e$-mod would force $R$ to be strongly graded (see Theorem 3.2.1). Example 3.2.4 establishes that an arbitrary equivalence of categories between $R$-gr and $R_e$-mod does not lead to $R$ being strongly graded in general.

In Section 3.4 we study the endomorphism ring $End_R(M)$ where $R$ is a strongly graded ring and $M$ is an $R$-module (not necessarily a graded one). Theorem 3.4.1 (Miyashita) is essential to this study. In particular it follows that $G$ has a natural action on $C_R(R_e)$ (that is the centralizer of $R_e$ in $R$). This theorem also allows the definition of a trace for an arbitrary morphism of $R$-modules. In section 3.5 we use this trace to arrive at an extension of Maschke's theorem for graded modules; Theorem 3.5.7, "essentially Maschke's theorem" closes the section. The idea of the proof of this result belongs to M. Lorenz and D. Passman (see [122]).

Section 3.6 is dedicated to $H$-regulated modules, where $H$ is a subgroup of $G$, a concept first introduced in representation theory of groups. This allows the introduction for strongly graded rings of J.A. Green's theory originating in the study of indecomposable representations of finite groups over a field. The fundamental results of this section are phrased in Theorems 3.7.6, 3.7.7, 3.7.8 and 3.7.10 Exercises in Section 3.8 extend the applicability of the results in this chapter.

### Some References

- P.R. Boisen [25]
- E. C. Dade [49], [55]
- J. A. Green, [88], [89]
- G. Karpilowsky [112]
- M. Lorenz, D. Passman [122]

- H. Maschke [127]
- A. Mărcus, [124], [126]
- C. Menini, C. Năstăsescu [130]
- C. Năstăsescu [141]
- C. Năstăsescu, F. Van Oystaeyen [146]
- D. Passman [169], [170]
- F. Van Oystaeyen [193], [195], [197]
- K.H. Ulbrich [183]

# Chapter 4

# Graded Clifford Theory

Throughout this section $R$ is a $G$-graded ring and $\Sigma = \oplus_{\sigma\in G}\Sigma_\sigma$ will be a simple object of $R$-gr.

## 4.1 The Category Mod$(R|\Sigma)$

The full additive subcategory generated by $\Sigma$ in $R$-mod will be denoted by $\mathrm{Mod}(R|\Sigma)$, that is the subcategory consisting of epimorphic images of direct sums of copies of $\Sigma$.

### 4.1.1 Lemma

Every $M \in \mathrm{Mod}(R|\Sigma)$ is semisimple as an $R_e$-module and every simple $R_e$-submodule $S$ of $M$ is $R_e$-isomorphic to $\Sigma_\sigma$ for some $\sigma \in \mathrm{sup}(\Sigma)$.

**Proof** Let $M$ be an epimorphic image of a direct sum $\Sigma^{(I)}$ for some index set $I$. Proposition 2.7.1 entails that $\Sigma^{(I)}$ is semisimple as an $R_e$-module and $\Sigma^{(I)} = \oplus_{\sigma\in G}\Sigma_\sigma^{(I)}$. Consequently $M$ is a semisimple $R_e$-module too and every simple $R_e$-submodule $S$ of $M$ is necessarily isomorphic to one of the $\Sigma_\sigma$. □

### 4.1.2 Lemma (The key lemma, cf. E. Dade [52])

For $M \in \mathrm{Mod}(R|\Sigma)$ and $\sigma \in \mathrm{sup}(\Sigma)$, any $R_e$-linear $\varphi : \Sigma_\sigma \to M$ extends to a unique $R$-linear $\varphi^e : \Sigma \to M$.

**Proof** First we check the uniqueness of $\varphi^e$. Assume that both $\psi$ and $\psi' : \Sigma \to M$ are $R$-linear maps with $\psi|\Sigma_\sigma = \psi'|\Sigma_\sigma$. Since $\Sigma_\sigma \neq 0$ there is an $x \in \Sigma_\sigma$, $x \neq 0$, and $Rx$ is a nonzero graded submodule of $\Sigma$. Therefore $\Sigma = Rx$ because $\Sigma$ is gr-simple. Any $y \in \Sigma$ is of the form $y = ax$ with $a \in R$.

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We calculate :

$$\psi(y) = \psi(ax) = a\psi(x) = a\psi'(x) = \psi'(ax) = \psi(y)$$

hence $\psi = \psi'$ follows. Now we establish the existence of an $R$-linear map $\varphi^e = \Sigma \to M$ such that $\varphi^e|\Sigma_\sigma = \varphi$. There exists an $R$-epimorphism, $\psi : \Sigma^{(I)} \to M$, for some index set $I$. As $\Sigma^{(I)}$ and $M$ are semisimple $R_e$-modules, see Lemma 4.1.1, there exists an $R_e$-linear map, $\theta : \Sigma_\sigma \to \Sigma^{(I)}$ such that $\varphi = \psi \circ \theta$. In order to show that $\varphi$ may be extended it suffices to extend $\theta$, hence we may assume that $M = \Sigma^{(I)}$. We may assume that $I$ is finite because $\Sigma_\sigma$ is a finitely generated $R_e$-module, say $I = \{1, 2, \ldots, n\}$. For $j \in \{1, \ldots, n\}$ let $i_j : \Sigma \to \Sigma^n$ and $\pi_j : \Sigma^n \to \Sigma$ be the canonical injection resp. projection. The identity $1_{\Sigma^n} : \Sigma^n \to \Sigma^n$ decomposes as $\sum_{j=1}^n i_j \circ \pi_j$, therefore $\varphi = \sum_{j=1}^n i_j \circ (\pi_j \circ \varphi)$, where for $j = 1, \ldots, n$, $\pi_j \circ \varphi : \Sigma_\sigma \to \Sigma$. If every $\pi_j \circ \varphi$ extends to $(\pi_j \circ \varphi)^e : \Sigma \to \Sigma$ then we may define $\varphi^e$ by $\varphi^e = \sum_{j=1}^n i_j \circ (\pi_j \circ \varphi)^e$. Obviously $\varphi^e$ is $R$-linear and also we have that $\varphi^e|\Sigma_\sigma = \varphi$. Consequently we may assume that $M = \Sigma$. Since $\Sigma_\sigma$ is a finitely generated $R_e$-module there exists a finite subset $F$ of $\sup(\Sigma)$ such that $\mathrm{Im}\varphi \subset \oplus_{\tau \in F}\Sigma_\tau$. For $\tau \in F$ put $i_\tau : \Sigma_\tau \to \oplus_{\gamma \in F}\Sigma_\gamma$, resp. $\pi_\tau : \oplus_{\gamma \in F}\Sigma_\gamma \to \Sigma_\tau$ for the canonical injection, resp. projection. Again the identity of $\oplus_{\gamma \in F}\Sigma_\gamma$ decomposes as $\sum_{\gamma \in F} i_\tau \circ \pi_\tau$ and we put $\varphi_\tau = \pi_\tau \circ \varphi$. This leads to $\varphi = \sum_{\tau \in F} i_\tau \circ \varphi_\tau$ with $\varphi_\tau : \Sigma_\sigma \to \Sigma_\tau$. Now, in view of Theorem 2.7.2 there exists a unique $R$-linear $\varphi_\tau^e = 1_R \overline{\otimes}_{R_e} \varphi_\tau$, $R\overline{\otimes}_{R_e}\Sigma_\sigma \to R\overline{\otimes}_{R_e}\Sigma_\tau$. Since $R\overline{\otimes}_{R_e}\Sigma_\sigma \simeq \Sigma(\sigma)$ and $R \otimes_{R_e} \Sigma_\tau \simeq \Sigma(\tau)$, we obtain an $R$-linear $\varphi_\tau^e : \Sigma \to \Sigma$. Moreover, $\varphi^e|\Sigma_\sigma = i_\tau \circ \varphi_\tau$. Putting $\varphi^e = \Sigma_{\tau \in F}\varphi_\tau^e$ we have $\varphi^e|\Sigma_\sigma = \sum_{\tau \in F} \varphi_\tau^e|\Sigma_\sigma = \sum_{\tau \in F} i_\tau \circ \varphi_\tau = \varphi$. □

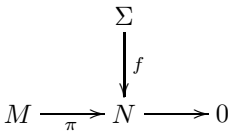
### 4.1.3 Propostion

The subcategory $\mathrm{Mod}(R|\Sigma)$ is closed under taking :$R$-submodules, $R$-quotient modules and direct sums. Thus $\mathrm{Mod}(R|\Sigma)$ is a closed subcategory. Moreover, $\Sigma$ is a finitely generated projective generator of $\mathrm{Mod}(R|\Sigma)$, hence the latter is a Grothendieck category.

**Proof**

It is clear that quotient objects and direct sums of objects of $\mathrm{Mod}(R|\Sigma)$ are again in $\mathrm{Mod}(R|\Sigma)$. Now let $N$ be a nonzero $R$-submodule of $M$ and $M$ in $\mathrm{Mod}(R|\Sigma)$. Since $M$ is semisimple as an $R_e$-module, so is $N$. Let $N$ be decomposed as $N = \oplus_{i \in I} S_i$, where $(S_i)_{i \in I}$ is a family of simple $R_e$-submodules of $N$ In view of Lemma 4.1.1, every $S_i$ is isomorphic to $\Sigma_\sigma$ for a suitable $\sigma \in \sup(\Sigma)$, let $\varphi : \Sigma_\sigma \to S_i$ be this $R_e$-isomorphism. It follows from Lemma 4.1.2 that there exists $\varphi_i^e : \Sigma \to M$ such that $\varphi_i^e|\Sigma_\sigma = \varphi_i$. Since $\Sigma = R\Sigma_\sigma$ we obtain : $\varphi_i^e(\Sigma) = \varphi_i^e(R\Sigma_\sigma) = R\varphi_i^e(\Sigma_\sigma) = R\varphi_i(\Sigma_\sigma) = RS_i = N$. Consequently, we may consider $\varphi_i^e : \Sigma \to N$. The family $\{\varphi_i^e, \in I\}$ defines an $R$-linear $\psi : \Sigma^{(I)} \to N$ such that $\mathrm{Im}\psi = \sum_{i \in I} RS_i = N$ and therefore $N \in \mathrm{Mod}(R|\Sigma)$. It is clear that $\Sigma$ is a finitely generated $R$-module and a generator for the

category Mod($R|\Sigma$). In order to establish projectivity of $\Sigma$ in Mod($R|\Sigma$) we consider the following diagram in the category Mod($R|\Sigma$) :




where $f \neq 0$. For some $\sigma \in \sup(\Sigma)$ we have $f|\Sigma_\sigma \neq 0$. Semisimplicity of both $M$ and $N$ as $R_e$-modules yields that there exists $g : \Sigma_\sigma \to M$ such that $\pi \circ g = f|\Sigma_\sigma$. In view of Lemma 4.1.2 there exists an $R$-linear map $g^e : \Sigma \to M$ such that $g^e|\Sigma_\sigma = g$. It is obvious that $\pi \circ g^e|\Sigma_\sigma = f|\Sigma_\sigma$ and from the uniqueness statement in Lemma 4.1.2 it follows that $\pi \circ g^e = f$. Therefore, $\Sigma$ is projective in Mod($R|\Sigma$). That Mod($R|\Sigma$) is a Grothendieck category follows from the fact that it is a closed subcategory of $R$-mod having a projective generator of finite type. □

Put $\Delta = \mathrm{End}_R(\Sigma) = \mathrm{END}_R(\Sigma)$; this is a $G$-graded ring with multiplication $f * g$ given by $g \circ f$ for $f$ and $g$ in $\Delta$. From Proposition 2.7.1, it follows that $\Delta$ is a gr-division ring and $\Delta = \oplus_{\tau \in G(\Sigma)} \Delta_\tau$ where $G(\Sigma)$ is the stabilizer subgroup of $\Sigma$. On $\Sigma$ there is a natural structure of a graded $R$-$\Delta$-bimodule and so we have the usual additive functors :

$$\begin{array}{ll} \mathrm{Hom}_R({}_R\Sigma_\Delta, -) : & \mathrm{Mod}(R|\Sigma) \to \Delta\text{−mod} \\ \Sigma \otimes_\Delta - : & \Delta\text{−mod} \to \mathrm{Mod}(R|\Sigma) \end{array}$$

Mitchel's theorem (cf. [181], Theorem ?) states that a Grothendieck category $\mathcal{A}$ with a small projective generator $U$ is equivalent to $A$-mod where $A = \mathrm{End}_{\mathcal{A}}(U)$. The equivalence is given by the functor $\mathrm{Hom}(U, -) : \mathcal{A} \to A$-mod. Recall that the multiplication of $A$ is given by the opposite of composition. Now we are ready to phrase the main result in this section, as a consequence of the foregoing results and observations.

### 4.1.4 Theorem

Let $\Sigma$ be simple in $R$-gr and $\Delta = \mathrm{End}_R(\Sigma)$. The categories Mod($R|\Sigma$) and $\Delta$-mod are equivalent via the functors :

$$\begin{array}{ll} \mathrm{Hom}_R({}_R\Sigma_\Delta, -) : & \mathrm{Hom}(R|\Sigma) \to \Delta\text{−mod} \\ {}_R\Sigma_\Delta \otimes_\Delta - \ : & \Delta\text{−mod} \to \mathrm{Mod}(R|\Sigma) \end{array}$$

### 4.1.5 Remarks

1. Notation is as in Section 2.6. Proposition 4.1.3 entails that Mod($R|\Sigma$) = $\sigma[\Sigma]$.

2. Consider a gr-semisimple $M = \oplus_{i\in I}\Sigma_i$. From Lemma 4.1.2. we derive a result similar to Proposition 4.1.3 i.e. $\mathrm{Mod}(R|M)$ is a closed subcategory of $R$-mod and $\{\Sigma_i, i \in I\}$ is a family of finitely generated and projective generators. So $M$ is a projective generator of $\mathrm{Mod}(R|\Sigma)$ but not finitely generated in general ! We obtain again : $\mathrm{Mod}(R|M) = \sigma[M]$.

## 4.2 The Structure of Objects of $\mathbf{Mod}(R|\Sigma)$ as $R_e$-modules

Let $\Sigma = \oplus_{\gamma\in G}\Sigma_\gamma$ be simple in $R$-gr and let $M \neq 0$ be in $\mathrm{Mod}(R|\Sigma)$. From Lemma 4.1.1 we retain that $M$ is semisimple as an $R_e$-module. For a simple $R_e$-submodule $S$ in $M$ there exists a $\sigma \in \sup(\Sigma)$ such that $S \cong \Sigma_\sigma$ as $R_e$-submodules.

### 4.2.1 Lemma

For $\tau \in G$, $R_\tau S \cong \Sigma_{\tau\sigma}$.

**Proof** There exists an $R_e$-linear $\varphi : \Sigma_\sigma \to M$ such that $\varphi(\Sigma_\sigma) = S$. We may consider $\varphi^e : \Sigma \to M$ such that $\varphi^e|\Sigma_\sigma = \varphi$ (see Lemma 4.2.2). Now we calculate : $\varphi^e(R_\tau\Sigma_\sigma) = R_\tau\varphi^e(\Sigma_\sigma) = R_\tau\varphi(\Sigma_\sigma) = R_\tau S$. Since $\Sigma_\sigma \neq 0$ we have $R\Sigma_\sigma = \Sigma$. Thus for any $\tau \in G$ we have $R_\tau\Sigma_\sigma = \Sigma_{\tau\sigma}$, hence $\varphi^e(\Sigma_{\tau\sigma}) = R_\tau S$. In case $\Sigma_{\tau\sigma} = 0$, then $R_\tau S = 0$ and again $R_\tau S \cong \Sigma_{\tau\sigma}$. If $\Sigma_{\tau\sigma} \neq 0$ then $R_{\tau^{-1}}\Sigma_{\tau\sigma} = \Sigma_\sigma$. If $\varphi^e(\Sigma_{\tau\sigma}) = 0$ then $\varphi^e(\Sigma_\sigma) = \varphi^e(R_{\tau^{-1}}\Sigma_{\tau\sigma}) = R_{\tau^{-1}}\varphi^e(\Sigma_{\tau\sigma}) = 0$, hence $\varphi^e(\Sigma_\sigma) = \varphi(\Sigma_\sigma) = S = 0$, a contradiction. Therefore we must have $\varphi^e(\Sigma_{\tau\sigma}) \neq 0$, so, putting $\psi = \varphi^e|\Sigma_{\tau\sigma}$, $\psi : \Sigma_{\tau\sigma} \to R_\tau S$ is an $R_e$-isomorphism. □

### 4.2.2 Lemma

If $S$ is a simple $R_e$-submodule of $M$ and $\tau \in G$, then either $R_\tau S = 0$ or $R_\tau S$ is a simple $R_e$-submodule of $M$. In the latter case, for all $\gamma \in G$ : $R_\gamma R_\tau S = R_{\gamma\tau}S$ and also $R_{\tau^{-1}}R_\tau S = S$.

**Proof** For some $\sigma \in \sup(\Sigma)$ we have $S \cong \Sigma_\sigma$. The first statement in the lemma follows from Lemma 4.2.1. Assume that $R_\tau S \neq 0$. Then $R_\tau S \cong \Sigma_{\tau\sigma} \neq 0$ and we may use Lemma 4.2.1 again to obtan that $R_\gamma R_\tau S = \Sigma_{\gamma\tau\sigma}$. On the other hand $R_{\gamma\tau}S \cong \Sigma_{\gamma\tau\sigma}$. Since $R_\gamma R_\tau \subset R_{\gamma\tau}$, $R_\gamma R_\tau S \subset R_{\gamma\tau}S$. Now if $\Sigma_{\gamma\tau\sigma} = 0$ then $R_\gamma R_\tau S = R_{\gamma\tau}S = 0$. If $\Sigma_{\gamma\tau\sigma} \neq 0$ then $R_\gamma R_\tau S \neq 0$ would lead to $R_\gamma R_\tau S = R_{\gamma\tau}S$ because we already know that $R_{\gamma\tau}S$ is a simple $R_e$-submodule of $M$. Finally, putting $\gamma = \tau^{-1}$ leads to $R_{\tau^{-1}}R_\tau S = S$. □

### 4.2.3 Lemma

For any $\sigma \in \sup(\Sigma)$ there exists a simple $R_e$-submodule of $M$ isomorphic to $\Sigma_\sigma$.

**Proof** Since $\Sigma$ generates Mod$(R|\Sigma)$ there exists a nonzero $R$-linear $u : \Sigma \to M$. Put $\varphi = u|\Sigma_\sigma, , \varphi : \Sigma_\sigma \to M$. If $\varphi = 0$ then $u(\Sigma_\sigma) = 0$ implies $u(\Sigma) = u(R\Sigma_\sigma) = Ru(\Sigma_\sigma) = 0$ and this leads to a contradiction $u = 0$. So we must accept that $\varphi \neq 0$ or $S = \varphi(\Sigma_\sigma)$ is a simple $R_e$-submodule of $M$. □

For any $M \in$ Mod$(R|\Sigma)$ we let $\Omega_{R_e}(M)$ be the set of isomorphism classes of simple $R_e$-submodules of $M$. We use the notation $w = [S] \in \Omega_{R_e}(M)$ to denote the class $\{S' \in R_e\text{-mod } S' \cong S\}$. For any nonzero object $M$ in Mod$(R|\Sigma)$ it is clear that $\Omega_{R_e}(M) = \Omega_{R_e}(\Sigma)$. For $M \in$ Mod$(R|\Sigma)$ and $w \in \Omega_{R_e}(\Sigma)$ look at $M_w = \Sigma S'$, the sum ranging over the simple $R_e$-submodules belonging to $w$. We call $M_w$ the **isotypic** $w$**-component** of $M$. In case $M = M_w$ we say that $M$ is $w$**-isotopic** (sometimes $S$**-primary**, where $S \in w$).

The semisimplicity of $M$ in $R_e$-mod entails that :

$$M = \oplus_{w \in \Omega_{R_e}(\Sigma)} M_w$$

However, for $w \in \Omega_{R_e}(\Sigma)$ there exists a $\sigma \in \sup(\Sigma)$ such that $\Sigma_\sigma \in w$. Consequently we may write : $M_w = \oplus_{i \in I} S_i$, where $S_i \cong \Sigma_\sigma$ for any $i \in I$.

For another decomposition $M_w = \oplus_{j \in J} S_j$ then $|I| = |J|$ and the latter number (cardinality) is called the **length of** $M_w$, denoted by $l_{R_e}(M_w)$.

### 4.2.4 Lemma

Assume tat $M_w = \oplus_{i \in I} S_i$ with $S_i \cong \Sigma_\sigma$. If $\gamma \in G$ then $R_\gamma M_w = \oplus_{i \in I} R_\gamma S_i$ with $R_\gamma S_i \cong \Sigma_{\gamma\sigma}$. Moreover, if $\gamma\sigma \notin \sup(\Sigma)$, then $R_\gamma M_w = 0$. When $\gamma\sigma \in \sup(\Sigma)$ then $R_\gamma M_w = M_{w'}$ where $w'$ is the class containing the simple $R_e$-submodule $\Sigma_{\gamma\sigma}$.

**Proof** Clearly, $R_\gamma M_w = \sum_{i \in I} R_\gamma S_i$ where $R_\gamma S_i \cong \Sigma_{\gamma\sigma}$ (see Lemma 4.2.1). Hence, if $\gamma\sigma \notin \sup(\Sigma)$ we have $R_\gamma M_w = 0$. Assume that $\gamma\sigma \in \sup(\Sigma)$, then we are about to establish that $\sum_{i \in I} R_\gamma S_i$ is a direct sum. Start from the assumption that $R_\gamma S_i \cap (\sum_{j \neq i} R_\gamma S_j) \neq 0$ for some $i \in I$. Since $R_\gamma S_i \cong \Sigma_{\gamma\sigma}$ is a simple $R_e$-submodule of $M$ it follows from the foregoing assumption that $R_\gamma S_i \subset \sum_{j \neq i} R_\gamma S_j$. From Lemma 4.2.2. it follows that $S_i \subset R_{\gamma^{-1}} R_\gamma S_i$ hence $S_i \subset \sum_{j \neq i} R_{\gamma^{-1}} R_\gamma S_j = \sum_{j \neq i} S_j$, a contradiction. Threfore we have established that $R_\gamma M_w = \oplus_{i \in I} R_\gamma S_i$. Furthermore, it is clear that $R_\gamma M_w \subset M_{w'}$. Hence we may write :

$$M_{w'} = R_\gamma M_w \oplus (\oplus_{j \in J} T_j)$$

where $T_j \cong \Sigma_{\gamma\sigma}$ for every $j \in J$.

The above argumentation yields : $R_{\gamma^{-1}} M_{w'} \subset M_w$. On the other hand $R_{\gamma^{-1}} M_{w'} = \oplus_{i\in I} R_{\gamma^{-1}} R_\gamma S_i \oplus (\oplus_{j\in J} R_{\gamma^{-1}} T_j)$. Therefore, $\oplus_{j\in J} R_{\gamma^{-1}} T_j = 0$, but $R_{\gamma^{-1}} T_j \simeq \Sigma_\sigma \neq 0$, hence $J = \emptyset$. Finally, we arrive at $M_{w'} = R_\gamma M_w$. □

### 4.2.5 Theorem

Consider a nonzero $M$ in $\mathrm{Mod}(R|\Sigma)$.

i) $M$ decomposes as a direct sum of isotypical components $M_w \neq 0$, $M = \oplus\{M_w, w \in \Omega_{R_e}(\Sigma)\}$.

ii) For any $w, w' \in \Omega_{R_e}(\Sigma)$ we have $l_{R_e}(M_w) = l_{R_e}(M_{w'})$.

iii) $|\Omega_{R_e}(\Sigma)| = [\mathrm{Sup}(\Sigma) : G\{\Sigma\}] \leq [G : G\{\Sigma\}]$.

iv) $M$ may be viewed, in a natural way, as an object of the category of $G/G\{\Sigma\}$-graded $R$-modules.

**Proof**

i) Clear enough.

ii) Consider $w$ and $w'$ in $\Omega_{R_e}(\Sigma)$. There exist $\sigma, \tau \in \sup(\Sigma)$ such that $\Sigma_\sigma \in w$ and $\Sigma_\tau \in w'$. Put $\gamma = \tau\sigma^{-1}$, then we have $\tau = \gamma\sigma$ and $l_{R_e}(M_w) = l_{R_e}(M_{w'})$ follows from Lemma 4.2.4

iii) Considering $\sigma$ and $\tau$ in $\sup(\Sigma)$, we may apply Theorem 2.7.2 and derive from it that $\Sigma_\sigma \cong \Sigma_\tau$ as $R_e$-modules if and only if $\Sigma(\sigma) \cong \Sigma(\tau)$ in $R$-gr, if and only if $\sigma^{-1}\tau \in G\{\Sigma\}$. It follows directly from this that : $|\Omega_{R_e}(\Sigma)| = [\sup(\Sigma) : G\{\Sigma\}]$.

iv) Take $C \in G/G\{\Sigma\}$, say $C = \sigma G\{\Sigma\}$, say $C = \sigma G\{\Sigma\}$ for some $\sigma \in G$. In case $\sigma \notin \sup(\Sigma)$, put $M_C = 0$, otherwise put $M_C = M_w$ where $\Sigma_\sigma \in w$. It is completely clear now that $M = \oplus_{C\in G/G\{\Sigma\}} M_C$ and for any $\gamma \in C$ we also have $R_\gamma M_C \subset M_{\gamma C}$ (using Lemma 4.2.2 once more). □

## 4.3 The Classical Clifford Theory for Strongly Graded Rings

In this section $R$ is strongly graded by $G$. For $\sigma \in G$ the functor $R_\sigma \otimes_{R_e} - : R_e$-mod $\to R_e$-mod, $X \mapsto R_\sigma \otimes_{R_e} X$ is an equivalence of categories with inverse : $R_{\sigma^{-1}} \otimes_{R_e} - : R_e$-mod$\to R_e$-mod.

We say that $X$ and $Y$ in $R_e$-mod are **$G$-conjugate** if there exists $\sigma \in G$ such that $Y \cong R_\sigma \otimes_{R_e} X$. An $R_e$-module $X$ is said to be **$G$-invariant** if and only

if $X \cong R_\sigma \otimes_{R_e} X$ for every $\sigma \in G$. For a simple $R_e$-module $X$ and for any $\sigma \in G$ the $R_e$-module $R_\sigma \otimes_{R_e} X$ is again simple. Recall that $\Omega_{R_e}$ stands for the set of isomorphism classes of simple left $R_e$-modules. It is easy to define an action of $G$ on $\Omega_{R_e}$ as follows :

$$G \times \Omega_{R_e} \to \Omega_{R_e}, (g, [S]) \mapsto [R_g \otimes_{R_e} S]$$

where $[S]$ is the class of $S$.

Consider a semisimple $R_e$-module $M$. We use the notation $\Omega_{R_e}(M)$ as in the foregoing section.

### 4.3.1 Proposition

For $M$ and $R$ as above we have :

i) For a simple $R_e$-submodule $N$ in $M$ and any $\sigma \in G$ we have : $R_\sigma \otimes_{R_e} N \cong R_\sigma N$.

ii) The $G$-action defined on $\Omega_{R_e}$ induces a $G$-action on $\Omega_{R_e}(M)$.

**Proof**

i) Look at the canonical $R_e$-linear $\alpha : R_\sigma \otimes_{R_e} N \to R_\sigma N$ given by $\alpha(\lambda \otimes x) = \lambda x$, for $\lambda \in R_\sigma, x \in N$. Since $\alpha$ is surjective, $R_\sigma N \neq 0$ and $R_\sigma \otimes_{R_e} N$ is a simple $R_e$-module, thus $\alpha$ is an isomorphism.

ii) Follows in a trivial way from i.

### 4.3.2 Theorem (Clifford)

Consider a strongly $G$-graded ring $R$ and a simple left $R$-module $M$. Assume that there exists a nonzero simple $R_e$-submodule $N$ of $M$.

a. We have $M = \sum_{\sigma \in G} R_\sigma N$ and $M$ is a semisimple $R_e$-module.

b. The $G$-action on $\Omega_{R_e}(M)$ is transitive.

c. Let us write $H$ for the subgroup $G\{N\}$ in $G$, then :

$$H = \{\sigma \in G, R_\sigma N \cong N\}$$

d. If $M = \oplus_{i \in I} M_{w_i}$, where the $M_{w_i}$ are the non-zero isotypical components of $M$ as an $R_e$-module, then we have :

i) $l(M_{w_i}) = l(M_{w_j})$ for any $i, j \in I$.

ii) $|I| = [G : H]$.

In particular, if $M$ is a finitely generated $R_e$-module, then $[G : H] < \infty$ and $l(M_{w_i}) < \infty$ for any $i \in I$.

e. Assume that $w \in \Omega_{R_e}(M)$ is such that $N \leq M_w$. Then $M_w$ is a simple $R_H$-module and $R \otimes_{R_H} M_w \simeq M$.

**Proof**
We consider $\Sigma = R \otimes_{R_e} N \in R$-gr. Since $N$ is a simple $R_e$-module, $\Sigma$ is gr-simple. On the other hand the canonical map :

$$\alpha : R \otimes_{R_e} N \to M, \alpha(\lambda \otimes x) = \lambda x, \lambda \in R, x \in N$$

Since $\alpha(R_\sigma \otimes_{R_e} N) = R_\sigma N$ we have $M = \sum_{\sigma \in G} R_\sigma N$. Hence $M \in \mathrm{Mod}(R|\Sigma)$ (in fact $M$ is a simple object in the category $\mathrm{Mod}(R|\Sigma)$). Since $R$ is strongly graded, $\sup(\Sigma) = G$ and $G\{\Sigma\} = H$. Now the assertions a., b. and d. follow from Theorem 4.2.5.

(b) If $w, w' \in \Omega_{R_e}(M)$, then there exist two simple $R_e$-submodules $S, S'$ of $M$ such that $w = [S]$ and $w' = [S']$. Assertion a. implies that there exist $\sigma, \tau$ in $G$ such that $S \cong R_\sigma N \cong R_\sigma \otimes_{R_e} N$ and $S' \cong R_\tau N \cong R_\tau \otimes_{R_e} N$. It is clear that $S' = R_{\tau\sigma^{-1}} \otimes_{R_e} S$, hence the action of $G$ on $\Omega_{R_e}(M)$ is transitive.

(e) Assertion iv. of Theorem 4.2.5 implies that $M = \oplus_{C \in G/H} M_C$ is a $G/H$-graded $R$-module. In fact $\{M_C, C \in G/H\}$ consists exactly of all isotypical components of $M$ as a semisimple $R_e$-module. Now exercise 9 in Section 2.12 allows to finish the proof. On the other hand, a direct proof using the same argument as in Dade's Theorem (Theorem 3.1.1) may also be given from hereon. □

### 4.3.3 Remarks

1. For $R$ and $M$ as in the theorem, the assumption that $M$ contains a nonzero simple $R_e$-module holds in the following situations :

   a. $R_e$ is a left Artinian ring.

   b. $G$ is a finite group.

   Indeed, the case a. is obvious and b. follows from Corollary 2.7.4.

2. Consider a field $K$ and a normal subgroup $H$ of $G$. The group ring $K[G]$ has a natural $G/H$-gradation, $K[G] = \oplus_{C \in G|H} K[G]_C$, where $K[G]_C = K[C] = \oplus_{g \in C} K_g$. The component of degree $\overline{e} \in G|H$ is exactly $K[H]$. An irreducible representation of $G$, $V$ say, corresponds to a simple $K[G]$-module $V$ with finite $K$-dimension. We may consider $V$ as a representation space for $H$ too, by restriction of scalars. Since $\dim_K V$ is finite, $V$ contains a nonzero simple (left) $K[H]$-module and so Theorem 4.3.2 is applicable to this case.

Given a strongly $G$-graded ring $R$ together with a simple left $R_e$-module $S$. One of the aims of classical Clifford theory is to provide a description of all simple left $R$-modules $M$ isomorphic to direct sums of copies of the given $S$ as $R_e$-modules i.e. to obtain all $S$-primary $R$-modules $M$ in an explicite way.

Put $\Sigma = R \otimes_{R_e} S$. Since $R$ is stronly graded, $\Sigma$ is gr-simple. Put $\Delta = \mathrm{End}_R(\Sigma)$. From Proposition 2.2. we retain that $\Delta$ is a gr-division ring and $\Delta = \oplus_{h\in H}\Delta_h$ where $H = G\{\Sigma\} = G\{S\}$. In case $S$ is $G$-invariant we have $H = G$.

### 4.3.4 Theorem

Consider a strongly graded ring $R$ together with a $G$-invariant simple $R_e$-module $S$. There is a one-to-one correspondence between the set of isomorhism classes of simple $\Delta$-modules and the set of isomorphisms classes of simple $R$-modules that are $S$-primary as $R_e$-modules.

**Proof** Consider $\mathrm{Mod}(R|\Sigma)$. Since $S$ is $G$-invariant it follows that every $M \in \mathrm{Mod}(R|\Sigma)$ is $S$-primary as an $R_e$-module. The result thus follows directly from Theorem 4.1.4 □

### 4.3.5 Example (E. Dade)

Let $A$ be a discrete valuation ring with field of fractions $K$. Let $w = Ap$ be the maximal ideal of $A$. Consider a cyclic group $< \sigma >= G$ of infinite order. The group ring $A[G]$ has the usual $G$-gradation. We may view $K$ as an $A[G]$-module in the following way : $\sigma^i(\lambda) = p^i\lambda$, for $i \in \mathbb{Z}$ and $\lambda \in K$. Let us verify that $K$ is a simple $R = A[G]$-module. Take $\lambda \neq 0$ in $K$. For $\mu \neq 0$ in $K$, $\lambda^{-1}\mu \in K$, say $\lambda^{-1}\mu = ab^{-1}$ where $a, b \in A$. Write $ab^{-1} = up^i$ where $u \in A - w$, hence $\mu = \lambda up^i = u\sigma^i(x)$ and $K = R\lambda$ or $K$ is simple.

Consider $K$ as an $A$-module via restrition of scalars for $A \to A[G]$, we obtain the structure of $K$ as an $A$-module (making it the fraction field of $A$). Assume that $K$ contains a simple $A$-module $X$, then $X \cap A \neq \{0\}$ and $X = X \cap A \subset A$. Since $A$ is a domain we must have $X = 0$. This example shows that the hypothesis in Theorem 4.3.2 is essential !

## 4.4 Application to Graded Clifford Theory

In this section we aim to elucidate further the structure of a gr-simple $\Sigma$ in $R$-gr when viewed as an ungraded module in $R$-mod. In Section 4.2. we have already described the structure of any $R$-module $M$ from $\mathrm{Mod}(R|\Sigma)$ considered as an $R_e$-module, we continue here using the same notation and conventions. We will obtain an answer to the problem posed above in case

$R$ is a $G$-graded ring of finite support, hence in particular when $G$ is a finite group.

First notation : for $M$ in $R$-mod we let $\mathrm{Spec}_R(M)$ be the set of isomorphism types $[S]$ of simple $R$-modules $S$ such that $S \cong P/Q$ where $Q \subset P \subset M$ are submodules. In a similar way we define $\mathrm{Spec}_{R_e}(M)$.

### 4.4.1 Proposition

Let $R$ be a graded ring of type $G$ and consider gr-simple $R$-modules $\Sigma$ and $\Sigma'$.

The following assertions are equivalent :

i) $\mathrm{Hom}_R(\Sigma, \Sigma') \neq 0$.

ii) There exists a $\sigma \in G$ such that $\Sigma' = \Sigma(\sigma)$.

iii) $\mathrm{Spec}_R(\Sigma)$ and $\mathrm{Spec}_R(\Sigma')$ have nontrivial intersection.

iv) We have an equality : $\mathrm{Spec}_R(\Sigma) = \mathrm{Spec}_R(\Sigma')$.

v) $\mathrm{Spec}_{R_e}(\Sigma)$ and $\mathrm{Spec}_{R_e}(\Sigma')$ have nontrivial intersection.

vi) We have an equality : $\mathrm{Spec}_{R_e}(\Sigma) = \mathrm{Spec}_{R_e}(\Sigma')$

**Proof**
i. $\Rightarrow$ ii. : If $\mathrm{Hom}_R(\Sigma, \Sigma') \neq 0$, there exists a nonzero $R$-homomorphism $f : \Sigma \to \Sigma'$. Since $\Sigma$ is a finitely generated $R$-module, then $\mathrm{Hom}_R(\Sigma, \Sigma') = \mathrm{HOM}_R(\Sigma, \Sigma')$, hence $f = \sum_{\sigma \in G} f_\sigma$ where $f_\sigma : \Sigma \to \Sigma'$ is a morphism of degree $\sigma$. Since $f \neq 0$, there exists a $\sigma \in G$ such that $f_\sigma \neq 0$. But $f_\sigma : \Sigma \to \Sigma'(\sigma)$ is a nonzero morphism in $R$-gr. Since $\Sigma$ and $\Sigma'(\sigma)$ are simple objects in $R$-gr we have $\Sigma'(\sigma) \simeq \Sigma$ or $\Sigma' \simeq \Sigma(\sigma^{-1})$.

The assertions ii. $\Rightarrow$ i., ii. $\Rightarrow$ iv., iv. $\Rightarrow$ iii. and vi. $\Rightarrow$ v. are easily verified. We prove iii. $\Rightarrow$ v. Let $S$ be a simple $R$-module such that $[S] \in \mathrm{Spec}_R(\Sigma) \cap \mathrm{Spec}_R(\Sigma')$.

In view of Lemma 4.1.1. it is clear that $\mathrm{Spec}_{R_e}(\Sigma) \cap \mathrm{Spec}_{R_1}(\Sigma') \neq \emptyset$. We now establish the implication v. $\Rightarrow$ ii. There exist $\sigma, \tau \in G$ such that $\Sigma_\sigma \simeq \Sigma'_\tau$. In this case, see Theorem 2.7.2., we have that $R\overline{\otimes}_{R_e}\Sigma_\sigma \simeq \Sigma(\sigma)$ and $R\overline{\otimes}_{R_e}\Sigma'_\tau \simeq \Sigma'(\tau)$. Hence $\Sigma(\sigma) \simeq \Sigma'(\tau)$ and therefore $\Sigma' \simeq \Sigma(\sigma)(\tau^{-1}) = \Sigma(\tau^{-1}\sigma)$. □

### 4.4.2 Remark

With notation as in Section 4.2. we have that $\mathrm{Spec}_{R_e}(\Sigma) = \Omega_{R_e}(\Sigma)$.

### 4.4.3 Corollary

Assume that $R$ has finite support and $S$ is a simple $R$-module, then there exists a gr-simple module $\Sigma$ such that $S$ is isomorphic to an $R$-submodule of $\Sigma$. Moreover $\Sigma$ is unique up to a $\sigma$-suspension i.e. if $S$ embeds in another gr-simple module $\Sigma'$, then $\Sigma' \simeq \Sigma(\sigma)$ for some $\sigma \in G$.

**Proof** The first part is a direct consequence of Corollary 2.7.4. The second part follows from Proposition 4.3.1.

### 4.4.4 Theorem

Let $R$ be a graded ring such that $\sup(R) < \infty$. If $\Sigma$ is a gr-simple module, then the following assertions hold :

i) $\Sigma$ has finite length in $R$-mod

ii) $G\{\Sigma\}$ is a finite subgroup of $G$

iii) $\Delta = \mathrm{End}_R(\Sigma)$ is a quasi-Frobenius ring.

iv) If $n = |G\{\Sigma\}|$ and $\Sigma$ is $n$-torsion free, then $\Sigma$ is semisimple of finite length in $R$-mod.

v) If $G$ is a torsionfree group, then $\Sigma$ is a simple $R$-module. Moreover, every simple $R$-module can be $G$-graded.

vi) If $S \in \mathrm{Spec}_R(\Sigma)$, then $S$ is isomorphic to a minimal $R$-submodule of $\Sigma$.

**Proof**

i) Clearly : $\sup(\Sigma) < \infty$. By Lemma 4.1.1. $\Sigma$ is a semisimple left $R_e$-module of finite length. So $\Sigma$ as an $R$-module is Noetherian and Artinian, then $\Sigma$ has finite length in $R$-mod.

ii) By Proposition 2.2.2., $G\{\Sigma\}$ is a finite subgroup of $G$ (since $\sup(\Sigma) < \infty$).

iii) Since $\Delta = \oplus_{\sigma \in G\{\Sigma\}} \Delta_\sigma$ where $\Delta_e$ is a division ring and $\Delta$ is a $G\{\Sigma\}$-crossed product, it follows that $\Delta$ is left and right Artinian. Moreover it is both left and right self-injective. Hence $\Delta$ is a quasi-Frobenius ring.

iv) Since $\Sigma$ is $n$-torsionfree, the morphism $\varphi_n : \Sigma \to \Sigma$ $\varphi_n(x) = nx$ is an isomorphism. Thus $n$ is invertible in $\Delta$. It follows from Maschke's theorem (Section 3.5) that $\Delta$ is a semi-simple Artinian ring. Now by 4.1.4 it follows that $\Sigma$ is semisimple in $\mathrm{Mod}(R|\Sigma)$ hence semisimple in $R$-mod too.

v) If $G$ is torsionfree, then $G\{\Sigma\} = 1$, and thus $\Delta = \Delta_e$, i.e. $\Delta$ is a division ring. As a consequence of Theorem 4.1.1 we obtain that $\Sigma$ is a simple $R$-module. The second part of this assertion derives from Corollary 4.3.3.

vi) From Proposition 4.1.3., if $S \in \mathrm{Spec}_R(\Sigma)$, there exists a nonzero morphism $f : \Sigma \to S$ which must be an epimorphism. Now, since $\Delta$ is a $QF$ ring, every simple $\Delta$-module is isomorphic to a minimal left ideal of $\Delta$. Then Theorem 4.1.1 entails that $S$ is isomorphic to a minimal submodule of $\Sigma$.

### 4.4.5 Corollary

Let $R = \oplus_{\sigma\in G} R_\sigma$ be a $G$-graded ring with $n = |G| < \infty$. We denote by $J(R)$ the Jacobson radical of $R$. If $a \in J(R)$ and $a = \sum_{\sigma\in G} a_\sigma$, $a_\sigma \in R_\sigma$ then $na_\sigma \in J(R)$ for any $\sigma \in G$. In particular if $n$ is invertible in $R$, then $J(R) = J^g(R)$.

**Proof** Let $\Sigma$ be a gr-simple left $R$-module. We put $m = |G\{\Sigma\}|$, then $m$ divides $n$. If $m\Sigma = 0$ also $n\Sigma = 0$, hence $(n \cdot a) = 0$. If $m\Sigma \neq 0$ then $m\Sigma = \Sigma$ and $\Sigma$ is $m$-torsion free. By assertion iv. of Theorem 4.3.4 it follows that $\Sigma$ is semisimple in $R$-mod. Since $a \in J(R)$, $a\Sigma = 0$, thus $(n.a)\Sigma = 0$. Therefore $na \in J^g(R)$. Since $na = \sum_{\sigma\in G} na_\sigma$ we obtain :$na_\sigma \in J^g(R)$. Now $J^g(R) \subseteq J(R)$ entails $na_\sigma \in J(R)$ for any $\sigma \in G$. The last statement of the Corollary is clear. □

### 4.4.6 Theorem

Let $R = \oplus_{\sigma\in G} R_\sigma$ be a $G$-graded ring where $G$ is a torsion free abelian group. Then the Jacobson radical $J(R)$ is a graded ideal of $R$.

**Proof** First assume that $G$ has finite rank, thus $G \simeq \mathbb{Z}^n (n \geq 1)$; we prove the assertion by recurrence on $n$. We start with the case $n = 1$. Let $a \in J(R)$ and write $a = a_{i_1} + \ldots + a_{i_s}$ where $a_{i_1}, \ldots, a_{i_s}$ are the nonzero homogeneous components of $a$. Assume $i_1 < i_1 < \ldots < i_s$. Clearly there exist two prime numbers $p, q$ such that $p$ and $q$ do not divide $i_l - i_r$ for any $1 \leq r < l \leq s$. In this case $a_{i_1}, \ldots, a_{i_s}$ remain the homogeneous components of $a$ when $R$ is considered as a $\mathbb{Z}_p$, respectively $\mathbb{Z}_q$-graded ring. Corollary 4.3.5 yields that $pa_{i_r}$ and $qa_{i_r}$ are in $J(R)$ for $1 \leq r \leq s$. Since $p$ and $q$ are prime and $p \neq q$ it follows that $(p, q) = 1$ and therefore $a_{i_r} \in J(R)$ for any $r = 1, \ldots, s$. Thus $J(R)$ is a graded ideal. Assume now that $G \simeq \mathbb{Z}^n$ with $n > 1$ and suppose that the assertion is true for $n-1$. We write $G = H \oplus K$ where $H$ and $K$ are two subgroups of $G$ such that $H \simeq \mathbb{Z}^{n-1}$ and $K \simeq \mathbb{Z}$. Let $a \in J(R)$ and $a = \sum_{g\in G} a_g$ where $a_g \in R_g$. Let $g \in G$ and consider $R$ with the grading of type $G|K \simeq H$. The induction hypothesis leads to $a_{g+K} = \sum_{x\in K} a_{g+x} \in J(R)$. If

$x, y \in K, x \neq y$ then we have $\widehat{g+x} \neq \widehat{g+y}$ in the quotient group $G/H \simeq K$. Since $K \simeq \mathbb{Z}$ the first step of induction establishes that $a_{g+x} \in J(R)$ for any $x \in K$. In particular for $x = 0$ we obtain $a_g \in J(R)$, hence $J(R)$ is a graded ideal of $R$. Now assume that $G$ is torsion free. Let $a \in J(R)$, $a = a_{g_1} + \ldots + a_{g_t}$ where $a_{g_1}, \ldots, a_{g_t}$ are nonzero homogeneous components of $a$. There exists a subgroup $H$ of $G$ such that $g_1, \ldots, g_t \in H$ and $H \simeq \mathbb{Z}^n$ for some $n \geq 1$. Clearly $R = \cup_{H \subseteq K} R_K$ where $K$ is an arbitrary finitely generated subgroup of $G$ such that $K$ contains $H$. Since $J(R) \cap R_K \subset J(R_K)$ and $a \in J(R) \cap R_K$ then $a \in J(R_K)$ and by the above argument $a_{g_i} \in J(R_K), (i \leq i \leq t)$. If $b \in R$), there exists a subgroup $K$, finitely generated such that $H \subset K$ and $b \in R_K$. So $1 - ba_{g_i}$ is invertible on $R_K$ so invertible in $R$. Hence $a_{g_i} \in J(R)$ for any $1 \leq i \leq t$.

### 4.4.7 Corollary

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a $G$-graded ring where $G$ is an abelian torsion free group. Then :

$$J(R) \subseteq J^g(R)$$

**Proof** Let $\Sigma$ be a gr-simple module. Since $\Sigma$ is finitely generated as an $R$-module, $J(R)\Sigma \neq \Sigma$. Since $J(R)$ is a graded ideal (Theorem 4.4.6), then $J(R)\Sigma = 0$ and $J(R) \subseteq J^g(R)$. □

### 4.4.8 Remark

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a $G$-graded ring where $G$ is a torsion free abelian group. If $M \in R$-gr is a finitely generated graded $R$-module then the above results are true for the Jacobson radical of $M$. So we have the following assertions :

i) $J(M)$ is a graded submodule of $M$.

ii) $J(M) \subseteq J^g(M)$

The proof of these assertions is identical to the proofs of Theorem 4.4.6 and Corollary 4.4.7

### 4.4.9 Corollary

Let $R$ be a $G$-graded ring, where $G$ is a torsionfree abelian group. If $\Sigma$ is a gr-simple module then $J(\Sigma) = 0$.

**Proof** By Remark 4.4.8 ii., we have $J(\Sigma) \subseteq J^g(\Sigma) = 0$.

### 4.4.10 Proposition

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a $G$-graded ring with $n = |G| < \infty$ and $M \in R$-gr. We denote by $s(M)$ (respectively by $s^{\mathrm{gr}}(M)$) the socle of $M$ in $R$-mod (respectively in $R$-gr). Then :

i) $s(M) \subseteq s^{\mathrm{gr}}(M)$

ii) $ns^{\mathrm{gr}}(M) \subseteq s(M)$

iii) If $n$ is invertible on $M$ then $s(M) = s^{\mathrm{gr}}(M)$

iv) If $m \in s(M)$, $m = \sum_{\sigma \in G}$, $m_\sigma \in M_\sigma$ then $nm_\sigma \in s(M)$ for any $\sigma \in G$.

**Proof**

i) Straight from Proposition 2.7.3. We have that $s^{\mathrm{gr}}(M)$ is the sum of all graded simple submodules $\Sigma$ of $M$. But $n\Sigma = 0$ or $\Sigma$ is semisimple in $R$-mod. Hence we have $n\Sigma \subseteq s(M)$ and therefore $ns^{\mathrm{gr}}(M) \subseteq s(M)$.

ii) Is clear and iv. is obvious from i. and ii.

### 4.4.11 Theorem

Let $R$ be a $G$-graded ring, where $G$ is a torsionfree abelian group. If $M \in R$-gr, then $s(M)$ is a graded submodule of $M$.

**Proof** Same proof as in Theorem 4.4.6, using Proposition 4.4.10.

## 4.5 Torsion Theory and Graded Clifford Theory

Let $R$ be a $G$-graded ring and $R$-gr the category of all (left) graded $R$-modules. Let $\mathcal{C}$ be a closed subcategory of $R$-gr (see Section 2.6); $\mathcal{C}$ is called **rigid** if for any $M \in \mathcal{C}$, we have $M(\sigma) \in \mathcal{C}$ for any $\sigma \in G$.

### 4.5.1 Examples

i) If $\mathcal{C}$ is the class of all semisimple objects of $R$-gr, then it is clear that $\mathcal{C}$ is a rigid closed subcategory (if $M$ is semisimple, then $M(\sigma)$ is also semisimple, because $T_\sigma$, the $\sigma$-suspension functor, is an isomorphism of categories.

ii) If $M \in R$-gr is a graded $G$-invariant, i.e. $M \simeq M(\sigma)$ in $R$-gr for any $\sigma \in G$, then it is easy to see that $\sigma[M]$ is a rigid closed subcategory of $R$-gr. Now for $M \in R$-gr, $\oplus_{\sigma \in G} M(\sigma)$ is a $G$-invariant graded module, putting :

$$\sigma^{\mathrm{gr}}[M] = \sigma\left[\oplus_{\sigma \in G} M(\sigma)\right]$$

we have that $\sigma^{\mathrm{gr}}[M]$ is also a rigid closed subcategory. In fact, it is the smallest rigid closed subcategory of $R$-gr containing $M$.

iii) There exist closed subcategories of $R$-gr which are not rigid. For example let $G \neq \{1\}$, take $\sigma \in G$ and let $\mathcal{C}_\sigma = \{M = \oplus_{\lambda \in G} M_\lambda \in R\text{−gr} | M_\sigma = 0\}$. Then $\mathcal{C}_\sigma$ is obviously a closed subcategory of $R$-gr (in fact it is a localizing subcategory) but is not rigid, unless $\mathcal{C}_\sigma = 0$. Let $\mathcal{C}$ be a rigid closed subcategory of $R$-gr. Denote the smallest closed subcategory of $R$-mod containing $\mathcal{C}$ by $\overline{\mathcal{C}}$.

### 4.5.2 Proposition

For $M \in R$-mod the following assertions are equivalent :

i) $M \in \overline{\mathcal{C}}$

ii) $F(M) \in \mathcal{C}$, where $F$ is a right adjoint functor for the forgetful functor $U : R\text{-gr} \rightarrow R\text{-mod}$.

iii) There exists $N \in \mathcal{C}$ such that $M$ is isomorphic to a quotient module of $N$ in $R$-mod.

**Proof** Put $\overline{\overline{\mathcal{C}}} = \{M \in R\text{−mod} | F(M) \in \mathcal{C}\}$. Since $F$ is an exact functor, $\overline{\overline{\mathcal{C}}}$ is a closed subcategory of $R$-mod. If $M \in \mathcal{C}$, $F(M) \simeq \oplus_{\sigma \in G} M(\sigma)$ then rigidity of $\mathcal{C}$ entails $M \in \overline{\overline{\mathcal{C}}}$, then $\mathcal{C} \subseteq \overline{\overline{\mathcal{C}}}$ and $\overline{\mathcal{C}} \subseteq \overline{\overline{\mathcal{C}}}$. If $M \in \overline{\overline{\mathcal{C}}}$, then $F(M) \in \mathcal{C} \subset \overline{\mathcal{C}}$. On the other hand $M$ is a homomorphic image of $F(M)$ in $R$-mod, hence $M \in \overline{\mathcal{C}}$. Therefore $\overline{\mathcal{C}} = \overline{\overline{\mathcal{C}}}$. So we have the equivalences i.$\Leftrightarrow$ ii. and i.$\Rightarrow$ iii. Assume now that $M$ is isomorphic to a quotient of $N$ where $N \in \mathcal{C}$. Consequently, we have the exact sequence :

$$N \rightarrow M \rightarrow 0$$

Since $F$ is exact, $F(N) \rightarrow F(M) \rightarrow 0$ is also an exact sequence. Since $N \in \mathcal{C}$, $F(N) \simeq \oplus_{\sigma \in G} N(\sigma) \in \mathcal{C}$ and thus $F(M) \in \mathcal{C}$ or $M \in \overline{\overline{\mathcal{C}}} = \overline{\mathcal{C}}$. Hence iii.$\Rightarrow$ i. □

### 4.5.3 Corollary

With notation as above we obtain : $\mathcal{C}$ is a localizing subcategory in $R$-gr if and only if $\overline{\mathcal{C}}$ is a localizing subcategory in $R$-mod.

**Proof** We apply Proposition 4.5.2 and the fact that the functor $F$ is exact. □

If $\mathcal{A}$ is a Grothendieck category and $M, N \in \mathcal{A}$ then $N$ is said to be **$M$-generated** if it is a quotient of a direct sum $M^{(I)}$ of copies of $M$. If each subobject of $M$ is $M$-generated, then we say that $M$ is a self-generator. It is easy to see that $M$ is self-generator if and only if, for any subobject $M' \subseteq M$ there exists a family $(f_i)_{i\in I}$ of elements of $\mathrm{End}_{\mathcal{A}}(M)$ such that $M' = \sum_{i\in I} f_i(M)$.

Assume now that $\mathcal{A} = R$-gr and $N, M \in R$-gr, we say that $N$ is **gr-generated** by $M$, if there exists a family $(\sigma_i)_{i\in J}$ of elements from $G$ such that $N$ is a homomorphic image of $\oplus_{i\in I} M(\sigma_i)$ in $R$-gr. Clearly $N$ is gr-generated by $M$ if and only if $N$ is generated by $\oplus_{\sigma\in G} M(\sigma)$ in $R$-gr. If each subobject of $M$ is gr-generated by $M$ then we say that $M$ is a **gr-self generator**. It is easy to see that $M \in R$-gr is a gr-self generator if and only if, for any graded submodule $M'$ of $M$ there exists a family $(f_i)_{i\in I}$ of elements of $\mathrm{END}_R(M)$ such that $M' = \sum_{i\in I} f_i(M)$.

### 4.5.4 Proposition

Let $\mathcal{C}$ be a rigid closed subcategory of $R$-gr and $M \in \mathcal{C}$. The following assertions hold :

a. If $M$ is a projective object in the category $\mathcal{C}$ then $M$ is a projective object in the category $\overline{\mathcal{C}}$.

b. If every object of $\mathcal{C}$ is gr-generated by $M$ then $M$ is a generator for the category $\overline{\mathcal{C}}$.

**Proof**

a. Proposition 4.5.2 entails that there are canonical functors $\overline{U} : \mathcal{C} \to \overline{\mathcal{C}}$ and $\overline{F} : \overline{\mathcal{C}} \to \mathcal{C}$ where $\overline{U}$ (respectively $\overline{F}$) is the restriction of the functor $U : R\text{−gr} \to R$-mod (resp. the restriction of functor $F : R$-mod $\to R$-gr). Obviously $\overline{F}$ is still a right adjoint of $\overline{U}$. Since $\overline{F}$ is an exact functor, $\overline{U}(M) = M$ is a projective object in $\overline{\mathcal{C}}$.

b. Direct from assertion iii. of Proposition 4.5.2.

When we consider $M \in R$-gr, it is clear that $\overline{\sigma^{\mathrm{gr}}[M]} = \sigma_R[M]$ where $\sigma_R[M]$ is the smallest closed subcategory of $R$-mod which contains the left $R$-module $M$. □

### 4.5.5 Proposition

Let $M$ be a graded $R$-module and assume that $M$ is finitely generated projective in $\sigma^{\mathrm{gr}}[M]$. Then the following conditions are equivalent.

i) $M$ is a gr-self generator.

ii) $M$ is a projective generator of $\sigma_R[M]$.

**Proof** Since $M$ is finitely generated it is easily seen that ii. $\Rightarrow$ i.

i. $\Rightarrow$ ii. Proposition 4.5.4 allows to reduce the problem to proving that every object of $\sigma^{\mathrm{gr}}[M]$ is gr-generated by $M$. Put $U = \oplus_{\sigma\in G} M(\sigma)$; $U$ is projective in $\sigma^{\mathrm{gr}}[M] = \sigma[U]$. Let $X$ be an object from $\sigma[U]$, then there exists an epimorphism $f : U^{(I)} \to X$ and a monomorphism $u : Y \to X$. If we put $Z = f^{-1}(U(Y))$ then $f$ induces an epimorphism $Z \xrightarrow{g} Y \longrightarrow 0$. In order to prove the assertion that $Y$ is $U$-generated it is sufficient to prove that $Z$ is $U$-generated. We may assume that $Y$ is a subobject of $U^{(I)}$. Since $Y = \sum_{J\subseteq I} Y \cap U^{(J)}$, $J$ ranging over all finite subsets of $I$, we may assume also that $I$ is a finite set. Now since $U = \oplus_{\sigma\in G} M(\sigma)$, by the same argument, it is sufficient to prove that $Y$ is $U$-generated for $Y$ a subobject of a direct sum $\oplus_{i=1}^{n} M(\sigma_i)$ where $\{\sigma_i, i = 1, \ldots, n\}$ is a system of elements from $G$.. Now we establish the assertion by induction. If $n = 1$, then $Y \subset M(\sigma)$, so $Y(\sigma^{-1}) \subseteq M$ and therefore, by the hypothesis, $Y(\sigma^{-1})$ is $U$-generated. Since $U$ is $G$-invariant it follows that $Y$ is $U$-generated. Assume that the assertion is true for $n-1$ we have $Y/Y \cap M(\sigma_1) \simeq Y + M(\sigma_1)/M(\sigma_1) \subset \oplus_{i=1}^{n} M(\sigma_i)/M(\sigma_1) \simeq \oplus_{i=2}^{n} M(\sigma_i)$. So, by the induction hypothesis it follows that $Y/Y \cap M(\sigma_1)$ is $U$-generated. Since $Y \cap M(\sigma_1) \hookrightarrow M(\sigma_1)$ we have that $Y \cap M(\sigma_i)$ is also $U$-generated. So we arrive at the diagram :

$$\begin{array}{ccccccccc}
 & & U^{(I)} & & & & U(J) & & \\
 & & \downarrow{\scriptstyle u} & & & & \downarrow{\scriptstyle v} & & \\
0 & \longrightarrow & Y \cap M(\sigma_1) & \longrightarrow & Y & \longrightarrow & Y/Y \cap M(\sigma_1) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

for some sets $I$ and $J$ where $u, v$ are epimorphisms. Now both $U^{(I)}$ and $U^{(J)}$ are projective on the category $\sigma[U]$ hence there exists an epimorphism

$$U^{(I)} \oplus U^{(I)} \xrightarrow{w} Y \longrightarrow 0$$

proving that $Y$ is $U$-generated. Finally it follows that $U$ is a generator in the category $\sigma[U]$. □

### 4.5.6 Remark

The implication i. $\Rightarrow$ ii. does not need the hypothesis for $M$ to be finitely generated.

Let $M \in R$-gr, we denote (as in Section 4.1) by $\mathrm{Mod}(R|M)$ the full subcategory of $R$-mod whose objects are the $R$-module which are $M$-generated in $R$-mod. We put $\Delta = \mathrm{End}_R(M)$.

### 4.5.7 Theorem

Let $M \in R$-gr be a finitely generated and projective object in $\sigma^{\mathrm{gr}}[M]$. If $M$ is a gr-self generator, then $\sigma_R[M] = \mathrm{Mod}(R|M)$ so $\mathrm{Mod}(R|M)$ is a closed subcategory of $R$-mod. Moreover the canonical functors :

$$\mathrm{Hom}_R(M, -) : \mathrm{Mod}(R|M) \longrightarrow \Delta\text{−mod}$$
$$M \otimes_\Delta - : \Delta\text{−mod} \longrightarrow \mathrm{Mod}(R|M)$$

are inverse equivalence of categories.

**Proof** From Proposition 4.4.5 it follows that $\sigma_R[M] = \mathrm{Mod}(R|M)$ and $M$ is a finitely generated projective genrator for the category $\mathrm{Mod}(R|M)$. Now from Mitchel's Theorem (cf.[179]) it follows that the above functors define an inverse equivalence of categories. □

### 4.5.8 Corollary

Let $\Sigma$ be a gr-semi-simple object of $R$-gr, such that $\Sigma$ is finitely generated as left $R$-module (for instance a gr-simple module). If $\Delta = \mathrm{End}_R(\Sigma) = \mathrm{END}_R(\Sigma)$ then $\mathrm{Hom}_R(M, -) : \mathrm{Mod}(R|\Sigma) \to \Delta$-mod and $\Sigma \otimes_\Delta : \Delta$-mod$\to \mathrm{Mod}(R|\Sigma)$ are inverse equivalence of categories.

**Proof** Since $\Sigma$ is gr-semisimple, it is clear that every object of $\sigma^{\mathrm{gr}}[\Sigma]$ is gr-semisimple (in fact every object has the form $\oplus_{i\in J}\Sigma(\sigma_i)$ where $(\sigma_i)_{i\in I}$ is a family of elements from $G$). Hence $\Sigma$ is projective and a self-generator in $\sigma^{\mathrm{gr}}[\Sigma]$. Now we apply Theorem 4.5.7. In particular, we obtain that Theorem 4.5.7 generalizes Theorem 4.1.4. □

## 4.6 The Density Theorem for gr-simple modules

Recall first that a graded ring $D = \oplus_{\sigma\in G} D_\sigma$ is called a gr-division ring when every nonzero homogeneous element of $D$ is invertible. It is clear that in this case $\sup(D) = G\{D\}$ and $D$ is an $e$-faithful graded ring.

### 4.6.1 Proposition

Let $D$ be a gr-division ring. If $V$ is a nonzero graded left $D$-module, then $V$ is a free $D$-module with a homogeneous basis. Moreover, every two homogeneous bases of $V$ have the same cardinality.

**Proof** Since $V \neq 0$, there exists a nonzero homogenous element $x_\sigma \in V$ for some $\sigma \in G$. Thus if $rx_\sigma = 0$ for some $r \in R$, $r = \sum_{\lambda \in G} r_\lambda, r_\lambda \in D_\lambda$, we have $r_\lambda x_\sigma = 0$ for every $\lambda \in G$, and, since $D$ is a gr-division ring this implies $r_\lambda = 0$, for every $\lambda \in G$, Hence $r = 0$. From Zorn's Lemma (as in the ungraded case) we get that $V$ has a homogeneous basis. Using the same argument as in the non-graded case we may conclude that two homogeneous bases of $V$ have the same cardinality. □

We denote by $\dim_D^{\mathrm{gr}}(V)$ the cardinality of a homogeneous basis of $V$. In the sequel we need the following obvious assertions.

If $D$ is a gr-division ring, then the following conditions are equivalent :

i) ${}_DV$ is finitely generated

ii) ${}_DV$ has a finite basis

iii) ${}_DV$ has a finite homogeneous basis

It is natural to ask whether the cardinality of a finite non-homogenous basis of ${}_DV$ must be equal to the cardinality of any finite homogeneous basis, i.e. whether a gr-division ring has the IBN (invariant basis number) property. For some particular cases we can give a positive answer to this question. For example, let $D = \oplus_{\sigma \in G} D_\sigma$ be a gr-division ring with $G$ being a locally finite group i.e., such that every subgroup generated by a finite number of elements is finite. Then $D$ has I.B.N. Indeed, assume first that $G$ is finite. If $V$ is a graded left $D$-module, with a basis $\{e_1, \ldots, e_n\}$ then, since $D_e$ is a division ring and $|G|$ is finite then $\sup(D)$ is a finite subgroup of $G$. Since $\dim_{D_e}(D)| = |\sup(D)|$ then $\dim_{D_e}(V) = n| \cdot |\sup(D)|$ and therefore $n = \dim_D(V)$ is independent of the chosen basis. Next, suppose that $G$ is just locally finite and let $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ be two finite bases of a free left $D$-module $V$. We may consider $V$ as a graded $D$-module (for example, with the grading induced by an isomorphism $V \cong D^m$). Let then $V = \oplus_{\sigma \in G} V_\sigma$ and for each $x \in V, x = \sum_{\sigma \in G} x_\sigma, x_\sigma \in V_\sigma$. Recall that $\sup(x) = \{g \in G | x_g \neq 0\}$. Since $G$ is locally finite, there exists a finite subgroup $H \leq G$ such that $\sup(e_i) \subseteq H (i \leq i \leq m)$ and $\sup(f_j) \subseteq H (i \leq j \leq n)$. We consider $V_H = \oplus_{\sigma \in H} V_\sigma$, then $V_H$ is a graded $D_H$-module. Furthermore, it is easy to see that $\{e_1, \ldots, e_m\}$ and $\{f_1, \ldots, f_n\}$ are, in fact, two bases of $V_H$ over the $G$-gr-division ring $D_H$. Since $H$ is finite then $m = n$. For some other cases consult the Exercises of Section 7.6.

Let $\Sigma = \oplus_{\sigma \in G} \Sigma_\sigma \in R$-gr be a gr-simple module. If $\Delta = \mathrm{End}({}_R\Sigma) = \mathrm{END}({}_D\Sigma)$ then $\Delta = \oplus_{\sigma \in G} \Delta_\sigma$ is a gr-division ring and $\Delta_1 = \mathrm{End}_{R-\mathrm{gr}}(\Sigma)$ is a division ring. Obviously, for any $\sigma \in \sup(\Sigma), \Sigma_\sigma$ is a right vector space over $\Delta_e$. On the other hand , in view of Proposition 2.2.2 we may write $\sup(\Sigma) = \cup_{i \in J} \sigma_i G\{\Sigma\}$, where $J \subseteq I$ and $(\sigma_i)_{i \in I}$ is a left transversal for $G\{\Sigma\}$

in $G$. Since $\Sigma$ is a right graded $\Delta$-module (in fact ${}_R\Sigma_\Delta$ is a graded $R-\Delta$-bimodule) we have the following result ($\Sigma_\Delta$ as a right $\Delta$-module is called the countermodule of $\Sigma$).

### 4.6.2 Proposition

$\dim^{\mathrm{gr}}_\Delta(\Sigma_\Delta) = \sum_{i\in J} \dim_{\Delta_e}(\Sigma_{\sigma_i})$.

**Proof** Let $i \in J$ and assume that $\{e_{i,r}\}_{r\in A_i}$ is a basis of the right $\Delta_e$-module $\Sigma_{\sigma_i}$. We clain that $B = \cup_{i\in J}\{e_{i,r}|r \in A_i\}$ is a homogeneous basis for $\Sigma_\Delta$. To verify this, we first prove that $B$ is linearly independent over $\Delta$. Consider an equality of the form :

$$\sum_{i\in J}\sum_{r\in A_i} e_{i,r}u_{i,r} = 0 \tag{1}$$

where $u_{i,r} \in \Delta$ and the family $\{u_{i,r}|i \in J, r \in A_i\}$ has finite support. Note that, since the $e_{i,r}$ are homogeneous, we may assume that the $u_{i,r}$ are homogeneous too. Since $\deg(e_{i,r}) = \sigma_i$ for each $r \in A_i$, and $\deg(u_{i,r}) \in G\{\Sigma\}$ and because $\sigma_i G\{\Sigma\} \cap \sigma_j G\{\Sigma\} = \emptyset$ for $i \neq j$, it follows from equality (1) that, for any $i \in J$,

$$\sum_{r\in A_i} e_{i,r}u_{i,r} = 0 \tag{2}$$

Now, for any $\sigma \in G\{\Sigma\}$ we consider $A_i^\sigma = \{r \in A_i|\deg(u_{i,r}) = \sigma\}$ so that $A_i = \cup_{\sigma\in G\{\Sigma\}} A_i^\sigma$ and $A_i^\sigma \cap A_i^\tau = \emptyset$ for $\sigma, \tau \in G\{\Sigma\}, \sigma \neq \tau$. Then it follows from (2) that we have :

$$\sum_{r\in A_i^\sigma} r_{i,r}u_{i,r} = 0, \text{ for any } \sigma \in G\{\Sigma\} \tag{3}$$

But, since $\Delta_\sigma \neq 0$, $\Delta_\sigma$ contains an invertible element $u_\sigma$ and from (3) we get the equality :

$$\sum_{r\in A_i^\sigma} e_{i,r}u_{i,r}u_\sigma^{-1} = 0 \tag{4}$$

Since $u_{i,r}u_\sigma^{-1} \in \Delta_e$, it follows from (4) that $u_{i,r}u_\sigma^{-1} = 0$ for every $r \in A_i^\sigma$ and so $u_{i,r} = 0$ for every $r \in A_i^\sigma$. This shows that $u_{i,r} = 0$ for every $r \in A_i$, so that in fact $u_{i,r} = 0$ for each $i \in J$ and each $r \in A_i$, completing the proof of the linear independence. It remains to establish that $B$ is a generating set for $\Sigma_\Delta$. To prove this, let $x_\sigma \in \Sigma_\sigma$ be a homogeneous element with $\sigma \in \sup(\Sigma)$. Then there exists an element $\sigma_i \in G(i \in J)$ such that $\sigma = \sigma_i h$ where $h \in G\{\Sigma\}$. Since $\Delta_h \neq 0$, there exists a nonzero (and hence invertible) element $u_h \in \Delta_h$ and therefore $x_\sigma u_h^{-1} \in \Sigma_{\sigma_i}$. Since $\{e_{i,r}|r \in A_i\}$ is a basis for $\Sigma_{\sigma_i}$ over $\Delta_e$ we have that $x_\sigma u_h^{-1} = \sum_{r\in A_i} e_{i,r}V_{i,r}$ for some $V_{i,r} \in \Delta_e$ and hence $x_\sigma = \Sigma_{r\in A_i} e_{i,r}V_{i,r}u_h$. Therefore $B$ is a generating set for $\Sigma_\Delta$ and this completes the proof. □

### 4.6.3 Corollary

Let $\Sigma$ be a gr-simple module. Then the following statements are equivalent :

i) The countermodule $\Sigma_\Delta$ is finitely generated

ii) a. $[\sup(\Sigma) : G\{\Sigma\}] < \infty$

b. For any $\sigma \in \sup(\Sigma)$ the countermodule $\Sigma_{\sigma_{\mathrm{End}_{R_e}(\Sigma_\sigma)}}$ is finitely generated.

**Proof** i. $\Rightarrow$ ii. If $\Sigma_\Delta$ is finitely generated, then $\Sigma_\Delta$ has a finite homogeneous basis as $\Lambda$ module. Then ii. follows from Proposition 4.5.2 using the fact that $\mathrm{End}_{R-\mathrm{gr}}(\Sigma) \simeq \mathrm{End}_{R_e}(\Sigma_\sigma)$.

ii. $\Rightarrow$ i. This follows from Proposition 4.5.2.

### 4.6.4 Corollary

If $\Sigma \in R$-gr is gr-simple module with $\Delta = \mathrm{End}({}_R\Sigma)$, then the following assertions are equivalent.

i) $\Sigma_\Delta$ is finitely generated

ii) $R/\mathrm{Ann}_R(\Sigma)$ is a gr-simple ring

iii) $R_e/\mathrm{Ann}_{R_e}(\Sigma)$ is a semisimple Artinian ring.

**Proof** i. $\Rightarrow$ ii. Let $\{x_1, \ldots, x_n\}$ be a generating set of $\Sigma_\Delta$. Then $\Sigma_\Delta = \sum_{i=1}^n x_i\Delta$ and we may assume that the $x_i$ are homogeneous with $\deg(x)_i) = \sigma_i$. Since $\mathrm{Ann}_R(\Sigma) = \cap_{i=1}^n \mathrm{Ann}_R(x_i)$, we have the exact sequence :

$$0 \longrightarrow R/\mathrm{Ann}_R(\Sigma) \longrightarrow \oplus_{i=1}^n \Sigma(\sigma_i)$$

and hence $R/\mathrm{Ann}_R(\Sigma)$ is a gr-simple ring.

ii. $\Rightarrow$ iii. Since $\mathrm{Ann}_{R_e}(\Sigma) = R_e \cap \mathrm{Ann}_R(\Sigma)$ and $R_e/\mathrm{Ann}_{R_e}(\Sigma) = (R/\mathrm{Ann}_R(\Sigma))_e$, it follows that $R_e/\mathrm{Ann}_{R_e}(\Sigma)$ is semisimple Artinian.

iii. $\Rightarrow$ i. Denote $I = \mathrm{Ann}_R(\Sigma)$ and $I_e = I \cap R_e$. The for every $\sigma \in \sup(\Sigma)$ we have that $I_e\Sigma_\sigma = 0$ and hence $\Sigma_\sigma$ is an $R_e/I_e$-module (in fact a simple $R_R/I_e$-module). Since $R_e/I_e$ is semisimple Artinian it follows that $\Sigma_\sigma$ is finitely generated right $\mathrm{End}_{R_e}(\Sigma_\sigma)$-module. On the other hand, using again the fact that $R_e/I_e$ is a semisimple Artinian ring we say that the family $\{\Sigma_\sigma | \sigma \in \sup(\Sigma)\}$ has only a finite number of nonisomorphic $R_e$-modules and hence $[\sup(\Sigma) : G\{\Sigma\}] < \infty$, since this index is equal to the cardinality of the set of isotopic components of the $R_e$-semisimple module $\Sigma$ (see Section 4.2). Therefore we may apply Corollary 4.6.3 to complete the proof. □

Let ${}_RM$ be a left graded $R$-module. We denote by $\Delta = \mathrm{END}({}_RM)$ and $\Lambda = \mathrm{End}({}_RM)$. We recall that $\Delta$ is a dense subring of $\Lambda$ (see Section 2.4). If $M$ is finitely gnerated then $\Delta = \Lambda$. Clearly $M$ is a right $\Delta$-module and also a right $\Lambda$-module if we put :

$$x.u = u(x), \text{ for any } x \in M, u \in \Lambda$$

Moreover $M_\Delta$ is a right graded $\Delta$-module. We define $\mathrm{BIEND}({}_RM) = \mathrm{END}(M_\Delta)$ and $\mathrm{Biend}({}_RM) = \mathrm{End}(M_\Lambda)$. $\mathrm{BIEND}({}_RM)$ is a graded ring with the grading :

$$\mathrm{BIEND}({}_RM)_\sigma = \{f \in \mathrm{END}(M_\Delta) | f(M_\lambda) \subseteq M_{\sigma\lambda} \forall \lambda \in G\}$$

for each $\sigma \in G$.

Since $\Delta$ is a dense subring of $\Lambda$ it follows that $\mathrm{BIEND}({}_RM)$ is a subring of $\mathrm{Biend}({}_RM)$ and we have a canonical morphism of rings :

$$\varphi : R \longrightarrow \mathrm{Biend}({}_RM), \varphi(r)(x) = rx \text{ for } r \in R, x \in M$$

Now, if $r_\sigma \in R_\sigma$, $\varphi(r_\sigma) \in \mathrm{BIEND}({}_RM)_\sigma$ thus $\mathrm{Im}\varphi \subseteq \mathrm{BIEND}({}_RM)$. If $M$ is a left $R$-module, then $R$ is said to **operate densely** on $M$ if for each finite set of elements $x_1, \ldots, x_n \in M$ and $\alpha \in \mathrm{Biend}({}_RM)$ there exist an $r \in R$ such that $rx_i = \alpha(x_i)(1 \leq i \leq n)$. We have the following result.

### 4.6.5 Proposition

Let ${}_RM$ be a left graded $R$-module such that ${}_RM$ is a gr-self generator and a projective object of $\sigma_R[M]$. Then $R$ operates densely on $M$.

**Proof** By Proposition 4.5.5 and Remark 4.5.6, $M$ is a projective generator of the category $\sigma_R[M]$. Let $x_1, x_2, \ldots, x_n \in M$. In $M^n$ we have the $R$-submodule $N = R(x_1, \ldots, x_n)$. Since $N$ is $M$-generated and finitely generated as an $R$-module, there exists $k > 0$ and an epimorphism :

$$M^k \xrightarrow{u} N \hookrightarrow M^n \tag{1}$$

Since the canonical map $\varphi : \mathrm{Biend}({}_RM) \longrightarrow \mathrm{Biend}(M^n)$ $\varphi(b)(y_1, \ldots, y_n) = (b(y_1), \ldots, b(y_n))$ is an isomorphism of rings, we derive from (1) that

$$\mathrm{Biend}_R(M)(N) \subseteq N$$

Therefore $\alpha.(x_1, \ldots, x_n) \in R.(x_1, \ldots, x_n)$ and hence there exists an $r \in R$ such that $\alpha(x_i) = rx_i \quad 1 \leq i \leq n$. □

### 4.6.6 Corollary

If $\Sigma \in R$-gr is a gr-simple module then $R$ operates densely on $\Sigma$.

### 4.6.7 Corollary (The graded version of Wedderburn's theorem)

Let $R$ be gr-simple ring. There exist a gr-division ring $D$ and a graded finite vectorspace $V_D$ such that :

$$R \simeq \mathrm{End}(V_D)$$

**Proof** Let $\Sigma$ be a gr simple $R$-module. We put $D = \mathrm{End}({}_R\Sigma) = \mathrm{END}({}_R\Sigma)$ and $V = \Sigma_D$. We have the canonical ring morphism :

$$\varphi : R \longrightarrow \mathrm{End}(V_D), \varphi(r)(x) = r.x$$

where $r \in R, x \in V$. Since $\mathrm{Ker}\varphi = \mathrm{Ann}_R(\Sigma) = 0$ ($R$ is a gr-simple ring i.e. ${}_RR = \oplus_{i=1}^n \Sigma(\sigma_i)$ for some elements $\sigma_1, \ldots, \sigma_n \in G$). Corollary 4.6.4 then entails that $V_D$ is finitely generated and Corollary 4.6.6 yields that $\varphi$ is also surjective, hence an isomorphism. □

## 4.7 Extending (Simple) Modules

For a $G$-graded ring $R$ we let $U(R)$, resp. $U^g(R)$, denote the set of all invertible, resp. invertible homogeneous elements of $R$. Obviously $U^g(R)$ is a subgroup of $U(R)$ and the sequence :

$$\mathcal{E}_R(R) : 1 \longrightarrow U(R_e) \longrightarrow U^g(R) \xrightarrow{\deg} G \longrightarrow e$$

is exact everywhere except possibly at $G$. Moreover, $\mathcal{E}_R(R)$ is exact if and only if $R$ is a crossed product. For $M \in R$-gr, $\mathrm{END}_R(M) = \mathrm{HOM}_R(M, M)$ is a $G$-graded ring with multiplication defined as in Chapter 2, Section 2.10, : $g.f = f \circ g$ for $f, g \in \mathrm{END}_R(M)$. Recall also from Section 2.10 that $M$ is $G$-invariant exactly when the sequence

$$\mathcal{E}_R(M) : I \to U(\mathrm{End}_{R-\mathrm{gr}}(M)) \to U^g(\mathrm{END}_R(M)) \to G \to e$$

is exact.

A splitting morphism $\gamma$ for $\mathcal{E}_R(M)$ is a group homomorphism $\gamma : G \to U^g(\mathrm{END}_R(M))$ for which $\deg\gamma(\sigma) = \sigma$ for all $\sigma \in G$.

If $M \in R_e$-mod then an **extension** $M^\odot$ of the $R_e$-module is an $R$-module yielding $M$ by restriction of scalars for the canonical ring morphism $R_e \to R$. Hence $M^\odot$ is $M$ as an additive group but with multiplication $\odot : R \times M^\odot \to M^\odot$ satisfying $r_e \odot m = r_e m$ for $r_e \in R_e$ and $m \in M^\odot$.

We now consider the case of a strongly graded ring $R$ and $M \in R_e$-mod. Let $N = R \otimes_{R_e} M$ be $G$-graded by putting $N_\sigma = R_\sigma \otimes_{R_e} M$, for $\sigma \in G$. For every $\sigma \in G$ we have a canonical group isomorphism for any $\tau \in G$ :

$$\mathrm{END}_R(N)_\sigma \cong \mathrm{Hom}_{R_e}(N_\tau, N_{\tau\sigma})$$

### 4.7.1 Theorem

Assume that $R = \oplus_{\sigma\in G} R_\sigma$ is a strongly graded ring and $M \in R_e$-mod. Then $M$ can be extended to an $R$-module if and only if the sequence $\mathcal{E}_R(N)$, where $N = R \otimes_{R_e} M$, is both exact and split. Indeed there is a one-to-one correspondence between extensions $M^\odot$ of $M$ and splitting homomorphisms $\gamma$ of $\mathcal{E}_R(N)$, in which $M^\odot$ correspond to $\gamma$ if and only if :

$$r_\sigma \odot m = \gamma(\sigma^{-1})(r_\sigma \otimes m) \in M$$

for all $m \in M, r_\sigma \in R_\sigma$.

**Proof** Suppose that $M^\odot$ is an extension of $M$ to an $R$-module. We have the multiplication :

$$\odot : R \times M^\odot \longrightarrow M^\odot$$

which yields the restriction :

$$R_\sigma \times M \longrightarrow M, (r_\sigma, x) = r_\sigma \odot x$$

If $e_e \in R_e$ we have : $(r_\sigma r_e) \odot x = r_\sigma \odot (r_e \odot x) = r_\sigma \odot r_e x$ and therefore we obtain $R_e$-homomorphism $u_\sigma$ of $N_\sigma = R_\sigma \otimes M$ into $M$, mapping $r_\sigma \otimes x$ to $r_\sigma \odot x$. Since $\mathrm{END}_R(N)_{\sigma^{-1}} \simeq \mathrm{Hom}_{R_e}(N_\sigma, N_e = M)$ there exists a unique $\gamma(\sigma^{-1}) \in \mathrm{END}_R(N)_{\sigma^{-1}}$ such that $\gamma(\sigma^{-1})|N_\sigma$ is $u_\sigma$, so $\gamma(\sigma^{-1})(r_\sigma \otimes x) = r_\sigma \odot x$. If $\sigma = e$ then $u_e = 1_M$ and therefore $\gamma(e) = 1$. For any $\sigma, \tau \in C$, the product $\gamma(\tau^{-1}).\gamma(\sigma^{-1})$ lies on $\mathrm{HOM}_R(N)_{\tau^{-1}}.\mathrm{HOM}_R(N)_{\sigma^{-1}} \subseteq \mathrm{HOM}_R(N)_{\tau^{-1}\sigma^{-1}} = \mathrm{HOM}_R(N)_{\tau^{-1}\sigma^{-1}} = \mathrm{HOM}_R(N)_{(\sigma\tau)^{-1}}$. If $x \in M, r_\sigma \in R_\sigma, r_\tau \in R_\tau$ we have :

$$\begin{aligned}
&[\gamma(\tau^{-1})\gamma(\sigma^{-1})][r_\sigma r_\tau \otimes x] = (\gamma(\sigma^{-1}) \circ \gamma(\tau^{-1}))[r_\sigma(r_\tau \otimes x)] \\
&= \gamma(\sigma^{-1}) \left(r_\sigma \gamma(\tau^{-1})(r_\tau \otimes x)\right) = \gamma(\sigma^{-1}) \left(r_\sigma(r_\sigma \odot x)\right) = \\
&= \gamma(\sigma^{-1})(r_\sigma \otimes (r_\tau \odot x)) = r_\sigma \odot (r_\tau \odot x) = (r_\sigma r_\tau) \odot x
\end{aligned}$$

Since $R_\sigma R_\tau = R_{\sigma\tau}$ we have that $(\gamma(\tau^{-1})\gamma(\sigma^{-1}))(\lambda \otimes x) = \lambda \odot x$ for any $\lambda \in R_\sigma R_\tau$ and $x \in M$. Therefore $\gamma(\tau^{-1}).\gamma(\sigma^{-1}) = \gamma(\tau^{-1}\sigma^{-1})$ and hence $\gamma$ is a homomorphism. In particular $\gamma(\sigma^{-1})$ is invertible and therefore the sequence $\mathcal{E}_R(N)$ is exact and splits. Conversely, let $\gamma$ be any splitting homomorphism for $\mathcal{E}_R(N)$. For any $\sigma \in G$, $x \in M$, $r_\sigma \in R_\sigma$ the element $\gamma(\sigma^{-1}) \in \mathrm{END}_R(N)_{\sigma^{-1}}$ and therefore $\gamma(\sigma^{-1})(r_\sigma \otimes x) \in N_R = R_e \otimes_{R_e} M = M$. Thus we define the multiplication $\odot : R \times M \longrightarrow M$ by $r_\sigma \odot x = \gamma(\sigma^{-1})(r_\sigma \otimes x)$. If $\sigma = e$ then $\gamma(e) = 1_N$. We have $r_e \odot x = 1_N(r_e \otimes x) = r_e \otimes x = r_e x$. Because $\gamma$ is a homomorphism, we have :

$$\begin{aligned}
&r_\sigma r_\tau \odot x = \gamma((\sigma\tau)^{-1})(r_\sigma r_\tau \otimes x) = \gamma(\tau^{-1}.\tau^{-1})(r_\sigma r_\tau \otimes x) \\
&= \gamma(\sigma^{-1})(\gamma(\tau^{-1}))(r_\sigma(r_\tau \otimes x)) = \\
&= \gamma(\sigma^{-1})(r_\sigma \gamma(\tau^{-1})(r_\tau \otimes x)) = \gamma(\sigma^{-1})(r_\sigma.(r_\tau \odot x)) \\
&= \gamma(\sigma^{-1})(r_\sigma \otimes (R_\tau \odot x)) = r_\sigma \odot (r_\tau \odot x)
\end{aligned}$$

The group $U(\mathrm{End}_{R-\mathrm{gr}}(N)) = U(\mathrm{End}_{R_e}(M))$ acts naturally on the set of all splitting homomorphisms for $\mathcal{E}_R(N)$. Indeed $u \in U(\mathrm{End}_{R-\mathrm{gr}}(N))$ acts by mapping any such homomorphism $\gamma$ to the conjugate splitting homomorphism $\gamma^u$ for $\mathcal{E}(N)$ defined by $\gamma^u(\sigma) = u^{-1}\gamma(\sigma)u$. □

### 4.7.2 Theorem

Two extensions of the $R_e$-module $M$ to $R$-modules are isomorphic as $R$-modules if and only if the corresponding splitting homomorphisms for $\mathcal{E}_R(N)$, as in Theorem 1, are $U(\mathrm{End}_{R-\mathrm{gr}}(N))$-conjugate. Thus the correspondence of Theorem 4.7.1 induces a one-to-one correspondence between all $R$-isomorphism classes of extensions of $M$ to $R$-modules and all $U(\mathrm{EnD}_{R-\mathrm{gr}}(N))$-conjugacy classes of splitting homomorphisms for $\mathcal{E}_R(R \otimes_{R_e} M)$.

**Proof** Let $M^{\odot}$ and $M^{\odot'}$ be two extensions of $M$ to $R$-modules and $\gamma$ and $\gamma'$ be their respective corresponding splitting homomorphisms for $\mathcal{E}_R(N) = \mathcal{E}_R(R \otimes_{R_e} M)$. Any $R$-isomorphism of $M^{\odot'}$ to $M^{\odot}$ is also an $R_e$-automorphism of $M$ i.e. a unit of $\mathrm{End}_{R-\mathrm{gr}}(R \otimes_{R_e} M) = \mathrm{End}_{R_e}(M)$. If $u : M^{\odot'} \longrightarrow M^{\odot}$ is $R$-isomorphism, we have $u \in \mathrm{End}_{R_e}(M)$ and moroever $u(r_\sigma \odot' x) = r_\sigma \odot u(r)$ or $u(\gamma'(\sigma^{-1})(r_\sigma \otimes x)) = \gamma(\sigma^{-1})(r_\sigma \otimes u(x))$. But $r_\sigma \otimes u(x) = \overline{u}(r_\sigma \otimes x)$ where $\overline{u}$ corresponds to $u$ in $\mathrm{End}_{R-\mathrm{gr}}(R \otimes_{R_e} M)\overline{u} = 1 \otimes u$. Hence $(\overline{u} \circ \gamma'(\sigma^{-1}))(r_\sigma \otimes x) = (\gamma(\sigma^{-1} \circ \overline{u})(r_\sigma \otimes x)$ and so $\overline{u} \circ \gamma'(\sigma^{-1}) = \gamma(\sigma^{-1}) \circ \overline{u}$ and therefore $\overline{u} \circ \gamma'(\sigma^{-1}) \circ \overline{u}^{-1} = \gamma(\sigma^{-1})$. Hence $\gamma' = \gamma^u$. The converse follows in a similar way. □

### 4.7.3 Corollary

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a $G$-strongly graded ring. Then the left $R_e$-module $R_e$ can be extended if and only if the sequence $\mathcal{E}_R(R)$ is exact and split. Indeed, there is a one-to-one correspondence between all extensions $R_e^{\odot}$ of $R_e$ to $R$-modules and all splitting homomorphisms $\gamma$ of $\mathcal{E}_R(R)$. The extension $R_e^{\odot}$ corresponds to $\gamma$ if and only if : $a_\sigma \odot a_e = a_\sigma a_e \gamma(\sigma^{-1})$ for all $a_e \in R_e, a_\sigma \in R_\sigma, \sigma \in G$. Moreover this correspondence induces a one-to-one correspondence between all $R$-isomorphism classes of such extension $R_e^{\odot}$ and all $U(R_e)$-conjugacy classes of such homomorphism $\gamma$.

**Proof** We apply Theorems 4.7.1 and 4.7.2 to the case $M =_{R_e} R_e$. In this case $N = R \otimes_{R_e} R_e \simeq R$ and $\mathrm{END}_R(N) \simeq R$.

### 4.7.4 Example

Let $A$ be a ring, $G$ a group and $\varphi : G \to \mathrm{Aut}(A)$ a group homomorphism. We denote by $R = A *_\varphi G$ the skew group ring associated to $A$ and $\varphi$. $R$ has multiplication defined by : $(ag)(bh) = a\varphi(g)(b)(gh)$ where $a, b \in A$ and $g, h \in G$. Moreover $R$ is a $G$-graded ring with grading $R = \oplus_{g \in G} R_g$ where

$R_g = Ag$. By Corollary 4.7.3, $A$ has a natural structure of left $R$-module given by the rule $(ag) \odot x = a\varphi(g)(x)$ where $a \in A, g \in G, x \in A$. If the action of $G$ on $A$ is trivial, i.e. $\varphi(g) = 1_A$ for any $g \in G$, then $R = A *_\varphi G$ is the group ring $A[G]$. If we denote by $\varepsilon : A[G] \to A$, the augmentation map i.e.$\varepsilon(g) = 1$ for any $g \in G$, then we have $\varepsilon \circ i = 1_A$ where $i : A \to A[G]$ is the inclusion morphism. In this case if $M \in A$-mod, then $M$ has a natural structure as an $A[G]$-module if for any $\alpha \in A[G]$ and $m \in M$ we put $\alpha \odot m = \varepsilon(\alpha).m$. For $\alpha = a \in A$ we obviously have $a \odot m = am$.

### 4.7.5 Remark

If $R = \oplus_{\sigma \in G} R_\sigma$ is a strongly graded ring and $M \in R_e$-mod is extended to an $R$-module then, by Theorem 4.7.1, $M$ is necessary $G$-invariant i.e. $M \simeq R_\sigma \otimes_{R_e} M$ for any $\sigma \in G$. Let $R = \oplus_{\sigma \in G} R_\sigma$ be a strongly graded ring and $S$ a left simple $R_e$-module. We put $\Sigma = R \otimes_{R_e} S$; $\Sigma$ is a gr-simple object in $R$-gr. Assume that $S$ is $G$-invariant so $\sigma$ is $G$-invariant in $R$-gr. We put by $\Delta = \mathrm{End}_R(\Sigma) = \mathrm{END}_R(\Sigma)$; $\Delta$ is a gr-division ring. If we put $\Delta = \oplus_{\sigma \in G} \Delta_\sigma$, then, since $S$ is $G$-invariant, $\Delta_\sigma \neq 0$ for any $\sigma \in G$. We denote by $\Delta^{*,\mathrm{gr}} = \cup_\sigma(\Delta_\sigma - \{0\})$; $\Delta^{*,\mathrm{gr}}$ is a subgroup of $U(\Delta)$. In this case $S$ is extended to $R$ if and only if there exists a morphism of groups $\gamma : G \to \Delta^{*,\mathrm{gr}}$ such that $\gamma(g) \in \Delta_g$ for any $g \in G$. Moreover if $S^\odot$ is an extension of $S$ to $R$, then $S^\odot$ is a simple $R$-module.

Now assume that $k$ is an algebraically closed field and $R$ is a $G$-graded finite dimensional $k$-algebra. Consider a left $G$-invariant simple $R_e$-module $S$. With notation as before, we now obtain from Schur's lemma that $\Delta_e = k$, because $k$ is algebraically closed.

Let us look at the case where $G =< g >$ is a finite cyclic group of order $n$, i.e. $g^n = e$. Pick $u_g \neq 0$ in $\Delta_g$. Since $u_g^n \in \Delta_{g^n} = \Delta_e = k$ there exists a $\xi \in k$ such that $u_g^n = \xi^n$. Putting $v_g = \xi^{-1} u_g \in \Delta_g$ leads to $v_g^n = 1$. Therefore we may define $\gamma : G \to \Delta^{*,\mathrm{gr}}$ by $\gamma(g^i) = v_g^i$. Since $\gamma$ is a group map, Theorem 4.7.1 yields that $S$ can be extended to an $R$-module.

For further application of the theory of extending simple modules in the representation theory of finite groups the reader may consult E. C. Dade's paper [50].

## 4.8 Exercises

1. Let $R$ be a $G$-graded ring where $G$ is a torsionfree abelian group. If $M$ is a maximal left ideal of $R$, then $(M)_g$ is the intersection of all maxinal left ideals containing it.

   **Hint :** If we put $J(R/(M)_g) = K/(M)_g$ then $K$ is a left graded ideal of $R$ such that $(M)_g \subseteq K \subseteq M$. Then $K = (M)_g$ and therefore

$J(R/(M)_g) = 0$.

2. Let $R$ be a $G$-graded ring, where $G$ is a free abelian group. If $I \leq R$ is a primitive (resp. semi-primitive) ideal of $R$, then $(I)_g$ is a semi-primitive ideal of $R$.

   **Hint :** If $I$ is a primitive ideal, there exists a left simple $R$-module $S$ such that $I = \mathrm{Ann}_R S$. Then $(I)_g = \cap_{x \in S}(\mathrm{Ann}_R(x))_g$. The above exercises may be used to prove that $J(R/(I)_g) = 0$.

3. Let $A$ be an arbitrary ring, $T$ a variable commuting with $A$; then $J(A(T]) = I[T]$ where $I = J(A[T]) \cap A$ is a nil ideal of $A$.

   **Hint :** If we consider the polynomial ring $A[T]$ with natural grading over the group $G = \mathbb{Z}$, then $J(A[T])$ is a graded ideal. So we can write $J(A[T]) = I_0 \oplus I_1 T \oplus I_2 T^2 \oplus \ldots \oplus I_n T^n \oplus \ldots$ such that $I_0 \subseteq I_1 \subseteq \ldots \subseteq I_n \subseteq \ldots$. Consider the automorphism $\varphi : A[T] \to A[T]$ such that $\varphi(T) = T + 1$. Since $\varphi(J(A[T])) = J(A[T])$ it follows easily that $I_0 = I_1 = \ldots = I_n = \ldots$. If we put $I = I_0$ then $I = J(A[T]) \cap A$. If $a \in I$, then it follows from $aT \in J(A[T])$ that $1 - aT$ is invertible in $A[T]$. That $a$ is a nilpotent element then follows easily.

4. (S. Amitsur) Let $A$ be an algebra over the feld $R$ and $R(T)$ be the field of rational functions in one variable over $R$. We have : $J(A \otimes_R R(T)) = I \otimes_R R(T)$, where $I = A \cap J(A \otimes_R R(T))$ is a nil ideal of $A$.

   **Hint :** (Following G. Bergman). Obviously : $I \otimes R(T) \subseteq J(A \otimes_R R(T))$. Conversely, any element of $J(A \otimes_R R(T))$ can be written as $p(T)^{-1}(a_m T^m + \ldots + a_0)$ where $p(T) \in R[T]$ is nonzero and $a_i \in A$. If $n > 1$, $R(T)$ is $\mathbb{Z}/n\mathbb{Z}$-graded if we put : $K(T)_i = T^i K(T^n)$ for $n > 0, i \in \mathbb{Z}/n\mathbb{Z}$. Looking at the induced gradation of $A \otimes_R R(T)$ and applying Corollary 4.3.5 we obtain for all $n > m$, $na_i T^i \in J(A \otimes_R R(T))$. Thus $a_i T^i \in J(A \otimes_R R(T))$ and therefore $a_i \in I$. Pick $x \in I$, then $(1 + xT)^{-1} = 1 - xT + x^2T^2 \ldots$ in $A \otimes_R R(T)$. However the coefficients have to sit in a finite dimensional $R$-subspace of $A$ i.e. $x$ is algebraic over $R$. If $x$ were no nilpotent then the polynomial equation satisfied by $x$ would give rise to a nonzero idempotent element in the Jacobson radical, a contradiction.

5. Let $A$ be a ring and $M$ a left finitely generated $A$-module. If $T$ is a commuting variable with $A$, prove that $J(M[T])$ has the form $N[T]$ where $N$ is a submodule of $M, N \neq M$ (here $M[T] = A[T] \otimes_A M$). In particular if $M$ is a simple $A$-module then $J(M[T]) = 0$.

   **Hint :** Similar to the proof of exercise 3.

6. Let $A$ be a local domain with maximal ideal $M \neq 0$. Prove that in the graded ring $A[X]$ with the natural grading we have

   i) $J(A[X]) = 0$

ii) $J^g(A[X]) = M[X]$

iii) $J(A[X]) \neq J^g(A[X])$

**Hint :** For assertion i. use the exercise and ii. ans iii. are obvious.

7. Let $R$ be a $G$-graded ring with $G$ a finite group. Show that if $M$ is gr-semi-simple then $M$ is **quasi-injective** in $R$-mod. We recall that $M$ is quasi-injective in $R$-mod, for any submodule $M'$ of $M$ and every $R$-homomorphism $f : M' \to M$, then $f$ is extended to the $R$-homomorphism $g : M \to M$, i.e. $g|M' = f$.

   **Hint :** We consider the closed subcategories $\sigma^{\mathrm{gr}}[M]$ of $R$-gr and $\sigma_R[M]$ of $R$-mod. Clearly $M$ is a injective object in the category $\sigma^{\mathrm{gr}}[M]$. Since the functor $F : R\text{-mod} \to R\text{-gr}$ is a left adjoint for the forgetful functor $U : R\text{-gr} \to R\text{-mod}$, resulting that $U(M) = M$ is an injective in the category $\sigma_R[M]$.

8. A $G$-graded ring $R = \oplus_{\sigma \in G} R_\sigma$ is said to be **gr-Von Neumann regular** (or **gr-regular**, for short) if, for any $\sigma \in G$ and any homogeneous element $a \in R_\sigma$, there exists $b \in R$ (which can be supposed to be also homogeneous such that $a = aba$. Then prove that the following assertions hold :

   i) If $R$ if gr-regular, then $R$ is e-faithful.

   ii) If $R$ is gr-regular and $\sigma \in G$, then $\sigma \in \sup(R)$ if and only if $\sigma^{-1} \in \sup(R)$.

   iii) If $D$ is a gr-division ring and $V_D$ a graded right $D$-module, then $\mathrm{END}(V_D)$ is regular.

   **Hint :**

   i) Let $a \in R_\sigma, a \neq 0$. Then there exists a homogeneous element $b \in R_{\sigma^{-1}}$ such that $a = aba$. Clearly $a \neq 0$ implies that $ab \neq 0$ and $ba \neq 0$ so $R_{\sigma^{-1}} a \neq 0$ and $aR_{\sigma^{-1}} \neq 0$ i.e. $R$ is e-faithful

   ii) Is clear.

   iii) Is proved as in the non-graded case.

9. A graded ring $R$ is called **gr-primitive**, if there exists a gr-simple module ${}_R\Sigma$ such that $\mathrm{Ann}_r({}_R\Sigma) = 0$. Then prove that if $R$ is gr-primitive, $R$ is e-faithful.

   **Hint :** If we put $\Delta = \mathrm{End}({}_R\Sigma)$ and $S = \mathrm{END}(\Sigma_\Delta)$ then $S$ is gr-regular. On the other hand, $R$ is dense in the ring $S$. Now we can apply exercise 8.

10. Let $R$ be a $G$-graded ring and $M \in R$-gr. If $(\sigma_i)_{i \in I}$ is a family of elements of the group $G$, prove that the map

$$\varphi : \mathrm{BIEND}_R(M) \longrightarrow \mathrm{BIEND}_R(\oplus_{i \in I} M(\sigma_i))$$

given by $\varphi(L)(x_i) = (b(x_i))_{i\in I}$ where $b \in \mathrm{BIEND}_R(M)$ and $(x_i)_{i\in I} \in \oplus_{i\in I} M(\sigma_i)$ is a isomorphism of rings.

**Hint :**

It is easy to see that if $\sigma \in G$, then $\mathrm{BIEND}_R(M) = \mathrm{BIEND}_R(M(\sigma))$. As in the non graded case we can prove first that $\varphi(b) \in \mathrm{BIEND}_R(\oplus_{i\in I} M(\sigma_i))$ and also $\varphi$ is a graded morphism of rings. Clearly $\varphi$ is an injective map and the fact that $\varphi$ is surjective it is analogue to the nongraded case.

11. Let $\Sigma$ be a gr-simple module in $R$-gr, with $\Delta = \mathrm{End}(_R\Sigma)$ and $H \subseteq G$ be a subgroup such that $H \cap \sup(\Sigma) \neq \emptyset$. Prove that the following assertions hold :

    i) $\Sigma_H$ is a simple object in $R_H$-gr.

    ii) $\sup(\Sigma_H) = H \cap \sup(\Sigma)$

    iii) $\mathrm{End}_{R_H}(\Sigma_H) = \Delta_H$.

    iv) $G\{\Sigma_H\} = H \cap G\{\Sigma\}$

    v) If the countermodule $\Sigma_\Delta$ is finitely generated, then the countermodule $(\Sigma_H)_{\Delta_H}$ is also finitely generated.

    **Hint :** i. and ii. are obvious.

    iii. We define $\varphi : \Delta_H \to \mathrm{End}_{R_H}(\Sigma_H), \varphi(f) = f|\Sigma_H$ where $f \in \Delta_H$. Clearly $\varphi$ is correctly defined and from i. yields that $\varphi$ is injective. To prove that $\varphi$ is surjective see exercises from Section 2.12;

    iv. follows from iii. and v. from Corollary 4.5.4.

12. With notation as in exercise 10, the functor :

$$R\overline{\otimes}_{R_H} - : R_H\text{−mod} \longrightarrow (G/H, R)$$

    (see exercise 9., Section 2.12) has the property that if $N \in \mathrm{Mod}(R_H|\Sigma_H)$ then $R\overline{\otimes}_{R_H} N \in \mathrm{Mod}(R|\Sigma)$.

    **Hint :** Use exercise 10, Section 2.12.

13. With notation as in exercise 11. If $H = G\{\Sigma\}$ then the functor : $R\overline{\otimes}_{R_H} - : \mathrm{Mod}(R_H/\Sigma_H) \longrightarrow \mathrm{Mod}(R/\Sigma)$ is an equivalence.

14. (Dade) Let $G$ be any group, $H$ a subgroup of $G$ and $(\sigma_i H)_{i\in I}$ a family of left cosets of $H$ in $G$. If we put $P = \cup_{i\in I}\sigma_i H$ prove that there exists a $G$-graded ring $R$, a left gr-simple $R$-module $\Sigma$ such that $\sup(\Sigma) = P$ and $G\{\Sigma\} = H$.

    **Hint :** First we show that we can assume that $H \subseteq P$. Indeed if there exists $M \in R$-gr such that $\sup(M) = P$ and $G\{M\} = H$, let $\sigma \in P$. If $H \not\subset P$ then $H \cap P = \emptyset$. Put $P' = P\sigma^{-1}$ and $H = \sigma H \sigma^{-1}$. Clearly $H' \subseteq P'$. In this case if $M' = M(\sigma)$, then $\sup(M') = P\sigma^{-1}$

and $G\{M'\} = H'$. So to prove the assertion in exercise 13. we may assume that $H \subset P$ i.e. there exists a $\sigma_i = 1$. Now we follow the proof of Proposition 8.1 of [52] and arrive at the following construction : Let $R$ be a field, $G$ a group and $A$ a left $G$-set, $A \neq \emptyset$. Consider the direct product $T = R^A$ so $T = \{(x_a)_{a\in A}, x_a \in R\}$. We define $\varphi : G \to \mathrm{Aut}(T)$ $\varphi(g)((x_a)_{a\in A}) = (x_{g^{-1}a})_{a\in A}$. It is easy to see that $\varphi$ is a homomorphism of groups. Then we can define the skew group ring $S = T * G$ with multiplication : $(tg)(t'g') = t\varphi(t')gg'$.

The ring $S$ is a $G$-graded ring with natural gradation $S_g = T \cdot g$ for any $g \in G$.

For any subset $B \subset A, B \neq \emptyset$ we denote by $e_B$ the idempotent element of $T$ : $C_B = (c_a)_{a\in A}$ where $c_a = 1$ if $a \in B$ and $x_a = 0$ if $a \notin B$. If $B \neq \emptyset$ we put $e_B = 0$. Clearly if $B, B'$ are two subjects of $A$ then $e_B.e_{B'} = e_{B'}.e_B = e_{B\cap B'}$. So in particular if $B \cap B' = \emptyset$ we have $e_B.e_{B'} = 0$. If $B = \{a\}$ we denote $e_B = e_a$. It is easy to see that in $T$ we have the equality $g.e_a = e_{g.a}$ for any $g \in G$. Let $B \subset A$, $B \neq \emptyset$, we denote by $R = e_BSe_B$, $R$ is $G$-graded ring where $R_g = e_BS_ge_B$. It is easy to see that $R_g.R_h \subseteq R_{gh}$ and $R_e = e_BTe_B = Re_B$. If $b \in B$ we put $\Sigma = Re_b$; clearly $\Sigma$ is a left graded $R$-module (in fact it is a left graded ideal of $R$).

The following assertions hold :

i) $\sup(\Sigma) = \{g \in G | gb \in B\}$.

ii) $G\{\Sigma\} = \mathrm{stab}_G(b) = \{g \in G | gb = b\}$.

iii) $\Sigma$ is gr-simple in $R$-gr.

Indeed if $g \in G$, we have $\Sigma_g = R_ge_b = e_B(T * g)e_B.e_b = e_B(T * g)e_b = (e_BTe_{gb})g = \begin{cases} (Re_{gb})g & \text{if } gb \in B \\ 0 & \text{if } gb \notin B \end{cases}$. Therefore $\sup(\Sigma) = \{g | gb \in B\}$. Now if $g \in G$ we have $\Sigma(g) \simeq \Sigma$ if and only if $\Sigma_g \simeq \Sigma_e$. Assume $g \in G\{\Sigma\}$, then we have an isomorphism $\alpha : \Sigma_g \to \Sigma_1$ of $R_e$-modules. Since $e_b \in R_e$ then $\alpha(e_b.x) = e_b.\alpha(x)$. Since $\Sigma_e = ke_b$, then $e_b\Sigma_e = \Sigma_e$ and hence $e_b\Sigma_g \neq 0$, so $e_b.e_{gb} = 0$ and therefore $gb = b$. Hence $G\{\Sigma\} \subseteq \mathrm{stab}_a(b)$. On the other hand if $g \in \mathrm{stab}_G(b)$ then $gb = b$. Clearly the map $\beta : \Sigma_e \to \Sigma_g, \beta(\lambda e_b) = \lambda e_{gb}g$ is an isomorphism of $R_e$-modules. Hence $\mathrm{stab}_G(b) \subseteq G\{\Sigma\}$ i.e. $G\{\Sigma\} = \mathrm{stab}_G(b)$. Let now $g \in \sup(\Sigma)$ and $x_g \in \Sigma_g$, $ax_g \neq 0$, then $x_g = (\lambda e_{gb})g$ where $gb \in B$ and $\lambda \in R, \lambda \neq 0$. If $h \in \sup(\Sigma)$ and $y_h \in \Sigma_h, y_h \neq 0$, then $y_h = (\mu e_{hb})h$ where $hb \in B$. Clearly we have $y_h = (e_B\mu\lambda^{-1}hG^{-1}e_B)(\lambda e_{gb}g)$ and therefore $\Sigma = Tx_g$ so $\Sigma$ is a gr-simple $T$-module.

Now, if we consider $A = G/H$; $A$ is a left $G$-set. Moreover if $b = \{H\}$ (we can assume $H \subset P$), we obtain the construction for exercise 13.

## 4.9 Comments and References for Chapter 4

The structure of simple objects in $R$-gr is given in Section 2.7. by Theorem 2.7.2, connecting the simples of $R$-gr to simple $R_e$-modules via the functor $R_e\overline{\otimes}(-)$. The purpose of Chapter 4 is to describe the structure of simple objects of $R$-gr when viewed as $R$-modules without gradation. For a simple $\Sigma$ in $R$-gr the category $\mathrm{Mod}(R|\Sigma)$ of $\Sigma$-generated $R$-modules is introduced. Its structure is analyzed in Theorem 4.1.4 by means of the ring $\Delta = \mathrm{End}_R(\Sigma)$. The key to this theorem is a lemma due to E.C. Dade (cf. Lemma 4.1.2).

In Section 4.2, objects of $\mathrm{Mod}(R|\Sigma)$ are viewed as $R_e$-modules by restriction of scalars, yielding Theorem 4.2.5. as a main result. Foregoing results are applied in Section 4.3 to strongly graded rings and their Clifford theory (see Theorem 4.3.2). The results of Section 4.1 are applied in Section 4.4 to yield a clarification of the structure of $\Sigma$ as an $R$-module in particular cases (cf. Theorem 4.4.4) and knowledge about this structure in turn is useful in the study of the Jacobson radical of $R$ (cf. Corollary 4.4.5. Theorems 4.4.6 and 4.4.11). Making use of torsion theories it is possible to extend the results of Section 4.1. (cf. Theorem 4.5.7).

A density theorem for simple objects of $R$-gr is obtained in Section 4.6 and a new graded version of Wedderburn's theorem is derived from this. At the end of Chapter 4 we present the theory of extending (simple) modules for strongly graded rings, following E.C. Dade [50]. The results in Chapter 4 provide useful general techniques, e.g. applications to Graded Clifford Theory and beyond establish that this theory is a powerful instrument in the theory of graded rings. This instrument stems from Representations of Groups Theory.

### Some References

- G. Bergman [18]
- M. Beattie, P. Stewart [12]
- A. H. Clifford [41]
- A.C. Dade [49], [50], [51], [52], [53]
- J.-L. Gomez Pardo, C. Năstăsescu [74], [76]
- C. Năstăsescu [139], [140]
- C. Năstăsescu, F. Van Oystaeyen [151], [153]
- C. Năstăsescu, B. Torrecillas [162], [163]

# Chapter 5

# Internal Homogenization

## 5.1 Ordered Groups

An **ordered group** or an **O-group** is a group $G$ together with a subset $S$ of $G$ (the set of positive elements) such that the following conditions hold :

OG1. $e \notin S$

OG2. If $a \in G$ then $a \in S$, $a = e$, or $a^{-1} \in S$

OG3. If $a, b \in S$ then $ab \in S$

OG4. $aSa^{-1} \subset S$ for any $a \in G$.

It is easy to check that OG4 is equivalent to $aSa^{-1} = S$ for any $a \in G$. For $a, b \in S$ write $b < a$ if $b^{-1}a \in S$. From OG4 we have $b(b^{-1}a)b^{-1} \in S$ so $ab^{-1} \in S$. Therefore $b^{-1}a \in S$ is equivalent to $ab^{-1} \in S$. From OG3 we see that $<$ is a transitive relation on $G$ and by OG2 we have that for $a, b \in G$ exactly one of $a < b$, $a = b$ or $b < a$ holds. If $b < a$ and $c \in G$ then $(cb)^{-1}(ca) = b^{-1}a \in S$, hence $cb < ca$ and in similar way $bc < ac$. Also if $b < a$ then $ab^{-1} \in S$, hence $(a^{-1})^{-1}b^{-1} \in S$ and thus $a^{-1} < b^{-1}$. Conversely, if the elements of $G$ are linearly ordered with respect to a relation $<$ such that $b < a$ implies that $bc < ac$ and $cb < ca$ for any $c \in G$, then the set $S = \{x \in G \mid e < x\}$ satisfies OG1-4.

### 5.1.1 Example

1. Any torsion-free nilpotent group is ordered.

2. Every free group is an ordered group.

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### 5.1.2 Remark

Recall from D. Passman ([167], pag 586) that a group G is said to be a **right ordered group** or an **RO-group** if the elements of G are linearly ordered with respect to the relation $<$ and if, for all $x, y, z \in G$, $x < y$ implies $xz < yz$. It is clear that if G is an O-group then G is a RO-group. From ([167], pag 587, Lemma 1.6), it follows that a group $G$ with finite normalizing series.

$$\{e\} = G_0 \lhd G_1 \lhd ... \lhd G_n = G$$

where consecutive quotients $G_{i+1}/G_i$ are torsion-free abelian groups, is an RO-group.

The following result ([167], Lemma 1.7, pag 588) will be useful : if G is an RO group and A and B are finite nonempty subsets of G, then there exists b' and b" $\in B$ such that the products $a_{\max}b'$ and $a_{\min}b''$ are uniquely represented in AB (here, $a_{\max}$ and $a_{\min}$ denote the largest and the smallest element in A, respectively).

## 5.2 Gradation by Ordered Groups. Elementary Properties

Throughout this section $G$ is an O-group. Let R be a graded ring of type G. An $\oplus_{\sigma\in G}M_\sigma$ is said to be **left limited** (or **right limited**) if there is a $\sigma_0 \in G$ such that $M_\sigma = 0$ for all $\sigma < \sigma_0$ (resp. $\sigma_0 < \sigma$). If $M_\sigma = 0$ for each $\sigma < e$, then $M$ is said to be positively graded and if $M_\sigma = 0$ for all $\sigma > e$ then $M$ is negatively graded. If $M = \oplus_{\sigma\in G}M_\sigma$ is a nonzero left graded module, $M$ is called **left (right) strongly limited** if there exists a $\sigma_0 \in G$ such that $M_{\sigma_0} \neq 0$ and $M_x = 0$ for any $x < \sigma_0$ (resp. $\sigma_0 < x$). Clearly if $M$ is left (right) limited and $\sup(M) \subseteq G$ has a least (upper) element then $M$ is left (right) strongly limited (for example when $G = \mathbb{Z}$). If $\sigma \in G$ then denote by $M_{\geq\sigma}$ (resp $M_{>\sigma}$) the sum $\oplus_{x\geq\sigma}M_x$ (resp $\oplus_{x>\sigma}M_x$). In the same way we define $M_{\leq\sigma}$ (resp $M_{<\sigma}$). Also we use $M_+$ for $M_{>e}$ (and $M_-$ for $M_{<e}$).

### 5.2.1 Proposition

Assume that R is left limited and $M \in R$-gr. Then the following properties hold :

1. If $M$ is finitely generated then M is left limited.
2. If $M$ is left limited, there exists a free resolution

$$... \to F_n \to ... \to F_2 \to F_1 \to F_0 \to M \to 0$$

where all $F_i$ are gr-free and left limited.

**Proof**

1. Assume that there exists $\sigma_0 \in G$, such that $R_\sigma = 0$ for any $\sigma < \sigma_0$. Also suppose that $M$ is generated by homogeneous elements $x_1, ..., x_n$ such that $\deg x_i = \tau_i$ and $\tau_1 \leq ... \leq \tau_n$. Then if $\sigma < \sigma_0\tau_1$ and $x \in M_\sigma$ we have $x = \sum_{i=1}^n a_i x_i$ where $a_1, ...a_n$ are homogeneous. Clearly $\deg a_i = \sigma\tau_i^{-1} < \sigma_0$ so $a_i = 0$ for all $i$, $1 \leq i \leq n$ and therefore $M_\sigma = 0$ for any $\sigma < \sigma_0\tau_1$.

2. Assume that there exist a $\tau_0 \in G$ such that $M_x = 0$ for any $x < \tau_0$. Let $(x_i)_{i\in I}$ a family of homogeneous generators of $M$. If we put $\tau_i = \deg x_i$ then we have $\tau_0 \leq \tau_i$. Consider the gr-free module

$$F_0 = \oplus_{i\in I} R(\tau_i^{-1})$$

   It is easy to see that $(F_0)_x = 0$ for any $x < \sigma_0\tau_1$ so $F_0$ is left limited. Clearly we have the canonical epimorphism $f_0 : F_0 \to M \to 0$ in R-gr. Since $K_0 = ker\, f_0$ is also left limited, we may repeat the argument. □

## 5.2.2 Proposition

Assume that R is strongly graded ring and $M = \oplus_{\sigma\in G} M_\sigma$ is left (right) limited. If $G \neq \{e\}$ then $M = 0$.

**Proof** Since $R$ is strongly graded ring we have $R_x M_y = M_{xy}$ for any $x, y \in G$. Since $M$ is left limited, there is $\sigma_0 \in G$ such that $M_x = 0$ for any $x < \sigma_0$. If $\sigma_0 < e$ then $\sigma_0^2 < \sigma_0$ and we have $M_{\sigma_0^2} = 0$ so $M = 0$. If $e < \sigma_0$ then $M_e = 0$ and therefore $M = 0$. If $\sigma_0 = e$ then since $G \neq \{e\}$ there is a $\sigma \in G$, $\sigma \neq e$. We may assume $\sigma < e$, (otherwise take $\sigma^{-1} < e$), so $M_\sigma = 0$ and thus $M = 0$. □

## 5.2.3 Proposition

Let $R$ be a positively graded ring and let $M \in R$-gr be a nonzero left strongly limited module. Then $R_{>0}M \neq M$.

**Proof** If $M \neq 0$ then there is $\sigma_0 \in G$ such that $M_{\sigma_0} \neq 0$ and $M_x = 0$ for any $x < \sigma_0$. Since $M_{\sigma_0} \cap R_{>0}M = 0$, we have $R_{>0}M \neq M$. □

## 5.2.4 Proposition

Let $R = \oplus_{\sigma\in G} R_\sigma$ be a positively graded ring. If $\Sigma$ is gr-simple then there is a $\sigma_0 \in G$ such that $\Sigma = \Sigma_{\sigma_0}$. In particular, we have that $J^g(R) = J(R_e) \oplus R_{>0}$.

**Proof** Since $\Sigma \neq 0$, there is a $\sigma_0 \in G$ such that $\Sigma_{\sigma_0} \neq 0$. Then $R\Sigma_{\sigma_0} = \Sigma$. But $R\Sigma_{\sigma_0} \subseteq \Sigma_{\geq \sigma_0}$ so $\Sigma_x = 0$ for any $x < \sigma_0$. On the other hand $\Sigma_{>\sigma_0}$ is a graded submodule of $\Sigma$. Since $\Sigma_{\sigma_0} \cap \Sigma_{>\sigma_0} = 0$ then $\Sigma_{>\sigma_0} = 0$ so we have $\Sigma_x = 0$ for any $x \neq \sigma_0$. Since $J^g(R) \cap R_e = J(R_e)$ then $J^g(R) \subseteq J(R_e) \oplus R_{>0}$. Now if $\Sigma$ is gr-simple then $\Sigma = \Sigma_{\sigma_0}$ for some $\sigma_0 \in G$. Clearly $R_{>0}\Sigma = R_{>0}\Sigma_{\sigma_0} = 0$ so we arrive at $R_{>0} \subseteq J^g(R)$. □

### 5.2.5 Proposition

Let $R$ be a $G$-graded ring, where $G$ is an $O$-group. If $\Sigma$ is a gr-simple module, then $J(\Sigma) = 0$. In particular we have $J(R) \subseteq J^g(R)$.

**Proof** From the graded Clifford theory (Section 4.1) it follows that it is enough to prove that $J(\Delta) = 0$, where $\Delta = \mathrm{End}_R(\Sigma)$. But $\Delta = \oplus_{\sigma \in G\{\Sigma\}} \Delta_\sigma$ and every nonzero homogeneous element of $\Delta$ is invertible. If $G\{\Sigma\} = \{e\}$, then $\Delta = \Delta_e$ is a division ring and so $J(\Delta) = 0$. On the other hand, if $G\{\Sigma\} \neq \{e\}$ then $G\{\Sigma\}$ is infinite since an $O$-group is torsion-free. Let $a \in \Delta$ an invertible element. We prove that $a$ is a homogeneous element. Indeed, assume that there is $b \in \Delta$ such that $ab = 1$. We can write $a = a_{\sigma_1} + ... + a_{\sigma_n}$ where $0 \neq a_{\sigma_i} \in \Delta_{\sigma_i}$ and $\sigma_1 < ... < \sigma_n$ and similarly $b = b_{\tau_1} + ...b_{\tau_m}$ where $0 \neq b_{\tau_i} \in \Delta_{\tau_i}$ and $\tau_1 < ... < \tau_m$. Then if $n \geq 2$, the product $ab$ has at least two nonzero homogeneous components $a_{\sigma_1} b_{\tau_1}$ and $a_{\sigma_n} b_{\tau_m}$ which contradicts the fact that $ab = 1$. Therefore we have $n = 1$ and $a$ is a homogeneous element. Assume now that $a$ is a nonzero element of $J(\Delta)$ such that $1 - ba$ is invertible and hence homogeneous for any $b \in \Delta$. If $a = a_{\sigma_1} + ... + a_{\sigma_n}$ with $a_{\sigma_i} \in \Delta_{\sigma_i}$ then since $G\{\Sigma\}$ is an infinite group, there is a homogeneous element $0 \neq b \in \Delta_\sigma$ with $\sigma \in G\{\Sigma\}$ such that $e \notin \{\sigma\sigma_1, ..., \sigma\sigma_n\}$ and hence $1 - ba$ has at least two nonzero homogeneous components. But on the other hand we have seen that $1 - ba$ is homogeneous and this is a contradiction which shows that $a = 0$, so $J(\Delta) = 0$.

### 5.2.6 Proposition

Let $R$ be a $G$-graded ring where $G$ is an $O$-group. Then :

1. If $P$ is a gr-prime ideal of $R$ then $P$ is prime. In particular we have $\mathrm{Spec}^g(R) \subseteq \mathrm{Spec}(R)$.

2. $\mathrm{rad}^g(R) = \mathrm{rad}(R)$, so $\mathrm{rad}(R)$ is a graded ideal of $R$.

3. $R$ is a domain if and only if $R$ has no homogeneous zero-divisors.

**Proof**

1. Let $a, b \in R$ such that $aRb \subset P$. Assume that $a \notin P$. We may write $a = a_{\sigma_1} + ... + a_{\sigma_n}$ with $0 \neq a_{\sigma_i} \in R_{\sigma_i}$ and $\sigma_1 < ... < \sigma_n$ and also $b = b_{\tau_1} + ... + b_{\tau_m}$ with $0 \neq b_{\tau_i} \in R_{\tau_i}$ and $\tau_1 < ... < \tau_m$. We may assume that $a_{\sigma_n} \notin P$ because otherwise we get $(a - a_{\sigma_n})Rb \subset P$ and in this case we can replace $a$ with $a - a_{\sigma_n}$. From $aRb \subset P$ we get $a_{\sigma_n} Rb_{\tau_m} \subset P$ and then $b_{\tau_m} \in P$ (because $a_{\sigma_n} \notin P$). Replacing $b$ with $b - b_{\tau_m}$ we may repeat the same argument and finally get that $b_{\tau_1}, ...b_{\tau_m} \in P$ so $b \in P$.
2. follows from i. and iii. is obvious. □

## 5.3 Internal Homogenization

Throughout this section $G$ will be an $O$-group and $R$ a graded ring of type $G$. Let $M \in R$-gr and $X \subset M$ a submodule. Any $x \in X$ may be written in a unique way as $x = x_{\sigma_1} + ... + x_{\sigma_n}$ with $\sigma_1 < ... < \sigma_n$. Denote by $X^{\sim}$, respectively $X_{\sim}$ the submodule of $M$ generated by $x_{\sigma_n}$, respectively $x_{\sigma_1}$, that is, $X^{\sim}$ is the submodule of $M$ generated by the homogeneous components of the highest degree of all the elements in $X$, a similar description holds for $X_{\sim}$.

### 5.3.1 Lemma

1. If $x \in X^{\sim}$ (resp $x \in X_{\sim}$) is a nonzero homogeneous element then there is a $y \in X$ with $y = y_{\tau_1} + ... + y_{\tau_n}$ such that $\tau_1 < ... < \tau_n$ and $x = y_{\tau_n}$ (resp $x = y_{\tau_1}$).
2. $X^{\sim}$ and $X_{\sim}$ are graded submodules of $M$.
3. $X = X^{\sim}$ (resp $X = X_{\sim}$) if and only if $X$ is a graded submodule.
4. $X = 0$ if and only if $X^{\sim} = 0$, and also if and only if $X_{\sim} = 0$.
5. If $X \subset Y \subset M$ then $X^{\sim} \subset Y^{\sim}$ and $X_{\sim} \subset Y_{\sim}$.
6. If $L$ is a left ideal of $R$ and $N$ is a $R$-submodule of $M$ then $L^{\sim} N^{\sim} \subseteq (LN)^{\sim}$ and $L_{\sim} N_{\sim} \subseteq (LN)_{\sim}$. Therefore if L is an ideal then $L^{\sim}$ and $L_{\sim}$ are ideals.

**Proof** i. Assume that $x \in X^{\sim}$ is a nonzero homogeneous element of $X^{\sim}$. Then we can find the elements $z_1, ..., z_s \in X$ such that $x = \lambda_1\alpha_1 + ... + \lambda_s\alpha_s$ where $\lambda_1, ..., \lambda_s$ are homogeneous elements from $R$ and $\alpha_i$ $(1 \leq i \leq s)$ are the components of the highest degree in $z_i$. We can assume that $\lambda_i\alpha_i \neq 0$ for all the $i$'s. In this case $\lambda_1\alpha_1 + ... + \lambda_s\alpha_s$ is the component of the highest degree of the element $\lambda_1 z_1 + ... + \lambda_s z_s \in X$. Now put $y = \lambda_1 z_1 + ... + \lambda_s z_s$ and this satisfies the required conditions.
The assertions (ii), (iii), (iv) and (v) are obvious and (vi) follows from (i). □

### 5.3.2 Proposition

Let $R$ be a graded ring of type $G$, ($G$ is an $O$-group as stated in the beginning of this section). Let $\sigma_0 \in G$ and assume that $sup(M) \cap \{x \in G \mid \sigma_0 \leq x\}$ is a well ordered subset of G. Consider the R submodules $X \subset Y \subset M$. The following assertions are equivalent :

1. $X = Y$.
2. $X^{\sim} = Y^{\sim}$ and $X \cap M_{<\sigma_0} = Y \cap M_{<\sigma_0}$

**Proof** i)⇒ii) is obvious.

ii)⇒i) Let $y \in Y$. Then $y = y_{\sigma_1} + ... + y_{\sigma_n}$ with $\sigma_1 < \sigma_2 < ... < \sigma_n$ and $y_{\sigma_n} \neq 0$. If $\sigma_n < \sigma_0$ then $y \in Y \cap M_{<\sigma_0} = X \cap M_{<\sigma_0}$ and so $y \in X$. Assume that $\sigma_n \geq \sigma_0$. Since $y_{\sigma_n} \in Y^{\sim} = X^{\sim}$, we have that there is $x \in X$ such that $x = x_{\tau_1} + ... + x_{\tau_{m-1}} + y_{\sigma_n}$ with $\tau_1 < ... < \tau_{m-1} < \sigma_n$. It follows that $y - x \in Y$ has a homogeneous decomposition in which the largest degree appearing is less than $\sigma_n$. Since $sup(M) \cap \{x \in G \mid \sigma_0 \leq x\}$ is a well ordered subset of $G$, after a finite number of steps we find $x^1, ..., x^k \in X$ such that $y - (x^1 + ... + x^k) \in Y \cap M_{<\sigma_0} = X \cap M_{<\sigma_0}$ and therefore $y \in X$. □

### 5.3.3 Corollary

If $M$ is a graded $R$-module such that sup$(M)$ is a well ordered subset of $G$, then for any two submodules $X \subset Y \subset M$ we have $X = Y$ if and only if $X^{\sim} = Y^{\sim}$.

For a left $R$-module ${}_RM$, we put : $Z({}_RM) = \{x \in M \mid \text{Ann}_R(x)$ is a left essential ideal of $R\}$. It is easy to see that $Z({}_RM)$ is a submodule of M; $Z_R(M)$ is called the **singular submodule** of $M$. If $Z({}_RM) = 0$ then ${}_RM$ is called nonsingular module.

### 5.3.4 Corollary

Let $R$ be a $G$-graded ring. Then :

1. If $I$ is a left essential of $R$ then $I^{\sim}$ and $I_{\sim}$ are left essential ideals of R.
2. $Z({}_RM)$ is a graded submodule of M.

**Proof**

1. Let $a \in R$ be a nonzero homogeneous element. Then there is a $b \in R$ such that $0 \neq ba \in I$ because $I$ is essential. If we write $b = b_{\sigma_1} + ... + b_{\sigma_n}$, $b_{\sigma_i} \in R_{\sigma_i}$, $b_{\sigma_i} \neq 0$ $(1 \leq i \leq n)$ such that $\sigma_1 < ... < \sigma_n$ then $ba = b_{\sigma_1}a + ... + b_{\sigma_n}a$. Since $ba \neq 0$, there is a maximal $k$, $1 \leq k \leq n$ such that $b_{\sigma_k}a \neq 0$, so $b_{\sigma_{k+1}}a = ... = b_{\sigma_n}a = 0$ and thus $b_{\sigma_k}a$ is the

homogeneous component of the highest degree appearing in $ba$. Since $ba \in I$ then $b_{\sigma_k} a \in I^{\sim}$ and $b_{\sigma_k} a \neq 0$, so by Proposition 2.3.6 we get that $I^{\sim}$ is essential in $R$. The fact that $I_{\sim}$ is essential may be proven in a similar way.

2. Let $x \in Z(_R M)$, $x \neq 0$. If we write $x = x_{\sigma_1} + ... + x_{\sigma_n}$ such that $x_{\sigma_i} \in M_{\sigma_i}$, $x_{\sigma_i} \neq 0$ and $\sigma_1 < ... < \sigma_n$. Since $I = \mathrm{Ann}_R(x)$ is an essential left ideal of $R$ it s easy to see that $I^{\sim} x_{\sigma_n} = 0$. By (i) $I^{\sim}$ is an essential left ideal, so $x_{\sigma_n} \in Z(_R M)$. Repeating the same argument for the element $x - x_{\sigma_n}$ instead of $x$ we obtain by induction that $x_{\sigma_1}, ..., x_{\sigma_n} \in Z(_R M)$. □

## 5.4 Chain Conditions for Graded Modules

Let $R$ be a graded ring of type $G$; $M \in R-gr$ is said to be left gr-Noetherian, respectively gr-Artinian (see Section 2.10), if $M$ satisfies the ascending, respectively descending chain condition for graded submodules of $M$. It is straightforward to see that $M$ is left gr-Noetherian if and only if every graded submodule is finitely generated and also if and only if each non-empty family of graded submodules of M has a maximal element. Dually, M is left gr-Artinian if and only if each non-empty family of graded submodules of M has a minimal element and also if and only if each intersection of graded submodules may be reduced to a finite intersection of the same family.

Let $G$ be an arbitrary group, H a subgroup of $G$ and $\{\sigma_i \mid i \in I\}$ a set of representatives for the right $H$-cosets of $G$. If $R$ is a $G$-graded ring and $M = \oplus_{x \in G} M_x$ is an object from $R$-gr, then for each $i \in I$ we put $M_{H\sigma_i} = \oplus_{h \in H} M_{h\sigma_i}$. It is clear that $M_{H\sigma_i}$ is a $H$-graded $R_H$-module with the gradation $(M_{H\sigma_i})_h = M_{h\sigma_i}$ for any $h \in H$.

### 5.4.1 Lemma

Let $P$ be a graded $R^{(H)}$ submodule of $M_{H\sigma_i}$. Then $RP \cap M_{H\sigma_i} = P$.

**Proof** If $y \in RP \cap M_{H\sigma_i}$ is a homogeneous element, then $y = \sum_{k=1}^{n} \lambda_k x_k$ with $\lambda_k \in h(R)$, $x_k \in h(P)$, $k = 1, ..., n$. Put $\tau_k = \deg x_k$ $(1 \leq x \leq n)$. Then $\tau_k = h_k \sigma_i$ for some $h_k \in H$. Let $\sigma = \deg y$. Since $y \in M_{H\sigma_i}$ then $\sigma = h\sigma_i$ for some $h \in H$. Then for any $1 \leq k \leq n$ we have $h\sigma_i = \deg \lambda_k \cdot (h_k \sigma_i)$ so $\deg \lambda_k \in H$. Therefore $\lambda_1, ..., \lambda_k \in R_H$ so $y \in P$ i.e. $RP \cap M_{H\sigma_i} \subseteq P$. Since the converse inclusion is clear, we have the equality $RP \cap M_{H\sigma_i} = P$.

### 5.4.2 Proposition

If $M$ is a left gr-Noetherian (resp gr-Artinian) module then for every $i \in I$, $M_{H\sigma_i}$ is a left gr-Noetherian (resp gr-Artinian) $R_H$. Conversely if $[G : H] < \infty$ and $M_{H\sigma_i}$ is left gr-Noetherian (resp gr-Artinian) for any $i \in I$, then $M$ is gr-Noetherian (resp Artinian).

**Proof** The first part follows easily from Lemma 5.4.1. If $[G : H] < \infty$ then $I$ is finite. Since $M = \oplus_{i\in I} M_{H\sigma_i}$ as left $R_H$-modules, then if $N$ is a gr-submodule of $M$ it follows that $N = \oplus_{i\in I} N_{H\sigma_i}$ so $M$ is gr-Noetherian (resp Artinian). □

### 5.4.3 Corollary

Let $G$ be a finite group and $R$ be a $G$-graded ring. If $M \in R - gr$ then the following assertions are equivalent :

1. $M$ is gr-Noetherian (resp gr-Artinian).
2. $M$ is Noetherian (resp Artinian) as an $R_e$ module.
3. $M$ is Noetherian (resp Artinian) in $R$-mod.

**Proof** It follows from Proposition 5.4.2 if we take $H = \{e\}$.

### 5.4.4 Corollary

If $R$ is gr-Noetherian (resp gr-Artinian) and $H \leq G$ is a subgroup of $G$ then $R_H$ is $H$-gr-Noetherian (resp. $H$-gr-Artinian). Here $G$ is an arbitrary group).

### 5.4.5 Proposition

Let $G$ be an $O$-group, $R$ a $G$-graded ring and $M \in R - gr$. The following assertions hold :

1. If $\sup(M)$ is a well ordered subset of $G$ then $M$ is gr-Noetherian (resp gr-Artinian) if and only if $M$ is Noetherian (resp Artinian) in $R$-mod.
2. Assume that $R$ is positively graded ring. If $M$ is gr-Noetherian then $\sup(M)$ is a well ordered subset of $R$ and in particular $M$ is strongly left limited and Noetherian in $R$-mod.

**Proof**

1. Follows from Corollary 5.3.3

2. Let $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq ...$ be a descending chain of elements from $\sup(M)$. Then $M_{\sigma_i} \neq 0$ for all $i$ and pick $x_n \in M_{\sigma_n}$, $x_n \neq 0$. We have the ascending chain of graduated submodules of $M$

$$Rx_1 \leq Rx_1 + Rx_2 \leq ... \leq Rx_1 + Rx_2 + ... + Rx_n \leq ...$$

Since $M$ is gr-Noetherian, there is an $n$ such that

$$Rx_1 + Rx_2 + ... + Rx_n = Rx_1 + Rx_2 + ... + Rx_n + Rx_{n+1}$$

So $x_{n+1} = a_1x_1 + ... + a_nx_n$ where $a_1, ..., a_n$ are homogeneous. Since $\deg a_i \geq e$, $(1 \leq i \leq n)$ then $\sigma_{n+1} = \deg x_{n+1} \geq \sigma_n$ so $\sigma_n = \sigma_{n+1}$. Hence $\sup(M)$ is a well-ordered subset of $G$. The last part of assertion (2) follows from (1). □

### 5.4.6 Proposition

Let $R$ be a graded ring of type $\mathbb{Z}$. If $M \in R-gr$ is left gr-Noetherian then :

1. $M_{\geq 0}$ is left Noetherian in $R_{\geq 0}$-mod.
2. $M_{\leq 0}$ is left Noetherian in $R_{\leq 0}$-mod.

**Proof**

1. By Proposition 5.4.5 it is enough to show that $M_{\geq 0}$ is gr-Noetherian in $R_{\geq 0}$-mod. Let $N$ be a graded $R_{\geq 0}$ submodule of $M_{\geq 0}$. Since $RN$ is finitely generated as a graded $R$-submodule we may assume that $RN = Rx_1 + ... + Rx_s$ with $x_1, ..., x_s \in N$ being homogeneous elements. Assume that $\deg x_i = n_i \geq 0\,(1 \leq i \leq s)$. Put $n = \max(n_1, ..., n_s)$. If $y_m \in N_m$, $m \geq n$ then there are some $\lambda_1, ..., \lambda_s \in h(R)$ such that $y_m = \sum_{i=1}^{s} \lambda_i x_i$. Since $m \geq n$, we get $\lambda_i \in R_{\geq 0}$ for any $i$, $1 \leq i \leq s$. On the other hand by Proposition 5.4.2, $M_i$ is a left $R_0$ Noetherian module so $\oplus_{0 \leq i < m} N_i$ is a finitely generated $R_0$ submodule of $\oplus_{0 \leq i < m} M_i$ and let $\{y_1, ..., y_r\}$ be a family of generators. Since $N = \oplus_{0 \leq i < m} N_i \oplus \oplus_{j \geq m} N_j$ it is clear that $\{x_1, ..., x_s, y_1, ..., y_r\}$ is a family of generators for $N$ over $R_{\geq 0}$. Therefore $M_{\geq 0}$ is a gr-Noetherian $R_{\geq 0}$-module.
2. is similar to (1). □

### 5.4.7 Theorem

Let $R$ be a graded ring of type $\mathbb{Z}$. If $M \in R-gr$ then the following assertions are equivalent :

1. $M$ is gr-Noetherian
2. $M$ is Noetherian in $R$-mod.

**Proof** As (2.)⇒(1.) is obvious, we show (1.)⇒(2.). Consider an ascending chain of $R$-submodules of $M$, $X_1 \subset X_2 \subset ... \subset X_p \subset ....$ By Proposition 5.4.6 there is an $n_0 \in \mathbf{N}$ such that $M_{\leq 0} \cap X_i = M_{\leq 0} \cap X_{i+1} = ...$ and $X_i^{\sim} = X_{i+1}^{\sim} = ...$ for $i \geq n_0$. By Proposition 5.3.2 it follows that $X_i = X_{i+1} = ... (i \geq n_0)$. □

A group $G$ is called **polycyclic-by-finite** if there is a finite series $\{e\} = G_0 \lhd G_1 \lhd ... \lhd G_n = G$ of subgroups such that each $G_{i-1}$ is normal in $G_i$ and $G_i/G_{i-1}$ is either finite or cyclic for each $i$. The main result of this section is the following

### 5.4.8 Theorem

Let $R$ be a strongly graded ring of type $G$, where $G$ is a polycyclic-by-finite group. Let $M = \oplus_{i \in G} M_\sigma$ be a left graded R-module. If $M_e$ is a left Noetherian $R_e$-module, then M is Noetherian as an R-module. In particular if $R_e$ is a left Noetherian ring, then R is a left Noetherian ring.

**Proof** Since $R$ is strongly graded, by Dade's Theorem (Section 3.1) we have that M is gr-Noetherian. Consider the subnormal series $\{e\} = G_0 \lhd G_1 \lhd ... \lhd G_n = G$. If $n = 1$ then $G = G_1$ is finite or cyclic and by Corollary 5.4.3 and Theorem 5.4.7 we obtain that $M$ is left Noetherian. We prove the statement by induction on $n$ assuming that it holds for subnormal series of length less than $n$. Put $H = G_{n-1}$, $G' = G/H$. By the induction hypothesis we have that $M_H$ is a Noetherian $R_H$-module. Since $R$ with the grading over $G/H$ is also strongly graded, since $G/H$ is finite or cyclic, it follows that $M$ is a Noetherian $R$-module. □

## 5.5 Krull Dimension of Graded Rings

For definitions and properties of Krull dimension of an object from an abelian category we refer to Appendix B. Let $R$ be a $G$-graded ring and $R - gr$ the category of graded left $R$-modules. If $M \in R - gr$ and $M$ has Krull dimension in the category $R - gr$ we denote by $K.\dim_R^{gr}(M)$ (or shortly $K.\dim^{gr}(M)$) its Krull dimension in this category and call it the **gr-Krull dimension**. We also denote by $K.\dim_R M$ the Krull dimension of M in $R$-mod. (in case it exists). If $M$ is $\alpha$-critical in $R - gr$ then we say that $M$ is **gr-$\alpha$-critical**.

### 5.5.1 Proposition

Let $R$ be a $G$-graded ring where $G$ is an $O$-group and $M \in R - gr$. Assume that $\sup(M)$ is a well ordered subset of G. Then :

1. $M$ has gr-Krull dimension if and only if $M$ has Krull dimension in $R$-mod and in this case we have

$$K.\dim^{gr}(M) = K.\dim(M)$$

2. $M$ is gr-$\alpha$-critical if and only if $M$ is $\alpha$-critical (in $R$-mod).

**Proof**

1. Apply Corollary 5.3.3 and Lemma I.1 (Appendix B).

2. It is enough to show only the "if"-implication. If $X \neq 0$ is any submodule of $M$, then $X^{\sim} \neq 0$ and hence $K.\dim(M/X^{\sim}) < \alpha$. By Corollary 5.3.3 and Lemma I.1 (Appendix B) we get that $K.\dim(M/X) \leq K.\dim_R^{gr}(M/X^{\sim}) < \alpha$. Consequently $M$ is $\alpha$-critical (in $R$-mod). □

## 5.5.2 Proposition

Let $R$ be a $G$ strongly graded ring and $M = \oplus_{\sigma \in G} M_\sigma \in R - gr$. Then $M$ has Krull dimension if and only if $M_e$ has Krull dimension in $R_e$-mod and if this is the case then $K.\dim^{gr} M = K.\dim_{R_e} M_e$.

**Proof** Apply Theorem 3.1.1. □

## 5.5.3 Proposition

Let $R$ be a $G$ graded ring and $H \leq G$ a subgroup of $G$. Let $\{\sigma_i \mid i \in I\}$ a set of representatives for the right $H$-cosets of $G$. If $M \in R$-gr has Krull dimension in $R_H$-gr and $K.\dim^{\mathrm{gr}}_{R_H} M_{\sigma_i} \leq K.\dim^{\mathrm{gr}}_R M$. If $H$ has finite index in $G$ then $K.\dim^{gr}_R M = \sup_{i \in I}\{K.\dim^{gr}_{R_H} M_{H\sigma_i}\}$.

**Proof** The first statement follows from Lemma 5.4.1. and Lemma I.1 (Appendix B). On the other hand, the lattice of graded submodules of $M$ maps into the product of lattices of graded submodules of $M_{H\sigma_i}$, $i \in I$ as follows :

$$N \mapsto (N_{H\sigma_i})_{i \in I}.$$

This map is strictly increasing and we may apply Lemma I.1 (Appendix B). □

## 5.5.4 Corollary

If $R$ is a ring graded by a finite group $G$ and $M \in R$-gr has gr-Krull dimension then $M$ has Krull dimension (in $R$-mod) and

$$K.\dim_R M = K.\dim^{gr}_R M = sup\{K.\dim_{R_e} M_\sigma \mid \sigma \in G\}$$

**Proof** It follows from Proposition 5.3.3 for $H = \{e\}$. □

### 5.5.5 Corollary

If the $G$-graded ring $R$ has left Krull dimension in $R$-gr, then for any subgroup $H$ of finite index in $G$, the ring $R_H$ has Krull dimension in $R_H$-gr and $K.\dim^{\mathrm{gr}}_R R = K.\dim^{\mathrm{gr}}_{R_H} R_H$.

**Proof** By Proposition 5.5.3 we get that $R_H$ has gr-Krull dimension (in $R$-gr). Since $K.\dim^{\mathrm{gr}}_{R_H} R_{H\sigma_i} \leq K.\dim^{\mathrm{gr}}_{R_H} R_H$ then by the same Proposition 5.5.3 we have $K.\dim^{gr}_R R = K.\dim^{gr}_{R_H} R_H$. □

### 5.5.6 Proposition

Let $R = \oplus_{i\in\mathbb{Z}} R_i$ be a graded ring of type $\mathbb{Z}$ and $M = \oplus_{i\in\mathbb{Z}} M_i$ a Noetherian graded left $R$-module. Assume that $K.\dim^{gr}_R M = \alpha$. Then :

1. $K.\dim_{R_{\geq 0}} M_{\geq 0} \leq \alpha + 1$ and $K.\dim_{R_{\leq 0}} M_{\leq 0} \leq \alpha + 1$.
2. $\alpha \leq K.\dim_R M \leq \alpha + 1$.

**Proof** By Theorem 5.4.7 $M$ is gr-Noetherian if and only if $M$ is Noetherian. It is clear that assertion (2) follows from Proposition 5.3.2 and Lemma I.1 (Appendix B).

We now prove assertion (1). We intend to show by transfinite induction on $\alpha$ that $K.\dim_{R\geq 0} M_{\geq 0} \leq \alpha + 1$. In of view Lemma I.7 (Appendix B) we may assume that $M$ is Noetherian and gr-$\alpha$-critic. If $\alpha = -1$, then $M = 0$ and the assertion is obvious. Assume $\alpha \geq 0$ and pick $x \neq 0$ a homogeneous element of $M_{\geq 0}$. We have $K.\dim^{\mathrm{gr}}_R (M/Rx) < \alpha$ since $M$ is gr-$\alpha$-critical. Assume that $\deg x = n \geq 0$. In this case we have $(M/Rx)_{\geq 0} = M_{\geq 0}/ \oplus_{i\geq -n} R_i x$. Since $R_{\geq 0}x \subset (Rx)_{\geq 0}$ we have the exact sequence

$$0 \to \frac{(Rx)_{\geq 0}}{R_{\geq 0}x} \to \frac{M_{\geq 0}}{R_{\geq 0}x} \to \left(\frac{M}{Rx}\right)_{\geq 0} \to 0$$

But $(Rx)_{\geq 0}/R_{\geq 0}x \simeq R_{-n}x \oplus ... \oplus R_{-1}x \subseteq \oplus^{-1}_{i=-n} M_i$. By Proposition 5.5.1 $K.\dim((Rx)_{\geq 0}/R_{\geq 0}x) \leq \alpha$ as $R_e$-module so $K.\dim_R((Rx)_{\geq 0}/R_{\geq 0}x) \leq \alpha$ as $R$-module. Since $K.\dim^{gr}_R(M/Rx) < \alpha$ by the induction hypothesis, we get that $K.\dim(M/Rx)_{\geq 0} \leq \alpha$. By the above exact sequence and using Lemma I.3 (Appendix B) we obtain that $K.\dim(M_{\geq 0}/R_{\geq 0}x) \leq \alpha$. By Lemma I.6 (Appendix B) we find that $K.\dim M_{\geq 0} \leq \alpha + 1$. In a similar way, $K.\dim_{R_{\leq 0}} M_{\leq 0} \leq \alpha + 1$.

□

We recall that a group G is **polycyclic-by-finite** if G has a subnormal series

$$\{e\} = G_0 \lhd G_1 \lhd ... \lhd G_n = G \quad (*)$$

such that $G_{i-1}$ is normal in $G$ and $G_i/G_{i-1}$ is either finite or cyclic infinite for each $i$. It is wellknown that the number of all the factors $G_i/G_{i-1}$ that are infinite is an invariant (it does not depend on the chosen subnormal series of $G$); this number is denoted by $h(G)$ and is called the **Hirsch number of** $G$. If $h(G) = n$ then $G$ is called **poly-infinite cyclic group**. If $H$ is a subgroup of $G$, then the series $\{e\} = H \cap G_o \lhd H \cap G_1 \lhd ... \lhd H \cap G_n = H$ is also a subnormal series of $G$. We clearly have that if $G$ is polycyclic-by-finite (resp poly-infinite cyclic) then H is polycyclic-by-finite (resp poly-infinite cyclic) (see [167] for details). The following is the main result of this section.

### 5.5.7 Theorem

Let $R$ be a $G$ strongly graded ring with $G$ a polycyclic-by-finite group. If $M = \oplus_{\sigma \in G} M_\sigma \in R-gr$ is gr-Noetherian (which is equivalent to the fact that $M_e$ is a left Noetherian $R_e$-module) and $K.\dim_{R_e} M_e = \alpha$ then

$$\alpha \leq K.\dim_R M \leq \alpha + h(G)$$

**Proof** Let $\{e\} = G_0 \lhd G_1 \lhd ... \lhd G_n = G$ be a subnormal series of the group $G$. We proceed by induction on $n$. If $n = 1$ then we have by Proposition 5.5.6, Proposition 5.5.2 and Corollary 5.5.4 that $K.\dim_R M \leq \alpha + 1$ if $G = G_1$ is cyclic infinite or $K.\dim_R M = \alpha$ if $G$ is a finite group. Assume that the assertion is true for $n-1$. If we put $H = G_{n-1}$ then we have $K.\dim_{R_H} M_H \leq \alpha + h(H)$ where $h(H)$ is the Hirsch number of the group $H$. Now consider the ring $R$ with gradation of type $G/H$, so $R = \oplus_{C \in G/H} R_C$, where $R_C = \oplus_{x \in C} R_x$ for any $x \in C$ and also $M = \oplus_{C \in G/H} M_C$ which is an $R$-graded module of type $G/H$. Since $H$ is the identity element of the group $G/H$, using the induction hypothesis for $n = 1$ we get that $K.\dim_R M \leq \alpha + h(H) + 1$ if $G/H$ is cyclic infinite or $K.\dim_R M \leq \alpha + h(H)$ if $G/H$ is finite. Since $h(G) = h(H) + 1$ if $G/H$ is cyclic infinite and $h(G) = h(H)$ if $G/H$ is finite, we finally obtain $K.\dim_R M \leq \alpha + h(G)$. The inequality $\alpha \leq K.\dim_R M$ is obvious by Lemma I.1 (Appendix B). □

### 5.5.8 Proposition

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a crossed product such that $R_e$ is a domain. If G is a poly-infinite cyclic group, then R is a domain.

**Proof** Let $\{e\} = G_0 \lhd G_1 \lhd ... \lhd G_n = G$ be a subnormal series of $G$ such that $G_{i+1}/G_i = \mathbb{Z}$ for every $0 \leq i \leq n$. If $n = 1$ then $G \simeq \mathbb{Z}$ and hence $R = \oplus_{i \in \mathbb{Z}} R_i$ is $\mathbb{Z}$-graded. Proposition 5.2.6 now implies that $R$ is a domain. The assertion now follows by induction on $n$. □

### 5.5.9 Theorem

Let $R$ be a $G$ graded ring and $\Sigma$ a gr-simple left $R$-module. If $G$ is a poly-infinite cyclic group, then $\Sigma$ has Krull dimension in $R$-mod. Moreover, $\Sigma$ is $k$-critical with $0 \leq k \leq h(G\{\Sigma\}) \leq h(G)$.

**Proof** Since $G\{\Sigma\}$ is a subgroup of $G$, it is poly-infinite cyclic too. Since the ring $\Delta = \mathrm{End}_R(\Sigma)$ is a crossed product of type $G\{\Sigma\}$ and $\Delta_e$ is a division ring, it follows by Theorem 5.5.7 that $\Delta$ has Krull dimension (on the left) and that $K.\dim_\Delta \Delta \leq h(G\{\Sigma\})$, by Lemma I.7 (Appendix B). We have now that $\Delta$ contains a $k$-critical left ideal $I$, with $k \leq h(G\{\Sigma\})$. Moreover by Proposition 5.5.8, $\Delta$ is a domain. Then if $0 \neq a \in I$, the map $\varphi : \Delta \to I$ given by $\varphi(\lambda) = \lambda a$ is a monomorphism and so $\Delta$ is also $k$-critical. It follows now from Graded Clifford Theory (Section 4.1) that $\Sigma$ is $k$-critical. □

## 5.6 Exercises

1. Let $\varphi$ be an automorphism of a ring $A$ and consider the skew polynomial rings $AX, \varphi]$ and $A[X, X^{-1}, \varphi]$. For an $A$-module $M$, we write $M[X, \varphi]$ for $A[X, \varphi] \otimes_A M$ and similarly for

$$M[X, X^{-1}, \varphi] = A[X, X^{-1}, \varphi] \otimes_A M$$

Establish the following claims :

i) If $M$ is left Noetherian then $M[X, \varphi]$ resp. $M[X, X^{-1}, \varphi]$, is a left Noetherian $A[X, \varphi]$- resp. $A[X, X^{-1}, \varphi]$-module.

ii) $M$ is left Noetherian if and only if $M[X]$ has Krull dimension over $A[X]$.

iii) Suppose that $M$ is left Noetherian. Then we have :

$$\mathrm{Kdim}_{A[X,\varphi]} M[X, \varphi] = \mathrm{Kdim}_A M + 1$$
$$\mathrm{Kdim}_{A[X,X^{-1},\varphi]} M[X, X^{-1}, \varphi] \leq \mathrm{Kdim}_A M + 1$$

Moreover if $M$ is $\alpha$-critical, then $M[X, \varphi]$ is an $(\alpha+1)$-critical $A[X, \varphi]$-module.

**Hint**

i) $M[X, \varphi]$ has the natural $\mathbb{Z}$-gradation by putting $M[X, \varphi]_n = \{X^n \otimes m, m \in M\}$ for $n \geq 0$, $M[X, \varphi]_n = 0$ if $n < 0$. It is clear how that the natural $\mathbb{Z}$-gradation of $M[X, X^{-1}, \varphi]$ has been defined. But $A[X, X^{-1}, \varphi]$ is strongly graded with $A[X, X^{-1}, \varphi]_0 = A$, so i. follows from Theorem 5.4.7.

ii) If $M$ is left Noetherian then $M[X]$ is left Noetherian as an $A[X]$-module and so it has Krull dimension. Conversely when $M[X]$ has Krull dimension look at a possibly infinite ascending chain : $M_0 \subset M_1 \subset \ldots \subset M_n \subset \ldots \subset M$ of $A$-submodules of $M$. It is clear that $K = M_1 \oplus XM_2 \oplus \ldots$, $L = M_0 + XM_1 + \ldots + X^n M_n + \ldots$, are $A[X]$-submodules of $M[X]$ such that $XK \subset L$ and $K/L = \frac{M_1}{M_0} \oplus \frac{M_2}{M_1} \oplus \ldots \oplus \frac{M_{n+1}}{M_n} \oplus \ldots$. It follows that $K/L$ has infinite Goldie dimension. On the other habd the fact that $M[X]$ has Krull dimension implies that $K/L$ has finite Goldie dimension (Appendix B); this contradicts the possibility that the original chain is infinite.

iii) Theorem 3.3.7 yields : $\mathrm{Kdim}_{A[X,X^{-1},\varphi]} M[X, X^{-1}, \varphi] \leq \mathrm{Kdim}_A M + 1$, and also : $\mathrm{Kdim}_{A[X,\varphi]} M[X.\varphi] \leq \mathrm{Kdim}_A M + 1$. Look at the infinite strictly decreasing sequence, putting

$$N = M[X, \varphi] : N \supset XN \supset X^2 N \supset \ldots \supset X^n N \supset \ldots$$

From the lattice isomorphisms

$$\mathcal{L}_{A[X,\varphi]} \left( \frac{X^n N}{X^{n+1} N} \right) \cong \mathcal{L}_{A[X,\varphi]} \left( \frac{N}{XN} \right) \cong \mathcal{L}_A(M)$$

it follows that $\mathrm{Kdim}_{[A[X,\varphi]} M[X, \varphi] \geq \mathrm{Kdim}_A M + 1$, so we have $\mathrm{Kdim}_{A[X,\varphi]} M[X, \varphi] = 1 + \mathrm{Kdim}_A M$.
Next, assume that $M$ is $\alpha$-critical. Since $N = M[X, \varphi]$ is $\mathbb{Z}$-graded we only have to establish that $N$ is $(\alpha + 1)$-critical in $R$gr, $R = [A[X, \varphi]$. Consider a homogeneous $x \in h(N)$, say $z = X^k m \neq 0$. We have the following sequence in $R$-gr : $N \supset XN \supset \ldots \supset X^k N \supset Rz$, where we have :

$$\mathrm{Kdim}_R \frac{X^v N}{X^{v+1} N}. = \alpha, \mathrm{Kdim}_R \frac{X^k N}{Rz} \leq \mathrm{Kdim}_R \left( \frac{M}{Am[X, \varphi]} \right) \leq \alpha,$$

the latter because $\mathrm{Kdim}_A M/Am < \alpha$. Thus $\mathrm{Kdim}_R \frac{M[X,\varphi]}{Rz} \leq \alpha$ and hence $N = M[X, \varphi]$ is $(\alpha + 1)$-critical.

2. Let $D = \oplus_{i \in \mathbb{Z}} D_i$ be a $\mathbb{Z}$-graded division ring. Prove that either $D = D_0$ or else there exists an automorphism $\varphi : D_0 \to D_0$ and an $X \in D$ with $\deg X > 0$ such that $D \cong D_0[X, X^{-1}, \varphi]$ where $D_0$ is a division ring and $X$ is a variable.

**Hint** Assume $D \neq D_0$, then there is an $i > 0$ such that $D_i \neq 0$. We may pick $n \in \mathbb{N}$, such that $D_n \neq 0$ and $n$ is minimal as such, and pick $a \neq 0$ in $D_n$. For any $\alpha \in \mathbb{Z}$, $D_{nd} = D_0 a^d = a_d D_0$ and $D_j = 0$ for $j \in \mathbb{Z}$ such that $n$ does not divide $j$. From $D_n = D_0 a = a D_0$ it follows that for a given $\lambda \in D_0$ there exists a unique $\varphi(\lambda) \in D_0$ such that $a\lambda = \varphi(\lambda) a$. The correspondence $\lambda \mapsto \varphi(\lambda)$ is an automorphism of the

division ring $D_0$. It is then easy enough to verify $D \cong D_0[X, X^{-1}, \varphi], X$ an indeterminate with $\deg X = n$.

3. Let $\Sigma$ be gr-simple over a $\mathbb{Z}$-graded ring $R$. Prove that any submodule of $\Sigma$ is generated by one element.

   **Hint** Suppose $M \neq 0$ is an $R$-submodule of $\Sigma$ and put $\Delta = \mathrm{End}_R(\Sigma)$. Clearly $\Delta$ is a gr-division ring and $\Delta$ is a left and right principal ideal ring because of exercise 2. Consider the functors $T, S$, obtained in Clifford theory (see Section 4.1), $(R/\Sigma)$-mod $\underset{S}{\overset{T}{\rightleftarrows}}$ $\Delta$-mod Then $T(M)$ is a left ideal of $\Delta$, thus since the latter ring is a principal ideal domain, there exists a nonzero morphism : $\Delta \xrightarrow{u} T(M) \longrightarrow 0$ in $\Delta$-mod. We arrive at the exact sequence : $\Sigma = S(\Delta) \xrightarrow{S(u)} ST(M) \longrightarrow 0$, where $ST(M) \cong M$. Put $v = S(u)$, then we arrive at a nonzero epimorphism $\Sigma \xrightarrow{v} M \longrightarrow 0$. Since $M \neq 0$, $\mathrm{Ker}v \neq \Sigma$ follows and hence there is an $n \in \mathbb{Z}$ such that $\Sigma_n \cap \mathrm{Ker}v = 0$. Pick a nonzero $x \in \Sigma_n$, then $v(x) \neq 0$. From $\Sigma = Rx$ it follows that $M = v(\Sigma) = v(Rx) = Rv(x)$.

4. With notation as in the foregoing exercise, put $R_{\geq 0} = \oplus_{i \geq 0} R_i$ and $\Sigma_{\geq 0} = \oplus_{i \geq 0} \Sigma_i$. Prove that every $R_{\geq 0}$-submodule of $\Sigma_{\geq 0}$ is generated by a simgle element.

   **Hint** Let $X \neq 0$ be a submodule of $\Sigma_{\geq 0}$. In case $X$ is a graded submodule then it is easy to see that there exists $p \geq 0$ in $\mathbb{N}$ such that $X = \Sigma_{\geq p} = \oplus_{i \geq p} \Sigma_i$. If we pick $x \in \Sigma_p$, $x \neq 0$, since $Rx = \Sigma$ it follows that for all $i \geq p$ : $\Sigma_i = R_{i-p}x$, hence $X = R_{\geq 0}x$. In case $X$ is not graded we may look at $X^\sim$ in $\Sigma_{\geq 0}$. Since $X^\sim$ is generated by one element, it follows that $X$ is generated by one element.

5. Let $M$ be a nonzero left $A$-module over an arbitrary ring $A$. A prime ideal of $A$ is said to be **associated** to $M$ if there is a nonzero submodule $N$ of $M$ such that $P = \mathrm{Ann}_A(N) = \mathrm{Ann}_A(N')$ for every nonzero submodule $N'$ of $N$. Let $\mathrm{Ass}(M)$ denote the set of prime ideals of $A$ associated to $M$. Consider an ordered group $G$ and let $R$ be a $G$-graded ring and $M \neq 0$ a graded $R$-module. Establish the following claims :

   i) If $P \in \mathrm{Ass}(M)$ then $P$ is a graded ideal.

   ii) If $R$ is left gr-Noetherian, then $\mathrm{Ass}(M) \neq \emptyset$.

   **Hint**

   i) Easy.

   ii) Let $a = a_{\sigma_1} + \ldots + a_{\sigma_m}$, $\sigma_1 < \ldots < \sigma_m$ be the homogeneous decomposition of $a \in P$. Pick $b \in h(R)$. Assume that $P = \mathrm{Ann}_R(N)$, $N$ a submodule of $M$. Since $ab \in P$ we have $abx = 0$ for $x \in N$. Write $x = x_{\tau_1} + \ldots + x_{\tau_n}$, $\tau_1 < \ldots < \tau_n$. Clearly

we have $a_{\sigma_m} b x_{\tau_n} = 0$. Let $c \in h(R)$ be such that $\deg c = \sigma$. Since $ac \in P$ we also have $acx = 0$. If $\sigma_m \sigma \tau_{n-1} = \sigma_{m-1} \sigma \tau_n$ we obtain $a_{\sigma_m} c x_{\tau_{n-1}} + a_{\sigma_{n-1}} c x_{\tau_n} = 0$; in case $\sigma_m \sigma \tau_{n-1} \neq \sigma_{m-1} \sigma \tau_n$ we have $a_{\sigma_m} c x_{\tau_{n-1}} = 0$. Now we may change $c$ to $c a_{\sigma_m} b$ and obtain in all cases that $(a_{\sigma_m} R)^2 x_{\tau_{n-1}} = 0$. It follows that for some $p \in \mathbb{N}$, $(a_{\sigma_m} R)^p x_{\tau_i} = 0$ for all $\tau_i, i = 1, \ldots, n$. Thus $(a_{\sigma_m} R)^p x = 0$ and thus $(a_{\sigma_m} R)^p Rx = 0$. Since $P \in \mathrm{Ass}(M)$, $P = \mathrm{Ann}_R(Rx)$ and hence $(a_{\sigma_m} R)^p \subset P$. Consequently, $a_{\sigma_m} \in P$ and it follows that $a - a_{\sigma_m} \in P$. By recurrence we obtain $a_{\sigma_v} \in P$ for $1 \leq v \leq m$ and this means thet $P$ is a graded ideal.

iii) Consider the set $\mathcal{Q} = \{\mathrm{Ann}_R(N), N \neq 0$ a graded submodule of $M\}$. We have $\mathcal{Q} \neq \emptyset$ and since $R$ is left gr-Noetherian the family $\mathcal{Q}$ has a maximal element, say $P = \mathrm{Ann}_R(N_0)$. It is easily checked that $P$ is a prime ideal and moreover : $P \in \mathrm{Ass}(M)$.

6. Let $R$ be a ring, $I$ an ideal of $R$. The Rees ring with respect to $I$ is the ring $R(I) = R + IX + \ldots + I^n X^n + \ldots \subset R[X]$, which is obviously a graded subring of $R[X]$. A ring $T$ is an overring of $R$ if $R \subset T$ and $1_T = 1_R$. An ideal $I$ of $R$ is an invertible ideal if there is an overring $T$ of $R$ containing an $R$-bimodule $J$ such that : $IJ = JI = R$. If $I$ is an invertible ideal of $R$ then we define the generalized Rees ring $\check{R}(I) = \oplus_{n \in \mathbb{Z}} I^n X^n$, where $I^{-1} = J$. It is clear that $\check{R}(I)$ is a graded subring of $R[X, X^{-1}]$. For an invertible ideal $I$ of $R$, prove the following statements :

i) $\check{R}(I)$ is a strongly graded ring.

ii) If $R$ is left Noetherian then $\check{R}(I)$ and $R(I)$ ate both left Noetherian rings.

iii) If $R$ is a Noetherian commutative ring and $I$ is an arbitrary ideal, then $R(I)$ is a Noetherian ring.

**Hint :**

i) Easy enough.

ii) We have $(\check{R}(I))_0 = R$ and $\check{R}(I)_{\geq 0} = R(I)$. Since $R$ is a left Noetherian ring, i. implies that $\check{R}(I)$ is left gr-Noetherian and so Theorem 5.4.8. applies, e.g. $\check{(}I)$ and $R(I)$ are left Noetherian rings.

iii) If $I$ is generated by $a_1, \ldots, a_n$ then $R(I)$ is a homomorphic image of a polynomial ring $R[X_1, \ldots, X_n]$.

7. Consider a prime left Noetherian ring $R$. Let $a \in R$ be a nonzero normalizing element, i.e. $aR = Ra$. Prove that $I = Ra$ is an invertible ideal of $R$.

**Hint :** If $ab = 0$ then $Rab = 0$ yields $aRb = 0$, hence $a = 0$ or $b = 0$ because $R$ is a prime ring, thus $b = 0$. Similarly, from $ba = 0$, $b = 0$ follows. We may conclude that $a$ is a (left and right) regular element. Goldie's theorem entails that $a$ is invertible in the ring of quotients $Q$ of $R$. Now putting $J = Ra^{-1} = a^{-1}R$ we obtain $IJ = JI = R$ as desired.

We include a few exercises involving filtrations.

8. Consider a filtration $FR$ on a ring $R$, given by an ascending chain of additive subgroups $F_nR$ of $R : \subset \ldots \subset F_nR \subset F_{n+1}R \subset \ldots \subset R$, (assume $F_mR = 0$ for $m < 0$) $F_nRF_mR \subset F_{n+m}R, 1 \in F_0R, R = \cup_n F_nR$. To $FR$ we associate the abelian group $G(R) = \oplus_{n\geq 0}\frac{F_nR}{F_{n-1}R}$. If $a \in F_PR$ let $a_{(P)}$ be the image of $a$ in $G(R)_p = \frac{F_pR}{F_{p-1}R}$. Define $a_{(i)}b_{(j)} = (ab)_{i+j}$ and extend it to a $\mathbb{Z}$-bilinear map $\mu : G(R) \times G(R) \to G(R)$. Prove the following statements :

   i) $G(R)$ is a $\mathbb{Z}$-graded ring with respect to $\mu$. In particular $G(R)_0 = F_0R$ and $F_0R$ is a subring of $R$.

   ii) If $G(R)$ is a left Noetherian ring then $R$ is a left Noetherian ring too.

   **Hint**

   i) Easy enough.

   ii) Let $L$ be a left ideal of $R$ and define :

$$G(L) = \oplus_{p\geq 0}\frac{L \cap F_pR}{L \cap F_{p-1}R}$$

   Verify that $G(L)$ is a left ideal of $G(R)$. Moreover if $L \subset L'$ are left ideals of $R$ then $G(L) \subset G(L')$ and $L = L'$ if and only if $G(L) = G(L')$. Now conclude that $R$ is left Noetherian if $G(R)$ is.

9. Let $\varphi$ be an automorphism of a ring $R$ and $\delta$ a $\varphi$-derivation i.e. $\delta : R \to R$ is additive and for $a, b \in R$, $\delta(ab) = \varphi(a)\delta(b) + \delta(a)b$. Define the skew polynomial ring $R[X, \varphi, \delta]$ by introducing multiplication according to the rule : $Xa = \varphi(a)X + \delta(a)$. Verify that $A = R[X, \varphi, \delta]$ is a filtered ring with filtration given by : $F_nA = \{$ polynomial expressions of degree $\leq n\}$. Prove that $A$ is left Noetherian when $R$ is left Noetherian.

   **Hint :** Apply exercise 8 and also exercise 1.

## 5.7 Comments and References for Chapter 5

Internal homogenization for $\mathbb{Z}$-graded rings appeared under a somewhat different form. For a $\mathbb{Z}$-graded module $M$ with an $R$-submodule $X$ one may induce a filtration (ascending) on $X$ by putting $F_nX = X \cap (\oplus_{i \geq -n} M_i)$ such that the associated graded module with respect to this filtration is exactly $X^{\sim}$. In a similar way $X_{\sim}$ may be realized by inducing a descending filtration on $X (F'_nX = X \cap (\oplus_{i \leq n} M_i))$.

This technique cannot be extended to gradation by arbitrary groups but it does extend successfully in case $G$ is an ordered group (Lemma 5.3.1). After some introductory facts about ordered groups in Section 5.1., the basic concepts of rings graded by ordered groups are introduced in Section 5.2. We can apply this theory for various types of $G$-graded rings, where $G$ is not necessarily an ordered group but is very close to an ordered group and that is, for example, when $G$ is a polycyclic-by-finite group. The Noetherian and Artinian objects in $R$-gr, in case $R$ is graded by an ordered group $G$, are topics under consideration in Section 5.4. As main results of the chapter let us quote Theorems 5.4.7. and 5.4.8. In Section 5.5. the Krull dimension of rings graded by an ordered group $G$ is studied, culminating in 5.5.7. Generalities on Krull dimension and related concepts have been included in Appendix B.

### Some References

- A. Bell [17]
- V. P. Camillo, K.R. Fuller [38]
- C. Năstăsescu [140], [139]
- C. Năstăsescu, F. Van Oystaeyen [150], [148]

# Chapter 6

# External Homogenization

## 6.1 Normal subsemigroup of a group

Let $G$ be a group with identity element $e \in G$ and let $S$ be a normal subsemigroup of $G$, i.e. if $x, y \in S$ then $xy \in S$. We say that $S$ is a **normal subsemigroup** of $G$ if $gSg^{-1} \subset S$ for any $g \in G$. We note that in this case we have $S \subset g^{-1}Sg$ for any $g \in G$ and in particular $S \subset (g^{-1})^{-1}Sg = gSg^{-1}$. Hence $S$ is a normal subsemigroup if and only if $gSg^{-1} = S$ for any $g \in G$.

### 6.1.1 Examples

1. If $S$ is a subsemigroup (resp normal subsemigroup) of $G$ then the set $S^{-1} = \{x^{-1} \mid x \in S\}$ is a subsemigroup (resp normal subsemigroup) too.
2. Clearly if $S$ is a subsemigroup (resp normal subsemigroup) of $G$ then $S \cup \{e\}$ is a subsemigroup (resp normal subsemigroup) of $G$.
3. If $H \triangleleft G$ is a normal subgroup of $G$, then $H$ is a normal subsemigroup of $G$. Now if $S$ is a normal subsemigroup then if we denote by $H =< S >$ the subgroup generated by $S$ then $H$ is a normal subgroup of $G$.
4. If $G$ is an abelian group then every subsemigroup of $G$ is normal.
5. Let $G$ be an O-group (section 5.1). Then the set $S$ of positive elements of $G$ is a normal subsemigroup of $G$.

## 6.2 External homogenization

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a $G$-graded ring and $S$ a normal subsemigroup of $G$. Also assume that $e \in S$. Denote by

$$R^{\mathrm{gr}}[S] = \oplus_{s \in S} R^s$$

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where $R^s = R$ for any $s \in S$. So we can identify

$$R^{\mathrm{gr}}[S] = \{\sum_{s\in S} r^s s,\ r^s \in R \text{ for any } s \in S\}$$

On $R^{\mathrm{gr}}[S]$ we define the multiplication as follows :

$$(a_\sigma s)(a_\tau t) = a_\sigma a_\tau((\tau^{-1}s\tau)t) \qquad (1)$$

where $a_\sigma \in R_\sigma$, $a_\tau \in R_\tau$ and $s, t \in S$. It is clear that the multiplication defined in (1) may be extended to any two elements from $R^{\mathrm{gr}}[S]$ by additivity. Also if $G$ is abelian then (1) is the classical multiplication for a semigroup algebra. For any $\sigma \in G$ put $R^{\mathrm{gr}}[S]_\sigma = \oplus_{s\in S} R_{\sigma s^{-1}} s$ (2). With this notation we have :

### 6.2.1 Proposition

The following assertions hold :

1. $R^{\mathrm{gr}}[S]$ with the multiplication (1) is $G$-graded ring (the grading is given by (2)).
2. The component of degree $e$ is $(R^{\mathrm{gr}}[S])_e = \oplus_{s\in S} R_{s^{-1}} s$. Also the map $\varphi : R_{S^{-1}} \to (R^{\mathrm{gr}}[S])_e$ defined by $\varphi(\Sigma_{s\in S} r_{s^{-1}}) = \Sigma_{s\in S} r_{s^{-1}} s$ is an isomorphism of rings (recall that $R_{S^{-1}} = \oplus_{x\in S^{-1}} R_x$).
3. If $S = G$ then $R^{\mathrm{gr}}[G]$ is a crossed product. Moreover if $S$ is a normal subsemigroup, then $R^{\mathrm{gr}}[S]$ is a graded subring of $R^{\mathrm{gr}}[G]$.

**Proof**

1. Consider the elements $a_{\sigma_i} s_i$, $i = 1, 2, 3$ where $a_{\sigma_i} \in R_{\sigma_i}$ and $s_i \in S$ $(1 \le i \le 3)$. Then we have :

$$\begin{aligned} a_{\sigma_1}s_1 \cdot (a_{\sigma_2}s_2 \cdot a_{\sigma_3}s_3) &= a_{\sigma_1}s_1 \cdot (a_{\sigma_2}a_{\sigma_3}\sigma_3^{-1}s_2\sigma_3 s_3) \\ &= a_{\sigma_1}a_{\sigma_2}a_{\sigma_3}(\sigma_3^{-1}\sigma_2^{-1}s_1\sigma_2) \\ &= a_{\sigma_1}a_{\sigma_2}a_{\sigma_3}(\sigma_3^{-1}\sigma_2^{-1}s_1\sigma_2 s_2\sigma_3 s_3) \end{aligned}$$

   On the other hand we have $(a_{\sigma_1}s_1 \cdot a_{\sigma_2}s_2) \cdot a_{\sigma_3}s_3 = (a_{\sigma_1}a_{\sigma_2}(\sigma_2^{-1}s_1\sigma_2 \cdot s_2))a_{\sigma_3}s_3 = a_{\sigma_1}a_{\sigma_2}a_{\sigma_3}(\sigma_3^{-1}\sigma_2^{-1}s_1\sigma_2 s_2\sigma_3 s_3)$ hence $a_{\sigma_1}s_1 \cdot (a_{\sigma_2}s_2 \cdot a_{\sigma_3}s_3) = (a_{\sigma_1}s_1 \cdot a_{\sigma_2}s_2) \cdot a_{\sigma_3}s_3$ and so the multiplication of $R^{\mathrm{gr}}[S]$ is associative. Since

$$\begin{aligned} (R_{\sigma s^{-1}}s)\cdot(R_{\tau t^{-1}}t) \subseteq R_{\sigma s^{-1}}R_{\tau t^{-1}}(\tau t^{-1})^{-1}s(\tau t^{-1})t \subseteq R_{\sigma s^{-1}\tau t^{-1}}t\tau^{-1}s\tau = \\ = R_{\sigma\tau u^{-1}}u \text{ where } u = t\tau^{-1}s\tau \in S \end{aligned}$$

   we get that $(R^{\mathrm{gr}}[S])_\sigma(R^{\mathrm{gr}}[S])_\tau \subseteq (R^{\mathrm{gr}}[S])_{\sigma\tau}$ so $R^{\mathrm{gr}}[S]$ is a $G$-graded ring with gradation given by (2). Obviously $R$ is a subring of $R^{\mathrm{gr}}[S]$.

2. Since $S$ is a basis of $R^{\mathrm{gr}}[S]$ as a left $R$-module and since $(r_{s^{-1}}s)(r_{t^{-1}}t) = r_{s^{-1}}r_{t^{-1}}ts^{-1}t^{-1}t = r_{s^{-1}}r_{t^{-1}}ts$ we obtain that $\varphi$ is an isomorphism of rings.

3. Clear. □

The graded ring $R^{\mathrm{gr}}[S]$ is called the **graded semigroup ring** associated to graded ring $R$ and to the semigroup $S$.

### 6.2.2 Remarks

1. The set $S$ is also a basis for $R^{\mathrm{gr}}[S]$ as right $R$-module. Indeed, consider the sum $\sum_{i=1}^{n} s_i a_i$ where $s_i \in S$, $a_i \in R$. Since $\{s_i \mid i = 1, ..., n\}$ are homogeneous elements we may suppose the $a_i$'s are homogeneous elements of $R$. Assume that $a_i = a_{\sigma_i} \in R_{\sigma_i}$. Then $\sum_{i=1}^{n} s_i a_{\sigma_i} = \sum_{i=1}^{n} a_{\sigma_i}(\sigma_i^{-1}s_i\sigma_i)$. Since $\deg(s_1 a_{\sigma_1}) = \deg(s_2 a_{\sigma_2}) = ... = \deg(s_n a_{\sigma_n})$ then $s_1\sigma_1 = s_2\sigma_2 = ... = s_n\sigma_n$ and therefore the elements $\{\sigma_i^{-1}s_i\sigma_i\}_{i=1,...,n}$ are pairwise distinct.
Then from $\sum_{i=1}^{n} a_{\sigma_i}(\sigma_i^{-1}s_i\sigma_i) = 0$ we obtain $a_{\sigma_i} = 0$, $i = 1, ..., n$.

2. Any element $s \in S$ commutes with any element of $(R^{\mathrm{gr}}[S])_e$, therefore $R^{\mathrm{gr}}[S]$ is the semigroup ring of $R^{\mathrm{gr}}[S])_e$ for the semigroup $S$ in the classical sense. Indeed let $t \in S$ and $a_{s^{-1}}s$ be an element from $(R^{\mathrm{gr}}[S])_e$. We have $t(a_{s^{-1}}s) = a_{s^{-1}}(s^{-1})^{-1}ts^{-1}s = a_{s^{-1}}st = (a_{s^{-1}}s)t$ and therefore $R^{\mathrm{gr}}[S]$ is the semigroup ring $(R^{\mathrm{gr}}[S])_e[S]$ in the classical sense.

Let now $M = \oplus_{\sigma\in G} M_\sigma$ be a left graded $R$-module. Denote by $M^{\mathrm{gr}}[S] = \oplus_{s\in S} M^s$ where $M^s = M$ for any $s \in S$. We identify $M^{\mathrm{gr}}[S] = \{\sum_{s\in S} m^s \cdot s \mid m^s \in M,\ s \in S\}$. We define on $M^{\mathrm{gr}}[S]$ a left $R^{\mathrm{gr}}[S]$ multiplication by :

$$(a_\sigma s)(m_\tau t) = (a_\sigma m_\tau)(\tau^{-1}s\tau t) \qquad (3)$$

where $a_\sigma \in R_\sigma$, $m_\tau \in M_\tau$ and $s, t \in S$. Also for any $\sigma \in G$ we put$(M^{\mathrm{gr}}[S])_\sigma = \oplus_{s\in S} M_{\sigma s^{-1}}s$.

### 6.2.3 Proposition

With notation as above we have :

1. $M^{\mathrm{gr}}[S]$ is a left graded $R^{\mathrm{gr}}[S]$-module with the gradation given by (4).

2. The correspondence $M \mapsto M^{\mathrm{gr}}[S]$ define an exact functor from the category $R$-gr to the category $R^{\mathrm{gr}}[S]$-gr.

3. $M^{\mathrm{gr}}[S]$ is isomorphic to $\oplus_{s\in S} M(s^{-1})$ in the category $R$-gr.

4. $(M^{\mathrm{gr}}[S])_e = \oplus_{s\in S} M_{s^{-1}}s$ and the mapping $\psi : M_{S^{-1}} \mapsto (M^{\mathrm{gr}}[S])_e$ where $\psi(\sum_{s\in S} m_{s^{-1}}) = \sum_{s\in S} m_{s^{-1}}s$ is a $\varphi$-isomorphism ($\varphi$ is the isomorphism from Proposition 6.2.1).

5. If $N \subset M$ is a graded $R$-submodule of $M$, then $N^{\mathrm{gr}}[S]$ is a $R^{\mathrm{gr}}[S]$-graded submodule of $M^{\mathrm{gr}}[S]$ and $N^{\mathrm{gr}}[S] \cap (M^{\mathrm{gr}}[S])_e = \psi(N_{S^{-1}})$.

6. $M^{\mathrm{gr}}[S]$ is isomorphic to the graded tensor product $R^{\mathrm{gr}}[S] \otimes_R M$.

**Proof**

1-4 For assertions (1) and (4) we have the same proof as in Proposition 6.2.1

2. is obvious.

3. If $\sigma \in G$ we have $(\oplus_{s\in S} M(s^{-1}))_\sigma = \oplus_{s\in S} M_{\sigma s^{-1}}$ and $(M^{\mathrm{gr}}[S])_\sigma = \oplus_{s\in S} M_{\sigma s^{-1}}s$ and it is clear that $M^{\mathrm{gr}}[S]$ is isomorphic to $\oplus_{s\in S} M(s^{-1})$ in $R$-gr.

5. is clear.

6. Define $\alpha : R^{\mathrm{gr}}[S] \otimes_R M \to M^{\mathrm{gr}}[S]$ by : $\alpha(a_\sigma s \otimes m_\lambda) = a_\sigma m_\lambda(\lambda^{-1}s\lambda)$, where $a_\sigma \in R_\sigma$, $s \in S$ and $m_\lambda \in M_\lambda$. It is easy to see that $\alpha$ is well defined. Now if $t \in S$ we have $\alpha(t(a_\sigma s \otimes m_\lambda)) = \alpha(ta_\sigma s \otimes m_\lambda) = \alpha(a_\sigma\sigma^{-1}t\sigma s \otimes m_\lambda) = a_\sigma m_\lambda\lambda^{-1}\sigma^{-1}t\sigma s\lambda = t\alpha(a_\sigma s \otimes m_\lambda)$. From this it follows that $\alpha$ is $R^{\mathrm{gr}}[S]$-linear.

   If $\sigma \in G$, then $(R^{\mathrm{gr}}[S] \otimes_R M)_\sigma$ is generated by elements of the form $u_\lambda \otimes m_\mu$ with $\lambda\mu = \sigma$ where $u_\lambda \in (R^{\mathrm{gr}}[S])_\lambda$ and $m_\mu \in M_\mu$. (see section 2.4), hence $\alpha$ is a graded morphism. It is obvious that $\alpha$ is surjective. By remark 6.2.2 (i) every element from $R^{\mathrm{gr}}[S] \otimes_R M$ has the form $y = \sum_{i=1}^n s_i \otimes m_i$ where $s_i \in S$, $m_i \in M$ for any $1 \le i \le n$. Assume that $\deg y = \sigma$. Then $m_i \in M_{s_i^{-1}\sigma}$. If $\alpha(y) = 0$ then we have $\alpha(y) = \sum_{i=1}^n m_i\sigma^{-1}s_i s_i s_i^{-1}\sigma = \sum_{i=1}^n m_i\sigma^{-1}s_i\sigma = 0$. Since $s_i \neq s_j$ for any $i \neq j$ then $\sigma^{-1}s_i\sigma \neq \sigma^{-1}s_j\sigma$ for $i \neq j$ so $m_i = 0$ for any $i = 1, ..., n$, hence $y = 0$ and $\alpha$ is an injective map too. □

Assume now that $S = G$; if $a \in R$ and $m \in M$ we put $\varphi(a) = a^*$ and $\psi(m) = m^*$. Clearly $a^*$ and and $m^*$ are homogeneous elements of degree $e$ in $R^{\mathrm{gr}}[G]$ and respectively $R^{\mathrm{gr}}[M]$. $a^*$ (resp $m^*$) is called the homogenized of $a$ (resp $m$).
If $N$ is an $R$-submodule of $M$, then $R^{\mathrm{gr}}[G]\psi(N)$ is denoted by $N^*$ and $N^*$ is a graded submodule of $M^{\mathrm{gr}}[G]$. In fact $N^*$ is generated by all elements $n^*$, $n \in N$. $N^*$ is called the homogenized of $N$. It is easy to see that if $a \in R$, $m \in M$ then $(am)^* = a^*m^*$. Now using Remark 6.2.2 we get that $N^* = R^{\mathrm{gr}}[G]\psi(N) = \sum_{g\in G} g\psi(N) = \oplus_{g\in G} g\psi(N)$. Define $\varepsilon_R : R^{\mathrm{gr}}[G] \to R$ (resp $\varepsilon_M : M^{\mathrm{gr}}[G] \to M$) the map $\varepsilon_R(x) = 1$ (resp $\varepsilon_M(x) = 1$) for any $x \in G$.

Now if $r \in R^{\mathrm{gr}}[G]$ and $u \in M^{\mathrm{gr}}[G]$ we denote by $r_* = \varepsilon_R(r)$ and $u_* = \varepsilon_M(u)$; $r_*$ (resp $u_*$) is called the dehomogenized of $r$ (resp $u$). Clearly $(ru)_* = r_* u_*$. Now suppose that $L$ is a graded submodule of $M^{\mathrm{gr}}[G]$ and write $L_* = \varepsilon_M(L)$. Then $L_*$ is an $R$ submodule of $M$; $L_*$ is called the dehomogenized of $L$.

### 6.2.4 Proposition

The correspondence $N \to N^*$ has the following properties :

1. $N = (N^*)_*$.
2. $N^* \cap M = (N)_g$.
3. If $L \subset N$, $L \neq N$ then $L^* \subset N^*$ and $L^* \neq N^*$.
4. If $N$ is a graded submodule then $N^* = N^{\mathrm{gr}}[G]$.
5. If $I$ is a left ideal of $R$ then $(IN)^* = I^* N^*$.

The correspondence $L \to L_*$ satisfies :

1. $(L_*)^* \supset L$
2. If $L \subset L'$ then $L_* \subset L'_*$
3. If $J$ is a left graded ideal of $R^{\mathrm{gr}}[G]$ then $(JL)_* = J_* L_*$.

**Proof**

- 1. and 3. are clear since $(n^*)_* = n$ for any $n \in N$.
- 2. If $n \in (N)_g$ is a homogeneous element of degree $\sigma$ then $n^* = n\sigma^{-1}$. But $\sigma n^* = n$ so $n \in N^* \cap M$ and therefore $(N)_g \subset N^* \cap M$. Conversely let $z \in N^* \cap M$ with $\deg z = \theta$. There is an $n \in N$ such that $z = \theta n^*$. Assume $n = \sum_{\sigma \in G} n_\sigma$ then $z = \theta n^* = \theta \sum_{\sigma \in G} n_\sigma \sigma^{-1} = \sum_{\sigma \in G} \sigma^{-1} \theta$. Since $z \in M_\theta$, we have $n_\sigma = 0$ for $\sigma \neq \theta$ and thus $n = n_\theta$ so $z = \theta n_\theta^* = \theta n_\theta \theta^{-1} = n_\theta$ so $n \in (N)_g$. Therefore we have $(N)_g = N^* \cap M$.
- The assertions 4. and 5. are obvious.
- 1. Let $l \in L$ be a homogeneous element of degree $x$, $x \in G$. Then $l = \sum_{\sigma \in G} m_{x\sigma^{-1}} \sigma$ where $m_x \in M_x$. Then $l_x = \sum_{\sigma \in G} m_{x\sigma^{-1}}$. We see that $(l_*)^* = \sum m_{x\sigma^{-1}} (x\sigma^{-1})^{-1} = \sum m_{x\sigma^{-1}} \sigma x^{-1}$ and therefore $l = x(l_*)^*$ so $l \in (L_*)^*$ and hence $L \subseteq (L_*)^*$.
- The assertions 2. and 3. are obvious. □

### 6.2.5 Example

Let $R = \oplus_{i\in\mathbf{Z}} R_i$ be a graded ring of type Z and $c > 0$ a natural number. Let $S = \{nc \mid n \in \mathbf{N}\}$ be the submonoid of Z generated by $c$. In this case $R^{\mathrm{gr}}[S]$ is the polynomial ring $R[T]$ where $\deg T = c$ and the grading is given by $R[T]_n = \sum_{i+cj=n} R_i T^j$ for any $n \in \mathbf{Z}$ and $j \geq 0$. In particular if $c = 1$ we obtain the polynomial ring $R[T]$ with the classical grading $R[T]_n = \sum_{i+j=n} R_i T^j$, $j \geq 0$.

## 6.3 A Graded Version of Maschke's Theorem. Applications

In this section $R = \oplus_{\sigma\in G} R_\sigma$ will be a graded ring of type $G$ where $G$ is a finite group with $\mid G \mid = n$ unless otherwise mentioned. Let $M = \oplus_{\sigma\in G} M_\sigma$ and $N = \oplus_{\sigma\in G} N_\sigma$ be two graded modules and $f \in \mathrm{Hom}_{R-\mathrm{gr}}(M, N)$. We define the map $\widetilde{f} : M \to N$ by $\widetilde{f}(x) = \sum_{g\in G} g^{-1} f(gx)$, for any $x \in M$.

### 6.3.1 Lemma

$\widetilde{f} \in \mathrm{Hom}_{R^{\mathrm{gr}}[G]-\mathrm{gr}}(M, N)$.

**Proof** If $x \in M_\sigma$ then $gx \in M_{g\sigma}$ hence $f(gx) \in N_{gx}$ and therefore $g^{-1}f(gx) \in N_\sigma$ so $\widetilde{f}(M_\sigma) \subseteq N_\sigma$. We show now that $\widetilde{f}$ is an $R^{\mathrm{gr}}[G]$-homomorphism i.e. $\widetilde{f}(ax) = a\widetilde{f}(x)$ for every $a \in R^{\mathrm{gr}}[G]$. It is enough to prove the assertion for $a = \lambda_\sigma\tau, \lambda_\sigma \in R_\sigma, \tau \in G$. We have

$$\begin{aligned}
\widetilde{f}((\lambda_\sigma\tau)x) &= \sum_{g\in G} g^{-1} f((g\lambda_\sigma\tau)x) \\
&= \sum_{g\in G} g^{-1} f(\lambda_\sigma(\sigma^{-1}g\sigma\tau)x) \\
&= \sum_{g\in G} g^{-1}\lambda_\sigma f((\sigma^{-1}g\sigma\tau)x) \\
&= \sum_{g\in G} \lambda_\sigma(\sigma^{-1}g^{-1}\sigma) f(\sigma^{-1}g\sigma\tau x).
\end{aligned}$$

Putting $h = \sigma g\sigma^{-1}\tau$, we have $\sigma^{-1}g\sigma = \tau h^{-1}$ and so

$$\widetilde{f}((\lambda_\sigma\tau)x) = \sum_{g\in G} \lambda_\sigma\tau h^{-1} f(hx) = \lambda_\sigma\tau \sum_{h\in G} h^{-1} f(hx) = (\lambda_\sigma\tau)\cdot\widetilde{f}(x)$$

□

### 6.3.2 Proposition

Let $M$ be a graded $R^{\mathrm{gr}}[G]$-module and let $N \leq M$ be an $R^{\mathrm{gr}}[G]$-graded submodule of $M$. Assume that $M$ has no $n$-torsion for $n =\mid G \mid$ (as stated in the beginning). If $N$ is a direct summand of $M$ in $R$-gr, then there is an $R^{\mathrm{gr}}[G]$-submodule $P$ of $M$ such that $N \oplus P$ is essential in $M$ as an $R$-module. Furthermore, if $M = nM$, then $N$ is a graded summand of $M$ as an $R^{\mathrm{gr}}[G]$-module.

**Proof** (Following the proof of Theorem 3.5.1)
There is an $f \in \mathrm{Hom}_{R-\mathrm{gr}}(M, N)$ such that $f(x) = x$ for any $x \in N$. Let $\widetilde{f} \in \mathrm{Hom}_{R^{\mathrm{gr}}[G]-\mathrm{gr}}(M, N)$ be as in Lemma 6.3.1. If $x \in N$ then we have $\widetilde{f}(x) = nx$. Put $P = \ker \widetilde{f}$. It is clear that $P$ is a graded $R^{\mathrm{gr}}[G]$-submodule of $M$. Now as in the proof of Theorem 3.5.1 we obtain that $nM \subset N \oplus P$ so $N \oplus P$ is essential in $M$ as an $R$-module. The last part of the statement is clear. □

### 6.3.3 Corollary

Let $M$ be a gr-semisimple module. If $M$ has no $n$-torsion, then $M$ is semisimple in $R$-mod.

**Proof** It suffices to prove the statement under the assumption that $M$ is gr-simple. Consider the graded $R^{\mathrm{gr}}[M]$-module $M^{\mathrm{gr}}[G]$. By Proposition 6.2.3, $M^{\mathrm{gr}}[G]$ is isomorphic to $\oplus_{\sigma \in G} M(\sigma^{-1})$ in the category $R$-gr. Hence $M^{\mathrm{gr}}[G]$ is gr-semisimple in $R$-gr. Since $M$ is a gr-simple and $M$ has no $n$-torsion, $M = nM$, hence $M^{\mathrm{gr}}[G] = nM^{\mathrm{gr}}[G]$. From Proposition 6.3.2 we obtain that $M^{\mathrm{gr}}[G]$ is a semisimple object in the category $R^{\mathrm{gr}}[G]$-gr. Now it follows from Proposition 6.2.1, since $(R^{\mathrm{gr}}[G])_e \simeq R$, that $M^{\mathrm{gr}}[G]_e$ is semisimple in $R$-mod. From $(M^{\mathrm{gr}}[G])_e \simeq M$ it follows that $M$ is semisimple. □

### 6.3.4 Corollary

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a graded ring of type $G$ such that $n =\mid G \mid$ is invertible in $R$. If $R$ is gr-semisimple then $R$ is a simple Artinian ring.

### 6.3.5 Corollary

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a $G$-graded ring where $n =\mid G \mid$ is invertible in $R$. If $R$ is a left gr-hereditary (resp gr-semi-hereditary, resp gr-regular von Neumann) ring then $R$ is a left hereditary (resp semi-hereditary, resp regular von Neumann) ring.

**Proof** Let $K$ be a left graded ideal (resp finitely generated graded left ideal) of $R^{\rm gr}[G]$. There is a gr-free module $L$ (resp a gr-free module with finite basis) in the category $R^{\rm gr}[G]$-gr such that

$$\varphi : L \to K \subset R^{\rm gr}[G]$$

where $\varphi \in {\rm Hom}_{R^{\rm gr}[G]-{\rm gr}}(L, K)$ and $\varphi$ is surjective. Since $R^{\rm gr}[G]$ is gr-free in $R$-gr then $L$ is gr-free in $R$-gr. Since $R^{\rm gr}[G] = \oplus_{\sigma\in G} R(\sigma)$ in $R$-gr by hypothesis we get that $K$ is gr-projective (resp. finitely generated gr-projective) in $R$-gr. Hence there is $\psi : K \to L$ a morphism in $R$-gr such that $\varphi \circ \psi = 1_K$. Using Lemma 6.3.1 we find $\varphi \circ \left(\frac{1}{n}\widetilde{\psi}\right) = 1_K$ and $\widetilde{\psi} : K \to L$ is a morphism in $R^{\rm gr}[G]$-gr. Therefore $K$ is projective in $R^{\rm gr}[G]$-gr and thus $R[G]$ is left gr-hereditary (resp gr-semi-hereditary, resp gr-von Neumann regular). □

We recall that if $R = \oplus_{\sigma\in G} R_\sigma$ is a graded ring, we denote by ${\rm gl.dim}\, R$ (resp ${\rm gr-gl.dim}\, R$) the left homological global dimension of the category $R$-mod (resp. $R$-gr).

## 6.3.6 Corollary

Let $R = \oplus_{\sigma\in G} R_\sigma$ be a $G$-graded ring such that $n =\mid G \mid$ is invertible in $R$. Then

$${\rm gr-gl.dim}\, R = {\rm gl.dim}\, R$$

**Proof** The inequality ${\rm gr-gl.dim}\, R \leq {\rm gl.dim}\, R$ is obvious. Suppose now that $t = {\rm gr-gl.dim}\, R < \infty$. Let $M \in R^{\rm gr}[G]$-gr and let

$$\ldots \longrightarrow P_n \xrightarrow{f_n} \ldots \longrightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \longrightarrow 0$$

be a projective resolution of $M$ in the category $R^{\rm gr}[G]$-gr. Since $R^{\rm gr}[G]$ is a free $R$-module, the $P_i$'s are projective $R$-modules for every $i \geq 0$. Since ${\rm gr-gl.dim}\, R = t$, $K = \ker f_{t-1} = {\rm Im}\, f_t$ is a projective $R$-module. Therefore there exists $g_t \in {\rm Hom}_{R-{\rm gr}}(K, P_t)$ such that $f_t \circ g_t = 1_K$. By Lemma 6.3.1 we have $f_t \circ \frac{1}{n}\widetilde{g_t} = 1_K$ so $K$ is gr-projective $R^{\rm gr}[G]$-module. Hence ${\rm gr-gl.dim}\, R^{\rm gr}[G] \leq t$. Now from 6.2.3 (4) we obtain that ${\rm gl.dim}\, R \leq t$. □

## 6.3.7 Remark

Corollaries 6.3.3 and 6.3.4 were also proved in Section 4.3 (Theorem 4.3.4) using Graded Clifford Theory.

If $M \in R$-gr and $K \leq M$ is a graded submodule of $M$, then $K$ is called **gr-superfluous** (or gr-small) in $M$ if and only if whenever $L \leq M$ is a graded submodule of $M$ such that $K + L = M$ we must have $L = M$.

### 6.3.8 Lemma

Let $(M_i)_{i=1,...,n}$ be graded $R$-modules and $K_i \subseteq M_i$ be gr-superfluous submodules in $M_i$ for every $i = 1, ..., n$. Then $\oplus_{i=1}^n K_i$ is gr-superfluous in $\oplus_{i=1}^n M_i$.

**Proof** Using induction it suffices to prove the assertion for $n = 2$. Let $L$ be a graded submodule of $M_1 \oplus M_2$ such that $(K_1 \oplus K_2) + L = M_1 \oplus M_2$. It is easy to see that $K_1 + (K_2 + L) \cap M_1 = M_1$. Since $K_1$ is gr-superfluous in $M_1$, $(K_2+L)\cap M_1 = M_1$ so $M_1 \subset K_2+L$. Since $K_1 \subset M_1 \subset (K_2+L) \subset M_1 \oplus M_2$, we have $K_2 + L = M_1 \oplus M_2$. Hence $K_2 + (L \cap M_2) = M_2$ and since $K_2$ is superfluous in $M_2$ we get $L \cap M_2 = M_2$ so $M_2 \subset L$. Similarly, we obtain that $M_1 \subset L$ and so $L = M_1 \oplus M_2$. □

### 6.3.9 Proposition

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a $G$-graded ring where $G$ is a finite group and let $M$ be a graded $R$-module and $K \leq M$ be a gr-superfluous submodule in $M$. Then $K$ is a superfluous submodule in $M$.

**Proof** From Lemma 6.3.8 and Proposition 6.2.3 we obtain that $K^{\mathrm{gr}}[G]$ is gr-superfluous in $M^{\mathrm{gr}}[G]$ as a graded $R$-module then $K^{\mathrm{gr}}[G]$ is gr-superfluous in $M^{\mathrm{gr}}[G]$ as a graded $R^{\mathrm{gr}}[G]$-module. Therefore Proposition 6.2.3 entails that $\psi(K) = (K^{\mathrm{gr}}[G])_e$ is a superfluous submodule in $\psi(M) = (M^{\mathrm{gr}}[G])_e$ as an $R^{\mathrm{gr}}[G]$-module. Hence $K$ is a superfluous submodule in $M$. □

### 6.3.10 Remarks

1. Proposition 6.3.9 is false if the group $G$ is infinite. For example if $k$ is a field and $R = k[X]$ is the polynomial ring with the natural gradation (i.e. $G = \mathbb{Z}$) then $(X)$ is small in the graded sense but not in $R$-mod.

2. If $M$ is finitely generated graded module, then the graded Jacobson radical $J^g(M)$ is the (unique) largest gr-superfluous submodule of $M$ so if $M \in R$-gr where $R$ is $G$-graded with $G$ a finite group then by Proposition 6.3.9 we obtain that $J^g(M) \subseteq J(M)$, a result which is known from Section 2.9.

### 6.3.11 Corollary

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a $G$-graded ring where $G$ is a finite group and let $M \in R$-gr. If

$$P \overset{f}{\longrightarrow} M \longrightarrow 0$$

is a projective cover of $M$ in the category $R$-gr, then it is a projective cover of $M$ in the category $R$-mod.

**Proof** The map $P \xrightarrow{f} M$ defines a projective cover in $R$-gr if and only if $\ker f$ is gr-superfluous. Now Proposition 6.3.9 applies and completes the proof. □

### 6.3.12 Lemma

Let $M$ be a $R^{\mathrm{gr}}[G]$-module and let $N \leq M$ be a graded $R$-submodule of $M$. If we put

$$\overline{N} = \cap_{\sigma \in G} \sigma N$$

then :

1. $\overline{N}$ is a graded $R^{\mathrm{gr}}[G]$-submodule of $M$.
2. If $N$ is an essential $R$-submodule of $M$, then $\overline{N}$ is essential in $M$ as $R$-module.

**Proof**

1. Clearly if $g \in G$, then $g\overline{N} \subset \overline{N}$. Now $a_\lambda \in R_\lambda$ yields $a_\lambda \overline{N} \subseteq a_\lambda \sigma N \subseteq \lambda\sigma\lambda^{-1}N$ so $a_\lambda \overline{N} \subseteq \cap_{\sigma \in G} \lambda\sigma\lambda^{-1}N = \overline{N}$. Therefore $\overline{N}$ is an $R^{\mathrm{gr}}[G]$-submodule submodule of $M$. Since every $\sigma N$ is graded as an abelian group, we have that $\overline{N}$ is also graded (as a group) so $\overline{N}$ is $R^{\mathrm{gr}}[G]$-submodule of $M$.

2. Assume that $N$ is an essential $R$-submodule of $M$. Let $0 \neq x_g \in M_g$ be a nonzero homogeneous element of $M$ and suppose $G = \{\sigma_1, ..., \sigma_n\}$. If we put $x = (\sigma_1 + ... + \sigma_n)x_g \in M$, then $x \neq 0$. By Proposition 2.3.6 there exists $a_\lambda \in R_\lambda$ such that $a_\lambda x \neq 0$ and $a_\lambda x \in N$. Hence $a_\lambda \sigma_i x_g = \lambda\sigma_i\lambda^{-1}a_\lambda x_g \in N$, so $a_\lambda x_g \in \lambda\sigma_i^{-1}\lambda^{-1}N$. Therefore $a_\lambda x_g \in \cap_{n\, i=1} \lambda\sigma_i^{-1}\lambda^{-1}N = \overline{N}$. Since $a_\lambda x \neq 0$, it is clear that $a_\lambda x_g \neq 0$. Thus we obtain that $\overline{N}$ is an essential $R$-submodule of $M$. □

### 6.3.13 Proposition

Let $N \leq M$ be two graded $R^{\mathrm{gr}}[G]$-modules having no $n$-torsion ($n = |G|$). Then :

1. There is a graded $R^{\mathrm{gr}}[G]$-submodule $P \subseteq M$ such that $N \oplus P$ is essential in $M$ as an $R$-module.
2. $N$ is essential in $M$ as an $R^{\mathrm{gr}}[G]$-module if and only if $N$ is essential in $M$ as an $R$-module.

**Proof** Using Lemma 6.3.12, we follow the lines of proof used in the proof of in Corollary 3.5.5.

### 6.3.14 Corollary

Let $R$ be a $G$-graded ring $n = |G|$, and $M \in R$-gr such that $M$ has no $n$-torsion. If $N \leq M$ is an essential $R$-submodule of $M$, then $(N)_g$ is essential in $M$.

**Proof** It is clear that $\psi(N)$ is essential in $\psi(M) = (M^{\mathrm{gr}}[G])_e$ as $(R^{\mathrm{gr}}[G])_e$-module. Since $R^{\mathrm{gr}}[G]$ is a crossed product then Dade's Theorem (section 3.1) implies that $R^{\mathrm{gr}}[G]\psi(N)$ is essential in $M^{\mathrm{gr}}[G]$ as an $R^{\mathrm{gr}}[G]$-module. Now by Proposition 6.3.13, $R^{\mathrm{gr}}[G]\psi(N)$ is essential in $M$ as an $R$-module. Hence $(N)_g = M \cap R^{\mathrm{gr}}[G]\psi(N)$ is essential in $M$ as an $R$-module. □

### 6.3.15 Corollary

Let $R$ be a $G$-graded ring with $n = |G|$ and $M \in R$-gr. Suppose that $R$ has no $n$-torsion. Then $Z(M)$, the singular submodule of $M$, is a graded submodule of $M$.

**Proof** Let $x \in Z(M)$ and $x = \sum_{g \in G} x_g$ where $x_g \in M_g$. If $I = Ann_R(x)$ then $I$ is an essential left ideal of $R$. Corollary 6.3.4 entails that $(I)_g$ is an essential left ideal of $R$. Since $Ix = 0$ it follows that $(I)_g x_\sigma = 0$ for any $\sigma \in G$. Hence $x_\sigma \in Z(M)$ and therefore $Z(M)$ is a graded submodule of $M$.

□

### 6.3.16 Corollary

Let $R$ be a $G$-graded ring such that $n = |G|$ is invertible in $R$. If $P \subset I$ are two ideals of $R$ such that $P \neq I$ and $P$ is a prime ideal then $(P)_g \neq (I)_g$.

**Proof** Let $P^*$ and respectively $I^*$ be the homogenized of $P$ respectively $I$. Clearly $P^* \subset I^*$, $P^* \neq I^*$ and $P^*$ is a gr-prime ideal in $R^{\mathrm{gr}}[G]$ and $I^*$ is a two sided ideal. It is easy to see that $I^*/P^*$ is an essential left ideal in $R[G]/P^*$. Using Corollary 6.3.14 we get that $I^*/P^*$ is essential in $R^{\mathrm{gr}}[G]$ as left $R$-module. Since $I^* \cap R = (I)_g$ and $P^* \cap R = (P)_g$, it follows that $(I)_g/(P)_g$ is essential in $R/(P)_g$. Therefore $(P)_g \neq (I)_g$, (and $(P)_g \subset (I)_g$).

□

### 6.3.17 Corollary

Let $R$ be a graded ring of type $G$, where $n = |G|$.

1. If $R$ has no torsion, then $rad(R) = rad^g(R)$ (see Section 2.11) In particular we obtain that $R$ is semiprime if and only if $R$ is gr-semiprime.
2. Moreover if $R$ is a strongly graded ring and $R_e$ is semiprime, then $R$ is a semiprime ring.

**Proof**

1. Since $\mathrm{rad}^g(R) = (\mathrm{rad}(R))_g$ we may replace $R$ with the ring $R/\mathrm{rad}^g(R)$. Now it is enough to prove that $R/\mathrm{rad}^g(R)$ has no $n$-torsion. Indeed if $n \cdot \widehat{a} = 0$ for some $a \in R$, then $n \cdot \widehat{a_g} = 0$ for all the homogeneous components of $a$. So $n \cdot a_g \in \mathrm{rad}^g(R)$. If $a_g \notin \mathrm{rad}^g(R)$ then there is a homogeneous sequence $a_0, a_1, ..., a_n, ...$ such that $a_0 = a_g$, $a_1 \in a_0 R a_0$,..., $a_n \in a_{n-1} R a_{n-1}$,... such that $a_n \neq 0$ for any $n \geq 0$. Consider the sequence $na_0$, $n^2 a_1$,...,$n^{k+1} a_k$,... Since $na_0 = na_g \in \mathrm{rad}^g(R)$, there is a $k$ such that $n^{k+1} a_k = 0$. Since $R$ has no $n$-torsion then $a_k = 0$, contradiction. Hence from $n \cdot a_g \in \mathrm{rad}^g(R)$, it follows that $a_g \in \mathrm{rad}^g(R)$ hence $a \in \mathrm{rad}^g(R)$. Therefore $R/\mathrm{rad}^g(R)$ has no $n$-torsion. Hence we may assume that $\mathrm{rad}^g(R) = 0$, We shall prove that $\mathrm{rad}^g(R^{\mathrm{gr}}[G]) = 0$ and therefore that $R^{\mathrm{gr}}[G]$ is gr-semiprime. Let $N$ be nilpotent graded ideal of $R^{\mathrm{gr}}[G]$ and put $I = Ann_{R^{\mathrm{gr}}[G]}(N)$. Then $I$ is a right ideal of $R^{\mathrm{gr}}[G]$ and moreover, $I$ is an essential right ideal in $R^{\mathrm{gr}}[G]$.Let $a_\sigma \in R_\sigma$, $a_\sigma \neq 0$. We may assume that $a_\sigma \notin I$ thus $a_\sigma N \neq 0$. Since $N$ is nilpotent, $a_\sigma N^t = 0$ for some $t > 1$ and $a_\sigma N^{t-1} \neq 0$. Thus $0 \neq a_\sigma N^{t-1} \subset I$ and therefore $I$ is a right essential ideal. From Proposition 6.3.3 we conclude that $I$ is right essential as an $R$-module. Hence $I \cap R$ is right essential in $R_R$. In particular we have $I \cap R \neq 0$. Since $\mathrm{rad}^g(R) = 0$ then $l_R(I \cap R) = r_R(I \cap R) = 0$. Indeed if we put $J = l_R(I \cap R)$ then if $K = J \cap (I \cap R) \neq 0$ then $K^2 = 0$ and thus $K = 0$, since $R$ is gr-semiprime. Hence $J \cap (I \cap R) = 0$ and since $I \cap R$ is right essential in $R$ and $J$ is an ideal we arrive at $J = 0$. Similarly we prove that $r_R(I \cap R) = 0$. Since $I \cap R$ is a graded ideal, it is easy to see that $l_{R^{\mathrm{gr}}[G]}(I \cap R) = r_{R^{\mathrm{gr}}[G]}(I \cap R) = 0$ and therefore $l_{R^{\mathrm{gr}}[G]}(I) = r_{R^{\mathrm{gr}}[G]}(I) = 0$. Since $N \subseteq r_{R^{\mathrm{gr}}[G]}(I)$, we obtain $N = 0$ and hence $R^{\mathrm{gr}}[G]$ is gr-semiprime. Thus $(R^{\mathrm{gr}}[G])_e$ is semiprime and $R$ is semiprime.

2. If $R$ is strongly graded and $R_e$ is semiprime then $\mathrm{rad}^g(R) = 0$ and thus $\mathrm{rad}(R) = 0$. □

## 6.4 Homogenization and Dehomogenization Functors

In this section $R = \oplus_{\sigma \in G} R_\sigma$ is a $G$-graded ring and $H \triangleleft G$ is a normal subgroup. As explained in Section 1.2, $R$ has a $G/H$ canonical gradation

$$R = \oplus_{C \in G/H} R_C \text{ where } R_C = \oplus_{x \in C} R_x \quad .(1)$$

We may consider the categories $G$-gr-$R$ (or simply gr-$R$) of all $G$-graded $R$-modules and $G/H$-gr-$R$, the category of all $G/H$-graded $R$-modules, where $R$ is considered with the $G/H$-gradation as in (1).

Fix $(g_C)_{C\in G/H} \subset G$ a set of representatives for the classes of $H$ in $G$, i.e. $C = g_C H = H g_C$. Consider $M \in R$-gr. From Section 6.1 it follows that $M^{\mathrm{gr}}[H]$ is a graded $R^{\mathrm{gr}}[H]$-module. Consider $m_C = \sum_{h\in H} m_{g_C h}$, a $G/H$ homogeneous element of $M$ i.e. $m_C \in M_C$ (here $M = \oplus_{C\in G/H} M_C$, $M_C = \oplus_{x\in C} M_x$). We define $m_C^* = \sum_{h\in H} m_{g_C h} h^{-1} \in M^{\mathrm{gr}}[H]$; $m_C^*$ is called the **homogenized** of $m_C$ in $M^{\mathrm{gr}}[H]$ (in fact $m_C^* \in (M^{\mathrm{gr}}[H])_{g_C}$). If $m_C, m'_C \in M_C$, clearly $(m_C + m'_C)^* = m_C^* + m_C'^*$. Also if $a_C \in R_C$ and $m_{C'} \in M_{C'}$ then we have : $(a_C \cdot m_{C'})^* = a_C^* \cdot m_{C'}^* (g_{CC'}^{-1} g_C g_{C'})^{-1}$ To check this it suffices to establish that $\deg (a_C \cdot m_{C'})^* = g_{CC'}$ and $\deg (a_C^* m_{C'}^*) = g_C g_{C'}$ thus $\deg (a_C^* \cdot m_{C'}^* (g_{CC'}^{-1} g_C g_{C'})) = g_{CC'}$. Let $M, N \in R$-gr and $u \in \mathrm{Hom}_{(G/H,R)-\mathrm{gr}}(M, N)$ i.e. $u(M_C) \subseteq N_C$ for any $C \in G/H$. We define $u^* : M^{\mathrm{gr}}[H] \to N^{\mathrm{gr}}[H]$ by

$$u^*(m_{g_C h} \cdot k) = (u(m_{g_C h}))^* \cdot (hk) \quad (2)$$

From (2) we have that if $m_C \in M_C$ then $u^*(m_C^*) = (u(m_C))^*$. Moreover if $u : M \to N$ is a morphism in $R$-gr, then $u^* = u^{\mathrm{gr}}[H]$ where $u\mathrm{gr}[H] : M^{\mathrm{gr}}[H] \to N^{\mathrm{gr}}[H]$, $u^{\mathrm{gr}}[H](m_\sigma \cdot h) = u(m_\sigma) \cdot h$ for any $m_\sigma \in M_\sigma$, $h \in H$. It is easily verified that $u^*$ is a morphism in $R^{\mathrm{gr}}[H]$-gr. Now let $M \in (G/H, R)$-gr. We may select an exact sequence

$$L_1 \xrightarrow{p} L_0 \xrightarrow{q} M \longrightarrow 0 \quad (3)$$

where the $L_1$, $L_0$ are free objects in $R$-gr and $p, q$ are morphisms in the category $(G/H, R)$-gr. From (3) we obtain the graded morphism in $R^{\mathrm{gr}}[H]$-gr :

$$L_1^{\mathrm{gr}}[H] \xrightarrow{p^*} L_0^{\mathrm{gr}}[H]$$

We put $M^* = \mathrm{coker}\, p^*$. It is clear that $M^* \in R^{\mathrm{gr}}[H]$-gr. $M^*$ is called the **homogenized** of the object $M \in (G/H, R)$-gr. If $u : M \to M'$ is a morphism in the category $(G/H, R)$-gr and we take an exact sequence

$$L'_1 \xrightarrow{p'} L'_0 \xrightarrow{q'} M' \longrightarrow 0 \quad (3')$$

where $L'_1$, $L'_0$ are free objects in $R$-gr and $p', q'$ are morphisms in the category $(G/H, R)$-gr then there exists a commutative diagram

$$\begin{array}{ccccccc} L_1 & \xrightarrow{p} & L_0 & \xrightarrow{q} & M & \longrightarrow & 0 \\ \downarrow{\scriptstyle u_1} & & \downarrow{\scriptstyle u_0} & & \downarrow{\scriptstyle u} & & \\ L'_1 & \xrightarrow{p'} & L'_0 & \xrightarrow{q'} & M' & \longrightarrow & 0 \end{array} \quad (4)$$

where $u_0, u_1$ are morphisms in $(G/H, R)$-gr. From (4) we obtain the diagram

$$\begin{array}{ccccccc} L_1^{\mathrm{gr}}[H] & \xrightarrow{p^*} & L_0^{\mathrm{gr}}[H] & \longrightarrow & M^* & \longrightarrow & 0 \\ \downarrow{\scriptstyle u_1^*} & & \downarrow{\scriptstyle u_0^*} & & \downarrow{\scriptstyle u^*} & & \\ L_1'^{\mathrm{gr}}[H] & \xrightarrow{p'^*} & L_0'^{\mathrm{gr}}[H] & \longrightarrow & M'^* & \longrightarrow & 0 \end{array} \quad (5)$$

where $u^* : M^* \to M'^*$ is a morphism in the category $R^{\mathrm{gr}}[H]$-gr. By standard arguments we can show that $u^*$ does not depend on the choice of the morphisms $u_0, u_1$. Moreover, if $u, v$ are morphisms in the category $(G/H, R)$-gr then $(v \circ u)^* = v^* \circ u^*$. Therefore the correspondence $M \to M^*$, $u \to u^*$ defines a covariant functor $(-)^* : (G/H, R)\text{-gr} \to G - R^{\mathrm{gr}}[H]\text{-gr}$. $(-)^*$ is called the **homogenization functor**. From diagram (3) it follows that, if $M \in R$-gr, then $M^* = M^{\mathrm{gr}}[H]$.

Now consider the augmentation morphism $\varepsilon : R^{\mathrm{gr}}[H] \to R$ where $\varepsilon(\sum_{i=1}^n a^i h_i) = \sum_{i=1}^n a^i$, where $a_i \in R$, $h_i \in H$ for any $1 \leq i \leq n$. So $\varepsilon(h) = 1$ for any $h \in H$. It is easy to see that $\varepsilon$ is a surjective morphism of rings and $\varepsilon((R^{\mathrm{gr}}[M]_C)) \subseteq R_C$ for any $C \in G/H$. So $\varepsilon \in \mathrm{gr}{-}G/H - RINGS$. Therefore $\ker \varepsilon$ is an object in $(G/H, R^{\mathrm{gr}}[H])$-gr. Also $\ker \varepsilon$ is generated as left (right) ideal by elements $1 - h$, $h \in H$. Using the morphism $\varepsilon : R^{\mathrm{gr}}[H] \to R$ we obtain the functor $R \otimes_{R^{\mathrm{gr}}[H]} - : (G, R^{\mathrm{gr}}[H]){-}\mathrm{gr} \to (G/H, R){-}\mathrm{gr}$, $M \to R \otimes_{R^{\mathrm{gr}}[H]} M = M/\ker \varepsilon \cdot M$. The functor $(-)_* = R \otimes_{R^{\mathrm{gr}}[H]} -$ is called the **dehomogenized functor**. The main result of this section is the following

### 6.4.1 Theorem

The functors $(-)^*$ and $(-)_*$ define an equivalence between the categories $(G/H, R)$-gr and $(G - R^{\mathrm{gr}}[H])$-gr.

**Proof** Since $R \otimes_{R^{\mathrm{gr}}[G]} M^{\mathrm{gr}}[G] \simeq M$, the exactness of the sequence (3) yields $(-)_* \circ (-)^* \simeq \mathrm{Id}_{(G/H|R)-\mathrm{gr}}$. On the other hand by Proposition 2. we have that $R^{\mathrm{gr}}[H] \otimes_R M \simeq M^{\mathrm{gr}}[H]$ and this implies that for any $\sigma \in G$ we have an isomorphism $M(\sigma)^{\mathrm{gr}}[H] \simeq (M^{\mathrm{gr}}[H])(\sigma)$. Therefore a gr-free module over the graded ring $R^{\mathrm{gr}}[H]$ is isomorphic to a $L^{\mathrm{gr}}[H]$ where $L$ is a gr-free module in $R$-gr. From the foregoing it easily follows that

$$(-)^* \circ (-)_* = Id_{R^{\mathrm{gr}}[H]-\mathrm{gr}}$$

Recall that for a $G$-graded ring $R$ we denote by $\mathrm{gr.dim}R$ (resp $\mathrm{gr.gl.dim}\, R$) the left homological global dimension of the category $R$-mod (resp. of the category $R$-gr). Using Theorem 6.4.1 we obtain the following application :

### 6.4.2 Corollary

Let $R$ be a $G$-graded ring with $G = \mathbf{Z^n}$. Then

$$\mathrm{gl.dim}\, R \leq \mathrm{gr.gl.dim}\, R + n$$

**Proof** There is a subgroup $H$ of $G$ such that $H \simeq \mathbf{Z}$ and $G/H \simeq \mathbf{Z^{n-1}}$. From Theorem 6.4.1 we obtain that the global dimension of the category $G/H - R$-gr is equal to the global dimension of the category $R[H]$-gr. Using induction it is enough to prove that $\mathrm{gr.gl.dim}\, R[H] \leq \mathrm{gr.gl.dim}\, R + 1$ where

$H$ is a subgroup of $G$ and $H \simeq \mathbf{Z}$. But it is easy to see that $R[H] \simeq R[T, T^{-1}]$ where $T$ is a variable over $R$. Since $R[T, T^{-1}] = S^{-1}R[T]$ where $S$ is the multiplicative system of homogeneous elements $S = \{1, T, T^2, ...\}$ we have $\mathrm{gr.gl.dim}\, R[T, T^{-1}] \leq \mathrm{gr.g.dim}\, R[T]$. Consider $M \in R[T]$-gr. Then the following sequence is exact in $R[T]$-mod :

$$0 \longrightarrow M[T] \overset{m_{T-1}}{\longrightarrow} M[T] \longrightarrow M \longrightarrow 0$$

where $m_{T-1}$ is the right multiplication by $T-1$. Assume that $\mathrm{gr.gl.dim}\, R = t < \infty$, since $M \in R$-gr then $\mathrm{gr.p.dim}\, M[T] = \mathrm{p.dim}\, M[T] \leq t$ so using the long exact $\mathrm{Ext}(-,-)$ sequence in $R[T]$-mod we obtain that $\mathrm{p.dim}_{R[T]} M \leq t+1$ and therefore $\mathrm{gr.p.dim}_{R[T]} M \leq \mathrm{p.dim}_{R[T]} M \leq t+1$.
Hence $\mathrm{gr.gl.dim}\, R[T, T^{-1}] \leq \mathrm{gr.gl.dim}\, R[T] \leq t+1$.

## 6.5 Exercises

1. Let $A$ be a Grothendieck category and $(M_i)_{i \in I}$ a family of objects from $A$. Assume that for every $i \in I$, $N_i \leq M_i$ is an essential subobject of $M_i$. Prove that $\oplus_{i \in I} N_i$ is essential in $\oplus_{i \in I} M_i$.

   **Hint :** Assume that $X$ is a nonzero subobject of $\oplus_{i \in I} M$. Since $A$ satisfies axiom AB5 (see Appendix A) iwe have that $X = \sum_{J \subset I} (X \cap \oplus_{j \in J} M_j)$ where $J$ ranges over all finite subsets of $I$. Thus we may assume that $I$ is a finite set. If $|I| = n$ then using induction it is enough to prove the assertion for $n = 2$. For $X \subset M_1 \oplus M_2$, $X \neq 0$ and $X \cap M_1 \neq 0$, then clearly $X \cap N_1 \neq 0$ so $X \cap (N_1 \oplus N_2) \neq 0$. If $X \cap M_1 = 0$ then $X$ is a subobject of $M_2$ so we get that $X \cap N_2 \neq 0$. Hence $N_1 \oplus N_2$ is essential in $M_1 \oplus M_2$.

2. Let $R$ be a $G$-graded ring, $M \in R$-gr and let $N$ be a graded submodule of $M$. Prove that $N$ is gr-essential in $M$ if and only if $N$ is essential $M$.

   **Hint :** $N^{\mathrm{gr}}[G] \subset M^{\mathrm{gr}}[G]$. Since $N^{\mathrm{gr}}[G] = \oplus_{\sigma \in G} N(\sigma)$ and $M^{\mathrm{gr}}[G] = \oplus_{\sigma \in G} M(\sigma)$ in the category $R$-gr, we may use exercise 1 and conclude that $N^{\mathrm{gr}}[G]$ is essential in $M^{\mathrm{gr}}[G]$ in the category $R$-gr and therefore $N^{\mathrm{gr}}[G]$ is essential in $M^{\mathrm{gr}}[G]$ in the category $R^{\mathrm{gr}}[G]$-gr. Hence $(N[G])_e$ is essential in $(M^{\mathrm{gr}}[G])_e$ as an $R^{\mathrm{gr}}[G]_e$ module, consequently $N$ is essential in $M$ in the category $R$-mod.

   **Remark** A different proof is given in Proposition 2.3.6. In fact the proof of exercise 2. also yields all the results in Proposition 2.3.6.

3. Let $Q \in R$-gr gr-injective and $\sup(Q) < \infty$. Prove that $Q$ is injective in $R$-mod.

**Hint :** Corollary 2.4.10 and the properties of adjoint functors (see Appendix A), entail that $HOM_R(R^{\mathrm{gr}}[G], Q)$ is an injective object in $R^{\mathrm{gr}}[G]$-gr. Since $\sup(Q) < \infty$ we have $\psi(Q) \simeq (\mathrm{HOM}_R(R^{\mathrm{gr}}[G], Q))_e = \mathrm{Hom}_{R-\mathrm{gr}}(R^{\mathrm{gr}}[G], Q) = \mathrm{Hom}_{R-\mathrm{gr}}(\oplus_{\sigma\in G} R(\sigma), Q) \simeq (Q^{\mathrm{gr}}[G])_e$ as $(R^{\mathrm{gr}}[G])_e$-modules (where $\psi : Q \to (Q^{\mathrm{gr}}[G])_e$ is the canonical morphism $\psi(x) = (\sum_{g\in G} x_g g^{-1})$ where $x = \sum_{g\in G} x_g$ and $x_g \in Q_g$ for $g \in G$). Hence $Q$ is injective in $R$-mod.

4. Let $R = \oplus_{\sigma\in G} R_\sigma$ be a $G$-graded ring with $G$ a finite group. If $M \in R^{\mathrm{gr}}[G]$-gr is a simple object, then $M$ is semisimple of finite length in $R$-gr.

**Hint :** It is easy to see that $M$ is finitely generated as an $R$-module. Let $N \subset M$ be a gr-maximal submodule of $M$. We define $\overline{N} = \cap_{\sigma\in G} \sigma N$. Then $\overline{N}$ is a graded $R^{\mathrm{gr}}[G]$-submodule of $M$. Since $\overline{N} \leq N$ we have $\overline{N} = 0$. Then from the exact sequence in $R_e$-mod :

$$0 \to M \to \oplus_{\sigma\in G} M/\sigma N$$

we derive that $M$ is semisimple of finite length in $R_e$-mod. Thus $M$ is gr-Noetherian and gr-Artinian (in $R$-gr). There exists a gr-simple submodule $P$ of $M$. It is clear that there is an exact sequence in $R^{\mathrm{gr}}[G]$-gr :

$$P^{\mathrm{gr}}[G] \to M \to 0$$

Since $P^{\mathrm{gr}}[G] = \oplus_{\sigma\in G} P[\sigma]$ in $R$-gr we finally get that $M$ is gr-semisimple of finite length in $R$-gr.

5. Let $M$ be a $G$-graded ring. Then :

   i) If $Q$ is a gr-prime ideal of $R^{\mathrm{gr}}[G]$, then $Q \cap R$ is gr-prime in $R$.

   ii) If $P$ is a gr-prime ideal of $R$, then there is a gr-prime ideal $Q$ of $R^{\mathrm{gr}}[G]$*such* that $P = Q \cap R$.

   iii) $\mathrm{rad}^g(R) = \mathrm{rad}^g(R^{\mathrm{gr}}[G]) \cap R$.

**Hint :**

   i) Is obvious.

   ii) Since $P^{\mathrm{gr}}[G] \cap R = P$, by Zorn's Lemma there is an ideal $Q$ of $R^{\mathrm{gr}}[G]$ which is maximal among ideals satisfying $Q \cap R = P$. It is easy to see now that $Q$ is a gr-prime ideal of $R^{\mathrm{gr}}[G]$.

   iii) From (ii) we get that $\mathrm{rad}^g(R^{\mathrm{gr}}[G]) \cap R \subseteq \mathrm{rad}^g(R)$. Conversely, we prove that $(\mathrm{rad}_g R)^{\mathrm{gr}}[G] \subseteq \mathrm{rad}^g(R^{\mathrm{gr}}[G])$. To do so it suffices to show that if $I$ is a nilpotent graded ideal, then

$I^{\mathrm{gr}}[G]$ is a nilpotent ideal of $R^{\mathrm{gr}}[G]$. Assume that $I^m = 0$; then if $a_{\sigma_1}, ..., a_{\sigma_m} \in I$ are homogeneous elements of $I$ and $\tau_1, ..., \tau_m \in G$ then $(a_{\sigma_1}\tau_1)...(a_{\sigma_m}\tau_m) = (a_{\sigma_1}...a_{\sigma_m})\tau = 0$ for some $\tau \in G$. Hence $(I^{\mathrm{gr}}[G])^m = 0$.

6. Let $R = \oplus_{\sigma\in G} R_\sigma$ be a $G$-graded ring with $n = |G| < \infty$. Let $M \in R$-gr. Prove that $M$ has finite Goldie dimension in $R$-gr if and only if $M$ has finite Goldie dimension in $R$-mod. Moreover if we denote by $\mathrm{gr.rank}_R M$ (resp $\mathrm{rank}_R M$) the Goldie dimension of $M$ in the category $R$-gr (resp. in $R$-mod) then we have

$$\mathrm{rank}_R M \leq n \cdot \mathrm{gr.rank}_R M$$

**Hint :** Let $(N_i)_{i\in I} a$ family of submodules of $M$ such that the sum $\sum_{i\in I} N_i$ is direct i.e. $N_i \cap (\sum_{j\neq i} N_j) = 0$ for any $i \in I$. Then we have the family of homogenized submodules $(N_i^*)_{i\in I}$ in $M^{\mathrm{gr}}[G]$ and the sum $\sum_{i\in I} N_i^*$ is direct. Since $M^{\mathrm{gr}}[G] = \oplus_{\sigma\in G} M(\sigma)$, if $M$ has finite dimension in $R$-gr then $M^{\mathrm{gr}}[G]$ has finite Goldie dimension in $R$-gr since $G$ is finite. Hence $|I| \leq n \cdot \mathrm{gr.rank}_R M$, and $\mathrm{rank}_R M \leq n \cdot \mathrm{gr.rank}_R M$.

## 6.6 Comments and References for Chapter 6

So-called "external homogenization" is a technique stemming from algebraic geometry related to embedding affine varieties in projective ones or to "blowing-up" techniques.

Ring theoretically, for a $\mathbb{Z}$-graded ring $R$, the relevant construction comes down to viewing $R$ as a graded subring of $R[X]$ with gradation defined by $R[X]_n = \sum_{i+j=n} R_i X^j$. For an arbitrary group $G$ and a normal sub-semigroup $S$ of $G$ we construct a $G$-graded ring $R^{\rm gr}[S]$ containing $R$ as a $G$-graded subring. The construction of $R^{\rm gr}[S]$ is similar to the semigroup ring $R[S]$, in fact when $G$ is abelian then those rings are isomorphic. The case $S = G$ allows to apply $R^{\rm gr}[G]$ in the theory of $R$-gr, making fruitful use of the embedding $\eta : R \hookrightarrow R^{\rm gr}[G], \eta(r) = \sum_{\sigma\in G} r_\sigma\sigma^{-1}$ for $r = \sum_{\sigma\in G} r_\sigma \in R$. Observe that $\eta$ is not a graded (degree zero) embedding, in fact $\eta$ defines an isomorphic between $R$ and $R^{\rm gr}[G]_e$. i.e. $R^{\rm gr}[G]_e = \oplus_{\sigma\in G} R_\sigma\sigma^{-1}$. The non-commutative technique stems from work on Rees ring extensions (e.g. [197] related to special subgroups of the Picard group of the degree zero part, subsequently used in the study of orders over arithmetically graded rings (cf. [156], and other references there).

Section 6.1. starts off with the definition and examples of normal sub-semigroups of a group. In Section 6.2. the technique of external homogenization is expounded. Since elements of $G$ became invertible homogeneous elements in $R^{\rm gr}[G]$ we may obtain a graded version of Maschke's theorem (cf. Lemma 6.3.1, Proposition 6.3.2, Corollary 6.3.3.)

Further interesting applications are presented in Section 6.3.. For a normal subgroup $H$ of $G$, $R^{\rm gr}[H]$ is a subring of $R^{\rm gr}[G]$. Unfortunately, if $H$ is not normal, then the foregoing need not hold. This is a drawback because external homogenization cannot be iterated for normal series : $\{e\} = H_0 \subset H_1 \subset \ldots \subset H_n = G$ where each $H_{i-1}$ is normal in $H_i$ but not necessarily in $G$.

The construction of $R^{\rm gr}[G]$ and all the results from Sections 6.1-6.3 appeared in the paper [142] and [184].

The main result in Section 6.4. is Theorem 6.4.1., stating that for a normal subgroup $H$ of $G$ the categories $(G/H, R)$-gr and $(G, R^{\rm gr}[H])$-gr equivalent via the homogenization and the dehomogenization functors.

Finally let us point out a positive property of external homogenization, in some sense making up for the drawback mentioned before. If $G$ is abelian and the $G$-graded ring $R$ is commutative then $R^{\rm gr}[G]$ is again commutative ! This makes the technique extra useful in commutative algebra (e.g.[128]).

## Some References

- B. Alfonsi [5]
- L. Le Bruyn, F. Van Oystaeyen [118]
- L. Le Bruyn, M. Van den Bergh, F. Van Oystaeyen [117]
- J. Matijevic [128]
- C. Năstăsescu [142]
- C. Năstăsescu, F. Van Oystaeyen [150], [151]
- M. Van den Bergh [184]
- F. Van Oystaeyen [190], [191]

# Chapter 7

# Smash Products

## 7.1 The Construction of the Smash Product

Let $R = \oplus_{x\in G} R_x$ be a $G$-graded ring. We denote by $M_G(R)$ the set of row and column finite nonzero elements matrices over $R$, with rows and columns indexed by the elements of $G$. If $\alpha \in M_G(R)$, then we write $\alpha(x, y)$ for the entry in the $(x, y)$ position of $\alpha$. For $\alpha, \beta \in M_G(R)$, the matrix product $\alpha\beta$ given by

$$(\alpha\beta)(x, y) = \sum_{z\in G} \alpha(x, z)\beta(z, y) \tag{1}$$

is correctly defined i.e. $\alpha\beta \in M_G(R)$. Therefore $M_G(R)$ is a ring with identity $I$, where $I$ is the matrix having $I(x, y) = \delta_{x,y}$ for any $x, y \in G$, where $\delta_{x,y}$ is the Kronecker symbol. If $x, y \in G$ we denote by $e_{x,y}$ the matrix with 1 in the $(x, y)$ position and zero elsewhere. We put $p_x = e_{x,x}$. Clearly $e_{x,y} \in M_G(R)$ for any $x, y \in G$ so in particular $p_x \in M_G(R)$. We also see that $e_{x,y}e_{u,v} = \delta_{y,u}e_{x,v}$. In particular the set $\{p_x \mid x \in G\}$ are orthogonal idempotents elements.

We denote by $M_G^*(R)$ the set of matrices of $M_G(R)$ with only finitely many nonzero entries. From (1) one easily obtains that $M_G^*(R)$ is a two sided ideal of $M_G(R)$. We have that $e_{x,y} \in M_G^*(R)$ for any $x, y \in G$ and also if $G$ is a finite group we have $M_G^*(R) = M_G(R)$.

Define the map

$$\eta : R \to M_G(R) \to by\ \eta(r) = \widetilde{r} \text{ where } \widetilde{r} = \sum_{x,y\in G} r_{xy^{-1}}e_{x,y} \tag{2}$$

where $r = \sum_{g\in G} r_g$, $r \in R$, $r_g \in Rg$ for any $g \in G$. In fact (2) implies that $\widetilde{r}$ is the matrix of $M_G(R)$ for which $\widetilde{r}(x, y) = r_{xy^{-1}}$, i.e. having the element $r_{xy^{-1}}$ in the $(x, y)$ position. In particular we have $\eta(1) = I$. When $G$ is an

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infinite group and $r \neq 0$, then $\widetilde{r} \notin M_G^*(R)$. For $r, s \in R$ we have :

$$\begin{aligned} \eta(r+s) &= \eta(r) + \eta(s) \\ \eta(r \cdot s) &= \eta(r) \cdot \eta(s) \end{aligned}$$

The first equality is obvious. For the second we write

$$\begin{aligned} \eta(r) \cdot \eta(s) &= (\sum_{x,y \in G} r_{xy^{-1}} e_{x,y})(\sum_{u,v \in G} s_{uv^{-1}} e_{u,v}) \\ &= \sum_{x,y,u,v} r_{xy^{-1}} s_{uv^{-1}} e_{x,y} e_{u,v} \\ &= \sum_{x,y,v} r_{xy^{-1}} s_{yv^{-1}} e_{x,y} \\ &= \sum_{x,v \in G} (rs)_{xv^{-1}} e_{x,v} \\ &= \eta(rs) \end{aligned}$$

On the other hand if $\eta(r) = \widetilde{r} = 0$, then we may apply (2) and obtain $r_{xy^{-1}} = 0$ for any $x, y \in G$, hence $r = 0$. Hence $\eta$ is an injective morphism of rings, called the **matrix embedding morphism** of the graded ring $R$. We put $\widetilde{R} = \mathrm{Im}\eta$ and we denote by $\widetilde{R}\#G$ or shortly $R\#G$ the subring of $M_G(R)$ generated by $\widetilde{R}$ and the set of orthogonal idempotents $\{p_x \mid x \in G\}$. The ring $\widetilde{R}\#G$ is called the **smash product** of $R$ by the group $G$. The group $G$ embeds in $M_G(R)$ as the group of permutation matrices; each $g \in G$ maps to $\overline{g}$ where :

$$\overline{g} = \sum_{x \in G} e_{x,xg} \tag{3}$$

Hence $\overline{g}$ is the matrix with 1 in the $(x, xg)$ position for any $x \in G$ and zero elsewhere. For $g, h \in G$ we obtain :

$$\overline{g} \cdot \overline{h} = \sum_{x \in G} e_{x,xg} \cdot \sum_{y \in G} e_{y,yh} = \sum_{x \in G} e_{x,xgh} = \overline{gh}$$

Therefore if we put $\overline{G} = \{\overline{g} \mid g \in G\}$ then $\overline{G}$ is a subgroup of the group units of $M_G(R)$, isomorphic to $G$.

### 7.1.1 Proposition

Let $R$ be a $G$-graded ring. The following assertions hold :

1. If $r, s \in R$ then $(\widetilde{r}p_x)(\widetilde{s}p_y) = \widetilde{r}\widetilde{s_{xy^{-1}}}p_y$ for any $x, y \in G$. If $r_\sigma \in R_\sigma$ then $p_x\widetilde{r_\sigma} = \widetilde{r_\sigma}p_{\sigma^{-1}x}$ for any $x \in G$.
2. If $G$ is an infinite group, then $\widetilde{R}\#G = \widetilde{R} \oplus (\oplus_{x \in G}\widetilde{R}p_x)$ or $\widetilde{R}\#G$ is a left and right free $\widetilde{R}$-module with basis $\{I\} \cup \{p_x \mid x \in G\}$. Furthermore

$\oplus_{x\in G}\widetilde{R}p_x$ is an ideal of $\widetilde{R}\#G$ which is essential both as a left and as a right ideal.

3. If $G$ is a finite group then $\widetilde{R}\#G = \oplus_{x\in G}\widetilde{R}p_x$.

**Proof**

1. It is enough to prove that $p_x(\widetilde{s}p_y) = \widetilde{s_{xy^{-1}}}p_y$. Since $\widetilde{s} = \eta(s) = \sum_{u,v} s_{uv^{-1}}e_{u,v}$ we have :

$$\begin{aligned} p_x(\widetilde{s}p_y) &= e_{x,x}(\sum_{u,v} s_{uv^{-1}}e_{u,v}e_{y,y}) = e_{x,x}\sum_{u\in G} s_{uy^{-1}}e_{u,y} \\ &= \sum_{u\in G} s_{uy^{-1}}e_{x,x}e_{u,y} = s_{xy^{-1}}e_{x,y} = \sum s_{xy^{-1}}e_{x,y}e_{y,y} = \\ &= \widetilde{s_{xy^{-1}}}p_y \end{aligned}$$

For the second equality we write $p_x\widetilde{r_\sigma} = e_{x,x}\sum_{y\in G} r_\sigma e_{\sigma y,y} = r_\sigma e_{x,\sigma^{-1}x} = (r_\sigma e_{x,\sigma^{-1}x})p_{\sigma^{-1}x} = \widetilde{r_\sigma}p_{\sigma^{-1}x}$.

2. It is clear that $\widetilde{R}\#G = \widetilde{R} + \sum_{x\in G}\widetilde{R}p_x$. Now if $r \in R$ is such that $\widetilde{r}p_x = 0$, then we have $0 = \widetilde{r}p_x = (\sum_{u,v\in G} r_{uv^{-1}}e_{u,v})p_x = \sum_{u\in G} r_{ux^{-1}}e_{u,x}$ so $r_{ux^{-1}} = 0$ for any $u \in G$ and hence $r = 0$ or $\widetilde{r} = 0$. Since $\widetilde{R}p_x \subseteq M_G^*(R)$ for any $x \in G$, it follows that $\sum_{x\in G}\widetilde{R}p_x \subseteq M_G^*(R)$. On the other hand we have $\widetilde{R} \cap M_G^*(R) = 0$ (because if $r \in R$, $r \neq 0$, $\widetilde{r} \notin M_G^*(R)$). If we had a relation $\widetilde{r}I + \sum_{x\in G}\widetilde{r^x}p_x = 0$ where $r \in R$, $r^x \in R$, then the foregoing remarks would entail that $\widetilde{r} = 0$ and $\sum_{x\in G}\widetilde{r^x}p_x = 0$. Since $\{p_x \mid x \in G\}$ is a set of orthogonal idempotents we have that $\widetilde{r_x}p_x = 0$ for any $x \in G$, so $\widetilde{r^x} = 0$. Hence $\{I\} \cup \{p_x \mid x \in G\}$ is a left basis of $\widetilde{R}\#G$ over $\widetilde{R}$. A similar argument may be used to establish that $\{I\} \cup \{p_x \mid x \in G\}$ is also a right basis of $\widetilde{R}\#G$ over $\widetilde{R}$. Using (1.) we may infer that $\oplus_{x\in G}\widetilde{R}p_x$ is an ideal of $\widetilde{R}\#G$. Clearly $\oplus_{x\in G}\widetilde{R}p_x$ is essential as a left ideal and also as a right ideal by assertion (1.).

3. If $G$ is finite, then $I = \sum_{x\in G} p_x$ and therefore we obtain that $\widetilde{R}\#G = \oplus_{x\in G}\widetilde{R}p_x$. □

### 7.1.2 Proposition

1. If $g \in G$ and $\alpha \in M_G(R)$ then

$$(\overline{g}^{-1}\alpha\overline{g})(x,y) = \alpha(xg^{-1}, yg^{-1})$$

for any $x, y \in G$. In particular, $\overline{g}^{-1}p_x\overline{g} = p_{xg}$ and $\overline{g}^{-1}\widetilde{r}\overline{g} = \widetilde{r}$, $r \in R$.

2. $G$ acts by conjugation on $\widetilde{R}\#G$ and $(\widetilde{R}\#G)^G = \widetilde{R}$.

3. $(\widetilde{R}\#G)\overline{G} = \sum_{g\in G}(\widetilde{R}\#G)\overline{g}$ is a direct sum of additive subgroups of $M_G(R)$.

4. $(\widetilde{R}\#G)\overline{G}$ is a subring of $M_G(R)$. Moreover, $(\widetilde{R}\#G)\overline{G}$ is a skew group ring of the group $\overline{G}$ over the ring $(\widetilde{R}\#G)$.

5. If $G$ is an infinte group, then

$$(\widetilde{R}\#G)\overline{G} = (\oplus_{g\in G}\widetilde{R_g}) \oplus M_G^*(R)$$

as additive groups and $M_G^*(R)$ is an ideal of $(\widetilde{R}\#G)\overline{G}$ which is essential both as a left and a right ideal.

6. If $G$ is a finite group then $(\widetilde{R}\#G)\overline{G} = M_G(R)$.

**Proof**

1. We have $(\overline{g}^{-1}\alpha\overline{g})(x,y) = \sum_{u,v\in G}\overline{g}^{-1}(x,u)\alpha(u,v)\overline{g}(v,y)$. Since $\overline{g}(v,y) = 1$ if and only if $v = y\overline{g}^{-1}$ and $\overline{g}^{-1}(x,u) = 1$ if and only if $u = xg^{-1}$, we have that $(\overline{g}^{-1}\alpha\overline{g})(x,y) = \alpha(xg^{-1}, yg^{-1})$. If we put $\alpha = p_x = e_{x,x}$ then we clearly have $\overline{g}^{-1}p_x\overline{g} = p_{xg}$. In fact $\overline{g}^{-1}e_{x,y}\overline{g} = e_{xg,yg}$. Thus if $r \in R$ then we have $\overline{g}^{-1}\widetilde{r}\overline{g} = \sum_{x,y\in G} r_{xy^{-1}}\overline{g}^{-1}e_{x,y}\overline{g} = \sum_{x,y\in G} r_{xy^{-1}}e_{xg,yg} = \sum_{x,y\in G} r_{(xg)(yg)^{-1}}e_{xg,yg} = \widetilde{r}$. Therefore $\widetilde{R} \subseteq (\widetilde{R}\#G)^G$.

2. Let $\alpha \in (\widetilde{R}\#G)^G$; then $\widetilde{g}^{-1}\alpha\widetilde{g} = \alpha$ for any $g \in G$. Since $\alpha \in \widetilde{R}\#G$ then $\alpha = \widetilde{r} + \sum_{x\in G}\widetilde{r^x}p_x$ where $r \in R$, $r^x \in R$ and the family $\{r^x \mid x \in G\}$ has only finitely many nonzero components. Since $\overline{g}^{-1}\alpha\overline{g} = \overline{g}^{-1}\widetilde{r}\overline{g} + \sum_{x\in G}(\overline{g}^{-1}\widetilde{r^x}\overline{g})(\overline{g}^{-1}p_x\overline{g}) = \widetilde{r} + \sum_{x\in G}\widetilde{r^x}p_{xg}$. Because $\overline{g}^{-1}\alpha\overline{g} = \alpha$ for any $g \in G$, we have that $\sum_{x\in G}\widetilde{r^x}p_x = \sum_{x\in G}\widetilde{r^xp_{xg}}$ for any $g \in G$, or $\widetilde{r^x} = \widetilde{r^{xg^{-1}}}$ for any $g \in G$. Now if $G$ is infinite, since only finitely many of the $\widetilde{r^x}$ are nonzero, we obtain that $\widetilde{r^x} = 0$ for any $x \in G$. Hence $\alpha = \widetilde{r} \in \widetilde{R}$. If $G$ is finite then $\widetilde{r^x} = \widetilde{r^y}$ for any $x, y \in G$. Thus if we put $s = r^x = r^y$ then we have : $\alpha = \widetilde{r} + \widetilde{s}(\sum_{x\in G}p_x) = \widetilde{r} + \widetilde{s} = \widetilde{r+s}$ (since $I = \sum_{x\in G}p_x$). Hence, $(\widetilde{R}\#G)^G = \widetilde{R}$.

3. Since $\sum_{x\in G}\widetilde{R}p_x = \sum_{x,y\in G}R_{xy^{-1}}e_{x,y}$, it follows that

$$\begin{aligned}(\widetilde{R}\#G)\overline{G} &= \sum_{g\in G}(\widetilde{R}\#G)\overline{g}\\ &= \sum_{g\in G}(\widetilde{R} + \sum_{x,y\in G}R_{xy^{-1}}e_{x,y})\overline{g}\\ &= \sum_{g\in G}\widetilde{R}\overline{g} + \sum_{x,y\in G}R_{xy^{-1}}e_{x,y}\overline{g}\end{aligned}$$

Note that $\sum_{x,y,g\in G} R_{xy^{-1}}e_{x,y}\overline{g} = \sum R_{xy^{-1}}e_{x,yg} = \sum_{x,u,g} R_{xgu^{-1}}e_{x,u} = \sum_{x,u\in G} Re_{x,u} = M_G^*(R)$. Thus $(\widetilde{R}\#G)\overline{G} = \sum_{g\in G}\widetilde{R\overline{g}} + M_G^*(R)$. If $r \in R$ then :

$$\widetilde{r\overline{g}} = \sum_{x,y\in G} r_{xy^{-1}}e_{x,y}\overline{g} = \sum_{x,y\in G} r_{xy^{-1}}e_{x,yg} = \sum_{x,u} r_{xgu^{-1}}e_{x,u} \qquad (4)$$

Consequently, every nonzero element from $\sum_{g\in G}\widetilde{R\overline{g}}$ has infinitely many nonzero entries, and therefore : $\sum_{g\in G}\widetilde{R\overline{g}} \cap M_G^*(R) = 0$. From the above equality (4), it follows that the sum $\sum_{g\in G}\widetilde{R\overline{g}}$ is direct, hence $\sum_{g\in G}\widetilde{R\overline{g}} = \oplus_{g\in G}\widetilde{R\overline{g}}$. Again from (4) we obtain that

$$(\widetilde{R}\#G)\overline{g} \cap (\sum_{h\neq g}(\widetilde{R}\#G)\overline{h}) = 0$$

and thus $(\widetilde{R}\#G)\overline{G} = \sum(\widetilde{R}\#G)\overline{g}$ is a direct sum.

4. Follows from (1.) and (2.).

5. Follows from (3.) and (6.) is clear. □

Let us write $R\{G\}$ for $(\widetilde{R}\#G)\overline{G}$. The foregoing proposition yields, for an infinite group $G$, that $R\{G\} = (\oplus_{g\in G}\widetilde{R\overline{g}}) \oplus M_G^*(R)$ and if $G$ is a finite group then $R\{G\} = M_G(R)$. More general, if $H \subseteq G$ is a subgroup of $G$ we put $R\{H\} = (\widetilde{R}\#G)\overline{H} = \oplus_{h\in H}(\widetilde{R}\#G)\overline{h}$. The arguments of Proposition 7.1.2, assertion (3.) yield that $R\{H\} = \oplus_{h\in H}\widetilde{R\overline{h}} \oplus R^*\{H\}$ where $R^*\{H\} = \{\alpha \in M_G^*(R) \mid \alpha(x,y) \in R_{xHy^{-1}}, \text{ for } x,y \in G\}$. If $G$ is a finite group then $R\{G\} = (\widetilde{R}\#G)\overline{G} = M_G(R)$. If $n =\mid G \mid$ then $M_G(R) \simeq M_n(R)$. Since $(\widetilde{R}\#G)\overline{G} \simeq (\widetilde{R}\#G) * G$ (cf. Proposition 7.1.2), we get that $(\widetilde{R}\#G) * G \simeq M_n(R)$. This isomorphism yields the duality for coactions' theorem stated in [43] by M.Cohen and S. Montgomery.

## 7.2 The Smash Product and the Ring $\mathrm{End}_{R-\mathrm{gr}}(U)$

Assume that $R = \oplus_{\sigma\in G}R_\sigma$ is a $G$-graded ring, $G$ a finite group. From Section 2.2 we retain that the set $\{R(\sigma) \mid \sigma \in G\}$ is a family of projective generators for the category $R$-gr. Put $U = \sum_{\sigma\in G} R(\sigma)$. Since $G$ is a group, $U$ a is finitely generated projective generator of $R-gr$. Since $U$ is finitely generated we have that $End_R(U) = END_R(U)$ (see Section 2.4), therefore $\mathrm{End}_R(U)$ is a $G$-graded ring where the multiplication is given by :

$$fg = g \circ f, \forall\, f,g \in \mathrm{End}(U)$$

If we put $G = \{g_1 = e, g_2, ..., g_n\}$, then from Section 2.10 we retain that $\mathrm{End}_R(U)$ is isomorphic to the matrix ring $M_n(R)$, considered with the grading given by $M_n(R) = \oplus_{\lambda \in G} M_n(R)_\lambda$ where :

$$\begin{pmatrix} R_{g_1\lambda g_1^{-1}} & R_{g_1\lambda g_2^{-1}} & ... & R_{g_1\lambda g_n^{-1}} \\ R_{g_2\lambda g_1^{-1}} & R_{g_2\lambda g_2^{-1}} & ... & R_{g_2\lambda g_n^{-1}} \\ ... & ... & ... & ... \\ R_{g_n\lambda g_1^{-1}} & R_{g_n\lambda g_2^{-1}} & ... & R_{g_n\lambda g_n^{-1}} \end{pmatrix}$$

In particular, the ring $\mathrm{End}_{R-\mathrm{gr}}(U)$ is isomorphic to the matrix ring

$$M_n(R)_e = \begin{pmatrix} R_e & R_{g_1g_2^{-1}} & ... & R_{g_1g_n^{-1}} \\ R_{g_2g_1^{-1}} & R_e & ... & R_{g_2g_n^{-1}} \\ ... & ... & ... & ... \\ R_{g_ng_1^{-1}} & R_{g_ng_2^{-1}} & ... & R_e \end{pmatrix}$$

### 7.2.1 Theorem

If $G = \{g_1 = e, g_2, ..., g_n\}$ is a finite group then :

1. The ring $\mathrm{End}_R(U)$ is isomorphic to the skew group ring $\mathrm{End}_{R-\mathrm{gr}}(U) * G$.
2. $\mathrm{End}_{R-gr}(U)$ is isomorphic to the smash product $\widetilde{R}\#G$.

**Proof**

1. Follows from Theorem 2.10.3

2. We put

$$T = \begin{pmatrix} R_e & R_{g_1g_2^{-1}} & ... & R_{g_1g_n^{-1}} \\ R_{g_2g_1^{-1}} & R_e & ... & R_{g_2g_n^{-1}} \\ ... & ... & ... & ... \\ R_{g_ng_1^{-1}} & R_{g_ng_2^{-1}} & ... & R_e \end{pmatrix}$$

Define $\varphi : R \longrightarrow T$ as follows : if $a = \sum_{g \in G} a_g$, $a_g \in R_g$ for all $g \in G$, then we put

$$\varphi(a) = \begin{pmatrix} a_e & a_{g_1g_2^{-1}} & ... & a_{g_1g_n^{-1}} \\ a_{g_2g_1^{-1}} & a_e & ... & a_{g_2g_n^{-1}} \\ ... & ... & ... & ... \\ a_{g_ng_1^{-1}} & a_{g_ng_2^{-1}} & ... & a_e \end{pmatrix}$$

It is easy to see that $\varphi$ is an injective morphism. In fact, since $M_n(R) \simeq M_G(R)$, the map $\varphi : R \longrightarrow T$ identifies with the map $\eta : R \longrightarrow M_G(R)$ given in Section 7.1. Consider the elements $p_{g_k}$ $(1 \leq k \leq n)$ to be the

matrix with 1 at $(k,k)$ and all its other entries being 0. If we put $S = \varphi(R)$ it is chear that $T = Sp_{g_1} + Sp_{g_2} + ... + Sp_{g_n} = p_{g_1}S + p_{g_2}S + ... + p_{g_n}S$. Moreover $\{p_{g_1}, ..., p_{g_n}\}$ is a left and right basis of $T$ over the subring $S$. Now if $g, h \in G$ we may assume that $g = g_m$ and $h = g_l$. If $s \in S$ we may assume that :

$$s = \begin{pmatrix} b_e & b_{g_1 g_2^{-1}} & ... & b_{g_1 g_n^{-1}} \\ b_{g_2 g_1^{-1}} & b_e & ... & b_{g_2 g_n^{-1}} \\ ... & ... & ... & ... \\ b_{g_n g_1^{-1}} & b_{g_n g_2^{-1}} & ... & b_e \end{pmatrix}$$

where $b = \sum_{i=1}^{n} b_{g_i}$, $b_{g_i} \in R_{g_i}$, is an element of the ring $R$. A matrix calculation yields $p_g(sp_h) = s_{gh^{-1}}p_h$. The latter equality establishes that $\widetilde{R}\#G \simeq T$.

## 7.2.2 Corollary

Assume that $R = \oplus_{\sigma \in G} R_\sigma$ is a $G$-strongly graded ring ($G$ finite group). Then $\mathrm{End}_{R_e}(R) \simeq \widetilde{R}\#G$. Moreover if $R$ is a crossed product then $\widetilde{R}\#G \simeq M_n(R_e)$ where $n = \mid G \mid$.

**Proof** First we show that if $\sigma, \tau \in G$ then $R_{\sigma^{-1}\tau} \simeq \mathrm{Hom}_{R_e}(R_\sigma, R_\tau)$ (as $R_e$-bimodules). Indeed define $\theta : R_{\sigma^{-1}\tau} \longrightarrow \mathrm{Hom}_{R_e}(R_\sigma, R_\tau)$ such that for $x \in R_{\sigma^{-1}\tau}$ and $y \in R_\sigma$ we put $\theta(x)(y) = yx$. If $a \in R_e$ then we have $\theta(ax)(y) = yax = (a\theta(x))(y)$ so $\theta(ax) = a\theta(x)$. In a similar way we may show that $\theta(xa) = \theta(x)a$ so $\theta$ is a morphism of $R_e$-bimodules. If $\theta(x) = 0$ we have $yx = 0$ for any $y \in R_\sigma$. Since $R_{\sigma^{-1}}R_\sigma = R_e$ we get that $x = 0$ thus $\theta$ is injective. Assume now that $u \in \mathrm{Hom}_{R_1}(R_\sigma, R_\tau)$. Since $R_\sigma^{-1}R_\sigma = R_e$ it follows that $1 = \sum_{i=1}^{n} a_i b_i$ where $a_i \in R_{\sigma^{-1}}$ and $b_i \in R_\sigma$. We define $x = \sum_{i=1}^{n} a_i u(b_i)$. Now if $y \in R_\sigma$ we have $\theta(x)(y) = \sum_{i=1}^{n} y a_i u(b_i) = \sum_{i=1}^{n} u(y a_i b_i) = u(y \sum_{i=1}^{n} a_i b_i) = u(y)$ so $\theta(x) = u$ and therefore $\theta$ is an isomorphism. Recall that if $M = M_1 \oplus ... \oplus M_n$ as $R$-modules then the ring $\mathrm{End}_R(M)$ is isomorphic to the matrix ring :

$$\begin{pmatrix} A_{11} & A_{12} & ... & A_{1n} \\ A_{21} & A_{22} & ... & A_{2n} \\ ... & ... & ... & ... \\ A_{n1} & A_{n2} & ... & A_{nn} \end{pmatrix}$$

where $A_{ij} = \mathrm{End}_R(M_i, M_j)$, $1 \leq i, j \leq n$. Now if we put $G = \{\sigma_1 = e, \sigma_2, ..., \sigma_n\}$ and $M_i = R_{\sigma_i^{-1}}$ for any $1 \leq i \leq n$, then we have that $R =$

$M_1 \oplus ... \oplus M_n$ and thus

$$\mathrm{End}_{R_e}(R) \simeq \begin{pmatrix} R_e & R_{\sigma_1\sigma_2^{-1}} & ... & R_{\sigma_1\sigma_n^{-1}} \\ R_{\sigma_2\sigma_1^{-1}} & R_e & ... & R_{\sigma_2\sigma_n^{-1}} \\ ... & ... & ... & ... \\ R_{\sigma_n\sigma_1^{-1}} & R_{\sigma_n\sigma_2^{-1}} & ... & R_e \end{pmatrix}$$

Theorem 7.2.1 (2.) entails that $\mathrm{End}_{R_e}(R) \simeq \widetilde{R}\#G$. If $R$ is a crossed product then $R_\sigma \simeq R_e$ as a left $R_e$ module for any $\sigma \in G$. In this case we have that $\mathrm{End}_{R_e}(R) \simeq M_n(R)$ and also $\widetilde{R}\#G \simeq M_n(R_e)$.

## 7.3 Some Functorial Constructions

Let $R$ be a $G$-graded ring and $\widetilde{R}\#G$ the smash product. If $M \in \widetilde{R}\#G$-mod we put by $M_0 = \sum_{x\in G} p_x M$; $M_0$ is an $\widetilde{R}\#G$-submodule of $M$. Indeed, it suffices to prove that $\widetilde{r_\sigma} M_0 \subseteq M_0$, where $r_\sigma \in R_\sigma$. But $\widetilde{r_\sigma} M_0 \subseteq \sum \widetilde{r_\sigma} p_x M \subseteq \sum p_{\sigma x} \widetilde{r_\sigma} M \subseteq \sum_{x\in G} p_{\sigma x} M = M_0$, since $\{p_x \mid x \in G\}$ is a family of orthogonal idempotents.

Denote now by $(R-gr)^\#$ the subclass of $\widetilde{R}\#G$-mod defined by the property $(R{-}\mathrm{gr})^\# = \{M \in \widetilde{R}\#G\text{-mod}, M = \sum_{x\in G} p_x M\}$.

### 7.3.1 Proposition

$(R\text{-gr})^\#$ is a localizing subcategory of $\widetilde{R}\#G$-mod (i.e. $(R\text{-gr})^\#$) is closed undertaking to subobjects, quotient objects, extensions and arbitrary direct sums). Moreover the radical associated to the localizing subcategory $(R\text{-gr})^\#$ is the functor defined by the correspondence $M \to M_0$, $M \in (\widetilde{R}\#G)$-mod.

**Proof** It is obvious that $(R\text{-gr})^\#$ is closed under quotient objects and arbitrary direct sums. Let $M \in (R\text{-gr})^\#$ and $N \subset M$ and $\widetilde{R}\#G$-submodule. If $n \in N$, then $n \in M = \oplus_{x\in G} p_x M$ and therefore $n = \sum_{x\in G} p_x m^x$. Thus $p_x n = p_x m^x$ and thus $n = \sum p_x n$, so $n \in \oplus_{x\in G} p_x N$, hence $N = \oplus_{x\in G} p_x N$ i.e. $N \in (R\text{-gr})^\#$.

Consider the exact sequence of $\widetilde{R}\#G$-modules

$$0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

where $M', M'' \in (R\text{-gr})^\#$. Let $m \in M$; then $v(m) \in M'' = \oplus_{x\in G} p_x M''$. Thus $v(m) = \sum_{x\in G} p_x v(m^x) = v(\sum_{x\in G} p_x m^x)$ and therefore $m - \sum_{x\in G} p_x m^x \in \ker(v) = \mathrm{Im}(u)$. Thus there exists $m' \in M'$ such that $m - \sum_{x\in G} p_x m^x = u(m')$. Since $M' \in (R\text{-gr})^\#$ we have $m' = \sum_{x\in G} p_x m'^x$ and thus $m - \sum_{x\in G} p_x m^x = u(\sum_{x\in G} p_x m'^x) = \sum_{x\in G} p_x u(m'^x)$. Hence $m = \sum_{x\in G} p_x(m^x +$

$u(m'^x))$, thus $M = M_0$ and $M \in (R\text{-gr})^{\#}$. Therefore $(R\text{-gr})^{\#}$ is a localizing subcategory. It is obvious that for $M \in (\widetilde{R}\#G)$-mod, $M_0 = \sum_{x\in G} p_x M$ is the largest submodule of $M$ belonging to $(R\text{-gr})^{\#}$. Hence the radical associated to the localizing subcategory $(R\text{-gr})^{\#}$ is the functor given by the correspondence $M \to M_0$.

Consider $M \in R$-gr. Then $M$ has a natural structure of $\widetilde{R}\#G$-module obtained by putting, for all $\widetilde{r} \in \widetilde{R}$ and $x \in G : \widetilde{r}m = rm$ and $p_x m = m_x$ where $m_x \in M_x$ is the $x$-homogeneous component of $m$. Indeed if $r, s \in R$ and $x, y \in G$, then we have $\widetilde{r}(\widetilde{s}m) = \widetilde{r}(sm) = r(sm) = (rs)m = \widetilde{rs}m$. We also have $(\widetilde{r}p_x)((\widetilde{s}p_y)m) = (\widetilde{r}p_x)(\widetilde{s}p_y)m = r(sm_y)_x$ where $(sm_y)_x$ is the homogeneous component of degree $x$ of the element $sm_y$. Since $sm_y = (\sum_{x\in G} s_x)m_y = \sum_{x\in G} s_x m_y$ then $(sm_y)_x = s_{xy^{-1}}m_y$. On the other hand, $((\widetilde{r}p_x)(\widetilde{s}p_y))m = (\widetilde{rs_{xy^{-1}}}p_y)m = (rs_{xy^{-1}})m_y = r(s_{xy^{-1}}m_y)$. Hence $M$ has a natural structure of $\widetilde{R}\#G$-module. We write $M^{\#}$ for the $\widetilde{R}\#G$-module $M$ with the the above defined structure. For $M, N \in R$-gr and $f \in \mathrm{Hom}_{R-gr}(M, N)$ the map $f : M^{\#} \longrightarrow N^{\#}$ is an $\widetilde{R}\#G$ linear morphism. An exact functor $(-)^{\#} : R$-gr $\to (\widetilde{R}\#G)$-mod may be defined by the correspondence $M \to M^{\#}$ and $(f)^{\#} = f$ if $f \in \mathrm{Hom}_{R-\mathrm{gr}}(M, N)$, $M, N \in R - gr$. If $M \in R$-gr, then $M^{\#} \in (R\text{−gr})^{\#}$ because $M_0 = \sum_{x\in G} p_x M = \oplus_{x\in G} M_x = M$. Therefore we may regard the functor $(-)^{\#}$ as a functor from $R$-gr to $(R\text{−gr})^{\#}$. Let $M$ be an $\widetilde{R}\#G$-module. We have seen that $M_0 = \sum_{x\in G} p_x M$ is an $\widetilde{R}\#G$-submodule of $M$. On $M_0$ a natural structure of $G$-graded $R$-module is defined by putting : $(M_0)_x = p_x M$ and we consider $M_0$ as an $R$-module via the morphism $\eta : R \longrightarrow \widetilde{R}\#G$. It is easy to see that the correspondence $M \to M_0$ defines an exact functor $H : \widetilde{R}\#G$-mod $\to R$-gr.

## 7.3.2 Proposition

With notation as above we have :

1. The functor $(-)^{\#}$ is a left adjoint of the functor $H$.
2. The corestriction of the functor $(-)^{\#} : R$-gr $\to (R\text{−gr})^{\#}$ is an isomorphism of categories.
3. If the group $G$ is finite then $(R\text{−gr})^{\#} = \widetilde{R}\#G$-mod and the functor $(-)^{\#} : R$-gr $\to \widetilde{R}\#G$-mod is an isomorphism of categories.

**Proof**

1. We define the functorial morphisms

$$\mathrm{Hom}_{\widetilde{R}\#G}((-)^{\#}, -) \xrightarrow{\alpha} \mathrm{Hom}_{R-\mathrm{gr}}(-, H) \text{ and}$$
$$\mathrm{Hom}_{R-\mathrm{gr}}(-, H) \xrightarrow{\beta} \mathrm{Hom}_{\widetilde{R}\#G}((-)^{\#}, -)$$

as follows :

If $M \in R$-gr, $N \in \widetilde{R}\#G$-mod then let $\alpha(M,N) : \mathrm{Hom}_{\widetilde{R}\#G}(M^{\#}, N) \to \mathrm{Hom}_{R-\mathrm{gr}}(M, H(N))$ be defined by $\alpha(M,N)(u)(x) = u(x)$ where $u : M^{\#} \to N$ and $x \in M$. Since $M^{\#} = \oplus_{\sigma\in G} p_\sigma M$, we have $u(M) = u(\oplus_{\sigma\in G} p_\sigma M) = \sum_{\sigma\in G} p_\sigma u(M) \subseteq N_0 = H(N)$.

We also define $\beta(M,N) : \mathrm{Hom}_{R-gr}(M, H(N)) \longrightarrow \mathrm{Hom}_{\widetilde{R}\#G}(M^{\#}, N)$ as follows : for $v \in \mathrm{Hom}_{R-\mathrm{gr}}(M, H(N))$, put $\beta(M,N)(v) = i \circ v$, where $i : H(N) = N_0 \hookrightarrow N$ is the inclusion map. It is obvious that $\alpha$ and $\beta$ are functorial morphisms and also $\alpha \circ \beta = Id$ and $\beta \circ \alpha = Id$. Consequently, $(-)^{\#}$ is a left adjoint of $H$.

2. Assume that $M \in R - gr$. Then $M = \oplus_{x\in G} M_x$. Since $p_x M^{\#} = M_x$ for any $x \in G$ we have that $H(M^{\#}) = M$ so $H \circ (-)^{\#} = Id_{R-\mathrm{gr}}$. Conversely if $N \in (R-\mathrm{gr})^{\#}$ then $N = N_0 = \oplus_{x\in G} p_x N$ so $N$ is a $G$-graded module with the grading $N = \oplus_{x\in G} N_x$ where $N_x = p_x N$. It is clear that $N^{\#} = N$, hence the functor $(-)^{\#} : R-\mathrm{gr} \longrightarrow (R-\mathrm{gr})^{\#}$ is an isomorphism of categories.

3. Follows from (2.). We have that $G$ is a finite group, in particular $\widetilde{R}\#G$ has a natural structure of a graded left $R$-module with $\widetilde{R}\#G = \oplus_{\sigma\in G}(\widetilde{R}\#G)_\sigma$ where $(\widetilde{R}\#G)_\sigma = p_\sigma(\widetilde{R}\#G) = \oplus_{xy=\sigma} \widetilde{R}_x p_y$. □

We may define another functor $(-)^{\#,\#} : R$-gr $\to \widetilde{R}\#G$-mod by $M^{\#,\#} = \prod_{x\in G} M_x$ for any $M = \oplus_{x\in G} M_x \in R - gr$, where $M^{\#,\#}$ has the following structure of an $\widetilde{R}\#G$-module: if $\widetilde{r}_\sigma \in \widetilde{R}_\sigma$, $r_\sigma \in R_\sigma$, $x \in G$ and $\overline{m} = (m_x)_{x\in G} \in \prod_{x\in G} M_x$ then put $\widetilde{r}_\sigma \overline{m} = \overline{n}$ where $\overline{n} = (n_y)_{y\in G}$, $n_y = r_\sigma m_{\sigma^{-1}y}$ and $p_x \overline{m} = \overline{m'} = (m'_y)_{y\in G}$, $m'_y = 0$ for $y \neq x$, $m'_x = m_x$. It is easy to see that $(M)^{\#,\#} = \prod_{x\in G} M_x$ is an $\widetilde{R}\#G$-module. It is also obvious that $(-)^{\#,\#}$ is an exact functor. Note that $(-)^{\#}$ is a subfunctor of $(-)^{\#,\#}$. Indeed for $M \in R$-gr, define the map $\alpha_M : M^{\#} \longrightarrow M^{\#,\#}$, $\alpha_M(m) = (m_x)_{x\in G}$ where $m = \sum_{x\in G} m_x$, $m_x \in M_x$ i.e. $\{m_x \mid x \in G\}$ are the homogeneous components of $m$. It is obvious that $\alpha_M$ is injective and it is also $\widetilde{R}\#G$-linear. When $G$ is a finite group we have $(-)^{\#} = (-)^{\#,\#}$. With notation as before we obtain :

### 7.3.3 Proposition

1. The functor $(-)^{\#,\#}$ is a right adjoint of the functor $H$.

2. If $M \in R$-gr and $\sup(M) < \infty$ then $\alpha_M : M^{\#} \longrightarrow M^{\#,\#}$ is an isomorphism.

**Proof**

1. We define the functorial morphisms :

$$\mathrm{Hom}_{R-\mathrm{gr}}(H(-),-) \xrightarrow{\gamma} \mathrm{Hom}_{\widetilde{R}\#G}(-,(-)^{\#,\#}) \text{ and}$$
$$\mathrm{Hom}_{\widetilde{R}\#G}(-,(-)^{\#,\#}) \xrightarrow{\delta} \mathrm{Hom}_{R-\mathrm{gr}}(H(-),-)$$

as follows :

If $M \in \widetilde{R}\#G$-mod, $N \in R$-gr then $\gamma(M,N) : \mathrm{Hom}_{R-\mathrm{gr}}(M_0,N) \to \mathrm{Hom}_{\widetilde{R}\#G}(M,N^{\#,\#})$ is defined by $\gamma(M,N)(u)(m) = (u(p_x m))_{x\in G}$ where $m \in M$, $u \in \mathrm{Hom}_{R-\mathrm{gr}}(M_0,N)$.

We show that $\gamma(M,N)(u) \in \mathrm{Hom}_{\widetilde{R}\#G}(M,N^{\#,\#})$. Indeed, if $\widetilde{r_\sigma} \in \widetilde{R}$, $r_\sigma \in R_\sigma$ we have :

$$\begin{aligned}
\gamma(M,N)(u)(\widetilde{r_\sigma m}) &= (u(p_x \cdot (\widetilde{r_\sigma} m)))_{x\in G} \\
&= (u(\widetilde{r_\sigma} p_{\sigma^{-1}x} m))_{x\in G} \\
&= (\widetilde{r_\sigma} u(p_{\sigma^{-1}x} \cdot m))_{x\in G} \\
&= \widetilde{r_\sigma}(u(p_x m))_{x\in G} \\
&= \widetilde{r_\sigma}\gamma(M,N)(u)(m)
\end{aligned}$$

(by the definition of the structure $N^{\#,\#}$).

We clearly have that $\gamma(M,N)(u)(p_y \cdot m) = p_y \gamma(M,N)(u)(m)$. Hence $\gamma(M,N)(u) \in \mathrm{Hom}_{\widetilde{R}\#G}(M,N^{\#,\#})$, and $\gamma(M,N)$ is a correctly defined functorial morphism. Now if $v \in \mathrm{Hom}_{\widetilde{R}\#G}(M,N^{\#,\#})$ we define $\delta(M,N)(v) \in \mathrm{Hom}_{R-gr}(M_0,N$ by the equality $\delta(M,N)(v)(m_0) = p_x v(m)$ where $m_0 = p_x m$ with $m \in M$. Note that $p_x v(m)$ is actually an element of $N$. Also $p_x v(m) \in N_x$ thus we obtain $\delta(M,N)(v)((M_0)_x) = \delta(M,N)(v)(p_x M) \subseteq N_x$. Hence $\delta(M,N)(v) \in \mathrm{Hom}_{R-\mathrm{gr}}(M_0,N)$. It is easy to see that $(\delta(M,N) \circ \gamma(M,N))(u) = u$ for any $u \in \mathrm{Hom}_{R-\mathrm{gr}}(M_0,N)$ so $(\delta(M,N) \circ \gamma(M,N)) = Id$. Let now $v \in \mathrm{Hom}_{\widetilde{R}\#G}(M,N^{\#,\#})$. If we put $(\gamma(M,N) \circ \delta(M,N))(v) = v'$ then we have, for any $m \in M$, $p_x v'(m) = p_x v(m)$. If we put $v(m) = (n_x)_{x\in G}$ and $v'(m) = (n'_x)_{x\in G}$, then we have that $n_x, n'_x \in M_x$ for any $x \in G$. Since $p_x v'(m) = p_x v(m)$, we obtain $n'_x = n_x$ for any $x \in G$ and therefore $v(m) = v'(m)$ so $v = v'$. Finally, we arrive at $\gamma(M,N) \circ \delta(M,N) = Id$.

2. Is obvious.

### 7.3.4 Corollary

The following assertions hold :

1. If $M \in R$-gr is gr-projective (resp. gr-flat) then $M^\#$ is projective (resp. flat) $\widetilde{R}\#G$-module.

2. If $M \in R$-gr is gr-injective and $\mid \sup(M) \mid < \infty$ then $M^{\#}$ is an injective $\widetilde{R}\#G$-module.

**Proof**

1. Since $(-)^{\#}$ is a left adjoint of $H$ and $H$ is an exact functor, then (cf. Appendix A) it follows from the gr-projective of $M$ that $M^{\#}$ is a projective left $\widetilde{R}\#G$-module.

2. Now if $M$ is left gr-flat, then $M$ is a direct limit of projective objects of the category $R$-gr. Since $(-)^{\#}$ commutes with direct limits (see Appendix A) we obtain that $M^{\#}$ is a flat $\widetilde{R}\#G$-module.

3. Since $(-)^{\#,\#}$ is a right adjoint of $H$ and $H$ is exact, (cf. Appendix A)it follows that the gr-injectivity of $M$ entails that $M^{\#,\#}$ is an injective $\widetilde{R}\#G$-module. On the other hand, since $M^{\#} \simeq M^{\#,\#}$ (because $\sup(M) < \infty$) we have that $M^{\#}$ is an injective $\widetilde{R}\#G$-module. □

### 7.3.5 Corollary

Assume that $Q \in R$-gr is gr-injective. If $\sup(Q) < \infty$, then $Q$ is injective in $R$-mod.

**Proof** Corollary 7.3.4 implies that $Q^{\#}$ is injective in $\widetilde{R}\#G$-mod. Since $\widetilde{R}\#G$ is free over the ring $\widetilde{R}$ it follows that $Q^{\#}$ is injective over $\widetilde{R}$ i.e. $Q$ is injective as a left $R$-module.

### 7.3.6 Remarks

1. The above result has already been proved by other arguments in Section 2.8.

2. Using Corollary 7.3.4 we obtain the following well-known result: if $P$ is gr-projective (resp. gr-flat) then $P$ is also projective (resp. flat) in $R$-mod.

### 7.3.7 Proposition

Let $M \in R - gr$ and $g \in G$. Then we have $M(g)^{\#} \simeq (\widetilde{R}\#G)\overline{g} \oplus_{\widetilde{R}\#G} M^{\#}$. In particular, we obtain that the inertial group $G\{M\}$ of $M$ in $R$-gr is equal to the inertial group $G\{M^{\#}\}$ with respect to the skew groupring $R\{G\} = (\widetilde{R}\#G)\overline{G}$.

**Proof** Since $(\widetilde{R}\#G)\overline{g} = \overline{g}(\widetilde{R}\#G)$, we have $(\widetilde{R}\#G)\overline{g} \oplus_{\widetilde{R}\#G} M^{\#} \simeq \overline{g} \oplus_{\widetilde{R}\#G} M^{\#}$. Define the map $\varphi : M(g)^{\#} \longrightarrow \overline{g} \oplus_{\widetilde{R}\#G} M^{\#}$ as follows: if $m_x \in M(g)_x$ then we put $\varphi(m_x) = \overline{g} \otimes m_x$. Since $\{\overline{g}\}$ is free over $\widetilde{R}\#G$, $\varphi$ is injective. It is obvious that $\varphi$ is surjective. So it is enough to prove that $\varphi$ is a morphism over the ring $\widetilde{R}\#G$. If $r_\sigma \in R_\sigma$ and $m_x \in M(g)$ then we have $\varphi(\widetilde{r_\sigma}) = \overline{g} \oplus \widetilde{r_\sigma} m = \overline{g}\widetilde{r_\sigma} \oplus m = \widetilde{r_\sigma}\overline{g} \otimes m = \widetilde{r_\sigma}\varphi(m)$. Since $m \in M(g)_x$, we have $p_y m = 0$ if $y \neq x$ and $p_y \cdot m = m$ if $y = x$. On the other hand $p_y\varphi(m) = p_y(\overline{g} \otimes m) = p_y\overline{g} \oplus m = \overline{g}p_{yg} \otimes m = \overline{g} \otimes p_{yg}m$. Since $m \in M_{xg}$ we have that $p_y\varphi(m) = 0$ if $y \neq x$ and $p_y\varphi(m) = \overline{g} \oplus m$ if $y = x$. Therefore $\varphi(p_y \cdot m) = p_y \cdot \varphi(m)$ i.e. $\varphi$ is an $\widetilde{R}\#G$-morphism. The last part of the proposition is clear. □

Let $\mathcal{C}$ be a closed (resp localizing) subcategory of $R - gr$. We denote by $\mathcal{C}^{\#} = \{\mathcal{M}^{\#} \mid \mathcal{M} \in \mathcal{C}\}$.

### 7.3.8 Proposition

The class $\mathcal{C}^{\#}$ is a closed (resp localizing) subcategory of $\widetilde{R}\#G$-mod. Moreover, if $\mathcal{C}$ is a rigid closed (resp rigid localizing) subcategory of $R$-gr then $\mathcal{C}^{\#}$ is a stable closed (resp stable localizing subcategory) of $\widetilde{R}\#G$-mod, with respect to the skew group ring $R\{G\}$. In particular $(R - gr)^{\#}$ is a stable localizing subcategory of $\widetilde{R}\#G$-mod.

**Proof** We apply the fact that the functor $(-)^{\#} : R - gr \to \widetilde{R}\#G$-mod is exact and Proposition 7.3.7. □

### 7.3.9 Proposition

Assume that $G$ is a finite group. Then the correspondence $\mathcal{C} \to \mathcal{C}^{\#}$ between the closed (resp localizing) subcategories of $R$-gr and the closed (resp localizing) subcategories of $\widetilde{R}\#G$-mod is bijective. Moreover, the above correspondence is bijective when considered between the rigid closed (resp rigid localizing) subcategories of $R$-gr and the stable closed (resp stable localizing) subcategories of $\widetilde{R}\#G$ with respect to the skew groupring $R\{G\}$.

For $M \in R$-mod we let by $\mathrm{Col}_G(M)$ be the set of all column matrices over $M$ with entries indexed by $G$ and only finitely many entries nonzero. Since the elements of $M_G(R)$ are both row and column finite, $\mathrm{Col}_G(M)$ is a left $M_G(R)$-module and hence a left $R\{G\}$-module. Hence we obtain the canonical exact functor

$$\mathrm{Col}_G(-) : R\text{−mod} \to R\{G\}\text{−mod}, M \to \mathrm{Col}_G(M)$$

### 7.3.10 Proposition

With notation as above, the following assertions hold :

1. If $\widetilde{m} \in \mathrm{Col}_G(M)$ and $A \in M_G(R)$, then there is a $B \in M_G^*(R)$ such that $A = B + C$ and $C\widetilde{m} = 0$.
2. Every $R\{G\}$ submodule of $\mathrm{Col}_G(M)$ has the form $\mathrm{Col}_G(N)$ where $N$ is an $R$-submodule of $M$.
3. The functor $\mathrm{Col}_G(-)$ is full and faithful.
4. For $M \in R - gr$, we have a canonical isomorphism of $R\{G\}$-modules :
$$\mathrm{Col}_G(M) \simeq R\{G\} \otimes_{\widetilde{R\#G}} M^{\#}$$
In particular we have that $\mathrm{Col}_G(M) \simeq \oplus_{\sigma \in G} M(\sigma)^{\#}$ as $\widetilde{R}\#G$-modules.
5. If $G$ is a finite group, then $\mathrm{Col}_G(-)$ is an equivalence of categories from $R$-mod to $R\{G\}$-mod.

**Proof**

1. We denote by $Y$ the set of elements $y \in G$ such that in the $y$ position of $\widetilde{m}$ there is a nonzero element i.e. $Y = \sup(\widetilde{m})$; $Y$ is a finite set. For $x \in G$ we denote by $A_x$ the "$x$-th" row of the matrix $A$ and by $Z_x = \{y \in G \mid A(x, y) \neq 0\}$. The column matrix may have a nonzero element at the $x$ position if $Z_x \cap Y \neq \emptyset$. It is easy to see that there are only finitely many elements $x \in G$ such that $Z_x \cap Y \neq \emptyset$, for otherwise there would exist a column $A$ with infinitely many nonzero entries, a contradiction. Let $U = \{x \in G \mid Z_x \cap Y \neq \emptyset\}$. Define the matrix $B$ as follows : if $x \in U$ then the $x$-th of $B$ is $A_x$; if $x \notin U$ then the entries of the $x$-th row of $B$ are all zero. We put $C = A - B$. It is clear that $B \in M_G^*(R)$, $A\widetilde{m} = B\widetilde{m}$ and $C\widetilde{m} = 0$.
2. Let $\widetilde{N}$ be an $R\{G\}$-submodule of $\mathrm{Col}_G(M)$. Since $R\{G\}$ contains $M_G^*(M)$, by assertion (1.) we have that $\widetilde{N}$ is also $M_G(R)$-submodule of $\mathrm{Col}_G(M)$. Put :
$$N = \{n \in M \mid \exists \widetilde{n} \in \widetilde{N}, \widetilde{n} = \begin{pmatrix} \dots \\ n \\ \dots \end{pmatrix} \leftarrow e \ \}$$
where $\widetilde{n}$ is a column with the property that the in the $e$-position we have the element $n$. It is easy to see that $N$ is an $R$-submodule of $M$. If
$$\widetilde{n} \in \widetilde{N}, \widetilde{n} = \begin{pmatrix} \dots \\ n_1 \\ \dots \\ n_2 \\ \dots \end{pmatrix} \begin{matrix} \\ \leftarrow x_1 \\ \\ \leftarrow x_2 \\ \end{matrix}$$

then we have $e_{e,x_i}\widetilde{n} \in \widetilde{N}$ and therefore $e_{e,x_i}\widetilde{n} \in N$. Also $\widetilde{n} = \sum_{i=1}^{s} e_{x_i,e} \cdot e_{e,x_i} \cdot \widetilde{n}$, so $\widetilde{n} \in \mathrm{Col}_G(N)$ and hence $\widetilde{N} \subseteq \mathrm{Col}_G(N)$. Since also $\mathrm{Col}_G(N) \subseteq \widetilde{N}$, we finally get $\mathrm{Col}_G(N) = \widetilde{N}$.

3. The map $f \longmapsto \mathrm{Col}_G(f)$ from $\mathrm{Hom}_R(M, M')$ to $\mathrm{Hom}_{R\{G\}}(\mathrm{Col}_G(M), \mathrm{Col}_G(M')$ is obviously injective. Let $\widetilde{f}$ be an $R\{G\}$-morphism from $\mathrm{Col}_G(M)$ to $\mathrm{Col}_G(M)$ to $\mathrm{Col}_G(M')$i. By assertion (1.) it follows that $\widetilde{f}$ is also an $M_G(R)$-morphism. Let $m \in M$ and consider the column matrix
$$\widetilde{m} = \begin{pmatrix} 0 \\ m \\ \cdots \end{pmatrix} \begin{matrix} \leftarrow x \\ \leftarrow e \end{matrix} \quad \text{where } x \neq e$$
Because $p_e\widetilde{m} = \widetilde{m}$ we have that $\widetilde{f}(\widetilde{m}) = \widetilde{f}(p_e\widetilde{m}) = p_e\widetilde{f}(\widetilde{m}) = \widetilde{m'}$ where
$$\widetilde{m'} = \begin{pmatrix} \cdots \\ m' \\ \cdots \end{pmatrix} \leftarrow e$$
$m' \in M'$ and we put $f(m) = m'$. It is easy to check that $f$ is an $R$-morphism and $\mathrm{Col}_G(f) = \widetilde{f}$. Thus the map $f \longmapsto \mathrm{Col}_G(f)$ is also surjective. Therefore the functor $\mathrm{Col}_G(-)$ is full and faithful.

4. If $M$ is a graded $R$-module, we put $M^* = \{v \in \mathrm{Col}_G(M) \mid v_x \in M_x\}$, where $v_x$ is the entry in the $x$-position of $v$. It is easy to see that $M^*$ is an $\widetilde{R\#G}$-submodule of $\mathrm{Col}_G(M)$, isomorphic to $M^{\#}$. Since $M = \oplus_{x\in G} M_x$, it is straightforward to verify that $\mathrm{Col}_G(M) = \sum_{g\in G} \overline{g}M^*$ and the sum $\sum_{g\in G}$ is direct. We obtain that $\mathrm{Col}_G(M)$ is isomorphic to $R\{G\} \otimes_{\widetilde{R\#G}} M^{\#}$. The last part of the assertion follows from Proposition 7.3.7.

5. By assertion (3.) it is enough to show that if $M$ is a left $M_G(R)$-module, there exists a left $R$-module $N$ such that $M \simeq \mathrm{Col}_G(N)$. We put $N = p_eM$. Since $R$ is a canonical subring of $M_G(R)$ then $N$ is an $R$-submodule of $M$. Now if $\widetilde{n} \in \mathrm{Col}_G(N)$ we denote by $n^x$ the element in the "$x$-th" position of the column $\widetilde{n}$. Then we define the map $\alpha : \mathrm{Col}_G(N) \longrightarrow M$ by $\alpha(\widetilde{n}) = \sum_{x\in G} e_{x,e}n^x$. Since $p_xe_{x,e} = e_{x,e}$ then $e_{x,e}n^x \in p_xM$. So if $\alpha(\widetilde{n}) = 0$, since $M = \oplus_{x\in G} p_xM$ we have $e_{x,e}n^x = 0$ for every $x \in G$. So $e_{e,x}e_{x,e}n^x = 0$ so $p_en^x = 0$. Since $n^x \in N = p_eM$, it follows that $n^x = 0$ for any $x \in G$ and therefore $\widetilde{n} = 0$. Hence $\alpha$ is injective. On the other hand if $m \in M$, we define the column $\widetilde{n}$ by
$$\widetilde{n} = \begin{pmatrix} \cdots \\ e_{e,x} \cdot m \\ \cdots \end{pmatrix} \leftarrow x$$

We have $e_{e,x}m = p_e e_{e,x}m$ so $\widetilde{n} \in \mathrm{Col}_G(N)$. But $\alpha(\widetilde{n}) = \sum_{x\in G} e_{x,e}e_{e,x}m = \sum_{x\in G} p_x m = m$ and hence $\alpha$ is an isomorphism. Since $\alpha(e_{u,v}\widetilde{n}) = e_{u,v}\alpha(\widetilde{n})$ for any $u, v \in G$, it follows that $\alpha$ is also an isomorphism of $M_G(R)$-modules.

## 7.4 Smash Product and Finiteness Conditions

We recall that a group $G$ is called **polycyclic-by-finite** if there exists a finite subnormal series $\{e\} = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ such that $G_{i-1} \lhd G_i$ and $G_i/G_{i-1}$ is a finite or infinite cyclic group for each $i, 1 \leq i \leq n$. The number of all infinite cyclic factor groups $G_i/G_{i-1}$ is called the Hirsch number $G$ and is denoted by $h(G)$ (it does not depend on the particular chosen series).

### 7.4.1 Theorem

Let $R$ be a $G$-graded ring where $G$ is a polycyclic-by-finite group. If $M \in R$-gr is gr-Noetherian, then $M$ is Noetherian as an $R$-module. Furthermore $K.\dim_R(M) \leq \mathrm{gr}.K.\dim_R(M) + h(G)$.

**Proof** We know $(R-\mathrm{gr})^{\#}$ is a localizing subcategory of $\widetilde{R}\#G$-mod then the results of Section 7.3 imply that $M^{\#}$ is Noetherian as left $\widetilde{R}\#G$-module and $\mathrm{gr}.K.\dim M = K.\dim_{\widetilde{R}\#G} M^{\#}$. As $R\{G\}$ is a skew groupring of the group $\overline{G}$ over the ring $\widetilde{R}\#G$, it is in particular a strongly graded ring with $(R\{G\})_e = \widetilde{R}\#G$. Theorem 5.4.8 yields that $R\{G\} \otimes_{\widetilde{R}\#G} M^{\#}$ is a Noetherian left $R\{G\}$-module. Moreover, from Theorem 5.5.7 it follows that $K.\dim_{R\{G\}}(R\{G\} \otimes_{\widetilde{R}\#G} M^{\#}) \leq K.\dim_{\widetilde{R}\#G} M^{\#} + h(G)$. Assertion 4., Proposition 7.3.10 yields that $\mathrm{Col}_G(M)$ is a Noetherian left $R\{G\}$ module and $K.\dim_{R\{G\}}\mathrm{Col}_G(M) \leq K.\dim_{\widetilde{R}\#G} M^{\#} + h(G)$. Finally assertion (2.) of Proposition 7.3.10 entails that $M$ is Noetherian in $R$-mod and $K.\dim_R M \leq K.\dim_{\widetilde{R}\#G} M^{\#} + h(G) = \mathrm{gr}.K.\dim(M) + h(G)$. The inequality $\mathrm{gr}.K.\dim(M) \leq K.\dim_R(M)$ is obvious. □

Let $N$ be a (not necessarily graded) $R$-submodule of $M$. We denote by $(N)_g$ (resp. $(N)^g$) the largest submodulle of $M$ contained in $N$ (resp. the smallest graded submodule of $M$ contaied in $N$) (see Section 2.1). As we have seen in Section 7.3, each $M \in R$-gr may be considered as an $\widetilde{R}\#G$-module. We have denoted this module by $M^{\#}$. In this case $N$ will be a subset of $M^{\#}$ such that $N$ is an $\widetilde{R}$-submodule of $M^{\#}$ by restriction of scalars ($\widetilde{R}$ is a subring of $\widetilde{R}\#G$).

### 7.4.2 Lemma

Assume that $G$ is a finite group. With notation as above :

1. $(N)_g = \bigcap_{x\in G}(N : p_x)$.

2. $(N)^g = (\widetilde{R}\#G)N = \oplus_{x\in G} p_x N$

**Proof**

1. Since $p_x\widetilde{r}_\sigma = \widetilde{r}_\sigma p_{\sigma^{-1}x}$, it follows that $\cap_{x\in G}(N : p_x)$ is an $\widetilde{R}\#G$-submodule of $M$ so $\cap_{x\in G}(N : p_x)$ is a graded submodule of $M$. Since $\sum_{x\in G} p_x = I$ is the identity of the ring $\widetilde{R}\#G$, we have that $\cap_{x\in G}(N : p_x) \subseteq N$ and thus $\cap_{x\in G}(N : p_x) \subseteq N$ and $\cap_{x\in G}(N : p_x) \subseteq (N)_g$. The converse inclusion $(N)_g \subseteq \cap_{x\in G}(N : p_x)$ is obvious.

2. Since $(\widetilde{R}\#G)N$ is a graded submodule of $M$ and $N \subset (\widetilde{R}\#G)N$, we get that $(\widetilde{R}\#G)N \supset (N)^g$. Now if $P$ is a graded submodule of $M$, such that $N \subset P$, we have $(\widetilde{R}\#G)N \subset (\widetilde{R}\#G)P = P$. Therefore $(\widetilde{R}\#G)N = (N)^g$. □

Assume that $M$ is in $R$-gr and $N$ is an $R$-submodule of $M$. We define the maps $\alpha : \mathrm{Col}_G(N) \to M^\#$ by $\alpha(\widetilde{n}) = \sum_{x\in G} p_x n^x$, where $\widetilde{n} \in \mathrm{Col}_G(N)$ and $n^x$ is the element of the column $\widetilde{n}$ in the $x$ position, $\beta : M^\# \to \mathrm{Col}_G(M/N)$ by $\beta(m) = (\widehat{p_x m})_{x\in G}$ i.e. $\beta(m)$ is the column having the element $p_x m$ modulo the submodule $N$ in the $x$ position.

## 7.4.3 Lemma

Assume that $G$ is a finite group. Then :

1. $\alpha$ is an $\widetilde{R}\#G$-morphism and $\mathrm{Im}\alpha = (N)^g$.

2. $\beta$ is an $\widetilde{R}\#G$-morphism and $\ker\beta = (N)_g$.

**Proof**

1. We must prove that $\alpha(\widetilde{r}_\sigma\widetilde{n}) = \widetilde{r}_\sigma \in R_\sigma, \sigma \in G$ and $\alpha(p_x\widetilde{n}) = p_x\alpha(\widetilde{n})$. Indeed if
$$\widetilde{n} = \begin{pmatrix} \cdots \\ n^y \\ \cdots \end{pmatrix} \leftarrow y$$
then we have
$$p_x\widetilde{n} = \begin{pmatrix} 0 \\ \cdots \\ n^x \\ 0 \\ \cdots \\ 0 \end{pmatrix} \leftarrow x$$

and therefore $\alpha(p_x\widetilde{n}) = p_x n^x$.

On the other hand, $p_x\alpha(\widetilde{n}) = p_x(\sum_{y\in G} p_y n^y) = p_x n^x$ so $\alpha(p_x\widetilde{n}) = p_x\alpha(\widetilde{n})$. Put $r = r_\sigma \in R_\sigma$; then $\widetilde{r} = \widetilde{r_\sigma} = \sum_{y\in G} r_\sigma e_{\sigma y,y} = r_\sigma \sum_{y\in G} e_{\sigma y,y}$. If we put $\widetilde{n}' = \widetilde{r}_\sigma\widetilde{n}$ then $\widetilde{n}'$ is the column matrix with the component $n'^x$ in the $x$-position where $n'^x = \sum_{u\in G} \widetilde{r_\sigma}(x,u)n^u = r_\sigma n^{\sigma^{-1}x}$. We have

$$\begin{aligned}\alpha(\widetilde{r_\sigma}\widetilde{n}) &= \sum_{x\in G} p_x(r_\sigma n^{\sigma^{-1}x}) = \sum_{x\in G} p_x\widetilde{r_\sigma}n^{\sigma^{-1}x} = \sum_{x\in G}(p_x\widetilde{r_\sigma})n^{\sigma^{-1}x}\\ &= \sum_{x\in G}\widetilde{r_\sigma}p_{\sigma^{-1}x}n^{\sigma^{-1}x} = \widetilde{r_\sigma}\sum_{x\in G} p_{\sigma^{-1}x}n^{\sigma^{-1}x} = \widetilde{r_\sigma}\alpha(n)\end{aligned}$$

Therefore $\alpha$ is an $\widetilde{R}\#G$-morphism. It is obvious that $\mathrm{Im}\alpha = (N)^g$.

2. We must show that $\beta(p_x m) = p_x\beta(m)$ and $\beta(\widetilde{r_\sigma}m) = \widetilde{r_\sigma}\beta(m)$ for any $m \in M, x \in G$ and $R_\sigma \in R_\sigma$. Indeed since $p_y(p_x m) = 0$ for any $y \neq x$, we have that

$$\beta(p_x m) = \begin{pmatrix} 0 \\ \dots \\ \widehat{p_x m} \\ 0 \\ \dots \\ 0 \end{pmatrix} \leftarrow x \quad = p_x\beta(m)$$

Now

$$\beta(\widetilde{r_\sigma}m) = \begin{pmatrix} \dots \\ \widehat{p_x\widetilde{r_\sigma}m} \\ \dots \end{pmatrix} \leftarrow x = \begin{pmatrix} \dots \\ \widehat{\widetilde{r_\sigma}r_{\sigma^{-1}x}m} \\ \dots \end{pmatrix} \leftarrow x =$$

$$= \widetilde{r_\sigma}\begin{pmatrix} \dots \\ \widehat{p_{\sigma^{-1}x}m} \\ \dots \end{pmatrix} \leftarrow x = \widetilde{r_\sigma}\beta(m)$$

Therefore $\beta$ is an $\widetilde{R}\#G$-morphism. Now $\beta(m) = 0$ if and only if $p_x m \in N$ for all $x \in G$ and therefore by Lemma 7.4.2 we have that $m \in \ker\beta$ if and only if $m \in \cap_{x\in G}(N : p_x) = (N)_g$. □

### 7.4.4 Theorem

Let $R = \oplus_{x\in G}R_x$ be a $G$-graded ring, $G$ a finite group. Let $M \in R$-gr and $N \subset M$ an $R$-submodule of $M$. Then the following assertions hold :

1. If $N$ is a Noetherian (resp. Artinian) submodule of $M$, then $(N)^g$ is a Noetherian (resp. Artinian) submodule of $M$.

2. If $N$ has Krull dimension then $(N)^g$ has Krull dimension. Moreover, $K.\dim_R(N) = K\dim_R(N)^g$.

3. If $N$ has Gabriel dimension, then $(N)^g$ has Gabriel dimension. Moreover $G.\dim_R(N) = G.\dim_R(N)^g$.

4. If $N$ is a simple submodule then $(N)^g$ is gr-semi-simple of finite length.

**Proof** Since $G$ is finite $R\{G\} = M_G(R)$. From Proposition 7.3.10 we may derive that the Noetherian property of $N$ (resp. Artinian, resp. has Krull dimension, resp. has Gabriel dimension, resp. is simple) it follows that $\mathrm{Col}_G(N)$ is a Noetherian $R\{G\}$-module (resp. Artinian, resp. has Krull dimension, resp[. has Gabriel dimension, resp. is simple). Now since $R\{G\}$ is a skew group ring over $R\#G$ for the finite group $\overline{G} \simeq G$, it follows that $\mathrm{Col}_G(N)$ is Noetherian (resp. Artinian, resp. has Krull dimension, resp. has Gabriel dimension, resp. is semi-simple of finite length) as an $\widetilde{R}\#G$-module. Now Lemma 7.4.3 and Proposition 7.3.2. imply that $(N)^g$ is gr-Noetherian (resp. gr-Artinian, resp. has gr. Krull dimansion, resp. has gr-Gabriel dimension, resp. is gr-semisimple of finite length). Moreover in case $N$ has Krull (Gabriel) dimension, we have that gr-$K$-$\dim(N)^g = K.\dim_{\widetilde{R}\#G}(N)^g \leq K.\dim_{\widetilde{R}\#G}(\mathrm{Col}_G(N)) = K.\dim_{R\{G\}}(\mathrm{Col}_G(N)) = K.\dim_R(N)$ (in a similar way, for the Gabriel dimensiuon we observe that gr-$G.\dim(N)^g \leq G.\dim_R(N)$). Since $G$ is a finite group, it follows from Corollary 5.4.3 and Corollary 5.5.4 that $N^g$ is Noetherian (resp. Artinian, resp. has Gabriel dimension) as $R$-module. Moreover, gr-$K.\dim(N)^g = K.\dim_R(N)^g$ (resp. gr-$G.\dim(N)^g = G.\dim_R(N)^g$) Hence $K.\dim(N)^g \leq K.\dim_R(N)$ (resp. $G.\dim_R(N)^g \leq G.\dim_R(N)$). Since $N \subset (N)^g$, then we also have $K.\dim_R(N) \leq K.\dim_R(N)^g$ (resp. $G.\dim_R(N) \leq G.\dim(N)^g$) so we obtain that $K.\dim_R N = K.\dim_R(N)^g$ (resp. $G.\dim_R(N) = G.\dim(N)^g$). □

### 7.4.5 Corollary

Assume that the group $G$ is finite. Then the following assertions hold :

1. If $M \in R$-mod is Noetherian (resp. Artinian), then $M$ is isomorphic to a submodule of a Noetherian (resp. Artinian) graded $R$-module.

2. If $M \in R$-mod has Krull dimension (resp. Gabriel dimension) then $M$ is isomorphic to a submodule of a graded $R$-module having the same Krull (resp. Gabriel) dimension.

3. If $M$ is a simple $R$-module, then $M$ is isomorphic to a submodule of a gr-simple module.

**Proof** Corollary 2.5.5. entails that $M$ is isomorphic to an $R$-submodule of a graded $R$-module. Now we apply Theorem 7.4.4.

From Lemma 7.4.3, assertion (2.) we obtain the dual of Theorem 7.4.4.

□

### 7.4.6 Theorem

Let $R = \oplus_{x\in G} R_x$ be a $G$-graded ring, where $G$ is a finite group. If $M$is a graded $R$-module and $N \subset M$ is a submodule, then the following assertions hold :

1. If $M/N$ is a Noetherian (resp. Artinian) module, then so is $M/(N)_g$.
2. If $M/N$ has Krull dimension, then so does $M/(N)_g$ and we have :
$$K.\dim_R(M/N) = K.\dim_R(M/(N)_g)$$
3. If $M/N$ has Gabriel dimension, then so does $M/(N)_g$ and we have :
$$G.\dim_R(M/N) = G.\dim_R(M/(N)_g)$$
4. If $M/N$ is a simple $R$-module, then $M/(N)_g$ is gr-semi-simple of finite length.

The foregoing result may be used to derive a result similar to Corollary 7.4.5.

### 7.4.7 Corollary

Assume that the group $G$ is finite.

1. If $M \in R$-mod is Noetherian (resp. Artinian), then $M$ is isomorphic to a quotient of a graded Noetherian (resp. Artinian) $R$-module.
2. If $M \in R$-mod has Krull dimension (resp. Gabriel dimension) then $M$ is isomorphic to a quotient graded $R$-module having the same Krull (resp. Gabriel) dimension.
3. If $M$ is a simple $R$-module, then $M$ is isomorphic to a quotient of a gr-simple module.

It is possible to extend the above results to $G$-graded rings of finite support where $G$ may be an infinite group, but having the so-called "finite embedding property". A group $G$ is an **FE-group** (i.e. has the "finite embedding" property) if for every finite subset $X$ of $G$, there is a finite group $(H, *)$ such that $X \subset H$ and for every $x, y \in X$ such that $xy \in X$ we have $x * y = xy$. Recall that a group is called **residually-finite** if the intersection if its normal subgroups of finite index reduces to $\{e\}$; $G$ is called locally residually finite if every finitely generated subgroup of $G$ is residually finite. Some examples of locally residually finite groups are : abelian groups, polycyclic-by-finite groups, nilpotent groups, solvable groups, free groups (see the book of D.I.S. Robinson "A Course in the Theory of Groups" [176]).

### 7.4.8 Proposition

A locally residually finite group is an $FE$-group.

**Proof** Let $X$ be a finite set contained in $G$. Replacing $G$ by the group $< X >$ generated by $X$ we may assume that $G$ is residually finite. For any $z \in Y = \{xy^{-1}, x, y \in X \text{ and } x \neq y\}$ there is a normal subgroup $N_z$ of $G$ such that $N_z$ has finite index and $z \notin N_z$. So we obtain $N = \cap_{z \in Y} N_z$, a normal subgroup of finite index in $G$ for which $N \cap Y = \emptyset$. Hence the finite group $H = G/N$ satisfies the conditions, up to identifying $X$ to its image in $H$. □

Now let $G$ be an FE-group, $R = \oplus_{g \in G} R_g$ a $G$-graded ring of finite support $X$ and $M = \oplus_{g \in G} M_g$ be a left graded $R$-module of finite support $Y$. In view of Proposition 7.4.2, there is a finite group $(H, *)$ such that $X \cup Y \subseteq H$ and for any $u, v \in X \cup Y$ such that $uv \in X \cup Y$ we have $u * v = uv$. Then we may view $R$ (resp. $M$) as an $H$-graded ring (resp. $M$ as an $H$-graded module) by putting $R_h = 0$ (resp. $M_h = 0$), when $h \notin X \cup Y$ and for $g \in X \cup Y, R_g$ (resp. $M_g$) is the homogeneous part of degree $g$. In this case $M$ is a $H$-graded module over the $H$-graded ring $R$. Also if $N$ is an $R$-submodule of $M$, then $N$ is a $G$-graded submodule if and only if $N$ is an $H$-graded submodule of $M$. The following remarks follow from these facts :

### 7.4.9 Remarks

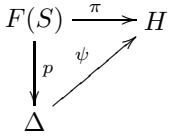
1. Consider an arbitrary $G$-graded ring $R$ where $G$ is an FE-group. If $M \in R$-gr has finite support then the assertions from Theorems 7.4.4 and 7.4.6 hold again. Indeed if $X = \sup(M)$, we put $J = R(\sum_{g \in XX^{-1}} R_g)R$ (here $X^{-1} = \{x^{-1} | x \in X\}$). Clearly, $J$ is a graded ideal and $JM = 0$ Moreover, the $G$-graded ring $S = R/J$ has finite support (in fact $\sup(S) \subseteq XX^{-1}$). (1.) now follows from the above considerations and the fact that $M$ is an $S$-graded module.

2. Assume that $\sup(R) < \infty$ and $G$ is an FE-group. If $M \in R$-mod, we may embed $M$ in the graded $R$-module $\mathrm{Coind}_R(M)$, which is exactly $\mathrm{Hom}_{R_e}(R, M)$. We recall that this embedding is given by the $R$-linear map $\varphi : M \longrightarrow \mathrm{Coind}_R(M)$, $\varphi(m)(r) = rm$ for any $m \in M$, $r \in R$. We also have that $\mathrm{Coind}_R(M)$ is a graded $R$-module of finite support. Then the assertions in Corollaries 7.4.5 and 7.4.7 also hold in this case.

### 7.4.10 Example

There are non-FE-groups ! Let $\Delta$ be an infinite simple finitely presented group (see [159]). If $\Delta$ is presented as $< S, R >$ with finite $S$ and $R$ then $\Delta = F(S)/R^{F(S)}$, where $F(S)$ is the free group generated by $S$ and $R^{F(S)}$ is the normalizer of the set $R$ in $F(S)$. Let $p : F(S) \longrightarrow \Delta$ be the canonical projection and put $X = \{p(\alpha) \mid \alpha \in S \text{ or } \alpha \text{ is a subword of an element of } R\}$. Suppose that $X$ is contained in a finite group $(H, *)$ such that $x * y = xy$ for every $x, y \in X$ for which $xy \in X$. If $\pi : F(S) \longrightarrow H$ is the natural group morphism then $R \subset \ker \pi$ and therefore $R^{F(S)} \subset \ker \pi$. Consequently, we

come up to the following commutative diagram :




Hence $\mathrm{Im}\psi \simeq \Delta/\mathrm{ker}\psi$ and since $S \subset \mathrm{Im}\psi$ we have $\mathrm{ker}\psi \neq \Delta$. Therefore $\mathrm{ker}\psi = e$ and $\mathrm{Im}\psi \simeq \Delta$, contradicting the finiteness of $H$.

## 7.5 Prime Ideals of Smash Products

Assume that $R = \oplus_{x\in G} R_x$ is a $G$-graded ring, where $G$ is a finite group. In this case we have the inclusions $\widetilde{R} \subset \widetilde{R}\#G \subset R\{G\} = M_G(R)$ where $R\{G\}$ is a skew group ring of the group $\overline{G}$ over the ring $\widetilde{R}\#G$. We recall (from Section 7.1) that there is an action of the group $G$ on $\widetilde{R}\#G$ given by $\varphi : G \longrightarrow \mathrm{Aut}(\widetilde{R}\#G), \varphi(g)(a) = \overline{g}^{-1}a\overline{g}$ where $g \in G$ and $a \in \widetilde{R}\#G$. In particular we have $\overline{g}^{-1}p_x\overline{g} = p_{xg}$. Also $(\widetilde{R}\#G)^G = \widetilde{R}$. If $I$ is a two sided ideal of $\widetilde{R}\#G$ we write $I^g = \overline{g}^{-1}I\overline{g}$ for any $g \in G$. $I^g$ is called the **$g$-conjugate of $I$**. From Section 2.11 we retain that $I^g = (\widetilde{R}\#G)\overline{g}^{-1}I \cdot (\widetilde{R}\#G)\overline{g}$ so $I^g$ is $g$-conjugate as in Section 2.11. If we have $I = I^g$ for any $g \in G$, $I$ is called $G$-invariant.

### 7.5.1 Example

Let $A$ be a graded ideal of $R$. Then the ideal of $\widetilde{R}\#G$ generated by $\widetilde{A}$ is $\widetilde{A}\#G$. Moreover we have that $\widetilde{A}\#G$ is $G$-invariant, $\widetilde{R} \cap (\widetilde{A}\#G) = \widetilde{A}$ and $\widetilde{R/A}\#G \simeq \widetilde{R}\#G/\widetilde{A}\#G$.

### 7.5.2 Proposition

Let $I$ be an ideal of $\widetilde{R}\#G$. Then :

1. $I \cap \widetilde{R} = (\cap_{g\in G} I^g) \cap \widetilde{R}$.
2. $I \cap \widetilde{R} = \widetilde{A}$ where $A$ is a graded ideal of $R$.
3. If $I$ is $G$-invariant then $I = \widetilde{A}\#G$.

**Proof**

1. The assertion of (1.) clearly follows by viewing $\widetilde{R}$ as the set of fixed elements in $\widetilde{R}\#G$.

2. It is clear that there is an ideal $A$ of $R$ such that $I \cap \widetilde{R} = \widetilde{A}$. We now prove that $A$ is a graded ideal. If $a \in A$ then we have $p_x \widetilde{a} p_y = \widetilde{a}_{xy^{-1}} p_y \in I$ for any $x, y \in G$ and $a = \sum_{z \in G} a_z$, $a_z \in R_z$. Since $x$ is an arbitrary element we have $\widetilde{a_g} p_y \in I$ for any $g, y \in G$ so $\sum_{y \in G} \widetilde{a_g} p_y \in I$ so $\widetilde{a_g} \in I \cap \widetilde{R} = \widetilde{A}$. So $a_g \in A$ and hence $A$ is a graded ideal.

3. The inclusion $\widetilde{A} \# G \subset I$ is obvious. Let $a = \sum_{x \in G} \widetilde{a^x} p_x$, where $a \in I$ and $a^x \in R$ for any $x \in G$. Since $I$ is an ideal we have $a p_y = \widetilde{a^x} p_y \in I$. Since $I$ is $G$-invariant we have $\overline{g}^{-1} \widetilde{a^x} p_y \overline{g} \in I$ for any $g \in G$ hence $\widetilde{a^x} p_{yg} \in I$ for any $g \in G$. Hence $\sum_{g \in G} a^x p_{yg} = \widetilde{a^x} \in I$. Hence $a^x \in A$ so $I \subseteq \widetilde{A} \# G$. □

In Section 2.11 it has been established that the Jacobson radical $J(\widetilde{R} \# G)$ and the prime radical $rad(\widetilde{R} \# G)$ are $G$-invariant. The following assertions provide a further characterization of these ideals.

### 7.5.3 Theorem

1. $J(R \# G) \cap \widetilde{R} = \widetilde{J^g(R)}$ where $J^g(R)$ is the graded Jacobson radical of $R$.
2. $J(R \# G) = \widetilde{J^g(R)} \# G$.
3. $J^g(R) \subset J(R)$.

**Proof**

1. Since the functor $(-)^{\#} : R - gr \to R \# G$-mod is an isomorphism of categories (see Proposition 7.3.2 (3.)) the simple $R \# G$-modules are all $\Sigma^{\#}$ where $\Sigma$ is a gr-simple module. Now (1.) follows easily because of $J(\widetilde{R} \# G) = \cap_{\Sigma} \mathrm{Ann}_{\widetilde{R} \# G}(\Sigma^{\#})$ and $J^g(R) = \cap_{\Sigma} \mathrm{Ann}_R(\Sigma)$.
2. Apply Proposition 7.5.2
3. This result has been proved in the more general case when $R$ has finite support (cf. Section 2.9). We include a different proof here, stemming from ideas of M. Cohen and S. Montgomery, cf. [43]. Let $a \in J^g(R)$. By assertion (1.), $1 - a$ is invertible in $J(R \# G)$ hence there is a $b = \sum_{x \in G} \widetilde{b^x} p_x$ such that $b^x \in R$ for any $x \in G$ and $(1 - \widetilde{a})b = b(1 - \widetilde{a}) = 1$. Therefore $\sum (1 - a) b^x p_x = \sum b^x \widetilde{(1 - a)} p_x = 1 = \widetilde{1} = \sum_{x \in G} \widetilde{p_x}$. Hence $\widetilde{(1 - a)} \widetilde{b^x} = \widetilde{b^x} \widetilde{(1 - a)} = 1$ for any $x \in G$. Consequently $(1 - a)$ is invertible also in $\widetilde{R}$ and therefore $1 - a$ is invertible in $R$, whence $J^g(R) \subset J(R)$. □

We recall that a graded ring $R$ is called graded semiprime (see Section 2.11) if $R$ has no nonzero nilpotent graded ideals.

### 7.5.4 Proposition

The following assertions are equivalent.

1. $\widetilde{R}\#G$ is semi-prime.
2. $R$ is graded semi-prime.
3. $R_e$ is semi-prime and $R$ is $e$-faithful.

**Proof**

(1.) $\Longrightarrow$ (2.) Let $I$ be a nilpotent graded ideal such that $I^2 = 0$. It is clear that $(\widetilde{I}\#G)^2 = 0$ and therefore, by hypothesis we have $\widetilde{I}\#G = 0$ so $\widetilde{I} = 0$ and hence $I = 0$.

(2.) $\Longrightarrow$ (3.) follows from Theorem 2.11.4.

(3.) $\Longrightarrow$ (1.) Assume that $\widetilde{R}\#G$ is not semiprime; then there is $0 \neq z \in \widetilde{R}\#G$ such that $z(\widetilde{R}\#G)z = 0$. Write $z = \sum_{x\in G} \widetilde{a^x}p_x$, $a^x \in R$. There is a $x \in G$ such that $a^x \neq 0$. Hence $zp_x = \widetilde{a^x}p_x \neq 0$. Then $\widetilde{a^x}p_x(\widetilde{R}\#G)\widetilde{a^x}p_x = 0$. In particular we have that $\widetilde{a^x}p_x(\widetilde{R}\#G)\widetilde{a^x}p_x = 0$, so $\widetilde{a^x}(p_x\widetilde{Ra^x}p_x) = 0$. But $p_x\widetilde{Ra^x}p_x = (\widetilde{Ra^x})_e p_x$ hence $a^x(\widetilde{Ra^x})_e = 0$, therefore $(\widetilde{Ra^x})_e^2 = 0$. Since $R_e$ is semiprime we have $(\widetilde{Ra^x})_e = 0$, and $(Ra^x)_e = 0$. Since $R$ is $e$-faithful we have $Ra^x = 0$ and thus $a^x = 0$. Hence $z = 0$, a contradiction. □

### 7.5.5 Theorem

$\mathrm{rad}(\widetilde{R}\#G) = \mathrm{rad}^g(R)\#G$ where $\mathrm{rad}^g(R)$ is the graded prime radical of $R$.

**Proof** Proposition 7.5.2 entails that $\mathrm{rad}(\widetilde{R}\#G) = \widetilde{A}\#G$ where $A$ is a graded ideal of $R$. Since $\widetilde{R}\#G/\mathrm{rad}(\widetilde{R}\#G) \simeq \widetilde{R/A}\#G$, it follows that $\widetilde{R/A}\#G$ is semiprime and Proposition entails that 7.5.4, $R/A$ is graded as well as a semi-prime ring. Hence $\mathrm{rad}^g(R) \subseteq A$. Since $R/\mathrm{rad}^g(R)$ is graded semiprime, we obtain that $R/\mathrm{rad}^g(R)\#G$ is also semi-prime and $\widetilde{R}\#G/\widetilde{\mathrm{rad}^g(R)}\#G$ is semiprime. Therefore we have $\mathrm{rad}(\widetilde{R}\#G) \subseteq \widetilde{\mathrm{rad}^g(R)}\#G$ thus $\widetilde{A}\#G \subseteq \widetilde{\mathrm{rad}^g(R)}\#G$ thus $\widetilde{A} \subseteq \widetilde{\mathrm{rad}^g(R)}$ and $A = \mathrm{rad}^g(R)$. □

### 7.5.6 Theorem

The following assertions hold :

1. If $P$ is a $G$-prime (and $G$-invariant) ideal of $\widetilde{R}\#G$ then $P \cap \widetilde{R} = \widetilde{Q}$, where $Q$ is a graded prime ideal of $R$. In particular it follows that if $P$ is a prime ideal of $\widetilde{R}\#G$ then $P \cap \widetilde{R} = \widetilde{Q}$, where $Q$ is a gr-prime ideal.

2. Conversely if $Q$ is a gr-prime ideal of $R$ then $\widetilde{Q}\#G$ is a $G$-prime ideal of $\widetilde{R}\#G$. Moreover the correspondence $Q \to \widetilde{Q}\#G$ between the set of all gr-prime ideals of $R$ and the set of all $G$-prime ideals of $\widetilde{R}\#G$ is bijective.

3. If $Q$ is a gr-prime ideal, then there is a prime ideal $P$ of $\widetilde{R}\#G$ such that $\widetilde{Q} = P \cap \widetilde{R}$.

4. If $P$ is a $G$-prime ($G$-invariant) ideal of $\widetilde{R}\#G$ then there is a prime ideal $Q$ of $R$ such that $P = \widetilde{(Q)_g}\#G$.

**Proof**

1. Let $I, J$ be two graded ideals of $R$ such that $IJ \subset Q$. Then $\widetilde{I}\widetilde{J} \subset \widetilde{Q}$ and therefore $(\widetilde{I}\#G)(\widetilde{J}\#G) \subseteq \widetilde{Q}\#G = P$ so $\widetilde{I}\#G \subset \widetilde{Q}\#G$ or $\widetilde{J}\#G \subseteq \widetilde{Q}\#G$. Thus $I \subseteq Q$ or $J \subseteq Q$ and therefore $Q$ is a gr-prime ideal. Assume now that $P$ is a prime ideal of $\widetilde{R}\#G$. Then $\cap_{g\in G}P^g$ is a $G$-prime ideal of $\widetilde{R}\#G$. Now by Proposition 7.5.2 (1.) we obtain that $P \cap \widetilde{R} = \widetilde{Q}$, where $Q$ is a gr-prime ideal of $R$.

2. Similar argumentaction establishes that $\widetilde{Q}\#G$ is a $G$-prime ideal of $\widetilde{R}\#G$.

3. Theorem 2.11.12 and Corollary 2.11.9 yield that there is a prime ideal $P$ of $\widetilde{R}\#G$ such that $\widetilde{Q}\#G = \cap_{\sigma\in G}P^\sigma$. Moreover, $P$ is minimal over $\widetilde{Q}\#G$. But $\widetilde{R} \cap (\widetilde{Q}\#G) = \widetilde{R} \cap (\cap_{\sigma\in G}P^\sigma)$. By Proposition 7.5.2 we have that $\widetilde{Q} = P \cap \widetilde{R}$.

4. Since $P \cap \widetilde{R}$ is a gr-prime ideal, cf. Proposition 2.11 there is a prime ideal $Q$ of $R$ such that $P \cap \widetilde{R} = \widetilde{(Q)_g}$. Thus $P = (P \cap \widetilde{R})\#G = \widetilde{(Q)_g}\#G$.

□

Following considerations depend on two results of M. Lorenz and D. Passman cf. "Prime Ideals in Crossed Products of finite groups", Israel Journal of Mathematics vol. 33, nr.2 (1979), called Theorem A and Theorem B. (see [123])

**Theorem A** Assume that $R = \oplus_{g\in G}R_g$ is a $G$-crossed product with respect to a finite group $G$. Let $I$ be a ideal of $R$ strictly containing a prime ideal $P$ then$P \cap R_e \neq I \cap R_e$.

**Theorem B** Assume that $R = \oplus_{g\in G}R_g$ is a $G$-crossed product where $G$ is finite. If $R_e$ is a $G$-prime ring, the following assertions hold :

1. A prime ideal $P$ of $R$ is minimal if and only if $P \cap R_e = 0$.

2. There are finitely many such minimal primes, $P_1, ..., P_n$ where $n \leq \mid G \mid$.

3. If we put $I = P_1 \cap ... \cap P_n$, then $I$ is the unique largest nilpotent ideal of $R$ such that $I^{|G|} = 0$.

From Theorem A, we obtain the following result for arbitrary $G$-graded rings.

### 7.5.7 Theorem

Let $R = \oplus_{g\in G} R_g$ be a $G$-graded ring where $G$ is a finite group. If $P \subseteq I$ and $P \neq I$ are two sided ideals of $R$ where $P$ is a prime ideal then $P \cap R_e \neq I \cap R_e$.

**Proof** Since $G$ is finite we have $M_G(P) \neq M_G(I)$ inside the matrix ring $M_G(R)$. Since $(\widetilde{R}\#G)\overline{G} = M_G(R)$ and $(\widetilde{R}\#G)\overline{G}$ is a skew groupring over the ring $\widetilde{R}\#G$ by $\overline{G} \simeq G$, then by Theorem A we have

$$(\widetilde{R}\#G) \cap M_G(P) \neq (\widetilde{R}\#G) \cap M_G(I)$$

Since the ideals $(\widetilde{R}\#G)\cap M_G(P)$ and $(\widetilde{R}\#G)\cap M_G(I)$ are $G$-invariants Proposition 7.5.2 yields that $\widetilde{R} \cap M_G(P) \neq \widetilde{R} \cap M_G(I)$. On the other hand $\widetilde{R} \cap M_G(I) = \widetilde{I_g}$. Indeed we clearly have $\widetilde{(I)_g} = \eta((I)_g) \subseteq \widetilde{R} \cap M_G(I)$. Conversely, if $\widetilde{r} = \eta(r) \in M_G(I)$, since $\widetilde{r} = \sum\limits_{x,y\in G} r_{xy^{-1}} e_{x,y}$ it follows that $r_{xy^{-1}} \in I$ for any $x, y \in G$ and therefore $r \in (I)_g$. Hence $\widetilde{R} \cap M_G(I) \subseteq \widetilde{(I)_g}$ and so $(P)_g \neq (I)_g$. Now consider the $G$-graded ring $S = R/(P)_g$. Since $(P)_g$ is a gr-prime ideal then $S$ is a gr-prime ring. Theorem 2.11.4 entails that $S$ is $e$-faithful. If we put $J = (I)_g/(P)_g$ then $J \neq 0$ and $J$ is a graded ideal of $S$, therefore $J \cap S_e \neq 0$. Since $S_e = R_e/P \cap R_e$ we obtain that $P \cap R_e \neq I \cap R_e$. □

### 7.5.8 Theorem (Cohen and Montgomery [43])

Let $R = \oplus_{g\in G} R_g$ be a $G$-graded ring, where $G$ is a finite group and $P$ a gr-prime ideal of $R$. Then the following assertions hold :

1. A prime ideal $Q$ of $R$ is minimal over $P$ if and only if $(Q)_g = P$.
2. There are finitely many such minimal prime ideals, say $Q_1, ..., Q_n$ where $n \leq \mid G \mid$.
3. If we put $I = Q_1 \cap ... \cap Q_n$, then $I^{|G|} \subseteq P$.

**Proof** By passing to the graded ring $R/P$ we may assume that $R$ is a gr-prime ring and $P = 0$. From Theorem 7.5.6 it follows that the ring $\widetilde{R}\#G$ is $G$-prime. A prime ideal $Q$ of $R$ is minimal if and only if $M_G(Q)$ is a minimal prime ideal in $M_G(R)$ and from Theorem B it follows that $Q$ is minimal if and only if $M_G(Q) \cap (\widetilde{R}\#G) = 0$. By Proposition 7.5.2 yields that we have that $M_G(Q) \cap (\widetilde{R}\#G) = 0$ if and only if $(Q)_g = M_G(Q) \cap \widetilde{R} =$

0. Thus $Q$ is a minimal prime ideal if and only if $(Q)_g = 0$ so we obtain assertion (1.). Assertion (2.) follows from Theorem B (assertion (2.)) and the above assertion (1.). If we put $I = Q_1 \cap ... \cap Q_n$ $(n \leq \mid G \mid)$ since $M_G(I) = M_G(Q_1) \cap ... \cap M_G(Q_n)$, then by assertion (3.) of theorem **B** we have $M_G(I)^{|G|} = 0$ so $M_G(I^{|G|}) = 0$ and therefore $I^{|G|} = 0$.

The smash product construction allows to connect the prime ideals of $R$ and the prime ideals of $R_e$. The following lemma will prove to be very important.

### 7.5.9 Lemma (see [123])

Let $R$ be a ring and $e \in R$ a nonzero idempotent of $R$. Let $\mathrm{Spec}_e(R)$ denote the set of primitive ideals of $R$ not containing $e$ and let $\mathrm{Spec}(eRe)$ be the set of primes of $eRe$. Then the map $\varphi : \mathrm{Spec}_e(R) \longrightarrow \mathrm{Spec}(eRe)$ defined by $\varphi(P) = P \cap eRe$ is bijective. Moreover if $P, Q \in \mathrm{Spec}_e(R)$ then $P \underset{\neq}{\subset} Q$ if and only if $\varphi(P) \underset{\neq}{\subset} \varphi(Q)$ and $P$ is primitive if and only if $\varphi(P)$ is primitive.

**Proof** First we prove that if $P \in \mathrm{Spec}_e(R)$ then $P \cap eRe$ is a prime ideal of $eRe$. Indeed if $a, b \in eRe$ such that $a, b \notin P$, we have $aRb \not\subseteq P$. Since $a = eae$ and $b = ebe$, then $aRb = eaeRebe = a(eRe)b$, so $a(eRe)b \not\subseteq P \cap eRe$. Therefore $P \cap eRe$ is a prime ideal of the ring $eRe$. Let $P, Q \in \mathrm{Spec}_e(R)$ such that $P \underset{\neq}{\subset} Q$. It is clear that $\varphi(P) \subset \varphi(Q)$. There is a $q \in Q$, $q \notin P$. Since $e \notin P$, $eRq \not\subseteq P$ entails that there is a $\lambda \in R$ such that $e\lambda q \notin P$. Hence $e\lambda qRe \not\subseteq P$ and there exists a $\mu \in R$ such that $e\lambda q\mu e \notin P \cap eRe$, or $\varphi(R) \underset{\neq}{\subset} \varphi(Q)$.

Conversely if $\varphi(P) \underset{\neq}{\subset} \varphi(Q)$, the argument above also yields that $P \underset{\neq}{\subset} Q$ and therefore $P \subset Q$. We now prove that $\varphi$ is injective. Assume that $\varphi(P) = \varphi(Q)$ with $P \neq Q$ and say $Q \not\subset P$. Using the argument above, we find a $q \in Q$, $q \notin P$ and the elements $\lambda, \mu \in R$ such that $e(\lambda q\mu)e \in Q \cap eRe = \varphi(Q)$ and $e(\lambda q\mu)e \notin P \cap eRe = \varphi(P)$, contradiction. We now prove that $\varphi$ is surjective. Assume that $Q \in \mathrm{Spec}(eRe)$. Since $(RQR) \cap (eRe) = Q$, using Zorn's Lemma we obtain an ideal $P$ of $R$ which is maximal with respect to the property that $P \cap (eRe) = Q$. Now we show that $P$ is prime. Let $I, J$ be ideals such that $I \not\subseteq P$, $J \not\subseteq P$ and $IJ \subset P$. Then if we put $A = (P + I) \cap (eRe)$, $B = (P + J) \subseteq (eRe)$ then we have $Q \underset{\neq}{\subset} A$ $Q \underset{\neq}{\subset} B$, so $AB \not\subseteq Q$. But $AB \subset P \cap eRe = Q$, a contradiction. Assume now that $P$ is primitive. There is a left simple $R$-module $S$ such that $P = \mathrm{Ann}_R(S)$. Since $e \notin P$, then $eS \neq 0$. If $ex \in eS$, $ex \neq 0$ then $R(ex) = S$ so $(eR)(sx) = eS$ and hence $(eRe)(ex) = eS$ and therefore $eS$ is a left $eRe$ simple module. It is obvious that $\varphi(P) = P \cap eRe = \mathrm{Ann}_{eRe}(eS)$ so $\varphi(P)$ is primitive. Conversely, let $Q$

be a primitive ideal of $eRe$ and let $T$ be a left simple $eRe$-module such that $Q = \mathrm{Ann}_{eRe}(T)$. Write $T \simeq eRe/X$ for some maximal left ideal $X$ of $eRe$. Since $R(1-e) \cap RX = 0$ we denote by $Y = RX \oplus R(1-e)$. If $Y = R$ then $1 = \sum_{i=1}^{n} \lambda_i x_i + \mu(1-e)$ for some elements $\lambda_i \in R, x_i \in X, (1 \leq i \leq n)$ and $\mu \in R$. So $e = \sum_{i=1}^{n} \lambda_i x_i e$. Since $x_i \in X \subseteq eRe$ then $ex_i = x_i e = ex_i e$ for any $1 \leq i \leq n$. So $e = e^2 = \sum_{i=1}^{n} e\lambda_i x_i e = \sum_{i=1}^{n} e\lambda_i e_i x_i \in X$. Hence $X = eRe$, a contradiction. Therefore $Y \neq R$ and denote by $M$ a maximal left ideal of $R$ such that $Y = M$. Let $S = R/M$; then $S$ is a simple left $R$-module. If $P = \mathrm{Ann}_R(S)$ then $P$ is a primitive ideal of $R$. Since $eRe + M/M \leq R/M$ and $eRe + M/M \simeq eRe/eRe \cap M \simeq T$, it follows that $P \cap (eRe) \subseteq \mathrm{Ann}_{eRe}(T) = Q$. Now let us prove the reverse inclusion. Since $Q$ is an ideal of $eRe$ we have $QRe = QeRe \subset X \subset Y \subseteq M$. Also $QR(1-e) \subseteq R(1-e) \subset M$. Hence $QR \subset M$ and so $RQR \subseteq RM = M$. Therefore $RQR \subseteq \mathrm{Ann}_R(S) = P$. Since $Q = RQR \cap eRe$ then $Q \subseteq P \cap eRe$. So $Q = \varphi(P)$. □

### 7.5.10 Theorem (Cohen and Montgomery [43])

Let $R = \oplus_{g \in G} R_g$ be a $G$-graded ring where $G$ is a finite group.

1. If $P$ is any prime ideal of $R$, then there are $k \leq \mid G \mid$ prime ideals $p_1, ..., p_k$ of $R_e$ minimal over $P \cap R_e$ and moreover $P \cap R_e = p_1 \cap ... \cap p_k$. The set $\{p_1, ..., p_k\}$ is uniquely determined by $P$.

2. Conversely if $p$ is a prime ideal of $R_e$, there exists a prime $P$ of $R$ such that $p$ is minimal over $P \cap R_e$. There are at most $m \leq \mid G \mid$ such primes $P_1, ..., P_m$ of $R$; they are precisely those prime ideals satisfying $(P_i)_g = (P)_g$.

**Proof** If $e$ is the identity element of the group $G$, then the idempotent element $p_e$ in $\widetilde{R}\#G$ has the property that $p_e(\widetilde{R}\#G)p_e \simeq R_e$.

1. Since $P \cap R_e = (P)_g \cap R_e$, we may pass to the graded ring $R/(P)_g$. So we may assume that $(P)_g = 0$ and therefore $R$ is gr-prime. In view of Theorem 7.5.6 we may select a prime ideal $Q$ of $\widetilde{R}\#G$ such that $Q \cap \widetilde{R} = 0$. If $Q^g$ is the $g$-conjugate of $Q$, since $\cap_{g \in G} Q^g$ is $G$-invariant and $\cap_{g \in G} Q^g \cap \widetilde{R} = 0$ then $\cap_{g \in G} Q^g = 0$. Lemma 7.5.9 yields $\cap_{g \in G} \varphi(Q^g) = 0$. It is clear that there is a system of elements $g_1, \ldots, g_k (k \leq |G|)$ such that $\cap_{i=1}^{k} \varphi(Q^{g_i}) = 0$ and the intersection is irreducible. If we put $p_i = \varphi(Q^{g_i})$ then $p_1, ..., p_k$ are prime ideals of $R_e$ such that $0 = \cap_{i=1}^{k} p_i$. Also since $Q$ is a minimal ideal of $\widetilde{R}\#G$, it follows that $p_1, ..., p_k$ are minimal ideals of $R_e$.

2. Consider a prime ideal $p$ of $R_e$. From Lemma 7.5.9 we retain that there is a unique prime ideal $Q$ of $\widetilde{R}\#G$ such that $p = \varphi(Q)$. Then $Q \cap \widetilde{R} = \widetilde{A}$ (Theorem 7.5.6), where $A$ is a graded prime ideal of $R$. In view of Proposition 2.11.1 there exists a prime ideal $P$ of $R$ such

that $A = (P)_g$, and Theorem 7.5.8 then learns that there are finitely many such primes $P_1, ..., P_m (m \leq \mid G \mid)$ such that $A = (P)_g = (P_i)_g$, being minimal primes over $(P)_g$. Applying part (1.), the primes of $R_e$ minimal over $P \cap R_e$ are precisely those in the set $\{\varphi(Q^g) \mid g \in G\}$. Since $p$ belongs to this set, $p$ is minimal over $P \cap R_e$. □

## 7.6 Exercises

1. Let $\mathcal{A}$ be an additive category. An object $A \in \mathcal{A}$ has an **invariant basis number** if and only if $A^n \simeq A^m$ implies $n = m$. Establish the following claims :

   i) If $F : \mathcal{A} \to \mathcal{B}$ is an additive functor which reflects isomorphisms then $F(A)$ has IBN if and only if $A$ has IBN.

   ii) $A \in \mathcal{A}$ has IBN if and only if $End_{\mathcal{A}}(A)$ has IBN.

   **Hint :** (ii) Put $R = \mathrm{End}_{\mathcal{A}}(A)$. If $R$ has IBN ($A^m \simeq A^n \Longrightarrow \mathrm{Hom}_{\mathcal{A}}(A, A^m) \simeq \mathrm{Hom}_{\mathcal{A}}(A, A^n) \to R^m \simeq R^n \Longrightarrow m = n$). Conversely assume $R_R^m \simeq R_R^n$. Consider the functor $S : \mathcal{A} \to \text{Mod-}R$ given by $S(X) = \mathrm{Hom}_{\mathcal{A}}(A, X)$.Then $S$ has a left adjoint functor $T$: mod-$R \to \mathcal{A}$ such that $T(R_R) \simeq A$. It is clear that $T(R_R^m) \simeq T(R_R^n) \Longrightarrow A^m \simeq A^n \Longrightarrow m = n$.

   Let $R$ be a $G$-graded ring. $R$ is called left gr-IBN if any two finite homogeneous bases of the same left gr-free module have the same number of elements i.e. if $\sigma_1, ..., \sigma_n$ and $\tau_1, ..., \tau_m$ are two systems of elements of $G$ such that $R(\sigma_1) \oplus ... \oplus R(\sigma_n) \simeq R(\tau_1) \oplus ... \oplus R(\tau_m)$ then $m = n$. It is clear that $R$ is left gr-IBN if and only if $R$ is right gr-IBN.

2. Let $R$ be a $G$-graded ring where $G$ is a finite group. We put $U = \oplus_{\sigma \in G} R(\sigma)$. The following are equivalent :

   i) $R$ has gr-IBN.

   ii) $U$ has IBN as an object in $R - gr$.

   iii) $R$ has IBN in $R$-mod.

   iv) The smash product $\widetilde{R}\#G$ has IBN.

   **Hint:**

   (i)$\Longrightarrow$(ii) is clear.

   (ii)$\Rightarrow$(iii) Assume $R^m \simeq R^n$. If $F : R-\mathrm{mod} \to R$-gr is the right adjoint of the forgetful funtor $U : R$-gr$\to R$-mod (see Chapter 2) then $F(R^m) \simeq F(R^n)$. But $F(R) \oplus_{\sigma \in G} R(\sigma) = U$. Hence $U^m \simeq U^n \Longrightarrow m = n$.

   (iii)$\Longrightarrow$(i) is clear.

   (ii)$\Longleftrightarrow$(iv) We have $End_{R-gr}(U) \simeq R\#G$ (Section 7.2).

3. Let $R$ be a $G$-graded ring (where $G$ is an arbitrary group). If $R$ has IBN in $R-gr$ then $R_e$ fas IBN.

   **Hint :** If $R_e^m \simeq R_e^n \Leftrightarrow R \otimes_{R_e} R_e^m \simeq R \otimes_{R_e} R_e^n \Longrightarrow R^m \simeq R^n \Longrightarrow m = n$.

4. Let $R$ be a $G$-strongly graded ring. If $R_e$ has finite Goldie dimension then $R$ has gr-IBN. Moreover if $G$ is finite then $R$ has IBN (by exercise 3).

   **Remark** In the paper [1], G.Abrams has proved that there exists a $G$-strongly graded ring $R = \oplus_{\sigma \in G} R_\sigma$ where $\mid G \mid = 2$, $R_e$ has IBN, but $R$ does not have IBN.

   Let $R = \oplus_{g \in G} R_g$ be a $G$ graded ring and $A$ a finite left $G$-set. We define the smash product of the graded ring $R$ by the $G$-set $A$, denoted by $R\#A$, as follows: $R\#A$ is the free left $R$-module with basis $\{p_x \mid x \in A\}$, and with multiplication defined by $(r_\sigma p_x)(s_\tau p_y) = (r_\sigma s_\tau)p_y$, if $\tau y = x$ and zero otherwise, for any $r_\sigma \in R_\sigma, r_\tau \in R_\tau, x, y \in A$. This makes $R\#A$ into a ring with identity $\sum_{x \in A} p_x$. Moreover the ring $R$ can be embeded in $R\#A$ via the map $\eta : R \longrightarrow R\#A$, $\eta(r) = \sum_{x \in A} r p_x$. We note that if $A = G$, then $R\#A$ is exactly the smash product defined in Section 7.1.

5. With notation as above prove that that ring $R\#A$ has the following properties :

   i) $\{p_x \mid x \in A\}$ is a set of orthogonal idempotents.

   ii) $p_x r_\sigma = r_\sigma p_{\sigma^{-1} x}$ for any $x \in A$, $\sigma \in G$, $r_\sigma \in R_\sigma$.

   iii) The set $\{p_x \mid x \in A\}$ is basis for $R$ as a right $R$-module.

   Denote by fin,$G$-set the category of finite left $G$-sets. If $\varphi : A \longrightarrow A'$ is a morphism in the category, we define the map $\varphi^* R\#G \longrightarrow R\#A$ by $\varphi^*(r_\sigma p_{x'}) = \sum r_\sigma \sum_{\varphi(x) = x'} p_x$ (with the convention that the sum of an empty family is zero).

6. Prove the following facts :

   i) $\varphi^*$ is a ring morphism.

   ii) The correspondence $A \to R\#A$ defines a contravariant functor from fin, $G$-Set to RINGS.

   iii) If $\varphi$ is injective (resp. surjective), then $\varphi^*$ is injective (resp. surjective).

7. Prove that the smash product $R\#A$ is a $G$-graded ring, with its $g$-homogeneous component $(R\#A)_g = \sum\limits_{x \in A} R_g p_x$.

8. Let $R$ be a $G$-graded ring and $A, B$ two finite $G$-sets. Prove that :

   i) There exists a ring isomorphism $R\#(A \times B) \simeq (R\#A)\#B$ where $A \times B$ has the $G$-set direct product structure.

   ii) If $A \cup B$ is the disjoint union of $A$ and $B$ with the natural $G$-structure, then $R\#(A \cup B) \simeq (R\#A) \times (R\#B)$.

   **Hint :**

   i) Define $\alpha : (R\#A)\#B \longrightarrow R\#(A \times B)$ by $\alpha((rp_x)p_y) = rp_{(x,y)}$ for any $r \in R$, $x \in A$, $y \in B$. It is easy to see that $\alpha$ is an isomorphism. Also $R\#A$ is a $G$-graded ring as in exercise 7).

   ii) Apply exercise 5.

   Let $A$ be a right $H$-set for some group $H$. We say that the action of $H$ on $A$ is faithful if $xh = x$ for all $x \in A$ implies $h = e$. The action of $H$ on $A$ is called fully faithful if $xh = x$ for some $x \in A$, $h \in H$ implies $h = e$ ($e$ is the unity element of $H$). If $G$ and $H$ are two groups, a $G$-$H$-set is a set which is a left $G$-set and right $H$-set such that $(gx)h = g(xh)$ for any $g \in G$, $x \in A$ and $h \in H$.

9. Prove that if $A$ is a $G$-$H$-set such that $G$ acts transitively on $A$ and the action of $H$ on $A$ is faithful, then the action of $H$ on $A$ is fully faithful.

10. Let $R$ be a $G$-graded ring and let $G$ and $H$ be two groups. Consider a finite $G - H$ set $A$ such that the action of $H$ is fully faithful. Prove that :

    i) There exists an action of $H$ on the ring $R\#A$.

    ii) If we denote by $\mathcal{O}_{\mathcal{H}}$ the set of $H$-orbits of $A$, then $\mathcal{O}_{\mathcal{H}}$ is a left $G$-set.

    iii) There exists a ring isomorphism :

    $$(R\#A) * H \simeq M_{|H|}(R\#\mathcal{O}_{\mathcal{H}})$$

    **Hint :** Since the action of $H$ on $A$ is fully faithful and $A$ is finite, $H$ is finite too. $\mathcal{O}_{\mathcal{H}}$ is a left $G$-set for the $G$-action defined by : $g(xH) = (gx)H$ for any $g \in G$, $x \in A$. The map $\varphi : A \longrightarrow \mathcal{O}_{\mathcal{H}}$ defined by $\varphi(x) = xH$ is a surjective morphism of $G$-sets. Therefore (cfr. Exercise 6.), $\varphi^* : R\#\mathcal{O}_{\mathcal{H}} \longrightarrow \mathcal{R}\#\mathcal{A}$ is an injective ring morphism. For any $h \in H$, the map $\alpha_h : A \longrightarrow A$ defined by $\alpha_h(x) = xh$ is an isomorphism of $G$-sets. Hence $\alpha_h^* : R\#A \longrightarrow R\#A$ is an automorphism of the ring $R\#A$. We have $\alpha_h^*(rp_x) = rp_{xh^{-1}}$. Hence $\theta : H \longrightarrow \mathrm{Aut}(R\#A)$, $\alpha(h) = \alpha_h^*$ is a group morphism that is clearly injective. A direct verification yields : $(R\#H)^H = \mathrm{Im}, \varphi^*$. Now let $\mathcal{O}_1, ..., \mathcal{O}_s$ be the $H$-orbits of $A$ and pick

$x_1 \in \mathcal{O}_1, ..., x_s \in \mathcal{O}_s$. For $h \in H$ consider the element $e_h = \sum_{i=1}^{s} p_{x_i h}$. The system $\{e_h \mid h \in H\}$ is a basis of the right $\mathrm{Im}\varphi^*$-module $R\#H$. Theorem 1.1 in the paper by F.R de Meyer *"Some notes on the general Galois theory of rings"*, Osaka J.Math. 2(1965), 117-127, cf. [36], leads to the existence of isomorphisms of rings $(R\#A) * H \simeq \mathrm{End}_{Im,\varphi^*}(R\#A) \simeq M_{|H|}(R\#\mathcal{O}_{\mathcal{H}})$.

11. Let $R$ be a $G$-graded ring, $G$ a finite group, and let $H$ be a subgroup of $G$. Prove that there is an algebra isomorphism $(R\#G) * H \simeq M_{|H|}(R\#G/H)$.

    **Hint :** Put $A = G$ in exercise 10), where $G$ is viewed as a $G$-$H$-set.

12. Assume that the group $G$ acts transitively on the finite set $A$. Show that $(R\#A) * \mathrm{Aut}_G(A) \simeq M_{|H|}(R\#\mathcal{O})$, where $\mathcal{O}$ is the set of orbits of $A$ with respect to the right action of $\mathrm{Aut}_G(A)$ on $A$.

    **Hint :** Take $H = \mathrm{Aut}_G(A)$ in exercise 10. Since $G$ acts transitively on $A$, $\mathrm{Aut}_G(A)$ has faithful right action on $A$.

13. Let $R$ be a $G$-graded ring and $H \triangleleft K \leq G$ subgroups of $G$ such that the index of $H$ in $G$ is finite. Prove that $(R\#G/H)*K/H \simeq M_{|K/H|}(R\#G/K)$.

    **Hint :** $K/H$ acts from the right on $G/H$ by $(gH)(kH) = (gk)H$ for any $g \in G$, $k \in K$. We also have that $\mathcal{O}_{\mathcal{K}/\mathcal{H}}(\mathcal{G}/\mathcal{H})$ is isomorphic to $G/K$ as a $G$-set. Now we can apply exercise 10.

14. Let $R$ be a $G$-graded ring and $A$ a finite left $G$-set. Consider the category $(G, A, R)$-gr of $A$ graded left modules (see Section 2.12). Show that $(G, A, R)$-gr is isomorphic to the category $(R\#A)$-mod.

    **Hint :** For an object $M = \oplus_{x\in A} M_x$ in the category $(G, A, R)$-gr we define $M^{\#}$ to be a left $R\#A$-module as follows: $M^{\#} = M$ as a group and if $m \in M$ and $ap_x \in R\#A$, $a \in R$ we put $(ap_x)m = a \cdot m_x$ where $m_x$ is the $x$-homogeneous component of $m$. The correspondence $M \to M^{\#}$ defines a covariant functor $(G, A, R) - gr \to (R\#A)$-mod. We can easily verify (as in Section 7.3) that this functor is an isomorphism of categories.

15. Let $R \oplus_{g\in G} R_g$ be a strongly $G$-graded ring. If $M, N$ are two right $R$-modules, then $G$ acts on $\mathrm{Hom}_{R_e}(M, N)$ as follows : if $f \in \mathrm{Hom}_{R_e}(M, N)$, $\sigma \in G$ and $(a_i)_i \in R_\sigma, (b_i)_i \in R_{\sigma^{-1}}$ are finite sets such that $\sum a_i b_i = 1$ then $f^\sigma(m) = \sum_i f(ma_i)b_i$ for any $m \in M$. Then $f^\sigma$ does not depend on the choice of the $a_i$'s and $b_i$'s, $f^\sigma$ is a morphism of $R_1$-modules and $(f^\sigma)^\tau = f^{\sigma\tau}$ for any $\sigma, \tau \in G$. Therefore $\mathrm{Hom}_{R_e}(M, N)$ becomes a left $G$-module by $\sigma \cdot f = f^\sigma$ for any $f \in Hom_{R_1}(M, N)$ and $\sigma \in G$. We also have $\mathrm{Hom}_{R_e}(M, N)^G = \mathrm{Hom}_R(M, N)$ (see Chapter 3). Assume now that $M = \oplus_{\sigma\in G} M_x$ is a $G$-graded module. We denote by

$\pi_x : M \longrightarrow M$ the projection on the $x$-th homogeneous component, i.e. $\pi_x(m) = m_x$ for any $m \in M$. For any $f \in Hom_{R_e}(M, N)$, define $\widetilde{f} : M \longrightarrow N$ by $\widetilde{f}(m_\tau) = f^{\tau^{-1}}(m_\tau)$. Show that :

i) If $f \in \mathrm{Hom}_{R_e}(M, N)$ and $x, \sigma \in G$ then $(f \circ \pi_x)^\sigma = f^\sigma \circ \pi_{x\sigma^{-1}}$. In particular $(\pi_x)^\sigma = \pi_{x\sigma^{-1}}$.

ii) $\widetilde{f}$ is $R$-linear.

iii) $f = \sum_{\sigma \in G} \widetilde{f^\sigma} \circ \pi_\sigma$.

iv) If $N = M$, then $\mathrm{End}_{R_e}(M) \simeq \mathrm{End}_R(M) \# G$ (in particular, for $M = R$, we obtain Corollary 7.2.2).

**Hint :**

i) If $m \in M$, we have $(f \circ \pi_x)^\sigma(m) = \sum_i (f \circ \pi_x)(ma_i)b_i = \sum_i f(m_{x\sigma^{-1}}a_i)b_i = (f^\sigma \circ \pi_{x\sigma^{-1}})(m)$.

ii) If $r_\sigma \in R_\sigma$ we have $\widetilde{f}(m_\tau r_\sigma) = f^{(\tau\sigma)^{-1}}(m_\tau r_\sigma) = (f^{\tau^{-1}})^{\sigma^{-1}}(m_\tau r_\sigma) = \sum_i f^{\tau^{-1}}(m_\tau r_\sigma a_i)b_i$ where $a_i \in R_{\sigma^{-1}}$, $b_i \in R_\sigma$ and $\sum a_i b_i = 1$. Since $f^{\tau^{-1}}$ is $R_e$-linear $\widetilde{f}(m_\tau r_\sigma) = \sum_i f^{\tau^{-1}}(m_\tau) r_\sigma a_i b_i = f^{\tau^{-1}}(m_\tau) r_\sigma = \widetilde{f}(m_r) r_\sigma$. Then $\widetilde{f} \in \mathrm{Hom}_R(M, N)$.

iii) We consider $m = m_x \in M_x$. We have $(\sum_{\sigma \in G} \widetilde{f^\sigma} \circ \pi_\sigma)(m) = \widetilde{f^x}(m_x) = (f^x)^{x^{-1}}(m_x) = f(m_x)$. So $f = \sum_{\sigma \in G} \widetilde{f^\sigma} \circ \pi_\sigma$.

iv) Follows from (i.) and (iii.).

16. We recall (see Appendix B) that if $M \in R$-mod has Krull (resp Gabriel) dimension then for any ordinal $\alpha \geq 0$ there is a largest submodule $\tau_\alpha(M)$ of $M$, having Krull (resp Gabriel) dimension less of equal to $\alpha$. Let $R = \sum_{\sigma \in G} R_\sigma$ be a $G$-graded ring, $G$ a graded finite group. Let $M \in R$-gr and assume that $M$ has Krull (resp. Gabriel) dimension. Show that for any ordinal $\alpha \geq 0$, $\tau_\alpha(M)$ is a graded submodule of $M$.

17. Let $R = \sum_{i \in \mathbf{Z}} R_\sigma$ be a $\mathbb{Z}$-graded ring. Let $M \in R-gr$ and assume that $M$ has *Krull* (resp Gabriel) dimension in $R$-mod (for example when $M$ is Noetherian). Prove that for any ordinal $\alpha \geq 0$, $\tau_\alpha(M)$ is a graded submodule of $M$.

**Hint :** If $n$ is a natural number, $n \geq 0$, let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and consider $R$ with gradation of type $\mathbb{Z}_n$. We denote by $(\tau_\alpha(M))^{g,n}$ the smallest graded submodule of $M$ containing $\tau_\alpha(M)$, where the gradation on $R$ and $M$ is the $\mathbb{Z}_n$-gradation. Exercise 16, yields $\tau_\alpha(M) = (\tau_\alpha(M))^{g,n}$ for any $n \geq 1$. Take now $x \in \tau_\alpha(M)$ and write $x = x_{-s} + ... + x_0 + ... + x_t$ where $x_{-s}, ..., x_0, ..., x_t$ are homogeneous components of $x$ in the initial gradation of $M$. Note that for any $n > s + t, x_{-s}, ..., x_0, ..., x_t$ are still homogeneous components of $x$ when $M$ is considered with its $\mathbb{Z}_n$ graduation. Since in this case $\tau_\alpha(M) = (\tau_\alpha(M))^{g,n}$, it follows that $x_{-s}, ..., x_t \in (\tau_\alpha(M))^{g,n} = \tau_\alpha(M)$. Thus $\tau_\alpha(M)$ is a graded submodule of $M$.

## 7.7 Comments and References for Chapter 7

The notion of smash product associated to a Hopf algebra goes back at least to M. Sweedler's book "Hopf Algebras", 1969. Since then other smash product concepts have been introduced, e.g. in [183] K. H. Ulbrich noticed that a $G$-graded algebra $R$ over a field $k$ may be viewed as a $k[G]$-comodule algebra where $k[G]$ is the Hopf algebra defined on the group ring. For a finite group $G$, $R$ is then a $k[G]^*$-module algebra where $k[G]^*$ is the dual Hopf algebra of the Hopf algebra $k[G]$. In this case we arrive at the smash product $A\#k[G]^*$. This has been used by M. Cohen, S. Montgomery in [43]. Other developments included an extension to infinite groups via a matrix ring point of view, cf. D. Quinn [174]. In Chapter 7 we develop the theory from this point of view. Smash products are introduced for arbitrary graded rings in Section 7.1; in the particular case when $G$ is finite it follows that D. Quinn's smash product is isomorphic to the smash product used by M. Cohen, S. Montgomery, [43]. In Section 7.2. we establish that the smash product is nothing but $\mathrm{End}_{\mathrm{gr}}(U)$ with respect to the canonical generator $U$ of $R$-gr ($G$ is a finite group now), cf. [141]. This makes it a very natural object, more so in view of the Barr-Mitchel theorem in category theory. The latter states that an Abelian category with arbitrary coproducts and having a small generator $P$ that is moreover a projective object, is canonically isomorphic to $A$-mod where $A$ is the endomorphism ring of $P$; observe that, when $G$ is a finite group the generator $U = \oplus_{\sigma\in G} R(\sigma)$ is small !

The connections between $R$-gr and $\overline{R}\#G$-mod, the latter corresponding to the smash product as defined by D. Quinn, are highlighted in Section 7.4. This is achieved via the construction of a series of adjoint functors. The main results, Theorem 7.4.4. and Theorem 7.4.6, have several interesting applications, e.g. Corollary 7.4.5., Corollary 7.4.7.

In Section 7.5., prime and gr-prime ideals of a $G$-graded ring $R$, $G$ a finite group, are related to prime ideals of the associated smash product; this section rests heavily on the work of M. Cohen and S. Montgomery.

Exercises are found in the final section, as usual.

Finally we remark that in general the smash product is a noncommutative ring, so for the study of commutative properties (for example Cohen Macaulay, Gorenstein, regular,... properties) it is not an efficient tool.

### Some References

- T. Albu, C. Năstăsescu [3]

- M. Beattie [9], [10], [11]

- W. Chin, D. Quinn [40]

- M. Cohen, S. Montgomery [43], [44]
- S. Dăscălescu, C. Năstăsescu, F. Van Oystaeyen, A. del Rio [59]
- M. Lorenz, D. Passman, [122], [123]
- C. Năstăsescu [145]
- C. Năstăsescu, F. Van Oystaeyen, S. Raianu [160]
- C. Năstăsescu, N. Rodino [154], [155]
- C. Năstăsescu, F.Van Oystaeyen, B.Torrecillas [164]
- C. Năstăsescu, F. Van Oystaeyen, Liu Shaoxue [161]
- D. Quinn [174]

# Chapter 8

# Localization of Graded Rings

## 8.1 Graded rings of fractions

Recall that if $R$ is a ring and $S$ is a multiplicatively closed subset of $R$ such that $1 \in S, 0 \notin S$ then the left ring of fractions $S^{-1}R$, with respect to $S$, exists if and only if $R$ satisfies the left Ore conditions with respect to $S$ :

$O_1$ If $s \in S$, $r \in R$ are such that $rs = 0$ then there is an $s' \in S$ such that $s'r = 0$.

$O_2$ For $r \in R$, $s \in S$ there is $r' \in R$, $s' \in S$ such that $s'r = r's$.

If $R$ is a left Noetherian ring then $O_1$ always holds and one only has to check $O_2$.

If the Ore conditions with respect to $S$ are being satisfied then $S^{-1}R = \left\{\frac{a}{s} \mid a \in R,\, s \in S\right\}$ where the operations are defined by $\frac{x}{s} + \frac{y}{t} = \frac{ax+by}{u}$ where $a, b \in R$ are such that $u = as = bt \in S$, $\frac{x}{s} \cdot \frac{y}{t} = \frac{x_1 \cdot y}{t_1 \cdot s}$ where $t_1 \in S$, $x_1 \in R$ are such that $t_1 x = x_1 t$.

Note that $\frac{x}{s} = \frac{y}{z}$ if and only if there is $a, b \in R$ such that $as = bt \in S$ and $ax = by$. When $R$ is a commutative ring we have that $\frac{x}{s} = \frac{y}{z}$ if and only if there is some $w \in S$ such that $w(tx - sy) = 0$.

Recall also that every $M \in R{-}mod$ allows the construction of fractions $S^{-1}M$ which is a left $S^{-1}R$-module. Actually $S^{-1}M \simeq S^{-1}R \otimes_R M$.

### 8.1.1 Lemma

Let $R$ be a graded ring of type $G$ and let $S$ be a multiplicatively closed subset contained in $h(R)$, i.e. consisting of homogeneous elements, then $R$ satisfies

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the left Ore conditions with respect to $S$ if and only if :

$O_1^g$ If $r \in h(R)$, $s \in S$ are such that $rs = 0$ then there is an $s' \in S$ such that $s'r = 0$.

$O_2^g$ For any $r \in h(R)$, $s \in S$, there is an $r' \in h(R), s' \in S$ such that $s'r = r's$.

**Proof** Clearly $O_1$ and $O_2$ imply $O_1^g$ and $O_2^g$. Conversely, consider $s \in S, r \in R$, where $r = r_{\sigma_1} + ... + r_{\sigma_n}$ with $r_{\sigma_i} \in h(R)$, $\sigma_i \in G$. If $n = 1$ then the left Ore conditions for $r, s$ clearly hold because $O_1^g$ and $O_2^g$ hold. Now we proceed by induction on $n$, supposing $O_1$ and $O_2$ hold for all $r \in R$ having the homogeneous decomposition of length less than $n$. By assumption there is $r^1 \in R$, $s^1 \in S$ such that $s^1(r_{\sigma_1} + ... + r_{\sigma_{n-1}}) = r^1 s$ and $s^2 \in S$, $r^2 \in R$ such that $s^2 r_{\sigma_n} = r^2 s$. By $O_1^g$ and $O_2^g$ there exist $u \in S$, $v \in h(R)$ such that $us^1 = vs^2 = t$ and $t \in S$ is non-zero. Then $tr = (ur^1 + vr^2)s$, and hence $O_2$ also holds if $r$ has a decomposition of length $n$. Furthermore, if $as = 0$ with $a = a_{\sigma_1} + ... + a_{\sigma_n}$, $a_{\sigma_i} \in R_{\sigma_i}$, then $a_{\sigma_n} s = 0$ and $(a_{\sigma_1} + ... + a_{\sigma_{n-1}})s = 0$. By the induction hypothesis we may pick $t_1 \in S$ such that $t_1(a_{\sigma_1} + ... + a_{\sigma_{n-1}}) = 0$. Now from $t_1 a_{\sigma_n} s = 0$ and $O_1^g$ it follows that there is a $t_2 \in S$ such that $t_2 t_1 a_{\sigma_n} = 0$. Putting $s' = t_2 t_1 \in S$ we see that $s'a = 0$ $s' \neq 0$ since $0 \notin S$).

If the graded ring $R$ satisfies the left Ore conditions with respect to some multiplicatively closed $S \subset h(R)$, then we can define a gradation on $S^{-1}R$ by putting $(S^{-1}R)_\lambda = \left\{ \frac{a}{s} | s \in S, a \in R \text{ such that } \lambda = (\deg, s)^{-1}\deg, a \right\}$.

## 8.1.2 Proposition

$S^{-1}R$ is a graded ring of type $G$.

**Proof** If $\frac{x}{s}, \frac{y}{t} \in (S^{-1}R)_\lambda$, then $\lambda = (\deg, s)^{-1}\deg, x = (\deg, t)^{-1}\deg, y$. Putting $u = as = bt \in S$ we get $\frac{x}{s} + \frac{y}{t} = \frac{ax+by}{u}$. Hence :

$$\begin{aligned} \deg\left(\frac{ax+by}{u}\right) &= (\deg u)^{-1}\deg(ax) \\ &= (\deg s)^{-1}(\deg a)^{-1}(\deg a)(\deg x) \\ &= (\deg s)^{-1}\deg x \end{aligned}$$

Therefore $(S^{-1}R)_\lambda$ is an additive subgroup of $S^{-1}R$ for each $\lambda \in G$. In a similar way one verifies $(S^{-1}R)_\lambda (S^{-1}R)_\mu \subset (S^{-1}R)_{\lambda\mu}$. Obviously $S^{-1}R = \sum_{\sigma \in G} (S^{-1}R)_\lambda$ and the common denominator theorem yields that the sum is direct.

## 8.1.3 Corollary

If $R$ is a strongly graded ring (resp crossed product) satisfying the left Ore conditions with respect to $S \subset h(R)$ then $S^{-1}R$ is a strongly graded ring. (resp crossed product).

**Proof** It is clear that $(S^{-1}R)_\sigma (S^{-1}R)_{\sigma^{-1}} = (s^{-1}R)_e$ follows from $R_{\sigma^{-1}}R_\sigma = R_\sigma R_{\sigma^{-1}} = R_e$. Now if $u_\sigma \in R_\sigma$ is invertible, it is clear that $\frac{u_\sigma}{1}$ is invertible in $S^{-1}R$, so if $R$ is a crossed product then $S^{-1}R$ is also a crossed product.

### 8.1.4 Proposition

Let $R$ be a $G$-graded ring, $G$ a finite group and let $S$ be a multiplicatively closed subset of $h(R)$, which satisfies the left Ore conditions. Consider $R\#G$ the smash product associated to $R$ and its $G$-gradation.

1. $R\#G$ satisfies the left Ore with respect to $S$;
2. $S^{-1}(R\#G) \simeq S^{-1}R\#G$.

**Proof**

1. Let $x = \sum_{g\in G} a^g p_g$ an arbitrary element from $R\#G$ and $s \in S$ such that $xs = 0$. Hence $0 = xs = \sum_{g\in G} a^g s p_{\sigma^{-1}g}$ where $\sigma = \deg, s$. Consequently $a^g s = 0$ for any $g \in G$. Assume that $\sup(x) = \{\sigma \in G \mid a^g \neq 0\} = \{g_1, ..., g_n\}$. Now by induction on $|\sup(x)|$ and using condition $O_1$ there exists an $s' \in S$ such that $s'a^g = 0$ for any $g \in \sup(x)$. Clearly $s'x = 0$. Take $x = \sum_{g\in G} a^g p_g \in R\#G$ and $s \in S$ with $deg\, s = \sigma$. For $g_1 \in \sup(x)$ there are $a'^{g_1}$ and $s'_1 \in S$ such that $s'_1 a^{g_1} = a'^{g_1} s$. Then $s'_1 x = s'_1 a^{g_1} p_{g_1} + y$ where $y = \sum_{g\neq g_1} s'_1 a^g p_g$, thus $s'_1 x = a'^{g_1} s p_{g_1} + y = a'_{g_1} p_{\sigma g_1} + y$. Since $|\sup(y)| < |\sup(x)$, and the induction on $|\sup(x)|$ there exist $s'_2 \in S$ and $y' \in R\#G$ such that $s'_2 y = y's$. In this case we have $s'_2 s'_1 x = s'_2 a'^{g_1} p_{\sigma g_1} s + s'_2 y = (s'_2 a'^{g_1} p_{\sigma g_1} + y')s$ and therefore $S$ also verifies the condition $O_2$ in $R\#G$.

2. As a consequence of (1.) the ring of fractions $S^{-1}(R\#G)$ does exist. Now it is easy to see that the canonical map

$$\varphi : S^{-1}R\#G \longrightarrow S^{-1}(R\#G)$$

defined by $\varphi(s^{-1}ap_x) = s^{-1}(ap_x)$ for all $s \in S, a \in R, x \in G$ is an isomorphism of rings. □

## 8.2 Localization of Graded Rings for a Graded Linear Topology

Let $R$ be a $G$-graded ring and $R$-gr the category of all (left) graded $R$-modules. Let $\mathcal{C}$ be a rigid closed subcategory of $R$-gr (see Section 2.6 and Section 4.4) We denote by $L(R)$ ($L^{gr}(R)$) the lattice of all left ideals (resp of all graded left ideals) of the graded ring $R$. We will say that a nonempty subset $H$ of $L^{gr}(R)$ is a **graded linear topology** (gr-linear topology) on $R$ if it is a filter

in $L^{gr}(R)$ and satisfies the following additional condition:
If $I \in H$ and $r \in h(R)$, then $(I : r) \in H$. Now, in a way similar to the correspondence between closed subcategories of $R$-mod and linear topology on $R$, it can be proved that there is a bijective correspondence between rigid closed subcategory of $R$-gr and graded linear topologies on $R$, given by

$$\mathcal{C} \longrightarrow H_{\mathcal{C}} = \{I \in L^{\mathrm{gr}}(R) \mid R/I \in \mathcal{C}\}$$

(Here $l_R(x) = \{a \in R \mid ax = 0\}$). If $H$ is a graded linear topology on $R$, then set $\overline{H} = \{I \in L(R) \mid \exists J \in H, J \subseteq I\}$ is a left linear topology on $R$ such that $H \subseteq \overline{H}$. Actually it is easy to see that $\overline{H}$ is the smallest linear topology on $R$ such that $H \subseteq \overline{H}$.

Let $\mathcal{C}$ be a rigid subcategory of $R$-gr. We denote by $\overline{\mathcal{C}}$ the smallest closed subcategory of $R$-mod such that $\mathcal{C} \subseteq \overline{\mathcal{C}}$. Assume that the graded linear topology associated to $\mathcal{C}$ is denoted by $H$. Then we complete Proposition 4.4.2. as follows :

### 8.2.1 Proposition

The following assertions are equivalent for $M \in R$-gr :

1. $M \in \overline{\mathcal{C}}$.
2. For any $x \in M$, $l_R(x) \in \overline{H}$.

**Proof**

1. $\Rightarrow$(2.) Proposition 4.4.2 applied to $M \in \overline{\mathcal{C}}$, yields that there is $N \in \mathcal{C}$ such that $M$ is isomorphic to a quotient module of $N$ in $R$-mod i.e. there is an epimorphism :

   $$N \xrightarrow{u} M \longrightarrow 0$$

   If $x \in M$, then there is $y \in N$ such that $u(y) = x$. It is clear that $l_R(y) \subseteq l_R(x)$. If we write $y = y_{\sigma_1} + ... + y_{\sigma_s}$ where $y_{\sigma_i} \in N_{\sigma_i}$ $(1 \leq i \leq s)$ then $I = \cap_{i=1}^{s} l_R(y_{\sigma_i}) \in H$. But $I \subset l_R(y) \subseteq l_R(x)$ so $l_R(x) \in \overline{H}$.

2. $\Rightarrow$ (1.) For any $x \in M$, we have that $l_R(x) \in \overline{H}$ and there is $I_x \in H$ such that $I_x \subseteq l_R(x)$. Thus we have an exact sequence in $R$-mod

   $$R/I_x \longrightarrow R/l_R(x) \longrightarrow 0$$

   and since $Rx \simeq R/l_R(x)$, we obtain an exact sequence

   $$\oplus_{x \in M} R/I_x \longrightarrow \oplus_{x \in M} Rx \longrightarrow 0$$

   Therefore setting $N = \oplus_{x \in M} R/I_x$, which obviously belongs to $\mathcal{C}$, we see that $M$ is a quotient of $N$ in $R$-mod as it is a quotient of $\oplus_{x \in M} Rx$. (in $R$-mod). □

### 8.2.2 Proposition

Let $\mathcal{C}$ and $\overline{\mathcal{C}}$ be as above and $t_{\mathcal{C}}$ and $t_{\overline{\mathcal{C}}}$ the corresponding left exact preradicals. If $M \in R-gr$, then $t_{\overline{\mathcal{C}}}(M) = t_{\mathcal{C}}(M)$.

**Proof** Since $\mathcal{C} \subseteq \overline{\mathcal{C}}$ it is clear that $t_{\mathcal{C}}(M) \subseteq t_{\overline{\mathcal{C}}}(M)$. On the other hand, if $x \in t_{\overline{\mathcal{C}}}(M)$, then there exists $J \in \overline{H}$ such that $Jx = 0$ and thus there exists $I \in H$ with $I \subseteq J$ and $Ix = 0$. If $x = \sum_{\sigma \in G} x_\sigma$, with $x_\sigma \in M_\sigma$ then $Ix_\sigma = 0$ for any $\sigma \in G$ (for $I$ is a graded ideal) and $x_\sigma \in t_{\mathcal{C}}(M)$ follows. Therefore $t_{\mathcal{C}}(M) = t_{\overline{\mathcal{C}}}(M)$. □

Assume now that $R = \oplus_{\sigma \in G} R_\sigma$ is a $G$-strongly graded ring. A closed subcategory $\mathcal{C}_e$ of $R_e - mod$ is called $G$-**stable** if and only if for any $M \in \mathcal{C}_e$ we have $R_\sigma \otimes_{R_e} M \in \mathcal{C}_e$ for every $\sigma \in G$. We denote by $\mathcal{C} = \{M = \oplus_{\sigma \in G} R_\sigma \in R\text{-gr}, M_e \in \mathcal{C}_e\}$. Since the functor $(-)_e$ is exact, then $\mathcal{C}$ is a closed subcategory. Moreover $\mathcal{C}$ is rigid. Indeed if $M \in \mathcal{C}$ then $M(\sigma)_e = M_\sigma \simeq R_\sigma \otimes_{R_e} M_e$ (by Dade's Theorem) and since $\mathcal{C}_e$ is $G$-stable we get that $M(\sigma) \in \mathcal{C}$. Moreover $\mathcal{C}_e$ is a localizing subcategory, then also $\mathcal{C}$ is a localizing subcategory of $R$-gr.

### 8.2.3 Proposition

Assume that $R$ is a strongly graded ring. With the above notations, the correspondence $\mathcal{C}_e \to \mathcal{C}$ between all closed (resp localizing) $G$-stable subcategories of $R_e$-mod and all rigid (resp localizing) subcategories of $R$-gr, is bijective.

**Proof** Assume $\mathcal{C}$ is a rigid closed subcategory of $R$-gr and put $\mathcal{C}_e = \{N \in R_e\text{−mod}, R \otimes_{R_e} N \in \mathcal{C}$. Since the functor $R \otimes_{R_e} - : R_e\text{−mod} \to R\text{-gr}$ is exact (in fact it is an equivalence) we get that $\mathcal{C}_e$ is a closed subcategory of $R_e$-mod. Since $R \otimes_{R_e} (R_\sigma \otimes_{R_e} N) \simeq (R \otimes_{R_e} N)(\sigma)$ it follows that $R_\sigma \otimes_{R_e} N \in \mathcal{C}_e$, so $\mathcal{C}_e$ is $G$-stable. Now it is clear that the above correspondence is bijective. On the other hand if $\mathcal{C}$ is a localizing subcategory of $R$-gr, then $\mathcal{C}_e$ is a localizing subcategory of $R_e$-mod.

Let $\mathcal{C}_e$ be a $G$-stable closed subcategory of $R_e$-mod. We denote by $H_e$ the linear topology associated to $\mathcal{C}_e$ i.e. $H_e = \{I \,\text{left ideal of}\, R_e, R_e/I \in \mathcal{C}_e\}$. Since $\mathcal{C}_e$ is $G$-stable, if $I \in H_e$ then for any $\sigma \in G$, $(R_\sigma I : \lambda_\sigma) \in H_e$ for every $\lambda_\sigma \in R_\sigma$. Indeed since $I \in H_e$ then $R_\sigma \otimes_{R_e} R_e/I \in \mathcal{C}_e$. But $R_\sigma \otimes_{R_e} R_e/I \simeq R_\sigma/R_\sigma I$ so $R_\sigma/R_\sigma I \in \mathcal{C}_e$ and therefore we obtain that $(R_\sigma I : \lambda_\sigma) \in H_e$ for any $\lambda_\sigma \in R_\sigma$. □

### 8.2.4 Examples

1. Let $S$ be a multiplicatively closed subset of $h(R)$ not containing zero. We denote by $\mathcal{C}_S = \{M \in R\text{−gr}, \text{for any}\, x \in h(M),\, l_R(x) \cap S \neq 0\}$. It is easy to see that $\mathcal{C}_S$ is a rigid localizing subcategory of $R$-gr. If we

denote by $H_S$ the graded linear topology associated to $\mathcal{C}_S$ then $H_S = \{L \in L^{\rm gr}(R) \mid (L : r) \cap S \neq \emptyset, \text{for all} \, r \in h(R)\}$. Clearly $\overline{\mathcal{C}}_S = \{M \in R\text{−mod} \mid l_R(x) \cap S \neq \emptyset, \text{for any } x \in M\}$.

2. Let $R$ be a $G$-graded ring and $M \in M$-gr. Consider $Z_g(M) = \{x \in M$ there exists an essential left graded ideal $I$ in $R$ such that $Ix = 0\}$. Obviously $Z_g(M)$ is a graded submodule of $M$; $Z_g(M)$ is called the **graded singular submodule** of $M$. It is clear that $Z_g(M) \subseteq Z(M)$ where $Z(M)$ is the singular submodule of $M$ in $R$-mod. On the other hand if $G$ is an ordered group then $Z_g(M) = Z(M)$ (see Section 5.2). We also define $Z_g^2(M)$ to be the graded submodule of $M$ such that $Z_g(M) \subseteq Z_g^2(M)$ and $Z_g^2(M)/Z_g(M) = Z(M/Z_g(M))$. The class $\mathcal{G} = \{M \in R\text{−gr}, M = Z_g^2(M)\}$ is a rigid localizing subcategory of $R$-gr. It is called the **Goldie torsion theory** of $R$-gr.

3. Let $Q$ be a gr-injective object in $R$-gr. Let $\mathcal{C}_Q = \{M \in R\text{−gr} \mid \mathrm{HOM}_R(M, Q) = 0\}$. Since the functor $HOM_R(-, Q)$ is exact we may conclude that $\mathcal{C}_Q$ is a rigid localizing subcategory of $R$-gr. Since

$$\begin{aligned} \mathrm{HOM}_R(M, Q) &= \oplus_{\sigma \in G} \mathrm{HOM}_R(M, Q)_\sigma \\ &= \oplus_{\sigma \in G} \mathrm{Hom}_{R-\mathrm{gr}}(M, Q(\sigma)) \end{aligned}$$

then $\mathcal{C}_Q = \{M \in R\text{−gr} \mid \mathrm{Hom}_{R-gr}(M, Q(\sigma)) = 0$ for any $\sigma \in G\}$. In particular if $Q$ is $G$ invariant we have that $\mathcal{C}_Q = \{M \in R\text{−gr} \mid \mathrm{Hom}_{R-\mathrm{gr}}(M, Q) = 0\}$. In fact every rigid localizing subcategory of $R$-gr has the form $\mathcal{C}_Q$ where $Q$ is some gr-injective object in $R$-gr, as is easily verified.

If $Q$ is gr-injective, we denote by $E(Q)$ the injective envelope of $Q$ in $R$-mod. We denote by $\mathcal{A}_{E(Q)} = \{M \in R\text{−mod}, \mathrm{Hom}_R(M, E(Q)) = 0\}$. $\mathcal{A}_{E(Q)}$ is a localizing subcategory of $R$-mod.

### 8.2.5 Proposition

With notation as before, let $Q$ be a gr-injective object of $R$-gr. Then

1. $\overline{\mathcal{C}}_Q \subseteq \mathcal{A}_{E(Q)}$.
2. If $M \in R$-gr and $M \in \mathcal{A}_{E(Q)}$ then $M \in \mathcal{C}_Q$ i.e. $\mathcal{C}_Q = \mathcal{A}_{E(Q)} \cap R$-gr.

**Proof**

1. Proposition 8.2.2 entails $t_{\overline{\mathcal{C}}_Q}(Q) = 0$ since $t_{\mathcal{C}_Q}(Q) = 0$. It is clear that $E(Q)$ is also $\overline{\mathcal{C}}_Q$-torsion free. Hence if $M \in \overline{C_Q}$, we have $\mathrm{Hom}_R(M, E(Q)) = 0$ and thus $\overline{C_Q} \subseteq \mathcal{A}_{E(Q)}$.
2. Let $M \in R$-gr such that $M \in \mathcal{A}_{E(Q)}$, then $\mathrm{Hom}_R(M, E(Q)) = 0$. Since $Q \leq E(Q)$, we have $\mathrm{Hom}(M, Q) = 0$. But $\mathrm{HOM}_R(M, Q) \subset \mathrm{Hom}_R(M, Q)$ and therefore $\mathrm{HOM}_R(M, Q) = 0$ and $M \in \mathcal{C}_Q$. □

### 8.2.6 Example

Let $k$ be a field and consider the ring of Laurent polynomials $R = k[X, X^{-1}]$ with the natural grading over the group $G = \mathbb{Z}$. $R$ is a gr-field so $Q =_R R$ is injective in $R - gr$. In this case $E(Q) = D$ the field of fractions of $R$. it is clear that $\mathcal{C}_Q = \{0\}$ and $\mathcal{A}_{E(Q)} is$ a localizing subcategory of torsion modules ($R$ is a principal ideal ring) Since $\overline{\mathcal{C}}_Q = \{0\}$ we have $\overline{\mathcal{C}}_Q \neq \mathcal{A}_{E(Q)}$.

Let $R = \oplus_{\sigma\in G} R_\sigma$ be a $G$ graded ring $Q \in R$-gr a gr-injective module of finite support. We know from Section 2.8 that $Q$ is also injective in $R$-mod. We write $H_Q$ (resp $F_Q$) for the linear topology associate to the localizing subcategory $\mathcal{C}_Q$ (resp $\mathcal{A}_Q$) of $R$-gr (resp of $R$-mod). Therefore

$$\mathcal{C}_Q = \{M \in R\text{−gr}, \mathrm{HOM}_R(M, Q) = 0\} \quad \text{and}$$
$$\mathcal{A}_Q = \{M \in R\text{−mod}, \mathrm{Hom}_R(M, Q) = 0\}$$

Consequently

$$H_Q = \{I \in L^{\mathrm{gr}}(R), \mathrm{Hom}_R(R/I, Q) = 0\} \quad \text{and}$$
$$F_Q = \{I \in L(R), \mathrm{Hom}_R(R/I, Q) = 0\}$$

### 8.2.7 Proposition

With hypotheses as before we have $F_Q = \overline{H_Q}$.

**Proof** Using the structure of gr-injective modules of finite support (Corollary 2.8.8) we may assume that $Q$ is $g$-faithful for some $g \in G$ and of finite support. In this case $Q(g) \simeq \mathrm{Coind}(Q_g) = \mathrm{Hom}_{R_1}(R, Q_g)$ where $Q_g$ is an injective $R_1$-module. Since $\mathrm{Hom}_R(R/I, Q) = \mathrm{Hom}_R(R/I, \mathrm{Hom}_{R_1}(R, Q_g)) \simeq \mathrm{Hom}_{R_e}(R \otimes_R R/I, Q_g) = \mathrm{Hom}_{R_e}(R/I, Q_g)$ we obtain that $I \in F_Q$ if and only if $\mathrm{Hom}_{R_e}(R/I, Q_g) = 0$. Since for any $\sigma \in G$ we have $0 \to R_\sigma/I \cap R_\sigma \to R/I$ we also obtain $\mathrm{Hom}_{R_e}(R_\sigma/I \cap R_\sigma, Q_g) = 0$ ($Q_g$ is injective in $R$-mod). If we put $J = \sum_{\sigma\in G} I \cap R_\sigma = (I)_g$ then $J$ is a left graded ideal of $R$ such that $J \subseteq I$. Furthermore

$$\begin{aligned}
\mathrm{Hom}_{R_e}(R/J, Q_g) &= \mathrm{Hom}_{R_e}(\oplus_{\sigma\in G} R_\sigma/(I \cap R_\sigma), Q_g) \\
&= \prod_{\sigma\in G} \mathrm{Hom}_{R_e}(R_\sigma/(I \cap R_\sigma), Q_g) \\
&= 0
\end{aligned}$$

whence $J \in F_Q$. Then $F_Q$ contains a cofinal set of graded left ideals. On the other hand $\mathrm{HOM}_R(R/J, Q) \leq \mathrm{Hom}_R(R/J, Q) = 0$ and $J \in H_Q$ follows. Finally we obtain $\overline{H_Q} = F_Q$. □

### 8.2.8 Corollary

Let $R$ be a $G$ strongly graded ring where $G$ is a finite group and $Q = \oplus_{\sigma\in G} Q_\sigma$ a gr-injective object.
We denote by $F_{Q_\sigma} = \{J \text{ left ideal of } R_e, \mathrm{Hom}_{R_e}(R/J, Q_\sigma) = 0\}$. Then $I \in F_Q$ if and only if $I \cap R_e \in \cap_{\sigma\in G} F_{Q_\sigma}$. Moreover if $Q$ is $G$-invariant then $I \in F_Q$ if and only if $I \cap R_e \in F_{Q_e}$.

**Proof** Obviously $Q_\sigma$ is an injective $R_e$-module. By the foregoing remarks : $I \in F_Q$ if and only if $\mathrm{Hom}_{R_e}(R/I, Q_\sigma) = 0$ for any $\sigma \in G$ and this clearly implies that $I \cap R_e \in F_{Q_\sigma}$ for any $\sigma \in G$. Conversely the statement follows from the fact that $J = R(I \cap R_e)$. Since $\mathrm{Hom}_R(R/J, Q) = \mathrm{HOM}_R(R/J, Q) = \oplus_{\sigma\in G}\mathrm{Hom}_{R-\mathrm{gr}}(R/J, Q(\sigma)) = \oplus_{\sigma\in G}\mathrm{Hom}_{R_e}(R_e/J \cap R_e, Q_\sigma) = 0$ we obtain that $J \in F_Q$ and since $J \subseteq I$ we have $I \in F_Q$.

## 8.3 Graded Rings and Modules of Quotients

The general localization theory in a Grothendieck category is well known (we may refer to [61] [162]). We restrict to recalling the fundamental concepts and basic results applied to the graded theory. Let $R$ be a $G$-graded ring and as usual $R$-gr the category of (left) graded $R$-modules. Let $\mathcal{C}$ be a rigid localizing subcategory of $R$-gr. We denote be $H$ the graded linear topology associated to $\mathcal{C}$ and by $t_\mathcal{C}$ the radical on $R - gr$ associated to $\mathcal{C}$. Then $H = \{I \in L^{gr}(R), R/I \in \mathcal{C}\}$ and if $M \in R$-gr then $t_\mathcal{C}(M)$ is the largest graded submodule of $M$ contained in $\mathcal{C}$ i.e. $t_\mathcal{C}(M) = \{x \in M \mid \exists I \in H,\, I \cdot x = 0\}$. We denote by $\overline{\mathcal{C}}$ (resp $\overline{H}$) the smallest localizing subcategory of $R$-mod (resp the smallest linear topology of $R$) that contains $\mathcal{C}$ (resp $H$). By Proposition 8.2.1 we have that $\overline{H} = \{J \in L(R) \mid \exists I \in H,\, I \subseteq J\}$. Now if $M \in R$-gr, we put

$$Q_H(M) = \varinjlim_{L\in\overrightarrow{H}} HOM_R(L, M/t_\mathcal{C}(M))$$

As in the non-graded situation, one can easily see that in $M =_R R$ then $Q_H(R)$ has a natural structure of $G$-graded ring, whereas for every $M \in R$-gr, $Q_H(M)$ turns out to be a graded $Q_H(R)$-module, i.e. $Q_H(M) \in Q_H(R)$-gr. For the linear topology $\overline{H}$ we have :

$$Q_{\overline{H}}(M) = \varinjlim_{I\in\overrightarrow{\overline{H}}} \mathrm{Hom}_R(I, M/t_{\overline{\mathcal{C}}}(M))$$

Since $t_\mathcal{C}(M) = t_{\overline{\mathcal{C}}}(M)$ and $H$ is cofinal in $\overline{H}$ we obtain

$$Q_{\overline{H}}(M) = \varinjlim_{L\in\overrightarrow{H}} \mathrm{Hom}_R(L, M/t_\mathcal{C}(M))$$

thus $Q_H(M) \subset Q_{\overline{H}}(M)$. In particular $Q_H(R) \subset Q_{\overline{H}}(R)$. In fact $Q_H(R)$ is a subring of $Q_{\overline{H}}(R)$. Also $R/t_\mathcal{C}(R)$ is a subring of $Q_H(R)$.

### 8.3.1 Proposition

1. If $H$ contains a cofinal system of left finitely generated graded ideals then for any $M \in R$-gr, then $Q_H(M) = Q_{\overline{H}}(M)$.
2. If the ring $R$ has finite support and $M \in R$-gr has finite support too, then $Q_H(M) = Q_{\overline{H}}(M)$.
3. If the ring $R$ has finite support then $Q_H(R) = Q_{\overline{H}}(R)$.

**Proof**

1. We calculate :

$$\begin{aligned} Q_{\overline{H}}(M) &= \varinjlim\{I \in \overline{H}, \mathrm{Hom}_R(I, M/t_{\mathcal{C}}(M))\} \\ &= \varinjlim\{J \text{ finitely generated in } H, \mathrm{Hom}_R(J, M/t_{\mathcal{C}}(M))\} \\ &= \varinjlim\{J \text{ finitely generated in } H, \mathrm{HOM}_R(I, M/t_{\mathcal{C}}(M))\} \\ &= Q_H(M) \end{aligned}$$

2. Similar to the Proof of 1., using result from Section 2.
3. Follows from 2.

### 8.3.2 Corollary

Assume that $R$ is left gr-Noetherian. With the notation as in Proposition 8.3.1, if $M \in R$-gr we have

$$Q_H(M) = Q_{\overline{H}}(M)$$

### 8.3.3 Example

Let $S$ be a multiplicatively closed subset of $R$ contained in $h(R)$. Assume that $1 \in S$, $0 \notin S$ and $S$ satisfies the left Ore conditions. Let $\mathcal{C}_S = \{M \in R\text{-gr} \mid \text{for any } x \in h(M), l_R(x) \cap S \neq \emptyset\}$ (see Example 8.2.4). In this case $H_S = \{L \in L^{\mathrm{gr}}(R), (L : r) \cap S \neq 0 \text{ for all } r \text{ in } h(R)\}$. In particular if $L \in H_S$ then $L \cap S \neq 0$ so there is $s \in S$ such that $Rs \subseteq L$. On the other hand if $I = Rs$, where $s \in S$ since $S$ verifies the Ore conditions then $(Rs : a) \cap S \neq \emptyset$ for all $a \in h(R)$, so $Rs \in H_S$. Proposition 8.3.1 entails for $M \in R$-gr that

$$Q_{H_S}(M) = Q_{\overline{H}_S}(M) = S^{-1}M$$

We recall that if $A$ is an arbitrary ring and we consider $E(A)$ the left injective envelope of ${}_AA$ in $A$-mod and denote by

$$F_{E(A)} = \{I \in L(A), \mathrm{Hom}_A(A/I, E(A)) = 0\}$$

then $F_{E(A)}$ is a linear topology which is associated to the localizing category $\mathcal{C}_{E(A)} = \{M \in A\text{−mod} \mid \mathrm{Hom}_A(M, E(A)) = 0\}$. In this case the quotient ring $Q_{F_{E(A)}}(A)$ is called the left maximal ring of $A$ and is denoted by $Q^l_{\max}(A)$ (or shortly $Q_{\max}(A)$). Clearly $A$ is a subring of $Q_{\max}(A)$.

### 8.3.4 Proposition

Let $R$ be a $G$-graded ring of finite support (in particular if $G$ is a finite group). Then the left maximal quotient ring $Q_{\max}(R)$ is endowed with a natural graded structure. Moreover, $Q_{\max}(R)$ has finite support too.

**Proof:** Let $E^g({}_RR)$ be the injective envelope of ${}_RR$ in $R$-gr. Assume that $\sup(R) = \{\sigma_1, ..., \sigma_n\}$. Let $\sigma \notin \{\sigma_i^{-1}\sigma_j, 1 \leq i, j \leq n\}$. If $(E^g({}_RR))_\sigma \neq 0$ there exist an element $x_\sigma \in (E^g({}_RR))_\sigma$, $x_\sigma \neq 0$. Since $E^g(R)$ is an essential extension of ${}_RR$, there is an element $a_\lambda \in R_\lambda$ such that $a_\lambda x_\sigma \in_R R$ and $a_\lambda x_\sigma \neq 0$. Hence $\lambda\sigma \in \sup(R))$ also since $a_\lambda \neq 0, \lambda \in \mathrm{supp}(R)$ we have that $\sigma \in \lambda^{-1}\sup(R)$, a contradiction. Hence $(E^g(R))_\sigma = 0$ and $E^g(R)$ has finite support. In this case $E^g(R) = E({}_RR)$ is an injective object in $R$-mod. From Proposition 8.2.7 and 8.3.1 we obtain that $Q_{\max}(R)$ is a graded ring. Since $Q_{\max}(R) \subset E({}_RR)$ we conclude that $Q_{\max}(R)$ has finite support. □

### 8.3.5 Proposition

Let $R = \oplus_{\sigma\in G} R_\sigma$ be a strongly graded ring, where $G$ is a finite group. Then :

1. The maximal (left) quotient ring $Q_{\max}(R)$ is a strongly graded ring of type $G$.
2. $(Q_{\max}(R))_e = Q_{\max}(R_e)$.
3. If $R$ is a crossed product, then $Q_{\max}(R)$ is a crossed product.

**Proof**

1. Since $R \subseteq Q_{\max}(R)$ and $R$ is a strongly graded ring we have $1 \in R_\sigma R_{\sigma^{-1}} \subseteq (Q_{\max}(R))_\sigma\ Q_{\max}(R))_{\sigma^{-1}}$, or $Q_{\max}(R)$ is a strongly graded ring.
2. Let $Q = E^g(R) = E(R)$ be the left injective envelope of ${}_RR$ in $R - gr$ (the same in $R$-mod). Since $t_Q(R) = 0$ we obtain :

$$Q_{\max}(R) = \varinjlim_{I\in H_Q} \mathrm{HOM}(I, R),$$

and therefore :

$$(Q_{\max}(R))_e \quad = \quad \varinjlim_{I\in H_Q} \mathrm{Hom}_{R-gr}(I, R)$$

$$= \varinjlim_{I \in H_Q} \mathrm{Hom}_{R_e}(I_e, R_e)$$

Corollary 8.2.8 implies that :

$$(Q_{\max}(R))_e = \varinjlim_{J \in F_{Q_e}} \mathrm{Hom}_{R_e}(J, R_e) = Q_{\max}(R_e)$$

(here $Q_e$ is the injective envelope of ${}_{R_e}R_e$).

3. Obvious. □

### 8.3.6 Theorem

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a graded ring of finite support. Let $P$ be a graded ideal of $R$ which is a prime ideal of $R$. If $I$ is an ideal of $R$ such that $P \subset I$, $P \neq I$, then $P \underset{\neq}{\subset} I_g$ and $P \cap R_e \neq I \cap R_e$.

**Proof** The problem can be reduced to the consideration of $R/P$. Hence $R$, we may assume that $R$ is a prime ring. Then $R$ is $e$-faithful (see Section 2.11). Put $Q = E({}_RR)$. Since $I \neq 0$ we have $\mathrm{Hom}_R(R/I, Q) = 0$. Indeed if $\mathrm{Hom}_R(R/I, Q) \neq 0$, then there is a nonzero $R$-morphism $f : R/I \to Q$. Thus $f(1) = x \in Q$ and $x \neq 0$; there exists $a \in R$ such that $ax \in R$ and $ax \neq 0$. Since $Ix = 0$, we have $I(ax) \subseteq Ix = 0$. Since $R$ is a prime ring we obtain $I = 0$, a contradiction. Since $I \in F_Q$ and in view of Proposition 8.2.7 we have $(I)_g \in F_Q$ and $\mathrm{Hom}_R(R/(I)_g, Q) = 0$. Hence $(I)_g \neq 0$. The results of Section 2.11 imply that : $I \cap R_e = (I)_g \cap R_e \neq 0$.

## 8.4 The Graded Version of Goldie's Theorem

It is well-known that Goldie's theorems are of great importance in Ring Theory, particularly in the study of left (right) Noetherian rings. A first study of Goldie's theorems for graded rings may be found in [94] and later in the book [136] for the $\mathbb{Z}$-graded case. A recent extension may be found in [73]. In order to state and prove Goldie's Theorem, some preliminary ingredients are needed.

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a $G$-graded ring. $R$ is said to be a **left graded Goldie ring** (sometimes written as gr-Goldie ring) if $R$ has finite Goldie dimension in $R$-gr and satisfies the ascending chain condition on graded left annihilators.

### 8.4.1 Lemma

Let $R$ be a graded ring satisfying the ascending chain condition for graded left annihilators. Then the left graded singular radical of $R$ is nilpotent.

**Proof** Write $J$ for the graded left singular radical of $R$. Then $J$ is generated as a left ideal by all homogeneous elements $a \in R$ such that $l_R(a)$ is an essential left ideal. Look at the ascending chain of graded left ideals :

$$l_R(J) \subset l_R(J^2) \subset ... \subset l_R(J^n) \subset ...$$

The hypothesis implies that $l_R(J^n) = l_R(J^{n+1})$ for some $n \in \mathbb{N}$. If $J^{n+1} \neq 0$ then $aJ^n \neq 0$ for some $a \in h(R)$ and we may chose $a$ such that $l(a)$ is maximal with respect to this property. If $b \in J \cap h(R)$ then $l_R(b) \cap Ra \neq 0$ because $l_R(b)$ is an essential left ideal of $R$. Thus there is $c \in h(R)$ such that $ca \neq 0$ and $cab = 0$, so $l_R(a) \underset{\neq}{\subset} l_R(ab)$. The hypothesis on $a$ entails that $abJ^n = 0$. Since $J$ is a graded ideal we have $aJ^{n+1} = 0$ i.e. $a \in l_R(J^{n+1}) = l_R(J^n)$, a contradiction. Therefore $J^{n+1} = 0$. □

### 8.4.2 Lemma

Let $R$ be a gr-semi-prime ring satisfying the ascending chain condition on graded left annihilators. If $I$ is a left (or right) ideal such that every element of $h(I)$ is nilpotent then $I = 0$.

**Proof** For $a \in h(R)$ every element of $h(Ra)$ is nilpotent if and only if every element of $h(aR)$ is nilpotent. It is therefore enough to prove the statement in case $I$ is a right ideal. If $I \neq 0$, there is $a \in I$, $a \neq 0$ such that $l_R(a)$ is maximal among the left annihilators of nonzero elements of $h(I)$. Let $\lambda \in h(R)$ such that $a\lambda \neq 0$. Then by hypothesis there is a $t > 0$ such that $(a\lambda)^t = 0$ and $(a\lambda)^{t-1} \neq 0$. From $l_R(a) \subset l_R((a\lambda)^{t-1})$ it follows that $l_R(a) = l_R((a\lambda)^{t-1})$. Therefore $a\lambda a = 0$ and hence $aRa = 0$, but this contradicts the fact that $R$ is a gr-semi-prime ring. □

### 8.4.3 Lemma (Goodearl, Stafford [79])

Assume that $R$ is a gr-semiprime left gr-Goldie ring. Let $a \in R$ be a homogeneous element such that $Ra$ is gr-uniform. Then its left annihilator $l_R(a)$ is maximal among all left annihilators of nonzero homogeneous elements of $R$.

**Proof** Assume that $l_R(a) \subset J = l_R(b)$ and $l_R(a) \neq l_R(b)$ for some homogeneous element $b \in R$. Then $Ja \neq 0$ and the gr-uniformity of $R$ implies that $Ja$ is gr-essential in $Ra$. Therefore $Ra/Ja$ is a gr-singular left $R$-module. But $Ra/Ja \simeq R/J \simeq Rb$ because $l_R(a) \subseteq J$. Hence $Rb$ is gr-singular. Lemma 8.4.1 allows to conclude that $b = 0$.

### 8.4.4 Theorem (Goodearl and Stafford [79])

Let $R$ be a $G$-graded ring $G$ an abelian group (semigroup). If $R$ is a gr-prime and left gr-Goldie ring then any essential graded left ideal $I$ of $R$ contains a homogeneous regular element.

**Proof** An element $a \in h(R)$ for which $Ra$ is gr-uniform is called **gr-uniform element**. Lemma 8.4.2 and the hypothesis imply that there is a non-nilpotent gr-uniform element $a_1 \in R$. By induction, suppose that we have found the non-nilpotent gr-uniform elements $a_1, ..., a_m \in I$ such that $a_i \in \cap_{1 \leq j \leq i-1} l_R(a_j)$ for $1 < i \leq m$. If $X = \cap_{1 \leq j \leq m} l_R(a_j) \neq 0$ then $I \cap X \neq 0$. Then Lemma 8.4.1 entails that there is a non-nilpotent gr-uniform element $a_{m+1} \in I \cap X$. Since $a_i \in l_R(a_j) = l_R(a_j^2)$ for $i > j$ (Lemma 8.4.3) we obtain that the sum $\sum_{i \geq 1} Ra_i$ is an internal direct sum. Since $R$ has finite graded Goldie dimension the process terminates. This implies that for some $n$ we have : $\cap_{i=1}^{n} l_R(a_i) = 0$. Since $R$ is gr-prime and the $a_i$ are non-nilpotent, $Ra_1^2 Ra_2^2 ... Ra_n^2 \neq 0$ thus $a_1^2 Ra_2^2 ... Ra_n^2 \neq 0$. Hence there exists homogeneous elements $s_2, ..., s_n \in R$ such that $a_1^2 s_2 a_2^2 s_3 ... s_n a_n^2 \neq 0$. From Lemma 8.4.2 we retain that there is a homogeneous element $s_1$ such that $c = s_1 a_1^2 s_2 a_2^2 ... s_n a_n^2$ is not nilpotent. For any $1 \leq i \leq n$ define $d_i = (a_i s_{i+1} a_{i+1}^2 ... s_n a_n^2)(s_1 a_1^2 ... s_i a_i)$. Note that the $d_i$ are sub-words of $c^2$ hence $d_i \neq 0$ for any $i = 1, ..., n$. Since $Rd_i \subseteq Ra_i$ we obtain that the sum $\sum_{i=1}^{n} Rd_i$ is direct. Lemma 8.4.3 yields that $l_R(d_i) = l_R(a_i)$, $1 \leq i \leq n$. Hence $l_R(d_1 + ... + d_n) = \cap_{i=1}^{n} l_R(d_i) = \cap l_R(a_i) = 0$. The fact that $G$ is abelian entails $\deg(d_i) = \deg(c)$ and therefore $d = d_1 + ... + d_n$ is homogeneous. Moreover $d \in I$ and $l_R(d) = 0$ hence $d$ is regular. □

### 8.4.5 Theorem (The Graded Version of Goldie's Theorem)

Let $R$ be a $G$-graded ring, where $G$ is an abelian (semigroup) group. If $R$ is a gr-prime left gr-Goldie ring then $R$ has a gr-simple, gr-Artinian left ring of fractions.

**Proof** The proof of this Theorem follows from Theorem 8.4.4 as in the ungraded case (see [72]).

### 8.4.6 Corollary

Let $R$ be a $G$-graded ring, where $G$ is an abelian (semigroup) group. If $R$ is gr-prime and left gr-Noetherian then $R$ has gr-simple, gr-Artinian ring of fractions.

**Proof** It is clear that if $R$ is left gr-Noetherian, then $R$ is a left gr-Goldie ring. □

In the ungraded context, there also exists a second Goldie's Theorem in case $R$ is a semi-prime left Goldie ring. Unfortunately a graded version of Goldie's Theorem for gr-semi-prime left gr-Goldie ring does not exist, as the following example shows.

### 8.4.7 Example [103]

Let $k$ be a field and let $R$ be the ring $k[X,Y]$ where the generators $X$ and $Y$ are subjected to the relation $XY = YX = 0$. Put $R_n = kX^n$ if $n \geq 0$ and $R_m = kY^m$ if $m \leq 0$. It is clear that each $x \neq 0$ in $h(R)$ with $\deg x \neq 0$ is non-regular. Moreover the ideal $(X,Y)$ is essential in $R$ but it does not contain a regular homogeneous element. This shows that, although $R$ is a gr-semi-prime gr-Goldie ring, it does not have a gr-semisimple grArtinian ring of fractions.

### 8.4.8 Remark

The book [150], Theorem I. 1.6 provides sufficient conditions for a gr-semi-prime gr-Goldie ring to have a gr-semisimple gr-Artinian ring of fractions. Finally we establish that for strongly graded rings, graded by by a finite group, the second Goldie Theorem does hold.

### 8.4.9 Theorem

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a strongly graded ring where $n = |G| < \infty$. If $R$ is a gr-semi-prime left gr-Goldie ring then $R_e$ is a semi-prime left Goldie ring. In this case there exists a left classical ring of fractions $Q_{cl}(R)$. In this case $Q_{cl}(R)$ is a strongly graded ring with $(Q_{cl}(R))_e = Q_{cl}(R_e)$. Moreover $Q_{cl}(R)$ is a gr-semisimple, gr-Artinian ring.

**Proof** That $R_e$ is a semi-prime ring follows from Theorem 2.11.4. Since $R$ is a strongly graded ring, $R_e$ is also a left Goldie ring thus there exists the left classical ring $Q_{cl}(R_e)$ where $Q_{cl}(R_e)$ a is semisimple Artinian ring. Now if $I$ is a left graded essential ideal of $R$, then $I \cap R_e$ is a left essential ideal of $R_e$. Thus there exist a regular element $s \in I \cap R_e$. We denote by $S$ the set of all regular elements of $R_e$. Every $s \in S$ is also a regular element in $R$. Indeed if $a_\sigma s = 0$ for $a_\sigma \in R_\sigma$ then since $R_\sigma R_{\sigma^{-1}} = R_e$, we have $1 = \sum_{i=1}^{n} a_i b_i$ where $a_i \in R_\sigma$, $b_i \in R_{\sigma^{-1}}$. Then for any $1 \leq i \leq n$ we have $(b_i a_\sigma)s = 0$. Since $b_i a_\sigma \in R_e$ we have $b_i a_\sigma = 0$ so $a_\sigma = \sum (a_i b_i) a_\sigma = 0$. Clearly if $a \in R$ is an element such that $as = 0$, we arrive at $a = 0$. In a similar way we obtain that $sa = 0$ implies $a = 0$. So $s$ in regular in $R$. Hence there exist $Q_{cl}(R) = S^{-1}R$ and $Q_{cl}(R)$ is a gr-semi-prime gr-Artinian ring. Also $(Q_{cl}(R))_e = S^{-1}R_e = Q_{cl}(R_e)$. □

## 8.5 Exercises

1. Let $R$ be a $G$-graded ring and $\mathcal{C}$ be a rigid localizing subcategory of $R$-gr. We denote by $H$ the graded linear topology associated to $\mathcal{C}$ and by $t_H$ the radical on $R$-gr associated to $\mathcal{C}$ (or to $H$). Denote by $\overline{\mathcal{C}}$ (resp $\overline{\mathcal{H}}$) the smallest localizing subcategory of $R$-mod (resp the smallest linear

topology on $R$) that contains $\mathcal{C}$ (resp $H$). We also denote by $t_{\overline{H}}$ the radical of $R$-mod associated to $H$. If $M \in R$-mod we denote by $Q_H(M)$ (resp $Q_{\overline{H}}(M)$) the quotient module of $M$ relative to $H$ (resp $\overline{H}$) (see section 8.3). Recall that we denote by $E^g(M)$ (resp $E(M)$) the injective envelope of $M$ in $R$-gr (resp $R$-mod). If $M \in R$-gr, $M$ is called gr-$H$-closed if the canonical homomorphisms

$$M \simeq \mathrm{HOM}_R(R, M) \to \mathrm{HOM}_R(I, M)$$

are isomorphisms for all $I \in H$. Prove that :

a) $M$ is gr-$H$-closed.

b) i) $M$ is $H$ gr-torsion free i.e. $t_H(M) = 0$

ii) If $N \in R$-gr and $P$ is a graded submodule of $N$ such that $N/P \in \mathcal{C}$, then the canonical homomorphism

$$\mathrm{Hom}_{R-\mathrm{gr}}(N, M) \to \mathrm{Hom}_{R-\mathrm{gr}}(P, M)$$

is an isomorphism.

c) The canonical morphism $\Psi(M) : M \to Q_H(M)$ is an isomorphism.

**Hint :** The same proof as the ungraded case (see [167], chapter IX)

2. With the notations above prove that if $\sup(R) < \infty$ and $\sup(M) < \infty$, then $Q_H(M) \simeq Q_{\overline{H}}(M)$ (see also Proposition 8.3.1)

**Hint :** It is necessary to prove that $X = Q_H(M)$ is $\overline{H}$-closed. Indeed if $J \in \overline{H}$, then there is an $I \in H$ such that $I \subseteq J$. Since $J/I \in \overline{\mathcal{C}}$ and $X$ is $\overline{H}$-torsion free, we have $\mathrm{Hom_R}(J/I, X) = 0$. On the other hand since $\mathrm{HOM}_R(I, M) = \mathrm{HOM}(I, M)$ it follows that $\mathrm{Hom}_R(I, M) = \mathrm{Hom}_R(J, M)$, so the canonical morphism $M \simeq \mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(J, M)$ is an isomorphism.

3. With the notations from exercise 1, if $M \in R$-gr is $H$-torsion free, then

$$Q_H(M) = \{x \in E^g(M) \mid \exists I \in H,\ Ix \subseteq M\}$$

**Hint :** It is easy to see that $Y = \{x \in E^g(M) \mid \exists I \in H,\ Ix \subseteq M\}$ is gr-$H$-closed.

4. Let $P \in R$-gr be a gr-projective module (i.e. $P$ is also projective in $R$-mod). We denote by

$$\mathcal{C}_P^{\mathrm{gr}} = \{M \in R - \mathrm{gr} \mid \mathrm{HOM}_R(P, M) = 0\}$$

$$\mathcal{C}_P = \{N \in R - \mathrm{mod} \mid \mathrm{Hom}_R(P, N) = 0\}$$

Prove that:

i)
$$\mathcal{C}_P^{\mathrm{gr}} = \{M \in R - \mathrm{gr} \mid \mathrm{Hom}_{R-\mathrm{gr}}(P(\sigma), M) = 0\}$$
for any $\sigma \in G$ and $\mathcal{C}_P^{\mathrm{gr}}$ is a rigid localizing subcategory of $R$-gr.

ii) $\mathcal{C}_P = \overline{\mathcal{C}_P^{\mathrm{gr}}}$

**Hint :**

i) (i) is clear

ii) (ii) $\mathcal{C}_P^{\mathrm{gr}} \subseteq \mathcal{C}_P$. Indeed if $M \in \mathcal{C}_P^{\mathrm{gr}}$ we have $\mathrm{HOM}_R(P, M) = 0$. Using Theorem 2.4.3 we have $\mathrm{Hom}_R(P, M) = 0$ so $\overline{\mathcal{C}_P^{\mathrm{gr}}} \subseteq \mathcal{C}_P$. Conversely, let $N \in \mathcal{C}_P$. Since $\mathrm{Hom}_{R-\mathrm{gr}}(P_\sigma, F(N)) = \mathrm{Hom}_R(U(P(\sigma)), N) = \mathrm{Hom}_R(P, N) = 0$, it follows that $F(N) \in \mathcal{C}_P^{\mathrm{gr}}$, so $N \in \overline{\mathcal{C}_P^{\mathrm{gr}}}$ (here $F$ is right adjoint to the forgetful functor $U : R - \mathrm{gr} \longrightarrow R - \mathrm{mod}$).

5. With notation as in exercise 1, prove that, for any $\sigma \in G$ :
$$Q_H(M(\sigma)) = Q_H(M)(\sigma)$$

**Hint :** If $\lambda \in G$, we have

$$\begin{aligned} Q_H(M(\sigma))_\lambda &= \varinjlim_{I \in H} \mathrm{HOM}_R(I, M(\sigma)/t_M(M(\sigma)))_\lambda \\ &= \varinjlim_{I \in H} \mathrm{Hom}_{R-\mathrm{gr}}(I, M(\sigma)(\lambda)/t_H(M(\sigma)(\lambda))) \\ &= \varinjlim_{I \in H} \mathrm{Hom}_{R-\mathrm{gr}}(I, M(\lambda\sigma)/t(M(\lambda\sigma))) = Q_H(M)_{\lambda\sigma} = \\ &= Q_H(M)(\sigma)_\lambda \end{aligned}$$

So $Q_H(M(\sigma)) = Q_H(M)(\sigma)$.

## 8.6 Comments and References for Chapter 8

Localization and local-global methods constitute a very effective tool, particularly in commutative algebra and algebraic geometry e.g. scheme theory. Geometrically speaking, passing to local data or phenomena valid in the neighbourhood of a point, is very natural and intuitive. Algebraically speaking this may be traced to the construction of a ring of fractions of some commutative ring (e.g. the coordinate ring of some algebraic variety) with respect to a prime ideal. This localization defines then a new ring of fractions that is indeed "local" in the sense that it has a unique maximal ideal and very element outside it has become invertible in the localized ring. In the noncommutative case however, even though localization techniques of a general nature do exist, the result is not always that satisfactory. For prime or semi-prime (left) Noetherian rings we have the well-known theory related to Goldie's theorems, but this is essentially dealing with a localization at the zero ideal ! In the presence of a gradation the situation becomes more complicated even, because in Goldie's theory the existence of regular elements in essential left ideals is necessary, but for homogeneous left ideals the presence of a regular element may not imply the presence of a homogeneous regular element. A study of Goldie's theorems for graded rings began in [103] and for $\mathbb{Z}$-graded rings in [150]. Localization theory of a more general type had been applied to $\mathbb{Z}$-graded rings by F. Van Oystaeyen (a. o. [189]) in an attempt to arrive at a projective scheme structure e.g. [187].

Section 8.1. contains a brief presentation of graded rings of fractions. In Section 8.2. and Section 8.3. the more general theory of localization, e.g. in the sense of P. Gabriel [67], is presented using a homogeneous linear Gabriel topology. We avoid repetition of the material already well documented in our book [150], but include some recent results and developments. For example, in Section 8.4. we include a recent result by K. Goodearl, J. Stafford [79] providing a generalization of the graded version of Goldie's theorem in the gr-prime case for an abelian group, or in the more general case of an abelian semigroup.

### Some References

- P. Gabriel [67]
- K. Goodearl, T. Stafford [79]
- I. D. Ion, C. Năstăsescu [103]
- A. Jensen, Jøndrup [105], [106]
- L. Le Bruyn, M. Van den Bergh, F. Van Oystaeyen [117]
- C. Năstăsescu [141], [143]

- C. Năstăsescu, F. Van Oystaeyen [150], [148]
- F. Van Oystaeyen [188], [192], [194]
- F. Van Oystaeyen, A. Verschoren [199], [200]

# Chapter 9

# Application to Gradability

## 9.1 General Descent Theory

Let $\mathcal{A}$ and $\mathcal{B}$ be two categories and let $F : \mathcal{A} \to \mathcal{B}$ be a functor. In the case where $\mathcal{A}$ and $\mathcal{B}$ are additive categories, we assume that $F$ is an additive functor. Following the classical descent theory (see [115]) we can introduce a descent theory relative to the functor $F$. Consider an object $N \in \mathcal{B}$. We have the following problems.

i) *Existence of $F$-descent objects :* does an object $M \in \mathcal{A}$ exist such that $N \simeq F(M)$ ?

ii) *Classification :* if such an $F$-descent object exists, classify (up to isomorphism) all objects $M$ for which $N \simeq F(M)$.

### 9.1.1 Remarks

i) If the functor $F$ is an equivalence of categories, then for any $N \in \mathcal{B}$ there exists an unique (up to isomorphism) $F$-descent object.

ii) Assume that we have two functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$. If for $Z \in \mathcal{C}$ there exists a $G \circ F$-descent object $X \in \mathcal{A}$, then $F(X)$ is a $G$-descent object for $Z$.

iii) If $F$ commutes with finite (arbitrary) coproducts or products, then any finite (arbitrary) coproduct or product of $F$-descent objects is also an $F$-descent object.

iv) Assume that $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F$ is a faithful and exact functor which preserves isomorphisms. If $M \in \mathcal{B}$ is a simple object and $N \in \mathcal{A}$ is an $F$-descent object for $M$, then $N$ is simple in $\mathcal{A}$. Indeed, if $i : X \to N$ is a non-zero monomorphism in $\mathcal{A}$, then $F(i) : F(X) \to F(N) \simeq M$ is a nonzero monomorphism,

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therefore $F(N) \simeq M$. Since $M$ is simple we see that $F(i)$ is an isomorphism, and then so is $i$. Thus $N$ is a simple object of $\mathcal{A}$.

### 9.1.2 Example

1. Let $R$ and $S$ be two rings and $\phi : R \to S$ be a ring morphism. Let $\mathcal{A} = R$-mod and $\mathcal{B} = S$-mod be the categories of modules. We have the following three natural functors :

   - $S \otimes_R - : R\text{-mod} \to S\text{-mod}$ (the induced functor)
   - $\mathrm{Hom}_R({}_RS, -) : R\text{-mod} \to S\text{-mod}$ (the coinduced functor)
   - $\varphi_* : S\text{-mod} \to R\text{-mod}$ (the restriction of scalars)

   When $S = l, R = k$ and $l$ is a commutative faithfully flat $k$-algebra, then the descent theory relative to the induced functor is exactly the classical descent theory.

2. Assume that $R \subseteq S$ is a ring inclusion. Then the descent theory relative to the functor $\phi_*$ (here $\phi : R \to S$ is the inclusion morphism) is exactly *the problem of extending the module structure*, i.e. of investigating whether for $M \in R$-mod there exists a structure of an $S$-module on $M$ which by the restriction of scalars to $R$ gives exactly the initial $R$-module structure on $M$. In particular, if $S = \oplus_{\sigma \in G} S_\sigma$ is a $G$-strongly graded ring and $R = S_e$, we obtain the theory of extending modules given in Section 4.7.

3. Let $R = \oplus_{\sigma \in G} R_\sigma$ be a $G$-graded ring. We consider the forgetful functor $U : R\text{-gr} \to R\text{-mod}$. If $M \in R$-mod and there exists an $U$-descent object $N \in R$-gr for $M$, i.e. $U(N) \simeq M$, then $M$ is called *a gradable module*. If $G$ is a finite group, we can consider the smash product $\tilde{R}\#G$ and the natural morphism $\eta : R \to \tilde{R}\#G$ (see Chapter 7). By Proposition 7.3.10, $M \in R$-mod is gradable if and only if $M$ has an extending relative to the morphism $\eta$.

Using the structure of gr-injective modules (Section 2.8) we have the following.

### 9.1.3 Proposition

Let $G$ be a finite group, $R$ a $G$-graded ring and $Q$ an injective $R$-module. The following assertions are equivalent.

i. $Q$ is gradable.

ii. There exists an injective $R_e$-module $N$ such that $Q \simeq Coind(N) = Hom_{R_e}(R, N)$ as $R$-modules.

**Proof** $(ii) \Rightarrow (i)$ Since $Coind(N)$ has a natural structure of an $R$-graded module, we have that $Q$ is gradable.
$(i) \Rightarrow (ii)$ If $Q$ is gradable, then there exists an injective object $M \in R$-gr such that $Q \simeq U(M)$ in $R$-mod. By Corollary 2.8.8 there exist $\sigma_1, \ldots, \sigma_s \in G$ and $N_1, \ldots, N_s$ injective $R_e$-modules such that $M \simeq \oplus_{i=1,s} Coind(N_i)(\sigma_i^{-1})$ in $R$-gr. If we take $N = \oplus_{i=1,s} N_i$, then we have $Q \simeq U(N)$ in $R$-mod. □

## 9.2 Good gradings on matrix algebras

Let $k$ be a field and $M_n(k)$ be the matrix algebra. We denote by $(E_{i,j})_{1 \leq i,j \leq n}$ the matrix units of $M_n(k)$, i.e. $E_{i,j}$ is the matrix having 1 on the $(i,j)$-position and 0 elsewhere.

### 9.2.1 Definition

A grading of $M_n(k)$ is called a good grading if all the matrix units $E_{i,j}$ are homogeneous elements.

### 9.2.2 Lemma

Let us consider a good $G$-grading on $M_n(k)$. Then $\deg(E_{i,i}) = e$,

$$\deg(E_{i,j}) = \deg(E_{i,i+1})\deg(E_{i+1,i+2}) \ldots \deg(E_{j-1,j}) \,\text{for i} < \text{j}$$

and

$$\deg(E_{i,j}) = \deg(E_{i-1,i})^{-1}\deg(E_{i-2,i-1})^{-1} \ldots \deg(E_{j,j+1})^{-1} \text{for i} > \text{j}$$

**Proof** Since $E_{i,i}$ is a homogeneous idempotent, we see that $\deg(E_{i,i}) = e$. The second relation follows from $E_{i,j} = E_{i,i+1}E_{i+1,i+2} \ldots E_{j-1,j}$ for any $i < j$. The third relation follows then from $E_{i,j}E_{j,i} = E_{i,i}$, which implies $\deg(E_{i,j}) = \deg(E_{j,i})^{-1}$. □

In the following we count the good gradings on $M_n(k)$.

### 9.2.3 Proposition

There is a bijective correspondence between the set of all good $G$-gradings on $M_n(k)$, and the set of all maps $f : \{1, 2, \ldots, n-1\} \to G$, such that to a good $G$-grading we associate the map defined by $f(i) = \deg(E_{i,i+1})$ for any $1 \leq i \leq n-1$.

**Proof** Lemma 9.2.2 shows that in order to define a good $G$-grading on $M_n(k)$, it is enough to assign some degrees to the elements

$$E_{1,2}, E_{2,3}, \ldots, E_{n-1,n}$$

The inverse of the correspondence mentioned in the statement takes a map $f : \{1, 2, \ldots, n-1\} \to G$ to the $G$-grading of $M_n(k)$ such that

$$\deg(E_{i,i}) = e,$$

$$\deg(E_{i,j}) = f(i)f(i+1)\cdots f(j-1), \text{ and}$$

$$\deg(E_{j,i}) = f(j-1)^{-1}f(j-2)^{-1}\cdots f(i)^{-1}$$

for any $1 \leq i < j \leq n$. □

### 9.2.4 Corollary

There exist $\mid G \mid^{n-1}$ good gradings on $M_n(k)$.

In Definition 9.2 good gradings are introduced from an interior point of view. There is an alternative way to define good gradings, from an exterior point of view. Let $R$ be a $G$-graded ring, and $V$ a right $G$-graded $R$-module. For any $\sigma \in G$ let

$$\mathrm{END(V)}_\sigma = \{\mathrm{f} \in \mathrm{End(V)} \mid \mathrm{f(V_g)} \subseteq \mathrm{V}_{\sigma\mathrm{g}} \text{ for any } \mathrm{g} \in \mathrm{G}\}$$

which is an additive subgroup of $\mathrm{End_R(V)}$. Note that

$$\mathrm{END(V)}_\sigma = \mathrm{Hom}_{\mathrm{R-gr}}(\mathrm{V}, (\sigma)\mathrm{V})$$

where $(\sigma)V$ is the $G$-graded right $R$-module which is just $V$ as an $R$-module, and has the shifted grading $(\sigma)V_g = V_{\sigma g}$ for any $g \in G$. Then the sum $\sum_{\sigma\in G} \mathrm{END(V)}_\sigma$ is direct, and we denote by $\mathrm{END_R(V)} = \oplus_{\sigma\in\mathrm{G}}\mathrm{END(V)}_\sigma$, which is a $G$-graded ring. A similar construction can be performed for left graded modules. If $R = k$ with the trivial $G$-grading, then a right graded $R$-module is just a vector space $V$ with a $G$-grading, i.e. $V = \bigoplus_{g\in G} V_g$ for some subspaces $(V_g)_{g\in G}$. In this situation we denote by $\mathrm{END(V)} = \mathrm{END_k(V)}$ and $\mathrm{End(V)} = \mathrm{End_k(V)}$. If $V$ has finite dimension $n$, then $\mathrm{END(V)} = \mathrm{End(V)} \simeq \mathrm{M_n(k)}$, and this induces a $G$-grading on $M_n(k)$. If $(v_i)_{1\leq i\leq n}$ is a basis of homogeneous elements of $V$, say $\deg(v_i) = g_i$ for any $1 \leq i \leq n$, let $(F_{i,j})_{1\leq i,j\leq n}$ be the basis of $\mathrm{End(V)}$ defined by $F_{i,j}(v_t) = \delta_{t,j}v_i$ for $1 \leq i, j, t \leq n$. Clearly $\deg(F_{i,j}) = g_i g_j^{-1}$. We have an algebra isomorphism between $\mathrm{End(V)}$ and $M_n(k)$ by taking $F_{i,j}$ to $E_{i,j}$ for $i, j$, and in this way $M_n(k)$ is endowed with a good $G$-grading. In fact any good $G$-grading can be produced this way.

### 9.2.5 Proposition

Let us consider a good $G$-grading on $M_n(k)$. Then there exists a $G$-graded vector space $V$, such that the isomorphism $\mathrm{END}(V) \cong \mathrm{End(V)} \cong \mathrm{M_n(k)}$ with respect to a homogeneous basis of $V$, is an isomorphism of $G$-graded algebras.

**Proof** We must find some $g_1, \ldots, g_n \in G$ such that $\deg(E_{i,j}) = g_i g_j^{-1}$ for any $i, j$. Proposition 9.2.3. shows that it is enough to check this for the pairs $(i,j) \in \{(1,2),(2,3),\ldots,(n-1,n)\}$, i.e. $g_i g_{i+1}^{-1} = \deg(E_{i,i+1})$ for any $1 \leq i \leq n-1$. But clearly $g_n = e$, $g_i = \deg(E_{i,i+1})\deg(E_{i+1,i+2}) \ldots \deg(E_{n-1,n})$ for any $1 \leq i \leq n-1$, is such a set of group elements. □

A $G$-grading of the $k$-algebra $M_n(k)$ is good if all $E_{i,j}$'s are homogeneous elements. However there exist gradings which are not good, but are isomorphic to good gradings, as the following example shows.

## 9.2.6 Example

Let $R = S = M_2(k)$ with the $C_2 = \{e, g\}$-grading defined by

$$R_e = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \quad R_g = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix},$$

$$S_e = \left\{ \begin{pmatrix} a & b-a \\ 0 & b \end{pmatrix} \middle| \; a, b \in k \right\}, \quad S_g = \left\{ \begin{pmatrix} d & c \\ d & -d \end{pmatrix} \middle| \; c, d \in k \right\}.$$

Then the map

$$f : R \to S, \; f\left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = \begin{pmatrix} a+c & b+d-a-c \\ c & d-c \end{pmatrix}.$$

is an isomorphism of $C_2$-graded algebras. The grading of $S$ is not good, since $E_{1,1}$ is not homogeneous, but $S$ is isomorphic as a graded algebra to $R$, which has a good grading. □

We see that in the previous example, although the grading of $S$ is not good, the element $E_{1,2}$ is homogeneous. A corollary of the following result shows that in general a grading of the algebra $M_n(k)$ is isomorphic to a good grading whenever one of the $E_{i,j}$'s is a homogeneous element.

## 9.2.7 Theorem

Let $R$ be the algebra $M_n(k)$ endowed with a $G$-grading such that there exists $V \in R$-gr which is simple as an $R$-module. Then there exists an isomorphism of graded algebras $R \cong S$, where $S$ is $M_n(k)$ endowed with a certain good grading.

**Proof** As a simple $M_n(k)$-module, $V$ must have dimension $n$. Let $\Delta = \mathrm{END}_R(V)$, as a $G$-graded algebra with multiplication the inverse map composition, hence $V$ is a $G$-graded right $\Delta$-module. We may consider $\mathrm{BIEND}_R(V) = \mathrm{END}_\Delta(V)$, a $G$-graded algebra with map composition as multiplication. Since $V$ is a simple $R$-module and $\Delta = \mathrm{END}_R(V) = \mathrm{End}_R(V) \cong k$, so $\Delta$ is

isomorphic to $k$ with the trivial grading. This shows that $\mathrm{BIEND_R(V)}$ is the endomorphism algebra of a $G$-graded $k$-vector space of dimension $n$, thus it is isomorphic to $M_n(k)$ with a certain good $G$-grading. On the other hand, the graded version of the Density Theorem (see Section 4.6) shows that the map $\phi : R \to \mathrm{BIEND_R(V)}$, $\phi(r)(v) = rv$ for $r \in R,\ v \in V$, is a surjective morphism of $G$-graded algebras. Because $\mathrm{Ann_R(V)} = 0$, we see that $\phi$ is injective, hence an isomorphism. □

### 9.2.8 Corollary

If $G$ is a torsion-free group, then a $G$-grading of $M_n(k)$ is isomorphic to a good grading.

**Proof** Let $V$ be a graded simple module (with respect to the given $G$-grading of $M_n(k)$). Then (see Section 4.4.) $V$ is a simple $R$-module, and the result follows from Theorem 9.2.7 □

### 9.2.9 Corollary

Let $R$ be the algebra $M_n(k)$ endowed with a $G$-grading such that the element $E_{i,j}$ is homogeneous for some $i, j \in \{1, \ldots, n\}$. Then there exists an isomorphism of graded algebras $R \cong S$, where $S$ is $M_n(k)$ endowed with a good grading.

**Proof** Since $E_{i,j}$ is a homogeneous element, then $V = RE_{i,j}$ is a $G$-graded $R$-submodule of $R$. Clearly $V$ is the set of the matrices with zero entries outside the $j^{\mathrm{th}}$ column, so $V$ is a simple $R$-module, and we apply Theorem 9.2.7. □

### 9.2.10 Example

There exist gradings isomorphic to good gradings, but where no $E_{i,j}$ is homogeneous. Let $R$ be the $C_2$-graded algebra from Example 9.2.6 and $S = M_n(k)$ with the grading

$$S_e = \left\{ \left( \begin{array}{cc} 2a-b & -2a+2b \\ a-b & -a+2b \end{array} \right) \middle| \ a, b \in k \right\},$$

$$S_g = \left\{ \left( \begin{array}{cc} a-2b & -a+4b \\ a-b & -a+2b \end{array} \right) \middle| \ a, b \in k \right\}.$$

Then the map

$$f : R \to S,\ f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} 2a+c-2b-d & -2a-c+4b+2d \\ a+c-b-d & -a-c+2b+2d \end{pmatrix}.$$

is an isomorphism of $C_2$-graded algebras. However, none of the elements $E_{i,j}$ is homogeneous in $S$. □

We will describe now the good $G$-gradings making $M_n(k)$ a strongly graded (respectively a crossed-product) algebra.

### 9.2.11 Proposition

Let us consider the algebra $\mathrm{End(V)} \cong M_n(k)$ with a good $G$-grading such that $\deg(E_{i,i+1}) = h_i$ for $1 \leq i \leq n-1$, where $V = \bigoplus_{g \in G} V_g$ is a graded vector space. The following assertions are equivalent :

i. $M_n(k)$ is a strongly graded algebra

ii. $V_g \neq 0$ for any $g \in G$

iii. All elements of $G$ appear in the sequence $e, h_1, h_1h_2, \ldots, h_1h_2 \cdots h_{n-1}$.

**Proof** We recall that $V$ is an object of $gr-k$, where $k$ is viewed as a $G$-graded algebra with the trivial grading. Then $\mathrm{END(V)} = \mathrm{End(V)}$ is strongly graded if and only if $V$ weakly divides $(\sigma)V$ for any $\sigma \in G$. This means that $V$ is isomorphic to a graded submodule of a finite direct sum of copies of $(\sigma)V$ in $gr-k$, and it is clearly equivalent to $V_g \neq 0$ for any $g \in G$. If $g_1, \ldots, g_n$ are the degrees of the elements of the basis of $V$ which induces the isomorphism $\mathrm{End(V)} \cong M_n(k)$, then

$$h_1 = g_1g_2^{-1},\ h_2 = g_2g_3^{-1}, \ldots, h_{n-1} = g_{n-1}g_n^{-1},\ \text{thus}$$

$$g_2 = h_1^{-1}g_1,\ g_3 = h_2^{-1}h_1^{-1}g_1, \ldots, g_n = h_{n-1}^{-1}h_{n-2}^{-1} \cdots h_1^{-1}g_1.$$

Then $V_g \neq 0$ for any $g \in G$ if and only if all the elements of $G$ appear in the sequence $g_1, g_2, \ldots, g_n$. Since

$$g_2 = h_1^{-1}g_1,\ g_3 = h_2^{-1}h_1^{-1}g_1, \ldots, g_n = h_{n-1}^{-1}h_{n-2}^{-1} \cdots h_1^{-1}g_1$$

this is equivalent to iii) in the statement. □

### 9.2.12 Corollary

If $M_n(k)$ has a good $G$-grading making it a strongly graded algebra, then $|G| \leq n$.

### 9.2.13 Corollary

Let $|G| = m \leq n$. Then the number of good gradings on $M_n(k)$ making it a strongly graded algebra is $m^{n-1}+(m-1)^{n-1}-\sum_{i=1,m-1}(-1)^{i+1}\begin{pmatrix} m \\ i \end{pmatrix}(m-i)^{n-1} - \sum_{i=1,m-2}(-1)^{i+1}\begin{pmatrix} m-1 \\ i \end{pmatrix}(m-i-1)^{n-1}$.

**Proof** Let $x_1 = h_1, x_2 = h_1h_2, \ldots, x_{n-1} = h_1h_2\cdots h_{n-1}$. Clearly the $(n-1)$-tuples $(h_1, h_2, \ldots, h_{n-1})$ and $(x_1, x_2, \ldots, x_{n-1})$ uniquely determine each other, so we have to count the number of maps $f : \{1, 2, \ldots, n-1\} \to G$ such that $G - \{e\} \subseteq \Im(f)$. This is $N_1 + N_2$, where $N_1$ (respectively $N_2$) is the number of surjective maps $f : \{1, 2, \ldots, n-1\} \to G - \{e\}$ (respectively $f : \{1, 2, \ldots, n-1\} \to G$). A classical combinatorial fact shows that

$$N_1 = m^{n-1} - \sum_{i=1,m-1}(-1)^{i+1}\begin{pmatrix} m \\ i \end{pmatrix} \quad \text{and}$$

$$N_2 = (m-1)^{n-1} - \sum_{i=1,m-2}(-1)^{i+1}\begin{pmatrix} m-1 \\ i \end{pmatrix}(m-i-1)^{n-1}$$

□

### 9.2.14 Proposition

Let $\mathrm{End(V)} \cong M_n(k)$ with a good $G$-grading, where $V = \bigoplus_{g\in G} V_g$ is a graded vector space. The following assertions are equivalent :

i. $M_n(k)$ is a crossed product

ii. $\dim(V) = |G| \cdot \dim(V_g)$ for any $g \in G$

iii. $\dim(M_n(k)_e) \cdot |G| = n^2$.

**Proof** $\mathrm{END(V)} = \mathrm{End(V)}$ is a crossed product if and only if $\mathrm{End(V)}_\sigma = \mathrm{Hom_{k-gr}(V, (\sigma)V)}$ contains an invertible element for any $\sigma \in G$, which means that $V \cong (\sigma)V$ as $k$-graded modules. This is equivalent to $\dim(V_g) = \dim(V_{\sigma g})$ for all $\sigma, g \in G$, which is just ii. Thus i.⇔ii.

Clearly i.⇒iii. Suppose now that iii. holds. As $\mathrm{End(V)_e} = \mathrm{End_{k-gr}(V)} = \bigoplus_{\mathrm{g}\in\mathrm{G}} \mathrm{End(V_g)}$, we find $\dim(\mathrm{End(V)_e}) = \sum_{\mathrm{g}\in\mathrm{G}}(\dim(\mathrm{V_g}))^2$. Then

$$|G|\sum_{g\in G}(\dim(V_g))^2 = n^2 = \left(\sum_{g\in G}\dim(V_g)\right)^2$$

and the Cauchy-Schwarz inequality shows that all $\dim(V_g)$, $g \in G$ must be equal. Thus $\dim(V) = |G| \cdot \dim(V_g)$ for any $g \in G$.

### 9.2.15 Corollary

Any two crossed product structures on $M_n(k)$ which are good $G$-gradings, are isomorphic as graded algebras.

**Proof** The two graded algebras are isomorphic to End(V), respectively End(W), where $\dim(V_g) = \dim(W_g) = \frac{n}{|G|}$ for any $g \in G$. Therefore $V \cong W$ as $k$-graded modules, and this shows that End(V) $\cong$ End(W) as graded algebras. □

As examples, we give a description of all good $C_2$-gradings on $M_2(k)$ and all good $C_2$-gradings on $M_3(k)$.

### 9.2.16 Example

Let $R = M_2(k)$, $k$ an arbitrary field. Then a good $C_2$-grading of $R$ is of one of the following two types :

i. The trivial grading, $R_e = M_2(k), R_g = 0$;

ii. $R_e = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, R_g = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$

### 9.2.17 Example

Let $R = M_3(k)$, $k$ an arbitrary field. Then a good $C_2$-grading of $R$ is of one of the following types :

(i) The trivial grading, $R_e = M_2(k), R_g = 0$;

(ii) $R_e = \begin{pmatrix} k & k & 0 \\ k & k & 0 \\ 0 & 0 & k \end{pmatrix}, \ R_g = \begin{pmatrix} 0 & 0 & k \\ 0 & 0 & k \\ k & k & 0 \end{pmatrix}$

(iii) $R_e = \begin{pmatrix} k & 0 & 0 \\ 0 & k & k \\ 0 & k & k \end{pmatrix}, \ R_g = \begin{pmatrix} 0 & k & k \\ k & 0 & 0 \\ k & 0 & 0 \end{pmatrix}$ ;

(iv) $R_e = \begin{pmatrix} k & 0 & k \\ 0 & k & 0 \\ k & 0 & k \end{pmatrix}, \ R_g = \begin{pmatrix} 0 & k & 0 \\ k & 0 & k \\ 0 & k & 0 \end{pmatrix}$

Using Proposition 9.2.11 and Proposition 9.2.14, we see that the examples (ii), (iii) and (iv) are strongly graded rings, but they are not crossed products.

Our aim is to classify the isomorphism classes of good $G$-gradings.

### 9.2.18 Theorem

Let $V$ and $W$ be finite dimensional $G$-graded $k$-vector spaces. Then the graded algebras $END(V)$ and $END(W)$ are isomorphic if and only if there exists $\sigma \in G$ such that $W \simeq V(\sigma)$.

**Proof** Let $v_1, \ldots, v_m$ be a basis of $V$ consisting of homogeneous elements, say of degrees $g_1, \ldots, g_m$. For any $1 \leq i, j \leq m$ define $F_{ij} \in \mathrm{END(V)}$ by $F_{ij}(v_t) = \delta_{t,j} v_i$. Then $F_{ij}$ is a homogeneous element of degree $g_i g_j^{-1}$ of END(V), the set $(F_{ij})_{1 \leq i,j \leq m}$ is a basis of END(V), and $F_{ij}F_{rs} = \delta_{j,r}F_{is}$ for any $i, j, r, s$. In particular $(F_{ii})_{1 \leq i \leq m}$ is a complete system of orthogonal idempotents of END(V). Let $u : \mathrm{END(V)} \to \mathrm{END(W)}$ be an isomorphism of $G$-graded algebras, and define $F'_{ij} = u(F_{ij})$ for any $1 \leq i, j \leq m$. Denote $Q_i = Im(F'_{ii})$, which is a graded vector subspace of $W$. Since $(F'_{ii})_{1 \leq i \leq m}$ is a complete system of orthogonal idempotents of END(W), we have that $W = \oplus_{1 \leq i \leq m} Q_i$ and $F'_{ii}$ acts as identity on $Q_i$ for any $i$. Combined with the relation $F'_{ij}F'_{ji} = F'_{ii}$, this shows that $F'_{ij}$ induces an isomorphism of degree $g_i g_j^{-1}$ from $Q_i$ to $Q_j$. In particular $Q_j \simeq (g_1 g_j^{-1})Q_1$ in $gr - k$ for any $j$. We obtain that $W \simeq \oplus_{1 \leq j \leq m}(g_1 g_j^{-1})Q_1$, showing that $Q_1$ has dimension 1. Repeating this argument for the identity isomorphism from END(V) to END(V), and denoting by $R_1 = Im(F_{11})$, we obtain that $V = \oplus_{1 \leq j \leq m}(g_1 g_j^{-1})R_1$. Since $Q_1$ and $R_1$ are graded vector spaces of dimension 1, we have that $Q_1 \simeq R_1(\sigma)$ for some $\sigma \in G$. Hence $W \simeq \oplus_{1 \leq j \leq m}(g_1 g_j^{-1})Q_1 \simeq \oplus_{1 \leq j \leq m}(g_1 g_j^{-1})(R_1(\sigma)) \simeq V(\sigma)$. Conversely, it is easy to see that $\mathrm{END(V)} \simeq \mathrm{END(V}(\sigma))$ as $G$-graded algebras. □

The previous theorem may be used to classify all good $G$-gradings on the matrix algebra $\mathrm{M}_m(k)$. Indeed, any such grading is of the form END(V) for some $G$-graded vector space $V$ of dimension $m$. To such a $V$ we associate an $m$-tuple $(g_1, \ldots, g_m) \in G^m$ consisting of the degrees of the elements in a homogeneous basis of $V$. Conversely, to any such a $m$-tuple, associate a $G$-graded vector space of dimension $m$. Obviously, the $G$-graded vector space associated to a $m$-tuple coincides with the one associated to a permutation of the $m$-tuple. In fact Theorem 9.2.18 may be reformulated in terms of $m$-tuples as follows. If $V$ and $W$ are $G$-graded vector spaces of dimension $m$ associated to the $m$-tuples $(g_1, \ldots, g_m)$ and $(h_1, \ldots, h_m)$, then $\mathrm{END(V)} \simeq \mathrm{END(W)}$ as $G$-graded algebras if and only if there exist $\sigma \in G$ and $\pi$ a permutation of $\{1, \ldots, m\}$ such that $h_i = g_{\pi(i)}\sigma$ for any $1 \leq i \leq m$. We have established the following.

### 9.2.19 Corollary

The good $G$-gradings of $\mathrm{M}_m(k)$ are classified by the orbits of the biaction of the symmetric group $\mathcal{S}_m$ (from the left) and $G$ (by translation from the right) on the set $G^m$.

In the particular case where $G = C_n =< c >$, a cyclic group with $n$ elements, we are able to count the orbits of this biaction. Let $F_{n,m}$ be the set of all $n$-tuples $(k_0, k_1, \ldots, k_{n-1})$ of non-negative integers with the property that $k_0 + k_1 + \ldots + k_{n-1} = m$. It is a well-known combinatorial fact that $F_{n,m}$ has $\binom{m+n-1}{n-1}$ elements. To any element $(g_1, \ldots, g_m) \in G^m$ we can associate an element $(k_0, k_1, \ldots, k_{n-1}) \in F_{n,m}$ such that $k_i$ is the number of appearances of the element $c^i$ in the $m$-tuple $(g_1, \ldots, g_m)$ for any $1 \leq i \leq n$. By Corollary 9.2.19 we see that the number of orbits of the $(\mathcal{S}_m, G)$-biaction on $G^m$ is exactly the number of orbits of the left action by permutations of the subgroup $H =< \tau >$ of $\mathcal{S}_n$ on the set $F_{n,m}$, where $\tau$ is the cyclic permutation $(1\ 2\ \ldots n)$. This number of orbits is the one we will effectively compute.
If $\alpha = (k_0, k_1, \ldots, k_{n-1}) \in F_{n,m}$ and $d$ is a positive divisor of $n$, then $\tau^d \alpha = \alpha$ if and only if $k_{i+d} = k_i$ for any $0 \leq i \leq n-1-d$, i.e. the first $d$ positions of $\alpha$ repeat $\frac{n}{d}$ times. In particular we must have that $\frac{n}{d}$ divides $m$. For any such $d$, let us denote by $A_d$ the set of all $\alpha \in F_{n,m}$ stabilized by $\tau^d$. Since for $\alpha \in A_d$ we have that $k_0 + k_1 + \ldots + k_{d-1} = \frac{md}{n}$, we see that $A_d$ has $\binom{\frac{md}{n}+d-1}{d-1}$ elements.
Let $\mathcal{D}(n, m)$ be the set of all positive divisors $d$ of $n$ with the property that $n/d$ divides $m$. If $d_1, d_2 \in \mathcal{D}(n, m)$, then $(d_1, d_2) \in \mathcal{D}(n, m)$. Indeed, since $n$ divides $d_1 m$ and $d_2 m$, then $n$ also divides $(d_1 m, d_2 m) = (d_1, d_2)m$, therefore $(d_1, d_2) \in \mathcal{D}(n, m)$. It follows that $\mathcal{D}(n, m)$ is a lattice with the order given by divisibility. For any $d \in \mathcal{D}(n, m)$ we denote by $\mathcal{D}(n, m, d)$ the set of all elements $d'$ of $\mathcal{D}(n, m)$ which divide $d$, and by $\mathcal{D}_0(n, m, d)$ the set of all maximal elements of $\mathcal{D}(n, m, d)$. The following is immediate.

### 9.2.20 Lemma

For any $d_1, d_2 \in \mathcal{D}(n, m)$ we have that $A_{d_1} \cap A_{d_2} = A_{(d_1, d_2)}$.

The following provides a description of the elements with the orbit of length $d$.

### 9.2.21 Lemma

For any $d \in \mathcal{D}(n, m)$ denote by $B_d$ the set of all the elements of $F_{n,m}$ having the orbit of length $d$. Then

$$B_d = A_d - \bigcup_{d' \in \mathcal{D}(n,m,d)} A_{d'} = A_d - \bigcup_{d \in \mathcal{D}_0(n,m,d)} A_{d'}$$

**Proof** The orbit of an element $\alpha$ has length $d$ if and only if the stabilizer of $\alpha$ is a subgroup with $\frac{n}{d}$ elements of $H$, thus equal to $< \tau^d >$. The result follows now from the definition of $A_d$. □

### 9.2.22 Corollary

Let $d \in \mathcal{D}(n,m)$ and $p_1, \ldots, p_s$ all the distinct prime divisors of $d$ such that $\frac{d}{p_1}, \ldots, \frac{d}{p_s} \in \mathcal{D}(n,m)$. Then

$$|B_d| = |A_d| + \sum_{1 \le t \le s} \sum_{1 \le i_1 < \ldots < i_t \le s} (-1)^t |A_{\frac{d}{p_{i_1} \cdots p_{i_t}}}|$$

and this is known taking into account the fact that for any $d'$ we have that $|A_{d'}| = \binom{\frac{md'}{n} + d' - 1}{d' - 1}$.

**Proof** We have that $\mathcal{D}_0(n,m,d) = \{\frac{d}{p_1}, \ldots, \frac{d}{p_s}\}$. The result follows now from Lemma 9.2.21 and by applying the principle of inclusion and exclusion. □

### 9.2.23 Theorem

The number of isomorphism types of good $C_n$-gradings of the algebra $\mathrm{M}_m(k)$ is

$$\sum_{d \in \mathcal{D}(n,m)} \frac{1}{d} |B_d| = \sum_{d \in \mathcal{D}(n,m)} \frac{1}{d} (|A_d| - |\bigcup_{d' \in \mathcal{D}_0(n,m,d)} A_{d'}|)$$

**Proof** The number of isomorphism types of the good gradings is the number of orbits of the action of $H$ on $F_{n,m}$. We have seen that if the orbit of an element $\alpha \in F_{n,m}$ has length $d$, then necessarily $d \in \mathcal{D}(n,m)$. The result follows since the number of orbits of length $d$ is $\frac{1}{d}|B_d|$. □

### 9.2.24 Example

Let $n = p^r$ with prime $p$ and let $m$ be a positive integer. We define $q$ by $q = r$ in the case where $n$ divides $m$, or $q$ is the exponent of $p$ in $m$ in the case where $n$ does not divide $m$. Then $\mathcal{D}(n,m) = \{p^i | r - q \le i \le r\}$ and the number of isomorphism types of good $C_n$-gradings on $\mathrm{M}_m(k)$ is

$$\frac{1}{p^{r-q}} \binom{\frac{m}{p^q} + p^{r-q} - 1}{p^{r-q} - 1} + \sum_{r-q < i \le r} \frac{1}{p^i} \left( \binom{\frac{m}{p^{r-i}} + p^i - 1}{p^i - 1} - \binom{\frac{m}{p^{r-i-1}} + p^{i-1} - 1}{p^{i-1} - 1} \right)$$

In particular, if $n = p$, then the number of isomorphism types of good $C_p$-gradings on $\mathrm{M}_m(k)$ is $1 + \frac{1}{p}(\binom{m+p-1}{p-1} - 1)$ if $p$ divides $m$, and $\frac{1}{p}\binom{m+p-1}{p-1}$ if $p$ does not divide $m$.

Over an algebraically closed field, any grading by a cyclic group on a matrix algebra is isomorphic to a good grading.

### 9.2.25 Theorem

Let $k$ be an algebraically closed field and $m, n$ positive integers. Then any $C_n$-grading of the matrix algebra $\mathrm{M}_m(k)$ is isomorphic to a good grading.

**Proof** Let $R = \mathrm{M}_m(k)$ be the matrix algebra endowed with a certain $C_n$-grading, and pick $\Sigma \in R$-gr a graded simple module. Then $\Delta = \mathrm{End}_{\mathrm{R}}(\Sigma) = \mathrm{END}_{\mathrm{R}}(\Sigma)$ is a $C_n$-graded algebra. Moreover, $\Delta$ is a crossed product when regarded as an algebra graded by the support of the $C_n$-grading of $\Delta$. The support is clearly a subgroup since it consists of all elements $g$ for which $\Sigma$ and $\Sigma(g)$ are isomorphic. Thus $\Delta \simeq \Delta_e \#_\sigma H$ for a cyclic group $H$ and a cocycle $\sigma$. On the other hand $\Delta_e = \mathrm{End}_{\text{R-gr}}(\Sigma)$ is a finite field extension of $k$, so $\Delta_e = k$. Hence $\Delta$ is a crossed product of $\Delta_e$, which is central in $\Delta$, and the cyclic group $H$, so $\Delta$ is commutative.
Since $R$ is a semisimple algebra with precisely one isomorphism type of simple module, say $S$, we have that $\Sigma \simeq S^p$ as $R$-modules for some positive integer $p$. But then $\Delta \simeq \mathrm{End}_{\mathrm{R}}(\mathrm{S^p}) \simeq \mathrm{M_p(k)}$, and the commutativity of $\Delta$ shows that $p$ must be 1. Then $\Sigma \simeq S$, so there exists a graded $R$-module which is simple as an $R$-module. By Theorem 9.2.7, the grading is isomorphic to a good one. □

## 9.3 Gradings over cyclic groups

Let $n$ be a positive integer and $C_n =< c >$ the cyclic group of order $n$. We assume that a primitive $n^{\text{th}}$ root of unity $\xi$ exists in $k$ (in particular this implies that the characteristic of $k$ does not divide $n$).

### 9.3.1 Theorem

Let $A = M_m(k)$, where $k$ is a field containing a primitive $n^{\text{th}}$ root of unity $\xi$. Then a $C_n$-grading of $k$-algebra $A$ is of the form $A_{c^i} = \{\ D + \xi^{-i}XDX^{-1} + \xi^{-2i}X^2DX^{-2} + \ldots + \xi^{-(n-1)i}X^{n-1}DX^{-n+1} \mid D \in M_m(k)\ \}$ where $X \in GL_n(k)$ is such that $X^n \in kI_m$. For any such matrix $X$ we will denote by $A(X)$ the algebra $A$ endowed with the grading induced by $X$.

**Proof** Let $A = \oplus_{i \in \mathbf{Z}_n} A_{c^i}$ be a $C_n$-grading of $A = M_m(k)$. Define the map $\Psi : A \to A$ by $\Psi(D) = \sum_{i \in \mathbf{Z}_n} \xi^i D_{c^i}$ for any $D \in A$. We obviously have that $\Psi$ is a linear map. Moreover, for any $D, B \in A$ we have that

$$\begin{aligned} \Psi(D)\Psi(B) &= (\sum_{i \in \mathbf{Z}_n} \xi^i D_{c^i})(\sum_{j \in \mathbf{Z}_n} \xi^j B_{c^j}) \\ &= \sum_{i,j \in \mathbf{Z}_n} \xi^{i+j} D_{c^i} B_{c^j} \end{aligned}$$

$$\begin{aligned} &= \sum_{s\in\mathbf{Z}_n}\sum_{i+j=s}\xi^s D_{c^i}B_{c^j} \\ &= \sum_{s\in\mathbf{Z}_n}\xi^s(DB)_{c^s} \\ &= \Psi(DB) \end{aligned}$$

showing that $\Psi$ is an algebra morphism. Moreover, for any $j$ we have that $\Psi^j(D)=\sum_{i\in\mathbf{Z}_n}\xi^{ji}D_{c^i}$ for any $D\in A$. In particular $\Psi^n=Id$, and $\Psi$ is an algebra automorphism of $R$. By the Skolem-Noether Theorem, an algebra automorphism $\Psi$ of $M_m(k)$ is of the form $\Psi(D)=XDX^{-1}$ for any $D\in M_m(k)$, where $X\in GL_m(k)$. In order to have $\Psi^n=Id$, we must impose the condition $X^n\in Z(M_m(k))=kI_m$, $I_m$ the identity matrix. It is possible to recover the grading from the automorphism $\Psi$. Indeed, let $j\in\mathbf{Z}_n$, and $D\in A$. Multiply the equations

$$D=\sum_{i\in\mathbf{Z}_n}D_{c^i},\ \Psi(D)=\sum_{i\in\mathbf{Z}_n}\xi^iD_{c^i},\ \ldots,\Psi^{n-1}(D)=\sum_{i\in\mathbf{Z}_n}\xi^{(n-1)i}D_{c^i}$$

by $1,\xi^{-j},\ \xi^{-2j},\ldots,\xi^{-(n-1)j}$ respectively, and then add the obtained equations. We find that

$$D+\xi^{-j}\Psi(D)+\ldots+\xi^{-(n-1)j}\Psi^{n-1}(D)=nD_{c^j}$$

therefore

$$\begin{aligned} D_{c^j} &= \frac{1}{n}(D+\xi^{-j}\Psi(D)+\ldots+\xi^{-(n-1)j}\Psi^{n-1}(D)) \\ &= \frac{1}{n}(D+\xi^{-i}XDX^{-1}+\xi^{-2i}X^2DX^{-2}+\ldots+\xi^{-(n-1)i}X^{n-1}DX^{-n+1}) \end{aligned}$$

□

### 9.3.2 Remark

The proof of Theorem 9.3.1 shows that for any $D\in M_m(k)$, the homogeneous components of $D$ in the grading defined by the matrix $X$ as in the statement are

$$D_{c^i}=\frac{1}{n}(D+\xi^{-i}XDX^{-1}+\xi^{-2i}X^2DX^{-2}+\ldots+\xi^{-(n-1)i}X^{n-1}DX^{-n+1})$$

for any $0\leq i\leq n-1$.

If the characteristic of $k$ divides $n$, we can not proceed in the same way for describing $C_m$-gradings of $M_m(k)$ since $n$ is not invertible in $A$. Nevertheless, we are able to produce an example of a $C_p$-grading of $M_p(\mathbf{Z}_p)$ which is not a good grading.

### 9.3.3 Proposition

Let $p \geq 3$ be a prime number, $A = M_p(\mathbf{Z}_p)$ and $a, b \in A$, $a = (a_{i,j})_{1 \leq i,j \leq p}$, $b = (b_{i,j})_{1 \leq i,j \leq p}$, where

$$a_{i,j} = \delta_{i+1,j} + \delta_{i,p}(-\delta_{j,1} + \delta_{j,2}), \text{ for all } 1 \leq i, j \leq p$$

$$b_{i,j} = \begin{pmatrix} i-1 \\ j-1 \end{pmatrix}, \text{ for all } 1 \leq i, j \leq p$$

($\delta_{i,j}$ denotes Kronecker's delta). Then $K = \mathbf{Z}_p[a]$ is a field, the sum $\sum_{i=0}^{p-1} Kb^i$ is direct and

$$A = K \oplus Kb \oplus \cdots \oplus Kb^{p-1}$$

is a $C_p$-graded division ring structure on $A$. In particular, this grading is not isomorphic to a good grading.

**Proof** Let $P(X) = X^p - X + 1 \in \mathbf{Z}_p[X]$. It is well known that $P$ is irreducible and it is the minimal polynomial of $a$, which is written in the Jordan canonical form. So, $K = \mathbf{Z}_p[a]$ is a field and it has $p^p$ elements. The matrix $b$ has zero entries above the diagonal and $b_{i,i} = 1$, for $i = 1, p$, thus the minimal polynomial of $b$ is $(X-1)^p = X^p - 1$, and $b$ is invertible. An easy computation shows that

$$ab = b(a + I_p),$$

so $Kb = bK$ and $b \notin K$. Now everything follows if we show that the sum

$$K + Kb + \cdots + Kb^{p-1}$$

is direct. We prove by induction that for any $0 \leq j \leq p-1$, $(\alpha_i)_{i=0,j} \subseteq K$ such that $\sum_{0 \leq i \leq j} \alpha_i (2b)^i = 0$, we have $\alpha_i = 0$, for every $0 \leq i \leq j$.
If $j = 0$, there is nothing to prove.
If $j = 1$ and

$$\alpha_0 + \alpha_1(2b) = 0$$

then $\alpha_1(2b) = -\alpha_0$. If $\alpha_1 \neq 0$, then $\alpha_1$ is invertible and $b = -2^{-1}\alpha_1^{-1}\alpha_0$. That means that $b \in K$ a contradiction. So $\alpha_1 = 0$ implying $\alpha_0 = 0$.
If $1 < j < p-1$ and

$$\sum_{i=0}^{j+1} \alpha_i (2b)^i = 0, \tag{9.1}$$

then multiplying this relation by $2b, (2b)^2, \ldots, (2b)^{p-1}$ and adding them, we obtain

$$\left( \sum_{i=0}^{j+1} \alpha_i \right) \left( I_p + (2b) + \cdots + (2b)^{p-1} \right) = 0.$$

Because $1 + 2b + \cdots + (2b)^{p-1} \neq 0$ and $\sum_{i=0}^{j+1} \alpha_i \in K$ we obtain

$$\sum_{i=0}^{j+1} \alpha_i = 0. \tag{9.2}$$

Substracting (9.2) from (9.1) we get

$$\sum_{i=0}^{j+1} \alpha_i((2b)^i - I_p) = 0$$

and since $2b - 1$ is invertible

$$\sum_{i=0}^{j+1} \alpha_i((2b)^{i-1} + \cdots + I_p) = 0,$$

which means that

$$\sum_{i=1}^{j+1} \alpha_i + \left(\sum_{i=2}^{j+1} \alpha_i\right)(2b) + \cdots + \alpha_{j+1}(2b)^j = 0$$

and $\alpha_{j+1} = 0$ by the induction hypothesis $\alpha_{j+1} = 0$. Now (9.1) yields $\alpha_i = 0$ for every $i = 0, j + 1$.
Finally we note that a good $C_p$-grading on $M_p(k)$ cannot be a graded division ring, since the elements $E_{i,j}$ are homogeneous, but not invertible. This establishes that the grading is not isomorphic to a good grading. □

### 9.3.4 Remark

We can also produce a $C_2$-graded division ring structure on $M_2(\mathbf{Z}_2)$. Indeed, let $A = M_2(\mathbf{Z}_2)$, and the matrices $D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $A_e = \{0, I_2, D, D^2\}$ and $A_c = \{0, B, DB, BD\}$ define a $C_2$-grading on $A$. Moreover, this clearly defines a gr-skewfield structure.

If $k$ is a field of characteristic $p > 0$, since $M_p(k) \simeq k \otimes_{\mathbf{Z}_p} M_p(\mathbf{Z}_p)$, the $C_p$-grading of $M_p(\mathbf{Z}_p)$ defined in Proposition 9.3.3 and Remark 9.3.4 extends to a $C_p$-grading of $M_p(k)$, which is not a good grading.

### 9.3.5 Remark

We can obtain other graded division ring structures on matrix rings in the following way. Let $A = M_n(k)$, and $S \in A$-mod the (unique) type of simple $A$-module. Assume that $A$ has a $G$-grading such that $S$ is not gradable. Then let $\Sigma$ be a graded simple $A$-module. Since $A$ is simple Artinian and $\Sigma$ is finite dimensional, we have $\Sigma \simeq S^t$ as $A$-modules, for some integer $t$. Since $S$ is not

gradable, we have $t > 1$. Then $M_t(k) \simeq End_R(S^t) \simeq End_A(\Sigma) = END_A(\Sigma)$. Since $\Sigma$ is a simple object in the category $A$-gr, then so is $\Sigma(\sigma)$ for any $\sigma \in G$. Thus any element of $END_A(\Sigma)_\sigma$ is either zero or invertible, hence $END_A(\Sigma)$ is a gr-skewfield. This transfers to a gr-skewfield structure on $M_t(k)$.

The following will be useful for the classification of gradings by cyclic groups.

### 9.3.6 Proposition

If $X,\ Y \in GL_m(k)$ with $X^n,\ Y^n \in kI_m$, then $A(X)$ is isomorphic to $A(Y)$ as $C_n$-graded algebras if and only if there exist $T \in GL_m(k)$ and $\lambda \in k$ such that $X = \lambda T^{-1}YT$. In particular, $A(X) \simeq A(J_X)$, where $J_X$ is the Jordan form of the matrix $X$.

**Proof** We have that $A(X) \simeq A(Y)$ as $C_n$-graded algebras if and only if they are isomorphic as left $kC_n$-module algebras. But this is equivalent to the existence of an algebra isomorphism $f : A(X) \to A(Y)$ with $f(XBX^{-1}) = Yf(B)Y^{-1}$ for any $B \in M_m(k)$. By the Skolem-Noether Theorem an automorphism $f$ of $M_m(k)$ is of the form $f(B) = TBT^{-1}$ for any $B \in M_m(k)$, where $T \in GL_m(k)$. Thus, the condition on $f$ becomes $TXBX^{-1}T^{-1} = YTBT^{-1}Y^{-1}$ for any $B \in M_m(k)$, which is equivalent to $T^{-1}Y^{-1}TX \in kI_m$. □

### 9.3.7 Theorem

(i) If $X = \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha_m \end{pmatrix}$ with $\alpha_i^n = 1$ for all $i = 1, \dots, m$, then $A(X)$ is a good grading.

(ii)Any good grading is isomorphic to a grading $A(X)$ for a diagonal matrix $X = \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha_m \end{pmatrix}$ with $\alpha_i^n = 1$ for all $i = 1, \dots, m$.

**Proof** (i) By Remark 9.3.2, a matrix in $A(X)_{c^s}$ is of the form

$$B_{c^s} = \frac{1}{n}(B + \xi^{-s}XBX^{-1} + \xi^{-2s}X^2BX^{-2} + \ldots + \xi^{-(n-1)s}X^{n-1}BX^{-n+1})$$

for a matrix $B \in M_m(k)$, $B = (b_{ij})_{1 \le i,j \le n}$. The element on the position $(i, j)$ in the matrix $B_{c^s}$ is

$$\beta_{ij} \quad = \quad \frac{1}{n}(b_{ij} + \xi^{-s}\alpha_i b_{ij}\alpha_j^{-1} + \xi^{-2s}\alpha_i^2 b_{ij}\alpha_j^{-2} + \ldots + \xi^{-(n-1)s}\alpha_i^{n-1}b_{ij}\alpha_j^{-n+1})$$

$$= \frac{b_{ij}}{n}[1 + \xi^{-s}\alpha_i\alpha_j^{-1} + (\xi^{-s}\alpha_i\alpha_j^{-1})^2 + \ldots + (\xi^{-s}\alpha_i\alpha_j^{-1})^{n-1}]$$
$$= \frac{b_{ij}}{n}P(\xi^{-s}\alpha_i\alpha_j^{-1})$$

where $P(t) = 1 + t + t^2 + \ldots + t^{n-1}$. If $P(\xi^{-s}\alpha_i\alpha_j^{-1}) = 0$, then obviously $\beta_{ij} = 0$. If $P(\xi^{-s}\alpha_i\alpha_j^{-1}) \neq 0$, then $\beta_{ij}$ ranges over the elements of $k$ when $b_{ij}$ does. Since $\xi^{-s}\alpha_i\alpha_j^{-1}$ is an $n$-th root of unity, we obtain that $\beta_{ij} \neq 0$ if and only if $\xi^{-s}\alpha_i\alpha_j^{-1} = 1$. In order to prove that $A(X)$ is a good grading, we have to establish for any pair $(i,j) \in \{1,\ldots,m\} \times \{1,\ldots,m\}$ that there exists a unique $s \in \{1,\ldots,n-1\}$ such that $\xi^{-s}\alpha_i\alpha_j^{-1} = 1$, or, equivalently, $\alpha_i\alpha_j^{-1} = \xi^s$. This is true because $\xi$ is a primitive $n$-th root of unity and $\alpha_i\alpha_j^{-1}$ is a root of unity. Thus, for any pair $(i,j)$ there exists a unique $s$ such that $P(\xi^{-s}\alpha_i\alpha_j^{-1}) \neq 0$. Now, taking $b_{ij} = nP(\xi^{-s}\alpha_i\alpha_j^{-1})^{-1}$ and $b_{rl} = 0$ for any $(r,l) \neq (i,j)$ we obtain that the matrix $E_{ij} \in A(X)_{c^s}$, so $A(X)$ is a good grading.

(ii) A good grading is obtained by assigning some degrees to the matrices $E_{12}$, $E_{23}$,..., $E_{m-1,m}$ (by Proposition 9.2.3). Thus, for given $s_1$, $s_2$,..., $s_{m-1}$ with $\deg(E_{i,i+1}) = s_i$, $1 \leq i \leq m-1$, we have to find a diagonal matrix $X = \begin{pmatrix} \alpha_1 & 0 & \ldots & 0 \\ 0 & \alpha_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \alpha_m \end{pmatrix}$ with $\alpha_i^n = 1$, $1 \leq i \leq m$, such that $A(X)$ is isomorphic to the initial good grading. We know from the proof of (i) that for any $1 \leq i \leq m-1$, $E_{i,i+1} \in A(X)_{c^{s_i}}$ if and only if $\xi^{-s_i}\alpha_i\alpha_{i+1}^{-1} = 1$, where $A(X)_{c^{s_i}}$ is the homogeneous part of degree $c^{s_i}$ of the grading $A(X)$. This is equivalent to $\alpha_i = \xi^{s_i}\alpha_{i+1}$ for any $1 \leq i \leq m-1$. Taking, for example, $\alpha_m = 1$, $\alpha_{m-1} = \xi^{s_{m-1}}$, $\alpha_{m-2} = \xi^{s_{m-2}+s_{m-1}}$,..., $\alpha_1 = \xi^{s_1+s_2+\ldots s_{m-1}}$ we obtain a diagonal matrix $X$ which satisfies the desired condition. □

As an application we obtain the following.

### 9.3.8 Corollary

If $k$ is an algebraically closed field such that $\mathrm{char}(k)$ does not divide $n$, then any $C_n$-grading on $M_m(k)$ is isomorphic to a good grading.

**Proof** The result follows immediately from the Theorem 9.3.7, since a polynomial of the form $X^n - a$, $a \in k - \{0\}$, has only simple roots, therefore a matrix $X \in GL_m(k)$ such that $X^n \in kI_m$ has a diagonal Jordan form. □

Now, the good $C_n$-gradings on $M_m(k)$ can be classified by the following.

### 9.3.9 Proposition

If $X = \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha_m \end{pmatrix}$ and $Y = \begin{pmatrix} \beta_1 & 0 & \dots & 0 \\ 0 & \beta_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \beta_m \end{pmatrix}$ such that $\alpha_i^n = \beta_i^n = 1$ for all $i = 1, \dots, m$, then $A(X) \simeq A(Y)$ if and only if there exists an $n$-th root of unity $\lambda$ such that the $m$-tuple $(\beta_1, \dots, \beta_m)$ is a permutation of the $m$-tuple $(\lambda\alpha_1, \dots, \lambda\alpha_m)$.

**Proof** By Proposition 9.3.6 we have that $A(X) \simeq A(Y)$ if and only if there exists $T \in GL_m(k)$ and $\lambda \in k$ such that $X = \lambda T^{-1} Y T$. But this is equivalent to the fact that the matrices $\lambda^{-1}X$ and $Y$ have the same Jordan form, and then $(\beta_1, \dots, \beta_m)$ can be obtained by a permutation from $(\lambda\alpha_1, \dots, \lambda\alpha_m)$. The condition $\beta_i = \lambda\alpha_j$ implies that $\lambda$ is an $n$-th root of unity. □

Let us consider the situation where $n = p$, a prime number. We determine all the isomorphism types of $C_p$-gradings on $M_m(k)$ and we count them.

### 9.3.10 Proposition

If $p$ divides $m$ then any $C_p$-graded algebra structure on $M_m(k)$ is isomorphic to $A(X)$, where $X$ is a matrix of one of the following two types.

(i) $X = \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha_m \end{pmatrix}$ with $\alpha_i^p = 1$ for any $1 \leq i \leq m$.

(ii) $X = Y_a$, for some $a \in k$ which is not a $p$-th power in $k$, where $Y_a$ is the matrix consisting of $\frac{m}{p}$ blocks of the form $\begin{pmatrix} 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ a & 0 & \dots & 0 \end{pmatrix}$ along the diagonal, and 0 elsewhere.

**Proof** We know from Theorem 9.3.1 that a $C_p$-graded algebra structure on $M_m(k)$ is of the form $A(X)$ for a matrix $X \in GL_m(k)$ with $X^p - aI_m = 0$ for some non-zero $a \in k$. We also know that $A(X) \simeq A(J_X)$ where $J_X$ is the Jordan form of the matrix $X$. There are two possibilities.

(i) If $a$ is a $p$-th power in $k$, then the polynomial $t^p - a$ has $p$ simple roots in $k$, and then $J_X = \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \alpha_m \end{pmatrix}$ for some $p$-th roots of unity $\alpha_1, \dots, \alpha_m$. Moreover, $A(J_X) \simeq A(a^{-\frac{1}{p}} J_X)$, and $A(a^{-\frac{1}{p}} J_X)$ is a diagonal matrix with diagonal entries $p$-th roots of unity.

(ii) If $a$ is not a $p$-th power in $k$, then the polynomial $t^p - a$ is irreducible over $k$, and then $t^p - a$ is the minimal polynomial of $X$, which implies that $J_X = Y_a$. □

### 9.3.11 Proposition

If $p$ does not divide $m$, then any $C_p$-grading on $M_m(k)$ is isomorphic to a good grading.

**Proof** Let $A(X)$ be a $C_p$-graded algebra structure on $M_m(k)$, where $X^p - aI_m = 0$ for some non-zero $a \in k$. Then $(det\ X)^p = a^m$. Since $p$ and $m$ are relatively prime there exist integers $u, v$ such that $up + vm = 1$. Then $a = a^{up+vm} = a^{up}(det\ X)^{vp} = (a^u(det\ X)^v)^p$, so $a$ is a $p$-th power in $k$, and the proof ends as in case (i) of Proposition 9.3.10. □

Recall by Example 9.2.24 that the number of all isomorphism types of good $C_p$-gradings on $M_m(k)$ is
(i) $1 + \frac{1}{p}(\binom{m+p-1}{p-1} - 1)$ if $p$ divides $m$,
(ii) $\frac{1}{p}\binom{m+p-1}{p-1}$ if $p$ does not divide $m$.

If we require now one more condition on $k$ then we can classify all the $C_p$-gradings on $M_m(k)$. The good gradings (type (i)) have been already classified by Proposition 9.3.9. Regarding the gradings which are not good (type (ii)), we have the following.

### 9.3.12 Proposition

If $p$ divides $m$ and $k$ contains a primitive $m$-th root of unity $\eta$, then for any $a, b$ which are not $p$-th powers in $k$, we have that $A(Y_a) \simeq A(Y_b)$ if and only if $\frac{a}{b}$ is a $p$-th power in $k$.

**Proof** Suppose that $A(Y_a) \simeq A(Y_b)$. Then, by Proposition 9.3.6, there exist $T \in GL_m(k)$ and $\lambda \in k$ such that $Y_a = \lambda T^{-1} Y_b T$. This implies $det\ Y_a = \lambda^m det\ Y_b$. Since $det\ Y_a = ((-1)^{p+1}a)^{\frac{m}{p}}$ we obtain that $(\frac{a}{b})^{m/p} = \lambda^m = (\lambda^p)^{m/p}$, so $\frac{a}{b} = \omega\lambda^p$ for some $(m/p)$-th root of unity $\omega$. Since $\eta$ is a primitive $m$-th root of unity, $\eta^p$ is a primitive $(m/p)$-th root of unity, so $\omega = (\eta^p)^s$ for some integer $s$. We obtain that $\frac{a}{b} = (\eta^s\lambda)^p$. Conversely, if $\frac{a}{b} = \lambda^p$ for some $\lambda \in k$, let $T \in GL_m(k)$ be the matrix having on the main diagonal $\frac{m}{p}$ blocks of the form $\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \lambda^{-1} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda^{-p+1} \end{pmatrix}$. Then $Y_a = \lambda T^{-1} Y_b T$, so $A(Y_a) \simeq A(Y_b)$. □

The classification of the $C_p$-graded algebra structures on $M_m(k)$ may be now stated as follows.

### 9.3.13 Theorem

Let $k$ be a field which contains a primitive $m$-th root of 1. Then the isomorphism types of $C_p$-graded algebra structures on the matrix algebra $M_m(k)$ are classified as follows.
(i) If $p$ does not divide $m$, then there exist $\frac{1}{p}\binom{m+p-1}{p-1}$ isomorphism types, all of them being good gradings.
(ii) If $p$ divides $m$, then there exist $1+\frac{1}{p}(\binom{m+p-1}{p-1}-1)$ isomorphism types of good gradings, and $|k^*/(k^*)^p|-1$ isomorphism types of non-good gradings, the last ones being the $A(Y_a)$'s, where $a$ ranges over a system of representatives of the non-trivial $(k^*)^p$-cosets of $k^*$.

Let us consider the particular case where $m=p=2$.

### 9.3.14 Corollary

Let $k$ be a field with $\mathrm{char}(k)\neq 2$. Then the (different) isomorphism types of $C_2$-graded algebra structures on $M_2(k)$ are the following.
i) The trivial grading $A_e=M_2(k)$, $A_c=0$;
ii) The good grading $A_e=\begin{pmatrix} k & 0\\ 0 & k\end{pmatrix}$, $A_c=\begin{pmatrix} 0 & k\\ k & 0\end{pmatrix}$
iii) The graded algebra $A(a)=M_2(k)$, with

$$A(a)_e=\{\begin{pmatrix} u & v\\ av & u\end{pmatrix}\mid u,v\in k\},\quad A(a)_c=\{\begin{pmatrix} u & v\\ -av & -u\end{pmatrix}\mid u,v\in k\}$$

where $a$ ranges over a system of representatives of the $(k^*)^2$-cosets of $k^*$ different from $(k^*)^2$.

For example, over the field $\mathbf{C}$ of complex numbers, there exist two isomorphism types of $C_2$-gradings of $M_2(k)$, namely the two good gradings. Over the field $\mathbf{R}$ of real numbers, there exist three isomorphism types of $C_2$-gradings, the two good gradings and the grading corresponding to the negatives if we look to $\mathbf{R}^*/(\mathbf{R}^*)^2$. More precisely, this third grading is $A(-1)$, where

$$A(-1)_e=\{\begin{pmatrix} u & v\\ -v & u\end{pmatrix}\mid u,v\in k\},\quad A(-1)_c=\{\begin{pmatrix} u & v\\ v & -u\end{pmatrix}\mid u,v\in k\}$$

Over the field $\mathbf{Q}$ of rational numbers there exist two good $C_2$-gradings on $M_2(\mathbf{Q})$, plus (countable) infinitely many non-good gradings.

## 9.4 $C_2$-gradings of $M_2(k)$

In the previous section we have found all the isomorphism types of $C_2$-gradings of $M_2(k)$, $k$ a field of characteristic different from 2. The purpose of this

section is to describe all $C_2$-gradings of $M_2(k)$. In this section we write $C_2 = \{e, g\}$.
We start with the situation where $\mathrm{char}(k) \neq 2$.

### 9.4.1 Theorem

Let $k$ be a field with $\mathrm{char}(k) \neq 2$, and $R = M_2(k)$. Then a $C_2$-grading of the $k$-algebra $R$ is of one of the following three types:
(i)

$$R_e = \left\{ \left( \begin{array}{cc} au+v & bu \\ cu & -au+v \end{array} \right) \middle| \; u, v \in k \right\},$$

$$R_g = \left\{ \left( \begin{array}{cc} -\frac{c}{2a}\delta - \frac{b}{2a}\gamma & \delta \\ \gamma & \frac{c}{2a}\delta + \frac{b}{2a}\gamma \end{array} \right) \middle| \; \gamma, \delta \in k \right\},$$

where $a, b, c \in k$, $a \neq 0$ and $a^2 + bc \neq 0$;
(ii)

$$R_e = \left\{ \left( \begin{array}{cc} v & bu \\ cu & v \end{array} \right) \middle| \; u, v \in k \right\}, \quad R_g = \left\{ \left( \begin{array}{cc} \gamma & b\delta \\ -c\delta & -\gamma \end{array} \right) \middle| \; \gamma, \delta \in k \right\},$$

where $b, c \in k - \{0\}$;
(iii) The trivial grading, $R_e = M_2(k), R_g = 0$.

**Proof** We start by finding all matrices $X \in M_2(k), X \neq 0$ such that $X^2 \in kI_2$. Let $X = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right)$. Then $X^2 = \alpha I_2$ for some $\alpha \in k$ if and only if

$$a^2 + bc = \alpha \ , \ d^2 + bc = \alpha \ , \ b(a+d) = 0 \ , \ c(a+d) = 0.$$

If $a + d \neq 0$, then $b = c = 0$, and $a = d$. If $a + d = 0$, then $d = -a$ and $a^2 + bc \neq 0$. Thus there are two types of matrix solutions: $X = \left( \begin{array}{cc} \beta & 0 \\ 0 & \beta \end{array} \right)$, with $\beta \in k - \{0\}$, and $X = \left( \begin{array}{cc} a & b \\ c & -a \end{array} \right)$, with $a, b, c \in k$, $a^2 + bc \neq 0$. For the first type, we obtain the trivial grading, since

$$R_e = \{A + XAX^{-1} |\ A \in M_2(k)\} = \{2A |\ A \in M_2(k)\} = M_2(k).$$

Let now $X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$, with $a^2+bc \neq 0$. If $A = \begin{pmatrix} x & y \\ z & t \end{pmatrix} \in M_2(k)$, then the homogeneous components of $A$ in the $C_2$-grading associated to $X$ are

$$\begin{aligned} A_e &= \frac{1}{2}(A + XAX^{-1}) \\ &= \frac{1}{2(a^2+bc)} \\ &\quad \begin{pmatrix} (2a^2+bc)x + acy + abz + bct & abx + bcy + b^2z - abt \\ acx + c^2y + bcz - act & bcx - acy - abz + (2a^2+bc)t \end{pmatrix} \end{aligned}$$

$$\begin{aligned} A_g &= \frac{1}{2}(A - XAX^{-1}) \\ &= \frac{1}{2(a^2+bc)} \\ &\quad \begin{pmatrix} bcx - acy - abz - bct & -abx + (2a^2+bc)y - b^2z + abt \\ -acx - c^2y + (2a^2+bc)z + act & -bcx + acy + abz + bct \end{pmatrix} \end{aligned}$$

We distinguish two possibilities. If $a \neq 0$, put

$$u = (ax + cy + bz - at)/2(a^2+bc)\ ,\ v = (x+t)/2.$$

Then $A_1 = \begin{pmatrix} au+v & bu \\ cu & -au+v \end{pmatrix}$, and $u, v$ can take any values in $k$, since the matrix $\begin{pmatrix} a & c & b & -a \\ 1 & 0 & 0 & 1 \end{pmatrix}$ has rank 2. Thus

$$R_e = \left\{ \left. \begin{pmatrix} au+v & bu \\ cu & -au+v \end{pmatrix} \right| u, v \in k \right\}.$$

On the other hand, putting

$$\gamma = (-acx - c^2y + (2a^2+bc)z + act)/2(a^2+bc)\ ,$$
$$\delta = (-abx + (2a^2+bc)y - b^2z + abt)/2(a^2+bc)$$

we have

$$A_g = \begin{pmatrix} -\frac{c}{2a}\delta - \frac{b}{2a}\gamma & \delta \\ \gamma & \frac{c}{2a}\delta + \frac{b}{2a}\gamma \end{pmatrix}$$

and the matrix $\begin{pmatrix} -ac & -c^2 & (2a^2+bc) & ac \\ -ab & (2a^2+bc) & -b^2 & ab \end{pmatrix}$ has rank 2, so

$$R_g = \left\{ \left. \begin{pmatrix} -\frac{c}{2a}\delta - \frac{b}{2a}\gamma & \delta \\ \gamma & \frac{c}{2a}\delta + \frac{b}{2a}\gamma \end{pmatrix} \right| \gamma, \delta \in k \right\}.$$

consequently the grading is of type (i).

If $a = 0$, then $bc \neq 0$. In this case $A_e = \begin{pmatrix} v & bu \\ cu & v \end{pmatrix}$, where $u = (cy+bz)/2bc$ and $v = (x+t)/2$ vary over the set of the elements of $k$. Then writing $\gamma = (x-t)/2$, $\delta = (cy-bz)/2bc$, we obtain that $A_g = \left\{ \begin{pmatrix} \gamma & b\delta \\ -c\delta & -\gamma \end{pmatrix} \middle| \ \gamma, \delta \in k \right\}$, and this leads to a grading of type (ii). □

### 9.4.2 Corollary

A $C_2$-algebra grading of $M_2(k)$, $\mathrm{char}(k) \neq 2$, different from the trivial grading, stems from a crossed product structure.

**Proof** It is enough to show that for any grading of type $(i)$ or $(ii)$ $R_g$ contains an invertible element. But this clearly follows from the fact that $m\delta^2 + n\delta\gamma + p\gamma^2 = 0$ for any $\gamma, \delta \in k$ if and only if $m = n = p = 0$. □

In the next two propositions we describe which $C_2$-gradings of $M_2(k)$ are isomorphic to a good grading.

### 9.4.3 Proposition

Let $b, c \in k - \{0\}$. Then the grading

$$R_e = \left\{ \begin{pmatrix} v & bu \\ cu & v \end{pmatrix} \middle| \ u, v \in k \right\},$$

$$R_g = \left\{ \begin{pmatrix} \gamma & b\delta \\ -c\delta & -\gamma \end{pmatrix} \middle| \ \gamma, \delta \in k \right\}$$

of $M_2(k)$ is isomorphic to a good grading if and only if $bc$ is a square in $k$.

**Proof** Let $S = M_2(k)$ with the trivial $C_2$-grading $S_e = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$, $S_g = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$. If $f : S \to R$ is an isomorphism of graded algebras, then there exists $Y \in GL_2(k)$ with $f(A) = YAY^{-1}$ for any $A \in S$. Let $Y = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$.

Then for $A = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \in S_g$ we have

$$f(A) = YAY^{-1} = \frac{1}{ps - qr} \begin{pmatrix} qsy - prx & -q^2y + p^2x \\ s^2y - r^2x & -qsy + prx \end{pmatrix} \in R_g$$

and this shows that $b(s^2y - r^2x) + c(-q^2y + p^2x) = 0$ for any $x, y \in k$. Thus $br^2 = cp^2$ and $bs^2 = cq^2$. We obtain that $bc = (\frac{cp}{r})^2$, a square in $k$.
Conversely, suppose that $bc = d^2$ for some $d \in k$. Let

$$Y = \begin{pmatrix} 1 & \frac{d}{2c} \\ -\frac{d}{b} & \frac{1}{2} \end{pmatrix} \in GL_2(k).$$

Then for $A = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \in R_e$ we have

$$YAY^{-1} = \begin{pmatrix} (x+y)/2 & d(y-x)/2c \\ d(y-x)/2b & (x+y)/2 \end{pmatrix}$$

and for $A = \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \in R_g$ we have

$$YAY^{-1} = \begin{pmatrix} (by + 4cx)/4d & (-by + 4cx)/4c \\ (by - 4cx)/4b & -(by + 4cx)/4d \end{pmatrix}.$$

These show that the map $f : S \to R, f(A) = YAY^{-1}$, is an isomorphism of graded algebras. □

### 9.4.4 Proposition

Let $a, b, c \in k$ such that $a \neq 0, a^2 + bc \neq 0$. Then the grading

$$R_e = \left\{ \begin{pmatrix} au + v & bu \\ cu & -au + v \end{pmatrix} \middle| \; u, v \in k \right\},$$

$$R_g = \left\{ \begin{pmatrix} -\frac{c}{2a}\delta - \frac{b}{2a}\gamma & \delta \\ \gamma & \frac{c}{2a}\delta + \frac{b}{2a}\gamma \end{pmatrix} \middle| \; \gamma, \delta \in k \right\}$$

is isomorphic to a good grading if and only if $a^2 + bc$ is a square in $k$.

**Proof** Keeping the notation from the proof of Proposition 9.4.3, suppose that $f : S \to R, f(A) = YAY^{-1}$ is an isomorphism of graded algebras. Since

$f\left(\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\right) \in R_g$, we find that

$$\begin{aligned} qsy - prx &= -\frac{c}{2a}(-q^2y + p^2x) - \frac{b}{2a}(s^2y - r^2x) \\ &= (\frac{c}{2a}q^2 - \frac{b}{2a}s^2)y - (\frac{c}{2a}p^2 - \frac{b}{2a}r^2)x \end{aligned}$$

for any $x, y \in k$. In particular

$$\frac{c}{2a}q^2 - \frac{b}{2a}s^2 = qs \text{ or } c(\frac{q}{s})^2 - 2a(\frac{q}{s}) - b = 0.$$

If $bc = 0$, then clearly $a^2 + bc$ is a square in $k$. If $bc \neq 0$, then in order to have roots in $k$ for the equation $ct^2 - 2at - b = 0$, we need $a^2 + bc$ to be a square. Conversely, suppose that $a^2 + bc$ is a square in $k$. We first consider the case where $bc \neq 0$. Let $t_1, t_2$ be the (distinct) roots of the equation $ct^2 - 2at - b = 0$, and $X = \begin{pmatrix} t_2 & t_1 \\ 1 & 1 \end{pmatrix}$. If $f : R \to S, f(A) = XAX^{-1}$, is the algebra isomorphism induced by $X$, then

$$f\left(\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}\right) = \frac{1}{t_1 - t_2}\begin{pmatrix} t_1y - t_2x & -t_1^2y + t_2^2x \\ y - x & -t_1y + t_2x \end{pmatrix} \in S_g$$

and

$$f\left(\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right) = \frac{1}{t_1 - t_2}\begin{pmatrix} t_2x - t_1y & \frac{b}{c}(x - y) \\ x - y & -t_1x + t_2y \end{pmatrix} \in S_e$$

showing that $f$ is an isomorphism of graded algebras. If $b = c = 0$, then $R = S$ as graded algebras. If $c = 0$ and $b \neq 0$, then $X = \begin{pmatrix} 1 & 1 \\ 0 & -\frac{2a}{b} \end{pmatrix}$ induces in a similar way a graded isomorphism between $R$ and $S$. Similarly for $c \neq 0, b = 0$, and this ends the proof. □

### 9.4.5 Corollary

If $k$ is an algebraically closed field of characteristic not 2, then any $C_2$-grading of the algebra $M_2(k)$ is isomorphic either to $R_e = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, R_g = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$ or to $R_e = M_2(k), R_g = 0$.

We turn now to the characteristic 2 case.

### 9.4.6 Theorem

Let $R = M_2(k)$, $k$ a field of characteristic 2. Then a $C_2$-grading of $R$ is of one of the following two types:
(i) The trivial grading, $R_e = M_2(k), R_g = 0$;

(ii) $R_e = \left\{ \left( \begin{array}{cc} x & \beta(x+y) \\ \alpha(x+y) & y \end{array} \right) \middle| \; x, y \in k \right\}$,

$R_g = \left\{ \left( \begin{array}{cc} \alpha x + \beta y & x \\ y & \alpha x + \beta y \end{array} \right) \middle| \; x, y \in k \right\}$ for some $\alpha, \beta \in k$.

**Proof** Let us consider a $C_2$-grading of $R$. Then for any $A, B \in R$ we have

$$\begin{aligned} (AB)_e &= A_e B_e + A_g B_g \\ &= A_e B_e + (A - A_e)(B - B_e) \\ &= AB + AB_e + A_e B \end{aligned}$$

Let $\phi : R \to R$, $\phi(A) = A_e$. A straightforward (but tedious) computation shows that the matrix of $\phi$ in the basis $E_{11}, E_{12}, E_{21}, E_{22}$ is of the form

$$X = \left( \begin{array}{cccc} 1 & \alpha & \beta & 0 \\ \beta & \gamma & 0 & \beta \\ \alpha & 0 & \gamma & \alpha \\ 0 & \alpha & \beta & 1 \end{array} \right)$$

for some $\alpha, \beta, \gamma \in k$ (to see this we let $A$ and $B$ run through elements of the basis in the previously displayed formula). Since $\phi^2 = \phi$, we must have $X^2 = X$, implying that either $\gamma = 1, \alpha = \beta = 0$ or $\gamma = 0$. In the first case $X = I_4$, thus $\phi = Id$, and we find the trivial grading. Let now

$$X = \left( \begin{array}{cccc} 1 & \alpha & \beta & 0 \\ \beta & 0 & 0 & \beta \\ \alpha & 0 & 0 & \alpha \\ 0 & \alpha & \beta & 1 \end{array} \right)$$

for $\alpha, \beta \in k$. If $A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_2(k)$, then

$$\begin{aligned} A_e = \phi(A) &= \left( \begin{array}{cc} a + b\alpha + c\beta & \beta(a+d) \\ \alpha(a+d) & d + b\alpha + c\beta \end{array} \right) \\ &= \left( \begin{array}{cc} x & \beta(x+y) \\ \alpha(x+y) & y \end{array} \right) \end{aligned}$$

where $x = a + b\alpha + c\beta, y = d + b\alpha + c\beta$. Also, if $A = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in M_2(k)$, then

$$\begin{aligned} A_g = A - A_e &= \left( \begin{array}{cc} b\alpha + c\beta & b + a\beta + d\beta \\ c + a\alpha + d\alpha & b\alpha + c\beta \end{array} \right) \\ &= \left( \begin{array}{cc} \alpha x + \beta y & x \\ y & \alpha x + \beta y \end{array} \right) \end{aligned}$$

where $x = b + a\beta + d\beta, y = c + a\alpha + d\alpha$. □

Using Theorem 9.4.6 we obtain the same result for the $\mathrm{char}(k) = 2$ case as for the $\mathrm{char}(k) \neq 2$ case (cf. Corollary 9.4.2), albeit by completely different methods.

### 9.4.7 Corollary

If $\mathrm{char}(k) = 2$, then any non-trivial $C_2$-grading of $M_2(k)$ is a crossed product.

**Proof** If $\alpha = \beta = 0$, then clearly $R_g$ contains invertible elements. If at least one of $\alpha$ and $\beta$, say $\alpha$, is non-zero, then $\begin{pmatrix} \alpha & 1 \\ 0 & \alpha \end{pmatrix}$ is an invertible element of $R_g$. □

### 9.4.8 Proposition

Let $\mathrm{char}(k) = 2$, and $R = M_2(k)$ with the grading

$$R_e = \left\{ \begin{pmatrix} x & \beta(x+y) \\ \alpha(x+y) & y \end{pmatrix} \middle| \; x, y \in k \right\},$$

$$R_g = \left\{ \begin{pmatrix} \alpha x + \beta y & x \\ y & \alpha x + \beta y \end{pmatrix} \middle| \; x, y \in k \right\}$$

Then this grading is isomorphic to a good grading if and only if there exists $t \in k$ such that $\alpha t^2 + t + \beta = 0$.

**Proof** We proceed as in the proof of Proposition 9.4.3. If $X = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ is an invertible matrix inducing an isomorphism of graded algebras between $R$ and $M_2(k)$ with the only non-trivial $C_2$-grading, then

$$\begin{aligned} \beta(ps + rq) &= pq \\ \alpha(ps + qr) &= rs \\ \alpha p^2 + \beta r^2 &= pr \\ \alpha q^2 + \beta s^2 &= qs. \end{aligned}$$

If $\alpha, \beta \neq 0$, then $p, q, r, s \neq 0$ (since $ps + qr = det(X) \neq 0$). Then $\alpha \left(\frac{p}{r}\right)^2 + \frac{p}{r} + \beta = 0$, and we take $t = \frac{p}{r}$. If $\alpha = 0$ or $\beta = 0$, then clearly there is $t \in k$ such that $\alpha t^2 + t + \beta = 0$.
Conversely, suppose that $\alpha t^2 + t + \beta = 0$ for some $t \in k$. If $\alpha \neq 0$, let $t_1, t_2 \in k$

be the (distinct) roots of this equation, and then the matrix $X = \begin{pmatrix} t_1 & t_2 \\ 1 & 1 \end{pmatrix}$ produces the required isomorphism. If $\alpha = 0$, we take $X = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$, which also induces an isomorphism as desired. □

### 9.4.9 Corollary

If $\mathrm{char}(k) = 2$ and $k$ is algebraically closed, then any $C_2$-grading of $M_2(k)$ is isomorphic to a good grading.

## 9.5 $C_2$-gradings on $M_2(k)$ in characteristic 2

Let $k$ be a field of characteristic 2. We keep the notation $C_2 = \{e, g\}$. All the $C_2$-gradings on the algebra $M_2(k)$ have been described in Section 9.5. In this section we classify the isomorphism types of such gradings.
Let $A = M_2(k)$. We have seen that a $C_2$-algebra grading of $A$ is of one of the following two types
(i) The trivial grading: $A_e = M_2(k)$, $A_g = 0$;
(ii) The grading

$$A_e = \left\{ \begin{pmatrix} x & \beta(x+y) \\ \alpha(x+y) & y \end{pmatrix} \middle| \, x, y \in k \right\}$$

$$A_g = \left\{ \begin{pmatrix} \alpha x + \beta y & x \\ y & \alpha x + \beta y \end{pmatrix} \middle| \, x, y \in k \right\}$$

for some $\alpha, \beta \in k$.
We denote by $A(\alpha, \beta)$ the $C_2$-grading of $A$ described by (ii).

### 9.5.1 Lemma

Let $\alpha, \beta, \alpha', \beta' \in k$. The following assertions are equivalent.
(a) The $C_2$-graded algebras $A(\alpha, \beta)$ and $A(\alpha', \beta')$ are isomorphic.
(b) There exist $u, v, w, t \in k$ such that $ut + vw \neq 0$ and

$$\beta'(ut + vw) = \beta u^2 + \alpha v^2 + uv \tag{9.3}$$

$$\alpha'(ut + vw) = \beta w^2 + \alpha t^2 + wt \tag{9.4}$$

(c) There exist $u, v, w, t \in k$ such that $ut + vw \neq 0$ and

$$\alpha(ut + vw) = \alpha' u^2 + \beta' w^2 + uw \tag{9.5}$$

$$\beta(ut + vw) = \alpha' v^2 + \beta' t^2 + vt \tag{9.6}$$

**Proof** $(b) \Rightarrow (c)$ Denote by $d = ut + vw \in k$. If we add the equation (9.3) multiplied by $\frac{w^2}{d}$ to the equation (9.4) multiplied by $\frac{u^2}{d}$ we obtain the equation (9.5). Similarly, adding the equation (9.3) multiplied by $\frac{t^2}{d}$ to the equation (9.4) multiplied by $\frac{v^2}{d}$ we obtain the equation (9.6).
$(c) \Rightarrow (b)$ The equation (9.3) can be obtained by adding the equation (9.5) multiplied by $\frac{v^2}{d}$ to the equation (9.6) multiplied by $\frac{u^2}{d}$. Finally, if we add (9.5) multiplied by $\frac{t^2}{d}$ to (9.6) multiplied by $\frac{w^2}{d}$ we obtain the equation (9.4).
We show now that $(a)$ is equivalent to $(b)$ (and $(c)$). Any isomorphism of $C_2$-graded algebras $\varphi : A(\alpha, \beta) \to A(\alpha', \beta')$ is, in particular, an automorphism of $M_2(k)$. Then, by Skolem-Noether Theorem, there exists $U \in GL_2(k)$ such that $\varphi(X) = UXU^{-1}$ for any $X \in M_2(k)$. Thus, $A(\alpha, \beta) \simeq A(\alpha', \beta')$ if and only if there exists $U = \begin{pmatrix} u & v \\ w & t \end{pmatrix} \in M_2(k)$ with $det\ U = ut + vw \neq 0$, such that

$$UA(\alpha, \beta)_e U^{-1} \subseteq A(\alpha', \beta')_e \tag{9.7}$$

and

$$UA(\alpha, \beta)_g U^{-1} \subseteq A(\alpha', \beta')_g \tag{9.8}$$

For $x, y \in k$ we have

$$U \begin{pmatrix} x & \beta(x+y) \\ \alpha(x+y) & y \end{pmatrix} U^{-1} = \frac{1}{ut+vw} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

where

$$\begin{aligned} b_{11} &= utx + \alpha vt(x+y) + \beta uw(x+y) + vwy \\ b_{12} &= uvx + \alpha v^2(x+y) + \beta u^2(x+y) + uvy \\ b_{21} &= wtx + \alpha t^2(x+y) + \beta w^2(x+y) + wty \\ b_{22} &= vwx + \alpha vt(x+y) + \beta uw(x+y) + uty \end{aligned}$$

Equation (9.7) holds if and only if for any $x, y \in k$ there exist $x', y' \in k$ such that $b_{11} = x', b_{12} = \beta'(x' + y')$, $b_{21} = \alpha'(x' + y')$, and $b_{22} = y'$. This is equivalent to

$$\beta'(b_{11} + b_{22}) = b_{12} \text{ and } \alpha'(b_{11} + b_{22}) = b_{21}$$

and these are equivalent to the conditions of $(b)$. Similarly, for $x, y \in k$ we have

$$U \begin{pmatrix} \alpha x + \beta y & x \\ y & \alpha x + \beta y \end{pmatrix} U^{-1} = \frac{1}{ut+vw} \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

where

$$\begin{aligned} d_{11} &= \alpha utx + \beta uty + vty + uwx + \alpha vwx + \beta vwy \\ d_{12} &= v^2 y + u^2 x \\ d_{21} &= t^2 y + w^2 x \\ d_{22} &= \alpha utx + \beta uty + vty + uwx + \alpha vwx + \beta vwy \end{aligned}$$

Equation (9.8) holds if and only if for any $x, y \in k$ there exist $x', y' \in k$ such that $d_{11} = \alpha' x' + \beta' y' = d_{22}, d_{12} = x'$ and $d_{21} = y'$. Or, equivalently,

$$\alpha' d_{12} + \beta' d_{21} = d_{11}$$

for any $x, y \in k$. This is equivalent to the two equations in $(c)$. □

### 9.5.2 Theorem

$A(\alpha, \beta)$ and $A(\alpha', \beta')$ are isomorphic as $C_2$-graded algebras if and only if there exists $z \in k$ such that $z^2 + z + \alpha\beta + \alpha'\beta' = 0$.

**Proof** Assume that $A(\alpha, \beta) \simeq A(\alpha', \beta')$. Then, by Lemma 9.5.1, there exist $u, v, w, t \in k$ such that $ut + vw \neq 0$ and the conditions (9.3), (9.4), (9.5) and (9.6) are satisfied. Since $ut + vw \neq 0$ we have either $u \neq 0$ or $v \neq 0$. Multiplying (9.3) by $\alpha$ and (9.5) by $\beta'$ we obtain

$$\alpha\beta u^2 + \alpha^2 v^2 + \alpha uv = \alpha'\beta' u^2 + \beta'^2 w^2 + \beta' uw$$

or, equivalently,

$$(\alpha\beta + \alpha'\beta')u^2 + (\alpha v + \beta' w)^2 + u(\alpha v + \beta' w) = 0$$

Thus, if $u \neq 0$ then $z = \frac{\alpha v + \beta' w}{u}$ satisfies $z^2 + z + \alpha\beta + \alpha'\beta' = 0$. On the other hand, if we multiply (9.3) by $\beta$ and (9.6) by $\beta'$ we obtain

$$\beta^2 u^2 + \alpha\beta v^2 + \beta uv = \alpha'\beta' v^2 + \beta'^2 t^2 + \beta' vt$$

or, equivalently,

$$(\alpha\beta + \alpha'\beta')v^2 + (\beta u + \beta' t)^2 + v(\beta u + \beta' t) = 0$$

Thus, if $v \neq 0$ then $z = \frac{\beta u + \beta' t}{v}$ satisfies $z^2 + z + \alpha\beta + \alpha'\beta' = 0$.
Conversely, suppose that there exists $z \in k$ such that $z^2 + z + \alpha\beta + \alpha'\beta' = 0$. Using again Lemma 9.5.1 it is enough to find $u, v, w, t \in k$ such that $ut + vw \neq 0$ and (9.3) and (9.4) (or (9.5) and (9.6)) are satisfied. We have the following cases.

1. $\alpha = 0$ and $\alpha' = 0$. Then we take $U = \begin{pmatrix} 1 & \beta + \beta' \\ 0 & 1 \end{pmatrix}$.

2. $\alpha \neq 0$ and $\alpha' \neq 0$. Then we take $U = \begin{pmatrix} 1 & \frac{z}{\alpha} \\ 0 & \frac{\alpha}{\alpha'} \end{pmatrix}$. $\alpha \neq 0$ and $\alpha' = 0$. Then we take
   $U = \begin{pmatrix} 0 & \frac{\beta'}{\alpha} \\ 1 & \frac{z}{\alpha} \end{pmatrix}$ in the case where $\beta' \neq 0$,
   $U = \begin{pmatrix} \alpha & 1 \\ 1 & 0 \end{pmatrix}$ in the case where $\beta' = 0$ and $\beta = 0$,

or $U = \begin{pmatrix} \frac{z_1}{\beta} & 1 \\ \frac{z_2}{\beta} & 1 \end{pmatrix}$ in the case $\beta' = 0$ and $\beta \neq 0$, where $z_1$ and $z_2$ are the distinct solutions of the equation $z^2 + z + \alpha\beta = 0$.

3. $\alpha = 0$ and $\alpha' \neq 0$. Similar to the case 3. □

In the following theorem we describe the isomorphism classes of $C_2$-gradings on $M_2(k)$ and classify them.

### 9.5.3 Theorem

Let $k$ be a field of characteristic 2. A $C_2$-grading of $M_2(k)$ is either the trivial grading $A_e = M_2(k)$, $A_g = 0$, or isomorphic to a grading $A(\alpha)$ given by

$$A_e = \left\{ \left( \begin{array}{cc} x & x+y \\ \alpha(x+y) & y \end{array} \right) \middle| \, x, y \in k \right\}$$

$$A_g = \left\{ \left( \begin{array}{cc} \alpha x + y & x \\ y & \alpha x + y \end{array} \right) \middle| \, x, y \in k \right\}$$

for some $\alpha \in k$. Moreover, two nontrivial $C_2$-gradings $A(\alpha)$ and $A(\alpha')$ are isomorphic if and only if there exists $z \in k$ such that $\alpha - \alpha' = z^2 + z$. Thus, there is a bijective correspondence between the isomorphism types of nontrivial $C_2$-gradings and the factor group $k/S(k)$, where $S(k) = \{ z^2 + z \mid z \in k \}$.

**Proof** Let us note first that for $\alpha, \beta \in k$ we have $A(\alpha, \beta) \simeq A(\alpha\beta, 1)$ (by Theorem 9.5.2). Thus, in every isomorphism class we can choose a representative of the form $A(\alpha, 1) = A(\alpha)$. Moreover, $A(\alpha, 1) \simeq A(\alpha', 1)$ if and only if there exists $z \in k$ such that $z^2 + z + \alpha + \alpha' = 0$, or, equivalently, $\alpha' - \alpha = z^2 + z \in S(k)$. □

### 9.5.4 Corollary

If $k$ is an algebraically closed field of characteristic 2 then there are two isomorphism types of $C_2$-gradings on $M_2(k)$. More precisely, any $C_2$-grading on $M_2(k)$ is either the trivial grading $A_e = M_2(k)$, $A_g = 0$, or isomorphic to the grading

$$A_e = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}, \quad A_g = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}$$

**Proof** Since $k$ is algebraically closed we have that $S(k) = k$, so by Theorem 9.5.3 we obtain that there exist precisely two isomorphism types of $C_2$-gradings on $M_2(k)$. Since the trivial grading is not isomorphic to the second grading in the statement, we obtain the result. □

## 9.6 Gradability of modules

In this section we use Clifford theory to prove that if $R = \oplus_{\sigma\in G} R_\sigma$ is a left Artinian ring graded by a torsion free group $G$, then a certain class of $R$-modules are gradable. We essentially follow the paper [38], where the results are given for $G \simeq \mathbf{Z}$.

### 9.6.1 Theorem

Let $R = \oplus_{\sigma\in G} R_\sigma$ be a left Artinian ring graded by the torsion free group $G$. Then $R$ has finite support.

**Proof** Since $R$ is a left Artinian ring, $R$ is an object of finite length in $R$-gr. Thus in order to show that $R$ has finite support it is enough to prove that for any graded simple $R$-module $\Sigma$ we have $sup(\Sigma) < \infty$.
Indeed, since $\Sigma$ is a cyclic left $R$-module and $R$ has finite length in $R$-mod, then so does $\Sigma$. Let $\Delta = \mathrm{End}({}_R\Sigma)$. Clearly $R_e$ is a left Artinian ring. By Theorem 4.2.5 we have $|\Omega_{R_e}(\Sigma)| = [\mathrm{sup}(\Sigma) : G\{\Sigma\}]$. Since $|\Omega_{R_e}(\Sigma)| < \infty$, we also have $[\mathrm{sup}(\Sigma) : G\{\Sigma\}] < \infty$. On the other hand $\Delta = \oplus_{\sigma\in G\{\Sigma\}}\Delta_\sigma$. By Theorem 4.1.4, $\Delta$ is a left Artinian ring. If $G\{\Sigma\} \neq e$, then there exists $\sigma \in G\{\Sigma\}$, $\sigma \neq e$. Let $H =< \sigma >$. Since $G$ is torsion free, we have $H \simeq \mathbf{Z}$. Clearly $\Delta_H = \oplus_{h\in H}\Delta_h$ is also a left Artinian ring. On the other hand $\Delta_H$ is a crossed product, so $\Delta_H \simeq \Delta_e[X, x^{-1}, \phi]$. In particular $\Delta_H$ is a domain and $\Delta_H$ is not a left Artinian ring, a contradiction. We conclude that $G\{\Sigma\} = \{e\}$ and thus $\mathrm{sup}(\Sigma) < \infty$. □

### 9.6.2 Corollary

Let $R = \oplus_{\sigma\in G} R_\sigma$ be a ring graded by the torsion free group $G$. Then $R$ is left Artinian if and only if $R_e$ is a left Artinian ring and ${}_{R_e}R$ is finitely generated.

**Proof** Assume that $R$ is a left Artinian ring. By Theorem 9.6.1, $\mathrm{sup}(R) < \infty$. If $\sigma \in \mathrm{sup}(R)$ and $X$ is an $R_e$-submodule of the left $R_e$-module $R_\sigma$, we have that $X = RX \cap R_\sigma$. It follows that $R_\sigma$ is a left $R_e$-module of finite length. In particular $R_e$ is a left Artinian ring and $R_\sigma$ is a finitely generated left $R_e$-module. Since $R = \oplus_{\sigma\in\mathrm{sup}(R)} R_\sigma$, we obtain that $R$ is finitely generated as a left $R_e$-module.
Conversely, since $R_e$ is a left Artinian ring, then $R$ has finite length as a left $R_e$-module. Hence ${}_{R_e}R$ is Artinian, and then so is ${}_RR$. □

### 9.6.3 Corollary

Assume that $R = \oplus_{\sigma \in G} R_\sigma$ is a semisimple Artinian ring graded by the torsion free group $G$. Then $\sup(R) < \infty$. Moreover, if $R_e$ is a simple Artinian ring, then $R_\sigma = 0$ for any $\sigma \neq e$.

**Proof** By Theorem 9.6.1 we have $\sup(R) < \infty$. Since $R$ is semisimple Artinian, we can write ${}_R R = L_1 \oplus \ldots \oplus L_s$, where $L_1, \ldots, L_s$ are gr-maximal left ideals of $R$. Using Theorem 4.2.5, since $R_e$ is a simple Artinian ring, we have $|\Omega_{R_e}(L_i)| = 1$. Since $G\{L_i\} = \{e\}$ ($G$ is torsion free), we have that $|\sup(L_i)| = 1$. Assume that for some $\sigma \in G$ we have $(L_i)_\sigma \neq 0$ and $(L_i)_x = 0$ for any $x \neq \sigma$. If $\sigma \neq e$, since $\sup(R) < \infty$ then there exists $t \geq 1$ such that $L_i^t = 0$. Since $J(R) = 0$ (the Jacoson radical) we have $L_i = 0$. Thus we must have $\sigma = e$, and then $\sup(L_i) = \{e\}$ for any $1 \leq i \leq s$, therefore $\sup(R) = \{e\}$. □

We recall that a ring $R$ is called semiprimary if the Jacobson radical $J(R)$ is nilpotent and $R/J(R)$ is a semisimple Artinian ring.

### 9.6.4 Proposition

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a $G$-graded ring of finite support, where $G$ is a torsion free group. Assume that $R$ is a semiprimary ring. Then $J(R) = J^g(R)$. If $R_e$ is a local ring (i.e. $R_e/J(R_e)$ is simple Artinian), then $(R/J(R))_\sigma = 0$ for any $\sigma \neq e$, i.e. $R/J(R) = R_e/J(R_e)$.

**Proof** Since $\sup(R) < \infty$, we have $J^g(R) \subseteq J(R)$. On the other hand, since $G$ is torsion free, we have by Theorem 4.4.4 that if $\Sigma$ is a graded simple $R$-module, then $\Sigma$ is simple as an $R$-module, therefore $J(R) \subseteq J^g(R)$. We conclude that $J(R) = J^g(R)$. If we denote $S = R/J(R) = R/J^g(R)$, then $S$ is a graded ring which is semisimple Artinian. By Corollary 9.6.3 we have $S = S_e$. □

### 9.6.5 Theorem

Let $R = \oplus_{\sigma \in G} R_\sigma$ be a $G$-graded ring, where $G$ is a torsion free group. Let $M \in R$-gr such that $\sup(M) < \infty$ and $M$ has finite length in the category $R$-gr. Then $M$ is indecomposable in $R$-gr if and only if it is indecomposable in $R$-mod.

**Proof**
If $M$ is indecomposable, then obviously it is gr-indecomposable. Assume that $M$ is gr-indecomposable. By Corollary 2.4.5 we have $END_R(M) = \mathrm{End}_R(M)$ and also $END_R(M)$ is a graded ring of finite support. If $M = \oplus_{\sigma \in \sup(M)} M_\sigma$, then $M_\sigma$ is an $R_e$-module of finite length, so $M$ is also an $R_e$-module of finite

length. This shows that $M$ has finite length as an $R$-module, and $\mathrm{End}_R(M)$ is a semiprimary ring. Since $M$ is gr-indecomposable, $\mathrm{End}_{R-\mathrm{gr}}(M)$ is a local ring. Now by Proposition 9.6.4 we see that $\mathrm{End}_R(M)$ is a local ring, so $M$ is indecomposable in $R$-mod. □

### 9.6.6 Corollary

Let $R$ be a left Artinian $G$-graded ring, $G$ a torsion free group. If $M \in R$-gr is finitely generated, then $M$ is indecomposable in $R$-gr if and only if $M$ is indecomposable in $R$-mod.

The main result of this section is the following.

### 9.6.7 Corollary

Let $R$ be a left Artinian $G$-graded ring, $G$ is a torsion free group. If $M, N \in I\!R$-gr are two finitely generated and indecomposable in $I\!R$-gr. Then $M \simeq N(\sigma)$ in $R$-gr if and only if $M \simeq N$ in $R$-mod.

**Proof** The implication $\Rightarrow$ is clear. Assume that $M \simeq N$ in $R$-mod. Then $F(M) \simeq F(N)$ in $R$-gr where $F$ is the right adjoint of the forgetful functor $U : R\text{-gr} \to R$. But $F(M) \cong \oplus_{\sigma\in G} M(\sigma)$ and $F(N) = \oplus_{\sigma\in G} N(\sigma)$. So $\oplus_{\sigma\in G} M(\sigma) \simeq \oplus_{\sigma\in G} N(\sigma)$. Since $M(\sigma), N(\sigma)$ are indecomposable of finite length, by the Krull-Remak-Schmidt-Azumaya Theorem we have $M \simeq N(\sigma)$ for some $\sigma \in G$. □

### 9.6.8 Theorem

Let $R$ be a left Artinian ring graded by the torsion free group $G$. The following assertions hold.

(i) Every semisimple left $R$-module is gradable.

(ii) If ${}_RM$ is a gradable finitely generated module, then any direct summand of $M$ is gradable.

(iii) Every projective left $R$-module is gradable.

(iv) Every injective left $R$-module is gradable.

**Proof** (i) It is enough to show that any simple $R$-module is gradable. But this is given in Theorem 4.4.4 assertion 5).
(ii) It follows from Corollary 5.6.6 and the Krull-Remak-Schmidt Theorem.
(iii) It is clear that any free left $R$-module is gradable. Since a projective module is a direct summand of a free module, we can apply again the Krull-Remak- Schmidt Theorem.

(iv) Let $Q$ be an injective object in $R-mod$. Since $R$ is also a left Noetherian ring, $Q$ is a direct sum of indecomposable injective modules. Thus we can reduce to the case where $Q$ is indecomposable injective. Since $R$ is left Artinian, there exists a simple left $R$-module $M$ such that $Q = E(M)$ (the injective envelope). But $M$ is gradable by assertion (i), so we can assume that $M$ is a graded module. Clearly $M$ has finite support, and then $E(M) \simeq E^{\rm gr}(M)$ (see Section 2.8). Hence $Q \simeq E^{\rm gr}(M)$, in particular $Q$ is gradable. Here $E^{\rm gr}(M)$ denoted the injective envelope of $M$ in $R$-gr. □

### 9.6.9 Remark

Since $R$ is a left Artinian ring and $Q \in R$-gr is an arbitrary object in $R$-gr, then $Q$ is injective in $R$-mod. Indeed, $Q$ is a direct summ of indecomposable injective objects from $R$-gr. Since $\sup(R) < \infty$, we have that any indecomposable injective object from $R$-gr is injective in $R$-mod (see Section 2.8). Since $R$ is left Noetherian, $Q$ is also an injective $R$-module.

## 9.7 Comments and References for Chapter 9.

Given a ring $R$ and a group $G$ the problem whether we can introduce a (nontrivial) $G$-gradation on $R$ arises. In this chapter we give necessary and sufficient conditions for some positive results. A detailed study for $M_n(k)$ is included; this is a problem of Zelmanov. The results obtained in Chapter 9 make it a tool for obtaining a series of examples and counterexamples in graded ring theory. This is done in Sections 2,3 and 4,5. Section 1 is a brief presentation of the descent theory. Section 6 provides sufficient conditions for a $R$-module over a $G$-graded ring $R$ to be "gradable", that is, whether we can introduce a $G$-gradation on this module making it into a graded $R$-module.

### Some References

- Crina Boboc, [22], [23]
- S. Dăscălescu, B. Ion, C. Năstăsescu and J. Rios Montes, [61]
- J.L. Gómez Pardo and C. Năstăsescu, [74], [76]
- C. Năstăsescu and F. Van Oystaeyen, [150]
- S. Caenepeel, S. Dăscălescu, C. Năstăsescu, [37]
- Yu. A. Bahturin, S. K. Sehgal, M. V. Zaicev [7]

# Appendix A. Some Category Theory

## A.1. The Categorical Language

A category $\mathcal{C}$ consists of a class of objects, and we agree to write $X \in \mathcal{C}$ to state that $X$ is an object of $\mathcal{C}$, together with sets (!) $\mathrm{Hom}_{\mathcal{C}}(A, B)$ for any pair $(A, B)$ of objects of $\mathcal{C}$. The elements of $\mathrm{Hom}_{\mathcal{C}}(A, B)$ are said to be the morphisms from $A$ to $B$. For $A \in \mathcal{C}$ there is a distinguished element $I_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$. For any triple $A, B, C$ of objects there is a composition map :

$$\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$$
$$(f, g) \mapsto g \circ f$$

satisfying the following properties :

1. For $f \in \mathrm{Hom}_{\mathcal{C}}(A, B), g \in \mathrm{Hom}_{\mathcal{C}}(B, C), h \in \mathrm{Hom}_{\mathcal{C}}(C, D)$ the associativity law holds, i.e. $h \circ (g \circ f) = (h \circ g) \circ f$.
2. For $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $f \circ I_A = f = I_B \circ f$.
3. Whenever $(A, B) \neq (A', B')$, the sets $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and $\mathrm{Hom}_{\mathcal{C}}(A', B')$ are disjoint.

It is customary to write $f : A \to B$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, and it is common to call $I_A$ the identity morphism of $A$. Some well-known examples include the categories :

- **Set**, the class of all sets and sets of maps for the morphisms.
- **Top**, the class of all topological spaces and sets of continuous functions for the morphisms.
- **Ring**, the class of all rings with identity and ring morphisms for the morphisms.
- **Ab**, the class of all abelian groups and sets of group morphisms for the morphisms.
- **Gr**, the class of all groups and sets of group morphisms for the morphisms.
- **R-mod**, for any ring $R$ this is the class of left $R$-modules with left $R$-linear map for the morphisms.

Let $\mathcal{C}$ be a category and $\mathcal{C}'$ a class of objects of $\mathcal{C}$. We say that $\mathcal{C}'$ is a **subcategory of** $\mathcal{C}$ if :

1. For $A, B \in \mathcal{C}'$, $\mathrm{Hom}_{\mathcal{C}'}(A, B) \subset \mathrm{Hom}_{\mathcal{C}}(A, B)$.

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2. Composition of morphisms in $C'$ is the same as in $\mathcal{C}$.

3. For $A \in \mathcal{C}'$, $I_A$ is the same in $\mathcal{C}'$ as in $\mathcal{C}$.

A subcategory is said to be **full subcategory** of $\mathcal{C}$ whenever for $A, B \in \mathcal{C}$ we have $\mathrm{Hom}_{\mathcal{C}}(A, B) = \mathrm{Hom}_{\mathcal{C}'}(A, B)$.

Consider a family of categories $(\mathcal{C}_i)_{i \in J}$ indexed by some index set $J$ supposed to be nonempty. The **direct product** of the family $(\mathcal{C}_i)_{i \in J}$ is the category $\mathcal{C}$ the objects of which are the families $(M_i)_{i \in J}$ of objects $M_i$ of $\mathcal{C}_i$ for $i \in J$. If $(N_i)_{i \in J}$ is another object then we define $\mathrm{Hom}_{\mathcal{C}}(M, N) = \{(f_i)_{i \in J}, f_i \in \mathrm{Hom}_{\mathcal{C}_i}(M_i, N_i), i \in J\}$. Composition of morphisms is defined componentwise. We denote this direct product category by $\prod_{i \in J} \mathcal{C}_i$; in case all $\mathcal{C}_i$ are $\mathcal{C}$ we also write $\mathcal{C}^J$ and if $J$ is finite, say $J = \{1, \ldots, n\}$ then we write $\mathcal{C}_1 \times \mathcal{C}_2 \times \ldots \times \mathcal{C}_n$ for the direct product.

A morphism $f : A \to B$ is called **monomorphism** if for any object $C$ of $\mathcal{C}$ and morphisms $h, g \in \mathrm{Hom}_{\mathcal{C}}(C, A)$ such that $f \circ h = f \circ g$ we have $g = h$. A morphism $f$ is called **epimorphism** if for any object $D$ of $\mathcal{C}$ and morphisms $h, g \in \mathrm{Hom}_{\mathcal{C}}(B, D)$ such $h \circ f = g \circ f$ we have $h = g$. An **isomorphism** of $\mathcal{C}$ is a morphism $f : A \to B$ for which there exists $g \in \mathrm{Hom}_{\mathcal{C}}(B, A)$ such that $g \circ f = I_A$ and $f \circ g = I_B$. One easily verifies that $g$ is unique if it exists; we call $g$ the **inverse** of $f$ and it will be denoted by $f^{-1}$. As an exercise one may check that an isomorphism is both a monomorphism and an epimorphism. Observe that the converse is false e.g. in **Ring** the inclusion $\mathbb{Z} \to Q$ is both a monomorphism and an epimorphism but not an isomorphism. The property of being monomorphism, resp. epimorphism, resp. isomorphism, is closed under composition. To $\mathcal{C}$ we may associate the category $\mathcal{C}^\circ$ having the smae objects as $\mathcal{C}$ but with $\mathrm{Hom}_{\mathcal{C}^\circ}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$; we call $\mathcal{C}^\circ$ the **dual category** of $\mathcal{C}$. If $f : A \to B$ is a morphism in $\mathcal{C}$ then $f$ is a monomorphism, resp. epimorphism, if and only if $f$ is an epimorphism, resp. monomorphism, when viewed as a morphism from $B \to A$ in the dual category $\mathcal{C}^\circ$. Thus, epimorphism is the dual notion for monomorphism.

Fix an object $A$ of $\mathcal{C}$. For monomorphisms $\alpha_1 : A_1 \to A$ and $\alpha_2 : A_2 \to A$ we define $\alpha_1 \leq \alpha_2$ if thgere exists a morphism $\gamma : A_1 \to A_2$ such that $\alpha_0 \circ \gamma = \alpha_1$. If such $\gamma$ exists it is necessarily unique and also a monomorphism. Monomorphisms $\alpha_1$ and $\alpha_2$ are called equivalent if $\alpha_1 \leq \alpha_2$ and $\alpha_2 \leq \alpha_1$; this defines an equivalence relation and the Zermelo axiom allows to choose a representative in every equivalence class. The resulting monomorphism is called a **subobject of** $A$. The notion of quotient object may be defined dually.

An object $I$, resp. $F$, of $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(I, A)$, resp. $\mathrm{Hom}_{\mathcal{C}}(A, F)$, is a singleton for every $A \in \mathcal{C}$, is called an **initial**, resp. **final** object of $\mathcal{C}$. It is not hard to verify that two initial, resp, final, objects of $\mathcal{C}$ are necessarily isomorphic. An object that is simultaneously initial and final is called a **zero**

**object** of $\mathcal{C}$. A morphism $f : A \to B$ is said to be a zero morphism whenever it factorizes through the zero object. When it exists a zero object is unique up to isomorphism and it will be denoted by 0. Then each set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ has precisely one zero morphism, that is denoted by $0_{AB}$ (or simply 0 when no confusion can arise).

For a family $(M_i)_{i\in J}$ of objects of $\underline{\mathcal{C}}$ we may define the product, denoted by $\prod_{i\in J} M_i$, by specifying a family of morphisms $\pi_j : \prod_{i\in J} M_i \to M_j$, $j \in J$, such that for every object $M \in \mathcal{C}$ and any family of morphisms $(f_i|_{i\in J}, f_i : M \to M_i$ for $i \in J$, there exists a unique morphism $f : M \to \prod_{i\in J} M_i$ such that $\pi_i \circ f = f_i$ for all $i \in J$. In fact we have to guarantee that this product exists, but if it does, then it is unique up to isomorphism. If $J$ is finite then the product will be denoted by $M_1 \times \ldots \times M_n$. The categories : **Set**, **Gr**, **Top**, $R$**--mod**, have products. By duality the notion of coproduct may be defined, i.e. it is the product in the dual category. It is denoted by $\coprod_{i\in J} M_i$ and in case $J$ is finite it is costumary to write $M_1 \oplus \ldots \oplus M_n$.

Categories cannot be related by "maps" in the set theoretical sense because we are in general not dealing with sets. A new notion takes the place of maps here i.e. functors. If $\mathcal{B}$ and $\mathcal{C}$ are categories then a **(covariant) functor** from $\mathcal{B} \to \mathcal{C}$ is obtained by associating to an object $B$ of $\mathcal{B}$ an object $F(B)$ of $\mathcal{C}$ and to $f \in \mathrm{Hom}_{\mathcal{B}}(B_1, B_2)$ a morphism $F(f) \in \mathrm{Hom}_{\mathcal{C}}(F(B_1)F(B_2))$ such that the following properties hold :

1. $F(I_B) = I_{F(B)}$ for any $B \in \mathcal{B}$
2. $F(g \circ f) = F(g) \circ F(f)$

Inspired by the set-theoretic notation we shall write $F : \mathcal{B} \to \mathcal{C}$, meaning that the correspondence associating $F(B) \in \mathcal{C}$ to $B \in \mathcal{B}$ is a covariant functor as defined above. A covariant functor $\mathcal{C}^\circ \to \mathcal{D}$ (or from $\mathcal{C}$ to $\mathcal{D}^\circ$) is then called a **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$.

A covariant functor $\mathcal{C} \to \mathcal{D}$ yields a map $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ for $A, B \in \mathcal{C}$, defined by $f \mapsto F(f)$. The functor $F$ is said to be faithful, resp. full, resp. full and faithful whenever the foregoing map is injective, resp. surjective, resp. bijective. For any category $\mathcal{C}$ we may define the identity functor $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ by $1_{\mathcal{C}}(A) = A$ for $A \in \mathcal{C}$ and morphism.

For functors $F, G : \mathcal{C} \to \mathcal{D}$ we may define a functorial morphism $\phi : F \to G$ by giving a family $\{\phi(A), A \in \mathcal{C}\}$ of morphisms such that $\phi(A) : F(A) \to G(A)$ for $A \in \mathcal{C}$, and for $f : A \to B$ we have $\phi(B) \circ F(f) = G(f) \circ \phi(A)$. We say that $\phi$ is a **functorial isomorphism** if for any $A \in \mathcal{C}$, $\phi(A)$ is an isomorphism; if such $\phi$ exists we write $F \simeq G$.

Functorial morphisms $\phi : F \to G, \psi : G \to H$ may be composed to $\psi \circ \phi : F \to H$, by putting $(\psi \circ \phi)(A) = \psi(A) \circ \phi(A)$ for all $A$. By $\mathrm{Hom}(F, G)$ we

denote the class of all functorial morphisms from $F$ to $G$. If $\mathcal{C}$ is a small category, that is if objects of $\mathcal{C}$ are sets then $\mathrm{Hom}(F,G)$ is also a set. Note that there is an identity functorial morphism $1_F : F \to F$ defined by taking $1_F(A) = 1_{F(A)}$ for any $A \in \mathcal{C}$.

A covariant functor $F : \mathcal{C} \to \mathcal{D}$ is an **equivalence of categories** if there is a covariant functor $G : \mathcal{D} \to \mathcal{C}$ such that $G \circ F \simeq 1_{\mathcal{C}}$, $F \circ G \simeq 1_{\mathcal{D}}$. In case we also have $G \circ F = 1_{\mathcal{C}}$, $F \circ G = 1_{\mathcal{D}}$ then $F$ is called an **isomorphism of category** or $\mathcal{C}$ and $\mathcal{D}$ are said to be isomorphic.

## A.1. Theorem

A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if :

1. $F$ is full and faithful.

2. For any $Y \in \mathcal{D}$ there is an $X \in \mathcal{C}$ such that $Y \simeq F(X)$.

A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ defining an equivalence between $\mathcal{C}^\circ$ and $\mathcal{D}$ is called a **duality**.

One of the most applicable results at this level of generality is the famous Yoneda lemma. In order to phrase it we have to introduce a few more notions and notation.

To an object $A$ of a category $\mathcal{C}$ we associate a contravariant functor $h_A : \mathcal{C} \to \underline{Set}$ by taking :

$$h_A(X) = \mathrm{Hom}_{\mathcal{C}}(X, A), h_A(u : X \to Y) : h_A(Y) \to h_A(X)$$

defined by $h_A(u)(f) = f \circ u$ for any $f \in h_A(Y)$.

## Theorem (The Yoneda Lemma)

For a contravariant functor $F : \mathcal{C} \to \underline{Set}$ and an object $A \in \mathcal{C}$, the natural map $\alpha$,

$$\alpha : \mathrm{Hom}(h_A, F) \to f(A), \phi \mapsto \phi(A)(1_A)$$

is a bijection.

## Corollary

$\mathrm{Hom}(h_A, F)$ is a set. Moreover if $A$ and $B$ are objects of $\mathcal{C}$ then $A \simeq B$ if and if only if $h_A \simeq h_B$.

## A.2. Abelian Categories and Grothendieck Categories

A category is said to be **pre-additive** if :

1. For $A, B \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is an abelian group, its zero element is denoted by $O_{AB}$ (simply $O$ if no confusion can arise) and it is called the zero morphism.

2. For $A, B, C \in \mathcal{C}$ and $u, u_1, u_2 \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, $v, v_1, v_2 \in \mathrm{Hom}_{\mathcal{C}}(B, C)$ it follows that :

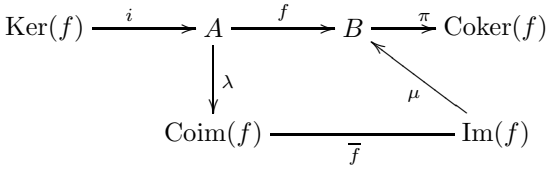
$$v \circ (u_1 + u_2) = v \circ u_1 + v \circ u_2$$
$$(v_1 + v_2) \circ u = v_1 \circ u + v_2 \circ u$$

3. There is an $X \in \mathcal{C}$ such that $1_X = O$. It is easily verified that such $X$ is exactly a zero object, which is unique up to isomorphism and usually denoted by $O$.

With notation as above $0 \to A$ is a monomorphism, $A \to 0$ is an epimorphism. It is clear that the dual of a preadditive category is also preadditive.

An **additive functor** $F : \mathcal{C} \to \mathcal{D}$ between preadditive categories is a functor such that $F(f + g) = F(f) + F(g)$ for $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and arbitrary $A, B \in \mathcal{C}$. If $O$ is the zero object of $\mathcal{C}$ then $F(0)$ is the zero object of $\mathcal{D}$. In a preadditive category we may define, for every morphism $f : A \to B$ in $\mathcal{C}$, $\mathrm{Ker}(f)$, $\mathrm{Coker}(f)$, $\mathrm{Im}(F)$ and $\mathrm{Coim}(f)$ (but these need not always exist).

A preadditive category $\mathcal{C}$ is said to satisfy $(AB1)$ if for any morphism $f : A \to B$ in $\mathcal{C}$ both $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ exist. Then $f$ allows a decomposition :




where $f = \mu \circ \overline{f} \circ \lambda$, $i$ and $\mu$ are monomorphisms and $\pi, \lambda$ are epimorphisms. The preadditive category $\mathcal{C}$ satisfies (AB2) if for every $f$ in $\mathcal{C}$, $\overline{f}$ is an isomorphism. In a category verifying (AB2) being an isomorphism is equivalent to being a monomorphism and an epimorphism.

Suppose $\mathcal{C}$ is preadditive and (AB1) and (AB2) hold.

A sequence $A \xrightarrow{f} B \xrightarrow{g} C$ is **exact** if $\mathrm{Im}(f) = \mathrm{Ker}(g)$ as subobjects of $B$. An arbitrary (long) sequence is said to be exact if every (short) subsequence of two consecutive morphisms is exact in the foregoing sense.

An additive functor $F : \mathcal{C} \to \mathcal{D}$ between preadditive categories with (AB1) and (AB2) is left, resp. right, exact if for any exact sequence of the type

$$0 \longrightarrow A \xrightarrow[f]{} B \xrightarrow[g]{} C \longrightarrow 0$$

the following sequence is exact :

$$0 \longrightarrow F(A) \xrightarrow[F(f)]{} F(B) \xrightarrow[F(g)]{} F(C)$$

resp

$$F(A) \xrightarrow[F(f)]{} F(B) \xrightarrow[F(g)]{} F(C) \longrightarrow 0$$

We say that $F$ is exact if it is both left and right exact.

An **additive category** is a preadditive category $\mathcal{C}$ with coproducts of any two objects; if moreover (AB1) and (AB2) hold then $\mathcal{C}$ is an **abelian category**. Let $A_1 \oplus A_2$ be the coproduct of $A_1$ and $A_2$ in an additive category $\mathcal{C}$ From the universal property of the coproduct if follows that there exist natural morphisms $i_n : A_n \to A_1 \oplus A_2$, $\pi_n : A_1 \oplus A_2 \to A_n$ for $n = 1, 2$, such that $\pi_n \circ i_n = 1_{A_n}$, $\pi_n \circ i_m = 0$ for $n \neq m$, $i_1 \circ \pi_1 + i_2 \circ \pi_2 = 1_{A_1 \oplus A_2}$.

The foregoing actually establishes that $(A_1 \oplus A_2, \pi_1, \pi_2)$ is a product of $A_1$ and $A_2$. Consequently, if $\mathcal{C}$ is additive, preabelian, abelian, then so is $\mathcal{C}^\circ$. Moreover a functor $F$ between additive categories is additive if and only if it commutes with finite coproducts.

In [92], A. Grothendieck introduced some extra axioms on abelian categories leading to the definition of what we now call Grothendieck categories.

**(AB3)** $\mathcal{C}$ has coproducts.

(AB3)$^*$ $\mathcal{C}$ has products.

If $\mathcal{C}$ satisfies (AB3) and $J$ is any nonempty index set, then we define a functor $\oplus_{i \in J} : \mathcal{C}^{(J)} \to \mathcal{C}$ associating to a family of $\mathcal{C}$-objects indexed by $J$ the coproduct (direct sum) of that family. This functor is always right exact.

**(AB4)** For any nonempty set $J$, $\oplus_{i \in J}$ is an exact functor.

(AB4)$^*$ For any nonempty set $J$, $\prod_{i \in J}$ is an exact functor.

In case $\mathcal{C}$ is abelian with (AB3) then for a family of subobjects $(A_i)_{i \in J} A_i$ of $A$ we may consider a "smallest" subobject $\sum_{i \in J} A_i$ of $A$ such that all $A_i$ are subobjects of the latter. That subobject is called the **sum** of the $(A)_{i \in J}$. Dually, if $\mathcal{C}$ is an abelian category satisfying (AB3)$^*$, then to a family of subobjects $(A_i)_{i \in J}$ we may associate a subobject $\cap_{i \in J} A_i$ the **intersection** of the family $(A_i)_{i \in J}$.

Since in an abelian category finite products do exist, the intersection of a family of two subobjects exists. We may now formulate a new axiom :

**(AB5)** Let $\mathcal{C}$ be a category with (AB3). For an $A \in \mathcal{C}$ and $(A_i)_{i\in J}$, $B$ subobjects of $A$ such that the family $(A_i)_{i\in J}$ is right filtered then $(\sum_{i\in J} A_1) \cap B = \sum_{i\in J}(A_i \cap B)$.

(AB5)$^*$ The dual version of (AB5). A category with (AB5) also has (AB4).

Consider an abelian category $\mathcal{C}$. A family $(U_i)_{i\in J}$ is called a **family of generators** of $\mathcal{C}$ if for any $A \in \mathcal{C}$ and any subobject $B$ of $A$ such that $B \neq A$ (as a subobject) there is an $i \in J$ and a morphism $\alpha : U_i \to A$ such that Im$\alpha$ is not a subobject of $B$. An object $U$ of $\mathcal{C}$ is a **generator** if $\{U\}$ is a family of generators. For an abelian category with (AB3) a family $(U_i)_{i\in J}$ in $\mathcal{C}$ is a family of generators if and only if $\oplus_{i\in J} U_i$ is a generator. An abelian category with (AB5) and having a generator is called a **Grothendieck category**.

Note that (AB5) and (AB5)$^*$ do not exist together, indeed an abelian category with (AB5) and $(AB5)^*$ must be the zero category (cf. loc cit). In particular, if $\mathcal{C}$ is a nonzero Grothendieck category then $\mathcal{C}^\circ$ is never a Grothendieck category.

Finally let us recall some basic facts about adjoint functors. Consider functors $F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C}$. We say that $F$ is a left adjoint of $G$ (or $G$ is a right adjoint of $F$) if there exists a functorial isomorphism :

$$\phi : \mathrm{Hom}_{\mathcal{D}}(F, -) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(-, G)$$

where : $\mathrm{Hom}_{\mathcal{D}}(F, -) : \mathcal{C}^\circ \times \mathcal{D} \to$ Set associates to $(A, B)$ the set $\mathrm{Hom}_{\mathcal{D}}(F(A), B)$, $\mathrm{Hom}_{\mathcal{C}}(-, G) : \mathcal{C}^\circ \times \mathcal{D} \to$ Set associates to $(A, B)$ the set $\mathrm{Hom}_{\mathcal{C}}(A, G(B))$.

When $\mathcal{C}$ and $\mathcal{D}$ are preadditive and $F, G$ are additive then we assume that $\phi(A, B)$ is an isomorphism of abelian groups for any $A \in \mathcal{C}, B \in \mathcal{D}$.

It is well known that if $\mathcal{C}$ is a Grothendieck category, then $\mathcal{C}$ has enough injective objects i.e., if $M \in \mathcal{C}$, there exists an injective object $Q$ in $\mathcal{C}$ and a monomorphism $0 \to M \to Q$. Also, as in the case of module category, if $M \in \mathcal{C}$ there exists a unique (up to isomorphism) injective object denoted by $E(M)$, such $M$ is a subobject of $E(M)$ and $E(M)$ is an essential extension of $M$; $E(M)$ is called the **injective envelope** of $M$.

With these conventions and notation as before we obtain :

## A.2. Theorem

Suppose $F$ is left adjoint to $G$, then :

1. The functor $F$ commutes with coproducts and $G$ commutes with products.

2. If $\mathcal{C}$ and $\mathcal{D}$ are abelian and $F, G$ are additive then $F$ is right exact, $G$ is left exact.

3. When $\mathcal{D}$ has enough injective objects, that is if for $B \in \mathcal{D}$ there exixts an injective $Q \in \mathcal{D}$ and a monomorphism $B \to Q$, then $F$ is exact if and only if $G$ preserves injectivity (if $Q$ is injective in $\mathcal{D}$ then $G(Q)$ is injective in $\mathcal{C}$).

4. When $\mathcal{C}$ has enough projective objects, that is if for $A \in \mathcal{C}$ there exists a projective $P \in \mathcal{C}$ and an epimorphism $P \to A$ in $\mathcal{C}$, then $G$ is exact if and only if $F$ preserves projectivity.

The abstract concept of "adjoint functors" is very fundamental and has applications in different areas of mathematics. A very well-known example of an adjoint pair of functors is provided by the Hom and $\otimes$ functors. More precisely, let $R$ and $S$ be associative rings, $R$-mod-$S$, $R$-mod and $S$-mod the respective module categories , then for $M \in R$-mod-$S$, the tensor functor $M \otimes_S - : S\text{-mod} \to R\text{-mod}$, is a left adjoint of $\mathrm{Hom}_R(M, -) : R\text{-mod} \to S\text{-mod}$.

# Appendix B. Dimensions in an Abelian Category

## B.1. Krull Dimension

The Krull dimension of ordered sets has been defined by P. Gabriel and R. Rentschler in [68], for finite ordinal numbers, and G. Krause generalized the notion to other numbers, cf. [116]. Let us recall some definitions and elementary facts.

Let $(E, \leq)$ be an ordered set. For $a, b \in E$ such that $a \leq b$ we denote by $[a, b]$ the set

$$\{x \in E | a \leq x \leq b\}$$

and we put $\Gamma(E) = \{(a, b) | a \leq b; a, b, \in E\}$. By transfinite recurrence we define on $\Gamma(E)$ the following filtration :

$$\Gamma_{-1}(E) = \{(a, b) | a = b\}, \Gamma_0(E) = \{(a, b) \in \Gamma(E) | [a, b] \text{ is Artinian}\}$$

supposing $\Gamma_\beta(E)$ has been defined for all $\beta < \alpha$, then $\Gamma_\alpha(E) = \{(a, b) \in \Gamma(E) | \forall b \geq b_1 \ldots \geq b_n \ldots \geq a$, there is an $n \in I\!N$ such that $(b_{i+1}, b_i) \in \cup_{\beta<\alpha} \Gamma_\beta(E)$ for all $i \geq n\}$.

We obtain an ascending chain $\Gamma_{-1}(E) \subset \Gamma_0(E) \subset \ldots \subset \Gamma_\alpha(E) \subset \ldots$. There exists an ordinal $\xi$ such that $\Gamma_\xi(E) = \Gamma_{\xi+1}(E) = \ldots$. If there exists an ordinal $\alpha$ such that $\Gamma(E) = \Gamma_\alpha(E)$ then $E$ is said to have **Krull dimension**. The smallest ordinal $\alpha$ with the property that $\Gamma_\alpha(E) = \Gamma(E)$ will be called the **Krull dimension** of $E$ and we denote it by $K.\dim E$.

### Lemma B.1.1.

Let $E, F$ be ordered sets and let $f : E \to F$ be a strictly increasing map. If $F$ has Krull dimension then $E$ has Krull dimension and $K.\dim E \leq K.\dim F$ (cf. [68]).

### Lemma B.1.2

If $E, F$ are ordered sets with Krull dimension then $E \times F$ has Krull dimension and $K.\dim(E \times F) = \sup(K.\dim E, K.\dim F)$ (note that $E \times F$ has the product ordering).

If $\mathcal{A}$ is an arbitrary abelian category and $M$ is an object of $\mathcal{A}$ then we consider the set $L(M)$ of all subobjects of $M$ in $\mathcal{A}$ ordered by inclusion. In fact $L(M)$ is a **modular lattice**. If $L(M)$ has Krull dimension then $M$ is said to have Krull dimension and we denote it by $K.\dim_{\mathcal{A}} M$ or simply $K.\dim M$ if no ambiguity can arise. In this case we may reformulate the definition of Krull dimension as follows : if $M = 0$, $K.\dim M = -1$; if $\alpha$ is an ordinal and

$K.\dim M \not< \alpha$ then $K.\dim M = \alpha$ provided there is no infinite descending chain $M \supset M_0 \supset M_1 \supset \ldots$ of subobjects $M_i$ of $M$ in $\mathcal{A}$ such that for $i \geq 1, K.\dim(M_{i-1}/M_i) \not< \alpha$. An object $M$ of $\mathcal{A}$ having $K.\dim M = \alpha$ is said to be $\alpha$**-critical** if $K.\dim(M/M') < \alpha$ for every non-zero subobject $M'$ of $M$ in $\mathcal{A}$. For example, $M$ is 0-critical if and only if $M$ is a simple object of $\mathcal{A}$. Also, it is obvious that any non-zero subobject of an $\alpha$-critical object is again $\alpha$-critical.

## Lemma B.1.3

If $M$ is an object of $\mathcal{A}$ and $N$ is a subobject of $M$ then

$$K.\dim M \leq \sup(K.\dim N, K.\dim(M/N))$$

and equality holds provided either side exists.

**Proof** Follows from Lemma B.1.2., cf [68]. □

## Lemma B.1.4

If $M \in \mathcal{A}$ has Krull dimension then it contains a critical subobject.

**Proof** Cf. [68] and [157]. □

## Lemma B.1.5

Every Noetherian object of $\mathcal{A}$ has Krull dimension.

**Proof** See Proposition 1.3. [85] or Corollary 3.1.8 [157]. □

## Lemma B.1.6.

Suppose that $\mathcal{A}$ is ab abelian category allowing infinite direct sums. If an object $M$ of $\mathcal{A}$ has Krull dimension then $M$ cannot contain an infinite direct sum of subobjects. In particular $M$ has finite Goldie dimension.

## Lemma B.1.7

Suppose that $\mathcal{A}$ is an abelian category allowing infinite direct sums and suppose that the object $M$ of $\mathcal{A}$ has Krull dimension. Put

$$\alpha = \sup\{K.\dim_{\mathcal{A}}(M/N) + 1 | N \text{ an essential subobject of} M\}$$

Then $K.\dim M \leq \alpha$.

**Proof** For B.1.6 and B.1.7 we refer to [68]. □

### Lemma B.1.8

If $M \in \mathcal{A}$ is a Noetherian object, then there exists a composition series $M \supset M_1 \supset \ldots \supset M_n = 0$ such that $M_{i-1}/M_i$ is a critical object for each $1 \leq i \leq n$. Moreover if $\alpha_i = K.\dim(M_{i-1}/M_i)$ then $K.\dim M = \sup\{\alpha_i | i = 1, \ldots, n\}$.

**Proof** See [80], [157].

### Lemma B.1.9

Assume that $U$ is a generator of the category $\mathcal{A}$ and $M \in \mathcal{A}$. If $U$ and $M$ have Krull dimension then $K.\dim M \leq K.\dim U$.

**Proof** See [80] or [157].

## B.2. Gabriel Dimension of a Grothendieck Category

This dimension is first defined by P. Gabriel in his thesis [67] but under the name of Krull dimension. The actual name "Gabriel dimension" is given by Gordon and Robson in [80]. In the book [157] the definition of Gabriel dimension is given for a modular lattice.

We follow the definition given by P. Gabriel using the notion of a localizing subcategory and its quotient category.

Let $\mathcal{A}$ be a Grothendieck category. An object $M \in \mathcal{A}$ is called **semi-Artinian** if for every subobject $M'$ of $M$ such that $M' \neq M$, $M/M'$ contains a simple subobject. It is easy to see that the full subcategory of all semi-Artinian objects is a localizing subcategory. In fact it is the smallest localizing subcategory that contains all simple objects of $\mathcal{A}$. Now, using transfinite recursion we define the ascending sequence of localizing subcategories of $\mathcal{A}$ :

$$\mathcal{A}_0 \subset \mathcal{A}_1 \subset \ldots \subset \mathcal{A}_\alpha \subset \ldots \qquad (B.2.1)$$

in the following way : $\mathcal{A}_0 = \{0\}; \mathcal{A}_1$ is the localizing subcategory of all semi-Artinian objects of $\mathcal{A}$. Let $\alpha$ be an ordinal and assume that for any $\beta < \alpha$ the localizing subcategory $\mathcal{A}_\beta$ is defined. If $\alpha$ is not a limit ordinal i.e. $\alpha = \beta + 1$, let $\mathcal{A}/\mathcal{A}_\beta$ be the quotient category of $\mathcal{A}$ by $\mathcal{A}_\beta$ and $T_\beta : \mathcal{A} \to \mathcal{A}/\mathcal{A}_\beta$ the canonical functor; $T_\beta$ is an exact functor (for detail see P. Gabriel [62]). Then $\mathcal{A}_\alpha$ is a localizing subcategory of all objects $M \in \mathcal{A}$, such that $T_\beta(M)$ is a semi-Artinian in $\mathcal{A}/\mathcal{A}_\beta$. If $\alpha$ is a limit ordinal, then $\mathcal{A}_\alpha$ is the smallest localizing subcategory that contains $\cup_{\beta,\alpha}\mathcal{A}_\beta$ (we remark that in general $\cup_{\beta<\alpha}\mathcal{A}_\beta \neq \mathcal{A}_\alpha$, because $\cup_{\beta<\alpha}\mathcal{A}$ need not be a localizing subcategory; in fact $\cup_{\beta<\alpha}A_\beta$ is not necessarily stable under arbitrary direct sums). It is easy to see that $M \in \mathcal{A}_\alpha$ if and only if for any subobject $N$ of $M$, $N \neq M$, $M/N$ contains a non-zero subobject isomorphic to some object from $\cup_{\beta<\alpha}\mathcal{A}_\beta$.

Since $\mathcal{A}$ has a generator there exists an ordinal $\xi$ such that $\mathcal{A}_\xi = \mathcal{A}_{\xi+1} = \ldots$. An object $M \in \mathcal{A}$ is said to have **Gabriel dimension** if there exists an ordinal $\alpha$ such that $M \in \mathcal{A}_\alpha$. If moreover $\alpha$ is the smallest ordinal such that $M \in \mathcal{A}_\alpha$ then we say that $M$ has Gabriel dimension $\alpha$ and denote this by $G.\dim M = \alpha$. In particular $G.\dim M = 1$ if and only if $M$ is a semi-Artinian object (if $M = 0$ we put $G.\dim M = 0$). We survey the main properties of Gabriel dimension.

## Lemma B.2.1

Let $M \in \mathcal{A}$ and $N \leq M$ is a subobjet. Then $M$ has Gabriel dimension if and only if $N$ and $M/N$ have Gabriel dimension. Moreover $G.\dim M = \sup(G.\dim N, G.\dim(M/N))$.

**Proof** See [67] and [157].

## Lemma B.2.2

If $(M_i)_{i\in I}$ is a family of objects of $\mathcal{A}$ such that each $M_i$ has Gabriel dimension for any $i \in I$, then $\oplus_{i\in I} M_i$ has Gabriel dimension and $G.\dim(\oplus_{i\in I}|M_i) = \sup_{i\in I}(G.\dim M_i)$.

**Proof** Cf. loc. cit.

## Lemma B.2.3

Let $M \in \mathcal{A}$ an object. Assume that $M$ has Krull dimension. Then $M$ has Gabriel dimension and $K.\dim M \leq G.\dim M \leq K.\dim M + 1$.

**Proof** Cf. loc. cit.

## Lemma B.2.4

If $M \in \mathcal{A}$ is a Noetherian object then $G.\dim M = K.\dim M + 1$.

A category $\mathcal{A}$ has **Gabriel dimension** if in the series (B.21) there exists an ordinal $\alpha$ such that $\mathcal{A} = \mathcal{A}_\alpha$; moreover the smallest ordinal $\alpha$ with this property is called the Gabriel dimension of the category $\mathcal{A}$ and we denote this by $G.\dim\mathcal{A} = \alpha$. If $U$ is a generator of the category $\mathcal{A}$, we have the following characterization : the category $\mathcal{A}$ has Gabriel dimension if and only if $U$ has Gabriel dimension and in this case $G.\dim\mathcal{A} = G.\dim U$.

An object $M \in \mathcal{A}$ is called $\alpha$-simple if

$$G.\dim M = G.\dim N = \alpha \text{ and } G.\dim(M/N) < \alpha,$$

for any non-zero subobject of $M$. This definition implies that for any $\alpha$-simple object $M$, $\alpha$ is a non-limit ordinal. We have in particular that $M$ is 1-simple if and only if $M$ is a simple object of $\mathcal{A}$.

An object $M$ is called **Gabriel simple** if $M$ is $\alpha$-simple for some ordinal $\alpha$.

## Lemma B.2.5

Let $M \in \mathcal{A}$ and assume that for any non-zero subobject $X$ of $M, M/X$ has Gabriel dimension. Then $M$ has Gabriel dimension and moreover $G.\dim M \leq \alpha + 1$ where $\alpha = \sup\{G.\dim(M/X) | X \subset M, X \neq 0\}$.

**Proof** Cf. loc. cit. □

## Lemma B.2.6

Let $M \in \mathcal{A}$, where $\mathcal{A}$ is a Grothendieck category. If $M$ has Krull (respectively Gabriel) dimension then for any ordinal $\alpha \geq 0$, there exists a largest subobject $\tau_\alpha(M)$ of $M$, having Krull (respectively Gabriel) dimension less than or equal to $\alpha$.

**Proof** Cf. loc. cit. □

# Bibliography


1. G. Abrams, Invariant Basis Number and Types for Strongly Graded Rings, J. Algebra 237(1), 2001, 32-37.

2. G. Abrams, C. Menini, Coinduction for Semigroup-graded Rings, Comm. Algebra 27(7), 1999, 3283-3301.

3. T. Albu, C. Năstăsescu, Infinite Group Graded Rings, Rings of Endomorphisms, and Localization, J. Pure Appl. Algebra 59, 1989, 125-150.

4. M. Asensio, M. Van den Bergh, F. Van Oystaeyen, New Algebraic Approach to Microlocalization of Filtered Rings, Trans. Amer. Math. Soc. 316, no. 2, 1989.

5. B. Alfonsi, Conditions Noetheriennes dans les anneaux gradués, Proceedings Antwerpen 1985, LNM Vol. 1197, Springer Verlag.

6. F. Anderson, K. Fuller, Rings and Categories of Modules, Springer Verlag, New York, 1974.

7. Bahturin, Yu. A., S. K. Sehgal, M. V. Zaicev, Group Gradings on Associative Algebras, J. Algebra, 241, 2001, 677-698.

8. M. Beattie, A Direct Sum Decomposition for the Brauer Group of $H$-Module Algebras, J. Algebra, 43, 1976, 686-693.

9. M. Beattie, A Generalization of the Smash Product of a Graded Ring, J. Pure Appl. Algebra 52, 1988, 219-226.

10. M. Beattie, Duality Theorems for Rings with Actions or Coactions, J. Algebra 115, 1988, 303-317.

11. M. Beattie, Duality Theorems for Group Actions and Gradings, LNM 1328, Granada, 1986, 28-32.

12. M. Beattie, P. Stewart, Graded Radicals of Graded Rings, Acta Math. Hungar. 58, 1991, 261-272.

13. M. Beattie, S. Dăscălescu, Categories of Modules Graded by $G$-sets. Applications, J. Pure Appl. Algebra 107, 1996, 129-139.

14. M. Beattie, S. Dăscălescu, C. Năstăsescu, A Note on Semilocal Graded Rings, Rev. Roumaine Math. Pures Appl. 40 (3-4), 1995, 253-258.

15. M. Beattie, A. Del Rio, The Picard Group of Category of Graded Modules, Comm. Algebra 24, 1996, 4397-4414.

16. M. Beattie, A. Del Rio, Graded Equivalences and Picard Groups, J. Pure Appl. Algebra 141, 1999, 131-157.


C. Năstăsescu and F. Van Oystaeyen: LNM 1836, pp. 291–302, 2004.
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17. A. Bell, Localization and Ideal Theory in Noetherian Strongly Group-graded rings, J. Algebra 105, 1987, 76-115.

18. G. Bergman, On Jacobson Radicals of Graded Rings, preprint.

19. G. M. Bergman, Groups Acting on Rings, groups - graded rings, and Beyond - some thoughts, preprint.

20. J. Bit David, J. C. Robson, Normalizing Extensions I, Ring Theory Antwerp 1980, LNM 825, Springer Verlag, Berlin 1981, 1-5.

21. J. Bit David, Normalizing Extensions II, Ring Theory Antwerp 1980, LNM 825, Springer Verlag, Berlin 1981, 6-9.

22. C. Boboc, Superalgebra Structures on $M_2(k)$ in Characteristic 2, Comm. Algebra., 30, 2002, 255-260.

23. C. Boboc, Grading of Matrix Algebras by the Klein Group, Comm. Algebra, 31, 2003, 2311-2326

24. C. Boboc, S. Dăscălescu, On Gradings of Matrix Algebras by Cyclic Groups, Comm. Algebra, 29, 2001, 5013-5021.

25. P. R. Boisen, The Representation Theory of Fully Group-graded Algebras, J. Algebra 151, 1992, 160-179.

26. P. R. Boisen, Graded Morita Theory, J. Algebra 164, 1994, 1-25

27. N. Bourbaki, Algèbre, Chap. 1-9, Act. Sci. Ind. Hermann, Paris.

28. N. Bourbaki, Algèbre commutative, Chap. 1-7, Act. Sci. Ind. Hermann, Paris.

29. D. Bulacu, S. Dăscălescu, L. Grünenfelder, Modules Graded by $G$-Sets : Duality and Finiteness Conditions, J. Algebra 195, 1997, 624-633.

30. D. Bulacu, Injective Modules Graded by $G$-Sets, Comm. Algebra 27 (7), 1999, 3537-3543.

31. S. Caenepeel, F. Van Oystaeyen, Crossed Products over gr-local Rings, LNM 917, Springer Verlag, Berlin, 1982, 25-43.

32. S. Caenepeel, A Cohomological Interpretation of the Graded Brauer Group I, Comm. Algebra 11, 1983, 2129-2149.

33. S. Caenepeel, A Cohomological Interpretation of the Graded Brauer Group II, J. Pure Appl. Algebra 38, 1985, 19-38.

34. S. Caenepeel, Brauer Groups, Hopf Algebras and Galois Theory, Kluwer Academic Publ., Dordrecht, 1998.

35. S. Caenepeel, On Brauer Groups of Graded Krull Domains and Positively Graded Rings, J. Algebra 99, 1986, 466-474.

36. S. Caenepeel, M. Van den Bergh, F. Van Oystaeyen, Generalized Crossed Products applied to Maximal Orders, Brauer Groups and Related Exact Sequences, J. Pure Appl. Algebra 33, 1984, 123-149.

37. S. Caenepeel, S. Dăscălescu, C. Năstăsescu, On Gradings of Matrix Algebras and Descent Theory, Comm. Algebra, 30, 2002, 5901-5920.

38. V. P. Camillo, K. R. Fuller, On Graded Rings with Finiteness Conditions, Proc. Amer. Math. Soc. 86, no. 1, 1982, 1-5.

39. H. Cartan, E. Eilenberg, Homological Algebra, Princeton Univ. Press, 1956.

40. W. Chin, D. Quinn, Rings Graded by Polycyclic-by-finite groups, Proc. Amer. Math. Soc. 102, no. 2, 1988, 235-241.

41. A. H. Clifford, Representations Induced from Invariant Subgroups, Ann. Math. 2, 38, 1937, 533-550.

42. M. Cohen, H. L. Rowen, Group Graded Rings, Comm. Algebra 11, no. 11, 1983, 1253-1270.

43. M. Cohen, S. Montgomery, Group Graded Rings, Smash Product and Group Actions, Trans. Amer. Math. Soc. 282, 1984, 237-258.

44. M. Cohen, S. Montgomery, Addendum; Group Graded Rings, Smash Product and Group Actions, Trans. Amer. Math. Soc. 300 (2), 1987, 810-811.

45. P. Cohn, Algebra II, J. Wiley, London, 1977.

46. C. W. Curtis and I. Reiner, Representation Theory of Finite Groups and Associative Algebras, New York - London, Intercience, 1962.

47. J. Cuadra, C. Năstăsescu and F. Van Oystaeyen, Graded Almost Noetherian Rings and Applications to Coalgebras, J. of Algebra, 256, 2002, 97-110.

48. E. C. Dade, Compounding Clifford's Theory, Ann. of Math. 2, 91, 1970, 236-270.

49. E. C. Dade, Group Graded Rings and Modules, Math. Zeitschrift 174, 3, 1980, 241-262.

50. E. C. Dade, Extending Irreducible Modules, J. Algebra 72, 1981, 374-403.

51. E. C. Dade, Clifford Theory for Group Graded Rings, J. Reine Angew. Math. 369, 1986, 40-86.

52. E. C. Dade, Clifford Theory for Group Graded Rings II, J. Reine Angew. Math. 387, 1988, 148-181.

53. E. C. Dade, Clifford Theory and Induction from Subgroups, Contemporary Mathematics 93, 1989, 113-144.

54. E. C. Dade, Blocks of Fully Graded Rings, Pacific Journal, Olga Taussky Todd, Memorial issues, 1997, 85-121.

55. E. C. Dade, The Equivalence of Various Generalizations of Group Rings and Modules, Math. Zeitschrift 181 (3), 1982, 335-344.

56. S. Dăscălescu, C. Năstăsescu, Graded $T$-rings, Comm. Algebra 17, 1989 (12), 3033-3042.

57. S. Dăscălescu, Graded Semiperfect Rings, Bull. Math. Soc. Sci. Math. Roumaine 36(84), 1992, 247-254.

58. S. Dăscălescu, A. del Rio, Graded $T$-rings with finite support, Comm. Algebra 21, 1993, 3619-3636.

59. S. Dăscălescu, C. Năstăsescu, A. Del Rio, F. Van Oystaeyen, Gradings of Finite Support. Application to Injective Objects, J. Pure Appl. Algebra 107, 1996, 193-206.

60. S. Dăscălescu, C. Năstăsescu, F. Van Oystaeyen, B. Torrecillas, Duality Theorems for Graded Algebras and Coalgebras, J. Algebra 192, 1997(1), 261-276.

61. S. Dăscălescu, B. Ion, C. Năstăsescu, J. Rios Montes, Group Gradings on Full Matrix Rings, J. Algebra 220, 1999, 709-728.

62. L. Dăus, C. Năstăsescu, On Dade's Conjecture, Comm. Algebra 29, 2001(6), 2541-2552.

63. F. R. DeMeyer, Some Notes on the General Galois Theory of Rings, Osaka J. Math. 2, 1965, 117-127.

64. Y. Fan, B. Külshammer, Group Graded Rings and Finite Block Theory, Pacific J. Math. 196 (1), 2000, 177-186.

65. R. Fossum, H. Foxby, The Category of Graded Modules, Math. Scand., Vol. 35, 1974, no. 2.

66. K. R. Fuller, Density and Equivalence, J. Algebra 29, 1974, 528-550.

67. P. Gabriel, Des Categories Abeliennes, Bull. Soc. Math. France 90, 1962, 323-448.

68. P. Gabriel, R. Rentschler, Sur la dimension des anneaux et ensembles ordonnes, C. R. Acad. Sci. Paris, A, 165 (1967), 712-715.

69. W. Gaschütz, Uber den Fundamentalsatz von Maschke zen Darstellungstheorie der endlichen Gruppen, Math. Z. 56, 1952, 376-387.

70. A. Goldie, Semi-prime Rings with Maximum Conditions, Proc. London Math. Soc. 10, 1960, 201-220.

71. A. Goldie, Localization in Non-commutative Noetherian Rings, J. Algebra 5, 1967, 89-105.

72. A. Goldie, The Stucture of Noetherian Rings, Lectures on Rings and Modules, Tulane Univ., LNM 246, Springer Verlag, Berlin, 1972.

73. J. L. Gomez Pardo, C. Năstăsescu, The Splitting Property for Graded Rings, Comm. Algebra 14, 1986 (3), 469-479.

74. J. L. Gomez Pardo, C. Năstăsescu, Relative Projectivity, Graded Clifford Theory, and applications, J. Algebra 141, 484-504, 1991.

75. J. L. Gomez Pardo, C. Năstăsescu, Topological Aspects of Graded Rings, Comm. Algebra 21 (12), 1993, 4481-4493.

76. J. L. Gomez Pardo, C. Năstăsescu, Graded Clifford Theory and Duality, J. Algebra 162, 1993, 28-45.

77. L. Gomez Pardo, G. Militaru, C. Năstăsescu, When is $\mathrm{HOM}_R(M,-)$ equal to $\mathrm{Hom}_R(M,-)$ in the Category $R$-gr ? Comm. Algebra 22(8), 3171-3181, 1994.

78. K. Goodearl, Von Neumann Regular Rings, Monographs and Studies in Math., vol. 4, Pitman, London, 1979.

79. K. Goodearl, T. Stafford, The Graded Version of Goldie's Theorem, Contemporary Math. 259, 2000, 237-240.

80. R. Gordon, J. C. Robson, Krull Dimension, Amer. Math. Soc. Memoirs, 1973, Vol. 133.

81. R. Gordon, J. C. Robson, The Gabriel Dimension of a Module, J. Algebra 29, 1974, 459-473.

82. R. Gordon, E. L. Green, Graded Artin Algebras, J. Algebra 76, 1982, 111-138.

83. R. Gordon, E. L. Green, Representation Theory of Graded Artin Algebras, J. Algebra 76, 1982, 138-152.

84. S. Goto, K. Watanabe, On Graded Rings I, J. Math. Soc. Japan, 30, 1978, 172-213.

85. S. Goto, K. Watanabe, On Graded Rings II, Tokyo J. Math. 1, 2, 1978.

86. E. L. Green, Graphs with Relations, Covering and Group-graded Algebras, Trans. Amer. Math. Soc., 279, 1983, 297-310.

87. E. L. Green and E. N. Marcos, Graded Quotients of Path Algebras : a Local Theory, J. Pure Appl. Algebra, 93 (1994), 195-226.

88. J. A. Green, On the Indecomposable Representations of a Finite Group, Math. Z. 70, 1959, 430-445.

89. J. A. Green, A Transfer Theorem for Modular Representations, J. Algebra 68, 1964, 73-84.

90. J. A. Green, Some Remarks on Defect Groups, Math. Z. 107, 1968, 133-150.

91. P. Greszczuk, On $G$-systems and $G$-graded Rings, Proc. Amer. Math. Soc. 95, 1985, 348-352.

92. A. Grothendieck, Sur quelques points d'algèbre homologique, Tohoku Math. J. 1958, 119-221.

93. J. Haefner, A Strongly Graded Ring that is Not Graded Equivalent to a Skew Groupring, Comm. Algebra 22 (12), 1994, 4795-4799.

94. J. Haefner, Graded Morita Theory for Infinite Groups, J. Algebra 169 (2), 1994, 552-586.

95. J. Haefner, Graded Equivalence Theory with Applications, J. Algebra 172 (2), 1995, 385-424.

96. A. Heinicke, J. Robson, Normalizing Extension, Prime Ideals and Incomparability, J. Algebra 72, 1981, 237-268.

97. D. G. Higman, Induced and Produced Modules, Can. J. Math. 7, 1955, 490-508.

98. Li Huishi, F. Van Oystaeyen, Sign Gradations on Group Ring Extensions of Graded Rings, J. Pure Appl. Algebra 85, no. 3, 1993, 311-316.

99. R. Khazal, Crina Boboc and S. Dăscălescu, Group Gradings of $M_2(k)$, preprint.

100. C. Iglesias, J. G. Torrecillas, C. Năstăsescu, Separable Functors in Graded Rings, J. Pure Appl. Algebra 127, 1998, no. 3, 219-230.

101. M. Ikeda, On a Theorem of Gaschütz, Osaka Math. J. 5, 1953, 53-58.

102. I. D. Ion, C. Năstăsescu, Anneaux gradués semi-simples, C. R. Acad. Sci. Paris Ser. A-B 283, 1976 (16), Ai, A1077-A1080.

103. I. D. Ion, C. Năstăsescu, Anneaux gradués semi-simples, Rev. Roum. Math. 4, 1978, 573-588.

104. A. V. Jategaonker, Left Principal Ideal Rings, LNM 123, Springer Verlag, Berlin, 1970.

105. A. Jensen, S. Jøndrup, Clasical Quotient Rings of Group Graded Rings, Comm. Algebra 20 (10), 1992, 2923-2936.

106. A. Jensen, S. Jøndrup, Smash Product, Group Action and Group Graded Rings, Comm. Algebra.

107. E. Jespers, J. Krempa, E. Puczylowski, On Radicals of Graded Rings, Comm. Algebra 10, 1987, 1849-1854.

108. E. Jespers, E. Puczylowski, The Jacobson and Brown-Mc Coy Radicals of Rings Graded by Free Groups, Comm. Algebra 19, 1991, 551-558.

109. T. Kanzaki, On Generalized Crossed Products and the Brauer Group, Osaka J. Math. 5, 1968, 175-188.

110. I. Kaplansky, Commutative Rings, Allyn and Bacon, 1970, Boston Mass.

111. G. Karpilowsky, The Jacobson Radical of Monoid Graded Algebra, Tsukuba J. 16, 1992, 19-52.

112. G. Karpilowsky, Projective Representations of Finite Groups, Monographs, Marcel Dekker, 1985.

113. A. V. Kelarev, Applications of Epigroups to graded Ring Theory, Semigroup Forum 50 (1995), 327-350.

114. M. A. Knus, Algebras Graded by a Group, Category Theory, Homology Theory Appl., Proc. Conf., Seattle, Res. Center Baltelle Mem. Inst. 1969, 2, 117-133.

115. M. A. Knus, M. Ojanguren, Theorie de la descente et algèbres d'Azumaya, LNM 389, Springer-Verlag, Berlin-New York, 1974.

116. G. Krause, On the Krull Dimension of Left Noetherian left Matlis Rings, Math. Z. 118, 1970, 207-214.

117. L. Le Bruyn, M. Van den Bergh, F. V. Oystaeyen, Graded Orders, Birkhäuser, Boston, 1988.

118. L. Le Bruyn, F. Van Oystaeyen, Generalized Rees Rings over Relative Maximal Orders, J. Algebra.

119. B. Lemonnier, Dimension de Krull et Codéviation. Application au Théorème d'Eakin, Comm. Algebra 6, 16, 1978.

120. J. Levitzki, A Theorem on Polynomial Identities, Proc. Amer. Math. Soc. 1 (1950), 334-341.

121. M. Lorenz, Primitive Ideals in Crossed Products and Rings with Finite Group Actions, Math. Z. 158, 1978, 234-285.

122. M. Lorenz, D. S. Passman, Two Applications of Maschke's Theorem, Comm. Algebra 8(19) 1980, 1853-1866.

123. M. Lorenz, D. S. Passman, Prime Ideals in Crossed Products of Finite Groups, Israel J. Math. Vol. 33, no. 2, 1979, 89-132.

124. A. Marcus, Representation Theory of Group Graded Algebras, Nova Science Publ. Inc. Commarck New York, 1999.

125. A. Marcus, Equivalence Induced by Graded Bimodules, Comm. in Algebra, 26, 1998, 713-731.

126. A. Marcus, Homology of Fully Graded Algebras. Morita and Derived Equivalences, J. Pure Appl. Algebra, 133, 1998, 209-218.

127. H. Maschke, Über den arithmetischen Charakter der Coefficienten der Substitutionen eulicher linearer Substitutions Gruppen, Math. Ann. 50, 1898, 482-498.

128. J. Matijevic, Three local conditions on a graded ring, Trans. Amer. Math. Soc. 205, 1975, 275-284.

129. J. McConnel, J. C. Robson, Noetherian Rings, Wiley, London.

130. C. Menini, C. Năstăsescu, When is $R$-gr equivalent to the Category of Modules, J. Pure Appl. Algebra 51, 1988 (3), 277-291.

131. C. Menini, C. Năstăsescu, Gr-simple Modules and gr-Jacobson Radical Applications I, Bull. Math. de la Soc. Sci. Math. de Roumanie 34 (82), 1990 (1), 25-36.

132. C. Menini, C. Năstăsescu, Gr-simple Modules and gr-Jacobson Radical Applications II, Bull. Math. de la Soc. Sci. Math. de Roumanie 34 (82), 1990 (2), 125-133.

133. C. Menini, C. Năstăsescu, When are Induction and Coinduction Functors Isomorphic ?, Bull. Belg. Math. Soc. Simon Stevin 1, 1994 (4), 521-558.

134. C. Menini, On the Injective Envelope of a Graded Module, Comm. Algebra 18 (5), 1990, 1461-1467.

135. Y. Miyashita, An Exact Sequence Associated with a Generalized Crossed Product, Nagoya Math. J. 49, 1973, 31-51.

136. S. Montgomery, Hopf Algebras and their Actions on Rings, CBMS Reg. Conf. Series 82, Providence, R. I., 1993.

137. E. Nauwelaerts, F. Van Oystaeyen, Zariski Extensions and Biregular Rings, Israel J. Math. 37(4) 1980, 315-326.

138. E. Nauwelaerts, F. Van Oystaeyen, Finite Generalized Crossed Products over Tame and Maximal Orders, J. Algebra 101 (1), 1986, 61-68.

139. C. Năstăsescu, Modules simples sur les anneaux gradués, C. R. Acad. Sci. Paris Ser A-B 283, 1976 (7), Ai, A425-A427.

140. C. Năstăsescu, Anneaux et Modules Gradués, Rev. Roum. Math. 21(7), 1976, 911-931.

141. C. Năstăsescu, Strongly Graded Rings of Finite Groups, Comm. Algebra 11, 1983 (10), 1033-1071.

142. C. Năstăsescu, Group Rings of Graded rings. Applications, J. Pure Appl. Algebra 33, 1984 (3), 313-335.

143. C. Năstăsescu, Modules graded by $G$-sets. Maschke type theorems, Green Theory and Clifford theory, 1986, INCREST, 129-137.

144. C. Năstăsescu, Some Constructions over Graded Rings. Applications, J. Algebra 120, 1989, 119-138.

145. C. Năstăsescu, Smash Product and Applications to Finiteness Conditions, Rev. Roum. Math. Pures et Appl. 34, 1989, no. 9, 825-837.

146. C. Năstăsescu, F. Van Oystaeyen, On Strongly Graded Rings and Crossed Products, Comm. Algebra 10, 1982, 2085-2106.

147. C. Năstăsescu, F. Van Oystaeyen, Graded and Filtered Rings and Modules, LNM 758, Springer Verlag, Berlin, 1979.

148. C. Năstăsescu, F. Van Oystaeyen, Jacobson Radicals and Maximal Ideals of Normalizing Extensions Applied to $\mathbb{Z}$-graded Rings, Comm. Algebra 10, 1982 (17), 1839-1847.

149. C. Năstăsescu, F. Van Oystaeyen, On Strongly Graded Rings and Crossed products, Comm. Algebra 10, 1982 (19), 2085-2106.

150. C. Năstăsescu, F. Van Oystaeyen, Graded Ring Theory, North-Holland Math. Library, Amsterdam, 1982.

151. C. Năstăsescu, F. Van Oystaeyen, The Strongly Prime Radical of Graded Rings, Bull. Soc. Math. Belg. Ser. B 36, 1984, 243-251.

152. C. Năstăsescu, F. Van Oystaeyen, Note on Graded Rings with Finiteness Conditions, Comm. Algebra 12, 1984 (13-14), 1647-1651.

153. C. Năstăsescu, F. Van Oystaeyen, A Note on the Socle of Graded Modules, Comm. Algebra 13, 1985 (3), 599-604.

154. C. Năstăsescu, N. Rodino, Group-graded Rings and Smash Products, Rend. Sem. Math. Univ. Padova 74, 1985, 129-137.

155. C. Năstăsescu, N. Rodino, Localization on Graded Modules, Relative Maschke's Theorem and Applications, Comm. Algebra 18, 1990 (3), 811-832.

156. C. Năstăsescu, E. Nauwelaerts, F. Van Oystaeyen, Arithmetically Graded Rings Revisited, Comm. Algebra 14, 1986 (10), 1991-2017.

157. C. Năstăsescu, F. Van Oystaeyen, Dimensions of Ring Theory, Reidel Publ. Company, 1987.

158. C. Năstăsescu, F. Van Oystaeyen, Clifford Theory for Subgroups of Grading Groups, Comm. Algebra 21, 1993 (7), 2583-2595.

159. C. Năstăsescu, M. Van den Bergh, F. Van Oystaeyen, Separable Functors Applied to Graded Rings, J. Algebra 123, 1989 (2), 397-413.

160. C. Năstăsescu, Ş. Raianu, F. Van Oystaeyen, Graded modules over $G$-sets, Math. Z. 203, 1990, 605-627.

161. C. Năstăsescu, L. Shaoxue, F. Van Oystaeyen, Graded Modules over $G$-sets II, Math. Z. 207, 1991, 341-358.

162. C. Năstăsescu, B. Torrecillas, Relative Graded Clifford Theory, J. Pure Appl. Algebra 83, 1992 (2), 177-196.

163. C. Năstăsescu, B. Torrecillas, Localization for Graded Rings and Modules. Applications to Finiteness Conditions, Comm. Algebra 21, 1993 (3), 963-974.

164. C. Năstăsescu, B. Torrecillas, F. Van Oystaeyen, IBN for graded rings, Comm. Algebra 2000(3), 1351-1360.

165. Orzech, M., On the Brauer Group of Algebras Having a Grading and an Action, Can. J. Math., 1976, 28, 533-552.

166. D. S. Passman, A course in Ring Theory, Wadsworth and Brooks, 1991.

167. D. Passman, The Algebraic Structure of Group Rings, Wiley Interscience Publ. 1977.

168. D. Passman, Semiprime and Prime Crossed Products, J. Algebra 83, 1983, 158-178.

169. D. Passman, Infinite Crossed Products and Group-Graded Rings, Trans. Amer. Math. Soc. Vol. 284, no. 2, 1984, 707-727.

170. D. Passman, Group Rings, Crossed Products and Galois Theory, C. B. M. S. no. 64, Amer. Math. Soc. 1986.

171. Picco, D. J., Platzeck, M.I. Graded Algebras and Galois Extensions, Revista Un. Mat. Argentina, 1971, 25, 401-415.

172. C. Procesi, Rings with Polynomial Identities, Marcel Dekker, New York, 1973.

173. E. R. Puczylowski, A Note on Graded Algebras, Proc. Amer. Math. Soc. 113, 1991, 1-3.

174. D. Quinn, Group Graded Rings and Duality, Trans. Amer. Math. Soc. 292, 1985, 155-167.

175. A. Del Rio, Weak Dimensions of Group Graded Rings, Publications Mathématiques, Vol. 34, 1990, 209-216.

176. D. Robinson, A Course in the Theory of Groups, Graduate Text in Mathematics, Springer Verlag 80, 1982.

177. L. Rowen, Rings Theory, Vol. I, II, Academic Press, San Diego, 1988.

178. P. Schmid, Clifford Theory of Simple Modules, J. Algebra 119, 1998, 185-212.

179. G. Sjöding, On Filtered Modules and their Associated Graded Modules, Math. Scand. 33, 1973, 229-249..

180. D. Ştefan, Cogalois Extensions via Strongly Graded Fields, Comm. Algebra 27 (1999), 5687-5702.

181. B. Stenström, Rings of Quotients, Springer Verlag, Berlin, 1975.

182. M. Teply, B. Torrecillas, Strongly Graded Rings with Bounded Splitting Property, J. Algebra 193, 1997, 1-11.

183. K. H. Ulbrich, Vollgraduierte Algebra, Ph. D. Thesis.

184. M. Van den Bergh, On a Theorem of S. Montgomery and M. Cohen, Proc. Amer. Math. Soc., 1985, 562-564.

185. J. P. Van Deuren, F. Van Oystaeyen, Arithmetically Graded Rings, Ring Theory, Antwerp, 1980, LNM 825, Springer Verlag, Berlin, 1981, 130-153.

186. J. Van Geel, F. Van Oystaeyen, On Graded Fields, Indag. Math. 43 (3), 1981, 273-286.

187. F. Van Oystaeyen, Prime Spectra in Non-commutative Algebra, LNM 444, Springer Verlag, Berlin, 1975.

188. F. Van Oystaeyen, Primes in Algebras over Fields, J. Pure Appl. Algebra, 5, 1977, 239-252.

189. F. Van Oystaeyen, On Graded Rings and Modules of Quotients, Comm. Algebra, 6, 1978, 1923-1959.

190. F. Van Oystaeyen, Graded and Non-graded Birational Extensions, Ring Theory 1977, Lecture Notes 40, Marcel Dekker, New York, 1978, 155-180.

191. F. Van Oystaeyen, Zariski Central Rings, Comm. Algebra 6, 1978, 799-821.

192. F. Van Oystaeyen, Birational Extensions of Rings, Ring Theory 1978, Lect. Notes 51, Marcel Dekker, New York, 1979, 287-328.

193. F. Van Oystaeyen, Graded Prime Ideals and the Left Ore Conditions, Comm. Algebra 8, 1980, 861-869.

194. F. Van Oystaeyen, Graded Azumaya Algebras and Brauer Groups, Ring Theory 1980, LNM 825 Springer Verlag, Berlin, 1981.

195. F. Van Oystaeyen, Graded P. I. Rings, Bull. Soc. Math. Belg. XXXII, 1980, 21-28.

196. F. Van Oystaeyen, Note on Graded Von Neumann Regular Rings, Rev. Roum. Math. 29, 1984, 263-271.

197. F. Van Oystaeyen, Generalized Rees Rings and Arithmetically Graded Rings, J. Algebra 82, 1983, 185-193.

198. F. Van Oystaeyen, Crossed Products over Arithmetically Graded Rings, J. Algebra 80, 1983, 537-551.

199. F. Van Oystaeyen, A. Verschoren, Fully Bounded Grothendieck Categories, II, Graded Modules, J. Pure Appl. Algebra 21, 1981, 189-203.

200. F. Van Oystaeyen, A. Verschoren, Noncommutative Algebraic Geometry, LNM 887, Springer Verlag, Berlin, 1982.

201. F. Van Oystaeyen, A. Verschoren, The Brauer Group of a Projective Variety, Israel J. of Math. 42, 1982, 37-59.

202. F. Van Oystaeyen, Derivation of Graded Rings and Clifford Systems, J. of Algebra 103, no. 1, 1986, 228-240.

203. P. Wauters, Strongly $G$-graded Rings with Component of Degree $e$ Semisimple Artinian Ring, J. Pure Appl. Algebra 35, 1985, 329-334.

204. C. A. Weibel, An Introduction to homological algebra, Cambridge Studies in Advanced Mathematics, Vol. 38 , 1994, Cambridge University Press.