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MONOMIAL ALGEBRAS

Rafael H. Villarreal

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Preface

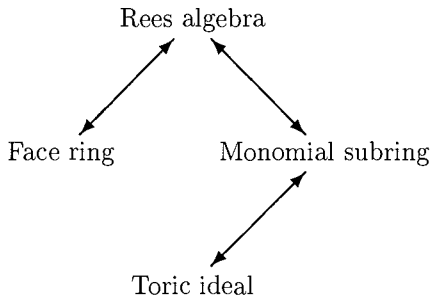
Let $R = k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be a polynomial ring in the indeterminates $x_1, \ldots, x_n$, over the field k. Let

$$f_i = x^{\alpha_i} = x_1^{\alpha_{i1}} \cdots x_n^{\alpha_{in}}, \ i = 1, \ldots, q,$$

be a finite set of monomials of R. We are interested in studying several algebras and ideals associated to these monomials. Some of these are:

- the monomial subring: $k[f_1, \ldots, f_q] \subset k[\mathbf{x}]$,
- the Rees algebra: $k[\mathbf{x}, f_1 t, \ldots, f_q t] \subset k[\mathbf{x}, t]$, which is also a monomial subring,
- the face ring or Stanley-Reisner ring: $k[\mathbf{x}]/(f_1, \ldots, f_q)$, if the monomials are square-free, and
- the toric ideal: the ideal of relations of a monomial subring.

In the following diagram we stress the most relevant relations between the properties of those algebras that will occur in this text.



If such monomials are square-free they are indexed by a hypergraph built on the set of indeterminates, which provides a second combinatorial structure in addition to the associated Stanley-Reisner simplicial complex.

This book was written with the aim of providing an introduction to the methods that can be used to study monomial algebras and their presentation ideals, with emphasis on square-free monomials. We have striven to provide methods that are effective for computations.

A substantial part of this volume is dedicated to the case of monomial algebras associated to graphs, that is, those defined by square-free quadratic monomials defining a simple graph. We will systematically use graph theory to study those algebras. Such a systematic treatment is a gap in the literature that we intend to fill. Two outstanding references for graph theory are [28] and [139].

In the text special attention is paid to providing means to determine whether a given monomial algebra or ideal is Cohen-Macaulay or normal. Those means include diverse characterizations and qualities of those two properties.

Throughout this work base rings are assumed to be Noetherian and modules finitely generated.

An effort has been made to make the book self contained by including a first chapter on commutative algebra that includes some detailed proofs and often points the reader to the appropriate references when proofs are omitted. However we make free use of the standard terminology and notation of homological algebra (including Tor and Ext) as described in [250] and [311].

The first goal is to present basic properties of monomial algebras. For this purpose in Chapter 2 we study affine and graded algebras. The topics include Noether normalizations and their applications, diverse attributes of Cohen-Macaulay graded algebras, Hilbert Nullstellensatz and affine varieties, some Gröbner bases theory and minimal resolutions.

In Chapter 3 a thorough presentation of complete and normal ideals is given. Here the systematic use of Rees algebras and associated graded algebras makes clear their importance for the area.

Chapter 4 deals with Hilbert series of graded modules and algebras, a topic that is quite useful in Stanley's proof of the upper bound conjecture for simplicial spheres. Here we introduce the h-vector and a-invariant of graded algebras and give several interpretations of the a-invariant when the algebra is Cohen-Macaulay. Some optimal upper bounds for the number of generators in least degree of Gorenstein and Cohen-Macaulay ideals are presented, which naturally leads to the notion of an extremal algebra. As an application the Koszul homology of Cohen-Macaulay ideals with pure resolutions is studied using Hilbert function techniques.

General monomial ideals and Stanley-Reisner rings are examined in Chapter 5. The first version of this chapter was some notes originally prepared to teach a short course during the *XXVII Congreso Nacional de la Sociedad Matemática Mexicana* in October of 1994. In this course we pre-

sented some applications of commutative algebra to combinatorics. We have expanded these notes to include a more complete treatment of shellable and Cohen-Macaulay complexes. The presentation of the last two sections of this chapter, discussing Hilbert series of face rings and the upper bound conjectures, was inspired by [25, 44] and [271].

Since monomial algebras defined by square-free monomials of degree two have an underlying graph theoretical structure it is natural that some interaction will occur between monomial algebras, graph theory and polyhedral theory. We have included three chapters that focus on monomial algebras associated to graphs. One of them is Chapter 6, where we present connections between graphs and ideals and study the Cohen-Macaulay property of the face ring. Another is Chapter 8, where we present a combinatorial description of the integral closure of the corresponding monomial subring and give some applications to graph theory. In Chapter 9 we consider monomial subrings and toric ideals of complete graphs with the aim of computing their Hilbert series, Noether normalizations and Gröbner bases.

The central topic of Chapter 7 is the normality of monomial subrings and ideals; some features of toric ideals are presented here.

At the end Chapter 10 is devoted to studying monomial curves and their toric ideals, where the focus of our attention will be on monomial space curves and monomial curves in four variables. Affine toric varieties and their toric ideals are studied in Chapter 11.

Most material in this textbook has been written keeping in mind a typical graduate student with a basic knowledge of abstract algebra and a non-expert who wishes to learn the subject. We hope that this book can be read by people from diverse subjects and fields, such as combinatorics, graph theory, and computer algebra. Various units are accessible to upper undergraduates.

In the last fifteen years a dramatic increase in the number of research articles and books in commutative algebra that stress its connections with computational issues in algebraic geometry and combinatorics has taken place. Excellent references for computational and combinatorial aspects that complement some of the material included here are [74, 92, 298], [44, 274, 278] and [39, 315].

A constant concern during the writing of this text was to give appropriate credits for the proofs and results that were adapted from printed material or communicated to us. We apologize for any inadvertent omission and would appreciate any comments and suggestions in this regard.

During the fall of 1999 a course on monomial algebras associated to graphs was given at the University of Messina covering Chapter 6 to Chapter 9 with the support of the *Istituto Nazionale Di Alta Matematica Francesco Severi*. It is a pleasure to thank Vittoria Bonanzinga, Marilena Crupi, Gaetana Restuccia, Rossana Utano, Maurizio Imbesi, Giancarlo Rinaldi, Fabio

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In the homepage "http://www.math.cinvestav.mx/profesores/vila" we will maintain an updated list of corrections.

Rafael H. Villarreal

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Chapter 1

Commutative Algebra

In this chapter some basic notions and results from commutative algebra will be introduced. All rings considered in this book are commutative and Noetherian and modules are finitely generated. Our main references are the works of W. Bruns and J. Herzog [44], D. Eisenbud [92], H. Matsumura [218] and W. Vasconcelos [298]. Some of the results presented below are just stated without giving proofs, if need be, the reader may locate the missing proofs in one of those references.

1.1 Module theory

Noetherian modules and localizations

A *Noetherian ring* R is a commutative ring with unit with the property that every ideal of I is finitely generated, that is, given an ideal I of R there exists a finite number of generators $f_1, \ldots, f_q$ such that

$$I = \left(\sum_{i=1}^{q} a_i f_i \,\middle|\, a_i \in R, \forall\, i \right).$$

Let R be a commutative ring with unit and let M be an R-module. Recall that M is called *Noetherian* if every submodule N of M is finitely generated, that is, $N = Rf_1 + \cdots + Rf_q$, for some $f_1, \ldots, f_q$ in N.

Proposition 1.1.1 *If M is a finitely generated R-module over a Noetherian ring R, then M is a Noetherian module.*

Corollary 1.1.2 *If R is a Noetherian ring and I is an ideal of R, then R/I and R^n are Noetherian R-modules. In particular any submodule of R^n is finitely generated.*

Theorem 1.1.3 (Hilbert basis theorem) *A polynomial ring $A[x]$ over a Noetherian ring A is Noetherian.*

One of the important examples of a Noetherian ring is a polynomial ring over a field k. Often we will denote a polynomial ring in several variables by $k[\mathbf{x}]$ and a polynomial ring in one variable by $k[x]$. Unless otherwise stated the letters k and K will always denote fields.

Theorem 1.1.4 *An R-module M is Noetherian if and only if M satisfies the ascending chain condition for submodules, that is, M is Noetherian if and only if for every ascending chain of submodules of M*

$$N_0 \subset N_1 \subset \cdots \subset N_n \subset N_{n+1} \subset \cdots \subset M$$

there exists an integer k such that $N_i = N_k$ for every $i \geq k$.

In this book by a *ring* (resp. module) we shall always mean a Noetherian ring (resp. finitely generated module).

The *spectrum* of a ring R will be denoted by $\operatorname{Spec}(R)$, it is the set of prime ideals of R. The *minimal primes* of R are the minimal elements of $\operatorname{Spec}(R)$ with respect to inclusion and the maximal ideals of R are the maximal elements of the set of proper ideals of R with respect to inclusion.

Let R be a ring and $X = \operatorname{Spec}(R)$. Given an ideal I of R, the set of all prime ideals of R containing I will be denoted by $V(I)$. It is not hard to verify that the pair $(X, \mathcal{Z})$ is a topological space, where $\mathcal{Z}$ is the family of open sets of X, and where U is in $\mathcal{Z}$ iff $U = X \setminus V(I)$, for some ideal I. This topology is called the *Zariski topology* of the prime spectrum of R.

A *local ring* $(R, \mathfrak{m}, k)$ is a Noetherian ring R with exactly one maximal ideal $\mathfrak{m}$, the field $k = R/\mathfrak{m}$ is called the *residue field* of R.

A *homomorphism of rings* is a map $\varphi\colon R \to S$ such that:

(i) $\varphi(a+b) = \varphi(a) + \varphi(b)$, $\forall\, a, b \in A$,

(ii) $\varphi(ab) = \varphi(a)\varphi(b)$, $\forall\, a, b \in A$, and

(iii) $\varphi(1) = 1$.

Let R be a ring and let $\varphi\colon \mathbb{Z} \to R$ be the canonical homomorphism

$$\varphi(a) = a \cdot 1_R,$$

then $\ker(\varphi) = n\mathbb{Z}$, for some $n \geq 0$. The integer n is called the *characteristic* of R and is denoted by $\operatorname{char}(R)$.

Proposition 1.1.5 *Let $(R, \mathfrak{m})$ be a local ring, then either* $\operatorname{char}(R) = 0$ *or* $\operatorname{char}(R) = p^k$, *for some prime number p and some integer $k \geq 1$.*

Proof. Let $n = \text{char}(R) \geq 2$. If $n = pqm$, where p, q are prime numbers with $q \neq p$, then $p \cdot 1_R$ and $q \cdot 1_R$ are both in $\mathfrak{m}$ because they are non invertible, but this is impossible because $1 = ap + bq$ and from the equality

$$1_R = (a \cdot 1_R)(p \cdot 1_R) + (b \cdot 1_R)(q \cdot 1_R),$$

one derives $1_R \in \mathfrak{m}$, a contradiction. □

Modules of fractions and localizations

Let R be a ring, M an R-module, and S a multiplicatively closed subset of R so that $1 \in S$. Then the *module of fractions* of M with respect to S, or the *localization* of M with respect to S, is defined by

$$S^{-1}(M) = \{m/s | \, m \in M, \, s \in S\},$$

where $m/s = m_1/s_1$ if and only if $t(s_1 m - s m_1) = 0$ for some $t \in S$. In particular $S^{-1}R$ has a ring structure given by the usual rules of addition and multiplication, and $S^{-1}M$ is a module over the ring $S^{-1}R$ with the operations:

$$\begin{aligned} m_1/s_1 + m_2/s_2 &= (s_2 m_1 + s_1 m_2)/s_1 s_2 \quad (s_i \in S, \; m_i \in M) \\ (a/s) \cdot (m_1/s_1) &= a m_1/s s_1 \qquad\qquad\quad (a \in R, \; s \in S). \end{aligned}$$

There is a canonical map $\varphi: M \to S^{-1}M$, where $\varphi(m) = m/1$. If $f: M \to N$ is a homomorphism of R-modules, then there is an induced homomorphism

$$S^{-1}f: S^{-1}M \longrightarrow S^{-1}N$$

of $S^{-1}R$-modules given by $f(m/s) = f(m)/s$.

Example 1.1.6 If f is in a ring R and $S = \{f^n | \, n \geq 0\}$, then $S^{-1}R$ is usually written R_f. For instance if $R = \mathbb{C}[x]$ is a polynomial ring in one variable over the field $\mathbb{C}$ of complex numbers, then

$$R_x = \mathbb{C}[x, x^{-1}]$$

is the ring of Laurent polynomials.

Definition 1.1.7 Let $\mathfrak{p}$ be a prime ideal of a ring R and $S = R \setminus \mathfrak{p}$. In this case $S^{-1}R$ is written $R_\mathfrak{p}$ and is called the *localization* of R at $\mathfrak{p}$.

Example 1.1.8 Let $\mathfrak{p}$ be a prime ideal of a ring R. The local ring

$$(R_\mathfrak{p}, \mathfrak{p}R_\mathfrak{p}, k(\mathfrak{p}))$$

is the prototype of a local ring, where $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ denotes the residue field of $R_\mathfrak{p}$.

Krull dimension and height

By a *chain* of prime ideals of a ring R we mean a finite strictly increasing sequence of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n,$$

the integer n is called the *length* of the chain. The *Krull dimension* of R, denoted by $\dim(R)$, is the supremum of the lengths of all chains of prime ideals in R. Let $\mathfrak{p}$ be a prime ideal of R, the *height* of $\mathfrak{p}$, denoted by $\mathrm{ht}\,(\mathfrak{p})$ is the supremum of the lengths of all chains of prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{p}$$

which end at $\mathfrak{p}$. Note $\dim(R_{\mathfrak{p}}) = \mathrm{ht}\,(\mathfrak{p})$. If I is an ideal of R, then $\mathrm{ht}\,(I)$, the *height* of I, is defined as

$$\mathrm{ht}(I) = \min\{\mathrm{ht}(\mathfrak{p}) |\, I \subset \mathfrak{p} \text{ and } \mathfrak{p} \in \mathrm{Spec}(R)\}.$$

In general $\dim(R/I) + \mathrm{ht}\,(I) \leq \dim(R)$. The difference $\dim(R) - \dim(R/I)$ is called the *codimension* of I.

Let M be an R-module. The *annihilator* of M is given by

$$\mathrm{ann}_R(M) = \{x \in R |\, xM = 0\},$$

if $m \in M$ the *annihilator* of m is $\mathrm{ann}\,(m) = \mathrm{ann}\,(Rm)$. It is convenient to generalize the notion of annihilator to ideals and submodules. Let N_1 and N_2 be two submodules of M, their *ideal quotient* or *colon* is defined as

$$(N_1 :_R N_2) = \{x \in R |\, xN_2 \subset N_1\}.$$

Let us recall that the *dimension* of an R-module M is

$$\dim(M) = \dim(R/\mathrm{ann}\,(M))$$

and the *codimension* of M is

$$\mathrm{codim}(M) = \dim(R) - \dim(M).$$

Theorem 1.1.9 *If $A[x]$ is a polynomial ring over a Noetherian ring A, then* $\dim A[x] = \dim(A) + 1$.

Primary decomposition of modules

Let I be an ideal of a ring R. The *radical* of I is

$$\mathrm{rad}\,(I) = \{x \in R |\, x^n \in I \text{ for some } n > 0\},$$

the radical is also denoted by $\sqrt{I}$. In particular $\sqrt{(0)}$, denoted by $\mathfrak{N}_R$, is the set of *nilpotent elements* of R and is called the *nilradical* of R. A ring is *reduced* if its nilradical is zero.

Proposition 1.1.10 *If I is a proper ideal of a ring R, then* $\operatorname{rad}(I)$ *is the intersection of all prime ideals containing I.*

Definition 1.1.11 Let M be a module over a ring R, the set of *associated primes* of M, denoted by $\operatorname{Ass}_R(M)$, is the set of all prime ideals $\mathfrak{p}$ of R with the property that there is a monomorphism ϕ of R-modules:

$$R/\mathfrak{p} \overset{\phi}{\hookrightarrow} M.$$

Note that $\mathfrak{p} = \operatorname{ann}(\phi(1))$.

If $M = R/I$ it is usual to say that an associated prime ideal of R/I is an *associated prime* ideal of I and to set $\operatorname{Ass}(I) = \operatorname{Ass}(R/I)$.

Proposition 1.1.12 *Let R be a ring and S a multiplicatively closed subset of R. If M is an R-module and $\mathfrak{p}$ is a prime ideal of R with $S \cap \mathfrak{p} = \emptyset$, then $\mathfrak{p}$ is an associated prime of M if and only if $S^{-1}\mathfrak{p}$ is an associated prime of $S^{-1}M$.*

Proof. If $\mathfrak{p}$ is in $\operatorname{Ass}(M)$, then $R/\mathfrak{p} \hookrightarrow M$. Hence

$$S^{-1}R/S^{-1}\mathfrak{p} \hookrightarrow S^{-1}M$$

and $S^{-1}\mathfrak{p}$ is an associated prime of $S^{-1}M$.

For the converse assume $S^{-1}\mathfrak{p} = \operatorname{ann}(m/1)$. Since R is Noetherian $\mathfrak{p}$ is generated by a finite set $a_1, \ldots, a_n$, hence for each i there is $s_i \in S$ such that $s_i a_i m = 0$. Set $s = s_1 \cdots s_n$. We claim $\mathfrak{p} = \operatorname{ann}(sm)$. Clearly one has $\mathfrak{p} \subset \operatorname{ann}(sm)$. To show the other containment take $x \in \operatorname{ann}(sm)$, then $xsm = 0$ and $x/1 \in \operatorname{ann}(m/1) = S^{-1}\mathfrak{p}$. Hence $x \in \mathfrak{p}$. □

Definition 1.1.13 Let M be an R-module, the *support* of M, denoted by $\operatorname{Supp}(M)$, is the set of all prime ideals $\mathfrak{p}$ of R such that $M_{\mathfrak{p}} \neq 0$.

A sequence $0 \to M' \overset{f}{\to} M \overset{g}{\to} M'' \to 0$ of R-modules is called a *short exact sequence* if f is a monomorphism, g is an epimorphism and $\operatorname{im}(f) = \ker(g)$. A sequence of R modules and homomorphisms:

$$\cdots \longrightarrow M_{i-1} \xrightarrow{f_{i-1}} M_i \xrightarrow{f_i} M_{i+1} \xrightarrow{f_{i+1}} \cdots$$

is said to be *exact* at M_i if $\operatorname{im}(f_{i-1}) = \ker(f_i)$, if the sequence is exact at each M_i it is called an *exact sequence.*

Lemma 1.1.14 *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules over a ring R, then*

$$\operatorname{Supp}(M) = \operatorname{Supp}(M') \cup \operatorname{Supp}(M'').$$

Proof. Let $\mathfrak{p}$ be a prime ideal of R. It suffices to observe that from the exact sequence

$$0 \to M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}} \to 0,$$

we get $M_{\mathfrak{p}} \neq 0$ if and only if $M'_{\mathfrak{p}} \neq 0$ or $M''_{\mathfrak{p}} \neq 0$. □

Theorem 1.1.15 *If M is an R-module, then there is a filtration of submodules*

$$(0) = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

and prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of R such that $M_i/M_{i-1} \simeq R/\mathfrak{p}_i$ for all i.

In general the primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ that occur in a filtration of the type described in the previous result are not associated primes of the module M; see [87] for a careful discussion of filtrations and some of their applications to combinatorics.

Lemma 1.1.16 *If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of modules over a ring R, then*

$$\operatorname{Ass}(M) \subset \operatorname{Ass}(M') \cup \operatorname{Ass}(M'').$$

Corollary 1.1.17 *If M is an R-module, then $\operatorname{Ass}_R(M)$ is a finite set.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be prime ideals as in Theorem 1.1.15. By a repeated use of Lemma 1.1.16 one has $\operatorname{Ass}_R(M) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subset \operatorname{Supp}(M)$. □

Let M be an R-module. An element $x \in R$ is a *zero divisor* of M if there is $0 \neq m \in M$ such that $xm = 0$. The set of zero divisors of M is denoted by $\mathcal{Z}(M)$. If x is not a zero divisor on M we say that x is a *regular element* of M.

Lemma 1.1.18 *If M is an R-module, then*

$$\mathcal{Z}(M) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathfrak{p}.$$

Proof. See [218, Theorem 6.1]. □

Definition 1.1.19 Let M be an R-module. A submodule N of M is said to be a *primary submodule* if $\operatorname{Ass}_R(M/N) = \{\mathfrak{p}\}$.

An ideal $\mathfrak{q}$ of a ring R is a *primary ideal* if $\operatorname{Ass}_R(R/\mathfrak{q}) = \{\mathfrak{p}\}$, thus $\mathfrak{q}$ is a primary ideal iff $xy \in \mathfrak{q}$ and $x \notin \mathfrak{q}$ implies $y^n \in \mathfrak{q}$ for some $n \geq 1$.

Definition 1.1.20 Let M be an R-module. A submodule N of M is said to be *irreducible* if N cannot be written as an intersection of two submodules of M that properly contain N.

Proposition 1.1.21 *Let M be an R-module. If $Q \neq M$ is an irreducible submodule of M, then Q is a primary submodule.*

Proof. Assume there are $\mathfrak{p}_1$ and $\mathfrak{p}_2$ distinct associated prime ideals of M/Q and pick $r_0 \in \mathfrak{p}_1 \setminus \mathfrak{p}_2$ (or vice versa). There is x_i in $M \setminus Q$ such that $\mathfrak{p}_i = \operatorname{ann}(\overline{x}_i)$, where $\overline{x}_i = x_i + Q$. We claim that

$$(Rx_1 + Q) \cap (Rx_2 + Q) = Q.$$

If z is in the intersection, then $z = \lambda_1 x_1 + q_1 = \lambda_2 x_2 + q_2$, for some $\lambda_i \in R$ and $q_i \in Q$. Note that $r_0 z \in Q$, hence $r_0 \lambda_2 x_2$ is in Q and consequently $r_0 \lambda_2 \in \mathfrak{p}_2$. Thus $\lambda_2 \in \mathfrak{p}_2$ and we get $\lambda_2 x_2 \in Q$. This shows $z \in Q$ and completes the proof of the claim. As Q is irreducible one has $Q = Rx_1 + Q$ or $Q = Rx_2 + Q$, which is a contradiction because $x_i \notin Q$ for $i = 1, 2$. □

Definition 1.1.22 Let M be an R-module and $N, N_1, \ldots, N_r$ submodules of M. A decomposition $N = N_1 \cap \cdots \cap N_r$ of N is said to be *irredundant* if

$$N \neq N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_r \text{ for all } i.$$

Theorem 1.1.23 *Let M be an R-module. If N is a submodule of M, then N has an irredundant primary decomposition $N = N_1 \cap \cdots \cap N_r$ so that:*

(a) $\operatorname{Ass}_R(M/N_i) = \{\mathfrak{p}_i\}$ *for all* i.

(b) $N \neq N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_r$ *for all* i.

(c) $\mathfrak{p}_i \neq \mathfrak{p}_j$ *if* $N_i \neq N_j$.

Remark 1.1.24 If $N \neq M$ and $N = N_1 \cap \cdots \cap N_r$ is an irredundant primary decomposition of N with $\operatorname{Ass}_R(M/N_i) = \{\mathfrak{p}_i\}$, then

$$\operatorname{Ass}_R(M/N) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\},$$

$\operatorname{ann}(M/N_i)$ is a $\mathfrak{p}_i$ primary ideal, and $\operatorname{rad}(\operatorname{ann}(M/N_i)) = \mathfrak{p}_i$ for all i.

Corollary 1.1.25 *If R is a Noetherian ring and I a proper ideal of R, then I has an irredundant primary decomposition*

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$$

such that $\mathfrak{q}_i$ is a $\mathfrak{p}_i$-primary ideal and $\operatorname{Ass}_R(R/I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$.

Proof. Let $(0) = I/I = (\mathfrak{q}_1/I) \cap \cdots \cap (\mathfrak{q}_r/I)$ be an irredundant decomposition of the zero ideal of R/I. Then $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ and $\mathfrak{q}_i/I$ is $\mathfrak{p}_i$-primary, that is, $\operatorname{Ass}_R((R/I)/(\mathfrak{q}_i/I)) = \operatorname{Ass}(R/\mathfrak{q}_i) = \{\mathfrak{p}_i\}$. Let us show that $\mathfrak{q}_i$ is a primary ideal. If $xy \in \mathfrak{q}_i$ and $x \notin \mathfrak{q}_i$, then y is a zero-divisor of $R/\mathfrak{q}_i$, but $\mathcal{Z}(R/\mathfrak{q}_i) = \mathfrak{p}_i$, hence $y \in \mathfrak{p}_i = \operatorname{rad}(\operatorname{ann}(R/\mathfrak{q}_i)) = \operatorname{rad}(\mathfrak{q}_i)$ and y^n is in $\mathfrak{q}_i$ for some $n > 0$. □

Definition 1.1.26 Let R be a ring and let S be the set of nonzero divisors of R. The ring $S^{-1}R$ is called the *total ring of fractions* of R. If R is a domain, $S^{-1}R$ is the *field of fractions* of R.

Proposition 1.1.27 *Let R be a ring and let K be the total ring of fractions of R. If R is reduced, then K is a direct product of fields.*

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal primes of R and $S = R \setminus \cup_{i=1}^{r} \mathfrak{p}_i$. Since R is reduced one has $(0) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ and $K = S^{-1}R$. Define

$$\phi\colon K \longrightarrow S^{-1}R/S^{-1}\mathfrak{p}_1 \times \cdots \times S^{-1}R/S^{-1}\mathfrak{p}_r$$

by $\phi(x) = (x + S^{-1}\mathfrak{p}_1, \ldots, x + S^{-1}\mathfrak{p}_r)$. As $S^{-1}\mathfrak{p}_1, \ldots, S^{-1}\mathfrak{p}_r$ are maximal ideals and its intersection is zero, it follows from the Chinese remainder theorem that ϕ is an isomorphism (see Exercise 1.1.38). □

Lemma 1.1.28 *Let M be an R-module and L an ideal of R. If $LM = M$, then there is $x \equiv 1 (\operatorname{mod} L)$ such that $xM = (0)$.*

Proof. Let $M = R\alpha_1 + \cdots + R\alpha_n$, $\alpha_i \in M$. As $LM = M$, there are b_{ij} in L such that $\alpha_i = \sum_{j=1}^{n} b_{ij}\alpha_i$. Set $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $H = (b_{ij}) - I$, where I is the identity matrix. Since $H\alpha^t = 0$ and $H\operatorname{adj}(H) = \det(H)I$, one concludes $\det(H)\alpha_i = 0$ for all i. Hence $xM = (0)$ and $x \equiv 1 (\operatorname{mod} L)$, where $x = \det(H)$. □

Lemma 1.1.29 (Nakayama) *Let R be a ring and N a submodule of an R-module M. If I is an ideal of R contained in the intersection of all the maximal ideals of R such that $M = IM + N$, then $M = N$.*

Proof. Note $I(M/N) = M/N$. By Lemma 1.1.28 there exists an element $x \equiv 1 (\operatorname{mod} I)$ such that $x(M/N) = (0)$. To finish the proof note that x is a unit; otherwise x belongs to some maximal ideal $\mathfrak{m}$ and this yields a contradiction because $I \subset \mathfrak{m}$. □

Let M be an R-module. The *minimum number of generators* of M will be denoted by $\nu(M)$. A consequence of Nakayama's lemma is an expression for $\nu(M)$, when the ring R is local (cf. Corollary 2.5.2).

Corollary 1.1.30 *If M is a module over a local ring $(R, \mathfrak{m})$, then*

$$\nu(M) = \dim_k(M/\mathfrak{m}M), \quad \textit{where } k = R/\mathfrak{m}.$$

Proof. Let $\alpha_1, \ldots, \alpha_q$ be a minimal generating set for M and $\overline{\alpha}_i = \alpha_i + \mathfrak{m}M$. After a permutation of the α_i one may assume that $\overline{\alpha}_1, \ldots, \overline{\alpha}_r$ is a basis for $M/\mathfrak{m}M$ as a k-vector space, for some $r \leq q$. Set $N = R\alpha_1 + \cdots + R\alpha_r$. Note the equality $M = N + \mathfrak{m}M$, then by Nakayama's Lemma $N = M$. Therefore $r = q$, as required. □

Modules of finite length

An R-module M has *finite length* if there is a *composition series*

$$(0) = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

where M_i/M_{i-1} is a nonzero *simple module* (that is, M_i/M_{i-1} has no proper submodules other than (0)) for all i. Note that M_i/M_{i-1} must be cyclic and thus isomorphic to $R/\mathfrak{m}$, for some maximal ideal $\mathfrak{m}$. The number n is independent of the composition series and is called the *length* of M, it is usually denoted by $\ell_R(M)$ or simply $\ell(M)$.

Proposition 1.1.31 *If M is an R-module, then*

$$\mathrm{Ass}(M) \subset \mathrm{Supp}(M) = V(\mathrm{ann}\,(M)),$$

and any minimal element of $\mathrm{Supp}(M)$ *is in* $\mathrm{Ass}(M)$.

Proof. If $\mathfrak{p}$ is an associated prime of M, then there is a monomorphism $R/\mathfrak{p} \hookrightarrow M$ and thus $0 \neq (R/\mathfrak{p})_\mathfrak{p} \hookrightarrow M_\mathfrak{p}$. Hence $\mathfrak{p}$ is in the support of M, this shows the first containment.

Let $\mathfrak{p} \in \mathrm{Supp}(M)$ and let $x \in \mathrm{ann}\,(M)$. If $x \notin \mathfrak{p}$, then $xm = 0$ for all $m \in M$ and $M_\mathfrak{p} = (0)$, which is absurd. Therefore $\mathfrak{p}$ is in $V(\mathrm{ann}\,(M))$. Conversely let $\mathfrak{p}$ be in $V(\mathrm{ann}\,(M))$ and let $m_1, \ldots, m_r$ be a finite set of generators of M. If $M_\mathfrak{p} = (0)$, then for each i there is $s_i \notin \mathfrak{p}$ so that $s_i m_i = 0$, therefore $s_1 \cdots s_r$ is in $\mathrm{ann}\,(M) \subset \mathfrak{p}$, which is impossible. Hence $M_\mathfrak{p} \neq 0$ and $\mathfrak{p}$ is in the support of M.

To prove the last part take a minimal prime $\mathfrak{p}$ in the support of M. As $M_\mathfrak{p} \neq (0)$ there is an associated prime $\mathfrak{p}_1 R_\mathfrak{p}$ of $M_\mathfrak{p}$, where $\mathfrak{p}_1$ is a prime ideal of R contained in $\mathfrak{p}$. Since $M_{\mathfrak{p}_1} \simeq (M_\mathfrak{p})_{\mathfrak{p}_1} \neq (0)$, we get that $\mathfrak{p}_1$ is in the support of M and $\mathfrak{p} = \mathfrak{p}_1$. Therefore using Proposition 1.1.12 one concludes $\mathfrak{p} \in \mathrm{Ass}(M)$. □

Proposition 1.1.32 *If $0 \to M' \to M \to M'' \to 0$ is an exact sequence of modules over a ring R, then*

$$\dim(M) = \max\{\dim(M'),\ \dim(M'')\}.$$

Proof. Set $d = \dim(M)$, $d' = \dim(M')$ and $d'' = \dim(M'')$. First note that $d = \dim(R/\mathfrak{p})$ for some prime $\mathfrak{p}$ containing $\mathrm{ann}\,(M)$, by Proposition 1.1.31 we obtain $M_\mathfrak{p} \neq (0)$. Therefore using Lemma 1.1.14 one has $M'_\mathfrak{p} \neq (0)$ or $M''_\mathfrak{p} \neq (0)$, thus either $\mathfrak{p}$ contains $\mathrm{ann}\,(M')$ or $\mathfrak{p}$ contains $\mathrm{ann}\,(M'')$. This proves $d \leq \max\{d', d''\}$. On the other hand $\mathrm{ann}\,(M)$ is contained in $\mathrm{ann}\,(M') \cap \mathrm{ann}\,(M'')$ and consequently $\max\{d', d''\} \leq d$. □

Let M be an R-module, the *minimal primes* of M are defined to be the minimal elements of $\mathrm{Supp}(M)$ with respect to inclusion. A minimal prime

of M is called an *isolated associated prime* of M. An associated prime of M which is not isolated is called an *embedded prime*.

If R is a ring and I is an ideal, note that the minimal primes of I are precisely the minimal primes of $\mathrm{Ass}_R(R/I)$. In particular the minimal primes of R are precisely the minimal primes of $\mathrm{Ass}_R(R)$.

Proposition 1.1.33 *If M is an R-module, then M has finite length if and only if every prime ideal in* $\mathrm{Supp}(M)$ *is a maximal ideal.*

Definition 1.1.34 A ring R is *Artinian* if $\ell_R(R) < \infty$.

Proposition 1.1.35 *Let R be a ring. Then R is Artinian if and only if R is Noetherian and every prime ideal of R is maximal.*

Definition 1.1.36 Let R be a ring with total ring of fractions Q. An R-module M is said to have *rank r* if $M \otimes_R Q$ is a free Q-module of rank equal to r.

Lemma 1.1.37 *If M is an R-module of positive rank r, then*

$$\dim(M) = \dim(R).$$

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the associated primes of R and $S = R \setminus \cup_{i=1}^n \mathfrak{p}_i$. By hypothesis $S^{-1}M \simeq (S^{-1}R)^r$. Since

$$M_{\mathfrak{p}_i} \simeq (S^{-1}M)_{\mathfrak{p}_i} \simeq [(S^{-1}R)^r]_{\mathfrak{p}_i} \simeq [(S^{-1}R)_{\mathfrak{p}_i}]^r \simeq (R_{\mathfrak{p}_i})^r \neq (0),$$

we obtain that all the minimal primes of R are in the support of M and consequently one has $\dim(M) = \dim(R)$. □

Exercises

1.1.38 (Chinese remainder theorem) Let $I_1, \ldots, I_r$ be ideals of a ring R. If $I_i + I_j = R$ for $i \neq j$, prove:

(a) $I_1 \cap \cdots \cap I_r = I_1 \cdots I_r$,

(b) The rings $R/I_1 \cap \cdots \cap I_r$ and $R/I_1 \times \cdots \times R/I_r$ are isomorphic.

1.1.39 Let $I_1, \ldots, I_n, \mathfrak{p}$ be ideals of a ring R. If $\mathfrak{p}$ is prime and $\cap_{i=1}^n I_i \subset \mathfrak{p}$, prove that $I_i \subset \mathfrak{p}$ for some i. In particular if $\cap_{i=1}^n I_i = \mathfrak{p}$, then $\mathfrak{p} = I_i$ for some i.

1.1.40 Let $I, P_1, \ldots, P_r$ be ideals of a ring R. If I is contained in $\cup_{i=1}^r P_i$ and P_i is a prime ideal for $i \geq 3$, prove that $I \subset P_i$ for some i (cf. Lemma 2.2.18).

1.1.41 Let R be a ring and I an ideal. If P is a prime ideal such that $I \subset P$, prove that $\mathrm{ht}(I) \leq \mathrm{ht}(I_P)$ and show an example where the strict inequality holds.

1.1.42 Let R be a ring and let I be an ideal of R. If all the minimal primes of I have the same height and P is a prime ideal such that $I \subset P$, prove that $\mathrm{ht}(I) = \mathrm{ht}(I_P)$.

1.1.43 Let I be an ideal of a ring R. Then a prime ideal $\mathfrak{p}$ of R is an associated prime of R/I if and only if $\mathfrak{p} = (I\colon x)$ for some $x \in R$.

1.1.44 Let M be an R-module and I an ideal of R contained in $\mathrm{ann}_R(M)$. Note that M inherits a structure of R/I-module. Prove that $\mathfrak{p} \in \mathrm{Ass}_R(M)$ if and only if $\mathfrak{p}/I \in \mathrm{Ass}_{R/I}(M)$.

1.1.45 Let $R' = k[x_2, \ldots, x_n]$ and $R = R'[x_1]$ be polynomial rings over a field k. If I' is an ideal of R' and $\mathfrak{p} \in \mathrm{Ass}_R(R/(I', x_1))$, then

(a) $\mathfrak{p} = x_1 R + \mathfrak{p}' R$, where $\mathfrak{p}'$ is a prime ideal of R', and

(b) $\mathfrak{p}'$ is an associated prime of R'/I'.

1.1.46 Let A be a ring and $\mathfrak{q}$ a proper ideal of A. Prove that $\mathfrak{q}$ is a primary ideal if and only if $\mathcal{Z}(A/\mathfrak{q}) \subset \mathfrak{N}_{A/\mathfrak{q}}$, where $\mathfrak{N}_{A/\mathfrak{q}}$ is the nilradical of $A/\mathfrak{q}$.

1.1.47 Let A be a ring and $\mathfrak{q}$ a proper ideal of A. If every element in $A/\mathfrak{q}$ is either nilpotent or invertible, prove that $\mathfrak{q}$ is a primary ideal.

1.1.48 Let A be a ring and $\mathfrak{q}$ an ideal of A such that $\mathrm{rad}(\mathfrak{q}) = \mathfrak{m}$ is a maximal ideal. Show that $\mathfrak{q}$ is a primary ideal.

Hint Note $\mathrm{Spec}(A/\mathfrak{q}) = \{\mathfrak{m}/\mathfrak{q}\}$ and $\mathfrak{N}_{A/\mathfrak{q}} = \mathfrak{m}/\mathfrak{q}$.

1.1.49 If $B = A[\mathbf{x}]$ is a polynomial ring over a ring A and $\mathfrak{q}$ a primary ideal of A, then $\mathfrak{q}B$ is a primary ideal of B.

Hint Note $(A/\mathfrak{q})[\mathbf{x}] \simeq B/\mathfrak{q}B$. Use induction together with the next exercise.

1.1.50 Let $A[x]$ be a polynomial ring in one variable over a ring A. Prove that a polynomial $f(x) \in A[x]$ is a zero divisor iff there exists $a \neq 0$ in A such that $af(x) = 0$.

1.1.51 Let M be an R-module and $S \subset T$ two multiplicatively closed subsets of R. Prove $T^{-1}M \simeq T^{-1}(S^{-1}M)$ as $T^{-1}R$ modules.

Hint Consider $\phi(m/t) = (m/1)/t$, which is induced by the canonical map from M to $S^{-1}M$.

1.1.52 Let M be an R-module and S a multiplicatively closed subset of R. If $\mathfrak{p}$ is a prime ideal such that $S \cap \mathfrak{p} = \emptyset$, prove $M_{\mathfrak{p}} \simeq (S^{-1}M)_{\mathfrak{p}}$.

1.1.53 Let R be a ring and I an ideal. If $x \in R \setminus I$, then there is an exact sequence of R-modules:

$$0 \longrightarrow R/(I:x) \xrightarrow{\psi} R/I \xrightarrow{\phi} R/(I,x) \longrightarrow 0,$$

where $\psi(\overline{r}) = x\overline{r}$ is multiplication by x and $\phi(\overline{r}) = \overline{r}$.

1.1.54 If $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ is an exact sequence of R-modules, prove that the following sequence is also exact

$$0 \longrightarrow S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M'' \longrightarrow 0.$$

1.1.55 Let N_1 and N_2 be submodules of an R-module M. If S is a multiplicatively closed subset of R, then

(a) $S^{-1}(N_1 + N_2) = S^{-1}(N_1) + S^{-1}(N_2)$,

(b) $S^{-1}(N_1 \cap N_2) = S^{-1}(N_1) \cap S^{-1}(N_2)$, and

(c) $S^{-1}(N_1/N_2) \simeq S^{-1}(N_1)/S^{-1}(N_2)$, as $S^{-1}(R)$-modules.

1.1.56 Let N_1 and N_2 be submodules of an R-module M. If S is a multiplicatively closed subset of R, then

(a) $S^{-1}(\operatorname{ann}(N_2)) = \operatorname{ann}(S^{-1}(N_2))$, and

(b) $S^{-1}(N_1:N_2) = (S^{-1}(N_1):S^{-1}(N_2))$.

Hint Use that N_2 is a finitely generated R-module.

1.1.57 If I and J are ideals of a ring R and S is a multiplicatively closed subset of R, then

(a) $\sqrt{S^{-1}(I)} = S^{-1}(\sqrt{I})$, and

(b) $S^{-1}(IJ) = S^{-1}(I)S^{-1}(J)$.

1.2 Graded modules

Let $(H, +)$ be an abelian semigroup. An *H-graded ring* is a ring R together with a decomposition

$$R = \bigoplus_{a \in H} R_a \quad \text{(as a } \mathbb{Z}\text{-module)},$$

such that $R_a R_b \subset R_{a+b}$ for all $a, b \in H$. A *graded ring* is by definition a $\mathbb{Z}$-graded ring.

If R is an H-graded ring and M is an R-module with a decomposition

$$M = \bigoplus_{a \in H} M_a,$$

such that $R_a M_b \subset M_{a+b}$ for all $a, b \in H$, we say that M is an *H-graded module*.

An element $f \in M$ is said to be *homogeneous* of degree a if $f \in M_a$, in this case we set $\deg(f) = a$. The elements in R_a are also called *forms* of degree a.

A map $\varphi\colon M \to N$ between H-graded modules is *graded* if $\varphi(M_a) \subset N_a$ for all $a \in H$. If $M = \oplus_{a \in H} M_a$ is an H-graded module and $N = \oplus_{a \in H} N_a$ is a graded submodule, that is, $N_a \subset M_a$ for all a, then M/N is an H-graded R-module with $(M/N)_a = M_a/N_a$.

Example 1.2.1 Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. A natural way to grade R is to assign degree d_i to the variable x_i, where $d_1, \ldots, d_n$ are positive integers. For $a = (a_1, \ldots, a_n)$ in $\mathbb{N}^n$ we set $x^a = x_1^{a_1} \cdots x_n^{a_n}$ and $|a| = a_1 d_1 + \cdots + a_n d_n$. The induced $\mathbb{N}$-grading is:

$$R = \bigoplus_{i=0}^{\infty} R_i, \text{ where } R_i = \bigoplus_{|a|=i} k x^a,$$

one may extend this grading to a $\mathbb{Z}$-grading by setting $R_i = 0$ for $i < 0$.

Let $R = k[x_1, \ldots, x_n]$ be polynomial ring over a field k endowed with an induced positive grading as above. In this case the homogeneous elements of R are sometimes called *quasi-homogeneous polynomials*. An ideal I of R is *homogeneous*, or *graded*, if there are homogeneous polynomials $f_1, \ldots, f_r$ such that $I = (f_1, \ldots, f_r)$. Set $\deg(f_i) = \delta_i$. The ideal I is graded by

$$I_i = I \cap R_i = f_1 R_{i-\delta_1} + \cdots + f_r R_{i-\delta_r}.$$

Hence R/I inherits a structure of $\mathbb{N}$-graded R-module whose components are given by $(R/I)_i = R_i/I_i$.

Definition 1.2.2 The *standard grading* or *usual grading* of a polynomial ring $k[x_1, \ldots, x_n]$ is the $\mathbb{N}$-grading induced by setting $\deg(x_i) = 1$ for all i.

Let us now recall some good properties of graded modules mainly in connection with primary decompositions.

Lemma 1.2.3 *If M is an $\mathbb{N}$-graded R-module and $\mathfrak{p} \in \operatorname{Ass}(M)$, then $\mathfrak{p}$ is a graded ideal and there is $m \in M$ homogeneous such that $\mathfrak{p} = \operatorname{ann}(m)$.*

Proof. See [44, Lemma 1.5.6]. □

Proposition 1.2.4 *Let M be an $\mathbb{N}$-graded R-module and Q a $\mathfrak{p}$-primary submodule of M. If $\mathfrak{p}$ is graded and Q^* is the submodule of M generated by the homogeneous elements in Q, then Q^* is again a $\mathfrak{p}$-primary submodule.*

Proof. See [217, Chapter 4]. □

Theorem 1.2.5 *Let M be an $\mathbb{N}$-graded R-module. If N is a graded submodule of M, then N has an irredundant primary decomposition*

$$N = N_1 \cap \cdots \cap N_r$$

such that:

(a) *N_i is a graded submodule and $\operatorname{Ass}_R(M/N_i) = \{\mathfrak{p}_i\}$ for all i,*

(b) *$N \neq N_1 \cap \cdots \cap N_{i-1} \cap N_{i+1} \cap \cdots \cap N_r$ for all i, and*

(c) *$\mathfrak{p}_i \neq \mathfrak{p}_j$ if $N_i \neq N_j$.*

Proof. Use Lemma 1.2.3, Proposition 1.2.4 and Theorem 1.1.23. □

Finding primary decompositions of graded ideals in polynomial rings over fields is a difficult task, the punch line is that there is not yet a best general strategy for computing primary decompositions [77]; for some of the main algorithms that can been used see [95, 127, 255, 313].

For a specially nice treatment of the principles of primary decomposition consult the book of Wolmer V. Vasconcelos [298, Chapter 3].

Lemma 1.2.6 (Graded Nakayama lemma) *Let M be an $\mathbb{N}$-graded R-module and let $\mathfrak{m} = R_+$. If N is a graded submodule of M and $I \subset \mathfrak{m}$ is a graded ideal of R such that $M = N + IM$, then $N = M$.*

Proof. Since $M/N = I(M/N)$, one may assume $N = (0)$. If $x \in M$ is a homogeneous element of degree r, then a recursive use of the equality $M = IM$ yields that $x \in I^{r+1}M$ and x must be zero. □

Hilbert polynomial Let $R = \oplus_{i=0}^{\infty} R_i$ be an $\mathbb{N}$-graded ring. We recall that R is a Noetherian ring if and only if R_0 is a Noetherian ring and $R = R_0[x_1, \ldots, x_n]$ for some $x_1, \ldots, x_n$ in R. If M is a finitely generated $\mathbb{N}$-graded R-module and R_0 is Artinian define the *Hilbert function* of M as

$$H(M, i) = \ell_{R_0}(M_i) \qquad (\ell_{R_0} = \text{length w.r.t } R_0).$$

Definition 1.2.7 An $\mathbb{N}$-graded ring $R = \oplus_{i=0}^{\infty} R_i$ is called a *homogeneous ring* if $R = R_0[x_1, \ldots, x_n]$, where $\deg(x_i) = 1$.

Theorem 1.2.8 (Hilbert) *Let $R = \oplus_{i=0}^{\infty} R_i$ be a homogeneous ring and let M be a finitely generated $\mathbb{N}$-graded R-module with $d = \dim(M)$. If R_0 is Artinian, then there is a unique polynomial $P_M(t) \in \mathbb{Q}[t]$ of degree $d-1$ such that $P_M(i) = H(M,i)$ for $i \gg 0$. The polynomial $P_M(t)$ is called the Hilbert polynomial of M.*

Proof. See [44, Theorem 4.1.3]. □

Definition 1.2.9 If $P_M(t) = a_{d-1}t^{d-1} + \cdots + a_0$ is the Hilbert polynomial of M, the integer $(d-1)!a_{d-1}$ is called the *multiplicity* of M and is denoted by $e(M)$.

Lemma 1.2.10 *Let $f(t) \in \mathbb{Q}[t]$ be a polynomial of degree $d-1$ such that $f(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$, then there are unique integers $a_0, \ldots, a_{d-1}$ such that*

$$f(t) = \sum_{i=0}^{d-1} a_i f_i(t), \quad \text{where} \quad f_i(t) = \binom{t+i}{i}.$$

Proof. The polynomials $f_i(t)$, $i \in \mathbb{N}$, are a basis for $\mathbb{Q}[t]$ as a $\mathbb{Q}$-vector space. Hence

$$f(t) = \sum_{i=0}^{d-1} a_i f_i(t),$$

for some $a_i \in \mathbb{Q}$. Using the Pascal triangle we get

$$f(t) - f(t-1) = \sum_{i=0}^{d-1} a_i \left[\binom{t+i}{i} - \binom{t+i-1}{i} \right] = \sum_{i=0}^{d-1} a_i \binom{t+i-1}{i-1},$$

thus by induction on the degree it follows that $a_i \in \mathbb{Z}$ for all i. □

Exercises

1.2.11 Let I be a graded ideal of a polynomial ring R. Prove that the radical of I is also graded.

1.2.12 Let $d, m \in \mathbb{N}$. Prove the equality

$$\binom{d+m}{d} = \sum_{j=0}^{m} \binom{j+d-1}{d-1}.$$

1.2.13 Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be a numerical function, f is said to be a *polynomial function* of degree d if there is a polynomial $P(t) \in \mathbb{Q}[t]$ such that $P(i) = f(i)$ for $i \gg 0$. Prove that f is a polynomial function of degree d if and only if the numerical function $g\colon \mathbb{Z} \to \mathbb{Z}$ given by $g(i) = f(i) - f(i-1)$ is a polynomial function of degree $d-1$.

1.2.14 Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and let $d_1, \ldots, d_n$ be a sequence of integers. Prove that R is a graded ring with the $\mathbb{Z}$-grading:

$$R_i = \bigoplus_{|a|=i} kx^a \quad (i \in \mathbb{Z}),$$

where $x^a = x_1^{a_1} \cdots x_n^{a_n}$ and $|a| = a_1 d_1 + \cdots + a_n d_n$ for $a = (a_i) \in \mathbb{N}^n$.

1.2.15 Let $R = \oplus_{i \in \mathbb{Z}} R_i$ be a graded ring. Prove that R_0 is a subring of R with $1 \in R_0$. A graded ring R with $R_i = 0$ for $i < 0$ is called a *graded R_0-algebra*.

1.2.16 Let $R = k[x_1, x_2]$ be a polynomial ring over a field k with the grading induced by $\deg(x_i) = (-1)^i$. Determine the subring R_0.

1.3 Cohen-Macaulay modules

Here we introduce some special types of rings and modules and present the following fundamental result of dimension theory.

Theorem 1.3.1 (Dimension theorem) *Let $(R, \mathfrak{m})$ be a local ring and let M be an R-module. Set*

$$\delta(M) = \min\{r | \text{ there are } x_1, \ldots, x_r \in \mathfrak{m} \text{ with } \ell_R(M/(x_1, \ldots x_r)M) < \infty\},$$

then $\dim(M) = \delta(M)$.

Proof. See [218, Theorem 13.4]. □

Definition 1.3.2 Let $(R, \mathfrak{m})$ be a local ring and let M be an R-module of dimension d. A *system of parameters* (s.o.p) of M is a set $\theta_1, \ldots, \theta_d \in \mathfrak{m}$ such that

$$\ell_R(M/(\theta_1, \ldots, \theta_d)M) < \infty.$$

Corollary 1.3.3 *Let $(R, \mathfrak{m})$ be a local ring and let M be an R-module of dimension d. If $h_1, \ldots, h_d \in \mathfrak{m}$ is a system of parameters of M, then*

$$\dim M/(h_1, \ldots, h_i)M = d - i \text{ for } 1 \leq i \leq d.$$

Proof. Set $\overline{M} = M/(h_1, \ldots, h_i)M$. First note that $h_{i+1}, \ldots, h_d$ is a s.o.p for $\overline{M}$, hence $\dim(\overline{M}) \leq d - i$ by Theorem 1.3.1. On the other hand if $\overline{\theta}_1, \ldots, \overline{\theta}_r$ is a s.o.p for $\overline{M}$, then $\theta_1, \ldots, \theta_r, h_1, \ldots, h_i$ is a s.o.p for M and again by the dimension theorem one has $d \leq r + i = \dim(\overline{M}) + i$. □

Definition 1.3.4 Let M be an R-module. A sequence $\underline{\theta} = \theta_1, \ldots, \theta_n$ in R is called a *regular sequence* of M or an *M-regular sequence* if $(\underline{\theta})M \neq M$ and $\theta_i \notin \mathcal{Z}(M/(\theta_1, \ldots, \theta_{i-1})M)$ for all i.

Proposition 1.3.5 *Let M be an R-module and I an ideal of R such that $IM \neq M$. If $\underline{\theta} = \theta_1, \ldots, \theta_r$ is an M-regular sequence in I, then $\underline{\theta}$ can be extended to a maximal M-regular sequence in I.*

Proof. By induction assume there is an M-regular sequence $\theta_1, \ldots, \theta_i$ in I for some $i \geq r$. Set $\overline{M} = M/(\theta_1, \ldots, \theta_i)M$. If $I \not\subset \mathcal{Z}(\overline{M})$ pick θ_{i+1} in I which is regular on $\overline{M}$. Since

$$(\theta_1) \subset (\theta_1, \theta_2) \subset \cdots \subset (\theta_1, \ldots, \theta_i) \subset (\theta_1, \ldots, \theta_{i+1}) \subset R$$

is an increasing sequence of ideals in a Noetherian ring R, this inductive construction must stop at a maximal M-regular sequence in I. □

Lemma 1.3.6 *Let M be a module over a local ring $(R, \mathfrak{m})$. If $\theta_1, \ldots, \theta_r$ is an M-regular sequence in $\mathfrak{m}$, then $r \leq \dim(M)$.*

Proof. By induction on $\dim(M)$. If $\dim(M) = 0$, then $\mathfrak{m}$ is an associated prime of M and every element of $\mathfrak{m}$ is a zero divisor of M. Using the second part of the proof of Lemma 1.3.10 one has $\dim(M/\theta_1 M) < \dim(M)$. Since $\theta_2, \ldots, \theta_r$ is a regular sequence on $M/\theta_1 M$ by induction one derives $r \leq \dim(M)$. □

Proposition 1.3.7 *Let M be an R-module and let I be an ideal of R.*

(a) $\operatorname{Hom}_R(R/I, M) = (0)$ *iff there is $x \in I$ which is regular on M.*

(b) $\operatorname{Ext}^r_R(R/I, M) \simeq \operatorname{Hom}_R(R/I, M/\underline{\theta}M)$, *where $\underline{\theta} = \theta_1, \ldots, \theta_r$ is any M-regular sequence in I.*

Proof. *(a)* $\Rightarrow$) Assume $I \subset \mathcal{Z}(M)$. Using Lemma 1.1.18 one has $I \subset \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_R(M)$. Hence there is a monomorphism $\psi\colon R/\mathfrak{p} \to M$. To derive a contradiction note that the composition

$$R/I \xrightarrow{\varphi} R/\mathfrak{p} \xrightarrow{\psi} M$$

is a nonzero map, where φ is the canonical map from R/I to $R/\mathfrak{p}$. The converse is left as an exercise.

(b) Consider the exact sequence

$$0 \longrightarrow M \xrightarrow{\theta_1} M \longrightarrow \overline{M} = M/\theta_1 M \longrightarrow 0.$$

According to [250, Theorem 7.3] there is a long exact sequence with natural connecting homomorphisms

$$\cdots \longrightarrow \operatorname{Ext}^{r-1}_R(R/I, M) \xrightarrow{\theta_1} \operatorname{Ext}^{r-1}_R(R/I, M) \longrightarrow \operatorname{Ext}^{r-1}_R(R/I, \overline{M}) \xrightarrow{\partial}$$
$$\operatorname{Ext}^r_R(R/I, M) \xrightarrow{\theta_1} \operatorname{Ext}^r_R(R/I, M) \longrightarrow \cdots$$

Since θ_1 is in I, using [250, Theorem 7.16] it follows that in the last exact sequence the maps given by multiplication by θ_1 are zero. Hence

$$\mathrm{Ext}_R^{r-1}(R/I, \overline{M}) \simeq \mathrm{Ext}_R^r(R/I, M),$$

and the proof follows by induction on r. □

Let $M \neq (0)$ be a module over a local ring $(R, \mathfrak{m})$. The *depth* of M, denoted by $\mathrm{depth}(M)$, is the length of any maximal regular sequence on M which is contained in $\mathfrak{m}$. From Proposition 1.3.7 one derives

$$\mathrm{depth}(M) = \inf\{r | \mathrm{Ext}_R^r(R/\mathfrak{m}, M) \neq (0)\}.$$

In general by Lemma 1.3.6 we have $\mathrm{depth}(M) \leq \dim(M)$.

Definition 1.3.8 An R-module M is called *Cohen-Macaulay* (C-M for short) if $\mathrm{depth}(M) = \dim(M)$, or if $M = (0)$.

Lemma 1.3.9 (Depth lemma) *If* $0 \to N \to M \to L \to 0$ *is a short exact sequence of modules over a local ring* R, *then*

(*a*) *If* $\mathrm{depth}(M) < \mathrm{depth}(L)$, *then* $\mathrm{depth}(N) = \mathrm{depth}(M)$.

(*b*) *If* $\mathrm{depth}(M) = \mathrm{depth}(L)$, *then* $\mathrm{depth}(N) \geq \mathrm{depth}(M)$.

(*c*) *If* $\mathrm{depth}(M) > \mathrm{depth}(L)$, *then* $\mathrm{depth}(N) = \mathrm{depth}(L) + 1$.

Proof. See [298, Corollary A.6.3]. □

Lemma 1.3.10 *If* M *is a module over a local ring* R *and* z *is a regular element of* M, *then*

(a) $\mathrm{depth}(M/zM) = \mathrm{depth}(M) - 1$, *and*

(b) $\dim(M/zM) = \dim(M) - 1$.

Proof. As $\mathrm{depth}\, M > \mathrm{depth}\, M/zM$ applying the depth lemma to the exact sequence

$$0 \longrightarrow M \xrightarrow{z} M \longrightarrow M/zM \longrightarrow 0$$

yields $\mathrm{depth}(M) = \mathrm{depth}(M/zM)+1$.

To prove the second equality first observe the validity of the inequality $\dim(M) > \dim(M/zM)$, otherwise there is a saturated chain of prime ideals

$$\mathrm{ann}\,(M) \subset \mathrm{ann}\,(M/zM) \subset \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_d,$$

where d is the dimension of M and $\mathfrak{p}_0$ is minimal over $\mathrm{ann}\,(M)$. According to Proposition 1.1.31 the ideal $\mathfrak{p}_0$ consists of zero divisors, a contradiction since $z \in \mathrm{ann}\,(M/zM) \subset \mathfrak{p}_0$. On the other hand the reverse inequality $\dim(M) \leq \dim(M/zM) + 1$ follows from Theorem 1.3.1. □

Proposition 1.3.11 *If $M \neq (0)$ is a Cohen-Macaulay R-module over a local ring $(R, \mathfrak{m})$ and $\mathfrak{p} \in \mathrm{Ass}(M)$, then*

$$\dim(R/\mathfrak{p}) = \mathrm{depth}(M).$$

Definition 1.3.12 Let $(R, \mathfrak{m})$ be a local ring of dimension d. A *system of parameters* (s.o.p) of R is a set $\theta_1, \ldots, \theta_d$ generating an $\mathfrak{m}$-primary ideal.

Theorem 1.3.13 *Let $(R, \mathfrak{m}, k)$ be a local ring and let*

$$\delta(R) = \min\{\nu(I) = \dim_k(I/\mathfrak{m}I) \mid I \text{ is an ideal of } R \text{ with } \mathrm{rad}\,(I) = \mathfrak{m}\}.$$

Then $\dim(R) = \delta(R)$.

Proof. This result is a particular instance of Theorem 1.3.1. □

Theorem 1.3.14 (Krull principal ideal theorem) *Let I be an ideal of a ring R generated by a sequence $h_1, \ldots, h_r$. Then*

(a) $\mathrm{ht}\,(\mathfrak{p}) \leq r$ *for any minimal prime $\mathfrak{p}$ of I.*

(b) *If $h_1, \ldots, h_r$ is a regular sequence, then* $\mathrm{ht}\,(\mathfrak{p}) = r$ *for any minimal prime $\mathfrak{p}$ of I.*

Proof. (a) Since $\sqrt{IR_{\mathfrak{p}}} = \mathfrak{p}R_{\mathfrak{p}}$ from Theorem 1.3.13 one has the inequality $\mathrm{ht}\,(\mathfrak{p}) = \dim(R_{\mathfrak{p}}) \leq r$.

(b) Set $J = (h_1, \ldots, h_{i-1})$ and $K = (h_1, \ldots, h_i)$. Assume $\mathrm{ht}\,(P) = i - 1$ for any minimal prime P over J. Since h_i is regular on R/J and K/J is a principal ideal one has $\mathrm{ht}\,(K/J) = 1$, thus there is a prime ideal $\mathfrak{p}_0$ minimal over J such that $J \subset \mathfrak{p}_0 \subset \mathfrak{p}$ and consequently $\mathrm{ht}\,(\mathfrak{p}) \geq i$. Using (a) one gets $\mathrm{ht}\,(\mathfrak{p}) = i$. □

Corollary 1.3.15 *If $(R, \mathfrak{m}, k)$ is a local ring, then* $\dim(R) \leq \dim_k(\mathfrak{m}/\mathfrak{m}^2)$ *and R has finite Krull dimension.*

Proof. Let $x_1, \ldots, x_q$ be a set of elements in $\mathfrak{m}$ whose images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis of this vector space. Then $\mathfrak{m} = (x_1, \ldots, x_q) + \mathfrak{m}^2$. Hence by Nakayama's lemma we get $\mathfrak{m} = (x_1, \ldots, x_q)$. Hence $\dim R \leq q$ by Theorem 1.3.13. □

Remark If R is a Noetherian ring, then $\dim R_{\mathfrak{p}} < \infty$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. However, there are examples of Noetherian rings of infinite Krull dimension.

Definition 1.3.16 A local ring $(R, \mathfrak{m}, k)$ is called *regular* if

$$\dim(R) = \dim_k(\mathfrak{m}/\mathfrak{m}^2).$$

A ring R is *regular* if $R_{\mathfrak{p}}$ is a regular local ring for every $\mathfrak{p} \in \mathrm{Spec}(R)$.

Cohen-Macaulay rings A local ring $(R, \mathfrak{m})$ is called *Cohen-Macaulay* if R is Cohen-Macaulay as an R-module. If R is non local and $R_{\mathfrak{p}}$ is a C-M local ring for all $\mathfrak{p} \in \operatorname{Spec}(R)$, then we say that R is a *Cohen-Macaulay ring*. An ideal I of R is *Cohen-Macaulay* if R/I is a Cohen-Macaulay R-module.

If R is a Cohen-Macaulay ring and S is a multiplicatively closed subset of R, then $S^{-1}(R)$ is a Cohen-Macaulay ring (see [44, Theorem 2.1.3]).

Proposition 1.3.17 *Let M be a module of dimension d over a local ring $(R, \mathfrak{m})$ and let $\underline{\theta} = \theta_1, \ldots, \theta_d$ be a system of parameters of M. Then M is Cohen-Macaulay if and only if $\underline{\theta}$ is an M-regular sequence.*

Proof. $\Rightarrow$) Let $\mathfrak{p}$ be an associated prime of M. By Proposition 1.3.11 one has $\dim(R/\mathfrak{p}) = d$. We claim that θ_1 is not in $\mathfrak{p}$. If $\theta_1 \in \mathfrak{p}$, then by Nakayama's lemma one has $(M/\theta_1 M)_{\mathfrak{p}} \neq (0)$. Hence $\mathfrak{p}$ is in the support of $M/\theta_1 M$, a contradiction because by Corollary 1.3.3 one has the equality $\dim(M/\theta_1 M) = d - 1$. Therefore $\theta_1 \notin \mathfrak{p}$ and θ_1 is regular on M. Thus the proof follows by induction, because according to Lemma 1.3.10 one has that $M/\theta_1 M$ is a C-M module of dimension $d - 1$ and the images of $\theta_2, \ldots, \theta_d$ in R/θ_1 form a system of parameters of $M/(\theta_1)M$.

$\Leftarrow$) By Corollary 1.3.3 one has $\dim(M/\underline{\theta}M) = 0$ and $M/(\underline{\theta})M$ is C-M. Therefore M is Cohen-Macaulay by Lemma 1.3.10. □

Lemma 1.3.18 *Let $(R, \mathfrak{m})$ be a local ring and let $(f_1, \ldots, f_r)$ be an ideal of height equal to r. Then there are $f_{r+1}, \ldots, f_d$ in $\mathfrak{m}$ such that $f_1, \ldots, f_d$ is a system of parameters of R.*

Proof. Set $d = \dim(R)$ and $I = (f_1, \ldots, f_r)$. One may assume $r < d$, otherwise there is nothing to prove. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal primes of I. Note that $\operatorname{ht}(\mathfrak{p}_i) = r$ for all i by Theorem 1.3.14. Hence if we pick f_{r+1} in $\mathfrak{m} \setminus \cup_{i=1}^{s} \mathfrak{p}_i$, one has the equality $\operatorname{ht}(I, f_{r+1}) = r + 1$, and the result follows by induction. □

Definition 1.3.19 Let R be a ring and let I be an ideal of R. If I is generated by a regular sequence we say that I is a *complete intersection.*

Definition 1.3.20 An ideal I of a ring R is called a *set-theoretic complete intersection* if there are $f_1, \ldots, f_r$ in I such that $\operatorname{rad}(I) = \operatorname{rad}(f_1, \ldots, f_r)$, where $r = \operatorname{ht}(I)$.

Definition 1.3.21 An ideal I of a ring R is *height unmixed* or *unmixed* if $\operatorname{ht}(I) = \operatorname{ht}(\mathfrak{p})$ for all $\mathfrak{p}$ in $\operatorname{Ass}_R(R/I)$.

Proposition 1.3.22 *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and let I be an ideal of R. If I is a complete intersection, then R/I is Cohen-Macaulay and I is unmixed.*

Proof. Set $d = \dim(R)$ and $r = \mathrm{ht}\,(I)$. By Lemma 1.3.10 R/I is Cohen-Macaulay and $\dim(R/I) = d - r$. Let $\mathfrak{p}$ be an associated prime of R/I, then using Proposition 1.3.11 yields

$$\dim(R) - \mathrm{ht}\,(\mathfrak{p}) \geq \dim(R/\mathfrak{p}) = \mathrm{depth}(R/I) = d - r.$$

As a consequence $\mathrm{ht}\,(\mathfrak{p}) \leq r$ and thus $\mathrm{ht}\,(\mathfrak{p}) = r$. Hence I is unmixed. □

Theorem 1.3.23 *Let R be a Cohen-Macaulay ring and I a proper ideal of R of height r. If I is generated by r elements $f_1, \ldots, f_r$, then I is unmixed.*

Proof. It is enough to prove that I has no embedded primes, because by Krull's theorem all minimal primes of I have height r. Let $\mathfrak{p}$ and $\mathfrak{q}$ be two associated primes of R/I and assume $\mathfrak{p} \subset \mathfrak{q}$, then $IR_{\mathfrak{q}} \subset \mathfrak{p}R_{\mathfrak{q}} \subset \mathfrak{q}R_{\mathfrak{q}}$. Since $IR_{\mathfrak{q}}$ has height r and is generated by r elements, one obtains (using Lemma 1.3.18) that $f_1/1, \ldots, f_r/1$ is part of a system of parameters of $R_{\mathfrak{q}}$. Therefore by Proposition 1.3.17 the sequence $f_1/1, \ldots, f_r/1$ is a regular sequence and thus Proposition 1.3.22 proves that $IR_{\mathfrak{q}}$ is unmixed. Noticing that $\mathfrak{p}R_{\mathfrak{q}}$ and $\mathfrak{q}R_{\mathfrak{q}}$ are both associated primes of $IR_{\mathfrak{q}}$, one derives $\mathfrak{p}R_{\mathfrak{q}} = \mathfrak{q}R_{\mathfrak{q}}$ and hence $\mathfrak{p} = \mathfrak{q}$. □

Proposition 1.3.24 *If I is an ideal of height r in a Cohen-Macaulay ring R, then there is a regular sequence $f_1, \ldots, f_r$ in I.*

Proof. By induction assume that $f_1, \ldots, f_s$ is a regular sequence in I such that $s < r$. Note that $(f_1, \ldots, f_s)$ is unmixed by Theorem 1.3.23. If $I \subset \mathcal{Z}(R/(f_1, \ldots, f_s))$, then $\mathrm{ht}\,(I) = s$, which is a contradiction. Therefore there is an element f_{s+1} in I which is regular modulo $(f_1, \ldots, f_s)$. □

Let A be a (Noetherian) ring one says that A is a *catenary* ring if for every pair $\mathfrak{p} \subset \mathfrak{q}$ of prime ideals $\mathrm{ht}\,(\mathfrak{q}/\mathfrak{p})$ is equal to the length of any maximal chain of prime ideals between $\mathfrak{p}$ and $\mathfrak{q}$. If A is a domain, then A is catenary if and only if $\mathrm{ht}\,(\mathfrak{q}/\mathfrak{p}) = \mathrm{ht}\,(\mathfrak{q}) - \mathrm{ht}\,(\mathfrak{p})$ for every pair of prime ideals $\mathfrak{p} \subset \mathfrak{q}$.

Theorem 1.3.25 *If R is a Cohen-Macaulay ring, then R is catenary.*

Proof. See [44, Theorem 2.1.12]. □

Gorenstein rings Let $M \neq (0)$ be a module over a local ring $(R, \mathfrak{m})$ and let $k = R/\mathfrak{m}$ be the residue field of R. The *socle* of M is defined as

$$\mathrm{Soc}(M) = (0 :_M \mathfrak{m}) = \{z \in M \,|\, \mathfrak{m}z = (0)\},$$

and the *type* of M is defined as

$$\mathrm{type}(M) = \dim_k \mathrm{Soc}(M/\underline{x}M),$$

where $\underline{x}$ is a maximal M-sequence in $\mathfrak{m}$. Observe that the type of M is well defined because by Proposition 1.3.7 one has:

$$\operatorname{Ext}_R^r(k, M) \simeq \operatorname{Hom}_R(k, M/\underline{x}M) \simeq \operatorname{Soc}(M/\underline{x}M),$$

where $r = \operatorname{depth}(M)$. The ring R is said to be *Gorenstein* if R is a Cohen-Macaulay ring of type 1, and an ideal $I \subset R$ is called *Gorenstein* if R/I is a Gorenstein ring.

For a thorough study of Gorenstein rings see [14, 44]; a vivid presentation of the ubiquity of Gorenstein rings is given in [184].

Exercises

1.3.26 Let $\phi\colon A \to B$ be an epimorphism of rings. If $\dim(A) = \dim(B) < \infty$ and A is a domain, prove that ϕ is injective.

1.3.27 Let R be a catenary ring. If $S \subset R$ is a multiplicatively closed set and I is an ideal of R, prove that R/I and $S^{-1}R$ are both catenary rings.

1.3.28 Let $M = \mathbb{Z}^r \times \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_m^{\alpha_m}}$, where $p_1, \ldots, p_m$ are prime numbers. Prove the following:

(a) M has finite length iff $r = 0$.

(b) $\ell(M) = \alpha_1 + \cdots + \alpha_m$, if $\ell(M) < \infty$.

(c) $\operatorname{Ass}(M) \neq \operatorname{Supp}(M)$, if $r \neq 0$.

1.3.29 If M, N are R-modules, prove $\operatorname{Ass}(M \oplus N) = \operatorname{Ass}(M) \cup \operatorname{Ass}(N)$.

1.3.30 (Cayley-Hamilton) Let $\varphi\colon M \to M$ be an endomorphism of an R-module M. If I is an ideal of R such that $\varphi(M) \subset IM$, then φ satisfies an equation:

$$\varphi^n + a_{n-1}\varphi^{n-1} + \cdots + a_1\varphi + a_0 = 0 \quad (a_i \in I).$$

Hint If $M = Rm_1 + \cdots + Rm_n$, write

$$\varphi(m_i) = \sum_{j=1}^{n} a_{ij} m_j$$

and consider the matrix $C = \varphi I - (a_{ij})$. Use that the entries of C are endomorphisms of M that commute with each other to conclude the equality $C\operatorname{adj}(C) = \operatorname{adj}(C)C = \det(C)I$. Note that C operates in M^n by matrix multiplication and $C(m_1, \ldots, m_n)^t = 0$.

1.3.31 Let I be an ideal of a ring R and let $x \in R \setminus I$. If I is unmixed and $I = \operatorname{rad}(I)$, then $(I\colon x)$ is unmixed and $\operatorname{ht}(I) = \operatorname{ht}(I\colon x)$.

1.3.32 Let M be a module over a local ring $(R, \mathfrak{m})$. If $\underline{\theta} = \theta_1, \ldots, \theta_r$ is an M-regular sequence in $\mathfrak{m}$, prove that $\underline{\theta}$ can be extended to a system of parameters of M.

1.3.33 Let M be an R-module and let I be an ideal of R. If there is $x \in I$ which is regular on M, show that $\mathrm{Hom}_R(R/I, M) = (0)$.

1.3.34 Let M be an R-module and let I be an ideal of R. If $J \subset I$, then:

$$\mathrm{Hom}_R(R/I, M/JM) \simeq (JM{:}_M I)/JM.$$

1.4 Normal rings

Let A and B be two rings. One says that B is an *A-algebra* if there is a homomorphism of rings

$$\varphi{:}\, A \longrightarrow B.$$

Note that B has an A-module structure (compatible with its ring structure) given by

$$a \cdot b = \varphi(a)b$$

for all $a \in A$ and $b \in B$.

Thus if $A \subset B$ is a *ring extension*, that is, A is a subring of B, then B has in a natural way an A-algebra structure induced by the inclusion map. If $A = K$ is a field and B is a K-algebra, then φ is injective, in this case one may always assume that $K \subset B$ is a ring extension.

Given $F = \{f_1, \ldots, f_q\}$ a finite subset of B, we denote the *subring of B generated* by F and $\varphi(A)$ by $A[F] = A[f_1, \ldots, f_q]$. Sometimes $A[F]$ is called the *A-subalgebra* or *A-subring* of B *spanned* by F. If $B = A[F]$, one says that B is an A-algebra of *finite type* or a *finitely generated* A-algebra, on the other hand one says that φ is *finite* or that B is a *finite A-algebra* if B is a finitely generated A-module.

A *homomorphism* $\phi{:}\, B \to C$ of *A-algebras* is a map ϕ which is both a homomorphism of rings and a homomorphism of A-modules. Note that $\phi(a \cdot 1) = a \cdot 1$ for all $a \in A$.

Definition 1.4.1 Let A be a subring of B. An element $b \in B$ is integral over A, if there is a monic polynomial $0 \neq f(x) \in A[x]$ such that $f(b) = 0$. The set $\overline{A}$ of all elements $b \in B$ which are integral over A is called the *integral closure* of A in B.

Let A be a subring of B, one says that $A \subset B$ is an *integral extension of rings* or that B is *integral over* A, if b is integral over A for all $b \in B$. A homomorphism of rings $\varphi{:}\, A \to B$ is called *integral* if B is integral over $\varphi(A)$, in this case one also says that B is *integral over* A.

Proposition 1.4.2 *Let A be a subring of B. If B is a finitely generated A-module, then B is integral over A.*

Proof. Let $\beta \in B$. There are $\alpha_1, \ldots, \alpha_n \in B$ such that

$$B = A\alpha_1 + \cdots + A\alpha_n.$$

One can write

$$\beta\alpha_i = \sum_{j=1}^{n} m_{ij}\alpha_j,$$

where $m_{ij} \in A$. Set $M = (m_{ij})$, $N = M - \beta I$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. Here I denotes the identity matrix. Since $N\alpha^t = 0$, one can use the formula $N\mathrm{adj}(N) = \det(N)I$ to conclude $\alpha_i \det(N) = 0$ for all i. Hence $\det(N) = 0$. To complete the proof note that

$$f(x) = (-1)^n \det(M - xI)$$

is a monic polynomial in $A[x]$ and $f(\beta) = (-1)^n \det(N) = 0$. □

Corollary 1.4.3 *If A is a subring of B, then the integral closure $\overline{A}$ of A in B is a subring of B.*

Proof. Let $\alpha, \beta \in \overline{A}$. If α and β satisfy monic polynomials with coefficients in A of degree m and n respectively, then $A[\alpha, \beta]$ is a finitely generated A-module with basis

$$\{\alpha^i\beta^j \mid 0 \leq i \leq m-1 \text{ and } 0 \leq j \leq n-1\}.$$

Hence by Proposition 1.4.2 $A[\alpha, \beta]$ is integral over A. In particular $\alpha \pm \beta$ and $\alpha\beta$ are integral over A. □

Corollary 1.4.4 *If B is an A-algebra of finite type, then B is integral over A if and only if B is finite over A.*

Proof. $\Rightarrow$) This implication follows using the arguments given in the proof of Corollary 1.4.3.

$\Leftarrow$) It follows from Proposition 1.4.2. □

Lemma 1.4.5 *Let $A \subset B$ be a ring extension. If B is a domain and b is an element of B which is integral over A, then $A[b]$ is a field if and only if A is a field.*

Proof. $\Rightarrow$) Let $a \in A \setminus \{0\}$ and $c = a^{-1}$ its inverse in $A[b]$. Since $A[b]$ is integral over A, there is an equation

$$c^n + a_{n-1}c^{n-1} + \cdots + a_1 c + a_0 = 0 \quad (a_i \in A),$$

multiplying by a^n it follows rapidly that $c \in A$.

$\Leftarrow$) Let $A[x]$ be a polynomial ring and let

$$\psi\colon A[x] \longrightarrow A[b]$$

be the epimorphism given by $\psi(f(x)) = f(b)$. Note that $\ker(\psi)$ is a nonzero prime ideal. As $A[x]$ is a principal ideal domain, $\ker(\psi)$ is a maximal ideal and consequently $A[b]$ is a field. □

Proposition 1.4.6 *Let $A \subset B$ be an integral extension of rings. If B is a domain, then A is a field if and only if B is a field.*

Proof. It follows from Lemma 1.4.5. □

Corollary 1.4.7 *Let $A \subset B$ be an integral extension of rings. If P is a prime ideal of B and $\mathfrak{p} = P \cap A$, then P is a maximal ideal of B if and only if $\mathfrak{p}$ is a maximal ideal of A.*

Proof. The result is a direct consequence of Proposition 1.4.6, because $A/\mathfrak{p} \subset B/P$ is an integral extension of rings. □

Corollary 1.4.8 *Let $A \subset B$ be an integral extension of rings. If $P \subset Q$ are two prime ideals of B such that $P \cap A = Q \cap A$, then $P = Q$.*

Proof. Set $\mathfrak{p} = P \cap A$. As $B_{\mathfrak{p}}$ is integral over $A_{\mathfrak{p}}$ and

$$PB_{\mathfrak{p}} \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}},$$

using Corollary 1.4.7 we get that $PB_{\mathfrak{p}}$ is maximal and hence $P = Q$. □

Lemma 1.4.9 *If $A \subset B$ is an integral extension of rings and $\mathfrak{p}$ is a maximal ideal of A, then $\mathfrak{p}B \neq B$ and $\mathfrak{p} = P \cap A$ for any maximal ideal P of B containing $\mathfrak{p}B$.*

Proof. If $\mathfrak{p}B = B$, one can write $1 = a_1b_1 + \cdots + a_qb_q$ with $a_i \in \mathfrak{p}$ and $b_i \in B$. Set $S = A[b_1, \ldots, b_q]$, the subring of B generated by $b_1, \ldots, b_q$. Since S is a finitely generated A-module and $\mathfrak{p}S = S$, then by Lemma 1.1.28 there is $x \equiv 1(\mathrm{mod}\,\mathfrak{p})$ such that $xS = (0)$. As $1 \in B$, one derives $x = 0$ and $1 \in \mathfrak{p}$, which is impossible. Hence $\mathfrak{p}B \neq B$. Note that this part of proof holds if $\mathfrak{p}$ is a proper ideal of A.

If $\mathfrak{p}B \subset P$ is a maximal ideal of B, then $\mathfrak{p} \subset P \cap A$ and thus $\mathfrak{p} = A \cap P$ by the maximality of $\mathfrak{p}$. □

Proposition 1.4.10 *If $A \subset B$ is an integral extension of rings and $\mathfrak{p}$ is a prime ideal of A, then there is a prime ideal P of B such that $\mathfrak{p} = P \cap A$.*

Proof. Since $A_{\mathfrak{p}} \subset B_{\mathfrak{p}}$ is an integral extension, then by Lemma 1.4.9 $\mathfrak{p}B_{\mathfrak{p}} \neq B_{\mathfrak{p}}$. Note that $P \cap A = \mathfrak{p}$ for any prime ideal P of B such that $PB_{\mathfrak{p}}$ is prime and contains $\mathfrak{p}B_{\mathfrak{p}}$ (see Exercise 1.4.23). □

Theorem 1.4.11 (Going-up) *Let $A \subset B$ be an integral ring extension. If $\mathfrak{p} \subset \mathfrak{q}$ are two prime ideals of A and $\mathfrak{p} = P \cap A$ for some P in* Spec(B), *then there is $Q \in$ Spec(B) such that $\mathfrak{q} = Q \cap A$ and $P \subset Q$.*

Proof. Since $A/\mathfrak{p} \subset B/P$ is an integral extension, by Proposition 1.4.10 one has $\mathfrak{q}/\mathfrak{p} = (Q/P) \cap (A/\mathfrak{p})$ for some prime ideal Q of B containing P. It follows readily that $\mathfrak{q} = A \cap Q$. □

Proposition 1.4.12 *Let $A \subset B$ be an integral extension of rings. If B is integral over A, then* $\dim(A) = \dim(B)$.

Proof. The formula follows using Theorem 1.4.11, Corollary 1.4.8 and Proposition 1.4.10. □

Let A be an integral domain and let K_A be its field of fractions. The *integral closure* or *normalization* of A, denoted $\overline{A}$, is the set of all $f \in K_A$ satisfying an equation of the form

$$f^n + a_{n-1}f^{n-1} + \cdots + a_1 f + a_0 = 0 \quad (a_i \in A \text{ and } n \geq 1).$$

By Corollary 1.4.3 the integral closure $\overline{A}$ of A is a subring of K_A. If $A = \overline{A}$ we say that A is *integrally closed* or *normal*. If A is not a domain we say that A is *normal* if $A_{\mathfrak{p}}$ is a normal domain for every prime ideal $\mathfrak{p}$ of A.

A useful fact is that a normal ring is a direct product of finitely many normal domains (see [92]).

Proposition 1.4.13 *Let $A \subset B$ be an integral extension of rings. If B is a domain and A is a normal domain, then*

(a) *(going-down) if $\mathfrak{p} \subset \mathfrak{q}$ are two prime ideals of A and $\mathfrak{q} = Q \cap A$ for some Q in* Spec(B), *then there is $P \in$ Spec(B) such that $\mathfrak{p} = P \cap A$ and $P \subset Q$.*

(b) $\operatorname{ht}(I) = \operatorname{ht}(I \cap A)$ *for any ideal I of B.*

Proof. See [217, Theorem 5] and [217, Theorem 20]. □

Proposition 1.4.14 *Let A be a domain and let x be an indeterminate over A. Then A is normal if and only if $A[x]$ is normal.*

Theorem 1.4.15 (Serre's criterion) *A ring A is normal if and only if*

(S_2) $\operatorname{depth}(A_{\mathfrak{p}}) \geq \inf\{2, \operatorname{ht}(\mathfrak{p})\}$ *for all* $\mathfrak{p} \in$ Spec(A), *and*

(R_1) $A_{\mathfrak{p}}$ *is regular for all* $\mathfrak{p} \in$ Spec(A) *with* $\operatorname{ht}(\mathfrak{p}) \leq 1$.

Theorem 1.4.16 (Auslander-Buchsbaum) *If $(A, \mathfrak{m})$ is a regular local ring, then A is a unique factorization domain.*

Flat and faithfully fiat algebras Let $\phi\colon A \to B$ be a homomorphism of rings and consider an exact sequence of A-modules

$$\cdots \longrightarrow N_{i-1} \xrightarrow{f_{i-1}} N_i \xrightarrow{f_i} N_{i+1} \xrightarrow{f_{i+1}} \cdots \qquad (N_\star)$$

If the sequence

$$\cdots \longrightarrow B \otimes_A N_{i-1} \xrightarrow{1\otimes f_{i-1}} B \otimes_A N_i \xrightarrow{1\otimes f_i} B \otimes_A N_{i+1} \xrightarrow{1\otimes f_{i+1}} \cdots \quad (B \otimes_A N_\star)$$

is exact for any $N_\star$, B is called a *flat A-algebra* and ϕ is said to be *flat*. The homomorphism ϕ is said to be *faithfully flat* if any sequence $N_\star$ of A-modules is exact if and only if $B \otimes_A N_\star$ is exact. In a similar way one can define flat and faithfully flat A-modules.

A basic property is that any free A-module (not necessarily finitely generated) is faithfully flat. In particular a polynomial ring $A[\mathbf{x}]$ in several variables with coefficients in A is a faithfully flat A-algebra.

Theorem 1.4.17 *Let $\phi\colon A \to B$ be a flat homomorphism of rings. The following holds:*

(a) $(I_1 \cap I_2)B = I_1B \cap I_2B$, *for any two ideals I_1, I_2 of A.*

(b) *The going down theorem holds for ϕ.*

(c) *If $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal of A such that $\mathfrak{p}B$ is prime, then $\mathfrak{q}B$ is a $\mathfrak{p}B$-primary ideal.*

Proof. See [217, Sections 3.H, 5.D and 9.C]. □

Theorem 1.4.18 *Let $\phi\colon A \to B$ be a faithfully flat homomorphism of rings. The following holds:*

(a) *The map $\phi^*\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$, $\phi^*(P) = \phi^{-1}(P)$ is surjective.*

(b) $IB \cap A = I$ *and* $\operatorname{ht}(I) = \operatorname{ht}(IB)$ *for any ideal I of A.*

Proof. [217, Sections 4.C and 13.B]. □

Exercises

1.4.19 Let K be a field and $\varphi\colon A \to B$ an isomorphism of K-algebras, note $\varphi(r) = r$ for all $r \in K$. If $A = \oplus_{i\geq 0} A_i$ is a graded K-algebra, then B is also a graded K-algebra graded by $B = \oplus_{i\geq 0}\varphi(A_i)$.

1.4.20 Let $\varphi\colon A \to B$ be an isomorphism between two integral domains. If K_A (resp. K_B) is the field of fractions of A (resp. B), then φ extends to an isomorphism $\overline{\varphi}$ from K_A to K_B such that $\overline{\varphi}(A) = B$.

1.4.21 If A is a unique factorization domain, show that A is normal.

1.4.22 Show an example of an integral extension of rings $A \subset B$, such that A is a field and B is not an integral domain (cf. Lemma 1.4.5).

1.4.23 Let $A \subset B$ be a ring extension and $\mathfrak{p}$ a prime ideal of A. If $S = A \setminus \mathfrak{p}$ and $B_{\mathfrak{p}} = S^{-1}B$, prove that the map $P \longmapsto PB_{\mathfrak{p}}$ gives a bijection between the set of prime ideals of P of B such that $P \cap A = \mathfrak{p}$ and the set of prime ideals of $B_{\mathfrak{p}}$ containing $\mathfrak{p}B_{\mathfrak{p}}$.

1.4.24 Let $B = A[\mathbf{x}]$ be a polynomial ring over a ring A and let I be an ideal of A. Prove $(A/I)[\mathbf{x}] \simeq B/IB$, where the left hand side is a polynomial ring with coefficients in A/I.

1.4.25 If $B = A[\mathbf{x}]$ is a polynomial ring over a ring A and $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal of A, then $\mathfrak{q}B$ is a $\mathfrak{p}B$-primary ideal of B.

Hint Use Theorem 1.4.17.

1.4.26 Let $B = A[\mathbf{x}]$ be a polynomial ring over a ring A and let I be an ideal of A. If $I = \cap_{i=1}^{r} \mathfrak{q}_i$ is a primary decomposition of I, then $IB = \cap_{i=1}^{r} \mathfrak{q}_i B$ is a primary decomposition of IB.

1.5 Koszul homology

Let a be an element of the ring R and $K_\star(a)$ be the complex defined as

$$K_i = \begin{cases} R & \text{for } i = 0, 1, \\ 0 & \text{otherwise,} \end{cases}$$

with $d_1 : K_1(a) \to K_0(a)$ being multiplication by a.

Let I be an ideal of R generated by the sequence $\underline{x} = \{x_1, \ldots, x_n\}$. The ordinary *Koszul complex* associated to $\underline{x}$ is defined as

$$K_\star(x; R) = K_\star(x_1) \otimes \cdots \otimes K_\star(x_n).$$

For an R-module M we shall write $K_\star(\underline{x}; M)$ for $K_\star(\underline{x}; R) \otimes M$. The Koszul complex $K_\star(\underline{x}; R)$ is then the exterior algebra complex associated to $E = R^n$ and the map

$$\theta : E \longrightarrow R,$$

defined as

$$\theta(z_1, \ldots, z_n) = z_1 x_1 + \cdots + z_n x_n.$$

That is, θ defines a differential $\partial = d\theta$ on the exterior algebra $\bigwedge(E)$ of E given in degree r by

$$\partial(e_1 \wedge \cdots \wedge e_r) = \sum_{i=1}^{r} (-1)^{i-1} \theta(e_i) e_1 \wedge \cdots \wedge \widehat{e}_i \wedge \cdots \wedge e_r.$$

A consequence of the definition of the differential of $K_\star(\underline{x}; R)$ is that if w and w' are homogeneous elements of $\bigwedge(E)$, of degrees p and q respectively, then

$$\partial(w \wedge w') = (-1)^p w \wedge \partial(w') + \partial(w) \wedge w'.$$

This implies that the cycles $\mathcal{Z}(K_\star)$ form a subalgebra of $\bigwedge(E)$, and that the boundaries $\mathcal{B}(K_\star)$ form a two-sided ideal of $\mathcal{Z}(K_\star)$. As a consequence the homology of the Koszul complex, $H_\star(\underline{x})$, inherits a skew commutative R-algebra structure.

One can also see that $H_\star(\underline{x})$ is annihilated by $I = (\underline{x})$. Indeed, if $e \in E$ and $w \in \mathcal{Z}_r(K_\star)$, we have from the last formula $\partial(e \wedge w) = \theta(e)w$.

The ordinary Koszul complex $K_\star(\underline{x}) = K_\star(\underline{x}; R)$ is simply the complex of free modules

$$K_\star(\underline{x}): \qquad 0 \to \bigwedge^n R^n \to \bigwedge^{n-1} R^n \to \cdots \to \bigwedge^1 R^n \to \bigwedge^0 R^n \to 0,$$

where $\wedge^k R^n$ is the *kth* exterior power of R^n; thus $\wedge^k R^n$ is a free R-module of rank $\binom{n}{k}$ with basis

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} | 1 \leq i_1 < \cdots < i_k \leq n\}.$$

Proposition 1.5.1 *If $\underline{x} = x_1, \ldots, x_n$ is a regular sequence in R, then the Koszul complex is acyclic, that is, the complex $K_\star(\underline{x})$ is exact.*

Proof. See [290, Theorem 2.3]. □

Sliding depth Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and let I be an ideal of R generated by $x_1, \ldots, x_n$, denote by $H_\star(\underline{x})$ the homology of the ordinary Koszul complex built on the sequence $\underline{x} = \{x_1, \ldots, x_n\}$.

Definition 1.5.2 (i) (*SD*) I satisfies *sliding depth* if

$$\text{depth } H_i(\underline{x}) \geq \dim(R) - n + i, \ \forall i \geq 0.$$

(ii) (*SCM*) I is *strongly Cohen-Macaulay* if $H_i(\underline{x})$ are C-M, $\forall i \geq 0$.

(Depths are computed with respect to maximal ideals. It is usual to set depth(0) equal to ∞.)

Remark 1.5.3 (a) The (SD) condition localizes [156], (b) If I satisfies (SD) with respect to some generating set, then it will satisfy (SD) with respect to any other generating set of I. This follows from the isomorphisms:

$H_i(\{x_1, \ldots, x_n, 0\}) \simeq H_i(\underline{x}) \oplus H_{i-1}(\underline{x})$, and

$H_i(\{x_1, \ldots, x_n, y\}) \simeq H_i(\{x_1, \ldots, x_n, 0\})$, where $y \in (\underline{x})$.

Linkage Let I and J be two ideals in a Cohen-Macaulay local ring R. The ideals I and J are said to be (algebraically) *linked* if there is an R-sequence $\underline{x} = \{x_1, \ldots, x_n\}$ in $I \cap J$ such that

$$I = (\underline{x}) : J) \quad \text{and} \quad J = (\underline{x}) : I),$$

if in addition I and J are unmixed ideals of the same height n without common components and such that $I \cap J = (\underline{x})$, then I and J are said to be *geometrically linked*.

When I and J are linked we shall write $I \sim J$. We say that J is in the *linkage class* of I if there are ideals $I_1, \ldots, I_m$ such that

$$I \sim I_1 \sim \cdots \sim I_m \sim J.$$

The ideal J is said to be in the *even linkage class* of I if m is odd.

Let R be a Gorenstein ring and let I be a Cohen-Macaulay ideal of R. If J is linked to I, then Peskine and Szpiro [233] showed that J is Cohen-Macaulay. A dramatic result of Huneke [179] proves that (SCM) is preserved under even linkage, his method can be adapted to prove the following.

Proposition 1.5.4 *Let I and J be two ideals in a Gorenstein local ring R of dimension d, and let $\underline{x} = \{x_1, \ldots, x_n\}$ be a generating set for I. Assume that J is evenly linked to I. If I satisfies the condition*

(SD_k) $\quad \text{depth } H_i(\underline{x}; R) \geq d - n + i, \quad 0 \leq i \leq k,$

then J satisfies the (SD_k) condition as well.

Exercises

1.5.5 Let $R = k[x_1, \ldots, x_6]$ be a polynomial ring over a field k and let $I = I_2(X)$ be the ideal of 2×2-minors of the symmetric matrix:

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix}.$$

Prove that I is linked to $I_2(X')$, where X' is the symmetric matrix

$$X' = \begin{bmatrix} x_1 & x_2 & -x_3 \\ x_2 & x_4 & x_5 \\ -x_3 & x_5 & x_6 \end{bmatrix}.$$

In particular if $\text{char}(k) = 2$ the ideal I is self-linked. For other characteristics this is an open question.

Chapter 2

Affine and Graded Algebras

A few topics connected with affine and graded algebras are studied in this chapter, e.g., Gröbner bases, Hilbert Nullstellensatz, minimal resolutions and Betti numbers. We present the affine and graded versions of the Noether normalization lemma and some of their applications to affine and Cohen-Macaulay graded algebras.

As before all base rings considered here are Noetherian and modules are finitely generated.

2.1 Noether normalizations

Affine algebras occur naturally in algebraic combinatorics and geometry. One of the key results to understand their behavior is the famous Noether normalization lemma, first we present its affine version.

Definition 2.1.1 Let k be a field and let S be a k-algebra. We say that S is an *affine k-algebra* if $S = k[y_1, \ldots, y_r]$ for some $y_1, \ldots, y_r \in S$.

Definition 2.1.2 Let α and β be in $\mathbb{N}^n$. The *lexicographical order* in $\mathbb{N}^n$ is obtained by declaring

$$\beta > \alpha$$

if the last nonzero entry of $\beta - \alpha$ is positive.

Notation The set of positive integers will be denoted by $\mathbb{N}_+$. If $\alpha, \beta \in \mathbb{R}^n$, here $\alpha \cdot \beta$ will denote the usual inner product of α and β.

Lemma 2.1.3 *Let $\alpha_1 > \cdots > \alpha_m$ be a sequence of m distinct points in $\mathbb{N}^n$ ordered lexicographically. Then there is $w = (w_1, \ldots, w_n) \in \mathbb{N}_+^n$ such that $w_1 = 1$ and $w \cdot \alpha_1 > w \cdot \alpha_i$ for $i \geq 2$.*

Proof. Let $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in})$ and $\beta_i = (\alpha_{i1}, \ldots, \alpha_{i(n-1)})$. We proceed by induction on $n \geq 2$. There is k so that $\alpha_{1n} = \cdots = \alpha_{kn}$ and $\alpha_{kn} > \alpha_{in}$ for $i > k$. One may assume $k < m$; for otherwise $\beta_1 > \cdots > \beta_m$ and one can use induction.

Since $\beta_1 > \cdots > \beta_k$ by induction there is $w' = (1, w_2, \ldots, w_{n-1})$ such that $w' \cdot \beta_1 > w' \cdot \beta_i$ for $2 \leq i \leq k$. On the other hand for every $i > k$ one can choose $\delta_i \in \mathbb{N}_+$ so that

$$w' \cdot \beta_1 + \alpha_{1n}\delta_i > w' \cdot \beta_i + \alpha_{in}\delta_i.$$

To finish the proof set

$$w_n = \max\{\delta_i |\, k < i \leq m\} \quad \text{and } w = (w', w_n)$$

and observe $w \cdot \alpha_1 > w \cdot \alpha_i$ for $i \geq 2$. □

Proposition 2.1.4 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and let f be a polynomial in $R \setminus k$. Then there is a change of variables*

$$x_i = x_1^{w_i} + y_i \quad (i \geq 2)$$

such that

$$f(x_1, x_1^{w_2} + y_2, \ldots, x_1^{w_n} + y_n) = c_0 x_1^r + c_1 x_1^{r-1} + \cdots + c_{r-1}x_1 + c_r,$$

where $0 \neq c_0 \in k$, $r > 0$, and $c_i \in k[y_2, \ldots, y_n]$ for $i \geq 1$.

Proof. The polynomial f can be written as

$$f = \sum_{i=1}^{m} b_i x^{\alpha_i},$$

where $0 \neq b_i \in k$ for all i. One may assume that $\alpha_1 > \cdots > \alpha_m$ are ordered lexicographically. By Lemma 2.1.3 there is $w \in \mathbb{N}_+^n$ such that $x_i = x^{w_i} + y_i$ satisfies the required properties. Note $r = w \cdot \alpha_1$. □

Lemma 2.1.5 *If $R = k[x_1, \ldots, x_n]$ is an affine k-algebra of dimension n, then R is a polynomial ring.*

Proof. One may assume $R \simeq B/I$, where B is a polynomial ring in n variables with coefficients in the field k and I is an ideal of B. Let

$$I \subset \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n$$

be a chain of prime ideals of B of length n. If $I \neq (0)$, then adding (0) to the chain yields a chain of length $n + 1$, which is impossible because $\dim(B) = n$. Hence $I = (0)$. □

Theorem 2.1.6 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and let $I \neq R$ be an ideal of R. Then there are $z_1, \ldots, z_n$ in R such that*

(a) *$k[x_1, \ldots, x_n]$ is integral over $k[z_1, \ldots, z_n]$, and*

(b) $I \cap k[z_1, \ldots, z_n] = z_1 k[z_1, \ldots, z_n] + \cdots + z_g k[z_1, \ldots, z_n]$.

Proof. The proof is by induction on n. If $n = 1$ and $I = (f(x_1)) \neq 0$, then one sets $z_1 = f(x_1)$. Assume $n \geq 2$ and $I \neq 0$. One may assume that I contains a monic polynomial in x_1. Otherwise take a nonzero polynomial g in I and apply Proposition 2.1.4 to get an isomorphism of k-algebras

$$k[x_1, \ldots, x_n] \xrightarrow{\varphi} k[x_1, y_2 \ldots, y_n]$$

induced by $\varphi(x_1) = x_1$ and $\varphi(x_i) = x_1^{w_i} + y_i$ for $i \geq 2$, such that $\varphi(g)$ is monic in x_1. Let

$$f = x_1^r + c_1 x_1^{r-1} + \cdots + c_{r-1} x_1 + c_r$$

be a polynomial in I with $c_i \in k[x_2, \ldots, x_n]$ and $r > 0$. Set

$$I_1 = I \cap k[x_2, \ldots, x_n].$$

By induction there are $z_2, \ldots, z_n$ such that:

(i) $k[x_2, \ldots, x_n]$ is integral over $k[z_2, \ldots, x_n]$, and

(ii) $I_1 \cap k[z_2, \ldots, z_n] = (z_2, \ldots, z_n)$.

Set $z_1 = f$. It is not hard to see that R is integral over $k[z_1, \ldots, z_n]$ and

$$k[z_1, \ldots, z_n] \cap I = (z_1, \ldots, z_g),$$

as required □

Corollary 2.1.7 *If $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k and $I \neq R$ is an ideal of R, then*

$$\dim(R/I) = \dim(R) - \operatorname{ht}(I).$$

Proof. One may assume that there are $z_1, \ldots, z_n$ in R and an integer g such that the conditions (a) and (b) of Theorem 2.1.6 are satisfied. We will show that g is equal to the height of I. By Lemma 2.1.5 the z_i's are algebraically independent. Hence Proposition 1.4.13 yields

$$\operatorname{ht}(I) = \operatorname{ht}(I \cap k[z_1, \ldots, z_n]) = g.$$

Note that there is an integral extension

$$k[z_{g+1}, \ldots, z_n] \simeq k[z_1, \ldots, z_n]/(z_1, \ldots, z_g) \overset{\varphi}{\hookrightarrow} k[x_1, \ldots, x_n]/I.$$

Therefore $n - g = \dim(R/I)$. □

Corollary 2.1.8 (Noether Normalization Lemma) *If $R = k[\mathbf{x}]$ is a polynomial ring over a field k and $I \neq R$ is an ideal, then there is an integral extension*

$$k[h_1, \ldots, h_d] \hookrightarrow R/I,$$

where $h_1, \ldots, h_d$ are in R and $d = \dim(R/I)$.

Proof. By Theorem 2.1.6 there is an integral extension

$$k[z_{g+1}, \ldots, z_n] \simeq k[z_1, \ldots, z_n]/(z_1, \ldots, z_g) \overset{\varphi}{\hookrightarrow} R/I.$$

To conclude the argument set $h_i = z_{g+i}$ for $i = 1, \ldots, d$. □

Corollary 2.1.9 *If $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k, then R is a catenary ring.*

Proof. Note $\mathrm{ht}\,(\mathfrak{q}/\mathfrak{p}) = \dim(R/\mathfrak{p}) - \dim(R/\mathfrak{q})$ for any two prime ideals $\mathfrak{p} \subset \mathfrak{q}$ in R. Hence by Corollary 2.1.7 we get $\mathrm{ht}\,(\mathfrak{q}/\mathfrak{p}) = \mathrm{ht}\,(\mathfrak{q}) - \mathrm{ht}\,(\mathfrak{p})$. □

Definition 2.1.10 Let $k \subset L$ be a field extension. A subset of L which is algebraically independent and is maximal with respect to inclusions is called a *transcendence basis* of L over k.

Theorem 2.1.11 *If $k \subset L$ is a field extension, then any two transcendence basis have the same cardinality.*

Proof. See [191, Theorem 8.35]. □

Definition 2.1.12 Let $k \subset L$ be a field extension. The *transcendence degree* of L over k, denoted $\mathrm{trdeg}_k(L)$, is the cardinality of any transcendence basis of L over k.

Corollary 2.1.13 *Let k be a field and let A be a finitely generated k-algebra. If A is a domain with field of fractions L, then*

$$\dim(A) = \mathrm{trdeg}_k(L).$$

Exercises

2.1.14 If A and B are affine algebras over a field k, then

$$\dim(A \otimes_k B) = \dim(A) + \dim(B).$$

Hint Use the normalization lemma.

2.2 Cohen-Macaulay graded algebras

In this section we will emphasize the relationship between graded Cohen-Macaulay rings and their homogeneous Noether normalizations. Then some useful characterizations of those rings will be given.

Definition 2.2.1 Let k be a field. A *standard algebra* or *homogeneous algebra* is a finitely generated $\mathbb{N}$-graded k-algebra

$$S = \bigoplus_{i=0}^{\infty} S_i = k[y_1, \ldots, y_r]$$

such that $y_i \in S_1$ for all i and $S_0 = k$. If we only require y_i homogeneous and $\deg(y_i) > 0$ for all i, we say that S is a *positively graded* k-algebra.

The *irrelevant maximal ideal* $\mathfrak{m}$ of S is defined by

$$\mathfrak{m} = S_+ = \bigoplus_{i=1}^{\infty} S_i.$$

Definition 2.2.2 Let k be a field and $S = k[y_1, \ldots, y_r]$ a positively graded k-graded with y_i homogeneous of degree d_i. There is a graded epimorphism

$$\varphi\colon R = k[x_1, \ldots, x_r] \longrightarrow S$$

given by

$$f(x_1, \ldots, x_r) \stackrel{\varphi}{\longmapsto} f(y_1, \ldots, y_r),$$

where R is a polynomial ring graded by $\deg(x_i) = d_i$, the *presentation* of S is the k-algebra $R/\ker(\varphi)$. The graded ideal $\ker(\varphi)$ is called the *ideal of relations* or the *presentation ideal* of S.

Proposition 2.2.3 *If S is a Cohen-Macaulay standard algebra over a field k and L is an ideal of S, then*

$$\dim(S) = \dim(S/L) + \operatorname{ht}(L).$$

Proof. One may assume $S = R/I$ and $L = J/I$, where R is a polynomial ring over k and J is an ideal of R containing I. Set $g = \operatorname{ht}(J/I)$ and $r = \operatorname{ht}(I)$. Let $\mathfrak{p}$ be a prime ideal such that $J \subset \mathfrak{p}$ and $g = \operatorname{ht}(\mathfrak{p}/I)$. There is a saturated chain of prime ideals

$$I \subset \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_g = \mathfrak{p}.$$

Since $\mathfrak{p}_0$ is a minimal prime ideal of I using Proposition 1.3.11 one obtains $\dim(R/I) = \dim(R/\mathfrak{p}_0)$, that is, $r = \operatorname{ht}(\mathfrak{p}_0)$. Hence there is a saturated chain of prime ideals

$$\mathfrak{q}_0 = (0) \subset \mathfrak{q}_1 \subset \cdots \subset \mathfrak{q}_r = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_g = \mathfrak{p}.$$

As R is a catenary domain one obtains $\mathrm{ht}\,(\mathfrak{p}) = r + g$ and consequently $\mathrm{ht}\,(J) \leq r + g$. It is not hard to verify that this inequality is equivalent to the inequality

$$\dim(S) \leq \mathrm{ht}\,(L) + \dim(S/L).$$

To conclude the proof recall that the inequality

$$\dim(S) \geq \mathrm{ht}\,(L) + \dim(S/L)$$

holds in general. □

Corollary 2.2.4 *Let R be a positively graded polynomial ring over a field k and I a graded ideal of R. If R/I is Cohen-Macaulay, then I is unmixed.*

Proof. It follows from Proposition 1.3.11 and Proposition 2.2.3. □

Definition 2.2.5 Let k be a field and let S be a positively graded k-algebra. A set of homogeneous elements $\underline{\theta} = \{\theta_1, \ldots, \theta_d\}$ is called a *homogeneous system of parameters* (h.s.o.p for short) if $d = \dim(S)$ and $\mathrm{rad}\,(\underline{\theta}) = S_+$.

Corollary 2.2.6 *Let S be a positively graded algebra over a field k and $h_1, \ldots, h_d$ a homogeneous system of parameters for S. Then*

$$\dim S/(h_1, \ldots, h_i) = \dim(S) - i,$$

for $1 \leq i \leq \dim(S)$.

Proof. Set $\overline{S} = S/(h_1, \ldots, h_i)$ and $d = \dim(S)$. First note that the set of images of $h_{i+1}, \ldots, h_d$ in $\overline{S}$ generate an $\overline{\mathfrak{m}}$-primary ideal, where $\mathfrak{m} = S_+$ is the irrelevant maximal ideal of S and $\overline{\mathfrak{m}} = \mathfrak{m}\overline{S}$. Hence

$$\dim(\overline{S}) \leq d - i$$

by the graded version of the dimension theorem (see Theorem 1.3.13). On the other hand if $\overline{\theta}_1, \ldots, \overline{\theta}_r$ is an homogeneous system of parameters for $\overline{S}$, then the sequence

$$\theta_1, \ldots, \theta_r, h_1, \ldots, h_i$$

generates an $\mathfrak{m}$-primary ideal of S, and applying the dimension theorem once again one has

$$d \leq r + i = \dim(\overline{S}) + i.$$

Thus $\dim(\overline{S}) = d - i$. □

Proposition 2.2.7 *Let S be a positively graded algebra over a field k and $\underline{\theta} = \theta_1, \ldots, \theta_d$ an h.s.o.p for S. Then S is Cohen-Macaulay if and only if $\underline{\theta}$ is a regular sequence.*

Proof. By Corollary 2.2.6 $\dim S/(\theta_1,\ldots,\theta_i) = d - i$. If S is C-M, then by Proposition 2.2.3 one has $\mathrm{ht}\,(\theta_1,\ldots,\theta_i) = i$. Assume $\theta_1,\ldots,\theta_{i-1}$ is a regular sequence. Next we show that θ_i is regular on $A = S/(\theta_1,\ldots,\theta_{i-1})$. Otherwise if θ_i is a zero divisor of A, then θ_i belongs to some minimal prime $\mathfrak{p}$ of $(\theta_1,\ldots,\theta_{i-1})$. Since $\mathrm{ht}\,(\mathfrak{p}) \leq i-1$ (see Theorem 1.3.14) we obtain $\mathrm{ht}\,(\theta_1,\ldots,\theta_i) \leq i-1$, which is impossible. Conversely if $\underline{\theta}$ is a regular sequence, then $\mathrm{depth}(S) = d$ and S is C-M. □

Proposition 2.2.8 *Let S be a positively graded algebra over a field k. If S is Cohen-Macaulay and $\underline{\theta} = \theta_1,\ldots,\theta_q$ is a regular sequence of homogeneous elements in S_+, then $S/(\underline{\theta})$ is Cohen-Macaulay.*

Proof. It suffices to prove the case $q = 1$, which is a direct consequence of Lemma 1.3.10. □

Lemma 2.2.9 *Let $V \neq \{0\}$ be a vector space over an infinite field K. Then V is not a finite union of proper subspaces of V.*

Proof. We proceed by contradiction. Assume that there are proper subspaces $V_1,\ldots,V_m$ of V such that

$$V = \bigcup_{i=1}^{m} V_i,$$

where m is the least positive integer with this property. Let

$$v_1 \in V_1 \setminus (V_2 \cup \cdots \cup V_m) \text{ and } v_2 \in V_2 \setminus (V_1 \cup V_3 \cup \cdots \cup V_m).$$

Pick $m+1$ distinct nonzero scalars $k_0,\ldots,k_m$ in K. Consider the vectors

$$\beta_i = v_1 - k_i v_2$$

for $i = 0,\ldots,m$. By the pigeon-hole principle there are distinct vectors β_r, β_s in V_j for some j. Since $\beta_r - \beta_s \in V_j$ we get $v_2 \in V_j$. Thus $j = 2$ by the choice of v_2. To finish the proof observe that $\beta_r \in V_2$ imply $v_1 \in V_2$, which contradicts the choice of v_1. □

Proposition 2.2.10 *Let $R = k[x_1,\ldots,x_n]$ be a polynomial ring over a field k with a positive grading and let I be a graded ideal of R. Then there are homogeneous polynomials $h_1,\ldots,h_d$ in R_+ such that*

$$\dim R/(I,h_1,\ldots,h_i) = d - i, \quad \textit{for } i = 1,\ldots,d,$$

where $\dim(R/I) = d$. If k is an infinite field and $\deg(x_i) = 1$ for all i, then $h_1,\ldots,h_d$ can be chosen in R_1.

Proof. Assume $d > 0$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the set of minimal primes of I of height $g = \operatorname{ht}(I)$. We claim that there is a homogeneous polynomial h_1 not in $\cup_{i=1}^{r} \mathfrak{p}_i$. To show it we use induction on r. Since $\mathfrak{p}_1 \cdots \mathfrak{p}_{r-1} \not\subset \mathfrak{p}_r$ and because $\mathfrak{p}_i$ is graded we can pick $f \in \mathfrak{p}_1 \cdots \mathfrak{p}_{r-1} \cap R_{d_1}$ and $f \notin \mathfrak{p}_r$, $d_1 > 0$. By induction there is $g \in R_{d_2}$ and $g \notin \cup_{i=1}^{r-1} \mathfrak{p}_i$, $d_2 > 0$. Assume $R_i \subset \cup_{i=1}^{r} \mathfrak{p}_i$ for all $i > 0$, hence $g \in \mathfrak{p}_r$. To complete the proof of the claim consider $h = f^{d_2} - g^{d_1}$ to derive a contradiction.

Note $\dim(R/I) > \dim R/(I, h_1)$, by the choice of h_1. Hence a repeated application of the claim rapidly yields a sequence $h_1, \ldots, h_s$ of homogeneous polynomials in R_+ with $s \leq d$ and such that $\operatorname{ht}(I, h_1, \ldots, h_s) = \dim(R)$. Therefore by Theorem 1.3.13 one concludes $d = s$, as required.

If k is infinite and $\deg(x_i) = 1$ for all i, then there is h_1 in R_1 and not in $\cup_{i=1}^{r} \mathfrak{p}_i$; for otherwise one has $R_1 = \cup_{i=1}^{r} (\mathfrak{p}_i)_1$ and since R_1 cannot be a finite union of proper subspaces (see Lemma 2.2.9) we derive $R_+ = \mathfrak{p}_i$ for some i, which is impossible. Thus one may proceed as above to get the required sequence. □

Theorem 2.2.11 (Noether normalization lemma) *Let k be a field and let R be a polynomial ring with coefficients in k. If R is positively graded and I is a graded ideal of R, then there are homogeneous polynomials $h_1, \ldots, h_d$ in R_+, with $d = \dim(R/I)$, and a natural embedding*

$$A = k[h_1, \ldots, h_d] \xhookrightarrow{\varphi} R/I$$

such that R/I is a finitely generated A-module. Moreover if k is infinite and $\deg(x_i) = 1$ for all i, then $h_1, \ldots, h_d$ can be chosen in R_1.

Proof. Let $R = k[x_1, \ldots, x_n]$ and set $\mathcal{B}_i = \{x^a | \deg(x^a) = i\}$. According to Proposition 2.2.10 there are homogeneous polynomials $h_1, \ldots, h_d$ in $\mathfrak{m} = R_+$ with $\operatorname{rad}(I, h_1, \ldots, h_d) = \mathfrak{m}$. Hence there is r such that

$$\mathfrak{m}^r \subset (I, h_1, \ldots, h_d).$$

Note that $\mathcal{B}_i \subset (I, h_1, \ldots, h_d)$ for $i \geq s = r\delta$, where δ is the maximum of the degrees of $x_1, \ldots, x_n$. Set $\mathcal{B} = \cup_{i=1}^{s} \mathcal{B}_i$. Using induction on i we now show that

$$\mathcal{B}_i \subset J = I + \mathcal{B}k[h_1, \ldots, h_d], \ \forall\, i \geq s.$$

The containment is clear for $i \leq s$. Assume $i > s$ and take x^a in $\mathcal{B}_i$. Let $f_1, \ldots, f_q$ be a homogeneous generating set for I. Since $(I, h_1, \ldots, h_d)$ is graded we can write

$$x^a = \sum_{j=1}^{q} b_j f_j + \sum_{j=1}^{d} c_j h_j,$$

where $b_1, \ldots, b_q$ and $c_1, \ldots, c_d$ are homogeneous and $\deg(c_j h_j) = i$. Hence all the monomials in the support of c_j have degree less than i. Thus by induction one gets c_j in J for all j and consequently $x^a \in J$.

Observe that the image of $\mathcal{B}$ in R/I generates R/I as an A-module. Next we verify that the canonical map φ form A to R/I is injective. Since R/I is integral over $\varphi(A)$ by Proposition 1.4.12 one obtains that d is equal to $\dim \varphi(A)$. Using $A/\ker(\varphi) \simeq \varphi(A)$ and the fact that A has dimension at most d we get $\ker(\varphi) = 0$. If k is infinite and $\deg(x_i) = 1$, then according to Proposition 2.2.10 one can choose $h_1, \ldots, h_d$ of degree one. □

Definition 2.2.12 Let R be a positively graded polynomial ring over a field k and $I \neq R$ a graded ideal. A *homogeneous Noether normalization* of $S = R/I$ is an integral extension

$$k[h_1, \ldots, h_d] \hookrightarrow S,$$

where $h_1, \ldots, h_d$ are homogeneous polynomials in R_+ and $d = \dim(S)$.

Proposition 2.2.13 (Stanley) *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and let I be a monomial ideal with $d = \dim(R/I)$. Then*

$$A = k[\sigma_1, \ldots, \sigma_d] \hookrightarrow S = R/I$$

is a Noether normalization, where σ_i is the ith symmetric polynomial.

Proof. It suffices to prove that S is integral over A. Let $\overline{x}_i = x_i + I$. Since $d = \dim(R/\mathrm{rad}(I))$, one has $\sigma_i \in \mathrm{rad}(I)$ for $i > d$. Hence from the equality

$$(x - x_1) \cdots (x - x_n) = x^n - \sigma_1 x^{n-1} + \cdots + (-1)^n \sigma_n$$

one has

$$\overline{x}_i^n - \overline{\sigma}_1 \overline{x}_i^{n-1} + \cdots + (-1)^d \overline{\sigma}_d \overline{x}_i^{n-d} = \overline{h},$$

where $\overline{h}$ is nilpotent in S. Hence if $h^m \in I$, raising the last equality to the m*th* power yields that $\overline{x}_i$ is integral over A. □

Proposition 2.2.14 *Let $R = k[x_1, \ldots, x_n]$ be a positively graded polynomial ring over a field k and let $I \neq R$ be a graded ideal of R. If $S = R/I$ is a Cohen-Macaulay ring and*

$$A = k[h_1, \ldots, h_d] \hookrightarrow S$$

is a homogeneous Noether normalization of S, then

$$S = A\overline{x}^{\beta_1} \oplus \cdots \oplus A\overline{x}^{\beta_m}$$

for any set of monomials $\mathcal{B} = \{x^{\beta_1}, \ldots, x^{\beta_m}\}$ whose image in the Artinian ring $\overline{S} = R/(I, h_1, \ldots, h_d)$ is a k-vector space basis of $\overline{S}$.

Proof. Set $\mathbb{M}_i = \{x^\alpha \in R | \deg(x^\alpha) = i\}$. First we show that the image of $\mathcal{B}$ generate S as an A-module. It suffices to prove $\mathbb{M}_i \subset J = I + A\mathcal{B}$ for all $i \geq 0$. Let $x^\alpha \in \mathbb{M}_i$. There are homogeneous polynomials $\mu_1, \ldots, \mu_d$ in R and $\lambda_1, \ldots, \lambda_m$ in k such that

$$x^\alpha = f + \sum_{j=1}^{m} \lambda_j x^{\beta_j} + \sum_{j=1}^{d} \mu_j h_j,$$

where $f \in R_i \cap I$ and $\deg(\mu_j h_j) = i$ if $\mu_j \neq 0$. Since $\deg \mu_j < i$, by induction one obtains that $\mu_j \in J$ for all j and consequently $x^\alpha \in J$.

We claim that if $c_1 x^{\beta_1} + \cdots + c_m x^{\beta_m}$ belongs to $(I, h_1, \ldots, h_{i-1})$ for some $2 \leq i \leq d$ and $c_1, \ldots, c_m$ in $k[h_i, \ldots, h_d]$, then $c_j = 0$ for all j. To prove the claim note that $h_1, \ldots, h_d$ is a regular sequence (see Proposition 2.2.7) and use descending induction on i starting with $i = d$.

Next we show

$$S = A\overline{x}^{\beta_1} \oplus \cdots \oplus A\overline{x}^{\beta_m}.$$

Assume

$$\sum_{i=1}^{m} a_i x^{\beta_i} \in I,$$

for some $a_1, \ldots, a_m$ in A. If $a_\ell \neq 0$ for some ℓ, write

$$a_i = \sum_{j=0}^{s_i} a_{ij} h_1^j,$$

where a_{ij} are in $k[h_2, \ldots, h_d]$. One may assume $a_{r0} \neq 0$ for some r; otherwise we may factor out h_1 to some power and apply that h_1 is regular on S. Hence $a_{10}x^{\beta_1} + \cdots + a_{m0}x^{\beta_m}$ is in (I, h_1), and applying the claim with $i = 2$ we derive $a_{r0} = 0$, which is a contradiction. Therefore $a_j = 0$ for all j, as required. □

Lemma 2.2.15 *Let k be an infinite field and let*

$$S = \bigoplus_{i=0}^{\infty} S_i$$

be a standard k-algebra. If S is Cohen–Macaulay, then there exists a h.s.o.p $\underline{\theta} = \{\theta_1, \ldots, \theta_d\}$ such that $\underline{\theta}$ is a regular sequence and $\theta_i \in S_1$ for all i.

Proof. Let $\operatorname{Ass}(S) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. Notice that $\operatorname{Ass}(S) = \operatorname{Min}(S)$, because S is Cohen–Macaulay. We may assume $d > 0$, otherwise there is nothing to prove. If $S_1 \subset \mathcal{Z}(S) = \cup_{i=1}^{s} \mathfrak{p}_i$, then $S_1 = \cup_{i=1}^{s} (\mathfrak{p}_i)_1$. Since k is infinite, by Lemma 2.2.9, we obtain that $s = 1$ and $\mathfrak{p}_1 = S_+$, that is, $\dim(S) = 0$, which is a contradiction. Hence there is $\theta_1 \in S_1$ which is regular on S.

By Proposition 2.2.3 $\dim S/(h_1) = d-1$. Applying the depth lemma (see Lemma 1.3.9) to the exact sequence

$$0 \longrightarrow S(-1) \xrightarrow{\theta_1} S \longrightarrow S/(\theta_1) \longrightarrow 0$$

yields depth $S/(\theta_1) = d-1$. Altogether $S/(\theta_1)$ is C-M and the result follows by induction. □

Lemma 2.2.16 *Let R be an $\mathbb{N}$-graded polynomial ring over a field k and I a graded ideal of R of height g. If I is a complete intersection, then there are homogeneous polynomials $f_1, \ldots, f_g$ such that $I = (f_1, \ldots, f_g)$.*

Proof. Let $h_1, \ldots, h_g$ be a regular sequence generating the ideal I. Using the graded version of Corollary 1.1.30 one has

$$r = \nu(I) = \dim_k(I/\mathfrak{m}I) \leq g,$$

where $\mathfrak{m} = R_+$. On the other hand, since I is graded, there are homogeneous polynomials $f_1, \ldots, f_r$ generating I and such that $\{f_i + \mathfrak{m}I\}_{i=1}^r$ is a k-basis for $I/\mathfrak{m}I$. By Theorem 1.3.14 one concludes $r = g$. □

Proposition 2.2.17 *Let R be a positively graded polynomial ring over a field k and let I be a graded ideal of R. If I is a complete intersection, then R/I is Cohen-Macaulay.*

Proof. By Lemma 2.2.16 the ideal I is generated by a sequence $f_1, \ldots, f_g$ consisting of homogeneous polynomials, where g denotes the height of I. According to Proposition 2.2.10 there is a sequence $h_1, \ldots, h_{n-g}$ consisting of homogeneous polynomials such that

$$\operatorname{rad}(f_1, \ldots, f_g, h_1, \ldots, h_{n-g}) = \mathfrak{m} = R_+.$$

Therefore $\{f_1, \ldots, f_g, h_1, \ldots, h_{n-g}\}$ is a homogeneous system of parameters for R and by Proposition 2.2.7 we derive that $f_1, \ldots, f_g$ is a regular sequence. Finally Lemma 1.3.10 yields that R/I is Cohen-Macaulay. □

Lemma 2.2.18 *Let $I, \mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be graded ideals of a polynomial ring R over a field k such that $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ are prime ideals. If $f \in \cup_{i=1}^m \mathfrak{p}_i$ for any $f \in I$ homogeneous, then $I \subset \mathfrak{p}_i$ for some i.*

Proof. By induction on m. If $m = 2$ and $I \not\subset \mathfrak{p}_i$ for $i = 1, 2$, then for each i pick a homogeneous polynomial $f_i \in I \setminus \mathfrak{p}_i$ of degree u_i. Since $f_1^{u_2} + f_2^{u_1}$ is homogeneous we readily derive a contradiction. Hence $I \subset \mathfrak{p}_i$ for some i.

Let G be the set of homogeneous elements in I and suppose $G \subset \cup_{i=1}^m \mathfrak{p}_i$. One may assume $\mathfrak{p}_i \not\subset \mathfrak{p}_j$ for $i \neq j$. If $I \not\subset \mathfrak{p}_i$ for all i, then by induction there is f_1 in $G \setminus \cup_{i=1}^{m-1} \mathfrak{p}_i$. On the other hand since $I\mathfrak{p}_1 \cdots \mathfrak{p}_{m-1} \not\subset \mathfrak{p}_m$ (see Exercise 1.1.39) there is a homogeneous polynomial f_2 in $I\mathfrak{p}_1 \cdots \mathfrak{p}_{m-1}$ and f_2 not in $\mathfrak{p}_m$. Since $f_1^{u_2} + f_2^{u_1}$, $\deg(f_i) = u_i$, is homogeneous we rapidly derive a contradiction. Thus $I \subset \mathfrak{p}_i$ for some i. □

Proposition 2.2.19 *Let R be a polynomial ring over a field k and let I be a graded ideal of height r, then there is a regular sequence $f_1, \ldots, f_r$ in I of homogeneous polynomials.*

Proof. We will proceed by induction. Assume that $f_1, \ldots, f_s$ is a regular sequence of homogeneous polynomials in I , where $s < r$. Note that the ideal $(f_1, \ldots, f_s)$ is Cohen-Macaulay by Proposition 2.2.17, and hence it is unmixed by Corollary 2.2.4. If all the homogeneous polynomials in I belong to $\mathcal{Z}(R/(f_1, \ldots, f_s))$, then by Lemma 2.2.18 the ideal I must be contained in an associated prime of $(f_1, \ldots, f_s)$ and consequently $\operatorname{ht}(I) \leq s$, which is impossible. Thus there is a homogeneous polynomial f_{s+1} in I such that f_{s+1} is regular modulo $(f_1, \ldots, f_s)$. □

Tensor product of affine algebras Let A, B be two affine algebras over a field k and consider presentations $A \simeq R_1/I_1$, $B \simeq R_2/I_2$, where $R_1 = k[\mathbf{x}]$, $R_2 = k[\mathbf{y}]$ are polynomial rings in disjoint sets of variables and I_i is an ideal of R_i. Set $R = k[\mathbf{x}, \mathbf{y}]$ and $I = I_1 + I_2$. The map

$$R \longmapsto R_1/I_1 \otimes_k R_2/I_2, \quad x_i \mapsto x_i \otimes 1, \quad y_i \mapsto 1 \otimes y_i,$$

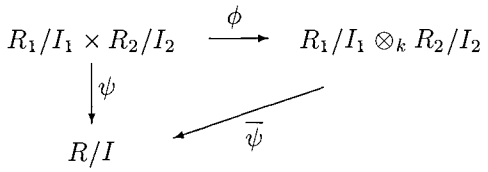
induces a k-algebra homomorphism:

$$\varphi\colon R/I \to R_1/I_1 \otimes_k R_2/I_2, \quad \varphi(\overline{f}(\mathbf{x})\overline{g}(\mathbf{y})) = \overline{f}(\mathbf{x}) \otimes \overline{g}(\mathbf{y}).$$

On the other hand there is a k-bilinear map

$$\psi\colon R_1/I_1 \times R_2/I_2 \to R/I,$$

given by multiplication $\psi(\overline{f}, \overline{g}) = \overline{fg}$. By the universal property of the tensor product there is a map $\overline{\psi}$ that makes commutative the following diagram



where ϕ is the canonical map and $\psi = \overline{\psi}\phi$. As a consequence $\overline{\psi}$ is the inverse of φ. Thus we have proved:

Proposition 2.2.20 *If A and B are two affine algebras over a field k with presentations $A \simeq k[\mathbf{x}]/I_1$ and $B \simeq k[\mathbf{y}]/I_2$, then*

$$A \otimes_k B \simeq k[\mathbf{x}, \mathbf{y}]/(I_1 + I_2).$$

Theorem 2.2.21 *If A and B are two standard algebras over a field k, then*

$$\operatorname{depth}(A \otimes_k B) = \operatorname{depth}(A) + \operatorname{depth}(B).$$

Proof. Pick a regular sequence $\underline{g} = g_1, \ldots, g_r$ (resp. $\underline{h} = h_1, \ldots, h_s$) on $A = k[\mathbf{x}]/I_1$ (resp. $B = k[\mathbf{y}]/I_2$) with $\underline{g} \subset (\mathbf{x})$ (resp. $\underline{h} \subset (\mathbf{y})$) and such that $\underline{g}$ (resp. $\underline{h}$) consist of forms, where r is the depth of A and s is the depth of B. As B is a faithfully flat k-algebra, applying the functor $(\cdot) \otimes_k B$ to the injective map

$$0 \longrightarrow k[\mathbf{x}]/(I_1, g_1, \ldots, g_{i-1}) \xrightarrow{g_i} k[\mathbf{x}]/(I_1, g_1, \ldots, g_{i-1})$$

gives a natural commutative diagram

$$\begin{array}{ccc} k[\mathbf{x}]/(I_1, g_1, \ldots, g_{i-1}) \otimes_k B & \xrightarrow{g_i \otimes 1} & k[\mathbf{x}]/(I_1, g_1, \ldots, g_{i-1}) \otimes_k B \\ \downarrow \simeq & & \downarrow \simeq \\ k[\mathbf{x}, \mathbf{y}]/(I_1, I_2, g_1, \ldots, g_{i-1}) & \xrightarrow{g_i} & k[\mathbf{x}, \mathbf{y}]/(I_1, I_2, g_1, \ldots, g_{i-1}) \end{array}$$

such that the map in the first row is injective. Since the vertical arrows are natural ismorphisms by Proposition 2.2.20, one concludes that the map in the second row is also injective. As a consequence $\underline{g}$ is a regular sequence on $k[\mathbf{x}, \mathbf{y}]/(I_1, I_2)$. Similarly if we tensor the injective map

$$0 \longrightarrow k[\mathbf{y}]/(I_2, h_1, \ldots, h_{i-1}) \xrightarrow{h_i} k[\mathbf{y}]/(I_2, h_1, \ldots, h_{i-1})$$

with the faithfully flat k-algebra $k[\mathbf{x}]/(I_1, \underline{g})$ it follows rapidly that $\underline{h}$ is a regular sequence on $k[\mathbf{x}, \mathbf{y}]/(I_1, I_2, \underline{g})$.

Altogether $\underline{g}, \underline{h}$ is a regular sequence on $k[\mathbf{x}, \mathbf{y}]/(I_1, I_2)$. To finish the proof note that $(\mathbf{x}, \mathbf{y})$ is an associated prime ideal of $k[\mathbf{x}, \mathbf{y}]/(I_1, I_2, \underline{g}, \underline{h})$ by Exercise 2.2.27, thus $r + s$ is the length of a maximal regular sequence in $(\mathbf{x}, \mathbf{y})$ and by Proposition 1.3.7 one obtains $\operatorname{depth}(A \otimes_k B) = r + s$. For an alternative proof see [115]. □

Corollary 2.2.22 *If A and B are two standard algebras over a field k, then $A \otimes_k B$ is Cohen-Macaulay if and only if A and B are Cohen-Macaulay.*

Proof. As the dimension is always greater or equal than the depth by Lemma 1.3.6, the result follows at once using Theorem 2.2.21 together with Exercise 2.1.14. □

A similar statement holds if one replaces Cohen-Macaulay by Gorenstein; indeed the type of the tensor products of two Cohen-Macaulay standard algebras is the product of their types [115].

Exercises

2.2.23 Let V be a vector space over an infinite field k and $W, V_1, \ldots, V_m$ vector subspaces of V. If $W \not\subset V_i$ for all i, prove $W \not\subset \cup_{i=1}^m V_i$.

Hint If $W \subset \cup_{i=1}^m V_i$, note $W = \cup_{i=1}^m (W \cap V_i)$ and use Lemma 2.2.9.

2.2.24 Let $I, I_1, \ldots, I_m$ be ideals of a ring R and let k be an infinite field. If $I \subset \cup_{i=1}^m I_i$ and k is a subring of R, then $I \subset I_i$ for some i.

Hint Note that $I, I_1, \ldots, I_m$ are vector spaces over k.

2.2.25 Let R be a polynomial ring over an infinite field k and I a graded ideal of height r. If I is minimally generated by forms of degree $p \geq 1$, prove that there are forms $f_1, \ldots, f_m$ of degree p in I such that $f_1, \ldots, f_r$ is a regular sequence and I is minimally generated by $f_1, \ldots, f_m$.

Hint Find a regular sequence $f_1, \ldots, f_r$ in $I_p = R_p \cap I$ and extend this sequence to a k-basis of I_p.

2.2.26 Let $R = k[\mathbf{x}]$ be a polynomial ring over a field k and $I \subset (\mathbf{x})$ an ideal. Note that k is an R-module with multiplication $f(x) \cdot \alpha = f(0)\alpha$, for $f(x) \in R$ and $\alpha \in k$. When is the canonical map

$$\varphi\colon k \longrightarrow R/I, \quad \alpha \stackrel{\varphi}{\longmapsto} \alpha + I$$

a homomorphism of R-modules?.

2.2.27 Let $A = k[\mathbf{x}]/I_1$ and $B = k[\mathbf{y}]/I_2$ be affine algebras over a field k. If $(\mathbf{x})$ (resp. $(\mathbf{y})$) is an associated prime of $k[\mathbf{x}]/I_1$ (resp. $k[\mathbf{y}]/I_2$), then $(\mathbf{x}, \mathbf{y})$ is as associated prime of $k[\mathbf{x}, \mathbf{y}]/(I_1, I_2)$.

Hint there are embeddings

$$k[\mathbf{x}]/(\mathbf{x}) \otimes_k k[\mathbf{y}]/(\mathbf{y}) \hookrightarrow k[\mathbf{x}]/(\mathbf{x}) \otimes_k B \text{ and } k[\mathbf{x}]/(\mathbf{x}) \otimes_k B \hookrightarrow A \otimes_k B.$$

2.2.28 Let $k[\mathbf{x}]$ be a polynomial ring and I an ideal (resp. graded). If $k \subset K$ is a field extension, then there is a natural (resp. graded) isomorphism of K-algebras:

$$K[\mathbf{x}]/IK[\mathbf{x}] \simeq k[\mathbf{x}]/I \otimes_k K.$$

2.2.29 Let A be an affine algebra over a field k and let $k \subset K$ be a field extension. Prove:

(a) $\dim(A) = \dim(A \otimes_k K)$.

(b) $\operatorname{depth}(A) = \operatorname{depth}(A \otimes_k K)$ if A is a standard algebra.

2.2.30 Let K be a field and let A be an affine K-algebra. Prove that A is Artinian if and only if $\dim_K(A) < \infty$.

2.3 Hilbert Nullstellensatz

Let k be a field and let $R = k[x_1, \ldots, x_n]$ be a polynomial ring, we define the *affine space* of dimension n over k, denoted by $\mathbb{A}_k^n$, to be the cartesian product $k^n = k \times \cdots \times k$.

Given an ideal I of R define the *zero set* or *variety* of I as

$$V(I) = \{\alpha \in k^n | \, f(\alpha) = 0, \forall f \in I\},$$

by the Hilbert basis theorem $V(I)$ is the zero locus of a finite collection of polynomials. Conversely for any $X \subset k^n$ define $I(X)$, the *ideal* of X, as the set of polynomials that vanish on X. An *affine variety* is the zero set of an ideal. The *dimension* of a variety X is the Krull dimension of its *coordinate ring* $R/I(X)$.

Proposition 2.3.1 *If $R = k[\mathbf{x}]$ is a polynomial ring over a field k, then*

(a) $V(I \cap J) = V(I) \cup V(J) = V(IJ)$, *for all ideals I and J of R.*

(b) $\cap V(I_\alpha) = V(\cup I_\alpha)$, *where $\{I_\alpha\}$ is any family of ideals of R.*

(c) $V(1) = \emptyset$ *and* $V(0) = \mathbb{A}_k^n$; *where n is the number of variables of R.*

Proof. It is left as an exercise. □

By the previous result the sets in the family

$$\tau = \{\mathbb{A}_k^n \setminus V(I) | \, I \subset R\}$$

are the open sets of a topology on $\mathbb{A}_k^n$, called the *Zariski topology.*

Definition 2.3.2 An affine variety $X \subset k^n$ is *reducible* if there are affine varieties $X_1 \neq X$ and $X_2 \neq X$ such that $X = X_1 \cup X_2$, otherwise X is *irreducible.*

Theorem 2.3.3 *Let k be a field and X an affine variety of $\mathbb{A}_k^n$, then X is irreducible if and only if $I(X)$ is a prime ideal.*

Proof. See [120, Chapter 1]. □

Proposition 2.3.4 *If $X \subset \mathbb{A}_k^n$ is an affine variety over a field k, then there are unique irreducible affine varieties $X_1, \ldots, X_r$ in $\mathbb{A}_k^n$ such that*

$$X = \bigcup_{i=1}^{r} X_i$$

and $X_i \not\subset X_j$ for all $i \neq j$.

Proof. See [141, Proposition 1.5]. □

The irreducible affine varieties $X_1, \ldots, X_r$ of Proposition 2.3.4 are called the *irreducible components* of X.

Theorem 2.3.5 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and let $\mathfrak{m}$ be a maximal ideal of R. If k is algebraically closed, then there are $a_1, \ldots, a_n \in k$ such that*

$$\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n).$$

Proof. According to Corollary 2.1.8 there is an integral extension

$$k[h_1, \ldots, h_d] \hookrightarrow R/\mathfrak{m},$$

where $d = \dim(R/\mathfrak{m}) = 0$. Hence the canonical map $\varphi: k \to R/\mathfrak{m}$ is an isomorphism because k is algebraically closed. To complete the proof choose $a_i \in k$ so that $\varphi(a_i) = \overline{a}_i = \overline{x}_i = \varphi(x_i)$. □

Proposition 2.3.6 *If I is an ideal of a ring R and $f \in R$, then $f \in \sqrt{I}$ if and only if $(I, 1 - tf) = R[t]$, where t is a new variable.*

Proof. $\Rightarrow$) Let f be an element in $\sqrt{I}$. If $(I, 1 - tf) \neq R[t]$, take a prime ideal $\mathfrak{p}$ containing $(I, 1 - tf)$. Since f must be in $\mathfrak{p}$, because f^n is in I for some $n > 0$, one concludes $1 \in \mathfrak{p}$ which is impossible.

$\Leftarrow$) If $1 = a_1 f_1 + \cdots + a_q f_q + a_{q+1}(1 - tf)$, where $f_i \in I$, $f \in R$ and $a_i \in R[t]$. Set $z = 1/t$ and note $1 - tf = (z - f)/t$, multiplying the first equation by z^m, with m large enough, one derives an equality

$$z^m = b_1 f_1 + \cdots + b_q f_q + b_{q+1}(z - f),$$

where $b_i \in R[z]$. As $R[z]$ is a polynomial ring over R, making $z = f$ gives $f^m \in I$ and $f \in \sqrt{I}$. □

Theorem 2.3.7 (Hilbert Nullstellensatz) *Let R be a polynomial ring over an algebraically closed field k and let I be an ideal of R, then*

$$I(V(I)) = \sqrt{I}.$$

Proof. One clearly has $I \subset I(V(I))$, hence $\sqrt{I} \subset I(V(I))$ because $I(V(I))$ is a radical ideal. For the other containment take $f \in I(V(I))$ and consider the ideal $J = (I, 1 - ft)$ of the ring $R[t]$, where $R = k[x_1, \ldots, x_n]$ and t is a new variable. Next we show the equality $J = R[t]$. If $J \neq R[t]$ by Theorem 2.3.5 one has

$$(I, 1 - ft) \subset (x_1 - a_1, \ldots, x_n - a_n, t - a_{n+1}), \quad a_i \in k.$$

Hence $\beta \in V(I)$, where $\beta = (a_1, \ldots, a_n)$. Using that $f \in I(V(I)$ gives $f(\beta) = 0$, but this is impossible because $1 - f(\beta)a_{n+1} = 0$. Hence $J = R[t]$. Thus $f \in \sqrt{I}$ by Proposition 2.3.6. □

Corollary 2.3.8 *Let $X = V(f_1, \ldots, f_q)$ be an affine variety, defined by polynomials $f_1, \ldots, f_q$ in n variables over an algebraically closed field k. Then $q \geq n - \dim(X)$.*

Proof. By the Nullstellensatz $I(X) = \operatorname{rad}(f_1, \ldots, f_q)$. Let $\mathfrak{p}$ be a minimal prime of $(f_1, \ldots, f_q)$ of height $n - \dim(X)$. Applying Theorem 1.3.14 we get $n - \dim(X) \leq q$. □

Proposition 2.3.9 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and let $\mathfrak{m}$ be a maximal ideal of R. If k is uncountable, then $R/\mathfrak{m}$ is a finite extension of k.*

Proof. Assume $\overline{x}_i$ is not algebraic over k, where, as usual, $\overline{r}$ denotes $r + \mathfrak{m}$ for any r in R. For simplicity we set $i = 1$. As $R/\mathfrak{m}$ has a countable basis consisting of monomials, one has that the (uncountable) set defined by

$$A = \left\{ \frac{\overline{1}}{(\overline{x}_1 - \overline{c})} \,\middle|\, c \in k \setminus \{0\} \right\}$$

is linearly dependent. Thus $\exists\ c, c_1, \ldots, c_q$ in k distinct and $\lambda_1, \ldots, \lambda_q$ in $k \setminus \{0\}$ such that:

$$\frac{\overline{1}}{(\overline{x}_1 - \overline{c})} = \sum_{i=1}^{q} \frac{\overline{\lambda}_i}{(\overline{x}_1 - \overline{c}_i)}.$$

Consider the polynomial

$$f(x) = \prod_{i=1}^{q}(x - \overline{c}_i) - (x - \overline{c}) \sum_{j=1}^{q} \prod_{i \neq j} (x - \overline{c}_i)\overline{\lambda}_i.$$

Since $f(\overline{x}_1) = 0$ and $\overline{x}_1$ is not algebraic over k we conclude that f is the zero polynomial, which is impossible because $f(\overline{c}) = \prod_{i=1}^{q}(c - c_i) \neq 0$. □

Exercises

2.3.10 Let I and J be two ideals in a polynomial ring R over a field k. If $\operatorname{rad}(I) = \operatorname{rad}(J)$, prove that $V(I) = V(J)$.

2.3.11 If k is a field and $X \subset \mathbb{A}_k^n$, then $X \subset V(I(X))$ with equality if X is an affine variety.

2.3.12 Let J be an ideal of a polynomial ring R over a field k. Prove that $J \subset I(V(J))$ with equality if J is the ideal of an affine variety.

2.3.13 Let k be an algebraically closed field. Prove that there is a one to one correspondence between affine varieties (resp. irreducible varieties) in k^n and radical ideals (resp. prime ideals) in the polynomial ring $k[x_1, \ldots, x_n]$.

2.4 Gröbner bases

In this section we review some basic facts and definitions on Gröbner bases. The reader may consult [1, 74, 243] for a detailed discussion of Gröbner bases and for the missing proofs of this section.

Let k be a field and let $R = k[x_1, \ldots, x_n]$ be a polynomial ring. We set

$$x^a = x_1^{a_1} \cdots x_n^{a_n} \text{ for } a = (a_1, \ldots, a_n) \in \mathbb{N}^n.$$

The monomials in

$$\mathcal{P}^n = \{x^a \mid a \in \mathbb{N}^n\}$$

are called the *terms*.

Definition 2.4.1 A total order $>$ of $\mathcal{P}^n$ is called a *monomial order* or *term order* if

(a) $p \geq 1$ for all $p \in \mathcal{P}^n$, and

(b) for all $p, q, r \in \mathcal{P}^n$, $q > p$ implies $qr > pr$.

Two examples of monomial orders in $\mathcal{P}^n$ are the *lexicographical order* or *lex order* defined as $x^b > x^a$ iff the last nonzero entry of $b - a$ is positive, and the *reverse lexicographical order* or *revlex order* given by $x^b > x^a$ iff the last nonzero entry of $b - a$ is negative.

In the sequel we assume that a monomial order for $\mathcal{P}^n$ has been fixed. Let f be a nonzero polynomial in R. One can write

$$f = \sum_{i=1}^{r} a_i M_i,$$

with $a_i \in k^* = k \setminus \{0\}$, $M_i \in \mathcal{P}^n$ and $M_1 > \cdots > M_r$. The *leading term* M_1 of f is denoted by $\mathrm{lt}(f)$. The *leading coefficient* a_1 of f and its *leading monomial* $a_1 M_1$ are denoted by $\mathrm{lc}(f)$ and $\mathrm{lm}(f)$ respectively.

Definition 2.4.2 Let I be an ideal of R. The *initial ideal* of I, denoted by $\mathrm{in}(I)$, is given by

$$\mathrm{in}(I) = (\{\mathrm{lt}(f) \mid f \in I\}).$$

Lemma 2.4.3 (Dickson) *If $\{M_i\}_{i=1}^{\infty}$ is a sequence in $\mathcal{P}^n$, then there is an integer k so that M_i is a multiple of some term in the set $\{M_1, \ldots, M_k\}$ for every $i > k$.*

Proof. We consider the ideal $I \subset k[x_1, \ldots, x_n]$ generated by $\{M_i\}_{i=1}^{\infty}$. By the Hilbert basis theorem I is finitely generated. It is readily seen that I can be generated by a finite set of terms $M_1, \ldots, M_k$. Hence for each $i > k$, there is $1 \leq j \leq k$ such that M_i is a multiple of M_j. □

Definition 2.4.4 Let $F = \{f_1, \ldots, f_q\} \subset R \setminus \{0\}$ be a set of polynomials in R. One says that f *reduces* to g *modulo* F, denoted $f \to_F g$, if

$$g = f - (au/\mathrm{lc}(f_i))f_i$$

for some $f_i \in F$, $u \in \mathcal{P}^n$, $a \in k^*$ such that $a \cdot u \cdot \mathrm{lt}(f_i)$ occurs in f with coefficient a.

Proposition 2.4.5 *The reduction relation "$\longrightarrow_F$" is Noetherian, that is, any sequence of reductions*

$$g_1 \longrightarrow_F g_2 \longrightarrow_F \cdots \longrightarrow_F g_i \longrightarrow_F \cdots$$

is stationary.

Proof. Notice that at the ith step of the reduction some term of g_i is replaced by terms of lower degree. Therefore if the sequence above is not stationary, then there is a never ending decreasing sequence of terms in $\mathcal{P}^n$, but this is impossible according to Dickson's lemma. □

Proposition 2.4.6 (Division algorithm) *If $f, f_1, \ldots, f_q$ are polynomials in R, then f can be written as*

$$f = a_1 f_1 + \cdots + a_q f_q + r,$$

where $a_i, r \in R$ and either $r = 0$ or $r \neq 0$ and no term of r is divisible by one of $\mathrm{lt}(f_1), \ldots, \mathrm{lt}(f_q)$. Furthermore if $a_i f_i \neq 0$, then $\mathrm{lt}(f) \geq \mathrm{lt}(a_i f_i)$.

Definition 2.4.7 The polynomial r in the division algorithm is called a *remainder* of f with respect to $F = \{f_1, \ldots, f_q\}$.

Definition 2.4.8 Let $I \neq (0)$ be an ideal of R and let $F = \{f_1, \ldots, f_r\}$ be a subset of I. The set F is called a *Gröbner basis* of I if

$$\mathrm{in}(I) = (\mathrm{lt}(f_1), \ldots, \mathrm{lt}(f_r)).$$

Definition 2.4.9 A Gröbner basis $F = \{f_1, \ldots, f_r\}$ of an ideal I is called a *reduced Gröbner basis* for I if:

(i) $\mathrm{lc}(f_i) = 1$ $\forall i$, and

(ii) none of the terms occurring in f_i belongs to $\mathrm{in}(F \setminus \{f_i\})$ $\forall i$.

Note that by the Hilbert basis theorem a Gröbner basis of I always exist. A nice feature of a reduced Gröbner basis is its uniqueness.

Definition 2.4.10 Let $f, g \in R$. The *S-polynomial* of f and g is given by

$$S(f,g) = \frac{[\mathrm{lt}(f), \mathrm{lt}(g)]}{\mathrm{lm}(f)} f - \frac{[\mathrm{lt}(f), \mathrm{lt}(g)]}{\mathrm{lm}(g)} g,$$

where $[f, g] = \mathrm{lcm}(f, g)$.

Given a set of generators of a polynomial ideal one can determine a Gröbner basis using the next fundamental procedure:

Theorem 2.4.11 (Buchberger) If $F = \{f_1, \ldots, f_q\}$ is a set of generators of an ideal I of R, then one can construct a Gröbner basis for I using the following algorithm:

Input: F
Output: a Gröbner basis G for I
Initialization: $G := F$, $B := \{\{f_i, f_j\} | \, f_i \neq f_j \in G\}$
while $B \neq \emptyset$ do
  pick any $\{f, g\} \in B$
  $B := B \setminus \{\{f, g\}\}$
  $r :=$ remainder of $S(f, g)$ with respect to G
  if $r \neq 0$ then
    $B := B \cup \{\{r, h\} | \, h \in G\}$
    $G := G \cup \{r\}$

Proof. See [1, 53, 243]. □

Proposition 2.4.12 *Let I be an ideal of R and let $F = \{f_1, \ldots, f_q\}$ be a generating set for I. If*

$$\mathcal{B} = \{\overline{u} \,|\, u \in \mathcal{P}^n \text{ and } u \notin (\mathrm{lt}(f_1), \ldots, \mathrm{lt}(f_n))\},$$

then $\mathcal{B}$ is a basis for the k-vector space R/I.

Proof. First we show that $\mathcal{B}$ is a generating set for R/I. For that take $\overline{f} \in R/I$. Because "$\longrightarrow_F$" is Noetherian we can write

$$f = \sum_{i=1}^{q} a_i f_i + \sum_{i=1}^{r} \lambda_i u_i,$$

where $\lambda_i \in k^*$ and such that every u_i is a term which is not a multiple of any of the terms $\mathrm{lt}(f_j)$. Accordingly $\overline{u}_i$ is in $\mathcal{B}$ for all i and $\overline{f}$ is a linear combination of the $\overline{u}_i$'s.

To prove that $\mathcal{B}$ is linearly independent assume $h = \sum_{i=1}^{s} \lambda_i u_i \in I$, where $u_i \in \mathcal{B}$ and $\lambda_i \in k$. We must show $h = 0$. If $h \neq 0$, then we can label the u_i's so that $u_1 > \cdots > u_s$ and $\lambda_1 \neq 0$. Hence $\mathrm{lt}(h) = \lambda_1 u_1 \in \mathrm{in}(I)$, but this is a clear contradiction because $\mathrm{in}(I) = (\mathrm{lt}(f_1), \ldots, \mathrm{lt}(f_q))$. Therefore $h = 0$, as required. □

Corollary 2.4.13 (Macaulay) *If I is a graded ideal of R, then R/I and $R/\mathrm{in}(I)$ have the same Hilbert function.*

Lemma 2.4.14 *Let f and g be two polynomials in R and let $F = \{f, g\}$. If $\mathrm{lm}(f)$ and $\mathrm{lm}(g)$ are relatively prime, then $S(f, g) \to_F 0$.*

Proof. Let $f = M_1 + \cdots + M_r$ and $g = N_1 + \cdots + N_s$, where M_i and N_i are monomials such that $M_1 > \cdots > M_r$, $N_1 > \cdots > N_s$. Set

$$h_i = S(f, g) - M_r g - \cdots - M_{r-i} g.$$

The monomials

$$\{N_1 M_2, \ldots, N_1 M_r, M_1 N_2, \ldots, M_1 N_s\}$$

are distinct because M_1 and N_1 are relatively prime, hence $S(f, g) \to_F h_0$. Since $N_1 M_{r-i-1} > M_k N_m$ for $k \geq r - i$ and $m \geq 2$, from the equality

$$\begin{array}{rcccccc} h_i = N_1 M_2 + \cdots + N_1 M_{r-i-1} & - & M_1 N_2 & - & \cdots & - & M_1 N_s \\ & - & M_r N_2 & - & \cdots & - & M_r N_s \\ & & & & \vdots & & \\ & - & M_{r-i} N_2 & - & \cdots & - & M_{r-i} N_s \end{array}$$

we obtain $S(f, g) \to_F h_0 \to_F h_1 \to_F \cdots \to_F h_{r-2}$. To complete the proof notice that $M_1 N_{i+1} > M_k N_m$, for $m \geq i + 1$. Hence using the identity

$$\begin{array}{rcccccc} h_{r-2} + N_2 f + \cdots + N_i f & = & - & M_1 N_{i+1} & - & \cdots & - & M_1 N_s \\ & & - & M_r N_{i+1} & - & \cdots & - & M_r N_s \\ & & & & & \vdots & & \\ & & - & M_2 N_{i+1} & - & \cdots & - & M_2 N_s \end{array}$$

we obtain

$$h_{r-2} \to_F h_{r-2} + N_2 f \to_F \cdots \to_F h_{r-2} + N_2 f + \cdots + N_s f = 0,$$

that is, $S(f, g) \to_F 0$. □

The next *criterion of Buchberger* will be used at several places to prove that a given set of polynomials form a Gröbner basis. Observe that this criterion is important in Buchberger algorithm.

Theorem 2.4.15 ([53]) *Let I be an ideal of R and let $F = \{f_1, \ldots, f_q\}$ be a set of generators of I, then F is a Gröbner basis for I if and only if*

$$S(f_i, f_j) \longrightarrow_F 0 \quad \textit{for all} \quad i \neq j.$$

Definition 2.4.16 Let I be an ideal of R generated by $F = \{f_1, \ldots, f_r\}$ and consider the homomorphism of R-modules

$$\varphi \colon R^r \longrightarrow I, \qquad e_i \longmapsto f_i,$$

where e_i is the i*th* unit vector. The kernel of φ, denoted by $Z(F)$, is called the *syzygy module* of I with respect to F.

Theorem 2.4.17 *Let $G = \{g_1, \ldots, g_r\}$ be a Gröbner basis and write*

$$S(g_i, g_j) = \frac{[\mathrm{lt}(g_i), \mathrm{lt}(g_j)]}{\mathrm{lm}(g_i)} \cdot g_i - \frac{[\mathrm{lt}(g_i), \mathrm{lt}(g_j)]}{\mathrm{lm}(g_j)} \cdot g_j = \sum_{k=1}^{r} a_{ijk} g_k,$$

where $a_{ijk} \in R$ and $\mathrm{lt}(S(g_i, g_j)) \geq \mathrm{lt}(a_{ijk} g_k)$ for all i, j, k. Then the set

$$\left\{ \frac{[\mathrm{lt}(g_i), \mathrm{lt}(g_j)]}{\mathrm{lm}(g_i)} \cdot e_i - \frac{[\mathrm{lt}(g_i), \mathrm{lt}(g_j)]}{\mathrm{lm}(g_j)} \cdot e_j - \sum_{k=1}^{r} a_{ijk} e_k \right\}_{1 \leq i,\, j \leq r}$$

generates the syzygy module of $I = (g_1, \ldots, g_r)$ with respect to G.

Proof. See [1, Theorem 3.4.1] and [75, Theorem 3.2]. □

Elimination of variables Let $k[z_1, \ldots, z_n]$ be a polynomial ring over a field k. A useful monomial order is the *elimination order* with respect to the variables $z_1, \ldots, z_i$. It is defined as

$$z^b > z^a$$

if and only if the total degree of z^b in the variables $z_1, \ldots, z_i$ is greater than that of z^a, or both degrees are equal, and the last nonzero component of $b - a$ is negative.

Theorem 2.4.18 *Let $S = k[x_1, \ldots, x_n, t_1, \ldots, t_q]$ be a polynomial ring over a field k with a term order such that terms in the x_i's are greater than terms in the t_i's. If I is an ideal of S with a Gröbner basis G, then $G \cap k[t_1, \ldots, t_q]$ is a Gröbner basis of $I \cap k[t_1, \ldots, t_q]$.*

Proof. Set $B = k[t_1, \ldots, t_q]$ and $I^c = I \cap B$. Assume $M \in \mathrm{in}(I^c)$. There is $f \in I^c$ with $\mathrm{lt}(f) = M$. Hence $M = \lambda\, \mathrm{lt}(g)$ for some $g \in G$, because G is a Gröbner basis. Since $M \in B$ and $x^\alpha > t^\beta$ for all α and β we obtain $g \in G \cap B$, that is, $M \in (\mathrm{in}(G \cap B))$. Thus

$$\mathrm{in}(I^c) = (\mathrm{in}(G \cap B)),$$

as required. □

Next we illustrate the previous result using the elimination order on S with respect to $x_1, \ldots, x_n$. Thus this order is given by

$$x^a t^c > x^b t^d$$

if and only if $\deg(x^a) > \deg(x^b)$, or both degrees are equal and the last nonzero entry of $(a, c) - (b, d)$ is negative.

Example 2.4.19 Let us assign $\deg(t_i) = 2$ and $\deg(x_j) = 1$ for all i, j. Using *Macaulay* [15] one rapidly obtains that the reduced Gröbner basis of the ideal

$$I = (t_1 - x_1x_2, t_2 - x_1x_3, t_3 - x_1x_4, t_4 - x_2x_3, t_5 - x_2x_4, t_6 - x_3x_4)$$

is equal to:

$$\begin{array}{lllll} t_3t_4 - t_1t_6, & t_2t_5 - t_1t_6, & x_4^2t_4 - t_5t_6, & x_4^2t_2 - t_3t_6, & x_4^2t_1 - t_3t_5, \\ x_3t_5 - x_4t_4, & x_3t_3 - x_4t_2, & x_3x_4 - t_6, & x_3^2t_1 - t_2t_4, & x_2t_6 - x_4t_4, \\ x_2t_3 - x_4t_1, & x_2t_2 - x_3t_1, & x_2x_4 - t_5, & x_2x_3 - t_4, & x_1t_6 - x_4t_2, \\ x_1t_5 - x_4t_1, & x_1t_4 - x_3t_1, & x_1x_4 - t_3, & x_1x_3 - t_2, & x_1x_2 - t_1, \\ x_3t_1t_6 - x_4t_2t_4. & & & & \end{array}$$

Hence $I \cap k[t_1, \ldots, t_6] = (t_3t_4 - t_1t_6, t_2t_5 - t_1t_6)$ by Theorem 2.4.18.

Proposition 2.4.20 *If I and J are two ideals of a polynomial ring R over a field k, then*

$$I \cap J = (t \cdot I, (1-t) \cdot J) \cap R,$$

where t is a new variable.

Proof. It is left as an exercise. □

Definition 2.4.21 Let I and J be two ideals of a ring R. One defines the *saturation* of I with respect to J as:

$$(I\colon J^\infty) = \bigcup_{i \geq 1} (I\colon_R J^i).$$

Proposition 2.4.22 *If I is an ideal of a ring R and $f \in R$, then*

$$(I\colon f^\infty) = \bigcup_{i \geq 1} (I\colon_R f^i) = (I, 1 - tf) \cap R,$$

where t is a new variable.

Proof. Let $g \in (I, 1 - tf) \cap R$. One can write

$$g = a_1f_1 + \cdots + a_qf_q + a_{q+1}(1 - tf),$$

where $f_i \in I$ and $a_i \in R[t]$. Making $t = 1/f$ in the last equation and multiplying by f^m, with m large enough, one derives an equality

$$gf^m = b_1f_1 + \cdots + b_qf_q,$$

where $b_i \in R$. Hence $gf^m \in I$ and $g \in (I\colon f^\infty)$.

Conversely let $g \in (I\colon f^\infty)$, hence there is $m \geq 1$ such that $gf^m \in I$. Since one can write

$$g = (1 - t^mf^m)g + t^mf^mg \text{ and } 1 - t^mf^m = (1 - tf)b,$$

for some $b \in R[t]$, one derives $g \in (I, 1 - tf) \cap R$. □

Lemma 2.4.23 ([98]) *Let R be a ring and let $x_1, \ldots, x_n \in R$. If I is an ideal of R, then the radical of I satisfies*

$$\sqrt{I} = \sqrt{(I\colon (x_1 \cdots x_n)^\infty)} \cap \sqrt{(I, x_1)} \cap \cdots \cap \sqrt{(I, x_n)}.$$

Proof. The left hand side is clearly a subset of the right hand side. To prove the other containment take a prime $\mathfrak{p}$ containing I, it suffices to prove that $\mathfrak{p}$ contains one of the ideals on the right hand side. If

$$\sqrt{(I\colon (x_1 \cdots x_n)^\infty)} \not\subset \mathfrak{p},$$

there is $f \in R \setminus \mathfrak{p}$ such that $f(x_1 \cdots x_n)^s \in I \subset \mathfrak{p}$ for some integer $s \geq 1$. Hence $x_i \in \mathfrak{p}$ for some i and $\mathfrak{p}$ contains $\sqrt{(I, x_i)}$, as required. □

Proposition 2.4.24 *Let $\mathfrak{p}$ be a prime ideal of a ring R and let $x_1, \ldots, x_n$ be a sequence in $R \setminus \mathfrak{p}$. If $I \subset \mathfrak{p}$ is an ideal, then $\sqrt{I} = \mathfrak{p}$ if and only if*

(a) $\mathfrak{p} = \sqrt{(I\colon (x_1 \cdots x_n)^\infty)}$, *and*

(b) $\sqrt{(I, x_i)} = \sqrt{(\mathfrak{p}, x_i)}$, *for all i.*

Proof. If J is any ideal of R and $y_1, \ldots, y_n \in R$, then by Lemma 2.4.23 the radical of J satisfies:

$$\operatorname{rad}(J) = \operatorname{rad}(J\colon (y_1 \cdots y_n)^\infty) \cap \operatorname{rad}(J, y_1) \cap \cdots \cap \operatorname{rad}(J, y_n).$$

Hence the result follows by applying this formula to I and $\mathfrak{p}$, together with the fact that $x_1 \cdots x_n$ is regular on $R/\mathfrak{p}$. □

The homogenization of an ideal Let $R = k[x_1, \ldots, x_n]$ and $S = R[w]$ be two polynomial rings over a field k. For $f \in R$ of degree e define

$$f^h = w^e f\left(\frac{x_1}{w}, \ldots, \frac{x_n}{w}\right),$$

that is, f^h is the homogenization of the polynomial f with respect to w. The *homogenization* of an ideal $I \subset R$ is the ideal I^h of $R[w]$ given by

$$I^h = (\{f^h \mid f \in I\}).$$

Proposition 2.4.25 *If I is an ideal of R, then $\sqrt{I^h} = \sqrt{I}^{\,h}$.*

Proof. For the inclusion $\sqrt{I}^{\,h} \subset \sqrt{I^h}$, we need only observe the equality

$$(f^h)^m = (f^m)^h$$

for $f \in R$ and $m \geq 0$.

Conversely let $g \in \sqrt{I^h}$. As $\sqrt{I^h}$ is a graded ideal one may assume that g is homogeneous. Write $g = w^s g_1^h$, where $g_1 \in R$. Since g^m is in I^h, it follows that $g_1 \in \sqrt{I}$, as required. □

Let $>$ be the elimination order on the monomials of $R[w]$ with respect to $x_1, \ldots, x_n, w$, this order extends the elimination order with respect to $x_1, \ldots, x_n$ on the monomials of R.

Proposition 2.4.26 *If I is an ideal of R generated by $F = \{f_1, \ldots, f_r\}$, then $\{f_1^h, \ldots, f_r^h\}$ is a Gröbner basis if and only if F is a Gröbner basis.*

Proof. $\Rightarrow$) Assume $\{f_1^h, \ldots, f_r^h\}$ is a Gröbner basis. Set $J = (f_1^h, \ldots, f_r^h)$. It suffices to prove $\text{in}(I) \subset (\text{lt}(f_1), \ldots, \text{lt}(f_r))$. Let M_1 be a monomial in the initial ideal of I. There is $f \in I \subset R$ of degree e such that $\text{lt}(f) = M_1$. We write

$$f = g_1 f_1 + \cdots + g_r f_r$$

and consider

$$\frac{f^h}{w^e} = \sum_{i=1}^{r} g_i\left(\frac{x_1}{w}, \ldots, \frac{x_n}{w}\right) f_i\left(\frac{x_1}{w}, \ldots, \frac{x_n}{w}\right)$$

multiplying both sides by w^ℓ for $\ell \gg 0$, one has

$$w^s f^h = h_1 f_1^h + \cdots + h_r f_r^h$$

for some s. Thus $w^s f^h \in J$. Since $\text{lt}(f^h) = \text{lt}(f)$ we have $\text{lt}(w^s f^h) = w^s M_1$. As $\{f_1^h, \ldots, f_r^h\}$ is a Gröbner basis one obtains

$$w^s M_1 \in (\text{lt}(f_1^h), \ldots, \text{lt}(f_r^h)).$$

Using $\text{lt}(f_i) = \text{lt}(f_i^h)$ yields $M_1 \in (\text{lt}(f_1), \ldots, \text{lt}(f_r))$.

$\Leftarrow$) Assume $\{f_1, \ldots, f_r\}$ is a Gröbner basis. Set $\overline{I} = (f_1^h, \ldots, f_r^h)$. We now show that $\text{in}(\overline{I}) \subset (\text{lt}(f_1^h), \ldots, \text{lt}(f_r^h))$. Let $M \in \text{in}(\overline{I})$, there is $f \in \overline{I}$ such that $M = \text{lt}(f)$. We may assume that f is a homogeneous polynomial. We can write

$$f = w^p(M_1 + M_2 w^{e_2} + \cdots + M_s w^{e_s}),$$

where $M_1, \ldots, M_s$ are monomials in R such that

$$M_1 > M_2 w^{e_2} > \cdots > M_s w^{e_s}.$$

Using that $M_i w^{e_i}$ all have the same degree we obtain

$$0 \leq e_2 \leq \cdots \leq e_s.$$

It follows that $f' = f(x_1, \ldots, x_n, 1)$ belongs to I and $\text{lt}(f') = M_1$. Therefore M_1 belongs to $\text{in}(I) = (\text{lt}(f_1), \ldots, \text{lt}(f_r))$. Since $\text{lt}(f_i) = \text{lt}(f_i^h)$ we obtain that M belongs to the ideal $(\text{lt}(f_1^h), \ldots, \text{lt}(f_r^h))$. □

Corollary 2.4.27 *Let $I = (f_1, \ldots, f_m)$ be an ideal of R and let*

$$I^h = (\{f^h | f \in I\})$$

be the homogenization of I. If $f_1^h, \ldots, f_m^h$ is a Gröbner basis, then

$$I^h = (f_1^h, \ldots, f_m^h).$$

Proof. It is enough to show $\{f^h | f \in I\} \subset (f_1^h, \ldots, f_m^h)$. Since $f_1, \ldots, f_m$ is a Gröbner basis any $f \in I$ can be written as

$$f = g_1 f_1 + \cdots + g_m f_m,$$

where $\mathrm{lt}(f) \geq \mathrm{lt}(g_i f_i)$ for all i. Notice that

$$e = \deg(f) \geq \deg(g_i f_i) = \deg(g_i) + \deg(f_i),$$

hence from:

$$\frac{f^h}{w^e} = \sum_{i=1}^{m} g_i \left(\frac{x_1}{w}, \ldots, \frac{x_n}{w}\right) f_i \left(\frac{x_1}{w}, \ldots, \frac{x_n}{w}\right)$$

we get

$$f^h = h_1 f_1^h + \cdots + h_m f_m^h,$$

that is, $f^h \in (f_1^h, \ldots, f_m^h)$. □

Projective closure Let k be a field. We define the *projective space* of dimension n over k, denoted by $\mathbb{P}_k^n$, to be the quotient space

$$(k^{n+1} \setminus \{0\}) / \sim$$

where two points α, β in $k^{n+1} \setminus \{0\}$ are equivalent if $\alpha = c\beta$ for some $c \in k$. It is usual to denote the equivalence class of α by $[\alpha]$.

Given a homogeneous ideal I of $S = k[x_1, \ldots, x_n, w]$ define the *zero set* of I as

$$V(I) = \{[\alpha] \in \mathbb{P}_k^n | f(\alpha) = 0, \forall \text{ homogeneous polynomials } f \in I\},$$

conversely for any $X \subset \mathbb{P}_k^n$ define $I(X)$, the *ideal* of X, as the ideal generated by the homogeneous polynomials in S that vanish on X. A *projective variety* is the zero set of a homogeneous ideal. It is not difficult to see that the members of the family

$$\tau = \{\mathbb{P}_k^n \setminus V(I) | I \subset S\}$$

are the open sets of a topology on $\mathbb{P}_k^n$, called the *Zariski topology*.

Definition 2.4.28 Let $Y \subset \mathbb{A}_k^n$. The *projective closure* of Y is defined as $\overline{Y} := \overline{\varphi(Y)}$, where φ is the map

$$\varphi\colon \mathbb{A}_k^n \longrightarrow \mathbb{P}_k^n, \qquad \alpha \longmapsto [(\alpha, 1)],$$

and $\overline{\varphi(Y)}$ is the closure of $\varphi(Y)$ in the Zariski topology of $\mathbb{P}_k^n$.

Proposition 2.4.29 *If Y is a subset of the affine space $\mathbb{A}_k^n$ over a field k, then the projective closure $\overline{Y}$ of Y is given by*

$$\overline{Y} = V(I(\varphi(Y))).$$

Proof. It is left as an exercise, see Proposition 7.1.14 for the affine case. □

Proposition 2.4.30 *Let $Y \subset \mathbb{A}_k^n$ and let $\overline{Y} \subset \mathbb{P}_k^n$ be its projective closure. If $f_1, \ldots, f_r$ is a Gröbner basis of $I(Y)$, then*

$$I(\overline{Y}) = (f_1^h, \ldots, f_r^h).$$

Proof. Note that $I(\overline{Y}) = I(Y)^h$ and use Corollary 2.4.27. □

Corollary 2.4.31 *Let $Y \subset \mathbb{A}_k^n$ and let $\overline{Y} \subset \mathbb{P}_k^n$ be its projective closure. Then the height of $I(Y)$ in R is equal to the height of $I(\overline{Y})$ in S.*

Exercises

2.4.32 Let I be an ideal of a ring R and f a nonzero element of R, then

$$(f)(I\colon f) = I \cap (f).$$

2.4.33 Let $R = k[\mathbf{x}]$ be a polynomial ring, where k is a field. Recall that f is a *binomial* in R if $f = x^\alpha - x^\beta$ for some α, β in $\mathbb{N}^n$. Use Buchberger's algorithm to prove that an ideal of R generated by a finite set of binomials has a Gröbner basis consisting of binomials.

2.4.34 Let I be an ideal of a ring R and $f \in R$. Prove that the following statements are equivalent:

(a) f is regular on R/I.

(b) $(I\colon_R f) = I$.

(c) $I = (I, 1 - tf) \cap R$, where t is a new variable.

2.4.35 If I is a prime ideal of a polynomial ring R, prove that I^h is also a prime ideal.

2.4.36 Let $\mathbb{N}^n$ be the set of n-tuples of non-negative integers endowed with the partial order given by

$$(a_1, \ldots, a_n) \geq (b_1, \ldots, b_n)$$

if and only if $a_i \geq b_i$ for all i. If $A \subset \mathbb{N}^n$, prove that A has only a finite number of minimal elements.

Hint Use Dickson's lemma.

2.4.37 Let $Y = \{(t^3, t^2, t) | \, t \in k\} \subset \mathbb{A}^3_k$ and let $\overline{Y}$ be its projective closure. If k is an infinite field prove that

$$I(\overline{Y}) = (x_3^2 - x_2 w, x_2 x_3 - x_1 w, x_2^2 - x_1 x_3).$$

2.5 Minimal resolutions

The aim of this section is to study homogeneous resolutions of positively graded modules over polynomial rings. In the sequel we shall be interested in the numerical data of those resolutions and in particular in the Betti numbers of those modules.

We begin with a result which is a consequence of the graded Nakayama's lemma (see Lemma 1.2.6).

Proposition 2.5.1 *Let $R = k[\mathbf{x}]$ be a polynomial ring over a field k and let M be an $\mathbb{N}$-graded R-module. If $F = \{f_1, \ldots, f_q\}$ is a set of homogeneous elements of M and $\mathfrak{m} = (\mathbf{x})$, then F is a minimal set of generators for M if and only if the image of F in $M/\mathfrak{m}M$ is a k-basis of $M/\mathfrak{m}M$.*

Corollary 2.5.2 *Let $R = k[\mathbf{x}]$ be a polynomial ring over a field k and let $\mathfrak{m} = (\mathbf{x})$. If M is an $\mathbb{N}$-graded R-module, then*

$$\nu(M) = \dim_k(M/\mathfrak{m}M),$$

where $\nu(M)$ is the minimum number of generators of M.

Lemma 2.5.3 *Let $R = k[\mathbf{x}]$ be a polynomial ring over a field k and let M be an $\mathbb{N}$-graded module over R with a presentation*

$$0 \longrightarrow K = \ker(\varphi) \longrightarrow R^q \xrightarrow{\varphi} M \longrightarrow 0, \qquad e_i \overset{\varphi}{\longmapsto} f_i.$$

If $\mathfrak{m} = (\mathbf{x})$ and $F = \{f_1, \ldots, f_q\}$ is a set of homogeneous elements, then F is a minimal set of generators for M if and only if $K \subset \mathfrak{m}R^q$.

Proof. $\Leftarrow$) Assume $K \subset \mathfrak{m}R^q$. By Proposition 2.5.1 we need only show that the image of F in $M/\mathfrak{m}M$ is linearly independent over k. If

$$\sum_{i=1}^{q} \lambda_i f_i \in \mathfrak{m}M,$$

for some $\lambda_i \in k$, then

$$(\lambda_1, \ldots, \lambda_q) - (a_1, \ldots, a_q) \in K \subset \mathfrak{m}R^q,$$

for some $a_1, \ldots, a_q$ in $\mathfrak{m}$. Hence $(\lambda_1, \ldots, \lambda_q)$ is in $\mathfrak{m}R^q$ and $\lambda_j = 0$ for all j.

$\Rightarrow$) If $K \not\subset \mathfrak{m}R^q$, pick $z = (z_1, \ldots, z_q) \in K$ such that $z \notin \mathfrak{m}R^q$. Then z_i is not in $\mathfrak{m}$ for some i. Since the f_i's are homogeneous this readily implies that f_i is a linear combination of elements in $F \setminus \{f_i\}$, a contradiction. □

Definition 2.5.4 Let $R = \oplus_{i=0}^{\infty} R_i$ be a positively graded ring and $a \in \mathbb{N}$. The graded R-module obtained by a *shift in the graduation* of R is given by

$$R(-a) = \bigoplus_{i=0}^{\infty} R(-a)_i,$$

where the *ith* graded component of $R(-a)$ is $R(-a)_i = R_{-a+i}$.

Proposition 2.5.5 *Let $R = \oplus_{i=0}^{\infty} R_i$ be a positively graded polynomial ring over a field k with maximal irrelevant ideal $\mathfrak{m} = R_+$ and M an $\mathbb{N}$-graded R-module. Then there is an exact sequence of graded free modules*

$$\mathbb{F}. \quad \cdots \longrightarrow \bigoplus_{i=1}^{b_k} R(-d_{ki}) \xrightarrow{\varphi_k} \cdots \longrightarrow \bigoplus_{i=1}^{b_1} R(-d_{1i}) \xrightarrow{\varphi_1} \bigoplus_{i=1}^{q} R(-d_i) \xrightarrow{\varphi_0} M \longrightarrow 0$$

where φ_k is a degree preserving map and $\operatorname{im}(\varphi_k) \subset \mathfrak{m}R^{b_{k-1}}$ for $k \geq 1$.

Proof. Let $f_1, \ldots, f_q$ be a set of homogeneous elements that minimally generate M. Set $d_i = \deg(f_i)$. There is an exact sequence

$$0 \longrightarrow Z_1 = \ker(\varphi_0) \xrightarrow{i} \bigoplus_{i=1}^{q} R(-d_i) \xrightarrow{\varphi_0} M \longrightarrow 0,$$

where φ_0 is a degree preserving homomorphism such that $\varphi_0(e_i) = f_i$ for all i. Note that $Z_1 \subset \mathfrak{m}R^q$ by Lemma 2.5.3. Since Z_1 is once again a finitely generated graded R-module one may iterate the process to obtain the required exact sequence $\mathbb{F}$. □

Theorem 2.5.6 (Hilbert syzygy theorem) *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and let M be an $\mathbb{N}$-graded R-module. Then M has a graded free resolution of length at most n.*

Proof. Set $\mathfrak{m} = R_+$. First notice that $\operatorname{Tor}_i^R(M, R/\mathfrak{m}) = 0$ for $i \geq n+1$, because the ordinary Koszul complex $K_\star(\underline{x})$ is a graded free resolution of $R/\mathfrak{m} = k$ of length n. On the other hand assume that

$$\cdots \longrightarrow F_k \xrightarrow{\varphi_k} \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow M \longrightarrow 0$$

is a graded free resolution of M as in Proposition 2.5.5. Applying the functor $(\cdot) \otimes_R R/\mathfrak{m}$ yields the complex

$$\cdots \longrightarrow F_k \otimes R/\mathfrak{m} \xrightarrow{\varphi_k \otimes 1} \cdots \longrightarrow F_1 \otimes R/\mathfrak{m} \xrightarrow{\varphi_1 \otimes 1} F_0 \otimes R/\mathfrak{m} \longrightarrow M \otimes R/\mathfrak{m} \longrightarrow 0.$$

Using $\operatorname{im}(\varphi_k) \subset \mathfrak{m}F_{k-1}$ for all $k \geq 1$ one obtains that all the maps $\varphi_k \otimes 1$ are zero. Hence

$$\operatorname{Tor}_i^R(M, R/\mathfrak{m}) \simeq F_i \otimes R/\mathfrak{m}$$

for $i \geq 1$. In particular $F_i \otimes R/\mathfrak{m} \simeq F_i/\mathfrak{m}F_i = 0$ for $i \geq n+1$ and by Nakayama's lemma one obtains $F_i = 0$ for all $i \geq n+1$. This proof was adapted from [92]. □

Corollary 2.5.7 *Let $R = \oplus_{i=0}^{\infty} R_i$ be a polynomial ring of dimension n over a field k and let M be an $\mathbb{N}$-graded R-module. Then there is a unique (up to complex isomorphism) exact sequence of graded modules*

$$0 \longrightarrow \bigoplus_{i=1}^{b_g} R(-d_{gi}) \xrightarrow{\varphi_g} \cdots \longrightarrow \bigoplus_{i=1}^{b_k} R(-d_{ki}) \xrightarrow{\varphi_k} \cdots \longrightarrow \bigoplus_{i=1}^{b_1} R(-d_{1i}) \xrightarrow{\varphi_1} \bigoplus_{i=1}^{b_0} R(-d_{0i}) \xrightarrow{\varphi_0} M \longrightarrow 0,$$

such that

(a) $g = \sup\{i \mid \operatorname{Tor}_i^R(M, k) \neq 0\} \leq n$, *and*

(b) $\operatorname{im}(\varphi_k) \subset \mathfrak{m}R^{b_{k-1}}$ *for all* $k \geq 1$, *where* $\mathfrak{m} = R_+$.

Proof. Notice the following isomorphisms:

$$\operatorname{Tor}_j^R(M, k) \simeq \bigoplus_{i=1}^{b_j} k(-d_{ji}) \simeq F_j/\mathfrak{m}F_j,$$

where F_j is the *jth* free module in the resolution of M. Hence the b_j's and the d_{ji}'s are uniquely determined and so is the length of the resolution. □

Remark 2.5.8 The entries of the matrices φ_k are in $\mathfrak{m}$. This condition is equivalent to require that at each stage we use a minimal generating set, see Lemma 2.5.3.

Remark 2.5.9 If $d_{k1} \leq \cdots \leq d_{kb_k}$ for all k, then

$$d_{11} < d_{21} < \cdots < d_{g1}.$$

Indeed if

$$\varphi_k(e_i) = \textstyle\sum_j r_{ij} e_j,$$

where $r_{ij} \in \mathfrak{m}$ and r_{ij} homogeneous, then $\deg(\varphi_k(e_i)) > d_{(k-1)1}$ because r_{ij} have positive degree. Hence one obtains $d_{k1} > d_{(k-1)1}$.

Definition 2.5.10 The integers $b_0, \ldots, b_g$ are the *Betti numbers* of M. The d_{ji}'s are the *twists*, they indicate a shift in the graduation.

In general the Betti numbers and twists of M may depend on the base field k, see for instance [108, Example 2.10] and [237, Remark 3].

Definition 2.5.11 The R-module $Z_k = \ker(\varphi_{k-1})$ is called the *k-syzygy module* of M.

Definition 2.5.12 The *homogeneous resolution* or *minimal resolution* of M is the unique graded resolution of M by free R-modules described in Corollary 2.5.7.

If M has a minimal free resolution as above, then note that $\mathrm{pd}_R(M)$, the *projective dimension* of M, is equal to g. In our situation the notions of "free resolution" and "projective resolution" coincide because of the theorem of Quillen-Suslin: *if k is a principal ideal domain, then all finitely generated projective $k[x_1, \ldots, x_n]$-modules are free* [250].

Theorem 2.5.13 (Auslander-Buchsbaum) *Let M be an R-module. If R is a regular local ring, then*

$$\mathrm{pd}_R(M) + \mathrm{depth}(M) = \dim(R).$$

Proof. See [109, Theorem 3.1] or [193]. □

Corollary 2.5.14 *If R is a polynomial ring over a field k and I is a graded ideal, then*

$$\mathrm{pd}_R(R/I) \geq \mathrm{ht}\,(I),$$

with equality if and only if R/I is a Cohen-Macaulay R-module.

Proof. The result follows from an appropriate graded version of the Auslander-Buchsbaum formula. □

Theorem 2.5.15 (Hilbert-Burch) *Let R be a polynomial ring over a field k and let I be a graded Cohen-Macaulay ideal of height two. If*

$$0 \longrightarrow R^{q-1} \xrightarrow{\varphi} R^q \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

is the minimal resolution of R/I, then I is generated by all the minors of size $q-1$ of the matrix φ.

Example Let $R = \mathbb{Q}[x, y, z, w]$ and let $I = (f_1, f_2, f_3)$, where

$$f_1 = y^2 - xz,\ f_2 = x^3 - yzw,\ f_3 = x^2y - z^2w.$$

Let us construct the minimal resolution of R/I. Consider the exact sequence

$$0 \longrightarrow \ker(\varphi_1) = Z_1 \xrightarrow{\text{id}} R(-2) \oplus R^2(-3) \xrightarrow{\varphi_1} I \longrightarrow 0, \quad e_i \xmapsto{\varphi_1} f_i.$$

Here Z_1 is the module of syzygies of I. Using *Macaulay* we find that Z_1 is generated as an R-modulo by the column vectors, ψ_1, ψ_2, of the matrix:

$$A = \begin{bmatrix} zw & x^2 \\ y & z \\ -x & -y \end{bmatrix}$$

Notice that $R(-2) \oplus R^2(-3)$ is graded by

$$(R(-2) \oplus R^2(-3))_i = R(-2)_i \oplus R(-3)_i \oplus R(-3)_i.$$

Hence $\psi_1, \psi_2 \in \left[R(-2) \oplus R^2(-3)\right]_4$. Next consider the exact sequence

$$0 \longrightarrow \ker(\varphi_2) = Z_2 \xrightarrow{\text{id}} R^2(-4) \xrightarrow{\varphi_2} Z_1 \longrightarrow 0, \quad e_i \xmapsto{\varphi_2} \psi_i.$$

Since $\psi_1 R + \psi_2 R$ is a free R-module, we have $Z_2 = (0)$. Altogether the minimal homogeneous resolution of R/I is:

$$0 \longrightarrow R^2(-4) \xrightarrow{\varphi_2} R(-2) \oplus R^2(-3) \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0.$$

In this example R/I is Cohen-Macaulay because $\operatorname{pd}_R(R/I) = 2 = \operatorname{ht}(I)$.

Pure and linear resolutions Let $R = \oplus_{i=0}^{\infty} R_i$ be a polynomial ring over a field k with its usual graduation and I a graded ideal of R. Let

$$0 \to \bigoplus_{i=1}^{b_g} R(-d_{gi}) \to \cdots \to \bigoplus_{i=1}^{b_1} R(-d_{1i}) \to R \to S = R/I \to 0 \qquad (2.1)$$

be the minimal graded resolution of S by free R-modules. The ideal I (or the algebra S) has a *pure resolution* if there are constants

$$d_1 < d_2 < \cdots < d_g$$

such that

$$d_{1i} = d_1, \ldots, d_{gi} = d_g$$

for all i. If in addition

$$d_i = d_1 + i - 1$$

for $2 \leq i \leq g$ the resolution is said to be d_1-*linear* .

Theorem 2.5.16 (Herzog-Kühl) *If S is a Cohen-Macaulay algebra with a pure resolution, then*

$$b_i = \prod_{j \neq i} \frac{d_j}{|d_j - d_i|} \qquad (1 \leq i \leq g).$$

Proof. See [154] for the proof and for a generalized version of this result valid for graded modules. □

Theorem 2.5.17 (Huneke-Miller) *Let S be a Cohen-Macaulay algebra. If S has a pure resolution, then the multiplicity $e(S)$ of S is given by*

$$e(S) = \frac{\left(\prod_{i=1}^{g} d_i\right)}{g!}.$$

Proof. See [185] for the proof and for a generalized version of this result valid not only for cyclic modules. For another simple proof see [158]. □

Since the Betti numbers and the twists are positive integers note that these formulae impose severe restrictions on the numbers $d_1, \ldots, d_g$ that can occur as the resolution type of a Cohen-Macaulay standard algebra with a pure resolution.

Exercises

2.5.18 Let R be a polynomial ring and let I be a graded Cohen-Macaulay ideal of height 3 with a pure resolution:

$$0 \to R^{b_3}(-(d+a+b)) \to R^{b_2}(-(d+a)) \to R^{b_1}(-d) \to I \to 0,$$

where $a, b \geq 1$. Assume $b_1 = 6$ and $b = 1$. Show that $x = a + 1$ and $y = d + a + 1$ is a solution of the diophantine equation

$$6x(x-1) = y(y-1).$$

Then prove that the integral solutions of this equation are given by

$$x = (X+1)/2 \text{ and } y = (Y+1)/2,$$

where $\sqrt{6}X + Y = \pm(1 \pm \sqrt{6})(5 + 2\sqrt{6})^n$, $n \in \mathbb{Z}$.

2.5.19 Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and let I be a graded ideal, then $(x_1, \ldots, x_n)$ is an associated prime of I if and only if $\mathrm{pd}_R(R/I) = n$.

2.5.20 Let $R = k[x, y, z]$ and let

$$I = (x^n + y^n - z^n | \, n \geq 2).$$

If $\mathrm{char}(k) \neq 2$, prove the equality

$$I = (x^2 + y^2 - z^2, x^3 + y^3 - z^3, x^2y^2).$$

2.5.21 Let $R = k[x, y, z]$ and let

$$I = (x^n + y^n - z^n | \, n \geq 2).$$

If $\mathrm{char}(k) \neq 2$, prove that

$$I = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \mathfrak{q}_3$$

is a primary decomposition of I, where

$\mathfrak{q}_1 = (y^2, x - z)$,

$\mathfrak{q}_2 = (x^2, y - z)$, and

$\mathfrak{q}_3 = (y^3, z^3, x^2y^2, x^2 + y^2 - z^2)$.

Chapter 3

Rees Algebras and Normality

In this part a detailed presentation of complete and normal ideals is given. The systematic use of Rees algebras and associated graded algebras will make clear their importance for the area. One of the outstanding references for blowup algebras is [296]. An important reference for the normality of Rees algebras is [40].

3.1 Symmetric algebras

Let M be an R-module. Given $n \geq 0$ we define

$$T^n(M) = \underbrace{M \otimes \cdots \otimes M}_{n-times} \quad \text{and} \quad T^0(M) = R.$$

Recall that the *tensor algebra* $T(M)$ of M is the non commutative graded algebra

$$T(M) = \bigoplus_{n=0}^{\infty} T^n(M),$$

where the product is induced by juxtaposition.

The *symmetric algebra* of M, denoted by $\mathrm{Sym}_R(M)$ or simply $\mathrm{Sym}(M)$, is defined as the quotient algebra

$$\mathrm{Sym}(M) = T(M)/\mathcal{J}$$

where $\mathcal{J}$ is the two sided ideal generated by the elements

$$xy - yx = x \otimes y - y \otimes x \in T^2(M)$$

with x and y running through M. Observe that $\mathrm{Sym}(M)$ is commutative.

Since $\mathcal{J}$ is a graded ideal generated by homogeneous elements of degree two, the symmetric algebra is graded by

$$\mathrm{Sym}_n(M) = T^n(M)/\mathcal{J} \cap T^n(M) \quad \text{and} \quad \mathrm{Sym}_0(M) = R.$$

Note

$$\mathrm{Sym}_2(M) = M \otimes M/(x \otimes y - y \otimes x),$$

with x and y running through M.

Exercises

3.1.1 If M is a free R-module of rank n prove that the symmetric algebra of M is a polynomial ring in n variables with coefficients in R.

3.2 Rees algebras and syzygetic ideals

Let I be an ideal of a ring R generated by $f_1, \ldots, f_q$. The *Rees algebra* of I, denoted by $\mathcal{R}(I)$ or $R[It]$, is the subring of $R[t]$ given by

$$\mathcal{R}(I) = R[f_1 t, \ldots, f_q t] \subset R[t],$$

where t is a new variable. Note

$$\mathcal{R}(I) = R \oplus It \oplus \cdots \oplus I^n t^n \oplus \cdots \subset R[t].$$

There is an epimorphism of R-algebras

$$\varphi\colon B = R[t_1, \ldots, t_q] \longrightarrow \mathcal{R}(I) \longrightarrow 0, \quad t_i \stackrel{\varphi}{\longmapsto} f_i t,$$

where B is a polynomial ring in the indeterminates $t_1, \ldots, t_q$ over the ring R. The kernel of φ, denoted by J, is the *presentation ideal* or *toric ideal* of $\mathcal{R}(I)$ with respect to $f_1, \ldots, f_q$. Notice that J is a graded ideal in the t_i-variables:

$$J = \bigoplus_{i=1}^{\infty} J_i,$$

where B has the standard grading induced by $\deg(t_i) = 1$.

The mapping

$$\psi\colon R^q \longrightarrow I$$

given by

$$\psi(z_1, \ldots, z_q) = \sum_{i=1}^{q} z_i f_i$$

induces an R-algebra epimorphism

$$\beta\colon R[t_1,\ldots,t_q] \longrightarrow \mathrm{Sym}_R(I).$$

Thus the *symmetric algebra* of I is:

$$\mathrm{Sym}_R(I) \simeq R[t_1,\ldots,t_q]/\ker(\beta),$$

where $\ker(\beta)$ is an ideal of $R[\mathbf{t}]$ generated by linear forms:

$$\ker(\beta) = \left(\left\{\sum_{i=1}^{q} b_i t_i \;\middle|\; \sum_{i=1}^{q} b_i f_i = 0 \text{ and } b_i \in R\right\}\right).$$

On the other hand the kernel of φ is generated by all forms $F(t_1,\ldots,t_q)$ such that $F(f_1,\ldots,f_q) = 0$. In particular, one may factor φ through $\mathrm{Sym}_R(I)$ and obtain the commutative diagram:

$$\begin{array}{ccc} R[t_1,\ldots,t_q] & \xrightarrow{\varphi} & \mathcal{R}(I) \\ \downarrow{\scriptstyle \beta} & {\scriptstyle \alpha}\nearrow & \\ \mathrm{Sym}_R(I) & & \end{array}$$

We say that I is an *ideal of linear type* if α is an isomorphism.

An important module-theoretic obstruction to "$\mathrm{Sym}_R(I) \simeq \mathcal{R}(I)$" is given by the following result.

Proposition 3.2.1 (Herzog-Simis-Vasconcelos) *Let I be an ideal of a ring R. If*

$$\mathrm{Sym}_R(I) \simeq \mathcal{R}(I),$$

then for each prime $\mathfrak{p}$ containing I, $I_{\mathfrak{p}}$ can be generated by $\mathrm{ht}\,(\mathfrak{p})$ *elements.*

Proof. See [155]. □

Syzygetic ideals Let I be an ideal of a ring R and let $H_1(I)$ be the first homology module of the Koszul complex $H_\star(\underline{x}, R)$ associated to a set $\underline{x} = x_1,\ldots x_n$ of generators of I. In [260] it is pointed out that $H_1(I)$ is related to I/I^2, the *conormal module* of I, by the following exact sequence:

$$H_1(I) \xrightarrow{f} R^n \otimes (R/I) \xrightarrow{h} I \otimes (R/I) = I/I^2 \longrightarrow 0. \qquad (*)$$

Here $f([z]) = z \otimes 1$ and $h(e_i \otimes 1) = x_i \otimes 1$. Set $\delta(I) = \ker(f)$.

Definition 3.2.2 The ideal I is called *syzygetic* if $\delta(I) = 0$.

Although $H_1(I)$ may depend on the set of generators for I, $\delta(I)$ depends only on I. Indeed A. Simis and W. Vasconcelos [260] proved that

$$\delta(I) \simeq \ker(\mathrm{Sym}_2(I) \to I^2),$$

where $\mathrm{Sym}_2(I) \to I^2$ is the surjection induced by the multiplication map.

Definition 3.2.3 An ideal I of a ring R is said to be *generically a complete intersection* if $IR_{\mathfrak{p}}$ is a complete intersection for all $\mathfrak{p} \in \mathrm{Ass}_R(R/I)$.

Remark 3.2.4 If I is unmixed and generically a complete intersection, then the R/I-torsion of $H_1(I)$ equals $\delta(I)$. To prove this, notice that $I_{\mathfrak{p}}$ is syzygetic for all $\mathfrak{p} \in \mathrm{Ass}_R(R/I)$, consequently $\delta(I)$ is a torsion module.

Computing the toric ideal of a Rees algebra Let I be a graded ideal of a polynomial ring R over a field k and $f_1, \ldots, f_q$ a generating set for I.

Consider the presentation of the Rees algebra:

$$\varphi\colon B = R[t_1, \ldots, t_q] \longrightarrow \mathcal{R}(I) \longrightarrow 0, \quad (t_i \longmapsto tf_i).$$

The kernel J of φ can be obtained as follows:

$$J = (t_1 - tf_1, \ldots, t_q - tf_q) \cap B.$$

Since J is a graded ideal in the t_i-variables, $J = \oplus_{i \geq 1} J_i$. The relationship between J and the Koszul homology of I is very tight. The exact sequence of Eq. $(*)$ can be made precise:

$$0 \longrightarrow J_2/B_1J_1 = \delta(I) \longrightarrow H_1(I) \longrightarrow (R/I)^q \longrightarrow I/I^2 \longrightarrow 0.$$

In particular one can decide whether I is syzygetic – that is, $J_2 = B_1J_1$–or of linear type–that is, $J = J_1B$.

Exercises

3.2.5 Let $R = A[\mathbf{x}]$ be a polynomial ring over a ring A and $f_1, \ldots, f_q$ forms of degree $d \geq 1$. Prove that there is a graded isomorphism of A-algebras

$$\varphi\colon A[f_1, \ldots, f_q] \longrightarrow A[tf_1, \ldots, tf_q], \quad \text{with } \varphi(f_i) = tf_i,$$

where t is a new variable and both rings have an appropriate grading such that $\deg(f_i) = 1$ and $\deg(tf_i) = 1$.

3.2.6 Let $I = I_2(X)$ be the ideal of 2×2-minors of the symmetric matrix:

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix},$$

where the entries of X are indeterminates over a field k. Prove that I is a syzygetic ideal that satisfies sliding depth. See [161].

3.3 Complete and normal ideals

Let us recall the notion of integrality of ideals. Let R be a ring and I an ideal of R, an element $z \in R$ is integral over I if z satisfies an equation

$$z^{\ell} + a_1 z^{\ell-1} + \cdots + a_{\ell-1} z + a_{\ell} = 0, \ a_i \in I^i,$$

the *integral closure* of I is the set of all elements $z \in R$ which are integral over I. This set will be denoted by $\overline{I}$ or I_a.

Definition 3.3.1 If $I = \overline{I}$, I is said to be *integrally closed* or *complete*. If all the powers I^k are complete, the ideal I is said to be *normal*.

Lemma 3.3.2 *Let R be a ring and I an ideal of R. An element $\alpha \in R$ is in the integral closure of I if and only if there is an ideal L of R such that*

(i) $\alpha L \subset IL$, *and*

(ii) $\alpha^r \operatorname{ann}(L) = (0)$ *for some integer* $r \geq 0$.

Proof. $\Rightarrow$) As α is in integral closure of I there is an equation:

$$\alpha^n = a_1 \alpha^{n-1} + \cdots + a_{n-1}\alpha + a_n, \quad \text{where } a_i \in I^i,$$

and $n \geq 1$ is some integer. The required L is obtained by setting

$$L = R\alpha^{n-1} + I\alpha^{n-2} + \cdots + I^{n-2}\alpha + I^{n-1}.$$

Indeed from the equation above

$$R\alpha^n \subset I\alpha^{n-1} + I^2\alpha^{n-2} + \cdots + I^{n-1}\alpha + I^n = IL,$$

and if $i \geq 1$ one has

$$\alpha(I^i \alpha^{n-i-1}) = I^i \alpha^{n-i} = I(I^{i-1}\alpha^{n-i}) \subset IL.$$

Hence $\alpha L \subset IL$. To finish this part of the proof note $\alpha^{n-1}\operatorname{ann}(L) = (0)$.

$\Leftarrow$) Since R is Noetherian $L = (f_1, \ldots, f_n)$. One can write

$$\alpha f_i = \sum_{j=1}^{n} b_{ij} f_j,$$

where $b_{ij} \in I$. Set $B = (b_{ij})$, $C = B - \alpha I_n$ and $f = (f_1, \ldots, f_n)$. Here I_n denotes the identity matrix. As $Cf^t = 0$, one can use the formula

$$C \cdot \operatorname{adj}(C) = \det(C) I_n$$

to conclude $f_i \det(C) = 0$ for all i. Therefore $\det(C)$ is in $\operatorname{ann}(L)$ and by hypothesis $\alpha^r \det(C) = 0$ for some r. Expanding in powers of α shows that α is integral over I. $\square$

Proposition 3.3.3 *If I is an ideal of a ring R, then $\overline{I}$ is an ideal of R.*

Proof. Let $\alpha_i \in \overline{I}$, $i = 1, 2$. By the proof of Lemma 3.3.2, there is an ideal L_i such that

(i) $\alpha_i L_i \subset IL_i$,

(ii) $\alpha_i^{r_i}\text{ann}(L_i) = (0)$ and $I^{r_i} \subset L_i$ for some $r_i \geq 0$.

Since α_i is integral over I, one has $\alpha_i^{n_i} \in I$ for some $n_i \geq 1$. Therefore for n large enough β^n is in L_1L_2, where $\beta = \alpha_1 + \alpha_2$, thus

$$\beta^n\text{ann}(L_1L_2) = (0).$$

On the other hand one clearly has the inclusion $\beta L_1L_2 \subset IL_1L_2$. Hence by Lemma 3.3.2 one concludes $\beta \in \overline{I}$.

To finish proving that $\overline{I}$ is an ideal note $x\alpha_1 \in \overline{I}$ for $x \in R$, this follows at once from the definition of $\overline{I}$. □

Proposition 3.3.4 *If I is an ideal of a ring R and S is a multiplicatively closed subset of R, then*

$$\overline{S^{-1}(I)} = S^{-1}(\overline{I}).$$

Proof. To prove $\overline{S^{-1}(I)} \subset S^{-1}(\overline{I})$ take $f \in \overline{S^{-1}(I)}$. One may assume $f = x/1$ with $x \in R$ because the integral closure of an ideal is again an ideal. There is an equation

$$f^n + \frac{a_1}{s_1}f^{n-1} + \cdots + \frac{a_{n-1}}{s_{n-1}}f + \frac{a_n}{s_n} = \frac{0}{1},$$

where $a_i/s_i \in S^{-1}(I)^i = S^{-1}(I^i)$. Thus one may assume $a_i \in I^i$ and $s_i \in S$ for all i. Clearing denominators and multiplying by an appropriate element of S one has an equality (in the ring R) of the form

$$sx^n + t_1a_1x^{n-1} + \cdots + t_{n-1}a_{n-1}x + t_na_n = 0 \quad (s, t_i \in S),$$

multiplying both sides of this equation by s^{n-1} yields sx is in $\overline{I}$. Therefore $f = (xs)/s$ is in $S^{-1}(\overline{I})$. The other containment follows rapidly using that localizations commute with powers of ideals. □

Lemma 3.3.5 *Let A be a domain and $x \in A \setminus \{0\}$ such that A_x is normal. Then A is normal if and only if (x) is a complete ideal.*

Proof. Assume A is normal. Let $z \in \overline{(x)}$. Since z satisfies a polynomial equation

$$z^m + (a_1x)z^{m-1} + \cdots + (a_{m-1}x^{m-1})z + (a_mx^m) = 0, \quad a_i \in A,$$

dividing by x^m yields that z/x is integral over A and $z \in (x)$.

Conversely assume (x) is complete. Let z be an element of the quotient field of A which is integral over A. As A and A_x have the same field of fractions and A_x is normal one may assume $z = a/x^r$, for some $a \in A$ and $r \geq 1$. To show $z \in A$ it suffices to verify $a \in (x)$. There is an equation

$$z^m + b_1 z^{m-1} + \cdots + b_{m-1} z + b_m = 0, \quad b_i \in A,$$

multiplying by x^{rm} yields the identity

$$a^m + (b_1 x^r) a^{m-1} + \cdots + (b_{m-1} x^{r(m-1)}) a + (b_m x^{rm}) = 0,$$

hence $a \in \overline{(x)} = (x)$, as required. □

Proposition 3.3.6 *If A is a domain and $x \in A \setminus \{0\}$, then*

$$A = A_x \bigcap \left(\bigcap_{\mathfrak{p} \in \operatorname{Ass} A/(x)} A_{\mathfrak{p}} \right).$$

Proof. Note that the left hand side of the equality is contained in the right hand side because A is a domain.

Conversely take an element $z \in A_x$ and $z \in A_{\mathfrak{p}}$, for all $\mathfrak{p} \in \operatorname{Ass} A/(x)$. Write $z = a/x^n$, $a \in A$ and $n \geq 1$. It is enough to prove that $a \in (x)$. If a is not in (x) note that

$$((x) : a) \subset \mathcal{Z}(A/(x)).$$

Hence $((x) : a) \subset \mathfrak{p}$, for some $\mathfrak{p} \in \operatorname{Ass} A/(x)$. Since $z \in A_{\mathfrak{p}}$ one may write $z = a/x^n = b/s$, $b \in A$ and $s \notin \mathfrak{p}$. Therefore $s \in ((x) : a) \subset \mathfrak{p}$, which yields a contradiction. □

Proposition 3.3.7 *If A is an integral domain and S is a multiplicatively closed subset of A, then*

$$\overline{S^{-1}(A)} = S^{-1}(\overline{A}).$$

Proof. Note that A and $S^{-1}(A)$ have the same field of fractions K. First we prove $\overline{S^{-1}(A)} \subset S^{-1}(\overline{A})$. Take any x in K integral over $S^{-1}(A)$. There is an equation

$$x^n + \frac{a_1}{s_1} x^{n-1} + \cdots + \frac{a_{n-1}}{s_{n-1}} x + \frac{a_n}{s_n} = \frac{0}{1},$$

where $a_i \in A$ and $s_i \in S$ for all i. Set $s = s_1 \cdots s_n$. If we multiply by s^n, it follows that $sx \in \overline{A}$ and $x \in S^{-1}(\overline{A})$.

Conversely take $x \in S^{-1}(\overline{A})$. There is $s \in S$ such that sx is integral over A. Hence sx satisfies

$$(sx)^n + a_1 (sx)^{n-1} + \cdots + a_{n-1}(sx) + a_n = 0,$$

for some $a_1, \ldots, a_n$ in A, dividing by s^n immediately yields that x is integral over $S^{-1}(A)$, as required. □

Corollary 3.3.8 *If A is a normal domain and is S a multiplicatively closed subset of A, then $S^{-1}(A)$ is a normal domain.*

Corollary 3.3.9 *Let A be a domain and $x \in A \setminus \{0\}$. Then A is normal if and only if A_x and $A_{\mathfrak{p}}$ are normal for every $\mathfrak{p} \in \operatorname{Ass} A/(x)$.*

Proof. If A is normal, then by the corollary above A_x and $A_{\mathfrak{p}}$ are normal for every $x \in A \setminus \{0\}$ and $\mathfrak{p} \in \operatorname{Spec}(A)$.

For the converse use Proposition 3.3.6 and observe that A_x and $A_{\mathfrak{p}}$ have the same quotient field as A. □

To state a useful criterion of normality of Rees algebras, it is convenient to introduce the *extended Rees algebra* of an ideal I in a ring R:

$$A = R[It, u], \ u = t^{-1}.$$

Part of its usefulness derives from the equality $A_u = R[t, t^{-1}]$.

Proposition 3.3.10 *Let I be an ideal of a ring R and $A = R[It, t^{-1}]$ its extended Rees algebra. If R is a normal domain, then A is normal if and only if $A_{\mathfrak{p}}$ is normal for each associated prime $\mathfrak{p}$ of $t^{-1}A$.*

Proof. Note $A_{t^{-1}} = R[t, t^{-1}]$. Hence $A_{t^{-1}}$ is a normal domain and one can use Corollary 3.3.9. □

Lemma 3.3.11 *Let A be a ring and x a regular element of A. If $A/(x)$ is reduced and $\mathfrak{p} \in \operatorname{Ass}_A A/(x)$, then $A_{\mathfrak{p}}$ is a normal domain.*

Proof. Let $\operatorname{Ass}_A A/(x) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$. Since $R/(x)$ is reduced

$$(x) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r,$$

hence $(x)A_{\mathfrak{p}_i} = \mathfrak{p}_i A_{\mathfrak{p}_i}$ for all i. Note that $\operatorname{ht}(\mathfrak{p}_i) = 1$, by Krull's principal ideal theorem. Therefore the maximal ideal of $A_{\mathfrak{p}_i}$ is generated by a system of parameters, that is, $A_{\mathfrak{p}_i}$ is a regular local ring and consequently $A_{\mathfrak{p}_i}$ is a normal domain by Theorem 1.4.16. □

Proposition 3.3.12 *Let A be a ring and x a regular element of A. If A_x is a normal domain and $A/(x)$ is reduced, then A is a normal domain.*

Proof. As A_x is a domain and x is a regular element on A one obtains that A is a domain. By Lemma 3.3.11 $A_{\mathfrak{p}}$ is normal for all $\mathfrak{p} \in \operatorname{Ass}_A A/(x)$, hence A must be normal according to Corollary 3.3.9. □

There is another graded algebra associated to an ideal I of a ring R whose properties are related to the normality of the Rees algebra of I, it is called the *associated graded algebra* of I and is defined as:

$$\mathrm{gr}_I(R) = R/I \oplus I/I^2 \oplus \cdots \oplus I^i/I^{i+1} \oplus \cdots,$$

with multiplication

$$(a + I^i)(b + I^j) = ab + I^{i+j-1} \quad (a \in I^{i-1},\ b \in I^{j-1}).$$

Given a generating set $f_1, \ldots, f_q$ of I, it is not difficult to verify that $\mathrm{gr}_I(R)$ is a graded algebra over R/I generated by the following elements of degree one:

$$\overline{f}_1 = f_1 + I^2, \ldots, \overline{f}_q = f_q + I^2.$$

Thus one has

$$\mathrm{gr}_I(R) = (R/I)[\overline{f}_1, \ldots, \overline{f}_q].$$

Lemma 3.3.13 *If I is an ideal of a ring R, then*

$$R[It]/IR[It] \simeq \mathrm{gr}_I(R) \ \text{ and } \ A/t^{-1}A \simeq \mathrm{gr}_I(R),$$

where $A = R[It, t^{-1}]$ is the extended Rees algebra of I.

Proof. It is left as an exercise. □

Theorem 3.3.14 *Let I be an ideal of a ring R. If I is generated by a regular sequence $f_1, \ldots, f_q$, then the epimorphism of graded algebras:*

$$\varphi\colon (R/I)[t_1, \ldots, t_q] \longrightarrow \mathrm{gr}_I(R) = (R/I)[\overline{f}_1, \ldots, \overline{f}_q],$$
$$\varphi(t_i) = \overline{f}_i = f_i + I^2,$$

is an isomorphism, where $t_1, \ldots, t_q$ are indeterminates over R/I.

Proof. The proof is by induction on q, the case $q = 1$ is easy to show. Let J be the ideal $(f_1, \ldots, f_{q-1})$ and consider the epimorphism:

$$\varphi'\colon (R/J)[t_1, \ldots, t_{q-1}] \longrightarrow \mathrm{gr}_J(R) = (R/J)[\overline{f}_1, \ldots, \overline{f}_{q-1}],$$

$\varphi'(t_i) = \overline{f}_i = f_i + J^2$. Note $(J\colon f_q) = J$ because f_q is regular on R/J, moreover since φ' is an isomorphism it follows (by induction on m) that one has the following equalities

$$(J^m\colon f_q) = J^m \text{ for all } m \geq 1.$$

Let $F \in R[\mathbf{t}] = R[t_1, \ldots, t_q]$ be a homogeneous polynomial of degree d such that its image in $(R/I)[\mathbf{t}]$ is in the kernel of φ, that is,

$$F(f) = F(f_1, \ldots, f_q) \in I^{d+1}.$$

As φ is graded it suffices to prove $F \in IR[\mathbf{t}]$. To show $F \in IR[\mathbf{t}]$ we proceed by induction on d, the case $d = 0$ is clear. There is $W \in R[\mathbf{t}]$ of degree $d+1$ with $F(f) = W(f)$, write

$$W = \sum_{i=1}^{q} t_i W_i,$$

where $W_i = 0$ or W_i is a homogeneous polynomial in $R[\mathbf{t}]$ of degree d. There are polynomials G in $R[t_1, \ldots, t_{q-1}]$ and H in $R[\mathbf{t}]$ of degrees d and $d-1$ respectively such that

$$F' = F - \sum_{i=1}^{q} f_i W_i = G + t_q H.$$

Observe that $F'(f) = 0$, hence $H(f) \in (J^d \colon f_q) = J^d \subset I^d$ and by induction on d one concludes that H has coefficients in I. It only remains to prove that G has coefficients in I. There is a polynomial $H' \in R[t_1, \ldots, t_{q-1}]$ of degree d with $H(f) = H'(f)$. Set

$$F'' = G + f_q H' \in R[t_1, \ldots, t_{q-1}],$$

noting $F''(f) = F'(f) = 0$ and using that φ' is an isomorphism one derives that F'' has coefficients in J, which implies that G has coefficients in I as required. □

Theorem 3.3.15 *Let I be an ideal of a ring R and $A = R[It, t^{-1}]$ its extended Rees algebra. If R is a normal domain and $\operatorname{gr}_I(R)$ is reduced, then A is normal.*

Proof. As $A/t^{-1}A \simeq \operatorname{gr}_I(R)$ is reduced and $A_{t^{-1}} = R[t, t^{-1}]$ is a normal domain, one may use Proposition 3.3.12. □

Definition 3.3.16 Let A be an integral domain and K its field of fractions. An element $x \in K$ is *almost integral* over A if there is $0 \neq a \in A$ such that $ax^n \in A$ for all $n \geq 0$.

Proposition 3.3.17 *Let A be an integral domain and let K be its field of fractions. Then an element $x \in K$ is integral over A if and only if x is almost integral over A.*

Proof. Let $x = c/d$, where c, d are in A and $d \neq 0$. If x is integral over A, then there is an equation

$$x^m + b_1 x^{m-1} + \cdots + b_{m-1} x + b_m = 0, \ b_i \in A.$$

Setting $a = d^m$ one has $ax^n \in A$ for all $n > 0$.

Conversely assume there is $0 \neq a \in A$ such that $ax^n \in A$ for all $n > 0$. Since $a^{-1}A$ is a Noetherian A-module and $A[x] \subset a^{-1}A$ one has that $A[x]$ is a finitely generated A-module. As A is a subring of $A[x]$ by Proposition 1.4.2 one derives that x is integral over A. □

Theorem 3.3.18 ([157]) *Let I be an ideal of a normal domain R. Then the following are equivalent:*

(a) *I is a normal ideal of R.*

(b) *The Rees algebra $R[It]$ is normal.*

(c) *The ideal $IR[It] \subset R[It]$ is complete.*

(d) *The ideal $(t^{-1}) \subset R[It, t^{-1}]$ is complete.*

(e) *The extended Rees algebra $R[It, t^{-1}]$ is normal.*

Proof. (a) $\Rightarrow$ (b) Set $A = R[It]$. Let $z \in \overline{A} \subset R[t]$ and write

$$z = \sum_{i=0}^{s} b_i t^i.$$

It suffices to prove $b_s t^s \in A$. First we prove that $b_s t^s \in \overline{A}$. As z is almost integral over A there is $0 \neq f \in A$ such that $fz^n \in A$ for all $n > 0$. Hence there is $0 \neq f_m \in I^m$ such that $(f_m t^m)(b_s t^s)^n$ is in A for all $n > 0$, that is, $b_s t^s$ is almost integral over A. As R is a Noetherian integral domain using Proposition 3.3.17 one derives that $b_s t^s$ is integral over A. Thus there is an equation

$$(b_s t^s)^m + a_1 (b_s t^s)^{m-1} + \cdots + a_{m-1}(b_s t^s) + a_m = 0,$$

where $a_i = \sum_{j=0}^{r_i} a_{ij} t^j$ and $a_{ij} \in I^j$. Grouping all the terms of t-degree equal to sm one has the equation

$$b_s^m + \sum_{i=1}^{m} a_{i,si} b_s^{m-i} = 0.$$

Thus b_s is integral over I^s and consequently $b_s \in I^s$, as required.

(b) $\Rightarrow$ (c) Let $z = b_0 + b_1 t + \cdots + b_s t^s$ be an element of $R[It]$ which is integral over $IR[It]$. Then z satisfies an equation of the form

$$z^m + a_1 z^{m-1} + \cdots + a_{m-1} z + a_m = 0, \ a_i \in I^i R[It],$$

multiplying by t^m one obtains that tz is integral over $R[It]$. Therefore $tz \in R[It]$, which proves that $z \in IR[It]$.

(c) $\Rightarrow$ (d) Set $B = R[It, t^{-1}]$. Let z be an element of B integral over $t^{-1}B$. As the negative part of the Laurent expansion of z is already in $t^{-1}B$ one may assume $z = \sum_{i=0}^{s} b_i t^i$, $s \geq 0$ and $b_i \in I^i$ for all $i \geq 0$. By descending induction on s it suffices to prove that $b_s \in I^{s+1}$. There is an equation

$$z^m + a_1 z^{m-1} + \cdots + a_{m-1} z + a_m = 0, \ a_i \in (t^{-1})^i B,$$

hence zt is almost integral over B. It follows rapidly that $b_s t^{s+1}$ is almost integral over B and thus $b_s t^{s+1}$ is integral over B. A straightforward calculation yields that $b_s t^{s+1}$ is integral over $IR[It]$, which shows that b_s is in I^{s+1}.

(d) $\Rightarrow$ (e) Set $u = t^{-1}$ and note $R[It, u]_u = R[t, u]$ is a normal domain. Therefore by Lemma 3.3.5 one concludes that $R[It, u]$ is normal.

(e) $\Rightarrow$ (a) If $z \in \overline{I^r}$, then z satisfies a polynomial equation

$$z^m + a_1 z^{m-1} + \cdots + a_{m-1} z + a_m = 0, \ a_i \in I^{ri},$$

multiplying by t^{rm} yields that zt^r is integral over the ring $R[It, t^{-1}]$. Hence $z \in I^r$. □

Corollary 3.3.19 *Let I be an ideal of a normal domain R and $R[It]$ its Rees algebra. If $\operatorname{gr}_I(R)$ is reduced, then $R[It]$ is normal.*

Proof. Use Theorem 3.3.15 and Theorem 3.3.18. □

An ideal I of a ring R is a *radical ideal* if $I = \operatorname{rad}(I)$. Note: (i) a proper ideal I is radical if and only if I is an intersection of finitely many primes, and (ii) $\overline{I} \subset \operatorname{rad}(I)$ and equality occurs if I is a radical ideal.

Corollary 3.3.20 *Let I be a radical ideal of a normal domain R. If I is generated by a regular sequence, then $\operatorname{gr}_I(R)$ is reduced and $R[It]$ is a normal domain.*

Proof. It follows from Theorem 3.3.14 and Corollary 3.3.19. □

Example 3.3.21 Let $R = \mathbb{Q}[x, y]$ be a polynomial ring in two variables over the field of rational numbers and $I = (x^2, y^2)$. Observe the equality

$$\overline{I} = (x^2, y^2, xy),$$

thus $R[It]$ is not normal because I is not even complete.

Definition 3.3.22 Let I be an ideal of a ring R and $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ the minimal primes of I. Given an integer $n \geq 1$, the *nth symbolic power* of I is defined to be the ideal

$$I^{(n)} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r,$$

where $\mathfrak{q}_i$ is the primary component of I^n corresponding to $\mathfrak{p}_i$.

Let $(R, \mathfrak{m})$ be a regular local ring and I an unmixed ideal. An interesting problem is whether $I^{(2)}$ is contained in $\mathfrak{m}I$, although the answer is negative in general, the problem remains open in characteristic zero. For some insight into this problem see [97, 186].

Proposition 3.3.23 *Let I be a proper ideal of a ring R and $S = R\backslash \cup_{i=1}^{r} \mathfrak{p}_i$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the minimal primes of I. Then*

$$I^{(n)} = S^{-1}I^n \cap R \quad \text{for } n \geq 1.$$

Proof. Let

$$I^n = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r \cap \mathfrak{q}_{r+1} \cap \cdots \cap \mathfrak{q}_s$$

be a primary decomposition, where $\mathfrak{q}_i$ is the primary component of $\mathfrak{p}_i$ for $i \leq r$ and $\mathfrak{q}_i$ is a primary component of the embedded prime $\mathfrak{p}_i$ for $i > r$. Since $\mathfrak{p}_i \cap S \neq \emptyset$ for $i > r$ one has $S^{-1}\mathfrak{q}_i = S^{-1}R$. Hence

$$S^{-1}I^n = S^{-1}\left(\bigcap_{i=1}^{s} \mathfrak{q}_i\right) = \bigcap_{i=1}^{r} S^{-1}\mathfrak{q}_i.$$

To finish the argument note that $S^{-1}\mathfrak{q}_i \cap R = \mathfrak{q}_i$ and intersect with R the equality above. □

Proposition 3.3.24 *Let I be a radical ideal of a ring R and $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ the minimal primes of I. Then*

$$I^{(n)} = \mathfrak{p}_1^{(n)} \cap \cdots \cap \mathfrak{p}_r^{(n)} \quad \text{for } n \geq 1.$$

Proof. Let

$$I^n = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r \cap \mathfrak{q}_{r+1} \cap \cdots \cap \mathfrak{q}_s$$

be a primary decomposition of I^n, where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary for $i \leq r$ and $\mathfrak{q}_i$ is an embedded primary component of I^n for $i > r$. Localizing at $\mathfrak{p}_i$ yields $I^n R_{\mathfrak{p}_i} = \mathfrak{q}_i R_{\mathfrak{p}_i}$ and from $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_r$ one obtains:

$$I^n R_{\mathfrak{p}_i} = (IR_{\mathfrak{p}_i})^n = (\mathfrak{p}_i R_{\mathfrak{p}_i})^n = \mathfrak{p}_i^n R_{\mathfrak{p}_i}.$$

Thus $\mathfrak{p}_i^n R_{\mathfrak{p}_i} = \mathfrak{q}_i R_{\mathfrak{p}_i}$ and contracting one has $\mathfrak{p}_i^{(n)} = \mathfrak{q}_i$, as required. □

An aspect of symbolic powers that has attracted attention is to describe when the symbolic and ordinary powers of a given ideal I coincide, see [173] and [230] for a detailed discussion. Next we present a few cases where equality of symbolic and ordinary can be described in terms of properties of the associated graded ring.

Definition 3.3.25 An ideal I of a ring R is called *normally torsion free* if $\operatorname{Ass}(R/I^i)$ is contained in $\operatorname{Ass}(R/I)$ for all $i \geq 1$ and $I \neq R$.

In [38] M. Brodman showed that when R is a Noetherian ring and I is an ideal of R the sets $\operatorname{Ass}(R/I^n)$ stabilize for large n. If I is a radical ideal which is normally torsion free, then $\operatorname{Ass}(R/I^n) = \operatorname{Ass}(R/I)$ for all $n \geq 1$, that is, in this case the notion of normally torsion free is a strong form of stability.

Proposition 3.3.26 *Let I be an ideal of a ring R. If I has no embedded primes, then I is normally torsion free if and only if $I^n = I^{(n)}$ for all $n \geq 1$.*

Proof. $\Rightarrow$) Note $\operatorname{Ass}(R/I^n) = \operatorname{Ass}(R/I)$ for $n \geq 1$, because any associated prime $\mathfrak{p}$ of I is a minimal prime of I^n and thus $\mathfrak{p} \in \operatorname{Ass}(R/I^n)$. As I^n has no embedded primes one concludes that I^n has a unique irredundant minimal primary decomposition and $I^n = I^{(n)}$ for $n \geq 1$.

$\Leftarrow$) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the associated primes of I. Since $\mathfrak{p}_i$ is a minimal prime of I for all i, one derives

$$\operatorname{Ass}(R/I^n) = \operatorname{Ass}(R/I^{(n)}) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\},$$

as required. □

Proposition 3.3.27 *Let I be an ideal of a Cohen-Macaulay ring R. If I is generated by a regular sequence, then $I^n = I^{(n)}$ for $n \geq 1$.*

Proof. Let $f_1, \ldots, f_r$ be an R-regular sequence such that $I = (f_1, \ldots, f_r)$. By Krull's principal ideal theorem $\operatorname{ht}(I) = r$, hence I is an unmixed ideal by Theorem 1.3.23. From Theorem 3.3.14 there is an isomorphism of graded rings

$$\varphi\colon B = (R/I)[t_1, \ldots, t_r] \longrightarrow \bigoplus_{i=0}^{\infty} I^i/I^{i+1}, \quad t_i \longmapsto f_i + I^2,$$

where B is a polynomial ring with coefficients in R/I. Therefore I^i/I^{i+1} is a free R/I-modulo because it is isomorphic to B_i, the *ith* graded component of B. Thus

$$\operatorname{Ass}_R(I^i/I^{i+1}) = \operatorname{Ass}_R(R/I).$$

Using the exact sequences

$$0 \longrightarrow I^i/I^{i+1} \longrightarrow R/I^{i+1} \longrightarrow R/I^i \longrightarrow 0,$$

it follows by induction that $\operatorname{Ass}_R(R/I^n) \subset \operatorname{Ass}_R(R/I)$ for $n \geq 1$. To finish the proof one may apply Proposition 3.3.26 to conclude the equality between the ordinary and symbolic powers of I. □

Definition 3.3.28 A proper ideal I of a ring R is said to be *locally a complete intersection* if $IR_{\mathfrak{p}}$ is a complete intersection for all $\mathfrak{p} \in V(I)$.

Proposition 3.3.29 *Let R be a Cohen-Macaulay ring and I a prime ideal. If I is locally a complete intersection, then $I^n = I^{(n)}$ for all $n \geq 1$.*

Proof. By Proposition 3.3.26 it suffices to prove $\operatorname{Ass}(R/I^n) \subset \operatorname{Ass}(R/I)$ for $n \geq 1$. If $\mathfrak{p}$ is an associated prime of R/I^n, then $\mathfrak{p}R_{\mathfrak{p}}$ is an associated prime of $R_{\mathfrak{p}}/I^n_{\mathfrak{p}}$. Using Proposition 3.3.27 yields that $I_{\mathfrak{p}}$ is normally torsion free, that is, the only associated prime of $R_{\mathfrak{p}}/I^n_{\mathfrak{p}}$ is $I_{\mathfrak{p}}$. Therefore $\mathfrak{p}R_{\mathfrak{p}} = I_{\mathfrak{p}}$ and $\mathfrak{p} = I$. □

Proposition 3.3.30 *Let $\mathfrak{p}$ be a prime ideal of a ring R such that $\mathfrak{p}R_{\mathfrak{p}}$ is a complete intersection, then $\mathfrak{p}^{(n)} = \mathfrak{p}^n$ for all $n \geq 1$ if and only if $\operatorname{gr}_{\mathfrak{p}}(R)$ is a domain.*

Proof. $\Rightarrow$) Since $\operatorname{gr}_{\mathfrak{p}}(R) = R[\mathfrak{p}t]/\mathfrak{p}R[\mathfrak{p}t]$, it is enough to show that $\mathfrak{p}R[\mathfrak{p}t]$ is a prime ideal of $R[\mathfrak{p}t]$. Let $x = a_0 + a_1t + \cdots + a_rt^r$ be an element in $R[\mathfrak{p}t]$ and $x' = (a_0/1) + (a_1/1)t + \cdots + (a_r/1)t^r$ the image of x in $R_{\mathfrak{p}}[\mathfrak{p}R_{\mathfrak{p}}t]$. Note that a_i is in $\mathfrak{p}^{i+1}$ if and only if $(a_i/1)$ is in $\mathfrak{p}^{i+1}R_{\mathfrak{p}}$, because the ordinary and symbolic powers of $\mathfrak{p}$ coincide, thus x is in $\mathfrak{p}R[\mathfrak{p}t]$ if and only if x' is in $\mathfrak{p}R_{\mathfrak{p}}[\mathfrak{p}R_{\mathfrak{p}}t]$. As a consequence $\mathfrak{p}R[\mathfrak{p}t]$ is prime if and only if $\mathfrak{p}R_{\mathfrak{p}}[\mathfrak{p}R_{\mathfrak{p}}t]$ is prime. To finish the argument use Theorem 3.3.14 and the hypothesis that $\mathfrak{p}R_{\mathfrak{p}}$ is generated by a regular sequence to get that $\operatorname{gr}_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is a domain.

$\Leftarrow$) First we prove $\operatorname{Ass}_R(\mathfrak{p}^i/\mathfrak{p}^{i+1}) = \{\mathfrak{p}\}$ for $i \geq 1$. Let $\mathfrak{p}_1$ be an associated prime of $\mathfrak{p}^i/\mathfrak{p}^{i+1}$, that is, $\mathfrak{p}_1 = \operatorname{ann}(x + \mathfrak{p}^{i+1})$, for some x in $\mathfrak{p}^i \setminus \mathfrak{p}^{i+1}$. Note $a \in \mathfrak{p}_1$ iff $ax \in \mathfrak{p}^{i+1}$, and since $\operatorname{gr}_{\mathfrak{p}}(R)$ is a domain, one readily concludes that $a \in \mathfrak{p}_1$ iff $a \in \mathfrak{p}$. Hence $\mathfrak{p}_1 = \mathfrak{p}$. There is an exact sequence:

$$0 \longrightarrow \mathfrak{p}^i/\mathfrak{p}^{i+1} \longrightarrow R/\mathfrak{p}^{i+1} \longrightarrow R/\mathfrak{p}^i \longrightarrow 0.$$

Hence using induction on $i \geq 1$ and Lemma 1.1.16, one rapidly derives, from the exact sequence above, that $\operatorname{Ass}(R/\mathfrak{p}^i) \subset \operatorname{Ass}(R/\mathfrak{p}) = \{\mathfrak{p}\}$. Thus $\mathfrak{p}$ is normally torsion free and by Proposition 3.3.26 one has $\mathfrak{p}^n = \mathfrak{p}^{(n)}$ for all $n \geq 1$. □

Theorem 3.3.31 ([262]) *Let R be a normal domain and I a radical ideal which is generically a complete intersection. If I is normally torsion free, then its Rees algebra $\mathcal{R}(I) = R[It]$ is a normal domain.*

Proof. By Corollary 3.3.19 we only have to show that the associated graded algebra $\operatorname{gr}_I(R) = R[It]/IR[It]$ is reduced. Let

$$\overline{f} = a_0 + a_1t + \cdots + a_st^s + IR[It]$$

be a nilpotent element of $\operatorname{gr}_I(R)$. Hence $(a_st^s)^m$ is in $IR[It]$ for some $m \geq 1$, by induction it suffices to verify that a_st^s is in $IR[It]$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the minimal primes of I. Since $IR_{\mathfrak{p}_i} = \mathfrak{p}_iR_{\mathfrak{p}_i}$ is a complete intersection one derives that $\operatorname{gr}_{\mathfrak{p}_iR_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i})$ is reduced (see Corollary 3.3.20). Therefore the image of a_st^s in $\operatorname{gr}_{\mathfrak{p}_iR_{\mathfrak{p}_i}}(R_{\mathfrak{p}_i})$ is zero for all i, and one readily concludes that a_s belongs to the following intersection:

$$(R \cap \mathfrak{p}_1^{s+1}R_{\mathfrak{p}_1}) \cap \cdots \cap (R \cap \mathfrak{p}_r^{s+1}R_{\mathfrak{p}_r}) = \mathfrak{p}_1^{(s+1)} \cap \cdots \cap \mathfrak{p}_r^{(s+1)}.$$

As I is normally torsion free one can write $I^n = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$, where $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary, localizing yields $I^nR_{\mathfrak{p}_i} = \mathfrak{p}_i^nR_{\mathfrak{p}_i} = \mathfrak{q}_iR_{\mathfrak{p}_i}$ and consequently $\mathfrak{q}_i = \mathfrak{p}_i^{(n)}$. Making n equal to $s+1$ proves that a_s is in I^{s+1}. □

For ideals, in polynomial rings, with small data there are effective computational tests of normality, see Section 3.5, [40] and [296, Section 10.6].

Descent of normality Let $A \subset B$ is an extension of rings such that B is normal, in general the normality of A is not inherit from B. We discuss some sufficient conditions for A to be normal.

Lemma 3.3.32 *Let $A \subset B$ be an extension of rings. If $B = A \oplus C$ (as A-modules), then $IB \cap A = I$ for every ideal I of A.*

Proof. Let $z \in IB \cap A$ and write $z = \sum_{i=1}^{q} b_i f_i$, where $b_i \in B$ and $f_i \in I$. By hypothesis $b_i = a_i + c_i$, $a_i \in A$ and $c_i \in C$. Since $z \in A$ it follows that $z = \sum_{i=1}^{q} a_i f_i \in I$. This proves the containment $IB \cap A \subset I$, the reverse containment is clear. □

Proposition 3.3.33 *Let $A \subset B$ be two integral domains with field of fractions K_A and K_B respectively. If $B = A \oplus C$ (as A-modules), then*

$$K_A \cap B \subset A.$$

In particular if B is normal, then A is normal.

Proof. By Lemma 3.3.32 one has $IB \cap A = I$ for every ideal I of A. Let $b = a/c \in B$, $a, c \in A$, then $a \in (c)B \cap A = (c)$, hence $a = \lambda c = bc$, with $\lambda \in A$. Therefore $b = \lambda \in A$. □

Exercises

3.3.34 Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and K_R its field of fractions. If $R' = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] \subset K_R$ is the ring of *Laurent polynomials*, prove that R' is a normal domain.

Hint R' is the localization of R at the multiplicative set of monomials.

3.3.35 Let R be a ring and I, J ideals of R. If I and J are integrally closed, prove that $I \cap J$ is integrally closed.

3.3.36 Let $\mathfrak{p}$ be a prime ideal of a ring R. Prove that the symbolic power $\mathfrak{p}^{(n)}$ is a primary ideal for $n \geq 1$.

Hint Note that $\mathfrak{p}^n R_{\mathfrak{p}}$ is a primary ideal because its radical is maximal.

3.3.37 Let R be a ring and S a multiplicatively closed subset of R. If $\mathfrak{p}$ is a prime ideal such that $S \cap \mathfrak{p} = \emptyset$, then

$$(S^{-1}\mathfrak{p})^{(n)} = S^{-1}\mathfrak{p}^{(n)} \quad \text{for} \quad n \geq 1.$$

3.3.38 Let I be an ideal of a Cohen-Macaulay ring R. If I is generated by a regular sequence, then

$$\operatorname{Ass}(R/I^n) = \operatorname{Min}(R/I) \quad \text{for} \quad n \geq 1.$$

3.3.39 Let I be an ideal of a ring R generated by a regular sequence. Prove that I^i/I^{i+1} is a free R/I-module for all i.

Hint Use Theorem 3.3.14.

3.3.40 Let I be a radical ideal of a ring R. Prove that I is normally torsion free if and only if $\mathrm{gr}_I(R)$ is torsion-free as an R/I-module (see [188] for other equivalent conditions).

Hint If $\mathrm{gr}_I(R)$ is torsion-free show

$$\mathrm{Ass}_R(I^i/I^{i+1}) \subset \mathrm{Ass}(R/I)$$

for $i \geq 1$ and use the exact sequence

$$0 \longrightarrow I^i/I^{i+1} \longrightarrow R/I^{i+1} \longrightarrow R/I^i \longrightarrow 0.$$

3.3.41 Let R be a ring and $\mathcal{F} = \{I_i\}_{i\in\mathbb{N}}$ a family of ideals in R. It is said that $\mathcal{F}$ is a *filtration* of R, if $I_{i+1} \subset I_i$, $I_0 = R$, and $I_iI_j \subset I_{i+j}$ for all $i, j \in \mathbb{N}$. If I is an ideal of R, prove that the following are filtrations of R:

(a) $I_i = I^i$, the ordinary powers of I,

(b) $I_i = \overline{I^i}$, the integral closure of I^i, and

(c) $I_i = I^{(i)}$, the symbolic powers of I.

Hint For (b) use the method of proof of Proposition 3.3.3.

3.3.42 Let R be a normal domain and $\mathcal{F} = \{I_i\}_{i\in\mathbb{N}}$ a filtration of R. If I_i is integrally closed for all i, prove that the *Rees algebra*

$$\mathcal{R}(\mathcal{F}) = \bigoplus_{i=0}^{\infty} I_i t^i \subset R[t]$$

of the filtration $\mathcal{F}$ is integrally closed. See [131] for a study of the Cohen-Macaulay and Gorenstein property of $\mathcal{R}(\mathcal{F})$.

3.3.43 Let R be a domain and I an ideal, then

$$\mathcal{R}_a(I) = \bigoplus_{i=0}^{\infty} \overline{I^i} t^i \subset \overline{R[It]}$$

with equality if R is a normal domain.

Hint To show equality use the very first part of the proof of Theorem 3.3.18.

3.3.44 Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and $\mathfrak{m}$ the maximal ideal $(x_1, \ldots, x_n)$. If $I = \mathfrak{m}^d$ and $J = (x_1^d, \ldots, x_n^d)$ for some positive integer d, then

$$\overline{\mathcal{R}(J)} = \mathcal{R}(I).$$

3.3.45 Let I and J be two ideals of a normal domain R. Define the multi Rees algebra of I and J as

$$\mathcal{R}(I \oplus J) = R[uI, vJ],$$

here u, v are new variables. Prove that

$$\overline{\mathcal{R}(I \oplus J)} = \oplus I_a^n J_a^m u^n v^m,$$

where $I_a^n = \overline{I^n}$ and $J_a^m = \overline{J^m}$.

3.4 A criterion of Jürgen Herzog

In this section we present an elegant and useful Cohen-Macaulay criterion (see Theorem 3.4.12). Once more we recall that all rings considered in this book are Noetherian and modules are finitely generated.

Quasi-regular sequences It is useful to generalize Theorem 3.3.14 to modules by introducing polynomial with coefficients in a module. Let M be an R-module and $R[t_1, \ldots, t_q]$ a polynomial ring over the ring R. Set

$$M[\mathbf{t}] = M[t_1, \ldots, t_q] = M \otimes_R R[t_1, \ldots, t_q],$$

and note that an element of $M[\mathbf{t}]$ can be regarded as a polynomial in the t_i variables with coefficients in M, thus $M[\mathbf{t}]$ is naturally graded. If $f_1, \ldots, f_q$ is a sequence in R and $I = (f_1, \ldots, f_q)$, there is a degree preserving map of additive groups

$$\varphi\colon (M/IM)[t_1, \ldots, t_q] \longrightarrow \mathrm{gr}_I(M) = \bigoplus_{i=0}^{\infty} I^i M/I^{i+1}M,$$

such that

$$\varphi(\overline{F(\mathbf{t})}) = F(f_1, \ldots, f_q) + I^{i+1}M,$$

for all $F(\mathbf{t})$ in $M[\mathbf{t}]$ homogeneous of degree i, where the notation $\overline{F(\mathbf{t})}$ means reducing the coefficients of $F(\mathbf{t})$ modulo IM. It is not hard to verify the equivalence between the following two conditions:

(a) φ is injective.

(b) For every homogeneous polynomial F in $M[\mathbf{t}]$ of positive degree n such that $F(\mathbf{f}) \in I^{n+1}M$, one has $F \in IM[\mathbf{t}]$.

Definition 3.4.1 If the map φ is an isomorphism and $IM \neq M$, the sequence $f_1, \ldots, f_q$ is called an *M-quasi-regular sequence*.

Theorem 3.4.2 *Let M be an R-module and $\mathbf{f} = f_1, \ldots, f_q$ a sequence in R. If $\mathbf{f}$ is an M-regular sequence, then $\mathbf{f}$ is an M-quasi-regular sequence.*

Proof. It follows adapting the proof of Theorem 3.3.14. □

Theorem 3.4.3 (Krull Intersection Theorem) Let M be an R-module and I an ideal of R. Then there is $a \in R$ such that

$$a \equiv 1 \bmod(I) \quad \text{and} \quad a \cdot \left(\bigcap_{i=1}^{\infty} I^i M \right) = 0.$$

Proof. See [218, Theorem 8.9] or [298, Appendix A]. □

Lemma 3.4.4 *Let $(R, \mathfrak{m})$ be a local ring (resp. $\mathbb{N}$-graded ring) and M an R-module (resp. $\mathbb{N}$-graded module). If N is a submodule of M (resp. graded submodule) and $I \subset \mathfrak{m}$ is an ideal of R (resp. graded ideal), then*

$$N = \bigcap_{i=1}^{\infty} (N + I^i M).$$

Proof. Using Krull's intersection theorem one has

$$\bigcap_{i=1}^{\infty} I^i M' = (0)$$

(this equality holds also in the graded case), where $M' = M/N$. Hence the required equality follows at once. □

Proposition 3.4.5 *Let $(R, \mathfrak{m})$ be a local ring (resp. $\mathbb{N}$-graded ring) and M an R-module (resp. $\mathbb{N}$-graded module). If $\mathbf{f} = f_1, \ldots, f_q$ is an M-quasi regular sequence of elements in $\mathfrak{m}$ (resp. homogeneous elements in $\mathfrak{m} = R_+$), then $\mathbf{f}$ is an M-regular sequence.*

Proof. Fix an integer $1 \leq r \leq q$ and set $I = (f_1, \ldots, f_q)$. To begin with we split the ideal I as $I = J + L$, where $J = (f_1, \ldots, f_{r-1})$ and $L = (f_r, \ldots, f_q)$, it is convenient to set $J = (0)$ if $r = 1$. It suffices to show the equality $(JM :_M f_r) = JM$, because this is equivalent to prove that f_r is not a zero divisor of M/JM. Let $m \in (JM :_M f_r)$, according to Lemma 3.4.4 one has

$$\bigcap_{n=1}^{\infty} (JM + L^n M) = JM,$$

thus the proof reduces to proving by induction on n that $m \in JM + L^nM$ for $n \geq 1$. Since $m \in (JM{:}_M f_r)$, there are m_i in M such that

$$mf_r = m_1f_1 + \cdots + m_{r-1}f_{r-1} \quad (m, m_i \in M). \tag{3.1}$$

Consider the polynomial

$$F = mt_r - m_1t_1 - \cdots - m_{r-1}t_{r-1} \in M[\mathbf{t}].$$

Note $\deg_t(F) = 1$ and $F(\mathbf{f}) = 0$. As $\mathbf{f}$ is an M-quasi regular sequence we get $m \in IM = JM + LM$. By induction assume $m \in JM + L^nM$, that is, there is $G \in M[t_r, \ldots, t_q]$ homogeneous of degree n such that

$$m = b_1f_1 + \cdots + b_{r-1}f_{r-1} + G(f_r, \ldots, f_q) \quad (b_i \in M). \tag{3.2}$$

Combining Eq. (3.1) and Eq. (3.2) yields

$$f_rG(f_r, \ldots, f_q) = a_1f_1 + \cdots + a_{r-1}f_{r-1} \quad (a_i \in M).$$

Using that $\mathbf{f}$ is an M-quasi regular sequence one may assume (after an induction argument) that $a_i \in I^nM$, and consequently $t_rG(t_r, \ldots, t_q)$ is in $IM[\mathbf{t}]$, that is, $G(t_r, \ldots, t_q)$ is in $IM[\mathbf{t}]$. One can write $G = G_1 + G_2$, where G_1, G_2 are homogeneous polynomials in $M[\mathbf{t}]$ of degree n with $G_1 \in JM[\mathbf{t}]$ and $G_2 \in LM[\mathbf{t}]$. Since $G_1(f_r, \ldots, f_q)$ is in JM and $G_2(f_r, \ldots, f_q)$ is in $L^{n+1}M$, from Eq. (3.2) we get that m is in $JM + L^{n+1}M$, as required. □

Multiplicities Let $(R, \mathfrak{m})$ be a local ring, M a finitely generated R-module and $\mathfrak{q}$ an ideal with $\operatorname{rad}(\mathfrak{q}) = \mathfrak{m}$. As $\operatorname{gr}_{\mathfrak{q}}(M)$ is a finitely generated $\operatorname{gr}_{\mathfrak{q}}(R)$-module and $A_0 = R/\mathfrak{q}$ is Artinian, by Theorem 1.2.8 there exists a polynomial $P(t) \in \mathbb{Q}[t]$ of degree equal to $d - \mathbf{1}$, where $d = \dim(\operatorname{gr}_{\mathfrak{q}}(M))$, such that

$$P(i) = \ell_{A_0}(\mathfrak{q}^iM/\mathfrak{q}^{i+1}M), \quad \text{for } i \geq n_0.$$

By Lemma 1.2.10 there are integers $a_0, \ldots, a_{d-1}$ such that

$$P(i) = \sum_{j=0}^{d-1} a_j \binom{i+j}{j}, \quad \text{for all } i \geq 0.$$

From the short exact sequences

$$0 \longrightarrow \mathfrak{q}^iM/\mathfrak{q}^{i+1}M \longrightarrow M/\mathfrak{q}^{i+1}M \longrightarrow M/\mathfrak{q}^iM \longrightarrow 0,$$

one derives $\ell(M/\mathfrak{q}^{i+1}M) - \ell(M/\mathfrak{q}^iM) = P(i)$ for $i \geq n_0$. Hence using the identity

$$\binom{d+m}{d} = \sum_{j=0}^{m} \binom{j+d-1}{d-1},$$

we get

$$\begin{aligned}\ell(M/\mathfrak{q}^{i+1}M) &= \ell(M/\mathfrak{q}^{n_0}M) + \sum_{j=n_0}^{i} P(j) \\ &= c_0 + \sum_{j=0}^{d-1} a_j \binom{j+i+1}{j+1},\end{aligned}$$

for $i \geq n_0$. Altogether the function

$$\chi_M^{\mathfrak{q}}(i) = \ell(M/\mathfrak{q}^{i+1}M)$$

is a polynomial function of degree d, called the *Samuel function* of M with respect to $\mathfrak{q}$. The integer a_{d-1}, is the *multiplicity* of M with respect to $\mathfrak{q}$ and is denoted by $e(\mathfrak{q}, M)$. Note that $e(\mathfrak{q}, M)/d!$ is the leading coefficient of $\chi_M^{\mathfrak{q}}(i)$.

We recall the following result in dimension theory that relates the degree of the Samuel function with the dimension of the module.

Theorem 3.4.6 *The Samuel function*

$$\chi_M^{\mathfrak{q}}(i) = \ell(M/\mathfrak{q}^{i+1}M)$$

is a polynomial function for $i \gg 0$ of degree equal to the dimension of M.

Proof. See [218, Theorem 13.4]. □

Definition 3.4.7 Let $(R, \mathfrak{m})$ be a local ring of dimension d, M an R-module and $\mathfrak{q}$ an ideal of R with $\operatorname{rad}(\mathfrak{q}) = \mathfrak{m}$. We define

$$e_d(\mathfrak{q}, M) = \begin{cases} e(\mathfrak{q}, M) & \text{if } \dim(M) = d, \\ 0 & \text{if } \dim(M) < d. \end{cases}$$

In order to show how the "multiplicity" behaves under short exact sequences we need to recall a result of E. Artin and D. Rees.

Theorem 3.4.8 (Artin-Rees lemma) *Let M be a module over a ring R. If N is a submodule of M and I an ideal of R, then there is a positive integer c such that*

$$I^n M \cap N = I^{n-c}(I^c M \cap N), \ \forall n > c.$$

Proof. See [218, Theorem 8.5]. □

Proposition 3.4.9 *Let $(R,\mathfrak{m})$ be a local ring of dimension d and let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of R. If*

$$0 \longrightarrow N \longrightarrow M \longrightarrow N' \longrightarrow 0$$

is an exact sequence of R-modules, then

$$e_d(\mathfrak{q}, M) = e_d(\mathfrak{q}, N) + e_d(\mathfrak{q}, N').$$

Proof. By Proposition 1.1.32 one has

$$\dim(M) = \max\{\dim(N), \dim(N')\},$$

hence one may assume $d = \dim(M)$. Tensoring with $R/\mathfrak{q}^{n+1}$ the exact sequence above yields an exact sequence

$$0 \to (N \cap \mathfrak{q}^{n+1}M)/\mathfrak{q}^{n+1}N \to N/\mathfrak{q}^{n+1}N \to M/\mathfrak{q}^{n+1}M \to N'/\mathfrak{q}^{n+1}N' \to 0.$$

Taking lengths with respect to $R/\mathfrak{q}$ gives

$$\chi^{\mathfrak{q}}_M(n) = \chi^{\mathfrak{q}}_N(n) + \chi^{\mathfrak{q}}_{N'}(n) - \ell(N \cap \mathfrak{q}^{n+1}M/\mathfrak{q}^{n+1}N). \qquad (*)$$

By the Artin-Rees lemma $N \cap \mathfrak{q}^{n+1}M \subset \mathfrak{q}^{n+1-c}N$, for some integer $c > 0$. Hence

$$\ell(N \cap \mathfrak{q}^{n+1}M/\mathfrak{q}^{n+1}N) \leq \ell(\mathfrak{q}^{n+1-c}N/\mathfrak{q}^{n+1}N) = \chi^{\mathfrak{q}}_N(n) - \chi^{\mathfrak{q}}_N(n-c),$$

as $\chi^{\mathfrak{q}}_N(n) - \chi^{\mathfrak{q}}_N(n-c)$ is a polynomial function of degree at most $d-1$, the result follows by dividing Eq. $(*)$ by n^d and taking limits when n goes to infinity. □

Proposition 3.4.10 *Let $(R,\mathfrak{m})$ be a local ring of dimension d and $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal. If M is a finitely generated R-module and $\mathcal{A}$ is the set of all prime ideals $\mathfrak{p}$ of R with $\dim(R/\mathfrak{p}) = d$, then*

$$e_d(\mathfrak{q}, M) = \sum_{\mathfrak{p}\in\mathcal{A}} \ell(M_{\mathfrak{p}}) e_d(\mathfrak{q}, R/\mathfrak{p}).$$

Proof. Set $\mathcal{B} = \mathcal{A} \cap \operatorname{Supp}(M)$. If $\mathcal{B} = \emptyset$, then $\dim(M) < d$ and $e_d(\mathfrak{q}, M)$ is equal to 0, thus in this case the identity above holds. Hence we may assume $\mathcal{B} \neq \emptyset$, this yields the equality $\dim(M) = d$. By Theorem 1.1.15 there are prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of R and a filtration of submodules:

$$(0) = M_0 \subset M_1 \subset \cdots \subset M_n = M,$$

such that $M_i/M_{i-1} \simeq R/\mathfrak{p}_i$ for all i. Observe

$$\mathcal{B} = \{\mathfrak{p}_i |\ \dim(R/\mathfrak{p}_i) = d\},$$

to show this equality note

$$\mathrm{Ass}(M) \subset \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \subset \mathrm{Supp}(M),$$

see Corollary 1.1.17 and its proof, and recall that the minimal elements of $\mathrm{Supp}(M)$ are in $\mathrm{Ass}(M)$. Using Lemma 3.4.9 and the exact sequences

$$0 \longrightarrow M_{i-1} \longrightarrow M_i \longrightarrow R/\mathfrak{p}_i \longrightarrow 0 \quad (i = 1, \ldots, n),$$

we get

$$e(\mathfrak{q}, M) = \sum e(\mathfrak{q}, R/\mathfrak{p}_i),$$

where the sum is taken over all $\mathfrak{p}_i$ in the multiset $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ such that $\dim(R/\mathfrak{p}_i) = d$. Let $\mathfrak{p} \in \mathcal{B}$, then

$$(M_i/M_{i-1})_{\mathfrak{p}} \simeq \begin{cases} (0) & \text{if} \quad \mathfrak{p} \neq \mathfrak{p}_i \\ R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} & \text{if} \quad \mathfrak{p} = \mathfrak{p}_i. \end{cases} \tag{3.3}$$

Since the second module is a field we get that the length of $M_{\mathfrak{p}}$ is equal to the number of times that $\mathfrak{p}$ occurs in the multiset $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Therefore

$$e(\mathfrak{q}, M) = \sum_{\mathfrak{p}_i \in \mathcal{B}} \ell(M_{\mathfrak{p}_i}) e(\mathfrak{q}, R/\mathfrak{p}_i),$$

as required. This proof was adapted from [44]. □

Proposition 3.4.11 *Let $(S, \mathfrak{m})$ be a local ring and M an S-module with a positive rank $r = \mathrm{rank}(M)$. If $\mathfrak{q}$ is an $\mathfrak{m}$-primary ideal, then*

$$e(\mathfrak{q}, M) = e(\mathfrak{q}, S)\mathrm{rank}(M).$$

Proof. First note $d = \dim(M) = \dim(S)$ by Lemma 1.1.37. Let $\mathcal{A}$ be the set of all prime ideals $\mathfrak{p}$ of S with $\dim(S/\mathfrak{p}) = d$. Since M has rank r one has $M_{\mathfrak{p}} \simeq (S_{\mathfrak{p}})^r$ for any associated prime $\mathfrak{p}$ of S, thus $M_{\mathfrak{p}} \simeq (S_{\mathfrak{p}})^r$ for any $\mathfrak{p} \in \mathcal{A}$. Therefore using Proposition 3.4.10 one obtains

$$\begin{aligned} e_d(\mathfrak{q}, M) &= \sum_{\mathfrak{p} \in \mathcal{A}} \ell(M_{\mathfrak{p}}) e_d(\mathfrak{q}, S/\mathfrak{p}) = \sum_{\mathfrak{p} \in \mathcal{A}} \ell(S_{\mathfrak{p}}^r) e_d(\mathfrak{q}, S/\mathfrak{p}) \\ &= \sum_{\mathfrak{p} \in \mathcal{A}} r\ell(S_{\mathfrak{p}}) e_d(\mathfrak{q}, S/\mathfrak{p}) = \mathrm{rank}(M) e_d(\mathfrak{q}, S). \end{aligned}$$

As $e(\mathfrak{q}, M) = e_d(\mathfrak{q}, M)$ and $e(\mathfrak{q}, S) = e_d(\mathfrak{q}, S)$, the proof is complete. □

In Chapter 4 we use the following important Cohen-Macaulay criterion of Jürgen Herzog to study the Koszul homology of graded ideals.

Theorem 3.4.12 ([148]) *Let $(S,\mathfrak{m})$ be a Cohen-Macaulay local ring and let M be a finitely generated S-module with a well defined and positive rank. If $\mathbf{y}=y_1,\ldots,y_d$ is a system of parameters of S, then*

$$\ell(S/(\mathbf{y}))\cdot \operatorname{rank}(M)\leq \ell(M/(\mathbf{y})M).$$

Furthermore equality holds if and only if M is Cohen-Macaulay.

Proof. By Lemma 1.1.37 one has $d=\dim(M)=\dim(S)$. Let $\mathfrak{q}$ be the $\mathfrak{m}$-primary ideal generated by $\mathbf{y}$. There is a (graded) epimorphism

$$\varphi\colon (M/\mathfrak{q}M)[\mathbf{t}]=(M/\mathfrak{q}M)[t_1,\ldots,t_d]\to \operatorname{gr}_{\mathfrak{q}}(M)=\bigoplus_{i=0}^{\infty}\mathfrak{q}^iM/\mathfrak{q}^{i+1}M \quad (3.4)$$

such that $\varphi(\overline{F})=F(y_1,\ldots,y_d)+\mathfrak{q}^{i+1}M$, for all F in $M[\mathbf{t}]$ homogeneous of degree i, where $\overline{F}$ means reducing the coefficients of F modulo $\mathfrak{q}M$. By restriction of φ to the ith graded component of $(M/\mathfrak{q}M)[\mathbf{t}]$ gives

$$\ell(M/\mathfrak{q}M)\binom{i+d-1}{d-1}\geq \ell(\mathfrak{q}^iM/\mathfrak{q}^{i+1}M)\quad \text{for } i\geq 0, \quad (3.5)$$

observe that both sides of the inequality are polynomial functions of degree $d-1$ and consequently $\ell(M/\mathfrak{q}M)\geq e(\mathfrak{q},M)$. On the other hand $\mathbf{y}$ is a regular sequence, because S is Cohen-Macaulay (see Proposition 1.3.17), this forces φ to be an isomorphism when $M=S$ (see Theorem 3.4.2). It follows rapidly that $\ell(S/\mathfrak{q})=e(\mathfrak{q},S)$. To finish the first part of the proof note that making use of Proposition 3.4.11 one obtains the required inequality.

At this point one should observe that, by the arguments above, the proof reduces to show that $\ell(M/\mathfrak{q}M)=e(\mathfrak{q},M)$ if and only if M is Cohen-Macaulay. Next we show both implications.

$\Rightarrow$) Taking into account Proposition 3.4.5 and Proposition 1.3.17 it suffices to prove that φ is injective; because this implies that $\mathbf{y}$ is an M-regular sequence and hence M is Cohen-Macaulay.

Using Eq. (3.4) we obtain an exact sequence

$$0\longrightarrow \ker(\varphi)_i\longrightarrow ((M/\mathfrak{q}M)[\mathbf{t}])_i\longrightarrow \mathfrak{q}^iM/\mathfrak{q}^{i+1}M\longrightarrow 0,$$

hence we get

$$\ell(\ker(\varphi)_i)=\ell(M/\mathfrak{q}M)\binom{i+d-1}{d-1}-\ell(\mathfrak{q}^iM/\mathfrak{q}^{i+1}M)\quad \text{for } i\geq 0.$$

As the polynomial functions on the right hand side have both degree $d-1$ and their leading terms cancel out, one concludes that $\ell(\ker(\varphi)_i)$ grows as a polynomial function of degree at most $d-2$.

Let $F = \sum m_\alpha t^\alpha$ be a homogeneous polynomial in $M[\mathbf{t}]$ of degree p, with m_α in M, and denote $\overline{m}_\alpha = m_\alpha + \mathfrak{q}M$. Assume $\overline{F}$ is in $\ker(\varphi)$ and $\overline{F} \neq 0$. Since $\mathfrak{m}^k \subset \mathfrak{q}$, there is an $s \geq 1$ such that $\mathfrak{m}^{s-1}\overline{F} \neq 0$ and $\mathfrak{m}^s\overline{F} = 0$, thus one may assume $\mathfrak{m}\overline{F} = 0$ and $\overline{F} \neq 0$. Hence there is a graded epimorphism ψ induced by multiplication by $\overline{F}$

$$\psi\colon (S/\mathfrak{m})[\mathbf{t}](-p) \longrightarrow (S/\mathfrak{q})[\mathbf{t}]\overline{F}.$$

We now show that ψ is injective. Let $g = \sum \overline{b}_\beta t^\beta$ be an element in the kernel of ψ, where $b_\beta \in S$ and $\overline{b}_\beta = b_\beta + \mathfrak{m}$. Thus $(\sum \widetilde{b}_\beta t^\beta)(\sum \overline{m}_\alpha t^\alpha) = 0$, where $\widetilde{b}_\beta = b_\beta + \mathfrak{q}$. One may assume that $\overline{b}_\beta t^\beta$ and $\overline{m}_\alpha t^\alpha$ are the leading terms of g and $\overline{F}$ respectively, w.r.t the lexicographical ordering of the t_i variables. Thus $\widetilde{b}_\beta \overline{m}_\alpha t^{\alpha+\beta}$ is the leading term of $g\overline{F}$ and $b_\beta m_\alpha \in \mathfrak{q}M$. Hence $b_\beta + \mathfrak{q}$ is a zero divisor of $M/\mathfrak{q}M$ and since $\operatorname{Ass}_{R/\mathfrak{q}}(M/\mathfrak{q}M) = \{\mathfrak{m}/\mathfrak{q}\}$, one has that $b_\beta + \mathfrak{q}$ is in $\mathfrak{m}/\mathfrak{q}$, which proves $b_\alpha \in \mathfrak{m}$ and $\overline{b}_\beta = 0$. Altogether one derives $g = 0$ and ψ is injective. Using that ψ is a graded isomorphism yields

$$\binom{i-p+d-1}{d-1} = \ell(((S/\mathfrak{q})[\mathbf{t}]\overline{F})_i) \leq \ell(\ker(\varphi)_i),$$

which is a contradiction because the left hand side is a polynomial function of degree $d-1$. Therefore $\overline{F} = 0$ and φ is injective, as required.

$\Leftarrow$) As M is Cohen-Macaulay, the sequence $\mathbf{y}$ is an M-regular sequence, an application of Theorem 3.4.2 yields that φ is an isomorphism. Therefore Eq. (3.5) becomes an equality and we get $\ell(M/\mathfrak{q}M) = e(\mathfrak{q}, M)$. □

3.5 Jacobian criterion

Here we introduce the Jacobian criterion and present some applications and examples to illustrate its use. Along the way some facts about regular local rings are discussed.

Let $(R, \mathfrak{m})$ be a local ring. By Nakayama's Lemma it follows rapidly that R is a regular local ring if and only if $\mathfrak{m}$ is generated by a system of parameters of R.

If R is a polynomial ring over a field k, then R is a regular ring, that is, $R_\mathfrak{p}$ is a regular local ring for all $\mathfrak{p}$ in $\operatorname{Spec}(R)$ or equivalently $R_\mathfrak{m}$ is a regular local ring for any maximal ideal $\mathfrak{m}$ of R (see [218, Theorem 19.5]).

Proposition 3.5.1 *If R is a regular local ring, then R is Cohen-Macaulay.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R and let $k = R/\mathfrak{m}$ be its residue field. Assume $x_1, \ldots, x_d$ is a set of generators of $\mathfrak{m}$, where $d = \dim(R)$. If

$k[t_1, \ldots, t_d]$ is a ring of polynomial over the field k, there is an epimorphism of graded k-algebras

$$\varphi\colon k[\mathbf{t}] = k[t_1, \ldots, t_d] \longrightarrow \operatorname{gr}_{\mathfrak{m}}(R) = \bigoplus_{i=0}^{\infty} \mathfrak{m}^i/\mathfrak{m}^{i+1}, \tag{3.6}$$

induced by $\varphi(t_i) = x_i + \mathfrak{m}^2$ for all i. It suffices to prove that the map φ is injective. Indeed if φ is injective, then $x_1, \ldots, x_d$ is a regular sequence by Proposition 3.4.5, and hence R is Cohen-Macaulay by Proposition 1.3.17.

If $I = \ker(\varphi) \neq 0$, pick a homogeneous polynomial f in I of degree $s \geq 1$. Using the isomorphism

$$k[\mathbf{t}](-s) \simeq fk[\mathbf{t}]$$

together with Eq. (3.6) one concludes:

$$\dim_k(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \dim_k k[\mathbf{t}]_i - \dim_k I_i \leq \binom{i+d-1}{d-1} - \binom{i-s+d-1}{d-1}.$$

By Theorem 3.4.6 the left hand side is a polynomial function of degree $d-1$ while the right hand side is a polynomial function of degree $d-2$. Therefore I must be zero and φ is injective. □

Corollary 3.5.2 *If $(R, \mathfrak{m})$ is a regular local ring, then R is a domain.*

Proof. By Proposition 3.5.1 $\mathfrak{m}$ is generated by a regular system of parameters. Thus $\operatorname{gr}_{\mathfrak{m}}(R)$ is a domain. Let $x, y \in R$ such that $xy = 0$. If $x \neq 0$ and $y \neq 0$, then by Theorem 3.4.3 there are $r, s \in \mathbb{N}$ with $x \in \mathfrak{m}^r \setminus \mathfrak{m}^{r+1}$ and $y \in \mathfrak{m}^s \setminus \mathfrak{m}^{s+1}$. Multiplying the images $\overline{x}, \overline{y}$ of x, y in $\operatorname{gr}_{\mathfrak{m}}(R)$, one obtains $\overline{x}\overline{y} = 0$. Hence $\overline{x} = 0$ or $\overline{y} = 0$, which is impossible. This proves R is a domain. □

Proposition 3.5.3 *Let $(R, \mathfrak{m})$ be a regular local ring and $I \neq R$ an ideal of R. If R/I is a regular local ring, then I is a complete intersection.*

Proof. Let $\overline{x}_1, \ldots, \overline{x}_d$ be a generating set of $\overline{\mathfrak{m}} = \mathfrak{m}/I$, where $\overline{x}_i = x_i + I$ and d is the dimension of R/I. Note that the set of images of $x_1, \ldots, x_d$ in $\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$ is a basis for $\overline{\mathfrak{m}}/\overline{\mathfrak{m}}^2$ as a vector space over $k = R/\mathfrak{m}$. Hence

$$x_1 + \mathfrak{m}^2, \ldots, x_d + \mathfrak{m}^2$$

are linearly independent in $\mathfrak{m}/\mathfrak{m}^2$. Set $n = \dim(R)$ and $\mathfrak{m}' = (x_1, \ldots, x_d)$. As R is a regular local ring $n = \dim_k(\mathfrak{m}/\mathfrak{m}^2)$, hence using the proof of Corollary 1.1.30 and the equality $\mathfrak{m} = \mathfrak{m}' + I$ one derives $\mathfrak{m} = \mathfrak{m}' + J$, for some ideal $J = (x_{d+1}, \ldots, x_n)$ contained in I.

By Proposition 3.5.1 $x_1, \ldots, x_n$ is a regular sequence, thus it suffices to show $J = I$, to prove this equality observe that R/J is a regular local ring

of dimension d, and therefore R/J is a domain by Corollary 3.5.2. Since R/I has also dimension d, we see that the canonical homomorphism

$$\varphi: R/J \to R/I$$

is an isomorphism, thus $I = J$, as asserted. □

Definition 3.5.4 Let $R = k[x_1, \ldots x_n]$ be a polynomial ring over a field k and $I = (f_1, \ldots, f_q)$ an ideal. The *Jacobian matrix* of I is the matrix

$$\mathfrak{J} = (\partial f_i / \partial x_j).$$

We denote the Jacobian matrix of I taken modulo an ideal P by

$$(\partial f_i / \partial x_j)(P).$$

Theorem 3.5.5 (Jacobian criterion) *Let $B = R/I$ be a quotient ring, where $R = k[x_1, \ldots, x_n]$ is a polynomial ring in n variables over a field k, and let $I = (f_1, \ldots, f_q) \subset R$ be an ideal. If $P \subset R$ is a prime ideal containing I and $\mathfrak{p} = P/I$, one has:*

(a) $\operatorname{rank}(\partial f_i / \partial x_j)(P) \leq \operatorname{ht}(I_P)$.

(b) *If* $\operatorname{rank}(\partial f_i / \partial x_j)(P) = \operatorname{ht}(I_P)$, *then $B_{\mathfrak{p}}$ is a regular ring.*

(c) *If k is a perfect field and $B_{\mathfrak{p}}$ is regular, then*

$$\operatorname{rank}(\partial f_i / \partial x_j)(P) = \operatorname{ht}(I_P).$$

Proof. See [217, p. 213]. □

Lemma 3.5.6 *Let R be a ring and I an ideal. If I is height unmixed and P is a prime ideal such that $I \subset P$, then I_P is also height unmixed and* $\operatorname{ht}(I) = \operatorname{ht}(I_P)$.

Proof. Let

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$$

be a primary decomposition of I, we may assume $\sqrt{\mathfrak{q}_i} \subset P$ for $i = 1, \ldots s$. Localizing at P gives a primary decomposition

$$IR_P = \bigcap_{i=1}^{s} \mathfrak{q}_i R_P.$$

Hence the associated primes of I_P are $\sqrt{\mathfrak{q}_i} R_P$, where $i \leq s$. To finish the proof note $\operatorname{ht}(\sqrt{\mathfrak{q}_i} R_P) = \operatorname{ht}(\sqrt{\mathfrak{q}_i}) = \operatorname{ht}(I)$ for $i \leq s$. □

Definition 3.5.7 Let k be a field and $I = (f_1, \ldots, f_q)$ an ideal of height g of $k[x_1, \ldots, x_n]$. The *Jacobian ideal* J of I is the ideal generated by the $g \times g$ minors of the Jacobian matrix $\mathfrak{J} = (\partial f_i / \partial x_j)$.

Corollary 3.5.8 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a perfect field k and $I = (f_1, \ldots, f_q) \subset R$ an unmixed ideal. If P is a prime ideal of R containing I and $\mathfrak{p} = P/I$, then $(R/I)_{\mathfrak{p}}$ is regular if and only if $J/I \not\subset \mathfrak{p}$, where J is the Jacobian ideal of I.*

Proof. $\Rightarrow$) By Theorem 3.5.5 and Lemma 3.5.6 one has

$$\operatorname{rank}(\partial f_i / \partial x_j)(P) = \operatorname{ht}(I),$$

hence there is $f \in J \setminus P$ and $J/I \not\subset P/I$.

$\Leftarrow$) If $(R/I)_{\mathfrak{p}}$ is not regular, then by Theorem 3.5.5 one has the inequality $\operatorname{rank}(\partial f_i / \partial x_j)(P) < \operatorname{ht}(I)$, hence $J/P = (0)$ and $J/I \subset P/I$, which is impossible. Note that this part of the proof is valid for any field k. □

Proposition 3.5.9 *Let R be a polynomial ring over a field k and I an unmixed ideal of height g. If J is the Jacobian ideal of I and $\operatorname{ht}(J, I) \geq g+2$, then $(R/I)_{\mathfrak{p}}$ is regular for any prime $\mathfrak{p}$ of R/I such that $\operatorname{ht}(\mathfrak{p}) \leq 1$.*

Proof. Let $B = R/I$ and $\mathfrak{p} = P/I$, where P is a prime ideal of R. If $B_{\mathfrak{p}}$ is not regular, then according to Corollary 3.5.8 one has $J/I \subset \mathfrak{p}$ and hence $(J, I) \subset P$. Since R is a catenary ring and I is unmixed one obtains $\operatorname{ht}(P) = g$ if $\operatorname{ht}(\mathfrak{p}) = 0$ and $\operatorname{ht}(P) = g + 1$ if $\operatorname{ht}(\mathfrak{p}) = 1$. Therefore $\operatorname{ht}(J, I) \leq \operatorname{ht}(P) \leq g + 1$, which is impossible. Thus $B_{\mathfrak{p}}$ must be regular, as required. □

Another consequence of the Jacobian criterion is the following radical test of [294]. Let k be a field and $I = (f_1, \ldots, f_q) \subset R = k[x_1, \ldots, x_n]$ an ideal of height g. Let J be the Jacobian ideal of I generated by the $g \times g$ minors of the Jacobian matrix $\mathfrak{J} = (\partial f_i / \partial x_j)$.

Theorem 3.5.10 (Vasconcelos) *If k is a perfect field and I is unmixed, then I is a radical ideal if and only if there is an element f in $J \setminus I$ such that $(I\colon f) = I$.*

Proof. $\Rightarrow$) Let $P_1, \ldots, P_r$ be the associated primes of I. If $J \subset \mathcal{Z}(R/I)$, then $J \subset P_i$ for some i because $\mathcal{Z}(R/I) = \cup_{i=1}^{r} P_i$. Hence by Corollary 3.5.8 one derives that $(R/I)_{P_i}$ is not a regular ring, which is a contradiction because $(R/I)_{P_i} = R_{P_i}/P_i R_{P_i}$ is a field.

$\Leftarrow$) Let $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ be an irredundant primary decomposition of I and $P_i = \sqrt{\mathfrak{q}_i}$ an associated prime ideal of I. Note $(R/I)_{P_i} \simeq (R/I)_{\mathfrak{p}_i}$ and $I_{P_i} = (\mathfrak{q}_i)_{P_i}$, where $\mathfrak{p}_i = P_i/I$. First we observe that $(R/I)_{\mathfrak{p}_i}$ is a regular ring; otherwise by Corollary 3.5.8 one has $J/I \subset P_i/I$, hence $f \in P_i$, which

is impossible because f is regular modulo I. Therefore $(R/I)_{P_i}$ is a regular local ring and thus an integral domain by Theorem 1.4.16. As $I_{P_i} = (\mathfrak{q}_i)_{P_i}$ must be prime, one concludes that $\mathfrak{q}_i$ is also prime and consequently $\mathfrak{q}_i = P_i$. Note that here the hypothesis k perfect is not being used. □

Example 3.5.11 Let $R = k[x_1, \ldots, x_7]$ and $I = (x_1x_7 - x_2x_6, x_3x_5 - x_4x_7)$, where k is a field. We now show that $B = R/I$ is a normal ring using Theorem 1.4.15. The Jacobian matrix of I is:

$$\mathfrak{J} = \begin{pmatrix} x_7 & -x_6 & 0 & 0 & 0 & -x_2 & x_1 \\ 0 & 0 & x_5 & -x_7 & x_3 & 0 & -x_4 \end{pmatrix}.$$

Note that the monomials

$$x_7^2,\ x_5x_6,\ x_3x_6,\ x_4x_6,\ x_2x_5,\ x_1x_5,\ x_2x_4$$

are in the Jacobian ideal J of I. Hence $\operatorname{ht}(J, I) \geq g+2$, where $g = \operatorname{ht}(I) = 2$. By Proposition 3.5.9 one has that B satisfies (R_1) and since I is a complete intersection it follows by Proposition 2.2.17 that B satisfies (S_2). Thus by Serre's normality criterion the ring B is normal.

Lemma 3.5.12 *Let R be a ring and I a prime ideal. If $f \in R \setminus I$ and $I_f^{(n)} = I_f^n$ for some $n \geq 1$, then*

$$I^{(n)} = (I^n \colon f^\infty).$$

Proof. If $z \in (I^n \colon f^\infty)$, then $zf^k \in I^n$ for some $k \geq 1$, hence $z \in I^{(n)}$. Conversely if $z \in I^{(n)}$, then $sz \in I^n$ for some $s \notin I$. Since $s/1 \notin I_f$ and $(s/1)(z/1) \in I_f^n$, then $z/1$ is in the nth symbolic power of I_f. Thus by hypothesis $z/1 \in I_f^n$ and this means that $z \in (I^n \colon f^\infty)$. □

There are a few methods to compute symbolic powers of prime ideals in polynomial rings, see [257] and [298, Chapter 3]. The following is a subtle application of the Jacobian criterion due to Wolmer Vasconcelos.

Proposition 3.5.13 Let R be a polynomial ring over a field k and I a prime ideal. If f is an element in the Jacobian ideal J of I which is not in the ideal I, then

$$I^{(n)} = (I^n \colon f^\infty) \quad \text{for } n \geq 1,$$

and such element f exist if k is a perfect field.

Proof. Since the localization R_f is Cohen-Macaulay, by Proposition 3.3.29 and Lemma 3.5.12 it is enough to prove that I_f is locally a complete intersection. Let P be a prime ideal such that $I \subset P$ and $f \notin P$. Set $\mathfrak{p} = P/I$. One has

$$(R_f/I_f)_{P_f} \simeq (R_f)_{P_f}/(I_f)_{P_f} \simeq R_P/I_P \simeq (R/I)_P \simeq (R/I)_{\mathfrak{p}}.$$

As $J/I \not\subset \mathfrak{p}$, using Corollary 3.5.8 we conclude that R_P/I_P is a regular ring and consequently $(I_f)_{P_f}$ is a complete intersection (see Proposition 3.5.3). The last assertion follows from Theorem 3.5.10. □

Example 3.5.14 Let $R = \mathbb{Q}[x, y, z]$ and

$$I = (x^3 - yz, y^2 - xz, z^2 - x^2y).$$

The ideal I is prime because it is the kernel of the homomorphism:

$$\varphi\colon \mathbb{Q}[x, y, z] \longrightarrow \mathbb{Q}[t], \quad \text{where } \varphi(h(x, y, z)) = h(t^3, t^4, t^5).$$

Using *CoCoA* [60] and Proposition 3.5.13 we get

$$I^{(2)} = (I^2 \colon f^\infty) = (I^2, \Delta),$$

where $f = 2y^2 + xz$ and $\Delta = -x^5 + 3x^2yz - xy^3 - z^3$. See [199].

Exercises

3.5.15 Use the following procedure in *CoCoA* [60] to verify the formula for the second symbolic power of Example 3.5.14

```
F1 := x^3 - y*z; -- variables must begin with capital letters
F2 := y^2 - x*z;
F3 := z^2 - x^2*y;
Jac := Jacobian([F1,F2,F3]);
J := Minors(2,Mat(Jac)); -- 2 x 2 minors of the Jacobian matrix
I := Ideal(F1,F2,F3);
H := Elim(t,I^2 + Ideal(1 - t(2*y^2 + x*z)));
Print(H);
```

3.5.16 Let R be a Cohen-Macaulay ring and I an ideal of R which is height unmixed. If I is a complete intersection, then I is locally a complete intersection.

Hint Use Lemma 1.3.18.

3.5.17 Let $R = \mathbb{Q}[x, y, z]$ and I the prime ideal generated by the set of binomials

$$f_1 = x^3 - yz, \ f_2 = y^2 - xz, \ f_3 = z^2 - x^2y.$$

Prove:

(a) $I_x = (f_1, f_2)$, that is, I_x is a complete intersection, and

(b) $I^{(n)} = (I^n \colon x^\infty)$ for $n \geq 1$.

3.5.18 Let R be a Cohen-Macaulay ring and I a prime ideal. If $f \in R \setminus I$ and I_f is a complete intersection, then

$$I^{(n)} = (I^n : f^\infty) \quad \text{for} \quad n \geq 1.$$

3.5.19 Let $R = \mathbb{Q}[x_1, \ldots, x_7]$ and let I be the principal ideal generated by the binomial

$$f = x_1x_3x_5x_7 - x_2x_4^2x_6.$$

Prove that R/I is a normal domain. Give a description of the irreducible binomials f such that $R/(f)$ is normal.

3.5.20 Let R be a Noetherian ring and a an element in the Jacobson radical of R. If a is regular and $R/(a)$ is an integral domain, prove that R is an integral domain. Recall that the *Jacobson radical* of R is the intersection of all the maximal ideals of R.

3.5.21 Let $I = I_2(X)$ be the ideal of 2×2-minors of the symmetric matrix:

$$X = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix},$$

where the entries of X are indeterminates over a field k. Prove that I_{x_1} is a complete intersection. Find a set of generators for $I^{(2)}$ and prove that $\operatorname{depth}(H_1) = 1$, where H_1 is the first Koszul homology module of I.

Chapter 4

Hilbert Series

Hilbert series of graded modules and algebras are introduced and studied in this chapter. The h-vector and a-invariant of a graded algebra S are defined through the Hilbert-Serre's theorem, when S is Cohen-Macaulay we present some of their properties. Some characteristics of Cohen-Macaulay and Gorenstein extremal algebras will be presented.

4.1 Hilbert-Serre's Theorem

Unless otherwise stated we shall always assume that modules are finitely generated and $\mathbb{N}$-graded.

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k with the grading induced by $\deg(x_i) = d_i$, where d_i is a positive integer. If

$$M = \bigoplus_{i=0}^{\infty} M_i$$

is a finitely generated $\mathbb{N}$-graded module over R, its *Hilbert function* and *Hilbert series* are defined by

$$H(M,i) = \ell(M_i) \text{ and } F(M,t) = \sum_{i=0}^{\infty} H(M,i)t^i$$

respectively, where $\ell(M_i)$ denotes the length of M_i as a k-module, in our case $\ell(M_i) = \dim_k(M_i)$. If $d \in \mathbb{N}$, then $M(-d)$ is the *regrading* of M obtained by a *shift* of the graduation of M, more precisely

$$M(-d) = \bigoplus_{i=0}^{\infty} M(-d)_i,$$

where $M(-d)_i = M_{-d+i}$. Note that we are assuming $M_i = 0$ for $i < 0$. In this way $M(-d)$ becomes an $\mathbb{N}$-graded R-module.

Lemma 4.1.1 $F(M(-d),t) = t^d F(M,t)$.

Proof. Since $M(-d)_i = M_{i-d}$ one has:

$$F(M(-d),t) = t^d \sum_{i=d}^{\infty} \ell(M_{i-d}) t^{i-d} = t^d F(M,t),$$

where the first equality follows using that $M_{i-d} = 0$ for $i = 0, \ldots, d-1$. □

Lemma 4.1.2 *Let M be a graded R-module. If $z \in R_d$, then there is a degree preserving exact sequence*

$$0 \longrightarrow (M/(0\colon z))(-d) \xrightarrow{z} M \xrightarrow{\varphi} M/zM \longrightarrow 0 \quad (\varphi(m) = m + zM),$$

where $(0\colon z) = \{m \in M \mid zm = 0\}$ and the first map is multiplication by z.

Proof. As the map $\psi\colon M(-d) \to M$, given by $\psi(m) = zm$, is a degree zero homomorphism one has that $(0\colon z)(-d)$ is a graded submodule of $M(-d)$. The exactness of the sequence above follows because ψ induces an exact sequence

$$0 \longrightarrow (0\colon z)(-d) \xrightarrow{\imath} M(-d) \xrightarrow{\psi} M \xrightarrow{\varphi} M/zM \longrightarrow 0,$$

where $\imath$ is an inclusion. □

Theorem 4.1.3 (Hilbert-Serre) *Let k be a field and $R = k[x_1, \ldots, x_n]$ a polynomial ring graded by $\deg(x_i) = d_i \in \mathbb{N}_+$. If M is a finitely generated $\mathbb{N}$-graded R-module, then the Hilbert series of M is a rational function that can be written as*

$$F(M,t) = \frac{h(t)}{\prod_{i=1}^{n}(1 - t^{d_i})} \quad \text{for some } h(t) \in \mathbb{Z}[t].$$

In particular if $d_i = 1$ for all i, then there is a unique polynomial $h(t) \in \mathbb{Z}[t]$ such that

$$F(M,t) = \frac{h(t)}{(1-t)^d} \quad \text{and} \quad h(1) \neq 0.$$

Proof. If $n = 0$, then

$$\dim_k \left(\bigoplus_{i=0}^{m} M_i \right) = \sum_{i=0}^{m} \dim_k(M_i) \leq \dim_k(M) < \infty, \text{ for all } m.$$

Hence the Hilbert function of M is zero for $i \gg 0$ and $F(M,t)$ is a polynomial. If $n > 0$, consider the exact sequence of graded modules

$$0 \longrightarrow (0\colon x_n)(-d_n) \longrightarrow M(-d_n) \xrightarrow{x_n} M \longrightarrow M/x_nM \longrightarrow 0.$$

Since the ends of this exact sequence are finitely generated modules over the polynomial ring $k[x_1,\ldots,x_{n-1}]$ the proof follows readily by induction on n. □

Remark 4.1.4 The conclusion of the Hilbert-Serre theorem is valid if we assume that $R = k[x_1,\ldots,x_n]$ is an $\mathbb{N}$-graded algebra over an Artinian ring k. See [9, Chapter 11] for details.

The integer d in the Hilbert-Serre's theorem will be denoted by $d(M)$; this integer will turn out to be equal to the Krull dimension of M, see Proposition 4.1.8 and its proof.

Definition 4.1.5 The degree of $F(M,t)$ as a rational function is denoted by $a(M)$, it is called the *a-invariant* of M.

The Hilbert polynomial One interesting invariant of a standard algebra is its degree or multiplicity, here we introduce multiplicities via Hilbert polynomials and explain how the a-invariant is useful to measure the difference between the Hilbert polynomial and the Hilbert function.

For the rest of this section we shall always assume that R is a polynomial ring over a field k with the usual grading.

Lemma 4.1.6 *Let $n \geq 1$ be an integer, then*

$$\frac{1}{(1-t)^n} = \sum_{m=0}^{\infty} \binom{m+n-1}{m} t^m.$$

Proof. As the case $n = 1$ is clear, by induction on n one has:

$$\begin{aligned} \frac{1}{(1-t)^{n+1}} &= \left[\frac{1}{n(1-t)^n}\right]' = \frac{1}{n}\left[\sum_{m=0}^{\infty}\binom{m+n-1}{m}t^m\right]' \\ &= \sum_{m=0}^{\infty}\frac{m+1}{n}\binom{m+n}{m+1}t^m. \end{aligned}$$

Using the identity

$$\frac{m+1}{n}\binom{m+n}{m+1} = \binom{m+n}{m}$$

the induction process is complete. □

Observe that Lemma 4.1.6 also follows rapidly from the classical *binomial expansion*:

$$(1+z)^r = \sum_{k\geq 0} \binom{r}{k} z^k, \ r \in \mathbb{R}.$$

Indeed recall that if $r \in \mathbb{R}$ and $k \in \mathbb{N}$, then

$$\binom{r}{k} = (-1)^k \binom{-r+k-1}{k}. \tag{4.1}$$

Using the binomial expansion for $(1-t)^{-n}$ one has

$$\frac{1}{(1-t)^n} = 1 + \sum_{m=1}^{\infty} (-1)^m \binom{-n}{m} t^m = 1 + \sum_{m=1}^{\infty} (-1)^{2m} \binom{m+n-1}{m} t^m.$$

Lemma 4.1.7 *If R is a polynomial ring in n variables with coefficients in a field k, then*

$$F(R,t) = \frac{1}{(1-t)^n} \quad \text{and} \quad H(R,m) = \binom{m+n-1}{n-1}.$$

Proof. Let x_1 be one of the variables. From the exact sequence

$$0 \longrightarrow R(-1) \xrightarrow{x_1} R \longrightarrow R/(x_1) \to 0,$$

we obtain

$$F(R,t) = F(R(-1),t) + F(R/(x_1),t)$$

Hence the formula for the Hilbert series follows by induction on n. To show the second equality use Lemma 4.1.6. □

Proposition 4.1.8 *Let R be a polynomial ring over a field k with the usual grading, and M a graded module over R. Then there are integers $a_{-d}, \ldots, a_{-1}$ so that*

$$H(M,i) = \sum_{j=0}^{d-1} a_{-(d-j)} \binom{i+d-j-1}{d-j-1}, \ \forall\ i \geq a(M)+1,$$

where $d = d(M)$ is the dimension of M and $a(M)$ is the degree of $F(M,t)$ as a rational function.

Proof. By the Hilbert-Serre's Theorem there is a polynomial $h(t) \in \mathbb{Z}[t]$ such that

$$F(M,t) = \frac{h(t)}{(1-t)^d}$$

and $h(1) \neq 0$. If $d = 0$, then $a(M)$ is equal to $\deg(h)$ and $H(M,i) = 0$ for $i \geq a(M) + 1$. Thus one may assume $d > 0$. Observe that by the division algorithm we can find $e(t) \in \mathbb{Z}[t]$ so that the Laurent expansion of $F(M,t) - e(t)$, in negative powers of $(1-t)$, is equal to

$$F(M,t) - e(t) = \sum_{j=0}^{d-1} \frac{a_{-(d-j)}}{(1-t)^{d-j}}, \text{ where } a_{-(d-j)} = \frac{(-1)^j h^{(j)}(1)}{j!}.$$

Next we expand $(1-t)^{d-j}$ in powers of t to obtain

$$F(M,t) = e(t) + \sum_{i=0}^{\infty} \left[\sum_{j=0}^{d-1} a_{-(d-j)} \binom{i+d-j-1}{d-j-1} \right] t^i = e(t) + \sum_{i=0}^{\infty} \varphi(i) t^i,$$

observe that $\varphi(i) = H(M,i)$ for $i \geq \deg e(t)+1$, where the degree of the zero polynomial is set equal to -1. To complete the proof note that $a_{-d}, \ldots, a_{-1}$ are integers by Lemma 1.2.10, and that $d = d(M)$ is the dimension of M by Theorem 1.2.8. □

Corollary 4.1.9 *If M is a graded R-module of dimension d, then there is a unique polynomial $\varphi_M(t) \in \mathbb{Z}[t]$ of degree $d-1$ so that $H(M,i) = \varphi_M(i)$ for $i \geq a(M) + 1$.*

Proof. Is an immediate consequence of Proposition 4.1.8. □

Definition 4.1.10 The polynomial

$$\varphi_M(t) = c_{d-1} t^{d-1} + \cdots + c_1 t + c_0$$

is the *Hilbert polynomial* of M and the integer $c_{d-1}(d-1)!$ is the *degree* or *multiplicity* of M and is denoted by $e(M)$. If $d = 0$, we set $e(M) = \ell(M)$, where $\ell(M)$ is the length M.

Remark 4.1.11 The leading coefficient of $\varphi_M(t)$ is equal to $h(1)/(d-1)!$, where $h(t)$ is the polynomial $F(M,t)(1-t)^d$. If $d = 0$, then

$$h(1) = \ell(M) = \dim_k(M).$$

Corollary 4.1.12 *Let M be a graded R-module. Set*

$$r_0 = \min\{r \in \mathbb{N} | H(M,i) = \varphi_M(i), \forall\, i \geq r\}.$$

Then $r_0 = 0$ if $a(M) < 0$ and $r_0 = a(M) + 1$ otherwise.

Corollary 4.1.13 *Let $S = R/I$ be a standard graded algebra of dimension d over a field k. If S is Cohen-Macaulay and $\underline{h} = \{h_1, \ldots, h_d\}$ a system of parameters of S consisting of linear forms, then*

$$e(S) = \ell(S/\underline{h}S).$$

Proof. By Proposition 2.2.7 $\{h_1, \ldots, h_d\}$ is a regular sequence. Thus there are exact sequences of k-vector spaces

$$0 \longrightarrow (S[-1])_i = S_{i-1} \xrightarrow{h_1} S_i \longrightarrow (S/h_1S)_i \longrightarrow 0.$$

Therefore one has the relation $\varphi_{\overline{S}}(i) = \varphi_S(i) - \varphi_S(i-1)$ between Hilbert polynomials for $i \gg 0$, where $\overline{S} = S/h_1S$. Let

$$\varphi_S(t) = \frac{e(S)}{(d-1)!}t^{d-1} + a_{d-2}t^{d-2} + \cdots + a_0,$$

$$\varphi_{\overline{S}}(t) = \frac{e(\overline{S})}{(d-2)!}t^{d-2} + b_{d-3}t^{d-3} + \cdots + b_0.$$

Now, observe that thanks to $\varphi_{\overline{S}}(i) = \varphi_S(i) - \varphi_S(i-1)$ one derives:

$$\varphi_{\overline{S}}(t) = \frac{e(S)(d-1)}{(d-1)!}t^{d-2} + \text{lower order terms},$$

consequently $e(\overline{S}) = e(S)$. Since $\overline{S}$ is Cohen-Macaulay of dimension $d-1$, see Lemma 1.3.10, the result follows by induction. □

Proposition 4.1.14 *If A and B are two standard algebras over a field k, then*

$$\dim(A \otimes_k B) = \dim(A) + \dim(B).$$

Proof. Since

$$A \otimes_k B = \bigoplus_{r=0}^{\infty} C_r,$$

where

$$C_r = \bigoplus_{i+j=r} A_i \otimes_k B_j,$$

one has the next equality between Hilbert series:

$$\begin{aligned} F(A,t)F(B,t) &= \left[\sum_{i=0}^{\infty}(\dim_k A_i)t^i\right]\left[\sum_{i=0}^{\infty}(\dim_k B_i)t^i\right] \\ &= \sum_{i=0}^{\infty}(\dim_k C_i)t^i = F(A \otimes_k B, t). \end{aligned}$$

Thus Theorem 4.1.3 yields the asserted equality. □

Making use of the Noether normalization lemma it is not difficult to prove that the result above holds for arbitrary affine algebras over a field k.

Computation of Hilbert series Let R be a polynomial ring over a field k and $I \subset R$ a graded ideal. Since R/I and $R/\operatorname{in}(I)$ have the same Hilbert function (see Corollary 2.4.13), the actual computation of the Hilbert series of R/I is a two step process:

- first one finds a Gröbner basis of I using Buchberger's algorithm, and
- second one computes the Hilbert series of $R/\operatorname{in}(I)$ using elimination of variables (see [16, 21]).

Another approaches to the computation of Hilbert series are through minimal resolutions and Stanley decompositions (see [276]). An alternative method to compute Hilbert series of Cohen-Macaulay $\mathbb{N}$-graded algebras is given by Proposition 2.2.14. Some exemplifications are given below.

Example 4.1.15 Let $R = k[x_1, x_2, x_3]$ be a polynomial ring and

$$I = (x_1^2, x_2^2 x_3, x_2^3).$$

Let us compute the Hilbert series of R/I using elimination. Pick any monomial involving more than one variable, say $x_2^2 x_3$. The idea is to eliminate x_2^2 from the monomials containing more than one variable. From the exact sequence of graded modules:

$$0 \longrightarrow R/(x_1^2, x_3, x_2)(-2) \xrightarrow{x_2^2} R/I \longrightarrow R/(x_1^2, x_2^2) \longrightarrow 0,$$

and applying Exercise 4.1.21, we get

$$F(R/I, t) = t^2 \left[\frac{(1-t)^2(1-t^2)}{(1-t)^3}\right] + \frac{(1-t^2)^2}{(1-t)^3} = \frac{-t^4 + 2t^2 + 2t + 1}{(1-t)}.$$

Example 4.1.16 Let $R = \mathbb{Q}[x, y, z, w]$ and $I = (f_1, f_2, f_3)$, where

$$f_1 = y^2 - xz, \ f_2 = x^3 - yzw, \ f_3 = x^2 y - z^2 w.$$

Using *Macaulay* one finds that the minimal resolution of R/I is:

$$0 \longrightarrow R^2(-4) \xrightarrow{\varphi_2} R(-2) \oplus R^2(-3) \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0.$$

Let us compute the Hilbert series of R/I using its minimal resolution:

$$\begin{aligned} F(R/I, t) &= F(R, t) - F(R(-2) \oplus R^2(-3), t) + F(R^2(-4), t) \\ &= \frac{1}{(1-t)^4} - \frac{t^2}{(1-t)^4} - \frac{2t^3}{(1-t)^4} + \frac{2t^4}{(1-t)^4} \\ &= \frac{1 + 2t + 2t^2}{(1-t)^2}. \end{aligned}$$

Example 4.1.17 Let $R = k[x_1, \ldots, x_5]$ and $I = (x_1x_2, x_3x_4x_5)$. Note that I is a complete intersection, hence by Exercise 4.1.21 one has

$$F(R/I,t) = \frac{(1-t^2)(1-t^3)}{(1-t)^5} = \frac{1+2t+2t^2+t^3}{(1-t)^3} = -1 + \frac{5t^2-t+2}{(1-t)^3}.$$

The Laurent expansion of $F(R/I,t)+1$, in negative powers of $(1-t)$ is

$$\frac{5t^2-t+2}{(1-t)^3} = \frac{a_{-3}}{(1-t)^3} + \frac{a_{-2}}{(1-t)^2} + \frac{a_{-1}}{(1-t)},$$

where $a_{-1} = 5$, $a_{-2} = -9$, $a_{-3} = 6$. Hence the Hilbert polynomial of $S = R/I$ is

$$\varphi_S(i) = 6\binom{i+2}{i} - 9\binom{i+1}{i} + 5\binom{i}{i} = 3i^2 + 2.$$

Exercises

4.1.18 Let k be a field and $R = k[x_1, \ldots, x_n]$ a polynomial ring graded by $\deg(x_i) = d_i \in \mathbb{N}_+$. Then the Hilbert series of R is

$$F(R,t) = \frac{1}{\prod_{i=1}^{n}(1-t^{d_i})}.$$

4.1.19 Let $f_1, \ldots, f_g$ be a regular sequence of forms of degree p in a polynomial ring R in n variables and $I = (f_1, \ldots, f_g)$. If $0 \leq i < p$ show:

$$\begin{aligned} H(p+i) &= g\binom{i+n-1}{n-1} \text{ and} \\ H(R/I, p+i) &= \binom{p+i+n-1}{n-1} - H(I, p+i). \end{aligned}$$

4.1.20 Let $R = k[x,y,z]$ be a polynomial ring over a field k and

$$I = (x^a, y^b, z^c, x^{a_1}y^{b_1}z^{c_1}).$$

If $a \geq a_1$, $b \geq b_1$ and $c \geq c_1$, then

$$e(R/I) = \dim_k(R/I) = abc_1 + (c-c_1)(a_1b + (a-a_1)b_1).$$

4.1.21 Let R be a polynomial ring in n variables over a field and $f_1, \ldots, f_r$ a homogeneous regular sequence in R. If $a_i = \deg(f_i)$, then

$$F(R/(f_1, \ldots, f_r), t) = \frac{\prod_{i=1}^{r}(1-t^{a_i})}{(1-t)^n}.$$

4.1.22 If $R = k[x_0, \ldots, x_n]$ is a polynomial ring over a field k and

$$\mathcal{B} = \{x_i^q x_j - x_i x_j^q | \, 0 \leq i < j \leq n\},$$

then $\mathcal{B}$ is a Gröbner basis for $I = (\mathcal{B})$ w.r.t the reverse lex ordering.

4.1.23 In Exercise 4.1.22 show that the a-invariant of R/I is $n(q-1)$ by proving that the Hilbert series $F(t) = F(R/I, t)$ can be written as:

$$F(t) = \frac{t^q(1 + w + \cdots + w^{n-2}) + w^{n-1}(w + t^q)}{(1-t)},$$

where $w = 1 + t + \cdots + t^{q-1}$.

4.1.24 Let $R = k[x_1, x_2, x_3]$ be a polynomial ring and

$$I = (x_1^2, x_2^2 x_3, x_2^3).$$

Prove that the a-invariant of R/I is 3 and $H(R/I, i) = 4$ for $i \geq 4$.

4.1.25 Let A, B be two standard algebras over a field k and define their *Segre product* as the graded algebra

$$S = A \otimes_S B = (A_0 \otimes_k B_0) \oplus (A_1 \otimes_k B_1) \oplus \cdots \subset A \otimes_k B,$$

where

$$(A \otimes_k B)_p = \sum_{i+j=p} A_i \otimes_k B_j.$$

Use Hilbert functions to prove

$$\dim(S) = \dim(A) + \dim(B) - 1.$$

Note $\dim(A_i \otimes_k B_i) = \dim(A_i)\dim(B_i)$.

4.1.26 Use the following procedure in *CoCoA* [60] to verify the formulas for the Hilbert series and Hilbert polynomial found in Example 4.1.17

```
Use R ::= Q[x[1..5]], Lex;
Q := Ideal(x[1] * x[2], x[3] * x[4] * x[5]);
Poincare(R/Q);
Hilbert(R/Q);
```

4.2 a-invariants and h-vectors

Our goal here is to relate the a-invariant of a Cohen-Macaulay $\mathbb{N}$-graded algebra with its minimal resolution and to prove that h-vectors of such algebras are non negative. We include Stanley's characterization of Cohen-Macaulay algebras in terms of Hilbert series.

We begin with a general inequality relating the a-invariant of a graded algebra with the shifts of its minimal resolution.

Proposition 4.2.1 *Let $R = k[x_1, \ldots, x_n]$ be a positively graded polynomial ring over a field k and I a graded ideal. If*

$$\mathbb{F}_\star : \quad 0 \to \bigoplus_{i=1}^{b_g} R(-d_{gi}) \xrightarrow{\varphi_g} \cdots \to \bigoplus_{i=1}^{b_k} R(-d_{ki}) \xrightarrow{\varphi_k} \cdots \to \bigoplus_{i=1}^{b_1} R(-d_{1i}) \xrightarrow{\varphi_1} R$$

is the minimal resolution of $S = R/I$, then

$$a(S) \leq \max\{d_{ki} | \, 1 \leq k \leq g \text{ and } 1 \leq i \leq b_k\} + a(R).$$

Proof. From the resolution of S one can write the Hilbert series of S as:

$$F(S,t) = \frac{\left(1 + \sum_{k=1}^{g} (-1)^k \left(\sum_{i=1}^{b_k} t^{d_{ki}}\right)\right)}{\prod_{i=1}^{n} (1 - t^{\deg(x_i)})},$$

hence $a(S) \leq \max\{d_{ki}\} + a(R)$. □

Definition 4.2.2 Let $R = k[x_1, \ldots, x_n]$ be a positively graded polynomial ring over a field k and I a Cohen-Macaulay graded ideal of height g. The *canonical module* of $S = R/I$ is defined as

$$\omega_S = \operatorname{Ext}_R^g(R/I, \omega_R),$$

where $\omega_R = R(-\delta)$ and $\delta = \sum_{i=1}^{n} \deg(x_i)$.

Proposition 4.2.3 *Let $R = k[x_1, \ldots, x_n]$ be a positively graded polynomial ring over a field k and I a Cohen-Macaulay graded ideal. If*

$$\mathbb{F}_\star : \quad 0 \to \bigoplus_{i=1}^{b_g} R(-d_{gi}) \xrightarrow{\varphi_g} \cdots \to \bigoplus_{i=1}^{b_k} R(-d_{ki}) \xrightarrow{\varphi_k} \cdots \to \bigoplus_{i=1}^{b_1} R(-d_{1i}) \xrightarrow{\varphi_1} R$$

is the minimal resolution of $S = R/I$, then $\max_i\{d_{1i}\} < \cdots < \max_i\{d_{gi}\}$ and the a-invariant of S is given by

$$a(S) = \max_{1 \leq i \leq b_g} \{d_{gi}\} + a(R) = -\min\{i \,|\, (\omega_S)_i \neq 0\}.$$

Proof. One may order the shifts as $d_{k1} \leq \cdots \leq d_{kb_k}$ for $k = 1, \ldots, g$. From the resolution above one can express the Hilbert series of S as:

$$F(S,t) = \frac{\left(1 + \sum_{k=1}^{g} (-1)^k \left(\sum_{i=1}^{b_k} t^{d_{ki}}\right)\right)}{\prod_{i=1}^{n} (1 - t^{\deg(x_i)})}.$$

In order to make use of this formula we first need to prove the assertion about the behaviour of the shifts d_{ki}.

Since S is Cohen-Macaulay, the height of I is equal g, this follows from Corollary 2.5.14. Recall that by Proposition 2.2.19 there exists a regular sequence $f_1, \ldots, f_g$ inside I, hence using Proposition 1.3.7 yields:

$$\mathrm{Ext}_R^i(S, \omega_R) = 0 \text{ for } i < g.$$

Set $\delta = \sum_{i=1}^{n} \deg(x_i)$ and $\omega_R = R(-\delta)$. Therefore dualizing $\mathbb{F}_\star$, with respect to the canonical module of R, one obtains the (exact) complex:

$$\mathrm{Hom}(\mathbb{F}_\star, \omega_R): \quad 0 \to \mathrm{Hom}(R, \omega_R) \to \cdots \to \mathrm{Hom}\left(\bigoplus_{i=1}^{b_g} R(-d_{gi}), \omega_R\right) \to 0.$$

Observe

$$\begin{aligned} \mathrm{Hom}\left(\bigoplus_{i=1}^{b_g} R(-d_{gi}), R(-\delta)\right) &\simeq \mathrm{Hom}\left(\bigoplus_{i=1}^{b_g} R(-d_{gi}), R\right)(-\delta) \\ &\simeq \bigoplus_{i=1}^{b_g} R(d_{gi})(-\delta) \\ &\simeq \bigoplus_{i=1}^{b_g} R(-(\delta - d_{gi})). \end{aligned}$$

Since $\mathrm{Hom}(\mathbb{F}_\star, \omega_R)$ is a minimal resolution of $\mathrm{Ext}_R^g(S, \omega_R)$, by Remark 2.5.9, one has $d_{1b_1} < d_{2b_2} < \cdots < d_{gb_g}$. As a consequence, from the expression for $F(S,t)$ given above, one concludes the equality $a(S) = d_{gb_g} + a(R)$. In particular one has a surjection

$$\bigoplus_{i=1}^{b_g} R(-(\delta - d_{gi})) \to \mathrm{Ext}_R^g(S, R(-\delta)) = \omega_S \to 0.$$

To complete the proof note that since $\delta - d_{gb_g} \leq \cdots \leq \delta - d_{g1}$, one has the equalities $\min\{i \mid (\omega_S)_i \neq 0\} = \delta - d_{gb_g} = -a(S)$. □

h-vectors Next we introduce the notion of h-vector of standard algebras which is based on the knowledge of the Hilbert-Serre theorem for Hilbert series (see Theorem 4.1.3).

Definition 4.2.4 Let S be a standard algebra and

$$h(t) = h_0 + h_1 t + \cdots + h_r t^r$$

the (unique) polynomial with integral coefficients such that $h(1) \neq 0$ and satisfying

$$F(S,t) = \frac{h(t)}{(1-t)^d},$$

where $d = \dim(S)$. The *h-vector* of S is defined by $h(S) = (h_0, \ldots, h_r)$.

Theorem 4.2.5 ([269]) *Let S be a standard algebra and $\theta_1, \ldots, \theta_d$ a homogeneous system of parameters for S with $a_i = \deg(\theta_i)$. Let A be the quotient ring $S/(\theta_1, \ldots, \theta_d)$. Then S is Cohen-Macaulay if and only if*

$$F(S,t) = \frac{F(A,t)}{\prod_{i=1}^{d}(1-t^{a_i})}. \tag{4.2}$$

Proof. $\Rightarrow$) Assume S is Cohen-Macaulay. It is enough to prove the equality

$$F(S/(\theta_1, \ldots, \theta_s), t) = \prod_{i=1}^{s}(1-t^{a_i})F(S,t), \tag{4.3}$$

for $1 \leq s \leq d$. We proceed by induction on s. Assume $s = 1$. From the exact sequence of graded S-modules

$$0 \longrightarrow S[-a_1] \xrightarrow{\theta_1} S \longrightarrow S/(\theta_1) \longrightarrow 0$$

we obtain

$$F(S/(\theta_1), t) = F(S,t) - F(S[-1], t).$$

Since

$$\begin{aligned} F(S[-a_1], t) &= \sum_{i=0}^{\infty} S[-a_1]_i t^i = \sum_{i=a_1}^{\infty} S[-a_1]_i t^i \\ &= t^{a_1} \sum_{i=a_1}^{\infty} S_{i-a_1} t^{i-a_1} \\ &= t^{a_1} F(S,t) \end{aligned}$$

we conclude

$$F(S/(\theta_1),t) = (1-t^{a_1})F(S,t),$$

which proves the case $s = 1$. Next assume $s > 1$ and that the equality (4.3) is true for $s-1$. From the exact sequence

$$0 \to (S/(\theta_1,\ldots,\theta_{s-1}))\,[-a_s] \xrightarrow{\theta_s} S/(\theta_1,\ldots,\theta_{s-1}) \longrightarrow S/(\theta_1,\ldots,\theta_s) \to 0$$

we get

$$F(S/(\theta_1,\ldots,\theta_s),t) = (1-t^{a_s})F(S/(\theta_1,\ldots,\theta_{s-1}),t),$$

and the induction hypothesis yields the required equality.

$\Leftarrow$) Let $f(t) = \sum_{i\geq 0} a_i t^i$ and $g(t) = \sum_{i\geq 0} b_i t^i$ be two formal power series, we say that $f(t) \geq g(t)$ if $a_i \geq b_i$ for all i. From the exact sequence

$$0 \longrightarrow (S/\mathrm{ann}\,(\theta_1))\,[-a_1] \xrightarrow{\theta_1} S \longrightarrow S/(\theta_1) \longrightarrow 0$$

we obtain

$$F(S/(\theta_1),t) - (1-t^{a_1})F(S,t) = t^{a_1}F(\mathrm{ann}\,(\theta_1),t).$$

Hence the power series $F(S/(\theta_1),t)$ is greater or equal than $(1-t^{a_1})F(S,t)$, and by induction it follows that

$$F(S/(\theta_1,\ldots\theta_r),t) \geq \prod_{i=1}^{r}(1-t^{a_i})F(S,t) \quad \text{for } 1 \leq r \leq d.$$

Set $S' = S/(\theta_1,\ldots,\theta_{d-1})$. Using the exact sequence

$$0 \longrightarrow (S'/\mathrm{ann}\,(\theta_d))[-a_d] \xrightarrow{\theta_d} S' \longrightarrow S'/(\theta_d) = S/(\theta_1,\ldots,\theta_d) \longrightarrow 0$$

we get

$$F(S'/(\theta_d),t) = (1-t)^{a_d}F(S',t) + t^{a_d}F(\mathrm{ann}\,(\theta_d),t).$$

On the other hand using the hypothesis we have

$$F(S'/(\theta_d),t) = \prod_{i=1}^{d}(1-t^{a_i})F(S,t).$$

Therefore

$$\underbrace{(1-t)^{a_d}}_{\geq 0}\underbrace{(F(S',t) - (1-t^{a_1})\cdots(1-t^{a_{d-1}})F(S,t))}_{\geq 0}$$
$$= -t^{a_d}F(\mathrm{ann}\,(\theta_d),t),$$

and $F(\mathrm{ann}\,(\theta_d),t) = 0$, which shows that θ_d is a regular element on S'. By induction it follows readily that $\theta_1,\ldots,\theta_d$ is a regular sequence. This proof was adapted from [269]. □

Theorem 4.2.6 *Let S be a Cohen-Macaulay standard algebra over a field k and $h(S) = (h_i)$ the h-vector of S, then $h_i \geq 0$ for all i.*

Proof. One may assume k infinite, otherwise one may change the coefficient field k using the functor $(\cdot) \otimes_k K$, where K is an infinite field extension of k. There is a h.s.o.p $\mathbf{y} = \{y_1, \ldots, y_d\}$ for S, where each y_i is a form in S of degree one. Set $\overline{S} = S/(\mathbf{y})S$. From Theorem 4.2.5 we derive

$$h_i = H(\overline{S}, i)$$

for all i and consequently $h_i \geq 0$. □

Exercises

4.2.7 Let R be a polynomial ring over a field k with the usual grading and M a finitely generated $\mathbb{N}$-graded module. If M is Cohen-Macaulay and $h(t) = h_0 + h_1 t + \cdots + h_r t^r$ the polynomial in $\mathbb{Z}[t]$ so that $h(1) \neq 0$ and

$$F(M,t) = \frac{h(t)}{(1-t)^d}, \text{ where } d = \dim(M).$$

Prove that $h_i \geq 0$ for all i.

4.3 Extremal algebras

The concept of an extremal Cohen-Macaulay or Gorenstein standard algebra (resp. ideal) came up in the works of J. Sally [251] and P. Schenzel [252]. Those algebras (resp. ideals) have the smallest possible reduction number (resp. smallest possible number of generators of least degree). We will study with certain detail those two distinguished classes of algebras.

The Cohen-Macaulay case

Let R be a polynomial ring over a field k with its usual grading and

$$I = \bigoplus_{i=p}^{\infty} I_i$$

a graded ideal with $I_p \neq (0)$. The integer p is the *initial degree* of I. As usual $\nu(I)$ denotes the minimal number of generators of I, and $\nu(I_p)$ stands for the minimal number of generators of I in degree p.

The next result, when applied to a Cohen-Macaulay ideal I generated by forms of the same degree, provides a bound for the minimum number of generators of I.

Proposition 4.3.1 *Let R be a polynomial ring over a field k and I a graded ideal in R of height g and initial degree p. If I is Cohen-Macaulay, then*

$$\nu(I_p) = H(I,p) \leq \binom{p+g-1}{p}.$$

Proof. We may assume that k is infinite, since a change of the coefficient field can be easily carried out using the functor $(\cdot) \otimes_k K$, where K is an infinite field extension of k. This change of the coefficient field preserves the hypothesis on I and leaves both the height of I and the dimensions of the vector spaces of forms of a given degree in I unchanged.

We set $S = R/I$. By Lemma 2.2.15, there is a homogeneous system of parameters $\mathbf{y} = \{y_1, \ldots, y_d\}$ for S, where each y_i is a linear form in R. If we tensor the minimal resolution of S with $\overline{R} = R/(\mathbf{y})$ it follows that

$$0 \leq H(\overline{S},p) = H(\overline{R},p) - H(I,p),$$

where $\overline{S} = S/(\mathbf{y})S$. To get the desired inequality it suffices to observe that $\overline{R}$ is a polynomial ring in g variables over the field k. □

Definition 4.3.2 If S is an Artinian positively graded algebra the *socle* of S is given by

$$\mathrm{Soc}(S) = (0 :_S S_+).$$

Lemma 4.3.3 ([118]) *Let $S = R/I$ be a quotient ring of a polynomial ring R over a field k modulo a graded ideal I. Let*

$$0 \to \bigoplus_{i=1}^{b_g} R(-d_{gi}) \to \cdots \to \bigoplus_{i=1}^{b_1} R(-d_{1i}) \to R \to S = R/I \to 0,$$

be the minimal resolution of S. If S is Artinian, then there is a degree zero isomorphism of graded k-vector spaces:

$$\mathrm{Soc}(S) \simeq \bigoplus_{i=1}^{b_g} k[g - d_{gi}].$$

Proof. Let $R = k[x_1, \ldots, x_g]$ and $\underline{x} = \{x_1, \ldots, x_g\}$. The ordinary Koszul complex $K_\star$ associated to the regular sequence $\underline{x}$ is an R-free resolution of $k = R/(\underline{x})$ and thus

$$\mathrm{Tor}_i^R(S,k) \simeq H_i(K_\star \otimes_R S).$$

On the other hand

$$\mathrm{Tor}_g^R(S,k) \simeq \bigoplus_{i=1}^{b_g} k[-d_{gi}]$$

(see the proof of Corollary 2.5.7). To complete the argument note

$$\begin{aligned} H_g(K_\star \otimes_R S)[g] &\simeq Z_g(K_\star \otimes_R S)[g] \\ &= \left\{ ze_1 \wedge \cdots \wedge e_g \,\middle|\, \sum_{i=1}^{g} (-1)^{i-1} z x_i e_1 \wedge \cdots \wedge \widehat{e}_i \wedge \cdots \wedge e_g = 0 \right\} [g] \\ &\simeq \operatorname{Soc}(S). \end{aligned}$$

Altogether one has

$$\operatorname{Soc}(S) \simeq \bigoplus_{i=1}^{b_g} k[g - d_{gi}],$$

as required. □

Proposition 4.3.4 *If S a Cohen-Macaulay positively graded k-algebra over a field k with presentation R/I, then the type of S is equal to the last Betti number in the minimal free resolution of R/I as an R-module.*

Proof. By making a change of coefficients using the functor $(\cdot) \otimes_k K$, where K is an infinite field extension of k, one may assume that k is infinite.

By Theorem 2.2.11 there exists a system of parameters $\mathbf{y} = \{y_1, \ldots, y_d\}$ for $S = R/I$ with each y_i a form of degree one of R. Since $\operatorname{Tor}_i(S, R/(\mathbf{y})) = 0$ for $i \geq 1$, the minimal resolution of $S/(\mathbf{y}S)$ as an $R/(\mathbf{y})$-module has the same twists and Betti numbers as the minimal resolution of S over R. Since $S/(\mathbf{y}S)$ is Artinian the result follows from Lemma 4.3.3. □

Corollary 4.3.5 *If R is a polynomial ring over a field k, then a graded ideal I of R is Gorenstein iff R/I is Cohen-Macaulay and the last Betti number in the minimal graded resolution of R/I is equal to* 1.

Corollary 4.3.6 *Let S be a Cohen-Macaulay positively graded k-algebra over a field k and ω_S its canonical module. Then*

$$\operatorname{type}(S) = \nu(\omega_S),$$

where $\nu(\omega_S)$ is the minimum number of generators of ω_S.

Proof. Use the proof of Proposition 4.2.3 and Proposition 4.3.4. □

Theorem 4.3.7 ([61]) *Let R be a polynomial ring over a field k and I a graded ideal in R of height g and initial degree p. If $S = R/I$ is a Cohen-Macaulay ring, then S has a p-linear resolution if and only if the following equality holds*

$$\nu(I_p) = H(I, p) = \binom{p+g-1}{p}. \tag{4.4}$$

Proof. Let

$$0 \to \bigoplus_{i=1}^{b_g} R(-d_{gi}) \to \cdots \to \bigoplus_{i=1}^{b_1} R(-d_{1i}) \to R \to S = R/I \to 0, \qquad (4.5)$$

be the minimal resolution of S. We order the shifts so that $d_{k1} \leq \cdots \leq d_{kb_k}$ for $k = 1, \ldots, g$; by the minimality of the resolution $d_{11} < d_{21} < \cdots < d_{g1}$. Observe that if the resolution of S is linear then the Herzog-Kühl formulas of Theorem 2.5.16 give the required equality.

Conversely assume the equality above. Since we may assume that k is infinite, there is a h.s.o.p $\mathbf{y} = \{y_1, \ldots, y_d\}$ for S, where each y_i is a form of degree one in R. Set $\overline{S} = S/(\mathbf{y})S$ and $\overline{R} = R/(\mathbf{y})$. From Theorem 4.2.5 we derive that the h-vector of S satisfies $h_p = 0$, and also get $h_i = H(\overline{S}, i)$. Therefore $h_i = 0$ for all $i \geq p$. Using Lemma 4.3.3 one obtains a degree zero isomorphism

$$\mathrm{Soc}(\overline{S}) \simeq \bigoplus_{i=1}^{b_g} k[g - d_{gi}]$$

of graded k-vector spaces, in particular the socle of $\overline{S}$ can only live in degrees $d_{gi} - g$, hence we conclude the inequality $d_{gi} - g \leq p - 1$. On the other hand the minimality of the resolution of $\overline{S}$ gives $p + (g - 1) \leq d_{gi}$ for all i. Altogether we have $d_{gi} = p + g - 1$. Because I is C-M, then by Proposition 4.2.3 we must have $p \leq d_{1b_1} < d_{2b_2} < \cdots < d_{gb_g} = p + g - 1$, which implies that the resolution is p-linear, as required. □

Example 4.3.8 ([237]) Let $R = k[a, \ldots, f]$ be a polynomial ring over a field k and

$$I = (abc, abd, ace, adf, aef, bcf, bde, bef, cde, cdf).$$

Then R/I is C-M if $\mathrm{char}(k) \neq 2$ and non C-M otherwise. Thus in the first case R/I has a 3-linear resolution.

Other examples of Cohen-Macaulay algebras with linear resolutions include rings of minimal multiplicity [251], the coordinate ring of a variety defined by the submaximal minors of a generic symmetric matrix [194], the coordinate ring of a variety defined by the maximal minors of a generic matrix [89], and some face rings [114].

Cohen-Macaulay rings with linear resolutions have been studied by Sally [251] for the case $p = 2$, and by Schenzel [252] for the general case; more general rings with linear resolutions have been examined in [93, 155].

The Gorenstein case

To give a sharp bound for the number of generators in least degree of a graded Gorenstein ideal is harder than the Cohen-Macaulay case, we deal with this problem in the next section.

The aim here is to introduce and characterize extremal Gorenstein rings in terms of their minimal resolutions and to determine their Betti numbers. Those Betti numbers are our "natural candidates" to bound the initial Betti numbers of graded Gorenstein ideals.

Proposition 4.3.9 *If $S = S_0 + \cdots + S_s$ is a positively graded Artinian Gorenstein algebra over a field k, then the multiplication maps*

$$S_i \times S_{s-i} \to S_s = \mathrm{Soc(S)} \simeq k$$

are a perfect pairing, that is, the homomorphisms:

$$\varphi_i : S_i \to \mathrm{Hom}_k(S_{s-i}, S_s) \simeq S_{s-i}, \quad \varphi_i(r)(x) = rx$$

are isomorphisms of k-vector spaces.

Proof. To show that φ_i is one to one take $r \in \ker(\varphi)$ and assume $r \neq 0$ and $i < s$. Since r is not in the socle of S, one may pick $z \in S$ homogeneous of maximal positive degree such that $zr \neq 0$. Note that $0 < \deg(z) < s - i$, because $rx = 0$ for $x \in S_{s-i}$. Hence zr has degree less than s, using that the socle of S lives only in degree s one obtains a homogeneous element w with $\deg(w) > 0$ such that $wzr = 0$, which contradicts the choice of z. Therefore $r = 0$ and φ is injective. Using the injectivity of φ_i one has $\dim_k S_i \leq \dim_k S_{s-i} \leq \dim_k S_i$, and thus φ_i is an isomorphism. □

Corollary 4.3.10 *Let $h = (h_0, \ldots, h_s)$ be the h-vector of a standard Gorenstein k-algebra, where k is a field. Then h is symmetric, that is,*

$$h_i = h_{s-i} \quad \textit{for all} \quad i.$$

Proposition 4.3.11 *Let R be a polynomial ring over a field k with the standard grading and I a graded Gorenstein ideal of initial degree p and height g. If the minimal resolution of $S = R/I$ by free R-modules is:*

$$\mathbb{F}_\star : \quad 0 \to \bigoplus_{i=1}^{b_g} R(-d_{gi}) \to \cdots \to \bigoplus_{i=1}^{b_k} R(-d_{ki}) \to \cdots \to \bigoplus_{i=1}^{b_1} R(-d_{1i}) \to R,$$

then

$$\bigoplus_{i=1}^{b_{g-k}} R(-(d_g - d_{(g-k)i})) \simeq \bigoplus_{i=1}^{b_k} R(-d_{ki}) \quad \textit{for} \quad 1 \leq k \leq g-1. \tag{4.6}$$

Proof. One may always order the shifts so that

$$d_{k1} \leq \cdots \leq d_{kb_k} \quad \text{for} \quad k = 1, \ldots, g.$$

As $b_g = 1$ and $d_{gi} = d_g$, by Proposition 4.2.3 one has that the a-invariant of S is $a(S) = d_g - n$, where n is the number of variables of R. Since $\mathrm{Hom}(\mathbb{F}_\star, \omega_R)$ is a minimal resolution of

$$\omega_S \simeq \mathrm{Ext}_R^g(S, \omega_R)$$

and using

$$\omega_S \simeq S(d_g - n),$$

one obtains that $\mathbb{F}_\star$ is self dual. □

With the same assumptions and notation of Proposition 4.3.11 one has:

Proposition 4.3.12 *Let $h(S) = (h_0, \ldots, h_s)$ be the h-vector of S. Then $s \geq 2(p-1)$ with equality if and only if the minimal resolution of S is of the form*

$$0 \to R(-(2p+g-2)) \to R^{b_{g-1}}(-(p+g-2)) \to \\ \cdots \to R^{b_2}(-(p+1)) \to R^{b_1}(-p) \to R.$$

Proof. As in the proof of Theorem 4.2.6 one may assume S Artinian and h_i equal to $H(S,i)$. Note that the socle of S lives in degree $d_g - g$ by Lemma 4.3.3, hence $s \geq d_g - g$. By Proposition 4.3.11 and the minimality of the resolution one concludes

$$s \geq d_g - g = d_{1b_1} + d_{(g-1)1} - g \geq p - g + d_{(g-1)1} \geq 2(p-1),$$

where the last inequality uses $d_{k1} \geq p + k - 1$ for $1 \leq k \leq g-1$. For the second assertion use the chain of inequalities above and compute the h-vector of S using its resolution. □

Definition 4.3.13 If $s = 2(p-1)$, the ring S is called an *extremal Gorenstein ring* and its resolution is called *pure and almost linear*.

Proposition 4.3.14 ([252]) *Let R be a polynomial ring over a field k and I a graded ideal of initial degree p and height g. If $S = R/I$ is an extremal Gorenstein algebra, then for $1 \leq i < g$ the ith Betti number of S is given by*

$$b_i = \binom{p+g-1}{p+i-1}\binom{p+i-2}{i-1} \\ + \binom{p+g-1}{i}\binom{p+g-i-2}{p-1} - \binom{g}{i}\binom{p+g-2}{p-1}.$$

Proof. As in the proof of Theorem 4.3.7 one may assume S Artinian and R a polynomial ring in g variables. Set $s = 2(p-1)$. From the minimal resolution of S

$$\left(\sum_{i=0}^{s} h_i z^i\right)(1-z)^g = 1 + \sum_{i=1}^{g-1}(-1)^i b_i z^{p+i-1} + (-1)^g b_g z^{s+g},$$

where $h_i = H(S,i) = h_{s-i}$. Therefore

$$b_m = \sum_{i=0}^{p+m-1} (-1)^{p-1-i}\binom{g}{p+m-1-i} h_i,\ 1 \le m \le g-1$$

Using the identity

$$\sum_{i=0}^{p-1}(-1)^{p-1-i}\binom{g}{p+m-1-i}\binom{g-1+i}{i} = \binom{p+g-1}{p+m-1}\binom{p+m-2}{m-1} \tag{4.7}$$

and the symmetry of the h-vector of S one has

$$b_m = \binom{p+g-1}{p+m-1}\binom{p+m-2}{m-1} + \sum_{i=p}^{p+m-1} (-1)^{p-1-i}\binom{g}{p+m-1-i} h_{s-i}.$$

Note

$$\sum_{i=p}^{p+m-1} (-1)^{p-1-i}\binom{g}{p+m-1-i} h_{s-i} = \sum_{i=0}^{m-1}(-1)^{i+1}\binom{g}{m-1-i}\binom{p+g-3-i}{g-1},$$

since the terms in the right hand side are zero for $i > p-1$ the last summation reduces to

$$\begin{aligned}
&\sum_{i=0}^{p-2}(-1)^{i+1}\binom{g}{m-1-i}\binom{p+g-3-i}{g-1} \\
&= \sum_{i=0}^{p-1}(-1)^{p-1-i}\binom{g}{g-m+p-1-i}\binom{g-1+i}{i} - \binom{g}{m}\binom{g+p-2}{p-1} \\
&= \binom{p+g-1}{m}\binom{p+g-m-2}{p-1} - \binom{g}{m}\binom{g+p-2}{p-1},
\end{aligned}$$

the last equality follows from Eq. (4.7) making m equal to $g-m$. □

Exercises

4.3.15 Let R be a polynomial ring over a field k and I a Cohen-Macaulay graded ideal in R of height g and initial degree p. Let $(h_0, \ldots, h_s)$ be the h-vector of R/I. Prove that $s \geq p-1$, with equality if and only if the minimal resolution of R/I is p-linear. If $s = p-1$ the algebra R/I is called an *extremal Cohen-Macaulay algebra.*

4.3.16 Let R be a polynomial ring over a field k and I a C-M graded ideal in R of height g and initial degree p. Prove that the Hilbert series of $S = R/I$ can be written as

$$F(S,t) = \frac{\sum_{i=0}^{p-1} \binom{i+g-1}{g-1} t^i}{(1-t)^d}, \quad \text{where } d = \dim S,$$

if and only if S has a p-linear resolution.

4.3.17 Let R be a polynomial ring over a field k and I a Cohen-Macaulay graded ideal in R of height g and initial degree p. If S has a linear resolution, show that the Betti numbers of $S = R/I$ and its multiplicity are given by

$$b_k = \binom{p+k-2}{k-1} \cdot \binom{p+g-1}{g-k} \quad \text{and} \quad e(S) = \binom{p+g-1}{g}.$$

4.3.18 Let $S = R/I$ be a standard Gorenstein algebra so that I has initial degree p and codimension g. Show that the multiplicity of S satisfies:

$$e(S) \geq e(g,p) = \binom{g+p-1}{g} + \binom{g+p-2}{g},$$

with equality if and only if S is an extremal algebra.

Hint The h-vector $h(S) = (h_0, \ldots, h_s)$ is symmetric and $s \geq 2(p-1)$. Then compute explicitly

$$h_{p-1} + 2\sum_{i=0}^{p-2} h_i.$$

4.3.19 Let R be a polynomial ring over a field k and I a graded Gorenstein ideal of height 4 generated by forms of degree p. Let $h(S) = (h_0, \ldots, h_s)$ be the h-vector of $S = R/I$. Prove that the minimal resolution of S has the form:

$$0 \to R(-(s+4)) \to R^b(-(s-p+4)) \to \bigoplus_{i=1}^{s-2p+3} R^{a_i}(-(p+i)) \to R^b(-p) \to R,$$

where $r = s - 2p + 3$ and $a_i = a_{r-i+1}$ for $i \geq 1$.

4.3.20 If $x > 1$ is an integer, prove that $x(x+8)$ is never a perfect square.

4.3.21 Let I be a graded Cohen-Macaulay ideal of a polynomial ring R with a minimal resolution:

$$0 \to R^{b_3}(-(p+a+b)) \to R^{b_2}(-(p+a)) \to R^{b_1}(-p) \to R \to R/I.$$

If $b_1 = 9$, use the Herzog-Kühl formulas to prove that I is Gorenstein.

Hint Note $b_3 - b_2 + b_1 - 1 = 0$ and prove $b_1 b_2 b_3 = 9b_3(b_3+8)$ is a perfect square, then use Exercise 4.3.20.

4.4 Initial degrees of Gorenstein ideals

Assume R is a polynomial ring over a field and I a homogeneous Gorenstein ideal of codimension $g \geq 3$ and initial degree $p \geq 2$. Let

$$e(g,p) = \binom{g+p-1}{g} + \binom{g+p-2}{g} \text{ and } \nu_0 = \binom{p+g-1}{g-1} - \binom{p+g-3}{g-1}.$$

A consequence of the symmetry of the h-vector of R/I is that the multiplicity of R/I satisfies

$$e(R/I) \geq e(g,p),$$

see Exercise 4.3.18. Given codimension g and initial degree p, a graded Gorenstein algebra R/J with multiplicity $e(R/J) = e(g,p)$ is extremal. This has strong structural implications: the minimal free resolution of R/J must be pure and almost linear; consequently all the Betti numbers $b_i(R/J)$ are determined, and in particular J is generated by ν_0 forms of degree p (see Section 4.3).

Results in the literature dealing with Cohen-Macaulay ideals (such as [106, 249]) give upper bounds for the minimal number of generators $\nu(I)$ of I in terms of codimension, initial degree, and multiplicity $e(R/I)$. One of the aims below is to elucidate the multiplicity information that is already determined by the codimension and initial degree (see Theorem 4.4.3). In general one expects no such conclusion, but for graded Gorenstein algebras the symmetry of the h-vector $h(R/I)$ can be effectively exploited.

Macaulay's Theorem

First we introduce the binomial expansion, and then state a famous result of Macaulay on the growth of Hilbert functions that will be needed later.

Lemma 4.4.1 *Let h, j be positive integers. Then h can be uniquely written*

$$h = \binom{a_j}{j} + \binom{a_{j-1}}{j-1} + \cdots + \binom{a_i}{i}, \tag{4.8}$$

where $a_j > a_{j-1} > \cdots > a_i \geq i \geq 1$.

Proof. By induction on j. If $j = 1$, then $h = \binom{h}{1}$. Assume $j > 1$, then there is a unique a_j such that $\binom{a_j}{j} \leq h < \binom{a_j+1}{j}$. Let $r = h - \binom{a_j}{j}$. By induction

$$r = h - \binom{a_j}{j} = \binom{a_{j-1}}{j-1} + \cdots + \binom{a_i}{i} \text{ where } a_{j-1} > \cdots > a_i \geq i \geq 1.$$

From the inequality

$$h = \binom{a_j}{j} + \binom{a_{j-1}}{j-1} + \cdots + \binom{a_i}{i} < \binom{a_j+1}{j} = \binom{a_j}{j} + \binom{a_j}{j-1}$$

we obtain $\binom{a_{j-1}}{j-1} < \binom{a_j}{j-1}$ and this implies $a_{j-1} < a_j$. Let us show the uniqueness. Assume h can be written as

$$h = \binom{b_j}{j} + \binom{b_{j-1}}{j-1} + \cdots + \binom{b_k}{k},$$

where $b_j > b_{j-1} > \cdots > b_k \geq k \geq 1$. By induction on j one has

$$\binom{b_j}{j} \leq h < \binom{b_j+1}{j}.$$

Since a_j is unique we derive $a_j = b_j$. Hence by induction hypothesis $k = i$ and $a_{j-1} = b_{j-1}, \ldots, a_i = b_i$. □

The unique expression for h in Eq. (4.8) is called the *binomial expansion* of h in base j. We set

$$h^{(j)} = \binom{a_j}{j+1} + \binom{a_{j-1}}{j} + \cdots + \binom{a_i}{i+1}, \tag{4.9}$$

$$h^{\langle j \rangle} = \binom{a_j+1}{j+1} + \binom{a_{j-1}+1}{j} + \cdots + \binom{a_i+1}{i+1}, \tag{4.10}$$

sometimes $h^{\langle j \rangle}$ is refer to as the *Macaulay symbol.*

Theorem 4.4.2 (Macaulay) *Let $h: \mathbb{N} \to \mathbb{N}$ be a numerical function and k a field. The following are equivalent*

(*a*) *there exists a homogeneous k-algebra S with $H(S, i) = h(i)$ for $i \geq 0$.*

(*b*) *$h(0) = 1$ and $h(i+1) \leq h(i)^{\langle i \rangle}$ for all $i \geq 1$.*

Proof. See [298, Appendix B]. □

Theorem 4.4.3 ([222]) *If I is a graded Gorenstein ideal of codimension $g \geq 3$ and initial degree $p \geq 2$, then*

$$\nu(I_p) \leq \nu_0 = \binom{p+g-1}{g-1} - \binom{p+g-3}{g-1},$$

and I is itself extremal if equality holds.

Proof. If either $\nu(I_p) > \nu_0$, or $\nu(I_p) = \nu_0$ and I is not extremal, then by the symmetry of the h-vector $h(R/I)$ there is some $j \geq p$ so that

$$h(R/I) = (h_0, h_1, \ldots, h_{p-1}, h_p, \ldots, h_j, h_{p-1}, \ldots, h_1, h_0),$$

where

$$h_j = h_p \leq \binom{p+g-3}{g-1} = h_{p-2}.$$

The idea of the argument is to use the Macaulay bound $h_j^{\langle j \rangle}$ for h_{j+1} to see that such a small value of h_j cannot grow to such a large value of

$$h_{j+1} = h_{p-1} = \binom{p+g-2}{g-1}.$$

Recall that this estimate is calculated from the binomial expansion for h_j:

$$h_j = \binom{a_j}{j} + \binom{a_{j-1}}{j-1} + \cdots + \binom{a_i}{i}, \tag{4.11}$$

where $a_j > a_{j-1} > \cdots > a_i \geq i \geq 1$. Then

$$h_{j+1} \leq h_j^{\langle j \rangle} = \binom{a_j+1}{j+1} + \binom{a_{j-1}+1}{j} + \cdots + \binom{a_i+1}{i+1}. \tag{4.12}$$

We may assume that $h_j > j$, for if not, then $h_j \leq j$ would imply that $a_\ell = \ell$ for all ℓ, and hence $h_j \geq h_{j+1}$, which contradicts our assumption. Notice that by grouping the terms of (4.11) according to the value of $a_\ell - \ell$ the binomial expansion for h_j can be written as

$$\begin{aligned} h_j &= \sum_{n=0}^{r} \left[\binom{j_n + k_n}{j_n} + \binom{j_n - 1 + k_n}{j_n - 1} + \cdots + \binom{j_n - i_n + k_n}{j_n - i_n} \right] \\ &= \sum_{n=0}^{r} \left[\binom{j_n + k_n + 1}{j_n} - \binom{j_n - i_n + k_n}{j_n - i_n - 1} \right], \end{aligned} \tag{4.13}$$

where $a_j - j = k_0 > k_1 > \cdots > k_r \geq 0$, $j = j_0 > j_1 > \cdots > j_r$, $j_r - i_r = i$, and $j_n = j_{n-1} - i_{n-1} - 1$ for $1 \leq n \leq r$. Set $k = k_0$. Since $p \leq j$ and

$$\binom{j+k}{j} \leq h_j \leq \binom{p+g-3}{g-1}$$

it follows that $k \leq g-2$. From (4.11), (4.12), and (4.13), together with Pascal's identity and

$$\binom{a+b+1}{b+1} = \frac{a+1}{b+1}\binom{a+b+1}{b},$$

we have

$$\begin{aligned} h_{j+1} - h_j &\leq \sum_{n=0}^{r} \left[\binom{j_n + k_n + 1}{j_n + 1} - \binom{j_n - i_n + k_n}{j_n - i_n} \right] \\ &= \sum_{n=0}^{r} \left[\frac{k_n + 1}{j_n + 1} \binom{j_n + k_n + 1}{j_n} - \frac{k_n + 1}{j_n - i_n} \binom{j_n - i_n + k_n}{j_n - i_n - 1} \right]. \end{aligned}$$

On the other hand from the upper bound on h_j and $h_{j+1} = h_{p-1}$ we get

$$\frac{g-1}{p-1} h_j \leq \binom{p+g-3}{g-2} \leq h_{j+1} - h_j.$$

Since $(g-1)/(p-1) > (k+1)/(j+1)$ it follows from (4.13) and the last two inequalities that $F_0 < 0$, where for $0 \leq s \leq r$ we set

$$\begin{aligned} F_s &= \sum_{n=s}^{r} \left[\left(\frac{k_s + 1}{j_s + 1} - \frac{k_n + 1}{j_n + 1} \right) \binom{j_n + k_n + 1}{j_n} \right] \\ &\quad + \sum_{n=s}^{r} \left[\left(\frac{k_n + 1}{j_n - i_n} - \frac{k_s + 1}{j_s + 1} \right) \binom{j_n - i_n + k_n}{j_n - i_n - 1} \right]. \end{aligned}$$

To derive a contradiction we are going to show the following inequalities

$$F_0 > F_1 > \cdots > F_r = \left(\frac{k_r + 1}{j_r - i_r} - \frac{k_r + 1}{j_r + 1} \right) \binom{a_i}{i - 1} > 0.$$

Assume $1 \leq s + 1 \leq r$. Notice that

$$\begin{aligned} 0 &\leq \sum_{n=s}^{r-1} \left[\binom{j_n - i_n + k_n}{j_n - i_n - 1} - \binom{j_{n+1} + k_{n+1} + 1}{j_{n+1}} \right] + \binom{j_r - i_r + k_r}{j_r - i_r - 1} \\ &= \binom{j_s - i_s + k_s}{j_s - i_s - 1} - \sum_{n=s+1}^{r} \left[\binom{j_n + k_n + 1}{j_n} - \binom{j_n - i_n + k_n}{j_n - i_n - 1} \right]. \end{aligned}$$

Therefore

$$\binom{j_s - i_s + k_s}{j_s - i_s - 1} \geq \sum_{n=s+1}^{r} \binom{j_n + k_n + 1}{j_n} - \sum_{n=s+1}^{r} \binom{j_n - i_n + k_n}{j_n - i_n - 1}.$$

Let A_s denote the first summation and B_s the second in this last inequality; clearly $A_s - B_s > 0$. Also note that

$$\frac{k_s + 1}{j_s - i_s} > \frac{k_s + 1}{j_s + 1} \qquad \text{and} \qquad \frac{k_s + 1}{j_s - i_s} > \frac{k_{s+1} + 1}{j_{s+1} + 1}.$$

Putting these together, we compute

$$\begin{aligned} F_s - F_{s+1} &= \left(\frac{k_s+1}{j_s-i_s} - \frac{k_s+1}{j_s+1}\right)\binom{j_s-i_s+k_s}{j_s-i_s-1} \\ &\quad + \left(\frac{k_s+1}{j_s+1} - \frac{k_{s+1}+1}{j_{s+1}+1}\right)A_s + \left(\frac{k_{s+1}+1}{j_{s+1}+1} - \frac{k_s+1}{j_s+1}\right)B_s \\ &\geq \left(\frac{k_s+1}{j_s-i_s} - \frac{k_s+1}{j_s+1}\right)(A_s - B_s) + \left(\frac{k_s+1}{j_s+1} - \frac{k_{s+1}+1}{j_{s+1}+1}\right)A_s \\ &\quad + \left(\frac{k_{s+1}+1}{j_{s+1}+1} - \frac{k_s+1}{j_s+1}\right)B_s \\ &= \left(\frac{k_s+1}{j_s-i_s} - \frac{k_{s+1}+1}{j_{s+1}+1}\right)(A_s - B_s) \quad > \quad 0. \end{aligned}$$

Hence $F_0 > F_r > 0$, which contradicts the observation that $F_0 < 0$. □

One might hope for even stronger estimates than suggested by the result above, for instance that $\nu(I) \leq \nu_0$. Next we present a counterexample showing that Theorem 4.4.3 cannot be extended to bound the number of generators in all degrees.

Example 4.4.4 Let

$$I = ((x_1^2, x_2^4, x_3^3, x_4^4) : (x_1x_2 - x_3x_4)).$$

Using Macaulay one readily obtains that I is given by:

$$(x_1^2,\ x_1x_2x_3 + x_3^2x_4,\ x_3^3,\ x_1x_3^2,\ x_2^4,\ x_4^4,\ x_1x_4^3,\ x_1x_2x_4^2 + x_3x_4^3,\ x_2^3x_3^2,\ x_2^3x_4^3).$$

Then R/I is a Gorenstein Artin algebra with h-vector $(1,4,9,13,13,9,4,1)$ and Betti sequence $(1,10,18,10,1)$, whereas the h-vector for an extremal Gorenstein algebra of codimension four and initial degree two is $(1,4,1)$ and the Betti sequence is $(1,9,16,9,1)$; notice that the multiplicity $e(R/I) = 54$ is far greater than the minimal value of six exhibited by an extremal algebra.

In height three one can bound the Betti numbers, thanks to the structure theorem of Buchsbaum and Eisenbud. For initial degree p, the extremal Gorenstein algebra of codimension three has $b_1 = b_2 = 2p+1$.

Theorem 4.4.5 *Let R be a polynomial ring over a field and I be a homogeneous Gorenstein ideal of height three. If p is the initial degree of I, then*

$$\nu(I) \leq 2p+1 \quad \textit{and} \quad b_2(R/I) \leq 2p+1.$$

Proof. (sketch) We may assume, without loss of generality, that R is equal to $k[x_1, x_2, x_3]$ and $A = R/I$ is Artinian and local with socle$(A) = A_\sigma$. Then by [54] the minimal free resolution of A has the form

$$0 \to R(-\sigma - 3) \to \bigoplus_{j=1}^{\nu} R(-n_j) \xrightarrow{Y} \bigoplus_{i=1}^{\nu} R(-m_i) \to R,$$

where Y is an alternating matrix, and the generators $f_1, \ldots, f_\nu$ of I are the maximal pfaffians of Y. If the theorem fails every generator has degree at least $(2p+1)/2 > p$, which is a contradiction since at least one minimal generator has degree p. □

Conjecture 4.4.6 (Miller-Villarreal) *Let $I \subset R$ be a graded Gorenstein ideal of initial degree p and height g. If $1 \leq i < g$ and γ_i is the ith initial Betti number of R/I, then*

$$\gamma_i \leq \binom{p+g-1}{p+i-1}\binom{p+i-2}{i-1} + \binom{p+g-1}{i}\binom{p+g-i-2}{p-1} - \binom{g}{i}\binom{p+g-2}{p-1}.$$

Proposition 4.4.7 *Let n be a given positive integer and $f, g\colon \mathbb{N} \to \mathbb{N}$ the numerical functions given by*

$$g(x) = n^{\langle x \rangle} \quad \text{and} \quad f(x) = x^{\langle n \rangle},$$

where $h^{\langle j \rangle}$ is the Macaulay symbol. Then $g(x)$ is non increasing and $f(x)$ is strictly increasing.

Proof. See [22] and [44, Lemma 4.2.13] respectively. □

There is a short and elegant argument, based on the behaviour of the combinatorial functions $g(x) = n^{\langle x \rangle}$ and $f(x) = x^{\langle n \rangle}$, to show the inequality of Theorem 4.4.3. We include the details of this argument in the proof below.

Corollary 4.4.8 *If R is a polynomial ring over a field k and I a graded Gorenstein ideal of R of codimension $g \geq 3$ and initial degree $p \geq 2$, then*

$$\nu(I_p) \leq \nu_0 = \binom{p+g-1}{g-1} - \binom{p+g-3}{g-1}.$$

Proof. First we make a change of coefficients using the functor $(\cdot) \otimes_k K$, where K is an infinite field extension of k. Hence we may assume that k is infinite. Let A be an *Artinian reduction* of $S = R/I$, that is,

$$A = S/(h_1, \ldots, h_d),$$

where $\underline{h} = h_1, \ldots, h_d$ is a system of parameters of S with h_i a form of degree 1 for all i. Note that one can write $A = \overline{R}/\overline{I}$, where $\overline{R} = R/(\underline{h})$ is a polynomial ring in g variables and $\overline{I} = I\overline{R}$ is a graded Gorenstein ideal of $\overline{R}$ of codimension g and initial degree p. By Proposition 4.3.9 the Hilbert function of A is symmetric. Since the h-vector $h(S) = (h_i)$ satisfy $h_i = \dim_k(A_i)$ there is some $j \geq p$ so that

$$h(S) = (h_0, h_1, \ldots, h_{p-1}, h_p, \ldots, h_j, h_{p-1}, \ldots, h_1, h_0),$$

thus

$$h_{j+1} = h_{p-1} = \binom{p+g-2}{g-1} \text{ and } h_j = h_p \tag{1}$$

The idea of the argument is to use the Macaulay bound

$$h_{j+1} \leq h_j^{\langle j \rangle}, \tag{2}$$

see Theorem 4.4.2. From the inequalities:

$$\begin{aligned} h_{j+1} &\overset{(1)}{=} h_{p-1} = \binom{p+g-2}{p-1} = \binom{p+g-3}{p-2}^{\langle p-2 \rangle} \\ &\overset{(2)}{\leq} h_j^{\langle j \rangle} \overset{(1)}{=} h_p^{\langle j \rangle} = \left[\binom{p+g-1}{p} - \nu(I_p)\right]^{\langle j \rangle} \\ &\overset{\text{use } g(x)}{\leq} \left[\binom{p+g-1}{p} - \nu(I_p)\right]^{\langle p-2 \rangle} \end{aligned}$$

we conclude

$$\binom{p+g-3}{p-2}^{\langle p-2 \rangle} \leq \left[\binom{p+g-1}{p} - \nu(I_p)\right]^{\langle p-2 \rangle}.$$

Using that $f(x)$ is non decreasing yields the required inequality. □

Exercises

4.4.9 Let I be a graded Gorenstein ideal of height 4 and initial degree p, prove $\nu(I_p) \leq (p+1)^2$. If $p = 4$ show the values $21 \leq \nu(I_4) \leq 24$ do not occur.

4.4.10 Let I be a graded Gorenstein ideal of height 4 generated by forms of degree p and R/J an extremal algebra of the same codimension and initial degree, prove that their Betti numbers satisfy

$$b_i(R/I) \leq b_i(R/J) \text{ for all } i.$$

4.4.11 Let $R = k[x_1, \ldots, x_g]$ be a polynomial ring over a field k and I a graded ideal of height g generated by forms of degree p. Prove that the Hilbert function of I satisfies:

$$H(I, p+i) \geq H(I,p) + i\binom{g}{2} \text{ for all } i \geq 0.$$

4.5 A symbolic study of Koszul homology

The first Koszul homology module of a Cohen-Macaulay graded ideal is studied here using the Cohen-Macaulay criterion of Jürgen Herzog introduced in Section 3.4, our approach is symbolic and makes use of Hilbert functions.

Lemma 4.5.1 *Let p and g be positive integers and let*

$$\psi(p) = \binom{p+g-1}{g-1}^2 - 2\binom{2p+g}{g}.$$

If $p \geq 2$ and $g \geq 6$ then $\psi(p) > 0$.

Proof. Notice the equality

$$6\psi(2) = (g+1)(g^3 - 3g^2 - 13g - 12),$$

which is certainly positive for $g \geq 6$. We proceed by induction on p. Assume $\psi(p) > 0$. It is easy to check that $\psi(p+1)$ is greater than

$$2\binom{2p+g}{g}\left[\frac{(p+1)(4(g-2)p^2 + (3g^2 - 3g - 8)p + g^2 - 3g - 2)}{(p+1)^2(2p+1)(2p+2)}\right].$$

Since the right hand side of this inequality is positive for $p \geq 2$ and $g \geq 3$, the induction step is complete. □

Theorem 4.5.2 ([240]) *Let $R = \oplus_{i=0}^{\infty} R_i$ be a polynomial ring with its usual graduation over an infinite field k and I a Cohen-Macaulay graded ideal of R of height g with a p-linear resolution*

$$0 \to R^{b_g}(-(p+g-1)) \to \cdots \to R^{b_k}(-(p+k-1)) \to \cdots \to R^{b_1}(-p) \to I \to 0.$$

If I is generically a complete intersection with $p \geq 2$ and $g \geq 3$, then $H_1(I)$, the first Koszul homology module of I, is not Cohen-Macaulay.

Proof. The module $H_1 = H_1(I)$ has a well defined rank equal to $b_1 - g$. Thus by Theorem 3.4.12, it suffices to prove that for some homogeneous system of parameters $\mathbf{y}$ of $S = R/I$, one has

$$\ell(H_1(I)/\mathbf{y}H_1(I)) > \operatorname{rank}(H_1(I)) \cdot \ell(S/\mathbf{y}S).$$

We start by making a specialization to the case of a polynomial ring of dimension g. Since k is infinite according to Theorem 2.2.11, there exists a system of parameters $\mathbf{y} = \{y_1, \ldots, y_d\}$ for S with each y_i a form of degree one of R. We make two observations: (i) Because $\operatorname{Tor}_i(S, R/(\mathbf{y})) = 0$ for $i \geq 1$, it is clear that the minimal resolution of $\overline{S} = S/(\mathbf{y}S)$ as an $\overline{R}$ $(= R/(\mathbf{y}))$-module has the same twists and Betti numbers as the minimal resolution of S over R. (ii) With H_1 Cohen–Macaulay it is easy to see that the first Koszul homology module of $I \otimes \overline{R}$ over $\overline{R}$ is precisely $H_1(I) \otimes \overline{R}$. We may then in the sequel assume that S is zero-dimensional.

We will compare the integer r=rank(original $H_1(I)$)$\cdot \ell(S)$, with partial Hilbert sums contributing to $\ell(H_1)$. First notice (see Exercise 4.3.17) that the length of S and its Betti numbers can be calculated using:

$$b_k = \binom{p+k-2}{k-1} \cdot \binom{p+g-1}{g-k} \text{ and } \ell(S) = \binom{p+g-1}{g}. \tag{4.14}$$

From the Koszul complex we obtain the exact sequences

$$0 \to B_1 \to Z_1 \to H_1 \to 0,$$

$$0 \to Z_2 \to R^{\binom{b_1}{2}}(-2p) \to B_1 \to 0,$$

where Z_i and B_i are the modules of cycles and boundaries defining $H_i(I)$. To simplify notation we set $\ell(M)_i = H(M, i)$, the dimension of the ith component of the graded module M. One may write

$$\ell(H_1)_i = \ell(Z_1)_i - \ell(B_1)_i = \ell(Z_1)_i - \binom{b_1}{2}\ell(R(-2p))_i + \ell(Z_2)_i,$$

and therefore

$$\ell(H_1) \geq \sum_{i=0}^{2p} \ell(H_1)_i \geq \sum_{i=0}^{2p} \ell(Z_1)_i - \binom{b_1}{2} \sum_{i=0}^{2p} \ell(R(-2p))_i. \tag{4.15}$$

From the minimal resolution of S we obtain

$$\ell(S)_i = \ell(R)_i - b_1\ell(R(-p))_i + \ell(Z_1)_i \geq 0.$$

Hence $\ell(Z_1)_i \geq b_1\ell(R(-p))_i - \ell(R)_i$ and $\ell(Z_1)_i = 0$ for $0 \leq i \leq p$, which leads to

$$\sum_{i=0}^{2p} \ell(Z_1)_i \geq b_1 \sum_{i=p+1}^{2p} \ell(R(-p))_i - \sum_{i=p+1}^{2p} \ell(R)_i. \tag{4.16}$$

Finally by (4.15) and (4.16) we have

$$\ell(H_1) \geq b_1 \sum_{i=p+1}^{2p} \ell(R(-p))_i - \sum_{i=p+1}^{2p} \ell(R)_i - \binom{b_1}{2}. \tag{4.17}$$

The proof now reduces to showing that the right hand side of (4.17) is greater than $\operatorname{rank}(H_1(I)) \cdot \ell(S) = (b_1 - g) \cdot \ell(S)$, that is, we must show that the following inequality holds for $p \geq 2$ and $g \geq 3$

$$f(p,g) = b_1 \sum_{i=p+1}^{2p} \ell(R(-p))_i - \sum_{i=p+1}^{2p} \ell(R)_i - \binom{b_1}{2} - (b_1 - g) \cdot \ell(S) > 0.$$

It is not hard to see that $f(p,g)$ simplifies to

$$f(p,g) = \binom{b_1 + 1}{2} + (1+g)\binom{p+g-1}{p-1} - \binom{2p+g}{g}.$$

Observe that from this equality we obtain the inequality:

$$2f(p,g) > \binom{p+g-1}{p}^2 - 2\binom{2p+g}{g}.$$

It is easy to check that $f(p,g) > 0$ for $g \in \{3,4,5\}$ and $p \geq 2$. The required inequality is now a direct consequence of Lemma 4.5.1. □

The result above complements one of Bernd Ulrich [286] in which the twisted conormal module $\omega_S \otimes_R I = \omega_S \otimes_R I/I^2$ was shown not to be Cohen-Macaulay; either result implies that I is not in the linkage class of a complete intersection.

Corollary 4.5.3 *Let I be a graded Cohen-Macaulay ideal of height $g \geq 3$ in polynomial ring R. If I is generically a complete intersection and has a p-linear resolution with $p \geq 2$, then I is not in the linkage class of a complete intersection.*

Proof. If I is in the linkage class of a complete intersection, then I is a strongly Cohen-Macaulay ideal, a contradiction because $H_1(I)$ is not Cohen-Macaulay. See [179] and [296, Corollary 4.2.4]. □

Chapter 5

Monomial Ideals and Stanley-Reisner Rings

This chapter is intended as an introduction to the study of *combinatorial commutative algebra*, some of the topics were chosen in order to motivate the subject. Part of our exposition was inspired by [25, 271].

The main objects of study in this chapter are monomial ideals, face rings and simplicial complexes. Here we emphasize the connection between algebraic properties of face rings and the reduced simplicial homology of the corresponding Stanley-Reisner complex. An understated goal here is to highlight some of the work of M. Hochster, G. Reisner and R. Stanley.

It turns out that some notions encountered while studying graded algebras, such as Cohen-Macaulay rings, Gorenstein rings and Hilbert series are useful to solve combinatorial problems. Some key results leading to the proofs of the upper bound conjectures for convex polytopes and simplicial spheres are presented, most of them without detailed proofs but giving the appropriate references. Our hope is to give a flavor of the methods and ideas of the area and to trace some routes for further reading.

5.1 Primary decomposition

General monomial ideals will be examined first, subsequently we specialize to the case of square-free monomial ideals and present some of their relevant properties.

Let $R = k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. To make notation simpler we set

$$x^{\alpha} = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \quad \text{for } \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n.$$

Definition 5.1.1 An ideal I of R is called a *monomial ideal* if there is $\mathcal{A} \subset \mathbb{N}^n$ such that I is generated by $\{x^\alpha | \alpha \in \mathcal{A}\}$. If I is a monomial ideal the quotient ring R/I is called a *monomial ring*.

Note that a monomial ideal is always generated by a finite set of monomials by Dickson's lemma.

Definition 5.1.2 A *face ideal* is an ideal $\mathfrak{p}$ of R generated by a subset of the set of variables, that is, $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_k})$ for some variables x_{i_j}.

Proposition 5.1.3 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. If $I \subset R$ is a monomial ideal, then every associated prime of I is a face ideal.*

Proof. By induction on the number of variables that occur in a minimal generating set of I consisting of monomials. Set $\mathfrak{m} = (x_1, \ldots, x_n)$. Let $\mathfrak{p}$ be an associated prime of I. If $\operatorname{rad}(I) = \mathfrak{m}$, then $\mathfrak{p} = \mathfrak{m}$. Hence we may assume $\operatorname{rad}(I) \neq \mathfrak{m}$. Pick a variable x_1 not in $\operatorname{rad}(I)$ and consider the ascending chain of ideals

$$I_0 = I \text{ and } I_{n+1} = (I_n : x_1) \qquad (n \geq 0).$$

Since R is Noetherian one has $I_k = (I_k : x_1)$ for some k. There are two cases to consider. If $\mathfrak{p}$ is an associated prime of (I_n, x_1) for some n, then by induction $\mathfrak{p}$ is a face ideal because one can write $(I_n, x_1) = (I'_n, x_1)$, where I'_n is an ideal minimally generated by a finite set of monomials in the variables $x_2, \ldots, x_n$ (cf. Exercise 1.1.45). Assume we are in the opposite case. By Lemma 4.1.2 for each n there is an exact sequence

$$0 \longrightarrow R/(I_n : x_1) \xrightarrow{x_1} R/I_n \longrightarrow R/(I_n, x_1) \longrightarrow 0,$$

hence making a recursive application of Lemma 1.1.16 one obtains that $\mathfrak{p}$ is an associated prime of I_n for all n. Since x_1 is regular on R/I_k one concludes that I_k is an ideal minimally generated by monomials in the variables $x_2, \ldots, x_n$, thus by induction $\mathfrak{p}$ is a face ideal. □

Definition 5.1.4 A monomial f in R is called *square-free* if

$$f = x_{i_1} \ldots x_{i_r}$$

for some $1 \leq i_1 < \cdots < i_r \leq n$.

Corollary 5.1.5 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. If I is an ideal of R generated by square-free monomials and $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ are the associated primes of I, then*

$$I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_s.$$

Proof. Set $J = \cap_{i=1}^{s} \mathfrak{p}_i$ and note $J = \operatorname{rad}(I)$. We only have to check $J \subset I$ because I is contained in any of its associated primes. Take a monomial f in $J = \operatorname{rad}(I)$ and write $f = x_{i_1}^{a_1} \cdots x_{i_r}^{a_r}$, where $i_1 < \cdots < i_r$ and $a_i > 0$ for all i. Since $f^k \in I$ for some $k \geq 1$, and using that I is generated by square-free monomials, we obtain $x_{i_1} \cdots x_{i_r} \in I$. Hence $f \in I$. To finish the proof observe that J is a monomial ideal because the intersection of monomial ideals is again a monomial ideal (see Exercise 5.1.24). □

Definition 5.1.6 Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring, the *support* of a monomial $x^a = x_1^{a_1} \cdots x_n^{a_n}$ in R is given by $\operatorname{supp}(x^a) = \{x_i \mid a_i > 0\}$.

Proposition 5.1.7 ([113]) *Let I be a monomial ideal in a polynomial ring R over a field k and $S = R/I$. Then there is a polynomial ring R' and a square-free monomial ideal I' of R' such that $S = S'/(\underline{h})$, where $S' = R'/I'$ and $\underline{h}$ is a regular sequence on S' of forms of degree one.*

Proof. Let $R = k[\mathbf{x}]$ and let $F = \{f_1, \ldots, f_r\}$ be a set of monomials that minimally generate I. Assume that one of the variables, say x_1, occurs in at least one of the monomials in F with multiplicity greater than 1. Thus one may write

$$f_1 = x_1^{a_1} g_1, \ldots, f_s = x_1^{a_s} g_s,$$

where $a_i \geq 1$ for all i, $a_1 \geq 2$, $x_1 \notin \operatorname{supp}(g_i)$ for all $i \leq s$ and $x_1 \notin \operatorname{supp}(f_i)$ for $i > s$. Set

$$I' = (x_0 x_1^{a_1 - 1} g_1, \ldots, x_0 x_1^{a_s - 1} g_s, f_{s+1}, \ldots, f_r) \subset R' = R[x_0],$$

where x_0 is a new variable. We claim that $x_0 - x_1$ is a nonzero divisor of $S' = R'/I'$. On the contrary assume that $x_0 - x_1$ belongs to an associated prime $\mathfrak{p}$ of I' and write $\mathfrak{p} = (I' : h)$, for some monomial h. Since $\mathfrak{p}$ is a face ideal, one has $x_i h \in I'$ for $i = 0, 1$. Hence using $h \notin I'$ gives $x_1 h = x_0 x_1^{a_i - 1} g_i h_1$ for some i and consequently $h = x_0 x_1^{a_i - 2} g_i h_1$. Hence

$$x_0 h = x_0^2 x_1^{a_i - 2} g_i h_1 = \begin{cases} x_0 x_1^{a_j - 1} g_j M & \text{or,} \\ f_j M & \text{for some } j > s. \end{cases}$$

In both cases one obtains $x_0 \in \operatorname{supp}(M)$ and $h \in I'$, a contradiction. Since

$$S'/(x_0 - x_1) = S$$

one can repeat the construction to obtain the asserted monomial ideal I'. This proof is due to R. Fröberg. □

The ideal I' constructed above is called the *polarization* of I. Thus any monomial ring is a deformation by linear forms of a monomial ring with square-free relations. Note that I is Cohen-Macaulay (resp. Gorenstein) iff I' is Cohen-Macaulay (resp. Gorenstein).

Proposition 5.1.8 *Let* $\mathfrak{q}$ *be a monomial ideal of* $R = k[x_1, \ldots, x_n]$. *Then* $\mathfrak{q}$ *is a primary ideal if and only if, after permutation of the variables,* $\mathfrak{q}$ *has the form:*

$$\mathfrak{q} = (x_1^{a_1}, \ldots, x_r^{a_r}, x^{b_1}, \ldots, x^{b_s}),$$

where $a_i \geq 1$ *and* $\cup_{i=1}^{s} \operatorname{supp}(x^{b_i}) \subset \{x_1, \ldots, x_r\}$.

Proof. If $\operatorname{Ass}(R/\mathfrak{q}) = \{\mathfrak{p}\}$, then by Proposition 5.1.3 one has that $\mathfrak{p}$ is equal to $(x_1, \ldots, x_r)$. Since $\operatorname{rad}(\mathfrak{q}) = \mathfrak{p}$ the ideal $\mathfrak{q}$ is minimally generated by a set of the form:

$$\{x_1^{a_1}, \ldots, x_r^{a_r}, x^{b_1}, \ldots, x^{b_s}\}.$$

Let $x_j \in \operatorname{supp}(x^{b_i})$, then $x^{b_i} = x_j N$, where N is a monomial not in $\mathfrak{q}$. Since $\mathfrak{q}$ is primary, a power of x_j is in $\mathfrak{q}$. Thus $x_j \in (x_1, \ldots, x_r)$ and consequently $1 \leq j \leq r$, as required.

For the converse note that any associated prime $\mathfrak{p}$ of $R/\mathfrak{q}$ can be written as $\mathfrak{p} = (\mathfrak{q}\colon f)$, for some monomial f. It follows rapidly that $(x_1, \ldots, x_r)$ is the only associated prime of $\mathfrak{q}$. □

Corollary 5.1.9 *If* $\mathfrak{p}$ *is a face ideal, then* $\mathfrak{p}^n$ *is a primary ideal for all* n.

Proposition 5.1.10 *If* I *is a monomial ideal of* $R = k[x_1, \ldots, x_n]$, *then* I *has an irredundant primary decomposition* $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$, *where* $\mathfrak{q}_i$ *is a primary monomial ideal for all* i *and* $\operatorname{rad}(\mathfrak{q}_i) \neq \operatorname{rad}(\mathfrak{q}_j)$ *if* $i \neq j$.

Proof. Let $f_1, \ldots, f_q$ be a set of monomials that minimally generate I. We proceed by induction on the number of variables that occur in the union of the supports of $f_1, \ldots, f_q$.

One may assume that one of the variables in $\cup_{i=1}^{q} \operatorname{supp}(f_i)$, say x_n, satisfy $x_n^i \notin I$ for all i, otherwise I is a primary ideal and there is nothing to prove. Next we permute the f_i in order to find integers $0 \leq a_1 \leq \cdots \leq a_q$, with $a_q \geq 1$, such that f_i is divisible by $x_n^{a_i}$ but by not higher power of x_n. If we apply this procedure to $(I, x_n^{a_q})$, instead of I, note that one must choose a variable different from x_n.

As $(I\colon x_n^{a_q})$ is generated by monomials in less than n variables and because of the equality

$$I = (I, x_n^{a_q}) \cap (I\colon x_n^{a_q})$$

one may apply the argument above recursively to both monomial ideals occurring in the intersection and use induction to obtain a decomposition of I into primary monomial ideals $\mathfrak{q}_1', \ldots, \mathfrak{q}_s'$. Finally we remove redundant primary ideals from $\mathfrak{q}_1', \ldots, \mathfrak{q}_s'$ and group those primary ideals with the same radical. □

Even for monomial ideals a minimal irredundant primary decomposition is not unique, what is unique is the number of terms in such a decomposition and also the primary components that correspond to minimal primes.

Example 5.1.11 If $I = (x^2, xy) \subset k[x, y]$, then

$$I = (x) \cap (x^2, xy, y^2) = (x) \cap (x^2, y),$$

are two minimal irredundant primary decompositions of I.

The computation of a primary decomposition of a monomial ideal can be carried out by successive elimination of powers of variables, as described in the proof of Proposition 5.1.10. Now we illustrate this procedure with a specific ideal.

Example 5.1.12 If $R = k[x, y, z]$ and $I = (yz^2, x^2z, x^3y^2)$.

$$\begin{array}{ccccc} & \nearrow & (I\colon x^3) = (z, y^2) & & \\ I & & & & \\ & \searrow & J = (I, x^3) = (x^3, zx^2, z^2y) & \begin{array}{c}\nearrow \\ \searrow\end{array} & \begin{array}{l}(J\colon z^2) = (x^2, y) \\ (J, z^2) = (z^2, x^3, x^2z).\end{array} \end{array}$$

Thus $I = (z, y^2) \cap (x^2, y) \cap (z^2, x^3, x^2z)$.

Corollary 5.1.13 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. If I is a monomial ideal, then I has a primary decomposition*

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m,$$

such that $\mathfrak{q}_i$ is generated by powers of variables for all i.

Proof. Let $\mathfrak{q}$ be a primary ideal minimally generated by the set of monomials $x_1^{a_1}, \ldots, x_r^{a_r}, f_1, \ldots, f_s$ and such that

$$\bigcup_{i=1}^{s} \operatorname{supp}(f_i) \subset \{x_1, \ldots, x_r\},$$

where $a_i > 0$ for all i. Note that if $f_1 = x_1^{b_1} \cdots x_r^{b_r}$ and $b_1 > 0$, then $a_1 > b_1$ and one has a decomposition:

$$\mathfrak{q} = (x_1^{b_1}, x_2^{a_2}, \ldots, x_r^{a_r}, f_2, \ldots, f_s) \cap (x_1^{a_1}, x_2^{a_2}, \ldots, x_r^{a_r}, x_2^{b_2} \cdots x_r^{b_r}, f_2, \ldots, f_s),$$

where in the first member of the intersection we have lower the degree of $x_1^{a_1}$ and have eliminated f_1, while in the second member we have eliminated the variable x_1 from f_1. Applying the same argument repeatedly it follows that one can write $\mathfrak{q}$ as an intersection of primary monomial ideals such that for each of those ideals the only minimal generators that contain x_1 are pure powers of x_1. Therefore by induction $\mathfrak{q}$ is the intersection of ideals generated by powers of variables. Hence the result follows from Proposition 5.1.10 and Proposition 5.1.8. □

Proposition 5.1.14 *Let I be an ideal of $k[x_1, \ldots, x_n]$ generated by monomials in the first variables $x_1, \ldots, x_r$. If*

$$I = I_1 \cap \cdots \cap I_s$$

is an irredundant decomposition of I into monomial ideals, then none of the I_i can contain a monomial in $k[x_{r+1}, \ldots, x_n]$.

Proof. Set $X = \{x_1, \ldots, x_r\}$ and $X' = X \setminus \{x_1, \ldots, x_r\}$. Assume some of the I_i contain monomials in $k[X']$. One may split the I_i's into two sets so that $I_1, \ldots, I_m$ do not contain monomials in $k[X']$ (note m could be zero), while $I_{m+1}, \ldots, I_s$ contain monomials in the set of variables X'. For $i \geq m+1$ pick a monomial g_i in I_i whose support is contained in X'. Since the decomposition of I is irredundant there is a monomial $f \in \cap_{i=1}^{m} I_i$ and $f \notin \cap_{i=m+1}^{s} I_i$, where we set $f = 1$ if $m = 0$. To derive a contradiction consider $f_1 = f g_{m+1} \cdots g_s$. As $f_1 \in I$, we get $f \in I$ which is absurd. □

Corollary 5.1.15 *Let $I \subset k[x_1, \ldots, x_n]$ be an ideal generated by monomials in the first variables $x_1, \ldots, x_r$. If*

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$$

is an irredundant primary decomposition into monomial ideals, then $\mathfrak{q}_i$ is generated by monomials in $k[x_1, \ldots, x_r]$.

Proof. It follows from Proposition 5.1.8 and Proposition 5.1.14. □

Proposition 5.1.16 *If $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k and I a monomial ideal, then I is irreducible if and only if up to permutation of the variables I can be written as*

$$I = (x_1^{a_1}, \ldots, x_r^{a_r}),$$

where $a_i > 0$ for all i.

Proof. $\Rightarrow$) Since I must be primary (see Proposition 1.1.21) from the proof of Corollary 5.1.13 one derives that I is generated by powers of variables.

$\Leftarrow$) If I is reducible, then by Corollary 5.1.13 there is an irredundant decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$, such that $\mathfrak{q}_i \neq I$ and $\mathfrak{q}_i$ is generated by some powers of the variables $x_1, \ldots, x_r$, for all i. Let $x_{j_i}^{c_i} \in \mathfrak{q}_i \setminus I$, where $1 \leq j_i \leq r$ and $c_i < a_{j_i}$. Set

$$f = \operatorname{lcm}\{x_{j_1}^{c_1}, \ldots, x_{j_r}^{c_r}\}.$$

Note that one can write $f = x_{m_1}^{e_1} \cdots x_{m_k}^{e_k}$, where $e_i < a_{m_i}$ and $m_1, \ldots, m_k$ distinct, but this is impossible since $f \in I$, thus I is irreducible. □

Theorem 5.1.17 *If I is a monomial ideal in a polynomial ring R over a field k, then there is a unique irredundant decomposition*

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$$

such that $\mathfrak{q}_i$ is an irreducible monomial ideal.

Proof. The existence follows from Corollary 5.1.13. For the uniqueness assume one has two irredundant decompositions:

$$\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r = \mathfrak{q}'_1 \cap \cdots \cap \mathfrak{q}'_s,$$

where $\mathfrak{q}_i$ and $\mathfrak{q}'_j$ are irreducible for all i, j. Using the arguments given in the proof of Proposition 5.1.16 one concludes that for each i, there is σ_i such that $\mathfrak{q}_{\sigma_i} \subset \mathfrak{q}'_i$ and vice versa for each j there is π_j such that $\mathfrak{q}'_{\pi_j} \subset \mathfrak{q}_j$. Therefore $r = s$ and $\mathfrak{q}_i = \mathfrak{q}'_{\rho_i}$ for some permutation ρ. □

There are various routines for *Macaulay*2 developed by S. Popescu [235] for dealing with monomial ideals, one of them computes the irredundant decomposition into irreducible monomial ideals of any monomial ideal.

For further information on the primary decomposition of more general monomial ideals, e.g. monomial ideals obtained from a regular sequence or with coefficients in a ring other than a field, consult [88, 143, 144] and [307].

Example 5.1.18 Let $I = (xy^2, x^2y) \subset k[x, y]$ and $V(I)$ the affine variety defined by I. From the primary decomposition

$$I = (x) \cap (y) \cap (x^2, y^2)$$

one has:

$$V(I) = \{(0,0)\} \cup V(x) \cup V(y) = V(x) \cup V(y) \subset \mathbb{A}^2_k,$$

where (x^2, y^2) corresponds to $\{(0,0)\}$ which is embedded in $V(x) \cup V(y)$. For this reason (x^2, y^2) is said to be an embedded primary component. If k is infinite, then the coordinate axes $V(x)$ and $V(y)$ are the irreducible components of $V(I)$.

Monomial ideals form a lattice Let L be a *lattice*, that is, L is a partially ordered set or poset in which any two elements x, y have a greatest lower bound or *meet* $x \wedge y$ and a lowest upper bound or *join* $x \vee y$. The lattice L is called *distributive* if

(a) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, and

(b) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, $\forall\, x, y, z$ in L.

Proposition 5.1.19 *If L is the family of monomial ideals, order by inclusion, of a polynomial ring R over a field k, then L is a distributive lattice under the operations $I \wedge J = I \cap J$ and $I \vee J = I + J$.*

Proof. Use Exercise 5.1.24 and Exercise 5.1.25. □

An interpretation of the minimal primes Let F be a set of monomials in the ring $k[y_1, \ldots, y_n]$ and $I = I(F)$ the ideal generated by F. Consider the hypergraph

$$\Upsilon(F) = \{\{y_{i_1}, \ldots, y_{i_r}\} | (y_{i_1}, \ldots, y_{i_r}) \text{ is a minimal prime of } I(F)\},$$

whose vertex set is

$$\bigcup_{f \in F} \operatorname{supp}(f).$$

An element in $\Upsilon(F)$ is called a *minimal vertex cover* of F.

In order to relate the set of minimal primes of I with the set of monomials occurring in a certain geometric link of I, consider a copy F_1 of F in the polynomial ring $k[x_1, \ldots, x_n]$ in a new set of variables $x_1, \ldots, x_n$.

Proposition 5.1.20 ([3]) *Let Υ_1 be the set of minimal vertex covers of F_1. If L is the ideal*

$$L = (\{x_{i_1} \cdots x_{i_r} | \{x_{i_1}, \ldots, x_{i_r}\} \in \Upsilon_1\}),$$

then $((\mathbf{z}): I(F)) = (\mathbf{z}, L)$, where $\mathbf{z} = \{x_1 y_1, \ldots, x_n y_n\}$.

Proof. Take $\{x_{i_1}, \ldots, x_{i_r}\} \in \Upsilon_1$ and $y^\alpha \in F$. Hence x_{i_j} divides x^α for some $1 \leq j \leq r$ and we obtain $x_{i_1} \cdots x_{i_r} y^\alpha \in (\mathbf{z})$. This shows $(\mathbf{z}, L) \subset ((\mathbf{z}): I(F))$. Conversely, assume M is a monomial in $((\mathbf{z}): I(F)) \setminus (\mathbf{z})$. Let y^α be any monomial in F. Since $My^\alpha = x_t y_t M_1$ and M is not in $(\mathbf{z})$, one has that x_t divides M and y_t divides y^α. Hence $\operatorname{supp}(M) \cap \operatorname{supp}(x^\alpha) \neq \emptyset$ for any x^α in F_1, and $M \in L$. □

Exercises

5.1.21 If $\mathbf{f} = f_1, \ldots, f_r$ is a sequence of monomials in R, then $\mathbf{f}$ is a regular sequence if and only if $\operatorname{supp}(f_i) \cap \operatorname{supp}(f_j) = \emptyset$ for $i \neq j$.

5.1.22 If I is a monomial ideal of R, then any associated prime $\mathfrak{p}$ of R/I can be written as $\mathfrak{p} = (I: f)$, for some monomial f.

5.1.23 If L is a lattice, then the following conditions are equivalent:

(a) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, $\forall x, y, z$ in L.

(b) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, $\forall x, y, z$ in L.

5.1.24 Let I and J be two ideals generated by finite sets of monomials F and G respectively, prove that the intersection $I \cap J$ is generated by the set

$$\{\operatorname{lcm}(f, g) | f \in F \text{ and } g \in G\}.$$

5.1.25 If I, J, K are monomial ideals, prove

$$I \cap (J + K) = (I \cap J) + (I \cap K).$$

5.1.26 If I and J are two monomial ideals, then $(I\colon J)$ is a monomial ideal.

5.1.27 Let $\mathfrak{q}$ be a primary ideal of $k[x_1, \ldots, x_n]$ generated by

$$\{x_1^{a_1}, \ldots, x_r^{a_r}, x^{b_1}, \ldots, x^{b_s}\}, \text{ where } \bigcup_{i=1}^{s} \operatorname{supp}(x^{b_i}) \subset \{x_1, \ldots, x_r\},$$

and $a_i \geq 1$ for all i. If $x^c = x_1^{c_1} \cdots x_r^{c_r}$ is a monomial of least degree in the variables $x_1, \ldots, x_r$ such that $x^c \notin \mathfrak{q}$, then $(x_1, \ldots, x_r) = (\mathfrak{q}\colon x^c)$.

5.1.28 If $\mathfrak{q}$ is a primary monomial ideal, then $\mathfrak{q}^n$ is a primary ideal and $\mathfrak{q}^n = \mathfrak{q}^{(n)}$ for $n \geq 1$.

Hint Use Proposition 5.1.8.

5.1.29 If $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ are primary monomial ideals of R with non comparable radicals and

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r,$$

then

$$I^{(n)} = \mathfrak{q}_1^n \cap \cdots \cap \mathfrak{q}_r^n.$$

Hint Proceed as in the proof of Proposition 3.3.24.

5.1.30 Let $I = (x_1^3, x_2^2 x_3^2, x_1 x_2^2 x_3)$, note that $I = \mathfrak{q}_1 \cap \mathfrak{q}_2$ is a primary decomposition of I, where $\mathfrak{q}_1 = (x_1^3, x_2^2)$ and $\mathfrak{q}_2 = (x_1^3, x_1 x_3, x_3^2)$. Prove $I^2 = I^{(2)}$ and $I^3 \neq I^{(3)}$.

Hint If $z = x_1^3 x_2^4 x_3^3$ and $a = x_1 + x_2^2 + x_3$, then $az \in I^3$ and $z \notin I^3$.

5.1.31 Use the next procedure in *CoCoA* to verify Exercise 5.1.30.

```
Use R ::= Q[x[1..3]]; -- DegRevLex is the default order
Q1 := Ideal(x[1]^3, x[2]^2);
Q2 := Ideal(x[1]^3, x[1] * x[3], x[3]^2);
I := Intersection(Q1, Q2);
S2 := Minimalized(Intersection(Q1^2, Q2^2));
I2 := Minimalized(I^2);
Len(S2); -- checking number of generators
S3 := Minimalized(Intersection(Q1^3, Q2^3));
EqSet(Gens(S2), Gens(I2)); -- checking equality
G := Gens(S3);
G[4] IsIn I^3;
```

5.1.32 Let $I = (y_1y_2^2, y_1^2y_2, y_1y_3^2, y_1^2y_3, y_2y_3^2, y_2^2y_3, y_1y_2y_3)$. Find a primary decomposition of I. Note (y_1, y_2, y_3) is an associated prime of I.

5.1.33 Use Proposition 5.1.20 to find the minimal primes of the ideal I generated by

$$y_1^3y_3y_5,\ y_1^3y_3y_6^2,\ y_1^3y_2^2y_3,\ y_1^3y_3y_4^2,\ y_1^4y_2^2,\ y_1^4y_4^2,\ y_1^2y_2^2y_3^2,\ y_2^3y_3^2y_5,\ y_2^3y_3^2y_6^5.$$

5.1.34 Show that $I = (y_1^2, y_1y_2)$ is a non primary ideal with a prime radical.

5.1.35 Let I be a monomial ideal in $k[x_1, \ldots, x_n]$ and $X = V(I) \subset \mathbb{A}^n(k)$ the associated monomial variety. If k is infinite, prove that X is irreducible if and only if $X = V(x_{i_1}, \ldots, x_{i_r})$.

Hint Note $I(V(x_1, \ldots, x_r)) = (x_1, \ldots, x_r)$.

5.2 Simplicial complexes and homology

A *finite simplicial complex* consists of a finite set V of vertices and a collection Δ of subsets of V called *faces* or *simplices* such that

(i) If $v \in V$, then $\{v\} \in \Delta$.

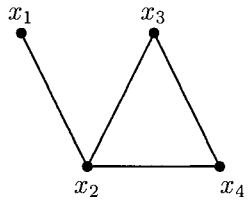
(ii) If $F \in \Delta$ and $G \subset F$, then $G \in \Delta$.

Let Δ be a simplicial complex and F a face of Δ. Define the *dimensions* of F and Δ by

$$\dim(F) = |F| - 1 \quad \text{and} \quad \dim(\Delta) = \sup\{\dim(F) | F \in \Delta\}$$

respectively. A face of dimension q is sometimes refer to as a q-face or as a q-simplex.

Example 5.2.1 Take the following triangulation of the disk with a whisker attached.



0-simplices: x_1, x_2, x_3, x_4
1-simplices: $x_1x_2, x_2x_3, x_3x_4, x_2x_4$
2-simplices: $x_2x_3x_4$

Let F be a q-simplex in Δ with vertices $v_0, v_1, \ldots, v_q$. We say that two total orderings of the vertices

$$v_{i_0} < \cdots < v_{i_q} \quad \text{and} \quad v_{j_0} < \cdots < v_{j_q}$$

are *equivalent* if $(i_0, \ldots, i_q)$ is an even permutation of $(j_0, \ldots, j_q)$. This is an equivalence relation because the set of even permutations form a group, and for $q > 1$, it partitions the total orderings of $v_0, \ldots, v_q$ into two equivalence classes. An *oriented q-simplex* of Δ is a q-simplex F with a choice of one of these equivalence classes. If $v_0, \ldots, v_q$ are the vertices of F, the oriented simplex determined by the ordering $v_0 < \cdots < v_q$ will be denoted by $[v_0, \ldots, v_q]$.

Suppose A is a commutative ring with unit. Let $C_q(\Delta)$ be the free A-module with basis consisting of the oriented q-simplices in Δ, modulo the relations

$$[v_0, v_1, v_2, \ldots, v_q] + [v_1, v_0, v_2, \ldots, v_q].$$

In particular $C_q(\Delta)$ is defined for any field k and $\dim_k C_q(\Delta)$ equals the number of q-simplices of Δ. For $q \geq 1$ we define the homomorphism

$$\partial_q \colon C_q(\Delta) \longrightarrow C_{q-1}(\Delta)$$

induced by

$$\partial_q([v_0, v_1, \ldots, v_q]) = \sum_{i=0}^{q} (-1)^i [v_0, v_1, \ldots, v_{i-1}, \widehat{v_i}, v_{i+1}, \ldots, v_q],$$

where $\widehat{v_i}$ means that the symbol v_i is to be deleted. Since $\partial_q \partial_{q+1} = 0$ we obtain the chain complex $C(\Delta) = \{C_q(\Delta), \partial_q\}$, which is called *the oriented chain complex* of Δ. The *augmented oriented chain complex* of Δ is the complex

$$0 \longrightarrow C_d(\Delta) \xrightarrow{\partial_d} C_{d-1}(\Delta) \longrightarrow \cdots \longrightarrow C_0(\Delta) \xrightarrow{\epsilon} C_{-1}(\Delta) = A \longrightarrow 0,$$

where $d = \dim(\Delta)$ and $\epsilon(v) = 1$ for every vertex v of Δ. This augmented chain complex is denoted by $(C(\Delta), \epsilon)$. Set $\partial_0 = \epsilon$ and $C_{-1}(\Delta) = A$.

Let $Z_q(\Delta, A) = \ker(\partial_q)$, $B_q = \operatorname{im}(\partial_{q+1})$ and

$$\widetilde{H}_q(\Delta; A) = Z_q(\Delta, A)/B_q(\Delta, A), \text{ for } q \geq 0.$$

The elements of $Z_q(\Delta, A)$ and $B_q(\Delta, A)$ are called *cycles* and *boundaries* respectively and $\widetilde{H}_q(\Delta; A)$ is the *qth reduced simplicial homology group* of Δ with coefficients in A.

Remark 5.2.2 Note that if $\Delta \neq \emptyset$, then $\widetilde{H}_i(\Delta; A) = 0$ for $i < 0$.

Proposition 5.2.3 *If Δ is a nonempty simplicial complex, then $\widetilde{H}_0(\Delta; A)$ is a free A-module of rank $r - 1$, where r is the number of connected components of Δ.*

Proof. Let $V = \{x_1, \ldots, x_n\}$ be the vertex set of Δ and $\Delta_1, \ldots, \Delta_r$ its connected components. Denote the vertex set of Δ_i by V_i and pick a vertex in each one of the V_i, one may assume $x_i \in V_i$ for $i = 1, \ldots, r$.

Set $\beta_i = x_i - x_r$ and $\overline{\beta}_i = \beta_i + \mathrm{im}(\partial_1)$ for $i = 1, \ldots, r-1$. We claim that the set $\overline{\mathcal{B}} = \{\overline{\beta}_1, \ldots, \overline{\beta}_{r-1}\}$ is an A-basis for $\widetilde{H}_0(\Delta; A) = \ker(\partial_0)/\mathrm{im}(\partial_1)$.

First we prove that $\overline{\mathcal{B}}$ is linearly independent. Assume $\sum_{i=1}^{r-1} a_i\beta_i$ is in $\mathrm{im}(\partial_1)$ for some a_i in A, by making an appropriate grouping of terms one can write

$$\sum_{i=1}^{r-1} a_i\beta_i = \sum_{x_i, x_j \in V_1} b_{ij}(x_i - x_j) + \cdots + \sum_{x_i, x_j \in V_r} b_{ij}(x_i - x_j),$$

for some b_{ij} in A. Since $C_0(\Delta)$ is a free A-module with basis V and the V_i are mutually disjoint one has

$$a_k x_k = \sum_{x_i, x_j \in V_k} b_{ij}(x_i - x_j) \quad \text{for} \quad 1 \leq k \leq r-1,$$

applying ∂_0 to both sides of this equality we obtain $a_k = 0$ for all k.

Next we show that $\overline{\mathcal{B}}$ is a set of generators. Let x_{i_1} and x_{j_1} be two vertices of Δ, it suffices to prove that $x_{i_1} - x_{j_1} + \mathrm{im}(\partial_1)$ is in the submodule generated by $\overline{\mathcal{B}}$ because $\ker(\partial_0)$ is generated by the elements of the form $x_i - x_j$. Assume $x_{i_1} \in V_m$ and $x_{j_1} \in V_s$. There is a path $x_{i_1}, \ldots, x_{i_t}, x_m$ in Δ_m and a path $x_{j_1}, \ldots, x_{j_p}, x_s$ in Δ_s such that every two consecutive vertices of those paths form a 1-face of Δ. From the equalities

$$\begin{aligned} (x_{i_1} - x_{i_2}) + (x_{i_2} - x_{i_3}) + \cdots + (x_{i_t} - x_m) &= x_{i_1} - x_m \\ (x_{j_1} - x_{j_2}) + (x_{j_2} - x_{j_3}) + \cdots + (x_{j_p} - x_s) &= x_{j_1} - x_s, \end{aligned}$$

one concludes $x_{i_1} - x_{j_1} + \mathrm{im}(\partial_1) = (x_m - x_r) - (x_s - x_r) + \mathrm{im}(\partial_1)$. □

Definition 5.2.4 Let Δ and Δ_1 be simplicial complexes on disjoint vertex sets V and W respectively. The *join*

$$\Delta * \Delta_1$$

is the simplicial complex on the vertex set $V \cup W$ with faces $F \cup G$ where $F \in \Delta$ and $G \in \Delta_1$. The *cone*

$$\mathrm{cn}(\Delta) = w * \Delta$$

of Δ is the join of a point $\{w\}$ with Δ.

Proposition 5.2.5 *If Δ is a simplicial complex and $\mathrm{cn}(\Delta) = w * \Delta$ its cone, then*

$$\widetilde{H}_p(\mathrm{cn}(\Delta); A) = 0 \text{ for all } p.$$

Proof. If $\sigma = [v_0, \ldots, v_p]$ is an oriented p-simplex, set $[w, \sigma]$ equal to $[w, v_0, \ldots, v_p]$. In general if $c_p = \sum n_i \sigma_i$ is a p-chain, set $[w, c_p]$ equal to $\sum n_i [w, \sigma_i]$. This *bracket operation* is a homomorphism from $C_p(\Delta)$ into $C_{p+1}(w * \Delta)$. If σ is an oriented p-simplex from the definition of the boundary operator one obtains

$$\partial[w, \sigma] = \begin{cases} \sigma - w & \text{if } \dim \sigma = 0, \\ \sigma - [w, \partial\sigma] & \text{if } \dim \sigma > 0. \end{cases}$$

As a consequence one has the more general formulas:

$$\partial[w, c_0] = c_0 - \epsilon(c_0)w, \tag{5.1}$$

$$\partial[w, c_p] = c_p - [w, \partial c_p] \text{ if } p > 0. \tag{5.2}$$

By Proposition 5.2.3 we obtain $\widetilde{H}_0(w * \Delta) = 0$, because $w * \Delta$ is connected. Assume $p > 0$. Let $z_p \in Z(w * \Delta)$ be a p-cycle of $w * \Delta$, we will show that z_p is a boundary. Write

$$z_p = c_p + [w, d_{p-1}],$$

where c_p consist of those terms of z_p with support in Δ, and d_{p-1} is a chain. It is enough to show that

$$z_p - \partial[w, c_p] = 0.$$

Using the identities (5.1) and (5.2) we obtain

$$z_p - \partial[w, c_p] = c_p + [w, d_{p-1}] - c_p + [w, \partial c_p] = [w, e_{p-1}],$$

where $e_{p-1} = d_{p-1} + \partial c_p$ is a chain in Δ. Applying ∂ to the equality above and using that z_p is a cycle yields $\partial[w, e_{p-1}] = 0$. By identities (5.1) and (5.2) one has

$$0 = \partial[w, e_{p-1}] = \begin{cases} e_{p-1} - \epsilon(e_{p-1})w & \text{if } p = 1, \\ e_{p-1} - [w, \partial e_{p-1}] & \text{if } p > 1. \end{cases}$$

Now, the part of this chain with support in Δ is e_{p-1}. Therefore $e_{p-1} = 0$ and $[w, \partial e_{p-1}] = 0$, which shows $z_p = \partial[w, c_p]$. □

Definition 5.2.6 The *reduced Euler characteristic* $\widetilde{\chi}(\Delta)$ of a simplicial complex Δ of dimension d is given by

$$\widetilde{\chi}(\Delta) = -1 + \sum_{i=0}^{d} (-1)^i f_i = -1 + \chi(\Delta),$$

where f_i is the number of i-faces in Δ and $\chi(\Delta)$ is its *Euler characteristic.*

Exercises

5.2.7 Let Δ be a simplicial complex of dimension d and f_i the number of i-faces in Δ. If k is a field, then

$$\sum_{i=-1}^{d}(-1)^i \text{rank}\,\widetilde{H}_i(\Delta;k) = -1 + \sum_{i=0}^{d}(-1)^i f_i.$$

5.2.8 Let Δ be a simplicial complex, prove that there is an exact sequence

$$0 \longrightarrow \widetilde{H}_0(\Delta;A) \longrightarrow H_0(\Delta;A) \longrightarrow A \longrightarrow 0$$

and conclude

$$H_0(\Delta;A) \simeq \widetilde{H}_0(\Delta,A) \oplus A.$$

5.2.9 Let Δ be a simplicial complex with vertex set $V = \{x_1,\ldots,x_n\}$ and

$$\cdots \xrightarrow{\partial_1} C_0(\Delta) \xrightarrow{\partial_0} C_{-1}(\Delta) = A \longrightarrow 0,$$

the right end of the augmented chain complex over a ring A. Prove that $\ker(\partial_0)$ is generated by the cycles of the form $x_i - x_j$.

5.2.10 Let Δ be a simplicial complex and f_i the number of i-faces of Δ. If

$$\cdots \longrightarrow C_1(\Delta) \xrightarrow{\partial_1} C_0(\Delta) \xrightarrow{\partial_0} C_{-1}(\Delta) = k \longrightarrow 0,$$

is the augmented chain complex of Δ over a field k, then $\dim_k(Z_1)$ is equal to $f_1 - f_0 + r$; where r is the number of connected components of Δ and $Z_1 = \ker(\partial_1)$.

5.3 Face rings

Let $R = k[x_1,\ldots,x_n]$ be a polynomial ring over a field k. If I is an ideal of R generated by square-free monomials, the *Stanley-Reisner simplicial complex* Δ_I associated to I has vertex set $V = \{x_i | \, x_i \notin I\}$ and its faces are defined by

$$\Delta_I = \{\{x_{i_1},\ldots,x_{i_k}\} |\; i_1 < \cdots < i_k\,,\; x_{i_1}\cdots x_{i_k} \notin I\}.$$

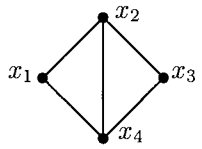
Conversely if Δ is a simplicial complex with vertex set V contained in $\{x_1,\ldots,x_n\}$, the *Stanley-Reisner ideal* I_Δ is defined as

$$I_\Delta = (\{x_{i_1}\cdots x_{i_r} |\; i_1 < \cdots < i_r,\, \{x_{i_1},\ldots,x_{i_r}\} \notin \Delta\})\,,$$

and its *Stanley-Reisner ring* $k[\Delta]$ is defined as the quotient ring R/I_Δ.

Definition 5.3.1 Let I be an ideal of R generated by square-free monomials. The quotient ring R/I is called a *face ring*.

Example 5.3.2 A simplicial complex Δ of dimension 1 and its Stanley-Reisner ideal:



$$I_\Delta = (x_1x_3, x_1x_2x_4, x_2x_3x_4)$$

Definition 5.3.3 Let Δ be a simplicial complex and $F \in \Delta$, define $\mathrm{lk}(F)$, the *link* of F, as

$$\mathrm{lk}(F) = \{H \in \Delta |\ H \cap F = \emptyset \text{ and } H \cup F \in \Delta\}.$$

Definition 5.3.4 Let k be a field. A simplicial complex Δ is said to be *Cohen-Macaulay* over k if the Stanley-Reisner ring $k[\Delta]$ is a Cohen-Macaulay ring.

Theorem 5.3.5 (Reisner) *Let Δ be a simplicial complex. If k is a field, then the following conditions are equivalent:*

(a) *Δ is Cohen-Macaulay over k.*

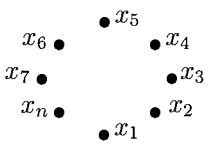
(b) *$\widetilde{H}_i(\mathrm{lk}\ F; k) = 0$ for $F \in \Delta$ and $i < \dim \mathrm{lk}\ F$.*

Proof. See the original paper [237], or [44] for a detailed proof that combines a vanishing theorem of Grothendieck [137] with a formula of M. Hochster for the Hilbert series of the local cohomology modules of $k[\Delta]$ with respect to the fine grading. □

We clarify that in particular a Cohen-Macaulay complex Δ, being the link of its empty face, must satisfy $\widetilde{H}_i(\Delta; k) = 0$ for $i < \dim \Delta$.

In general the Cohen-Macaulay property of a face ring may depend on the characteristic of the base field, classical examples of this dependence are triangulations of the projective plane (see Exercise 5.3.31).

Example 5.3.6 If Δ is a discrete set with n vertices



then Δ is Cohen-Macaulay and $I_\Delta = (x_ix_j | 1 \leq i < j \leq n)$.

Corollary 5.3.7 If Δ is a 1-dimensional simplicial complex, then Δ is connected if and only if Δ is Cohen-Macaulay.

Proof. By Reisner theorem Δ is Cohen-Macaulay iff $\widetilde{H}_0(\Delta; k) = 0$, because the link of a vertex of Δ consists of a discrete set of vertices. To complete the argument apply Proposition 5.2.3. □

Proposition 5.3.8 ([174]) *If k is a field, then a simplicial complex Δ is Cohen-Macaulay over k if and only if $\operatorname{lk}(F)$ is Cohen-Macaulay over k for all $F \in \Delta$.*

Proof. $\Rightarrow$) Let F be a fixed face in Δ and $\Delta' = \operatorname{lk}(F)$. If $G \in \Delta'$, using the equality

$$\operatorname{lk}_{\Delta'}(G) = \operatorname{lk}_{\Delta}(G \cup F)$$

and Reisner criterion one infers $\widetilde{H}_i(\operatorname{lk}_{\Delta'}(G), k) = 0$ for $i < \dim \operatorname{lk}_{\Delta'}(G)$, thus Δ' is Cohen-Macaulay.

$\Leftarrow$) Since all the links are Cohen-Macaulay, it suffices to take $F = \emptyset$ and observe $\Delta = \operatorname{lk}(\emptyset)$ to conclude that Δ is Cohen-Macaulay. □

Definition 5.3.9 A face F of a simplicial complex Δ is said to be a *facet* if F is not properly contained in any other face of Δ.

Proposition 5.3.10 *If Δ is a simplicial complex with vertices $x_1, \ldots, x_n$, then the primary decomposition of the Stanley-Reisner ideal of Δ is:*

$$I_\Delta = \bigcap_F P_F,$$

where the intersection is taken over all facets F of Δ, and P_F denotes the face ideal generated by all x_i such that $x_i \notin F$.

Proof. Let F be a face of Δ and P_F the face ideal generated by the x_i such that $x_i \notin F$. It is not hard to prove that P_F is a minimal prime of I_Δ if and only if F is a facet. Therefore the result follow from Corollary 5.1.5. □

Corollary 5.3.11 *Let Δ be a simplicial complex of dimension d on the vertex set $V = \{x_1, \ldots, x_n\}$ and k a field. Then*

$$\dim k[\Delta] = d + 1 = \max\{s \mid x_{i_1} \cdots x_{i_s} \notin I_\Delta \text{ and } i_1 < \cdots < i_s\}.$$

Proof. Let F be a facet of Δ with $d + 1$ vertices. Note that by Proposition 5.3.10 the face ideal P_F generated by the variables not in F has height equal to the height of I_Δ. Hence $\operatorname{ht}(I_\Delta) = n - d - 1$, taking into account Proposition 2.2.3 one has $\dim k[\Delta] = d + 1$. □

Corollary 5.3.12 *A Cohen-Macaulay simplicial complex Δ is pure, that is, all its maximal faces have the same dimension.*

Proof. Use Proposition 5.3.10 and Corollary 2.2.4. □

Definition 5.3.13 The *q-skeleton* of a simplicial complex Δ is the simplicial complex Δ^q consisting of all p-simplices of Δ with $p \leq q$.

Proposition 5.3.14 Let Δ be a simplicial complex of dimension d and Δ^q its q-skeleton. If Δ is Cohen-Macaulay over a field k, then Δ^q is Cohen-Macaulay for $q \leq d$.

Proof. Set $\Delta' = \Delta^q$ and take $F \in \Delta'$ with $\dim(F) < q \leq d$. Note

$$\mathrm{lk}_{\Delta'}(F) = (\mathrm{lk}_{\Delta}(F))^{q-|F|}.$$

Since Δ is pure there is a facet F' of Δ of dimension d containing F, hence $F' \setminus F$ is a face of $\mathrm{lk}_{\Delta}(F)$ of dimension $d-|F|$. It follows that the dimension of $\mathrm{lk}_{\Delta'}(F)$ is $q - |F|$. Using Proposition 5.3.8 we get

$$\widetilde{H}_i(\mathrm{lk}_{\Delta}(F); k) = 0 \text{ for } i < \dim(\mathrm{lk}_{\Delta}(F)) = d - |F|,$$

and consequently

$$\widetilde{H}_i((\mathrm{lk}_{\Delta}(F))^{q-|F|}; k) = \widetilde{H}_i(\mathrm{lk}_{\Delta}(F); k) = 0 \text{ for } i < q - |F| \leq d - |F|.$$

Observe that the equality between the homology modules follows simply because the augmented oriented chain complexes of $(\mathrm{lk}_{\Delta}(F))^{q-|F|}$ and $\mathrm{lk}_{\Delta}(F)$ are equal up to degree $q - |F|$ and thus their homologies have to agree up to degree one less. Hence by Reisner criterion Δ^q is Cohen-Macaulay. □

Example 5.3.15 ([114]) Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and Δ the $(p-2)$-skeleton of a simplex of dimension $n-1$, where $p \geq 2$. Then I_{Δ} is a Cohen-Macaulay ideal and the face ring

$$k[\Delta] = R/(\{x_{i_1} \cdots x_{i_p} | \, 1 \leq i_1 < \cdots < i_p \leq n\})$$

has a p-linear resolution by Theorem 4.3.7.

Face rings with pure and linear resolutions have been studied in [46] and [114, 116]; special results are obtained when the corresponding Stanley-Reisner ideal is generated in degree two.

Proposition 5.3.16 *If Δ_1 and Δ_2 are simplicial complexes, then their join $\Delta_1 * \Delta_2$ is Cohen-Macaulay if and only if Δ_i is Cohen-Macaulay for $i = 1, 2$.*

Proof. Since one has the equality

$$I_{\Delta_1 * \Delta_2} = I_{\Delta_1} + I_{\Delta_2},$$

there is a graded isomorphism of k-algebras

$$k[\Delta_1 * \Delta_2] \simeq k[\Delta_1] \otimes_k k[\Delta_2],$$

see Proposition 2.2.20. To complete the proof use Corollary 2.2.22. □

Given a simplicial complex Δ and $\Delta_1, \ldots, \Delta_s$ sub complexes of Δ, we define their *union*

$$\bigcup_{i=1}^{s} \Delta_i$$

as the simplicial complex with vertex set $\cup_{i=1}^{s} V_i$ and such that F is a face of $\cup_{i=1}^{s} \Delta_i$ if and only if F is a face of Δ_i for some i. The intersection can be defined similarly.

Definition 5.3.17 A pure simplicial complex Δ of dimension d is *shellable* if the facets of Δ can be order $F_1, \ldots, F_s$ such that

$$\overline{F}_i \bigcap \left(\bigcup_{j=1}^{i-1} \overline{F}_j \right)$$

is pure of dimension $d-1$ for all $i \geq 2$. Here

$$\overline{F}_i = \{\sigma \in \Delta | \, \sigma \subset F_i\}.$$

If Δ is shellable, $F_1, \ldots, F_s$ is called a *shelling*.

Theorem 5.3.18 *Let Δ be a simplicial complex. If Δ is shellable, then Δ is Cohen-Macaulay over any field k.*

Proof. Our proof is based on a refined argument of T. Hibi [162]. Let $V = \{x_1, \ldots, x_n\}$ be the vertex set of Δ and $R = k[\mathbf{x}]$ a polynomial ring over a field k. Assume that $F_1, \ldots, F_s$ is a linear order of the facets of Δ such that $\overline{F}_i \cap (\cup_{j=1}^{i-1} \overline{F}_j)$ is pure of dimension $d-1$ for all $i \geq 2$, where d is the dimension of Δ. Set

$$\sigma = \left\{ x_i \in F_s \;\middle|\; F_s \setminus \{x_i\} \in \overline{F}_s \bigcap \left(\bigcup_{j=1}^{s-1} \overline{F}_j \right) \right\},$$

one may assume $\sigma = \{x_1, \ldots, x_r\}$. There is a short exact sequence

$$0 \longrightarrow R/(I\colon f) \xrightarrow{f} R/I \longrightarrow R/(I, f) \longrightarrow 0, \qquad (*)$$

where $f = x_1 \cdots x_r$ and $I = I_\Delta$ is the Stanley-Reisner ideal of Δ.

First we show the equality

$$R/(I\colon f) = k[\overline{F}_s],$$

note that $k[\overline{F}_s]$ is a polynomial ring in $d+1$ variables. By Proposition 5.3.10 one can write $I = \cap_{i=1}^{s} \mathfrak{p}_i$, where $\mathfrak{p}_i$ is generated by $V \setminus F_i$ for all i. Hence

$$(I\colon f) = \bigcap_{i=1}^{s} (\mathfrak{p}_i\colon f) = \bigcap_{\sigma \subset F_i} (\mathfrak{p}_i\colon f).$$

Observe that if $\sigma \subset F_i$ for some i, then $i = s$; otherwise if $i < s$, then σ belongs to $\overline{F}_s \cap (\cup_{j=1}^{s-1} \overline{F}_j)$, but since this simplicial complex is pure of dimension $d-1$, there is $v \in \sigma$ such that $\sigma \subset F_s \setminus \{v\}$, a contradiction. Hence $i = s$ and $(I\colon f) = (\mathfrak{p}_s\colon f) = \mathfrak{p}_s$, as asserted.

Next let us show

$$R/(I, f) = k[\overline{F}_1 \cup \cdots \cup \overline{F}_{s-1}].$$

Note that $f \in \mathfrak{p}_i$ for $i = 1, \ldots, s-1$; otherwise if $f \notin \mathfrak{p}_i$ for some $i < s$, then $\sigma \subset F_i$ and the previous argument yields a contradiction. If $\mathfrak{p}$ is a minimal prime of (I, f) different from $\mathfrak{p}_1, \ldots, \mathfrak{p}_{s-1}$, then $\mathfrak{p}_s \subset \mathfrak{p}$ and $x_i \in \mathfrak{p}$ for some $x_i \in \sigma$, thus by construction of σ one has $F_s \setminus \{x_i\} \subset F_k$ for some $k < s$. Therefore $V \setminus F_k \subset \{x_i\} \cup (V \setminus F_s) \subset \mathfrak{p}$, and by the minimality of $\mathfrak{p}$ one derives $\mathfrak{p}_k = \mathfrak{p}_s$, which is impossible. Altogether we conclude that $\mathfrak{p}_1, \ldots, \mathfrak{p}_{s-1}$ are the minimal primes of (I, f) and

$$(I, f) = \bigcap_{i=1}^{s-1} \mathfrak{p}_i,$$

as required.

Using induction on s and the exact sequence $(*)$ it follows that R/I is Cohen-Macaulay by a direct application of the depth lemma. To apply the depth lemma one uses that the extremes of the exact sequence $(*)$ are both Cohen-Macaulay R-modules of dimension $d+1$. □

Exercises

5.3.19 Let R be a polynomial ring over a field k. Prove that the family of ideals of R generated by square-free monomials is a sublattice of the lattice of monomial ideals.

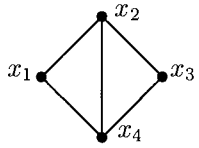
5.3.20 A monomial ideal I, with coefficients in a field, is generated by square-free monomials iff any of the following conditions hold:

(a) I is an intersection of prime ideals.

(b) $\operatorname{rad}(I) = I$.

(c) A monomial f is in I iff $x_1 \cdots x_r \in I$, where $\operatorname{supp}(f) = \{x_i\}_{i=1}^r$.

5.3.21 If Δ is the simplicial complex



prove the equality

$$I_\Delta = (x_1, x_2) \cap (x_1, x_3) \cap (x_1, x_4) \cap (x_2, x_3) \cap (x_3, x_4)$$

by using the correspondence between facets and minimal primes.

5.3.22 Let Δ be a Cohen-Macaulay simplicial complex and $F \in \Delta$. If $\operatorname{lk}(F)$ is not a discrete set of vertices, prove that $\operatorname{lk}(F)$ is connected.

5.3.23 Let Δ be a simplicial complex on the vertex set V and I its Stanley-Reisner ideal. Take $x \in V$ and set $J = (I\colon x)$.

(a) Prove that $J = \cap_{x \notin \mathfrak{p}_i} \mathfrak{p}_i$, where $\operatorname{Ass}(I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$.

(b) Find a relation between Δ and Δ_J.

5.3.24 Let R be a polynomial ring over a field k and I an ideal minimally generated by square-free monomials $f_1, \ldots, f_q$. If R/I is Cohen-Macaulay and x is a variable not in I such that $x \in \cup_{i=1}^q \operatorname{supp}(f_i)$, then $R/(I\colon x)$ is Cohen-Macaulay.

5.3.25 Let Δ be a pure simplicial complex on the vertex set V and $I = I_\Delta$. Assume that x is a variable in $\cup_{i=1}^q \operatorname{supp}(f_i)$, where the f_i are monomials that minimally generate I. If $(I\colon x)$ and (I, x) are Cohen-Macaulay, then I is Cohen-Macaulay.

Hint Show $\operatorname{ht}(I) = \operatorname{ht}(I\colon x) = \operatorname{ht}(I, x)$.

5.3.26 Let Δ a simplicial complex and $\sigma \in \Delta$, define the *star* of σ as

$$\operatorname{star}(\sigma) = \{G \in \Delta | \, \sigma \cup G \in \Delta\}.$$

Prove that the facets of $\operatorname{star}(\sigma)$ are the facets of Δ that contain σ.

5.3.27 Let $R = k[\mathbf{x}]$ be a polynomial ring over a field k and Δ a simplicial complex with vertex set $\mathbf{x}$. If $\sigma = \{x_1, \ldots, x_r\} \in \Delta$, prove the equality

$$k[\operatorname{star}(\sigma)] = R/(I_\Delta\colon f),$$

where $f = \prod_{i=1}^r x_i$ and $\operatorname{star}(\sigma)$ is the star of σ.

5.3.28 If Δ is a Cohen-Macaulay complex and F a face of Δ, then $\operatorname{star}(F)$ is a Cohen-Macaulay complex.

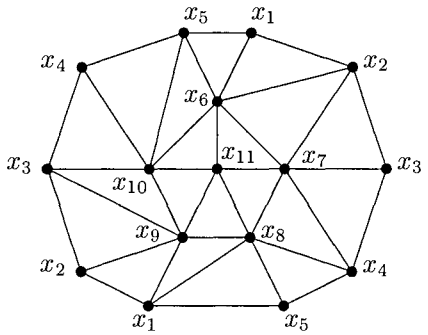
Hint $\operatorname{star}(F) = \overline{F} * \operatorname{lk}(F)$.

5.3.29 A pure simplicial complex Δ is shellable if and only if the facets of Δ can be listed $F_1, \ldots, F_s$ such that for all $1 \leq j < i \leq s$, there exist some $v \in F_i \setminus F_j$ and some $k \in \{1, \ldots, i-1\}$ with $F_i \setminus F_k = \{v\}$.

5.3.30 Let Δ be a pure simplicial complex. Prove that Δ is shellable if and only if the facets of Δ can be ordered $F_1, \ldots, F_s$ such that for all $1 \leq j < i \leq s$, there is $v \in F_i \setminus F_j$ and $k < i$ with $F_i \cap F_j \subset F_i \cap F_k = F_i \setminus \{v\}$.

The next exercise is a subdivision of the minimal triangulation of the projective plane given by G. Reisner [237, Remark 3].

5.3.31 (N. Terai) If k is a field and Δ is the following triangulation of the real projective plane $\mathbb{P}^2$



then the link of any vertex is a cycle and $k[\Delta]$ is Cohen-Macaulay if and only if $\operatorname{char}(k) \neq 2$.

Hint $\widetilde{H}_1(\Delta; k) \simeq \widetilde{H}_1(\mathbb{P}^2; k) = k/2k$ and use Theorem 5.3.5.

5.3.32 Let R be a ring and I_1, I_2 ideals of R. Prove that there is an exact sequence of R-modules:

$$0 \longrightarrow R/(I_1 \cap I_2) \xrightarrow{\varphi} R/I_1 \oplus R/I_2 \xrightarrow{\phi} R/(I_1 + I_2) \longrightarrow 0,$$

where $\varphi(\overline{r}) = (\overline{r}, -\overline{r})$ and $\phi(\overline{r}_1, \overline{r}_2) = \overline{r_1 + r_2}$.

5.3.33 ([172]) Let $R = k[\mathbf{x}]$ be a polynomial ring over a field k and Δ_1, Δ_2 simplicial complexes whose vertex sets are contained in $\mathbf{x}$. Prove that there is an exact sequence of R-modules:

$$0 \longrightarrow k[\Delta_1 \cup \Delta_2] \longrightarrow k[\Delta_1] \oplus k[\Delta_2] \longrightarrow k[\Delta_1 \cap \Delta_2] \longrightarrow 0.$$

5.4 Hilbert series of face rings

For an Stanley-Reisner ring $k[\Delta]$ there are explicit formulas for its Hilbert function and Hilbert series in terms of the combinatorial data of the complex, we begin with a direct approach to compute these formulae.

Definition 5.4.1 Let k and n be two positive integers. A *k-partition* of n is an element $\alpha = (\alpha_1, \ldots, \alpha_k)$ in $\mathbb{N}_+^k$ so that $n = \alpha_1 + \cdots + \alpha_k$.

Lemma 5.4.2 *The number $c(n,k)$ of k-partitions of n is equal to*

$$c(n,k) = \binom{n-1}{k-1}.$$

Proof. If $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a k-partition of n define $\psi(\alpha)$ by

$$\psi(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{k-1}\} \subset \{1, \ldots, n\}.$$

To finish the proof note that the map ψ induces a bijection between the set of k-partitions of n and the set of subsets of $\{1, \ldots, n-1\}$ with $k-1$ elements. □

Definition 5.4.3 Given a simplicial complex Δ of dimension d its *f-vector* is the $(d+1)$-tuple:

$$f(\Delta) = (f_0, \ldots, f_d),$$

where f_i is the number of i-faces of Δ. Note $f_{-1} = 1$.

Proposition 5.4.4 *If Δ is a simplicial complex of dimension d and (f_i) its f-vector, then the Hilbert function of $k[\Delta]$ is:*

$$H(k[\Delta], j) = \sum_{i=0}^{d} \binom{j-1}{i} f_i, \quad \text{for } j \geq 1 \text{ and } H(k[\Delta], 0) = 1.$$

Proof. Let $x_1, \ldots, x_n$ be the vertices of Δ. Given $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ we denote its image in $k[\Delta]$ by $\overline{x}^\alpha$. The face ideal I_Δ is generated by square-free monomials, hence a monomial x^α is in I_Δ if and only if $\operatorname{supp}(x^\alpha) = \{x_i | \, \alpha_i > 0\}$ is not a face of Δ. Therefore the set of all monomials $\overline{x}^\alpha$ such that $\operatorname{supp}(x^\alpha)$ is a face in Δ form a basis for $k[\Delta]$ as a vector space over k.

If $\{x_{i_1}, \ldots, x_{i_r}\}$ is a face of Δ of dimension $r-1$, we can use Lemma 5.4.2 to count the number of monomials of the form $x_{i_1}^{\alpha_1} \cdots x_{i_r}^{\alpha_r}$ such that $(\alpha_1, \ldots, \alpha_r)$ is an r-partition of j. The desired formula follows at once. □

Corollary 5.4.5 *If Δ is a simplicial complex and $f(\Delta) = (f_0, \ldots, f_d)$ its f-vector, then the Hilbert series of $k[\Delta]$ is given by*

$$F(k[\Delta], z) = \sum_{i=-1}^{d} \frac{f_i z^{i+1}}{(1-z)^{i+1}}, \quad d = \dim(\Delta).$$

Proof. By the proposition above we have

$$F(z) = 1 + \sum_{j=1}^{\infty} \left(\sum_{i=0}^{d} \binom{j-1}{i} f_i \right) z^j = 1 + \sum_{i=0}^{d} \left(\sum_{j=1}^{\infty} \binom{j-1}{i} z^j \right) f_i.$$

Hence using the identities

$$\sum_{j=1}^{\infty} \binom{j-1}{i} z^j = z^{i+1} \left(\sum_{j=0}^{\infty} \binom{i+j}{i} z^j \right) = \frac{z^{i+1}}{(1-z)^{i+1}},$$

the formula is established. □

A result of Kruskal-Katona

Kruskal and Katona [202] showed that $(f_0, f_1, \ldots, f_d) \in \mathbb{Z}^{d+1}$ is the f-vector of some d-dimensional simplicial complex if and only if

$$0 < f_{i+1} \leq f_i^{(i+1)}, \ 0 \leq i \leq d-1,$$

where $f_i^{(i+1)}$ is defined according to Equation 4.9.

Simplicial complexes and their h-vectors As an application we derive formulas for $h(\Delta)$, the h-vector of $k[\Delta]$. If Δ is Cohen-Macaulay, we present an important numerical constraint for $h(\Delta)$ which is basic for some applications of Commutative Algebra to Combinatorics.

Theorem 5.4.6 *If Δ is a simplicial complex of dimension d and (h_i) the h-vector of $k[\Delta]$, then $h_k = 0$ for $k > d+1$ and*

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{k-i} f_{i-1} \quad \text{for} \quad 0 \leq k \leq d+1\,. \tag{5.3}$$

Proof. The idea is to write the Hilbert series of $k[\Delta]$ in two ways. By Corollary 5.4.5 we have:

$$F(k[\Delta], z) = \sum_{i=-1}^{d} \frac{f_i z^{i+1}}{(1-z)^{i+1}}\,. \tag{5.4}$$

On the other hand by the Hilbert-Serre Theorem 4.1.3 there is a (unique) polynomial $h(z) = h_0 + h_1 z + \cdots + h_r z^r$ with integral coefficients so that $h(1) \neq 0$ and satisfying

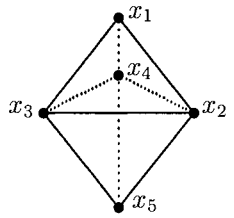
$$F(k[\Delta], z) = \frac{h(z)}{(1-z)^{d+1}}\,. \tag{5.5}$$

Comparing (5.4) and (5.5) yields the asserted equalities. □

Definition 5.4.7 The h-vector of a simplicial complex Δ, denoted by $h(\Delta)$, is defined as the h-vector of the standard graded algebra $k[\Delta]$.

Computation of the h-vector of face rings

To compute the h-vector (of face rings) the following procedure of R. Stanley [271] can be used. Consider the 3-dimensional convex polytope P consisting of two solid tetrahedrons joined by a 2-face:



one has $f(P) = (5, 9, 6)$. Write down the f-vector on a diagonal, and put a 1 to the left of f_0.

$$\begin{array}{ccccc} 1 & & 5 & & \\ & & & 9 & \\ & & & & 6 \end{array}$$

Complete this array constructing a *table*, by placing below a pair of consecutive entries their difference, and placing a 1 on the left edge:

$$\begin{array}{rccccc} & & & 1 & 5 & \\ & & 1 & & 4 & 9 \\ & 1 & & 3 & & 5 & 6 \\ \hline h(P) = & (1 & & 2 & & 2 & 1) \end{array}$$

After completing the table the next row of differences will be the h-vector. Hence the boundary simplicial complex $\Delta = \Delta(P)$ is 2-dimensional, $k[\Delta]$ has multiplicity 6, and $I_\Delta = (x_1x_5, x_2x_3x_4)$.

Now we present a result that plays a role in the proof of the upper bound conjecture for spheres.

Theorem 5.4.8 *Let Δ be a simplicial complex of dimension d with n vertices. If $k[\Delta]$ is Cohen-Macaulay and k is an infinite field, then the h-vector of Δ satisfies*

$$0 \leq h_i(\Delta) \leq \binom{i+n-d-2}{i} \quad \textit{for} \quad 0 \leq i \leq d+1. \tag{5.6}$$

Proof. Set $S = k[\Delta] = R/I$, where $R = k[x_1, \ldots, x_n]$ and $I = I_\Delta$. By Lemma 2.2.15 there exists a regular homogeneous system of parameters $\underline{\theta} = \theta_1, \ldots, \theta_{d+1}$ for S such that each θ_i has degree one. Since $A = S/(\underline{\theta})S$ is Artinian, in this case Theorem 4.2.5 says that $h_i(\Delta) = H(A, i)$.

Note that $S/(\underline{\theta})S \simeq \overline{R}/\overline{I}$, where $\overline{R} = R/(\underline{\theta})$ is a polynomial ring in $n - d - 1$ variables and $\overline{I}$ is the image of I in $\overline{R}$. Therefore we have

$$h_i(\Delta) = H(A, i) = H(\overline{R}/\overline{I}, i) \leq H(\overline{R}, i) = \binom{i + n - d - 2}{i},$$

as required. □

Example 5.4.9 Let Δ be a 2-dimensional complex with $f(\Delta) = (5, 6, 2)$. Thus $k[\Delta]$ is not Cohen-Macaulay because its h vector has a negative entry:

$$\begin{array}{rcccccc} & & & & 1 & 5 & \\ & & & 1 & & 4 & & 6 \\ & & 1 & & 3 & & 2 & & 2 \\ \hline h(\Delta) = & (1 & & 2 & & -1 & & 0). \end{array}$$

A shellable complex Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and Δ a simplicial complex with vertices $x_1, \ldots, x_n$. Associated to the Stanley-Reisner ideal I_Δ we define another ideal

$$I'(\Delta) = (I_\Delta, x_1 y_1, \ldots, x_n y_n) \subset k[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

and denote by $S(\Delta)$ the Stanley–Reisner complex corresponding to this ideal $I'(\Delta)$.

Proposition 5.4.10 *The simplicial complex $S(\Delta)$ is shellable.*

Proof. To prove that $S(\Delta)$ is shellable we will use the criterion of Exercise 5.3.29. Let us begin by describing the facets of $S(\Delta)$. It is not hard to show that $S(\Delta)$ is pure. Notice that the facets of $S(\Delta)$ are of the form $F' = \{y_{r_1}, \ldots, y_{r_k}\} \cup F$, where $F = \{x_{s_1}, \ldots, x_{s_\ell}\}$ is a face of Δ so that $k + \ell = n$, $\{r_1, \ldots, r_k\} \cap \{s_1, \ldots, s_\ell\} = \emptyset$. Therefore $F \mapsto F'$ defines a one to one correspondence between the faces of Δ and the facets of $S(\Delta)$. To construct a shelling for $S(\Delta)$ we first linearly order the faces of Δ according to its dimension, and then linearly order the facets of $S(\Delta)$ in such a way that if F, G are faces of Δ and $F < G$ then $F' < G'$. Let F', G' be facets of $S(\Delta)$, with $F' < G'$, there is $x_j \in G' \setminus F'$, setting $H' = (G' \cup \{y_j\}) \setminus \{x_j\}$ we obtain $H' < G'$ and $G' \setminus H' = \{x_j\}$, as required. □

Corollary 5.4.11 *$S(\Delta)$ is a Cohen-Macaulay complex and the f-vector of Δ is related to the h-vector of $S(\Delta)$ by*

$$f_i(\Delta) = h_{i+1}(S(\Delta)) \quad \textit{for all} \quad i.$$

Proof. The first assertion follows from Theorem 5.3.18 and the previous result. For the other claim set $J = (I_\Delta, x_1^2, \ldots, x_n^2)$ and note that the face ring $k[S(\Delta)] = k[\mathbf{x}, \mathbf{y}]/I'_\Delta$ reduces to $k[\mathbf{x}]/J$ modulo $\mathbf{z} = \{y_1 - x_1, \ldots, y_n - x_n\}$. Thus $\mathbf{z}$ is a system of parameters for $k[S(\Delta)]$. Since $S(\Delta)$ is Cohen-Macaulay, $\mathbf{z}$ must be a regular sequence on $k[S(\Delta)]$ according to Proposition 2.2.7. Using Theorem 4.2.5 one has that the h-vector of $S(\Delta)$ is then equal to the h-vector of $k[\mathbf{x}]/J$, which is nothing other than $f(\Delta)$ shifted by one. □

Exercises

5.4.12 Let Δ be a simplicial complex of dimension d and $h(\Delta) = (h_i)$ its h-vector. Prove

$$h_{d+1} = (-1)^d \widetilde{\chi}(\Delta),$$

where $\widetilde{\chi}(\Delta)$ is the reduced Euler characteristic of Δ.

5.4.13 Let $S = k[\Delta]$ be the Stanley-Reisner ring of a simplicial complex Δ over a field k and $\varphi(t)$ the Hilbert polynomial of S. Prove:

(a) $\varphi(i) = \dim_k(S_i)$ for all $i \geq 1$, and

(b) $\varphi_S(0) = 1$ if and only if $\widetilde{\chi}(\Delta) = 0$.

5.4.14 Let Δ be a simplicial complex of dimension d and $(f_0, \ldots, f_d)$ its f-vector. Prove that f_d is equal to $e(k[\Delta])$, the multiplicity of $k[\Delta]$.

5.4.15 Let $S = k[\Delta]$ be the Stanley-Reisner ring of a simplicial complex Δ with vertices $x_1, \ldots, x_n$ and $A = S/(x_1^2, \ldots, x_n^2)$. Prove that Δ is pure if and only if all the nonzero elements of the socle of A have the same degree.

5.5 Upper bound conjectures

It is our intention here to give flashes of some of the steps in the proof of the upper bound conjecture for simplicial spheres. An excellent reference for a detailed proof of this conjecture is [44].

Convex polytopes A *convex polytope* $P \subset \mathbb{R}^s$, or simply a *polytope*, is the *convex hull* of a finite set of points $x_1, \ldots, x_n$ in $\mathbb{R}^s$, that is, P is the set of all *convex combinations* of those points:

$$P = \{a_1 x_1 + \cdots + a_n x_n | \, a_i \geq 0, \, a_1 + \cdots + a_n = 1, \, a_i \in \mathbb{R}\}.$$

For $A \subset \mathbb{R}^s$, let $\mathrm{aff}(A)$ be the affine space generated by A. Recall that $\mathrm{aff}(A)$ is the set of all *affine combinations* of points in A:

$$\mathrm{aff}(A) = \{a_1 p_1 + \cdots + a_r p_r | \, p_i \in A, \, a_1 + \cdots + a_r = 1, \, a_i \in \mathbb{R}\}.$$

One can represent $\mathrm{aff}(A)$ as $\mathrm{aff}(A) = x_0 + V$, where $x_0 \in \mathbb{R}^s$ and V is a (unique) linear subspace of $\mathbb{R}^s$. The *dimension* of A is defined as

$$\dim(A) = \dim_{\mathbb{R}}(V).$$

Given $y \in \mathbb{R}^s \setminus \{0\}$ and $\alpha \in \mathbb{R}$, define the hyperplane

$$H(y,\alpha) = \{x \in \mathbb{R}^s \,|\, \langle x, y\rangle = \alpha\},$$

where $\langle\,,\rangle$ is the usual inner product in $\mathbb{R}^s$. The two *closed halfspaces* bounded by $H(y,\alpha)$ are

$$H^+(y,\alpha) = \{x \in \mathbb{R}^s \,|\, \langle x, y\rangle \geq \alpha\} \text{ and } H^-(y,\alpha) = H^+(-y,-\alpha).$$

Definition 5.5.1 A *proper face* of a convex polytope P is a subset $F \subset P$ such that there is a *supporting hyperplane* $H(y,\alpha)$ satisfying:

(a) $F = P \cap H(y,\alpha) \neq \emptyset$,

(b) $P \not\subset H(y,\alpha)$ and $P \subset H^-(y,\alpha)$.

The *improper faces* of a polytope P are P itself and $\emptyset$. It is usual to call *vertex* to a face of P of dimension zero; it is a result of polyhedral geometry that P is in fact the convex hull of its vertices, see [39, 315].

Proposition 5.5.2 *If $P \subset \mathbb{R}^s$ is a convex polytope, then P is a compact convex subset of $\mathbb{R}^s$ with a finitely many faces and any face of P is a convex polytope.*

Proof. See [39]. □

A set of points $y_1, \ldots, y_m \in \mathbb{R}^s$ is *affinely independent* if the relation

$$\sum_{i=1}^{m} a_i y_i = 0 \text{ with } a_1 + \cdots + a_m = 0 \text{ and } a_i \in \mathbb{R}$$

can hold only if $a_i = 0$ for all i.

Definition 5.5.3 A *q-simplex* is a polytope generated by $q+1$ affinely independent points. A polytope is called a *simplicial* if every proper face is a simplex.

Let P be a polytope of dimension $d+1$ and denote by f_i the number of its faces of dimension i, thus f_0 is the number of vertices of P. The *f-vector* of P is the vector $f(P) = (f_0, \ldots, f_d)$ and we set $f_{-1} = 1$.

Simplicial spheres Let Δ be a finite simplicial complex with vertices $x_1, \ldots, x_n$ and e_i the *ith* unit vector in $\mathbb{R}^n$. Given a face $F \in \Delta$ set

$$|F| = \text{conv}\{e_i | \, x_i \in F\},$$

where "conv" denotes convex hull. Define the *geometric realization* $|\Delta|$ of Δ as

$$|\Delta| = \bigcup_{F \in \Delta} |F|,$$

thus Δ is a topological space with the induced usual topology of $\mathbb{R}^n$.

The *boundary complex* $\Delta(P)$ of a simplicial polytope of dimension $d+1$ is by definition the "abstract simplicial complex" whose vertices are the vertices of P and whose faces are those sets of vertices that span a proper face of P, thus $\Delta(P)$ has dimension d. Note that P and $\Delta(P)$ have the same f-vector and accordingly one defines the *h-vector* of P as the h-vector of $\Delta(P)$. The geometric realization $|\Delta(P)|$ is therefore homeomorphic to a d-sphere S^d. This suggest the following more general concept.

Definition 5.5.4 A simplicial complex Δ of dimension d is called a *simplicial sphere* if

$$|\Delta| \simeq S^d$$

and in this case Δ is said to be a *triangulation* of S^d.

Theorem 5.5.5 *Let Δ be a simplicial complex of dimension d. If $|\Delta| \cong S^d$, then*

$$\widetilde{H}_i(\text{lk}\, F; k) \cong \begin{cases} k & \text{for } i = \dim(\text{lk}(F)), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. See [44, Chapter 5]. □

Proposition 5.5.6 *If Δ is a simplicial sphere, then Δ is Cohen-Macaulay and the following equality holds*

$$\widetilde{\chi}(\text{lk}\ F) = (-1)^{\dim \text{lk}\ F} \quad \text{for} \quad F \in \Delta.$$

Proof. Use Reisner's criterion and Theorem 5.5.5. □

Remark 5.5.7 If P is a simplicial d-polytope, then $\Delta(P)$ is a simplicial sphere of dimension $d-1$. Hence from Proposition 5.5.6 we obtain the Euler formula

$$\widetilde{\chi}(P) = \widetilde{\chi}(\Delta(P)) = \sum_{i=-1}^{d-1} (-1)^i f_i = (-1)^{d-1} = -1 + \chi(P),$$

where $f = (f_0, \ldots, f_{d-1})$ is the f-vector of P. The Euler formula is valid for any d-polytope, not just the simplicial ones (see [39, Theorem 16.1]).

Corollary 5.5.8 *If Δ is a simplicial sphere, then $k[\Delta]$ is a Gorenstein ring.*

Proof. It follows from [274, Theorem 5.1] and Theorem 5.5.5. □

Theorem 5.5.9 *If Δ is a simplicial sphere of dimension d, then its h-vector satisfies*

$$h_i = h_{d+1-i} \quad \text{for} \quad i = 0, \ldots, d+1.$$

Proof. Since $k[\Delta]$ is a Gorenstein ring, its h-vector is symmetric according to Exercise 4.3.9. □

Remark 5.5.10 If Δ is a simplicial sphere, then the linear relations

$$h_i = h_{d+1-i}$$

of its h-vector contain the Euler formula as a particular case (make $i = 0$). Those relations are called the *Dehn-Sommerville equations.*

Corollary 5.5.11 *If $h = (h_0, \ldots, h_{d+1})$ is the h-vector of a simplicial polytope of dimension $d+1$, then h satisfies*

$$h_i = h_{d+1-i} \quad \text{for} \quad 0 \leq i \leq d+1.$$

Let us mention two conjectures raised around 1993 in connection with the problem of bounding the Betti numbers of graded Gorenstein ideals [222].

Conjecture 5.5.12 (Miller-Villarreal) *Let Δ be a simplicial sphere and I the Stanley-Reisner ideal of Δ. If I has initial degree p and height g, then for $1 \leq i < g$ the ith Betti number β_i of $k[\Delta]$ satisfies:*

$$\beta_i \leq \binom{p+g-1}{p+i-1}\binom{p+i-2}{i-1} + \binom{p+g-1}{i}\binom{p+g-i-2}{p-1} - \binom{g}{i}\binom{p+g-2}{p-1}.$$

Conjecture 5.5.13 (Miller-Villarreal) *Let Δ be a simplicial sphere and I its Stanley-Reisner ideal. If I has initial degree p and height g, then*

$$\nu(I) \leq \binom{p+g-1}{g-1} - \binom{p+g-3}{g-1}.$$

Cyclic polytopes In order to formulate the upper bound conjecture for simplicial spheres and the upper bound conjectures for convex polytopes we need to introduce some results on cyclic polytopes, the reader is referred to [39] for a detailed discussion of this distinguished class of polytopes.

Consider the *monomial curve* $\Gamma \subset \mathbb{R}^{d+1}$ given parametrically by

$$\Gamma = \{(\tau, \ldots, \tau^{d+1}) | \, \tau \in \mathbb{R}\}.$$

A *cyclic polytope*, denoted by $C(n, d+1)$, is the convex hull of any n distinct points in Γ such that $n > d+1$. The combinatorial type or f-vector of $C(n, d+1)$ depends only on n and d and not on the points chosen and dim $C(n, d+1) = d+1$.

One of the relevant properties of cyclic polytope is that they are simplicial, thus the boundary $\partial C(n, d+1)$ defines an abstract simplicial complex $\Delta(n, d+1) = \Delta(C(n, d+1))$ so that $|\Delta(n, d+1)| \cong S^d$. Notice that $\partial C(n, d+1)$ is a union of proper faces (in fact of facets) of $C(n, d+1)$. A polytope P has the highest possible number of i-faces when every set of $i+1$ vertices is the vertex set of a proper face of P. In this case it is said that P is $(i+1)$-*neighbourly*.

Theorem 5.5.14 *A cyclic polytope $C(n, d+1)$ is $\lfloor (d+1)/2 \rfloor$-neighbourly. In particular*

$$f_i \leq \binom{n}{i+1} \quad \textit{for} \quad 0 \leq i < \left\lfloor \frac{d+1}{2} \right\rfloor.$$

Proof. See [39]. □

Corollary 5.5.15 *If $h = (h_0, \ldots, h_{d+1})$ is the h vector of $C(n, d+1)$, then*

$$h_k = \binom{n-d+k-2}{k} \quad \textit{for} \quad 0 \leq k \leq \left\lfloor \frac{d+1}{2} \right\rfloor.$$

Proof. Let $0 \leq k \leq \lfloor (d+1)/2 \rfloor$. One has

$$\begin{aligned} h_k &= \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{k-i} f_{i-1} = \sum_{i=0}^{k} (-1)^{2(k-i)} \binom{k-d-2}{k-i} \binom{n}{i} \\ &= \binom{n-d+k-2}{k} \end{aligned}$$

The second equality uses Eq. (4.1) and the last equality follows using induction on k, or setting $a = n$ and $c = k-d-2$ in:

$$\sum_{i=0}^{k} \binom{a}{i} \binom{c}{k-i} = \binom{a+c}{k}, \text{ for } a, c \in \mathbb{Z}.$$

For a proof of this formula see [39, Appendix 3]. □

Upper bound conjecture for simplicial spheres Next we present the upper bound conjecture for simplicial spheres. It was conjectured by V. Klee and proved in 1975 by R. Stanley [268]. It was motivated by the performance of the simplex algorithm in linear programming.

Conjecture 5.5.16 (UBCS) *Let Δ be simplicial complex of dimension d with n vertices. If $|\Delta| \cong S^d$, then*

$$f_i(\Delta) \leq f_i(\Delta(C(n, d+1))) \quad \textit{for} \quad i = 0, \ldots, d.$$

Proof. (sketch) Consider the following four conditions:

(a) $h_i(\Delta) \leq h_i(\Delta(C(n, d+1)))$ for $0 \leq i \leq \lfloor (d+1)/2 \rfloor$.

(b) $h_i(\Delta(C(n, d+1))) = h_{d+1-i}(\Delta(C(n, d+1)))$ for $0 \leq i \leq d+1$.

(c) $h_i(\Delta) \leq \binom{i+n-d-2}{i}$ for all i.

(d) $h_i(\Delta) = h_{d+1-i}(\Delta)$ for $0 \leq i \leq d+1$.

Observe that (a), (b) and (d) imply the upper bound conjecture. To show it notice that if $\lfloor (d+1)/2 \rfloor < i \leq d+1$, then $0 \leq d+1-i \leq \lfloor (d+1)/2 \rfloor$ and

$$h_i(\Delta) = h_{d+1-i}(\Delta) \leq h_{d+1-i}(\Delta(C(n, d+1))) = h_i(\Delta(C(n, d+1))),$$

hence $h_i(\Delta) \leq h_i(\Delta(C(n, d+1)))$ for $0 \leq i \leq d+1$. Therefore

$$\begin{aligned} f_{k-1}(\Delta) &= \sum_{i=0}^{d+1} \binom{d+1-i}{k-i} h_i(\Delta) \leq \sum_{i=0}^{d+1} \binom{d+1-i}{k-i} h_i(\Delta(C(n, d+1))) \\ &= f_{k-1}(\Delta(C(n, d+1))) \quad \text{for } 1 \leq k \leq d+1, \end{aligned}$$

and the proof of the observation is complete. By Corollary 5.5.15 (c) $\Rightarrow$ (a), and by Corollary 5.5.11 (b) is satisfied because $C(n, d+1)$ is a simplicial polytope. As a consequence the upper bound conjecture is reduced to prove (c) and (d). To complete the proof notice that (c) follows from Theorem 5.4.8, and (d) follow from Theorem 5.5.9. □

As a particular case of the UBCS one obtains the upper bound conjecture for convex polytopes, this conjecture was posed by Motzkin in 1957 and proved in 1970 by McMullen [221].

Corollary 5.5.17 (UBCP) *If P is a convex polytope of dimension d with n vertices, then*

$$f_i(P) \leq f_i(C(n, d)) \quad \textit{for} \quad 0 \leq i \leq d-1.$$

Proof. (sketch) By a process known as "pulling the vertices" P can be transformed into a simplicial polytope with the same number of vertices as P, and at least as many faces of higher dimension. Hence we may assume that P is simplicial. Since P is simplicial the boundary complex $\Delta(P)$ is a simplicial sphere, that is, $|\Delta(P)| \cong S^{d-1}$. Using the UBCS we obtain

$$f_i(P) = f_i(\Delta(P)) \leq f_i(\Delta(C(n,d))) = f_i(C(n,d)),$$

for $i = 0, \ldots, d$. □

Remark Not every triangulation of a sphere is the boundary complex of some simplicial polytope. There are examples, found by Grünbaum in 1965, with $d = 4$ and $n = 8$ satisfying $|\Delta| \cong S^{d-1}$ and Δ not isomorphic to $\Delta(P)$ for every d-polytope. Moreover it follows from a result of Steinitz (a graph is the 1-skeleton of a 3-polytope in $\mathbb{R}^3$ if and only if it is a 3-connected planar graph) that such an example does not exist for $d = 3$.

Exercises

5.5.18 If Δ is a connected simplicial complex having all its vertices of degree two, then Δ is a cycle.

5.5.19 If Δ is a simplicial sphere on n vertices whose geometric realization $|\Delta|$ is isomorphic to the unit circle S^1, then

(a) Δ is a cycle of length.

(b) $\nu(I_\Delta) = \binom{n}{2} - n$ for $n \geq 4$.

5.5.20 ([18]) Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. If I is a monomial Gorenstein ideal and $\operatorname{rad}(I) = (x_1, \ldots, x_n)$, prove that

$$I = (x_1^{a_1}, \ldots, x_n^{a_n})$$

Hint Prove that I is irreducible. See [43] and [218, Theorem 18.1].

5.5.21 Give an example of a monomial Gorenstein ideal which is not a complete intersection.

Chapter 6

Edge Ideals

Edge ideals are the most simple polynomial ideals after face ideals, they are generated by square-free monomials of degree two and they can be associated to graphs and to Stanley-Reisner complexes. At first sight the use of graph theory to study edge ideals is convenient because there are common objects of study, but in fact some remarkable applications of graph theoretical results will take place in this chapter. There are many works in the literature relating graph theory with other branches of algebra, see for instance [17, 24, 135, 136, 164, 262, 263].

In the first section the connection between ideals and graphs is established through the notion of minimal vertex cover. Some graph theory results and terminology are recollected and various versions of a theorem due to König are discussed. As we learned graph theory from [28] and [139], those remain as our main references for the subject and they are highly recommended.

We study some properties of edge ideals, such as the Cohen-Macaulay property, and stress that some of those properties can be better understood using graph theoretical notions. A structure theorem for Cohen-Macaulay trees is presented in Section 6.3; this leads to a deeper result showing that Cohen-Macaulay bipartite graphs can be defined recursively.

The notion of "ideal of covers" of a monomial ideal is introduced in Section 6.7, and we study the connection between properties of an edge ideal and properties of its ideal of covers, the notion of triangulated graph is used to link properties of those two ideals.

6.1 Graphs and ideals

A *simple graph* G consists of a finite set V of *vertices* and a collection E of subsets of V called *edges* such that every edge of G is a pair $\{v_i, v_j\}$ for

some v_i, v_j in V. Note that a simple graph G has no loops, that is, we are requiring $v_i \neq v_j$ for all edges $\{v_i, v_j\}$ of G.

When working with several graphs it is convenient to write $V(G)$ and $E(G)$ for the vertex set and edge set of G respectively. Let H and G be two graphs, it is said that H is a *subgraph* of G if $V(H) \subset V(G)$ and $E(H) \subset E(G)$. A *spanning subgraph* is a subgraph H of G containing all the vertices of G.

Let G be a graph on the vertex set V. If $z = \{v_i, v_j\}$ is an edge of G one says that the vertices v_i and v_j are *adjacent* or *connected* by z, in this case it is also usual to say that the edge z is incident with the vertex v_i. The *degree* of a vertex v in V, denoted by $\deg(v)$, is the number of edges incident with v. A vertex of degree one (resp. degree zero) is called an *end vertex* (resp. *isolated vertex*). If all the vertices of G are isolated, G is called a *discrete graph*.

A *walk* of *length* n in G is an alternating sequence of vertices and edges

$$w = \{v_0, z_1, v_1, \ldots, v_{n-1}, z_n, v_n\},$$

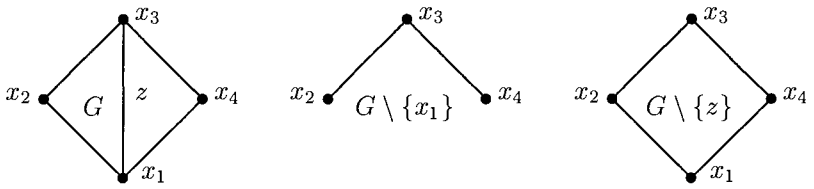
where $z_i = \{v_{i-1}, v_i\}$ is the edge joining v_{i-1} and v_i. A walk may also be written $\{v_0, \ldots, v_n\}$ with the edges understood, or $z_1, z_2, \ldots, z_n$ with the vertices understood. If $v_0 = v_n$, the walk w is called a *closed walk*. A *path* is a walk with all its vertices distinct.

We say that G is *connected* if for every pair of vertices v_1 and v_2 there is a path from v_1 to v_2. Note that G has a vertex disjoint decomposition

$$G = \bigcup_{i=1}^{r} G_i,$$

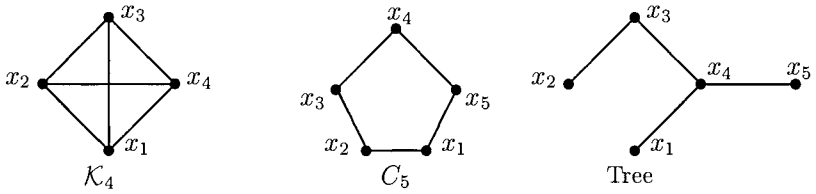
where $G_1, \ldots, G_r$ are the maximal (with respect to inclusion) connected subgraphs of G, the G_i are called the *connected components* of G.

If z is a edge, we denote by $G \setminus \{z\}$ the spanning subgraph of G obtained by deleting z and keeping all the vertices of G. The removal of a vertex v from a graph G results in that subgraph $G \setminus \{v\}$ of G consisting of all the vertices in G except v and all the edges not incident with v. For instance:



If A is a set of vertices of G one can define $G \setminus A$ recursively by removing one vertex at a time.

A *cycle* of *length* n is a closed walk $\{v_0, \ldots, v_n\}$ in which $n \geq 3$ and the vertices $v_1, \ldots, v_n$ are distinct. A cycle is *even* (resp. odd) if its length is even (resp. odd). We denote by C_n the graph consisting of a cycle with n vertices, C_3 will be called a *triangle*, C_4 a *square* and so on. A *tree* is a connected graph without cycles, and a *forest* is an acyclic graph. The *complete graph* $\mathcal{K}_n$ has every pair of its n vertices adjacent. For instance:



A graph G is *bipartite* if its vertex set V can be partitioned into disjoint subsets V_1 and V_2 such that every edge of G joins V_1 with V_2.

The *distance* $d(u, v)$ between two vertices u and v of a graph G is defined to be the minimum of the lengths of all possible paths from u to v. If there is no path joining u and v, then $d(u, v) = \infty$.

Proposition 6.1.1 *A graph G is bipartite if and only if all the cycles of G are even.*

Proof. $\Rightarrow$) Let V_1 and V_2 be a partition of $V(G)$ such that every edge of G joins V_1 with V_2. If $\{v_0, \ldots, v_n\}$ is a cycle of G, one may assume $v_0 \in V_1$. Note $v_1 \in V_2$, it follows at once that v_i is in V_1 if and only if i is even, thus n must be even.

$\Leftarrow$) It suffices to prove that each connected component of G is bipartite, thus one may assume G connected. Pick a vertex $v_0 \in V$. Set

$$V_1 = \{v \in V(G) | d(v, v_0) \text{ is even}\} \text{ and } V_2 = V(G) \setminus V_1.$$

It follows that no two vertices of V_i are adjacent for $i = 1, 2$, otherwise G would contain an odd cycle. Therefore G is bipartite. □

Corollary 6.1.2 *If T is a forest, then T is a bipartite graph.*

Definition 6.1.3 A set of edges in a graph G is *independent* if no two of them have a vertex in common.

Let G be a graph on the vertex set $V(G)$. Given a subset $U \subset V(G)$, the *neighbor set* of U, denote $N_G(U)$ or simply $N(U)$, is defined as

$$N(U) = \{v \in V(G) \mid v \text{ is adjacent to some vertex in } U\}.$$

Notice that if C_r is a cycle, then $C_r \subset N(C_r)$.

Proposition 6.1.4 *Let G be a bipartite graph with vertex sets V_1, V_2 having m and n vertices respectively. If*

$$|A| \leq |N(A)|$$

for all $A \subset V_1$, then there are m independent edges in G.

Proof. By induction on m. If $m = 1$, then there is at least one vertex in V_2 connected to V_1, thus there is one independent edge. Assume $m \geq 2$.

Case (I): First assume $|A| < |N(A)|$ for all $A \subset V_1$ with $|A| < m$. Let $x_1 \in V_1$ and $y_1 \in V_2$ be two adjacent vertices. Consider the subgraph

$$H = G \setminus \{x_1, y_1\}$$

obtained from G by removing x_1 and y_1. Note that for every $A \subset V_1 \setminus \{x_1\}$ one has $N_H(A) = N(A)$ if $y_1 \notin N(A)$ and $N_H(A) = N(A) \setminus \{y_1\}$ otherwise. Hence by induction there are $z_2, \ldots, z_m$ independent edges in H, which together with $z_1 = \{x_1, y_1\}$ yield the required number of independent edges.

Case (II): Next assume $|A| = |N(A)|$ for some $A \subset V_1$ with $|A| < m$. Set $r = |A|$. Consider the subgraph

$$H = G \setminus (V_1 \setminus A).$$

Since $N_H(S) = N(S)$ for all $S \subset A$, by induction there are independent edges $z_1, \ldots, z_r$ in H.

On the other hand consider $F = G \setminus (A \cup N(A))$. If $S \subset V_1 \setminus A$, then

$$|N(A \cup S)| \geq |A \cup S| = |S| + |A|,$$

since

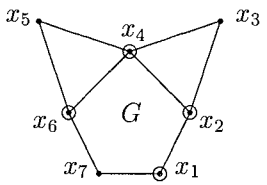
$$N(S \cup A) = N(A) \cup N_F(S)$$

and $r = |N(A)|$ one derives $|N_F(S)| \geq |S|$. Applying induction there are $w_1, \ldots, w_{m-r}$ independent edges in F. Altogether it follows that there are m independent edges in G, as required. This proof is due to Halmos and Vaughn. □

Corollary 6.1.5 *Let G be a bipartite graph with vertex sets V_1, V_2. If there is an integer $p \geq 0$ such that $|A| - p \leq |N(A)|$ for all $A \subset V_1$, then there are $m - p$ independent edges in G, where $m = |V_1|$.*

Proof. By Proposition 6.1.4 the assertion holds for $p = 0$. Consider the graph F obtained from G by adding a new vertex y to V_2 and joining every vertex of V_1 to y. Let $A \subset V_1$, since $N_F(A) = N(A) \cup \{y\}$ one concludes $|N_F(A)| \geq |A| - (p-1)$. Hence by induction there are $m-p+1$ independent edges in F, it follows that there are $m - p$ independent edges in G. □

Let G be a graph with vertex set V. A subset $A \subset V$ is a *minimal vertex cover* for G if: (i) every edge of G is incident with one vertex in A, and (ii) there is no proper subset of A with the first property. If A satisfies condition (i) only, then A is called a *vertex cover* of G and A is said to cover all the lines of G.



$A = \{x_1, x_2, x_4, x_6\}$
minimal vertex cover of G.

It is convenient to regard the empty set as a minimal vertex cover for a graph with all its vertices isolated. A set of vertices in G is *independent* if no two of them are adjacent. Notice that a set in G is a maximal independent set iff its complement is a minimal vertex cover for G. The *vertex covering number* of G, $\alpha_0(G)$, is the smallest number of vertices in any minimal vertex cover.

Definition 6.1.6 The *independence number* of a graph G, denoted by $\beta_1(G)$, is the maximum number of independent edges in G.

Theorem 6.1.7 (König) *If G is a bipartite graph, then $\beta_1(G) = \alpha_0(G)$.*

Proof. Let $V = V_1 \cup V_2$ be a partition of the vertex set of G such that every edge of G joins V_1 with V_2. Set $r = \beta_1(G)$ and $m = |V_1|$. Since there are r independent edges in G one obtains $\alpha_0(G) \geq r$.

Note there is $A \subset V_1$ such that $|N(A)| \leq |A| - (m - r)$, otherwise if $|N(A)| > |A| - (m - r)$ for all $A \subset V_1$, then by Corollary 6.1.5 one would have $r+1$ independent edges which is impossible. Therefore $N(A) \cup (V_1 \setminus A)$ is a vertex cover of G with at most r elements, thus $\alpha_0(G) \leq r$. □

A pairing off of all the vertices of a graph G is called a *perfect matching*. Thus G has a perfect matching if and only if G has an even number of vertices and there is a set of independent lines "covering" (containing) all the vertices.

There are several results equivalent to König theorem (see [28, 139, 223]), next we present its connection with perfect matchings; see Proposition 8.8.11 for a generalization of the marriage problem.

Theorem 6.1.8 (Marriage problem) *If G is a bipartite graph with vertex set V, then the following are equivalent*

(a) *G has a perfect matching.*

(b) *$|A| \leq |N(A)|$ for all $A \subset V$ independent set of vertices.*

Proof. Let V_1, V_2 be a partition of V such that every edge of G joins V_1 with V_2. Set $r = \beta_1(G)$. It is clear that (a) implies (b).

(b) $\Rightarrow$ (a): Since V_i is an independent set for $i = 1, 2$, one readily derives $|V_1| = |V_2|$. By König theorem there are $z_1, \ldots, z_r$ independent edges and a minimal vertex cover A of G with r elements, hence $z_i \cap A$ has exactly one vertex for any i and A is an independent set. Using that also $V \setminus A$ is an independent set of vertices and the equality $N(V \setminus A) = A$ we conclude

$$|V \setminus A| \leq |N(V \setminus A)| = |A|.$$

It follows that $|V_1| = r$, thus $z_1, \ldots, z_r$ yields a perfect matching. □

The cover complexity Given a graph G, the family of its minimal vertex covers will be denoted by $\Upsilon(G)$. One may call the integer $|\Upsilon(G)|$ the *cover complexity* of the graph G. We are interested in algebraic ways of grasping this important number. The best known estimate is due to E. Sperner (cf. [112, Theorem 3.1]):

Theorem 6.1.9 ([265]) *If G is a graph with n vertices, then*

$$|\Upsilon(G)| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Proposition 6.1.10 *Let*

$$\gamma_n = \sup \{|\Upsilon(G)| \mid G \text{ has } n \text{ vertices}\}.$$

If $\{F_n\}$ denotes the Fibonacci sequence given by

$$F_0 = F_1 = 1 \quad \text{and} \quad F_n = F_{n-1} + F_{n-2},$$

then $\gamma_n \leq F_n$ for $n \geq 1$.

Proof. The proof is by induction on n. The required inequality is clearly satisfied if $n \leq 4$. Assume $\gamma_k \leq F_k$ for $k < n$ and $n \geq 5$. Let G be a graph with n vertices. We write $G = G_1 \cup G_2$ where G_1 is a connected component of G with at least two vertices. Let x be a vertex in G_1 and $x_1, \ldots, x_r$ the vertices of G adjacent to x. Let Υ_1 be the set of minimal covers of G that contain x and $\Upsilon_2 = \Upsilon(G) \setminus \Upsilon_1$. If $A \in \Upsilon_1$ notice that $A \setminus \{p\}$ is a (possible empty) minimal vertex cover for $G \setminus \{x\}$, hence $|\Upsilon_1| \leq F_{n-1}$. If $A \in \Upsilon_2$, then $A \setminus \{x_1, \ldots, x_r\}$ is a (possible empty) minimal vertex cover for $G \setminus \{x, x_1, \ldots, x_r\}$, which implies $|\Upsilon_2| \leq F_{n-r-1}$. Therefore $\gamma_n \leq F_n$ as required. □

Proposition 6.1.11 *Let $F_0 = F_1 = 1$, $F_2 = 2$ and $F_n = F_{n-2} + F_{n-3}$ for $n \geq 3$. If P_n is the path $\{x_1, \ldots, x_n\}$, then*

$$F_n = |\Upsilon(P_n)|.$$

Proof. To show it note: (i) The minimal vertex covers of P_n not containing x_1 are in one to one correspondence with the minimal vertex covers of the graph $P_n \setminus \{x_1, x_2\}$, and (ii) the minimal vertex covers of P_n containing x_1 are in one to one correspondence with the minimal vertex covers of the graph $P_n \setminus \{x_1, x_2, x_3\}$. □

Proposition 6.1.12 *If T is a forest, then $|\Upsilon(T)| \geq \alpha_0(T) + 1$.*

Proof. The proof is by induction on the number of vertices. Assume the inequality for all forests with less than n vertices and assume T is a forest with n vertices.

Let z be a vertex of degree one in T and x the vertex in T adjacent to z. Set $T_z = T \setminus \{z\}$. Let $A \in \Upsilon(T_z)$. If $x \in A$ one has $A \in \Upsilon(T)$, and if $x \notin A$, then $A \cup \{z\} \in \Upsilon(T)$. Hence $|\Upsilon(T)| \geq |\Upsilon(T_z)|$. If $\alpha_0(T) = \alpha_0(T_z)$ the inequality follows by induction hypothesis. If

$$\alpha_0(T_z) < \alpha_0(T),$$

then $\alpha_0(T_z) = \alpha_0(T) - 1$. Therefore there is a minimal vertex cover A_1 of T so that $|A_1| = \alpha_0(T)$ and $z \in A_1$. Notice that

$$A_2 = (A_1 \setminus \{z\}) \cup \{x\}$$

is not a minimal vertex cover for T_z because $A_1 \setminus \{z\}$ is a minimal cover for T_z. As a consequence A_2 is a cover of T which is not a lift of a cover of T_z. The required inequality now follows using the induction hypothesis. □

Some types of graphs For convenience let us recall the notion of Cohen-Macaulay standard algebra. Let R be a polynomial ring over a field k with irrelevant maximal ideal $\mathfrak{m}$ and I a graded ideal of R. The *depth* of R/I, denoted by depth R/I, is the largest integer r so that there is a homogeneous sequence $f_1, \ldots, f_r$ in $\mathfrak{m}$ with f_i not a zero-divisor of $R/(I, f_1, \ldots, f_{i-1})$ for all $1 \leq i \leq r$, here we set $f_0 = 0$. This number r is bounded by dim R/I according Lemma 1.3.6.

Definition 6.1.13 The quotient ring R/I is called *Cohen-Macaulay* (C-M for short) if

$$\operatorname{depth}(R/I) = \dim(R/I),$$

and the ideal I is said to be *Cohen-Macaulay* if R/I is Cohen-Macaulay.

Let G be a graph with vertices $v_1, \ldots, v_n$ and

$$R = k[x_1, \ldots, x_n]$$

a polynomial ring over a field k, with one variable x_i for each vertex v_i, we will often identify the vertex v_i with the variable x_i.

Definition 6.1.14 The *edge ideal* $I(G)$ associated to the graph G is the ideal of R generated by the set of square-free monomials x_ix_j such that v_i is adjacent to v_j, that is,

$$I(G) = (\{x_ix_j | \{v_i, v_j\} \in E(G)\}) \subset R.$$

If all the vertices of G are isolated we set $I(G) = (0)$.

Note that the non zero edge ideals are precisely the ideals of R generated by square-free monomials of degree two.

Definition 6.1.15 The graph G is said to be *Cohen-Macaulay* over the field k (C-M graph for short) if $R/I(G)$ is a Cohen-Macaulay ring.

According to Theorem 6.4.7 the Cohen-Macaulay property of a bipartite graph is independent of the field k, but in general this property may depend on the field k (see Example 5.3.31).

The next result establishes a one to one correspondence between the minimal vertex covers of a graph and the minimal primes of the corresponding edge ideal.

Proposition 6.1.16 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and G a graph with vertices $x_1, \ldots, x_n$. If $\mathfrak{p}$ is an ideal of R generated by $A = \{x_{i_1}, \ldots, x_{i_r}\}$, then $\mathfrak{p}$ is a minimal prime of $I(G)$ if and only if A is a minimal vertex cover of G.*

Proof. Note that $I(G) \subset \mathfrak{p}$ if and only if A is a vertex cover of G. Assume that A is a minimal vertex cover of G, hence $I(G) \subset \mathfrak{p}$. By Proposition 5.1.3 any minimal prime of $I(G)$ is a face ideal, thus $\mathfrak{p}$ is a minimal prime of $I(G)$. The converse is clear. □

Definition 6.1.17 A graph G is said to be an *unmixed graph* if any two minimal vertex covers of G have the same cardinality.

Corollary 6.1.18 *If G is a graph and $I(G)$ its edge ideal, then the vertex covering number $\alpha_0(G)$ is equal to the height of the ideal $I(G)$.*

Proof. It follows at once from Proposition 6.1.16. □

Definition 6.1.19 An edge x of a graph G is critical if

$$\alpha_0(G \setminus x) < \alpha_0(G).$$

If all the edges in G are critical G is called *edge-critical* . A vertex v of G is critical if

$$\alpha_0(G \setminus v) < \alpha_0(G),$$

and G is called *vertex-critical* if all its vertices are critical.

Proposition 6.1.20 *Let G be a graph. If $\alpha_0(G \setminus v) < \alpha_0(G)$ for some vertex v of G, then $\alpha_0(G \setminus v) = \alpha_0(G) - 1$.*

Proof. Set $r = \alpha_0(G \setminus v)$ and note $r + 1 \leq \alpha_0(G)$. Pick a minimal vertex cover A of $G \setminus v$ with r vertices. Since v cannot be an isolated vertex $A \cup \{v\}$ is a minimal vertex cover of G. Hence $\alpha_0(G) \leq r + 1$ and one has the asserted equality. □

If G is a graph without isolated vertices, then the relationships between the various notions defined above are the following:

$$\text{C-M} \Rightarrow \text{unmixed} \Rightarrow \text{vertex-critical} \Leftarrow \text{edge-critical},$$

where none of the arrows can be reversed in general. We now prove the first two implications and leave the third implication as an exercise.

Proposition 6.1.21 *If G is a Cohen-Macaulay graph, then G is unmixed.*

Proof. By Corollary 5.1.5 $I(G)$ is the intersection of its minimal primes. Therefore $I(G)$ is an unmixed ideal by Corollary 2.2.4, thus G is unmixed by the correspondence between minimal covers and minimal primes. □

Proposition 6.1.22 *If G is an unmixed graph without isolated vertices, then G is vertex-critical.*

Proof. Let $v \in V(G)$. Since G is unmixed there is a minimal vertex cover A of G containing v and such that $|A| = \alpha_0(G)$. As $A \setminus \{v\}$ is a minimal vertex cover of $G \setminus v$, one has $\alpha_0(G \setminus v) < \alpha_0(G)$. □

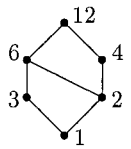
Definition 6.1.23 A *partially ordered set* or *poset* is a pair $P = (V, \leq)$, where V is a finite set of vertices and $\leq$ is a binary relation on V satisfying:

(a) $u \leq u$, $\forall u \in V$ (reflexivity).

(b) $u \leq v$ and $v \leq u$, imply $u = v$ (antisymmetry).

(c) $u \leq v$ and $v \leq w$, imply $u \leq w$ (transitivity).

A poset $P = (V, \leq)$ with vertex set $V = \{x_1, \ldots, x_n\}$ can be displayed by an inclusion diagram, as with the diagram



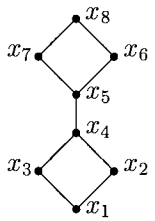
of the divisors of 12. In such a diagram x_i is joined to x_j by raising a line if $x_i < x_j$ and there is no other vertex in between.

Given a poset P with vertex set $V = \{x_1, \ldots, x_n\}$ its *order complex*, denoted by $\Delta(P)$, is the simplicial complex on V whose faces are the chains (linearly ordered sets) in P, thus

$$k[\Delta(P)] = k[V]/(x_i x_j | \, x_i \not\sim x_j)$$

is the Stanley-Reisner ring of $\Delta(P)$. Here $x_i \not\sim x_j$ means that x_i is not comparable to x_j. The Cohen-Macaulay property of those rings associated to posets has been extensively studied, see [12, 26, 44, 123, 162, 274] and the references there. Those simplicial complexes that occur as order complexes have been nicely characterized by Stanley [270], see also [162].

Example 6.1.24 Let $P = (V, \leq)$ be a partially ordered set with vertex set $V = \{x_1, \ldots, x_8\}$ ordered according to the next inclusion diagram. The Stanley-Reisner ring of the order complex $\Delta(P)$ is $k[V]/(x_6x_7, x_2x_3)$.



facets of $\Delta(P)$

$\{x_1, x_2, x_4, x_5, x_6, x_8\}$
$\{x_1, x_2, x_4, x_5, x_7, x_8\}$
$\{x_1, x_3, x_4, x_5, x_6, x_8\}$
$\{x_1, x_3, x_4, x_5, x_7, x_8\}$

Simplicial complexes of edge ideals Given a graph $\overline{G}$ on the vertex set V one defines a simplicial complex $\mathrm{Simp}(\overline{G})$ on the same set of vertices:

$$\mathrm{Simp}(\overline{G}) = \{F | \, \exists \, H \text{ a subgraph of } \overline{G} \text{ such that } F = V(H) \text{ and } H \simeq \mathcal{K}_r\},$$

where $\mathcal{K}_r$ denotes a complete graph on r vertices. Conversely a simplicial complex Δ defines a graph $\mathrm{Skel}(\Delta)$, the 1-skeleton of Δ, on the same set of vertices:

$$\mathrm{Skel}(\Delta) = \{F \in \Delta | \, \dim(F) \leq 1\}.$$

In general one has $\Delta \subset \mathrm{Simp}(\mathrm{Skel}(\Delta))$.

Proposition 6.1.25 *Let Δ be a simplicial complex and I_Δ its Stanley-Reisner ideal. Then $\Delta = \mathrm{Simp}(\mathrm{Skel}(\Delta))$ if and only if I_Δ is generated by square-free monomials of degree two.*

Proof. Let $V = \{x_1, \ldots, x_n\}$ be the vertex set of Δ. Assume I_Δ is generated by square-free monomials of degree two. Take $F \in \mathrm{Simp}(\mathrm{Skel}(\Delta))$, for simplicity set $F = \{x_1, \ldots, x_r\}$, that is, $\{x_i, x_j\}$ for all $1 \leq i < j \leq r$. If $F \notin \Delta$, then $x_1 \cdots x_r$ is in I_Δ, and by assumption $x_i x_j$ is in I_Δ for some $1 \leq i < j \leq r$. Hence $\{x_i, x_j\} \notin \Delta$, which is a contradiction. This shows $\Delta = \mathrm{Simp}(\mathrm{Skel}(\Delta))$. The converse also follows rapidly. □

Simplicial complexes associated to edge ideals have been studied in the literature, they are called *flag complexes*, see [274, Chapter III] and the references there for a series of amusing related problems.

Exercises

6.1.26 Prove that $\overline{G} = \text{Skel}(\text{Simp}(\overline{G}))$, for any graph $\overline{G}$.

6.1.27 Let G be a graph with vertex set $V = \{x_1, \ldots, x_n\}$, prove Euler's identity:

$$2|E(G)| = \sum_{i=1}^{n} \deg(x_i).$$

6.1.28 (P. Hall) Let X be a finite set and $S_1, \ldots, S_m$ a collection of subsets of X. If

$$\left| \bigcup_{j=1}^{k} S_{i_j} \right| \geq k$$

for all $i_1, \ldots, i_k$ distinct, prove that there are $x_1, \ldots, x_m$ in X distinct such that $x_i \in S_i$ for all i.

Hint Consider the bipartite graph with vertex sets $V_1 = \{S_1, \ldots, S_m\}$ and $V_2 = X$, where $x \in X$ is adjacent to S_j if $x \in S_j$.

6.1.29 Let H be a subgroup of G of index n and G/H (resp. $(G/H)_L$) the set of right (resp. left) cosets. Then there are $g_1, \ldots, g_n$ in G such that

$$G/H = \{Hg_1, \ldots, Hg_n\} \text{ and } (G/H)_L = \{g_1H, \ldots, g_nH\}.$$

Hint If $G/H = \{H_1, \ldots, H_n\}$ and $(G/H)_L = \{H'_1, \ldots, H'_n\}$, define S_i as the set of j such that $H_i \cap H'_j \neq \emptyset$ and write

$$B_r = \bigcup_{i=1}^{n} \left(B_r \bigcap H'_i \right),$$

where B_r is equal to

$$\bigcup_{s=1}^{r} H_{i_s}$$

with $i_1, \ldots, i_r$ distinct. Note that at least r of the $B_r \cap H'_i$ are non empty.

6.1.30 If G is a connected graph and d the distance function, then:

(a) $d(x, y) \geq 0$, and $d(x, y) = 0 \Leftrightarrow x = y$,

(b) $d(x, y) = d(y, x)$ (symmetry),

(c) $d(x, y) \leq d(x, z) + d(z, y)$ (the triangle inequality).

6.1.31 Let G be a graph. If $\alpha_0(G \setminus x) < \alpha_0(G)$ for some edge x of G, prove that $\alpha_0(G \setminus x) = \alpha_0(G) - 1$.

6.1.32 If G is an edge-critical graph without isolated vertices, prove that G is vertex-critical.

6.1.33 In [234] there is a list of all the known edge-critical graphs with fewer than nine vertices, prove that all these graphs are unmixed. Can you give an example of an edge-critical graph which is not unmixed?.

6.2 Cohen-Macaulay graphs

Let G be a graph on the vertex set $V = \{x_1, \ldots, x_n\}$. We define the *complementary simplicial complex* Δ_G of a graph G by

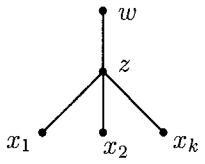
$$\Delta_G = \{A \subset V | \text{ A is an independent set in } G\},$$

notice that Δ_G is precisely the Stanley-Reisner simplicial complex of $I(G)$.

According to Reisner's Theorem 5.3.5 the Cohen-Macaulay property of a given graph can be decided once the reduced homology of the links of Δ_G is computed. Unfortunately the simpler the graph G the more complicated the complex Δ_G. Following [262, 302] we will study C-M graphs by looking at both the complementary simplicial complex and the graph itself.

Construction of Cohen-Macaulay graphs

One of the purposes here is to show how large classes of C-M graphs can be produced and to show some obstructions for a graph to be C-M.



First construction Let H be a graph with vertex set $V(H) = \{x_1, \ldots, x_n, z, w\}$ and J its edge ideal. Assume that z is adjacent to w with $\deg(z) \geq 2$ and $\deg(w) = 1$. We label the vertices of H such that $x_1, \ldots, x_k, w$ are the vertices of H adjacent to z, as shown in the figure. The next two results describe how the Cohen-Macaulay property of H relates to that of the two subgraphs $G = H \setminus \{z, w\}$ and $F = G \setminus \{x_1, \ldots, x_k\}$. One has the equalities

$$J = (I, x_1 z, \ldots, x_k z, zw) \text{ and } (I, x_1, \ldots, x_k) = (L, x_1 \ldots, x_k),$$

where I and L are the edge ideals of G and F respectively.

Assume H is unmixed with height of J equal to $g + 1$. Since z is not isolated, there is a minimal prime $\mathfrak{p}$ over I containing $\{x_1, \ldots, x_k\}$ and such that $\operatorname{ht}(I) = \operatorname{ht}(\mathfrak{p}) = g$. It is not difficult to prove that $k < n$ and $\deg(x_i) \geq 2$ for $i = 1, \ldots, k$.

Proposition 6.2.1 *If H is a C-M graph, then F and G are C-M graphs.*

Proof. Set $A = k[x_1, \ldots, x_n]$ and $R = A[z, w]$. By Proposition 9.3.2 there exists a homogeneous system of parameters $\{f_1, \ldots, f_d\}$ for A/I, where $f_i \in A_+$ for all i. Because of the hypothesis and the equalities

$$z(z-w) + zw = z^2 \text{ and } w(w-z) + zw = w^2,$$

the set $\{f_1, \ldots, f_d, z-w\}$ is a regular system of parameters for R/J. Hence $f_1, \ldots f_d$ is a regular sequence on R/I, that is, G is C-M. To show the corresponding property for F we consider the sequence

$$0 \longrightarrow R/(I, x_1, \ldots, x_k, w)[-1] \xrightarrow{z} R/J \xrightarrow{\psi} R/(I, z) \longrightarrow 0$$

where the first map is multiplication by z and ψ is induced by a projection. The exactness of this sequence follows from Corollary 5.1.5. Taking depths w.r.t the maximal ideal of R, by the depth lemma one has

$$n - g + 1 \leq \text{depth } R/(I, x_1, \ldots, x_k, w), \text{ where } g = \text{ht}(I).$$

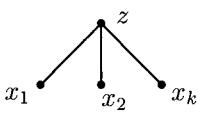
Since $(I, x_1, \ldots, x_k, w) = (L, x_1, \ldots, x_k, w)$, F is C-M. □

Proposition 6.2.2 *If F and G are C-M graphs and $\{x_1, \ldots, x_k\}$ form a part of a minimal vertex cover for G, then H is a C-M graph.*

Proof. Consider the exact sequence of Proposition 6.2.1. Since the ends of this sequence have R-depth equal to $\dim(R/J)$, by the depth lemma H is a Cohen-Macaulay graph. □

Corollary 6.2.3 *If G is C-M and $\{x_1, \ldots, x_k\}$ is a minimal vertex cover for G, then H is C-M.*

Proof. Note that in this case F is Cohen-Macaulay because $I(F) = (0)$. □



Second construction For the discussion of the second construction we change our notation. Let H be a graph on the vertex set $V(H) = \{x_1, \ldots, x_n, z\}$. Let $\{x_1, \ldots, x_k\}$ be the vertices of H adjacent to z, as shown in the figure. Taking into account the first construction one may assume $\deg(x_i) \geq 2$ for $i = 1, \ldots, k$ and $\deg(z) \geq 2$. Setting $G = H \setminus \{z\}$ and $F = G \setminus \{x_1, \ldots, x_k\}$, notice that the ideals J, I and L associated to H, G and F respectively are related by the equalities $J = (I, x_1 z, \ldots x_k z)$ and $(I, x_1, \ldots, x_k) = (L, x_1, \ldots, x_k)$.

Proposition 6.2.4 *If H is C-M, then F is C-M.*

Proof. We set $R = k[x_1, \ldots, x_n, z]$, $A = k[x_1, \ldots, x_n]$ and $\text{ht}(J) = g + 1$. The polynomial $f = z - x_1 - \cdots - x_k$ is regular on R/J because it is

clearly not contained in any associated prime of J. Therefore there is a sequence $\{f, f_1, \ldots, f_m\}$ regular on R/J so that $\{f_1, \ldots, f_m\} \subset A_+$, where $m = n - g - 1$. Observe that $\{f_1, \ldots, f_m\}$ is in fact a regular sequence on A/I, which gives $\operatorname{depth}(A/I) \geq n - g - 1$. Next, we use the exact sequence

$$0 \longrightarrow R/(I, x_1, \ldots, x_k)[-1] \xrightarrow{z} R/J \longrightarrow R/(I, z) \longrightarrow 0$$

and $\operatorname{ht}(I, x_1, \ldots, x_k) = g + 1$ to conclude that F is C-M. □

Proposition 6.2.5 *Assume $x_1, \ldots, x_k$ do not form a part of a minimal vertex cover for G and $\operatorname{ht}(I, x_1, \ldots, x_k) = \operatorname{ht}(I) + 1$. If F and G are C-M, then H is C-M.*

Proof. The assumption on $\{x_1, \ldots, x_k\}$ forces $\operatorname{ht}(J) = \operatorname{ht}(I) + 1$. From the exact sequence

$$0 \longrightarrow R/(I, x_1, \ldots, x_k)[-1] \xrightarrow{z} R/J \longrightarrow R/(I, z) \longrightarrow 0$$

we obtain that H is C-M. □

Corollary 6.2.6 *If G is C-M and $\{x_1, \ldots, x_{k-1}\}$ is a minimal vertex cover for G, then H is C-M.*

Connected components of Cohen-Macaulay graphs A good property of Cohen-Macaulay graphs is its additivity with respect to connected components.

Lemma 6.2.7 *Let $R_1 = k[\mathbf{x}]$ and $R_2 = k[\mathbf{y}]$ be two polynomial rings over a field k and $R = k[\mathbf{x}, \mathbf{y}]$. If I_1 and I_2 are graded ideals in R_1 and R_2 respectively, then*

$$\operatorname{depth}(R_1/I_1) + \operatorname{depth}(R_2/I_2) = \operatorname{depth} R/(I_1 + I_2).$$

Proof. We give two proofs. First proof: the formula is a direct consequence of Proposition 2.2.20 and Theorem 2.2.21. Second proof: by a change of base $(R_2/I_2) \otimes_k R_1 \simeq R/I_2R$ is a flat R_1-module. The equality now follows from a general property of tensor products [217, Theorem 50]. □

Proposition 6.2.8 *If G is a graph and $G_1, \ldots, G_n$ its connected components, then G is C-M if and only if G_i is C-M for all i.*

Proof. The result follows from Lemma 6.2.7. □

Complementary complex versus the complement

The *complement* $\overline{G}$ of a graph G has vertex set $V(G)$ and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. Along the chapter we will encounter the complement $\overline{G}$, and witness its usefulness to encapsulate information of the edge ideal $I(G)$.

Proposition 6.2.9 *Let G be a graph and Δ_G its complementary simplicial complex. Then $\overline{G} \subset \Delta_G$, with equality if and only if $\overline{G}$ has no triangles.*

Proof. If $x = \{v_1, v_2\}$ is an edge of $\overline{G}$, then $\{v_1, v_2\}$ is an independent set of G and $x \in \Delta_G$. Hence $\overline{G} \subset \Delta_G$. Assume $\overline{G}$ has no triangles. If $A = \{v_1, \ldots, v_n\}$ is an independent set of G, then $n = 1$ or $n = 2$; otherwise if $n > 2$, then $\overline{G}$ would contain the triangle $\{v_1, v_2, v_3\}$. Therefore A is either a point or an edge of $\overline{G}$, thus $\overline{G} = \Delta_G$. The converse also follows rapidly. □

In general a Cohen-Macaulay graph G can have all the subgraphs $G \setminus \{v\}$ failing to be C-M, for v in G; see Example 6.2.24. Under special circumstances we will study the existence and distribution of the vertices of G whose removal preserve the C-M property, the case of bipartite graphs is analyzed in Section 6.4.

Definition 6.2.10 A *cutpoint* of a graph is a vertex whose removal increases the number of connected components.

Proposition 6.2.11 *If u, v are two vertices at maximum distance in a non discrete connected graph G, then u, v are not cutpoints.*

Proof. Assume v is a cutpoint and write $G \setminus \{v\} = G_1 \cup \cdots \cup G_s$, where $G_1, \ldots, G_s$ are the connected components of $G \setminus \{v\}$. One may assume $u \in G_1$, since $s \geq 2$ there is $w \in G_i$ for some $i \geq 2$. Note that v is in any path joining u and w, thus $d(u, w) > d(u, v)$, a contradiction. This argument is due to Harary. □

Definition 6.2.12 A set of vertices D of a graph G is called *dominant* if every vertex of G is adjacent with at least one vertex in D.

Lemma 6.2.13 *Let G be a connected graph. If $\overline{G}$ is connected and has no triangles, then*

$$D = \{v \in V(\overline{G}) \mid \overline{G} \setminus \{v\} \text{ is connected}\}$$

is a dominant set of vertices of G.

Proof. Assume $\overline{G} \setminus \{v\}$ is disconnected for some $v \in V(G)$. We set

$$V(G) = \{v, x_1, \ldots, x_k, x_{k+1}, \ldots, x_m\}$$

where $x_1, \ldots, x_k$ are the vertices of G adjacent to v. Since $\overline{G}$ has no triangles $d(x_i, x_j) = 2$ for $k+1 \leq i < j \leq m$, where d is the distance in $\overline{G}$. Notice that at least one of the vertices $\{x_{k+1}, \ldots, x_m\}$ has degree greater or equal than two in $\overline{G}$, say x_m. We order the vertex set so that x_k and x_m are adjacent in $\overline{G}$. Let $G_1, \ldots, G_\ell$ be the connected components of $\overline{G} \setminus \{v\}$. We may also assume that x_k and x_m are in $V(G_1)$. Notice that $d(x_k, b) \geq 3$ for all b in $V(G_2)$. Using Proposition 6.2.11 one has that any pair of vertices at maximum distance in $\overline{G}$ must be in D. Therefore $\overline{G} \setminus \{x_i\}$ is connected for some $1 \leq i \leq k$, that is, x_i is a vertex in D adjacent to v in G. □

Proposition 6.2.14 *Let G be a connected graph and $\overline{G}$ its complement. If $\overline{G}$ is connected and has no triangles, then*

$$D = \{z \in V(G) | \, G \setminus \{z\} \text{ is C-M}\}$$

is a dominant set.

Proof. Let Δ_G be the complementary simplicial complex of G. Observe that $\Delta_G = \overline{G}$, hence by Reisner's Theorem G is C-M. The rest of the proof is an immediate consequence of Lemma 6.2.13. □

Exercises

6.2.15 Show that a cycle C_n of length $n \geq 3$ is unmixed if and only if $n = 3, 4, 5, 7$.

6.2.16 Let G be a graph such that $\overline{G}$ is connected and has no triangles. Show that $I(G)$ is Gorenstein if and only if $\overline{G}$ is a cycle.

Hint It follows from [274, Theorem 5.1] that $I(G)$ is Gorenstein iff $\overline{G}$ is a pseudo manifold. Since $\overline{G}$ is a graph it is a pseudo manifold iff it is a cycle.

6.2.17 ([309]) Let G_1, G_2 be two graphs with vertex set $\mathbf{x}$. If their complements are triangle free and they have the same number of edges, then the Hilbert series of $k[\mathbf{x}]/I(G_1)$ and $k[\mathbf{x}]/I(G_2)$ are identical.

6.2.18 Let G be a graph on the vertex set $V = \{x_1, \ldots, x_p\}$ such that $\mathrm{ht}(I(G)) = p - 2$. Prove the following:

(a) G is Cohen-Macaulay if and only if Δ_G is connected, and

(b) if $I(G)$ is not Cohen-Macaulay, then $I(G)$ is unmixed if and only if Δ_G has no isolated vertices.

6.2.19 If G is a C-M graph with q edges, prove that

$$q \leq (g^2 + g)/2,$$

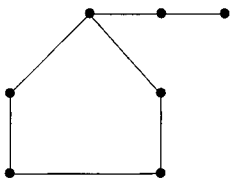
where g is the height of $I(G)$.

6.2.20 A graph G with q edges is said to be *saturated* if $q = g(g+1)/2$, where g is the height of $I(G)$, and G is C-M. Let G be a graph with connected components $G_1, \ldots, G_m$. Prove that G is saturated if and only if G_i is saturated for all i.

6.2.21 If G is a saturated graph, prove that at most one of its connected components has more than one vertex.

6.2.22 Let G be a Cohen-Macaulay graph over a field k. Prove that G is saturated if and only if $R/I(G)$ has a 2-linear resolution.

6.2.23 Prove that the following graph is Cohen-Macaulay.



6.2.24 Prove that the ideal $I = I_\Delta$ of Example 5.3.31 is generated by

$$\begin{array}{llllllll} x_1x_3, & x_1x_4, & x_1x_7, & x_1x_{10}, & x_1x_{11}, & x_2x_4, & x_2x_5, & x_2x_8, \\ x_2x_{10}, & x_2x_{11}, & x_3x_5, & x_3x_6, & x_3x_8, & x_3x_{11}, & x_4x_6, & x_4x_9, \\ x_4x_{11}, & x_5x_7, & x_5x_9, & x_5x_{11}, & x_6x_8, & x_6x_9, & x_7x_9, & x_7x_{10}, \\ x_8x_{10}. & & & & & & & \end{array}$$

Let $R_i = k[x_1, \ldots, \widehat{x}_i, \ldots, x_{11}]$ and $G_i = G \setminus \{x_i\}$. Note $I = I(G)$ for some graph G. Prove that the f-vector of Δ_{G_i} is $(10, 25, 15)$ or $(10, 24, 14)$. Thus the h-vector of $R_i/I(G_i)$ is $(1, 7, 8, -1)$ or $(1, 7, 7, -1)$. Use Theorem 4.2.6 to conclude that G_i is not a C-M graph for all i.

6.2.25 If G is a Cohen-Macaulay graph and x_1, x_2 are two vertices of degree 1, prove that they cannot have an adjacent vertex in common.

6.2.26 Let G be a connected graph and $\{z_1, \ldots, z_n\}$ a path with $n \geq 4$. If $\deg(z_1) = 1$, $\deg(x_i) = 2$ for $i = 2, \ldots n-1$ and $\deg(z_n) \geq 3$, prove that G cannot be unmixed.

6.3 Trees

In this section we give an effective description of Cohen-Macaulay trees and show an interesting family of graphs containing all Cohen-Macaulay trees. The first Koszul homology module of a Cohen-Macaulay forest is also studied.

Proposition 6.3.1 *Let S be a positively graded algebra of dimension d over a field k. If $A = k[h_1, \ldots, h_d] \hookrightarrow S$ is a homogeneous Noether normalization of S, then S is a Cohen-Macaulay ring if and only if S is a free A-module.*

Proof. By Theorem 2.5.13 $\operatorname{pd}_A(S) + \operatorname{depth}(S) = \dim(A)$, where the depth of S is taken with respect to A_+. Assume S is a free A-module, then $\operatorname{depth}(S) = d$ and thus the depth of S with respect to S_+ is also equal d. Therefore S is Cohen-Macaulay.

Conversely assume S Cohen-Macaulay. Note that A is a polynomial ring. Let $\mathfrak{p} \in \operatorname{Spec} S$ be a minimal (graded) prime over A_+S. There is an integral extension

$$k = A/A_+ \hookrightarrow S/\mathfrak{p},$$

hence $S/\mathfrak{p}$ is a zero dimensional domain and consequently $\mathfrak{p}$ is a maximal ideal. Therefore $\operatorname{rad}(A_+S) = S_+$, that is, $h_1, \ldots, h_d$ is a h.s.o.p for S. Now use Proposition 2.2.7 to conclude that $h_1, \ldots, h_d$ is a regular sequence in S, that is, $\operatorname{depth} S = d$. Another application of the Auslander-Buchsbaum formula yields that S is a free A-module. □

Next we apply the above characterization of Cohen-Macaulay graded algebras to give a proof, due to Wolmer Vasconcelos, of the following key result:

Proposition 6.3.2 *Let $R = k[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be a polynomial ring over a field k and $M \subset \{(i,j) |\ 1 \leq i < j \leq n\}$. Then the ideal*

$$I = (\{x_i y_i, y_j y_\ell |\ i = 1, \ldots, n \text{ and } (j,\ell) \in M\}),$$

is a Cohen-Macaulay ideal.

Proof. We start by doing a change of variables $z_i = x_i - y_i$ for $i = 1, \ldots, n$. Its effect is that

$$A = k[z_1, \ldots, z_n, y_1, \ldots, y_n]/I$$

becomes an integral graded algebra over $k[z_1, \ldots, z_n]$, where

$$I = (y_i^2 + y_i z_i, y_j y_\ell).$$

Observe that the images of $y_{i_1} \cdots y_{i_r}$ $(1 \leq i_1 < \cdots < i_r \leq n)$ not divisible by any of the monomials $y_j y_\ell$ form a generating set for the $k[z_1, \ldots, z_n]$-module A. To complete the proof we now show that they are in fact a basis. We set $z = (z_1, \ldots, z_n)$ and $y = (y_1, \ldots, y_n)$. Assume the following equality

$$\sum f_i(z)y_i + \sum f_{i_1 i_2}(z)y_{i_1}y_{i_2} + \cdots + \sum f_{i_1 \ldots i_m}(z)y_{i_1} \cdots y_{i_m} =$$

$$\sum g_i(y,z)(y_i^2 + y_i z_i) + \sum h_{j\ell}(y,z)y_j y_\ell$$

Using induction we will prove that $f_i = f_{i_1 i_2} = \cdots = f_{i_1 \ldots i_m} = 0$. First we show $f_i = 0$ for $1 \leq i \leq n$. Making $y_j = 0$ for $j \neq i$ in the equation above we obtain $f_i(z) = g_i(y_i, z)(y_i + z_i)$, which forces $f_i = 0$. We set the next induction step by assuming $f_i = f_{i_1 i_2} = \ldots = f_{i_1 \ldots i_{r-1}} = 0$. The substitutions $y_j = 0$ for $y_j \notin \{y_{i_1}, \ldots, y_{i_r}\}$ and $y_{i_p} = -z_{i_p}$ for $1 < p \leq r$ into the first equality gives

$$\pm f_{i_1 \ldots i_r}(z)z_{i_2} \cdots z_{i_r} = g_{i_1}(y_{i_1}, z)(y_{i_1} + z_{i_1}),$$

therefore $f_{i_1 \ldots i_r}(z) = 0$. Altogether A is a free module over $k[z_1, \ldots, z_n]$ and this implies that I is C-M by Proposition 6.3.1.

Let us give a quite different second proof pointed out by Jürgen Herzog. Note that the polarization of the ideal

$$J = (\{x_i^2, x_j x_\ell | \, 1 \leq i \leq n \text{ and } (j,\ell) \in M\})$$

is equal to I and apply Proposition 5.1.7. □

Lemma 6.3.3 *Let T be a tree. If v and w are two adjacent vertices of degree at least two, then there is a minimal vertex cover for T containing both v and w.*

Proof. Let x be the edge joining v and w. Since $T \setminus \{x\}$ has exactly two components, say T_1 and T_2, there are minimal vertex covers A and B for T_1 and T_2 respectively so that $v \in A$ and $w \in B$. Therefore $A \cup B$ is the required cover for T. □

Theorem 6.3.4 *Let T be a tree with vertex set V and edge set E. Then T is C-M if and only if $|V| \leq 2$ or $2 < |V| = 2r$ and there are vertices $x_1, \ldots, x_r, y_1, \ldots, y_r$ so that $\deg(x_i) = 1$, $\deg(y_i) \geq 2$, and $\{x_i, y_i\} \in E$ for $i = 1, \ldots, r$.*

Proof. $\Rightarrow$) As T is a bipartite graph there are disjoint sets V_1 and V_2 such that $V = V_1 \cup V_2$ and every edge of T joins V_1 with V_2. Since V_1 and V_2 are minimal vertex covers for T, and T is unmixed, we have

$$r = |V_1| = |V_2| = \alpha_0(T).$$

By Theorem 6.1.7 there are r independent edges. Therefore we may assume $V_1 = \{z_1, \ldots, z_r\}$, $V_2 = \{w_1, \ldots, w_r\}$, and $\{z_i, w_i\} \in E$ for $i = 1, \ldots, r$. To complete the proof is enough to show that for each i either z_i or w_i has degree one. Assume $\deg(z_i) \geq 2$ and $\deg(w_i) \geq 2$ for some i. By Lemma 6.3.3 there is a minimal vertex cover for T, say A, containing z_i and w_i. Since $\{z_j, w_j\} \bigcap A \neq \emptyset$ for $j \neq i$ we conclude $|A| \geq r + 1$, which is a contradiction.

$\Leftarrow$) The sufficiency follows from Proposition 6.3.2. □

Corollary 6.3.5 *If G is a tree, then G is C-M if and only if G is unmixed.*

Proof. If G is unmixed, using the proof above it follows that $I(G)$ is a C-M ideal of the kind described in Proposition 6.3.2. □

Corollary 6.3.6 *The only C-M cycles are a triangle and a pentagon.*

Proof. Let $C_n = \{x_0, x_1, \ldots, x_n = x_0\}$ be a C-M cycle of length n. The cases $n = 4$, 6, and 7 can be treated separately, so we assume $n \geq 8$. By Proposition 6.2.4 $C_n \setminus \{x_0, x_1, x_{n-1}\}$ is a C-M path of length $n - 3$ which is impossible by Theorem 6.3.4 □

Proposition 6.3.7 *If $I = I(T) \subset R$ is the edge ideal of a Cohen-Macaulay forest with $2n$ vertices and without isolated vertices, then*

$$\operatorname{depth}(R/I^2) \geq n - 1.$$

Proof. By Theorem 6.3.4 there are vertices $x_1, \ldots, x_n, y_1, \ldots, y_n$ with $\deg(x_i) = 1$, $\deg(y_i) \geq 1$ for all i, such that I has a generating set:

$$\{x_1y_1, \ldots, x_ny_n, y_jy_\ell |\ (j, \ell) \in M\}, \quad \text{where } M \subset [1, n] \times [1, n]\,.$$

Set $R = k[\mathbf{x}, \mathbf{y}]$, where k is a field. We proceed by induction on n. If I is a complete intersection or $1 \leq n \leq 2$ the result follows from Theorem 3.3.14 together with the exact sequence

$$0 \longrightarrow I/I^2 \longrightarrow R/I^2 \longrightarrow R/I \longrightarrow 0.$$

Notice that if I is not a complete intersection, then T has a vertex of degree two, its existence follows from Theorem 6.3.4 and the fact the sum of the degrees of the vertices of a graph is twice the number of its edges. We order the variables so that $\deg(y_n) = 2$ and $y_{n-1}y_n \in I$ and assume the result true for CM forest with less than $2n$ vertices.

The ideal of R obtained by evaluation of I at $y_n = 0$ will be denoted by J, while L will denote the ideal in R obtained by evaluation of J at $y_{n-1} = 0$. We also set $K = (J^2, y_{n-1}y_n, x_ny_n)$.

In the sequel we shall also make use of the fact that, if I is a monomial ideal, and f is a monomial, then $(I\colon f)$ is a monomial ideal. Consider the complex (easily shown to be exact by the preceding remark):

$$0 \longrightarrow R/(L^2, y_{n-1}) \xrightarrow{y_n} R/(J^2, y_{n-1}y_n) \longrightarrow R/(J^2, y_n) \longrightarrow 0. \quad (1)$$

By an appropriate induction hypothesis

$$\operatorname{depth}(R/(L^2, y_{n-1})) \geq n \quad \text{and} \quad \operatorname{depth}(R/(J^2, y_n)) \geq n-1.$$

Hence from (1) $\operatorname{depth}(R/(J^2, y_{n-1}y_n)) \geq n-1$, which together with the exact sequence

$$0 \longrightarrow R/(L^2, y_{n-1}) \xrightarrow{x_n y_n} R/(J^2, y_{n-1}y_n) \longrightarrow R/K \longrightarrow 0 \quad (2)$$

gives $\operatorname{depth}(R/K) \geq n-1$. Because of the equality $K = (I^2, y_{n-1}y_n, x_n y_n)$ we shift our attention to the exact sequence

$$0 \longrightarrow R/(J, y_{n-1}y_n, x_n) \xrightarrow{y_{n-1}y_n} R/(I^2, x_n y_n) \longrightarrow R/K \longrightarrow 0. \quad (3)$$

To estimate the depth of the module on the left, we use the exact sequence

$$0 \longrightarrow R/(L, y_{n-1}) \xrightarrow{y_n} R/(J, y_{n-1}y_n) \longrightarrow R/(J, y_n) \longrightarrow 0 \quad (4)$$

Taking into account that L and J are attached to CM forests, we have that the left end of the last sequence has depth equal to $n+1$ while its right end has depth equal to n. Hence a combination of (3) and (4) gives

$$\operatorname{depth}(R/(I^2, x_n y_n)) \geq n-1.$$

To complete the proof observe that from the sequence

$$0 \longrightarrow R/I \xrightarrow{x_n y_n} R/I^2 \longrightarrow R/(I^2, x_n y_n) \longrightarrow 0 \quad (5)$$

we get $\operatorname{depth}(R/I^2) \geq n-1$. □

Exercises

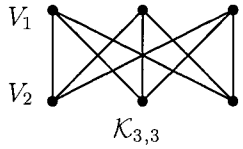
6.3.8 Let T be a tree with $2r$ vertices. If $r \geq 2$, prove that T is Cohen-Macaulay iff $\operatorname{ht} I(T) = r$ and T has exactly r vertices of degree 1.

6.3.9 Let G be a graph with vertex set $V = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and edge set $X = \{\{x_k, y_k\}, \{y_i, y_j\} | \, k = 1, \ldots, n \text{ and } 1 \leq i < j \leq n\}$. Show that G is a saturated graph over any field k.

6.4 Bipartite graphs

Let G be a Cohen-Macaulay bipartite graph and $I = I(G)$ its edge ideal. We prove that $G \setminus \{v\}$ is Cohen-Macaulay for some vertex v in G. Then we show that the Stanley-Reisner simplicial complex of I is shellable.

Cohen-Macaulay Bipartite graphs A graph G is a *complete bipartite graph* if its vertex set V can be partitioned into disjoint subsets V_1 and V_2 such that G contains every edge joining V_1 and V_2. If V_1 and V_2 have m and n vertices respectively, we denote such a complete bipartite graph by $\mathcal{K}_{m,n}$. In the following illustration V_1 (resp. V_2) is the set of vertices in the top (resp. bottom) row.



$\mathcal{K}_{3,3}$

Lemma 6.4.1 The complete bipartite graph $\mathcal{K}_{m,n}$ is Cohen-Macaulay if and only if $m = n = 1$.

Proof. By Exercise 5.3.22 it is enough to observe that the complementary complex of $\mathcal{K}_{m,n}$ is disconnected for $m + n \geq 3$. □

Lemma 6.4.2 *Let G be an unmixed bipartite graph and $I(G)$ its edge ideal. If $I(G)$ has height g, then there are disjoint sets of vertices $V_1 = \{x_1, \ldots, x_g\}$ and $V_2 = \{y_1, \ldots, y_g\}$ such that:* (i) *$\{x_i, y_i\}$ is an edge of G for all i, and* (ii) *every edge of G joins V_1 with V_2.*

Proof. The statement follows rapidly if one uses that g is equal to the maximum number of independent edges of G, see Theorem 6.1.7. □

Definition 6.4.3 Let G be a graph and $v \in V(G)$. The *neighbor set* $N(v)$ of v is the set of vertices of G adjacent to v.

Theorem 6.4.4 *Let G be a Cohen-Macaulay bipartite graph. If G is not a discrete graph, then there is a vertex $v \in V(G)$ such that* $\deg(v) = 1$.

Proof. We may harmlessly assume that G has no isolated vertices. Set $g = \operatorname{ht} I(G)$. Let V_1 and V_2 be as in Lemma 6.4.2. We proceed by contradiction. Assume $\deg(v) \geq 2$ for all $v \in V_1$. Let $v \in V_1$ be a vertex of minimal degree. We may assume $v = x_1$. If $\deg(v) = g$, then $G = \mathcal{K}_{g,g}$ is a complete bipartite graph, but this is impossible by Lemma 6.4.1. Therefore we may also assume $2 \leq \deg(x_1) \leq g - 1$. Set $r = \deg(x_1)$. Let $N(x_1)$ be the set of vertices in V adjacent to x_1, say $N(x_1) = \{y_1, \ldots, y_r\}$.

We claim that $\deg(x_i) = r$, for all $i = 1, \ldots, r$. For simplicity we assume $i = 2$. If $\deg(x_2) > r$, then x_2 is adjacent to y_j for some $r+1 \leq j \leq g$. Let $G' = G \setminus \{x_1, y_1, \ldots, y_r\}$. By Proposition 6.2.4 we derive that G' is C-M, and hence G' is unmixed. Notice that $\{y_{r+1}, \ldots, y_g\}$ is a minimal vertex cover for G' and there is also a minimal vertex cover for G' containing $\{x_2, x_{r+1} \ldots, x_g\}$, which is a contradiction since G' is unmixed. Thus $\deg(x_2) = r$. The previous argument also shows that $N(x_i) = \{y_1, \ldots, y_r\}$, for all $1 \leq i \leq r$. Hence $N(y_i)$ is a subset of $\{x_{r+1}, \ldots, x_g\}$, for all $r+1 \leq i \leq g$. Consider the graph

$$H = G \setminus (\{y_{r+1}, \ldots, y_g\} \cup N(y_{r+1}) \cup \cdots \cup N(y_g)).$$

To complete the proof observe that a repeated use of Proposition 6.2.4 shows that H is a C-M bipartite graph. This is impossible since $H = \mathcal{K}_{r,r}$ is a complete bipartite graph and $r \geq 2$. □

Corollary 6.4.5 *If G is a Cohen-Macaulay bipartite graph, then $G \setminus \{v\}$ is C-M for some vertex v in G.*

Proof. Let $\{v, z\}$ be an edge of the graph G such that $\deg(v) = 1$. Then by Proposition 6.2.1 we obtain that $G \setminus \{z\}$ is C-M. □

The results above yield a recursive algorithm to construct all the C-M bipartite graphs with a given number of vertices, see exercises.

Shellability The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. Recall that the faces of Δ_G, the complementary simplicial complex of G, are precisely the independent sets of G (together with the empty set). Notice that the *facets* (faces of maximal dimension) of Δ_G are the maximal independent sets of G.

Let G be a graph on the vertex set $V = \{z_1, \ldots, z_n, x, y\}$. Assume that x is adjacent to y and $\deg(y) = 1$. We label G so that $y, z_1, \ldots, z_k$ are the vertices of G adjacent to x.

Lemma 6.4.6 *Let $H = G \setminus \{x, y\}$ and $A \subset V$. If G is Cohen-Macaulay, then A is a facet of Δ_G if and only if A can be written as*

(a) *$A = F \cup \{y\}$, where $F \subset V(H)$ is a facet of Δ_H, or*

(b) *$A = F \cup \{x\}$, where $F \subset V(H)$ is a facet of Δ_H and $z_1, \ldots, z_k$ are not in F.*

Proof. Let A be a facet of Δ_G. Suppose $x \notin A$, then $y \in A$. Set $W = V(H)$ and $F = A \setminus \{y\} \subset W$. We must show that F is a facet of Δ_H. First notice that F is an independent set of H, because $E(H) \subset E(G)$. Next let us show that F is not contained in any other independent set of H. Let $v \in W \setminus F$.

Since A is a facet of Δ_G, we obtain that there is an edge L in $E(G)$ so that $L \subset A \cup \{v\}$. As $x \notin A$ and $\deg(y) = 1$, we have $L \in E(H)$ and $L \subset F \cup \{v\}$. Hence F is a facet of Δ_H and we are in case (a). On the other hand if $x \in A$, then we have $y, z_1, \ldots, z_k \notin A$. Set $F = A \setminus \{x\}$. Let C be a facet of Δ_H containing F. Since $C \cup \{y\}$ is an independent set of G, and G is C-M, we obtain that $|C| + 1 \leq |A| = |F| + 1$. Hence $C = F$ and we are in case (b), as required. The converse also follows readily using similar arguments. □

Theorem 6.4.7 *Let G be a Cohen-Macaulay graph. If G is bipartite, then the Stanley-Reisner simplicial complex Δ_G of $I(G)$ is shellable.*

Proof. We will use the shellability criterion of Exercise 5.3.29. One may assume that G has at least one vertex of positive degree, for otherwise Δ_G is a simplex with a unique facet. Set $g = \operatorname{ht} I(G)$. We argue by induction on g. Let V_1 and V_2 be as in Lemma 6.4.2. By Theorem 6.4.4 we may assume that $\deg(y_1) = 1$. Let us label the vertices of G so that $y_2, \ldots, y_t$ be the vertices of G (if any) adjacent to x_1. According to Proposition 6.2.1 the graph $H = G \setminus \{x_1, y_1\}$ is C-M, hence by induction Δ_H is shellable. Let $F_1, \ldots, F_m$ be a shelling of Δ_H. By Lemma 6.4.6 there are integers $1 \leq r_1 < \cdots < r_s \leq m$ so that the facets of Δ_G can be linearly ordered as follows

$$F_1 \cup \{y_1\}, \ldots, F_m \cup \{y_1\}, F_{r_1} \cup \{x_1\}, \ldots, F_{r_s} \cup \{x_1\},$$

where $F_{r_1}, \ldots, F_{r_s}$ are the facets of H not containing $y_2, \ldots y_t$ (or equivalently containing $x_2, \ldots, x_t$). We claim that the subcomplex Δ' of Δ_G generated by $F_{r_1} \cup \{x_1\}, \ldots, F_{r_s} \cup \{x_1\}$ is shellable, with shelling given by $F_{r_1} \cup \{x_1\}, \ldots, F_{r_s} \cup \{x_1\}$. To show the claim assume $1 \leq j < i \leq s$. Since Δ_H is shellable there is $v \in F_{r_i} \setminus F_{r_j}$ and $\ell < r_i$ so that $F_{r_i} \setminus F_\ell = \{v\}$. It follows that $x_2, \ldots, x_t \in F_\ell$. Hence $F_\ell = F_{r_k}$, for some $1 \leq k < i$. Therefore Δ' is shellable. It now follows readily that the linear ordering of the facets of Δ_G given above is a shelling for Δ_G. □

Computing the type Let G be a C-M bipartite graph on the vertex set V. Assume G has no isolated vertices. There are disjoint sets V_1 and V_2 so that $V = V_1 \cup V_2$ and every edge of G joins V_1 with V_2. Since V_1 and V_2 are minimal vertex covers for G, and G is unmixed, we have $g = |V_1| = |V_2| = \operatorname{height} I(G)$. By König's theorem there are g independent edges. Therefore we may assume

$$V_1 = \{y_1, \ldots, y_g\},\ V_2 = \{w_1, \ldots, w_g\},\ \text{and } y_i w_i \in I(G) \text{ for all } i.$$

Notice that $\mathbf{w} = \{y_i - w_i\}_{i=1}^g$ is a regular system of parameters for $R/I(G)$, where $R = k[\mathbf{y}, \mathbf{w}]$, and $R/I(G)$ modulo $(\mathbf{w})$ reduces to:

$$A = k[\mathbf{y}]/(y_1^2, \ldots, y_g^2, I(H)),$$

where H is a graph on the vertex set V_1. The algebra A is generated, as a k-vector space, by the image of 1 and the images of all the monomials $y_{i_1} \cdots y_{i_r}$ such that $\{y_{i_1}, \ldots, y_{i_r}\}$ is an independent set of H. Observe that the type of $R/I(G)$ is equal to $|\Upsilon(H)|$, because $\operatorname{Soc}(A)$ is generated by the images of all the monomials $y_{i_1} \cdots y_{i_r}$ such that $\{y_{i_1}, \ldots, y_{i_r}\}$ is a maximal independent set of H.

Example 6.4.8 Let G be the C-M bipartite graph whose edge ideal is:

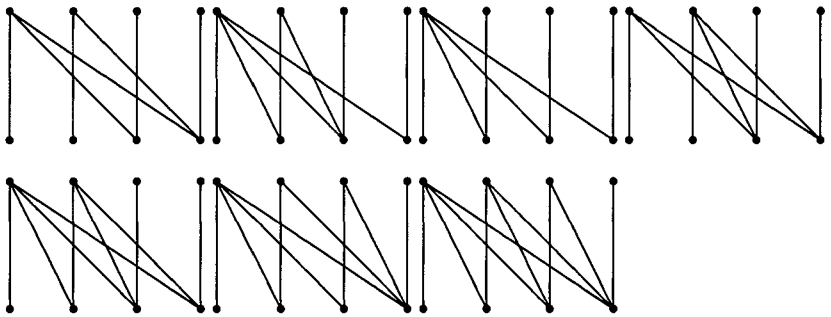
$$I(G) = (y_1w_1, y_1w_2, y_1w_3, y_1w_4, y_2w_2, y_2w_3, y_2w_4, y_3w_3, y_4w_4).$$

The graph H obtained by reduction has edges $y_1y_2, y_1y_3, y_1y_4, y_2y_3, y_2y_4$. Since this graph has only three minimal vertex covers, the type of $R/I(G)$ is equal to three.

Exercises

6.4.9 Let G be a C-M bipartite graph. Show that the h-vector of $R/I(G)$ has length at most $g = \operatorname{ht} I(G)$. Can we replace C-M by unmixed?

6.4.10 Prove that the following is the full list of Cohen-Macaulay connected bipartite graphs with eight vertices, in each case determine the type and the multiplicity of the edge ring.



6.4.11 Let $I = (x_iy_j | \, 1 \leq i \leq j \leq n)$, draw a picture of the bipartite graph G such that $I = I(G)$ and prove that I is a Cohen-Macaulay ideal.

6.4.12 Let G be a Cohen-Macaulay bipartite graph. If G is not a discrete graph, prove that G has at least two vertices of degree one.

6.5 Links of some edge ideals

In this section we give graph theoretical interpretations for the links of certain ideals attached to graphs.

Let G_0 be a graph on the vertex set $V = \{y_1, \ldots, y_n\}$ and $I = I(G_0)$. Take a new set of variables $x_1, \ldots, x_n$ and define $S(G_0)$ to be the new graph obtained by attaching to each vertex y_i a new vertex x_i and the edge $\{x_i, y_i\}$. We set

$$K = I(S(G_0)) = (x_1y_1, \ldots, x_ny_n, I) \subset R = k[\mathbf{x}, \mathbf{y}],$$

where k is a field. The S in the notation is intended for *suspension* or attaching *whiskers* to the base graph. The significance of this construction lies partly in the following restatement of Theorem 6.3.4:

Theorem 6.5.1 *If G is a graph, then G is a Cohen-Macaulay tree if and only if $G = S(G_0)$ for some tree G_0.*

Of course, all the structure of $S(G_0)$ lies in the graph G_0. The following simple computation conforms to this view.

Proposition 6.5.2 *Let G_0 be a graph and $G = S(G_0)$ its suspension. Then the multiplicity of $R/I(G)$ is*

$$\sum_{i=-1}^{d} f_i,$$

where f_i is the number of i-faces of the Stanley-Reisner complex of $I(G_0)$.

Proof. Let $y_1, \ldots, y_n$ be the vertices of G_0. By Proposition 6.3.2, the set of linear forms $\{y_i - x_i\}_{i=1}^n$ is a regular system of parameters for $R/I(G)$. The reduction of $R/I(G)$ modulo these forms is

$$k[\mathbf{y}]/(y_1^2, \ldots, y_n^2, I(G_0)),$$

whose k-vector space basis is formed by the standard monomials, that is, the faces of the attached simplicial complex. □

The next proposition describes the first link of $S(G_0)$ relative to the regular sequence $\mathbf{z} = \{x_1y_1, \ldots, x_ny_n\}$.

Proposition 6.5.3 *Let G_0 and G_1 be two copies of the same graph on disjoint sets of vertices $\{y_1, \ldots, y_n\}$ and $\{x_1, \ldots, x_n\}$ respectively. Let Υ be the set of minimal vertex covers of G_1 and*

$$L = (\{x_{i_1} \cdots x_{i_s} \,|\, \{x_{i_1}, \ldots, x_{i_s}\} \in \Upsilon(G_1)\}).$$

Then $((\mathbf{z}) : K) = (\mathbf{z}, L)$, where $K = I(S(G_0))$.

Proof. Take $\{x_{i_1}, \ldots, x_{i_r}\} \in \Upsilon$ and $y_ky_\ell \in I$. Since $i_s \in \{k, \ell\}$ for some s we obtain $x_{i_1} \cdots x_{i_r}y_ky_\ell \in (\mathbf{z})$. This shows $(\mathbf{z}, L) \subset ((\mathbf{z}) : K)$. Conversely, assume M is a monomial and $M \in ((\mathbf{z}) : K) \setminus (\mathbf{z})$. Let $\{y_k, y_\ell\}$ be any edge in G_0. Since $My_ky_\ell = x_ty_tM_1$, either x_k divides M or x_ℓ divides M; in either case, $M \in L$. □

Corollary 6.5.4 *The type of* $R/I(S(G_0))$ *is equal to* $|\Upsilon(G_0)|$.

Proof. Since R/K is a Cohen-Macaulay ring and $\mathbf{z}$ is a maximal regular sequence in K, where $K = I(S(G_0))$, using Proposition 1.3.7 one has:

$$\operatorname{Ext}_R^n(R/K, R) \simeq \operatorname{Hom}_R(R/K, R/(\mathbf{z})) \simeq ((\mathbf{z}): K)/(\mathbf{z}).$$

Thus by Corollary 4.3.6 one obtains $\operatorname{type}(R/K) = \nu(((\mathbf{z}): K)/(\mathbf{z}))$ and the assertion follows from Proposition 6.5.3. □

Observe that these results can be used to compute the primary decomposition of an edge ideal.

Example 6.5.5 Let $I(G_0) = (y_1y_3, y_1y_4, y_2y_4, y_2y_5, y_3y_5, y_3y_6, y_4y_6)$. Note that the ideal $((\mathbf{z}): I(S(G_0))$ is generated by $\mathbf{z} = \{x_1y_1, \ldots, x_6y_6\}$ together with:

$$x_2x_3x_4,\ x_3x_4x_5,\ x_1x_2x_5x_6,\ x_1x_2x_3x_6,\ x_1x_4x_5x_6,$$

and these monomials correspond to the minimal primes of $I(G_0)$. The type of $R/I(S(G_0))$ is 5 and the multiplicity of the face ring of $I(G_0)$ is 2.

Definition 6.5.6 A *star* on n vertices is a complete bipartite graph of the form $\mathcal{K}_{1,n-1}$ and is denoted by $\operatorname{star}(n)$.

Corollary 6.5.7 *If* $G_0 = \operatorname{star}(n)$ *is the star on* n *vertices, then* $I(S(G_0))$ *lies in the linkage class of a complete intersection.*

Proof. Let $y_1, \ldots, y_n$ be the vertices of G_0 and y_1 its center; then the only minimal vertex covers of G_1 are $\{x_1\}$ and $\{x_2, \ldots, x_n\}$, so that, in the notation above, the first link is

$$\begin{aligned} J &= ((\mathbf{z}): I(S(G_0))) = (x_1y_1, \ldots, x_ny_n, x_1, x_2 \cdots x_n) \\ &= (x_1, x_2y_2, \ldots, x_ny_n, x_2 \cdots x_n). \end{aligned}$$

Take now its link with respect to the regular sequence formed by the first n elements:

$$(x_1, x_2y_2, \ldots, x_ny_n): J = (x_1, y_2, \ldots, y_n),$$

which is a complete intersection. □

Exercises

6.5.8 Let G_0 and G_1 be two copies of the same graph on disjoint sets of vertices $\{y_1, \ldots, y_n\}$ and $\{x_1, \ldots, x_n\}$ respectively. Let Υ be the set of minimal vertex covers of G_1 and

$$L = (\{x_{i_1} \cdots x_{i_s} \mid \{x_{i_1}, \ldots, x_{i_s}\} \in \Upsilon(G_1)\}).$$

Then $(K, \mathbf{z})$ and $(\mathbf{z}, L)$ are geometrically linked, where $K = I(S(G_0))$.

6.5.9 Let $I = (x_iy_j \mid 1 \leq i \leq j \leq n) \subset R = k[\mathbf{x}, \mathbf{y}]$. Prove that R/I has type n and its a-invariant is $-(n-1)$.

6.6 First syzygy module of an edge ideal

Let G be a graph and $I(G) \subset R$ its edge ideal. An interesting problem is to express some of the first initial Betti numbers of $I(G)$ in terms of the graph theoretical data of G; below we give an indication that this might be possible, see [103] for further information.

Let R be a polynomial ring over a field K with the usual grading and I a graded ideal of R. The minimal graded resolution of R/I by free R-modules can be expressed as:

$$0 \longrightarrow \bigoplus_{i=1}^{c_g} R^{b_{gi}}(-d_{gi}) \xrightarrow{\varphi_g} \cdots \longrightarrow \bigoplus_{i=1}^{c_1} R^{b_{1i}}(-d_{1i}) \xrightarrow{\varphi_1} R \longrightarrow R/I \longrightarrow 0,$$

where all the maps are degree preserving and the d_{ij} are positive integers. One may assume $d_{j1} < \cdots < d_{jc_j}$ for all $j = 1, \ldots, g$. The integer g is equal to $\mathrm{pd}_R(R/I)$, the *projective dimension* of R/I. To simplify notation set

$$F_j = \bigoplus_{i=1}^{c_j} R^{b_{ji}}(-d_{ji}).$$

The rank of F_j is the *jth Betti number* of I. From the minimality of the resolution $d_{11} < \cdots < d_{g_1}$, see Remark 2.5.9. We define the *jth initial Betti number* of I as $\gamma_j = b_{j1}$ and the *jth initial virtual Betti number* of I as

$$v_j = \dim_K (F_j)_{d_{11}+j-1}.$$

Note that some of the initial virtual Betti numbers may be zero.

For a general monomial ideal I there is an explicit and elegant description of a minimal set of generators for the first module of syzygies of I due to S. Eliahou [101]; see also [43].

Definition 6.6.1 The *edge graph* of G, denoted by $L(G)$, has vertex set equal to $E = E(G)$ with two vertices of $L(G)$ adjacent whenever the corresponding edges of G have exactly one common vertex.

Proposition 6.6.2 *If G is a graph with vertices $x_1, \ldots, x_n$ and edge set $E(G)$, then the number of edges of the edge graph $L(G)$ is given by*

$$|E(L(G))| = \sum_{i=1}^{n} \binom{\deg(x_i)}{2} = -|E(G)| + \sum_{i=1}^{n} \frac{\deg^2 x_i}{2}, \qquad (6.1)$$

Proof. To prove the first equality note that each vertex x_i of G of degree d_i contributes with $\binom{d_i}{2}$ edges to the number of edges of $L(G)$.

The second equality follows from the Euler's identity

$$2|E(G)| = \sum_{i=1}^{n} \deg(x_i),$$

see Exercise 6.1.27. □

Let us now present a formula, in graph theoretical terms, for the second initial Betti number of $I(G)$.

Proposition 6.6.3 ([103]) *Let $I = I(G) \subset R$ be the edge ideal of a graph G and V the vertex set of G. If*

$$\cdots \longrightarrow R^c(-4) \oplus R^b(-3) \longrightarrow R^q(-2) \xrightarrow{\psi} R \longrightarrow R/I \longrightarrow 0$$

is the minimal graded resolution of R/I. Then $b = |E(L(G))| - N_t$, where N_t is the number of triangles of G and $L(G)$ is the edge graph of G.

Proof. Assume $I = (f_1, \ldots, f_q)$ and $\psi(e_i) = f_i$. Let Z_1' be the set of elements in $\ker(\psi)$ of degree 3. We may regard the f_i as the vertices of $L(G)$. Every edge $e = \{f_i, f_j\}$ in $L(G)$ determines the syzygy $\text{syz}(e) = x_j e_i - x_i e_j$, where $f_i = x_i z$ and $f_j = x_j z$ for some z, x_i, x_j in V. According to Theorem 2.4.17 the set of those syzygies generate Z_1'. Given a triangle $C_3 = \{z_1, z_2, z_3\}$ in G we set

$$\phi(C_3) = \{z_1 e_j - z_3 e_i, z_1 e_j - z_2 e_k, z_2 e_k - z_3 e_i\},$$

where $f_i = z_1 z_2$, $f_j = z_2 z_3$, and $f_k = z_1 z_3$. Notice that $\phi(C_3) \cap \phi(C_3') = \emptyset$ if $C_3 \neq C_3'$. For every triangle C_3 in G choose an element $\rho(C_3) \in \phi(C_3)$. It is not hard to show that the set

$$B = \{\text{syz}(e) | \, e \in L(G)\} \setminus \{\rho(C_3) | \, C_3 \text{ is a triangle in } G\}$$

is a minimal generating set for Z_1'. □

It is a fact that the second Betti number of a Stanley-Reisner ideal (resp. monomial ideal) is independent of the field [169] (resp. [43, 101]), moreover for edge ideals the third and fourth Betti numbers are also independent of the field [168].

An application to Hilbert series Let Δ be a simplicial complex of dimension d and $f = (f_0, \ldots, f_d)$ its f-vector, where f_i is the number of faces of dimension i in Δ and $f_{-1} = 1$. By Corollary 5.4.5 the Hilbert Series of the Stanley-Reisner ring $S = R/I_\Delta$ can be expressed as

$$F(z) = f_{-1} + \frac{f_0 z}{(1-z)} + \cdots + \frac{f_d z^{d+1}}{(1-z)^{d+1}}. \tag{6.2}$$

We now use this formula to compute a particular instance.

Corollary 6.6.4 *Let G be a graph with q edges, $V = \{x_1, \ldots, x_n\}$ its vertex set and $F(z)$ the Hilbert series of $S = R/I(G)$. If S has dimension 3, then $F(z)(1-z)^3$ is equal to*

$$1 + gz + \left(\binom{g+1}{2} - q\right) z^2 + \left(\frac{2g + 3g^2 + g^3 - 6q - 6gq - 6N_t + 3v}{6}\right) z^3,$$

where g is the height of $I(G)$, N_t is the number of triangles of G, and

$$v = \sum_{i=1}^{n} \deg^2 x_i.$$

Proof. Let $\Delta = \Delta_G$ be the Stanley-Reisner complex of $I = I(G)$. The first entries of the f-vector of Δ are given by

$$f_{-1} = 1, \quad f_0 = n, \quad f_1 = \frac{n(n-1)}{2} - q.$$

To compute f_2 notice that the resolution above yields

$$\dim(R/I)_3 = \binom{n+2}{n-1} - qn + b$$

and

$$f_2 = \binom{n+2}{n-1} - qn + b - 2f_1 - n,$$

where the last expression for f_2 uses Proposition 5.4.4. The desired formula follows by substitution of the f_i in Eq. (6.2) and using Eq. (6.1). □

The number of triangles of a graph Let G be a graph with q edges and vertex set $V = \{x_1, \ldots, x_n\}$. As the number of edges of the edge graph $L(G)$ of G is given by the formula:

$$|E(L(G))| = \sum_{i=1}^{n} \binom{\deg(x_i)}{2} = -q + \sum_{i=1}^{n} \frac{\deg^2 x_i}{2},$$

observe that Proposition 6.6.3 provides a method to compute the number of triangles of a graph G by computing syzygies with *Macaulay* or *CoCoA*. An alternative method using linear algebra is recalled next.

The *adjacency matrix* of G is the $n \times n$ matrix $A = (a_{ij})$ whose entries are given by

$$a_{ij} = \begin{cases} 1 & \text{if } x_i \text{ is adjacent } x_j, \\ 0 & \text{otherwise.} \end{cases}$$

It follows directly that A is symmetric and that its trace is equal zero. If

$$f(x) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + c_3 x^{n-3} + \cdots + c_n,$$

is the characteristic polynomial of A, then $c_1 = 0$, $-c_2$ is the number of edges of G, and $-c_3$ is twice the number of triangles of G. For an interpretation of all the c_i's consult [24].

Let G be the graph whose complement is the union of two disjoint copies of a path of length two. The edge ideal of G is equal to:

$$I(G) = (x_1x_4, x_1x_5, x_1x_6, x_2x_3, x_2x_4, x_2x_5, x_2x_6, x_3x_4, x_3x_5, x_3x_6, x_5x_6),$$

and part of its minimal resolution is:

$$\cdots \to R^{24}(-3) \to R^{11}(-2) \to R \to R/I(G) \to 0.$$

Hence the number of triangles N_t of G is equal to 6. Of course the same number is obtained noticing that

$$f(x) = x^6 - 11x^4 - 12x^3 + 3x^2 + 4x - 1$$

is the characteristic polynomial of the adjacency matrix of G.

Exercises

6.6.5 Let G be a graph with vertex set $\mathbf{x}$ and $F_G(z)$ the Hilbert series of $k[\mathbf{x}]/I(G)$, where k is a field. For any vertex $x \in V(G)$ one has:

$$F_G(z) = F_{G\setminus\{x\}}(z) + \frac{z}{1-z}F_{G\setminus(N(x)\cup\{x\})}(z), \quad \text{with } F_\emptyset(z) = 1.$$

6.6.6 If $K_{r,s}$ is a complete bipartite graph, then

$$F_{K_{r,s}}(z) = \frac{1}{(1-z)^r} + \frac{1}{(1-z)^s} - 1.$$

6.6.7 ([309]) Let P_n be a path graph with vertices $\mathbf{x} = x_1, \ldots, x_n$. Prove that the Hilbert series of $k[\mathbf{x}]/I(P_n)$ can be express as:

$$F_{P_n}(z) = \sum_{j=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-j+1}{j} \sum_{i=0}^{\infty} \binom{j+i-1}{i} z^{j+i}$$

6.6.8 Let P_n be a path graph with vertices $\mathbf{x} = x_1, \ldots, x_n$. Prove that the Hilbert function of $k[\mathbf{x}]/I(P_n)$ is given by

$$H_{L_n}(m) = \sum_{j=0}^{\min\left(m, \lfloor \frac{n+1}{2} \rfloor\right)} \binom{n-j+1}{j}\binom{m-1}{m-j}.$$

Notice that for $m \geq \lfloor \frac{n+1}{2} \rfloor$ this is the Hilbert polynomial.

6.6.9 Let $G = C_7$ be an heptagon. Prove the formula:

$$F_G(z) = \frac{-z^3 + 3z^2 + 4z + 1}{(1-z)^3}.$$

What is the number of minimal primes of $I(G)$?

6.6.10 Let $G = C_7$ be an heptagon and Δ_G its complementary simplicial complex. Prove that Δ_G is a triangulation of a Möbius band whose reduced Euler characteristic is $\widetilde{\chi}(\Delta_G) = -1$.

6.6.11 Let $I = I(G) \subset R$ be the edge ideal of a graph G. If

$$\cdots \longrightarrow R^c(-4) \oplus R^b(-3) \longrightarrow R^q(-2) \xrightarrow{\psi} R \longrightarrow R/I \longrightarrow 0$$

is the minimal graded resolution of R/I. Prove that c is equal to the number of unordered pairs of lines $\{f_i, f_j\}$ such that f_i and f_j are independent lines that cannot be joined by an edge.

6.7 Edge rings with linear resolutions

A noteworthy use of graph theoretical methods in commutative algebra is implicit in Lyubeznik thesis [212] and explicit in the work of R. Fröberg [116]. In both cases the central notion is that of a triangulated graph. It will turn out that this is the right notion to express some homological properties of edge ideals and the Cohen-Macaulay property of unmixed ideals of height two generated by square-free monomials.

Given a graph G an interesting duality between $I(G)$ and $I_c(G)$, the ideal of covers of G, will take place in this section. Under special circumstances $I(G)$ reflects properties of $I_c(G)$ and viceversa.

The Cohen-Macaulay case of dimension two Let us show a family of flag complexes where shellability is equivalent to Cohen-Macaulayness. We begin with:

Lemma 6.7.1 *Let Δ be a simplicial complex of dimension 2 on the vertex set V. If Δ is Cohen-Macaulay, then there are facets $F_1, \ldots, F_s$ such that*

(i) $V = \cup_{i=1}^{s} F_i$,

(ii) $|F_1 \cup \cdots \cup F_i| = 2 + i$ *for* $1 \leq i \leq s$, *and*

(iii) $(F_1 \cup \cdots \cup F_{k-1}) \cap F_k$ *is a face in* Δ *of dimension* 1 *for* $k \geq 2$.

Proof. We will proceed by induction on k. Assume that there are facets $F_1, \ldots, F_k$ in Δ with $|F_1 \cup \cdots \cup F_i| = 2 + i$ for $1 \leq i \leq k$, and such that $(F_1 \cup \cdots \cup F_{i-1}) \cap F_i$ has dimension 1 for $2 \leq i \leq k$. Set $U_i = F_1 \cup \cdots \cup F_i$.

Assume $U_k \neq V$, otherwise the proof is complete. Choose $F \not\subset U_k$ so that $|F \cap U_k|$ has maximal dimension. Set $G = F \cap U_k$ and $r = |G|$. We will show $r = 2$. Assume $r = 1$, i.e., $G = \{p\}$. Notice that $p \in F_i \subset U_k$ for some $i \leq k$. We pick $z_0 \in F \setminus U_k$ and $x_0 \in F_i$, $x_0 \neq p$. Observe that z_0 and x_0 are both in $\operatorname{lk}(G)$ and that $\operatorname{lk}(G)$ is connected, hence there is a path

$$\{z_0 = w_0, \ldots, w_m = x_0\}$$

in $\operatorname{lk}(G)$ joining z_0 and x_0. We pick j so that $w_j \notin U_k$ and $w_{j+1} \in U_k$. Since $\{w_j, w_{j+1}\} \in \operatorname{lk}(G)$ we obtain that $F' = \{w_j, w_{j+1}, p\}$ is a facet of Δ so that $F' \not\subset U_k$ and $F' \cap U_k = \{p, w_{j+1}\}$, which contradicts the choice of F. Thus $r = 2$, and we can set $F = F_{k+1}$ to complete the induction. □

Proposition 6.7.2 ([116]) *Let Δ be a simplicial complex of dimension 2. If the Stanley-Reisner ring $k[\Delta]$ has a 2-linear resolution, then Δ is a Cohen-Macaulay complex if and only if Δ is shellable.*

Proof. Assume that $k[\Delta]$ is a Cohen-Macaulay ring with a 2-linear resolution. The f-vector of Δ is given by $f_0 = n$, $f_1 = 2n - 3$ and $f_2 = n - 2$, where n is the number of vertices of Δ (see Exercise 6.7.17).

By Lemma 6.7.1 there are facets $F_1, \ldots, F_s$ in Δ so that $V = \cup_{i=1}^{s} F_i$, $|F_1 \cup \cdots \cup F_i| = 2 + i$ for $1 \leq i \leq s$, and $(F_1 \cup \cdots \cup F_{k-1}) \cap F_k$ is a face of dimension 1 of Δ for $2 \leq k \leq s$. Notice that in our case $s = n - 2$, hence $F_1, \ldots, F_s$ are precisely the facets of Δ. Set

$$U_i = F_1 \cup \cdots \cup F_i \text{ and } \Delta_i = \overline{F}_1 \cup \cdots \cup \overline{F}_i,$$

where $\overline{F}_i = \{\sigma \in \Delta \,|\, \sigma \subset \overline{F}_i\}$. We claim that $\Delta_i = \overline{U}_i$. Since Δ_i is a subcomplex of $\overline{U}_i$, it suffices to show that they have the same f-vector. Observe that

$$f_1(\overline{U}_i) + 2 \leq f_1(\overline{U}_{i+1})$$

and

$$2i + 1 \leq f_1(\Delta_i) \leq f_1(\overline{U}_i).$$

Using $f_1(\overline{U}_s) = 2n - 3$ yields $f_1(\overline{U}_i) = f_1(\Delta_i) = 2i + 1$ for all i. A similar argument shows $f_2(\Delta_i) = f_2(\overline{U}_i)$ for all i.

To complete the proof notice that

$$G = F_k \cap U_{k-1} \in \overline{F}_k \cap \Delta_i = \overline{F}_k \cap \overline{U}_{k-1},$$

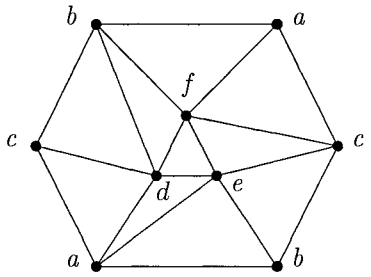
accordingly $G \subset F_k \cap F_j$ for some $j < k$, using that $|G| = 2$ we get

$$F_k \cap F_j = F_k \setminus \{x\}$$

for some $x \in V$. Thus Δ is shellable by Exercise 5.3.29. The converse is true quite generally (see Theorem 5.3.18). □

Corollary 6.7.3 *If $k[\Delta]$ is a Cohen-Macaulay ring of dimension 3 with a 2-linear resolution over a field k, then $K[\Delta]$ is a Cohen-Macaulay ring with a 2-linear resolution for any other field K.*

Example 6.7.4 Let $R = k[a, \ldots, f]$ be a polynomial ring over a field k and Δ the following triangulation of the real projective plane $\mathbb{P}^2$:



Note that $I_\Delta = (abc, abd, ace, adf, aef, bcf, bde, bef, cde, cdf)$. The face ring $k[\Delta]$ was studied by Reisner [237]. One has that $k[\Delta]$ is Cohen-Macaulay and has a 3-linear resolution if k has characteristic other than 2, and is not Cohen-Macaulay and has a non linear resolution otherwise (compare with [114, Example 3]). Hence Δ is not shellable and $\dim \Delta = 2$.

The general case Let G be a graph and $\overline{G}$ its complement. The graph $\overline{G}$ is said to be *triangulated* or *chordal* if every cycle C_n in $\overline{G}$ of length $n \geq 4$ has a chord in $\overline{G}$. A *chord* of C_n is an edge joining two non adjacent vertices of C_n.

The following result provides a graph theoretical characterization of the rings $k[\Delta]$ having a 2-linear resolution.

Theorem 6.7.5 ([116]) *Let G be a graph and Δ the Stanley-Reisner complex of $I(G)$. Then $k[\Delta]$ has a 2-linear resolution if and only if $\overline{G}$ is a triangulated graph.*

Next we introduce the notion of d-tree and state an interesting consequence due to R. Fröberg.

Definition 6.7.6 A *d-tree* is defined inductively: (i) a complete graph K_{d+1} is a d-tree, (ii) let G be a d-tree and H a subgraph of G with $H \simeq K_d$, if v is a new vertex connected to all vertices in H, then $G \cup \{v\}$ is a d-tree.

Theorem 6.7.7 ([116]) *Let G be a graph and Δ the Stanley-Reisner complex of $I(G)$. If G is Cohen-Macaulay over a field k, then $k[\Delta]$ has a 2-linear resolution if and only if $\overline{G}$ is a d-tree, where $d = \dim \Delta$.*

Corollary 6.7.8 *Let G be a connected Cohen-Macaulay bipartite graph with q edges. If q is equal to $g(g+1)/2$, where $g = \operatorname{ht} I(G)$, then*

$$I(G) = (x_i y_j | \, 1 \leq i \leq j \leq g).$$

Proof. First notice that $R/I(G)$ has a linear resolution by Theorem 4.3.7. Hence, according to Theorem 6.7.7, the 1-skeleton of Δ_G is a $(g-1)$-tree, which implies that there is a vertex w of G of degree g. We now use the notation (and similar arguments) of the proof of Theorem 6.4.4 to conclude that $\deg(x_i) = 1$ and $\deg(y_j) = 1$, for some i, j. To finish the proof consider the graph $G \setminus \{w\}$ and use induction. □

The more general problem of determining the simplicial complexes Δ whose Stanley-Reisner ideal I_Δ has a pure resolution has been addressed by W. Bruns and T. Hibi [46, 47], where a very concrete solution can be found if I_Δ is generated by square-free monomials of degree two.

Edge ideals and their ideals of vertex covers Triangulated graphs have been extensively studied, they can be constructed according to a result of G. A. Dirac, see [84, 285]. Those graphs have appeared in the commutative algebra literature in the works of R. Froberg [116] and G. Lyubeznik [213], here we point out a connection between these two works that leads to an interaction between some properties of an edge ideal and some properties of the ideal of minimal covers of the graph.

Lemma 6.7.9 *Let G be a connected graph with vertex set $V(G)$ and*

$$d = \min\{\deg(v) | \, v \in V(G)\}.$$

If G is not a complete graph, then there is $S \subset V(G)$ such that $G \setminus S$ is disconnected and $d = |S|$.

Proof. Let v be a vertex of G of degree d and $S = N(v)$ the neighbor set of v. Since G is not a complete graph and v has minimum degree there is a vertex $w \notin S \cup \{v\}$. Thus $G \setminus S$ has at least two components. □

Definition 6.7.10 Let G be a graph and S a set of vertices, the *induced subgraph* $<S>$ is the maximal subgraph of G with vertex set S. Thus two vertices of S are adjacent in $<S>$ if and only if they are adjacent in G.

Proposition 6.7.11 ([84]) *If H is a triangulated non complete graph, then it can be constructed out of two smaller disjoint triangulated graphs H_1 and H_2 by identifying two (possibly empty) complete subgraphs of the same size in H_1 and H_2.*

Proof. One may assume that H is a connected graph with n vertices. Since H has at least one vertex of degree less than $n-1$, by Lemma 6.7.9 there is a minimal set of vertices S such that $H \setminus S$ is disconnected and $|S| \leq n-2$. Hence we can write $H \setminus S = F_1 \cup F_2$, where F_1 and F_2 are disjoint subgraphs of H. Set

$$H_1 = <V(F_1) \cup S> \quad \text{and} \quad H_2 = <V(F_2) \cup S>,$$

where $<A>$ is the induced subgraph with vertex set A. Note $H_1 \cup H_2 = H$ and $H_1 \cap H_2 = <S>$.

Next we show that $<S>$ is a complete graph. Assume there are vertices x_1 and x_2 in S not connected by an edge of H. Since $H \setminus (S \setminus \{x_i\})$ is connected for $i = 1, 2$, there are paths p_1 and p_2 in H_1 and H_2 respectively joining x_1 with x_2, whose only vertices in S are x_1 and x_2. If p_1 and p_2 are the shortest paths with this property, then $p_1 \cup p_2$ is a cycle of length greater or equal than four without a chord, a contradiction. □

Lemma 6.7.12 ([285]) *Let H be a triangulated graph and K a complete subgraph of H. If $K \neq H$, then there is $x \notin V(K)$ such that the subgraph $<N(x)>$ induced by the neighbor set $N(x)$ of x is a complete subgraph.*

Proof. By induction on $V(H)$. If H is complete, any $x \notin V(K)$ would satisfy the requirement. Otherwise, using Proposition 6.7.11 $H = H_1 \cup H_2$, where H_1, H_2 are triangulated graphs smaller than H such that $H_1 \cap H_2$ is a complete subgraph. It follows that K is a subgraph of H_i for some i, say $i = 1$. As $H_1 \cap H_2$ is a proper complete subgraph of H_2, by induction x can be chosen in $H_2 \setminus (H_1 \cap H_2)$. □

Let $R = k[x_1, \ldots x_n]$ be a polynomial ring over a field k and I a monomial ideal of I. The *ideal of minimal covers* or simply the *ideal of covers* of I, denoted I_c , is the ideal of R generated by all the monomials $x_{i_1} \cdots x_{i_k}$ such that the ideal $(x_{i_1}, \ldots, x_{i_k})$ is an associated prime ideal of I. This terminology is intended to emphasize that a minimal prime "covers" or contains all the monomials in I. If I is a Stanley-Reisner ring one has a duality:

$$\begin{array}{ccccc} I & = & (x_{i_1} \cdots x_{i_r}, \ldots,) & = & (x_{j_1}, \ldots, x_{j_s}) \cap \cdots \\ & & \updownarrow & & \updownarrow \\ I_c & = & (x_{i_1}, \ldots, x_{i_r}) \cap \cdots & = & (x_{j_1} \cdots x_{j_s}, \ldots,) \end{array}$$

Let G be a graph and $E = E(G)$ its edge set, in this case we denote the ideal of minimal covers of $I(G)$ by $I_c(G)$. One has

$$I_c(G) = \bigcap_{\{x,y\} \in E} (x, y).$$

Thus $I_c(G)$ is an unmixed ideal of height two. The Cohen-Macaulay property of this ideal was successfully studied by G. Lyubeznik.

Theorem 6.7.13 ([213]) *Let G be a graph and $\overline{G}$ its complement. Then the ideal of minimal covers $I_c(G)$ is Cohen-Macaulay if and only if $\overline{G}$ is a triangulated graph.*

Proposition 6.7.14 *Let G be a graph and $I_c(G)$ its ideal of covers. If $I_c(G)$ is Cohen-Macaulay, then*

$$\nu(I_c(G)) \leq g + 1,$$

where $\nu(I_c(G))$ is the minimum number of generators of $I_c(G)$ and g is the height of $I(G)$.

Proof. By the Hilbert-Burch Theorem 2.5.15, $I_c(G)$ is generated by the $m-1$ minors of an $m \times (m-1)$ matrix A with homogeneous entries, where $m = \nu(I_c(G))$. Take f in $I_c(G)$ of degree g, since any $m-1$ minor of A has degree at least $m-1$, one obtains $\deg(f) = g \geq m-1$. □

Proposition 6.7.15 *Let G be a graph whose complement $\overline{G}$ is a triangulated graph. Then $I(G)$ is Cohen-Macaulay if and only if $I_c(G)$ has a linear resolution.*

Proof. $\Rightarrow$) By the previous results one has that $I_c(G)$ is Cohen-Macaulay and the following inequality holds

$$\nu(I_c(G)) \leq \operatorname{ht} I(G) + 1,$$

according to Theorem 4.3.7 it suffices to prove that one has equality.

Note that $\nu(I_c(G))$ is equal to the multiplicity of $R/I(G)$, because all the minimal primes of $I(G)$ are of height $g = \operatorname{ht} I(G)$. On the other hand the multiplicity of $R/I(G)$ is equal to $g+1$ (see Exercise 6.7.17).

$\Leftarrow$) By induction on the height of $I(G)$. One may assume

$$g = \operatorname{ht} I(G) \geq 1,$$

otherwise $I(G) = 0$. Note that the hypotheses imply that G is an unmixed graph such that the multiplicity of $R/I(G)$ is equal to $e = g+1$. Hence any two maximal complete subgraphs of $\overline{G}$ are isomorphic. Note

$$d + 1 = n - g,$$

where n is the number of vertices of G and $d+1$ is the dimension of $R/I(G)$. Let K be a proper complete subgraph of $\overline{G}$ on $d+1$ vertices, by Lemma 6.7.12 there is $x \notin V(K)$ such that the subgraph $< N(x) >$ induced by the neighbor set $N(x)$ of x is a complete subgraph. Since x belongs to a complete subgraph L of $\overline{G}$ on $d+1$ vertices, it follows rapidly that $\deg_{\overline{G}}(x) = d$ or equivalently $\deg_G(x) = g$. Hence by induction $I(G \setminus \{x\})$ is C-M of height $g-1$. Using Proposition 6.2.5 one concludes that $I(G)$ is C-M. □

One can use a duality criterion described next to see that this result is valid in a more general setting and without the hypothesis that the graph $\overline{G}$ is triangulated (see Exercise 6.7.21).

Definition 6.7.16 Let Δ be a simplicial complex on the vertex set V, the *Alexander dual* Δ^* of Δ is the simplicial complex given by

$$\Delta^* = \{G \subset V | V \setminus G \notin \Delta\}.$$

According to [90] the Stanley-Reisner ideal of a simplicial complex has a linear resolution if and only if its Alexander dual is Cohen-Macaulay. This has been recently generalized [151] replacing linear resolution by the notion of *componentwise linear ideal*, and Cohen-Macaulay by *sequentially Cohen-Macaulay.*

Exercises

6.7.17 Let Δ be a simplicial complex of dimension d with n vertices and $f = (f_0, \ldots, f_d)$ its f-vector. If k is a field and $k[\Delta]$ is a Cohen-Macaulay ring with a 2-linear resolution, then

$$f_k = \binom{d+1}{k+1} + (n-d-1)\binom{d}{k}, \ 0 \leq k \leq d.$$

6.7.18 Let Δ be a simplicial complex and k a field. If the ring $k[\Delta]$ is Cohen-Macaulay and has a 2-linear resolution, then the reduced Euler characteristic $\tilde{\chi}(\Delta)$ is equal to 0 and $\tilde{H}_d(\Delta) = 0$, where $d = \dim \Delta$.

6.7.19 Let Δ be a simplicial complex. If $k[\Delta]$ is a Cohen-Macaulay ring with a 2-linear resolution over a field k, prove that Δ is shellable.

Hint Use Theorem 6.7.7.

6.7.20 Show that the complement of a triangulated graph cannot have a chordless cycle with five or more vertices.

6.7.21 Let Δ be a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$ and $\Delta^\star$ its Alexander dual. Prove that the Stanley-Reisner ideal of Δ^* is equal to the ideal of minimal covers of $I = I_\Delta$, that is:

$$I_{\Delta^\star} = (\{x_{i_1} \cdots x_{i_r} | \, (x_{i_1}, \ldots, x_{i_r}) \text{ is a minimal prime of } I\}).$$

6.7.22 Prove that the third Betti number of the ideal

$$I = (abc, abd, ace, adf, aef, bcf, bde, bef, cde, cdf)$$

depends on the base field, and show that I is equal to its ideal of vertex covers. See Example 6.7.4.

6.7.23 Let k be a field of characteristic zero or two. Use*Macaulay* to verify that the initial Betti numbers and initial degrees of the Stanley-Reisner ideal I of Exercise 6.2.24 are as follows:

$$10:\underbrace{11}_{\gamma_8}\quad 9:\underbrace{80}_{\gamma_7}\quad 8:\underbrace{245}_{\gamma_6}\quad 6:\underbrace{6}_{\gamma_5}\quad 5:\underbrace{40}_{\gamma_4}\quad 4:\underbrace{95}_{\gamma_3}\quad 3:\underbrace{80}_{\gamma_2}\quad 2:\underbrace{25}_{\gamma_1}$$

where the number before the colon indicates initial degree. What are the initial virtual Betti numbers of I?

Chapter 7

Monomial Subrings

Two of the main objects of study here are subalgebras generated by monomials and the corresponding Rees algebras. In general one would like to relate properties of those two algebras, such as the normality property or the Cohen-Macaulay property.

Some of the relevant topics of the chapter are the descriptions of the integral closure of ideals and monomial subrings and how the various notions of normality for algebras and ideals relate to each other.

Special attention is given to showing the relation between the normality of the Rees algebra and properties of other related objects.

We present some degree bounds for the generators of the integral closure of Rees algebras and monomial subrings. Upper bounds for the a-invariant of some subrings will also be given.

7.1 Basic properties

Let $R = k[\mathbf{x}] = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k in the indeterminates $x_1, \ldots, x_n$ and $F = \{f_1, \ldots, f_q\}$ a finite set of distinct monomials in R such that $f_i \neq 1$ for all i. For $c \in \mathbb{N}^n$ we set $x^c = x_1^{c_1} \cdots x_n^{c_n}$. The *monomial subring* spanned by F is the k-subalgebra

$$k[F] = k[f_1, \ldots, f_q] \subset R.$$

The exponent vector of $f_i = x^{\alpha_i}$ is denoted by $\log(f_i) = \alpha_i$ and $\log(F)$ denotes the set of exponent vectors of the monomials in F. To try to avoid cumbersome notation we set $f^c = f_1^{c_1} \cdots f_q^{c_q}$ if $c \in \mathbb{N}^q$.

Note that $k[F]$ is equal to the *semigroup ring*

$$k[C] = k[\{x^\alpha | \, \alpha \in C\}],$$

where $C = \mathbb{N}\log(f_1)+\cdots+\mathbb{N}\log(f_q)$ is the subsemigroup of $\mathbb{N}^n$ generated by $\log(F)$. Thus as a k-vector space $k[F]$ is generated by the set of monomials of the form x^α, with $\alpha \in C$.

An important feature of $k[F]$ is that it is a graded subring of R with the grading $k[F]_i = k[F] \cap R_i$. There is a graded epimorphism of k-algebras:

$$\varphi\colon B = k[t_1, \ldots, t_q] \longrightarrow k[F] \longrightarrow 0, \text{ induced by } \varphi(t_i) = f_i,$$

where B is a polynomial ring graded by $\deg(t_i) = \deg(f_i)$. Note that the map φ is given by

$$\varphi(h(t_1, \ldots, t_q)) = h(f_1, \ldots, f_q), \text{ for all } h \in B.$$

The kernel P_F of φ is called the *presentation ideal* or *toric ideal* of $k[F]$ with respect to $f_1, \ldots, f_q$.

Definition 7.1.1 A *binomial* f in $B = k[t_1, \ldots, t_q]$ is a difference of two monomials, that is, $f = t^\alpha - t^\beta$, for some α, β in $\mathbb{N}^q$. An ideal generated by binomials is called a *binomial ideal*.

Proposition 7.1.2 *The presentation ideal P_F of $k[F]$ is a graded prime ideal generated by a finite set of binomials.*

Proof. Observe $B/P_F \simeq k[F]$ and that $k[F]$ is a Noetherian domain. If a polynomial $h_0 + \cdots + h_s$ is in P_F with $h_i \in B_i$, then $\varphi(h_i) = 0$ because φ is graded. Hence $h_i \in (P_F)_i$ for all i, and P_F is a graded ideal.

Consider the ideal L of B generated by the binomials in P_F. Since P_F has a finite basis of homogeneous polynomials, to show $L = P_F$ it suffices to prove $B_d \cap P_F \subset L$ for all d. Let $h \in B_d \cap P_F$, write

$$h = \sum_{i=1}^{r} a_{\gamma_i} t^{\gamma_i}$$

with $a_{\gamma_i} \in k$ and $t^{\gamma_i} \in B_d$. We show that $h \in L$ by induction on r. Assume $r \geq 3$. Making the substitution $x_i = t_1$ in the equality $\sum_{i=1}^r a_{\gamma_i} f^{\gamma_i} = 0$ one obtains $(\sum_{i=1}^r a_{\gamma_i}) t_1^d = 0$ and $\sum_{i=1}^r a_{\gamma_i} = 0$. One may assume $f^{\gamma_i} = f^{\gamma_j}$ for some $i \neq j$, otherwise $a_{\gamma_i} = 0$ for all i. For simplicity assume $i = 1$ and $j = 2$. Since $a_{\gamma_2} = -a_{\gamma_1} + \beta$, with $\beta \in k$ one has

$$h = a_{\gamma_1}(t^{\gamma_1} - t^{\gamma_2}) + \beta t^{\gamma_2} + \sum_{i=3}^{r} a_{\gamma_i} t^{\gamma_i} = a_{\gamma_1}(t^{\gamma_1} - t^{\gamma_2}) + g.$$

By induction one gets $g \in L$ and $h \in L$, as required. □

Definition 7.1.3 If $F = \{f_1, \ldots, f_q\}$ is a set of monomials in $k[\mathbf{x}]$, the *associated matrix* of $k[F]$ is the $n \times q$ matrix M whose columns are the exponent vectors $\log(f_1), \ldots, \log(f_q)$.

Recall that the support of a vector $\beta \in \mathbb{R}^q$ is $\text{supp}(\beta) = \{i \mid \beta_i \neq 0\}$. Note that any vector α in $\mathbb{R}^n$ can be written as $\alpha = \alpha_+ - \alpha_-$, where α_+ and α_- are two non negative vectors with disjoint support.

Corollary 7.1.4 *If P_F is the toric ideal of the monomial subring $k[F]$ and M is the associated matrix of $k[F]$, then*

$$P_F = (\{t^{\alpha+} - t^{\alpha-} \mid \alpha \in \mathbb{Z}^q \text{ and } M\alpha = 0\}).$$

Proof. Let $t^\alpha - t^\beta$ be a binomial in P_F. If $\text{supp}(\alpha) \cap \text{supp}(\beta) \neq \emptyset$, then we can write $f^\alpha - f^\beta = f^\gamma(f^{\alpha'} - f^{\beta'}) = 0$, where $\gamma_i = \min\{\alpha_i, \beta_i\}$. Note that α' and β' have disjoint support and $f^\alpha - f^\beta = 0$ iff $M(\alpha - \beta) = 0$. To complete the argument use that P_F is generated by binomials. □

Corollary 7.1.5 *The toric ideal P_F of a monomial subring $k[F]$ has a Gröbner basis consisting of binomials with respect to any monomial ordering of the polynomial ring B.*

Proof. Let E be a finite set of generators of P_F consisting of binomials and let $f, g \in E$. Since the S-polynomial $S(f, g)$ is again a binomial and the remainder of $S(f, g)$ with respect to E is also a binomial, it follows that the output of the Buchberger algorithm (see Theorem 2.4.11) is a Gröbner basis of P_F consisting of binomials. □

Definition 7.1.6 Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and $k \geq 1$ an integer. The *monomial subring of k-products* or simply the *subring of k-products* is:

$$K[V_k] \subset R, \text{ where } V_k = \{x_{i_1} \cdots x_{i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}.$$

The *kth Veronese subring* of R, denoted by $R^{(k)}$, is by definition:

$$R^{(k)} = \bigoplus_{i=0}^{\infty} R_{ik} \subset R,$$

where an element in R_{ik} is said to have *normalized degree i*. Thus $R^{(k)}$ is again a standard algebra generated by elements of normalized degree 1. Sometimes the subring $K[V_k]$ is called the *square-free kth Veronese subring* of R, because it is contained in $R^{(k)}$.

Corollary 7.1.7 ([111]) *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and let P be the toric ideal of the subring of k-products $K[V_k]$. If $k \geq 2$, then P is generated by homogeneous binomials of degree two.*

Proof. Let $V_k = \{x_{i_1} \cdots x_{i_k} | \, 1 \leq i_1 < \cdots < i_k \leq n\}$ be the set of square-free monomials of degree k in R. To begin with consider a polynomial ring:

$$B = K[\{t_{i_1 \cdots i_k} | \, 1 \leq i_1 < \cdots < i_k \leq n\}],$$

with one variable $t_{i_1 \cdots i_k}$ for each monomial $x_{i_1} \cdots x_{i_k}$ in V_k. Let φ be the graded homomorphism

$$\varphi \colon B \longrightarrow K[V_k], \text{ induced by } t_{i_1 \cdots i_k} \stackrel{\varphi}{\longmapsto} x_{i_1} \cdots x_{i_k},$$

where B has the usual grading and $K[V_k]$ has the normalized grading. Thus $P = \ker(\varphi)$ is the toric ideal of $K[V_k]$.

By Proposition 7.1.2 the toric ideal $P = \bigoplus_{s=2}^{\infty} P_s$ is a graded ideal generated by homogeneous binomials, hence it suffices to show that all the binomials in P_s are in (P_2) for $s \geq 2$. We proceed by induction on s. Assume $s \geq 3$ and that all the binomials in $P_2, \ldots, P_{s-1}$ are in P_2. Let

$$\mathcal{T} = \{t_{i_1 \cdots i_k} | \, 1 \leq i_1 < \cdots < i_k \leq n\} \text{ and } F = t_1 \cdots t_s - t_1' \cdots t_s' \in P_s,$$

where t_i, $t_i' \in \mathcal{T}$ for all i. If $\varphi(t_i) = f_i$ and $\varphi(t_i') = f_i'$, then after reordering the variables we may write $f_1 = x_1 \cdots x_r x_{r+1} \cdots x_k$, where $x_1, \ldots, x_r$ are in $\cap_{i=1}^{s} \operatorname{supp}(f_i) = \cap_{i=1}^{s} \operatorname{supp}(f_i')$ and $x_{r+1}, \ldots, x_k$ are not in $\cap_{i=1}^{s} \operatorname{supp}(f_i)$. Set h equal to $x_1 \cdots x_r$ and write $f_1' = h x_{i_{r+1}} \cdots x_{i_k}$. If $x_{r+1} \notin \operatorname{supp}(f_1')$, then $f_p' = h x_{r+1} x_{p_{r+2}} \cdots x_{p_k}$ for some $p \geq 2$. By our assumption there is f_ℓ' so that $x_{i_{r+1}}$ is not in $\operatorname{supp}(f_\ell')$ for some $\ell \geq 2$. First consider the case $x_{r+1} \in \operatorname{supp}(f_\ell')$. Write $f_\ell' = h x_{r+1} x_{\ell_{r+2}} \cdots x_{\ell_k}$. Note that $f_1' f_\ell' = f_1'' f_\ell''$, where $f_1'' = (x_{r+1} f_1')/x_{i_{r+1}}$ and $f_\ell'' = (x_{i_{r+1}} f_\ell')/x_{r+1}$. Since f_1'' and f_ℓ'' are in V_k, there are t_1'' and t_ℓ'' in $\mathcal{T}$ mapping under φ to f_1'' and f_ℓ'' respectively. Let $G = t_1' t_\ell' - t_1'' t_\ell''$. From the equality

$$\begin{aligned} F_1 &= (t_1 \cdots t_s - t_1' \cdots t_s') + (t_1' t_\ell' - t_1'' t_\ell'')(t_2' \cdots t_{\ell-1}' t_{\ell+1}' \cdots t_s') \\ &= t_1 \cdots t_s - t_1'' t_2' \cdots t_{\ell-1}' t_\ell'' t_{\ell+1}' \cdots t_s = F + G t_2' \cdots t_{\ell-1}' t_{\ell+1}' \cdots t_s' \end{aligned}$$

we get $F \in (P_2)$ if and only if $F_1 \in (P_2)$. On the other hand if x_{r+1} is not in $\operatorname{supp}(f_\ell')$, write $f_\ell' = h x_{\ell_{r+1}} \cdots x_{\ell_k}$. Note that $\{x_{\ell_{r+1}}, \ldots, x_{\ell_k}\}$ is not contained in $\mathcal{A} = \{x_{p_{r+2}}, \ldots, x_{p_k}\}$, say $x_{\ell_{r+1}}$ is not in $\mathcal{A}$. Setting $f_\ell'' = (x_{r+1} f_\ell')/x_{\ell_{r+1}}$ and $f_p'' = (x_{\ell r+1} f_p')/x_{r+1}$, we get $G_2 = t_\ell' t_p' - t_\ell'' t_p'' \in P_2$, where t_ℓ'', t_p'' map under φ to f_ℓ'', f_p'' respectively. From the equality

$$\begin{aligned} F_2 &= F + G_2(t_1' \cdots t_{\ell-1}' t_{\ell+1}' \cdots t_{p-1}' t_{p+1}' \cdots t_s') \\ &= t_1 \cdots t_s - t_\ell'' t_1' \cdots t_{\ell-1}' t_{\ell+1}' \cdots t_{p-1}' t_p'' t_{p+1}' \cdots t_s' \end{aligned}$$

we obtain $F \in (P_2)$ if and only if $F_2 \in (P_2)$. Set $f_\ell''' = (f_\ell'' x_{i_{r+1}})/x_{r+1}$, $f_1'' = (f_1' x_{r+1})/x_{i_{r+1}}$ and $G_3 = t_1' t_\ell'' - t_1'' t_\ell'''$. From the identity

$$\begin{aligned} F_3 &= F_2 + G_3(t_2' \cdots t_{\ell-1}' t_{\ell+1}' \cdots t_{p-1}' t_p'' t_{p+1}' \cdots t_s') \\ &= t_1 \cdots t_s - t_1'' t_2' \cdots t_{\ell-1}' t_\ell''' t_{\ell+1}' \cdots t_{p-1}' t_p'' t_{p+1}' \cdots t_s' \end{aligned}$$

we obtain $F_2 \in (P_2)$ if and only if $F_3 \in (P_2)$. Hence from the outset one may assume

$$f_1 = hx_{r+1}\cdots x_k \text{ and } f_1' = hx_{r+1}x_{i_{r+2}}\cdots x_{i_k}.$$

Using the same argument as above with x_{r+2} and $x_{i_{r+2}}$ playing the role of x_{r+1} and $x_{i_{r+1}}$, and so on, we derive that there is a binomial $F_m \in P_s$ of the form $F_m = t_1\cdots t_s - t_1u_2\cdots u_s$, with $u_2,\cdots,u_s \in \mathcal{T}$, so that $F \in (P_2)$ if and only if $F_m \in (P_2)$. To complete the proof notice that F_m is in (P_2) by the induction hypothesis. □

Corollary 7.1.8 *Let $R = K[x_1,\ldots,x_n]$ be a polynomial ring over a field K and P the toric ideal of the Veronese subring $R^{(k)}$. If $k \geq 2$, then P is generated by homogeneous binomials of degree two.*

Proof. Consider a polynomial ring:

$$B = K[\{t_\alpha |\, \alpha \in \mathbb{N}^n \text{ and } |\alpha| = k\}],$$

with one variable t_α for each monomial x^α of degree k. Let φ be the graded homomorphism

$$\varphi\colon B \longrightarrow R^{(k)}, \text{ induced by } t_\alpha \stackrel{\varphi}{\longmapsto} x^\alpha,$$

where B has the usual grading and $R^{(k)}$ has the normalized grading. By Proposition 7.1.2 the toric ideal $P = \ker(\varphi)$ is a graded ideal $P = \bigoplus_{s=2}^{\infty} P_s$ generated by homogeneous binomials. Hence we need only show that all the binomials in P_s are in (P_2) for $s \geq 2$. We proceed by induction on s. Let

$$\mathcal{T} = \{t_\alpha |\, \alpha \in \mathbb{N}^n \text{ and } |\alpha| = k\} \text{ and } F = t_1\cdots t_s - u_1\cdots u_s \in P_s,$$

where t_i, $u_i \in \mathcal{T}$ for all i. If $\varphi(t_i) = f_i$ and $\varphi(u_i) = g_i$, then one has the equality $f_1\cdots f_s = g_1\cdots g_s$. Write $f_1 = x_1^{a_1}\cdots x_n^{a_n}$ and $g_1 = x_1^{c_1}\cdots x_n^{c_n}$. To show that F belongs to (P_2) we consider a few simple cases.

Case (a): Assume $c_i < k$ and $x_i \in \operatorname{supp}(g_j)$ for some $j \neq 1$. For simplicity of notation set $i = 1$ and $j = 2$. Pick $x_\ell \neq x_1$ in the support of g_1. Note that $g_1g_2 = h_1h_2$, where $h_1 = x_1g_1/x_\ell$ and $h_2 = x_\ell g_2/x_1$. Since h_1 and h_2 have degree k, there are u_1' and u_2' in $\mathcal{T}$ mapping under φ to h_1 and h_2 respectively. From the equality

$$\begin{aligned} F_1 &= (t_1\cdots t_s - u_1\cdots u_s) + u_3\cdots u_s(u_1u_2 - u_1'u_2') \\ &= t_1\cdots t_s - u_1'u_2'u_3\cdots u_s \end{aligned}$$

we conclude that $F \in (P_2)$ if and only if $F_1 \in (P_2)$.

Case (b): x_i^k divides $g_1\cdots g_s$ for some i, say $i = 1$. By a repeated use of case (a) one may assume $g_1 = x_1^k$. Hence by symmetry one may also assume $f_1 = x_1^k$, that is, $F = t_1\cdots t_s - t_1u_2\cdots u_s$, which by induction is in (P_2).

Case (c): x_i^k does not divide $g_1 \cdots g_s$ for all i. Let

$$\operatorname{supp}(g_1) = \{x_{i_1}, \ldots, x_{i_r}\}.$$

Note $c_{i_j} < k$ for all j. Thus using case (a) one may assume $g_1 = x_{i_1}^{c_{i_1}} \cdots x_{i_r}^{c_{i_r}}$, where $x_{i_1} \notin \cup_{i \neq 1} \operatorname{supp}(g_i)$. A repeated application of case (a) yields that one may assume $g_1 = x_{i_1}^{c_{i_1}} \cdots x_{i_r}^{c_{i_r}}$, where $x_{i_j} \notin \cup_{i \neq 1} \operatorname{supp}(g_i)$ for all j. By symmetry one may write $f_1 = x_{i_1}^{c_1} \cdots x_{i_r}^{c_r}$, where $x_{i_j} \notin \cup_{i \neq 1} \operatorname{supp}(f_i)$, that is, one can reduce to the case $F = t_1 \cdots t_s - t_1 u_2 \cdots u_s$, which by induction gives $F \in (P_2)$. □

Proposition 7.1.9 *If R is a polynomial ring over a field k and $f_1, \ldots, f_q$ are in R, then the kernel of the homomorphism of k-algebras*

$$\varphi\colon B = k[t_1, \ldots, t_q] \longrightarrow k[f_1, \ldots, f_q], \text{ induced by } \varphi(t_i) = f_i,$$

is the ideal $(t_1 - f_1, \ldots, t_q - f_q) \cap B$.

Proof. Let $F \in \ker(\varphi)$. Making $t_i = (t_i - f_i) + f_i$ we can write F as:

$$F = F(t_1, \ldots, t_q) = \textstyle\sum_\gamma a_\gamma (t_1 - f_1)^{\gamma_1} \cdots (t_q - f_q)^{\gamma_q} + \sum_\beta a_\beta f^\beta.$$

Since $F(f_1, \ldots, f_q) = 0$ one derives $\sum_\beta a_\beta f^\beta = 0$, and consequently F is in $(t_1 - f_1, \ldots, t_q - f_q) \cap B$. The other containment also follows readily. □

Let us illustrate a technique used to compute the toric ideal P_F of a monomial subring $k[F]$ using Gröbner bases and elimination of variables.

Example 7.1.10 Let $F = \{x_i x_j | \, 1 \leq i < j \leq 4\}$ and $\varphi\colon k[t_{ij}] \to k[F]$, the graded epimorphism of k-algebras with $\varphi(t_{ij}) = x_i x_j$ and $\deg(t_{ij}) = 2$. By Proposition 7.1.9 one has

$$P_F = \ker(\varphi) = L \cap k[t_{ij}], \; L = (t_{ij} - f_{ij} | \, 1 \leq i < j \leq 4).$$

Using Macaulay [15] one obtains that the reduced Gröbner basis of L w.r.t the elimination ordering in the first variables $x_1, \ldots, x_4$ is equal to:

$$\begin{array}{llll}
t_{14}t_{23} - t_{12}t_{34}, & t_{13}t_{24} - t_{12}t_{34}, & x_4^2 t_{23} - t_{24}t_{34}, & x_4^2 t_{13} - t_{14}t_{34}, \\
x_4^2 t_{12} - t_{14}t_{24}, & x_3 t_{24} - x_4 t_{23}, & x_3 t_{14} - x_4 t_{13}, & x_3 x_4 - t_{34}, \\
x_3^2 t_{12} - t_{13}t_{23}, & x_2 t_{34} - x_4 t_{23}, & x_2 t_{14} - x_4 t_{12}, & x_2 t_{13} - x_3 t_{12}, \\
x_2 x_4 - t_{24}, & x_2 x_3 - t_{23}, & x_1 t_{34} - x_4 t_{13}, & x_1 t_{24} - x_4 t_{12}, \\
x_1 t_{23} - x_3 t_{12}, & x_1 x_4 - t_{14}, & x_1 x_3 - t_{13}, & x_1 x_2 - t_{12}, \\
x_3 t_{12} t_{34} - x_4 t_{13} t_{23}. & & &
\end{array}$$

Therefore $P_F = (t_{14}t_{23} - t_{12}t_{34}, t_{13}t_{24} - t_{12}t_{34})$ by Theorem 2.4.18.

Another method to compute P_F can be found in [83]; this method uses Buchberger's algorithm [53] adapted for the specialized situation of toric ideals. In [23] some algorithms are devised, that in many respects improve existing algorithms for the computation of toric ideals. For a general study of binomial ideals we refer the reader to [98].

Lemma 7.1.11 *Let f be a polynomial in $k[x_1, \ldots, x_n]$. If k is an infinite field and $f(a) = 0$ for all $a \in k^n$, then f is the zero polynomial.*

Proof. By induction on n. The case $n = 1$ is clear because any nonzero polynomial in $k[x_1]$ of degree r has at most r roots. If $n \geq 2$ write

$$f = f_0 + f_1 x_n + \cdots + f_r x_n^r,$$

where $f_i \in k[x_1, \ldots, x_{n-1}]$ for all i. Fix $c = (c_1, \ldots, c_{n-1})$ in k^{n-1} and consider the polynomial

$$g(x_n) = f(c, x_n) = f_0(c) + f_1(c)x_n + \cdots + f_r(c)x_n^r.$$

As g vanishes on k and g is in $k[x_n]$ one obtains that g is the zero polynomial. Hence $f_i(c) = 0$ for all i. Since c was an arbitrary point in the affine space k^{n-1} by induction one has that f_i is the zero polynomial for all i, thus f is the zero polynomial. □

Corollary 7.1.12 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and $X \subset k^q$ a set given parametrically by $t_i = f_i(x_1, \ldots, x_n)$, where $F = \{f_1, \ldots, f_q\} \subset R$. If k is infinite, then the ideal $I(X)$ of polynomials $f(t_1, \ldots, t_q)$ that vanish on X is equal to P_F, the presentation ideal of $k[F]$.*

Proof. As $P_F = (t_1 - f_1, \ldots, t_q - f_q) \cap k[t_1, \ldots, t_q]$, one has $P_F \subset I(X)$. Conversely take $f \in I(X)$ and consider the revlex ordering such that $t_i > x_j$ for all i, j. By the division algorithm $f(t_1, \ldots, t_q) = \sum_i g_i(t_i - f_i) + r$, where r is a polynomial in R. Hence for any $\alpha \in k^n$ one has

$$f(f_1(\alpha), \ldots, f_q(\alpha)) = \sum_i g_i(\alpha, x)(f_i(\alpha) - f_i(x_1, \ldots, x_n)) + r(x_1, \ldots, x_n) = 0.$$

This proves $r(\alpha) = 0$ for any $\alpha \in k^n$, thus r is the zero polynomial because k is an infinite field (see Lemma 7.1.11). □

Definition 7.1.13 Given a subset X of the affine space $\mathbb{A}_k^n$ over a field k, the *Zariski closure* of X, denoted by $\overline{X}$, is the closure of X in the Zariski topology of $\mathbb{A}_k^n$. Thus $\overline{X}$ is the smallest affine variety of $\mathbb{A}_k^n$ containing X.

Proposition 7.1.14 *If k is a field and X a subset of the affine space $\mathbb{A}_k^n$, then $\overline{X} = V(I(X))$, where $\overline{X}$ is the Zariski closure of X.*

Proof. As $V(I(X))$ is closed and $X \subset V(I(X))$, one has $\overline{X} \subset V(I(X))$. For the other containment note $\overline{X} = V(J)$, where J is an ideal of a polynomial ring R over k. From $X \subset V(J)$ one derives $I(V(J)) \subset I(X)$, and since one also has the containment $J \subset I(V(J))$ by transitivity $J \subset I(X)$. Therefore $V(I(X)) \subset V(J)$ and consequently $V(I(X)) \subset \overline{X}$. □

Definition 7.1.15 An *affine toric variety* V is defined as the zero set of a toric ideal, that is, $V = V(P_F)$ for some toric ideal P_F.

Corollary 7.1.16 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and $X \subset k^q$ a set given parametrically by $t_i = f_i(x_1, \ldots, x_n)$, where $F = \{f_1, \ldots, f_q\}$ is a subset of R. If k is infinite, then the Zariski closure of X is equal to $V(P_F)$.*

Proof. By Corollary 7.1.12 and Proposition 7.1.14 one has the equalities $I(X) = P_F$ and $\overline{X} = V(I(X))$. □

Proposition 7.1.17 *If $k[F]$ is a monomial subring over a field k generated by a finite set F of monomials and M is its associated matrix, then*

$$\dim k[F] = \operatorname{rank}(M),$$

where $\dim k[F]$ *is the Krull dimension of* $k[F]$.

Proof. Let $F = \{f_1, \ldots, f_q\}$ and $f_i = x^{\alpha_i}$, where $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{in})$ for $i = 1, \ldots, q$. Let $M = (\alpha_{ji})$ be the $n \times q$ matrix of $k[F]$. Set $r = \operatorname{rank}(M)$. By the Noether normalization lemma $\dim k[F] = \operatorname{trdeg}_k k[F]$, therefore it suffices to prove the equality $r = \operatorname{trdeg}_k k[F]$. We may assume $\alpha_1, \ldots, \alpha_r$ is a basis for the column space of M. We now show that $f_1, \ldots, f_r$ are algebraically independent over k. If $f_1, \ldots, f_r$ satisfy a polynomial relation, then there are

$$a = (a_1, \ldots, a_r, 0, \ldots, 0) \in \mathbb{N}^q \text{ and } c = (c_1, \ldots, c_r, 0, \ldots, 0) \in \mathbb{N}^q$$

such that $f^a = f^c$. Hence $M(a) = M(c)$ and $\gamma_1\alpha_1 + \cdots + \gamma_r\alpha_r = 0$, where $\gamma_i = a_i - c_i$. Therefore $a_i = c_i$, that is $f^a = f^c$ is a trivial relation. To show that f_{r+i} is integral over the quotient field $k(f_1, \ldots, f_r)$ pick $s \in \mathbb{N}$ such that $s\alpha_{r+i} = \lambda_1\alpha_1 + \cdots + \lambda_r\alpha_r$, where $\lambda_j \in \mathbb{Z}$ for all j. Therefore $f_{r+i}^s = f_1^{\lambda_1} \cdots f_r^{\lambda_r}$, as required. □

Exercises

7.1.18 Let $k = \mathbb{Z}_2$ and $X \subset k^3$ the space curve given parametrically by $t_1 = x_1^7$, $t_2 = x_1^8$, $t_3 = x_1^9$. Prove that

$$P_F = (t_1^5 - t_2t_3^3, t_2^2 - t_1t_3, t_1^4t_2 - t_3^4)$$

and $P_F \neq I(X)$.

7.1.19 If P is the toric ideal of the subring of k-products $S_{n,k} = K[V_k]$ and $n \geq 2k \geq 4$, then the minimal number of generators of P is given by:

$$\nu(P) = \sum_{i=0}^{k-2} \binom{n}{i} \binom{n-i}{2(k-i)} \left[\binom{2(k-i)-1}{k-i-1} - 1 \right].$$

7.1.20 Use Theorem 2.4.18 together with Proposition 7.1.9 to give a second proof of the fact that toric ideals of monomial subrings have Gröbner bases consisting of binomials.

7.1.21 Let $R = K[x_1, x_2, y_1, \ldots, y_5]$ be a polynomial ring over a field K and $S = K[F]$ the monomial subring spanned by

$$F = \{x_1 y_i y_j, x_2 y_i y_j | \, 1 \leq i < j \leq 5\}.$$

Prove that $(x_1 y_1 y_2 :_S x_2 y_3 y_4) = (x_1 y_1 y_2, x_1 x_2 y_1 y_2 y_5^2, x_1^2 y_1 y_2 y_5^2)$.

7.1.22 Let $R = k[x_1, x_2, x_3]$ and $R^{(2)}$ the second Veronese subring. If φ is the homomorphism of k-algebras

$$\varphi : k[t_{ij}] \longrightarrow R^{(2)}$$

induced by $\varphi(t_{ij}) = x_i x_j$, prove that the toric ideal $P = \ker(\varphi)$ is generated by the 2-minors of the symmetric matrix

$$X = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{12} & t_{22} & t_{23} \\ t_{13} & t_{23} & t_{33} \end{pmatrix}.$$

7.1.23 Let $R = k[x_1, x_2, x_3]$ and let $S = R^{(2)}$ be the second Veronese subring endowed with the normalized grading

$$S = \bigoplus_{i=0}^{\infty} S_i,$$

where $S_i = R_{2i}$. Prove that the Hilbert series of $R^{(2)}$ is equal to

$$F(S, z) = \frac{(1+3z)}{(1-z)^3}.$$

7.2 Integral closure of subrings

Here we present a description of the integral closure of a monomial subring, which links polyhedral geometry and integer programming techniques with the problem of finding integral closures of monomial subrings. The problem of the effective determination of the integral closure of a monomial subring has a satisfactory answer given by an algorithm of W. Bruns and R. Koch; see [48] for details on this algorithm.

Polyhedral sets and cones An *affine space* or *linear variety* in $\mathbb{R}^n$ is by definition a translation of a linear subspace of $\mathbb{R}^n$. Let $A \subset \mathbb{R}^n$. Let us recall that $\text{aff}(A)$, the *affine space generated* by A, is the set of all *affine combinations* of points in A:

$$\text{aff}(A) = \{a_1p_1 + \cdots + a_rp_r | \, p_i \in A,\ a_1 + \cdots + a_r = 1,\ a_i \in \mathbb{R}\}.$$

There is a unique linear subspace V of $\mathbb{R}^n$ such that

$$\text{aff}(A) = x_0 + V,$$

for some $x_0 \in \mathbb{R}^n$. The *dimension* of A is defined as $\dim A = \dim_{\mathbb{R}}(V)$.

A point $x \in \mathbb{R}^n$ is called a *convex combination* of $p_1, \ldots, p_r \in \mathbb{R}^n$ if there are non negative real numbers $a_1, \ldots, a_r$ such that

$$x = a_1p_1 + \cdots + a_rp_r \text{ and } a_1 + \cdots + a_r = 1.$$

Let $A \subset \mathbb{R}^n$. The *convex hull* of A, denoted by $\text{conv}(A)$, is the set of all convex combinations of points in A. If $A = \text{conv}(A)$ we say that A is a *convex set.*

Definition 7.2.1 A *convex polytope* $P \subset \mathbb{R}^n$ is the convex hull of a finite set of points $p_1, \ldots, p_r$ in $\mathbb{R}^n$, that is, $P = \text{conv}(p_1, \ldots, p_r)$.

The *inner product* of two vector $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ is defined by

$$\langle x, y \rangle = x_1y_1 + \cdots + x_ny_n.$$

Given $a \in \mathbb{R}^n \setminus \{0\}$ and $c \in \mathbb{R}$, define the hyperplane

$$H(a, c) = \{x \in \mathbb{R}^n | \, \langle x, a \rangle = c\}.$$

Note that every hyperplane of $\mathbb{R}^n$ is of this form. The two *closed halfspaces bounded* by $H(a, c)$ are

$$H^+(a, c) = \{x \in \mathbb{R}^n | \, \langle x, a \rangle \geq c\} \text{ and } H^-(a, c) = H^+(-a, -c).$$

Definition 7.2.2 A *polyhedral set* or *convex polyhedron* is a set Q in $\mathbb{R}^n$ which is the intersection of a finite number of closed halfspaces of $\mathbb{R}^n$.

Since the intersection of an arbitrary family of convex sets in $\mathbb{R}^n$ is convex, one derives that any polyhedral set is convex and closed. It is a fact that P is a convex polytope in $\mathbb{R}^n$ if and only if P is a bounded polyhedral set; see [310, Theorem 3.2.5].

Definition 7.2.3 If $C \subset \mathbb{R}^n$ is closed under linear combinations with non negative real coefficients, we say that C is a *convex cone* or simply a *cone.* A *polyhedral cone* is a convex cone which is also a polyhedral set.

If $\mathcal{A}$ is a set of points in $\mathbb{R}^n$, the cone generated by $\mathcal{A}$, denoted by $\mathbb{R}_+\mathcal{A}$ or cone($\mathcal{A}$), is defined as

$$\mathbb{R}_+\mathcal{A} = \left\{ \sum_{i=1}^{q} a_i\beta_i \,\middle|\, a_i \in \mathbb{R}_+ \text{ and } \beta_i \in \mathcal{A}, \text{ for all } i \right\},$$

where $\mathbb{R}_+$ is the set of non negative real numbers. We say that C is a *finitely generated cone* in $\mathbb{R}^n$ if $C = \mathbb{R}_+A$, for some finite set of points $\mathcal{A} \subset \mathbb{R}^n$.

Theorem 7.2.4 *If $C \subset \mathbb{R}^n$, then C is a polyhedral cone in $\mathbb{R}^n$ if and only if C is a finitely generated cone.*

Proof. See [310, Theorem 4.1.1]. □

One of the fundamental results on polyhedral geometry is the following remarkable representation theorem.

Theorem 7.2.5 (Finite basis theorem) *If Q is a set in $\mathbb{R}^n$, then Q is a polyhedral set if and only if Q can be expressed as $Q = P + C$, where P is a convex polytope and C is a finitely generated cone.*

Proof. See [64, Theorem 16.2] and [310, Theorem 4.1.3]. □

Thus according to the finite basis theorem a convex polyhedron has two representations. We have found it very convenient to use the computer program *Polyhedron Representation Transformation Algorithms* (PORTA for short) to switch between the two representations. See [63]; for the case of polyhedral cones see [48].

Definition 7.2.6 Let Q be a closed convex set in $\mathbb{R}^n$. A hyperplane H of $\mathbb{R}^n$ is called a *supporting hyperplane* of Q if Q is contained in one of the two closed halfspaces bounded by H and $Q \cap H \neq \emptyset$.

Definition 7.2.7 A *proper face* of a polyhedral set Q is a set $F \subset Q$ such that there is a supporting hyperplane $H(y, c)$ satisfying the conditions:

(a) $F = Q \cap H(y, c) \neq \emptyset$,

(b) $Q \not\subset H(y, c)$, and $Q \subset H^+(y, c)$ or $Q \subset H^-(y, c)$.

The *improper faces* of a polyhedral set Q are Q itself and $\emptyset$.

Proposition 7.2.8 *If Q is a polyhedral set in $\mathbb{R}^n$ and F_1, F_2 are faces of Q, then their intersection $F = F_1 \cap F_2$ is a face of Q.*

Proof. Let $H_i = H(a_i, c_i)$ be supporting hyperplanes of Q, where $c_i \in \mathbb{R}$ and $0 \neq a_i \in \mathbb{R}^n$, such that $F_i = Q \cap H_i$ and $Q \subset H_i^+$ for $i = 1, 2$. Let us prove the equality

$$F = Q \cap H(a_1 + a_2, c_1 + c_2).$$

The left hand side is clearly contained in the right hand side. On the other hand if $x \in Q \cap H(a_1 + a_2, c_1 + c_2)$, then using $\langle x, a_i \rangle \geq c_i$ one has:

$$c_1 + c_2 = \langle x, a_1 + a_2 \rangle = \langle x, a_1 \rangle + \langle x, a_2 \rangle \geq c_1 + c_2,$$

hence $\langle x, a_i \rangle = c_i$ for $i = 1, 2$ and $x \in F$. Since

$$Q \subset H^+(a_1 + a_2, c_1 + c_2)$$

the set F is a face of Q. □

Definition 7.2.9 Let Q be a polyhedral set and $x_0 \in Q$. The point x_0 is called a *vertex* or an *extreme point* of Q if $\{x_0\}$ is a proper face of Q.

Definition 7.2.10 Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ its *Euclidean norm* is

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Lemma 7.2.11 *Let Q be a convex polyhedron in $\mathbb{R}^n$ and $x_0 \in Q$. Then x_0 is a vertex of Q if and only if $Q \setminus \{x_0\}$ is a convex set.*

Proof. $\Rightarrow$) Let $H(a, c)$ be a proper supporting hyperplane of Q such that $\{x_0\} = Q \cap H(a, c)$ and $Q \subset H^+(a, c)$. Take $x, y \in Q \setminus \{x_0\}$ and consider $z = ty + (1 - t)x$ with $0 < t < 1$. Note that x, y are not in $H(a, c)$, thus

$$\langle z, a \rangle = t\langle y, a \rangle + (1 - t)\langle x, a \rangle > tc + (1 - t)c = c.$$

Hence $\langle z, a \rangle > c$, that is, $z \neq x_0$ and $z \in Q$, as required.

$\Leftarrow$) Let $Q = \cap_{i=1}^r H^+(a_i, c_i)$ be a decomposition of Q as an intersection of closed halfspaces, where $0 \neq a_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$ for all i. First observe that x_0 is in $H(a_i, c_i)$ for some i, otherwise if $\langle x_0, a_i \rangle > c_i$ for all i, there is an open ball $B_\delta(x_0)$ in $\mathbb{R}^n$, of radius δ centered at x_0, whose closure lies in Q. Hence taking two antipodal points x_1, x_2 in the boundary of $B_\delta(x_0)$ one obtains $x_0 \in Q \setminus \{x_0\}$, a contradiction. One may now assume there is $k \geq 1$ such that $x_0 \in H(a_i, c_i)$ for $i \leq k$ and $\langle x_0, a_i \rangle > c_i$ for $i > k$. Set

$$A = H(a_1, c_1) \cap \cdots \cap H(a_k, c_k).$$

We claim that $A = \{x_0\}$. If there is $x_1 \in A \setminus \{x_0\}$, pick $B_\delta(x_0)$ whose closure is contained in $H^+(a_i, c_i)$ for all $i > k$. Since $z = tx_0 + (1 - t)x_1$ is in A for all $t \in \mathbb{R}$, making $t = 1 + \delta/\|x_1 - x_0\|$ one derives $z \in A$ and $\|z - x_0\| = \delta$, thus $z \in Q$. Note that if $k = r$, then x_1 is already in Q

and in this case we set $z = x_1$. Making $t = -1$ in $z_1 = tz + (1-t)x_0$ one concludes $z_1 \in A$ and $\|z_1 - x_0\| = \delta$. Altogether z, z_1 are in $Q \setminus \{x_0\}$ and $x_0 = (z + z_1)/2$, a contradiction because $Q \setminus \{x_0\}$ is a convex set.

From the equality $\{x_0\} = A = A \cap Q$ one has that $\{x_0\}$ is an intersection of faces. Using Proposition 7.2.8 yields that $\{x_0\}$ is a face, as required. □

Proposition 7.2.12 *Let Q be a polyhedral set in $\mathbb{R}^n$ which is not an affine space and*

$$Q = \bigcap_{i=1}^{r} H^+(a_i, c_i),$$

where $0 \neq a_i \in \mathbb{R}^n$ and $c_i \in \mathbb{R}$ for all i. If $x_0 \in \mathbb{R}^n$, then x_0 is a vertex of Q if and only if

(a) $\{x_0\} = \cap_{i \in I} H(a_i, c_i)$ *for some* $I \subset \{1, \ldots, r\}$, *and*

(b) $\langle x_0, a_i \rangle \geq c_i$ *for all* $i = 1, \ldots, r$.

Proof. $\Rightarrow$) This direction follows at once from the proof of Lemma 7.2.11.

$\Leftarrow$) Note $x_0 \in Q$. Since the intersection of faces of Q is a face, see Proposition 7.2.8, one has that $\{x_0\}$ is a face. □

If $a \in \mathbb{R}^n$, $a \neq 0$, then the set H_a will denote the *hyperplane* of $\mathbb{R}^n$ through the origin with normal vector a, that is, $H_a = \{x \in \mathbb{R}^n \,|\, \langle x, a \rangle = 0\}$. This hyperplane determines two *closed halfspaces* denoted by

$$H_a^+ = \{x \in \mathbb{R}^n \,|\, \langle x, a \rangle \geq 0\} \quad \text{and} \quad H_a^- = \{x \in \mathbb{R}^n \,|\, \langle x, a \rangle \leq 0\}.$$

Proposition 7.2.13 *If C is a closed convex cone in $\mathbb{R}^n$ and H a supporting hyperplane of C, then H is a hyperplane passing through the origin.*

Proof. Let $H = H(a, c)$. Since $0 \in C \subset H^+$, one has $c \leq 0$. If $C \cap H = \{0\}$, then $c = 0$ as required. Assume $C \cap H \neq \{0\}$ and pick $0 \neq z \in C$ such that $\langle z, a \rangle = c$. Using $tz \in C \subset H^+(a, c)$ for all $t \geq 0$, one derives

$$tc = t\langle z, a \rangle = \langle tz, a \rangle \geq c,$$

for every $t \geq 0$. Thus $c = 0$. □

Proposition 7.2.14 *Let C be a cone in $\mathbb{R}^n$ and let*

$$C = H_{a_1}^+ \cap \cdots \cap H_{a_r}^+$$

be a representation of C. If $a_i \in \mathbb{Q}^n \setminus \{0\}$ for all i, then C is a finitely generated rational cone, that is, there are $\beta_i \in \mathbb{Q}^n$ such that

$$C = \mathbb{R}_+ \beta_1 + \cdots + \mathbb{R}_+ \beta_s.$$

Proof. Consider the polyhedral set

$$P = \bigcap_{a \in A} H^-(a, 1),$$

where

$$A = \{\varepsilon_1 e_1 + \cdots + \varepsilon_n e_n | \, \varepsilon_i \in \{1, -1\} \text{ for all } i\}.$$

Note that P is a bounded set and the origin is an interior point of P. Hence P is a polytope and consequently the set $Q = C \cap P$ is also a polytope. Thus by [39, Theorem 7.2] one can represent Q as $Q = \text{conv}(\beta_1, \ldots, \beta_s)$, where $\beta_1, \ldots, \beta_s$ are the vertices of Q. By Proposition 7.2.12 each vertex of Q is the unique solution of a system of linear equations with rational coefficients. Thus all the β_i have rational entries.

Since $\beta_i \in C$ for all i one has $\mathbb{R}_+\beta_1 + \cdots + \mathbb{R}_+\beta_s \subset C$. Conversely if $x \in C$, there is an scalar $t > 0$ such that $tx \in Q$, and one rapidly obtains that x is in the cone generated by $\beta_1, \ldots, \beta_s$. □

Definition 7.2.15 A proper face F of a polyhedral set $Q \subset \mathbb{R}^n$ is called a *facet* of Q if

$$\dim(F) = \dim(Q) - 1.$$

Theorem 7.2.16 *Let Q be a polyhedral set in $\mathbb{R}^n$ which is not an affine space such that*

$$Q = \text{aff}(Q) \cap H^+(a_1, c_1) \cap \cdots \cap H^+(a_r, c_r)$$

where a_i is in $\mathbb{R}^n \setminus \{0\}$, $c_i \in \mathbb{R}$ for all i and none of the halfspaces $H^+(a_i, c_i)$ can be omitted in the intersection. Then the facets of Q are precisely the sets $F_1, \ldots, F_r$, where $F_i = Q \cap H(a_i, c_i)$, and each proper face of Q is the intersection of those facets of Q that contain it.

Proof. See [310, Theorem 3.2.1]. □

Definition 7.2.17 If a polyhedron Q in $\mathbb{R}^n$ is represented as

$$Q = \bigcap_{i=1}^{r} H^+(a_i, c_i) \qquad (*)$$

and satisfies

$$Q \neq \bigcap_{\substack{1 \leq i \leq r \\ i \neq j}} H^+(a_i, c_i)$$

for all j, we say that $(*)$ is an *irreducible representation* of Q.

We shall always assume that an affine space in $\mathbb{R}^n$ has the topology induced by the usual topology of $\mathbb{R}^n$. By the *relative interior* of a set X in $\mathbb{R}^n$, denoted by $\mathrm{ri}(X)$, we mean the interior of X in $\mathrm{aff}(X)$.

Theorem 7.2.18 *Let Q be a polyhedral set in $\mathbb{R}^n$ with $\dim(Q) = n$ and such that $Q \neq \mathbb{R}^n$. Let*

$$Q = H^+(a_1, c_1) \cap \cdots \cap H^+(a_r, c_r) \qquad (*)$$

be a representation of Q with $H^+(a_1, c_1), \ldots, H^+(a_r, c_r)$ distinct, where a_i is in $\mathbb{R}^n \setminus \{0\}$ for all i. Set $F_i = Q \cap H(a_i, c_i)$, for $i = 1, \ldots, r$. Then

(a) $\mathrm{ri}(Q) = \{x \in \mathbb{R}^n \mid \langle x, a_1 \rangle > c_1, \ldots, \langle x, a_r \rangle > c_r\}$.

(b) *Each facet F of Q is of the form $F = F_i$ for some i.*

(c) *Each F_i is a facet of Q if and only $(*)$ is irreducible.*

Proof. See [39, Theorem 8.2] and [310, Theorem 3.2.1]. □

Corollary 7.2.19 *If $Q \neq \mathbb{R}^n$ is a polyhedral set of dimension n, then Q has a unique irreducible representation*

$$Q = H^+(a_1, c_1) \cap \cdots \cap H^+(a_r, c_r),$$

as an intersection of closed halfspaces.

Proof. Follows from parts (b) and (c) of Theorem 7.2.18. Note that if the set $F = Q \cap H(a, c)$ is a facet of Q, then $\mathrm{aff}(F) = H(a, c)$. □

Corollary 7.2.20 *If $C \neq \mathbb{R}^n$ is a polyhedral cone of dimension n, then there is a unique irreducible representation*

$$C = H^+_{a_1} \cap \cdots \cap H^+_{a_r}, \text{ where } a_i \in \mathbb{R}^n \setminus \{0\}.$$

Proof. Note that according to Proposition 7.2.13 a set F is a proper face of C if there is a supporting hyperplane H_a of C such that

$$F = C \cap H_a \neq \emptyset \text{ and } C \not\subset H_a.$$

In particular the facets of C are defined by hyperplanes through the origin, therefore the irreducible representation of C has the required form. □

The following two results are quite useful for determining the facets of a polyhedral cone.

Proposition 7.2.21 *Let $\mathcal{A}$ be a finite set of points in $\mathbb{Z}^n$. If F is a nonzero face of $\mathbb{R}_+\mathcal{A}$, then $F = \mathbb{R}_+\mathcal{A}'$ for some $\mathcal{A}' \subset \mathcal{A}$.*

Proof. Let $F = \mathbb{R}_+ \mathcal{A} \cap H_a$ with $\mathbb{R}_+ \mathcal{A} \subset H_a^-$. Then F is equal to the cone generated by the set $\mathcal{A}' = \{\alpha \in \mathcal{A} | \langle \alpha, a \rangle = 0\}$. □

Corollary 7.2.22 *Let $\mathcal{A}$ be a finite set in $\mathbb{Z}^n$ and F a face of $\mathbb{R}_+ \mathcal{A}$.*

(a) *If $\dim(F) = 1$ and $\mathcal{A} \subset \mathbb{N}^n$, then $F = \mathbb{R}_+ \alpha$ for some $\alpha \in \mathcal{A}$.*

(b) *If $\dim(\mathbb{R}_+ \mathcal{A}) = n$ and F is a facet defined by the supporting hyperplane H_a, then H_a is generated by a linearly independent subset of $\mathcal{A}$.*

Definition 7.2.23 Let Q be a polyhedral cone in $\mathbb{R}^n$ with $\dim(Q) = n$ and such that $Q \neq \mathbb{R}^n$. Let

$$Q = \bigcap_{i=1}^{r} H_{a_i}^+$$

be the irreducible representation of the cone Q. If $a_i = (a_{i1}, \ldots, a_{in})$ we shall call

$$a_{i1}x_1 + \cdots + a_{in}x_n = 0 \quad (i = 1, \ldots, r)$$

a set of *equations of the cone Q*.

Proposition 7.2.24 *If Q is a proper cone in $\mathbb{R}^n$ that can be written as*

$$Q = \mathbb{R}_+ \alpha_1 + \cdots + \mathbb{R}_+ \alpha_q \neq \mathbb{R}^n$$

with $\alpha_i \in \mathbb{Q}^n$ for all i and $\dim(Q) = n$, then there are unique (up to sign) $a_1, \ldots, a_r$ in $\mathbb{Z}^n$ with relatively prime entries and such that

$$Q = \bigcap_{i=1}^{r} H_{a_i}^+$$

is the irreducible representation of Q.

Proof. First note that if H_b is a supporting hyperplane of Q generated by a set of $n-1$ linearly independent vectors in $\{\alpha_1, \ldots, \alpha_q\}$, then by the Gram-Schmidt process H_b has an orthogonal basis of vectors in $\mathbb{Q}^n$, and consequently there is a normal vector a to H_b such that $a \in \mathbb{Q}^n$ and $H_a = H_b$. Hence multiplying a by a suitable integer and then dividing by the greatest common divisor of the entries, one may assume $H_a = H_b$, where a is in $\mathbb{Z}^n$ and has relative prime entries. Observe that a, b are linearly dependent because the orthogonal complement of H_a is one dimensional. It is readily seen that a is uniquely determined up to sign.

To complete the proof use Corollary 7.2.22 (b) to see that any facet of Q is defined by a supporting hyperplane H_b as above. □

Definition 7.2.25 Given $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, we set

$$|a| = |a_1| + \cdots + |a_n|.$$

Theorem 7.2.26 *Let $\mathcal{A}$ be a finite set of nonzero points in $\mathbb{N}^n$ and $\mathcal{A}'$ the set of $\alpha \in \mathcal{A}$ such that $\mathbb{R}_+\alpha$ is a face of $\mathbb{R}_+\mathcal{A}$ of dimension 1. Then*

$$\mathbb{R}_+\mathcal{A} = \mathbb{R}_+\mathcal{A}'.$$

Proof. Set $\mathcal{A} = \{\alpha_1, \ldots, \alpha_q\}$ and $S = \operatorname{conv}(e_1, \ldots, e_n)$. Consider the convex polytope $P = S \cap \mathbb{R}_+\mathcal{A}$. Note that $P = \operatorname{conv}(\beta_1, \ldots, \beta_q)$, where $\beta_i = \alpha_i/|\alpha_i|$. Therefore by [39, Theorem 7.2] the set of vertices $V(P)$ of P is contained in $\{\beta_1, \ldots, \beta_q\}$ and P is the convex hull of $V(P)$. Since $\mathbb{R}_+\mathcal{A} = \mathbb{R}_+V(P)$ it suffices to show that $\mathbb{R}_+\beta$ is a face of the cone spanned by $\mathcal{A}$ for every β in $V(P)$.

Let $\beta \in V(P)$. There is a hyperplane $H = \{x \in \mathbb{R}^n \,|\, \langle x, a\rangle = c\}$ such that $P \cap H = \{\beta\}$ and $P \subset H^-$. Set $a = (a_1, \ldots, a_n)$. To complete the proof we now show that

$$\mathbb{R}_+\mathcal{A} \subset H_b^- \quad \text{and} \quad \mathbb{R}_+\mathcal{A} \cap H_b = \mathbb{R}_+\beta,$$

where $b = (a_1 - c, \ldots, a_n - c)$. Since $\alpha_i/|\alpha_i| \in P \subset H^-$ for all i one has

$$\left\langle \frac{\alpha_i}{|\alpha_i|}, a \right\rangle \leq c \implies \langle \alpha_i, a\rangle \leq c|\alpha_i| = \langle \alpha_i, (c, \ldots, c)\rangle,$$

thus $\langle \alpha_i, b\rangle \leq 0$ for all i. This proves $\mathbb{R}_+\mathcal{A} \subset H_b^-$. There is i such that $\beta = \alpha_i/|\alpha_i|$. Next we show $\mathbb{R}_+\beta \subset \mathbb{R}_+\mathcal{A} \cap H_b$. It suffices to prove $\beta \in H_b$. Since $\beta \in H$ we obtain

$$\left\langle \frac{\alpha_i}{|\alpha_i|}, a \right\rangle = c = \left\langle \frac{\alpha_i}{|\alpha_i|}, (c, \ldots, c) \right\rangle.$$

Hence $\beta \in H_b$. To prove the reverse inclusion take $0 \neq x \in \mathbb{R}_+\mathcal{A} \cap H_b$. Then

$$\langle x, a\rangle = \langle x, (c, \ldots, c)\rangle = c|x| \implies \left\langle \frac{x}{|x|}, a \right\rangle = c.$$

Thus $x/|x| \in H$. Observe that $x/|x| \in S$ and $x/|x| \in \mathbb{R}_+\mathcal{A}$, hence

$$\frac{x}{|x|} \in S \cap \mathbb{R}_+\mathcal{A} = P.$$

Altogether

$$\frac{x}{|x|} \in P \cap H = \{\beta\},$$

this proves $x \in \mathbb{R}_+\beta$. □

Notation Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we write $x \geq 0$ if $x_i \geq 0$ for all i.

Theorem 7.2.27 (Farkas) *Let A an $n \times q$ matrix with entries in a field K and $z \in K^n$. Assume $K = \mathbb{Q}$ or $K = \mathbb{R}$. Then either there exists $x \in K^q$ with $Ax^t = z^t$ and $x \geq 0$ or there exists $v \in K^n$ with $vA \geq 0$ and $vz^t < 0$, but not both. Here x^t denotes the transpose of x.*

Proof. See [315, Proposition 1.8] for the case $K = \mathbb{R}$ and [207, 216] for the case $K = \mathbb{Q}$. □

Normality of monomial subrings One of the issues to be discussed in first place is a description of the normalization of a monomial subring. For convenience recall that if A is a domain with field of fractions K_A, the *normalization* or *integral closure* of A is the subring $\overline{A}$ consisting of all the elements of K_A which are integral over A.

Theorem 7.2.28 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and let F be a finite set of monomials in R. Then*

$$\overline{k[F]} = k[\{x^a \,|\, a \in \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A}\}],$$

where $\mathcal{A} = \log(F)$ is the set of exponent vectors of the monomials in F.

Proof. Set

$$B = k[\{x^a \,|\, a \in \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A}\}].$$

First we show $\overline{k[F]} \subset B$. Take $z \in \overline{k[F]}$, since $k[F] \subset R$ and R is normal we get $\overline{k[F]} \subset R$ and $z \in R$. Hence one can write

$$z = \sum_{i=1}^{r} d_i x^{\gamma_i},$$

where $d_i \in k \setminus \{0\}$ and $\gamma_1, \ldots, \gamma_r$ distinct nonzero points in $\mathbb{N}^n$. Write $\gamma_i = n_i\beta_i$, with n_i equal to the gcd of the entries of γ_i (observe that $\beta_1, \ldots, \beta_r$ are not necessarily distinct). Consider the cone C spanned by $\mathcal{A}$ and $\{\beta_1, \ldots, \beta_r\}$. One must show $\beta_i \in \mathbb{R}_+\mathcal{A}$ for all i. Assume on the contrary that C is not equal to $\mathbb{R}_+\mathcal{A}$. By Theorem 7.2.26 one may assume that $\mathbb{R}_+\beta_1$ is a face of C not contained in $\mathbb{R}_+\mathcal{A}$. Let H_a be a hyperplane so that $H_a \cap C = \mathbb{R}_+\beta_1$ and $C \subset H_a^-$. There is $1 \leq \ell \leq r$ such that

$$n_\ell = \sup_{1 \leq j \leq r} \{n_j \,|\, \gamma_j = n_j\beta_1\}.$$

Since z is integral over $k[F]$ it satisfies a monic polynomial f of degree m with coefficients in $k[F]$. To derive a contradiction we claim that the term

$x^{m\gamma_\ell}$ occurs only once in the expansion of $f(z)$ as a sum of monomials. Assume the equality

$$(x^{\gamma_\ell})^m = x^\alpha (x^{\gamma_{i_1}})^{m_{i_1}} \cdots (x^{\gamma_{i_t}})^{m_{i_t}},$$

where $m_{i_j} > 0$ for all j, $m \geq \sum_{j=1}^t m_{i_j}$ and $\alpha \in \mathbb{R}_+\mathcal{A}$. As $\gamma_\ell = n_\ell \beta_1$, one has $\langle m\gamma_\ell, a\rangle = 0$ and from this equality one rapidly derives

$$\langle \alpha, a\rangle = \langle \beta_{i_1}, a\rangle = \cdots = \langle \beta_{i_t}, a\rangle = 0.$$

Hence $\alpha, \beta_{i_i}, \ldots, \beta_{i_t} \in H_a \cap C = \mathbb{R}_+\beta_1$. Note $\beta_1 \notin \mathbb{R}_+\mathcal{A}$. Thus $\alpha = 0$ and $\beta_{i_j} = \beta_1$ for all j. Therefore $t = 1$ and $\gamma_\ell = \gamma_{i_1}$, as claimed. Altogether we get $f(z) \neq 0$, which is impossible. This proves that $\gamma_i \in \mathbb{R}_+\mathcal{A}$ for all i. On the other hand using that z belongs to the field of fractions of $k[F]$ it follows that $\gamma_i \in \mathbb{Z}\mathcal{A}$ for all i.

Let us now show the other containment $B \subset \overline{k[F]}$. Note the equality

$$\mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A} = \mathbb{Z}\mathcal{A} \cap \mathbb{Q}_+\mathcal{A},$$

which follows from Theorem 7.2.27. An straightforward calculation rapidly shows that x^α is in the field of fractions of $k[F]$ if $\alpha \in \mathbb{Z}\mathcal{A}$, and x^α is an integral element over $k[F]$ if $\alpha \in \mathbb{Q}_+\mathcal{A}$. Hence $B \subset \overline{k[F]}$. □

Corollary 7.2.29 *Let R be a polynomial ring over a field k and F a finite set of monomials in R. Then $k[F]$ is normal if and only if*

$$\mathbb{N}\mathcal{A} = \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A},$$

where $\mathcal{A} = \log(F)$.

Proof. It suffices to observe the equality

$$k[F] = k[\{x^\alpha | \alpha \in \mathbb{N}\mathcal{A}\}]$$

and to use the description of $\overline{k[F]}$ given above. □

Theorem 7.2.30 (E. Noether) *Let $B = k[t_1, \ldots t_q]$ be a polynomial ring over a field k and $\mathfrak{p}$ a prime ideal of B. Then the integral closure of $B/\mathfrak{p}$ is a finitely generated $B/\mathfrak{p}$ module.*

Proof. See [92, Corollary 13.13]. □

In general it is hard to compute the integral closure of an affine domain; see [295] and [298, Chapter 6].

Corollary 7.2.31 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and F a finite set of monomials in R. Then*

$$\overline{k[F]} = x^{\beta_1}k[F] + \cdots + x^{\beta_m}k[F],$$

for some $x^{\beta_1}, \ldots, x^{\beta_m}$ in $\overline{k[F]}$.

Proof. Use Theorem 7.2.28 and the finiteness of the integral closure of an affine domain, see Theorem 7.2.30. This result also follows from the next lemma. □

Lemma 7.2.32 *Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_q\}$ be a finite set of points in $\mathbb{Z}^n$. Then there exist $\gamma_1, \ldots, \gamma_m$ in $\mathbb{Z}^n$ such that*

$$\mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A} = \mathbb{N}\gamma_1 + \cdots + \mathbb{N}\gamma_m$$

and $\gamma_i \in \left[-q \max_{1\leq i\leq q} |\alpha_i^-|, q \max_{1\leq i\leq q} |\alpha_i^+|\right]^n$ *for all i.*

Proof. Set $M = q \max_{1\leq i\leq q} |\alpha_i^+|$ and $N = -q \max_{1\leq i\leq q} |\alpha_i^-|$. Recall the equality

$$C = \mathbb{Z}\mathcal{A} \cap \mathbb{R}_+\mathcal{A} = \mathbb{Z}\mathcal{A} \cap \mathbb{Q}_+\mathcal{A}.$$

Let $\beta \in C$. One can write

$$\beta = \sum_{i=1}^{q} \left(\frac{x_i}{y_i}\right) \alpha_i,$$

where $x_i \in \mathbb{N}$ and $0 \neq y_i \in \mathbb{N}$. By the division algorithm there are non negative integers r_i, n_i such that $x_i = n_i y_i + r_i$ and $0 \leq r_i < y_i$. Therefore one can write

$$\beta = \sum_{i=1}^{q} n_i\alpha_i + \sum_{i=1}^{q} a_i\alpha_i,$$

where $n_i \in \mathbb{N}$ and $a_i \in [0,1] \cap \mathbb{Q}$. Since $\sum_{i=1}^{q} a_i\alpha_i$ is in $C \cap [N,M]^n$ one rapidly obtains that

$$\mathcal{A} \cup (C \cap [N,M]^n)$$

is a generating set for C with the required property. Our argument was based on the proof of Gordan's Lemma given in [44]. □

Definition 7.2.33 Let R be a polynomial ring over a field k and F a finite set of monomials in R. A decomposition

$$k[F] = \bigoplus_{i=0}^{\infty} k[F]_i$$

of the k-vector space $k[F]$ is an *admissible grading* if $k[F]$ is a positively graded k-algebra with respect to this decomposition and each component $k[F]_i$ has a finite k-basis consisting of monomials.

Theorem 7.2.34 (Danilov, Stanley) *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and F a finite set of monomials in R. If $k[F]$ is normal, then the canonical module $\omega_{k[F]}$ of $k[F]$, with respect to an arbitrary admissible grading, can be expressed as*

$$\omega_{k[F]} = (\{x^a | \, a \in \mathbb{N}\mathcal{A} \cap \mathrm{ri}(\mathbb{R}_+ \mathcal{A})\}), \tag{7.1}$$

where $\mathcal{A} = \log(F)$ and $\mathrm{ri}(\mathbb{R}_+ \mathcal{A})$ denotes the relative interior of $\mathbb{R}_+ \mathcal{A}$.

The formula above represents the canonical module of $k[F]$ as an ideal of $k[F]$ generated by monomials. Soon it will become clear how to apply it to estimate some a-invariants. For a comprehensive treatment of the Danilov-Stanley formula see [44, Theorem 6.3.5] and [76].

Theorem 7.2.35 (Hochster) *Let R be a polynomial ring over a field k and F a finite set of monomials in R. If $k[F]$ is normal, then $k[F]$ is Cohen-Macaulay.*

Proof. See [30] and [172]. □

Theorem 7.2.36 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and F a finite set of monomials in R. If*

$$A = k[h_1, \ldots, h_d] \hookrightarrow k[F]$$

is a homogeneous Noether normalization of $k[F]$, then $\overline{k[F]}$ is a free A-module whose generators have degree at most $\sum_{i=1}^{d} \deg(h_i)$.

Proof. Let $S = k[F]$ endowed with the induced grading $S_i = S \cap R_i$. Note that the composition

$$A \hookrightarrow S \hookrightarrow \overline{S}$$

is a homogeneous Noether normalization. Since $\overline{S}$ is Cohen-Macaulay by Theorem 7.2.35, one can then use Proposition 2.2.14 to obtain a direct sum decomposition

$$\overline{S} = A\overline{x}^{\beta_1} \oplus \cdots \oplus A\overline{x}^{\beta_m}.$$

Therefore the Hilbert series of $\overline{S}$ can be expressed as

$$F(\overline{S}, t) = \sum_{i=1}^{m} F(Ax^{\beta_i}, t) = \frac{\sum_{i=1}^{m} t^{\deg \beta_i}}{\prod_{i=1}^{d} (1 - t^{\deg h_i})}.$$

Applying Theorem 7.2.34 one derives that $a(\overline{S})$, the a-invariant of $\overline{S}$, is negative. Using that $a(\overline{S})$ is equal to the degree of $F(\overline{S}, t)$ as a rational function (see Proposition 4.2.3) one concludes $\deg x^{\beta_i} \leq \sum_{i=1}^{m} \deg h_i$. □

An algorithm to compute normalizations Let us describe some steps of an algorithm of Winfried Bruns and Robert Koch [48] that effectively computes normalizations. Let $\mathcal{A}$ be a finite set of points in $\mathbb{N}^n$. First one finds $w_1, \ldots, w_d$ in $\mathbb{Z}^n$ such that

$$\mathbb{Z}\mathcal{A} = \mathbb{Z}w_1 \oplus \cdots \oplus \mathbb{Z}w_d.$$

There exists an isomorphism of $\mathbb{Z}$-modules

$$T\colon \mathbb{Z}\mathcal{A} \longrightarrow \mathbb{Z}^d$$

with $T(w_i) = e_i$. Because of Lemma 7.2.32 there are $\beta_1, \ldots \beta_r$ in a bounding box $[N, M]^d$ such that

$$\mathbb{Z}^d \cap \mathbb{R}_+ T(\mathcal{A}) = \mathbb{N}\beta_1 + \cdots + \mathbb{N}\beta_r.$$

The determination of the β_i's can be accomplished by finding the irreducible representation of the cone $\mathbb{R}_+ T(\mathcal{A})$ as an intersection of closed half spaces (see Corollary 7.2.20), and then testing which elements in

$$[N, M]^d \cap \mathbb{Z}^d$$

lie in the cone spanned by $T(\mathcal{A})$. To finish the description note

$$\mathbb{Z}\mathcal{A} \cap \mathbb{R}_+ \mathcal{A} = \mathbb{N}T^{-1}(\beta_1) + \cdots + \mathbb{N}T^{-1}(\beta_r).$$

Observe that the computation of the irreducible representation of $\mathbb{R}_+ T(\mathcal{A})$ can be performed using Corollary 7.2.22; in several specific examples we have used [48] or PORTA [63] to rapidly determine such representation.

Example 7.2.37 Let $\mathcal{A} \subset \mathbb{N}^7$ be the set of points

$$\begin{array}{llll} \alpha_1 = e_1 + e_2, & \alpha_2 = e_2 + e_3, & \alpha_3 = e_1 + e_3, & \alpha_4 = e_3 + e_4, \\ \alpha_5 = e_4 + e_5, & \alpha_6 = e_5 + e_6, & \alpha_7 = e_6 + e_7, & \alpha_8 = e_5 + e_7. \end{array}$$

Note

$$\mathbb{Z}\mathcal{A} = \mathbb{Z}w_1 \oplus \cdots \oplus \mathbb{Z}w_7,$$

where the w_i's are the row vectors of the matrix:

$$P = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}.$$

Next we consider the affine transformation T determined by $T(w_i) = e_i$. An straightforward computation shows that the points in $T(\mathcal{A})$ correspond to the rows of the matrix:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & -2 & 2 & -2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}.$$

Using [48] or PORTA [63] we obtain that a point $x = (x_1, \ldots, x_7)$ is in $\mathbb{R}_+ T(\mathcal{A})$ if and only if x satisfies the system of linear inequalities

$$\begin{array}{rrrrrrrl}
 & & & & -x_6 & -2x_7 & \leq 0 & \\
 & & & -x_5 & -x_6 & & \leq 0 & \\
 & & -x_4 & & & -2x_7 & \leq 0 & \\
 & & -x_4 & -2x_5 & & +2x_7 & \leq 0 & \\
 & -x_3 & & & -2x_6 & -2x_7 & \leq 0 & \\
 & -x_3 & -x_4 & & & & \leq 0 & \\
 & & & -x_5 & & +x_7 & \leq 0 & \\
-x_2 & & & & & -x_7 & \leq 0 & \\
-x_2 & & & -x_5 & & +x_7 & \leq 0 & \\
-2x_2 & +x_3 & & & & & \leq 0 & \\
-x_1 & & & & -x_6 & -x_7 & \leq 0 & \\
-2x_1 & & -x_4 & & & & \leq 0 & \\
-x_1 & -x_2 & & & & & \leq 0 & \\
 & & & & & -x_i & \leq 0, & i = 1, 3, 7.
\end{array}$$

It follows that

$$\mathbb{Z}^d \cap \mathbb{R}_+ T(\mathcal{A}) = \mathbb{N}T(\mathcal{A}) + \mathbb{N}\beta,$$

where $\beta = (1, 0, 1, -1, 2, -1, 1)$. Hence

$$\mathbb{Z}\mathcal{A} \cap \mathbb{R}_+ \mathcal{A} = \mathbb{N}\mathcal{A} + \mathbb{N}(1, 1, 1, 0, 1, 1, 1).$$

The linear inequalities shown above define the facets of the cone spanned by $T(\mathcal{A})$.

Homogeneous subrings The class of homogeneous monomial subrings has been studied with detail in [278]. Examples that fall into this class are Rees algebras of ideals generated by monomials of the same degree, subrings generated by monomials of the same degree, and polytopal subrings.

Definition 7.2.38 Let $F = \{x^{\alpha_1}, \ldots, x^{\alpha_q}\}$ be a set of monomials in the ring of polynomials $R = K[x_1, \ldots, x_n]$, where K is a field. The monomial subring $K[F]$ is said to be *homogeneous* if there is $v \in \mathbb{Q}^n$ such that

$$\langle \alpha_i, v \rangle = 1 \quad \text{for} \quad i = 1, \ldots, q.$$

Proposition 7.2.39 *Let $F = \{x^{\alpha_1}, \ldots, x^{\alpha_q}\}$ be a set of monomials in the ring $R = K[x_1, \ldots, x_n]$, where K is a field. Then $K[F]$ is a homogeneous subring if and only if $K[F]$ is a standard graded algebra with the grading*

$$K[F]_i = \sum_{|c|=i} K(x^{\alpha_1})^{c_1} \cdots (x^{\alpha_q})^{c_q}, \quad \textit{where} \quad |c| = c_1 + \cdots + c_q.$$

Proof. $\Rightarrow$) Let $v \in \mathbb{Q}^n$ such that $\langle \alpha_i, v \rangle = 1$ for all i. It suffices to prove

$$K[F] = \bigoplus_{i=0}^{\infty} K[F]_i,$$

because one clearly has

$$K[F]_i K[F]_j \subset K[F]_{i+j} \quad \text{for all} \quad i, j.$$

First let us prove that $K[F]_i \cap K[F]_j = \{0\}$ for $i \neq j$. If this intersection is not zero, one has an equality

$$(x^{\alpha_1})^{a_1} \cdots (x^{\alpha_q})^{a_q} = (x^{\alpha_1})^{c_1} \cdots (x^{\alpha_q})^{c_q},$$

with $|a| = i$ and $|c| = j$. Hence

$$i = |a| = \left\langle \sum_{i=1}^{q} a_i \alpha_i, v \right\rangle = \left\langle \sum_{i=1}^{q} c_i \alpha_i, v \right\rangle = |c| = j,$$

a contradiction. To finish assume $f_{i_1} + \cdots + f_{i_r} = 0$, with $f_{i_j} \in K[F]_{i_j}$ and $i_1 < \cdots < i_r$. If $f_{i_r} \neq 0$, using that the monomials of R form a K-basis, one has $K[F]_{i_r} \cap K[F]_{i_j} \neq \{0\}$ for some $j < r$, a contradiction. Thus $f_{i_r} = 0$. By induction all the f_{i_j} must be zero, as required.

$\Leftarrow$) Set $A = \operatorname{aff}_{\mathbb{Q}}(\alpha_1, \ldots, \alpha_q)$ and $r = \dim(A)$. To begin with we claim that 0 is not in A. Otherwise if $0 \in A$, write $0 = a_1\alpha_1 + \cdots + a_q\alpha_q$ with $a_i \in \mathbb{Q}$ for all i and $\sum_{i=1}^{q} a_i = 1$. There is $0 \neq p \in \mathbb{Z}$ such that $pa_i \in \mathbb{Z}$ for all i. One may assume, by reordering the α_i, that $pa_i \geq 0$ for $i \leq s$ and $pa_i \leq 0$ for $i > s$. As $K[F]$ is graded, from the equality

$$(x^{\alpha_1})^{pa_1} \cdots (x^{\alpha_s})^{pa_s} = (x^{\alpha_{s+1}})^{-pa_{s+1}} \cdots (x^{\alpha_q})^{-pa_q}$$

one derives

$$p(a_1 + \cdots + a_s) = -p(a_{s+1} + \cdots + a_q),$$

that is, $p\sum_{i=1}^{q} a_i = 0$, a contradiction. Thus $0 \notin A$, and consequently $r < n$. It is well known that a linear variety A in $\mathbb{Q}^n$ of dimension r can be written as an intersection of hyperplanes

$$A = \bigcap_{i=1}^{n-r} H(y_i, c_i),$$

where $0 \neq y_i \in \mathbb{Q}^n$ and $c_i \in \mathbb{Q}$ for all i; see [310, Corollary 1.4.2]. Since $0 \notin A$, one has $c_j \neq 0$ for some j. To finish it suffices to set $v = y_j/c_j$. □

Corollary 7.2.40 *Let $F \subset K[\mathbf{x}]$ be a finite set of monomials and P_F the toric ideal of $K[F]$. Then P_F is a graded ideal with respect to the standard grading if and only if $K[F]$ is a homogeneous subring.*

Proof. As P_F is generated by a finite set of binomials, the result follows rapidly. □

Proposition 7.2.41 *Let $F = \{f_1, \ldots, f_q\}$ be a set of monomials in the polynomial ring $R = K[\mathbf{x}]$ over a field K and $T_1, \ldots, T_q, T$ a new set of variables. If φ, ψ are the maps of K-algebras defined by the diagram*

$$\begin{array}{ccc} K[T_1, \ldots, T_q] & \xrightarrow{\psi} & K[F] \\ \downarrow{\scriptstyle \varphi} & \overset{\overline{\psi}}{\nearrow} & \\ K[TF] & & \end{array} \qquad \begin{array}{ccc} T_i & \overset{\psi}{\rightsquigarrow} & f_i \\ \downarrow{\scriptstyle \varphi} & & \\ Tf_i & & \end{array}$$

then there is a unique an epimorphism $\overline{\psi}$ such $\psi = \overline{\psi}\varphi$. In addition $\overline{\psi}$ is an isomorphism if and only if $K[F]$ and $K[TF]$ have the same Krull dimension.

Proof. To show the existence of $\overline{\psi}$ it suffices to prove $\ker(\varphi) \subset \ker(\psi)$. This containment follows at once because any binomial in $\ker(\varphi)$ clearly belongs to $\ker(\psi)$, and $\ker(\varphi)$ is a binomial ideal.

Note that $\ker(\varphi)$ and $\ker(\psi)$ are both prime ideals. The map $\overline{\psi}$ is an isomorphism if and only if $\ker(\varphi) = \ker(\psi)$, thus to finish we need only observe that the last equality holds if and only if $K[F]$ and $K[TF]$ have the same dimension. □

With the notation above one has

Corollary 7.2.42 *The map $\overline{\psi}\colon K[TF] \to K[F]$ is a isomorphism if and only if $K[F]$ is a homogeneous subring.*

Proof. If $\overline{\psi}$ is an isomorphism, then by the proof above $\ker(\psi) = \ker(\varphi)$ and $K[F]$ is homogeneous because $\ker(\varphi)$ is homogeneous with the standard grading of $K[T_1, \ldots, T_q]$.

Conversely assume $\ker(\psi)$ homogeneous with the standard grading. Note that any homogeneous binomial in $\ker(\psi)$ is also in $\ker(\varphi)$, thus one has $\ker(\psi) = \ker(\varphi)$ and $\overline{\psi}$ is an isomorphism. □

Ehrhart function In this paragraph we consider a fixed set of distinct monomials $F = \{x^{\alpha_1}, \ldots, x^{\alpha_q}\}$ in a polynomial ring $R = K[x_1 \ldots, x_n]$ over a field K. Let

$$P = \text{conv}(\mathcal{A})$$

be the convex hull of the set $\mathcal{A} = \{\alpha_1, \ldots, \alpha_q\}$. The *normalized Ehrhart ring* of P is the graded algebra

$$A_P = \bigoplus_{i=0}^{\infty} (A_P)_i \subset R[T],$$

where the i*th* component is given by

$$(A_P)_i = \sum_{\alpha \in \mathbb{Z}\mathcal{A} \cap iP} K x^{\alpha} T^i.$$

Note that because of the convexity of P one has

- $(A_P)_i (A_P)_j \subset (A_P)_{i+j}$ and
- $(A_P)_i \cap (A_P)_j = \{0\}$ for $i \neq j$.

Thus A_P is in fact a graded K-algebra.

Proposition 7.2.43 $\overline{K[FT]} \subset A_P$, *with equality if* $K[F]$ *is homogeneous.*

Proof. Set $\mathcal{A}' = \{(\alpha_1, 1), \ldots, (\alpha_q, 1)\} \subset \mathbb{Z}^{n+1}$. First we show:

$$\mathbb{Z}\mathcal{A}' \cap \mathbb{R}_+ \mathcal{A}' \subset \{(\alpha, i) | \, \alpha \in \mathbb{Z}\mathcal{A} \cap iP \text{ and } i \in \mathbb{N}\}. \qquad (*)$$

Take $z = (\alpha, i)$ in $\mathbb{Z}\mathcal{A}' \cap \mathbb{R}_+ \mathcal{A}'$ and write

$$\begin{aligned} z &= n_1(\alpha_1, 1) + \cdots + n_q(\alpha_q, 1) \\ &= \lambda_1(\alpha_1, 1) + \cdots + \lambda_q(\alpha_q, 1), \end{aligned}$$

where $n_j \in \mathbb{Z}$ and $\lambda_j \geq 0$ for all j. Note $\alpha = 0$ if $i = 0$, and $\alpha/i \in P$ if $i \geq 1$. Hence z is in the right hand side of $(*)$. By Theorem 7.2.28 one concludes the required inclusion.

Assume $K[F]$ is a homogeneous subring. Let v be a vector in $\mathbb{Q}^n$ such that $\langle \alpha_j, v \rangle = 1$ for all j. It suffices to prove that equality holds in $(*)$. Take $z = (\alpha, i)$ in the right hand side of $(*)$ and write

$$\alpha = n_1\alpha_1 + \cdots + n_q\alpha_q = i(\lambda_1\alpha_1 + \cdots + \lambda_q\alpha_q),$$

where $n_j \in \mathbb{Z}$, $\lambda_j \geq 0$, and $\lambda_1 + \cdots + \lambda_q = 1$. Hence

$$\begin{aligned} \langle \alpha, v \rangle &= n_1 + \cdots + n_q = i, \text{ and} \\ z = (\alpha, i) &= n_1(\alpha_1, 1) + \cdots + n_q(\alpha_q, 1) \\ &= i\lambda_1(\alpha_1, 1) + \cdots + i\lambda_q(\alpha_q, 1). \end{aligned}$$

Thus $z \in \mathbb{Z}\mathcal{A}' \cap \mathbb{R}_+ \mathcal{A}'$, as required. □

Corollary 7.2.44 *If $K[F]$ is homogeneous, then $K[F]$ is normal if and only if $K[FT] = A_P$.*

Proof. Use Corollary 7.2.42 and Proposition 7.2.43. □

The *normalized Ehrhart function* of P is defined as

$$E_P(i) = \dim_K(A_P)_i = |\mathbb{Z}\mathcal{A} \cap iP|.$$

Corollary 7.2.45 *If $K[F]$ is a homogeneous subring of R and h is the Hilbert function of $K[F]$, then*

(a) *$h(i) \leq E_P(i)$ for all $i \geq 0$, and*

(b) *$h(i) = E_P(i)$ for all $i \geq 0$ if and only if $K[F]$ is normal.*

Proposition 7.2.46 *If $K[F]$ is a homogeneous subring of dimension d, then E_P is a polynomial function of degree $d - 1$.*

Proof. As the Ehrhart ring A_P is equal to $\overline{K[FT]}$, by Corollary 7.2.31 one has that A_P is a finitely generated K-algebra. Hence A_P is a finitely generated graded $K[FT]$-module.

Using Theorem 1.2.8 and Proposition 1.4.12 one concludes that the Hilbert function of A_P is a polynomial function of degree $d - 1$. □

Corollary 7.2.47 *If $K[F]$ is normal and homogeneous, then the Hilbert polynomial of $K[F]$ is equal to the normalized Ehrhart polynomial of P.*

Example 7.2.48 Let $F = \{x^4, x^3y, xy^3, y^4\}$ and $P = \text{conv}(\mathcal{A})$, where $\mathcal{A} = \log(F)$. The normalized Ehrhart ring is

$$A_P = K[x^4T, x^3yT, xy^3T, y^4T, x^2y^2T].$$

The subrings $K[F]$ and A_P have different Hilbert functions, but they have the same Hilbert polynomial, which is equal to $4t + 1$.

Proposition 7.2.49 ([278]) *If $K[F]$ is a homogeneous monomial subring whose toric ideal has a square-free initial ideal with respect to some term order, then $K[F]$ is normal.*

Proof. See [278, Proposition 13.15]. □

Normal polytopes Some special types of monomial subrings associated to polytopes are introduced next. Our exposition has been influenced by the beautiful paper [41], that presents an appealing study of normal polytopal subrings.

Let P be a *lattice polytope* in $\mathbb{R}^n$, that is, P has all its vertices in $\mathbb{Z}^n$. For simplicity we assume that the vertices of P are in $\mathbb{N}^n$. Given a field K the monomial subring

$$K[P] = K[\{x^\alpha T | \alpha \in P \cap \mathbb{Z}^n\}] \subset K[x_1, \ldots, x_n, T]$$

is called the *polytopal subring* of P. The polytope P is said to be *normal* if $K[P]$ is normal.

Proposition 7.2.50 *If P is a lattice polytope in $\mathbb{R}^n$ and K a field, then*

$$\dim K[P] = \dim(P) + 1.$$

Proof. Let $\alpha_1, \ldots, \alpha_q \in \mathbb{N}^n$ such that $P = \operatorname{conv}(\alpha_1, \ldots, \alpha_q)$. First we observe the equality

$$\operatorname{aff}(P) = \operatorname{aff}(\alpha_1, \ldots, \alpha_q) = \alpha_1 + V,$$

where $V = \mathbb{R}(\alpha_2 - \alpha_1) + \cdots + \mathbb{R}(\alpha_q - \alpha_1)$. Thus one has $\dim(P) = \dim_{\mathbb{R}}(V)$. On the other hand using the identity

$$\operatorname{rank} \begin{pmatrix} \alpha_1 & 1 \\ \vdots & \vdots \\ \alpha_q & 1 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \alpha_2 - \alpha_1 & 0 \\ \vdots & \vdots \\ \alpha_q - \alpha_1 & 0 \end{pmatrix} + 1,$$

the asserted formula follows from Proposition 7.1.17. □

Proposition 7.2.51 *Let $A_0, \ldots, A_n$ be a set of affinely independent points in $\mathbb{R}^n$ and $\Delta = \operatorname{conv}(A_0, \ldots, A_n)$. Then the volume of the simplex Δ is*

$$\operatorname{vol}(\Delta) = \frac{\left|\det \begin{pmatrix} A_0 & 1 \\ \vdots & \vdots \\ A_n & 1 \end{pmatrix}\right|}{n!} = \frac{\left|\det \begin{pmatrix} A_1 - A_0 \\ \vdots \\ A_n - A_0 \end{pmatrix}\right|}{n!}.$$

Proof. The result follows using linear algebra or applying the change of variables formula. See [206, Chapter XII] or [310, Chapter 6]. □

Let Δ be an n-dimensional *lattice simplex* in $\mathbb{R}^n$, that is, Δ is the convex hull of a set of $n+1$ affinely independent point in $\mathbb{Z}^n$. The *normalized volume* of Δ is defined as $n!\operatorname{vol}(\Delta)$. From the result above one has

$$\operatorname{vol}(\Delta) \geq \frac{1}{n!},$$

recall that Δ is called a *unimodular simplex* if equality holds.

Proposition 7.2.52 *Let Δ be an n-dimensional lattice simplex in $\mathbb{R}^n$ with vertices $A_0, \ldots, A_n$ in $\mathbb{Z}^n$. Then Δ is unimodular if and only if either of the following two equivalent conditions hold*

(a) $\mathbb{Z}^{n+1} = \mathbb{Z}(A_0, 1) + \cdots + \mathbb{Z}(A_n, 1)$.

(b) $\mathbb{Z}^n = \mathbb{Z}(A_1 - A_0) + \cdots + \mathbb{Z}(A_n - A_0)$.

Proof. Recall that a set of vectors $\alpha_1, \ldots, \alpha_n$ in $\mathbb{Z}^n$ generate $\mathbb{Z}^n$ as a $\mathbb{Z}$-module if and only if the $n \times n$ matrix A whose rows are the α_i has determinant ± 1. Thus the result follows from Proposition 7.2.51. □

Proposition 7.2.53 *If Δ is a unimodular simplex in $\mathbb{R}^n$ with vertex set $\mathcal{A} = \{\alpha_0, \ldots, \alpha_n\}$, then $\Delta \cap \mathbb{Z}^n = \mathcal{A}$.*

Proof. It suffices to show that $\Delta \cap \mathbb{Z}^n$ is contained in $\mathcal{A}$. Let $\alpha \in \Delta \cap \mathbb{Z}^n$. There are λ_i in $\mathbb{R}_+$ such that $\alpha = \lambda_0\alpha_0 + \cdots + \lambda_n\alpha_n$ and $\sum_{i=0}^n \lambda_i = 1$. If $\lambda_0 > 0$, from the equalities

$$1 \leq \left|\det \begin{pmatrix} \alpha & 1 \\ \alpha_1 & 1 \\ \vdots & \vdots \\ \alpha_n & 1 \end{pmatrix}\right| = \left|\det \begin{pmatrix} \sum_{i=0}^n \lambda_i(\alpha_i, 1) \\ \alpha_1 \quad 1 \\ \vdots \\ \alpha_n \quad 1 \end{pmatrix}\right| = \lambda_0 n! \mathrm{vol}(\Delta) = \lambda_0,$$

one derives $\lambda_0 = 1$, and consequently $\lambda_i = 0$ for all $i \geq 1$. Hence $\alpha = \alpha_0$. Similarly if $\lambda_i > 0$ for some $i > 0$, then $\alpha = \alpha_i$. □

Corollary 7.2.54 *Let K be a field and P a unimodular simplex in $\mathbb{R}^n$ with vertex set $\mathcal{A} = \{\alpha_0, \ldots, \alpha_n\}$. Then*

$$K[P] = K[x^{\alpha_0}T, \ldots, x^{\alpha_n}T]$$

and $K[P]$ is a polynomial ring.

Proof. The asserted equality is a direct consequence of the result above. As $K[P]$ has dimension $n + 1$, using Lemma 2.1.5 gives that $K[P]$ is a polynomial ring. □

Proposition 7.2.55 ([41]) *Let $\{C, C_i\}$ be a family of finitely generated subsemigroups of $\mathbb{N}^n$ such that $C = \cup_i C_i$. If $\mathbb{Z}\mathbb{C} = \mathbb{Z}C_i$ and $K[C_i]$ is normal for all i, then $K[C]$ is normal.*

Proof. By Corollary 7.2.29 it suffices to show that $\mathbb{Z}C \cap \mathbb{R}_+ C$ is contained in C. Take $z \in \mathbb{Z}C \cap \mathbb{R}_+ C$. There is an integer $s \geq 1$ such that $sz \in C$, thus $sz \in C_i$ for some i. As $\mathbb{Z}C = \mathbb{Z}C_i$ and $K[C_i]$ is normal we get $z \in C_i \subset C$, as required. □

Definition 7.2.56 Let P be a convex polytope in $\mathbb{R}^n$. A collection $\{\Delta_i\}$ of unimodular lattice simplices in $\mathbb{R}^n$ is called a *unimodular covering* of P if $P = \cup_i \Delta_i$.

Theorem 7.2.57 ([41]) *Let P be a lattice polytope in $\mathbb{R}^n$ of dimension n with vertices in $\mathbb{N}^n$. If P has a unimodular covering and K is a field, then $K[P]$ is normal.*

Proof. Let $\{\Delta_\ell\}$ be a unimodular covering of P and $\alpha_1, \ldots, \alpha_q$ the vertices of P. By Proposition 7.2.55 we need only show that $C = \cup_\ell C_\ell$, where C and C_ℓ are the subsemigroups of $\mathbb{N}^{n+1}$ defined as:

$$C = \mathbb{N}\{(\alpha, 1) | \, \alpha \in \mathbb{Z}^n \cap P\} \quad \text{and} \quad C_\ell = \mathbb{N}\{(\alpha, 1) | \, \alpha \in \mathbb{Z}^n \cap \Delta_\ell\}.$$

One clearly has the containment $\cup_\ell C_\ell \subset C$, because $P = \cup_\ell \Delta_\ell$. Conversely take $z \in C$

$$z = (\beta, s) = n_1(\beta_1, 1) + \cdots + n_r(\beta_r, 1),$$

where $n_i \in \mathbb{N}$ and $\beta_i \in \mathbb{Z}^n \cap P$ for all i. There are scalars $\mu_{ij} \geq 0$ such that

$$\beta_i = \mu_{i1}\alpha_1 + \cdots + \mu_{iq}\alpha_q \quad \text{and} \quad \mu_{i1} + \cdots + \mu_{iq} = 1 \ \forall i.$$

Note that β/s is in P, where $\beta = \sum_i n_i\beta_i$ and $s = \sum_i n_i$. Hence β/s is in Δ_ℓ for some ℓ. We can write $\Delta_\ell = \text{conv}(\gamma_0, \ldots, \gamma_n)$, with $\{\gamma_0, \ldots, \gamma_n\}$ the vertex set of Δ_ℓ. Thus

$$\frac{\beta}{s} = \mu_0\gamma_0 + \cdots + \mu_n\gamma_n,$$

for some $\mu_i \geq 0$ satisfying $\sum_i \mu_i = 1$. Hence

$$z = (\beta, s) = s\mu_0(\gamma_0, 1) + \cdots + s\mu_n(\gamma_n, 1) \in \mathbb{R}_+\{(\gamma_0, 1), \ldots, (\gamma_n, 1)\},$$

by the unimodularity of Δ_ℓ one has

$$z \in \mathbb{Z}\{(\gamma_0, 1), \ldots, (\gamma_n, 1)\} = \mathbb{Z}^{n+1}.$$

Therefore using that $K[\Delta_\ell]$ is normal (see Corollary 7.2.54) yields $z \in C_\ell$, as required. This proof is due to W. Bruns, J. Gubeladze and N. Trung. □

Exercises

7.2.58 Let $A \subset \mathbb{R}^n$. Prove that A is an affine variety in $\mathbb{R}^n$ if and only if $\lambda_1 A + \lambda_2 A \subset A$ for all λ_1, λ_2 in $\mathbb{R}$ such that $\lambda_1 + \lambda_2 = 1$.

7.2.59 If $K[F]$ is a homogeneous monomial subring over a field K, use the Danilov-Stanley formula to show $a(\overline{K[F]}) < 0$.

7.2.60 Let $\alpha_1, \ldots, \alpha_q \in \mathbb{N}^n$ and $C = (\mathbb{R}_+\alpha_1 + \cdots + \mathbb{R}_+\alpha_q) \cap \mathbb{N}^n$. Then

(a) $C = \mathbb{N}\gamma_1 + \cdots + \mathbb{N}\gamma_r$, for some $\gamma_1, \ldots, \gamma_r$ in $\mathbb{N}^n$.

(b) $k[C] = k[\{x^\alpha | \alpha \in C\}] \subset k[\mathbf{x}]$ is a normal domain, where k is a field.

7.2.61 Let $A \subset \mathbb{R}^n$. Prove that A is an affine variety in $\mathbb{R}^n$ if and only if $\lambda_1 A + \lambda_2 A \subset A$ for all λ_1, λ_2 in $\mathbb{R}$ such that $\lambda_1 + \lambda_2 = 1$.

7.2.62 Determine the facets of the convex polytope

$$P = \text{conv}(\pm e_1, \ldots, \pm e_n) \subset \mathbb{R}^n.$$

7.2.63 Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_q\} \subset \mathbb{N}^n$ and $C = \mathbb{N}\alpha_1 + \cdots + \mathbb{N}\alpha_q$. Prove that the semigroup ring $k[C]$ is normal if and only if any element $\alpha \in \mathbb{Z}\mathcal{A}$ satisfying $r\alpha \in C$ for some integer $r \geq 1$ belongs to C.

7.2.64 If P is a unimodular lattice simplex in $\mathbb{R}^n$ with vertices $\alpha_0, \ldots, \alpha_n$, prove that the polytopal subring $K[P]$ is not isomorphic (as a ring) to the monomial subring $K[x^{\alpha_0}, \ldots, x^{\alpha_n}] \subset K[\mathbf{x}]$, where K is a field.

7.2.65 Let $1 \leq k \leq n$ be two integers and

$$\mathcal{A} = \{e_{i_1} + \cdots + e_{i_k} | \, 1 \leq i_1 < \cdots < i_k \leq n\},$$

where e_i is the ith unit vector in $\mathbb{R}^n$. If $P = \text{conv}(\alpha_1, \ldots, \alpha_q)$, for some $\alpha_1, \ldots, \alpha_q \in \mathcal{A}$, then $P \cap \mathbb{Z}^n = \{\alpha_1, \ldots, \alpha_q\}$.

7.2.66 Let $F = \{x^{\alpha_1}, \ldots, x^{\alpha_q}\}$ be a finite set of square-free monomials of degree k in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K. If $P = \text{conv}(\alpha_1, \ldots, \alpha_q)$, then $K[P] \simeq K[F]$.

7.2.67 Let $F = \{x^{\alpha_1}, \ldots, x^{\alpha_q}\}$ be a set of monomials in a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K. Prove that $K[F]$ is a homogeneous subring of R if and only if $0 \notin A = \text{aff}_{\mathbb{Q}}(\alpha_1, \ldots, \alpha_q) \subset \mathbb{Q}^n$.

Hint A is an intersection of $n - r$ rational hyperplanes, where $r = \dim(A)$.

7.2.68 Let $\{\alpha_1, \ldots, \alpha_q\}$ be a set of vectors in $\mathbb{Q}^n$ and $A = \text{aff}_{\mathbb{Q}}(\alpha_1, \ldots, \alpha_q)$. Prove that $0 \notin A$ if and only if

$$\text{rank} \begin{pmatrix} 1 & \alpha_1 \\ \vdots & \vdots \\ 1 & \alpha_q \end{pmatrix} < \text{rank} \begin{pmatrix} 1 & \alpha_1 \\ \vdots & \vdots \\ 1 & \alpha_q \\ 1 & 0 \end{pmatrix}.$$

7.2.69 Is the integral closure of a homogeneous subring homogeneous?

7.2.70 Is the monomial subring $K[x^2, xy, y^3] \subset K[x, y]$ homogeneous?

7.2.71 Let $F = \{x^{\alpha_1}, \ldots, x^{\alpha_q}\} \subset K[\mathbf{x}]$. Prove that $K[F]$ is homogeneous if and only if

$$\operatorname{rank} \begin{pmatrix} \alpha_1 & 1 \\ \vdots & \vdots \\ \alpha_q & 1 \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_q \end{pmatrix}.$$

7.2.72 Let $F = \{x^{\alpha_1}, \ldots, x^{\alpha_q}\}$ be a set of monomials in $R = K[x_1, \ldots, x_n]$ and consider $P = \operatorname{conv}(\log(F))$. If $\mathcal{A}' = \{(p, 1) | \, p \in \mathbb{Z}^n \cap P\}$, then

$$A(P) = K[x^\alpha T^i | \, (\alpha, i) \in \mathbb{R}_+ \mathcal{A}' \cap \mathbb{Z}^{n+1}] = K[x^\alpha T^i | \, \alpha \in \mathbb{Z}^n \cap iP],$$

and $K[FT] \subset K[P] \subset A(P)$.

The ring $A(P)$ is called the *Ehrhart ring* of P. In the following problems we keep the notations of the previous exercise. For more details on Ehrhart rings consult [44, 163, 272]. See [48] for related algorithms.

7.2.73 Prove that the Ehrhart ring $A(P)$ is a finitely generated K-algebra and a normal domain.

7.2.74 Prove that

$$A(P) = \bigoplus_{i=0}^{\infty} A(P)_i$$

is a graded K-algebra with its i*th* component given by

$$A(P)_i = \sum_{\alpha \in \mathbb{Z}^n \cap iP} K x^\alpha T^i.$$

7.2.75 Prove that the Ehrhart ring $A(P)$ is integral over $K[FT]$, and that the Hilbert function of $A(P)$ is a polynomial function of degree $d = \dim(P)$. The corresponding polynomial is called the *Ehrhart polynomial* of P.

7.2.76 If $d = \dim(P)$, then according to [272, Proposition 4.6.30] one has:

$$\operatorname{vol}(P) = \lim_{i \to \infty} \frac{|\mathbb{Z}^n \cap iP|}{i^d}.$$

Thus $\operatorname{vol}(P)$ is the leading coefficient of the Ehrhart polynomial of P. Prove this formula for the case $n = \dim(P)$. In practice one can compute volumes of lattice polytopes using [48].

7.3 Integral closure of monomial ideals

As noted earlier the simplest kind of integrally closed monomial ideals are the Stanley-Reisner ideals and the powers of face ideals. Below we give a few basic properties of the integral closure of monomial ideals.

Lemma 7.3.1 *Let R be a polynomial ring over a field k and let I be an ideal of R. Assume that $M_1, \ldots, M_q$ is a set of linearly independent monomials over k such that*

$$M_j^{s_j} = b_{m_j} \prod_{i \neq j} M_i^{r_{ji}} \notin k, \quad \text{where} \quad s_j = m_j + \sum_{i \neq j} r_{ji} > 0,$$

$m_j \in \mathbb{N}$ and $r_{ji} \in \mathbb{N}$ for $j = 1, \ldots, q$. If b_{m_j} is a monomial in I^{m_j} for $m_j > 0$ and $b_{m_j} \in k$ for $m_j = 0$, then $M_q^{\ell} \in I^{\ell}$ for some $\ell > 0$.

Proof. We argue by induction on q. The case $q = 1$ is clear. Assume $q \geq 2$. For $j \geq 2$ we substitute $M_1^{s_1}$ into $M_j^{s_j s_1}$ to obtain:

$$M_j^{s_j s_1 - r_{1j} r_{j1}} = b_{m_j}^{s_1} b_{m_1}^{r_{j1}} \prod_{1 \neq i \neq j} M_i^{s_1 r_{ji} + r_{1i} r_{j1}}.$$

Notice that in these equations M_1 has been eliminated. Hence applying the induction hypothesis to $M_2, \ldots, M_q$ yields $M_q^{\ell} \in I^{\ell}$ for some $\ell > 0$. □

As a consequence of Lemma 7.3.1 we derive the following result using a short algebraic argument.

Proposition 7.3.2 ([198]) *Let R be a polynomial ring over a field k and let I be a monomial ideal in R. Then $\overline{I}$, the integral closure of I, is a monomial ideal.*

Proof. Let $z \in \overline{I}$ and let $z = M_1 + \cdots + M_q$, where the M_i's are linearly independent monomials over k. There is an equation

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0 \quad (a_i \in I^i).$$

By induction it suffices to verify that $M_i \in \overline{I}$ for some i. If $M_i^n \in I^n$, then $M_i \in \overline{I}$. Hence we may assume $M_i^n \notin I^n$ for all i. Since I^i is a monomial ideal, all the monomials in the support of a_i belong to I^i. Using the equation above we obtain (after cancelling out common terms) equations of the form

$$M_j^{s_j} = b_{m_j} \prod_{i \neq j} M_i^{r_{ji}}, \quad \text{where} \quad s_j = m_j + \sum_{i \neq j} r_{ji} > 0,$$

$m_j \in \mathbb{N}$ and $r_{ji} \in \mathbb{N}$ for $j = 1, \ldots, q$, and such that b_{m_j} are monomials in I^{m_j} for $m_j > 0$, and $b_{m_j} \in k$ for $m_j = 0$. As a consequence Lemma 7.3.1 yields $M_q \in \overline{I}$. □

Proposition 7.3.3 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and $I \subset R$ a monomial ideal. Then the integral closure of I is given by*

$$\overline{I} = (x^{\alpha} | \, x^{m\alpha} \in I^m \text{ for some } m \geq 1).$$

Proof. If $x^{m\alpha} \in I^m$, then x^{α} satisfies a polynomial of the form $z^m + a_m$, with $a_m \in I^m$, hence x^{α} is in $\overline{I}$. On the other hand if $z = x^{\alpha} \in \overline{I}$, then there is an equation

$$z^r + a_1 z^{r-1} + \cdots + a_{r-1} z + a_r = 0 \quad (a_i \in I^i).$$

Since I is a monomial ideal one obtains $z^m \in I^m$ for some $m \geq 1$. Noting that $\overline{I}$ is a monomial ideal the asserted equality follows. □

A geometric description of the integral closure Let $\alpha \in \mathbb{Q}_+^n$, where $\mathbb{Q}_+$ is the set of non negative rational numbers. We define the *upper right corner* or *ceiling* of α as the vector $\lceil \alpha \rceil$ whose entries are given by

$$\lceil \alpha \rceil_i = \begin{cases} \alpha_i & \text{if } \alpha_i \in \mathbb{N}, \\ \lfloor \alpha_i \rfloor + 1 & \text{if } \alpha_i \notin \mathbb{N}, \end{cases}$$

where $\lfloor \alpha_i \rfloor$ stands for the integral part of α_i. Let $a_i = (a_{i1}, \ldots, a_{in}) \in \mathbb{N}^n$ and let $\operatorname{conv}(a_1, \ldots, a_r)$ be its convex hull (over the rationals), that is,

$$\operatorname{conv}(a_1, \ldots, a_r) = \left\{ \sum_{i=1}^{r} \lambda_i a_i \,\middle|\, \sum_{i=1}^{r} \lambda_i = 1, \ \lambda_i \in \mathbb{Q}_+ \right\}$$

is the set of all *convex combinations* of $a_1, \ldots, a_r$.

Proposition 7.3.4 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. If $I \subset R$ is an ideal generated by monomials $x^{a_1}, \ldots, x^{a_r}$, then its integral closure is:*

$$\overline{I} = \left(\left\{ x^{\lceil \alpha \rceil} \,\middle|\, \alpha \in \operatorname{conv}(a_1, \ldots, a_r) \right\} \right).$$

Proof. We set

$$F = \left\{ x^{\lceil \alpha \rceil} \,\middle|\, \alpha \in \operatorname{conv}(a_1, \ldots, a_r) \right\}.$$

Let $x^{\lceil \alpha \rceil} \in F$ and let $\alpha = \sum_{i=i}^{r} \lambda_i a_i$ be a convex combination of $a_1, \ldots, a_r$. Since $\lceil \alpha \rceil \geq \alpha$, there is $\beta \in \mathbb{Q}^n$, with $\beta \geq 0$, such that $\lceil \alpha \rceil = \beta + \alpha$. Hence there exists $p > 0$ so that $p\beta \in \mathbb{N}^n$ and $p\lambda_i \in \mathbb{N}$ for all i. Therefore

$$x^{p\lceil \alpha \rceil} = x^{p\beta} x^{p\alpha} = x^{p\beta} (x^{a_1})^{p\lambda_1} \cdots (x^{a_r})^{p\lambda_r} \in I^p,$$

showing $x^{\lceil \alpha \rceil} \in \overline{I}$ and $(F) \subset \overline{I}$.

Conversely let $x^\gamma \in \overline{I}$, that is, $x^{p\gamma} \in I^p$ for some $p > 0$. There are non-negative integers s_i satisfying

$$x^{p\gamma} = x^\delta \, (x^{a_1})^{s_1} \cdots (x^{a_r})^{s_r} \text{ and } s_1 + \cdots + s_r = p.$$

Hence

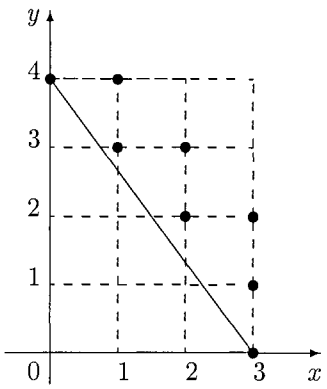
$$\gamma = \frac{\delta}{p} + \sum_{i=1}^{r} \left(\frac{s_i}{p}\right) a_i.$$

Set $\alpha = \sum_{i=1}^{r} (s_i/p) a_i$. By dividing the entries of δ by p, we can write $\gamma = \theta + \beta + \alpha$, where $0 \leq \beta_i < 1$ for all i and $\theta \in \mathbb{N}^n$. Notice $\beta + \alpha \in \mathbb{N}^n$. To complete the argument we claim that $\lceil \alpha \rceil = \beta + \alpha$. If $\beta_i = 0$, then $\alpha_i \in \mathbb{N}$ and $\lceil \alpha \rceil_i = \beta_i + \alpha_i$. Assume $0 < \beta_i < 1$, since $\beta_i + \alpha_i \in \mathbb{N}$ we have $\alpha_i \notin \mathbb{N}$ and $\lceil \alpha \rceil_i = \lfloor \alpha_i \rfloor + 1$. If $\lfloor \alpha_i \rfloor + 1 < \alpha_i + \beta_i < \alpha_i + 1$, then $\lfloor \alpha_i \rfloor + 1 < \alpha_i$, which is impossible, hence $\alpha_i < \alpha_i + \beta_i \leq \lfloor \alpha_i \rfloor + 1$ and $\beta_i + \alpha_i = \lfloor \alpha_i \rfloor + 1$. Consequently $x^\gamma \in (F)$, as required. □

Corollary 7.3.5 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and $I \subset R$ an ideal generated by monomials $x^{a_1}, \ldots, x^{a_r}$. Then $\overline{I}$ is generated by monomials of degree at most $d+n-1$, where d is the maximum of the degrees of $x^{a_1}, \ldots, x^{a_r}$.*

Proof. Let $\alpha \in \operatorname{conv}(a_1, \ldots, a_r)$ and $\beta = \lceil \alpha \rceil$. As $\beta_i = \alpha_i + \delta_i$, for some δ_i in $[0, 1)$, one has $\deg(x^\beta) < d + n$, as required. □

Example 7.3.6 Let $I = (x^3, y^4) \subset k[x, y]$. The integral closure of I is generated by the monomials of the points marked in black dots in the figure:



Therefore

$$\overline{I} = (x^3, y^4, xy^3, xy^4, x^2y^2, x^2y^3, x^3y, x^3y^2) = (x^3, y^4, xy^3, x^2y^2).$$

Proposition 7.3.7 *Let k be a field and $R = k[x_1, x_2]$ a polynomial ring in two variables. If I is generated by monomials $x^{a_1}, \ldots, x^{a_q}$ of degree d, then $\overline{I}$ is also generated by monomials of degree d.*

Proof. One may assume $a_{11} < \cdots < a_{q1}$, where $a_i = (a_{i1}, a_{i2})$. We claim that $\overline{I}$ is generated by the set $\mathcal{A}$ of all x^c such that $c_1 + c_2 = d$ and $a_{11} \leq c_1 \leq a_{q1}$.

If $x^c \in \mathcal{A}$, then $c_1 = ta_{11} + (1-t)a_{q1}$ for some $0 \leq t \leq 1$. Hence $c = ta_1 + (1-t)a_q$ and $c \in \operatorname{conv}(a_1, a_q)$, that is, $x^c \in \overline{I}$. Conversely let $x^\beta \in \overline{I}$, by Proposition 7.3.4 one may assume $\beta = \lceil \alpha \rceil$ for some α in $\operatorname{conv}(a_1, \ldots, a_q)$. Set $c = (\lceil \alpha_1 \rceil, d - \lceil \alpha_1 \rceil)$, by the previous argument x^c is in $\mathcal{A}$. Since $\beta \geq c$ one has $x^\beta \in (\mathcal{A})$. □

Proposition 7.3.8 Let $R = k[x_1, x_2]$ and I an ideal of R generated by monomials of the same degree d. If I is integrally closed, then I is normal.

Proof. The ideal I is of the form $I = zJ$, where z is the greatest common divisor of the monomials in I. Since I is integrally closed if and only if J is integrally closed, one may assume I is $\mathfrak{m}$-primary, where $\mathfrak{m} = (x_1, x_2)$. As x_1^d and x_2^d belong to I, one rapidly derives $I = \mathfrak{m}$. □

In polynomial rings in two variables any integrally closed ideal is normal; more generally one has the following result of Zariski.

Theorem 7.3.9 *If R is a regular local ring of dimension two and $I_1, \ldots, I_r$ are integrally closed ideals, then $I_1 \cdots I_r$ is integrally closed.*

Proof. See [314, Appendix 5, Theorem 2']. □

Example 7.3.10 Let $R = k[x_1, x_2, x_3]$ and $I = (x_1^3, x_2 x_3^2)$. Using the program *Normaliz* [48] one can compute the integral closure of I. The input and part of the output file are:

`example.in`

```
2          (number of generators)
3          (dimension of ambient lattice)
3  0  0    (generators)
0  1  2
3          (option or mode)
```

`example.out`

integral closure of I

```
3  0  0
0  1  2
2  1  1
```

Definition 7.3.11 If x^a is a monomial in $R = k[\mathbf{x}]$ we set

$$\log(x^a) = a.$$

Given a set F of monomials, the *log set* of F, denoted $\log(F)$, consists of all $\log(x^a)$ with $x^a \in F$.

Proposition 7.3.12 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and $I \subset R$ a monomial ideal. Then the integral closure of I is given by*

$$\overline{I} = (x^\alpha | \, \alpha \in \operatorname{conv}(\log(I)) \cap \mathbb{Z}^n).$$

Proof. Set

$$J = (x^\alpha | \, \alpha \in \operatorname{conv}(\log I)).$$

Let $\alpha \in \operatorname{conv}(\log I)$. One can write

$$\alpha = \sum_{i=1}^{q} \mu_i \beta_i,$$

where $\mu_i \geq 0$, $\sum_{i=1}^{q} \mu_i = 1$ and $x_i^\beta \in I$. Note that $A\mu^t = (1, \alpha)^t$, where A is the $(n+1) \times q$ matrix whose columns are $(1, \beta_i)^t$, $i = 1, \ldots, q$, hence by Farkas's Lemma one may assume $\mu_i \in \mathbb{Q}$. By choosing $m \in \mathbb{N}_+$ such that $m\mu_i \in \mathbb{N}$ for all i one obtains $x^{m\alpha} \in I^m$ and $x^\alpha \in \overline{I}$. Conversely let us show $\overline{I} \subset J$. Since $\overline{I}$ is generated by monomials for any $x^\alpha \in \overline{I}$ there is $m \in \mathbb{N}_+$ such that $x^{m\alpha} \in I^m$. It follows readily that $\alpha \in \operatorname{conv}(\log I)$. □

Proposition 7.3.13 *Let F be a finite set of monomials of degree k in a polynomial ring R over a field K. If $I = (F)$ is complete, then*

$$(\mathcal{P}) \qquad \operatorname{conv}(\log(F)) \cap \mathbb{Z}^n = \log(F).$$

Proof. Set $F = \{x^{\alpha_1}, \ldots, x^{\alpha_q}\}$. Let $\beta \in \operatorname{conv}(\log F) \cap \mathbb{Z}^n$, then

$$\beta = \sum_{i=1}^{q} \mu_i \alpha_i,$$

where $\mu_i \geq 0$ and $\sum_{i=1}^{q} \mu_i = 1$. Since $|\beta| = k$ and x^β is in $\overline{I} = I$ (see Proposition 7.3.12), then $x^\beta = x^{\alpha_i}$ for some i and $\beta = \alpha_i$. □

Condition $(\mathcal{P})$ says that $\log(F)$ is the set of all lattice points in the convex polytope spanned by itself. We may interpret it as asserting that the integral closedness of I is not violated in the lowest possible degree. If F consists of square-free monomials, then $\mathcal{P}$ is certainly satisfied.

Symbolic Rees algebras of monomial ideals For a monomial ideal I generated by square-free monomials it is possible to compare the integral closure of the powers of I with the symbolic powers of I, to make the comparison we begin with the following description of the symbolic powers.

Proposition 7.3.14 *Let R be a polynomial ring over a field K and let I be an ideal of R generated by square-free monomials. Then*

$$I^{(n)} = \mathfrak{p}_1^n \cap \cdots \cap \mathfrak{p}_r^n,$$

for all $n \geq 1$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the minimal primes of I.

Proof. From Proposition 5.1.8 one has $\mathfrak{p}_i^n = \mathfrak{p}_i^{(n)}$. Since I is a radical ideal the result follows from Proposition 3.3.24. □

Corollary 7.3.15 *Let R be a polynomial ring over a field K. If I is an ideal of R generated by square-free monomials, then $I^{(n)}$ is integrally closed for $n \geq 1$.*

Proof. Set $J = I^{(n)}$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the associated primes of I. If $f \in \overline{J}$, then there is $m \geq 1$ so that $f^m \in \mathfrak{p}_i^{nm}$ for all i. By Corollary 3.3.20 $\mathfrak{p}_i^n$ is complete, thus $f \in \mathfrak{p}_i^n$ for all i. Hence Proposition 7.3.14 shows $f \in J$. □

Corollary 7.3.16 *Let R be a polynomial ring over a field K and let I be a monomial ideal. If I is a radical ideal, then $\overline{I^n} \subset I^{(n)}$ for $n \geq 1$.*

Proof. Let $x \in \overline{I^n}$, there is $k \geq 1$ so that $x^k \in (I^n)^k \subset (I^{(n)})^k$, thus x is in $\in \overline{I^{(n)}}$. Note that $I^{(n)}$ is integrally closed by Corollary 7.3.15, hence $x \in I^{(n)}$. □

Let us present the monomial version of Theorem 3.3.31, which can be proved directly in the monomial case.

Proposition 7.3.17 *Let R be a polynomial ring with coefficients in a field. If I is a radical monomial ideal of R which is normally torsion free, then its Rees algebra $R[It]$ is a normal domain.*

Proof. By Proposition 3.3.26 $I^n = I^{(n)}$ for $n \geq 1$. Hence thanks to Corollary 7.3.16 I^n is integrally closed for $n \geq 1$, and consequently $R[It]$ is normal. □

Proposition 7.3.18 ([214]) *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and I an ideal generated by square-free monomials. Then the symbolic Rees algebra*

$$\mathcal{R}_s(I) = R + It + I^{(2)}t^2 + \cdots + I^{(n)}t^n + \cdots \subset R[t]$$

is a finitely generated R-algebra.

Proof. We follow Lyubeznik's original argument. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the minimal primes of I. Given a monomial $y = x_1^{r_1} \cdots x_n^{r_n}$, set $\deg_{x_i}(y) = r_i$ and $\deg_{\mathfrak{p}_i}(y) = \sum_{x_j \in \mathfrak{p}_i} \deg_{x_j}(y)$. To every monomial y in $I^{(m)}$ we associate the vector:

$$w(y) = (\deg_{x_1}(y), \ldots, \deg_{x_n}(y), \deg_{\mathfrak{p}_1}(y) - m, \ldots, \deg_{\mathfrak{p}_s}(y) - m).$$

By Proposition 7.3.14 a monomial y is in $I^{(m)}$ if and only if $\deg_{\mathfrak{p}_i}(y) \geq m$ for all i. Hence $w(y) \in \mathbb{N}^{n+s}$ for all $y \in I^{(m)}$.

Note that the set $\mathbb{N}^{n+s}$ is partially ordered by: $(a_i) \geq (b_i)$ if and only if $a_i \geq b_i$ for all i. Denote by $\mathcal{A}$ the set of all $w(y)$ such that y is a monomial which is part of a minimal generating set of $I^{(m)}$ for some $m \geq 1$. By Dickson's lemma $\mathcal{A}$ has only a finite number of minimal elements, and for each c in $\mathcal{A}$ there is a minimal element d of $\mathcal{A}$ with $c = b + d$ for some $b \in \mathbb{N}^{n+s}$. Let $y_1, \ldots, y_p$ be the monomials in R such that $w(y_1), \ldots, w(y_p)$ are the minimal elements of $\mathcal{A}$. Thus y_i is part of a minimal generating set of $I^{(m_i)}$, for some $m_i \geq 1$.

We claim $\mathcal{R}_s(I) = R[y_1 t^{m_1}, \ldots, y_p t^{m_p}]$. Let $yt^m \in I^{(m)} t^m$, where y is a monomial in $I^{(m)}$, note that one can write $w(y) = b + w(y_j)$ for some j and some $b \in \mathbb{N}^{n+s}$. One has:

(i) $y = x^a y_j$, and

(ii) $\deg_{\mathfrak{p}_i}(y) - m \geq \deg_{\mathfrak{p}_i}(y_j) - m_j$ for all i.

Combining (i) and (ii) yields:

$$\deg_{\mathfrak{p}_i}(x^a) = \deg_{\mathfrak{p}_i}(y) - \deg_{\mathfrak{p}_i}(y_j) \geq m - m_j \quad \text{for all } i.$$

Hence $x^a t^{m-m_j} \in I^{(m-m_j)} t^{m-m_j}$, and since $yt^m = (x^a t^{m-m_j})(y_j t^{m_j})$ it follows by induction that y is in the subring $R[y_1 t^{m_1}, \ldots, y_p t^{m_p}]$. □

Exercises

7.3.19 Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and $\mathfrak{m}$ its irrelevant maximal ideal, then $\mathfrak{m}^d$ is the integral closure of the ideal $(x_1^d, \ldots, x_n^d)$.

7.3.20 Prove $\overline{I \cap J} \neq \overline{I} \cap \overline{J}$, where $I = (x^5, y^3)$ and $J = (y^5, z^2)$.

7.3.21 Let $R = k[x_1, \ldots, x_m, y_1, \ldots, y_n]$ and $L = I_1 J_r + I_r J_1$, where J_t (resp. I_s) denote the ideal of R generated by the square-free monomials of degree t (resp. s) in the y_i (resp. x_i) variables. If $r \geq 3$, prove that L^3 is not integrally closed.

Hint Set $f = x_1^2 \cdots x_r^2 y_1^2 \cdots y_r^2$ and prove $f \in \overline{L^3} \setminus L^3$.

7.4 Normality of some Rees algebras

There are several interesting examples of normal ideals generated by square-free monomials, here we present some of them.

Since the normality of an ideal I, in a polynomial ring R over a field k, is equivalent to the normality of the Rees algebra of I (see Theorem 3.3.18), it is often the case that is much easier to verify the normality of the ideal than the normality of its Rees algebra.

An advantage of proving normality of the Rees algebra of I occurs when the ideal is generated in a single degree p, in this case one automatically obtains the normality of the subring $k[I_p]$, by the next general result on descent of normality. See Example 8.7.14 for a counter-example if I is generated in different degrees.

Proposition 7.4.1 *Let R be a polynomial ring over a field k and I an ideal of R generated by homogeneous polynomials $f_1, \ldots, f_q$. Assume* $\deg(f_i) = d$ *for all i. If $\mathcal{R}(I)$ is normal, then $k[f_1, \ldots, f_q]$ is also normal.*

Proof. Let $\mathfrak{m} = R_+$ be the irrelevant maximal ideal of the polynomial ring R and $A = k[Tf_1, \ldots, Tf_q]$. Observe that there is a decomposition of A-modules:

$$\mathcal{R}(I) = R[Tf_1, \ldots, Tf_q] = k[Tf_1, \ldots, Tf_q] \oplus \mathfrak{m}\mathcal{R}(I).$$

As $A \simeq k[f_1, \ldots, f_q]$ (as rings) the result follows from Proposition 3.3.33. □

Proposition 7.4.2 *Let $R = k[x_1, \ldots, x_n]$ and $R[x]$ be polynomial rings over a field k. If I is a normal ideal of R generated by square-free monomials of the same degree $t \geq 2$, then the ideal $J = I + (xx_1, \ldots, xx_n)$ is normal.*

Proof. Set $V = \{x_1, \ldots, x_n\}$ and $J = I + (xx_1, \ldots, xx_n)$. Here to make the notation simpler we denote the integral closure of J^p by J_a^p. By induction on p we will show that $J_a^p = J^p$ for all $p \geq 1$. If $p = 1$ then J, being an ideal generated by square-free monomials, is complete (see Corollary 5.1.5). Assume $J_a^i = J^i$ for $i < p$ and $p \geq 2$. Using Proposition 7.3.3 we have

$$J_a^p = (\{y \,|\, y \text{ is a monomial in } R[x] \text{ and } y^m \in J^{mp} \text{ for some } m \geq 1\}).$$

Let y be a monomial in J_a^p, then $y^m \in J^{mp}$, $m > 0$. Let us show $y \in J^p$. Since $J_a^p \subset J_a^{p-1} = J^{p-1}$ we can write

$$y = x^{\ell} M(xw_1) \cdots (xw_s) f_1 \cdots f_{p-s-1},$$

for some $0 \leq s \leq p-1$, where M is a monomial with $x \notin \operatorname{supp}(M)$, the f_i's are degree t monomials in J with $x \notin \operatorname{supp}(f_i)$ for all i, and $w_i \in V$ for $1 \leq i \leq s$. Likewise we can write

$$y^m = N(xz_1) \cdots (xz_r) g_1 \cdots g_{mp-r},$$

for some $0 \leq r \leq mp$, where N is a monomial, the g_i's are monomials of degree t in J with $x \notin \operatorname{supp}(g_i)$ for all i, and $z_i \in V$ for all $1 \leq i \leq r$. To clarify notation set $xw_0 = xz_0 = 1$ and $f_0 = g_0 = 1$. Altogether we obtain:

$$\begin{aligned} y^m &= x^{m(\ell+s)} M^m (w_1 \cdots w_s f_1 \cdots f_{p-s-1})^m \\ &= N x^r z_1 \cdots z_r g_1 \cdots g_{mp-r}, \end{aligned} \tag{7.2}$$

making $x = 1$ and taking degrees one readily derives the inequality

$$m(ts + t - s) \leq r(t-1) + m\deg(M), \tag{7.3}$$

where $r \leq \min\{mp, m\ell + ms\}$.

(a) If $\deg(M) = 0$, the inequality above gives $\ell \geq 2$ since $r \leq m(\ell + s)$, and $s \leq p - 2$ since $r \leq mp$. Therefore $x^\ell f_1 \in J^2$ and $y \in J^p$.

(b) Assume $\deg(M) > 0$, notice that $\ell = 0$ and $r \leq ms$, otherwise $y \in J^p$. Let y_1 be the result of evaluating y at $x = 1$. From Eq.(7.2) we derive $y_1^m \in I^{mp-r} \subset I^{m(p-s)}$, and $y_1 \in I_a^{p-s} = I^{p-s}$. Using Eq.(7.2) and Eq.(7.3) we obtain

$$\deg(y) \geq 2s + t(p - s),$$

as $y = x^s y_1$ by a degree argument it follows that $y \in J^p$. □

The next example shows that Proposition 7.4.2 does not extend to monomial ideals with minimal generators in different degrees.

Example 7.4.3 Let $R = k[x_1, \ldots, x_8]$ be a polynomial ring over a field k of characteristic zero and let

$$f_1 = x_1x_2x_3x_4, \quad f_2 = x_5x_6x_7x_8, \quad f_3 = x_1x_2x_4x_5,$$
$$f_4 = x_3x_4x_6x_8, \quad f_5 = x_1x_3x_7, \quad f_6 = x_2x_7x_8.$$

If I is the ideal of R generated by $f_1, \ldots, f_6$, then $\mathcal{R}(I)$ is a normal domain, whereas the ideal

$$J = I + (xx_1, \ldots, xx_8)$$

is not a normal ideal of the polynomial ring $R[x]$.

Let us briefly explain how to verify the assertions in Example 7.4.3. Let P be the toric ideal of $\mathcal{R}(I)$ with respect to $f_1, \ldots, f_6$ and let L be its Jacobian ideal, that is, L is generated by the $g \times g$ minors of the Jacobian matrix of P, where g is the height of P (in our case $g = 5$). Using the computer algebra system *Macaulay* [15] we readily obtain the equality

$$\mathrm{pd}_B(B/P) = \mathrm{height}(P), \text{ where } B = R[T_1, \ldots, T_6].$$

Therefore $\mathcal{R}(I)$ is Cohen-Macaulay by Corollary 2.5.14. Again making use of *Macaulay* we derive that (L, P) is an ideal of height 7. Therefore the normality of $\mathcal{R}(I)$ follows from the Serre's normality criterion together with the Jacobian criterion for regularity in polynomial rings; see Example 3.5.11 for details.

On the other hand observe that J^3 is not complete because $xf_1f_2 \notin J^3$ and $(xf_1f_2)^2 = (xx_5)(xx_6)f_3f_4f_5f_6 \in J^6$.

Proposition 7.4.4 *Let $R = k[x_1, \ldots, x_n]$ and $R[x]$ be polynomial rings over a field k. If I is an ideal of R generated by square-free monomials and I is normal, then $J = (I, xx_1)$ is a normal ideal of $R[x]$.*

Proof. By induction on p we will show $J_a^p = J^p$ for all $p \geq 1$. If $p = 1$, then J^p is integrally closed by Corollary 5.1.5. Assume $J_a^i = J^i$ for $i < p$ and $p \geq 2$. Let y be a monomial in J_a^p, then $y^m \in J^{pm}$, $m > 0$. Since $J_a^p \subset J_a^{p-1} = J^{p-1}$ we can write

$$y = x^t(xx_1)^r M f_1 \cdots f_{p-r-1} = x^t(xx_1)^r z,$$

where M is a monomial with $x \notin \operatorname{supp}(M)$ and the f_i's are monomials in J with $x \notin \operatorname{supp}(f_i)$ for all i. By Proposition 7.3.2 it suffices to show $y \in J^p$. Since $y^m \in J^{pm}$ we have

$$y^m = x^{tm}(xx_1)^{rm} z^m = N(xx_1)^s g_1 \cdots g_{mp-s}, \tag{7.4}$$

where N is a monomial, $x \notin \operatorname{supp}(g_i)$ for all i, and the g_i's are monomials in J. We distinguish two cases

(a) Assume $t = 0$, then $s \leq rm$. Since x_1^{rm} divides y^m from Eq. (7.4) we obtain $z^m \in I^{(p-r)m}$, hence $z \in I_a^{p-r} = I^{p-r}$ and $y = (xx_1)^r z \in J^p$.

(b) If $t > 0$, we may assume $x_1 \notin \operatorname{supp}(M)$, otherwise $y \in J^p$. We may also assume $x_1 \notin \operatorname{supp}(f_i)$ for all i, otherwise it is not hard to see that we are back in case (a). Notice that $s \leq rm$, because $x_1 \notin \operatorname{supp}(z)$. Since x_1^{rm} divides y^m it follows that $z \in I_a^{p-r} = I^{p-r}$ and $y = x^t(xx_1)^r z \in J^p$. □

Proposition 7.4.5 *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. If I_t is the ideal of R generated by the square-free monomials of degree t, then I_t is a normal ideal.*

Proof. We will proceed by induction on t. Assume that I_i is a normal ideal for $i < t$ and let $I = I_t$. We set $\mathfrak{m} = (x_1, \ldots, x_n)$. It is enough to show (by induction on $p \geq 0$) that I^p is integrally closed for all p. Assume $p > 0$ and I^i is integrally closed for $i < p$. In our case the integral closure of I^p is a monomial ideal given by

$$I_a^p = (y \in R | \, y^j \in I^{jp} \text{ for some } j),$$

see Proposition 7.3.2. Assume $M = I_a^p / I^p$ is non-zero. We claim that M has finite length. If $\mathfrak{p} \in \operatorname{Ass}_R M \setminus \{\mathfrak{m}\}$, say $x_1 \notin \mathfrak{p}$, then $I_{\mathfrak{p}} = (I_{t-1})_{\mathfrak{p}}$, where I_{t-1} is the ideal of R generated by the square-free monomials of degree $t-1$ in $x_2, \ldots, x_n$. By the first induction hypothesis we get $M_{\mathfrak{p}} = (0)$, which is impossible. Hence $\operatorname{Ass}_R M = \{\mathfrak{m}\}$. Let y be a monomial in $I_a^p \setminus I^p$ which is conducted by $\mathfrak{m}$ into I^p. Note that, by the above description of I_a^p, any monomial in I_a^p has degree at least tp. We have $y \in I_a^p \subset I_a^{p-1} = I^{p-1}$, so we can write

$$y = z_1 \cdots z_{p-1} u,$$

where $z_1, \ldots, z_{p-1}$ are monomials of degree t in I and u is a monomial with $\deg(u) \geq t$. Notice that $|\operatorname{supp}(u)| \leq t - 1$, otherwise $y \in I^p$. We can write,

after permutation of variables, $u = x_1^{b_1} \cdots x_\lambda^{b_\lambda}$, $b_i \geq 1$ all i. If $x_1 \notin \operatorname{supp}(z_i)$ for some i, then there is $x_{j_1} \in \operatorname{supp}(z_i) \setminus \operatorname{supp}(u)$, and we can re-write

$$y = z_1 \cdots z_{i-1} z_i' z_{i+1} \cdots z_{p-1} x_1^{b_1-1} x_2^{b_2} \cdots x_\lambda^{b_\lambda} x_{j_1},$$

where $z_i' = (z_i/x_{j_1})x_1$. We have three cases to consider.
(i) The number of z_i's not containing x_1 is greater or equal than b_1, then we can re-write

$$y = w_1 \cdots w_{p-1} x_2^{b_2} \cdots x_\lambda^{b_\lambda} x_{j_1} \cdots x_{j_{b_1}},$$

where w_i are monomials in $I \setminus \mathfrak{m}I$, the variables $x_2, \ldots, x_\lambda, x_{j_1}, \ldots, x_{j_{b_1}}$ distinct, and $\lambda + b_1 \leq t$.
(ii) The number q of z_i's not containing x_1 is less than b_1, then we can write

$$y = w_1 \cdots w_{p-1} x_1^{b_1-q} x_2^{b_2} \cdots x_\lambda^{b_\lambda} x_{j_1} \cdots x_{j_q},$$

where $x_1, x_2, \ldots, x_\lambda, x_{j_1}, \ldots, x_{j_q}$ are distinct, w_i are monomials in $I \setminus \mathfrak{m}I$ and x_1 in $\operatorname{supp}(w_i)$ for all i.
(iii) All the z_i's contain x_1, which reduces to case (ii).

Applying the same arguments to $x_2, \ldots, x_\lambda$ yields that we can write

$$y = y_1 \cdots y_{p-1} x_{i_1}^{a_1} \cdots x_{i_r}^{a_r} x_{j_1} \cdots x_{j_s},$$

where y_i are monomials in $I \setminus \mathfrak{m}I$, $a_i \geq 1$ for all i, $x_{i_1}, \ldots, x_{i_r}, x_{j_1}, \ldots, x_{j_s}$ distinct and

$$x_{i_1}, \ldots, x_{i_r} \in \bigcap_{i=1}^{p-1} \operatorname{supp}(y_i).$$

Note that $r + s \leq t - 1$, otherwise $y \in I^p$. We set $x_{j_s} = 1$ if $s = 0$. We substitute $x_{i_1} = \cdots = x_{i_r} = 1$ in $x_{i_1} y$ and denote by g the monomial obtained from $x_{i_1} y$ after evaluation. To derive a contradiction observe that

$$\deg(g) = (p-1)(t-r) + s,$$

while on the other hand, since $x_{i_1} y \in I^p$ we have $\deg(g) \geq p(t-r)$. Hence $r + s \geq t$, which is impossible. □

Some of the arguments given in Proposition 7.4.5 may apply for other groups of square-free monomials whose elements are of degree t or less. Next we show a result in this direction.

Theorem 7.4.6 *Let $R = k[x_1, \ldots x_n]$ and $R[x]$ be polynomial rings over a field k and let*

$$L_t = I_t + xI_{t-2},$$

where I_t is the ideal of R generated by the square-free monomials of degree $t \geq 2$. Then L_t is a normal ideal of $R[x]$.

Proof. We use induction on t. If $t = 2$, then Proposition 7.4.5 yields that $L_2 = I_2 + xR$ is normal. Assume that L_i is a normal ideal for $i < t$ and set $\mathfrak{m} = (x_1, \ldots, x_n, x)$, $L = L_t$, $I = I_{t-2}$, and $J = I_t$. By induction on $p \geq 1$ we now show that L^p is integrally closed for all p. Recall that L is an intersection of prime ideals because L is generated by square-free monomials (see Corollary 5.1.5), hence $L = L_a$. Assume that L^i is integrally closed for $i < p$ and $p \geq 2$. Let us show $L_a^p = L^p$. If on the contrary $L^p \neq L_a^p$, we set $(0) \neq M = L_a^p/L^p$. Observe that for any $\mathfrak{p} \in \operatorname{Ass}_A M \setminus \{\mathfrak{m}\}$ one has $L_\mathfrak{p} = (L_{t-1})_\mathfrak{p}$ if $x_i \notin \mathfrak{p}$ for some i and $L_\mathfrak{p} = (I_{t-2})_\mathfrak{p}$ if $x \notin \mathfrak{p}$, therefore by induction hypothesis and Proposition7.4.5 it follows that the only associated prime of M is $\mathfrak{m}$. Hence there is a monomial $y \in L_a^p \setminus L^p$ so that $\mathfrak{m}y \subset L^p$, since $L_a^p \subset L_a^{p-1} = L^{p-1}$ we can write

$$y = u(xh_1)\cdots(xh_s)f_1\cdots f_{p-s-1},$$

where the h_i's are monomials in I of degree $t-2$ and the f_j's are monomials in J of degree t. Observe that if $x \in \operatorname{supp}(u)$, then we may re-write

$$y = \left(\frac{x_i x_j u}{x}\right)(xh_1)\cdots(xh_s)\left(x\left(\frac{f_1}{x_i x_j}\right)\right)f_2\cdots f_{p-s-1},$$

for some $x_i, x_j \in \operatorname{supp}(f_1)$ and $i \neq j$, a repeated use of this rearrangement of the factors of y yields that either $y = u(xh_1)\cdots(xh_s)f_1\cdots f_{p-s-1}$ with $x \notin \operatorname{supp}(u)$, or $y = u(xh_1)\cdots(xh_{p-1})$ with $x \in \operatorname{supp}(u)$. In the second case using that $y^m \in L^{mp}$ for some $m \geq 1$, together with Proposition 7.4.5, rapidly gives $y \in L^p$. As a consequence from the start we may assume $x \notin \operatorname{supp}(u)$; notice that $\deg(u) \geq t$. Let $u = x_1^{a_1}\cdots x_\lambda^{a_\lambda}$, $1 \leq \lambda \leq t-1$, $a_1 \geq 2$. There are three cases to consider:

(a) Assume $|\operatorname{supp}(u)| = t - 1$. In this case

$$x_1 \in \bigcap_{i=1}^{p-s-1} \operatorname{supp}(f_i),$$

otherwise $y \in L^p$. We may assume $x_1 \notin \operatorname{supp}(h_\ell)$ for some ℓ, otherwise by induction on t we obtain $y \in L^p$. We claim that

$$\operatorname{supp}(h_i) \subset \operatorname{supp}(u) \subset \operatorname{supp}(f_j), \text{ for all } i, j.$$

If the first containment is false, then pick $x_r \in \operatorname{supp}(h_i) \setminus \operatorname{supp}(u)$. If x_1 is not in $\operatorname{supp}(h_i)$, then we re-write

$$y = \left(\frac{ux_r}{x_1}\right)(xh_1)\cdots(xh_{i-1})\left(\frac{xx_1h_i}{x_r}\right)(xh_{i+1})\cdots(xh_s)f \in L^p,$$

where $f = f_1\cdots f_{p-s-1}$, which contradicts the choice of y. On the other hand if $x_1 \in \operatorname{supp}(h_i)$, then $\operatorname{supp}(h_\ell) \not\subset \operatorname{supp}(h_i) \setminus \{x_1\}$ and we may pick

$z \in \operatorname{supp}(h_\ell)$ and $z \notin \operatorname{supp}(h_i)$. Therefore

$$y = \left(\frac{ux_r}{x_1}\right)(xh_1)\cdots(xh_{\ell-1})\left(\frac{x_1xh_\ell}{z}\right)(xh_{\ell+1}) \cdots(xh_{i-1})\left(\frac{xzh_i}{x_r}\right)\cdots(xh_s)f,$$

where $f = f_1 \cdots f_{p-s-1}$, hence $y \in L^p$, and again this is a contradiction. To show the second containment note that we can write $h_\ell = x_2 \cdots x_{t-1}$. If $\operatorname{supp}(u) \not\subset \operatorname{supp}(f_j)$, pick x_r in $\operatorname{supp}(u) \setminus \operatorname{supp}(f_j)$ and re-write

$$y = u(xh_1)\cdots(xh_{\ell-1})(xh'_\ell)(xh_{\ell+1}) \cdots(xh_s)f_1 \cdots f_{j-1}f'_jf_{j+1}\cdots f_{p-s-1},$$

where $h'_\ell = (x_1h_\ell)/x_r$ and $f'_j = (x_rf_j)/x_1$, since x_1 is not in $\operatorname{supp}(f'_j)$ and $\operatorname{supp}(f'_j) \not\subset \operatorname{supp}(u)$ we get $y \in L^p$, and the proof of the claim is complete. Let g be the result of evaluating y at $x_1 = \cdots = x_{t-1} = 1$, notice that $\deg(g) = p-1$, while by evaluating $x_1y \in L^p$ at the same points we obtain $\deg(g) \geq p$, which is a contradiction. Altogether in this case $y \in L^p$.

(b) Assume $|\operatorname{supp}(u)| = t-2$. If

$$x_1 \in \left(\bigcap_{i=1}^{s} \operatorname{supp}(h_i)\right) \bigcap \left(\bigcap_{i=1}^{p-s-1} \operatorname{supp}(f_i)\right),$$

then by induction hypothesis on t we derive $y \in L^p$, otherwise we reduce to a case similar to (a).

(c) If $|\operatorname{supp}(u)| \leq t-3$. Since $\deg(h_i) = t-2$, $\operatorname{supp}(h_i) \not\subset \operatorname{supp}(u)$ and $\operatorname{supp}(f_j) \not\subset \operatorname{supp}(u)$ for all i, j. Therefore we may apply the method of proof of Proposition 7.4.5 to conclude that (after at most $a_1 + \cdots + a_\lambda$ rearrangements) we reduce to a case similar to (a), (b), or there is some x_j in the support of u, h_i, and f_j for all i, j, which implies (by induction on t) that $y \in L^p$. □

Definition 7.4.7 Let I and J be two monomial ideals of the polynomial rings $k[x_0, \ldots, x_m]$ and $k[y_0, \ldots, y_n]$ respectively. The *join* of I and J is:

$$I * J = I + J + K,$$

where $K = (x_iy_j \mid 0 \leq i \leq m \text{ and } 0 \leq j \leq n)$.

Theorem 7.4.8 ([72]) *Let $R = k[x_0, \ldots, x_m]$ and $S = k[y_0, \ldots, y_n]$ be polynomial rings over a field k, and let I, J be two monomial ideals of R and S respectively. If I and J are normal ideals generated by square-free monomials of the same degree $t \geq 2$, then their join $I * J$ is normal.*

Proof. Set $X = \{x_0, \ldots, x_m\}$, $Y = \{y_0, \ldots, y_n\}$ and $L = I + J + K$, where $K = (X)(Y)$. By induction on p we will show that $L_a^p = L^p$ for all $p \geq 1$, where L_a^p denotes the integral closure of L^p. If $p = 1$ then L is a radical ideal (see Corollary 5.1.5), hence L is integrally closed. Assume $L_a^i = L^i$ for $i < p$ and $p \geq 2$. Using Proposition 7.3.2, we have

$$L_a^p = (\{z \,|\, z \text{ is a monomial in } k[X,Y] \text{ and } z^q \in L^{qp} \text{ for some } q \geq 1\}).$$

Let z be a monomial in L_a^p, then $z^q \in L^{qp}$, $q > 0$. Let us show $z \in L^p$. Since $L_a^p \subset L_a^{p-1} = L^{p-1}$ we can write

$$z = Mh_1 \cdots h_s g_1 \cdots g_r f_1 \cdots f_{p-r-s-1},$$

where M is a monomial, the h_i's are monomials of degree two in K, the g_i's and f_i's are degree t monomials in J and I respectively. Likewise we can write

$$z^q = Nh_1' \cdots h_{s_1}' g_1' \cdots g_{r_1}' f_1' \cdots f_{qp-r_1-s_1}',$$

where N is a monomial, $\deg(h_i') = 2$ and h_i' is a monomial in K for all i, g_i' and f_j' are degree t monomials in J and I respectively for all i, j. From the last two equalities we have

$$\begin{aligned} z^q &= M^q(h_1 \cdots h_s)^q(g_1 \cdots g_r)^q(f_1 \cdots f_{p-r-s-1})^q \\ &= Nh_1' \cdots h_{s_1}' g_1' \cdots g_{r_1}' f_1' \cdots f_{qp-r_1-s_1}'. \end{aligned} \tag{7.5}$$

From Eq. (7.5) one readily derives the inequality

$$(qs - s_1)(t-2) + qt \leq q \deg(M). \tag{7.6}$$

We may assume $M = x^\alpha$ or $M = y^\alpha$, otherwise $z \in L^p$. By symmetry we may assume $M = x^\alpha$, where $\alpha \geq 0$.

(a) If $t \geq 3$, then $r = 0$ or $p - r - s - 1 = 0$, otherwise $g_1 f_1 \in L^3$ and hence $z \in L^p$.

First we treat the case $r = 0$. By taking degrees in Eq. (7.5) w.r.t. the variables $y_0, \ldots, y_n$ one has $s_1 + tr_1 \leq qs$, which together with Eq. (7.6) yields $\deg M \geq t$. Let z_1 be the result of evaluating z at $y_i = 1$. From Eq. (7.5) we derive

$$z_1^q \in I^{qp-s_1-r_1} \subset I^{q(p-s)},$$

and $z_1 \in I_a^{p-s} = I^{p-s}$. We may write $z = y^\theta z_1$ and $z_1 = x^\beta w$, where $\deg(y^\theta) = s$ and w is a monomial in I^{p-s} of degree $t(p-s)$. Since

$$\deg(z_1) = \deg(M) + s + t(p - s - 1)$$

we obtain $\deg(z_1) \geq s + t(p-s)$, hence $\deg(x^\beta) \geq s$. Altogether we derive $z = y^\theta x^\beta w \in L^p$. Next we consider the case $r = p - s - 1 \geq 1$, observe that

$\deg(M) \leq 1$, otherwise $z \in L^p$. Therefore either $z = h_1 \cdots h_s g_1 \cdots g_r$, or we may rewrite

$$z = y^\beta h_1 \cdots h_s h_{s+1} g_1 \cdots g_{r-1},$$

where $\deg(h_{s+1}) = 2$ and $h_{s+1} \in K$, interchanging the x_i and y_i variables we may apply the arguments above to conclude $z \in L^p$.

(b) Assume $t = 2$. Using $z^q \in L^{qp}$ one rapidly obtains $\deg(M) \geq 2$, hence we may assume $r = 0$ (otherwise $z \in L^p$) and the arguments of case (a) can be applied to conclude $z \in L^p$. □

Corollary 7.4.9 ([256]) *Let $X = \{x_0, \ldots, x_m\}$ and $\{y_0, \ldots, y_n\}$ be two disjoint sets of indeterminates over a field k. Let I be a normal ideal of $k[X]$ generated by square-free monomials of degree t and let $L = I + K$, where $K = (X)(Y)$. Then L is a normal ideal.*

Proof. Proceed as in the proof of Theorem 7.4.8 and notice that in this case $r = p - s - 1$. □

Exercises

7.4.10 Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring and $I = (f_1, \ldots, f_q)$, where k is a field. If f_i is homogeneous of degree $d \geq 1$ for all i. Prove

(a) $\mathcal{R}(I) \simeq \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I) \oplus \mathfrak{m}\mathcal{R}(I)$ as $\mathcal{R}(I)$ modules, where $\mathfrak{m} = R_+$,

(b) $k[f_1, \ldots, f_q] \simeq \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$ as k-algebras,

(c) if $F = \{f_1, \ldots, f_q\}$ is a set of monomials, then $\mathcal{R}(I) \simeq k[F, t\mathbf{x}]$ as k-algebras, where $\mathbf{x} = \{x_1, \ldots, x_n\}$.

Hint For (c) consider

$\psi\colon k[\mathbf{x}, t_j] \to \mathcal{R}(I)$, where $\psi(x_i) = x_i$, $\psi(t_j) = tf_j$, and

$\phi\colon k[\mathbf{x}, t_j] \to k[F, t\mathbf{x}]$, where $\phi(x_i) = tx_i$, $\phi(t_j) = f_j$.

Use Proposition 7.1.2 to show that ϕ and ψ have the same kernel.

7.4.11 Let $R = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ and $L = I_s J_t$, where I_s (resp. J_t) denote the ideal of R generated by the square-free monomials of degree s (resp. t) in the x_i (resp. y_i) variables. Prove that L is normal.

7.4.12 Let $R = K[\mathbf{x}]$ be a polynomial ring over a field K. If I is an ideal of R generated by square-free monomials of degree 2, then I and I^2 are both integrally closed.

7.5 Ideals of mixed products

Let $R = K[\mathbf{x}, \mathbf{y}]$ be a polynomial ring in two disjoint sets of variables $\mathbf{x}$, $\mathbf{y}$ over a field K. We will study *ideals of mixed products*

$$L = I_k J_r + I_s J_t$$

such that $k+r = s+t$, where I_k (resp. J_r) denotes the ideal of R generated by the square-free monomials of degree k (resp. r) in the $\mathbf{x}$ (resp. $\mathbf{y}$) variables. The notion of ideal of mixed products was introduced in [241].

The aim is to present a complete classification of the normal ideals of mixed products, the main result (see Theorem 7.5.8 and [241]) generalizes the normality of various previously known normal ideals and contain as a particular case the normality of generalized graph ideals associated to complete bipartite graphs. The dimension of the monomial subring $K[F]$ will be examined, where F denotes the minimal generating set of L consisting of monomials.

Powers of ideals of mixed products

Let $R = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ a polynomial ring over a field K. Given non negative integers k, r, s, t such that $k+r = s+t$ consider the square-free monomial ideal of *mixed products* given by

$$L = I_k J_r + I_s J_t,$$

where I_k (resp. J_r) is the ideal of R generated by the square-free monomials of degree k in the variables $x_1, \ldots, x_m$ (resp. $y_1, \ldots, y_n$).

Note that L has a *dual ideal* L' obtained by an interchange of the two sets of variables:

$$L' = I_t J_s + I_r J_k \subset R' = K[x_1, \ldots, x_n, y_1, \ldots, y_m].$$

It is convenient to set $I_0 = J_0 = R$. By symmetry there are essentially two cases:

(i) $L = I_k J_r + I_s J_t$ with $0 \leq k < s$, or

(ii) $L = I_k J_r$ with $k \geq 1$ or $r \geq 1$.

One of the results that will be used throughout this section is the fact that the powers of the ideals I_k and J_r are complete, see Proposition 7.4.5.

Remark 7.5.1 (a) If $A = R[y_i^{-1}]$ is the Laurent polynomial ring for some y_i, then

$$J_t A = J'_{t-1} A,$$

where J'_{t-1} is the ideal of R generated by the square-free monomials of degree $t-1$ in the variables $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n$. In fact one clearly has $J_t \subset J'_{t-1}$, thus $J_t A \subset J'_{t-1} A$. On the other hand consider a monomial f in J'_{t-1}, then $y_i f \in J_t$ and $f \in J_t A$. Hence $J'_{t-1} A \subset J_t A$.

(b) If a variable y_i is not in a prime ideal $\mathfrak{p} \subset R$, then the localization of J_t at $\mathfrak{p}$ is the same as the localization of J'_{t-1} at $\mathfrak{p}$. A similar statement holds if a variable x_i is not in $\mathfrak{p}$ and one consider I_k (resp. I'_{k-1}) instead of J_t (resp. J'_{t-1}). We need this observation only when $\mathfrak{p}$ is a face ideal.

For use below recall that products of ideals and quotients of modules commute with localizations. We also recall that the integral closure of ideals commute with localizations, see Proposition 3.3.4. For basic facts on monomial ideals see Chapter 5.

As usual a monomial $f = x_1^{a_1} \cdots x_m^{a_n} y_1^{b_1} \cdots y_n^{b_n}$ will be denoted simply by $f = x^\alpha y^\beta$, where $\alpha = (a_1, \ldots a_m)$ and $\beta = (b_1, \ldots, b_n)$. The support of f, denoted by $\operatorname{supp}(f)$, is:

$$\operatorname{supp}(f) = \{x_i | \, a_i > 0\} \cup \{y_i | \, b_i > 0\}.$$

We begin by treating some special cases.

Lemma 7.5.2 *Let $R = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be a polynomial ring over a field K. If*

$$L = J_r + I_m,$$

then L is a normal ideal for any r.

Proof. Let us prove directly that $\overline{L^p} = L^p$ for all $p \geq 1$. Take $f = x^\alpha y^\beta$ in $\overline{L^p}$, where $x^\alpha = x_1^{a_1} \cdots x_m^{a_m}$. Thus f^i is in L^{pi} for some $i \geq 1$. If $a_j = 0$ for some j, then $y^{i\beta} \in J_r^{ip}$ and $y^\beta \in \overline{J_r^p} = J_r^p$. Hence one may assume that $a_1 \geq 1$ is the smallest value of the a_i. Note that $p \leq a_1$ implies $x^\alpha \in I_m^p$. Assume $p > a_1$. It follows that $y^{i\beta} \in J_r^{i(p-a_1)}$. Using that J_r is normal yields $y^\beta \in J_r^{p-a_1}$, which together with $x^\alpha \in I_m^{a_1}$ proves $f \in L^p$. □

Example 7.5.3 If $m = n = 3$, then $L = J_2 + I_2$ is not a normal ideal because $f \in \overline{L^3} \setminus L^3$, where f is the product of all the variables.

Proposition 7.5.4 *Let $R = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be a polynomial ring over a field K. If*

$$L = J_r + I_m J_t$$

with $r = m + t$, then L is a normal ideal.

Proof. By induction on r. If $r = 1$ or $t = 0$ the result is clear because of Lemma 7.5.2. Next we show by induction on p that $\overline{L^p} = L^p$ for all $p \geq 1$; the case $p = 1$ is clear because L is a radical ideal. Assume L^i is integrally

closed for $1 \leq i < p$ and set $M = \overline{L^p}/L^p$. If $M \neq (0)$, take an associated prime ideal $\mathfrak{p}$ of M. As

$$M \hookrightarrow R/L^p,$$

the prime ideal $\mathfrak{p}$ is a face ideal. Note that y_i is in $\mathfrak{p}$ for all i; for otherwise if $y_i \notin \mathfrak{p}$, then by Remark 7.5.1 one concludes

$$L_{\mathfrak{p}} = (L_1)_{\mathfrak{p}},$$

where

$$L_1 = J'_{r-1} + I_m J'_{t-1}$$

and J'_{r-1} is the ideal generated by the square-free monomials of degree $r-1$ in the variables $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n$. Thus by induction on r, the immediate consequence is that the localization of M at $\mathfrak{p}$ is zero, that is,

$$M_{\mathfrak{p}} = (\overline{L^p}/L^p)_{\mathfrak{p}} = (0),$$

a contradiction because $\mathfrak{p}$ is in the support of the R-module M. Hence y_i is in $\mathfrak{p}$ for all i.

As $\mathfrak{p}$ is generated by a subset of the variables, there is a monomial f in $\overline{L^p} \setminus L^p$ such that $\mathfrak{p} = (L^p\colon f)$. Since $y_i f$ is in L^p and $p \geq 2$, one can pick a variable y_1 in $\operatorname{supp}(f)$ such that $\deg_{y_1}(f) \geq \deg_{y_i}(f)$ for all i. Therefore one can write

$$y_1 f = g w_1 \cdots w_p, \tag{1}$$

where $w_1, \ldots, w_p$ are monomials of degree r in L and g is some monomial of R. Note $\deg(g) > 0$, because $f^i \in L^{pi}$ for some $i \geq 1$, and $y_1 \notin \operatorname{supp}(g)$, because $f \notin L^p$.

Case (a): First we assume y_j divides g for some j, note $j \neq 1$ because $f \notin L^p$. Set $c = \deg_{y_1}(f)$. Since y_1^{c+1} divides $y_1 f$ one has that y_1^{c+1} divides $w_1 \cdots w_p$, thus one may assume y_1 is in the support of w_i for $i = 1, \ldots, c+1$. In order to derive a contradiction note that $y_j \in w_i$ for all $i = 1, \ldots, c+1$, this follows by observing that if $y_j \notin w_i$, then from the equality

$$y_1 f = \left(\frac{y_1 g}{y_j}\right) w_1 \cdots \left(\frac{w_i y_j}{y_1}\right) \cdots w_{c+1} \cdots w_p$$

one derives $f \in L_p$. Therefore y_j^{c+1} divides f, which contradicts the choice of c.

Case (b): Next we assume $g = x^\alpha$. One may order the w_i such that $w_\ell \in I_m J_t$ for $\ell \leq j$ and $w_\ell \in J_r$ for $\ell > j$, where j is some fixed integer between 0 and p. Therefore from Eq. (1) one has

$$\deg_y(f) = \deg_y(y_1 f) - 1 = jt + (p-j)r - 1. \tag{2}$$

On the other hand since $f^i \in L^{pi}$ for some positive integer i, one can find monomials $z_1, \ldots, z_{pi}$ in L of degree r such that

$$f^i = hz_1 \cdots z_{pi}, \tag{3}$$

for some monomial h in R. We order the z_i such that $z_\ell \in I_m J_t$ for $\ell \leq j_1$ and $z_\ell \in J_r$ for $\ell > j_1$, where j_1 is some integer between 0 and pi. Using Eq.(2) and taking degrees in Eq.(3) gives

$$\begin{aligned} i(jt + (p-j)r - 1) \geq i \deg_y(f) \geq j_1 t + r(pi - j_1) \\ \iff (r-t)(j_1 - ij) \geq i. \end{aligned} \tag{4}$$

Hence $j < p$ and consequently $|\mathrm{supp}(x^\alpha)| < m$, because otherwise $y \in L^p$. Pick a variable $x_1 \notin \mathrm{supp}(x^\alpha)$ and note that Eq. (1) gives $\deg_{x_1}(f^i) = ij$, which together with Eq. (3) yields $j_1 \leq ij$, but this contradicts Eq. (4). □

Proposition 7.5.5 *Let $R = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ and let*

$$L = J_r + I_s J_t.$$

If $t \geq 1$, $r = s + t$ and $2 \leq s < m$, then L^2 is not integrally closed.

Proof. Since $s < m$ it makes sense to set

$$f = (x_1 x_2 \cdots x_{s+1})(y_1^2 \cdots y_{t-1}^2 y_t \cdots y_r).$$

In order to prove that $f \in \overline{L^2}$, set $y_0 = 1$ and observe that from the equalities:

$$\begin{aligned} f^s &= \underbrace{(x_1 x_2 \cdots x_{s+1})^s}_{\text{belongs to } I_s^{s+1}} y_1^{2s} \cdots y_{t-1}^{2s} y_t^s \cdots y_r^s \\ y_1^{2s} \cdots y_{t-1}^{2s} y_t^s \cdots y_r^s &= (y_1 \cdots y_r)^{s-1} (y_1^{s+1} \cdots y_{t-1}^{s+1} y_t \cdots y_r) \\ y_1^{s+1} \cdots y_{t-1}^{s+1} y_t \cdots y_r &= \underbrace{(y_1 \cdots y_{t-1} y_t) \cdots (y_1 \cdots y_{t-1} y_r)}_{s+1 \text{ terms in } J_t} \end{aligned}$$

we obtain $f^s \in L^{2s}$, as required. By counting degrees, and using $s \geq 2$, it follows that $f \notin L^2$. □

Proposition 7.5.6 *Let $R = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ and let*

$$L = I_k J_r + I_s J_t.$$

If $s > k + 1 \geq 2$, $t \geq 1$ and $k + r = s + t$, then L^3 is not integrally closed.

Proof. Set

$$f = x_1^3 x_2^3 \cdots x_{k-1}^3 x_k^2 \cdots x_s^2 y_1^3 y_2^3 \cdots y_{t-1}^3 y_t^2 \cdots y_r^2,$$

where $x_0 = y_0 = 1$. To show that f is in the integral closure of L^3, it suffices to observe that from the equalities.

$$\begin{aligned} f^2 &= x_1^6 x_2^6 \cdots x_{k-1}^6 x_k^4 \cdots x_s^4 y_1^6 y_2^6 \cdots y_{t-1}^6 y_t^4 \cdots y_r^4 \\ x_1^6 \cdots x_{k-1}^6 x_k^4 \cdots x_s^4 &= (x_1 \cdots x_{k-1} x_k)(x_1 \cdots x_{k-1} x_{k+1}) \cdot \\ &\quad (x_1 \cdots x_{k-1} x_{k+2}) \cdot \\ &\quad (x_1^3 x_2^3 \cdots x_{k-1}^3 x_k^3 x_{k+1}^3 x_{k+2}^4 x_{k+3}^4 \cdots x_s^4) \\ y_1^6 \cdots y_{t-1}^6 y_t^4 \cdots y_r^4 &= (y_1^3 \cdots y_r^3)(y_{t+3} \cdots y_r) \cdot \\ &\quad (y_1 \cdots y_{t-1} y_t)(y_1 \cdots y_{t-1} y_{t+1})(y_1 \cdots y_{t-1} y_{t+2}) \end{aligned}$$

it follows that f^2 is in L^6. As $\deg_x(f) = 2s + k - 1$ and $\deg_y(f) = 2r + t - 1$, by a counting degree argument it follows that f is not in L^3. □

Proposition 7.5.7 *Let $R = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ and let*

$$L = I_s + J_s.$$

If $2 \leq s < \inf\{m, n\}$, then L^{s+1} is not integrally closed.

Proof. Let $f = (x_1^{s-1} \cdots x_{s+1}^{s-1})(y_1 \cdots y_{s+1})$. To show that f lies in the integral closure of L^{s+1} note that from the equality:

$$f^s = \underbrace{((x_1 \cdots x_{s+1})^s)^{s-1}}_{\text{is in } I_s^{(s+1)(s-1)}} \underbrace{(y_1 \cdots y_{s+1})^s}_{\text{is in } J_s^{s+1}}$$

it follows that f^s is in $L^{(s+1)s}$. As $s \geq 2$, by a counting degree argument, it follows that f is not in L^{s+1}. □

Theorem 7.5.8 ([241]) *Let $R = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be a polynomial ring over a field K. If*

$$L = I_k J_r + I_s J_t \neq R$$

is an ideal of mixed products with $k + r = s + t$, then L is normal if and only if L can be written (up to permutation of k, s and r, t) in one of the following forms:

(a) $L = I_k J_r + I_{k+1} J_{r-1}$, $k \geq 0$ *and* $r \geq 1$.

(b) $L = I_k J_r$, $k \geq 1$ *or* $r \geq 1$.

(c) $L = I_k J_r + I_s J_t$, $0 = k < s = m$, *or* $0 = t < r = n$, *or* $k = t = 0$, $s = 1$.

Proof. $\Rightarrow$) If $k = s$, then $r = t$ and we are in case (b). By symmetry one may assume $0 \leq k < s \leq m$. Taking into account the previous results one can readily conclude the following classification. If $k = t = 0$, then

$$L = \begin{cases} J_1 + I_1, & s = 1, & \text{case (c).} \\ J_s + I_s, & s = \inf\{m, n\}, & \text{case (c).} \\ J_s + I_s, & 2 \leq s < \inf\{m, n\}, & \text{not normal.} \end{cases}$$

If $k = 0$ and $t \geq 1$, then

$$L = \begin{cases} J_r + I_1 J_t, & s = 1, & \text{case (a).} \\ J_r + I_m J_t, & s = m, & \text{case (c).} \\ J_r + I_s J_t, & 2 \leq s < m, & \text{not normal.} \end{cases}$$

If $k \geq 1$ and $t = 0$, then

$$L = \begin{cases} I_k J_1 + I_{k+1}, & r = 1, & \text{case (a).} \\ I_k J_n + I_s, & r = n, & \text{case (c).} \\ I_k J_r + I_s, & 2 \leq r < n, & \text{not normal.} \end{cases}$$

If $k \geq 1$ and $t \geq 1$, then

$$L = \begin{cases} I_k J_r + I_{k+1} J_{r-1}, & s = k + 1, & \text{case (a).} \\ I_k J_r + I_s J_t, & s \geq k + 2, & \text{not normal.} \end{cases}$$

Altogether one obtains that L has the required form.

$\Leftarrow$) (a): By induction on $k + r$ we show that L is normal. If $k + r = 1$, then $L = J_1 + I_1$ and L is normal. Assume $k + r \geq 2$. Next we use induction on p to prove that $\overline{L^p} = L^p$ for all $p \geq 1$; the case $p = 1$ follows because L is a radical ideal. Assume L^i is integrally closed for $1 \leq i < p$ and set $M = \overline{L^p}/L^p$. If $M \neq (0)$, take an associated prime ideal $\mathfrak{p}$ of M. Since $M \hookrightarrow R/L^p$, the prime ideal $\mathfrak{p}$ is generated by a subset of the variables. Note that by Remark 7.5.1 one has $L_{\mathfrak{p}} = (L_1)_{\mathfrak{p}}$, under the following conditions

$$L_1 = \begin{cases} I_k J'_{r-1} + I_{k+1} J'_{r-2}, & \text{if} \quad y_i \notin \mathfrak{p} \text{ and } r \geq 2, \\ I'_{k-1} J_r + I'_k J_{r-1}, & \text{if} \quad x_i \notin \mathfrak{p} \text{ and } k \geq 1, \\ J'_{r-1} + I_1 J'_{r-2}, & \text{if} \quad y_i \notin \mathfrak{p} \text{ and } k = 0, \\ J_{r-1}, & \text{if} \quad x_i \notin \mathfrak{p} \text{ and } k = 0, \\ I'_{k-1} J_1 + I'_k, & \text{if} \quad x_i \notin \mathfrak{p} \text{ and } r = 1, \\ I_k, & \text{if} \quad y_i \notin \mathfrak{p} \text{ and } r = 1. \end{cases}$$

Here J'_{r-1} and I'_{k-1} have the same meaning as in Remark 7.5.1. Thus if $\mathfrak{p}$ is different from the maximal ideal $\mathfrak{m} = R_+$, using the induction hypothesis and the fact that I_k and J_{r-1} are normal, one concludes that $M_{\mathfrak{p}} = (0)$, a contradiction because $\mathfrak{p}$ is in the support of M. Hence $\mathfrak{m}$ is the only associated prime ideal of M and there is a monomial f in $\overline{L^p} \setminus L^p$ such that

$\mathfrak{m} = (L^p\colon f)$. Observe that the support of f contains one of the y_i variables; because if $f = x^\alpha$, then $f^i \in L^{pi}$ for some $i \geq 1$ but this can only take place if $r = 1$ and $f^i \in (I_{k+1})^{pi}$, as I_{k+1} is normal one has $f \in I_{k+1}^p \subset L^p$ which is a contradiction.

Pick a variable y_1 in $\operatorname{supp}(f)$ such that $\deg_{y_1}(f) \geq \deg_{y_i}(f)$ for all i, then one can write

$$y_1 f = g w_1 \cdots w_p, \tag{1}$$

where $w_1, \ldots, w_p$ are monomials of degree $k+r$ in L and g is some monomial of R. Note $\deg(g) > 0$ because $f^i \in L^{pi}$ for some $i \geq 1$, and y_1 is not in $\operatorname{supp}(g)$ because $f \notin L^p$.

Case (I): First assume y_j divides g for some j fixed, note $j \neq 1$ because $f \notin L^p$. Set $c = \deg_{y_1}(f)$. Since y_1^{c+1} divides $y_1 f$ one has that y_1^{c+1} divides $w_1 \cdots w_p$, thus one may assume y_1 is in the support of w_i for $i = 1, \ldots, c+1$. Note that y_j is in $\operatorname{supp}(w_i)$ for all $i = 1, \ldots, c+1$; because if $y_j \notin \operatorname{supp}(w_i)$, then from the equality

$$y_1 f = w_1 \cdots (w_i y_j / y_1) \cdots w_{c+1} \cdots w_p (y_1 g / y_j)$$

one derives $f \in L_p$. Hence y_j^{c+1} divides f, which contradicts the choice of y_1.

Case (II): Next assume $g = x^\alpha$ and x_j divides g, for some fixed j. We divide this case into two subcases.

Subcase (A): First assume that there is a term w_ℓ of the form

$$w_\ell = (x_{i_1} \cdots x_{i_k})(y_1 y_{j_2} \cdots y_{j_r}), \quad \text{with} \quad y_1 \in \operatorname{supp}(w_\ell),$$

observe that x_j is in $\operatorname{supp}(w_\ell)$, otherwise write

$$y_1 f = w_1 \cdots w_{\ell-1} (x_{i_1} \cdots x_{i_k} x_j)(y_{j_2} \cdots y_{j_r}) w_{\ell+1} \cdots w_p (y_1 g / x_j),$$

to obtain $f \in L^p$. We claim that $x_j \notin \cap_{i=1}^p \operatorname{supp}(w_i)$, because if x_j is in the support of w_i for all i, then x_j^{p+2} divides $x_j f$, but since $x_j f \in L^p$ one readily derives that f is also in L^p, a contradiction. Then there is w_q such that

$$w_q = \begin{cases} \text{(i)} & (x_{s_1} \cdots x_{s_{k+1}})(y_{t_1} \cdots y_{t_{r-1}}), \quad x_j \notin \operatorname{supp}(w_q), \text{ or} \\ \text{(ii)} & (x_{s_1} \cdots x_{s_k})(y_{t_1} \cdots y_{t_r}), \quad x_j \notin \operatorname{supp}(w_q). \end{cases}$$

If (i) occurs $\{x_{s_1}, \ldots, x_{s_{k+1}}\} \not\subset \{x_{i_1}, \ldots, x_{i_k}\}$, say $x_{s_1} \notin \{x_{i_1}, \ldots, x_{i_k}\}$. From

$$y_1 f = g w_\ell w_q \prod_{i \neq \ell, q} w_i = (y_1 g / x_j)(x_{s_1} w_\ell / y_1)(x_j w_q / x_{s_1}) \prod_{i \neq \ell, q} w_i,$$

one derives $f \in L^p$. If (ii) occurs $\{y_{t_1}, \ldots, y_{t_r}\} \not\subset \{y_{j_2}, \ldots, y_{j_r}\}$, say y_{t_1} is not in $\{y_{j_2}, \ldots, y_{j_r}\}$. From

$$y_1 f = g w_\ell w_q \prod_{i \neq \ell, q} w_i = (y_1 g / x_j)(y_{t_1} w_\ell / y_1)(x_j w_q / y_{t_1}) \prod_{i \neq \ell, q} w_i,$$

one derives $f \in L^p$, again a contradiction.

Subcase (B): Now assume all the terms w_ℓ that contain y_1 in their support (and there is at least one of such terms) are of the form

$$w_\ell = (x_{i_1} \cdots x_{i_{k+1}})(y_1 y_{j_2} \cdots y_{j_{r-1}}), \quad y_1 \in \operatorname{supp}(w_\ell). \tag{$*$}$$

Consider the following two possibilities for the terms $w_1, \ldots, w_q$

$$w_q = \begin{cases} \text{(i)} & (x_{s_1} \cdots x_{s_k})(y_{t_1} \cdots y_{t_r}), & \text{for some } 1 \leq q \leq p, \text{ or} \\ \text{(ii)} & (x_{s_1} \cdots x_{s_{k+1}})(y_{t_1} \cdots y_{t_{r-1}}), & \text{for all } 1 \leq q \leq p. \end{cases}$$

If (i) occurs and w_ℓ is any term as in Eq. $(*)$, pick

$$y_{t_1} \in \{y_{t_1}, \ldots, y_{t_r}\} \setminus \{y_{j_2}, \ldots, y_{j_{r-1}}\} \text{ and}$$
$$x_{i_1} \in \{x_{i_1}, \ldots, x_{i_{k+1}}\} \setminus \{x_{s_1}, \ldots, x_{s_k}\}.$$

Note x_j is in $\operatorname{supp}(w_q)$ because otherwise from

$$y_1 f = g w_\ell w_q \prod_{i \neq \ell, q} w_i = (y_1 g / x_j)(y_{t_1} w_\ell / y_1)(x_j w_q / y_{t_1}) \prod_{i \neq \ell, q} w_i,$$

one derives $f \in L^p$. Assume $x_j \notin \operatorname{supp}(w_\ell)$ for some w_ℓ (that contains y_1 in its support) and note that from

$$y_1 f = (y_1 g / x_j)(y_{t_1} x_j w_\ell / y_1 x_{i_1})(x_{i_1} w_q / y_{t_1}) \prod_{i \neq \ell, q} w_i,$$

one derives $f \in L^p$. Hence one may now assume $x_j \in \operatorname{supp}(w_\ell)$ for all w_ℓ containing y_1 in its support. From subcase (A) it follows that $x_j \notin \cap_{i=1}^p \operatorname{supp}(w_i)$. Hence x_j is not in the support of w_θ for some $1 \leq \theta \leq p$. If w_θ has the form $w_\theta = x_{\lambda_1} \cdots x_{\lambda_k} y_{\mu_1} \cdots y_{\mu_r}$, one may assume $y_{\mu_1} \notin \{y_{j_1}, \ldots, y_{j_{r-1}}\}$ and use

$$y_1 f = (y_1 g / x_j)(x_j w_\theta / y_{\mu_1})(y_{\mu_1} w_\ell / y_1) \prod_{i \neq \ell, q} w_i,$$

to derives $f \in L^p$. If w_θ has the form $w_\theta = x_{\lambda_1} \cdots x_{\lambda_{k+1}} y_{\mu_1} \cdots y_{\mu_{r-1}}$, choose $x_{\lambda_1} \in \{x_{\lambda_1}, \ldots, x_{\lambda_{k+1}}\} \setminus \{x_{s_1}, \ldots, x_{s_k}\}$, then from

$$y_1 f = (y_1 g / x_j)(x_{\lambda_1} w_q / y_{t_1})(x_j w_\theta / x_{\lambda_1})(y_{t_1} w_\ell / y_1) \prod_{i \neq \ell, q} w_i,$$

one derives $f \in L^p$. Thus the argument for case (i) is complete.

Assume (ii), that is, all the w_i have the form $w_i = x^a y^b$, where the degree of x^a (resp. y^b) is equal to $k+1$ (resp. $r-1$). Using Eq.(1) and the fact that $g = x^\alpha$ one concludes $\deg_y(f) = p(r-1) - 1$. On the other

hand, since $f^j \in L^{jp}$ for some $j \geq 1$, we get $j \deg_y(f) \geq (r-1)pj$. Hence $\deg_y(f) \geq (r-1)p$ which is impossible. This completes the proof of the normality in case (a).

To finish the proof of the theorem note that the normality of case (b) follows readily using that I_k and J_r are normal, and the normality of case (c) follows from Proposition 7.5.4. □

Definition 7.5.9 Let G be a graph with vertices $x_1, \ldots, x_n$, the *generalized graph ideal*, denoted by $I_q(G)$, is the ideal of $K[x_1, \ldots x_n]$ generated by the square-free monomials $x_{i_1} \cdots x_{i_q}$ such that the x_{i_j} is adjacent to $x_{i_{j+1}}$ for all $1 \leq j \leq q-1$.

Corollary 7.5.10 *If G is a complete bipartite graph, then the generalized graph ideal $I_q(G)$ is normal for all $q \geq 2$.*

Proof. Let $x_1, \ldots, x_m, y_1, \ldots, y_n$ be the vertex set of G, one may assume that the edges of G are precisely the pairs of the form $\{x_i, y_j\}$. Therefore

$$I_q(G) = \begin{cases} I_t J_{t+1} + I_{t+1} J_t & \text{if} \quad q = 2t+1 \\ I_t J_t & \text{if} \quad q = 2t, \end{cases}$$

and the result follows from Theorem 7.5.8. □

Segre products and a dimension formula

Let us begin with two results that are useful. The first is the following dimension formula, and the second result states that certain monomial subrings of mixed products are isomorphic to Segre products.

Proposition 7.5.11 *Let B and C be two standard algebras over a field K, then their Segre product:*

$$B \otimes_S C = (B_0 \otimes_K C_0) \oplus (B_1 \otimes_K C_1) \oplus \cdots \subset B \otimes_K C,$$

has Krull dimension equal to $\dim(B) + \dim(C) - 1$.

Proof. See Exercise 4.1.25. □

Definition 7.5.12 Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. The *set of k-products* in the x_i variables, denoted by V_k, is defined as:

$$V_k = \{x_{i_1} \cdots x_{i_k} | \, 1 \leq i_1 < \cdots < i_k \leq n\},$$

where k is a positive integer.

Proposition 7.5.13 *Let $V_k \subset K[\mathbf{x}]$ and $V'_r \subset K[\mathbf{y}]$ be the sets of k and r products respectively in disjoint sets of variables. Then the Segre product of $K[V_k] \subset K[\mathbf{x}]$ and $K[V'_r] \subset K[\mathbf{y}]$ is isomorphic to*

$$A = K[\{fg | f \in V_k \text{ and } g \in V'_r\}].$$

Proof. We grade $K[V_k]$, $K[V'_r]$ and A with the normalized grading, thus their generators have normalized degree 1. It is no hard to see that there is a well defined graded epimorphism

$$\varphi: A \to K[V_k] \otimes_S K[V'_r],$$

such that $\varphi(fg) = f \otimes g$ for all $f \in V_k$ and $g \in V'_r$, where

$$K[V_k] \otimes_S K[V'_r] = (K[V_k]_0 \otimes_K K[V'_r]_0) \oplus (K[V_k]_1 \otimes_K K[V'_r]_1) \oplus \cdots.$$

Since

$$\dim_K(A_i) = \dim_K(K[V_k]_i \otimes_K K[V'_r]_i)$$

for all i one obtains that the map φ is an isomorphism. □

Lemma 7.5.14 *Let A be the matrix with rows*

$$e_{i_1} + \cdots + e_{i_r}, -e_1 + e_2, -e_2 + e_3, \ldots, -e_{n-1} + e_n,$$

where e_i is the ith unit vector in $\mathbb{R}^n$. Then $\det(A) = r$ for $n \geq 2$.

Proof. Expanding by the first column of A and using induction on the size n of the matrix, the result follows rapidly. □

Proposition 7.5.15 *If L is the ideal of mixed products in m and n variables and F the minimal generating set of L consisting of square-free monomials, then the Krull dimension d of $K[F]$ is given by*

$$d = \begin{cases} m+n-1, & \text{if } L = I_k J_r \quad 1 \leq k < m \text{ and } 1 \leq r < n, \\ m+n, & \text{if } L = I_k J_r + I_s J_t \quad 1 \leq k < s \leq m, \\ & \qquad\qquad 1 \leq t < r \leq n, \text{ and} \\ & \qquad\qquad k + r = s + t. \end{cases}$$

Proof. The first formula follows at once using Proposition 7.5.11 and Proposition 7.5.13. To prove the second formula consider the linear subspace W of $\mathbb{R}^{m+n}$ generated by the set of exponent vectors of the monomials that minimally generate L. Note that the exponent of x_i is e_i and the exponent of y_i is e_{m+i}, where e_i is the ith unit vector. Recall that

$$\dim K[F] = \dim_K(W),$$

that is, the proof reduces to showing $\dim_K W = m + n$.

Give two integers $1 \leq i < j \leq m$, pick integers $j_1, \ldots, j_{k-1}$ between 1 and m such that $i, j, j_1, \ldots, j_{k-1}$ are all distinct, this choice is possible because $1 \leq k < m$. Hence subtracting the vectors

$$e_i + e_{j_1} + \cdots + e_{j_{k-1}} + e_{m+1} + \cdots + e_{m+r}$$
$$e_j + e_{j_1} + \cdots + e_{j_{k-1}} + e_{m+1} + \cdots + e_{m+r}$$

we get $e_i - e_j$ in W for $1 \leq i < j \leq m$. A symmetric argument using $1 \leq t < n$ shows that $e_i - e_j$ is in W for $m+1 \leq i < j \leq m+n$. Similarly subtracting

$$e_{i_1} + \cdots + e_{i_s} + e_{j_1} + \cdots + e_{j_t} \quad (1 \leq i_\ell \leq m) \quad (m < j_\ell \leq m+n)$$
$$e_{i_1} + \cdots + e_{i_k} + e_{j_1} + \cdots + e_{j_r} \quad (1 \leq i_\ell \leq m) \quad (m < j_\ell \leq m+n)$$

yield that $e_{i_{k+1}} + \cdots + e_{i_s} - e_{j_{t+1}} - \cdots - e_{j_r}$ are in W for all possible choices, or equivalently the vectors $e_{i_1} + \cdots + e_{i_{s-k}} - e_{j_1} - \cdots - e_{j_{r-t}}$ are in W for all possible choices. By adding

$$\sum_{\ell=2}^{s-k}(e_{i_1} - e_{i_\ell}) - \sum_{\ell=2}^{r-t}(e_{j_1} - e_{j_\ell})$$

to any of these vectors we conclude $(s-k)e_{i_1} - (r-t)e_{j_1} \in W$, but by assumption $s - k = r - t$, thus $e_{i_1} - e_{j_1}$ are in W for $1 \leq i_1 \leq m$ and $m < j_1 \leq m+n$.

Altogether $e_i - e_j$ are in W for all $1 \leq i < j \leq m+n$. To finish the argument apply Lemma 7.5.14 to the matrix having first row

$$e_1 + \cdots + e_k + e_{m+1} + \cdots + e_{m+r}$$

to conclude $\dim_K(W) = n + m$. □

Remark 7.5.16 There is a notion of sequentially Koszul algebra that has been recently studied [7] and that is likely to hold for some members of the family of monomial subrings associated to mixed products. See also [152] for the related notion of strongly Koszul algebra.

Exercises

7.5.17 Let $V_k \subset K[\mathbf{x}]$ and $V'_r \subset K[\mathbf{y}]$ be the sets of k and r products respectively, in disjoint sets of variables, and

$$A = K[\{fg | f \in V_k \text{ and } g \in V'_r\}].$$

If $K[V_k]$, $K[V_r]$ and A have the normalized grading, then for all d one has the equality

$$\dim_K(A_d) = \dim_K(K[V_k]_d \otimes_K K[V'_r]_d).$$

7.5.18 Let $L = I_k J_r + I_s J_t$ be an ideal of mixed products with $k+r = s+t$ and F the minimal generating set of L consisting of monomials. Prove that $\dim K[F] = m + n$ in the following three "extreme" cases:

$$\begin{array}{lllll} (a) & 0 \leq k < s < m & \text{and} & 1 \leq t < r \leq n, \\ (b) & 1 \leq k < s \leq m & \text{and} & 0 \leq t < r < n, \\ (c) & 0 = k < s < m & \text{and} & 0 = t < r < n. \end{array}$$

7.5.19 Let $L = I_1 J_4 + I_4 J_1$ be an ideal of mixed products in the polynomial ring $K[x_1, \ldots, x_4, y_1, \ldots, y_4]$. Prove that the integral closure of L^3 is not generated by monomials of degree 15.

7.6 Degree bounds for some integral closures

Let S be an affine domain and $\overline{S}$ its integral closure or normalization. When S is graded, it is rarely possible to have estimates of the degrees of the generators of $\overline{S}$ unless there are very circumscribed situations. Here we discuss one of these when S is a monomial subring permitting a sharp regrading. In this section we use some techniques introduced by W. Bruns, J. Gubeladze and N. Trung [41].

Theorem 7.6.1 ([51]) *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over an infinite field K and F a finite set of monomials in R of the same degree k. Let I be the ideal of R generated by F and*

$$\mathcal{R} = \bigoplus_{i=0}^{\infty} I^i t \subset R[t]$$

its Rees algebras. Then the normalization $\overline{\mathcal{R}}$ is generated as an $\mathcal{R}$-module, and thus as an $\mathcal{R}$-algebra, by elements $g \in R[t]$ of t-degree at most n.

Proof. Let $F = \{f_1, \ldots, f_q\}$ and $\mathcal{R} = K[x_1, \ldots, x_n, f_1 t, \ldots, f_q t]$. We introduce a new grading δ on $R[t]$ by setting

$$\delta(x_i) = 1 \quad \text{and} \quad \delta(t) = -(k-1).$$

Note that $\mathcal{R}$ is generated as a K-algebra by elements of degree 1 because $\delta(f_i t) = \delta(f_i) + \delta(t) = 1$. Consider the subsemigroup

$$C = \mathbb{N}\log(x_1) + \cdots + \mathbb{N}\log(x_n) + \mathbb{N}\log(f_1 t) + \cdots + \mathbb{N}\log(f_q t)$$

of $\mathbb{N}^{n+1}$ generated by $e_1, \ldots, e_n, \log(f_1 t), \ldots, \log(f_q t)$. Since $\mathbb{Z}C = \mathbb{Z}^{n+1}$, according to Theorem 7.2.28 the integral closure of $\mathcal{R}$ can be expressed as:

$$\overline{\mathcal{R}} = K[\{x^a t^b | \, (a,b) \in \mathbb{Z}^{n+1} \cap \mathbb{R}_+ C\}],$$

where $\mathbb{R}_+C$ is the cone spanned by C in $\mathbb{R}^{n+1}$. If $M = x^a t^b$ with $(a,b) \neq 0$ and $(a,b) \in \mathbb{Z}^{n+1} \cap \mathbb{R}_+C$, then it is not hard to show that $\delta(M) \geq b$. Therefore $\overline{\mathcal{R}}$ is generated as a K-algebra by monomials of positive degree. There is a Noether normalization of $\overline{\mathcal{R}}$

$$A = K[z_1, \ldots, z_{n+1}] \overset{\varphi}{\hookrightarrow} \mathcal{R} \overset{\psi}{\hookrightarrow} \overline{\mathcal{R}},$$

where $z_1, \ldots, z_{n+1} \in \mathcal{R}_1$. Note that

$$A \xrightarrow{\psi\varphi} \overline{\mathcal{R}}$$

is a Noether normalization of $\overline{\mathcal{R}}$. By Theorem 7.2.35 $\overline{\mathcal{R}}$ is Cohen-Macaulay. Hence by Proposition 2.2.14 $\overline{\mathcal{R}}$ is a free module over A and one may write

$$\overline{\mathcal{R}} = AM_1 \oplus \cdots \oplus AM_r,$$

where $M_i = x^{\beta_i} t^{b_i}$. Using that the length is additive one has the following expression for the Hilbert series

$$F(\overline{\mathcal{R}}, z) = \sum_{i=0}^{r} \frac{z^{\delta(M_i)}}{(1-z)^{n+1}} = \frac{h_0 + h_1 z + \cdots + h_s z^s}{(1-z)^{n+1}},$$

where $h_i = |\{j \mid \delta(M_j) = i\}|$. Note that $a(\overline{\mathcal{R}})$, the a-invariant of $\overline{\mathcal{R}}$, is equal to $s - (n+1)$. On the other hand by Proposition 4.2.3 one has

$$a(\overline{\mathcal{R}}) = -\min\{i \mid (\omega_{\overline{\mathcal{R}}})_i \neq 0\},$$

where $\omega_{\overline{\mathcal{R}}}$ is the canonical module of $\overline{\mathcal{R}}$. An application of Theorem 7.2.34 yields

$$a(\overline{\mathcal{R}}) = s - (n+1) < 0$$

and $s \leq n$. Altogether one has $\delta(M_i) \leq n$. An straightforward computation shows that if $M_i = x^{\beta_i} t^{b_i}$, then the t-degree of M_i is less or equal than n, that is, $b_i \leq n$. □

For a more general version of Theorem 7.6.1 see [51]. We give a variant of the previous theorem for the subalgebra generated by monomials of the same degree k. A homogeneous polynomial of degree ik is said to have *normalized degree* i.

Theorem 7.6.2 ([51]) *Let F be a finite set of monomials of degree k and*

$$S = K[F] \subset R = K[x_1, \ldots, x_n]$$

the monomial subring generated by F. Then the normalization $\overline{S}$ of S is generated as an S-module, and thus as an S-algebra, by elements $g \in R$ of normalized degree at most $\dim(S) - 1$.

Proof. Let

$$R^{(k)} = \bigoplus_{i=0}^{\infty} R_{ik}$$

be the kth Veronese subalgebra of the polynomial ring R. Both S and $\overline{S}$ are contained in $R^{(k)}$. We grade $\overline{S}$ by setting $\overline{S}_i = \overline{S} \cap R_{ik}$. Then we may adapt the arguments of the previous proof. □

7.7 Degree bounds in the square-free case

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and F a finite set of square-free monomials of the same degree k. One of the goals is to present bounds for the generators of the normalization of the monomial subring $K[F]$. In this case there are embeddings

$$K[F] \subset S \subset R^{(k)},$$

where

$$S = K[\{x_{i_1} \cdots x_{i_k} \,|\, 1 \leq i_1 < \cdots < i_k \leq n\}],$$

and $R^{(k)}$ is the kth Veronese subring of R. All the constructions of $K[F]$ take place inside of one of these algebras. Our exposition will follow closely that of [51].

Especially we want to present an explicit generating set for the canonical module ω_S of S that appears in [51]. In particular we will fully describe the algebras S with the Gorenstein property and compute the a-invariant of S. The analysis below will determine the Cohen-Macaulay type of S but also leads to the control of degrees in $\overline{K[F]}$.

A description of the canonical module

Let us fix some of the notation that will be used throughout the rest of the section.

Let $n \geq 2k \geq 4$ be two integers (this is not an essential restriction; see Remark 7.7.12). We set

$$\mathcal{A} = \{e_{i_1} + \cdots + e_{i_k} \,|\, 1 \leq i_1 < \cdots < i_k \leq n\},$$

where $e_1, \ldots, e_n$ are the canonical vectors in $\mathbb{R}^n$. The affine subsemigroup of $\mathbb{N}^n$ generated by $\mathcal{A}$ will be denoted by C, that is, we have

$$C = \sum_{\alpha \in \mathcal{A}} \mathbb{N}\alpha.$$

The *cone generated* by C will be denoted by $\mathbb{R}_+C$, thus

$$\mathbb{R}_+C = \left\{ \sum_{i=1}^{r} a_i\alpha_i \,\middle|\, a_i \in \mathbb{R}_+,\ \alpha_i \in \mathcal{A},\ r \in \mathbb{N} \right\}.$$

If $0 \neq a \in \mathbb{R}^n$, then the set H_a will denote the *hyperplane* of $\mathbb{R}^n$ through the origin with *normal vector* a. Thus $H_a = \{x \in \mathbb{R}^n \,|\, \langle x, a\rangle = 0\}$, and H_a determines two *closed half-spaces*

$$H_a^+ = \{x \in \mathbb{R}^n \,|\, \langle x, a\rangle \geq 0\} \quad \text{and} \quad H_a^- = \{x \in \mathbb{R}^n \,|\, \langle x, a\rangle \leq 0\}.$$

Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. Observe that the K-subring of R spanned by the set $\{x^a \,|\, a \in \mathcal{A}\}$ is equal to the *affine semigroup ring* $K[C]$.

Remark 7.7.1 Let φ be the matrix whose columns are the vectors of $\mathcal{A}$. By Proposition 7.1.17, we obtain

$$\operatorname{rank}(\varphi) = \dim K[C] = n.$$

Hence the vector subspace generated by $\mathcal{A}$ is equal to $\mathbb{R}^n$ and $\dim \mathbb{R}_+C = n$.

The equations of the cone In order to describe the canonical module of S we will use the Danilov-Stanley formula. Thus the equations of the cone $\mathbb{R}_+C$ should be determined first.

Lemma 7.7.2 *Set* $N_1 = \{-e_1, \ldots, -e_n\}$, $N = N_1 \cup N_2$ *and*

$$N_2 = \{-e_1 - \cdots - e_{i-1} + (k-1)e_i - e_{i+1} - \cdots - e_n \,|\, 1 \leq i \leq n\}.$$

If H is a supporting hyperplane of the cone $\mathbb{R}_+C$ such that H contains a set $\alpha_1, \ldots, \alpha_{n-1}$ of linearly independent vectors in $\mathcal{A}$, then $H = H_a$ for some vector $a \in N$.

Proof. Let $0 \neq a = (a_1, \ldots, a_n)$, such that $H = H_a$. Let M be the $(n-1) \times n$ matrix whose rows are the vectors $\alpha_1, \ldots, \alpha_{n-1}$.

First note that if the j-column of M is equal to zero, then $\langle \alpha_i, e_j\rangle = 0$ for all i and $H = H_{e_j}$. On the other hand, if all the entries of the j-column of M are equal to 1, then $H = H_a$ for some $a \in N_2$. Therefore we may assume that all the columns of M have some zero entries and also some entries equal to 1. Set

$$\mathcal{I} = \{i \,|\, a_i > 0\} \quad \text{and} \quad \mathcal{J} = \{i \,|\, a_i < 0\},$$

without loss of generality we may assume

$$A = \{a_i \,|\, i \in \mathcal{I}\} = \{a_1, \ldots, a_\ell\},$$

where $\ell = |\mathcal{I}|$. By symmetry the proof can be easily reduced to the following four cases.

(a) Assume $\mathcal{I} = \emptyset$. Using that all the columns of M have at least one entry equal to 1 we obtain that $a = 0$; hence this case cannot occur.

(b) Assume $2 \leq \ell \leq k-1$. We claim that there exists α_i so that in the set of the first ℓ entries of α_i there is at least one entry equal to zero and at least one entry equal to 1. Otherwise, using that $\operatorname{rank}(M) = n-1$ and row-reducing M to its normal form we obtain $\ell = 2$, and hence $a = a_1 e_1 - a_1 e_2$. Setting

$$\beta = e_1 + e_3 + \cdots + e_{k+1} \quad \text{and} \quad \gamma = e_2 + e_3 + \cdots + e_{k+1}$$

we get

$$\langle \beta, a \rangle = a_1 > 0 \quad \text{and} \quad \langle \gamma, a \rangle = -a_1 < 0,$$

which is impossible because H_a is a supporting hyperplane of $\mathbb{R}_+ C$ and the proof of the claim is complete. For simplicity we assume

$$\alpha_1 = e_{i_1} + e_{i_2} + \cdots + e_{i_m} + e_{j_1} + \cdots + e_{j_r},$$

$$e_{i_1} = e_1,\ m + r = k,$$

$$3 \leq i_2 < \cdots < i_m \leq \ell < j_1 < \cdots < j_r \leq n \text{ and } r, m \geq 1.$$

Let us show $\langle \beta, a \rangle > 0$, where $\beta = e_1 + \cdots + e_\ell + e_{j_1} + \cdots + e_{j_{k-\ell}}$. Note that

$$\begin{aligned} \langle \beta, a \rangle &> a_1 + a_3 + \cdots + a_\ell + a_{j_1} + \cdots + a_{j_{k-\ell}} \\ &\geq a_{i_1} + a_{i_2} + \cdots + a_{i_m} + a_{j_1} + \cdots + a_{j_{k-\ell}} \geq 0, \end{aligned}$$

because $\langle \alpha_1, a \rangle = 0$. As a consequence $\beta \in \mathcal{A}$ and $\langle \beta, a \rangle > 0$, as required. If $2 \leq |\mathcal{J}| \leq k-1$, we may apply similar arguments to prove that there is a $\gamma \in \mathcal{A}$ so that $\langle \gamma, a \rangle < 0$, which contradicts that H is a supporting hyperplane. Therefore we may further assume that $|\mathcal{J}|$ is either equal to 1 or $|\mathcal{J}| \geq k$; in this situation we can rapidly find $\gamma \in \mathcal{A}$ so that $\langle \gamma, a \rangle < 0$, which again yields a contradiction.

(c) Assume $\ell = 1$ and $|\mathcal{J}| \geq k$. Set $r = |\mathcal{J}|$ and $B = \{a_i | i \in \mathcal{J}\}$, say $B = \{a_2, \ldots, a_{r+1}\}$. If $r = n-1$, then using $\langle \alpha_i, a \rangle = 0$ for all i we derive that all the entries of the first column of M are equal to 1, hence $H = H_a$ for some $a \in N_2$. Next we assume $r \leq n-2$. Let α_i be a vector with its first entry equal to zero. Since $\langle \alpha_i, a \rangle = 0$ it follows that the first $r+1$ entries of α_i are equal to zero and $n > r + k$. Setting

$$\beta = e_1 + e_{r+2} + \cdots + e_{r+k} \quad \text{and} \quad \gamma = \beta - e_1 + e_2$$

we obtain

$$\langle \beta, a \rangle > 0 \quad \text{and} \quad \langle \gamma, a \rangle < 0,$$

which is impossible. The case $\ell = 1$ and $|\mathcal{J}| = 1$ can be treated similarly.

(d) Assume $\ell \geq k$ and $|\mathcal{J}| \geq k$. This case cannot occur because one can rapidly find vectors β, γ in $\mathcal{A}$ so that $\langle \beta, a \rangle > 0$ and $\langle \gamma, a \rangle < 0$. □

Remark 7.7.3 The converse of Lemma 7.7.2 is also true because we are assuming $n \geq 2k \geq 4$, and this implies $n \geq k+2$. Note that if $n = 3$ and $k = 2$, then the cone $\mathbb{R}_+C$ has only three facets.

Proposition 7.7.4 ([51]) *A point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is in $\mathbb{R}_+C$ if and only if x is a feasible solution of the system of linear inequalities*

$$\begin{array}{rcll} -x_i & \leq & 0, & i = 1, \ldots, n \\ (k-1)x_i - \sum_{j \neq i} x_j & \leq & 0, & i = 1, \ldots, n. \end{array}$$

Proof. Let $\mathbb{R}_+C = H_{b_1}^- \cap \cdots \cap H_{b_m}^-$ be the irreducible representation of $\mathbb{R}_+C$ as an intersection of closed half spaces. By Theorem 7.2.18 the set $H_{b_i} \cap \mathbb{R}_+C$ is a facet of $\mathbb{R}_+C$. Note that H_{b_i} is generated by a set of linearly independent vectors in $\mathcal{A}$, see Corollary 7.2.22. Therefore by Lemma 7.7.2 one obtains $H_{b_i} = H_a$, for some $a \in N$, here N denotes the set defined in Lemma 7.7.2. □

A generating set for the canonical module Let X be an arbitrary subset of $\mathbb{R}^n$. The *relative interior* of X, denoted $\operatorname{ri}(X)$, is the interior of X relative to $\operatorname{aff}(X)$, the affine hull of X.

Lemma 7.7.5 *Let a be a vector in $C \cap \operatorname{ri}(\mathbb{R}_+C)$ and set $A = \{i \mid a_i \geq 2\}$. If $|A| \geq k$ and $i_1, \ldots, i_k$ are distinct integers in A, then $a' = a - e_{i_1} - \cdots - e_{i_k}$ also belongs to $C \cap \operatorname{ri}(\mathbb{R}_+C)$.*

Proof. Without loss of generality one may assume $a_1 \geq a_2 \geq \cdots \geq a_n$, $a_k \geq 2$ and $a' = a - e_1 - \cdots - e_k$. We claim that $a' \in \operatorname{ri}(\mathbb{R}_+C)$. First recall that $\mathbb{R}_+C$ has dimension n; thus one has

$$\operatorname{ri}(\mathbb{R}_+C) = \operatorname{int} \mathbb{R}_+C = \operatorname{int} \bigcap_{c \in N} H_c^- = \bigcap_{c \in N} \operatorname{int} H_c^- = \bigcap_{c \in N} H_c^- \setminus H_c \qquad (7.7)$$

by Proposition 7.7.4. Hence, using that $a \in \operatorname{ri}(\mathbb{R}_+C)$ we readily obtain

$$(k-1)(a_i - 1) \leq -1 + \sum_{\substack{1 \leq j \leq k \\ j \neq i}} (a_j - 1) + \sum_{k+1 \leq j} a_j, \quad \text{for } 1 \leq i \leq k. \qquad (7.8)$$

On the other hand, if $k < i \leq n$ we have

$$\begin{aligned} \sum_{j=1}^{n} a_j &\geq a_1 + (k-1)a_i + \sum_{k+1 \leq j} a_j = a_1 + ka_i + \sum_{\substack{k+1 \leq j \\ j \neq i}} a_j \\ &\geq ka_i + n - k + 1. \end{aligned}$$

As $n \geq 2k \geq 4$ we obtain

$$\sum_{j=1}^{n} a_j \geq ka_i + k + 1,$$

or equivalently, one has

$$(k-1)a_i \leq -1 + \sum_{j=1}^{k}(a_j - 1) + \sum_{\substack{k+1 \leq j \\ j \neq i}} a_j, \quad \text{for} \quad k+1 \leq i \leq n. \tag{7.9}$$

Altogether making use of (7.7), (7.8), and (7.9) we obtain $a' \in \mathrm{ri}(\mathbb{R}_+ C)$. By Proposition 7.4.5 the subring $K[C]$ is a normal domain, and therefore $C = \mathbb{Z}C \cap \mathbb{R}_+ C$. Since $a' \in \mathbb{Z}C$ we conclude $a' \in C$. □

Let $R = \bigoplus_{i=0}^{\infty} R_i$ be the standard grading of R and let

$$R^{(k)} = \bigoplus_{i=0}^{\infty} R_{ki}$$

be the *kth Veronese subring* of R graded by

$$(R^{(k)})_i = R_{ki}.$$

Notice that $K[C]$ is a graded subring of $R^{(k)}$ with the *normalized grading*:

$$K[C] = \bigoplus_{i=0}^{\infty} (K[C])_i,$$

where

$$(K[C])_i = K[C] \cap (R^{(k)})_i.$$

In the sequel we shall assume that $K[C]$ has the normalized grading.

Theorem 7.7.6 ([51]) *Set $S = K[C]$. Let ω_S be the canonical module of S and let $\mathfrak{B}$ be the set of monomials $M = x_1^{a_1} \cdots x_n^{a_n}$ satisfying the following conditions:*

(a) $a_i \geq 1$ *and* $(k-1)a_i \leq -1 + \sum_{j \neq i} a_j$, *for all* i.

(b) $\sum_{i=1}^{n} a_i \equiv 0 \bmod (k)$.

(c) $|\{\, i \mid a_i \geq 2\}| \leq k - 1$.

If $n \geq 2k \geq 4$, then $\mathfrak{B}$ is a generating set for ω_S.

Proof. According to Theorem 7.2.34 we have

$$\omega_S = (\{x^a \mid a \in C \cap \mathrm{ri}(\mathbb{R}_+ C)\}).$$

Taking into account the arguments of the proof of Lemma 7.7.5 and by a repeated use of Lemma 7.7.5 it is enough to prove that $\mathfrak{B} \subset \omega_S$. Let $M \in \mathfrak{B}$; without loss of generality we may assume $M = x_1^{a_1} \cdots x_{k-1}^{a_{k-1}} x_k \cdots x_n$, where $a_1 \geq \cdots \geq a_{k-1} \geq 1$. The monomial $N = x_1^{a_1} \cdots x_{k-1}^{a_{k-1}}$ can be factored as

$$N = \prod_{i=1}^{k-1} N_i, \text{ where } N_i = \begin{cases} (x_1 \cdots x_i)^{a_i - a_{i+1}}, & \text{if } 1 \leq i \leq k-2 \\ (x_1 \cdots x_{k-1})^{a_{k-1}}, & \text{if } i = k-1. \end{cases}$$

On the other hand, by the properties (a) and (b) it follows that we can write

$$\prod_{i=k}^{n} x_i = N' \prod_{i=1}^{k-1} N_i',$$

where

$$\deg(N_i') = \begin{cases} (k-i)(a_i - a_{i+1}), & \text{if } 1 \leq i \leq k-2 \\ a_{k-1}, & \text{if } i = k-1, \end{cases}$$

and $\deg(N') \equiv 0 \bmod (k)$. Hence

$$M = N' \prod_{i=1}^{k-1} (N_i N_i')$$

is in $K[C]$, which readily implies $M \in \omega_S$. □

Applications of the description of the canonical module

To find the presentation of $K[C]$ as a quotient of a polynomial ring B modulo a prime ideal P, consider the polynomial ring over the field K

$$B = K[\{T_{i_1 \ldots i_k} \mid 1 \leq i_1 < \cdots < i_k \leq n\}],$$

with one variable $T_{i_1 \ldots i_k}$ for each monomial $x_{i_1} \cdots x_{i_k}$. Here B has the usual grading. There is a graded homomorphism of K-algebras

$$\psi \colon B \longrightarrow K[C], \text{ induced by } T_{i_1 \ldots i_k} \overset{\psi}{\longmapsto} x_{i_1} \cdots x_{i_k},$$

the ideal $P = \ker(\psi)$ is the *presentation ideal* or *toric ideal* of $K[C]$. Thus the *Cohen-Macaulay type* of the ring $K[C]$, denoted by $\mathrm{type}(K[C])$, is the last Betti number in the minimal free resolution of B/P as a B-module.

Remark 7.7.7 Set $S = K[C]$. By Corollary 4.3.6 the type of S is equal to the minimal number of generators of the canonical module ω_S of S. To compute the type of S notice that a monomial $M = x_1^{a_1} \cdots x_{k-1}^{a_{k-1}} x_k \cdots x_n$ is in $\mathfrak{B}$ if and only if for all $1 \leq i \leq k-1$ one has

$$\sum_{j=1}^{k-1} a_j = mk - n + k - 1 \quad \text{and} \quad 1 \leq a_i \leq m-1, \quad \text{for some} \quad m \geq 2.$$

These two conditions imply

$$\frac{n}{k} \leq m \leq n - 2k + 2.$$

Therefore, by Theorem 7.7.6, the computation of the type of S reduces to counting partitions of positive integers.

Corollary 7.7.8 *Let ω_S be the canonical module of $S = K[C]$. Assume $k = 2$ and $n \geq 2k$. If n is odd, then*

$$\omega_S = (\{x_1 \cdots x_{j-1} x_j^{2i} x_{j+1} \cdots x_n | \, 1 \leq j \leq n, \ 1 \leq i \leq (n-3)/2\}).$$

Corollary 7.7.9 *If $k = 2$ and $n \geq 2k$, then*

$$\operatorname{type}(K[C]) = \begin{cases} \dfrac{n(n-3)}{2} & \text{if } n \text{ is odd,} \\ \dfrac{(n^2 - 4n + 2)}{2} & \text{if } n \text{ is even.} \end{cases}$$

Proof. The details are left as an exercise. □

Corollary 7.7.10 *If $n = 2k + 1 \geq 5$, then*

$$\operatorname{type}(K[C]) = \binom{n}{k-1}.$$

Proof. Left as an exercise. □

Let S be a Cohen-Macaulay positively graded K-algebra over a field K and ω_S the canonical module of S. We recall that $a(S)$, the a-invariant of S, is given by

$$a(S) = -\min\{ i \, | \, (\omega_S)_i \neq 0\}.$$

Corollary 7.7.11 ([51]) *If $n \geq 2k \geq 4$, then the a-invariant of $K[C]$ is:*

$$a(K[C]) = -\left\lceil \frac{n}{k} \right\rceil,$$

where $\lceil x \rceil$ is the least integer greater or equal than x.

Proof. Set $S = K[C]$ and $m = \lceil \frac{n}{k} \rceil$. It follows from Remark 7.7.7 that the degree of the generators in least degree of ω_S is at least m. To complete the proof we exhibit some generators of ω_S living in degree m. Write $n = qk+r$, $0 \leq r < k$; note $q \geq 2$. If $r \geq 1$, observe that the monomials

$x_1^2 \cdots x_{k-r}^2 x_{k-r+1} \cdots x_{k-1} x_k \cdots x_n$ and

$x_1 \cdots x_{n-k+r} x_{n-k+r+1}^2 \cdots x_{n-1}^2 x_n^2$

belong to $(\omega_S)_m$. In particular, S cannot be a Gorenstein ring in this case. If $r = 0$, then the monomial $M = x_1 \cdots x_n$ satisfies $M \in (\omega_S)_m$. □

Remark 7.7.12 (Duality) Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over the field K, and k an integer such that $1 \leq k \leq n-1$. Consider

$$S_{n,k} = K[\{x_{i_1} \cdots x_{i_k} \,|\, 1 \leq i_1 < \cdots < i_k \leq n\}],$$

the K-subring of R spanned by the $x_{i_1} \cdots x_{i_k}$'s. Observe that there is a graded isomorphism of K-algebras of degree zero:

$$\rho : S_{n,k} \longrightarrow S_{n,n-k}, \text{ induced by } \rho(x_{i_1} \cdots x_{i_k}) = x_{j_1} \cdots x_{j_{n-k}},$$

where $\{j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\} \setminus \{i_1, \ldots, i_k\}$. In particular if $n \leq 2k$, then

$$a(S_{n,k}) = a(S_{n,n-k}) = -\left\lceil \frac{n}{n-k} \right\rceil.$$

Because of this *duality* one may always assume that $n \geq 2k$.

The next corollary was shown by De Negri and Hibi [81] using different methods.

Corollary 7.7.13 ([51]) *The subring $S_{n,k}$ is a Gorenstein ring if and only if $k \in \{1, n-1\}$ or $n = 2k$.*

Proof. By duality one may assume $n \geq 2k \geq 4$. Set $S = S_{n,k}$. If S is Gorenstein, then by the proof of Corollary 7.7.11 we may assume $n = qk$. If $q \geq 3$, then $x_1 \cdots x_n$ and $x_1^3 x_2^2 \cdots x_{k-1}^2 x_k \cdots x_n$ belong to $(\omega_S)_q$ and $(\omega_S)_{q+1}$ respectively, which is impossible. Therefore $q = 2$, as required.

Conversely assume $n = 2k$. Let $\mathfrak{B}$ be as in Theorem 7.7.6. Take a monomial M in $\mathfrak{B}$; it suffices to verify that M is equal to $x_1 \cdots x_n$. One may assume that $M = x_1^{a_1} \cdots x_{k-1}^{a_{k-1}} x_k \cdots x_n$, where $a_i \geq a_{i+1} \geq 1$. By hypothesis one has

$$\sum_{j=1}^{k-1} a_j = k(m-1) - 1, \tag{7.10}$$

for some $m \geq 2$. On the other hand one has

$$ka_i \leq k + \sum_{j=1}^{k-1} a_j, \text{ for all } 1 \leq i \leq k-1. \tag{7.11}$$

Next we combine (7.10) and (7.11) to obtain $a_i \leq m-1$ for all i. Therefore using (7.10) again we rapidly derive

$$k(m-1)-1 \leq (k-1)(m-1),$$

which yields $m \leq 2$. As a consequence $a_i = 1$ for all i, as required. □

Proposition 7.7.14 ([51]) *Let F be a finite set of square-free monomials of degree k in R and let $A = K[F]$ be the K-subring of R spanned by F. If* $\dim(A) = n$, *then*

$$a(\overline{A}) \leq a(K[C]),$$

where $\overline{A}$ denotes the integral closure of A and C is the subsemigroup of $\mathbb{N}^n$ generated by $\mathcal{A} = \{e_{i_1} + \cdots + e_{i_k} \,|\, 1 \leq i_1 < \cdots < i_k \leq n\}$.

Proof. Let C and C_F be the subsemigroups of $\mathbb{N}^n$ generated by $\mathcal{A}$ and $\log(F)$ respectively. Set $S = K[C]$. Since S is normal by Proposition 7.4.5, we obtain $\overline{A} \subset S$. Let

$$M_1 = \{x^a \,|\, a \in \overline{C_F} \cap \mathrm{ri}(\mathbb{R}_+\overline{C_F})\} \text{ and } M_2 = \{x^a \,|\, a \in C \cap \mathrm{ri}(\mathbb{R}_+ C)\},$$

where

$$\overline{C_F} = \mathbb{Z}C_F \cap \mathbb{R}_+ C_F.$$

Notice $\mathbb{R}_+\overline{C_F} = \mathbb{R}_+ C_F$ and $\mathrm{aff}(\mathbb{R}_+ C_F) = \mathbb{R}^n$. Therefore the relative interior of $\mathbb{R}_+\overline{C_F}$ equals its interior in $\mathbb{R}^n$. For similar reasons we have $\mathrm{ri}(\mathbb{R}_+ C) = (\mathbb{R}_+ C)^\circ$. Hence $\mathrm{ri}(\mathbb{R}_+\overline{C_F}) \subset \mathrm{ri}(\mathbb{R}_+ C)$. Altogether we obtain $M_1 \subset M_2$. Let x^b be an element of minimal degree in M_1 so that $\deg(x^b) = -a(\overline{A})$. Set $r = -a(\overline{A})$. Since x^b is in M_2 and $x^b \in \overline{A}_r \subset S_r$, we conclude

$$-a(S) = \min\{\deg(x^a) | a \in M_2\} \leq r = -a(\overline{A}).$$

Hence $a(\overline{A}) \leq a(S)$, as required. □

Corollary 7.7.15 ([51]) *Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and F a finite set of square-free monomials of the same degree k. If* $\dim K[F] = n$, *then*

$$a(\overline{K[F]}) \leq \begin{cases} -\left\lceil \dfrac{n}{k} \right\rceil, & \text{if } n \geq 2k, \\ -\left\lceil \dfrac{n}{n-k} \right\rceil, & \text{if } n \leq 2k,\ n \neq k. \end{cases}$$

Proof. Use Corollary 7.7.11, Proposition 7.7.14, and duality. □

Remark 7.7.16 It is not hard to verify the equality

$$a(R^{(k)}) = -\left\lceil \frac{n}{k} \right\rceil,$$

where $R^{(k)}$ is the kth Veronese subring of R. Let $K[C]$ be the subring generated by the square-free monomials of degree k. Notice that

$$a(K[C]) \leq a(R^{(k)}).$$

For $n = 5$ and $k = 3$ we have

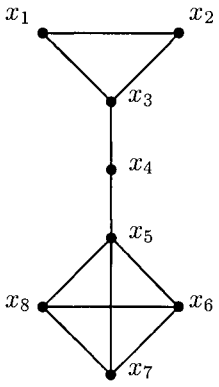
$$a(K[C]) = -3 < -2 = a(R^{(k)}).$$

Because of this, to keep better control of the a-invariant, it is preferable to embed $K[F]$ into $K[C]$, instead of $R^{(k)}$.

Corollary 7.7.17 *Let $R = K[x_1, \ldots, x_n]$ a polynomial ring over a field K and F a finite set of square-free monomials of degree k in R. If $K[F]$ has dimension n and $n \geq 2k \geq 4$, then $\overline{K[F]}$ is generated as a K-algebra by elements of normalized degree less or equal than $n - \lceil \frac{n}{k} \rceil$.*

Proof. Proceed along the lines of the proof of Theorem 7.6.2 and use Corollary 7.7.15. □

Example 7.7.18 Let $K[F]$ be the subring of $R = K[x_1, \ldots, x_8]$ spanned by the monomials of R defining the edges of the graph shown below.



$$\begin{aligned}
f_1 &= x_1x_2, \\
f_2 &= x_2x_3, \\
f_3 &= x_1x_3, \\
f_4 &= x_3x_4, \\
f_5 &= x_4x_5, \\
f_6 &= x_5x_6, \\
f_7 &= x_6x_7, \\
f_8 &= x_5x_7, \\
f_9 &= x_7x_8, \\
f_{10} &= x_5x_8, \\
f_{11} &= x_6x_8,
\end{aligned}$$

$$\begin{aligned}
f_{12} &= x_1x_2x_3x_5x_6x_8, \\
f_{13} &= x_1x_2x_3x_6x_7x_8, \\
f_{14} &= x_1x_2x_3x_5x_6x_7, \\
f_{15} &= x_1x_2x_3x_5x_7x_8.
\end{aligned}$$

$$F := \{f_1, \ldots, f_{11}\},$$

$$\overline{K[F]} = K[F][f_{12}, f_{13}, f_{14}, f_{15}].$$

The generators of $\overline{K[F]}$ can be computed using the program [48], see also [298, Section 7.3]. A Noether normalization for $K[F]$ is given by

$$A_0 = K[h_1, \ldots, h_8] \hookrightarrow K[F] \hookrightarrow \overline{K[F]},$$

where

$$h_1 = f_1, \quad h_3 = f_8 - f_{11}, \quad h_5 = f_2 - f_3, \quad h_7 = f_5 - f_7 - f_9 - f_{11},$$
$$h_2 = f_6, \quad h_4 = f_9 - f_{10}, \quad h_6 = f_3 - f_5, \quad h_8 = f_1 - \textstyle\sum_{i=2}^{11} f_i.$$

Since $\overline{K[F]}$ is Cohen-Macaulay we obtain a decomposition

$$\begin{aligned}\overline{K[F]} \;=\; & A_0 1 \oplus A_0 f_7 \oplus A_0 f_{10} \oplus a_0 f_{11} \oplus \\ & A_0 f_7^2 \oplus A_0 f_{10}^2 \oplus A_0 f_{10} f_{11} \oplus A_0 f_7 f_{11} \oplus \\ & A_0 f_7^3 \oplus A_0 f_7^2 f_{11} \oplus A_0 f_{10}^3 \oplus A_0 f_{10}^2 f_{11} \oplus \\ & A_0 f_{12} \oplus A_0 f_{13} \oplus A_0 f_{14} \oplus A_0 f_{15} \oplus A_0 f_{10}^3 f_{11}.\end{aligned}$$

Therefore the Hilbert series of $\overline{K[F]}$ is equal to

$$H(\overline{K[F]}, z) = \frac{1 + 3z + 4z^2 + 8z^3 + z^4}{(1-z)^8} \quad \text{and} \quad a(\overline{K[F]}) = -4.$$

Altogether $\overline{K[F]}$ is generated as an A_0-module by monomials of normalized degree less or equal than $\dim R + a(\overline{K[F]}) = 4$. However, as a K-algebra it is generated by square-free monomials of normalized degree at most 3.

Next we show a general statement on degrees valid when S and $\overline{S}$ are both Cohen-Macaulay.

Proposition 7.7.19 *Let S be a standard graded algebra and let $M \subset N$ be finitely generated graded S-modules of the same dimension d and multiplicity e. If M and N are Cohen-Macaulay and*

$$H_M(t) = \frac{f(t)}{(1-t)^d}, \qquad H_N(t) = \frac{g(t)}{(1-t)^d}$$

are their Hilbert series, then $\deg f(t) \geq \deg g(t)$.

Proof. Consider the exact sequence

$$0 \to M \longrightarrow N \longrightarrow P \to 0$$

of graded modules. If $M \neq N$, P is a module of dimension $< d$ since M and N have the same multiplicity. Since M and N are Cohen-Macaulay, standard depth chasing (see Lemma 1.3.9) implies that P is Cohen-Macaulay of dimension $d-1$.

We have the equality of Hilbert series,

$$\frac{g(t)}{(1-t)^d} = \frac{f(t)}{(1-t)^d} + \frac{h(t)}{(1-t)^{d-1}},$$

and therefore

$$g(t) - f(t) = (1-t)h(t).$$

The assertion follows since the h-vectors of these two modules are positive (see Chapter 4 and [44, Corollary 4.1.10]). □

Lemma 7.7.20 *If $K[F] \subset K[\mathbf{x}]$ is a monomial subring over a field K, then there is a positive integer n_0 such that $(x^\alpha)^{n_0} \in K[F]$ for all $x^\alpha \in \overline{K[F]}$.*

Proof. By Lemma 7.2.32 there are $\gamma_1, \ldots, \gamma_r \in \mathbb{N}^n$ such that

$$\overline{K[F]} = K[x^{\gamma_1}, \ldots, x^{\gamma_r}].$$

For each i one can find a positive integer n_i such that $x^{n_i\gamma_i} \in K[F]$. It is readily seen that the integer $n_0 = \operatorname{lcm}(n_1, \ldots, n_r)$ satisfies the required condition, because any $x^\alpha \in \overline{K[F]}$ can be written as $x^\alpha = x^{a_1\gamma_1} \cdots x^{a_r\gamma_r}$, for some $a_1, \ldots, a_r \in \mathbb{N}$. □

Proposition 7.7.21 *If $A = K[F] \subset K[\mathbf{x}]$ is a monomial subring over a field K and I is the ideal*

$$I = (A :_A \overline{A}) = \{a \in A \mid a\overline{A} \subset A\},$$

then there is a monomial $x^\gamma \in I$. In particular $I \neq 0$.

Proof. Let $\gamma_1, \ldots, \gamma_r \in \mathbb{N}^n$ such that $\overline{A} = K[x^{\gamma_1}, \ldots, x^{\gamma_r}]$. One can write $x^{\gamma_i} = x^{\beta_i}/x^{\delta_i}$, where x^{β_i} and x^{δ_i} are in A. By Lemma 7.7.20 there is $0 \neq n_0 \in \mathbb{N}$ such that $x^{n_0\alpha} \in A$ for all $x^\alpha \in \overline{A}$. Set $x^\gamma = x^{n_0\delta_1} \cdots x^{n_0\delta_r}$. Take x^α in $\overline{A}$ and write $x^\alpha = (x^{\gamma_1})^{a_1} \cdots (x^{\gamma_r})^{a_r}$, where $a_i \in \mathbb{N}$ for all i. By the division algorithm there are integral equations $a_i = q_i n_0 + c_i$, where $0 \leq c_i < n_0$ for all i. From the equality

$$x^\gamma x^\alpha = \underbrace{\prod_{i=1}^{r} (x^{n_0\gamma_i})^{q_i}}_{\text{is in } A} \underbrace{\prod_{i=1}^{r} (x^{\delta_i+\gamma_i})^{c_i}}_{\text{is in } A} \underbrace{\prod_{i=1}^{r} (x^{\delta_i})^{n_0-c_i}}_{\text{is in } A}$$

one concludes $x^\gamma x^\alpha \in A$. Since x^α was an arbitrary monomial in $\overline{A}$, we get $x^\gamma \overline{A} \subset A$ and $x^\gamma \in I$. □

Proposition 7.7.22 *If $A = K[F] \subset K[\mathbf{x}]$ is a homogeneous monomial subring over a field K, then $\overline{A}$ is a positively graded K-algebra and the multiplicity of $\overline{A}$ is equal to the multiplicity of A.*

Proof. Set $\mathcal{A} = \log(F)$ and $P = \operatorname{conv}(\mathcal{A})$. As A is homogeneous, using Proposition 7.2.43 one has that

$$\overline{A} = \bigoplus_{i=0}^{\infty} \overline{A}_i$$

is a graded K-algebra whose ith component is given by

$$\overline{A}_i = \sum_{\alpha \in \mathbb{Z}\mathcal{A} \cap iP} K x^\alpha.$$

Recall that A is graded as in Proposition 7.2.39. Let h and E be the Hilbert functions of A and $\overline{A}$ respectively. Since A and $\overline{A}$ have the same dimension d, one can write

$$h(i) = a_0 i^{d-1} + \text{terms of lower degree},$$
$$E(i) = c_0 i^{d-1} + \text{terms of lower degree},$$

for $i \gg 0$. One clearly has $a_0 \leq c_0$ because $A_i \subset \overline{A}_i$ for all i. On the other hand by Proposition 7.7.21 there is $x^\gamma \in A$ of degree m such that $x^\gamma \overline{A} \subset A$. Thus for each i there is a one to one map

$$\overline{A}_i \xrightarrow{x^\gamma} A_{i+m}.$$

Hence $E(i) \leq h(i+m)$ for all i, and consequently $c_0 \leq a_0$. Altogether $a_0 = c_0$. Therefore

$$e(A) = a_0(d-1)! = c_0(d-1)! = e(\overline{A}),$$

as required. □

Corollary 7.7.23 ([51]) *If K is a field and $K[F]$ is a Cohen-Macaulay homogeneous monomial subring, then*

$$a(\overline{K[F]}) \leq a(K[F]).$$

Proof. Use Proposition 7.7.19 and Proposition 7.7.22. □

Exercises

7.7.24 Let F be a finite set of square-free monomials of degree k in the polynomial ring $K[x_1, \ldots, x_n]$, K a field. If $n < 2k$ and $K[F]$ is not a normal domain, prove that $\overline{K[F]}$ cannot be generated as a K algebra by square-free monomials.

7.7.25 ([130, 219]) Let $R^{(k)}$ be the *kth* Veronese subring of a polynomial ring R in n variables over a field K. Prove that $R^{(k)}$ is Gorenstein if and only if k divides n.

Hint The interior of the cone spanned by the exponent vectors in $R^{(k)}$ is the "first open quadrant" of $\mathbb{R}^n$ and use the Danilov-Stanley formula.

7.7.26 Let $R^{(k)}$ be the *kth* Veronese subring of a polynomial ring R in n variables over a field K. Prove that the a-invariant of $R^{(k)}$ is given by

$$a(R^{(k)}) = -\left\lceil \frac{n}{k} \right\rceil.$$

7.7.27 If $K[F]$ is a two dimensional homogeneous monomial subring, prove that the Hilbert polynomial of $K[FT]$ is equal to the Hilbert polynomial of $\overline{K[FT]}$, where T is a new variable.

7.7.28 Let $A = K[F]$ be a Cohen-Macaulay homogeneous monomial subring. If $\dim(A) = 2$ and $a(A) < 0$, prove that A is normal.

7.7.29 Let S be the *kth* square-free Veronese subring of a polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K. If K_S is the field of fractions of S. Use the equality

$$\frac{\prod_{i=2}^{k+1} x_1 \cdots x_{i-1}\widehat{x_i}x_{i+1} \cdots x_{k+1}}{(x_2 \cdots x_{k+1})^{k-1}} = x_1^k,$$

to derive $x_1^k \in K_S$. Using a similar equality prove $K[x_1^k, \ldots, x_n^k] \subset K_S$, then conclude $\text{tr.deg}_K(S) = n$.

7.7.30 Let F be the set of square-free monomials of degree k in the variables $x_1, \ldots, x_n$ and $S = K[F]$ the monomial subring spanned by F endowed with the normalized grading. If $n = 2k+1 \geq 5$, prove that the canonical module of S is given by

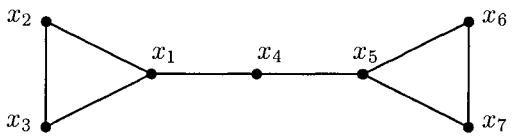
$$\omega_S = (\{x^a | \text{ card}(\{i|a_i = 2\}) = k-1 \text{ and } \text{card}(\{i|a_i = 1\}) = n-k+1\}).$$

Recall that a standard Artin algebra A over a field K is called a *level algebra* [27] if all the nonzero elements of the socle of A have the same degree. Prove that if $h_1, \ldots, h_n$ is a system of parameters for $S = K[C]$ with each h_i a form of degree 1, then

$$S/(h_1, \ldots, h_n)$$

is a level algebra.

Consider the homogeneous subring $A = K[F]$, where F is the set of all monomials x_ix_j such that x_i and x_j are connected by a line of the graph:



Note that A is a graded subring of $\overline{A}$, where both rings are endowed with the normalized grading. In the following problems we indicate how to compute some invariants of A and $\overline{A}$.

7.7.31 Use *CoCoA*, and the procedures indicated below, to prove that the Hilbert series and Hilbert polynomial of A and $\overline{A}$ are given by:

$$F(A,t) = \frac{1+t+t^2+t^3+t^4}{(1-t)^7}, \quad F(\overline{A},t) = \frac{1+t+t^2+2t^3}{(1-t)^7}$$

$$\varphi(A,t) = \frac{1}{144}t^6 + \frac{1}{16}t^5 + \frac{55}{144}t^4 + \frac{65}{48}t^3 + \frac{47}{18}t^2 + \frac{31}{12}t + 1,$$

$$\varphi(\overline{A},t) = \frac{1}{144}t^6 + \frac{17}{240}t^5 + \frac{55}{144}t^4 + \frac{21}{16}t^3 + \frac{47}{18}t^2 + \frac{157}{60}t + 1.$$

In particular $e(A) = e(\overline{A}) = 5$, $a(A) = -3$ and $a(\overline{A}) = -4$.

7.7.32 Use this *CoCoA* procedure to compute the Hilbert series and Hilbert function of A. Note that we are first computing the toric ideal P of A, through the usual procedure of elimination of variables.

```
Use R ::= Q[t[1..8], x[1..7]];
Q := Ideal(x[1]*x[2] - t[1], x[2]*x[3] - t[2], x[1]*x[3] - t[3],
x[3]*x[4] - t[4], x[4]*x[5] - t[5], x[5]*x[6] - t[6],
x[6]*x[7] - t[7], x[5]*x[7] - t[8]);
P := Elim(x, Q);
Use R ::= Q[t[1..8]];
P := Ideal(t[2]*t[3]*t[5]^2*t[7] - t[1]*t[4]^2*t[6]*t[8]);
Poincare(R/P);
Hilbert(R/P);
```

7.7.33 Use *Normaliz* to verify that $\overline{A} = A[x_1x_2x_3x_5x_6x_7]$. This step is required for the next computation.

7.7.34 Use the following *CoCoA* procedure to compute the Hilbert series of $\overline{A}$. Note that we are also computing the toric ideal P of $\overline{A}$.

```
Use R ::= Q[t[1..9], x[1..7]];
Q := Ideal(x[1]*x[2] - t[1], x[2]*x[3] - t[2], x[1]*x[3] - t[3],
x[3]*x[4] - t[4], x[4]*x[5] - t[5], x[5]*x[6] - t[6],
x[6]*x[7] - t[7], x[5]*x[7] - t[8],
x[1]*x[2]*x[3]*x[5]*x[6]*x[7] - t[9]);
P := Elim(x, Q);
Use R ::= Q[t[1..9]], Weights([1,1,1,1,1,1,1,1,3]);
P := Ideal(t[1]*t[4]*t[6]*t[8] - t[5]*t[9],
t[2]*t[3]*t[5]*t[7] - t[4]*t[9],
t[1]*t[2]*t[3]*t[6]*t[7]*t[8] - t[9]^2);
Poincare(R/P);
```

7.7.35 Verify the assertions about $\overline{A}$ using the following input file for the program *Normaliz*, and running *Normaliz* with the option $-h$.

```
bowtie.in
 8                                  (number of generators)
 7                                  (dimension of ambient lattice)
 1  1  0  0  0  0  0   (generators)
 0  1  1  0  0  0  0
 1  0  1  0  0  0  0
 0  0  1  1  0  0  0
 0  0  0  1  1  0  0
 0  0  0  0  1  1  0
 0  0  0  0  0  1  1
 0  0  0  0  1  0  1
 2                                  (option or mode)
```

7.8 Some lexicographical Gröbner bases

A *lexicographical order* of the terms in a polynomial ring $K[z_1, \ldots, z_r]$ is an order obtained by representing terms as $z_{\pi_1}^{a_1} \cdots z_{\pi_r}^{a_r}$ for a fixed permutation π of the variables and then using the lexicographical order. The Gröbner bases obtained by lexicographical orders are called *lexicographical*.

There are several classical examples of graded prime ideals that are minimally generated by a lexicographical Gröbner basis [59, 66, 159, 276]. Below we study some special linear orders and relate them to toric ideals of subrings of k-products.

Some linear orders of the variables Given two integers $n \geq 2k \geq 4$ we consider a "label set":

$$\mathfrak{I}_k^n = \{[i_1, \ldots, i_k] | \, 1 \leq i_1 < \cdots < i_k \leq n\}$$

and a set of variables:

$$\mathbf{T} = \{T[i_1, \ldots, i_k] | \, [i_1, \ldots, i_k] \in \mathfrak{I}_k^n\}.$$

Let $A = (a_{ij})$ be a non-singular matrix of order k with entries in $\mathbb{R}$ and

$$u_i = (a_{i1}, \ldots, a_{ik})$$

its ith row. This matrix defines a linear ordering in $\mathbb{Z}^k$ given by

$$x = (x_1, \ldots, x_k) >_A y = (y_1, \ldots, y_k) \text{ iff } Ax >_{\text{lex}} Ay,$$

that is $x >_A y$ if the first non-zero entry of $(x \cdot u_1 - y \cdot u_1, \ldots, x \cdot u_k - y \cdot u_k)$ is positive. Thus we can linearly order the variables in $\mathbf{T}$ by:

$$T[i_1, \ldots, i_k] > T[j_1, \ldots, j_k] \text{ iff } (i_1, \ldots, i_k) >_A (j_1, \ldots, j_k).$$

Example 7.8.1 If $n = 5$ and $k = 2$, then the linear ordering induced by the matrix:

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is $T[1,5]T[2,5]T[3,5]T[4,5]T[1,4]T[2,4]T[3,4]T[1,3]T[2,3]T[1,2]$.

A distinguished group of matrices Consider the set $\mathcal{A}$ of matrices of order k of the form

$$\begin{pmatrix} \pm e_{i_1} \\ \vdots \\ \pm e_{i_k} \end{pmatrix},$$

such that the set $\{e_{i_1}, \ldots, e_{i_k}\}$ is a permutation of the set of canonical vectors $\{e_1, \ldots, e_k\}$ in $\mathbb{R}^k$. Note that $\mathcal{A}$ is a group of order $2^k k!$. As the rows of any matrix in $\mathcal{A}$ are orthonormal, we see that $\mathcal{A}$ is a subgroup of the orthogonal group $O_k(\mathbb{R})$.

Finding lexicographical quadratic Gröbner bases Let K be a field and $R = K[x_1, \ldots, x_n]$ a polynomial ring. Let V_k be the set of square-free monomials of degree k in R. Consider the presentation:

$$\varphi\colon K[\mathbf{T}] \longrightarrow K[V_k], \text{ induced by } T[i_1 \cdots i_k] \overset{\varphi}{\longmapsto} x_{i_1} \cdots x_{i_k},$$

thus $P = \ker(\varphi)$ is the toric ideal of $K[V_k]$.

Remark 7.8.2 It has been shown by Bernd Sturmfels that there exist a quadratic Gröbner basis, with respect to some term order, for the toric ideal P of $K[V_k]$, see [278].

The examples below can be verified using *Macaulay*. We thank Eduardo Virueña Silva for writing some Pascal programs to generate all the matrices in $\mathcal{A}$, and the corresponding linear orderings of the variables.

Example 7.8.3 The matrices below are precisely the matrices in $\mathcal{A}$ that induce a quadratic Gröbner bases for the toric ideal of $\mathbb{Q}[V_k]$ corresponding to $k = 2$ and $n = 6$.

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad A_4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A_7 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad A_8 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Example 7.8.4 For $n = 6$ and $k = 3$ the full list of matrices in $\mathcal{A}$ that induce quadratic Gröbner bases are:

$$A_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad A_9 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A_{13} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad A_{15} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A_{16} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad A_{39} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_{41} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad A_{45} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A_{47} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \qquad A_{48} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

Example 7.8.5 For $n = 8$ and $k = 4$ the complete list of matrices in $\mathcal{A}$ that give quadratic Gröbner bases is:

$$A_{25} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad A_{79} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

$$A_{89} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad A_{351} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{367} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad A_{377} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Conjecture 7.8.6 *If P is the toric ideal of $K[V_k]$ and $n \geq 2k \geq 4$, then there exists a linear order of the variables in $\mathbf{T}$ so that the reduced Gröbner basis of P, with respect to the lexicographical order, is generated by quadrics.*

Exercises

7.8.7 If we order $\mathfrak{I}_k^n$, this induces an order on $\mathfrak{I}_{n-k}^n$. Prove that $P(V_k)$ has a lexicographical quadratic Gröbner basis if and only if $P(V_{n-k})$ does. Here $P(V_k)$ denotes the toric ideal of $K[V_k]$.

7.8.8 (V. Bonanzinga) Let $n \geq d \geq 1$ be two integers. If $\mathfrak{I}_k^n$ has the lex order and $N[i_1, \ldots, i_d]$ represents the position of $[i_1, \ldots, i_d]$ with respect to this order. Prove the formula:

$$N[i_1, \ldots, i_d] = \binom{n}{d} - \sum_{k=1}^{d} \sum_{j=i_k+1}^{n-d+k} \binom{n-j}{d-k} = \binom{n}{d} - \sum_{k=1}^{d} \binom{n-i_k}{d-k+1}.$$

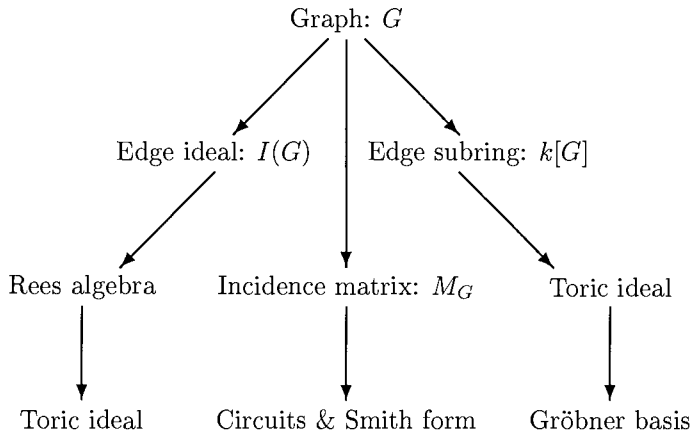
See [29] for more details about this formula.

Chapter 8

Monomial Subrings of Graphs

Here we study the monomial subrings associated to graphs and their toric ideals. A full description of the integral closure of those subrings will be presented along with some connections with graph theory and polyhedral geometry.

Let G be a graph. The following diagram gives an idea of the objects associated to G that will be studied along this chapter.



The edge subring $k[G]$ associated to a graph G is the monomial subring generated by the monomials corresponding to the edges of G, where k is a field. The description of the integral closure of $k[G]$ will be given in

terms of special circuits of the graph (see Theorem 8.7.9). This description links the normality property of $k[G]$ with the combinatorics of the graph G (see Proposition 8.8.11). The incidence matrix of G plays an important role because it is related to the toric ideal of $k[G]$ and its rank can be interpreted in graph theoretical terms.

8.1 The subring associated to a graph

Let G be a graph on the vertex set $V = \{v_1, \ldots, v_n\}$ and

$$R = k[x_1, \ldots, x_n] = \bigoplus_{i=0}^{\infty} R_i$$

a polynomial ring over a field k with the standard grading. To simplify notation sometimes one identifies the indeterminate x_i with the vertex v_i. The *monomial subring* or *edge subring* of the graph G is the k-subalgebra

$$k[G] = k[\{x_i x_j \,|\, v_i \text{ is adjacent to } v_j\}] \subset R.$$

If $F = \{f_1, \ldots, f_q\}$ is the set of monomials $x_i x_j$ such v_i is adjacent to v_j, the elements in $k[G]$ are polynomial expressions in F with coefficients in k. A monomial $f = x_i x_j$ in R is said to be an *edge generator* of G if v_i is adjacent to v_j.

As $k[G]$ is a standard k-algebra with the normalized grading

$$k[G]_i = k[G] \cap R_{2i},$$

there is a graded epimorphism of k-algebras

$$\varphi \colon B = k[T_1, \ldots, T_q] \longrightarrow k[G], \quad T_i \longmapsto f_i,$$

where B is a polynomial ring graded by $\deg(T_i) = 1$ for all i. The kernel of φ, denoted by $P(G)$, is a graded ideal of B called the *toric ideal* of $k[G]$ with respect to $f_1, \ldots, f_q$.

Definition 8.1.1 Let $w = \{v_0, v_1, \ldots, v_r = v_0\}$ be an even closed walk in G such that $f_i = x_{i-1}x_i$. As

$$f_1 f_3 \cdots f_{r-1} = f_2 f_4 \cdots f_r,$$

the binomial

$$T_w = T_1 \cdots T_{r-1} - T_2 \cdots T_r$$

is in $P(G)$. One says that T_w is the *binomial associated* to w.

Next we identify a set of generators for $P(G)$ corresponding to the even closed walks of G. A closed walk of even length will be called a *monomial walk*.

Notation For use below $\mathcal{I}_s$ denotes the set of all non-decreasing sequences $\alpha = (i_1, \cdots, i_s)$ of length s. Let $a_1, \ldots, a_q$ be a sequence and $\alpha = (i_1, \cdots, i_s)$ in $\mathcal{I}_s$, then we set $a_\alpha = a_{i_1} \cdots a_{i_s}$.

Proposition 8.1.2 ([303]) *If G is a graph and $P = P(G)$ the toric ideal of the edge subring $k[G]$, then*

(a) $P = (\{T_w | \, w \text{ is an even closed walk}\})$, *and*

(b) $P = (\{T_w \, | \, w \text{ is an even cycle}\})$ *if G is bipartite.*

Proof. Let $f_1, \ldots, f_q$ be the edge generators of G and P the kernel of the homomorphism of k-algebras

$$\varphi \colon B \longrightarrow k[G],$$

where B is the polynomial ring $k[T_1, \ldots, T_q]$ and $\varphi(T_i) = f_i$. Recall that P is graded and

$$P = B \cdot \left(\bigcup_{s=2}^{\infty} P_s \right).$$

First we set

$$\mathcal{B} = \{T_w | \, w \text{ is an even closed walk}\}.$$

One clearly has $(\mathcal{B}) \subset P$. To prove the other containment we use induction on s. First recall that P is a binomial ideal by Proposition 7.1.2. If $s = 2$, it is enough to note that the binomials in P_2 come from squares. Assume $P_{s-1} \subset (\mathcal{B})$. To show $P_s \subset (\mathcal{B})$, take $T_\alpha - T_\beta$ in P_s, where $\alpha = (i_1, \ldots, i_s)$ and $\beta = (j_1, \ldots, j_s)$. Define G_1 as the subgraph of G having vertex set

$$V_1 = \{x_i \in V | \, x_i \text{ divides } f_\alpha\}$$

and edge set

$$E_1 = \{\{x, y\} \in E(G) | \, f_\ell = xy \text{ for some } \ell \in \{i_1, \ldots, i_s, j_1, \ldots, j_s\}\}.$$

Note that if $f_{i_1} \cdots f_{i_m} = f_{j_1} \cdots f_{j_m}$ for some $m < s$ and for some ordering of the generators then $T_\alpha - T_\beta \in (\mathcal{B})$; to prove it, notice that $T_\alpha - T_\beta$ can be written as

$$\begin{aligned} T_\alpha - T_\beta = T_{i_{m+1}} \cdots T_{i_s} (T_{i_1} \cdots T_{i_m} - T_{j_1} \cdots T_{j_m}) \\ + T_{j_1} \cdots T_{j_m} (T_{i_{m+1}} \cdots T_{i_s} - T_{j_{m+1}} \cdots T_{j_s}) \end{aligned}$$

and use induction hypothesis. Now we can assume $f_{i_1} \cdots f_{i_m} \neq f_{j_1} \cdots f_{j_m}$ for $m < s$ and for any re-ordering of the f_i's. Observe that this forces G_1

to be connected. Take $v_0 \in V_1$. Since $f_\alpha = f_\beta$, it is easy to check that after re-ordering we can write $f_{i_k} = v_{2k-2}v_{2k-1}$ and $f_{j_k} = v_{2k-1}v_{2k}$, where $1 \leq k \leq s$ and $v_0 = v_{2s}$. Therefore the monomial walk $w = \{v_0, \ldots, v_{2n}\}$ satisfies $T_w = T_\alpha - T_\beta$ and the induction is complete. This proves part (a).

Assume G is bipartite. The second part of the proposition can be proved similarly if we notice that this time the condition $f_{i_1} \cdots f_{i_m} \neq f_{j_1} \cdots f_{j_m}$ for $m < s$ and for any re-ordering of the f_i's implies that the graph G_1 is an even cycle. □

Definition 8.1.3 Let F be a finite set of monomials in a polynomial ring $k[\mathbf{x}]$ and let P be the toric ideal of $k[F]$. A binomial

$$T^\alpha - T^\beta \in P$$

is called *primitive* if there is no other binomial $T^\gamma - T^\delta \in P$ such that T^γ divides T^α and T^δ divides T^β.

According to a result of [278], the set of all primitive binomials in a toric ideal P is finite and is called a *Graver basis* of P.

Corollary 8.1.4 *If G is a graph and $f = T^\alpha - T^\beta$ is a primitive binomial in $P(G)$, then $f = T_w$ for some even closed walk w of G.*

Proof. It follows from the proof of Proposition 8.1.2. □

Corollary 8.1.5 *If G is a bipartite graph and $f = T^\alpha - T^\beta$ is a primitive binomial in $P(G)$, then $f = T_w$ for some even cycle w of G.*

Proof. It follows from the proof of Proposition 8.1.2. □

Proposition 8.1.6 *Let G be a graph and let $P(G)$ be the toric ideal of the edge subring $k[G]$. If $f = T^\alpha - T^\beta$ is a primitive binomial in $P(G)$, then the entries of α satisfy $\alpha_i \leq 2$ for all i.*

Proof. By Corollary 8.1.4 one can write

$$f = T_w = T_1T_3 \cdots T_{\ell-1} - T_2T_4 \ldots T_\ell,$$

where ℓ is even, and w is an even closed walk in G of length ℓ:

$$w = \{v_0, v_1, \ldots, v_{\ell-1}, v_\ell = v_0\}$$

such that $f_1f_3 \cdots f_{\ell-1} = f_2f_4 \cdots f_\ell$ and $f_i = v_{i-1}v_i$ for all i. Note that there may be some repetitions of the variables T_i and accordingly some repetitions in the monomials f_i. As usual we are identifying the vertices of G with variables. It suffices to verify that v_0 occurs at most three times

(including the end vertices) in the closed walk w. If v_0 occurs more than three times in w, one can write

$$w = \{v_0, v_1, \ldots, v_{t_1} = v_0, v_{t_1+1}, \ldots, v_{t_1+t_2} = v_0, v_{t_1+t_2+1}, \ldots, v_\ell = v_0\},$$

note that t_1 must be odd; otherwise using $w_1 = \{v_0, v_1, \ldots, v_{t_1} = v_0\}$ one has that $T_{w_1} = T^\gamma - T^\delta$, where T^γ divides $T_1 T_3 \cdots T_{\ell-1}$ and T^δ divides $T_2 T_4 \ldots T_\ell$, a contradiction because f is primitive. By a similar argument t_2 must be odd. Thus $t_1 + t_2$ is even, to derive a contradiction note that the closed walk

$$w' = \{v_{t_1+t_2} = v_0, v_{t_1+t_2+1}, \ldots, v_\ell = v_0\}$$

is of even length, and apply the previous argument. □

Lemma 8.1.7 *Let P be the toric ideal of a monomial subring $K[F]$. If $f = T^\alpha - T^\beta$ is a binomial in the reduced Gröbner basis of P with respect to some term order, then f is a primitive binomial.*

Proof. See [278, Lemma 4.6]. □

Lemma 8.1.8 *Let G be a graph and let $P = P(G)$ be the toric ideal of $k[G]$. If f is a polynomial in any reduced Gröbner bases of P, then*

(a) *f is a primitive binomial and $f = T_w$ for some even closed walk w of the graph G.*

(b) *If G is bipartite, then f is primitive and $f = T_w$ for some even cycle w of the graph G.*

Proof. Let $\mathcal{A}$ be a reduced Gröbner basis of P and take $f \in \mathcal{A}$. Using Corollary 7.1.5 together with the fact that normalized reduced Gröbner bases are uniquely determined, we obtain $f = T^\alpha - T^\beta$. By Lemma 8.1.7 f is primitive, thus by Corollary 8.1.4 and Corollary 8.1.5 f has the required form. □

Proposition 8.1.9 *If G is a graph and $P = P(G)$ is the toric ideal of $k[G]$, then the set*

$$\{T_w | \, T_w \text{ is primitive and } w \text{ is an even closed walk}\}$$

is a universal Gröbner basis of P.

Proof. It follows from Lemma 8.1.8. □

Proposition 8.1.10 *If G is a bipartite graph and $P = P(G)$ is the toric ideal of $k[G]$, then the set*

$$\{T_w | \, w \text{ is an even cycle}\}$$

is a universal Gröbner basis of P.

Proof. It follows from Lemma 8.1.8. □

Exercises

8.1.11 Let G be a graph with q edges and $P(G)$ the toric ideal of the edge subring $k[G]$. If f is a primitive binomial in $P(G)$ prove:

$$\deg(f) \leq \begin{cases} q & \text{if } q \text{ is even, and} \\ q-1 & \text{otherwise.} \end{cases}$$

8.1.12 Let G be a bipartite graph and $k[G]$ its edge ideal. Prove that $k[G]$ is a normal domain.

8.2 Rees algebras of edge ideals

We will look closely at the presentation ideal or toric ideal of the Rees algebra of an edge ideal, and then show a few applications.

Notation For f and g in a unique factorization domain we define

$$[f, g] = \operatorname{lcm}(f, g).$$

As before $\mathcal{I}_s$ denotes the set of all non-decreasing sequences $\alpha = (i_1, \cdots, i_s)$ of length s. Let $a_1, \ldots, a_q$ be a sequence and let $\alpha = (i_1, \cdots, i_s) \in \mathcal{I}_s$, then we set

$$a_\alpha = a_{i_1} \cdots a_{i_s} \quad \text{and} \quad \widehat{a}_{\alpha_t} = a_{i_1} \cdots \widehat{a}_{i_t} \cdots a_{i_s}.$$

Theorem 8.2.1 *Let I be the edge ideal of a graph G and $F = \{f_1, \ldots, f_q\}$ the set of edge generators of I. If B/J is the presentation of the Rees algebra of I with respect to F, then*

$$J = BJ_1 + B \cdot \left(\bigcup_{s=2}^{\infty} P_s \right),$$

where $P_s = \{T_\alpha - T_\beta |\ f_\alpha = f_\beta$, for some $\alpha, \beta \in \mathcal{I}_s\}$.

Proof. Let $x_1, \ldots, x_n$ be the set of vertices of G and R a polynomial ring in n variables over a field k. Here we identify the vertices of G with the variables of R. Let $\mathcal{R}(I) = R[IT]$ be the Rees algebra of I. The toric ideal J of $\mathcal{R}(I)$ is the kernel of the graded epimorphism:

$$\varphi\colon B = R[T_1, \ldots, T_q] \longrightarrow \mathcal{R}(I) \longrightarrow 0 \quad (T_i \longmapsto f_i T).$$

It is sufficient to show the recursive formula

$$J_s = B_1 J_{s-1} + RP_s \quad \text{for} \quad s \geq 2.$$

According to Theorem 2.4.17 J_s is generated by polynomials of the form

$$\left(\frac{[f_\alpha, f_\beta]}{f_\beta}\right) T_\beta - \left(\frac{[f_\alpha, f_\beta]}{f_\alpha}\right) T_\alpha, \tag{8.1}$$

where $\alpha = (i_1, \ldots, i_s)$, $\beta = (j_1, \ldots, j_s)$ are in $\mathcal{I}_s$. It suffices to prove that all those generators are in $B_1 J_{s-1} + B_{s-1} J_1 + RP_s$. Since for $f_\alpha = f_\beta$ expression (8.1) simplifies to $T_\alpha - T_\beta \in RP_s$, we may assume $f_\alpha \neq f_\beta$. In this case there is a variable $z \in \{x_1, \ldots, x_n\}$ so that $f_\alpha = z^a h$ and $f_\beta = z^b g$, with $a \neq b$. Assume $b > a \geq 0$, it is not hard to see that there are integers m and t so that $[f_\alpha, f_\beta]$ is a multiple of $f_{j_m} \widehat{f}_{\alpha_t}$. Notice that the generators of J_s can be re-written as

$$\left(\left(\frac{[f_\alpha, f_\beta]}{f_\beta}\right) \widehat{T}_{\beta_m} - A\widehat{T}_{\alpha_t}\right) T_{j_m} + \left(AT_{j_m} - \left(\frac{[f_\alpha, f_\beta]}{f_\alpha}\right) T_{i_t}\right) \widehat{T}_{\alpha_t}.$$

The proof now reduces to showing that there are polynomials λ, μ, A in R and integers m and t so that

$$\left(\frac{[f_\alpha, f_\beta]}{f_\beta}\right) \widehat{T}_{\beta_m} - A\widehat{T}_{\alpha_t} = \lambda \left(\left(\frac{[\widehat{f}_{\alpha_t}, \widehat{f}_{\beta_m}]}{\widehat{f}_{\beta_m}}\right) \widehat{T}_{\beta_m} - \left(\frac{[\widehat{f}_{\alpha_t}, \widehat{f}_{\beta_m}]}{\widehat{f}_{\alpha_t}}\right) \widehat{T}_{\alpha_t}\right)$$

and

$$AT_{j_m} - \left(\frac{[f_\alpha, f_\beta]}{f_\alpha}\right) T_{i_t} = \mu \left(\left(\frac{[f_{i_t}, f_{j_m}]}{f_{j_m}}\right) T_{j_m} - \left(\frac{[f_{i_t}, f_{j_m}]}{f_{i_t}}\right) T_{i_t}\right).$$

To complete the proof observe that such polynomials exist if $[f_\alpha, f_\beta]$ is a multiple of $f_{j_m} \widehat{f}_{\alpha_t}$ for some integers m and t. □

The example below proves that Theorem 8.2.1 do not extends to ideals generated by square-free monomials of higher degree.

Example 8.2.2 Let $R = k[x_1, \ldots, x_7]$ be a polynomial ring and let I be the ideal of R generated by

$$f_1 = x_1x_2x_3, \ f_2 = x_2x_4x_5, f_3 = x_5x_6x_7, \ f_4 = x_3x_6x_7.$$

Then using Macaulay [15] we get that the toric ideal of $\mathcal{R}(I)$ with respect to $F = \{f_1, \ldots, f_4\}$ is minimally generated by the binomials

$$x_6x_7T_2 - x_2x_4T_3, \quad x_6x_7T_1 - x_1x_2T_4, \quad x_3T_3 - x_5T_4,$$
$$x_4x_5T_1 - x_1x_3T_2, \quad x_4T_1T_3 - x_1T_2T_4.$$

Corollary 8.2.3 *Let $I = I(G)$ be the edge ideal of a graph G. Then I is a syzygetic ideal if and only if G contains no squares.*

Proof. Notice that according to Proposition 8.2.1 I is syzygetic if and only if $RP_2 = (0)$. Hence it is sufficient to observe that if I is generated by $f_1, \ldots, f_q$ then a non trivial relation of the form $f_i f_j = f_k f_\ell$ can only occur if $f_i f_j$ is a square-free monomial. □

Corollary 8.2.4 ([303]) *Let G be a connected graph and let I be its edge ideal. Then I is an ideal of linear type if and only if G is a tree or G has a unique cycle of odd length.*

Proof. Let $V = \{x_1, \ldots, x_n\}$ be the vertex set of G and $I = (f_1, \ldots, f_q)$. If I is an ideal of linear type then by Proposition 3.2.1 G is a tree or G has a unique cycle of odd length.

Conversely assume G is either a tree or G has a unique cycle of length. We claim that $RP_s = (0)$ for $s \geq 2$, we proceed by induction on n. If $n \leq 3$ this is easily verified, assume the claim true for graphs with less than n vertices. Using induction on s we now show $RP_s = (0)$ for $s \geq 2$, notice that by Corollary 8.2.3 $RP_2 = (0)$. Assume $RP_{s-1} = (0)$ and $RP_s \neq (0)$, take a nontrivial relation $f_{i_1} \cdots f_{i_s} = f_{j_1} \cdots f_{j_s}$. Since a tree has a vertex of degree one, by induction hypothesis it follows that G must be a cycle. Therefore, using induction again, we may write $\{f_{i_1}, \ldots, f_{i_s}\} = \{g_1, \ldots, g_s\}$, where $\gcd(g_i, g_j) = 1$ for $i \neq j$ and $g_1 \cdots g_k = x_1 \cdots x_n$. Notice that this equality cannot occur if n is odd, hence $RP_s = (0)$ and the proof of the claim is complete. That I is an ideal of linear type is now a direct consequence of Proposition 8.2.1. □

Let us show an application of Proposition 6.3.7 that lends some support to the following open question [291]:

Conjecture 8.2.5 (Vasconcelos) *Let I be a Cohen-Macaulay ideal in a regular local ring R. If I is syzygetic and I/I^2 is Cohen-Macaulay, then I is a Gorenstein ideal.*

Corollary 8.2.6 *Let T be a tree with vertex set V and let I be its edge ideal. If I is Cohen-Macaulay and $m = |V| \geq 3$, then*

$$\operatorname{depth}(I/I^2) = \frac{m}{2} - 1.$$

Proof. First note that I has the description given in Theorem 6.3.4, here we use the notation of this theorem. Set

$$R = k[\mathbf{x}, \mathbf{y}] \text{ and } \mathbf{z} = \{x_i - y_i\}_{i=1}^r,$$

where $m = 2r$. Since I is syzygetic (see Corollary 8.2.3), we have an exact sequence

$$0 \longrightarrow H_1(I) \longrightarrow (R/I)^{m-1} \longrightarrow I/I^2 \longrightarrow 0.$$

If I/I^2 is Cohen-Macaulay, then

$$\operatorname{Tor}_1^R(I/I^2,\, R/(\mathbf{z})) = 0.$$

Therefore if we tensor this exact sequence with $R/(\mathbf{z})$ we get that the ideal

$$(\{y_i^2 | \, 1 \leq i \leq r\}) + (\{y_j y_\ell | \, x_j \text{ adjacent to } y_\ell\})$$

obtained from I by making $x_i = y_i$ is syzygetic, which is contradicted by the relations $(y_j y_\ell)^2 = y_j^2 y_\ell^2$. □

Corollary 8.2.7 *Let $G = \mathcal{K}_n$ be the complete graph on n vertices and let I be its edge ideal. Then the toric ideal of $\mathcal{R}(I)$ is generated by binomials of degree two.*

Proof. Let $P = \ker(B = k[T_1, \ldots T_q] \xrightarrow{\varphi} k[G])$, where $\varphi(T_i) = f_i$. Recall that we can write

$$P = B\left(\bigcup_{s=2}^{\infty} P_s\right).$$

By Theorem 4.3.7 (cf. [251]) the edge ideal I has a linear resolution, hence using Theorem 8.2.1 the proof reduces to showing $P = (P_2)$.

Using induction on s we will show $P_s \subset P_2$ for all s. Assume $P_{s-1} \subset P_2$. Let $f(T) = T^\alpha - T^\beta \in P_s$. By the induction hypothesis we may assume α and β have disjoint support and that $f(T)$ comes from a monomial walk. In particular we may assume that α_1, α_2, and β_3 are positive integers so that $f_1 = x_1x_2$, $f_2 = x_2x_3$ and $f_3 = x_3x_4$, where $x_1, \ldots, x_4$ are distinct vertices. Note that $f_k = x_1x_4$ for some k and $g(T) = T_1T_2 - T_3T_k \in P_2$. Using

$$\begin{aligned} f(T) \;=\; & T_1^{\alpha_1 - 1} T_2^{\alpha_2 - 1} T_3^{\alpha_3} \cdots T_q^{\alpha_q} g(T) \\ & + T_3 \left(T_1^{\alpha_1 - 1} T_2^{\alpha_2 - 1} T_3^{\alpha_3} \cdots T_q^{\alpha_q} T_k - T_3^{\beta_3 - 1} T_4^{\beta_4} \cdots T_q^{\beta_q} \right), \end{aligned}$$

the induction assumption shows $f(T) \in P_s$, as required. □

The next result uses a formula of Huneke and Rossi [187, 261] for the Krull dimension of the symmetric algebra of a module. For another proof of this formula based on slightly different ideas, as well as for a nice survey on symmetric algebras, see [293].

Theorem 8.2.8 ([303]) *Let G be a graph with n vertices and q edges and $I = I(G)$ its edge ideal. If G is connected, then*

$$\dim(\operatorname{Sym}(I)) = \sup\{n + 1, q\}.$$

Proof. According to the Huneke-Rossi formula we have

$$\dim \operatorname{Sym}(I) = \sup\{\nu(I_\wp) + \dim (R/\wp) |\ \wp \in \operatorname{Spec}(R)\}\ ,$$

where $\nu(I_\wp)$ denotes the minimal number of generators of the localization of I at $\wp$. By this formula it is clear that the Krull dimension of $\operatorname{Sym}(I)$ is at least the given bound. To prove the converse, take a prime ideal $\wp$ of height $n-i$ containing I and set $\mathcal{B} = \{v \in V | v \notin \wp\}$. It follows that $\wp$ contains a face ideal Q, that is, Q is an ideal generated by a subset of the vertices of G. We may assume $Q = (\mathcal{A})$, where $\mathcal{A}$ is a minimal vertex cover for G. Define now

$$\mathcal{C} = \{x \in \mathcal{A} |\ x \text{ is adjacent to some vertex in } \mathcal{B}\}$$

and

$$Y = \{\{x,y\} \in E(G) |\ \{x,y\} \cap \mathcal{C} = \emptyset\}.$$

Notice that $\nu(I_\wp) \leq |\mathcal{C}| + |Y|$. Since $|\mathcal{B}| \geq i$ we obtain

$$\nu(I_\wp) + \dim (R/\wp) \leq |\mathcal{C}| + |Y| + |\mathcal{B}|.$$

The formula is now a consequence of Lemma 8.2.9 below. □

Using the notation above one has:

Lemma 8.2.9 *Let G be a connected (n,q)-graph with vertex set $V = V(G)$ and edge set $E = E(G)$. Let $\mathcal{A}$ be a minimal vertex cover for G and $\mathcal{B}$ a subset of $V \backslash \mathcal{A}$. Then $|\mathcal{C}| + |Y| + |\mathcal{B}| \leq \sup\{n+1, q\}$.*

Proof. Let Y' be the set of edges cover by $\mathcal{C}$, that is, $Y' = E(G) \backslash Y$. Consider the subgraph G' of G with edge set equal to Y' and vertex set

$$V(G') = \{z \in V(G) |\ z \text{ lies in some edge in } Y'\}.$$

If $Y = \emptyset$, then $|\mathcal{B}| + |\mathcal{C}| + |Y| \leq n$. Assume $Y \neq \emptyset$, and denote the connected components of G' by $G_1, \ldots, G_m$. Set $\mathcal{B}_i = \mathcal{B} \cap V(G_i)$ and $\mathcal{C}_i = \mathcal{C} \cap V(G_i)$; notice $\mathcal{B} = \cup \mathcal{B}_i$ and $\mathcal{C} = \cup \mathcal{C}_i$.

We claim that $|\mathcal{B}_i| + |\mathcal{C}_i| \leq n_i - 1$, for all i, where $n_i = |V(G_i)|$. For that, fix an edge $\{x,y\} \in Y$. If $x \in V(G_i)$, then $x \notin \mathcal{B}_i \cup \mathcal{C}_i$; using that $\mathcal{B}_i$ and $\mathcal{C}_i$ are disjoint we get the asserted inequality. On the other hand, if $x \notin V(G_i)$, we choose a vertex $z \in V(G_i)$ at a minimum distance from x. This yields a path $\{x = x_0, x_1, \ldots, x_r = z\}$. As $x_{r-1} \notin V(G_i)$, $z \notin \mathcal{B}_i \cup \mathcal{C}_i$, which gives $|\mathcal{B}_i| + |\mathcal{C}_i| \leq n_i - 1$. Altogether we have

$$|\mathcal{B}| + |\mathcal{C}| + |Y| \leq \sum_{i=1}^{m} (n_i - 1) + |Y| = \sum_{i=1}^{m} (n_i - 1) + (q - |Y'|).$$

But Y' is the disjoint union of $E(G_i)$ for $i = 1, \cdots, m$, and thus

$$|Y'| = \sum_{i=1}^{m} q_i,$$

here $q_i = |E(G_i)|$. This permits writing the last inequality as

$$|\mathcal{B}| + |\mathcal{C}| + |Y| \leq \sum_{i=1}^{m} (n_i - q_i - 1) + q$$

Since G_i is a connected (n_i, q_i)-graph we have $n_i - 1 \leq q_i$. therefore

$$|\mathcal{B}| + |\mathcal{C}| + |Y| \leq q,$$

establishing the claim. □

A dimension formula Let us describe the *cycle space* of a graph G over the two element field $F = \mathbb{Z}_2$. We denote the edge set of G by

$$E(G) = \{f_1, \ldots, f_q\}$$

and the vertex set of G by

$$V = \{x_1, \ldots, x_n\}.$$

Let C_0 and C_1 denote the vector spaces over F of 0-chains and 1-chains respectively. Recall that a 0-chain of G is a formal linear combination

$$\sum a_i x_i$$

of points and a 1-chain is a formal linear combination

$$\sum b_i f_i$$

of edges, where $a_i \in F$ and $b_i \in F$. The boundary operator ∂ is the linear transformation defined by

$$\partial(f_k) = \partial(\{x_i, x_j\}) = x_i + x_j.$$

The *cycle space* $\mathcal{Z}(G)$ of G over F is equal to $\ker(\partial)$. The vectors in $\mathcal{Z}(G)$ can be regarded as a set of edge-disjoint cycles. A *cycle basis* for G is a basis for $\mathcal{Z}(G)$ which consist entirely of cycles, such a basis can be constructed as follows:

Remark 8.2.10 If G is connected, then G has an spanning tree T. The subgraph of G consisting of T and any edge in G has exactly one cycle, the collection of all the cycles obtained in this way form a cycle basis for G. See [139] for details. In particular

$$\dim_F \mathcal{Z}(G) = q - n + 1,$$

see Exercise 5.2.10.

Lemma 8.2.11 *Let G be a connected graph and $\mathcal{Z}_e(G)$ the subspace of $\mathcal{Z}(G)$ of all cycle vectors of G with an even number of terms. Then*

$$\dim_F \mathcal{Z}_e(G) = \begin{cases} q-n+1 & \text{if } G \text{ is bipartite, and} \\ q-n & \text{otherwise.} \end{cases}$$

Proof. Let $c_1, \ldots, c_l, c_{l+1}, \ldots, c_m$ be a cycle basis for G, where $c_1, \ldots, c_l$ are even cycles and $c_{l+1}, \ldots, c_m$ odd cycles. It is clear that a bases for $\mathcal{Z}_e(G)$ is given by $\{c_1, \ldots, c_l, c_{l+1} + c_m, \ldots, c_{m-1} + c_m\}$. □

The next result shows how the Krull dimension of $k[G]$ is related to the cycle space of its graph.

Proposition 8.2.12 ([303]) *Let G be a connected graph and let P be the toric ideal of the edge subring $k[G]$. Then*

$$\operatorname{ht}(P) = \dim_F \mathcal{Z}_e(G).$$

Proof. Let G be a connected graph with q edges and n vertices. Assume that $I(G)$ is minimally generated by the monomials $f_1, \ldots, f_q$. There is an spanning tree T of G so that (after re-ordering) $I(T) = (f_1, \ldots, f_{n-1})$. Since $I(T)$ is an ideal of linear type

$$\dim k[G] = \operatorname{tr.deg}_k k[G] \geq n - 1.$$

If G is bipartite notice that $f_k \in k(f_1, \ldots, f_{n-1})$ for $k \geq n$; to prove it we write $f_k = xz$ and observe that the graph $T \cup \{x, z\}$ has a unique cycle of even length. This shows the equality $\dim k[G] = n - 1$.

If G is not bipartite, then for some $f_k = xz$, $k \geq n$. Note that the graph $T \cup \{x, z\}$ has a unique cycle of odd length. By Corollary 8.2.4 the ideal $(f_1, \ldots, f_{n-1}, f_k)$ is of linear type, hence $\dim k[G] \geq n$; to show equality recall that $\operatorname{tr.deg}_k k[G] \leq n$. To finish the proof use Lemma 8.2.11. □

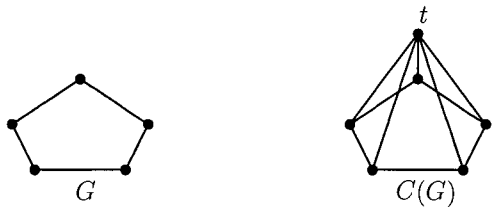
Corollary 8.2.13 *If G is a connected graph with n vertices and $k[G]$ its edge subring, then*

$$\dim(k[G]) = \begin{cases} n & \text{if } G \text{ is not bipartite, and} \\ n-1 & \text{otherwise.} \end{cases}$$

Proof. It follows from the proof of Proposition 8.2.12. □

The role of the cone of a graph The *cone* $C(G)$, over the graph G, is obtained by adding a new vertex t to G and joining every vertex of G to t.

Example 8.2.14 A pentagon and its cone:



Proposition 8.2.15 *Let G be a graph and let $C(G)$ be the cone over G. Then there is an isomorphism*

$$\mathcal{R}(I(G)) \stackrel{\varphi}{\simeq} k[C(G)].$$

Proof. Let $V = \{x_1, \ldots, x_n\}$ be the vertex set of G and

$$\mathcal{R}(I(G)) = k[\{x_1, \ldots, x_n, tf_i | \, 1 \leq i \leq q\}]$$

the Rees algebra of the ideal $I(G) = (f_1, \ldots, f_q)$. We assume that $f_1, \ldots, f_q$ are the monomials in the x_i's corresponding to the edges of G. Let

$$k[C(G)] = k[\{tx_i, f_j | \, 1 \leq i \leq n, \, 1 \leq j \leq q\}]$$

be the monomial subring of the cone. As these two algebras are integral domains of the same dimension by Proposition 7.1.17 and Proposition 8.2.12 (cf. [94]), it follows that there is an isomorphism

$$\varphi: \mathcal{R}(I(G)) \longrightarrow k[C(G)], \text{ induced by } \varphi(x_i) = tx_i \text{ and } \varphi(tf_i) = f_i.$$

That is $\mathcal{R}(I(G)) \stackrel{\varphi}{\simeq} k[C(G)]$. □

Exercises

8.2.16 Let G be a connected graph with vertices $x_1, \ldots, x_n$ and $f_1, \ldots, f_q$ the edge generators of G. Prove that

(a) $k(f_1, \ldots, f_q, x_1) = k(x_1, \ldots, x_n)$, and

(b) $\dim k[G][x_1^2] = n$.

Here $k(F)$ denotes the field of fractions of $k[F]$.

8.2.17 If G is any non discrete graph with n vertices and $C(G)$ its cone, prove that the edge subring $k[C(G)]$ has dimension $n+1$.

8.2.18 Let G be a graph with vertices $x_1, \ldots, x_n$ and $I \subset R = k[x_1, \ldots, x_n]$ its edge ideal. If φ, ψ are the maps of k-algebras defined by the diagram

$$\begin{array}{ccc} R[T_1, \ldots, T_q] & \xrightarrow{\varphi} & \mathcal{R}(I) \\ \downarrow \psi & & \\ k[C(G)] & & \end{array} \qquad \begin{array}{ccc} T_i & \overset{\varphi}{\rightsquigarrow} & Tf_i \\ \wr \psi & & \\ f_i & & \end{array} \qquad \begin{array}{ccc} x_i & \overset{\varphi}{\rightsquigarrow} & x_i \\ \wr \psi & & \\ xt_i & & \end{array}$$

where $C(G)$ is the cone of G and $f_1, \ldots, f_q$ are the edge generators, then $\ker(\varphi) = \ker(\psi)$.

8.3 Incidence matrix of a graph

One of the most interesting matrices that have been associated to a graph is its incidence matrix, here we present some properties of this important matrix. The reader is encouraged to consult [136] for a thorough study of the minors of an incidence matrix.

Let G be a simple graph with vertex set $V = \{x_1, \ldots, x_n\}$ and edge set $E = \{z_1, \ldots, z_q\}$, where every edge z_i is an unordered pair of distinct vertices $z_i = \{x_{i_j}, x_{i_k}\}$. The *incidence matrix* $M_G = [b_{ij}]$ associated to G is the $n \times q$ matrix defined by

$$b_{ij} = \begin{cases} 1 & \text{if } x_i \in z_j, \text{ and} \\ 0 & \text{if } x_i \notin z_j. \end{cases}$$

Note that each column of M_G has exactly two 1's and the rest of its entries equal to zero. If $z_i = \{x_{i_j}, x_{i_k}\}$ define $\alpha_i = e_{i_j} + e_{i_k}$, where e_i is the ith canonical vector in $\mathbb{R}^n$. Thus the columns of M_G are precisely the vectors $\alpha_1, \ldots, \alpha_q$ regarded as column vectors. As a simple example consider a triangle G with vertices x_1, x_2, x_3. In this case:

$$M_G = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix},$$

with the vectors $\alpha_1 = e_1 + e_2$, $\alpha_2 = e_2 + e_3$, and $\alpha_3 = e_1 + e_3$ corresponding to the edges $z_1 = \{x_1, x_2\}$, $z_2 = \{x_2, x_3\}$, and $z_3 = \{x_1, x_3\}$.

Remark 8.3.1 Let A be a square submatrix of M_G. In [136] it is shown that either $\det(A) = 0$ or $\det(A) = \pm 2^k$, for some integer k such that $0 \leq k \leq \tau_0$, where τ_0 is the maximum number of vertex disjoint odd cycles in G. Moreover for any such value of k there exists a minor equal to $\pm 2^k$.

Recall that a graph G is bipartite if all its cycles are of even length. Thus any tree and in particular any point is a bipartite graph. The number of bipartite connected components of G will be denoted by c_0, and the number of non bipartite connected components will be denoted by c_1. Thus $c = c_0 + c_1$ is the total number of components of G.

Lemma 8.3.2 ([136]) *If G is a graph with n vertices and M_G its incidence matrix, then* $\operatorname{rank}(M_G) = n - c_0$.

Proof. Let $G_1, \ldots, G_c$ be the connected components of G, and n_i the number of vertices of G_i. After permuting the vertices we may assume that M_G is a "diagonal" matrix

$$M_G = \operatorname{diag}(M_{G_1}, \ldots, M_{G_c}),$$

where M_{G_i} is the incidence matrix of G_i. By Proposition 8.2.12 the rank of M_{G_i} is equal to $n_i - 1$ if G_i is bipartite, and is equal to n_i otherwise. Hence the rank of M_G is equal to $n - c_0$. □

Definition 8.3.3 Let U be a square matrix with entries in a commutative ring R. The matrix U is called *unimodular* if $\det(U)$ is a unit in R.

Theorem 8.3.4 ([136]) *If G is a graph with n vertices and M_G is the incidence matrix of G, then there are unimodular integral matrices U, V such that*

$$S = UM_GV = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$D = \operatorname{diag}(1, 1, \ldots, 1, 2, 2, \ldots, 2),$$

$n - c$ is the number if 1's and c_1 is the number of 2's.

Proof. See [136, Theorem 3.3]. □

The matrix S is called the *Smith normal form* of M_G and the numbers in the diagonal matrix D are the *invariant factors* of M_G.

Corollary 8.3.5 *Let G be a graph with vertices $x_1, \ldots, x_n$ and $\mathcal{A}_G$ the set of all vectors $e_i + e_j$ such that $\{x_i, x_j\}$ is an edge of G. Then*

$$\mathbb{Z}^n/\mathbb{Z}\mathcal{A}_G \simeq \mathbb{Z}^{n-r} \times \mathbb{Z}_2^{c_1} = \mathbb{Z}^{c_0} \times \mathbb{Z}_2^{c_1},$$

where $r = n - c_0$ is the rank of M_G.

Proof. It follows at once from the fundamental structure theorem of finitely generated modules over a principal ideal domain. See [190, Chapter 3]. □

Exercises

8.3.6 Let G be a cycle of length n and M_G its incidence matrix. Prove

$$\det(M_G) = \begin{cases} \pm 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

8.3.7 Let U be a square matrix with entries in a commutative ring R. Prove that U is unimodular if and only if U is invertible.

8.4 Circuits of a graph and Gröbner bases

Let G be a graph. In this section we study the toric ideal $P(G)$ of the edge subring $k[G]$ from the view point of Gröbner bases, and examine the elementary integral vectors of the incidence matrix of G. For general results on Gröbner bases of toric ideals a standard reference is [278, Chapter 4].

The elementary integral vectors The notion of elementary integral vector occurs in convex analysis [247] and in the theory of toric ideals of graphs [278, 303].

If $\alpha \in \mathbb{R}^q$, its *support* is defined as

$$\operatorname{supp}(\alpha) = \{i \mid \alpha_i \neq 0\}.$$

Note that $\alpha = \alpha_+ - \alpha_-$, where α_+ and α_- are two non negative vectors with disjoint support.

Remark 8.4.1 If α is a linear combination of $\beta_1, \ldots, \beta_r \in \mathbb{R}^q$, then

$$\operatorname{supp}(\alpha) \subset \bigcup_{i=1}^{r} \operatorname{supp}(\beta_i).$$

Definition 8.4.2 Let N be a linear subspace of $\mathbb{Q}^q$. An *elementary vector* of N is a nonzero vector α in N whose support is minimal with respect to inclusion, i.e. $\operatorname{supp}(\alpha)$ does not properly contain the support of any other nonzero vector in N.

The concept of an elementary vector arises in graph theory when N is the kernel of the incidence matrix of a graph G. The reader is referred to [247, Section 22] for a precise interpretation of the vector in N as flows of G which are conservative at every vertex.

Lemma 8.4.3 *If N is a linear subspace of $\mathbb{Q}^q$ and α, β are two elementary vectors of N with the same support, then $\alpha = \lambda\beta$ for some $\lambda \in \mathbb{Q}$.*

Proof. If $i \in \text{supp}(\alpha)$, then one can write $\alpha_i = \lambda\beta_i$ for some scalar λ. Since $\text{supp}(\alpha - \lambda\beta) \subsetneq \text{supp}(\alpha)$, one concludes $\alpha - \lambda\beta = 0$, as required. □

Definition 8.4.4 Let N be a linear subspace of $\mathbb{Q}^q$. An *elementary integral vector* or *circuit* of N is an elementary vector of N with relatively prime integral entries.

Corollary 8.4.5 *If N is a linear subspace of $\mathbb{Q}^q$, then number of circuits of N is finite.*

Proof. It follows from Lemma 8.4.3. □

Definition 8.4.6 Two vectors $\alpha = (\alpha_i)$ and $\beta = (\beta_i)$ in $\mathbb{Q}^q$ are in *harmony* if $\alpha_i\beta_i \geq 0$ for every i.

Lemma 8.4.7 ([246]) *Let N be a linear subspace of $\mathbb{Q}^q$. If $0 \neq \alpha \in N$, then there is an elementary vector $\gamma \in N$ in harmony with α such that* $\text{supp}(\gamma) \subset \text{supp}(\alpha)$.

Proof. Let $\beta = (\beta_1, \ldots, \beta_q)$ be an elementary vector of N whose support is contained in $\text{supp}(\alpha)$. By replacing β by $-\beta$ one may assume $\alpha_i\beta_i > 0$ for some i. Consider

$$\lambda_j = \frac{\alpha_j}{\beta_j} = \min_{1 \leq i \leq q} \left\{ \frac{\alpha_i}{\beta_i} \middle| \, \alpha_i\beta_i > 0 \right\}.$$

Note that $z_i = (\alpha_i - \lambda_j\beta_i)\alpha_i \geq 0$ for all i. Indeed if $\alpha_i\beta_i \leq 0$, then clearly $z_i \geq 0$, and if $\alpha_i\beta_i > 0$ by the minimality of λ_j one has $z_i \geq 0$. Thus the vector $\alpha - \lambda_j\beta$ is in harmony with α and its support is strictly contained in the support of α. If $\alpha - \lambda_j\beta = 0$ we are done, otherwise we apply the same argument with $\alpha - \lambda_j\beta$ playing the role of α. Since being in harmony is an equivalence relation, the result follows by a recursive application of this procedure. □

Theorem 8.4.8 ([246]) *If N is a vector subspace of $\mathbb{Q}^q$ and $\alpha \in N \setminus \{0\}$, then α can be written as*

$$\alpha = \sum_{i=1}^{r} \beta_i$$

for some elementary vectors $\beta_1, \ldots, \beta_r$ of N with $r \leq \dim N$ such that

(i) *$\beta_1, \ldots, \beta_r$ are in harmony with α,*

(ii) $\text{supp}(\beta_i) \subset \text{supp}(\alpha)$ *for all i, and*

(iii) $\text{supp}(\beta_i)$ *is not contained in the union of the supports of $\beta_1, \ldots, \beta_{i-1}$ for all $i \geq 2$.*

Proof. By induction on the number of elements in the support of α. If α is not an elementary vector of N, then by Lemma 8.4.7 there is an elementary vector $\alpha_1 \in N$ in harmony with α such that $\text{supp}(\alpha_1) \subset \text{supp}(\alpha)$. Using the proof of Lemma 8.4.7 it follows that there is a positive scalar $\lambda_1 > 0$ such that $\alpha - \lambda_1\alpha_1$ is in harmony with α and the support of $\alpha - \lambda_1\alpha_1$ is strictly contained in $\text{supp}(\alpha)$. Therefore the result follows applying induction. This proof uses the original argument given in [246]. □

Corollary 8.4.9 *If N is a linear subspace of $\mathbb{Q}^q$, then the circuits of N generate N as a $\mathbb{Q}$-vector space.*

Theorem 8.4.10 *Let M be an $n \times q$ integral matrix and ψ the $\mathbb{Z}$-linear homomorphism*

$$\psi\colon \mathbb{Z}^q \longrightarrow \mathbb{Z}^n$$

given by $\psi(\alpha) = M(\alpha)$. If M has rank n, then $\ker(\psi)$ is generated as a $\mathbb{Z}$-module by the elementary integral vectors of $\ker(\psi)$.

Proof. Set $N = \ker(\psi)$. Let $\alpha_1, \ldots, \alpha_q$ be the column vectors of M and e_i the i*th* unit vector of $\mathbb{Z}^q$. By the proof of [278, Lemma 4.9], the set of circuits of N is the set of vectors of the form

$$\frac{\pm 1}{h}\sum_{j=1}^{n+1}(-1)^j \cdot \det(\alpha_{i_1}, \ldots, \alpha_{i_{j-1}}, \alpha_{i_{j+1}}, \ldots, \alpha_{i_{n+1}}) \cdot e_{i_j} = \frac{\pm 1}{h} \cdot v,$$

where $1 \leq i_1 < \cdots < i_{n+1} \leq q$, and h is the greatest common divisor of the entries of the vector v, if v is nonzero. For convenience we introduce the notation $v = v(i_1, \ldots, i_{n+1})$ and

$$[i_1, \ldots, \widehat{i_j}, \ldots, i_{n+1}] = \det(\alpha_{i_1}, \ldots, \alpha_{i_{j-1}}, \alpha_{i_{j+1}}, \ldots, \alpha_{i_{n+1}}).$$

Set d equal to the greatest common divisor of all the nonzero integers having the special form $[i_1, \ldots, \widehat{i_j}, \ldots, i_{n+1}]$ and

$$\mathcal{B} = \left\{ \frac{v(i_1, \ldots, i_{n+1})}{d} \neq 0 \,\middle|\, 1 \leq i_1 < \cdots < i_{n+1} \leq q \right\}.$$

Note $\mathbb{Z}\mathcal{B} \subset L \subset N$, where L is the $\mathbb{Z}$-module generated by the elementary integral vectors of N.

We claim that $\mathbb{Z}\mathcal{B} = N$. First observe that L and N have both rank $q - n$, see Corollary 8.4.9. Since $\mathbb{Z}\mathcal{B}$ and L have equal rank, one concludes that $N/\mathbb{Z}\mathcal{B}$ is a finite group. Thus to show $\mathbb{Z}\mathcal{B} = N$ it suffices to prove that $N/\mathbb{Z}\mathcal{B}$ is torsion-free or equivalently that $\mathbb{Z}^q/\mathbb{Z}\mathcal{B}$ is torsion-free of rank n.

Consider the matrix A whose rows are the vectors of $\mathcal{B}$. By the fundamental theorem of finitely generated abelian groups the proof reduces to

showing that the ideal $I_{q-n}(A)$ of $\mathbb{Z}$ generated by the $(q-n)$-minors of A is equal to $\mathbb{Z}$.

Fix $\Delta = [i_1, \ldots, \widehat{i_j}, \ldots, i_{n+1}]/d$. For simplicity of notation one may consider the case $\Delta = [1, \ldots, n]/d$. Note that Δ occurs in $v(1, \ldots, n, i)/d$ with a $\pm$ sign for $i = n+1, \ldots, q$. Thus the matrix

$$A' = \begin{pmatrix} a_{11} & \cdots & a_{1n} & \Delta & 0 & 0 & \cdots & 0 \\ a_{21} & \cdots & a_{2n} & 0 & \Delta & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{(q-n)1} & \cdots & a_{(q-n)n} & 0 & 0 & 0 & \cdots & \Delta \end{pmatrix}$$

whose ith row is the vector $v(1, \ldots, n, i)/d$ is a submatrix of A. Therefore Δ^{q-n} belongs to $I_{q-n}(A)$. If $I_{q-n}(A) \neq \mathbb{Z}$, there is a prime number p dividing Δ^{q-n} for all Δ. Thus p divides Δ for all Δ, and this implies that pd divides d, a contradiction. □

On the circuits of a graph Let G be a graph on the vertex set V with edge set E and incidence matrix M. We set $N = \ker(M)$, the kernel of M in $\mathbb{Q}^q$. Note that a vector $\alpha \in N \cap \mathbb{Z}^q$ determine the subgraph G_α of G having vertex set

$$V_\alpha = \{x \in V \,|\, x \text{ divide } f^{\alpha+}\}$$

and edge set

$$E_\alpha = \{\{x, y\} \in E \,|\, xy \in \operatorname{supp}(f^{\alpha+}) \cup \operatorname{supp}(f^{\alpha-})\},$$

where $f_1, \ldots, f_q$ are the square-free monomials corresponding to the edges of G, that is, the edge generators of G. Note that we are using the following notation:

Notation For $\alpha = (\alpha_1, \ldots, \alpha_q) \in \mathbb{N}^q$ and $f_1, \ldots, f_q$ in a commutative ring with identity we set $f^\alpha = f_1^{\alpha_1} \cdots f_q^{\alpha_q}$. The *support* of f^α is defined as the set $\operatorname{supp}(f^\alpha) = \{f_i \,|\, \alpha_i \neq 0\}$.

The next result gives a geometric description of the elementary vectors of the kernel of the incidence matrix of a graph G.

Proposition 8.4.11 ([303]) *Let G be a graph with incidence matrix M and let N be the kernel of M in $\mathbb{Q}^q$. Then a vector $\alpha \in \mathbb{Z}^q \cap N$ is an elementary vector of N if and only if*

(i) *G_α is an even cycle, or*

(ii) *G_α is a connected graph consisting of two edge disjoint odd cycles joined by a path.*

Proof. Assume $\alpha \in \mathbb{Z}^q \cap N$ is an elementary vector of N so that G_α is not an even cycle. From the equation $f^{\alpha+} = f^{\alpha-}$ we obtain a walk in G_α

$$w = \{x_0, x_1, \ldots, x_n = x_0, x_{n+1}, \ldots, x_\ell\},$$

so that $x_1, \ldots, x_{\ell-1}$ are distinct vertices, n is odd and x_ℓ is in $\{x_1, \ldots, x_{\ell-2}\}$. The walk w can be chosen so that $x_i x_{i+1} \in \operatorname{supp}(f^{\alpha+})$ if i is even and $x_i x_{i+1} \in \operatorname{supp}(f^{\alpha-})$ otherwise. We claim that x_ℓ is not in $\{x_1, \ldots, x_{n-1}\}$. Assume $x_\ell = x_m$ for some $1 \leq m \leq n-1$. If m and $\ell - n$ are odd then the monomial walk

$$w_1 = \{x_0, x_1, \ldots, x_m, x_{\ell-1}, x_{\ell-2}, \ldots, x_n = x_0\}$$

yields a relation $f^\delta = f^\gamma$ with $\operatorname{supp}(\delta - \gamma)$ properly contained in $\operatorname{supp}(\alpha)$, which is a contradiction. If m is odd and $\ell - m$ is even then we can consider the walk

$$w_1 = \{x_n, x_{n+1}, \ldots, x_\ell = x_m, x_{m+1}, \ldots, x_n\}$$

and derive a contradiction, the remaining cases are treated similarly. We may now assume $x_\ell = x_m$ for $n \leq m \leq \ell - 2$. The cycle

$$C_1 = \{x_m, \ldots, x_\ell = x_m\}$$

must be odd, otherwise C_1 would give a relation $f^\delta = f^\gamma$ with $\operatorname{supp}(\delta - \gamma)$ a proper subset of $\operatorname{supp}(\alpha)$. The walk w gives a relation $f^\delta = f^\gamma$ with

$$\operatorname{supp}(f^\delta) \subset \operatorname{supp}(f^{\alpha+}) \quad \text{and} \quad \operatorname{supp}(f^\gamma) \subset \operatorname{supp}(f^{\alpha-}).$$

Hence $\operatorname{supp}(\delta - \gamma)$ is a subset of $\operatorname{supp}(\alpha)$, and since the support of α is minimal one has the equality $\operatorname{supp}(\delta - \gamma) = \operatorname{supp}(\alpha)$. Therefore

$$V_\alpha = \{x_1, \ldots, x_\ell\}$$

and G_α is as required. The converse follows from Corollary 8.2.4. □

Definition 8.4.12 A *circuit* of a graph G is either an even cycle or a subgraph consisting of two edge disjoint odd cycles joined by a path.

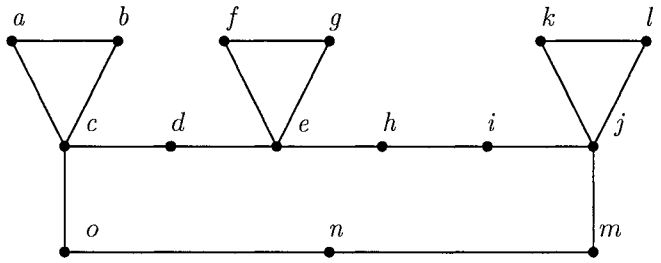
Proposition 8.4.13 *If G is a graph with incidence matrix M and P is the toric ideal of $k[G]$, then*

$$P = (\{T^{\alpha+} - T^{\alpha-} \mid \alpha \in \mathbb{Z}^q \text{ and } M\alpha = 0\}).$$

Proof. The result is a consequence of Corollary 7.1.4. □

Let G be a graph with incidence matrix M and let $N = \ker(M) \subset \mathbb{Q}^q$. The next example was computed using Macaulay [15], it shows that the set of elementary integral vectors of N does not determine the presentation ideal of $k[G]$.

Example 8.4.14 Let $R = k[a, \ldots, o]$ be a polynomial ring over a field k and let G be the labeled graph



the toric ideal P of $k[G]$ has height three and $k[G]$ is not Cohen-Macaulay. The toric ideal P is minimally generated by

$$t_7t_9^2t_{11}^2t_{13} - t_6t_8t_{10}^2t_{12}t_{14}, \quad t_1t_3t_6t_8t_{10}^2t_{15}^2t_{17}^2 - t_2t_7t_9^2t_{11}^2t_{16}^2t_{18}^2$$

$$t_4t_9t_{11}t_{13}t_{15}t_{17} - t_5t_{10}t_{12}t_{14}t_{16}t_{18}, \quad t_1t_3t_{13}t_{15}^2t_{17}^2 - t_2t_{12}t_{14}t_{16}^2t_{18}^2$$

$$t_4t_6t_8t_{10}t_{15}t_{17} - t_5t_7t_9t_{11}t_{16}t_{18}, \quad t_1t_3t_5t_{10}t_{15}t_{17} - t_2t_4t_9t_{11}t_{16}t_{18}$$

$$t_2t_4^2t_9^2t_{11}^2t_{13} - t_1t_3t_5^2t_{10}^2t_{12}t_{14}, \quad t_4^2t_6t_8t_{13}t_{15}^2t_{17}^2 - t_5^2t_7t_{12}t_{14}t_{16}^2t_{18}^2$$

$$t_1t_3t_5^2t_7 - t_2t_4^2t_6t_8, \quad t_1t_3t_5t_7t_9t_{11}t_{13}t_{15}t_{17} - t_2t_4t_6t_8t_{10}t_{12}t_{14}t_{16}t_{18}.$$

Remark 8.4.15 By Proposition 8.4.11 there are, up to sign, exactly nine elementary integral vectors of N. They are in one to one correspondence with the exponent vectors of the first nine generators given above. Therefore the set

$$\{T^{\alpha+} - T^{\alpha-} \mid \alpha \text{ is an elementary integral vector of N}\}$$

is not a generating set for the presentation ideal of $k[G]$.

Corollary 8.4.16 *Let G be a graph with incidence matrix M. If $\lambda = (\lambda_i)$ is an elementary integral vector of $N = \ker(M)$, then $|\lambda_i| \leq 2$ for all i.*

Proof. Let λ be an elementary integral vector of N. Since G_λ is an even cycle or two edge disjoint cycles joined by a path we obtain an elementary integral vector α with $\text{supp}(\alpha) = \text{supp}(\lambda)$ and so that $|\alpha_i| \leq 2$ for all i. Because any two elementary vectors with the same support are dependent we obtain $\lambda = \alpha$. □

Theorem 8.4.17 *Let G be a connected graph with q edges and let P be the toric ideal of $k[G]$. Then the total degree of a polynomial in any reduced Gröbner basis of P is less or equal than $2q \dim Z_e(G)$.*

Proof. Let M be the incidence matrix of G and let

$$N = \{\alpha \in \mathbb{Q}^q \mid M\alpha = 0\}.$$

Assume F is the reduced Gröbner basis of P and take $f = f(T) \in F$. Using Corollary 7.1.5 and Corollary 8.4.13, together with the fact that normalized reduced Gröbner basis are uniquely determined, we obtain $f(T) = T^{\alpha} - T^{\beta}$. Notice that $\operatorname{supp}(\alpha) \cap \operatorname{supp}(\beta) = \emptyset$, otherwise take i in the intersection and write $f(T) = T_i(T^a - T^b)$. Since $T^a - T^b$ reduces w.r.t F it follows that $f(T)$ reduces w.r.t $F \setminus \{f\}$ which is impossible. We can now write $f(T) = T^{\alpha_+} - T^{\alpha_-}$ for some $\alpha \in N \cap \mathbb{Z}^q$. By Theorem 8.4.8 we can write $\alpha = \sum_{i=1}^{r} a_i\lambda_i$ for some $a_i \in \mathbb{Q}^+$, and for some elementary integral vectors λ_i each in harmony with α and such that $\operatorname{supp}(\lambda_i) \subset \operatorname{supp}(\alpha)$ for all i and $r \leq \dim N$. Hence

$$\alpha_+ = \sum_{i=1}^{r} a_i(\lambda_i)_+ \quad \text{and} \quad \alpha_- = \sum_{i=1}^{r} a_i(\lambda_i)_-.$$

If $a_i > 1$ for some i then α_{i+} and α_{i-} are componentwise larger than λ_{i+} and λ_{i-} respectively, note that this is not possible by the previous argument. Hence $0 < a_i \leq 1$. Using Corollary 8.4.16 it now follows that the entries of α satisfy $|\alpha_i| \leq 2 \dim Z_e(G)$ for all i, which gives the asserted inequality. □

Let G be a graph with incidence matrix M and $N = \ker(M)$ the kernel of M in $\mathbb{Q}^q$. Let $\lambda_1, \ldots, \lambda_s$ are the elementary integral vectors of N. Define the *elementary zonotope* of N as $\mathcal{E}_N = [0, \lambda_1] + \cdots + [0, \lambda_s]$.

Corollary 8.4.18 *If N is the kernel of the incidence matrix of a graph G and P is the toric ideal of $k[G]$, then*

$$U = \{T^{\alpha_+} - T^{\alpha_-} \mid \alpha \in \mathcal{E}_N \cap \mathbb{Z}^q\}$$

is a universal Gröbner basis for P.

Exercises

8.4.19 Prove that the elementary integral vector of the kernel of the matrix M are precisely the row vectors of the matrix A with a $\pm$ sign

$$M = \begin{pmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{pmatrix} \qquad A = \begin{pmatrix} 2 & -3 & 1 & 0 \\ 3 & -4 & 0 & 1 \\ 0 & 1 & -3 & 2 \\ 1 & 0 & -4 & 3 \end{pmatrix}.$$

8.5 Edge subrings of bipartite planar graphs

It is the goal of this section to extract information about the edge subring $k[G]$, where G is a bipartite planar graph, by studying the particular geometry of the graph G.

Our exposition follows closely that of [85], all the main results and proofs in this section are due to Luisa Doering and Tor Gunston.

Atomic cycles Let G be a bipartite planar graph with vertex set

$$V(G) = \{x_1, \ldots, x_n\}$$

and

$$R = k[x_1, \ldots, x_n]$$

a polynomial ring over a field k. If $f_1, \ldots, f_q$ are the edge generators of G, that is, the monomials that correspond to the edges of G, then the presentation ideal $P(G)$ of the edge subring $k[G]$ is the kernel of the homomorphism of k-algebras

$$S_1 = k[T_1, \ldots, T_q] \longrightarrow k[G]$$

induced by the assignment $T_i \mapsto f_i$. By abuse of notation, we will sometimes refer to "the edge T_i".

Fix an embedding of the graph G in the plane. Let c be an even closed walk, $c = T_{i_1}, \ldots, T_{i_{2r}}$. For any such walk, write

$$T_c = T_{i_1} T_{i_3} \cdots T_{i_{2r-1}} - T_{i_2} T_{i_4} \cdots T_{i_{2r}}.$$

By Proposition 8.1.2, we have that the generators of $P(G)$ correspond to cycles in G:

$$P(G) = (\{T_c | \, c \text{ is a cycle in } G\}).$$

Since G is embedded in the plane, G divides the plane into regions. If c is a closed walk that bounds a region (not including the unbounded region) we say that c is an *atomic cycle* (even though it need not have distinct vertices). Let $A(G) \subset P(G)$ be the ideal generated by

$$\{T_c | \, c \text{ is an atomic cycle in } G\}.$$

As G is embedded in the plane, every cycle in G has a locus in the plane. We denote cycles by lower case letters, and use the same symbol for its locus; the corresponding upper case letter denotes the open region in the plane: $a = \partial A$. Similarly, for each atomic cycle c_i, we denote the open region it bounds by C_i.

If G can be written as $G = G_1 \cup G_2$, where G_1 and G_2 are subgraphs with at most one vertex in common, then

$$P(G) = (P(G_1), P(G_2)),$$

and

$$k[G] \cong k[G_1] \otimes_k k[G_2].$$

Therefore, we will limit ourselves to graphs which are 2-connected (that is, connected graphs with no cutvertices). Under this assumption, *atomic cycles will actually be cycles.*

For any vector $v = (v_1, \ldots, v_q)$ we will write T^v for $T_1^{v_1} \cdots T_q^{v_q}$. If u and v are vectors with non-negative components, then we define $\langle u, v \rangle$ to be the vector whose ith component is given by $\langle u, v \rangle_i = \max(0, u_i - v_i)$.

In the sequel, G is a 2-connected bipartite planar graph with a fixed embedding in the plane, n vertices, q edges and r regions.

Polarizations The concept of polarization will allow us to define term orders which in some respects are the "right ones" to study $P(G)$ and some homomorphic images of $P(G)$.

Definition 8.5.1 Let c be an even closed walk of G, $c = T_{i_1}, \ldots, T_{i_t}$. A *polarization* of c is the set

$$\{T_{i_1}, T_{i_3}, \ldots, T_{i_{t-1}}\}$$

or the set

$$\{T_{i_2}, T_{i_4}, \ldots, T_{i_t}\}.$$

Let $\{c_1, \ldots, c_r\}$ be the atomic cycles of G. A *polarization* $\mathcal{P}$ of G is a set

$$\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_r\}$$

where $\mathcal{P}_i$ is a polarization of c_i for all i such that

(a) for $i \neq j$ we have $\mathcal{P}_i \cap \mathcal{P}_j = \emptyset$, and

(b) $\{T_i | \, T_i \notin \mathcal{P}_j \text{ for } j = 1, \ldots, r\}$ is a polarization of w, where w is the boundary of the unbounded region.

Proposition 8.5.2 ([85]) *G has a polarization.*

Proof. Let $\{x_1, \ldots, x_n\}$ be the vertex set of G. Form a new graph G' as follows. G' has vertex set

$$\{x_1, \ldots, x_n, y_1, \ldots, y_r\}$$

and edge set

$$\{\{x_i, x_j\} | \, \{x_i, x_j\} \text{ is an edge in } G\}$$
$$\bigcup \{\{x_i, y_j\} | \, x_i \text{ is in the atomic cycle } c_j\}.$$

In other words, G' is obtained from G by adding a vertex in each bounded region and connecting it to each vertex in the boundary of that region. We view G as a subgraph of G'. It is clear that G has a polarization if the plane map determined by G' can be 2-colored. In order to verify this, we let G'' be the graph with vertex set

$$\{v_i | \, v_i \text{ is a bounded region of } G'\}$$

and edge set

$$\{\{v_i, v_j\} | \, v_i \text{ is adjacent to } v_j \text{ in } G'\}$$

We claim that G'' is bipartite. G'' has an embedding in the plane induced from the embedding of G, so we need only check that every atomic cycle of G'' is even. The atomic cycles of G'' are of two sorts: those corresponding to a vertex x_i and those corresponding to a vertex y_i. A cycle corresponding to a vertex y_i is even because G is bipartite. It is easy to check that a cycle corresponding to a vertex x_i is even also. Thus G'' is bipartite. □

Lemma 8.5.3 *If $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_r\}$ is a polarization of G, then by relabeling one may assume that $T_i \in \mathcal{P}_i$ and that if $1 \leq i < j \leq r$, T_j does not appear in the cycle c_i.*

Proof. We induct on r, the number of atomic cycles. If $r = 1$ the lemma is clear. Assume $r > 1$. It is not hard to see that there is al least one edge that appears in only one atomic cycle, that we label c_r, and is contained in $\mathcal{P}_r$. Label this edge T_r. Let G' be the graph obtained from G by deleting all edges and vertices that do not appear in any atomic cycle except c_r. Then G' is planar and bipartite, and is oriented by

$$\mathcal{P}' = \{\mathcal{P}_1, \ldots, \mathcal{P}_{r-1}\}.$$

Apply the lemma, labeling with T_1 through T_{r-1}, and we are done. □

Proposition 8.5.4 *If $c_1, \ldots, c_r$ are the atomic cycles, then $T_{c_1}, \ldots, T_{c_r}$ is a regular sequence.*

Proof. Assume we have a labeling as in the previous lemma. Let $>$ be the lexicographical ordering on S_1 with

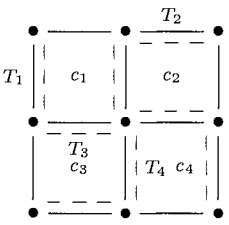
$$T_1 > T_2 > \cdots > T_r > \cdots > T_q$$

and let $\text{in}(-)$ refer to the initial term or ideal with respect to this ordering. Let $m_i = \text{in}(T_{c_i})$. Note that

$$m_i = \prod \{T_k \in \mathcal{P}_i\},$$

so for $i \neq j$ we have $\gcd(m_i, m_j) = 1$. Hence the height of $(T_{c_1}, \ldots, T_{c_r})$ is r, as required. □

Example 8.5.5 Let G be the following graph and consider the polarization obtained by taking pairs of dashed lines in each atomic cycle.



Thus with the lex ordering $T_1 > \cdots > T_4 > T_5 > \cdots > T_{12}$ the binomials $T_{c_1}, \ldots, T_{c_4}$ form a Gröbner basis for the ideal they generate.

Fixing more notation Choose a polarization $\mathcal{P} = \{\mathcal{P}_1, \ldots, \mathcal{P}_r\}$ of G. Assume that the edges are labeled such that $T_i \in \mathcal{P}_i$. We employ the following notation:

$$\begin{aligned} \mathcal{P}^c &= \{T_1, \ldots, T_q\} \setminus \bigcup \mathcal{P}_i \,, \\ \mathcal{P}_i^c &= \{T_j | \, T_j \text{ appears in } c_i\} \setminus \mathcal{P}_i \end{aligned}$$

and for any cycle a, with $a = \partial A$:

$$\mathcal{P}_a = \{T_j | \, T_j \text{ appears in } a \text{ and } T_j \in \mathcal{P}_i \text{ for some } C_i \subset A\}.$$

Let

$$N = \mathcal{P}^c \cup \{T_i - T_j | \, 1 \leq i \leq r \text{ and } T_j \in \mathcal{P}_i\}.$$

Set $S_2 = k[U_1, \ldots, U_r]$ and define a homomorphism ϕ of k-algebras by:

$$\phi \colon S_1 \longrightarrow S_2 = k[U_1, \ldots, U_r]$$

$$T_k \overset{\phi}{\longmapsto} \begin{cases} 0 & \text{if } T_k \in \mathcal{P}^c \\ U_i & \text{if } T_k \in \mathcal{P}_i. \end{cases}$$

Note $\ker(\phi) = (N)$ and $S_2 = k[U_1, \ldots, U_r] \cong S_1/(N)$.

Term orders Define a term order in S_2 as follows: Assume that the atomic cycles of G are labeled $c_1, \ldots, c_r$ in such a way that for every atomic cycle c_i, there is a sequence of atomic cycles

$$c_i = c_{i_1}, \ldots, c_{i_l} \tag{8.2}$$

such that $i_1 < i_2 < \ldots < i_l$ and

$$\begin{aligned} \mathcal{P}_{i_j}^c \cap \mathcal{P}_{i_{j+1}} &\neq \emptyset \text{ for } j = 1, \ldots, l-1 \\ \mathcal{P}_{i_l}^c \cap \mathcal{P}^c &\neq \emptyset. \end{aligned}$$

To see that this can always be done, choose an atomic cycle w such that

$$\mathcal{P}_w^c \cap \mathcal{P}^c \neq \emptyset$$

and label that walk $w = c_r$. Let G' be obtained from G by deleting all those edges that appear in no atomic cycle other than c_r. Inductively, label the cycles of G' with $c_1, \ldots, c_{r-1}$.

Let $>_{S_2}$ be the reverse lexicographic ordering with respect to this ordering, with $U_1 >_{S_2} U_2 >_{S_2} \cdots >_{S_2} U_r$.

Now we define a term order on S_1 as follows: for monomials a and b in S_1, say $a \succ b$ if $\phi(a) >_{S_2} \phi(b)$. Let $>_{S_1}$ be a term order on S_1 that is a refinement of $\succ$.

Remark. For a cycle a, we write $T_a = T^{a^+} - T^{a^-}$ where $T^{a^+} = \text{in}_{>_{S_1}}(T_a)$. It is easy to see that

$$T^{a^+} = \prod_{T_i \in \mathcal{P}_a} T_i.$$

Gröbner Bases In this part we find Gröbner bases for $\phi(P(G))$ and $\phi(A(G))$ with respect to the term orders introduced in the previous paragraph. We write $g \rightsquigarrow h$ for the reduction of g to h with respect to a set of polynomials and a term order. For any two cycles a and b, we will write

$$T_{a,b} = T^{\langle b^+, a^+ \rangle} T_a - T^{\langle a^+, b^+ \rangle} T_b = T^{\langle a^+, b^+ \rangle + b^-} - T^{\langle b^+, a^+ \rangle + a^-}.$$

Note that $T_{a,b}$ is the S-polynomial of T_a and T_b with respect to $>_{S_1}$. If A is a region in the plane, we write $\overline{A}$ for the closure of A.

Proposition 8.5.6 *Let a and b be cycles. Suppose the connected components of $A \setminus \overline{B}$ are $A_1, \ldots, A_s$ and the connected components of $B \setminus \overline{A}$ are $B_1, \ldots, B_t$. Then $T_{a,b} \in (T_{a_1}, \ldots, T_{a_s}, T_{b_1}, \ldots, T_{b_t})$.*

Proof. The proof is left as an exercise. □

Note that since the initial terms of the T_{a_i} and the T_{b_i} are all relatively prime, they form a Gröbner basis for the ideal they generate; see Lemma 2.4.14. Thus, it follows from Proposition 8.5.6 that $T_{a,b} \rightsquigarrow 0$ with respect to $T_{a_1}, \ldots, T_{a_s}, T_{b_1}, \ldots, T_{b_t}$.

Lemma 8.5.7 *Let $A = k[x_1, \ldots, x_n]$ and let $>$ be a term order on A. Suppose $mg \rightsquigarrow 0$ with respect to $f_1, \ldots, f_s$, where $g, f_1, \ldots, f_s \in A$ and m is a monomial such that $\gcd(m, \text{in}_>(f_i)) = 1$ for $1 \leq i \leq s$. Then $g \rightsquigarrow 0$ with respect to $f_1, \ldots, f_s$.*

Proof. The proof is left as exercise. □

Theorem 8.5.8 ([85]) *$M = M(G) = \{\phi(T_c) | \, c$ is a cycle $\}$ is a Gröbner basis for $\phi(P(G))$ with respect to $>_{S_2}$.*

Proof. For an exponent v of T we will write $\phi(v)$ for the vector such that $\phi(T^v) = U^{\phi(v)}$. Let a and b be cycles. Then

$$U_a = U^{\phi(a^+)} - U^{\phi(a^-)} \quad \text{and} \quad U_b = U^{\phi(b^+)} - U^{\phi(b^-)}.$$

Let

$$\begin{aligned} U_{a,b} &= U^{(\phi(b^+),\phi(a^+))}U_a - U^{(\phi(a^+),\phi(b^+))}U_b \\ &= U^{(\phi(a^+),\phi(b^+))+\phi(b^-)} - U^{(\phi(b^+),\phi(a^+))+\phi(a^-)}. \end{aligned}$$

By Buchberger's criterion, it suffices to show that $U_{a,b} \rightsquigarrow 0$ with respect to $\{U_c = \phi(T_c) | \, c$ is a cycle $\}$.

By Proposition 8.5.6, $T_{a,b} \rightsquigarrow 0$ with respect to $T_{a_1}, \ldots, T_{a_s}, T_{b_1}, \ldots, T_{b_t}$. Since every T_{a_i} and T_{b_i} is a binomial or monomial, every step in the reduction $T_{a,b} \rightsquigarrow 0$ is a binomial or monomial. Further, since $T^u >_{S_1} T^v$ implies $U^{\phi(u)} \geq_{S_2} U^{\phi(v)}$, for any binomial $f \in S_1$ we have $\phi(f) = 0$ or $\mathrm{in}_{S_2}(\phi(f)) = \phi(\mathrm{in}_{S_1}(f))$. Thus

$$\phi(T_{a,b}) = T^{\phi((a^+,b^+))+\phi(b^-)} - T^{\phi((b^+,a^+))+\phi(a^-)} \rightsquigarrow 0$$

with respect to $U_{a_1}, \ldots, U_{a_s}, U_{b_1}, \ldots, U_{b_t}$.

It can be easily seen that $U_{a,b} | \overline{\phi(T_{a,b})}$ and the quotient is a monomial U^v such that $U_i | U^v$ implies $C_i \subset \overline{A \cap B}$. Thus, by Lemma 8.5.7, $U_{a,b} \rightsquigarrow 0$ with respect to $\{U_c | \, c$ is a cycle $\}$. □

Note that it follows from this and from Buchberger's criterion that

$$\{T_c | \, c \text{ is a cycle in } G\}$$

is a Gröbner basis for $P(G)$ with respect to $>_{S_1}$. In fact, it is actually a *universal* Gröbner basis, even in the case where G is not planar according to Proposition 8.1.10.

Theorem 8.5.9 ([85]) *$B = B(G) = \{\phi(T_c) | \, c$ is an atomic cycle $\}$ is a Gröbner basis for $\phi(A(G))$ with respect to $>_{S_2}$.*

Proof.

$$U_{c_i} = \phi(T_{c_i}) = \begin{cases} U_i^{|\mathcal{P}_i|} & \text{if } c_i \text{ contains any edges in } \mathcal{P}^c, \\ U_i^{|\mathcal{P}_i|} - U_{j_1} \cdots U_{j_k} & \text{if } c_i \text{ doesn't contain any edges in } \mathcal{P}^c, \end{cases}$$

where $c_{j_1}, \ldots, c_{j_k}$ are the atomic cycles that share an edge with $\mathcal{P}_i^c$, counting multiplicities. Since $\mathrm{in}_{>_{S_2}}(U_{c_i}) = U_i^{|\mathcal{P}_i|}$, the result is clear. □

Applications As applications of the results of the previous paragraph, we show that N is a system of parameters for $k[G]$.

Corollary 8.5.10 height $A(G) = r$.

Corollary 8.5.11 ([85]) *The image of N in $S_1/A(G)$ forms a system of parameters.*

Proof. Both corollaries follow from the fact that

$$\text{in}_{>_{S_2}}(\phi(A(G))) = (U_1^{|\mathcal{P}_1|}, \ldots, U_r^{|\mathcal{P}_r|})$$

is $(U_1, \ldots, U_r)$-primary, and the fact that N has the expected number of elements. From

$$|N| = |\mathcal{P}^c| + \sum_i (|\mathcal{P}_i| - 1) = |\mathcal{P}^c| + \sum_i |\mathcal{P}_i| - r = q - r,$$

one obtains that N has $q - r$ elements. □

Proposition 8.5.12 height $P(G) = r$.

Proof. It follows from Proposition 8.2.12. □

Corollary 8.5.13 ([85]) *The image of N in $S_1/P(G) \cong k[G]$ forms a system of parameters.*

Proposition 8.5.14 *If a is a cycle and $T_a \in (T_c |\, c$ is a cycle, $c \neq a)$, then a has a chord.*

Proof. If $T_a \in (T_c |\, c$ is a cycle, $c \neq a)$, then each term of T_a must be divisible by a term of T_c for some $c \neq a$. □

For the purposes of computing Hilbert functions, we clarify that the various rings involved are graded by

$$\deg(T_i) = \deg(U_i) = 1 \text{ and } \deg(x_i) = 1/2.$$

Thus the isomorphism

$$S_1/P(G) \cong k[G] \subset k[x_1, \ldots, x_n]$$

is a *graded* isomorphism of standard k-algebras.

Corollary 8.5.15 *The Hilbert series of $k[G]$ can be written as*

$$F_{k[G]}(t) = \frac{Q(t)}{(1-t)^{n-1}},$$

where $Q(t)$ is a polynomial with positive integer coefficients and

$$\deg Q(t) \leq \sum_i |\mathcal{P}_i| - r.$$

Proof. Let $\overline{k[G]} = S_1/(N + P(G)) \cong S_2/\phi(P(G))$. Since the edge subring $k[G]$ is Cohen-Macaulay, the image of N in $k[G]$ is a regular sequence. Thus

$$F_{k[G]}(t) = \frac{F_{\overline{k[G]}}(t)}{(1-t)^{n-1}}.$$

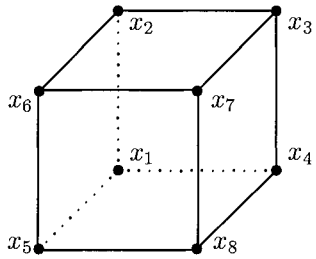
We have that

$$\text{in}_{>_{S_2}}\phi(P(G)) = (U_1^{|\mathcal{P}_1|}, \ldots, U_r^{|\mathcal{P}_r|}, \text{ other monomials }).$$

Then we have $Q(t) = H_{\overline{k[G]}}(t)$ with the desired properties. □

Exercises

8.5.16 Let G be the bipartite graph



and $P(G)$ the toric ideal of $k[G]$ obtained by mapping t_{ij} to x_ix_j. Prove that $P(G)$ is a Cohen-Macaulay ideal of height 5. If we order the t_{ij} variables in reverse lexicographical order:

$$t_{14} > t_{23} > t_{12} > t_{56} > t_{37} > t_{26} > t_{34} > t_{78} > t_{15} > t_{48} > t_{67} > t_{58},$$

use this order to show that $I = (h_1, h_2, h_3, h_4, h_5)$ is a complete intersection inside $P(G)$, where

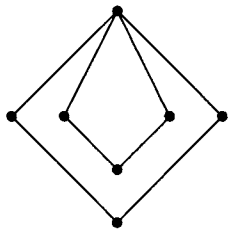
$$h_1 = t_{15}t_{48} - t_{14}t_{58}, \quad h_2 = t_{37}t_{26} - t_{23}t_{67}, \quad h_3 = t_{12}t_{56} - t_{26}t_{15},$$
$$h_4 = t_{14}t_{23} - t_{12}t_{34}, \quad h_5 = t_{34}t_{78} - t_{37}t_{48}.$$

Find an appropriate embedding of G in the plane to explain how this order was obtained and why this order is useful to find a maximal regular sequence of binomials inside $P(G)$.

8.5.17 Let G be a graph that can be written as $G = G_1 \cup G_2$, where G_1 and G_2 are subgraphs with at most one vertex in common. If G_1 is bipartite, then $P(G) = (P(G_1), P(G_2))$ and

$$k[G] \cong k[G_1] \otimes_k k[G_2].$$

8.5.18 Show that the following embedding in the plane of the bipartite graph consisting of two edge disjoint squares that meet at a point has atomic cycles which are not cycles.



8.6 Normality of bipartite graphs

In this section we characterize the torsion freeness of an edge ideal in graph theoretical terms and study the integral closure of the powers of an edge ideal. Two key notions to our examination are the even cycles and certain class of configurations of the graph closely related to the circuits of the graph.

Torsion freeness of bipartite graphs

We shall prove here that edge ideals of bipartite graphs are normally torsion free. In particular, by Proposition 7.3.17 and Theorem 7.2.35, their Rees algebras are normal Cohen-Macaulay domains.

The main technical device is the following:

Lemma 8.6.1 *Let G be a bipartite graph and let $I = I(G)$ be its edge ideal. If x_1 is a vertex in G, then $x_1 I^s \cap (I^{s+1} : x_1) \subset I^{s+1}$ for all $s \geq 0$.*

Proof. The proof is by induction on s. The case $s = 0$ is easy to verify. Assume the containment true for all integers less than s. Assume M is a monomial belonging to $x_1 I^s \cap (I^{s+1} : x_1) \setminus I^{s+1}$. There are square-free monomials of degree two $M_1, \ldots, M_s, N_1, \ldots, N_{s+1}$ in I so that $M = M_1 \cdots M_s \overline{M}$ and $x_1 M = N_1 \cdots N_{s+1} L$, for some monomial L and for some monomial $\overline{M}$ divisible by x_1. We set $\overline{M} = x_1 \overline{M}_1$. By an induction hypothesis we may assume $M_{i_1} \cdots M_{i_r} \neq N_{j_1} \cdots N_{j_r}$ for all increasing sequences $\{i_k\}$, $\{j_k\}$ having values in $\{1, \ldots, s\}$ and for all $r \leq s$.

For convenience, set $M_{s+1} = N_{s+2} = 1$. Given $1 \leq k \leq s+1$ we claim that there are distinct vertices $x_1, \ldots, x_{2k}$ of G so that (after re-ordering the N_i's and M_i's) $M_i = x_{2i} x_{2i+1}$ for $i \leq k-1$, $N_i = x_{2i-1} x_{2i}$ for $i \leq k$, and $M_k \cdots M_s \overline{M} = x_{2k} N_{k+1} \cdots N_{s+1} H$, for some monomial H.

The proof of the claim is by induction on k. If $k = 1$, from the equation $x_1M = N_1 \cdots N_{s+1}H$ and using that $M \notin I^{s+1}$, we obtain $N_i = x_1x_2$ for some i. We reorder the N_i's such that $N_1 = x_1x_2$, which shows that $M = x_2N_2 \cdots N_{s+1}H$. Assume that the claim is valid for k and consider the equality

$$M_k \cdots M_s\overline{M} = x_{2k}N_{k+1} \cdots N_{s+1}H.$$

Observe that if x_{2k} divides $\overline{M}$, then the equation

$$\begin{aligned} M &= (x_2x_3) \cdots (x_{2k-2}x_{2k-1})M_k \cdots M_s(x_1x_{2k})\overline{M}_1 \\ &= N_1 \cdots N_kM_k \cdots M_s\overline{M}_1 \end{aligned} \tag{8.3}$$

implies $M \in I^{s+1}$. Hence x_{2k} divides M_i for some $k \leq i \leq s$; as before we re-order the M_i's so that $M_k = x_{2k}z$ for some vertex z. If $z \in \{x_1, \cdots, x_{2k}\}$, then either $z = x_i$ with i odd and in this case we have an odd cycle, or i is even and we have a relation of the form $M_{i_1} \cdots M_{i_r} = N_{j_1} \cdots N_{j_r}$: using that G is a bipartite graph together with our previous assumptions we see that none of these cases can occur. Therefore $z = x_{2k+1}$ satisfies:

$$z \notin \{x_1, \ldots, x_{2k}\} \text{ and } x_{2k+1}M_{k+1} \cdots M_s\overline{M} = N_{k+1} \cdots N_{s+1}H.$$

Since $M \notin I^{s+1}$ the last equation shows that x_{2k+1} divides N_i for some i, say $N_{k+1} = x_{2k+1}w$ for some vertex w. We set $w = x_{2k+2}$. Notice that by the last argument $w \notin \{x_1, \ldots x_{2k+1}\}$, and the induction on k is complete. We now apply the claim for $k = s+1$ to obtain $\overline{M} = x_{2s+2}H$. To complete the induction on s, if $k = s+1$ then an application of Eq. (8.3) above leads to $M \in I^{s+1}$, which is a contradiction to the initial choice of M. □

Lemma 8.6.2 *Let G be a graph and $U = \{x_1, \ldots, x_r\}$ a subset of the vertex set of G. If A is a minimal vertex cover of G, then $N_G(U) \not\subset A$, where $N_G(U)$ is the neighbor set of U.*

Proof. Set $N = N_G(x_1, \ldots, x_r)$. Assume A is a minimal vertex cover of G such that $N \subset A$. Take an edge e of G. If $x_1 \in e$, then $e = \{x_1, z\}$ with $z \in N$, thus $z \in A \setminus \{x_1\}$ and $e \cap (A \setminus \{x_1\}) \neq \emptyset$. If $x_1 \notin e$, then using $e \cap A \neq \emptyset$, one has $e \cap (A \setminus \{x_1\}) \neq \emptyset$. Altogether $A \setminus \{x_1\}$ is a vertex cover of G, which contradicts the minimality of A. Therefore N cannot be contained in A. □

Theorem 8.6.3 ([262]) *Let G be a graph and let $I = I(G)$ be its edge ideal. The following conditions are equivalent:*

(i) *G is bipartite.*

(ii) *I is normally torsion free.*

Proof. Let $V = \{x_1, \ldots, x_n\}$ be the vertex set of G and let

$$R = k[\mathbf{x}] = k[x_1, \ldots, x_n]$$

be a polynomial ring over a field k. We set $\mathbf{m} = (\mathbf{x})$.

(i) $\Rightarrow$ (ii) We will proceed by induction on n. The ideal I is clearly normally torsion free for $n = 1$ and $n = 2$. Assume I is normally torsion free for bipartite graphs with fewer than n vertices. Let $P \in \operatorname{Ass}(R/I^s)$ for some s. First notice that by Lemma 8.6.1

$$\mathbf{m} \neq (I^s : M)$$

for every monomial M in the $\mathbf{x}$ variables and for all $s \geq 1$. As a consequence $\mathbf{m}$ is not an associated prime of R/I^s for all s. Therefore, P is a face ideal properly contained in $\mathbf{m}$, hence I_P can be written (after a permutation of the variables) as

$$I_P = (x_1, \ldots, x_k, I(G_1))_P,$$

where G_1 is a bipartite subgraph of G. Using the induction hypothesis we conclude that

$$PR_P \in \operatorname{Ass}(R_P/I_P),$$

which gives $P \in \operatorname{Ass}(R/I)$ and the proof is complete.

(ii) $\Rightarrow$ (i) Assume G is not bipartite and pick an odd cycle with vertices $x_1, \ldots, x_r$. Set $k = (r+1)/2$ and $f = x_1 \cdots x_r$. Note $f \notin I^k$, because any monomial in I^k has degree at least $2k$. Let

$$N = \{x_1, \ldots, x_r, x_{r+1}, \ldots, x_s\}$$

be the neighbor set of $\{x_1, \ldots, x_r\}$. Note that $f \in I^{(k)}$, because according to Lemma 8.6.2 the linear form $s = x_1 + \cdots + x_s$ is a nonzero divisor of R/I and $sf \in I^k$. Therefore $I^k \neq I^{(k)}$, a contradiction to Proposition 3.3.26. □

Let G be a non bipartite graph and $I = I(G)$ its edge ideal. According to [38] the sets $\operatorname{Ass}(R/I^n)$ will stabilize for large n. A recent article [62] gives a constructive method to find this stable set and present an upper bound on where the stable set will occur.

Corollary 8.6.4 ([262]) *Let G be a graph and $I = I(G)$ its edge ideal. If G is bipartite, then the Rees algebra $\mathcal{R}(I)$ is a normal domain.*

Proof. Note that $I(G)$ is a radical ideal which is generically a complete intersection and use Theorem 3.3.31. □

Integral closure

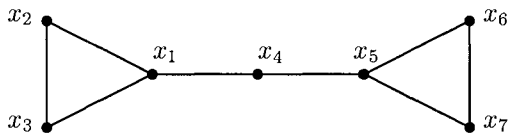
We discuss here some typical behavior of the integral closure of the powers of an edge ideal.

Proposition 8.6.5 *If I is the edge ideal of a graph, then I^2 is integrally closed.*

Proof. Is left as an exercise. □

M. Hochster pointed out the next example showing that the proposition does not extend to the next power of I.

Example 8.6.6 (Hochster's configuration) Let G be the graph



Note that $f = (x_1x_2x_3)(x_5x_6x_7)$ satisfies $f \in \overline{I^3} \setminus I^3$.

This example leads naturally to the following concept.

Definition 8.6.7 Let G be a graph. A *Hochster configuration of order t* (*H-configuration* for short) in G consists of two odd cycles C_{2r+1} and C_{2s+1} satisfying the following conditions:

(i) $C_{2r+1} \cap N(C_{2s+1}) = \emptyset$ and $t = r + s + 1$.

(ii) No chord of either C_{2r+1} or C_{2s+1} is an edge of G.

Given such a configuration, we consider the product of the monomials corresponding to the two cycles to produce new generators of the integral closure of the appropriate powers of $I(G)$.

Proposition 8.6.8 *If I is the edge ideal of a graph G admitting an H-configuration of order t, then $\overline{I^t} \neq I^t$.*

Proof. The monomial obtained by multiplying all the variables in the two cycles of the configuration is an element of $\overline{I^t} \setminus I^t$. Indeed, let

$$\delta = z_1 \cdots z_{2r+1} \quad \text{and} \quad \gamma = w_1 \cdots w_{2s+1}$$

be the monomials corresponding to the two cycles in the configuration. By definition, we have $r + s + 1 = t$. Therefore $\delta\gamma \in I^{t-1} \setminus I^t$. On the other hand $\delta^2 \in I^{2r+1}$ and $\gamma^2 \in I^{2s+1}$, thus $(\delta\gamma)^2 \in I^{2(r+s+1)} = I^{2t}$. This shows that $\delta\gamma \in \overline{I^t}$. □

Conjecture 8.6.9 ([262]) *The edge ideal of a graph G is normal if and only if G admits no H-configurations.*

As it will be seen in the next section this conjecture has a positive answer, but first we need to find a good description for the integral closure of the edge subring of a graph.

Exercises

8.6.10 Let $R = k[x_1, x_2, x_3, y_1, y_2, y_3, y_4]$ and $I = I_1 J_3 + I_3 J_1$, where I_r (resp. J_r) denote the ideal of R generated by the square-free monomials of degree r in the x_i variables (resp. y_i variables). If $z = x_1 x_2 x_3 y_1^2 y_2 y_3 y_4$, then $z^2 \in I^4$ and $z \in \overline{I^2} \setminus I^2$.

8.6.11 Let G be a graph. Explain the differences between a circuit of G and an H-configuration of G.

8.7 The integral closure of an edge subring

Let G be a graph and let $k[G]$ be the edge subring of G over an arbitrary field k. In general, it is difficult to certify whether $k[G]$ has a given algebraic property or to obtain a measure of its numerical invariants. This arises because the number q of monomials is usually large. It is similarly difficult to carry out manipulations such as those required in the Noether normalization process. Our goal here is to unfold a construction for the integral closure or normalization $\overline{k[G]}$ of $k[G]$, for this purpose the underlying graph theoretic aspects are very helpful.

A combinatorial description of normalizations

Let us begin by proving the normality of edge subrings associated to graphs which are nearly bipartite.

Proposition 8.7.1 *Let G be a connected graph and k a field. If G has at most one odd cycle, then $k[G]$ is normal.*

Proof. We may assume that G has a unique odd cycle, for otherwise G is a bipartite graph and by Theorem 8.6.3 $k[G]$ is normal. Pick an edge e of G which is on the unique odd cycle and set $H = G \setminus \{e\}$. Notice that H is a connected bipartite graph because e is not a bridge (it is on a cycle). Hence $k[H]$ is a normal domain of dimension $\dim k[G] - 1$ (see Proposition 8.2.12). Consider the epimorphism

$$k[H][t] \xrightarrow{\varphi} k[G] \longrightarrow 0, \text{ induced by } \varphi(t) = f_e,$$

where t is a new indeterminate and f_e is the monomial corresponding to e. Since $k[H][t]$ and $k[G]$ are both Noetherian domains of the same dimension, φ is an isomorphism. Therefore $k[G]$ is normal. □

Lemma 8.7.2 *Let G be a graph without even cycles. Then the intersection of any two cycles is either a point or the empty set.*

Proof. Let $Z_1 = \{z_0, z_1, \ldots, z_n = z_0\}$ and $Z_2 = \{w_0, w_1, \ldots, w_m = w_0\}$ be two distinct cycles of G such that $Z_1 \cap Z_2 \neq \emptyset$. We may assume $z_0 = w_0$ and $w_1 \notin Z_1$. Let

$$j = \min\{i \geq 1 | \, w_i \in Z_1\},$$

and write $w_j = z_s$. If $0 < s < n$, considering the cycles

$$\{w_0, w_1, \ldots, w_j = z_s, z_{s-1}, \ldots, z_0\}$$
$$\text{and } \{w_0, w_1, \ldots, w_j = z_s, z_{s+1}, \ldots, z_n\}$$

of lengths $s + j$ and $j + n - s$, respectively, we obtain that n is even, which is a contradiction. Therefore $s = 0$ or $s = n$. To conclude observe that $Z_2 = \{w_0, w_1, \ldots, w_{j-1}, z_0\}$, hence $Z_1 \cap Z_2 = \{z_0\}$. □

Definition 8.7.3 A graph G having just one cycle is said to be *unicyclic*.

Definition 8.7.4 Let G be a graph without even cycles. A cycle Z of G is said to be a *terminal cycle* if $N(Z) = Z \cup \{v\}$, for some $v \notin Z$.

Lemma 8.7.5 *Let G be a connected graph without even cycles. Assume that $Z_1 \cap N(Z_2) = \emptyset$, for any two cycles Z_1, Z_2 of G. If G is not a unicyclic graph, and $\deg(v) \geq 2$ for all $v \in V(G)$, then G has at least two terminal cycles.*

Proof. Let $Z_1, \ldots, Z_r$ be the cycles of the graph G. We now construct a new graph T having vertex set

$$V(T) = \{Z_1, \ldots, Z_r\} \bigcup \left(V(G) \setminus \bigcup_{i=1}^{r} Z_i \right).$$

Thus the vertices of T are the cycles of G, together with the vertices of G not lying on any cycle of G. Two points Y_1, Y_2 of T are adjacent if and only if $Y_1 \cap N(Y_2) \neq \emptyset$, here we regard a vertex v of G as the one point set $\{v\}$. Notice that T is a tree and that the terminal cycles of G correspond to the end points of T. Hence G must have at least two terminal cycles. □

The integral closure in terms of bow ties Let G be a graph with vertex set $V = \{x_1, \ldots, x_n\}$ and let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k.

The set $\{f_1, \ldots, f_q\}$ will denote the set of all monomials $x_i x_j$ in R such that $\{x_i, x_j\}$ is an edge of G. Note

$$k[G] = k[f_1, \ldots, f_q] \subset R$$

In general it is difficult to predict which elements lie in the integral closure of an affine domain, but in the case of $k[G]$ one has the following general description: $\overline{k[G]}$ is also a monomial subalgebra generated by monomials

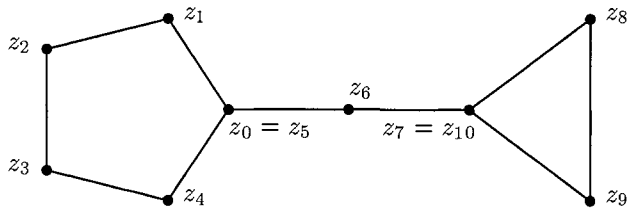
$$f = x_1^{\beta_1} \cdots x_n^{\beta_n} = \mathbf{x}^{\beta}, \quad \beta \in \mathbb{N}^n,$$

with the following two properties (see Theorem 7.2.28):

- $f = \prod f_i^{a_i}, \quad a_i \in \mathbb{Z},$
- $f^m = \prod f_i^{b_i}, \quad m, b_i \in \mathbb{N}$ and $m \geq 1$.

The first condition asserts that f lies in the field of fractions of $k[G]$, the other being the special form the integrality condition takes for such elements.

To explain the description of $\overline{k[G]}$ we begin by exhibiting some monomials in the integral closure of $k[G]$. A *bow tie* of a graph G is an induced subgraph w of G consisting of two edge disjoint odd cycles



$$Z_1 = \{z_0, z_1, \ldots, z_r = z_0\} \text{ and } Z_2 = \{z_s, z_{s+1}, \ldots, z_t = z_s\},$$

joined by a path $\{z_r, \ldots, z_s\}$. In this case one sets $M_w = z_1 \cdots z_r z_{s+1} \cdots z_t$. We observe that Z_1 and Z_2 are allowed to intersect and that only the variables in the cycles occur in M_w, not those in the path itself.

If w is a bow tie of a graph G, as above, then M_w is in the integral closure of $k[G]$. Indeed if $g_i = z_{i-1} z_i$, then

$$M_w = \prod_{i \text{ odd}} g_i \prod_{\substack{i \text{ even} \\ r < i \leq s}} g_i^{-1} \quad \text{and} \quad M_w^2 = g_1 \cdots g_r g_{s+1} \cdots g_t.$$

Hence $M_w \in \overline{k[G]}$.

Lemma 8.7.6 *Let G be a connected graph without even cycles, V the vertex set of G, and $f_1, \ldots, f_q$ the monomials defining the edges of G. Assume the conditions:*

(a) $Z_1 \cap N(Z_2) = \emptyset$, *for any two cycles Z_1, Z_2 of G, and*

(b) $\deg(v) \geq 2$ *for all $v \in V$ and G has an even number of vertices.*

Then $\prod_{v \in V} v$ is in $k[\mathcal{B}]$, where $\mathcal{B} = \{f_1, \ldots, f_q\} \cup \{M_w | \, w$ is a bow tie$\}$.

Proof. The proof is by induction on the number of cycles of the graph G. Set $z = \prod_{v \in V} v$. If G has exactly two cycles, then G is a bow tie and we can write $z = f\delta$, where $f \in k[G]$ and $\delta = M_G$ is the product of the vertices in the two cycles. We may now assume that G has at least three cycles.

By Lemma 8.7.5 there are two terminal cycles

$$Z_1 = \{x_0, x_1, \ldots, x_r = x_0\}, \quad Z_2 = \{y_0, y_1, \ldots, y_s = y_0\}.$$

There is a path $\{x_r, x_{r+1}, \ldots, x_{r_1}\}$, so that $r + 1 \leq r_1$, $\deg(x_j) = 2$ for all $r < j < r_1$, and $\deg(x_{r_1}) \geq 3$. Likewise there is a path $\{y_s, y_{s+1}, \ldots, y_{s_1}\}$, so that $s + 1 \leq s_1$, $\deg(y_j) = 2$ for all $s < j < s_1$, and $\deg(y_{s_1}) \geq 3$. We may assume r_1 and s_1 even, for otherwise, if r_1 is odd, we write

$$z = (x_1x_2)(x_3x_4)\cdots(x_rx_{r+1})\cdots(x_{r_1-2}x_{r_1-1}) \prod_{v \in V_1} v,$$

where $V_1 = V \setminus \{x_1, \ldots, x_{r_1-1}\}$, and then use induction. It is not hard to show that

$$\{x_1, \ldots, x_{r_1}\} \cap \{y_1, \ldots, y_{s_1-1}\} = \{x_1, \ldots, x_{r_1-1}\} \cap \{y_1, \ldots, y_{s_1}\} = \emptyset.$$

Set

$$\sigma_1 = (x_{r+1}x_{r+2})\cdots(x_{r_1-2}x_{r_1-1}) \quad \text{and} \quad \sigma_2 = (y_{s+1}y_{s+2})\cdots(y_{s_1-2}y_{s_1-1}).$$

If $x_{r_1} \neq y_{s_1}$ then we write

$$z = \sigma_1\sigma_2 \prod_{i=1}^{s} y_i \prod_{i=1}^{r} x_i \prod_{v \in V_1} v,$$

where $V_1 = V \setminus \{x_1, \ldots, x_{r_1-1}, y_1, \ldots, y_{s_1-1}\}$, hence using induction we get $z \in k[\mathcal{B}]$. We may now assume that r_1, s_1 are even and $x_{r_1} = y_{s_1}$. Consider the subgraph

$$H = G \setminus \{x_1, \ldots, x_{r_1-1}, y_1, \ldots, y_{s_1-1}\}.$$

If $\deg(x_{r_1}) \geq 4$, then applying the induction hypothesis to H we obtain $z \in k[\mathcal{B}]$. Assume $\deg(x_{r_1}) = 3$. Note that H has a terminal cycle

$$Z_3 = \{z_0, z_1, \ldots, z_t = z_0\}$$

and there is a path $\{z_t, z_{t+1}, \ldots, z_{t_1}\}$, so that $t+1 \leq t_1$, $\deg(z_j) = 2$ for all $t < j < t_1$, and $\deg(z_{t_1}) \geq 3$, by the previous arguments we are reduced to the case r_1, s_1, t_1 even and $x_{r_1} = y_{s_1} = z_{t_1}$. Observe that this forces the equality

$$V = \{x_1, \ldots, x_{r_1}, y_1, \ldots, y_{s_1-1}, z_1, \ldots, z_{t_1-1}\}.$$

To complete the proof we use the identity

$$z = (z_1 z_2)(z_3 z_4) \cdots (z_{t_1-1} z_{t_1}) \sigma_1 \sigma_2 \prod_{i=1}^{r} x_i \prod_{i=1}^{s} y_i,$$

to derive $z \in k[\mathcal{B}]$. □

Given a graph G we set $\mathcal{A}_G = \{\log(f_1), \ldots, \log(f_q)\}$, where $f_1, \ldots, f_q$ are the monomials corresponding to the edges of G and $\log(f_i)$ is the exponent vector of f_i. The affine subsemigroup of $\mathbb{N}^n$ generated by $\mathcal{A}_G$ will be denoted by C_G. Thus

$$C_G = \sum_{\alpha \in \mathcal{A}_G} \mathbb{N}\alpha.$$

The *cone generated* by C_G will be denoted by $\mathbb{R}_+ C_G$, it is equal to

$$\mathbb{R}_+ C_G = \left\{ \sum_{i=1}^{r} a_i \alpha_i \,\middle|\, a_i \in \mathbb{R}_+,\ \alpha_i \in \mathcal{A}_G,\ r \in \mathbb{N} \right\}.$$

Note that $k[G]$ is equal to the *affine semigroup ring* $k[C_G]$. This cone will be studied closely in Section 8.8.

Lemma 8.7.7 *Let G_1 and G_2 be two finite sets of monomials in disjoint sets of indeterminates. Then $k[G_1 \cup G_2] \simeq k[G_1] \otimes_k k[G_2]$ and*

$$\overline{k[G_1 \cup G_2]} \simeq \overline{k[G_1]} \otimes_k \overline{k[G_2]} \simeq \overline{k[G_1]}\ \overline{k[G_2]},$$

is the subring generated by the generators of $\overline{k[G_1]}$ and $\overline{k[G_2]}$.

Proof. Set $G = G_1 \cup G_2$. By Theorem 7.2.28 we have

$$\overline{k[G]} = k[\{x^a \,|\, a \in \mathbb{Z}C_G \cap \mathbb{R}_+ C_G\}].$$

On the other hand there is a decomposition

$$\mathbb{Z}C_G \cap \mathbb{R}_+ C_G = (\mathbb{Z}C_{G_1} \cap \mathbb{R}_+ C_{G_1}) \oplus (\mathbb{Z}C_{G_2} \cap \mathbb{R}_+ C_{G_2}).$$

Using this equality the assertion follows readily. □

Corollary 8.7.8 *Let G be a graph and $G_1, \ldots, G_r$ its connected components. Then $k[G]$ is normal if and only if $k[G_i]$ is normal for each i.*

Theorem 8.7.9 ([263]) *Let G be a graph and k a field. Then the integral closure $\overline{k[G]}$ of $k[G]$ is generated as a k-algebra by the set*

$$\mathcal{B} = \{f_1, \ldots, f_q\} \cup \{M_w | \, w \text{ is a bow tie}\},$$

where $f_1, \ldots, f_q$ denote the monomials defining the edges of G.

Proof. Let $V = \{x_1, \ldots, x_n\}$ and $E = \{e_1, \ldots, e_q\}$ be the vertex set and edge set of G, respectively. According to Theorem 7.2.28 one has:

$$\overline{k[G]} = k[\{x^a | \, a \in \mathbb{Z}C_G \cap \mathbb{R}_+ C_G\}].)$$

The proof is by induction on q, the number of edges of G. The case $q = 1$ is clear. Assume $q \geq 2$ and that the result is true for graphs with less than q edges. If $e_i = \{x_j, x_k\}$, we set $f_i = x_j x_k$.

First we remove all the isolated vertices of G, without affecting the subring $k[G]$. By Lemma 8.7.7 and the induction hypothesis we may assume that G is a connected graph. We may also assume that $\deg(v) \geq 2$ for all $v \in V$; because if $\deg(v) = 1$ and $f_i = vx_j$, then setting $G' = G \setminus \{e_i\}$ we have $k[G] = k[G'][f_i]$, the latter being a polynomial ring over $k[G']$, hence

$$\overline{k[G]} = \overline{k[G']}[f_i]$$

and f_i is not part of a bow tie, which by induction yields the required equality.

Let $z = x^a \in \overline{k[G]}$ be a minimal generator of the integral closure of $k[G]$. There is a positive integer m so that

$$z^m = f_1^{m\lambda_1} \cdots f_q^{m\lambda_q} = f_1^{b_1} \cdots f_q^{b_q}, \text{ where } \lambda_i \in \mathbb{Z} \text{ and } b_i \in \mathbb{Z}_+, \ \forall i. \quad (8.4)$$

Observe that if $b_i = \lambda_i = 0$ for some i, then each bowtie of $G \setminus \{e_i\}$ is a bowtie of G and the induction hypothesis yields $z \in k[\mathcal{B}]$. Hence we may further assume that all the monomials $f_1, \ldots, f_q$ must be present in Eq. (8.4). There are two cases to consider.

Case (I): Assume that G contains an even cycle Z. We set $v_i = \log(f_i)$. For convenience we use the edges of G to represent the cycle Z by the sequence $\{e_{i_1}, \ldots, e_{i_k}\}$, where two consecutive edges are adjacent. If $b_{i_r} = 0$ for some $1 \leq r \leq k$, then using the relation

$$\sum_{j=1}^{k} (-1)^j v_{i_j} = \sum_{j=1}^{k} (-1)^j \log(f_{i_j}) = 0, \quad (8.5)$$

that comes from the even cycle, we may rewrite Eq. (8.4) as

$$z^m = f_1^{m\mu_1} \cdots f_q^{m\mu_q} = f_1^{b_1} \cdots f_q^{b_q}, \text{ where } \mu_i \in \mathbb{Z} \ \forall i, \text{ and } \mu_{i_r} = b_{i_r} = 0,$$

which by induction yields $z \in k[\mathcal{B}]$. We may now assume $b_{i_r} > 0$ for all $1 \leq r \leq k$. We may harmlessly assume that $b_{i_1} \leq \cdots \leq b_{i_k}$. Using Eq. (8.4) and Eq. (8.5) we obtain

$$ma = \sum_{i=1}^{q} b_i v_i = b_{i_1} \sum_{j=1}^{k} v_{i_j} + \sum_{i \neq i_1} c_i v_i = 2b_{i_1} \sum_{j \text{ even}} v_{i_j} + \sum_{i \neq i_1} c_i v_i,$$

where the c_i's are non negative integers. The net effect is that now we may rewrite Eq. (8.4) as

$$z^m = f_1^{m\delta_1} \cdots f_q^{m\delta_q} = f_1^{d_1} \cdots f_q^{d_q}, \text{ where } \delta_i \in \mathbb{Z}, d_i \in \mathbb{Z}_+, \text{ for all } i,$$

and $d_{i_1} = \delta_{i_1} = 0$. Hence by induction $z \in k[\mathcal{B}]$.

Case (II): Assume that all the cycles of G are odd. Let Z_1 and Z_2 be two distinct cycles of G, represented by the sequences of edges $\{e_{i_1}, \ldots, e_{i_k}\}$ and $\{e_{\ell_1}, \ldots, e_{\ell_s}\}$, respectively. If $Z_1 \cap Z_2 \neq \emptyset$, then according to Lemma 8.7.2 $Z_1 \cap Z_2 = \{x\}$, for some $x \in V$, say $x \in \operatorname{supp}(f_{i_1}) \cap \operatorname{supp}(f_{\ell_1})$. Using the identity

$$\sum_{j \text{ odd}} v_{i_j} + \sum_{j \text{ even}} v_{\ell_j} = \sum_{j \text{ even}} v_{i_j} + \sum_{j \text{ odd}} v_{\ell_j}$$

we may proceed as in Case (I). If $Z_1 \cap Z_2 = \emptyset$ and $N(Z_1) \cap Z_2 \neq \emptyset$, say $f_t = xy$, with $x \in \operatorname{supp}(f_{i_k})$ and $y \in \operatorname{supp}(f_{\ell_1})$. Again using the equality

$$\sum_{j \text{ odd}} v_{i_j} + \sum_{j \text{ odd}} v_{\ell_j} = 2v_t + \sum_{j \text{ even}} v_{i_j} + \sum_{j \text{ even}} v_{\ell_j},$$

we may proceed as in Case (I) to derive $z \in k[\mathcal{B}]$.

Therefore we are reduced to the case $N(Z_1) \cap Z_2 = \emptyset$, for any two cycles Z_1, Z_2. If x_j is not in the support of $f_1^{b_1} \cdots f_q^{b_q}$, for some j, consider the set

$$D = \{i \mid x_j \in \operatorname{supp}(f_i^{\pm\lambda_i})\}.$$

Then $\sum_{i \in D} \lambda_i = 0$ and $b_i = 0$, for all $i \in D$. Note that, for any $i \in D$, we may eliminate the monomial f_i from Eq. (8.4), hence by induction we get $z \in k[\mathcal{B}]$. As a consequence we are reduced to the case $\operatorname{supp}(z^m) = V$, that is, $z = x_1^{a_1} \cdots x_n^{a_n}$, with $a_i \geq 1$ for all i. By Corollary 7.7.17 we obtain that $\overline{k[G]}$ is generated as a k-algebra by monomials of degree at most $2(n - \lceil \frac{n}{2} \rceil)$. Altogether we get

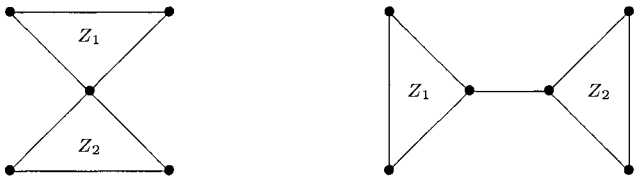
$$n \leq \deg(z) \leq 2\left(n - \left\lceil \frac{n}{2} \right\rceil\right),$$

hence n must be even and $z = x_1 \cdots x_n$. Using Lemma 8.7.6 we conclude that $z \in k[\mathcal{B}]$, as required. □

There is a version of Theorem 8.7.9, due to Hibi and Ohsugi [164], that allows loops in the graph G.

Next we clarify that there are two special types of bowties that are irrelevant for the description of the integral closure of $k[G]$.

Proposition 8.7.10 *Let w be a bowtie consisting of two edge disjoint odd cycles Z_1, Z_2 that meet at a vertex or are connected by an edge, that is, w has essentially one of the forms*:



then M_w is in $k[G]$.

Proof. Left as an exercise. □

As an immediate consequence of Theorem 8.7.9 and Proposition 8.7.10 one has the following full characterization of when $k[G]$ is integrally closed.

Corollary 8.7.11 *Let G be a connected graph. Then $k[G]$ is normal if and only if for any two edge disjoint odd cycles Z_1, Z_2 either Z_1 and Z_2 have a common vertex, or Z_1 and Z_2 are connected by an edge.*

Thus the smallest connected graph G such that $k[G]$ is non normal is the Hochster configuration of Example 8.6.6, consisting of a graph G with two disjoint triangles joined by a 2-path.

The corollary above prompts the following:

Definition 8.7.12 A graph G is said to satisfy the *odd cycle condition* if every two vertex disjoint odd cycles of G can be joined by an edge in G.

The *odd cycle condition* has come up in the literature in connection with the normality of edge subrings [164, 262, 263] and the description of the circuits of a graph [247, 303]; it also occurred before [119]. It could be said that this notion has been rediscovered during the study of monomial subrings of graphs.

Corollary 8.7.13 *Let G be a connected graph and $I = I(G)$ its edge ideal. Then $k[G]$ is normal if and only if the Rees algebra $\mathcal{R}(I)$ is normal.*

Proof. $\Rightarrow$) Let $C(G)$ be the cone over the graph G; it is obtained by adding a new vertex t to G and joining every vertex of G to t. According to Proposition 8.2.15 there is a isomorphism

$$\mathcal{R}(I(G)) \simeq k[C(G)].$$

Let M_w be a bowtie of $C(G)$ with edge disjoint cycles Z_1 and Z_2 joined by the path P, it is enough to verify that $M_w \in k[C(G)]$. If $t \notin Z_1 \cup Z_2 \cup P$, then w is a bowtie of G and $M_w \in k[G]$. Assume $t \in Z_1 \cup Z_2$, say $t \in Z_1$. If $Z_1 \cap Z_2 \neq \emptyset$, then $M_w \in k[C(G)]$. On the other hand if $Z_1 \cap Z_2 = \emptyset$, then $M_w \in k[C(G)]$ because in this case Z_1 and Z_2 are joined by the edge $\{t, z\}$, where z is any vertex in Z_2. It remains to consider the case $t \notin Z_1 \cup Z_2$ and $t \in P$, since G is connected there is a path in G joining Z_1 with Z_2. Therefore $M_w = M_{w_1}$ for some bowtie w_1 of G and $M_w \in k[G]$.

$\Leftarrow$) Conversely if $\mathcal{R}(I(G))$ is normal, then by Proposition 7.4.1 we obtain that $k[G]$ is normal. □

In the corollary above the hypothesis that G is connected is essential to prove that $k[G]$ normal imply $\mathcal{R}(I)$ normal; for instance if G consists of two disjoint triangles, then $k[G]$ is a polynomial ring and $\mathcal{R}(I)$ is non normal.

We give the following counter-example to show that in general it is not true that normality of the Rees algebra implies normality of the monomial subring.

Example 8.7.14 Let $F = \{f_1, \ldots, f_6\}$ and $I = (F)$, where

$$f_1 = x_1x_2, f_2 = x_3x_4x_5, f_3 = x_1x_3, f_4 = x_2x_4, f_5 = x_2x_5, f_6 = x_1x_5.$$

Then

$$k[F] \simeq k[t_1, \ldots, t_6]/(t_1^3t_2^2 - t_3^2t_4^2t_5t_6)$$

is clearly not normal but $\mathcal{R}(I)$ is normal by the Jacobian criterion.

Proposition 8.7.15 *Let G be a graph and let $G_1, \ldots, G_r$ be the connected components of G. Then $\mathcal{R}(I(G))$ is normal if and only if either:*

(i) *G is bipartite, or*

(ii) *exactly one of the components, say G_1, is non bipartite and $K[G_1]$ is a normal domain.*

Proof. One may proceed as in the proof of Corollary 8.7.13. □

Corollary 8.7.16 ([262]) *The edge ideal of a graph G is normal if and only if G admits no H-configurations.*

Remark 8.7.17 (a) Let G be a graph consisting of two disjoint triangles. Note that G itself is an H-configuration but G is not contained in a bowtie, since the two triangles cannot be joined by a path.

(b) If two disjoint odd cycles Z_1, Z_2 of a graph G are in the same connected component, then the two cycles form an H-configuration if and only if Z_1, Z_2 cannot be connected by an edge of G and Z_1, Z_2 have no chords in G

Integral closure of Rees algebras Let G be a graph with vertices $x_1, \ldots, x_n$ and $I = I(G)$ its edge ideal. Consider the endomorphism ϕ of the field $k(x_1, \ldots, x_n, t)$ defined by $x_i \mapsto x_i t$, $t \mapsto t^{-2}$. It induces an isomorphism

$$R[It] \stackrel{\phi}{\longmapsto} k[C(G)], \quad x_i \mapsto t x_i, \quad x_i x_j t \mapsto x_i x_j,$$

where $R = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k and $C(G)$ is the cone over G obtained by adding a new variable t.

We thank Wolmer Vasconcelos for showing us how to get the description of the integral closure of $R[It]$ given below. First we describe the integral closure of $k[C(G)]$.

If $Z_1 = \{x_1, x_2, \ldots, x_r = x_1\}$ and $Z_2 = \{z_1, z_2, \ldots, z_s = z_1\}$ are two edge disjoint odd cycles in $C(G)$. Then one has

$$M_w = x_1 \cdots x_r z_1 \cdots z_s$$

for some bow tie w of $C(G)$, this follows readily because $C(G)$ is a connected graph. In particular M_w is in the integral closure of $k[C(G)]$. Observe that if t occurs in Z_1 or Z_2, then $M_w \in k[C(G)]$ because in this case either Z_1 and Z_2 meet at a point, or Z_1 and Z_2 are joined by an edge in $C(G)$; see Proposition 8.7.10. Thus in order to compute the integral closure of $k[C(G)]$ the only bow ties that matter are those defining the set:

$$\mathcal{B} = \{M_w | \, w \text{ is a bowtie in } C(G) \text{ such that } t \notin \operatorname{supp}(M_w)\}.$$

Therefore by Theorem 8.7.9 we have proved:

Proposition 8.7.18 $\overline{k[C(G)]} = k[C(G)][\mathcal{B}]$.

To obtain a description of the integral closure of the Rees algebra of I note that using ϕ together with the equality

$$M_w = \frac{(tz_1)\prod_{i=1}^{\frac{r+1}{2}}(x_{2i-1}x_{2i})\prod_{i=1}^{\frac{s-1}{2}}(z_{2i}z_{2i+1})}{tx_1},$$

where $x_{r+1} = x_1$, we see that M_w is mapped back in the Rees algebra in the element

$$M_w t^{\frac{r+s}{2}} = x_1 x_2 \cdots x_r z_1 z_2 \cdots z_s t^{\frac{r+s}{2}}.$$

If $\mathcal{B}'$ is the set of all monomials of this form, then one obtains:

Proposition 8.7.19 $\overline{R[It]} = R[It][\mathcal{B}']$.

Exercises

8.7.20 Prove that $k[G]$ is normal for every connected graph G with six vertices, where k is a field.

8.7.21 Let G be a graph on the vertex set $V = \{x_1, \ldots, x_n\}$ and choose a new set of vertices $y_1, \ldots, y_n$, define $S(G)$, the *suspension* of G, to be the graph obtained by attaching to each vertex x_i the new vertex y_i and the edge $\{x_i, y_i\}$. If $I(G)$ is a normal ideal, then $I(C(G))$ and $I(S(G))$ are normal ideals.

8.7.22 Let G be a graph and let $m = (r+s)/2$ be the least positive integer such that G has two vertex disjoint odd cycles of lengths r and s. If $I = I(G)$ is the edge ideal of G, prove that I^i is integrally closed for $i < m$.

8.8 The equations of the edge cone

Let G be a graph on the vertex set $V = \{v_1, \ldots, v_n\}$ and $R = k[x_1, \ldots, x_n]$ a polynomial ring over a field k. The *edge cone* of G is the cone $\mathbb{R}_+\mathcal{A} \subset \mathbb{R}^n$ spanned by the set $\mathcal{A}$ of all vectors $e_i + e_j$ such that v_i is adjacent to v_j, where e_i denotes the ith unit vector.

In this section we present a combinatorial description of the *facets* (faces of maximal dimension) of $\mathbb{R}_+\mathcal{A}$ when $k[G]$ has dimension n. We show some estimates for the a-invariant of $k[G]$, for certain G. The problem of finding upper bounds for the a-invariant of more general normal monomial subrings generated by square-free monomials of the same degree was addressed in Chapter 7.

The irreducible representation of the edge cone

The computation of the *equations* defining the facets of the edge cone of a graph G is a crucial step toward the effective determination of the canonical module and the a-invariant of $k[G]$, whenever $k[G]$ is a normal domain.

For simplicity we have focus our attention on connected non bipartite graphs, but the methods can be applied to bipartite graphs as well.

Definition 8.8.1 The *support* of $\alpha \in \mathbb{R}^n$ is defined as

$$\operatorname{supp}(\alpha) = \{\alpha_i \,|\, \alpha_i \neq 0\}.$$

Lemma 8.8.2 *Let G be a graph with connected components $G_1, \ldots, G_r$ and M its incidence matrix. If G_1 is a tree with at least two vertices and $G_2, \ldots, G_r$ are unicyclic non bipartite graphs, then* $\ker(M^t) = (\beta)$ *for some vector β whose support is contained in* $\{0, 1, -1\}$.

Proof. Let $V = \{v_1, \ldots, v_n\}$ be the vertex set of G. If G is a tree, using induction on the number of vertices it is not hard to show that $\ker(M^t) = (\beta)$, where $\operatorname{supp}(\beta) = \{1, -1\}$. On the other hand if $G = G_i$ for some $i \geq 2$, then M^t is non singular and $\ker(M^t) = (0)$. The general case follows readily by writing M^t as a "diagonal" matrix

$$M^t = \operatorname{diag}(M_{G_1}^t, \ldots, M_{G_m}^t),$$

where M_{G_i} is the incidence matrix of the graph G_i. Observe the equality $V(G_1) = \{v_i \in V \,|\, \beta_i = \pm 1\}$. □

The facets of the edge cone Let us introduce some more terminology and fix the notation that will be used throughout. To ease the reading we recall some of the relevant notation on polyhedral geometry; see Chapter 7.

We set $\mathcal{A}_G$ (or simply $\mathcal{A}$ if G is understood) equal to the set $\{\alpha_1, \ldots, \alpha_q\}$ of row vectors of the transpose of the incidence matrix M_G of G. The *edge cone* $\mathbb{R}_+\mathcal{A}$ of G is defined as the *cone generated* by $\mathcal{A}$, that is:

$$\mathbb{R}_+\mathcal{A} = \left\{ \sum_{i=1}^{q} a_i\alpha_i \,\middle|\, a_i \in \mathbb{R}_+ \text{ for all } i \right\}.$$

By Lemma 8.3.2 one has

$$n - c_0(G) = \dim k[G] = \operatorname{rank}(M_G) = \dim \mathbb{R}_+\mathcal{A},$$

where $c_0(G)$ is the number of bipartite components of G.

If $a \in \mathbb{R}^n$, $a \neq 0$, then the set H_a will denote the *hyperplane* of $\mathbb{R}^n$ through the origin with normal vector a, that is,

$$H_a = \{x \in \mathbb{R}^n \,|\, \langle x, a\rangle = 0\}.$$

This hyperplane determines two *closed half-spaces*

$$H_a^+ = \{x \in \mathbb{R}^n \,|\, \langle x, a\rangle \geq 0\} \quad \text{and} \quad H_a^- = \{x \in \mathbb{R}^n \,|\, \langle x, a\rangle \leq 0\}.$$

A subset $F \subset \mathbb{R}^n$ is a *proper face* of the edge cone of G if there is a supporting hyperplane H_a such that

$$F = \mathbb{R}_+\mathcal{A} \cap H_a \neq \emptyset, \quad \mathbb{R}_+\mathcal{A} \subset H_a^-, \quad \text{and} \quad \mathbb{R}_+\mathcal{A} \not\subset H_a.$$

Definition 8.8.3 A proper face F of the edge cone is a *facet* if

$$\dim F = \dim \mathbb{R}_+\mathcal{A} - 1.$$

When the edge cone is properly embedded into $\mathbb{R}^n$ it will turn out that its facets are of two kinds: those defined by independent sets (see definition below) and those defined by supporting hyperplanes of the form H_{e_i}.

Lemma 8.8.4 *If the incidence matrix of G has maximal rank n, then the set $F = H_{e_i} \cap \mathbb{R}_+\mathcal{A}$ is a proper face of the edge cone.*

Proof. The proof is left as an exercise. □

Let A be an independent set of vertices of G. The supporting hyperplane of the edge cone defined by

$$\sum_{v_i \in A} x_i = \sum_{v_i \in N(A)} x_i$$

will be denoted by H_A.

Lemma 8.8.5 *If the incidence matrix of G has maximal rank n and A is an independent set of vertices of G, then the set $F = \mathbb{R}_+\mathcal{A} \cap H_A$ is a proper face of the edge cone.*

Proof. The proof is left as an exercise. □

To determine whether H_A defines a facet of $\mathbb{R}_+\mathcal{A}$ consider the subgraph $L = L_1 \cup L_2$, where L_1 is the subgraph of G with vertex set and edge set

$$V(L_1) = A \cup N(A) \text{ and } E(L_1) = \{z \in E(G) | \, z \cap A \neq \emptyset\}$$

respectively, and $L_2 = <S>$ is the subgraph of G induced by $S = V \setminus V(L_1)$. The vectors in $\mathcal{A} \cap H_A$ correspond precisely to the edges of L.

Remark 8.8.6 Let F be a facet of $\mathbb{R}_+\mathcal{A}$ defined by the hyperplane H_a. If the rank of M_G is n, then it is not hard to see that H_a is generated by a linearly independent subset of $\mathcal{A}$. See Corollary 7.2.22.

Theorem 8.8.7 ([305]) *Let G be a graph with vertex set $V = \{v_1, \ldots, v_n\}$ and M its incidence matrix. If* rank$(M) = n$, *then F is a facet of $\mathbb{R}_+\mathcal{A}$ if and only if either*

(*a*) *$F = H_{e_i} \cap \mathbb{R}_+\mathcal{A}$ for some i, where all the connected components of $G \setminus \{v_i\}$ are non bipartite graphs, or*

(*b*) *$F = H_A \cap \mathbb{R}_+\mathcal{A}$ for some independent set $A \subset V$ such that L_1 is a connected bipartite graph, and the connected components of L_2 are non bipartite graphs.*

Proof. Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_q\}$ be the set of column vectors of the incidence matrix of G. Assume that F is a facet of $\mathbb{R}_+\mathcal{A}$. By Remark 8.8.6 we may assume that there is $a \in \mathbb{R}^n$ such that

(i) $F = \mathbb{R}_+\mathcal{A} \cap H_a$, $\langle a, \alpha_i \rangle \leq 0 \ \forall i \in \{1, \ldots, q\}$, and

(ii) $\alpha_1, \ldots, \alpha_{n-1}$ are linearly independent vectors in H_a.

Consider the subgraph D of G whose edges correspond to $\alpha_1, \ldots, \alpha_{n-1}$ and its vertex set is the union of the vertices in the edges of D. Let Γ be the transpose of the incidence matrix of D, and $D_1, \ldots, D_r$ the connected components of D. Without loss of generality one may assume

$$\Gamma = \begin{bmatrix} \Gamma_1 & 0 & \ldots & 0 \\ 0 & \Gamma_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \Gamma_r \end{bmatrix}, \ \operatorname{rank}(\Gamma) = n - 1,$$

where Γ_i is the transpose of the incidence matrix of D_i. We set n_i and q_i equal to the number of vertices and edges of D_i respectively. Let Θ be the matrix with rows $\alpha_1, \ldots, \alpha_{n-1}$. We consider two cases.

Case (I): Assume Θ has a zero column, hence it has exactly one zero column and D has vertex set $V(D) = V \setminus \{v_i\}$, for some i. This implies

$$\sum_{i=1}^{r} n_i = \sum_{i=1}^{r} q_i = n - 1.$$

Since $n_i \leq q_i$ for all i, one obtains $n_i = q_i$ for all i. Note that $c_0(D) = 0$, by Lemma 8.3.2. Therefore $D_1, \ldots, D_r$ are unicyclic non bipartite graphs. As any component of $G \setminus \{v_i\}$ contains a component of D, we are in case (a).

Case (II): Assume that all the columns of Θ are non zero, that is, D has vertex set equal to V. Note that in this case

$$n = \sum_{i=1}^{r} n_i = 1 + \sum_{i=1}^{r} q_i. \tag{8.6}$$

Since $\operatorname{rank}(\Gamma) = n - c_0(D) = n - 1$, we get that D has exactly one bipartite component, say D_1. Hence $n_1 - 1 \leq q_1$ and $n_i \leq q_i$ for all $i \geq 2$, which together with Eq. (8.6) yields $q_1 = n_1 - 1$ and $n_i = q_i$ for all $i \geq 2$. Altogether D_1 is a tree and $D_2, \ldots, D_r$ are unicyclic non bipartite graphs. By Lemma 8.8.2 there is $\beta \in \mathbb{R}^n$ such that

$$\ker(\Gamma) = (\beta) \ \text{ and } \ \operatorname{supp}(\beta) \subset \{0, 1, -1\}.$$

We may harmlessly assume $a = \beta$, because $H_a = H_\beta$ and $a \in (\beta)$. Set

$$A = \{v_i \in V \,|\, a_i = 1\} \ \text{ and } \ B = \{v_i \in V \,|\, a_i = -1\}.$$

Note that $A \neq \emptyset$, because D has no isolated vertices and its vertex set is V. We will show that A is an independent set in G and $B = N(A)$, where the neighbor set of A is taken w.r.t. G.

On the contrary if $\{v_i, v_j\}$ is an edge of G for some v_i, v_j in A, then $\alpha_k = e_i + e_j$ and by (i) we get $\langle a, \alpha_k \rangle \leq 0$, which is impossible because $\langle a, \alpha_k \rangle = 2$. This proves that A is an independent set.

Next we show $N(A) = B$. If $v_i \in N(A)$, then $\alpha_k = e_i + e_j$ for some v_j in A, using (i) we obtain $\langle a, \alpha_k \rangle = a_i + 1 \leq 0$ and $a_i = -1$, hence $v_i \in B$. Conversely if $v_i \in B$, since D has no isolated vertices, there is $1 \leq k \leq n-1$ so that $\alpha_k = e_i + e_j$, for some j, by (ii) we obtain $\langle a, \alpha_k \rangle = -1 + a_j = 0$, which shows that $v_j \in A$ and $v_i \in N(A)$.

From the proof of Lemma 8.8.2 we obtain $V(D_1) = A \cup N(A)$. Therefore $\alpha_1, \ldots, \alpha_{n-1}$ are in H_A and $H_a = H_A$. Observe that L_1 (see notation before Remark 8.8.6) cannot contain odd cycles because A is independent and every edge of L_1 contains a vertex in A. Hence L_1 is bipartite, that L_1 is connected follows from the fact that D_1 is an spanning tree of L_1. Note that every connected component of L_2 is non bipartite because it contains some D_i, with $i \geq 2$.

Conversely assume that F is as in (a). Since the vectors in $\mathcal{A} \cap H_{e_i}$ correspond precisely to the edges of $G \setminus \{v_i\}$ one obtains that F is a facet of the edge cone. If F is as in (b), a similar argument shows that F is also a facet. □

Corollary 8.8.8 ([305]) *Let G be a connected non bipartite graph with vertex set $V = \{v_1, \ldots, v_n\}$. Then a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is in $\mathbb{R}_+\mathcal{A}$ if and only if x is a feasible solution of the system of linear inequalities*

$$\begin{aligned} -x_i &\leq 0, \quad i = 1, \ldots, n \\ \textstyle\sum_{v_i \in A} x_i - \sum_{v_i \in N(A)} x_i &\leq 0, \quad \text{for all independent sets } A \subset V. \end{aligned}$$

Proof. Let

$$\mathbb{R}_+\mathcal{A} = H^-_{b_1} \cap \cdots \cap H^-_{b_m}$$

be the irreducible representation of the edge cone as an intersection of closed half-spaces. By Theorem 7.2.18 the set $H_{b_i} \cap \mathbb{R}_+\mathcal{A}$ is a facet of $\mathbb{R}_+\mathcal{A}$, and we may apply Theorem 8.8.7. □

Proposition 8.8.9 ([305]) *Let G be a connected non bipartite graph with n vertices. If $k[G]$ is a normal domain, then a monomial $x_1^{\beta_1} \cdots x_n^{\beta_n}$ belongs to $k[G]$ if and only if the following two conditions hold*

(i) *$\beta = (\beta_1, \ldots, \beta_n)$ is in the edge cone of G, and*

(ii) *$\sum_{i=1}^n \beta_i$ is an even integer.*

Proof. Set $\mathcal{A} = \mathcal{A}_G = \{\alpha_1, \ldots, \alpha_q\}$. Assume $\beta \in \mathbb{R}_+\mathcal{A}$ and $\deg(x^\beta)$ even. We proceed by induction on $\deg(x^\beta)$. Using Corollary 8.3.5 one has the isomorphism

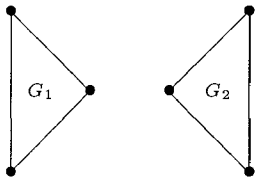
$$\mathbb{Z}^n/(\alpha_1, \ldots, \alpha_q) \simeq \mathbb{Z}_2,$$

hence $2\beta \in \mathbb{R}_+\mathcal{A} \cap \mathbb{Z}\mathcal{A} = \mathbb{N}\mathcal{A}$ and one may write

$$2\beta = 2\sum_{i=1}^{q} s_i\alpha_i + \sum_{i=1}^{q} \epsilon_i\alpha_i, \quad s_i \in \mathbb{N} \text{ and } \epsilon_i \in \{0,1\},$$

by induction one may assume $\sum_{i=1}^{q} s_i\alpha_i = 0$. Therefore from the equality above one concludes that the subgraph whose edges are defined by the set $\{\alpha_i |\, \epsilon_i = 1\}$ is an edge disjoint union of cycles $Z_1, \ldots, Z_r$. By induction one may further assume that all the Z_i's are odd cycles. Note $r \geq 2$, because $\deg(x^b)$ is even. As G is connected, using Corollary 8.7.11 it follows that $x^\beta \in k[G]$. The converse is clear because $k[G]$ is normal. □

Remark 8.8.10 Let us give a simple example to see that the connectivity hypothesis is essential in Proposition 8.8.9. Consider the graph $G = G_1 \cup G_2$ with six vertices consisting of two disjoint triangles:



The vector $\beta = (1, \ldots, 1)$ belongs to the edge cone because $2\beta \in \mathbb{N}\mathcal{A}$, but x^β is not in $k[G]$.

Let G be a connected non bipartite graph with $k[G]$ normal. The last results can be used to give conditions for a graph to have a perfect matching.

Proposition 8.8.11 *Let G be a connected graph with n vertices. If n is even and $k[G]$ is normal of dimension n, then $x_1 \cdots x_n$ is in $k[G]$ if and only if $|A| \leq |N(A)|$ for every independent set of vertices A of G.*

Proof. $\Rightarrow$) Since $k[G]$ is normal:

$$\mathbb{R}_+\mathcal{A}_G \cap \mathbb{Z}\mathcal{A}_G = \mathbb{N}\mathcal{A}_G\,.$$

Hence $a = (1, \ldots, 1) = (a_1, \ldots, a_n)$ is in $\mathbb{R}_+\mathcal{A}_G$. Using Corollary 8.8.8 we get that the vector a satisfies the inequalities:

$$|A| = \sum_{v_i \in A} a_i \leq \sum_{v_i \in N(A)} a_i = |N(A)|$$

for every independent set of vertices A of G, as required.

$\Leftarrow$) First we use Corollary 8.8.8 to conclude that a is in the edge cone, then apply Proposition 8.8.9 to get $a \in k[G]$. □

Corollary 8.8.12 (Generalized marriage problem) *Let G be a graph with an even number of vertices. If G is connected and satisfies the odd cycle condition, then the following are equivalent*

(a) *G has a perfect matching.*

(b) *$|A| \leq |N(A)|$ for all A independent set of vertices of G.*

Proof. The proof follows using Theorem 6.1.8 and Proposition 8.8.11. □

Corollary 8.8.13 ([51]) *Let $G = \mathcal{K}_n$ be the complete graph on n vertices. Then a vector $x \in \mathbb{R}^n$ is in $\mathbb{R}_+\mathcal{A}$ if and only if $x = (x_1, \ldots, x_n)$ satisfies*

$$\begin{array}{rcll} -x_i & \leq & 0, & i = 1, \ldots, n \\ x_i - \sum_{j \neq i} x_j & \leq & 0, & i = 1, \ldots, n. \end{array}$$

In addition if $n \geq 4$, these equations define all the facets of $\mathbb{R}_+\mathcal{A}$.

Example 8.8.14 *If G is a triangle with vertices $\{x_1, x_2, x_3\}$, then the edge cone of G has three facets defined by*

$$x_1 \leq x_2 + x_3, \quad x_2 \leq x_1 + x_3, \quad x_3 \leq x_1 + x_2.$$

Note that $x_1 \geq 0$, $x_2 \geq 0$ and $x_3 \geq 0$ define proper faces of dimension 1.

The a-invariant of some normal subrings

Let X be an arbitrary subset of $\mathbb{R}^n$. The *relative interior* of X, denoted $\operatorname{ri}(X)$, is the interior of X relative to $\operatorname{aff}(X)$, the affine hull of X. As usual the interior of X in $\mathbb{R}^n$ is denoted by X°.

We grade the edge subring $k[G] \subset R$ of a graph G with the *normalized grading*

$$k[G]_i = k[G] \cap R_{2i},$$

where

$$R = \bigoplus_{i=0}^{\infty} R_i$$

is a polynomial ring over the field k with the standard grading.

Recall that $k[G]$ is normal if and only if $\mathbb{N}\mathcal{A}_G = \mathbb{R}_+\mathcal{A}_G \cap \mathbb{Z}\mathcal{A}_G$, this follows from Corollary 7.2.29. If $k[G]$ is normal, then by Theorem 7.2.34, the canonical module $\omega_{k[G]}$ of $k[G]$ is given by

$$\omega_{k[G]} = (\{x^a | \, a \in \mathbb{N}\mathcal{A}_G \cap \operatorname{ri}(\mathbb{R}_+\mathcal{A}_G)\}). \tag{8.7}$$

This formula will be used throughout to estimate some a-invariants.

Remark 8.8.15 Let G be a connected non bipartite graph and take a vector β in $\mathrm{ri}(\mathbb{R}_+\mathcal{A}_G)$. If F is a proper face of $\mathbb{R}_+\mathcal{A}_G$, then $\beta \notin F$. This observation holds for all cones because a proper face is always contained in the boundary [39, Theorems 5.3]. In particular $\beta_i > 0$ for all i, because $H_{e_i} \cap \mathbb{R}_+\mathcal{A}_G$ is a proper face.

Definition 8.8.16 The *join* $G_1 * G_2$ of two disjoint graphs G_1 and G_2 consists of $G_1 \cup G_2$ and all the lines joining G_1 with G_2.

Proposition 8.8.17 *Let G_1 and G_2 be two connected graphs on n vertices. If $k[G_1]$ and $k[G_2]$ are normal, then*

$$a(k[G_1 * G_2]) = -n.$$

Proof. Let $V_1 = \{v_1, \ldots, v_n\}$ and $V_2 = \{v_{n+1}, \ldots, v_{2n}\}$ be the vertex set of G_1 and G_2 respectively. One may assume $n \geq 2$. Set $G = G_1 * G_2$. Note that the vector $\beta = (1, \ldots, 1) \in \mathbb{R}^{2n}$ satisfies

$$\sum_{v_i \in A} x_i \leq -1 + \sum_{v_i \in N(A)} x_i \quad \text{and} \quad x_i \geq 1,$$

for all independent sets A of G, because $|A| \leq n - 1 < |N(A)|$. Hence by Corollary 8.8.8 and Lemma 8.3.2 we get $\beta \in (\mathbb{R}_+\mathcal{A}_G)^\circ \cap \mathbb{N}\mathcal{A}_G$. Since $k[G]$ is normal by Corollary 8.7.11, we apply Eq. (8.7) to get $x^\beta \in \omega_{k[G]}$. This implies $-n \leq a(k[G])$. On the other hand by Corollary 7.7.15, we conclude $a(k[G]) = -n$. □

Proposition 8.8.18 *Let G be a connected non bipartite graph on n vertices. If $k[G]$ is normal, then*

$$a(k[G]) - 1 \leq a(k[C(G)]) \leq -\left\lceil \frac{n+1}{2} \right\rceil,$$

where $\lceil x \rceil$ is the least integer greater or equal than x.

Proof. Let ω_1 and ω_2 be the canonical modules of $k[G]$ and $k[C(G)]$ respectively. Set $\mathcal{A} = \mathcal{A}_G$ and $V = V(G)$. By Lemma 8.3.2 one has

$$\mathrm{ri}(\mathbb{R}_+\mathcal{A}) = (\mathbb{R}_+\mathcal{A})^\circ \subset \mathbb{R}^n \quad \text{and} \quad \mathrm{ri}(\mathbb{R}_+\mathcal{A}_{C(G)}) = (\mathbb{R}_+\mathcal{A}_{C(G)})^\circ \subset \mathbb{R}^{n+1}.$$

As $k[G]$ is normal and according to Eq.(8.7), we may pick $\beta \in (\mathbb{R}_+\mathcal{A})^\circ \cap \mathbb{N}\mathcal{A}$ so that $\deg(x^\beta)$ is equal to $-a(k[G])$. Set $\gamma = (\beta_1 + 1, \beta_2, \ldots, \beta_n, 1)$. To prove the first inequality we will show that γ is in $(\mathbb{R}_+\mathcal{A}_{C(G)})^\circ \cap \mathbb{N}\mathcal{A}_{C(G)}$, that is, one must show that γ is not on any facet of $\mathbb{R}_+\mathcal{A}_{C(G)}$.

Let F be a facet of $\mathbb{R}_+\mathcal{A}_{C(G)}$ and H the hyperplane defining F. By Theorem 8.8.7 there are three cases to consider. If $H = H_{e_i}$ for some

$1 \leq i \leq n$, then $\gamma \notin H$; for otherwise β would be on the proper face of $\mathbb{R}_+\mathcal{A}$ determined by H_{e_i}, which contradicts the choice of β (see Remark 8.8.15). If $H = H_{e_{n+1}}$, note $\langle \gamma, e_{n+1} \rangle = 1$ and $\gamma \notin H$. Assume that $H = H_A$, for some independent set in $C(G)$. If $v_{n+1} \in A$, then γ satisfies

$$x_{n+1} \leq -1 + \sum_{v_i \in V} x_i$$

and $\gamma \notin H_A$. Assume $v_{n+1} \notin A$, since β satisfies the inequality

$$\sum_{v_i \in A} x_i \leq -1 + \sum_{v_i \in N_G(A)} x_i,$$

it follows that γ satisfies

$$\sum_{v_i \in A} x_i \leq -1 + \sum_{v_i \in N_{C(G)}(A)} x_i.$$

Hence $\gamma \notin H_A$. Observe that $k[C(G)]$ is normal by Corollary 8.7.11. Thus using Eq.(8.7) one obtains $x^\gamma \in \omega_2$ and the proof of the first inequality is complete. The second inequality follows from Corollary 7.7.15. □

Proposition 8.8.19 *Let G_1 and G_2 be two connected non bipartite graphs. If $k[G_1]$ and $k[G_2]$ are normal, then*

$$a(k[G_1]) + a(k[G_2]) \leq a(k[G_1 * G_2]).$$

Proof. Choose vectors a and b in

$$(\mathbb{R}_+\mathcal{A}_{G_1})^\circ \cap \mathbb{N}\mathcal{A}_{G_1} \quad \text{and} \quad (\mathbb{R}_+\mathcal{A}_{G_2})^\circ \cap \mathbb{N}\mathcal{A}_{G_2}$$

respectively, such that

$$\deg(x^a) = -a(k[G_1]) \quad \text{and} \quad \deg(x^b) = -a(k[G_2]).$$

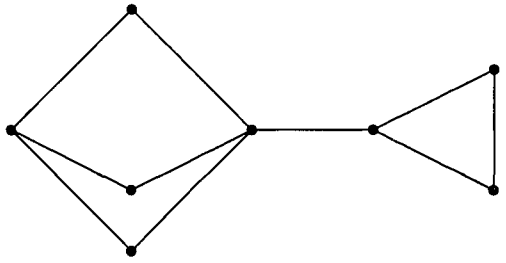
Using arguments similar to those given in the proof of Proposition 8.8.18 it follows that (a, b) is an interior point of $\mathbb{R}_+\mathcal{A}_{G_1 * G_2}$, which implies the asserted inequality. □

Exercises

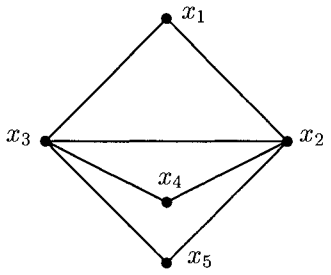
8.8.20 Let $G = G_1 \cup G_2$ be a graph with two non bipartite connected components G_1 and G_2. Find a relation between the irreducible representation of the cone $\mathbb{R}_+\mathcal{A}_G$ and the irreducible representations of the cones $\mathbb{R}_+\mathcal{A}_{G_1}$ and $\mathbb{R}_+\mathcal{A}_{G_2}$.

8.8.21 Let G be a connected non bipartite graph with n vertices. If $k[G]$ is a normal domain, then $x_1 \cdots x_n$ is in $k[G]$ if and only if $(1, \ldots, 1)$ is in the edge cone of G and n is even.

8.8.22 Prove that the following graph does not have a perfect matching by exhibiting an independent set A such that $|A| \not\leq |N(A)|$.



8.8.23 Prove that the following are the equations of the edge cone of the graph G shown on the left.



$$\begin{aligned} -x_1 &\leq 0 \\ -x_4 &\leq 0 \\ -x_5 &\leq 0 \\ x_3 &\leq x_1 + x_2 + x_4 + x_5 \\ x_2 &\leq x_1 + x_3 + x_4 + x_5 \\ x_1 + x_4 + x_5 &\leq x_2 + x_3 \end{aligned}$$

Show that the a-invariant of $k[G]$ is equal to -4.

Chapter 9

Semigroup Rings of Complete Graphs

The purpose of this chapter is to study some of the properties of edge subrings of complete graphs, including the bipartite case. Special attention is given to the computation of their Hilbert series, Gröbner bases and Noether normalizations.

9.1 Monomial subrings of bipartite graphs

Here we treat two special types of monomial subrings attached to complete bipartite graphs and compute their Hilbert series and a-invariant. First we study the edge subring of a complete bipartite graph, and then we examine the Rees algebra of the edge ideal of a complete bipartite graph.

The monomial subring of a bipartite graph

One of the results that simplify the study of an edge subring associated to a complete bipartite graphs is the fact that its toric ideal is a determinantal prime ideal. Let us begin with a classical formula for determinantal ideals.

Theorem 9.1.1 *Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a field K and $I_t(X)$ the ideal generated by the t-minors of X. If $m \leq n$ and $1 \leq t \leq m+1$, then*

$$\operatorname{height}(I_t(X)) = (m-t+1)(n-t+1).$$

Proof. See [52, Theorem 2.5]. □

Proposition 9.1.2 *Let K be a field and $\mathcal{K}_{m,n}$ the complete bipartite graph with vertex set $V = \{x_1, \ldots, x_m, y_1, \ldots, y_n\}$. Then the toric ideal P of $K[\mathcal{K}_{m,n}]$ is generated by the 2×2 minors of a generic $m \times n$ matrix.*

Proof. Let

$$A = K[\{T_{ij} | \, 1 \leq i \leq m, \ 1 \leq j \leq n\}]$$

be a polynomial ring over a field K in the indeterminates T_{ij} and consider the graded homomorphism of K-algebras

$$\psi \colon A \longrightarrow K[\mathcal{K}_{m,n}] = K[x_i y_j\text{'s}], \quad \psi(T_{ij}) = x_i y_j.$$

We claim that the kernel of ψ is generated by the 2×2 minors of a generic $m \times n$ matrix. Indeed let $X = (T_{ij})$. It is clear that the ideal $I_2(X)$, generated by the 2×2 minors of X, is contained in $P = \ker(\psi)$. On the other hand, since the graph is bipartite one has

$$\dim K[\mathcal{K}_{m,n}] = m + n - 1,$$

see Proposition 8.2.12. Therefore

$$\operatorname{height}(P) = (mn) - (m + n - 1) = (m - 1)(n - 1) = \operatorname{height}(I_2(X)),$$

the latter equality by Theorem 9.1.1. Since they are both prime ideals, we have $I_2(X) = P$. □

The Cohen-Macaulayness and Gorensteiness of algebras that include $K[\mathcal{K}_{m,n}]$, has been dealt with in great detail in [50].

Proposition 9.1.3 *Let $\mathcal{K}_{m,n}$ be the complete bipartite graph and A/P the toric ideal of $K[\mathcal{K}_{m,n}]$. If $n \geq m$, then the Hilbert series of A/P is given by*

$$F(A/P, z) = \frac{\displaystyle\sum_{i=0}^{m-1} \binom{m-1}{i} \binom{n-1}{i} z^i}{(1-z)^{n+m-1}}.$$

Proof. Let $V_1 = \{x_1, \ldots, x_m\}$ and $V_2 = \{y_1, \ldots, y_n\}$ be two disjoint sets of vertices, and $R = K[V_1 \cup V_2]$ a polynomial ring over a field K. Let

$$A = K[\{T_{ij} | \, 1 \leq i \leq m, \ 1 \leq j \leq n\}]$$

be a polynomial ring in the indeterminates T_{ij} and consider the graded homomorphism of K-algebras

$$\psi \colon A \longrightarrow K[\mathcal{K}_{m,n}] = K[x_i y_j\text{'s}], \quad \psi(T_{ij}) = x_i y_j.$$

By Proposition 9.1.2 the toric ideal $P = \ker(\psi)$ is equal to $I_2(X)$, the ideal generated by the 2-minors of the $m \times n$ matrix $X = (T_{ij})$. Note that the

dimension of $K[\mathcal{K}_{m,n}]$ is equal to $m+n-1$. The set of the 2×2 minors of X form a Gröbner basis for $I_2(X)$ with respect to the lexicographical ordering induced by

$$T_{11} > \cdots > T_{1n} > \cdots > T_{m1} > \cdots > T_{mn},$$

see [59, 276]. Note that $\text{in}(P)$, the initial ideal of P, is generated by the 2-diagonals of X, that is, one has

$$\text{in}(P) = (T_{ij}T_{k\ell} | \, 1 \leq i < k \leq m, \; 1 \leq j < \ell \leq n).$$

By Corollary 2.4.13 one has

$$F(A/P, z) = F(A/\text{in}(P), z),$$

thus the proof reduces to compute $F(A/\text{in}(P), z)$. Consider the arrays:

$$X_\ell = \begin{array}{ccccc} T_{11} & \cdots & T_{1\ell} & \cdots & T_{1n} \\ \vdots & & \vdots & & \vdots \\ T_{(m-1)1} & \cdots & T_{(m-1)\ell} & \cdots & T_{(m-1)n} \\ T_{m1} & \cdots & T_{m\ell} & & \end{array}$$

and

$$Y_\ell = \begin{array}{ccc} T_{1\ell} & \cdots & T_{1n} \\ T_{2\ell} & \cdots & T_{2n} \\ \vdots & & \vdots \\ T_{(m-1)\ell} & \cdots & T_{(m-1)n}. \end{array}$$

Let I_ℓ and J_ℓ be the ideals of A generated by the 2-diagonals of X_ℓ and Y_ℓ respectively. Note $\text{in}(P) = I_n$. For $2 \leq \ell \leq n$ there are graded exact sequences

$$0 \longrightarrow A/K_\ell[-1] \xrightarrow{T_{m\ell}} A/(T_{m(\ell+1)}, \ldots, T_{mn}, I_\ell) \longrightarrow A/(T_{m\ell}, \ldots, T_{mn}, I_{\ell-1}) \longrightarrow 0,$$

where

$$K_\ell = (T_{ij} | \, i < m, \; j < \ell) + J_\ell + (T_{m(\ell+1)}, \ldots, T_{mn}).$$

As $T_{m1}, \ldots, T_{m\ell}$ is a regular sequence on A/K_ℓ, it is not hard to see that one can write

$$F(A/K_\ell, z) = \frac{h_{(m-1)(n-\ell+1)}}{(1-z)^{m+n-1}},$$

where

$$F(A_\ell/J_\ell, z) = \frac{h_{(m-1)(n-\ell+1)}}{(1-z)^{n+m-\ell-1}} \text{ and } A_\ell = K[T_{ij} | \, i < m, \; \ell \leq j],$$

that is, the polynomial $h_{(m-1)(n-\ell+1)}$ determines the h-vector of A_ℓ/J_ℓ. Hence

$$\begin{aligned} F(A/I_n,z) &= \left(\sum_{\ell=2}^{n} F(A/K_\ell,z)\right) z + F(A/(T_{m2},\ldots,T_{mn},I_1),z) \\ &= \frac{\left(1+h_{(m-1)2}+\cdots+h_{(m-1)(n-1)}\right) z + h_{(m-1)n}}{(1-z)^{m+n-1}}. \end{aligned}$$

Solving this recursive formula for the Hilbert series of A/I_n rapidly leads to the required identity. □

Corollary 9.1.4 *Let $\mathcal{K}_{m,n}$ be the complete bipartite graph. If $n \geq m$, then the a-invariant and the multiplicity of $K[\mathcal{K}_{m,n}]$ are given by:*

$$a(K[\mathcal{K}_{m,n}]) = -n \quad \text{and} \quad e(K[\mathcal{K}_{m,n}]) = \binom{m+n-2}{m-1}.$$

Proof. Use Remark 4.1.11. □

Another approach to the computation of the a-invariant is through the theory of Segre products, and then appealing directly to the results of [132]. Useful methods to compute the a-invariant of some interesting families of algebras, e.g. algebras with straightening laws or Stanley-Reisner rings, are introduced in [42].

Let us state two general results on determinantal rings generalizing Proposition 9.1.3 and Corollary 9.1.4 respectively

Theorem 9.1.5 ([69]) *Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a field K with $m \leq n$ and $I_{r+1}(X)$ the ideal generated by the $(r+1)$-minors of X. Then the Hilbert series of $R_{r+1} = K[x_{ij}]/I_{r+1}(X)$ is:*

$$F(R_{r+1},z) = \frac{\det\left(\sum_k \binom{m-i}{k}\binom{n-j}{k} z^k\right)_{i,j=1,\ldots,r}}{z^{\binom{r}{2}}(1-z)^d} \quad (d = \dim R_{r+1}).$$

Theorem 9.1.6 ([133, 42]) *Let $X = (x_{ij})$ be an $m \times n$ matrix of indeterminates over a field K and $I_r(X)$ the ideal of $K[x_{ij}]$ generated by the r-minors of the matrix X. If $m \leq n$, then*

$$a(K[x_{ij}]/I_r(X)) = -(r-1)n,$$

Proposition 9.1.7 *Let G be an spanning connected subgraph of $\mathcal{K}_{m,n}$. If $n \geq m$, then the a-invariant of $K[G]$ satisfies*

$$a(K[G]) \leq a(K[\mathcal{K}_{m,n}]) = -n.$$

Proof. From Corollary 9.1.4 one has

$$a(K[\mathcal{K}_{m,n}]) = -n.$$

Hence one may proceed as in the proof of Proposition 7.7.14 to rapidly derive the asserted inequality. □

An application of Dedekind-Mertens formula If R is a commutative ring and $f = f(t) \in R[t]$ is a polynomial, say

$$f = a_0 + \cdots + a_m t^m,$$

the *content* of f is the R-ideal $(a_0, \ldots, a_m)$. It is denoted by $c(f)$. Given another polynomial g, the *Gaussian ideal* of f and g is the R-ideal

$$c(fg).$$

This ideal bears a close relationship to the ideal $c(f)c(g)$, one aspect of which is expressed in the classical lemma of Gauss: If R is a principal ideal domain, then

$$c(fg) = c(f)c(g).$$

In general, these two ideals are very different but one aspect of their relationship is given by a formula originally due to Dedekind-Mertens (see [91] and [231]):

$$c(fg)c(g)^m = c(f)c(g)^{m+1}. \tag{9.1}$$

We consider the ideal $c(fg)$ in the case when f and g are generic polynomials. It turns out that some aspects of the theory of Cohen-Macaulay rings show up very naturally when we closely examine $c(fg)$.

One path to our analysis and its applications to Noether normalizations of some semigroup rings, starts by multiplying both sides of (9.1) by $c(f)^m$; we get

$$c(fg)[c(f)c(g)]^m = c(f)c(g)[c(f)c(g)]^m. \tag{9.2}$$

It will be shown that this last formula is sharp in terms of the exponent $m = \deg(f)$.

To make this connection, we recall the notion of a *reduction* of an ideal. Let R be a ring and I an ideal. A *reduction* of I is an ideal $J \subset I$ such that, for some non-negative integer r, the equality

$$I^{r+1} = JI^r$$

holds. The smallest such integer is the *reduction number* $r_J(I)$ of I *relative* to J, and the *reduction number* $r(I)$ of I is the smallest reduction number amongst all reductions J of I.

Thus (9.2) says that $J = c(fg)$ is a reduction for $I = c(f)c(g)$, and that the reduction number is at most $\min\{\deg(f), \deg(g)\}$.

The following result solves the question of Noether normalizations for monomial subrings of complete bipartite graphs.

Theorem 9.1.8 ([91]) *Let $R = k[x_0, \ldots, x_m, y_0, \ldots, y_n]$ be a polynomial ring over a field k and let*

$$h_q = \sum_{i+j=q} x_i y_j.$$

Then

$$A = k[h_0, h_1, \ldots, h_{m+n}] \hookrightarrow S = k[\{x_i y_j \mid 0 \leq i \leq m,\ 0 \leq j \leq n\}]$$

is a Noether normalization of S.

Proof. Let $R[t]$ be a polynomial ring in a new variable t and $f, g \in R[t]$ the generic polynomials

$$f = f(t) = \sum_{i=0}^{m} x_i t^i \quad \text{and} \quad g = g(t) = \sum_{j=0}^{n} y_j t^j.$$

Set $I = c(f)c(g)$ and $J = c(fg)$. Note that J is a reduction of I by Eq. (9.2) and more precisely $JI^m = I^{m+1}$. Therefore

$$R[Jt] \hookrightarrow R[It]$$

is an integral extension; indeed $R[It]$ is generated as an $R[Jt]$-module by a finite set of elements of t-degree at most m. On the other hand, as I and J are generated by homogeneous polynomials of the same degree, one has an integral embedding:

$$R[Jt] \otimes_R R/\mathfrak{m} = k[h_0, \ldots, h_{m+n}] \hookrightarrow R[It] \otimes R/\mathfrak{m} = k[x_i y_j\text{'}s],$$

where $\mathfrak{m} = (x_0 \ldots, x_m, y_0, \ldots, y_n)$. To complete the proof note that the dimension of $k[x_i y_j\text{'}s]$ is equal to $n + m + 1$, by Lemma 8.3.2. □

Theorem 9.1.9 ([72]) *Let $R = k[x_0, \ldots, x_m, y_0, \ldots, y_n]$ be a polynomial ring over a field k and let $f, g \in R[t]$ be the generic polynomials*

$$f = \sum_{i=0}^{m} x_i t^i \quad \text{and} \quad g = \sum_{j=0}^{n} y_j t^j$$

with $n \geq m$. If $I = c(f)c(g)$ and $J = c(fg)$, then $r_J(I) = m$.

Proof. By Theorem 9.1.8 there is a Noether normalization:

$$A = k[h_0, h_1, \ldots, h_{m+n}] \hookrightarrow S = k[x_i y_j\text{'}s],$$

where

$$h_q = \sum_{i+j=q} x_i y_j.$$

Set $I = (f_1, \ldots, f_s)$. We note that the ideal $I = (x_i y_j\text{'}s)$ is the edge ideal associated to a complete bipartite graph G which is the join of two discrete graphs, one with $m+1$ vertices and another with $n+1$ vertices. Since $k[G]$ is Cohen-Macaulay (see Corollary 8.7.11), using Proposition 2.2.14 we get

$$k[G] = Af^{\beta_1} \oplus \cdots \oplus Af^{\beta_k}, \ \ \beta_i \in \mathbb{N}^s.$$

By Corollary 9.1.4 the a-invariant of $k[G]$ is equal to $-n-1$, we can compute the Hilbert series of $k[G]$ using the decomposition above to conclude that

$$\max_{1 \le i \le k} \{\deg(f^{\beta_i})\} = m,$$

where the degree is taken with respect to the normalized grading.

On the other hand assume $r = r_J(I)$ and $JI^r = I^{r+1}$, since we already know the inequality $r \le m$. It suffices to prove $r \ge m$. Note

$$k[G] = \sum_{\alpha \in \mathcal{I}} Af^{\alpha},$$

where

$$\mathcal{I} = \left\{ \alpha \;\middle|\; \alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{N}^s \text{ and } |\alpha| = \sum_{i=1}^{s} \alpha_i \le r \right\}.$$

If $r < m$ this rapidly leads to a contradiction because there is $\mathcal{J} \subset \mathcal{I}$ such that $k[G]$ can be written as

$$k[G] = \bigoplus_{\alpha \in \mathcal{J}} Af^{\alpha}.$$

See Exercise 9.1.18. □

Multiproducts In order to see a different explanation of Eq. (9.2), we extend it to the product of 3 (or more) polynomials, but using the theory of Segre products as a tool.

Let $X = \{x_0, \ldots, x_m\}$, $Y = \{y_0, \ldots, y_n\}$, and $Z = \{z_0, \ldots, z_p\}$ be 3 sets of indeterminates. Defining the polynomials

$$f = \sum_{i=0}^{m} x_i t^i, \quad g = \sum_{j=0}^{n} y_j t^j, \quad \text{and} \quad h = \sum_{k=0}^{p} z_k t^k,$$

one has that $J = c(fgh)$ is a reduction of $I = c(f)c(g)c(h)$ by Dedekind-Mertens formula. If $m \leq n \leq p$, a simple calculation will show

$$r_J(I) \leq m + n.$$

We now resolve this inequality.

Proposition 9.1.10 ([72]) *Let $X = \{x_0, \ldots, x_m\}$, $Y = \{y_0, \ldots, y_n\}$, and $Z = \{z_0, \ldots, z_p\}$ be sets of distinct indeterminates, let $R = k[X, Y, Z]$ be a polynomial ring over a field k, and let*

$$I = (x_i y_j z_k | \, x_i \in X, \, y_j \in Y, \, z_k \in Z).$$

Then I is a normal ideal of R.

Proof. We will show that I^q is complete for all $q \geq 1$. Let I_a^q be the integral closure of I^q and let $f \in I_a^q$ be a monomial. We write

$$f = x_{i_1}^{a_1} \cdots x_{i_r}^{a_r} y_{j_1}^{b_1} \cdots y_{j_s}^{b_s} z_{k_1}^{c_1} \cdots z_{k_t}^{c_t}.$$

Since $f^w \in I^{sq}$ for some $w > 0$ we can write

$$f^w = x_{q_1}^{d_1} \cdots x_{q_\lambda}^{d_\lambda} M,$$

where M is a monomial whose support is contained in $Y \cup Z$. Therefore we obtain

$$w \sum_{i=1}^{r} a_i = \sum_{i=1}^{\lambda} d_i \geq wq,$$

which implies

$$\sum_{i=1}^{r} a_i \geq q.$$

A similar argument shows

$$\sum_{i=1}^{s} b_i \geq q \quad \text{and} \quad \sum_{i=1}^{t} c_i \geq q.$$

Therefore $f \in I^q$. □

By Theorem 7.2.35, the Rees algebra $R[It]$ is a Cohen-Macaulay ring. Since

$$\mathcal{F}(I) = k[x_i y_j z_k | \, x_i \in X, \, y_j \in Y, \, z_k \in Z]$$

is a normal domain by Proposition 7.4.1, and therefore Cohen-Macaulay by Theorem 7.2.35. We may thus more easily compute the reduction number of the subring $\mathcal{F}(I)$.

Theorem 9.1.11 *The reduction number $r_J(I)$ of the ideal I above is $m+n$.*

Proof. Since $\mathcal{F}(I)$ is a Cohen-Macaulay algebra, its reduction number can also be obtained from the degrees of the generators of its canonical module. But $\mathcal{F}(I)$ is a Segre product of standard Cohen-Macaulay algebras and the canonical module is given by an explicit formula from the canonical modules of the factors. Entering the data in [132, Theorem 4.3.1], we get $r_J(I) = m + n$. □

Rees algebras of complete bipartite graphs

The following result show that Rees algebras of edge ideals associated to complete bipartite graphs have some structure; its proof makes use of general facts about ladder ideals that can be found in the literature.

Proposition 9.1.12 *Let B/Q be the presentation of $\mathcal{R}(I)$, the Rees algebra of the edge ideal $I = (x_iy_j | 1 \leq i \leq m, 1 \leq j \leq n)$. Then the toric ideal Q is generated by the 2-minors of the ladder*

$$Y = \begin{array}{c} \\ x_1 \\ \vdots \\ x_m \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & \cdots & y_n \end{array} \\ \begin{array}{cccc} T_{11} & T_{12} & \cdots & T_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ T_{m1} & T_{m2} & \cdots & T_{mn} \end{array} \end{array}$$

and the 2-minors of Y form a Gröbner basis w.r.t the lex ordering induced by $y_1 > \cdots > y_n > x_1 > T_{11} > \cdots > T_{1n} > x_2 > \cdots > T_{mn}$.

Proof. Consider the presentation of the Rees algebra of I:

$$\psi : B = K[x_i\text{'s}, y_j\text{'s}, T_{ij}\text{'s}] \longrightarrow \mathcal{R}(I), \quad \psi(T_{ij}) = Tx_iy_j,$$

Note $I_2(Y) \subset Q = \ker(\psi)$. As they are both prime ideals of height $mn - 1$ by [67, Section 4] and [228] (cf. [65, Proposition 3.3.1]), one has equality.

The assertion that the 2-minors are a Gröbner basis of $I_2(Y)$ is a general fact of ladder ideals [228]. □

Proposition 9.1.13 *Let B/Q be the presentation of $\mathcal{R}(I)$, the Rees algebra of the edge ideal $I = (x_iy_j | 1 \leq i \leq m, 1 \leq j \leq n)$. If $n \geq m$, then the Hilbert series of B/Q can be written as*

$$F(B/Q, z) = \frac{1 + (mn-1)z + \sum_{i=2}^{m} \binom{n}{i}\binom{m}{i} z^i}{(1-z)^{n+m+1}}.$$

In particular $a(B/Q) = -(n+1)$ and $e(B/Q) = \binom{m+n}{m} - 1$.

Proof. Let

$$B = K[x_i, y_j, T_{ij} | \, 1 \leq i \leq m, \, 1 \leq j \leq n]$$

be a polynomial ring over a field K and consider the ladders

$$M_\ell = \begin{array}{c|cccc} & y_1 & y_2 & \cdots & y_n \\ & T_{11} & T_{12} & \cdots & T_{1n} \\ & \vdots & \vdots & & \vdots \\ x_\ell & T_{\ell 1} & T_{\ell 2} & \cdots & T_{\ell n} \\ \vdots & \vdots & \vdots & & \vdots \\ x_m & T_{m1} & T_{m2} & \cdots & T_{mn} \end{array} \qquad N_\ell = \begin{array}{cccc} y_1 & y_2 & \cdots & y_n \\ T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ T_{\ell 1} & T_{\ell 2} & \cdots & T_{\ell n}. \end{array}$$

Let I_ℓ and J_ℓ be the ideals generated by the 2-diagonals of M_ℓ and N_ℓ respectively, where a 2-diagonal is the product of the diagonal entries of a 2×2 submatrix.

By Proposition 9.1.12 and Corollary 2.4.13 one has that the Hilbert series of B/Q is equal to the Hilbert series of B/I_1. Hence the proof reduces to compute $F(B/I_1, z)$.

For $1 \leq \ell \leq m-1$ consider the graded exact sequences

$$0 \longrightarrow B/K_\ell[-1] \xrightarrow{x_\ell} B/(x_1, \ldots, x_{\ell-1}, I_\ell) \longrightarrow B/(x_1, \ldots, x_\ell, I_{\ell+1}) \longrightarrow 0,$$

where

$$K_\ell = (T_{ij} | \, \ell + 1 \leq i) + J_\ell + (x_1, \ldots, x_{\ell-1}).$$

Therefore

$$F(B/I_1, z) = \left(\sum_{\ell=1}^{m-1} F(B/K_\ell, z) \right) z + F(B/(x_1, \ldots, x_{m-1}, I_m), z).$$

As $x_\ell, \ldots, x_m$ is a regular sequence on B/K_ℓ using Proposition 9.1.3 one readily obtains

$$F(B/K_\ell, z) = \frac{\sum_{i=0}^{\ell} \binom{\ell}{i} \binom{n-1}{i} z^i}{(1-z)^{m+n+1}},$$

and consequently

$$F(B/I_1, z) = \frac{\left(\sum_{\ell=1}^{m-1} \left(\sum_{i=0}^{\ell} \binom{\ell}{i} \binom{n-1}{i} z^i \right) \right) z + \sum_{i=0}^{m} \binom{m}{i} \binom{n-1}{i} z^i}{(1-z)^{m+n+1}}.$$

Using the formula

$$\sum_{i=0}^{n} \binom{x+i}{i} = \binom{x+n+1}{n}$$

yields the required equality. □

Theorem 9.1.14 ([306]) *Let*

$$S = K[\{x_iy_j, x_ix, y_jx | \, 1 \leq i \leq m, \; 1 \leq j \leq n\}]$$

be the monomial subring of the cone over a complete bipartite graph $\mathcal{K}_{mn}$. If $n \geq m \geq 2$, then the canonical module ω_S of S is minimally generated as follows:

(a) *If $n = m$, then*

$$\omega_S = (x_1 \cdots x_n y_1 \cdots y_n x^{2i} | \, 1 \leq i \leq n-1).$$

(b) *If $n \geq m+1$, then ω_S is minimally generated by the set of monomials*:

(i) $x_1 \cdots x_m y_1 \cdots y_n x^b$, *where* $n - m + 1 \leq b \leq n + m - 1$ *and* $n + m + b \in 2\mathbb{N}$, *or*

(ii) $x_1^{a_1} \cdots x_m^{a_m} y_1 \cdots y_n x^b$, *where*

$$\sum_{i=1}^{m} a_i = n - b + 2,$$

$b = 2, 3, \ldots, n - m + 1$, *and* $a_i \geq 1$ *for all* i.

Proof. For a monomial $M = x_1^{a_1} \cdots x_m^{a_m} y_1^{b_1} \cdots y_n^{b_n} x^b$, set

$$\log(M) = (a_1, \ldots, a_m, b_1, \ldots, b_n, b).$$

Define

$$\mathcal{A} = \{\log(M) | \, M \in \{x_iy_j, x_ix, y_jx | 1 \leq i \leq m, 1 \leq j \leq n\}\}.$$

As S is normal of dimension $n+m+1$ (see Theorem8.7.9 and Lemma 8.3.2) one can use the Danilov-Stanley formula (see Theorem 7.2.34) to express its canonical module as:

$$\omega_S = (\{M = x_1^{a_1} \cdots x_m^{a_m} y_1^{b_1} \cdots y_n^{b_n} x^b | \, \log(M) \in \mathbb{N}\mathcal{A} \cap (\mathbb{R}_+\mathcal{A})^\circ\}),$$

where $(\mathbb{R}_+\mathcal{A})^\circ$ is the interior of the cone generated by $\mathcal{A}$ and S has the normalized grading. Let

$$M = x_1^{a_1} \cdots x_m^{a_m} y_1^{b_1} \cdots y_n^{b_n} x^b$$

be a minimal generator of the ideal ω_S. According to Theorem 8.8.7 a vector $\beta \in \mathbb{N}^{m+n+1}$ is in $(\mathbb{R}_+\mathcal{A})^\circ$ if and only if β satisfies the inequalities:

$$\begin{aligned} 1 &\leq \beta_i,\ i = 1, \ldots, m+n \\ \sum_{i=1}^{m} \beta_i &\leq -1 + \beta_{m+n+1} + \sum_{i=m+1}^{m+n} \beta_i \\ \sum_{i=m+1}^{m+n} \beta_i &\leq -1 + \beta_{m+n+1} + \sum_{i=1}^{m} \beta_i \\ \beta_{m+n+1} &\leq -1 + \sum_{i=1}^{m+n} \beta_i, \end{aligned} \tag{9.3}$$

Therefore one has:

$$\begin{aligned} 1 \leq a_i,\quad 1 \leq b_j,\quad 1 \leq i \leq m,\quad 1 \leq j \leq n \\ \sum_{i=1}^{m} a_i &\leq -1 + b + \sum_{i=1}^{n} b_i \\ \sum_{i=1}^{n} b_i &\leq -1 + b + \sum_{i=1}^{m} a_i \\ b &\leq -1 + \sum_{i=1}^{m} a_i + \sum_{i=1}^{n} b_i \end{aligned} \tag{9.4}$$

Using that $\alpha \in \mathbb{N}\mathcal{A}$ one also obtains:

$$b + \sum_{i=1}^{m} a_i + \sum_{i=1}^{n} b_i \in 2\mathbb{N} \tag{9.5}$$

Observe that (9.4) and (9.5) yield $b \geq 2$. We consider the following cases:

I. Assume $a_i \geq 2$ for some i. We claim that $b_i = 1$, for all i. For simplicity assume $a_1 \geq 2$ and $b_1 \geq 2$. Write:

$$b = -i + \sum_{i=1}^{m} a_i + \sum_{i=1}^{n} b_i,\ i \geq 1.$$

If $1 \leq i \leq 2$, write $M = M' y_1 x$. Since $\log M = \log M' + \log y_1 x$ one has that $\log M'$ is in $\mathbb{Z}\mathcal{A}$. It is not hard to see that $\log M'$ satisfies all the equations in (9.3), thus $\log M' \in (\mathbb{R}_+\mathcal{A})^\circ$. By the normality of S one concludes

$\log M' \in \mathbb{N}\mathcal{A}$, hence M' is in ω_S, which contradicts the minimality of M. If $i \geq 3$, consider $M = M'x_1y_1$, as before one readily obtains $M' \in \omega_S$, which again contradicts the choice of M.

II. Assume $n = m$. First let us show $a_i = 1$ and $b_j = 1$, for all i, j. By the previous case and symmetry if $a_i \geq 2$ for some i (resp. $b_i \geq 2$), then $b_i = 1$ for all i (resp. $a_i = 1$ for all i). If $a_i \geq 2$ for some i, say $i = 1$, then $M' \in \omega_S$, where $M = M'x_1x$, and this is impossible by the choice of M. Next, using (9.4) and (9.5), it follows that $b = 2i$ for some $1 \leq i \leq n-1$, and we are in case (a).

III. Assume $n \geq m+1 \geq 3$. We claim that $b_i = 1$, for all i. If $b_i \geq 2$, say $i = 1$, then by case I one has $a_i = 1$ for all i. As before one readily derives a contradiction by writing $M = M'y_1x$. Altogether one can write:

$$M = x_1^{a_1} \cdots x_m^{a_m} y_1 \cdots y_n x^b, \quad b \geq 2.$$

If $a_i = 1$ for all i, then using (9.4) and (9.5) one has

$$n - m + 1 \leq b \leq m + n - 1 \quad \text{and} \quad m + n + b \in 2\mathbb{N},$$

and we are in case (b). Next assume $a_i \geq 2$ for some i, say $i = 1$. Combining (9.4) and (9.5) one has

$$n \leq \sum_{i=1}^{m} a_i + b - 2.$$

If the inequality

$$n \leq \sum_{i=1}^{m} a_i + b - 3$$

holds, one derives a contradiction by considering $M = M'x_1x$. Therefore

$$m + 1 \leq \sum_{i=1}^{m} a_i = n - b + 2$$

for $b = 2, \ldots, n - m + 1$, and we are in case (b).

Thus any minimal generator of ω_S must be as in (a) or (b). To complete the argument observe that any of the monomials occurring in (a) or (b) are in ω_S and they are not multiple of each other. □

Remark 9.1.15 If $n > m = 1$, then assertion (b) above still holds. To show it replace the very first set of inequalities in (9.3) by $\beta_i \geq 1$, $i = 2, \ldots, n+m$. On the other hand if $n = m = 1$, then $\omega_S = (x_1^2y_1^2x^2)$.

Corollary 9.1.16 *Let $S = \mathcal{R}(I)$ be the Rees algebra of the edge ideal I of the complete bipartite graph $\mathcal{K}_{mn}$. If $n \geq m \geq 2$ or $n > m = 1$, then*

$$\operatorname{type}(S) = \binom{n}{m} + (m-2).$$

Proof. If $n = m \geq 2$ or if $n > m = 1$ the formula follows rapidly from Theorem 9.1.14(a) and Remark 9.1.15. Assume $n \geq m+1 \geq 3$. Observe that there are $m-1$ generators of ω_S as in Theorem 9.1.14(i), to count the remaining generators of ω_S as in Theorem 9.1.14(ii) note that the sum from $b = 2$ to $b = n - m + 1$ of the m-partitions of $n - b + 2$ is equal to

$$\sum_{b=2}^{n-m+1} \binom{n-b+1}{m-1} = \sum_{i=m+1}^{n} \binom{i-1}{m-1}$$
$$= -1 + \sum_{i=0}^{n-m} \binom{m-1+i}{m-1} = -1 + \binom{n}{m}.$$

Hence

$$\text{type}(S) = (m-1) + \binom{n}{m} - 1 = \binom{n}{m} + (m-2),$$

as required. □

Exercises

9.1.17 Let $R[t]$ be a polynomial ring over a ring R. If $f = f(t)$ and $g = g(t)$ are two polynomials in $R[t]$, prove the Dedekind-Mertens formula

$$c(fg)c(g)^m = c(f)c(g)^{m+1} \quad (m = \deg(f)).$$

9.1.18 Let M be a graded free module over a polynomial ring A such that $M = Af_1 \oplus \cdots \oplus Af_r$, where f_i is homogeneous for all i. If M is generated by homogeneous elements $g_1, \ldots, g_s$, then $M = Ag_{i_1} \oplus \cdots \oplus Ag_{i_r}$, for some $i_1, \ldots, i_r$.

9.1.19 Let $A = k[x_1, \ldots, x_m]$ and $R = k[y_1, \ldots, y_n]$ be two polynomial rings over a field k, prove that the Segre product

$$(A_0 \otimes_k R_0) \oplus (A_1 \otimes_k R_1) \oplus \cdots \subset A \otimes_k R$$

is isomorphic to $S = k[x_i y_j\text{'s}] \subset k[x_i\text{'s}, y_j\text{'s}]$.

9.1.20 Let A and R be two standard algebras over a field k of dimensions m and n respectively. If S is the Segre product of A and R, prove the following relation between the multiplicities

$$e(S) = e(A)e(R)\binom{m+n-2}{m-1}.$$

9.1.21 Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and $\mathfrak{m} = (x_1, \ldots, x_n)$. Prove that the toric ideal of the Rees algebra $\mathcal{R}(\mathfrak{m})$ is the ideal generated by the 2-minors of the following matrix of indeterminates:

$$X = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \\ T_1 & T_2 & \cdots & T_n \end{pmatrix}.$$

9.1.22 Let $R = K[x_1, \ldots, x_4]$ be a polynomial ring over a field K and I the 3rd square-free Veronese subring of R generated by the monomials

$$x_1x_2x_3,\ x_1x_2x_4,\ x_1x_3x_4,\ x_2x_3x_4.$$

Prove that the Rees algebra $\mathcal{R}(I)$ is a complete intersection.

9.1.23 Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and $I = (x_1, \ldots, x_n)^k$. Set $s = -\lceil n/k \rceil + n$, and write $n = qk + r_1$, for some $q, r_1 \in \mathbb{N}$ such that $0 \leq k < r_1$. By a "counting degrees" argument prove:

$$\begin{aligned} x_1^{k-1} \cdots x_n^{k-1} &\in I^s \qquad \text{if } r_1 = 0, \\ x_1^{k-1} \cdots x_q^{k-1} x_{q+1}^{r_1-1} x_{q+2}^{k-1} \cdots x_n^{k-1} &\in I^s \qquad \text{if } r_1 > 0. \end{aligned}$$

9.1.24 Let $R = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K and $\mathfrak{m} = (x_1, \ldots, x_n)$. If $I = \mathfrak{m}^k$ and $J = (x_1^k, \ldots, x_n^k)$, then J is a reduction of I and the reduction number of I relative to J is given by

$$r_J(I) = -\left\lceil \frac{n}{k} \right\rceil + n, \quad \text{where } \lceil x \rceil \text{ is the ceiling of } x.$$

9.1.25 Let S be a standard algebra over an infinite field K and

$$A = K[h_1, \ldots, h_d] \hookrightarrow S$$

a Noether normalization of S with $h_i \in S_1$ for all i. If $b_1, \ldots, b_t$ is a minimal set of homogeneous generators of S and as an A-module, prove

$$(h_1, \ldots, h_d)S_r = S_{r+1},$$

where $r = \max_i\{\deg(b_i)\}$ and that $r \geq 0$ is the minimum integer where equality occurs. The *reduction number* of S, denote by $r(S)$, is the minimum r taken over all Noether normalizations. See [298, Chapter 9] for an study of several degrees of complexity associated to a graded module.

9.1.26 Let S be a standard algebra over an infinite field K and $a(S)$ its a-invariant. If S is Cohen-Macaulay prove:

$$r(S) = a(S) + \dim(S).$$

9.1.27 Let $\phi: B \to S$ be a graded epimorphism of standard K-algebras, where K is an infinite field. If $\dim(B) = \dim(S)$ and

$$A = K[h_1, \ldots, h_d] \overset{i}{\hookrightarrow} B$$

is a Noether normalization of B with $h_i \in S_1$ for all i, prove that S is integral over $A' = K[\phi(h_1), \ldots, \phi(h_d)]$ and that the composition

$$A = K[h_1, \ldots, h_d] \overset{i}{\hookrightarrow} B \xrightarrow{\phi} S$$

is a Noether normalization of S.

9.1.28 Let $\phi: B \to S$ be a graded epimorphism of standard K-algebras, where K is an infinite field. If $\dim(B) = \dim(S)$, prove $r(B) \geq r(S)$.

9.2 Monomial subrings of complete graphs

Lexicographical Gröbner bases and Hilbert functions of toric ideals of edge subrings associated to complete graphs will be considered here.

Let us recall some terminology and fix some of the notation that will be used throughout the section.

The letters i, j, k, ℓ, n, r will denote positive integers. We denote by G_{σ_n}, the graph with vertex set

$$V_n = \{T_{ij} | \, (i,j) \in \sigma_n\}$$

and edge set

$$E = \{\{T_{ij}, T_{k\ell}\} | \, (i,j), (k,\ell) \in \sigma_n, \{i,j\} \cap \{k,\ell\} = \emptyset, i < k \text{ and } j < \ell\},$$

where

$$\sigma_n = \{(i,j) | \, 1 \leq i < j \leq n\}.$$

Given a set $D \subset V_n$, we denote by G_D the maximal subgraph of G_{σ_n} with vertex set D.

Let $A_n = k[V_n]$ be a polynomial ring over a field k and G a graph on the vertex set V_n. The *edge-ideal* $I(G)$, of the graph G, is the ideal of A_n generated by the square free monomials $T_{ij}T_{k\ell}$ so that T_{ij} is adjacent to $T_{k\ell}$ in G. If all the vertices of a graph G are isolated we set $I(G) = (0)$.

The graph G is called Cohen-Macaulay (C-M for short) if $I(G)$ is a C-M ideal. A subset $C \subset V_n$ is a *minimal vertex cover* for G if every edge of G is incident with some vertex in C and there is no proper subset of C with this property.

Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over the field k, and $\mathcal{K}_n$ the complete graph on the vertex set $X = \{x_1, \ldots, x_n\}$. For $i < j$ we set $f_{ij} = x_i x_j$. Let

$$k[\mathcal{K}_n] = k[\{f_{ij} \mid 1 \leq i < j \leq n\}]$$

be the k-subring of R spanned by the f_{ij}. Consider the homomorphism

$$\psi\colon A_n = k[\{T_{ij} \mid 1 \leq i < j \leq n\}] \to k[\mathcal{K}_n], \text{ induced by } T_{ij} \to f_{ij}.$$

The ideal $P_n = \ker(\psi)$ is the *presentation ideal* or *toric ideal* of $k[\mathcal{K}_n]$.

Proposition 9.2.1 ([304]) *Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and P_n the presentation ideal of $k[\mathcal{K}_n]$. Set*

$$\mathcal{B} = \{T_{ij}T_{k\ell} - T_{i\ell}T_{jk},\ T_{ik}T_{j\ell} - T_{i\ell}T_{jk} \mid 1 \leq i < j < k < \ell \leq n\}.$$

If the terms in $A_n = k[\{T_{ij} \mid 1 \leq i < j \leq n\}]$ are ordered lexicographically by $T_{12} < \cdots < T_{1n} < T_{23} < \cdots < T_{2n} < \cdots < T_{(n-1)n}$, then $\mathcal{B}$ is a minimal generating set for P_n and $\mathcal{B}$ is a reduced Gröbner basis for P_n.

Proof. By Corollary 7.1.7 it follows that $\mathcal{B}$ is a minimal generating set for P_n. Let $f, g \in \mathcal{B}$. Taking into account Theorem 2.4.15, it suffices to verify that $S(f,g)$, the S-polynomial of f and g, reduces to zero with respect to $\mathcal{B}$. There are two sequences of positive integers $i < j < k < \ell$ and $p < q < r < s$ so that

$$f = \begin{cases} \text{(a)} & T_{ij}T_{k\ell} - T_{i\ell}T_{jk} \quad \text{or,} \\ \text{(b)} & T_{j\ell}T_{ik} - T_{i\ell}T_{jk}. \end{cases} \qquad g = \begin{cases} \text{(c)} & T_{pq}T_{rs} - T_{ps}T_{qr} \quad \text{or,} \\ \text{(d)} & T_{pr}T_{qs} - T_{ps}T_{qr}. \end{cases}$$

Assume f and g as in (a) and (c) respectively, the others cases can be treated similarly. We may assume that the leading terms of f and g have common factors, for otherwise $S(f,g)$ reduces to zero w.r.t $\{f,g\}$ by Lemma 2.4.14. First we assume $T_{ij} = T_{pq}$, in this case $i = p$ and $j = q$, and the S-polynomial of f and g is equal to $S(f,g) = -T_{rs}T_{p\ell}T_{qk} + T_{k\ell}T_{ps}T_{qr}$. Observe that

$$S(f,g) = \begin{cases} T_{qr}(T_{\ell r}T_{ps} - T_{rs}T_{p\ell}) & \text{if } k = r, \\ T_{ps}(T_{qr}T_{s\ell} - T_{rs}T_{q\ell}) + T_{rs}(-T_{p\ell}T_{qs} + T_{ps}T_{q\ell}) & \text{if } k = s, \\ T_{kr}(T_{ps}T_{qr} - T_{pr}T_{qs}) - T_{pr}(T_{rs}T_{qk} - T_{qs}T_{kr}) & \text{if } \ell = r, \\ T_{ps}(-T_{rs}T_{qk} + T_{sk}T_{qr}) & \text{if } \ell = s. \end{cases}$$

Hence $S(f,g) \to_{\mathcal{B}} 0$ if $\{k,\ell\} \cap \{r,s\} \neq \emptyset$. We may then assume k and ℓ not in $\{r,s\}$. If $\ell > s$ and $r > k$ then

$$S(f,g) - T_{ps}(T_{k\ell}T_{qr} - T_{q\ell}T_{kr}) - T_{kr}(T_{ps}T_{q\ell} - T_{p\ell}T_{qs}) + T_{p\ell}(T_{rs}T_{qk} - T_{kr}T_{qs}) = 0.$$

On the other hand if $\ell > s$ and $k > r$ we have

$$S(f,g) - T_{ps}(T_{k\ell}T_{qr} - T_{q\ell}T_{rk}) - T_{rk}(T_{ps}T_{q\ell} - T_{p\ell}T_{qs}) \\ +T_{p\ell}(T_{rs}T_{qk} - T_{rk}T_{qs}) = 0.$$

In both cases $S(f,g) \to_{\mathcal{B}} 0$. Assume $s > \ell$ and $k > r$. From the equality

$$S(f,g) - T_{ps}(T_{qr}T_{k\ell} - T_{q\ell}T_{rk}) + T_{p\ell}(T_{qk}T_{rs} - T_{qs}T_{rk}) \\ -T_{rk}(T_{ps}T_{q\ell} - T_{p\ell}T_{qs}) = 0,$$

we derive $S(f,g) \to_{\mathcal{B}} 0$. To complete the argument consider the case $s > \ell$ and $r > k$, using the equality

$$S(f,g) + T_{p\ell}(T_{rs}T_{qk} - T_{qs}T_{kr}) + T_{kr}(T_{p\ell}T_{qs} - T_{ps}T_{q\ell}) \\ -T_{ps}(T_{k\ell}T_{qr} - T_{kr}T_{q\ell}) = 0,$$

we obtain $S(f,g) \to_{\mathcal{B}} 0$.

Now consider the case $T_{ij} = T_{rs}$, note that the case $T_{k\ell} = T_{pq}$ is symmetric. We have $i = r$, $j = s$ and $S(f,g) = T_{k\ell}T_{ps}T_{qr} - T_{pq}T_{r\ell}T_{sk}$. Since

$$S(f,g) - T_{qr}(T_{ps}T_{k\ell} - T_{p\ell}T_{sk}) - T_{sk}(T_{qr}T_{p\ell} - T_{pq}T_{r\ell}) = 0,$$

we obtain $S(f,g) \to_{\mathcal{B}} 0$. Finally we show the case $T_{k\ell} = T_{rs}$, notice $k = r$, $\ell = s$ and

$$S(f,g) = T_{ij}T_{ps}T_{qr} - T_{pq}T_{is}T_{jr}.$$

If $i \geq q$ then

$$S(f,g) + T_{is}(T_{pq}T_{jr} - T_{pr}T_{qj}) - T_{qj}(T_{qs}T_{pr} - T_{ps}T_{qr}) = 0,$$

and $S(f,g) \to_{\mathcal{B}} 0$. If $i < q$, then

$$S(f,g) + T_{is}(T_{pq}T_{jr} - T_{pr}T_{qj}) - T_{ps}(T_{ij}T_{qr} - T_{ir}T_{qj}) \\ -T_{qj}(T_{ps}T_{ir} - T_{is}T_{pr}) = 0,$$

and $S(f,g) \to_{\mathcal{B}} 0$. □

Example 9.2.2 If $k[\{T_{ij} | 1 \leq i < j \leq 5\}]$ has the lex ordering induced by

$$T_{45} < T_{34} < T_{35} < T_{23} < T_{24} < T_{25} < T_{12} < T_{13} < T_{14} < T_{15}.$$

Then the reduced Gröbner basis for the toric ideal P of $k[\mathcal{K}_5]$ is equal to:

$$\begin{array}{lll} T_{35}T_{24} - T_{45}T_{23}, & T_{34}T_{25} - T_{45}T_{23}, & T_{45}T_{23}T_{13} - T_{34}T_{35}T_{12}, \\ T_{25}T_{13} - T_{35}T_{12}, & T_{35}T_{14} - T_{45}T_{13}, & T_{23}T_{14} - T_{34}T_{12}, \\ T_{34}T_{15} - T_{45}T_{13}, & T_{23}T_{15} - T_{35}T_{12}, & T_{24}T_{15} - T_{45}T_{12}. \\ T_{24}T_{13} - T_{34}T_{12} & T_{25}T_{14} - T_{45}T_{12}. & \end{array}$$

Corollary 9.2.3 *If P_n is the toric ideal of the subring $k[\mathcal{K}_n]$ and $n \geq 4$, then P_n is the ideal $I_2(Y)$ generated by the 2-minors of the symmetric array*

$$Y = \begin{array}{ccccccc} \bullet & T_{12} & T_{13} & \cdots & T_{1(n-2)} & T_{1(n-1)} & T_{1n} \\ T_{12} & \bullet & T_{23} & \cdots & T_{2(n-2)} & T_{2(n-1)} & T_{2n} \\ T_{13} & T_{23} & \bullet & \cdots & T_{3(n-2)} & T_{3(n-1)} & T_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ T_{1(n-2)} & T_{2(n-2)} & T_{3(n-2)} & \cdots & \bullet & T_{(n-2)(n-1)} & T_{(n-2)n} \\ T_{1(n-1)} & T_{2(n-1)} & T_{3(n-1)} & \cdots & T_{(n-2)(n-1)} & \bullet & T_{(n-1)n} \\ T_{1n} & T_{2n} & T_{3n} & \cdots & T_{(n-2)n} & T_{(n-1)n} & \bullet \end{array},$$

and the 2-minors of Y form a Gröbner basis for $I_2(Y)$ w.r.t the lex ordering $T_{12} < \cdots < T_{1(n-1)} < T_{1n} < T_{23} < \cdots < T_{2(n-1)} < T_{2n} < \cdots < T_{(n-1)n}$.

Lemma 9.2.4 *Let $n \geq 4$ be a fixed integer, and let r be positive integer so that $1 \leq r \leq n-1$. Set*

$$D'_r = \{T_{ij} \in V_n | \, r \leq i \leq n-2, r+1 \leq j \leq n-1\},$$

$$D''_{kr} = \{T_{ir} \in V_n | \, k \leq i \leq r-1\} \cup \{T_{in} | \, k \leq i \leq r-1\}, \text{ and}$$

$$D_{kr} = D'_r \cup D''_{kr}.$$

If $I(G_{\sigma_i})$ is a Cohen-Macaulay ideal of height

$$\frac{i(i-3)}{2}$$

for all $i < n$, then $I(G_{D_{kr}})$ is a Cohen-Macaulay ideal of height

$$\binom{n-r-1}{2} + r - k - 1$$

for $1 \leq k \leq r$.

Proof. We proceed by descending induction on k. If $k = r$ we have $D''_{rr} = \emptyset$ and $G_{D_{rr}} \simeq G_{\sigma_{n-r}}$. Assume $k \leq r-1$ and $G_{D_{(k+1)r}}$ Cohen-Macaulay of the appropriate height. Let

$$N = \{T_{ij} \in D_{kr} | \, T_{ij} \text{ is adjacent to } T_{kr}\}.$$

Since N is equal to

$$\{T_{ij} \in D_{kr} | \, r+1 \leq i \leq n-2, r+2 \leq j \leq n-1\} \cup \{T_{in} | \, k+1 \leq i \leq r-1\},$$

it follows that

$$|N| = \binom{n-r-1}{2} + r - k - 1$$

and that N is a minimal vertex cover for $G_{D_{kr}}$. Therefore

$$(I(G_{D_{kr}}), N) = (N) \text{ and } (I(G_{D_{kr}}), T_{kr}) = (I(G_{D_{(k+1)r}}), T_{kr}).$$

Observe that $T_{(k+1)n}$ is an isolated vertex in $G_{D_{(k+1)r}}$, hence N can not be contained in a minimal vertex cover for $G_{D_{(k+1)r}}$ and

$$\operatorname{ht} I(G_{D_{kr}}) = \operatorname{ht} I(G_{D_{(k+1)r}}) + 1.$$

Consider the exact sequence

$$0 \longrightarrow A_n/(I(G_{D_{kr}}), N) \xrightarrow{T_{kn}} A_n/I(G_{D_{kr}}) \longrightarrow A_n/(I(G_{D_{kr}}), T_{kr}) \longrightarrow 0.$$

Since the ends of this sequence are both Cohen-Macaulay rings of dimension equal to $\dim A_n/I(G_{D_{kr}})$ we derive that $A_n/I(G_{D_{kr}})$ is CM, as required. □

Corollary 9.2.5 ([280]) *Let*

$$I(G_{D_{1(n-1)}}) = (\{T_{i(n-1)}T_{jn} | \, 1 \leq i < j < n-1\}).$$

Then $I(G_{D_{1(n-1)}})$ is Cohen-Macaulay of height $n-3$ for $n \geq 4$.

Proof. Take $r = n-1$ and $k = 1$ in Lemma 9.2.4. □

Theorem 9.2.6 *The edge ideal $I(G_{\sigma_n})$ is Cohen-Macaulay of height equal to $n(n-3)/2$.*

Proof. The proof is by induction on n. The assertion is not hard to verify for $n \leq 5$. Assume the result for all graphs isomorphic to G_{σ_i} for $i < n$. Set

$$C_r = V_{n-1} \cup \{T_{1n}, T_{2n}, \ldots, T_{rn}\}.$$

We claim that G_{C_r} is Cohen-Macaulay with

$$\operatorname{ht} I(G_{C_r}) = (r-1) + \frac{(n-1)(n-4)}{2},$$

for $1 \leq r \leq n-1$. The proof of the claim is by induction on r. If $r = 1$ then $I(G_{C_1}) = I(G_{\sigma_{n-1}})$ and, by the first induction hypothesis, $I(G_{C_1})$ is CM of height $(n-1)(n-4)/2$. Assume G_{C_i} CM with

$$\operatorname{ht} I(G_{C_i}) = (i-1) + \frac{(n-1)(n-4)}{2}$$

for $i < r$. Let

$$U_r = \{T_{ij} \in C_r | \, T_{ij} \text{ is adjacent to } T_{rn}\},$$

and let D_{1r} be as in Lemma 9.2.4. Observe that

$$|U_r| = (r-1)(n-r-1) + \frac{(r-2)(r-1)}{2}.$$

Set $L_r = (I(G_{C_r}), U_r)$ and $J_r = (I(G_{C_r}), T_{rn})$. Since

$$(I(G_{C_r}), U_r) = (I(G_{D_{1r}}), U_r) \text{ and } (I(G_{C_r}), T_{rn}) = (I(G_{C_{r-1}}), T_{rn}),$$

an application of Lemma 9.2.4 yields that L_r is Cohen-Macaulay and

$$\begin{aligned} \operatorname{ht}(L_r) &= \operatorname{ht} I(G_{D_{1r}}) + |U_r| \\ &= \binom{n-r-1}{2} + (r-2) + (r-1)(n-r-1) + \binom{r-1}{2}. \end{aligned}$$

Hence

$$\operatorname{ht}(L_r) = (r-1) + \frac{(n-1)(n-4)}{2}.$$

On the other hand by the second induction hypothesis $\operatorname{ht}(J_r) = \operatorname{ht}(L_r)$ and J_r is CM. Altogether we have that A/J_r and A/L_r are Cohen-Macaulay rings with the same dimension. Since L_r is also equal to $(I(G_{C_{r-1}}), U_r)$ we have

$$\operatorname{ht}(U_r, I(G_{C_{r-1}})) = \operatorname{ht} I(G_{C_{r-1}}) + 1.$$

Therefore U_r can not be a subset of a minimal cover for $G_{C_{r-1}}$, and thus

$$\operatorname{ht} I(G_{C_r}) = \operatorname{ht} I(G_{C_{r-1}}) + 1.$$

From the exact sequence

$$0 \longrightarrow (A_n/L_r)[-1] \xrightarrow{T_{rn}} A_n/I(G_{C_r}) \longrightarrow A_n/J_r \longrightarrow 0.$$

we obtain $I(G_{C_r})$ Cohen-Macaulay. In particular for $r = n-1$ we conclude that $I(G_{\sigma_n})$ is Cohen-Macaulay of height equal to $n(n-3)/2$. □

Proposition 9.2.7 *Let $F_n(z) = h_n(z)(1-z)^{-1}$ be the Hilbert series of the ring $A_n/I(G_{\sigma_n})$. Then the polynomial $h_n(z)$ satisfies the difference equation*

$$h_n(z) = 2h_{n-1}(z) + (z-1)h_{n-2}(z) + (n-2)z^2$$

for $n \geq 5$ with boundary conditions $h_3(z) = 1$ and $h_4(z) = 1 + 2z + z^2$.

Proof. Set $A = A_n$ and $C_r = V_{n-1} \cup \{T_{1n}, T_{2n}, \ldots, T_{rn}\}$. Let D_{1r} and U_r be as in Lemma 9.2.4 and Theorem 9.2.6 respectively. Consider the exact sequences

$$0 \longrightarrow (A/L_{n-1})[-1] \stackrel{T_{(n-1)n}}{\longrightarrow} A/I(G_{C_{n-1}}) \longrightarrow A/J_{n-1} \longrightarrow 0,$$

and

$$0 \longrightarrow (A/L_r)[-1] \stackrel{T_{rn}}{\longrightarrow} A/J_{r+1} \longrightarrow A/J_r \longrightarrow 0,$$

where

$$L_r = (I(G_{D_{1r}}), U_r, T_{(r+1)n}, \ldots, T_{(n-1)n})$$

and

$$J_{r+1} = (I(G_{C_r}), T_{(r+1)n}, \ldots, T_{(n-1)n}).$$

Using these sequences recursively, for $r = n-2, \ldots, 3, 2$ yields

$$F_n(z) = z\,[F(A/L_{n-1}, z) + \cdots + F(A/L_2, z)] + F(A/J_2, z),$$

where $J_2 = (T_{2n}, \ldots, T_{(n-1)n}, I(G_{\sigma_{n-1}}))$. On the other hand, by induction on r, we readily obtain

$$(1-z)^n F(A/L_r, z) = \begin{cases} (r-2)z + 1 & \text{if } r \in \{n-1, n-2\}, \\ (r-1)z + h_{n-r}(z) & \text{if } 2 \leq r \leq n-3. \end{cases}$$

Therefore

$$h_n(z) = z\left[\sum_{k=n-2}^{n-1}(1 + (k-2)z) + \sum_{k=4}^{n-1}((n-k)z + h_{k-1}(z))\right] + h_{n-1}(z)$$

and

$$h_{n-1}(z) = z\left[\sum_{k=n-3}^{n-2}(1 + (k-2)z) + \sum_{k=4}^{n-2}((n-k-1)z + h_{k-1}(z))\right] + h_{n-2}(z).$$

subtracting the last two equations we obtain

$$h_n(z) = 2h_{n-1}(z) + (z-1)h_{n-2}(z) + (n-2)z^2.$$

The boundary conditions follow readily since $I(G_{\sigma_3}) = (0)$. □

Corollary 9.2.8 ([262]) *The ring $k[\mathcal{K}_n]$ is a Cohen-Macaulay ring.*

Proof. Let A_n/P_n be the presentation of $k[\mathcal{K}_n]$. Note that $\text{in}(P)$, the initial ideal of P, is equal to $I(G_{\sigma_n})$. By Theorem 9.2.4 we obtain that $A/\text{in}(P)$ is Cohen-Macaulay, and this implies (see [133, 208]) that A/P is Cohen-Macaulay as well. □

Remark 9.2.9 Let

$$\mathcal{R}(I) = k[\{x_1, \ldots, x_n, Tx_ix_j | \, 1 \leq i < j \leq n\}]$$

be the Rees algebra of the edge-ideal $I = I(\mathcal{K}_n)$ and

$$k[\mathcal{K}_{n+1}] = k[\{x_ix_j | \, 1 \leq i < j \leq n+1\}].$$

Then by Proposition 8.2.15 there is an isomorphism

$$\varphi: \mathcal{R}(I(\mathcal{K}_n)) \to k[\mathcal{K}_{n+1}]$$

induced by

$$\varphi(x_i) = x_ix_{n+1} \text{ and } \varphi(Tx_ix_j) = x_ix_j.$$

Therefore by Lemma 9.2.8 we obtain that the Rees algebra of the edge-ideal of a complete graph is Cohen-Macaulay. In fact such algebras are even normal by Proposition 7.4.5.

Theorem 9.2.10 ([304]) *Let $\mathcal{K}_n$ be the complete graph on n vertices and A_n/P_n be the presentation of $k[\mathcal{K}_n]$. If $n \geq 3$, then the Hilbert series $F_n(z)$ of A_n/P_n satisfies*:

$$(1-z)^n F_n(z) = 1 + \frac{n(n-3)}{2}z + \sum_{m=2}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2m} z^m$$

Proof. Let $F_n(z) = F(A_n/P_n, z)$. Since $\text{in}(P_n)$, the initial ideal of P_n, equals $I(G_{\sigma_n})$, by Corollary 2.4.13 the Hilbert series of $A_n/I(G_{\sigma_n})$ is equal to $F_n(z)$. Let

$$(1-z)^n F_n(z) = h_n(z) = a_{0n} + a_{1n}z + \cdots + a_{\theta(n)n}z^{\theta(n)},$$

and let

$$G_m(x) = \sum_{n=3}^{\infty} a_{mn}x^n$$

be the generating function of the sequence $(a_{mn})_{n \geq 3}$. By Proposition 9.2.7 we have

$$a_{mn} = \begin{cases} 2a_{m(n-1)} + a_{(m-1)(n-2)} - a_{m(n-2)} & \text{if } n \geq 5, \ m \notin \{0, 2\}, \\ 2a_{m(n-1)} + a_{(m-1)(n-2)} - a_{m(n-2)} & \\ +(n-2) & \text{if } n \geq 5, \ m = 2. \end{cases}$$

Recall that

$$G_0(x) = \sum_{n=3}^{\infty} x^n, \quad h_3(z) = 1, \quad \text{and} \quad h_4(z) = 1 + 2z + z^2.$$

Therefore for $m \geq 3$ we obtain

$$\begin{aligned} G_m(x) &= \sum_{n=5}^{\infty} a_{mn}x^n \\ &= 2\sum_{n=5}^{\infty} a_{m(n-1)}x^n + \sum_{n=5}^{\infty} a_{(m-1)(n-2)}x^n - \sum_{n=5}^{\infty} a_{m(n-2)}x^n \\ &= 2xG_m(x) + x^2G_{m-1}(x) - x^2G_m(x), \end{aligned}$$

and consequently

$$G_m(x) = \left(\frac{x^2}{(1-x)^2}\right) G_{m-1}(x)$$

for $m \geq 3$. By a similar argument we derive

$$G_1(x) = \frac{x^4(2-x)}{(1-x)^3} = \sum_{n=3}^{\infty} \frac{n(n-3)}{2}x^n, \text{ and } G_2(x) = x^4/(1-x)^5.$$

Hence $a_{1n} = n(n-3)/2$ for $n \geq 3$. To complete the proof observe that

$$G_m(x) = \frac{x^m}{(1-x)^{2m+1}} = \sum_{n=2m}^{\infty} \binom{n}{2m} x^n$$

for $m \geq 2$. Hence $a_{mn} = \binom{n}{2m}$ for $m \geq 2$ and $n \geq 3$. □

Remark 9.2.11 The Hilbert function of $k[\mathcal{K}_n]$ can be computed using the fact that the Ehrhart polynomial of the second hypersimplex of order n is equal to the Hilbert function of $k[\mathcal{K}_n]$. See Chapter 7 and [79, 277].

Corollary 9.2.12 *The multiplicity $e(k[\mathcal{K}_n])$, of the ring $k[\mathcal{K}_n]$, is equal to $2^{n-1} - n$.*

Exercises

9.2.13 Let k be a field and $A = k[\{T_{ij} | 1 \leq i < j \leq 5\}]$. Consider the permutation of the variables:

$$\phi = \begin{pmatrix} T_{12} & T_{13} & T_{14} & T_{15} & T_{23} & T_{24} & T_{25} & T_{34} & T_{35} & T_{45} \\ T_{45} & T_{34} & T_{35} & T_{23} & T_{24} & T_{25} & T_{12} & T_{13} & T_{14} & T_{15} \end{pmatrix}.$$

Prove that the toric ideal P of $k[\mathcal{K}_5]$ is not invariant under the automorphism of A induced by ϕ.

9.2.14 If Q is the toric ideal of the Rees algebra $\mathcal{R}(I)$ of the monomial ideal $I = (x_i x_j | 1 \leq i < j \leq n-1)$, then Q is the ideal $I_2(Y)$ generated by the 2-minors of the symmetric array

$$Y = \begin{array}{ccccccc} \bullet & T_{12} & T_{13} & \cdots & T_{1(n-2)} & T_{1(n-1)} & x_1 \\ T_{12} & \bullet & T_{23} & \cdots & T_{2(n-2)} & T_{2(n-1)} & x_2 \\ T_{13} & T_{23} & \bullet & \cdots & T_{3(n-2)} & T_{3(n-1)} & x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ T_{1(n-2)} & T_{2(n-2)} & T_{3(n-2)} & \cdots & \bullet & T_{(n-2)(n-1)} & x_{n-2} \\ T_{1(n-1)} & T_{2(n-1)} & T_{3(n-1)} & \cdots & T_{(n-2)(n-1)} & \bullet & x_{n-1} \\ x_1 & x_2 & x_3 & \cdots & x_{n-2} & x_{n-1} & \bullet \end{array},$$

and the 2-minors of Y form a Gröbner basis for $I_2(Y)$ w.r.t the lex ordering $T_{12} < \cdots < T_{1(n-1)} < x_1 < T_{23} < \cdots < T_{2(n-1)} < x_2 < \cdots < x_{n-1}$.

Hint Set $B = K[T_{ij} | 1 \leq i < j \leq n]$ and $x_i = T_{in}$ for $1 \leq i \leq n-1$. There is a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & K[\mathcal{K}_n] \\ \downarrow{\scriptstyle \psi} & \overset{\alpha}{\nearrow} & \\ \mathcal{R}(I) & & \end{array}$$

$$\begin{aligned} \psi(T_{in}) &= T_{in},\ 1 \leq i < j \leq n-1 \\ \psi(T_{ij}) &= T x_i x_j,\ 1 \leq i < j \leq n-1 \\ \alpha(x_i) &= x_i x_n,\ 1 \leq i < n \\ \alpha(T x_i x_j) &= x_i x_j,\ 1 \leq i < j \leq n-1. \end{aligned}$$

Since α is a isomorphism one has $Q = \ker(\psi) = \ker(\varphi)$, then use the description of $\ker(\varphi)$ given in Proposition 9.2.1.

9.3 Noether normalizations of edge subrings

In this section two explicit Noether normalization of the edge subring of a complete graph are presented, it will turn out that those normalizations can serve as a model to deal with some specific graphs. The results of this section are due to A. Alcántar [2, 3].

There are few classes of graded algebras where homogeneous Noether normalizations are known, such classes include quotients of monomials ideals [298, Proposition 2.3.3], edge subrings of complete bipartite graphs [72] and edge subrings of bipartite planar graphs [85].

Systems of parameters of edge subrings Let G be a graph on the vertex set $V = \{x_1, \ldots, x_n\}$ and $R = K[x_1, \ldots, x_n]$ a polynomial ring over a field K. One can express the toric ideal of the edge subring $K[G]$ as the kernel of the epimorphism of K-algebras:

$$\psi\colon A = K[\{T_{ij} | i < j \text{ and } x_i \text{ is adjacent to } x_j\}] \to K[G]$$

induced by $\psi(T_{ij}) = x_i x_j$, where $\{T_{ij}\}$ is a new set of variables in one to one correspondence with the edges of G.

We shall always assume that A and R have the usual grading and that $K[G]$ has the normalized grading

$$K[G]_i = R_{2i} \cap K[G],$$

thus ψ is a graded map. In the sequel we denote $\ker(\psi)$ by $P(G)$. An standard *Noether normalization* of $A/P(G)$ is an integral extension

$$K[h_1, \ldots, h_d] \hookrightarrow A/P(G), \qquad (*)$$

where $h_1, \ldots, h_d$ are homogeneous polynomials of degree one and d is the Krull dimension of $K[G]$. Note that $(*)$ is a Noether normalization if and only if

$$\operatorname{rad}(h_1, \ldots, h_d, P(G)) = \mathfrak{M},$$

where $\mathfrak{M}$ is the maximal ideal of A generated by all the variables T_{ij}. See Chapter 2.

Edge subrings of complete graphs The *complete graph* with vertices $x_1, \ldots, x_n$, denoted by $\mathcal{K}_n$, has every pair of its n vertices adjacent. Let

$$K[\mathcal{K}_n] = K[x_i x_j | \, 1 \leq i < j \leq n\}] \subset R$$

be the edge subring of R associated to $\mathcal{K}_n$. Thus the *toric ideal* $P(\mathcal{K}_n)$ of $K[\mathcal{K}_n]$ is the kernel of the homomorphism of K-algebras

$$\psi \colon A_n = K[\{T_{ij} | \, 1 \leq i < j \leq n\}] \to K[\mathcal{K}_n]$$

induced by $T_{ij} \to x_i x_j$.

Notation For use below we set $P_n = P(\mathcal{K}_n)$. If $j < i$, we set $T_{ji} := T_{ij}$. The homogeneous maximal ideal of A_n will be denoted by $\mathfrak{M}_n$.

Lemma 9.3.1 ([3]) *Let $Q_n = (P_n, J_n) \subset A_n$ be the ideal of A_n generated by P_n and $J_n = (h_1, \ldots, h_n)$, where*

$$h_k = \sum_{1 \leq i < k} T_{ik} + \sum_{k < i \leq n} T_{ki}.$$

Let P be an associated prime ideal of Q_n. Assume Q_p is an $\mathfrak{M}_p$-primary ideal of A_p for $4 \leq p < n$. If P contains a variable T_{rs}, then $P = \mathfrak{M}_n$.

Proof. Let P be an associated prime ideal of Q_n so that $T_{rs} \in P$, to simplify notation assume $r = n - 1$ and $s = n$. Since $T_{k(n-1)} T_{jn} - T_{kj} T_{(n-1)n} \in P_n$,

we obtain $T_{k(n-1)}T_{jn} \in P$ for $j \neq k$. If $T_{k(n-1)} \notin P$ for some k, then $T_{jn} \in P$ for all $j \neq k$, $j \leq n-1$. As $h_n \in P$ we have

$$\{T_{1n}, \ldots, T_{(n-1)n}\} \subset P,$$

hence $Q_{n-1} \subset P$. Using the hypothesis yields $T_{k(n-1)} \in P$, a contradiction. Therefore $T_{k(n-1)} \in P$ for all $k \neq n-1$. Let

$$Q = (Q_n, T_{1(n-1)}, \ldots, T_{(n-2)(n-1)}, T_{(n-1)n}).$$

Because $Q \subset P$, from the hypothesis we derive that P contains all variables not in $\{T_{k(n-1)}\}_{k \neq n-1}$. Therefore $P = \mathfrak{M}_n$. □

Theorem 9.3.2 ([3]) *If K is either a field of characteristic zero or K is a field of characteristic $p \geq 2$ and $2, \ldots, n-1$ are relatively prime to p, then the ideal Q_n is an $\mathfrak{M}_n$-primary ideal of A_n.*

Proof. We proceed by induction on n. It is not hard to verify that

$$\mathrm{ht}(Q_i) = \frac{i(i-1)}{2}$$

for $3 \leq i \leq 6$ (see arguments below). Assume $n \geq 7$ and the result true for $m < n$. Let P be an associated prime ideal of Q_n. By Lemma 9.3.1 it suffices to prove that P contains one of the T_{ij}'s variables. Set

$$L_n = T_{(n-1)n} - T_{2(n-1)} \quad \text{and} \quad g = T_{1n}T_{2(n-1)} - T_{12}T_{(n-1)n}.$$

Consider the equality

$$g - T_{2(n-1)}h_n + T_{(n-1)n}h_2 + \sum_{i=3}^{n-2}(T_{2(n-1)}T_{in} - T_{2i}T_{(n-1)n}) = T_{2n}L_n \quad (9.6)$$

and define the following permutations of $\{1, \ldots, n\}$

$$\sigma_1 = \begin{pmatrix} 1 & 2 & n-2 & n-1 & n \\ n-2 & n & 2 & 1 & n-1 \end{pmatrix}$$

$$\sigma_2 = \begin{pmatrix} 1 & 2 & n-3 & n-2 & n-1 & n \\ n-3 & n-1 & 2 & n & 1 & n-2 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 2 & n-3 & n-2 & n-1 & n \\ n & n-2 & n-1 & 2 & 1 & n-3 \end{pmatrix}$$

$$\sigma_4 = \begin{pmatrix} 1 & 2 & k & k+1 & k+2 & n-1 & n \\ k+2 & k+1 & 2 & n & n-1 & 1 & k \end{pmatrix}$$

Observe that a permutation σ of $\{1,\ldots,n\}$ induces an automorphism

$$\sigma' : A_n \longrightarrow A_n$$

such that $\sigma'(T_{ij}) = T_{\sigma(i)\sigma(j)}$. Since

$$\sigma'(P_n) \subset P_n \quad \text{and} \quad \sigma'(J_n) \subset J_n$$

we have $\sigma'(Q_n) \subset Q_n$. Applying σ_i' to Eq. (9.6) yields $\sigma_i'(T_{2n}L_n) \in Q_n$. Note that

$$\sigma_i'(T_{2n}L_n) = \begin{cases} T_{k(k+1)}(T_{1k} - T_{1(k+1)}) & \text{if } i = 1,\ n \geq 5,\ k = n-1 \\ T_{k(k+1)}(T_{1k} - T_{1(k+1)}) & \text{if } i = 2,\ n \geq 6,\ k = n-2 \\ T_{k(k+1)}(T_{1k} - T_{1(k+1)}) & \text{if } i = 3,\ n \geq 6,\ k = n-3 \\ T_{k(k+1)}(T_{1k} - T_{1(k+1)}) & \text{if } i = 4,\ n-4 \geq k \geq 2. \end{cases}$$

If $T_{k(k+1)} \notin P$ for all $2 \leq k \leq n-1$, then $T_{1i} - T_{1j} \in P$ for $i \neq j$. Using the identity

$$h_1 + \sum_{j=3}^{n}(T_{12} - T_{1j}) = (n-1)T_{12}$$

we obtain $T_{12} \in P$. □

Corollary 9.3.3 ([3]) *If K is a field of characteristic zero, then*

$$K[h_1,\ldots,h_n] \hookrightarrow A_n/P_n \simeq K[\mathcal{K}_n]$$

is a Noether normalization of $K[\mathcal{K}_n]$.

Proof. Note that the monomial subring $K[\mathcal{K}_n]$ is integral over the subring $K[h_1,\ldots,h_n]$ by Theorem 9.3.2. □

Corollary 9.3.4 *If K is a field of characteristic zero, then A_n/P_n is a finitely generated free module over $K[h_1,\ldots,h_n]$.*

Proof. Since $K[\mathcal{K}_n]$ is a Cohen-Macaulay graded algebra of dimension n, by Proposition 6.3.1 we obtain that A_n/P_n is a finitely generated free module over $K[h_1,\ldots,h_n]$. Note that $\{h_1,\ldots,h_n\}$ is a regular system of parameters for A_n/P_n. □

Another Noether normalization Let us present here a simpler Noether normalization of the monomial subring of the complete graph. Since any variable T_{ij} occurs only in h_i and h_j, one has:

$$2h = h_1 + \cdots + h_n, \text{ where } h = \sum_{1 \leq i < j \leq n} T_{ij}.$$

Setting

$$g_i = \begin{cases} -h + h_1 + h_{i+1} & \text{for } i = 1, 2, \\ h_{i+1} & \text{for } 3 \leq i \leq n-1, \\ h - h_1 & \text{for } i = n, \end{cases}$$

it is not hard to see that $F = \{g_1, \ldots, g_n\}$ is a Gröbner basis for $(h_1, \ldots, h_n)$ w.r.t. the lex ordering $T_{12} > \cdots > T_{1n} > T_{23} > \cdots > T_{(n-1)n}$. Thus one may try to eliminate some of the first variables from F with the intention of producing some "triangulated" Noether normalization whose first elements are variables. The next result proves that one may replace g_1 by T_{12} and still get a normalization.

Proposition 9.3.5 *Let K be a field of characteristic p. If $2, \ldots, n-1$ are relatively prime to p, then*

$$K[T_{12}, g_2, \ldots, g_n] \hookrightarrow A_n/P_n \simeq K[\mathcal{K}_n]$$

is a Noether normalization of $K[\mathcal{K}_n]$.

Proof. From the equalities

$$h = (h_1 - h) + h_2 + \cdots + h_n \tag{9.7}$$

and

$$Kh_1 + \cdots + Kh_n = Kg_1 + \cdots + Kg_n,$$

one has

$$K[h_1, \ldots, h_n] = K[g_1, \ldots, g_n].$$

As $K[\mathcal{K}_n]$ is Cohen-Macaulay (see Proposition 7.4.5) by Theorem 9.3.2 one concludes that $g_1, \ldots, g_n$ is a regular system of parameters for A_n/P_n. In particular $g_2, \ldots, g_n$ is a regular sequence on A_n/P_n.

To finish the proof it suffices to show that T_{12} is a regular element on

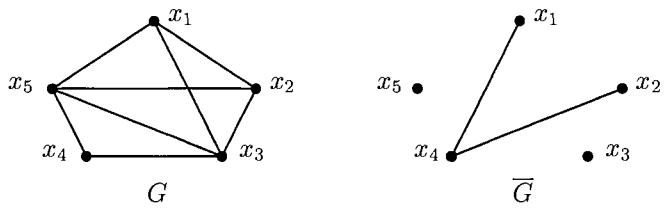
$$S = A_n/(P_n, g_2, \ldots, g_n).$$

Let Q be an associated prime of S and assume T_{12} is in Q. If $h_1 \in Q$, then $h_3, \ldots, h_n, h$ are in Q, which together with Eq. (9.7) gives $h_2 \in Q$. Hence $g_1 \in Q$; a contradiction to the fact that g_1 is regular on S. Therefore h_1 cannot be in Q and consequently there is $k \geq 3$ so that $T_{1k} \notin Q$.

If $\{1,k\} \cap \{2,j\} = \emptyset$ and $j \neq 2$, then $T_{1k}T_{2j} - T_{12}T_{kj}$ is in P_n and thus T_{2j} is in Q for all $j \notin \{2,k\}$. Note T_{2k} is not in Q; for otherwise $h_2, \ldots, h_n$ and $h - h_1$ are in Q and from Eq. (9.7) $h \in Q$, and hence $h_1 \in Q$ which is impossible. By considering $T_{2k}T_{ij} - T_{2j}T_{ik}$, one obtains $T_{ij} \in Q$ for all ij so that i, j are not in $\{2,k\}$ and $i \neq j$. It follows readily that all the variables T_{ij} not occurring in h_k belong to Q. Using Eq. (9.7) and noticing that T_{ik} occurs exactly twice in $h_1 + \cdots + h_n$, one gets $2(h - h_k) \in Q$, but since h_k is also in Q one derives $h, h_2 \in Q$ and consequently g_1 is in Q, which is again a contradiction. Altogether T_{12} is not in the union of the associated primes of S and is therefore a regular element on S. □

Examples and one open question The minimal vertex covers of the complement $\overline{G}$ of a graph G seem to be related to the problem of finding a Noether normalization of the edge subring $K[G]$.

Example 9.3.6 Let G be the following graph and $\overline{G}$ its complement.



Using Macaulay one has $P(G) = (g_1, g_2, g_3, g_4)$, where

$$g_1 = t_{13}t_{25} - t_{15}t_{23}, \qquad g_2 = t_{13}t_{45} - t_{15}t_{34},$$
$$g_3 = t_{12}t_{35} - t_{15}t_{23}, \qquad g_4 = t_{23}t_{45} - t_{25}t_{34}.$$

By Corollary 8.7.11 it follows that the edge subring $K[G]$ is a normal Cohen-Macaulay domain. To find a Noether normalization for $K[G]$ over a field K of characteristic zero consider the sequence:

$$\theta_1 = t_{12} + t_{13} + t_{15}, \qquad \theta_2 = t_{12} + t_{23} + t_{25},$$
$$\theta_3 = t_{13} + t_{23} + t_{34} + t_{35}, \qquad \theta_4 = t_{34} + t_{45},$$
$$\theta_5 = t_{15} + t_{25} + t_{35} + t_{45},$$

unfortunately $\operatorname{ht}(P(G), \theta_1, \ldots, \theta_5) = 7$, but we can modify the coefficients of the variables in the support of θ_i to obtain the required height 8, more precisely a Noether normalization for $K[G]$ is:

$$K[h_1, \ldots, h_5] \hookrightarrow K[G] \simeq K[t_{ij}]/P(G),$$

where

$$h_1 = t_{12} + t_{13} + t_{15}, \qquad h_2 = t_{12} + t_{23} + t_{25},$$
$$h_3 = t_{13} + t_{23} + 3t_{34} + t_{35}, \qquad h_4 = 3t_{34} + t_{45},$$
$$h_5 = t_{15} + t_{25} + t_{35} + t_{45}.$$

Note that the coefficients of the variables occurring in θ_i were modified by first finding a minimal vertex cover $C = \{x_4\}$ for $\overline{G}$ and then changing t_{ij} by it_{ij}, if x_j is in C.

Remark 9.3.7 To verify the assertion that $\mathrm{ht}(P(G), h_1, \ldots, h_5) = 8$ in the Example 9.3.6, note that $h_1, \ldots, h_5$ specializes nicely, that is,

$$h'_1, \ldots, h'_{i-1}, h'_{i+1}, \ldots, h'_5$$

is a system of parameters of $K[G \setminus \{x_i\}]$ for each $1 \leq i \leq 5$, where h'_j is obtained making $t_{ik} = 0$ in the sequence $h_1, \ldots, h_5$. Hence we may proceed essentially as in the proof of Proposition 9.3.5.

Take an associated prime Q of $(P(G), h_1, \ldots, h_5)$. If $t_{12} \in Q$, then $t_{15}t_{23} \in Q$. Note one may assume $t_{23} \in Q$; otherwise t_{15} and t_{13} are in Q and from:

$$(P(G), h_1, \ldots, h_5, t_{12}, t_{13}, t_{15}) = P(G\setminus\{x_1\}), h'_2, h'_3, h'_4, h'_5, t_{12}, t_{13}, t_{15}) \subset Q,$$

one concludes $\mathrm{ht}(Q) = 8$. Using:

$$(P(G), h_1, \ldots, h_5, t_{12}, t_{23}, t_{25}) = P(G\setminus\{x_2\}), h'_1, h'_3, h'_4, h'_5, t_{12}, t_{23}, t_{25}) \subset Q,$$

again one concludes $\mathrm{ht}(Q) = 8$. Similar arguments show that if t_{13} or t_{15} are in Q, then $\mathrm{ht}(Q) = 8$. Hence one may assume t_{12}, t_{13}, t_{15} not in Q. From the equalities:

$$-g_1 + t_{25}h_1 - t_{15}h_2 = t_{12}(-t_{15} + t_{25})$$
$$-3g_2 - g_3 + t_{35}h_1 - t_{15}h_3 = t_{13}(-t_{15} + t_{35} - 3t_{45})$$
$$g_3 - g_1 - g_2 + (t_{15} - t_{35})h_1 + t_{13}h_5 = t_{15}(t_{12} + 2t_{13} + t_{15} + t_{34} - t_{35}),$$

one readily derives that Q is the irrelevant maximal ideal of $K[t_{ij}]$.

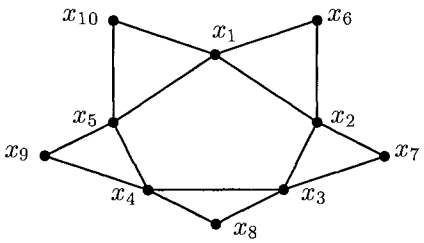
All the normalizations of monomial subrings of graphs that we have found, through the procedure of Example 9.3.6, suggest the following:

Question 9.3.8 Let G be a graph with n vertices and $A/P(G)$ the presentation of $K[G]$ over a field K of characteristic zero. Assume $K[G]$ is a normal subring of dimension n such that $\deg(v) \geq 2$ and $\dim K[G \setminus \{v\}] = n - 1$ for any vertex v of G. Is there a Noether normalization of the form:

$$K[h_1, \ldots, h_n] \hookrightarrow A/P(G), \; h_j = \sum_{i \neq j} \epsilon_{ij} T_{ij} \text{ for all } 1 \leq j \leq n, \; \epsilon_{ij} \in K\,?$$

The next instance of a graph is rather more complicated than the graph given in Example 9.3.6, but exhibits the same sensitivity to modifying the coefficients linked to subsets of minimal covers of the complement of the graph.

Example 9.3.9 Let G be the following graph on 10 vertices and $\overline{G}$ its complement.



ht $P(G) = 5$.
Minimal cover for $\overline{G}$:
$C = \{x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}$.

By the description of the integral closure of $K[G]$ given in Corollary 8.7.11 it follows that $K[G]$ is a normal domain. It is not hard to verify that a Noether normalization for $K[G]$ over a field K of characteristic zero is:

$$K[h_1, \ldots, h_{10}] \hookrightarrow K[G] \simeq K[t_{i,j}]/P(G),$$

where

$$\begin{aligned}
h_1 &= t_{1,2} + t_{1,5} + t_{1,6} + t_{1,10}, & h_2 &= t_{1,2} + 2t_{2,3} + t_{2,6} + 2t_{2,7},\\
h_3 &= 2t_{2,3} + 3t_{3,4} + 3t_{3,7} + 3t_{3,8}, & h_4 &= 3t_{3,4} + 4t_{4,5} + 4t_{4,8} + 4t_{4,9}\\
h_5 &= t_{1,5} + 4t_{4,5} + 5t_{5,9} + 5t_{5,10}, & h_6 &= t_{1,6} + t_{2,6}\\
h_7 &= 2t_{2,7} + 3t_{3,7}, & h_8 &= 3t_{3,8} + 4t_{4,8}\\
h_9 &= 4t_{4,9} + 5t_{5,9}, & h_{10} &= t_{1,10} + 5t_{5,10}.
\end{aligned}$$

The coefficients of the variables occurring in h_i were chosen using a minimal vertex cover C for $\overline{G}$ and then modifying the coefficient of t_{ij}, if x_j is in C.

Remark 9.3.10 The monomial subring of Example 9.3.9 was studied by Hibi and Ohsugi [165] to show a normal subring $K[G]$ of a graph G such that none of the initial ideals of $P(G)$ is squarefree.

Chapter 10

Monomial Curves

In this chapter we present some material on toric ideals of monomial curves and symmetric semigroups. The emphasis will be on space curves and monomial curves in the affine space of dimension four whose toric ideals are generated by critical binomials.

We should point out that Chapter 10 and Chapter 11 overlap at certain points. A few almost identical results will be proven in both chapters with slightly different proofs.

10.1 Defining equations of monomial curves

Let k be a field and

$$S = d_1\mathbb{N} + \cdots + d_n\mathbb{N}$$

the semigroup generated by a finite sequence $d_1, \ldots, d_n$ of relatively prime positive integers. A *monomial curve* Γ in the affine space $\mathbb{A}_k^n$ over k is given parametrically by

$$x_i = t^{d_i},$$

that is, we have

$$\Gamma = \left\{(t^{d_1}, \ldots, t^{d_n}) \in \mathbb{A}_k^n \,|\, t \in k\right\}.$$

Let $R = k[x_1, \ldots, x_n]$ and $k[t]$ be two polynomial rings over k graded by $\deg(x_i) = d_i$ for all i and $\deg(t) = 1$ respectively. Let ϕ be the graded homomorphism of k-algebras

$$\phi\colon R \longrightarrow k[t], \quad x_i \longmapsto t^{d_i}.$$

The image of ϕ, denoted by $k[S]$ or $k[\Gamma]$, is the *semigroup ring* associated with S and the homogeneous ideal $P = \ker(\phi)$ is the *presentation ideal* or *toric ideal* of $k[S]$.

By Proposition 7.1.2 the toric ideal P is generated by a finite set of binomials. Hence the exact sequence

$$0 \longrightarrow K = \ker(\psi) \longrightarrow \mathbb{Z}^n \xrightarrow{\psi} \mathbb{Z} \longrightarrow 0, \ \psi(e_i) = d_i$$

is related to the presentation ideal P of $k[S]$ in the following manner: if $g = x^\alpha - x^\beta$ is a binomial, then $g \in P$ if and only if $\alpha - \beta \in \ker(\psi)$. Given a binomial $g = x^a - x^b$, we set $\widehat{g} = a - b$.

Given a subset $I \subset R$ we denote its zero set in $\mathbb{A}_k^n$ by $V(I)$, and given a subset $X \subset \mathbb{A}_k^n$ we denote its vanishing ideal in R by $I(X)$.

If $h_1, \ldots, h_r \in \mathbb{Z}^n$, we denote by

$$\langle h_1, \ldots, h_r \rangle$$

the subgroup of $\mathbb{Z}^n$ generated by these elements.

Let $G \subset \mathbb{Z}^n$ be a subgroup. We associate to G an equivalence relation on the monomials in $R = k[x_1, \ldots, x_n]$, by $x^\alpha \sim_G x^\beta$ if $\alpha - \beta \in G$. Note that this relation is compatible with the product, i.e., $x^\alpha \sim_G x^\beta$ implies $x^\gamma x^\alpha \sim_G x^\gamma x^\beta$ for all $\gamma \in \mathbb{N}^n$.

Definition 10.1.1 A nonzero polynomial

$$f = \sum_\alpha \lambda_\alpha x^\alpha$$

in R is *simple* (with respect to $\sim_G$) if all its monomials, *i.e.*, those x^α with nonzero coefficient λ_α, are equivalent under $\sim_G$.

Given any polynomial $f \in R \setminus \{0\}$, we can group together its monomials by equivalence classes under $\sim_G$, thereby obtaining a decomposition

$$f = h_1 + \cdots + h_m$$

with the property that each summand h_i is simple, and that no monomial in h_i is equivalent with a monomial in h_j if $j \neq i$. Such a decomposition of f as a sum of maximal simple subpolynomials is unique up to order. We will refer to the summands h_i as the *simple components* of f (with respect to $\sim_G$).

Lemma 10.1.2 *Let $g_1, \ldots, g_r$ be binomials in R and $G \subset \mathbb{Z}^n$ the subgroup generated by $\widehat{g}_1, \ldots, \widehat{g}_r$. If $f \in (g_1, \ldots, g_r)$, then every simple component of f with respect to $\sim_G$ also belongs to $(g_1, \ldots, g_r)$.*

Proof. First, each generator $g_i = x^{\alpha_i} - x^{\beta_i}$ is simple, as its two constituting monomials $x^{\alpha_i}, x^{\beta_i}$ are equivalent under $\sim_G$ by construction. Moreover, $x^\gamma g_i$ remains simple for any $\gamma \in \mathbb{N}^n$, since the relation $\sim_G$ on monomials

is compatible with the product. Let now f be any element in the ideal $(g_1,\dots,g_r)$. Then, f is a linear combination of polynomials of the form $x^\gamma g_i$, with $i \in \{1,\dots,r\}$ and $\gamma \in \mathbb{N}^n$, which are simple. Hence, every simple component of f is also a linear combination of some $x^\gamma g_i$ and therefore belongs to $(g_1,\dots,g_r)$. □

Theorem 10.1.3 ([104]) *Let $g_1,\dots,g_r$ be a system of binomials in the toric ideal P and $I = (g_1,\dots,g_r)$. If* $\operatorname{char}(k) = p \neq 0$ *(resp.* $\operatorname{char}(k) = 0$*), then* $\operatorname{rad}(I) = P$ *if and only if*

(a) $p^m\ker(\psi) \subset \langle \widehat{g}_1,\dots,\widehat{g}_r\rangle$ *for some* $m \in \mathbb{N}$ *(resp.* $\ker(\psi) = \langle \widehat{g}_1,\dots,\widehat{g}_r\rangle$*).*

(b) $V(g_1,\dots,g_r,x_i) = \{0\}$*, for all* i*.*

Proof. ⇒) (a) First we consider the case $\operatorname{char}(k) = p \neq 0$. By hypothesis, there exists $m \geq 0$ such that $f^{p^m} \in I$ for all $f \in P$. Let $\gamma \in \ker(\psi)$ and write $\gamma = \alpha - \beta$ with $\alpha, \beta \in \mathbb{N}^n$ and $g = x^\alpha - x^\beta$. Thus g belongs to P and $\widehat{g} = \alpha - \beta = \gamma$. Set

$$G = \langle \widehat{g}_1,\dots,\widehat{g}_r\rangle \subset \mathbb{Z}^n.$$

We know that $g^{p^m} \in I$ and $g^{p^m} = x^{\alpha p^m} - x^{\beta p^m}$. Since the simple components of g^{p^m} belong to I and hence to P (see Lemma 10.1.2), and since $x^{\alpha p^m}$ does not belong to I because it does not belong to P, it follows that g^{p^m} is simple, i.e., $p^m\alpha \sim_G p^m\beta$. Hence $p^m\alpha - p^m\beta \in G$. This shows $p^m\ker(\psi) \subset G$.

Next we treat the case $\operatorname{char}(k) = 0$; this case was shown in [100] using simple components. Let $G = \langle \widehat{g}_1,\dots,\widehat{g}_r\rangle$ and $0 \neq \eta \in \ker(\psi)$. Note that one can write $\eta = \alpha - \beta$, where α and β are in $\mathbb{N}^n$ and have disjoint support, thus $f = x^\alpha - x^\beta \in P$ and f^q is in I for some prime $q \gg 0$. Hence

$$f^q = \sum_{i=1}^{m} c_i(x^{\gamma_i} - x^{\delta_i}),$$

where $\gamma_i - \delta_i$ are in G and $c_i \in k \setminus \{0\}$, for all i. Set

$$h = \sum_{i\in\mathcal{I}} c_i(x^{\gamma_i} - x^{\delta_i}), \quad \text{where } \mathcal{I} = \{i \mid q\alpha - \gamma_i \in L\}.$$

Observe that h equals the sum of all the monomials $c_i x^{\theta_i}$ occurring in f^q such that $q\alpha - \theta_i \in G$, because $q\alpha - \gamma_i \in G$ iff $q\alpha - \delta_i \in G$. Expanding f^q one has:

$$f^q = x^{q\alpha} - x^{q\beta} + \sum_{i=1}^{q-1}(-1)^i\binom{q}{i}x^{\alpha(q-i)}x^{i\beta}.$$

If $q\alpha - q\beta \notin G$, then

$$h = x^{q\alpha} + \sum_{i\in\mathcal{J}}(-1)^i\binom{q}{i}x^{\alpha(q-i)}x^{i\beta},$$

for some $\mathcal{J}$ contained in $\{1,\ldots,q-1\}$. As q is prime, q must divide $\binom{q}{i}$ for $0 < i < q$. Thus making $x_i = 1$ yields $1_k = (sq)1_k$, $s \in \mathbb{Z}$, but since $\mathrm{char}(k) = 0$ one has $1 = qs$, which is impossible. Hence $q(\alpha - \beta) \in G$. Pick a prime $r \neq q$ such that $f^r \in I$, by the previous argument $r(\alpha - \beta) \in G$. Since $\gcd(q,r) = 1$, we get $\alpha - \beta \in G$.

(b) Let $\alpha \in V(g_1,\ldots,g_r,x_i)$, since $V(P) = V(I)$ and $x_i^{d_j} - x_j^{d_i}$ are in P for all $j \neq i$, it follows that $\alpha = 0$.

$\Leftarrow$) First we assume $\mathrm{char}(k) = p \neq 0$. Let $f = x^c - x^e$ be a binomial in P and write $g_i = x^{\alpha_i} - x^{\beta_i}$. By (a) there are integers s_i such that

$$(x^{cp^m}/x^{ep^m}) - 1 = \left(x^{\alpha_1}/x^{\beta_1}\right)^{s_1} \cdots \left(x^{\alpha_r}/x^{\beta_r}\right)^{s_r} - 1,$$

one may assume $s_i \geq 0$ by replacing x^{α_i}/x^{β_i} by its inverse if necessary. Hence writing

$$x^{\alpha_i}/x^{\beta_i} = ((x^{\alpha_i}/x^{\beta_i}) - 1) + 1$$

and using the binomial theorem it follows that Mf^{p^m} is in I, for some monomial M. Since P is generated by a finite set of binomials, there is a monomial x^γ such that $x^\gamma f^{p^m} \in I$, for all $f \in P$. Let P_1 be a minimal prime of I, we will show that P_1 contains P. As I is graded observe that P_1 is also graded (see Lemma 1.2.3). If none of the variables in the support of x^γ are in P_1, then $P \subset P_1$. Otherwise, after a permutation of the variables and the binomials, one has the inclusions:

$$I \subset (x_1,\ldots,x_m,G') \subset P_1 \subset (x_1,\ldots,x_n),$$

where $G' = \{g_1,\ldots,g_q\}$ is a subset of $\{g_1,\ldots,g_r\}$ such that the set of variables occurring in any binomial in G' is disjoint from $x_1,\ldots,x_m$ and the binomials $g_{q+1},\ldots,g_r$ belong to the ideal $(x_1,\ldots,x_m)$. Set $a_i = 0$ for $1 \leq i \leq m$ and $a_i = 1$ for $i > m$. Since

$$(a_1,\ldots,a_n) \in V(g_1,\ldots,g_r,x_1)$$

one derives that G' must be the empty set. If $m < n$, then (a_i) is again in $V(g_1,\ldots,g_r,x_1)$, which yield $m = n$ and thus $P \subset P_1$. Hence $P \subset \mathrm{rad}(I)$ and $\mathrm{rad}(I) = P$. Note that this part of the proof works if $\mathrm{char}(k) = 0$ by making $p^m = 1$. □

Remark 10.1.4 The condition (a) above is equivalent to require that the group $\ker(\psi)/G$ is a finite p-group, and this condition can be readily validated by noticing that $\ker(\psi)/G$ is a finite p-group if and only if there is an isomorphism

$$\mathbb{Z}^n/G \simeq \mathbb{Z} \times H,$$

where H is a finite p-group.

Example 10.1.5 Consider the monomial curve

$$\Gamma = \{(t,t) | \, t \in k\} \subset \mathbb{A}^2_k,$$

where k is a field of characteristic $p > 0$. If $g = x^p - y^p \in k[x,y]$, then

$$g = (x-y)^p \quad \text{and} \quad \Gamma = V(g).$$

Note that $\widehat{g}$ does not generate the subgroup $\ker(\psi) = \langle (1,-1) \rangle$ of $\mathbb{Z}^2$.

Proposition 10.1.6 *If $g_1, \ldots, g_r$ is a set of binomials generating the toric ideal P of $k[S]$, then $\widehat{g}_1, \ldots, \widehat{g}_r$ generate* $\ker(\psi)$.

Proof. Let $L = (\widehat{g}_1, \ldots, \widehat{g}_r)$ and $0 \neq \eta \in \ker(\psi)$. Note $\eta = \alpha - \beta$, where α and β are two non negative vectors with disjoint support, thus $f = x^\alpha - x^\beta \in P$ and one can write

$$f = \sum_{i=1}^{m} c_i (x^{\gamma_i} - x^{\delta_i}),$$

where $\gamma_i - \delta_i$ are in L and $c_i \in k$, for all i. Set

$$\mathcal{I} = \{i | \, \alpha - \gamma_i \in L\} = \{i | \, \alpha - \delta_i \in L\} \text{ and}$$
$$\mathcal{J} = \{i | \, \beta - \gamma_i \in L\} = \{i | \, \beta - \delta_i \in L\}.$$

Since $\sum_{i \notin \mathcal{I} \cup \mathcal{J}} c_i (x^{\gamma_i} - x^{\delta_i}) = 0$, one has

$$f = \sum_{i \in \mathcal{I}} c_i (x^{\gamma_i} - x^{\delta_i}) + \sum_{i \in \mathcal{J}} c_i (x^{\gamma_i} - x^{\delta_i}).$$

Using $x^{\gamma_i} - x^{\delta_i} = (x^\theta - x^{\delta_i}) - (x^\theta - x^{\gamma_i})$, with $\theta = \alpha$ or $\theta = \beta$ one may rewrite f as

$$f = \sum_{i=1}^{s} c_i (x^\alpha - x^{\gamma_i}) + \sum_{j=1}^{m} e_j (x^\beta - x^{\delta_j}), \quad c_i, e_j \in k,$$

where $\alpha - \gamma_i, \beta - \delta_j \in L \setminus \{0\}$, for all i and j. One may assume $\beta \neq \gamma_i$ and $\alpha \neq \delta_j$ for all i, j; otherwise one has $\alpha - \beta \in L$. Hence

$$\sum_{i=1}^{s} c_i = 1, \quad \sum_{j=1}^{m} e_j = -1 \quad \text{and} \quad \sum_{i=1}^{s} c_i x^{\gamma_i} = -\sum_{j=1}^{m} e_j x^{\delta_j},$$

consequently $x^{\gamma_i} = x^{\delta_j}$ for some i and j, and this shows $\eta = \alpha - \beta \in L$. □

Lemma 10.1.7 *Let*

$$\psi\colon \mathbb{Z}^n \longrightarrow \mathbb{Z}$$

be the homomorphism induced by $\psi(e_i) = d_i$, *where* $d_1, \ldots, d_n$ *is a sequence of relatively prime integers. If* L *is a submodule of* $\ker(\psi)$, *then* $L = \ker(\psi)$ *if and only if* $\mathbb{Z}^n/L$ *is a free* $\mathbb{Z}$*-module of rank* 1.

Proof. Notice that $\ker(\psi)/L$ is torsion free if and only if $\mathbb{Z}^n/L$ is torsion free. Therefore one has the equality $\ker(\psi) = L$ if and only if $\mathbb{Z}^n/L$ is a free $\mathbb{Z}$-module of rank 1. □

Proposition 10.1.8 *Let* $d_1, \ldots, d_n$ *be a sequence of relatively prime integers and* $\psi\colon \mathbb{Z}^n \longrightarrow \mathbb{Z}$ *the linear map induced by* $\psi(e_i) = d_i$, *then the set*

$$L = \left\{ \frac{d_j}{(d_i, d_j)} e_i - \frac{d_i}{(d_i, d_j)} e_j \,\middle|\, 1 \leq i < j \leq n \right\}$$

is a generating set for $\ker(\psi)$. *Here* $(d_i, d_j) = \gcd(d_i, d_j)$ *and* e_i *is the* i*th unit vector.*

Proof. By Lemma 10.1.7 it suffices to prove that $\mathbb{Z}^n/(L)$ is free of rank equal to 1. Let A be the matrix with rows

$$\frac{d_j}{(d_i, d_j)} e_i - \frac{d_i}{(d_i, d_j)} e_j$$

and let I_i be the ideal generated by the i-rowed minors of A. Let us show that $I_{n-1} = \mathbb{Z}$. For a fix k consider the $(n-1) \times n$ submatrix B of A with rows

$$\frac{d_k}{(d_i, d_k)} e_i - \frac{d_i}{(d_i, d_k)} e_k,$$

$i \neq k$. It is not hard to see that the integers

$$a_k = \prod_{i \neq k} \frac{d_k}{(d_k, d_i)}$$

occur as $n-1$ minors of B. If a prime p divides a_k, then it divides d_k, therefore $I_{n-1} = \mathbb{Z}$. Since $I_i \subset I_{i-1}$ all the invariant factors of A must be equal to 1 and the proof is complete. □

Corollary 10.1.9 *Let* I *be the ideal generated by the set of binomials*

$$x_i^{d_j/(d_i,d_j)} - x_j^{d_i/(d_i,d_j)} \qquad (1 \leq i < j \leq n).$$

Then $\operatorname{rad}(I) = P$, *where* P *is the toric ideal of* $k[S]$.

Corollary 10.1.10 *If* $I = (\{x_i^{d_j} - x_j^{d_i} | \ 1 \leq i < j \leq n\})$, *then* $\mathrm{rad}(I) = P$.

Proof. Using the method of proof of Proposition 10.1.8 it follows that $\ker(\psi)$ is generated by

$$\{d_j e_i - d_i e_j | \, 1 \leq i < j \leq n\},$$

thus $\mathrm{rad}(I) = P$ by Theorem 10.1.3. □

Remark 10.1.11 Let $g_1, \ldots, g_r$ be a finite set of binomials in the toric ideal P and I the ideal generated by $g_1, \ldots, g_r$. In general the algebraic equality

$$\mathrm{rad}(I) = P$$

and the geometric equality

$$V(P) = V(I)$$

are not equivalent (see Exercise 10.1.20), but they are related as follows:

- if k is any field and $\mathrm{rad}(I) = P$, then $V(I) = V(P)$,
- on the other hand if k is algebraically closed and $V(I) = V(P)$, then by the Nullstellensatz $\mathrm{rad}(I) = P$.

Now, we want to investigate what it means geometrically for a set of binomials $g_1, \ldots, g_r$ in P to satisfy only condition (a) in Theorem 10.1.3. The next result is a refinement of [100, Lemma 3] in positive characteristic.

Proposition 10.1.12 ([104]) *Let* $G = \langle \widehat{g}_1, \ldots, \widehat{g}_r \rangle$, *where* $g_1, \ldots, g_r$ *is a set of binomials in* P. *If* k *is a field of* $\mathrm{char}(k) = p \neq 0$ (*resp. any field* k) *and* $p^m \ker(\psi) \subset G$ (*resp.* $\ker(\psi) = G$), *then*

$$V(g_1, \ldots, g_r) \subset \Gamma \cup V(x_1 \cdots x_n).$$

Proof. Assume $\mathrm{char}(k) \neq 0$; for the other case see [100]. Let $a = (a_1, \ldots, a_n)$ be in $V(g_1, \ldots, g_r)$, and suppose $a_1 \cdots a_n \neq 0$. One has

$$g_i(a) = a^{\alpha_i} - a^{\beta_i} = 0 \ \ \forall i,$$

where $g_i = x^{\alpha_i} - x^{\beta_i}$. Hence $a^{\alpha_i} = a^{\beta_i}$. Since all the entries of a are distinct from zero, this can also be expressed as $a^{\alpha_i - \beta_i} = 1$.

Now, if $g = x^\alpha - x^\beta \in P$, then $\alpha - \beta \in \ker(\psi)$ and $p^m(\alpha - \beta) \in G$. Thus one can write $p^m(\alpha - \beta) = \sum_i \ell_i(\alpha_i - \beta_i)$, for some $\ell_i \in \mathbb{Z}$. Hence

$$a^{p^m(\alpha-\beta)} = a^{\sum_i \ell_i(\alpha_i - \beta_i)} = \prod_i (a^{\alpha_i} - a^{\beta_i})^{\ell_i} = 1$$

and $a^{p^m \alpha} = a^{p^m \beta}$. Therefore $a^{p^m \alpha} - a^{p^m \beta} = (a^\alpha - a^\beta)^{p^m} = 0$, which implies $g(a) = a^\alpha - a^\beta = 0$. As P is generated by binomials it follows that a is in $V(P) = \Gamma$, as claimed. □

Monomial curves are affine toric varieties An interesting property of monomial curves is that they are affine varieties defined by binomials. In Section 11.2 we give a criterion to decide when a set defined parametrically by monomials has this property.

Lemma 10.1.13 *Let k be a field and $a, b \in k\backslash\{0\}$. If n, m are two relatively prime integers and $a^n = b^m$, then the equation $z^m = a$ has a solution in k.*

Proof. Since $1 = ns + qm$, one has

$$a = (a^n)^s a^{qm} = (b^m)^s a^{qm} = (b^s a^q)^m = z^m,$$

as required. □

Proposition 10.1.14 *Let I be the ideal of R generated by the set*

$$\left\{ x_i^{d_j} - x_j^{d_i} \,\middle|\, 1 \leq i < j \leq n \right\}.$$

Then $\Gamma = V(I)$, where $V(I)$ is the zero set of I.

Proof. It is enough to prove the containment $V(I) \subset \Gamma$, because clearly one has $\Gamma \subset V(I)$. Let $a = (a_1, \ldots, a_n)$ be an element in $V(I)$. One may assume $a_i \neq 0$ for all i, otherwise $a = 0$.

Assume $n = 2$. As $a_1^{d_2} = a_2^{d_1}$ by Lemma 10.1.13 there is $t \in k$ such that $a_1 = t^{d_1}$. Note that $u = a_2/t^{d_2}$ satisfies $u^{d_1} = 1$ and consequently $\gcd(o(u), d_2) = 1$. Thus there is an integer s so that $sd_2 \equiv 1 \bmod o(u)$, setting $t_0 = tu^s$ readily gives $a_i = t_0^{d_i}$, for $i = 1, 2$. Hence $a \in \Gamma$.

Set $h = \gcd(d_1, \ldots, d_{n-1})$ and for $1 \leq i \leq n-1$ write $d_i = d_i' h$, where $\gcd(d_1', \ldots, d_{n-1}') = 1$. Let I' be the ideal generated by the binomials:

$$x_i^{d_j'/(d_i', d_j')} - x_j^{d_i'/(d_i', d_j')} \qquad (1 \leq i < j \leq n-1).$$

By induction assume

$$V(I') \subset \Gamma' = \left\{ (t^{d_1'}, \ldots, t^{d_{n-1}'}) \,\middle|\, t \in k \right\}.$$

Using the equalities:

$$x_i^{d_j/(d_i, d_j)} - x_j^{d_i/(d_i, d_j)} = x_i^{d_j'/(d_i', d_j')} - x_j^{d_i'/(d_i', d_j')}, \quad 1 \leq i < j \leq n-1,$$

one concludes that $a' = (a_1, \ldots, a_{n-1})$ is in Γ'. Hence there $t \in k$ such that $a_i = t^{d_i'}$, for all $i < n$. Set $b = a_n^h$ and $z = b/t^{d_n}$. From the identities

$$a_n^{d_i/(d_i, d_n)} = a_i^{d_n/(d_i, d_n)} \quad \text{and} \quad (d_i, d_n) = (d_i' h, d_n) = (d_i', d_n), \; i < n,$$

one readily derives $z^{d_i'/(d_i',d_n)} = 1$, thus $o(z)$ divides d_i' for $i < n$, that is, $o(z) = 1$ and $t^{d_n} = a_n^h$. By Lemma 10.1.13 there is $t_1 \in k$ satisfying $t = t_1^h$. Set $v = a_n/t_1^{d_n}$, since $v^h = 1$ the order of v divides h and this implies $\gcd(o(v), d_n) = 1$. Hence $sd_n \equiv 1 \bmod o(v)$, for some integer s. To complete the proof set $t_0 = t_1 v^s$ and note

$$t_0^{d_n} = (t_1 v^s)^{d_n} = t_1^{d_n} v = a_n$$

and

$$t_0^{d_i} = t_1^{d_i} v^{sd_i} = t^{d_i'} v^{sd_i} = a_i \qquad (1 \leq i < n),$$

where the last equality uses that $o(v)$ divides h and that h divides d_i for $i < n$. □

For an alternative shorter argument to prove Proposition 10.1.14 one can use the next elementary fact.

Proposition 10.1.15 *Let $a_1, \ldots, a_n$ be a sequence of non zero elements of a field k. If $d_1, \ldots, d_n$ are relatively prime integers such that $a_i^{d_j} = a_j^{d_i}$ for all i, j, then there is $t \in k$ with $a_i = t^{d_i}$ for all i.*

Proof. One can write $1 = c_1 d_1 + \cdots + c_n d_n$, for some $c_i \in \mathbb{Z}$. From the equalities

$$(a_1^{c_1} \cdots a_n^{c_n})^{d_i} = a_1^{c_1 d_i} \cdots a_i^{c_i d_i} \cdots a_n^{c_n d_i} = a_i^{d_1 c_1} \cdots a_i^{c_i d_i} \cdots a_i^{c_n d_n} = a_i,$$

one derives that $t = a_1^{c_1} \cdots a_n^{c_n}$ is the required element. □

Remark 10.1.16 Assume k algebraically closed, then Proposition 10.1.14 is valid without assuming $\gcd(d_1, \ldots, d_n) = 1$. More generally, if X is the curve in $\mathbb{A}_k^n$ given parametrically by

$$x_i = f_i(t),$$

where $f_i(t) \in k[t] \setminus k$ for $i = 1, \ldots, n$, then by the extension theorem it follows that

$$V(P) = X,$$

where P is the toric ideal of $k[f_1(t), \ldots, f_n(t)]$. See [74, Chapter 3] for a beautiful exposition of the extension theorem.

Proposition 10.1.17 *Let $\Gamma = \{(t^{d_1}, \ldots, t^{d_n}) \mid t \in k\}$ be a monomial curve in the affine space $\mathbb{A}_k^n$. If $d_1 > d_2 > \cdots > d_n$ and k is the field of complex numbers, then the projective closure $\overline{\Gamma}$ of Γ is equal to*

$$\overline{\Gamma} = \left\{ [t^{d_1}, t^{d_2} u^{d_1 - d_2}, \ldots, t^{d_n} u^{d_1 - d_n}, u^{d_1}] \in \mathbb{P}_k^n \,\middle|\, u, t \in k \right\}. \tag{10.1}$$

Proof. Let Γ' be the right hand side of Eq. (10.1). By Proposition 2.4.29 one has $\overline{\Gamma} = V(I(\varphi(\Gamma)))$, where φ is the map

$$\varphi\colon \mathbb{A}_k^n \longrightarrow \mathbb{P}_k^n$$

given by $\varphi(a) = [(a, 1)]$. Let

$$\alpha = [a_1, \ldots, a_n, b] \in \overline{\Gamma},$$

then $f(a_1, \ldots, a_n, b) = 0$ for any polynomial f in $k[x_1, \ldots, x_n, w]$ such that $f(t^{d_1}, \ldots, t^{d_n}, 1) = 0$ for all $t \in k$ and f homogeneous. One may assume $b \neq 0$, otherwise if $b = 0$, then $f = x_i^{d_1} - x_1^{d_i} w^{d_1 - d_i}$ is in $I(\varphi(\Gamma))$ for $i \geq 2$, hence making $u = 0$ and $t = {a_1}^{1/d_1}$ we get $\alpha \in \Gamma'$. Since $b \neq 0$ we readily get $f(a_1/b, \ldots, a_n/b) = 0$ for every binomial f in the toric ideal P of $k[t^{d_1}, \ldots, t^{d_n}]$, hence $(a_1/b, \ldots, a_n/b) \in V(P)$. As $\Gamma = V(P)$, there is $t_1 \in k$ such that $a_i/b = t_1^{d_i}$ for all i. Making $t = b^{1/d_1} t_1$ and $u = b^{1/d_1}$ one concludes $\alpha \in \Gamma'$.

Conversely let

$$\alpha = [t^{d_1}, t^{d_2} u^{d_1 - d_2}, \ldots, t^{d_n} u^{d_1 - d_n}, u^{d_1}]$$

be an element of Γ' and f a homogeneous polynomial of degree r such that f vanishes on $\varphi(\Gamma)$. From the equality

$$f(x_1, \ldots, x_n, w) = w^r f(x_1/w, \ldots, x_n/w, 1)$$

one readily derives that $f(\alpha) = 0$, that is, $\alpha \in \overline{Y}$. □

Exercises

10.1.18 Let $d_1, \ldots, d_n$ be a sequence of relatively prime integers and

$$\psi\colon \mathbb{Z}^n \longrightarrow \mathbb{Z}$$

the linear map induced by $\psi(e_i) = d_i$. If G is a subgroup of $\ker(\psi)$, then

$$\text{Torsion}(\mathbb{Z}^n/G) \subset \ker(\psi)/G,$$

with equality if $\ker(\psi)/G$ is a finite group.

10.1.19 Let $D = (d_{ij})$ be an $m \times n$ matrix with integer entries and

$$\psi\colon \mathbb{Z}^n \to \mathbb{Z}^m$$

the linear map induced by $\psi(e_i) = D_i$, where D_i is the *ith* column of D. If G is a subgroup of $\ker(\psi)$, prove that $G = \ker(\psi)$ if and only if $\mathbb{Z}^n/G$ is torsion free of rank equal to $\text{rank}(D)$.

10.1.20 Let P be the toric ideal of the subring $k[t^6, t^8, t^9]$ and $I = (g_2, g_3)$, where $g_2 = x_1^3 - x_3^2$, $g_3 = x_2^9 - x_3^8$. Then

(a) $P = (g_2, x_2^3 - x_1 x_3^2)$ for any field k,

(b) $\mathbb{Z}^3/\langle \widehat{g}_2, \widehat{g}_3 \rangle \simeq \mathbb{Z} \times \mathbb{Z}_3$,

(c) $\operatorname{rad}(I) = P$ if $\operatorname{char}(k) = 3$, and $\operatorname{rad}(I) \neq P$ if $\operatorname{char}(k) = 2$,

(d) $\Gamma = \{(t^6, t^8, t^9) | \, t \in k\} = \{(0,0,0), (1,1,1)\} = V(I)$, if $k = \mathbb{Z}_2$.

10.1.21 Let P be the toric ideal of $\mathbb{Z}_2[t^6, t^8, t^9]$ and let

$$g_1 = x_1^4 - x_2^3, \;\; g_2 = x_1^3 - x_3^2, \;\; g_3 = x_2^9 - x_3^8.$$

Then

(a) $P = \operatorname{rad}(g_1, g_3) = \operatorname{rad}(g_1^2, g_2^2, g_3^2)$ and g_i^2 is a binomial for $i = 1, 2, 3$.

(b) $I(X) = (x_1 + x_3, x_2 + x_3, x_3 + x_3^2)$, where $X = \{(t^6, t^8, t^9) | \, t \in \mathbb{Z}_2\}$.

10.1.22 Let $d_1, \ldots, d_n$ be a sequence of nonzero integers and ψ the linear map from $\mathbb{Z}^n$ to $\mathbb{Z}$ induced by $\psi(e_i) = d_i$. Prove that the set

$$L = \left\{ \left(\frac{d_j}{d}\right) e_i - \left(\frac{d_i}{d}\right) e_j \,\middle|\, 1 \leq i < j \leq n \right\}$$

is a generating set for $\ker(\psi)$, where $d = \gcd(d_1, \ldots, d_n)$.

10.1.23 An affine variety X in k^n is said to be a *set-theoretic complete intersection* if its vanishing ideal $I(X)$ is a set-theoretic complete intersection. If k is algebraically closed prove that X is a set-theoretic complete intersection if and only if X can be defined by $n - \dim(X)$ polynomials in n variables with coefficients in k.

10.2 Symmetric semigroups

Let $S \neq (0)$ be a *semigroup* of $\mathbb{N}$, that is, S is a subset of $\mathbb{N}$ which is closed under addition and $0 \in S$. Note that there exists a sequence $d_1 < \cdots < d_n$ of positive integers such that

$$S = d_1\mathbb{N} + \cdots + d_n\mathbb{N}.$$

If $\gcd(d_1, \ldots, d_n) = 1$, it is said that S is a *numerical semigroup*.

Lemma 10.2.1 *If $S \subset \mathbb{N}$ is a numerical semigroup, then*

$$c + \mathbb{N} \subset S$$

for some $c \in S$.

Proof. Let $d_1 < \cdots < d_n$ be a sequence of positive integers generating S and such that $\gcd(d_1, \ldots, d_n) = 1$. To determine c write

$$1 = r_1 d_1 + \cdots + r_n d_n$$

and set

$$c = (|r_1|d_1 + \cdots + |r_n|d_n)d_n.$$

It is not hard to see that c satisfies the required property. □

Definition 10.2.2 Let S be a numerical semigroup and c the greatest integer not in S, this number is called the *Frobenius number* of S.

Definition 10.2.3 Let S be a numerical semigroup and c its Frobenius number, the semigroup S is said to be *symmetric* if $c - z \in S$ for all $z \notin S$.

Proposition 10.2.4 *Let S be a numerical semigroup of $\mathbb{N}$ and let c be its Frobenius number. Then S is symmetric if and only if*

$$(c+1)/2 = |\mathbb{N} \setminus S|.$$

Proof. Consider the map

$$\varphi: \mathbb{N} \setminus S \to [0, c] \cap \mathbb{N},$$

given by $\varphi(z) = c - z$. Observe that S is symmetric if and only if

$$\varphi(\mathbb{N} \setminus S) = S \cap [0, c).$$

Because of the decomposition

$$[0, c] = (\mathbb{N} \setminus S) \cup (S \cap [0, c)),$$

one obtains that S is symmetric if and only if $c + 1 = 2|\mathbb{N} \setminus S|$. □

Definition 10.2.5 A *discrete valuation* of a field K is a mapping

$$v : K \longrightarrow \mathbb{Z} \cup \{\infty\}$$

with the following properties. For all $a, b \in K$:

(a) $v(ab) = v(a) + v(b)$,

(b) $v(a + b) \geq \min\{v(a), v(b)\}$,

(c) $v(a) = \infty$ if and only if $a = 0$.

The *valuation ring* of v is the set of all $a \in K$ with $v(a) \geq 0$.

Example 10.2.6 Let $K = k(t)$ be the field of rational functions in one variable over a field k. For $f(t) \in k[t]$ write $f(t) = t^m g(t)$, where $m \in \mathbb{N}$ and $g(t)$ is a polynomial with $g(0) \neq 0$. Define $v(f) = m$, and extend v to a discrete valuation of K by setting

$$v(f/g) = v(f) - v(g).$$

The valuation v is the *t-adic valuation* of K.

Proposition 10.2.7 *Let $(A, \mathfrak{m}, k)$ be Cohen-Macaulay local ring with total ring of fractions Q and $x \in \mathfrak{m}$ a nonzero divisor. If $\dim(A) = 1$, then there is an isomorphism of k-vector spaces*

$$\mathrm{Soc}(A/xA) \simeq \mathfrak{m}^{-1}/A,$$

where $\mathfrak{m}^{-1} = \{z \in Q | \, z\mathfrak{m} \subset A\}$.

Proof. The "multiplication by x" map

$$\varphi: \mathfrak{m}^{-1}/A \to \mathrm{Soc}(A/xA),$$

$\varphi(z + A) = xz + xA$ yields the required isomorphism. □

One of the motivations to study symmetric semigroups comes from the following result. Here we adapt a proof of E. Kunz to make it work for the graded case.

Theorem 10.2.8 ([203]) *Let $A = k[t^{d_1}, \ldots, t^{d_n}]$ be the monomial subring of $k[t]$ generated by $t^{d_1}, \ldots, t^{d_n}$ over the field k and $S = v(A)$, where v is the t-adic valuation on $K = k(t)$. If $S \subset \mathbb{N}$ is a numerical semigroup, then S is symmetric if and only if A is a Gorenstein ring.*

Proof. Assume S is symmetric. Set $\mathfrak{m} = A_+$ and

$$\mathfrak{m}^{-1} = \{x \in K | \, x\mathfrak{m} \subset A\}.$$

Let $h \in \mathfrak{m}^{-1} \setminus A$. If $v(h) \in S$, then $p = v(h) \geq 1$. Hence $h \in k[t]$ and we can write $h = h_1 + h_2$ where $h_1 \in A$ and $v(h_2) \notin S$. We claim that $v(h_2) = c$, where c is the greatest integer not in S. If $v(h_2) < c$, then since S is symmetric $0 < p = c - v(h_2) \in S$. Choose $g \in A$ with $v(g) = p$, then

$$c = v(h_2) + v(g) = v(h_2 g),$$

therefore $h_2 g \notin A$, contradicting $h_2 \in \mathfrak{m}^{-1}$. Thus $v(h_2) \geq c$. To complete the proof of the claim notice that $v(h_2) \leq c$, for otherwise $v(h_2) \in S$.

Next we show that $\mathfrak{m}^{-1} = A + Af$ for any $f \in \mathfrak{m}^{-1} \setminus A$. We may assume $v(f) \notin S$, otherwise write $f = f_1 + f_2$ with $f_1 \in A$ and $f_2 \notin S$ and notice

$A + Af = A + Af_2$. Hence $v(f) = c$ and f is a polynomial in t. We may assume, by dividing f by an appropriate constant, that $f(t) = t^c g(t)$ where $g(0) = 1$ and $g(t) \in k[t]$. Take $x \in \mathfrak{m}^{-1} \setminus A$, if $v(x) \in S$ write $x = x_1 + x_1$ with $x_1 \in A$ and $v(x_2) = c$ and notice $x \in A + Af$ if and only if $x_2 \in A + Af$. Hence we may assume $v(x) \notin S$, in this case $v(x) = c$ and x is a polynomial in t of the form $x(t) = t^c g_1(t)$ with $g_1(0) = 1$. As a consequence $v(x - f) > c$, which gives $x - f \in A$ and $x \in A + Af$. Therefore the length of $\mathfrak{m}^{-1}/A$ as an A-module is 1 and according to Proposition 10.2.7 this implies that A is Gorenstein. Note that the proof shows $\mathfrak{m}^{-1} \subset k[t]$.

Conversely assume A is a Gorenstein ring. Let c be the greatest integer not in S and

$$\begin{aligned} B &= \{t^j + t^{c-1}A \mid 0 \leq j \leq c,\ j \in S\}, \\ C &= \{t^{c+1+i} + t^{c+1}A \mid 0 \leq i \leq c,\ i \notin S\}. \end{aligned}$$

First note $\dim_k A/t^{c+1}A = c + 1$, because the set $B \cup C$ is a k-basis of $A/t^{c+1}A$. Define

$$\varphi\colon B \to C \quad \text{by} \quad \varphi(t^j + t^{c+1}A) = t^{2c+1-j} + t^{c+1}A.$$

We now show that φ is onto. Let $0 \leq i \leq c$ and $i \notin S$, it suffices to prove $c - i \in S$. If $c - i$ is not in S, pick the largest i with $c - i \notin S$. Since the socle of $A/t^{c+1}A$ lives only in degree $2c + 1$, there is $s \in S \setminus \{0\}$ such that

$$t^s(t^{c+1+i} + t^{c+1}A) \neq t^{c+1}A,$$

hence $s + i$ is not in S and thus $i < s + i \leq S$ and $c - (i + s) \notin S$, which contradicts the choice of i. Hence $c - i$ must be in S and φ is onto. Since φ is clearly injective we obtain

$$\frac{c+1}{2} = |[0, c] \cap S|$$

and S is symmetric by Proposition 10.2.4. □

A nice generalization of Theorem 10.2.8 due to R. Fröberg will be presented below; the idea is to find an appropriate expression for the Cohen-Macaulay type of A in terms of the semigroup using an Artinian reduction of A.

Proposition 10.2.9 ([117]) *Let*

$$T(S) = \{x \in \mathbb{N} \setminus S \mid x + s \in S,\ \forall s \in S,\ s > 0\},$$

where S is a numerical semigroup of $\mathbb{N}$ with Frobenius number c. Then S is symmetric if and only if $T(S) = \{c\}$.

Proof. $\Rightarrow$) Let $x \in T(S)$ and assume $x < c$, then $0 < c - x \in S$ and $x + (c - x) = c$ is in S, which is impossible, hence $x = c$.

$\Leftarrow$) Let $z \notin S$, $z > 0$, one must show $c - z \in S$. If $c - z$ is not in S, choose the least positive integer z such that $c - z$ is not in S. Since $c - z \neq c$, by definition of $T(S)$ there is $s \in S$, $s > 0$ with $(c - z) + s$ not in S, hence $z - s > 0$, which contradicts the choice of z. □

Theorem 10.2.10 ([117]) *Let $A = k[S]$ be the semigroup ring of a numerical semigroup S over a field k and*

$$T(S) = \{x \in \mathbb{N} \setminus S \mid x + s \in S, \forall s \in S,\ s > 0\}.$$

Then $|T(S)|$ is equal to the Cohen–Macaulay type A.

Proof. Let $d_1 < \cdots < d_n$ be a minimal generating set of S with $d_1, \ldots, d_n$ relatively prime and $s \in S$, $s > 0$. Consider the map

$$\varphi: T(S) \to \operatorname{Soc}(A/t^s A),\ \varphi(x) = t^{x+s} + t^s A.$$

It is enough to verify that the image of $T(S)$ is a k-vector space basis of $\operatorname{Soc}(A/t^s A)$, because φ is clearly injective. Note that $\operatorname{Soc}(A/t^s A)$ has a k-basis consisting of monomials of the form $t^w + t^s A$, with $w - s \notin S$ and $t^{d_i}(t^w + t^s A) = t^s A$ for all i, write $d_i + w = s + s_i$, where s_i is in S.

We claim $w - s \in T(S)$. It follows that $(w - s) + s'$ is in S for all $s' \in S$, $s' > 0$. Observe that $w - s > 0$; otherwise from $w - s = s_1 - d_1 = s_2 - d_2$ we derive $s_1 = 0$ and $d_2 = s_2 + d_1$, hence $d_2 = \mu d_1$, which is impossible because d_1 and d_2 are part of a minimal generating set of S. Hence $w - s \in T(S)$ and $\varphi(w - s) = w$. To finish note that $\varphi(T(S))$ is linearly independent. □

Arithmetical symmetric semigroups

Let S be a numerical semigroup and c its Frobenius number, that is, c is the greatest integer not in S. The *problem of Frobenius* consists in determining the integer c; in non-mathematical terms this problem can be explained as follows. Given a sufficient supply of coins of various denominations find the largest amount that cannot be formed with these coins. The problem of computing and estimating the Frobenius number has been examined by several authors [117, 248, 254].

Lemma 10.2.11 *Let $u, v \in \mathbb{N}_+$ and assume $\gcd(u, v) = 1$, then*

$$c = uv - u - v$$

is the largest integer not in $U = u\mathbb{N} + v\mathbb{N}$.

Proof. It is easy to see that c is not in U. Indeed if $c \in U$, then

$$uv - u - v = r_1 u + r_2 v$$

for some r_1, r_2 in $\mathbb{N}$, that is,

$$u(v - r_1 - 1) = v(r_2 + 1)$$

and since $\gcd(u, v) = 1$ one derives $v - r_1 - 1 = qv \geq v$, which is impossible.

Next we show $c + i \in U$ for $i \in \mathbb{N}_+$. One has $1 = \mu_1 u + \mu_2 v$, hence

$$c + i = (v - 1 + i\mu_1)u + (-1 + i\mu_2)v,$$

and by the division algorithm one may write $c + i = b_1 u + b_2 v$, where $0 \leq b_2 < u$. From the identity

$$c - b_1 u - b_2 v = (-b_1 - 1)u + (u - 1 - b_2)v = -i \notin U,$$

one concludes $b_1 \geq 0$ and consequently $c + i \in U$. □

Lemma 10.2.12 *Let $n \geq 3$ and $v \geq 1$ be two integers and S the semigroup*

$$S = d_1\mathbb{N} + \cdots + d_n\mathbb{N},$$

where $d_i = d_1 + (i - 1)v$ and $d_1 = r + k(n - 1)$ for some $k, r \in \mathbb{N}$. If $\gcd(d_1, v) = 1$ and $2 \leq r \leq n$, then the greatest integer not in S is equal to $(k + v)d_1 - v$.

Proof. Set $c = (k + v)d_1 - v$, we will show $c + i \in S$ for $i \geq 1$. One may assume $1 \leq i \leq d_1$. Notice

$$c + i = (k + 1)d_1 + \lambda + i,$$

where $\lambda = vd_1 - d_1 - v$. Consider $U = d_1\mathbb{N} + v\mathbb{N}$. By Lemma 10.2.11 λ is the greatest integer not in U. Hence $\lambda + i = ad_1 + bv$, for some a, b in $\mathbb{N}$. One has

$$vd_1 + i = (a + 1)d_1 + (b + 1)v.$$

Hence $b \leq (n - 1)(k + a + 1)$ and we can write $b = j(k + a + 1) + \ell$, for some $1 \leq \ell \leq k + a + 1$ and $0 \leq j \leq n - 2$. Therefore the equality

$$c + i = (k + a + 1)d_1 + bv = (k + a + 1 - \ell)d_{j+1} + \ell d_{j+2},$$

gives $c + i$ in S. It remains to show $c \notin S$. If $c \in S$, then

$$c = \sum_{i=1}^{n} a_i d_i,$$

for some a_i in $\mathbb{N}$. Setting

$$s = \sum_{i=1}^{n} a_i,$$

note that one can rewrite c as:

$$c = (k+1)d_1 + \lambda = \sum_{i=1}^{n} a_i d_i = sd_1 + pv,$$

for some $p \in \mathbb{N}$. Since $\lambda \notin U$, one derives $s \leq k$ and $c \leq sd_n \leq kd_n$, which is a contradiction since $c = kd_n + (r-1)v$. □

Theorem 10.2.13 ([3, 107, 195]) *Let $r, k, n, v \in \mathbb{N}$ and let*

$$d_1 = r + k(n-1) \quad \text{and} \quad d_i = d_1 + (i-1)v$$

be an arithmetical sequence. If $2 \leq r \leq n \neq 2$ and $\gcd(d_1, v) = 1$, then $r = 2$ if and only if the semigroup

$$S = d_1\mathbb{N} + \cdots + d_n\mathbb{N}$$

is symmetric.

Proof. $\Rightarrow$) Let $d_1 = 2 + (n-1)k$. Set

$$c = (k+v)d_1 - v = kd_n + v \quad \text{and} \quad U = d_1\mathbb{N} + v\mathbb{N}.$$

Assume $z \notin S$. First consider the case $z \in U$, write $z = rd_1 + sv$, $r, s \in \mathbb{N}$. By the Euclidean algorithm and Lemma 10.2.12 one can write $z = ad_n + bv$, where a, b are integers so that $0 \leq a \leq k$ and $1 \leq b < d_1$. Writing $a = k - i$, where $0 \leq i \leq k$, we claim that $1 \leq b \leq (n-1)i + 1$. If $b > (n-1)i + 1$, then $z = x_1 d_1 + x_2 v$, with $x_1 = k + v - i$ and $x_2 = b - (n-1)i - 2$. Since $0 \leq x_2 \leq (n-1)x_1$ it is not hard to show that $z \in S$, which is impossible. Hence $1 \leq b \leq (n-1)i + 1$. Set $y_1 = i$ and $y_2 = (n-1)i - b + 1$. Since $0 \leq y_2 \leq (n-1)y_1$ from the equality $c - z = y_1 d_1 + y_2 b$, one gets $c - z \in S$. Now consider the case $z \notin U$. One can write $z = ad_1 + bv$, with $1 \leq b < d_1$ and $a < 0$. Set $w_1 = k + 1$ and $w_2 = d_1 - b - 1$. Using $c - z = w_1 d_1 - (1 + a)d_1 + w_2 v$ and $0 \leq w_2 \leq (n-1)w_1$, gives $c - z \in S$. Therefore S is symmetric.

$\Leftarrow$) Assume S is symmetric and keep the same notations as above. Let $2 \leq r \leq n$ and $d_1 = r + k(n-1)$. As $v \notin S$ one has $c - v \in S$ and

$$c - v = \sum_{i=1}^{n} b_i d_i,$$

where $b_i \in \mathbb{N}$. Set $a = \sum_{i=1}^{n} b_i$ and notice that

$$c - v = (k+1)d_1 + \lambda - v = ad_1 + qv,$$

for some $q \in \mathbb{N}$. Therefore $a \leq k$, otherwise one obtains $\lambda \in U$. From the inequality $c - v \leq ad_n \leq kd_n$, it follows that $r = 2$. □

Exercises

10.2.14 Let $S \neq (0)$ be a semigroup of $\mathbb{N}$. Prove that the following are equivalent:

(a) S is a numerical semigroup.

(b) $c + \mathbb{N} \subset S$ for some $c \in S$.

(c) $|\mathbb{N} \setminus S| < \infty$.

10.2.15 Let S be a numerical semigroup of $\mathbb{N}$. Prove that there is a unique minimal set of generators $\{d_1, \ldots, d_n\}$ of S with $\gcd(d_1, \ldots, d_n) = 1$.

10.2.16 Prove that the semigroup

$$S = 4\mathbb{N} + 5\mathbb{N} + 6\mathbb{N}$$

is symmetric.

10.2.17 Let S be the numerical semigroup

$$S = 5\mathbb{N} + 6\mathbb{N} + 7\mathbb{N} + 8\mathbb{N}$$

and k a field. Prove that $k[S]$ is a Gorenstein ring which is not a complete intersection.

10.2.18 Let $S \subset \mathbb{N}$ be a numerical semigroup with Frobenius number c and $k[S]$ its semigroup ring over a field k. If

$$A = k[S] \subset k[t]$$

has the induced grading, show that the degree of the Hilbert series of $A/t^{c+1}A$ is $2c + 1$.

10.2.19 Let k be a field and let

$$B = k[x_1, x_2]/(x_1^{d_2} - x_2^{d_1}, x_1^{a_1} x_2^{a_2}),$$

where x_i has degree $d_i \in \mathbb{N}_+$. If $\gcd(d_1, d_2) = 1$ and $c + 1 = a_1 d_1 + a_2 d_2$, show that the Hilbert series of B is

$$\frac{(1 - z^{c+1})(1 - z^{d_1 d_2})}{(1 - z^{d_1})(1 - z^{d_2})},$$

where c is the Frobenius number of the proper semigroup $S = d_1\mathbb{N} + d_2\mathbb{N}$. Note $B \simeq A/t^{c+1}\mathrm{A}$, where $A = k[t^{d_1}, t^{d_2}]$.

10.3 Ideals generated by critical binomials

Let us begin by fixing some notation to be used throughout this section. We will always assume that

$$S = \mathbb{N}d_1 + \cdots + \mathbb{N}d_n$$

is a numerical semigroup generated by a sequence $d_1, \ldots, d_n$ of relatively prime positive integers and that Γ is the corresponding monomial curve.

The kernel of the homomorphism of k-algebras

$$\phi: R = k[x_1, \ldots, x_n] \longrightarrow k[S] = k[t^{d_1}, \ldots, t^{d_n}],$$

induced by $\phi(x_i) = t^{d_i}$, is the toric ideal of $k[S]$ and will be denoted by P, where R is a polynomial ring in n variables over a field k.

Definition 10.3.1 A binomial

$$f_i = x_i^{m_i} - \prod_{j \neq i} x_j^{a_{ij}} \in P$$

is called a *critical binomial* with respect to x_i if m_i is the least positive integer such that

$$m_i d_i \in \sum_{j \neq i} d_j \mathbb{N}.$$

Definition 10.3.2 A set $\{f_1, \ldots, f_n\}$ is called a *full set* of critical binomials if f_i is a critical binomial with respect to x_i for all i.

Given a full set F of critical binomials, a main problem is to determine when the ideal generated by F is equal to P.

The interest in studying ideals generated by critical binomials comes from a result of J. Herzog [146] showing that the ideal P of a monomial space curve is generated by a full set of critical binomials (see Theorem 10.3.10). This is not longer the case for toric ideals of monomial curves in higher dimensions:

Example 10.3.3 ([99]) Let m be a positive integer and let

$$(d_1, d_2, d_3, d_4) = (m^2, 2m^2 - 1, 3m^2 + m, 4m^2 + m - 1),$$

then $\nu(P) \geq m$, where $\nu(P)$ is the minimum number of generators of P.

If S is symmetric and generated non redundantly by four integer, then P is minimally generated by 3 or 5 elements [34]; thus even in this case P is not generated by critical binomials.

For simplicity of notation f_i will stand for a critical binomial with respect to x_i, for $i = 1, \ldots, n$. Note that we do not assume $\pm f_1, \ldots, \pm f_n$ distinct.

Lemma 10.3.4 ([146]) *Let*

$$f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}, f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3}, f_3 = x_3^{c_3} - x_1^{c_1}x_2^{c_2}$$

be a full set of critical binomials. If $b_i > 0, c_i > 0$ for all i, then $a_i > 0$ for all i and $a_1 = b_1 + c_1, b_2 = a_2 + c_2, c_3 = a_3 + b_3$.

Proof. If $a_2 = 0$, then $a_3 \geq c_3$. Using the equality

$$f_1 + x_3^{a_3 - c_3} f_3 = x_1^{a_1} - x_1^{c_1}x_2^{c_2}x_3^{a_3 - c_3}$$

we derive a contradiction, hence $a_2 > 0$, by a similar argument $a_3 > 0$.

Define

$$\alpha = (-a_1, a_2, a_3),\ \beta = (b_1, -b_2, b_3),\ \gamma = (c_1, c_2, -c_3).$$

and $(v_1, v_2, v_3) = \alpha + \beta + \gamma$. Next we show $v_i \geq 0$ for all i. Note $a_1 > b_1$, otherwise using $x_1^{b_1 - a_1}x_3^{b_3} f_1 + f_2$ one derives a contradiction. Hence using $x_3^{b_3} f_1 + x_1^{a_1 - b_1} f_2$, one derives $a_3 + b_3 \geq c_3$ and $v_3 \geq 0$. Similarly one shows $v_1, v_2 \geq 0$. Since $v_1 d_1 + v_2 d_2 + v_3 d_3 = 0$ we have $v_i = 0$ for all i. □

Proposition 10.3.5 *Let*

$$f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3},\ f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3},\ f_3 = -x_1^{c_1}x_2^{c_2} + x_3^{c_3}$$

be a full set of critical binomials and $I = (f_1, f_2, f_3)$. If $b_i > 0, c_i > 0$ for all i, then $\{f_1, f_2, f_3\}$ is a Gröbner basis for I with respect to the revlex ordering $x_1 > x_2 > x_3$.

Proof. Use Lemma 10.3.4 and Buchberger's criterion Theorem 2.4.15. □

Proposition 10.3.6 *Let*

$$f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3},\ f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3},\ f_3 = -x_1^{c_1}x_2^{c_2} + x_3^{c_3}$$

be a full set of critical binomials. If $b_i > 0, c_i > 0$ for all i, then the ideal (f_1, f_2, f_3) is equal to the toric ideal P of $k[S]$.

Proof. Let f be a binomial in P and assume we order the monomials with the revlex ordering $x_1 > x_2 > x_3$. As f_1, f_2, f_3 are binomials, by the division algorithm Proposition 2.4.6 we can write

$$f = h_1 f_1 + h_2 f_2 + h_3 f_3 + r, \text{ where } r = x_1^{r_1}x_2^{r_2}x_3^{r_3} - x_1^{s_1}x_2^{s_2}x_3^{s_3} \in P,$$

$r_1, s_1 < a_1$, $r_2, s_2 < b_2$ and such that none of the monomials occurring in r is divisible by the leading term of f_3. To agree with Proposition 10.3.5 one must show $r = 0$.

Assume $r \neq 0$. As $x_3^{r_3}$ or $x_3^{s_3}$ is a factor of r, one may assume that $s_3 = 0$. If $s_1 \leq r_1$ or $s_2 \leq r_2$ we get a contradiction with the minimality of b_2 or a_1 respectively, hence $s_1 > r_1$ and $s_2 > r_2$. From the equality $r = x_1^{r_1} x_2^{r_2} h$, where $h = x_3^{r_3} - x_1^{s_1 - r_1} x_2^{s_2 - r_2}$, one concludes $h \in P$ and $r_3 \geq c_3$. Using the identity

$$h - x_3^{r_3 - c_3} f_3 = x_3^{r_3 - c_3} x_1^{c_1} x_2^{c_2} - x_1^{s_1 - r_1} x_2^{s_2 - r_2}$$

and the minimality of b_2 and a_1 allow us to conclude $s_1 - r_1 > c_1$ and $s_2 - r_2 > c_2$, a contradiction because $x_1^{s_1} x_2^{s_2}$ is not divisible by $x_1^{c_1} x_2^{c_2}$. □

Remark 10.3.7 The proof of Proposition 10.3.6 does not use the hypothesis that the integers d_1, d_2, d_3 are relatively prime.

Proposition 10.3.8 *If*

$$f_1 = x_1^{a_1} - x_2^{a_2} = -f_2, \quad f_3 = x_3^{c_3} - x_1^{c_1} x_2^{c_2}$$

is a full set of critical binomials, then the ideal (f_1, f_3) is equal to the toric ideal P of $k[S]$.

Proof. Let f be a binomial in P and assume we order the monomials with the revlex ordering $x_3 > x_1 > x_2$. As f_1, f_3 are binomials, by the division algorithm we can write

$$f = h_1 f_1 + h_3 f_3 + r,$$

where $r = x_1^{r_1} x_2^{r_2} x_3^{r_3} - x_1^{s_1} x_2^{s_2} x_3^{s_3} \in P$, such that $r_1, s_1 < a_1$ and $r_3, s_3 < c_3$.

Assume $r \neq 0$. As $x_2^{r_2}$ or $x_2^{s_2}$ is a factor of r, one may assume that $s_2 = 0$. Note $s_1 > r_1$ and $s_3 > r_3$. From the equality $r = x_1^{r_1} x_3^{r_3} h$, where $h = x_2^{r_2} - x_1^{s_1 - r_1} x_3^{s_3 - r_3}$, one concludes $h \in P$ and $r_2 \geq a_2$. Therefore the identity

$$h + x_2^{r_2 - a_2} f_1 = x_1^{a_1} x_2^{r_2 - a_2} - x_1^{s_1 - r_1} x_3^{s_3 - r_3},$$

rapidly leads to contradiction with the minimality of c_3 because one has the inequality $a_1 > s_1 - r_1$. □

Definition 10.3.9 The *support of a binomial* $f = x^\alpha - x^\beta$, denoted by $\operatorname{supp}(f)$, is defined as $\operatorname{supp}(x^\alpha) \cup \operatorname{supp}(x^\beta)$, where $\operatorname{supp}(x^\alpha) = \{x_i | \alpha_i > 0\}$.

Theorem 10.3.10 ([146]) *If*

$$f_1 = x_1^{a_1} - x_2^{a_2} x_3^{a_3}, \quad f_2 = x_2^{b_2} - x_1^{b_1} x_3^{b_3}, \quad f_3 = x_3^{c_3} - x_1^{c_1} x_2^{c_2}$$

is a full set of critical binomials, then the ideal (f_1, f_2, f_3) is equal to the toric ideal P of $k[S]$.

Proof. By Proposition 10.3.6 one may assume $\text{supp}(f_i) \neq \{x_1, x_2, x_3\}$ for at least two f_i. Hence one may assume $a_3 = 0$ and the cases to consider are the following: (i) $b_1 = 0$, (ii) $b_3 = 0$, (iii) $c_1 = 0$ and (iv) $c_2 = 0$.

(i) Note $b_3 \geq c_3$, if $b_3 = c_3$, then f_2 is a critical binomial w.r.t x_3 and by Proposition 10.3.8 one concludes that P is generated by $\{f_1, f_2\}$. If $b_3 > c_3$, from the identity

$$f_2 + x_3^{b_3-c_3} f_3 = x_2^{b_2} - x_3^{b_3-c_3} x_1^{c_1} x_2^{c_2},$$

we get $c_2 = 0$. Thus

$$f_2' = f_2 + x_3^{b_3-c_3} f_3 = x_2^{b_2} - x_3^{b_3-c_3} x_1^{c_1} \text{ and}$$
$$f_3' = f_3 + x_1^{c_1-a_1} f_1 = x_3^{c_3} - x_1^{c_1-a_1} x_2^{a_2}.$$

Applying Lemma 10.3.4 to f_1, f_2', f_3' yield $c_1 = a_1$. Hence f_3 is critical w.r.t x_1, using Proposition 10.3.8 one concludes that P is generated by $\{f_2, f_3\}$.

(ii) As $a_2 \geq b_2$ and $b_1 \geq a_1$, from $f_1 - x_2^{a_2-b_2} f_2 = x_1^{a_1} - x_2^{a_2-b_2} x_1^{b_1}$ we get $f_1 = -f_2$ and $P = (f_1, f_3)$ by Proposition 10.3.8.

One may now assume $b_1 > 0$ and $b_3 > 0$, (iii) using $x_2^{c_2-b_2} f_2 + f_3$ we readily see that this case cannot occur, (iv) using $x_2^{a_2-b_2} f_2 + f_1$ we derive a contradiction. This proof is due to A. Alcántar [3]. □

Proposition 10.3.11 ([146]) *If P is the toric ideal of $k[t^{d_1}, t^{d_2}, t^{d_3}]$ over a field k and d_1, d_2, d_3 are relatively prime, then P is a set theoretic complete intersection.*

Critical binomials in four variables

Let I be an ideal generated by a full set of critical binomials $f_1, \ldots, f_n$, one f_i for each variable x_i. It would be desirable to find effective means for determining when I is equal to the toric ideal P. Here we address the case $n = 4$ and $\text{char}(k) = 0$ by presenting the elements of an algorithm, having $f_1, \ldots, f_4$ as input, that decides when $I = P$ occurs.

In the remaining of this Section we study ideals generated by critical binomials in four variables. Our aim is to determine when the ideal generated by a full set of critical binomials is prime and in this case determine its type. If the presentation ideal P of a monomial curve in $\mathbb{A}_k^4$ is generated by a full set of critical binomials, produced by any critical-binomial-algorithm, we predict how these generators should look like (see Theorem 10.3.15).

Some of the results presented here can be shown using different methods introduced by W. Gastinger, E. Kunz and R. Waldi [124, 125], and by H. Bresinsky [37]. A complete description of almost complete intersection monomial curves in $\mathbb{A}_k^4$ was given in [124].

Lemma 10.3.12 *Let*

$$f_i = x_i^{a_i} - x_j^{a_j} x_k^{a_k} x_\ell^{a_\ell}, f_\ell = x_i^{b_i} x_j^{b_j} x_k^{b_k} - x_\ell^{b_\ell}, f_j = x_j^{c_j} - x_i^{c_i} x_k^{c_k} x_\ell^{c_\ell}$$

be a sequence of critical binomials. One has

(i) *If* $a_j > 0, a_\ell > 0, b_i > 0$, *then* $a_i > b_i, b_\ell > a_\ell$.

(ii) *If* $a_j > 0, a_\ell > 0, b_i > 0$ *and* $a_k = b_k = 0$, *then* $a_j + b_j \geq c_j$.

Proof. (i) If $b_i \geq a_i$, the following equality implies that f_ℓ is not critical

$$x_i^{b_i - a_i} x_k^{b_k} x_j^{b_j} f_i - f_\ell = x_\ell^{b_\ell} - x_i^{b_i - a_i} x_j^{a_j + b_j} x_k^{a_k + b_k} x_\ell^{a_\ell}.$$

Hence $a_i > b_i$. By symmetry $b_\ell > a_\ell$. (ii) The minimality of c_j together with the identity $x_j^{b_j} f_i - x_i^{a_i - b_i} f_\ell = x_i^{a_i - b_i} x_\ell^{b_\ell} - x_j^{a_j + b_j} x_\ell^{a_\ell}$ yields the inequality $a_j + b_j \geq c_j$. □

Lemma 10.3.13 *Let* $f_1, \ldots, f_4$ *be the full set of critical binomials:*

$$f_1 = x_1^{a_1} - x_2^{a_2} x_3^{a_3} x_4^{a_4}, \quad f_2 = x_2^{b_2} - x_1^{b_1} x_3^{b_3} x_4^{b_4}, \quad f_3 = x_3^{c_3} - x_1^{c_1} x_2^{c_2} x_4^{c_4},$$
$$f_4 = x_4^{e_4} - x_1^{e_1} x_2^{e_2} x_3^{e_3}.$$

If $a_i > 0, b_i > 0$ *for all* i, *then* $I = (f_1, f_2, f_3, f_4)$ *is not a prime ideal.*

Proof. Assume I is prime. As in Lemma 10.3.12 we have $a_1 > b_1$ and $b_2 > a_2$. From the identity

$$x_3^{b_3} x_4^{b_4} f_1 + x_1^{a_1 - b_1} f_2 = x_2^{a_2}(x_1^{a_1 - b_1} x_2^{b_2 - a_2} - x_3^{a_3 + b_3} x_4^{a_4 + b_4}) = x_2^{a_2} g_1$$

we derive $g_1 \in I$ and there is a monomial M so that

$$x_1^{a_1 - b_1} x_2^{b_2 - a_2} = \begin{cases} \text{(i)} & M x_1^{c_1} x_2^{c_2} x_4^{c_4} \quad \text{or,} \\ \text{(ii)} & M x_1^{e_1} x_2^{e_2} x_3^{e_3}. \end{cases} \tag{10.2}$$

Therefore $c_4 = 0$ or $e_3 = 0$. First we consider the case $c_4 = 0$. If $e_1 > 0$, then $a_1 > e_1$ and $e_4 > a_4$. From the identity

$$x_2^{e_2} x_3^{e_3} f_1 + x_1^{a_1 - e_1} f_4 = x_4^{a_4}(x_1^{a_1 - e_1} x_4^{e_4 - a_4} - x_2^{a_2 + e_2} x_3^{a_3 + e_3}) = x_4^{a_4} g_2$$

we obtain $g_2 \in I$ and there is a monomial M so that

$$x_1^{a_1 - e_1} x_4^{e_4 - a_4} = \begin{cases} \text{(i)} & M x_1^{c_1} x_2^{c_2} x_4^{c_4} = M x_1^{c_1} x_2^{c_2} \quad \text{or,} \\ \text{(ii)} & M x_1^{e_1} x_2^{e_2} x_3^{e_3}. \end{cases} \tag{10.3}$$

If (i) holds, then $c_2 = 0$ and $a_1 - e_1 \geq c_1$, since f_1 is critical and $f_3 = x_3^{c_3} - x_1^{c_1}$ we have a contradiction. If (ii) holds then, $e_2 = e_3 = 0$ and $a_1 - e_1 \geq e_1$, which is impossible because f_1 is critical and $f_4 = x_4^{e_4} - x_1^{e_1}$. Hence $e_1 = 0$.

Next we show $e_2 = 0$. Assume $e_2 > 0$, then $b_2 > e_2$ and $e_4 > b_4$. From the identity

$$x_1^{e_1}x_3^{e_3}f_2 + x_2^{b_2-e_2}f_4 = x_4^{b_4}(x_2^{b_2-e_2}x_4^{e_4-b_4} - x_1^{b_1+e_1}x_3^{b_3+e_3}) = x_4^{b_4}g_3$$

we obtain $g_3 \in I$ and there is a monomial M so that

$$x_2^{b_2-e_2}x_4^{e_4-b_4} = \begin{cases} \text{(i)} & Mx_1^{c_1}x_2^{c_2}x_4^{c_4} = Mx_1^{c_1}x_2^{c_2} \\ \text{(ii)} & Mx_1^{e_1}x_2^{e_2}x_3^{e_3} = Mx_2^{e_2}x_3^{e_3} \end{cases} \text{ or,} \tag{10.4}$$

If (i) occurs, then $c_1 = 0$ and $b_2 - e_2 \geq c_2$, which is a contradiction since f_2 is critical and $f_3 = x_3^{c_3} - x_2^{c_2}$. If (ii) occurs, then $e_1 = e_3 = 0$ and $b_2 - e_2 \geq e_2$, which is also a contradiction since f_2 is critical and $f_4 = x_4^{e_4} - x_2^{e_2}$. Hence $e_2 = 0$. We claim that $c_1 = 0$. If $c_1 > 0$, then $a_1 > c_1$ and $c_3 > a_3$. Using the equality

$$x_2^{c_2}x_4^{c_4}f_1 + x_1^{a_1-c_1}f_3 = x_3^{a_3}(x_1^{a_1-c_1}x_3^{c_3-a_3} - x_2^{a_2+c_2}x_4^{a_4+c_4}) = x_3^{a_3}g_4$$

we see that $g_4 \in I$ and there is a monomial M so that

$$x_1^{a_1-c_1}x_3^{c_3-a_3} = \begin{cases} \text{(i)} & Mx_1^{c_1}x_2^{c_2}x_4^{c_4} = Mx_1^{c_1}x_2^{c_2} \\ \text{(ii)} & Mx_1^{e_1}x_2^{e_2}x_3^{e_3} = Mx_3^{e_3} \end{cases} \text{ or,} \tag{10.5}$$

If (i) holds, then $c_2 = 0$ and $a_1 > c_1$, which is impossible since f_1 is critical and $f_3 = x_3^{c_3} - x_1^{c_1}$. If (ii) holds, then $c_3 > e_3$, which is impossible because f_3 is critical and $f_4 = x_4^{e_4} - x_3^{e_3}$. Therefore $c_1 = 0$. Altogether we have $c_1 = c_4 = 0$ and $e_1 = e_2 = 0$. We claim that $b_2 > c_2$, otherwise if $c_2 \geq b_2$, then from the equality

$$x_2^{c_2-b_2}f_2 + f_3 = x_3^{c_3} - x_1^{b_1}x_2^{c_2-b_2}x_3^{b_3}x_4^{b_4}$$

and using the minimality of c_3 we obtain a contradiction. To complete the proof for the case $c_4 = 0$ notice that the inequality $b_2 > c_2$ also yields a contradiction since $f_3 = x_3^{c_3} - x_2^{c_2}$ and f_2 is critical.

It remains to consider the case $e_3 = 0$. We may assume $c_4 > 0$. We claim that $c_1 = 0$. If $c_1 > 0$, then by (10.5) either (i) or (ii) hold. Since $c_4 > 0$ (ii) holds, as a result $e_2 = 0$ and $a_1 > e_1$, which is a contradiction since f_1 is critical and $f_4 = x_4^{e_4} - x_1^{e_1}$. Hence $c_1 = 0$. Next we show $e_1 = 0$. Assume $e_1 > 0$, by (10.3) we have that (i) or (ii) hold. If (i) holds, then $c_2 = 0$ and $e_4 > c_4$, which is a contradiction since f_4 is critical and $f_3 = x_3^{c_3} - x_4^{c_4}$. If (ii) holds, then $e_2 = 0$ and $a_1 > e_1$, which is a contradiction since f_1 is critical and $f_4 = x_4^{e_4} - x_1^{e_1}$. Hence $e_1 = 0$. Altogether we have $c_1 = 0$ and $e_1 = e_3 = 0$. We claim that $b_2 > e_2$, if on the contrary $e_2 \geq b_2$ from the identity

$$x_2^{e_2-b_2}f_2 + f_4 = x_4^{e_4} - x_1^{b_1}x_2^{e_2-b_2}x_3^{b_3}x_4^{b_4}$$

we contradict the minimality of e_4. To complete the proof notice that $b_2 > e_2$ also leads to a contradiction since f_2 is critical and $f_4 = x_4^{e_4} - x_2^{e_2}$. Therefore I is not a prime ideal. □

Proposition 10.3.14 *Let $I = (f_1, f_2, f_3, f_4)$ be an ideal generated by a full set of critical binomial such that $f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}x_4^{a_4}$, $a_i > 0$ for all i and $f_k = x_k^{b_k} - x_j^{b_j}$ for some k. If I is prime, then we can re-order the variables so that f_2, f_3, f_4 can be written in one of the following forms*

(a) *$f_2 = x_2^{b_2} - x_3^{b_3}$, $f_3 = -f_2$ and $f_4 = x_4^{e_4} - x_2^{e_2}x_3^{e_3}$, where $e_i \geq 0$.*

(b) *$f_2 = x_2^{b_2} - x_3^{c_3}, f_3 = x_3^{c_3} - x_4^{e_4}, f_4 = x_4^{e_4} - x_2^{b_2}$.*

Proof. As $a_i > 0$ for all i we may assume $f_2 = x_2^{b_2} - x_3^{b_3}$. Write

$$f_3 = x_3^{c_3} - x_1^{c_1}x_2^{c_2}x_4^{c_4} \quad \text{and} \quad f_4 = x_4^{e_4} - x_1^{e_1}x_2^{e_2}x_3^{e_3}.$$

If $c_1 > 0$, then Lemma 10.3.12 and Lemma 10.3.13 yield $c_2 = 0$, $a_1 > c_1$, $c_3 > a_3$ and $c_4 > 0$. Using the equality

$$x_4^{c_4}f_1 + x_1^{a_1-c_1}f_3 = x_3^{a_3}(x_1^{a_1-c_1}x_3^{c_3-a_3} - x_2^{a_2}x_4^{a_4+c_4}) = x_3^{a_3}g$$

we obtain $g \in I$ and $x_1^{a_1-c_1}x_3^{c_3-a_3} = Mx_1^{e_1}x_2^{e_2}x_3^{e_3}$ for some monomial M. Therefore $e_2 = 0$, and by Lemma 10.3.12 and Lemma 10.3.13 we obtain $e_1 > 0$ and $e_3 > 0$. From the equation

$$x_1^{e_1}f_3 + x_3^{c_3-e_3}f_4 = x_4^{c_4}(x_3^{c_3-e_3}x_4^{e_4-c_4} - x_1^{e_1+c_1}) = x_4^{c_4}g$$

we obtain $g \in I$ and this is a contradiction since $x_3^{c_3-e_3}x_4^{e_4-c_4}$ is not a multiple of any of the monomials that occur in $f_1, \ldots, f_4$. Therefore $c_1 = 0$. Assume $c_4 > 0$, note that Lemma 10.3.12 yields $c_2 = 0$. If $e_3 > 0$, then $e_1 = e_2 = 0$ and it follows (see argument below) that $f_4 = -f_3$, as required. On the other hand if $e_3 = 0$, we have $e_2 > 0$ and from the equation

$$x_2^{e_2}f_1 + x_1^{a_1-e_1}f_4 = x_4^{a_4}(x_1^{a_1-e_1}x_4^{e_4-a_4} - x_2^{a_2+e_2}x_3^{a_3}) = x_4^{a_4}g$$

we obtain $g \in I$ and there is a monomial M so that $x_1^{a_1-e_1}x_4^{e_4-a_4} = Mx_1^{a_1}$. Hence $e_1 = 0$ and we are in case (b). We may now assume $c_1 = c_4 = 0$. By minimality $c_2 \geq b_2$ and since $f_3 + x_2^{c_2-b_2}f_2 = x_3^{c_3} - x_2^{c_2-b_2}x_3^{b_3}$ we obtain the identity $c_3d_3 = (c_2 - b_2)d_2 + b_3d_3$, which implies $f_3 = -f_2$. Write $f_4 = x_4^{e_4} - x_1^{e_1}x_2^{e_2}x_3^{e_3}$, we claim that $e_1 = 0$. If $e_1 > 0$, then $a_1 > e_1$ and $e_4 > a_4$. Therefore from the equality

$$x_2^{e_2}x_3^{e_3}f_1 + x_1^{a_1-e_1}f_4 = x_4^{a_4}(x_1^{a_1-e_1}x_4^{e_4-a_4} - x_2^{a_2+e_2}x_3^{a_3+e_3}) = x_4^{a_4}g$$

we have $g \in I$. There is a monomial M so that $x_1^{a_1-e_1}x_4^{e_4-a_4} = Mx_1^{e_1}x_2^{e_2}x_3^{e_3}$, it follows that $e_2 = e_3 = 0$, which is a contradiction since f_1 is critical. Therefore $e_1 = 0$ and the proof is complete. □

Next we present a classification theorem.

Theorem 10.3.15 ([5]) *Let f_i be a critical binomial with respect to the variable x_i for all i and $I = (f_1, f_2, f_3, f_4)$. If I is a prime ideal of height 3, then, up to permutation of the variables, f_1, f_2, f_3, f_4 can be written in one of the following forms*

(A) $f_1 = x_1^{a_1} - x_3^{a_3}x_4^{a_4}, f_2 = x_2^{b_2} - x_1^{b_1}x_4^{b_4}, f_3 = x_3^{c_3} - x_2^{c_2}x_4^{c_4}$,
$f_4 = x_1^{e_1}x_2^{e_2}x_3^{e_3} - x_4^{e_4}$,
where $a_i > 0, b_i > 0, c_i > 0, e_i > 0$ for all i.

(B) $f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}x_4^{a_4}, f_2 = x_2^{b_2} - x_3^{b_3}, f_3 = -f_2, f_4 = x_4^{e_4} - x_2^{e_2}x_3^{e_3}$.

(B1) $f_1 = x_1^{a_1} - x_2^{a_2}, f_2 = x_2^{a_2} - x_3^{b_3}, f_3 = x_3^{b_3} - x_1^{a_1}, f_4 = x_4^{e_4} - x_1^{e_1}x_2^{e_2}x_3^{e_3}$.

(B2) $f_1 = x_1^{a_1} - x_2^{a_2}, f_2 = x_2^{a_2} - x_3^{b_3}, f_3 = x_3^{b_3} - x_4^{c_4}, f_4 = x_4^{c_4} - x_1^{a_1}$.

(C) $f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}, f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3}, f_3 = x_3^{c_3} - x_1^{c_1}x_2^{c_2}$,
$f_4 = x_4^{e_4} - x_1^{e_1}x_2^{e_2}x_3^{e_3}$,
where $a_i > 0, b_i > 0, c_i > 0$ and $e_i \geq 0$ for all i.

(D) $f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}x_4^{a_4}, f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3}, f_3 = x_3^{c_3} - x_1^{c_1}x_2^{c_2}$,
$f_4 = x_4^{e_4} - x_2^{e_2}x_3^{e_3}$,
where $a_3 > 0, a_4 > 0$ and $b_i > 0, c_i > 0$ for all i.

(E) $f_1 = x_1^{a_1} - x_4^{a_4}, f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3}, f_3 = x_3^{c_3} - x_1^{c_1}x_2^{c_2}, f_4 = x_4^{e_4} - x_2^{e_2}x_3^{e_3}$,
where $b_i > 0$, $c_i > 0$ for all i and $e_3 > 0$.

Proof. We set $S_i = \operatorname{supp}(f_i)$. Assume that $|S_i| = 4$ for some i, say $|S_1| = 4$. Let $f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}x_4^{a_4}$, $a_i > 0$ for all i. By Lemma 10.3.13 we have $|S_i| \leq 3$ for $i = 2, 3, 4$. Assume that $|S_i| = 3$ for $i \geq 2$. If $x_1 \notin (S_2 \cup S_3 \cup S_4)$, then (C) occurs, hence we may further assume $x_1 \in S_i$ for some $i \geq 2$, say $x_1 \in S_2$. We claim that $x_1 \in S_3 \cup S_4$. If $x_1 \notin S_3 \cup S_4$, then (after permutation of variables) we have

$$f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}x_4^{a_4}, f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3}, f_3 = x_3^{c_3} - x_2^{c_2}x_4^{c_4}, f_4 = x_4^{e_4} - x_2^{e_2}x_3^{e_3},$$

where $a_i > 0, b_i > 0, c_i > 0, e_i > 0$ for all i. Using the identity

$$x_2^{e_2}f_3 + x_3^{c_3-e_3}f_4 = x_4^{c_4}(x_3^{c_3-e_3}x_4^{e_4-c_4} - x_2^{c_2+e_2}) = x_4^{c_4}g$$

we obtain $g \in I$ and this is a contradiction since $x_3^{c_3-e_3}x_4^{e_4-c_4}$ is not a multiple of any of the monomials that occur in the f_i's. Hence $x_1 \in S_3 \cup S_4$. Assume that $x_1 \in S_3 \setminus S_3 \cap S_4$ (the case $x_1 \in S_4 \setminus S_3 \cap S_4$ is similar). Notice that $S_2 = S_3$, otherwise after permutation of variables we have

$$f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}x_4^{a_4}, f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3}, f_3 = x_3^{c_3} - x_1^{c_1}x_4^{c_4}, f_4 = x_4^{e_4} - x_2^{e_2}x_3^{e_3},$$

and, as before, this leads to a contradiction. Thus $S_2 = S_3$ and we are in case (D). Now we may also assume $x_1 \in S_3 \cap S_4$, by similar arguments it

is not hard to see that case (A) must occur. This completes the proof for the case $|S_1| = 4$ and $|S_i| = 3$ for $i = 2,3,4$. Notice that if $|S_1| = 4$ and $|S_2| = 2$, then by Proposition 10.3.14 we are in case (B) or (B1).

The remaining cases are: (i) $|S_i| = 3$ for all i, (ii) $|S_i| = 2$ for all i and (iii) $|S_i| = 3$ for some i, $|S_j| = 2$ for some j and $|S_i| \leq 3$ for all i.

Assume (ii). If $S_i = S_j$, then $f_i = -f_j$ and it follows that we are in case (B). Hence we may assume $S_i \neq S_j$ for $i \neq j$. Up to re-ordering of variables there are two possibilities:

(α) $f_1 = x_1^{a_1} - x_2^{a_2}, f_2 = x_2^{b_2} - x_3^{b_3}, f_3 = x_3^{c_3} - x_1^{c_1}, f_4 = x_4^{e_4} - x_k^{e_k}$.

(β) $f_1 = x_1^{a_1} - x_2^{a_2}, f_2 = x_2^{b_2} - x_3^{b_3}, f_3 = x_3^{c_3} - x_4^{c_4}, f_4 = x_4^{e_4} - x_1^{e_1}$.

Assume (β) and consider the equality

$$h = x_1^{e_1-a_1}(f_1 + x_2^{a_2-b_2}f_2) + f_4 = x_4^{e_4} - x_1^{e_1-a_1}x_2^{a_2-b_2}x_3^{b_3}.$$

Applying Lemma 10.3.12 to f_3 and h yields $a_1 = e_1$ and $a_2 = b_2$. Because of the equality $f_1+f_2+f_4 = x_4^{e_4}-x_3^{b_3}$ it also follows $e_4 = c_4$ and $b_3 = c_3$, that is, we are in case (B2). The same argument can be used in (α) to obtain (B1), more precisely, if (α) holds we can use $h = x_1^{c_1-a_1}f_1 + f_3 = x_3^{c_3} - x_1^{c_1-a_1}x_2^{a_2}$ and f_2 to show that $c_1 = a_1$, and $f_1 + f_3 = x_3^{c_3} - x_2^{a_2}$ and f_2 to show that $c_3 = b_3$ and $a_2 = b_2$.

Assume (i). There are essentially the following six possibilities:

(a) $f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}, f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3}, f_3 = x_3^{c_3} - x_1^{c_1}x_4^{c_4}, f_4 = x_4^{e_4} - x_1^{e_1}x_3^{e_3}$.

(b) $f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}, f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3}, f_3 = x_3^{c_3} - x_1^{c_1}x_4^{c_4}, f_4 = x_4^{e_4} - x_2^{e_2}x_3^{e_3}$.

(c) $f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}, f_2 = x_2^{b_2} - x_3^{b_3}x_4^{b_4}, f_3 = x_3^{c_3} - x_2^{c_2}x_4^{c_4}, f_4 = x_4^{e_4} - x_2^{e_2}x_3^{e_3}$.

(d) $f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}, f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3}, f_3 = x_3^{c_3} - x_1^{c_1}x_4^{c_4}, f_4 = x_4^{e_4} - x_1^{e_1}x_2^{e_2}$.

(e) $f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}, f_2 = x_2^{b_2} - x_1^{b_1}x_4^{b_4}, f_3 = x_3^{c_3} - x_1^{c_1}x_4^{c_4}, f_4 = x_4^{e_4} - x_2^{e_2}x_3^{e_3}$.

(f) $f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}, f_2 = x_2^{b_2} - x_1^{b_1}x_4^{b_4}, f_3 = x_3^{c_3} - x_2^{c_2}x_4^{c_4}, f_4 = x_4^{e_4} - x_1^{e_1}x_3^{e_3}$.

Observe that (c) and (d) correspond to (C) and (D) respectively. It is not hard to check that in the other cases, using f_1 and f_2, I is not prime.

Assume (iii). By symmetry we may assume $f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}$, $a_i > 0$ for all i and $j = 2$ or $j = 4$. First we consider the case $j = 4$, that is, we can write $f_4 = x_4^{e_4} - x_k^{e_k}$ for some k. Set $f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3}x_4^{b_4}$ and $f_3 = x_3^{c_3} - x_1^{c_1}x_2^{c_2}x_4^{c_4}$. If $b_1 > 0$, then $a_1 > b_1$, $b_2 > a_2$ and from the identity

$$x_3^{b_3}x_4^{b_4}f_1 + x_1^{a_1-b_1}f_2 = x_2^{a_2}(x_1^{a_1-b_1}x_2^{b_2-a_2} - x_3^{a_3+b_3}x_4^{b_4}) = x_2^{a_2}g$$

we obtain $g \in I$ and $x_1^{a_1-b_1}x_2^{b_2-a_2} = Mx_1^{c_1}x_2^{c_2}x_4^{c_4}$ for some monomial M. Hence $a_1 > c_1$, $b_2 > c_2$ and $c_4 = 0$, it follows readily that $c_1 > 0$ and $c_2 > 0$. A similar argument shows $b_4 = 0$ and we are in case (C). The case $c_1 > 0$

is symmetric since we can interchange x_2 and x_3. We may now assume $b_1 = c_1 = 0$. If $b_4 = c_4 = 0$, then $f_3 = -f_2$ and (B) occurs. If $b_4 > 0$ and $c_4 > 0$, then $k = 2, 3$ lead to case (B) and $k = 1$ lead to case (E). If $b_4 > 0$ and $c_4 = 0$, then $k = 2$ or $k = 3$ and (B) or (B1) occur. On the other hand if $b_4 = 0$ and $c_4 > 0$, then $c_2 = 0$ and $k = 1$ cannot occur, observe that $k = 2$ and $k = 3$ lead to cases (B1) and (B) respectively. To complete the proof of (iii) it remains to consider the case $j = 2$. Assume $j = 2$, in this case we can write $f_2 = x_2^{b_2} - x_k^{b_k}$ for some k and $f_4 = x_4^{e_4} - x_1^{e_1}x_2^{e_2}x_3^{e_3}$, note that $k \neq 1$. First assume $c_1 > 0$, using f_1 and f_3 we obtain $x_1^{a_1-c_1}x_3^{c_3-a_3} = Mx_1^{e_1}x_2^{e_2}x_3^{e_3}$, hence $e_2 = 0$, $a_1 > e_1$ and $c_3 > e_3$. Notice that $e_1 > 0$ and $e_3 > 0$, otherwise we are in case $j = 4$. Therefore $c_4 = 0$, $k = 4$ and we are in (E). We may now assume $c_1 = 0$, notice $k \neq 1$. If $k = 3$, then (B) or (B1) occur. If $k = 4$ and $e_1 = e_3 = 0$, then (B) occurs. If $k = 4$, $e_1 = 0$ and $e_3 > 0$, then $e_2 = c_2 = 0$ and (B) occurs. We are going to show, by contradiction, that the case $k = 4$, $c_1 = 0$ and $e_1 > 0$ cannot occur. If $k = 4$, $c_1 = 0$ and $e_1 > 0$, then using f_2 and f_4 gives $e_2 = 0$ and we can write

$$f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}, f_2 = x_2^{b_2} - x_4^{b_4}, f_3 = x_3^{c_3} - x_2^{c_2}x_4^{c_4}, f_4 = x_4^{e_4} - x_1^{e_1}x_3^{e_3}$$

where $a_i > 0$ for all i and $e_1 > 0$. Observe that $b_4 \geq e_4$ and consider

$$f_4' = x_4^{b_4-e_4}f_4 + f_2 = x_2^{b_2} - x_1^{e_1}x_3^{e_3}x_4^{b_4-e_4} \in I.$$

It follows, using f_4' and f_1, that $b_2 > a_2$ and $a_1 > e_1$. From the identity

$$x_3^{a_3}f_4' + x_2^{b_2-a_2}f_1 = x_1^{e_1}(x_1^{a_1-e_1}x_2^{b_2-a_2} - x_3^{a_3+e_3}x_4^{b_4-e_4}) = x_1^{e_1}g \in I,$$

we obtain $g \in I$ and there is a monomial M so that

$$x_1^{a_1-e_1}x_2^{b_2-a_2} = \begin{cases} \text{(i)} & Mx_2^{c_2}x_4^{c_4} \\ \text{(ii)} & Mx_1^{e_1}x_3^{e_3} \end{cases} \text{ or,}$$

If (i) occurs, then $c_4 = 0$ and $b_2 > c_2 > 0$, which contradicts the minimality of b_2. On the other hand if (ii) occurs, then $e_3 = 0$ and $a_1 > e_1 > 0$, which contradicts the minimality of a_1. Therefore in both cases we obtain a contradiction and the proof is complete. □

Remark 10.3.16 Here we keep the assumptions and notation of the theorem above. (i) In (E) $a_4 \geq e_4$ and $e_2 = 0$ is not possible. If this is the case we have

$$f_1 = x_1^{a_1} - x_4^{a_4}, f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3}, f_3 = x_3^{c_3} - x_1^{c_1}x_2^{c_2}, f_4 = x_4^{e_4} - x_3^{e_3}.$$

This gives the polynomial $x_4^{a_4-e_4}f_4 + f_1 = x_1^{a_1} - x_4^{a_4-e_4}x_3^{e_3} \in I$ from which $x_1^{a_1} - x_1^{c_1}x_2^{c_2}x_3^{e_3-c_3}x_4^{a_4-e_4} \in I$ since $e_3 \geq c_3$. Thinking about it as a diophantine equation, $a_1 > c_1$ since $c_2 > 0$. Thus $x_1^{a_1-c_1} - x_2^{c_2}x_3^{e_3-c_3}x_4^{a_4-e_4} \in I$ and

$a_1 - c_1 > 0$ since $c_2 > 0$, a contradiction to a_1 being minimal. Therefore in (E) we have $b_i > 0$, $c_i > 0$ for all i, $e_3 > 0$ and $e_2 > 0$. In (E) if $a_4 > e_4$, then $I = (f_1', f_2, f_3, f_4)$, where $f_1' = f_1 + x_4^{a_4 - e_4} f_4 = x_1^{a_1} - x_2^{e_2} x_3^{e_3} x_4^{a_4 - e_4}$, that is, I has a generating set as in (D). In (E) if $a_4 = e_4$, then $I = (f_1', f_2, f_3, f_4)$, where $f_1' = f_1 + f_4 = x_1^{a_1} - x_2^{e_2} x_3^{e_3}$ and I has a generating set as in (C).

(ii) Assume (D), then it follows from the proof of Proposition 10.3.23 that I has a generating set as in (C).

(iii) If (B2) holds, then $I = (f_1, f_2, f_3', f_4)$, where $f_3' = f_3 + f_4 = x_3^{b_3} - x_1^{a_1}$ and I has a generating set as in (B1). If (B1) holds, then $I = (f_1, f_2, f_3', f_4)$, where $f_3' = f_1 + f_3 = -f_2$, after permutation of x_1 and x_4 we obtain that I has a generating set as in (B).

From this remark we obtain the following

Corollary 10.3.17 *Let I be an ideal generated by four critical binomials, one for each variable. If I is a prime ideal of height 3, then there are critical binomials f_1, f_2, f_3, f_4 with respect to x_1, x_2, x_3, x_4 generating I so that, up to permutation of the variables, f_1, f_2, f_3, f_4 can be written in one of the following forms.*

(A) $f_1 = x_1^{a_1} - x_3^{a_3} x_4^{a_4}, f_2 = x_2^{b_2} - x_1^{b_1} x_4^{b_4}, f_3 = x_3^{c_3} - x_2^{c_2} x_4^{c_4}$,
$f_4 = x_1^{e_1} x_2^{e_2} x_3^{e_3} - x_4^{e_4}$,
where $a_i > 0, b_i > 0, c_i > 0, e_i > 0$ for all i.

(B) $f_1 = x_1^{a_1} - x_2^{a_2} x_3^{a_3} x_4^{a_4}, f_2 = x_2^{b_2} - x_3^{b_3}, f_3 = -f_2, f_4 = x_4^{e_4} - x_2^{e_2} x_3^{e_3}$.

(C) $f_1 = x_1^{a_1} - x_2^{a_2} x_3^{a_3}, f_2 = x_2^{b_2} - x_1^{b_1} x_3^{b_3}, f_3 = x_3^{c_3} - x_1^{c_1} x_2^{c_2}$,
$f_4 = x_4^{e_4} - x_1^{e_1} x_2^{e_2} x_3^{e_3}$,
where $a_i > 0, b_i > 0, c_i > 0$ and $e_i \geq 0$ for all i.

Let f_1, f_2, f_3, f_4 be a full set of critical binomials having one of the forms (A)–(E) as in the theorem above, and $I = (f_1, \ldots, f_4)$. The rest of the section is devoted to find the arithmetical conditions, on the initial data, for I to be prime.

Proposition 10.3.18 ([37, 124]) *Let $I = (f_1, f_2, f_3, f_4)$ be an ideal generated by a full set of critical binomials:*

$$f_1 = x_1^{a_1} - x_3^{a_3} x_4^{a_4}, f_2 = x_2^{b_2} - x_1^{b_1} x_4^{b_4}, f_3 = x_3^{c_3} - x_2^{c_2} x_4^{c_4}, f_4 = x_1^{e_1} x_2^{e_2} x_3^{e_3} - x_4^{e_4},$$

so that $a_i > 0, b_i > 0, c_i > 0, e_i > 0$ for all i. Then I is prime if and only if

(a) $a_1 = b_1 + e_1$, $b_2 = c_2 + e_2$, $c_3 = a_3 + e_3$, $e_4 = a_4 + b_4 + c_4$, *and*

(b) $d_4 = a_1 b_2 c_3 - a_3 b_1 c_2$.

Proof. Assume I is prime. By Lemma 10.3.12 we have

$$a_1 > e_1, \quad b_2 > e_2, \quad c_3 > e_3, \quad e_4 > a_4, \quad e_4 > b_4, \quad e_4 > c_4.$$

From the equalities

$$\begin{array}{lcll} x_2^{e_2}x_3^{e_3}f_1 - x_1^{a_1-e_1}f_4 & = & x_4^{a_4}(-x_2^{e_2}x_3^{a_3+e_3} + x_1^{a_1-e_1}x_4^{e_4-a_4}) & = & x_4^{a_4}g_1 \\ x_1^{e_1}x_3^{e_3}f_2 - x_2^{b_2-e_2}f_4 & = & x_4^{b_4}(-x_1^{e_1+b_1}x_3^{e_3} + x_2^{b_2-e_2}x_4^{e_4-b_4}) & = & x_4^{b_4}g_2 \\ x_1^{e_1}x_2^{e_2}f_3 - x_3^{c_3-e_3}f_4 & = & x_4^{c_4}(-x_1^{e_1}x_2^{c_2+e_2} + x_3^{c_3-e_3}x_4^{e_4-c_4}) & = & x_4^{c_4}g_3 \end{array}$$

we obtain $g_1, g_2, g_3 \in I$. Let M be the set of monomials that occur in f_i for $i = 1, 2, 3, 4$. Notice that any monomial p that occur in g_i can be written as $p = p_1p_2$, where p_1 is a monomial and p_2 is in M. As a consequence $a_1 = b_1 + e_1, b_2 = c_2 + e_2, c_3 = a_3 + e_3$. Set

$$\begin{array}{ll} \alpha = (-a_1, 0, a_3, a_4), & \beta = (b_1, -b_2, 0, b_4), \\ \gamma = (0, c_2, -c_3, c_4), & \delta = (e_1, e_2, e_3, -e_4), \end{array}$$

and consider the exact sequence.

$$0 \longrightarrow K = \ker(\psi) \longrightarrow \mathbb{Z}^4 \xrightarrow{\psi} \mathbb{Z} \longrightarrow 0, \ \psi(e_i) = d_i.$$

Since $v = \alpha+\beta+\gamma+\delta \in K$ we obtain $e_4 = a_4+b_4+c_4$. By Theorem 10.1.3 K is generated by $\alpha, \beta, \gamma, \delta$. Hence $I_3(A) = \mathbb{Z}$, where A is the matrix with rows $\alpha, \beta, \gamma, \delta$ and $I_3(A)$ is the ideal generated by the minors of order 3 of A. The equality $-\delta = \alpha + \beta + \gamma$ shows $I_3(A) = (A_{41}, A_{42}, A_{43}, A_{44})$, where A_{4j} are the cofactors of A corresponding to the last row. Since the column vectors $\zeta = (A_{41}, A_{42}, A_{43}, A_{44})$ and $\eta = (d_1, d_2, d_3, d_4)$ are in the kernel of A we have $\ker(A) = \zeta\mathbb{Z} = \eta\mathbb{Z}$. Therefore $d_4 = a_1b_2c_3 - a_3b_1c_2$ and the first part of the proof is complete.

Conversely assume (a) and (b) hold. Observe that $\zeta \in \ker(A) = \eta\mathbb{Z}$ and write $\zeta = a\eta$, $a \in \mathbb{Z}$. Using (b) gives $|a| = 1$, that is, $\mathbb{Z}^4/(\alpha, \beta, \gamma, \delta)$ is a torsion free $\mathbb{Z}$-module. This readily implies $\ker(\psi) = (\alpha, \beta, \gamma, \delta)$. An application of Theorem 10.1.3 gives $\operatorname{rad}(I) = P$. We claim that f_1, f_2, f_3, f_4 form a Gröbner basis with respect to the reverse lexicographical ordering $x_1 > x_2 > x_3 > x_4$, it is enough to show that the S–polynomial of f_i and f_j reduces to zero. As we may assume that the leading terms of f_i and f_j have common factors the claim follows from the identities

$$\begin{array}{ll} S(f_1, f_4) = -x_2^{e_2}x_4^{a_4}f_3 - x_4^{a_4+c_4}f_2, & S(f_2, f_4) = -x_3^{e_3}x_4^{b_4}f_1 - x_4^{a_4+b_4}f_3 \\ S(f_3, f_4) = -x_1^{e_1}x_4^{c_4}f_2 - x_4^{c_4+b_4}f_1. & \end{array}$$

From the claim we derive that x_4 is a regular element on R/I, i.e., R/I is a one dimensional Cohen–Macaulay ring. Altogether P is the only associated

prime of I. Consider the cofactor J_{42} of the Jacobian matrix J of I, it is given by

$$\begin{aligned} f &= a_1c_3b_4x_1^{a_1+b_1-1}x_3^{c_3-1}x_4^{b_4-1} + a_3b_1c_4x_1^{b_1-1}x_2^{c_2}x_3^{a_3-1}x_4^{e_4-1} \\ &\quad + a_4b_1c_3x_1^{b_1-1}x_3^{a_3+c_3-1}x_4^{a_4+b_4-1}. \end{aligned}$$

Since $f \notin P$, Theorem 3.5.10 shows $\operatorname{rad}(I) = I$. Therefore $I = P$. □

In the following corollary we keep the same assumptions and notation as in the proposition above.

Corollary 10.3.19 *If I is prime, then R/I is a Cohen–Macaulay ring of type* 3.

Proof. Since I is prime $a_1 > e_1 \geq 1, b_2 > e_2 \geq 1, c_3 > e_3 \geq 1$. Set $S = R/I$ and $A = S/x_4S$. Notice $A \simeq R_1/I_1$, where $R_1 = k[x_1, x_2, x_3]$ and $I_1 = (x_1^{a_1}, x_2^{b_2}, x_3^{c_3}, x_1^{e_1}x_2^{e_2}x_3^{e_3})$. Set

$$V = \operatorname{Socle}(A) = (I_1 : \mathfrak{m})/I_1.$$

It is readily seen that the socle of A is equal to

$$V = (I_1, x_1^{a_1-1}x_2^{e_2-1}x_3^{c_3-1}, x_1^{e_1-1}x_2^{b_2-1}x_3^{c_3-1}, x_1^{a_1-1}x_2^{b_2-1}x_3^{e_3-1})/I_1.$$

Therefore $\dim_k(V) = 3$ and the type of S is 3. □

Proposition 10.3.20 *Let*

$$f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}x_4^{a_4}, f_2 = x_2^{b_2} - x_3^{b_3}, f_3 = -f_2, f_4 = x_4^{e_4} - x_2^{e_2}x_3^{e_3}$$

be a set of critical binomials with respect to x_1, x_2, x_3, x_4 respectively. Let $I = (f_1, f_2, f_4)$. Then I is a prime ideal if and only if $d_3 = a_1b_2e_4$.

Proof. The proof is the same as that of Proposition 10.3.18. It is enough to make two observations: (i) f_1, f_2, f_4 form a Gröbner bases with the rev. lex. ordering $x_1 > x_4 > x_3 > x_2$, hence x_2 is regular on R/I. (ii) The polynomial

$$f = a_1b_3e_4x_1^{a_1-1}x_3^{b_3-1}x_4^{e_4-1}$$

is an element in the Jacobian ideal of I which is regular on R/I. □

Proposition 10.3.21 *Let*

$$f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}, f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3}, f_3 = x_3^{c_3} - x_1^{c_1}x_2^{c_2}, f_4 = x_4^{e_4} - x_1^{e_1}x_2^{e_2}x_3^{e_3}$$

be a set of critical binomials with respect to x_1, x_2, x_3, x_4 respectively. Let $I = (f_1, f_2, f_3, f_4)$. If $a_i > 0, b_i > 0, c_i > 0$ and $e_i \geq 0$ for all i, then I is a prime ideal if and only if $d_1 = e_4(b_2c_3 - b_3c_2)$.

Proof. We proceed as in the proof of Proposition 10.3.18. It is enough to make three observations: (i) By Remark 10.3.7 $I_1 = (f_1, f_2, f_3)$ is a prime ideal of height two and f_4 is regular on R/I_1. Hence R/I is Cohen–Macaulay. (ii) The polynomial

$$f = e_4 x_4^{e_4-1}(b_1c_3x_1^{b_1-1}x_3^{b_3+c_3-1} + c_1b_3x_1^{c_1+b_1-1}x_2^{c_2}x_3^{b_3-1})$$

is an element in the Jacobian ideal of I which is regular on R/I. (iii) By the previous lemma $a_1 = b_1 + c_1$, $b_2 = a_2 + c_2$ and $c_3 = a_3 + b_3$. □

Let I be a prime ideal as in the proposition above, then we can also determine its type. We have

Corollary 10.3.22 *If I is prime, then R/I is a Cohen–Macaulay ring of type 2.*

Proof. Set $S = R/I$, $A = S/x_1S$, and $I_1 = (x_2^{b_2}, x_3^{c_3}, x_4^{e_4}, x_2^{a_2}x_3^{a_3})$. Since

$$\text{Socle}(A) \simeq (I_1, x_2^{a_2-1}x_3^{c_3-1}x_4^{e_4-1}, x_2^{b_2-1}x_3^{a_3-1}x_4^{e_4-1})/I_1 \text{ for } e_4 \geq 1$$

S is a Cohen–Macaulay algebra of type 2. □

Proposition 10.3.23 *Let*

$$f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}x_4^{a_4}, f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3}, f_3 = x_3^{c_3} - x_1^{c_1}x_2^{c_2}, f_4 = x_4^{e_4} - x_2^{e_2}x_3^{e_3}$$

be a set of critical binomials w.r.t x_1, x_2, x_3, x_4 respectively. If $a_3 > 0, a_4 > 0$ and $b_i > 0, c_i > 0$ for all i, then $I = (f_1, f_2, f_3, f_4)$ is a prime ideal if and only if $d_1 = e_4(b_2c_3 - b_3c_2)$ and $a_4 = qe_4$ for some $q \in \mathbb{N}$.

Proof. Assume I is prime. Consider the identity

$$x_2^{c_2}f_1 + x_1^{a_1-c_1}f_3 = x_3^{a_3}(x_1^{a_1-c_1}x_3^{c_3-a_3} - x_2^{a_2+c_2}x_4^{a_4}) = x_3^{a_3}g.$$

Since I is prime $g \in I$ and there is a monomial M so that

$$x_1^{a_1-c_1}x_3^{c_3-a_3} = \begin{cases} \text{(i)} & Mx_1^{b_1}x_3^{b_3} \\ \text{(ii)} & Mx_2^{e_2}x_3^{e_3}. \end{cases} \quad \text{or,}$$

Since (ii) can not occur we have $a_1 \geq b_1 + c_1$, hence Lemma 10.3.12 yields $a_1 = b_1 + c_1$. Let a_1' be the least positive integers so that $a_1'd_1 \in d_2\mathbb{N} + d_3\mathbb{N}$ and let $f_1'' = x_1^{a_1'} - x_2^{a_2'}x_3^{a_3'}$. By Lemma 10.3.4

$$a_1' = b_1 + c_1,\ b_2 = a_2' + c_2,\ c_3 = a_3' + b_3,$$

in particular $a_1' = a_1$. Notice $a_2' \geq a_2$ and $a_3' \geq a_3$ (see argument below). From

$$f_1'' - f_1 = x_2^{a_2}x_3^{a_3}(x_4^{a_4} - x_2^{a_2'-a_2}x_3^{a_3'-a_3})$$

we obtain $a_4 \geq e_4$. By the division algorithm we can write $a_4 = qe_4 + r$, with $0 \leq r < e_4$ and $q \geq 1$. The equality

$$f_1' = x_1^{a_1} - x_2^{a_2+qe_2}x_3^{a_3+qe_3}x_4^r = f_1 + f_4\left(\sum_{i=1}^{q} x_2^{a_2+(i-1)e_2}x_3^{a_3+(i-1)e_3}x_4^{a_4-ie_4}\right)$$

shows $I' = (f_1', f_2, f_3, f_4) = I$. We now prove $r = 0$. Since

$$f_1' - f_1'' = x_2^{a_2'}x_3^{a_3'} - x_2^{a_2+qe_2}x_3^{a_3+qe_3}x_4^r$$

we conclude

$$\begin{aligned} rd_4 &= (a_2' - a_2 - qe_2)d_2 + (a_3' - a_3 - qe_3)d_3 \\ &= (b_2 - c_2 - a_2 - qe_2)d_2 + (c_3 - a_3 - b_3 - qe_3)d_3. \end{aligned}$$

To prove $r = 0$ it is enough to show that the coefficients of d_2 and d_3 are non-negative. From the equality

$$x_1^{a_1-c_1}f_3 + x_2^{c_2}f_1' = x_3^{a_3+qe_3}(x_3^{c_3-a_3-qe_3}x_1^{a_1-c_1} - x_2^{a_2+c_2+qe_2}x_4^r)$$

it follows that there is a monomial M so that

$$x_1^{a_1-c_1}x_3^{c_3-a_3-qe_3} = \begin{cases} \text{(i)} & Mx_1^{b_1}x_3^{b_3} \\ \text{(ii)} & Mx_2^{e_2}x_3^{e_3}. \end{cases} \quad \text{or,}$$

As (ii) can not occur we obtain $a_3' - a_3 - qe_3 \geq 0$. Similarly using f_1' and f_2 yields $a_2' - a_2 - qe_2 \geq 0$. As in the proof of Theorem 10.3.18 we readily obtain $d_1 = e_4(b_2c_3 - b_3c_2)$. For the converse note that by Proposition 10.3.21 and Lemma 10.3.4 I' is a prime ideal. □

Corollary 10.3.24 *Let*

$$f_1 = x_1^{a_1} - x_4^{a_4}, f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3}, f_3 = x_3^{c_3} - x_1^{c_1}x_2^{c_2}, f_4 = x_4^{e_4} - x_2^{e_2}x_3^{e_3}$$

be a set of critical binomials so that f_i is critical w.r.t x_i for all i. If $b_i > 0$, $c_i > 0$ for all i and $e_3 > 0$, then $I = (f_1, f_2, f_3, f_4)$ is a prime ideal if and only if $d_1 = e_4(b_2c_3 - b_3c_2)$ and $a_4 = qe_4$ for some $q \in \mathbb{N}$.

Proof. Let $f_1' = f_1 + x_4^{a_4-e_4}f_4 = x_1^{a_1} - x_2^{e_2}x_3^{e_3}x_4^{a_4-e_4}$. As $I = (f_1', f_2, f_3, f_4)$ the proof readily follows from Propositions 10.3.21 and 10.3.23. □

10.4 An algorithm for critical binomials

The critical binomials in the next example were computed using the algorithm below. As an application of the results in Section 3 we have

Example (a) If $d_1 = 7$, $d_2 = 8$, $d_3 = 9$, $d_4 = 17$, then

$$f_1 = x_1^5 - x_3^2 x_4,\ f_2 = x_2^2 - x_1 x_3,\ f_3 = x_3^4 - x_1^4 x_2,\ f_4 = x_4 - x_2 x_3$$

is a set of critical binomials generating a height 3 prime ideal of type 2.

(b) If $d_1 = 204$, $d_2 = 855$, $d_3 = 1216$, $d_4 = 1260$, then

$$f_1 = x_1^{47} - x_2^4 x_3^3 x_4^2,\ f_2 = x_2^8 - x_1^{15} x_4^3,\ f_3 = x_3^6 - x_1^{19} x_2^4,\ f_4 = x_4^5 - x_1^{13} x_3^3$$

is a set of critical binomials generating a height 3 prime ideal of type 3.

(c) If $d_1 = 9$, $d_2 = 12$, $d_3 = 18$, $d_4 = 19$, then

$$f_1 = -f_3 = x_1^2 - x_3,\ f_2 = x_2^3 - x_3^2,\ f_4 = x_4^3 - x_1 x_2 x_3^2$$

is a complete intersection prime ideal.

Let us present an algorithm to compute a full set $f_1, \ldots, f_n$ of critical binomials, one f_i for each variable x_i. Let $S = d_1\mathbb{N} + \cdots + d_n\mathbb{N} \subset \mathbb{N}$ and P the toric ideal of $k[S]$, where $\gcd(d_1, \ldots, d_n) = 1$.

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we set $\deg(\alpha) = \sum_{i=1}^n \alpha_i$. Let $\alpha, \beta \in \mathbb{N}^n$. The graded reverse lexicographical ordering in $\mathbb{N}^n$ is defined by $\alpha < \beta$ if $\deg(\alpha) < \deg(\beta)$, or $\deg(\alpha) = \deg(\beta)$ and the first nonzero entry of $\beta - \alpha$ is negative. Since $x_i^{d_j} - x_j^{d_i} \in P$ for $j \neq i$, an upper bound for m_i is given by $d = \min\{d_1, \ldots, \widehat{d_i}, \ldots, d_n\}$. For each i search for $\alpha \in \mathbf{N}^{n-1}$ so that

$$\alpha \cdot (d_1, \ldots, \widehat{d_i}, \ldots, d_n) \equiv 0 \bmod d_i, \tag{10.6}$$

where $\cdot$ denotes the usual inner product, all this from degree zero up to degree d. For the α's satisfying (10.6) compute

$$m_\alpha = \alpha \cdot (d_1, \ldots, \widehat{d_i}, \ldots, d_n)/d_i$$

and take

$$m_i = m_{\alpha_0} \doteq \min\{m_\alpha \mid \alpha \text{ satisfies (10.6)}\}.$$

If we search the α's in increasing order with respect to $<$ we can take α_0 minimum with respect to this ordering. To generate the α's in increasing order, from $(0, \ldots, 0)$ up to a certain degree g, can be accomplished with the following:

Algorithm 10.4.1

```
for i_1 = 0 to g do
    for i_2 = 0 to i_1 do
        ...
            for i_{n-1} = 0 to i_{n-2} do
                α = (i_1 - i_2, i_2 - i_3, ..., i_{n-2} - i_{n-1}, i_{n-1}).
```

Exercises

10.4.2 Let P be the toric ideal of $k[t^{d_1}, t^{d_2}, t^{d_3}]$, where k is a field and d_1, d_2, d_3 are relatively prime positive integers. Prove that the following are equivalent:

(a) P is a complete intersection.

(b) P is Gorenstein.

(c) $S = d_1\mathbb{N} + d_2\mathbb{N} + d_3\mathbb{N}$ is symmetric.

10.4.3 If f_1, f_2, f_3 is a full set of critical binomials, prove the equality

$$\langle \widehat{f_1}, \widehat{f_2}, \widehat{f_3} \rangle = \ker(\psi),$$

where $\psi\colon \mathbb{Z}^3 \to \mathbb{Z}$ is the $\mathbb{Z}$-linear map induced by $\psi(e_i) = d_i$. Recall $\widehat{f} = \alpha - \beta$ if $f = x^\alpha - x^\beta$.

10.4.4 Let $f_1 = x_1^{a_1} - x_2^{a_2}x_3^{a_3}, f_2 = x_2^{b_2} - x_1^{b_1}x_3^{b_3}, f_3 = -x_1^{c_1}x_2^{c_2} + x_3^{c_3}$ be a full set of critical binomials. If $b_i > 0, c_i > 0$ for all i, prove the equalities $d_1 = a_2b_3 + a_3b_2$, $d_2 = a_1b_3 + a_3b_1$, $d_3 = a_1b_2 - a_2b_1$.

10.4.5 Let $g_1, \ldots, g_r$ be a set of binomials in $R = k[x_1, \ldots, x_n]$ such that $g_i(0) = 0$ for all i. If $V(g_1, \ldots, g_r, x_i) = \{0\}$ for all i, prove:

(a) $\cup_{i=1}^r \operatorname{supp}(g_i) = \{x_1, \ldots, x_n\}$

(b) $(\cup_{i\in J} \operatorname{supp}(g_i)) \cap (\cup_{i\notin J} \operatorname{supp}(g_i)) \neq \emptyset$, $\forall\, J \subset \{1, \ldots, r\}$.

Chapter 11

Affine Toric Varieties and Toric Ideals

We will study general toric sets and toric ideals over arbitrary fields. The aim is to examine when a given set of binomials define a toric ideal up to radical and to give a criterion to characterize when a given toric set is an affine toric variety. Here we generalize in a natural way some of the results of Chapter 10.

11.1 Systems of binomials in toric ideals

Let k be any field and D a fixed $m \times n$ matrix with non negative integer entries d_{ij} and with nonzero columns. The *toric set* Γ determined by the matrix D is the set in the affine space $\mathbb{A}_k^n$ given parametrically by

$$x_i = t_1^{d_{1i}} \cdots t_m^{d_{mi}} \qquad (i = 1, \ldots, n).$$

Thus one has

$$\Gamma = \left\{ (t_1^{d_{11}} \cdots t_m^{d_{m1}}, \ldots, t_1^{d_{1n}} \cdots t_m^{d_{mn}}) \in \mathbb{A}_k^n \;\middle|\; t_1, \ldots, t_m \in k \right\}.$$

Let

$$R = k[x_1, \ldots, x_n] \quad \text{and} \quad B = k[t_1, \ldots, t_m]$$

be two polynomial rings over k, graded by

$$\begin{aligned} \deg(x_i) &= d_{1i} + \cdots + d_{mi} \text{ for all } i, \text{ and} \\ \deg(t_j) &= 1 \text{ for all } j, \end{aligned}$$

respectively. Let ϕ be the graded homomorphism of k-algebras:

$$\phi\colon R \longrightarrow k[t_1, \ldots, t_m], \qquad x_i \stackrel{\phi}{\longmapsto} t^{D_i},$$

where $D_i = (d_{1i}, \ldots, d_{mi})$ is the transpose of the ith column of D and

$$t^{D_i} = t_1^{d_{1i}} \ldots t_m^{d_{mi}}.$$

We denote the image of ϕ by $k[\Gamma]$. The kernel of ϕ, denoted by P, is the *toric ideal* of $k[\Gamma]$.

Closely related to the map ϕ is the homomorphism

$$\psi: \mathbb{Z}^n \longrightarrow \mathbb{Z}^m$$

determined by the matrix D in the standard bases of $\mathbb{Z}^n$ and $\mathbb{Z}^m$. Indeed, we have

$$\phi(x^\alpha) = t^{\psi(\alpha)}$$

for all $\alpha \in \mathbb{N}^n$. As a consequence, a binomial $g = x^\alpha - x^\beta$ belongs to $P = \ker(\phi)$ if and only if $\widehat{g} = \alpha - \beta$ belongs to $\ker(\psi)$.

To prove the following two results we will use the notion of simple component of a polynomial with respect to a subgroup of $\mathbb{Z}^n$; see Section 10.1.

Proposition 11.1.1 *If $g_1, \ldots, g_r$ is a set of binomials generating the toric ideal P of $k[\Gamma]$, then $\widehat{g}_1, \ldots, \widehat{g}_r$ generate* $\ker(\psi)$.

Proof. Set $G = \langle \widehat{g}_1, \ldots, \widehat{g}_r \rangle \subset \mathbb{Z}^n$. Consider the equivalence relation $\sim_G$ on the monomials in R given by

$$x^\alpha \sim_G x^\beta$$

if $\alpha - \beta \in G$. Let $\gamma \in \ker(\psi)$ and write $\gamma = \alpha - \beta$, where $\alpha, \beta \in \mathbb{N}^n$. Proving that $\gamma \in G$ amounts to prove that $x^\alpha \sim_G x^\beta$, *i.e.*, that the binomial $g = x^\alpha - x^\beta$ is simple. Now, g belongs to the toric ideal $P = (g_1, \ldots, g_r)$, hence the simple component h of g containing x^α also belongs to P, by Lemma 10.1.2. Since $x^\alpha \notin P$, it follows that $h = g$, hence that g is simple, as required. □

Definition 11.1.2 Let I be an ideal of a ring R and $f \in R$, the *saturation* of I with respect to f is

$$(I: f^\infty) = \bigcup_{i=1}^{\infty} (I: f^i) = \{r \in R \mid rf^i \in I, \text{ for some } i \geq 1\}.$$

If R is a polynomial ring the saturation can be computed using Gröbner bases and the equality

$$(I: f^\infty) = (I, 1 - tf) \cap R,$$

where t is a new variable. See Chapter 2.

For toric ideals the condition (a) of Proposition 2.4.24 will be replaced by a condition which is much easier to verify.

Proposition 11.1.3 ([104]) *Let $g_1, \ldots, g_r$ a set of binomials in the toric ideal P of $k[\Gamma]$ and $I = (g_1, \ldots, g_r)$. If $\mathrm{char}(k) = p \neq 0$ (resp. $\mathrm{char}(k) = 0$), then the following two conditions are equivalent:*

(a_1) $P = \mathrm{rad}(I : z^\infty)$, *where* $z = x_1 \cdots x_n$ *and* $(I : z^\infty) = \cup_{i \geq 1}(I : z^i)$.

(a_2) $p^u \ker(\psi) \subset \langle \widehat{g}_1, \ldots, \widehat{g}_r \rangle$ *for some* $u \in \mathbb{N}$ (*resp.* $\ker(\psi) = \langle \widehat{g}_1, \ldots, \widehat{g}_r \rangle$).

Proof. (a_1) $\Rightarrow$ (a_2): First we consider the case $\mathrm{char}(k) = p \neq 0$. By hypothesis, there exists $u \geq 0$ such that

$$f^{p^u} \in (I : z^\infty) \text{ for all } f \in P.$$

Let $\gamma \in \ker(\psi)$. Write $\gamma = \alpha - \beta$ with $\alpha, \beta \in \mathbb{N}^n$ and $g = x^\alpha - x^\beta$, thus $g \in P$ and $\widehat{g} = \alpha - \beta = \gamma$. Set

$$G = \langle \widehat{g}_1, \ldots, \widehat{g}_r \rangle \subset \mathbb{Z}^n.$$

We know that $x^\delta g^{p^u} \in I$ for some $\delta \in \mathbb{N}^n$ and

$$x^\delta g^{p^u} = x^{\alpha p^u + \delta} - x^{\beta p^u + \delta}.$$

Since the simple components of $x^\delta g^{p^u}$ belong to I and hence to P (see Lemma 10.1.2), and since $x^\delta x^{\alpha p^u}$ does not belong to I because it does not belong to P, it follows that $x^\delta g^{p^u}$ is simple, *i.e.*,

$$\delta + p^u \gamma \sim_G p^u \beta + \delta.$$

Hence $p^u \alpha - p^u \beta \in G$. This shows $p^u \ker(\psi) \subset G$.

Now we consider the case $\mathrm{char}(k) = 0$. Let $G = \langle \widehat{g}_1, \ldots, \widehat{g}_r \rangle$ and let $0 \neq \eta \in \ker(\psi)$. Note $\eta = \alpha - \beta$, where α and β are in $\mathbb{N}^n$, thus $f = x^\alpha - x^\beta$ is in P and $x^\gamma f^q$ is in I for some prime $q \gg 0$ and for some $\gamma \in \mathbb{N}^n$. Expanding f^q one has:

$$x^\gamma f^q = x^{q\alpha + \gamma} - x^{q\beta + \gamma} + \sum_{i=1}^{q-1} (-1)^i \binom{q}{i} x^{\alpha(q-i)} x^{i\beta + \gamma}.$$

Let h be the simple component of $x^\gamma f^q$ with respect to G containing $x^{q\alpha+\gamma}$. If $q\alpha - q\beta \notin G$, then $x^{q\beta+\gamma}$ does not occur in h. Hence

$$h = x^{q\alpha+\gamma} + \sum_{i \in \mathcal{J}} (-1)^i \binom{q}{i} x^{\alpha(q-i)} x^{i\beta+\gamma},$$

for some $\mathcal{J}$ contained in $\{1, \ldots, q-1\}$. Note that $h \in I$ by Lemma 10.1.2. As q is prime, q must divide $\binom{q}{i}$ for $0 < i < q$. Thus making $x_i = 1$ yields $1_k = (sq)1_k$, $s \in \mathbb{Z}$, but since $\mathrm{char}(k) = 0$ one has $1 = qs$, which is

impossible. Hence $q(\alpha-\beta) \in G$. Pick a prime $r \neq q$ such that $x^{\gamma} f^{r} \in I$, by the previous argument $r(\alpha-\beta) \in G$. Since $\gcd(q,r)=1$, we get $\alpha-\beta \in G$.

$(a_1) \Leftarrow (a_2)$: It suffices to prove

$$P \subset \operatorname{rad}(I: z^{\infty}),$$

because the other contention is clear. First we assume $\operatorname{char}(k)=p \neq 0$. Let

$$f = x^{c} - x^{e}$$

be a binomial in P and write

$$g_i = x^{\alpha_i} - x^{\beta_i}.$$

By (a_2) there are integers s_i such that

$$\left(\frac{x^{cp^u}}{x^{ep^u}}\right) - 1 = \left(\frac{x^{\alpha_1}}{x^{\beta_1}}\right)^{s_1} \cdots \left(\frac{x^{\alpha_r}}{x^{\beta_r}}\right)^{s_r} - 1,$$

where we may assume $s_i \geq 0$ by replacing x^{α_i}/x^{β_i} by its inverse if necessary. Hence writing

$$\frac{x^{\alpha_i}}{x^{\beta_i}} = \left(\left(\frac{x^{\alpha_i}}{x^{\beta_i}}\right) - 1\right) + 1$$

and using the binomial theorem it follows that $x^{\gamma} f^{p^u}$ is in I, for some monomial x^{γ}. Therefore $f \in \operatorname{rad}(I: z^{\infty})$.

In the case of characteristic zero, the same arguments as above will work, after replacing p^u by 1 throughout. □

Theorem 11.1.4 ([104]) *Let Γ be a toric set and $g_1, \ldots, g_r$ a set of binomials in the toric ideal P of $k[\Gamma]$. Set $I = (g_1, \ldots, g_r)$. If $\operatorname{char}(k) = p \neq 0$ (resp. $\operatorname{char}(k) = 0$), then $\operatorname{rad}(I) = P$ if and only if*

(a) $p^u \ker(\psi) \subset \langle \widehat{g}_1, \ldots, \widehat{g}_r \rangle$ *for some $u \in \mathbb{N}$ (resp. $\ker(\psi) = \langle \widehat{g}_1, \ldots, \widehat{g}_r \rangle$)*

(b) $\operatorname{rad}(I, x_i) = \operatorname{rad}(P, x_i)$ *for all i.*

Proof. It is a consequence of Proposition 2.4.24 and Proposition 11.1.3. □

Corollary 11.1.5 ([104]) *Let $g_1, \ldots, g_r$ be a set of binomials in the ideal P of the monomial curve Γ. Set $I = (g_1, \ldots, g_r)$. If $\operatorname{char}(k) = p \neq 0$ (resp. $\operatorname{char}(k) = 0$), then $\operatorname{rad}(I) = P$ if and only if*

(a) $p^m \ker(\psi) \subset \langle \widehat{g}_1, \ldots, \widehat{g}_r \rangle$ *for some $m \in \mathbb{N}$ (resp. $\ker(\psi) = \langle \widehat{g}_1, \ldots, \widehat{g}_r \rangle$)*

(b) $V(g_1, \ldots, g_r, x_i) = \{0\}$, *for all i.*

Proof. $\Rightarrow$) By Theorem 11.1.4 one derives condition (a) and

$$\mathrm{rad}(I, x_i) = \mathrm{rad}(P, x_i)$$

for all i. On the other hand

$$\mathrm{rad}(P, x_i) = (x_1, \ldots, x_n),$$

because the height of P is $n-1$ and x_i is regular on R/P. Therefore

$$V(I, x_i) = V(x_1, \ldots, x_n) = \{0\}.$$

$\Leftarrow$) By Proposition 11.1.3 one has

$$P = \mathrm{rad}(I : z^\infty), \quad \text{where} \quad z = x_1 \cdots x_n.$$

Hence there is monomial x^δ and an integer N such that $x^\delta P^N \subset I$.

In order to prove that $\mathrm{rad}(I) = P$, we will show that every prime ideal Q containing I also contains P. If Q contains no variable, then the inclusions

$$x^\delta P^N \subset I \subset Q$$

imply that $P \subset Q$, as desired. If Q contains at least one variable, then we claim that Q contains *all* the variables, implying

$$Q = (x_1, \ldots, x_n) \supset P.$$

This is the point in the proof where we need a special argument, compensating for the fact that the condition (b) above is weaker than that in Theorem 11.1.4. Indeed, up to a renumbering of the variables $x_1, \ldots, x_n$, one may assume that $\{x_1, \ldots, x_s\}$ is the list of all the variables contained in the prime ideal Q, for some integer $1 \leq s \leq n$. Let $a = (a_i)$ be the point in affine space given by

$$a_1 = \cdots = a_s = 0, \quad a_{s+1} = \cdots = a_n = 1.$$

We claim that $a \in V(I)$. Indeed, let $g = x^\alpha - x^\beta$ be any binomial belonging to I. Since g belongs to Q, it easily follows that $x^\alpha \in (x_1, \ldots, x_s)$ if and only if $x^\beta \in (x_1, \ldots, x_s)$. As a consequence, $a^\alpha = a^\beta$, this common value being 0 if $x^\alpha \in (x_1, \ldots, x_s)$, or 1 otherwise. In either case, we have $g(a) = 0$. It follows that $a \in V(I)$, as claimed. But $a_1 = 0$, hence a belongs to $V(I, x_1)$, which is $\{0\}$ by condition (b). Thus $a = 0$, which by definition of a implies that $s = n$. Hence, Q contains all the variables and consequently contains P, as desired. □

Remark 11.1.6 In the proof of Corollary 11.1.5 the underlying hypothesis $\gcd(d_1, \ldots, d_n) = 1$ is unnecessary.

Corollary 11.1.7 *Let $g_1, \ldots, g_r$ be a set of binomials in a toric ideal P. If k is a field of* $\operatorname{char}(k) = p \neq 0$ (resp. $\operatorname{char}(k) = 0$) *and the containment $p^m \ker(\psi) \subset G = \langle \widehat{g}_1, \ldots, \widehat{g}_r \rangle$ holds for some $m \geq 0$* (resp. $\ker(\psi) = G$), *then*

$$V(g_1, \ldots, g_r) \subset V(P) \cup V(x_1 \cdots x_n).$$

Proof. By Proposition 11.1.3, there is a monomial x^δ and an integer N such that $x^\delta P^N \subset I$. It follows that

$$V(I) \subset V(P^N) \cup V(x^\delta) \subset V(P) \cup V(x_1 \cdots x_n),$$

as required. □

Exercises

11.1.8 Let Γ be a toric set and P the toric ideal of $k[\Gamma]$. If k is an infinite field, prove that

$$P = I(\Gamma) \text{ and } V(P) = \overline{\Gamma},$$

where $\overline{\Gamma}$ is the Zariski closure Γ.

11.1.9 Let P be the ideal of a toric set Γ. If I is an ideal of R and $\sqrt{I} = P$, prove that

$$\sqrt{IR'} = PR',$$

where $R' = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is the ring of Laurent polynomials.

Hint Use that R' is the localization of R at the multiplicative set of monomials.

11.1.10 Let I be an ideal generated by a set of binomials in the ideal P of the toric set Γ. Prove that the following conditions are equivalent:

(a_1) $\operatorname{rad}(I \cdot R') = P \cdot R'$, where $R' = k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$

(a_2) $x^\delta P^N \subset I$ for some monomial $x^\delta \in R$ and some integer $N \geq 1$

11.1.11 Let D be an $m \times n$ integral matrix and $\psi\colon \mathbb{Z}^n \to \mathbb{Z}^m$ the homomorphism determined by D. If G is a subgroup of $\ker(\psi)$, prove that

$$\mathrm{T}(\mathbb{Z}^n/G) = \mathrm{T}(\ker(\psi)/G),$$

where T denotes torsion.

11.1.12 Let Γ be an arbitrary toric set and P the toric ideal of $k[\Gamma]$. If $g_1, \ldots, g_r$ is a set of binomials in P such that

$$V(g_1, \ldots, g_r, x_i) = \{0\}$$

for all i, prove that $\dim k[\Gamma] = \dim(R/P) = 1$.

Hint Use the proof of Corollary 11.1.5 to show $\operatorname{rad}(P, x_i) = (x_1, \ldots, x_n)$.

11.1.13 Let D be an $m \times n$ integral matrix with nonzero columns and $\psi\colon \mathbb{Z}^n \to \mathbb{Z}^m$ the homomorphism determined by D. If G is a subgroup of $\ker(\psi)$ and $p \geq 2$ a prime number, prove that the following are equivalent:

(a_1) $p^u \ker(\psi) \subset G$ for some $u \in \mathbb{N}$.

(a_2) $\ker(\psi)/G$ is a finite p-group or $\ker(\psi)/G = (0)$.

(a_3) $\mathbb{Z}^n/G \simeq \mathbb{Z}^s \times H$, where H is a finite p-group or $H = (0)$, and s is equal to $\operatorname{rank}(D)$.

11.1.14 (Rational quartic curve in $\mathbb{P}^3$) Let k be a field and P the toric ideal of $k[\Gamma]$, where Γ is the toric set defined by the matrix:

$$D = \begin{pmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{pmatrix}.$$

Prove $\operatorname{rad}(x_2x_3 - x_1x_4,\ x_3^3 - x_2x_4^2,\ x_2^3 - x_1^2x_3) = P$ and note $x_1x_3^2 - x_2^2x_4 \in P$.

11.2 Affine toric varieties

Our aim here is to use linear algebra to characterize when a toric set Γ is an affine toric variety in terms of the existence of certain roots in the base field and a vanishing condition. We will make use of the fact that any integral matrix is equivalent to a diagonal matrix which is in Smith normal form.

First we fix some notation. Let Γ be a toric set defined by an $m \times n$ matrix $D = (d_{ij})$. According to [229, Theorem II.9], there are invertible integral matrices $U = (u_{ij})$ and $Q = (q_{ij})$ of orders m and n respectively such that

$$L = UDQ = \operatorname{diag}(\lambda_1, \ldots, \lambda_s, 0, \ldots, 0),$$

where s is the rank of D and $\lambda_1, \ldots, \lambda_s$ are the invariant factors of D, that is, λ_i divides λ_{i+1} and $\lambda_i > 0$ for all i.

Notation For use below set

$$U^{-1} = (f_{ij}) \quad \text{and} \quad Q^{-1} = (b_{ij}).$$

In the sequel e_i will denote the i*th* unit vector in $\mathbb{Z}^n$.

Proposition 11.2.1 *If $\psi\colon \mathbb{Z}^n \to \mathbb{Z}^m$ is the homomorphism determined by D in the standard bases, then*

$$\ker(\psi) = \mathbb{Z}q_{s+1} \oplus \cdots \oplus \mathbb{Z}q_n,$$

where q_i corresponds to the ith column of Q.

Proof. Let $x \in \mathbb{Z}^n$ and make the change of variables $y = Q^{-1}x$. As

$$L = UDQ$$

it follows that $Dx = 0$ if and only if $Ly = 0$. Set $y = (y_1, \ldots, y_n)$.

First note $q_i \in \ker(\psi)$ for $i \geq s+1$, because $LQ^{-1}q_i = Le_i$ where e_i is the ith unit vector in $\mathbb{Z}^n$. On the other hand if x is in $\ker(\psi)$, then $\lambda_i y_i = 0$ for $i = 1, \ldots, s$. Thus

$$x = Qy = \sum_{i=s+1}^{n} y_i q_i.$$

To complete the proof observe that the columns of Q are a basis for $\mathbb{Z}^n$. □

Proposition 11.2.2 *Let*

$$\psi: \mathbb{Z}^n \to \mathbb{Z}^m$$

be the linear map determined by D and

$$v_j = \sum_{i=1}^{s} b_{ij} q_i,$$

where q_i correspond to the ith column of Q. If $e_1, \ldots, e_n$ is the standard basis of $\mathbb{Z}^n$, then

$$\{v_i - e_i\}_{i=1}^{n}$$

is a generating set for $\ker(\psi)$.

Proof. Set $Q^{-1} = (b_{ij})$. Note

$$e_j = \sum_{i=1}^{n} b_{ij} q_i$$

for all j, because $QQ^{-1} = I$. Hence one can write

$$v_j = \sum_{i=1}^{s} b_{ij} q_i = e_j - \sum_{i=s+1}^{n} b_{ij} q_i \qquad (j = 1, 2, \ldots, n),$$

and using Proposition 11.2.1 we obtain $v_j - e_j \in \ker(\psi)$ for $j = 1, \ldots, n$. On the other hand from the equality above:

$$\begin{aligned} \sum_{j=1}^{n} q_{jk}(e_j - v_j) &= \sum_{j=1}^{n} q_{jk} \left(\sum_{i=s+1}^{n} b_{ij} q_i \right) = \sum_{i=s+1}^{n} q_i \left(\sum_{j=1}^{n} q_{jk} b_{ij} \right) \\ &= \sum_{i=s+1}^{n} q_i \delta_{ik} = q_k, \end{aligned}$$

for all $k \geq s+1$. As usual $\delta_{ik} = 1$ if $i = k$ and $\delta_{ik} = 0$ otherwise. □

Theorem 11.2.3 ([242]) *Let k be a field and P the toric ideal of $k[\Gamma]$, where $\Gamma \subset k^n$ is the toric set defined by the matrix D. Then $\Gamma = V(P)$ if and only if the following two conditions are satisfied:*

(a) *If $(a_i) \in V(P)$ and $a_i \neq 0$ $\forall i$, then $a_1^{q_{1i}} \cdots a_n^{q_{ni}}$ has a λ_i-root in k for $i = 1, \ldots, s$.*

(b) *$V(P, x_i) \subset \Gamma$ for $i = 1, \ldots, n$.*

Proof. $\Leftarrow$) One invariably has $\Gamma \subset V(P)$. To prove the other containment take a point $a = (a_1, \ldots, a_n)$ in $V(P)$, by condition (b) one may assume $a_i \neq 0$ for all i. Thus using (a) there are $t_1', \ldots, t_s'$ in k such that

$$(t_i')^{\lambda_i} = a_1^{q_{1i}} \cdots a_n^{q_{ni}} = a^{q_i} \qquad (i = 1, \ldots, s). \tag{11.1}$$

For convenience of notation we extend the definition of t_i' by putting $t_i' = 1$ for $i = s+1, \ldots, m$ and $t' = (t_1', \ldots, t_m')$. Set

$$t_j = (t_1')^{u_{1j}} \cdots (t_m')^{u_{mj}} \qquad (j = 1, \ldots, m). \tag{11.2}$$

We claim that $t^{d_k} = t_1^{d_{1k}} \cdots t_m^{d_{mk}} = a_k$ for $k = 1, \ldots, n$. Setting $U^{-1} = (f_{ij})$ and comparing columns in the equality $U^{-1}L = DQ$ one has:

$$\lambda_i f_i = \sum_{j=1}^{n} q_{ji} d_j \qquad (i = 1, 2, \ldots, s), \tag{11.3}$$

where $f_i = (f_{1i}, \ldots, f_{mi})$ and $d_j = (d_{1j}, \ldots, d_{mj})$ denote the ith and jth columns of U^{-1} and D respectively. Next we compare columns in

$$D = (U^{-1}L)Q^{-1}$$

to get:

$$d_k = \sum_{j=1}^{s} \lambda_j b_{jk} f_j \qquad (k = 1, 2, \ldots, n), \tag{11.4}$$

where $Q^{-1} = (b_{ij})$. Using $UU^{-1} = I$ and Eq.(11.2) we rapidly conclude:

$$t^{f_k} = t_k' \qquad (k = 1, \ldots, m). \tag{11.5}$$

From Proposition 11.2.2 we derive $Dv_j = De_j = d_j$ for $j = 1, \ldots, n$, where

$$v_j = \sum_{i=1}^{s} b_{ij} q_i = \left(\sum_{\ell=1}^{s} q_{1\ell} b_{\ell j}, \ldots, \sum_{\ell=1}^{s} q_{n\ell} b_{\ell j} \right) \qquad (j = 1, \ldots, n). \tag{11.6}$$

Hence $D(v_j)_+ = D(e_j + (v_j)_-)$, that is, $x^{(v_j)_+} - x^{e_j + (v_j)_-}$ belongs to the toric ideal P. Using that $a \in V(P)$ yields $a^{(v_j)_+} = a^{e_j + (v_j)_-}$, thus

$$a^{v_j} = a^{e_j} = a_j \qquad (j = 1, \ldots, n). \tag{11.7}$$

Therefore putting altogether

$$t^{d_k} \overset{11.4}{=} t^{\sum_{j=1}^{s} \lambda_j b_{jk} f_j} = (t^{f_1})^{\lambda_1 b_{1k}} \cdots (t^{f_s})^{\lambda_s b_{sk}} \overset{11.5}{=} (t'_1)^{\lambda_1 b_{1k}} \cdots (t'_s)^{\lambda_s b_{sk}}$$
$$\overset{11.1}{=} (a^{q_1})^{b_{1k}} \cdots (a^{q_s})^{b_{sk}} = a^{q_1 b_{1k} + \cdots + q_s b_{sk}} \overset{11.6}{=} a^{v_k} \overset{11.7}{=} a_k$$

for $k = 1, \ldots, n$. Thus $a \in \Gamma$, as required.

$\Rightarrow$) It is clear that (b) holds because $V(P, x_i) \subset V(P)$. To prove (a) take (a_i) in $V(P)$ with $a_i \neq 0$ for all i, then by definition of Γ there are $t_1, \ldots, t_m$ in k such that $a_j = t^{d_j}$ for $j = 1, \ldots, n$. Therefore by Eq. (11.3) one has:

$$t^{\lambda_i f_i} = t^{q_{1i} d_1} \cdots t^{q_{ni} d_n} = a_1^{q_{1i}} \cdots a_n^{q_{ni}}.$$

Thus $(t^{f_i})^{\lambda_i} = a_1^{q_{1i}} \cdots a_n^{q_{ni}}$, as required. □

Corollary 11.2.4 *If k is an algebraically closed field, then*

$$V(P) \subset \Gamma \cup V(x_1 \cdots x_n).$$

Proof. Let $a = (a_i) \in V(P)$ such that $a_i \neq 0$ for all i. Since k is algebraically closed condition (a) above holds. Therefore one may proceed as in the first part of the proof of Theorem 11.2.3 to get $a \in \Gamma$. □

Next we present another consequence that can be used to prove that monomial curves over arbitrary fields are affine toric varieties and generalizes Proposition 10.1.14.

Corollary 11.2.5 *If the columns of D generate $\mathbb{Z}^m$ as $\mathbb{Z}$-module, then the equality $\Gamma = V(P)$ holds if and only if $V(P, x_i) \subset \Gamma$ for all i.*

Proof. Since

$$\mathbb{Z}d_1 + \cdots + \mathbb{Z}d_n = \mathbb{Z}^m,$$

one has $\lambda_i = 1$ for all i, thus condition (a) holds. Therefore Γ is an affine toric variety if and only if (b) holds. □

Corollary 11.2.6 *If k is algebraically closed, then $\Gamma = V(P)$ if and only if $V(P, x_i) \subset \Gamma$ for all i.*

Proof. If k is algebraically closed, then (a) is satisfied, thus Γ is a toric variety if and only if $V(P, x_i) \subset \Gamma$ for all i. □

As a more concrete application we now show that *Veronese toric sets* are affine toric varieties.

Corollary 11.2.7 *Let d be a positive integer and*

$$A = \{(a_1, \ldots, a_m) \in \mathbb{N}^m \mid a_1 + \cdots + a_m = d\}.$$

If k is an algebraically closed field and D the matrix whose columns are the vectors in A, then the toric set Γ determined by D is an affine toric variety.

Proof. Let

$$\mathcal{B} = \{t^a | \, a \in A\} = \{f_1, \ldots, f_m, f_{m+1}, \ldots, f_s\},$$

where

$$s = \binom{d+m-1}{m-1}.$$

One can order the f_i such that $f_i = t_i^d$ for $i = 1, \ldots, m$ and $|\text{supp}(f_i)| \geq 2$ for $i > m$, where $\text{supp}(t^a) = \{t_i | \, a_i > 0\}$.

Fix an integer $1 \leq i \leq s$, it suffices to prove

$$V(P, x_i) \subset \Gamma,$$

where P is the toric ideal associated with D. We use induction on m. Take $a \in V(P, x_i)$. If $i > m$ and $\phi(x_i) = f_i = t_1^{r_1} \cdots t_m^{r_m}$, note that the binomial

$$x_i^d - x_1^{r_1} \cdots x_m^{r_m}$$

belongs to P, hence $a \in V(P, x_j)$ for some $1 \leq j \leq m$. Therefore one may harmlessly assume $1 \leq i \leq m$ and $\phi(x_i) = t_i^d$, for simplicity of notation we assume $i = 1$. Observe that for every $f_j = t_1^{r_1} \cdots t_m^{r_m}$ with $r_1 > 0$ one has $a_j = 0$; indeed since $x_j^d - x_1^{r_1} \cdots x_m^{r_m}$ belongs to P and $a \in V(P, x_1)$ one has $a_j = 0$. Let D' be the submatrix of D obtained by removing the first row and all the columns with nonzero first entry (from top to bottom), and P' the toric ideal of D'. The vector $a' = (a_i | t_1 \notin \text{supp}(f_i))$ is in $V(P')$, because $P' \subset P$. Since $V(P') \subset \Gamma' \cup V(x_2 \cdots x_s)$, where Γ' is the toric set associated with D', by induction one readily obtain $a \in \Gamma$. □

In the light of Exercise 11.2.10, a natural question is whether Γ can be a variety but not a toric variety, to clarify consider:

Example 11.2.8 Let $k = \mathbb{Z}_3$ and $D = (2, 4)$, then

$$\Gamma = \{(0,0)\} \cup \{(1,1)\} = V(x_1 - x_2, x_2^2 - x_2).$$

On the other hand $P = (x_1 - x_2^2)$ and $(1, 2) \in V(P)$. Thus $\Gamma \neq V(P)$

Exercises

11.2.9 Let D be an $m \times n$ integral matrix such that there are invertible integral matrices U and Q such that $UDQ = \text{diag}(\lambda_1, \ldots, \lambda_s, 0 \ldots, 0)$. If Q_1 (resp. Q_2) denote the submatrix of Q (resp. Q^{-1}) obtained by fixing the first s columns (resp. s rows) of Q (resp. Q^{-1}), prove the equality $DQ_1Q_2 = D$.

11.2.10 Let Γ be a toric set and P the toric ideal of $k[\Gamma]$. If k is an infinite field and Γ is a variety defined by an ideal J, then

$$J \subset P \text{ and } \Gamma = V(J) = V(P).$$

11.2.11 Let k be an algebraically closed field and Γ the toric set associated with the matrix:

$$D = \begin{pmatrix} 4 & 3 & 1 & 0 \\ 0 & 1 & 3 & 4 \end{pmatrix}.$$

Prove that Γ is an affine toric variety.

11.2.12 Prove that Corollary 11.2.4 and Corollary 11.2.6 are valid assuming condition (a) of Theorem 11.2.3, instead of assuming k algebraically closed.

11.2.13 (Segre variety) Let m,n be two positive integers. If

$$\Gamma = \{(t_i u_j) \in k^{mn} | \, t_i \in k, \, u_j \in k\} \subset \mathbb{A}_k^{mn}$$

is the toric set associated with the incidence matrix D of a complete bipartite graph $\mathcal{K}_{m,n}$, prove that Γ is an affine toric variety over any field k.

Hint Use that D is totally unimodular (i.e., every minor is 0 or ± 1).

11.3 Curves in positive characteristic

Let I be an ideal of a ring R the *arithmetical rank* of I, denoted by

$$r = \operatorname{ara}(I),$$

is the least positive integer r such that there are $f_1, \ldots, f_r$ with

$$\operatorname{rad}(f_1, \ldots, f_r) = \operatorname{rad}(I).$$

By Krull's principal ideal theorem one has $\operatorname{ara}(I) \geq \operatorname{ht}(I)$, note that equality occurs if I is a set-theoretic complete intersection.

Let k be a field and P the associated toric ideal of a monomial curve Γ in the affine space $\mathbb{A}_k^n$.

In characteristic zero is an open problem whether P is a set theoretic complete intersections, several authors have contributed toward the solution of this problem, see [99, 215, 224, 282], the case $n = 3$ is treated in [35].

If $\operatorname{char}(k) > 0$, then by [73, Theorem 2] and [73, Remark 1] it follows that P is a set-theoretic complete intersection; a result of T. T. Moh [224] shows that P is indeed generated up to radical by $n - 1$ binomials.

Our aim is to give a different constructive proof of Moh's result due to Alcántar, Reyes and Zárate [3, 4] by identifying special generating sets for the solutions of linear diophantine equations.

Lemma 11.3.1 *Let $D = (d_1, \ldots, d_n)$ be a $1 \times n$ matrix with $d_i \in \mathbb{Z} \setminus \{0\}$, then there is a unimodular matrix $Q = (q_2, \ldots, q_n, q_1)$ with entries in $\mathbb{Z}$ whose first $n-1$ columns are of the form:*

$$q_i = \begin{pmatrix} q_{i1} \\ \vdots \\ q_{ii} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad i \geq 2 \text{ and } q_{ii} > 0,$$

such that

(a) $DQ = (0, \ldots, 0, \lambda)$, *and*

(b) $\ker(D) = q_2\mathbb{Z} + \cdots + q_n\mathbb{Z}$, *where λ is equal to* $\gcd(d_1, \ldots, d_n)$.

Proof. The idea is to use an algorithm similar to the one used to diagonalize a matrix with entries in an Euclidean domain. To "diagonalize" D we use elementary operations on the columns of the following matrix:

$$\begin{pmatrix} D \\ I \end{pmatrix},$$

where I is the identity matrix of order n. Note that using elementary operations one can obtain the following reductions:

$$\begin{pmatrix} D \\ I \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & r_2 & d_3 & \cdots & d_n \\ q_{21} & b_{12} & 0 & \cdots & 0 \\ q_{22} & b_{22} & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 0 & 0 & r_3 & d_4 & \cdots & d_n \\ q_{21} & q_{31} & b_{13} & 0 & \cdots & 0 \\ q_{22} & q_{32} & b_{23} & 0 & \cdots & 0 \\ 0 & q_{33} & b_{33} & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

where $r_2 = \gcd(d_1, d_2)$ and $r_3 = \gcd(r_2, d_3)$. In the first reduction we have used elementary operations on the first two columns only, those operations come from the Euclidean algorithm used to compute the greatest common

divisor of d_1 and d_2. Similarly the second reduction can be obtained by performing elementary operations on the second and third columns only, where those operations come from the Euclidean algorithm to compute $\gcd(r_2, d_3)$. Thus by induction we rapidly get the required matrix Q.

To finish the proof use Proposition 11.2.1 to conclude that $q_2, \ldots, q_n$ generate $\ker(D)$ as $\mathbb{Z}$-module. □

Proposition 11.3.2 *Let $D = (d_1, \ldots, d_n)$ be a matrix with $d_i \in \mathbb{Z} \setminus \{0\}$. Then there are $f_2, \ldots, f_n \in \mathbb{Z}^n$ such that*

(a) $f_i = \begin{pmatrix} f_{i1} \\ \vdots \\ f_{ii} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$,

(b) *for each f_i with $i \geq 2$, it holds $f_{ii} > 0$ and $f_{ij} \leq 0$ for $2 \leq j \leq i-1$,*

(c) $\ker(D)$ *is generated by $f_2, \ldots, f_n$ as $\mathbb{Z}$-module.*

Proof. Let $q_2, \ldots, q_n$ be as in Lemma 11.3.1 and let $2 \leq k \leq n$. It will be shown recursively that there are $f_2, \ldots, f_k$ satisfying conditions (a),(b), and

$$\mathbb{Z}q_2 + \cdots + \mathbb{Z}q_k = \mathbb{Z}f_2 + \cdots + \mathbb{Z}f_k.$$

We proceed by induction on k. For $k = 2$, take $f_2 = q_2$. Assume $2 \leq k < n$ and that we have constructed $f_2, \ldots, f_k$ satisfying (a), (b), and such that

$$\mathbb{Z}q_2 + \cdots + \mathbb{Z}q_k = \mathbb{Z}f_2 + \cdots + \mathbb{Z}f_k. \qquad (*)$$

Next define $f_{k+1,k+1} = q_{k+1,k+1} > 0$ and $f_{k+1,k} = q_{k+1,k} - \lambda_k f_{k,k}$, with λ_k large enough to have $f_{k+1,k} \leq 0$. In general for $2 \leq i \leq k$ and haven chosen $\lambda_{i+1}, \ldots, \lambda_k$, define

$$f_{k+1,i} = q_{k+1,i} - \lambda_k f_{k,i} - \cdots - \lambda_{i+1} f_{i+1,i} - \lambda_i f_{i,i},$$

choosing $\lambda_i \gg 0$ to have $f_{k+1,i} \leq 0$. Finally define

$$f_{k+1,1} = q_{k+1,1} - \lambda_k f_{k,1} - \cdots - \lambda_2 f_{2,1},$$

note that we are not requiring $f_{k+1,1} \leq 0$. If we set

$$f_{k+1} = \begin{pmatrix} f_{k+1,1} \\ \vdots \\ f_{k+1,k+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

clearly $f_2, \ldots, f_{k+1}$ satisfy the hypotheses (a), (b) of the proposition. On the other hand

$$f_{k+1} = q_{k+1} - \lambda_k f_k - \cdots - \lambda_2 f_2,$$

using Eq. $(*)$ this implies:

$$f_{k+1} \in \mathbb{Z}q_2 + \cdots + \mathbb{Z}q_{k+1},$$

in addition $q_{k+1} \in \mathbb{Z}f_2 + \cdots + \mathbb{Z}f_{k+1}$, therefore again by Eq. $(*)$ one obtains $\mathbb{Z}f_2 + \cdots + \mathbb{Z}f_{k+1} = \mathbb{Z}q_2 + \cdots + \mathbb{Z}q_{k+1}$. □

Lemma 11.3.3 *Let $D = (d_1, \ldots, d_n)$ be a matrix with $d_i \in \mathbb{Z} \setminus \{0\}$ and $f_2, \ldots, f_n$ in $\mathbb{Z}^n$ such that*

(a) $f_i = \begin{pmatrix} f_{i1} \\ \vdots \\ f_{ii} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$,

(b) $f_{ii} \neq 0$ *for all i, and*

(c) $\ker(D)$ *is generated by $f_2, \ldots, f_n$ as $\mathbb{Z}$-module.*

Then f_{ii} divides d_1 for all $i \geq 2$.

Proof. Since $d_1 e_i - d_i e_1$ belong to $\ker(D)$ for $i \geq 2$, one obtains

$$d_1 e_i - d_i e_1 = \mu_2 f_2 + \cdots + \mu_n f_n, \ \mu_i \in \mathbb{Z}.$$

Clearly one has $d_1 = \mu_n f_{nn}$ if $i = n$ and $\mu_n = 0$ if $i < n$. By reverse induction follows that $\mu_n = \mu_{n-1} = \cdots = \mu_{i+1} = 0$ and $d_1 = \mu_i f_{ii}$. □

Theorem 11.3.4 ([224]) *Let $D = (d_1, \ldots, d_n)$ be an integral matrix with positive entries and P the toric ideal of $k[t^{d_1}, \ldots, t^{d_n}]$. If $\operatorname{char}(k) = p > 0$, then $P = \operatorname{rad}(g_1, \ldots, g_{n-1})$ for some binomials $g_1, \ldots, g_{n-1}$.*

Proof. Let $f_2, \ldots, f_n$ be as in Proposition 11.3.2. First we prove recursively that for $2 \leq k \leq n$ there are $w_2, \ldots, w_k \in \mathbb{Z}^n$ and $r_k \in \mathbb{N}$ so that

(i) $w_i = \begin{pmatrix} w_{i1} \\ \vdots \\ w_{ii} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ with $w_{ij} \leq 0$ for $j = 1, \ldots, i-1$ y $w_{ii} > 0$,

(ii) $p^{r_k}(\mathbb{Z}f_2 + \cdots + \mathbb{Z}f_k) \subset \mathbb{Z}w_2 + \cdots + \mathbb{Z}w_k$.

To prove this we proceed by induction on k, for $k = 2$ it is enough to take $w_2 = f_2$ and $r_2 = 0$. Let $2 \leq k < n$ and assume there are $w_2, \ldots, w_k \in \mathbb{Z}^n$ satisfying (i) and $r_k \in \mathbb{N}$ such that

$$p^{r_k}(\mathbb{Z}f_2 + \cdots + \mathbb{Z}f_k) \subset \mathbb{Z}w_2 + \cdots + \mathbb{Z}w_k. \qquad (**)$$

By Lemma 11.3.3, we have $f_{ii}|d_1$ for $i = 2, \ldots, n$. Then $d_1 = f_{k+1,k+1}\ell$ for some ℓ, let $p^{\alpha_{k+1}}$ be the greatest power of p dividing ℓ, that is, we can write

$$d_1 = f_{k+1,k+1}p^{\alpha_{k+1}}s_{k+1}, \qquad (1)$$

where p does not divide s_{k+1}, hence using that the image $\overline{p}$ of p in $\mathbb{Z}_{s_{k+1}}$ is non zero, one obtains that $\overline{p}$ has finite order. Therefore there is $\beta_{k+1} \in \mathbb{N}_+$ such that $p^{\beta_{k+1}} \equiv 1 \bmod s_{k+1}$, hence $p^{\beta_{k+1}u} \equiv 1 \bmod s_{k+1}$, for all $u \in \mathbb{N}$, consequently there is $c_{k+1} \in \mathbb{N}$ (depending on u) such that

$$p^{\beta_{k+1}u} = s_{k+1}c_{k+1} + 1, \qquad (2)$$

choose u large enough to guarantee $p^{\alpha_{k+1}}f_{k+1,1} - d_{k+1}c_{k+1} < 0$ and define the entries of w_{k+1} as:

$$\begin{array}{ll} w_{k+1,1} = p^{\alpha_{k+1}}f_{k+1,1} - d_{k+1}c_{k+1} & \\ w_{k+1,i} = p^{\alpha_{k+1}}f_{k+1,i} \;\; i = 2, \ldots, k & \\ w_{k+1,k+1} = p^{\alpha_{k+1}}f_{k+1,k+1} + d_1c_{k+1} & \\ w_{k+1,j} = 0 & \text{for } j > k+1 \end{array}$$

Observe that $w_{k+1,i} \leq 0$ for $i = 1, \ldots, k$ and $w_{k+1,k+1} > 0$. Note the equality

$$w_{k+1} = p^{\alpha_{k+1}}f_{k+1} + (d_1c_{k+1})e_{k+1} - (d_{k+1}c_{k+1})e_1.$$

Therefore

$$Dw_{k+1} = p^{\alpha_{k+1}}Df_{k+1} + (d_1c_{k+1})De_{k+1} - (d_{k+1}c_{k+1})De_1 = 0,$$

as a result $w_{k+1} \in \ker(D)$, hence

$$\begin{array}{ll} w_{k+1,k+1} & = p^{\alpha_{k+1}}f_{k+1,k+1} + d_1c_{k+1} \\ & \overset{(1)}{=} p^{\alpha_{k+1}}f_{k+1,k+1} + (p^{\alpha_{k+1}}f_{k+1,k+1})(s_{k+1}c_{k+1}) \\ & = p^{\alpha_{k+1}}f_{k+1,k+1}(1 + s_{k+1}c_{k+1}) \\ & \overset{(2)}{=} p^{\alpha_{k+1}+\beta_{k+1}u}f_{k+1,k+1} \end{array}$$

setting $q_{k+1} = \alpha_{k+1} + \beta_{k+1}u$, one concludes

$$w_{k+1,k+1} = p^{q_{k+1}}f_{k+1,k+1}. \qquad (3)$$

Now, since only the first $k+1$ entries of w_{k+1} and f_{k+1} can possible be different from zero (all the remaining entries are zero), we get from Eq. (3) that only the first k entries of $w_{k+1} - (p^{q_{k+1}})f_{k+1}$ can possible be different from zero (the rest are zero), hence using that $\ker(D)$ is generated by $f_2, \ldots, f_n$, one derives:

$$w_{k+1} - (p^{q_{k+1}})f_{k+1} \in \mathbb{Z}f_2 + \cdots + \mathbb{Z}f_k,$$

which together with Eq.(**) gives $p^{r_k}(w_{k+1} - (p^{q_{k+1}})f_{k+1}) \in \mathbb{Z}(w_2, \ldots, w_k)$, and consequently

$$(p^{q_{k+1}+r_k})f_{k+1} \in \mathbb{Z}(w_2, \ldots, w_k, w_{k+1})$$

setting $r_{k+1} = q_{k+1} + r_k$, one concludes

$$p^{r_{k+1}}\mathbb{Z}(f_2, \ldots, f_{k+1}) \subset \mathbb{Z}(w_2, \ldots, w_{k+1}).$$

Altogether this terminates the induction argument and ends the first step of the proof.

Making $k = n$, we obtain that there are $w_2, \ldots, w_n$ such that the first $i-1$ entries of w_i are negative or zero, the ith entry of w_i is positive, and for $j > i$ the entries of w_i are zero. Furthermore there is p^{r_n} such that

$$p^{r_n}\mathbb{Z}(f_2, \ldots, f_n) \subset \mathbb{Z}(w_2, \ldots, w_n).$$

Set $g_i = \widetilde{w}_{i+1}$, this means that g_i is a binomial such that $\widehat{g}_i = w_{i+1}$. Let $I = (g_1, \ldots, g_{n-1})$. Take $a \in V(I, x_i)$, with $a = (a_1, \ldots, a_n)$, and suppose that $a \neq 0$ and let a_j be the first entry of a equal to zero (such entry exists because at least $a_i = 0$). If $j \neq 1$, one would have

$$g_{j-1} = \widetilde{w}_j = x_j^{w_{jj}} - x_1^{w_{j1}} \cdots x_{j-1}^{w_{j,j-1}}$$

and $a_j = 0$, then some $a_r = 0$ for $r < j$, which is not possible, thus j must be equal to 1. Let us see that $a = (0, \ldots, 0)$ by induction on the coordinates of a, assume $a_1 = \cdots = a_k = 0$ for $1 \leq k < n$. One has:

$$g_k = x_{k+1}^{w_{k+1,k+1}} - x_1^{w_{k+1,1}} \cdots x_k^{w_{k+1,k}}$$

and since $a \in V(I, x_i) \subset V(I)$ and $g_k \in I$, one obtains $a_{k+1} = 0$. Therefore $a = (0, \ldots, 0)$. As a result $V(I, x_i) = 0$ for $i = 1, \ldots, n$, this fact together with

$$p^{r_n}\ker(D) \subset \mathbb{Z}(w_2, \ldots, w_n)$$

prove that the conditions (a) and (b) of Theorem 10.1.3 are satisfied, and consequently $P = \mathrm{rad}(g_1, \ldots, g_{n-1})$. □

Remark 11.3.5 Working over an algebraically closed field k of characteristic zero S. Eliahou [100] proved that the toric ideal P of a monomial curve in $\mathbb{A}_k^n$ is the radical of an ideal generated by n binomials.

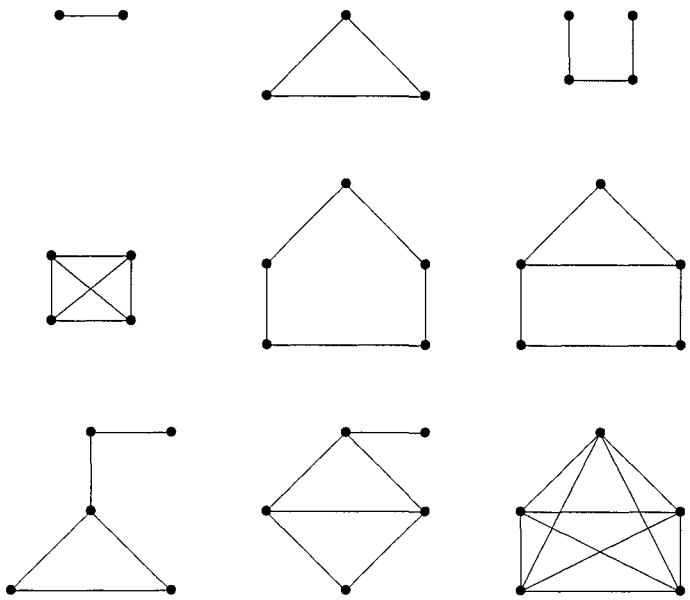
Appendix A

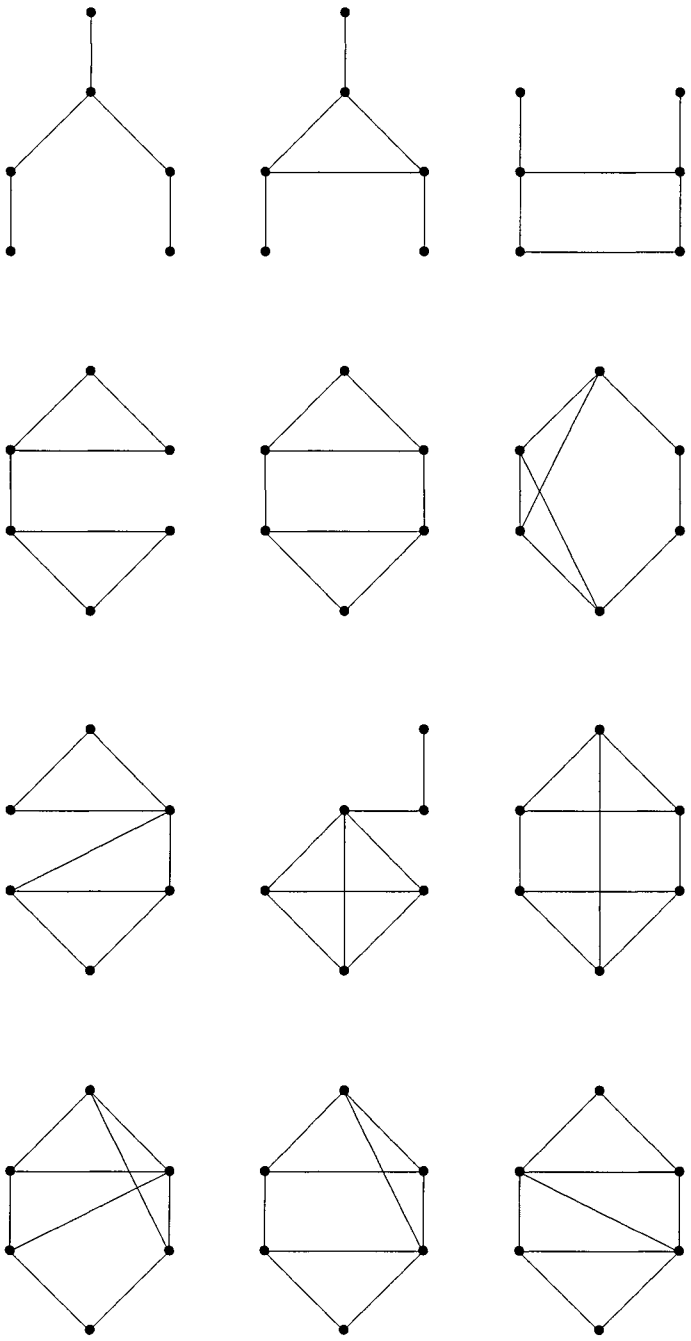
Graph Diagrams

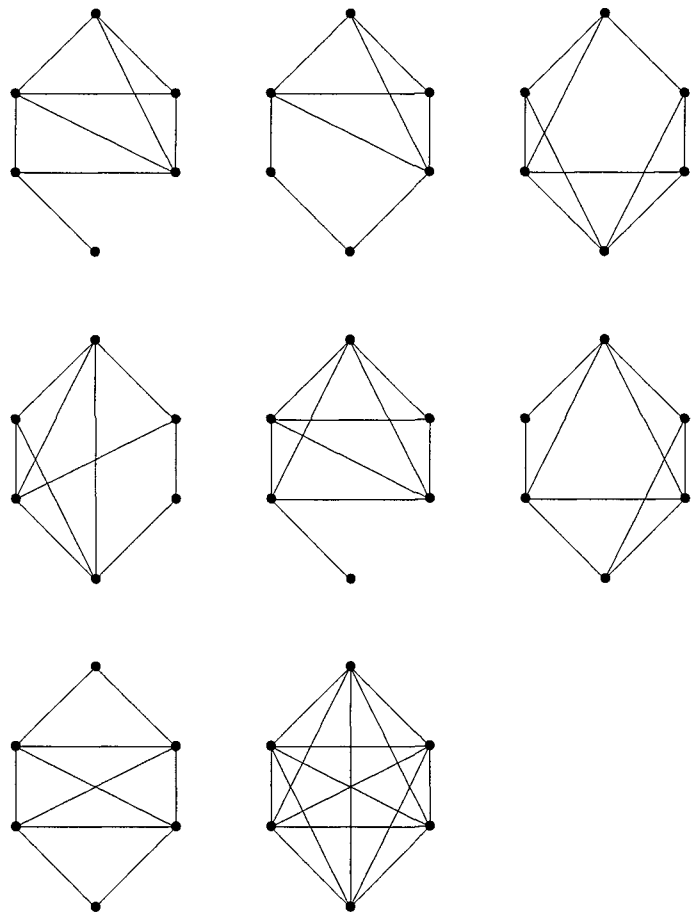
For convenience we will display some Cohen-Macaulay graphs and unmixed graphs with small number of vertices.

A.1 Cohen-Macaulay graphs

The complete list of Cohen-Macaulay connected graphs with at most six vertices is:

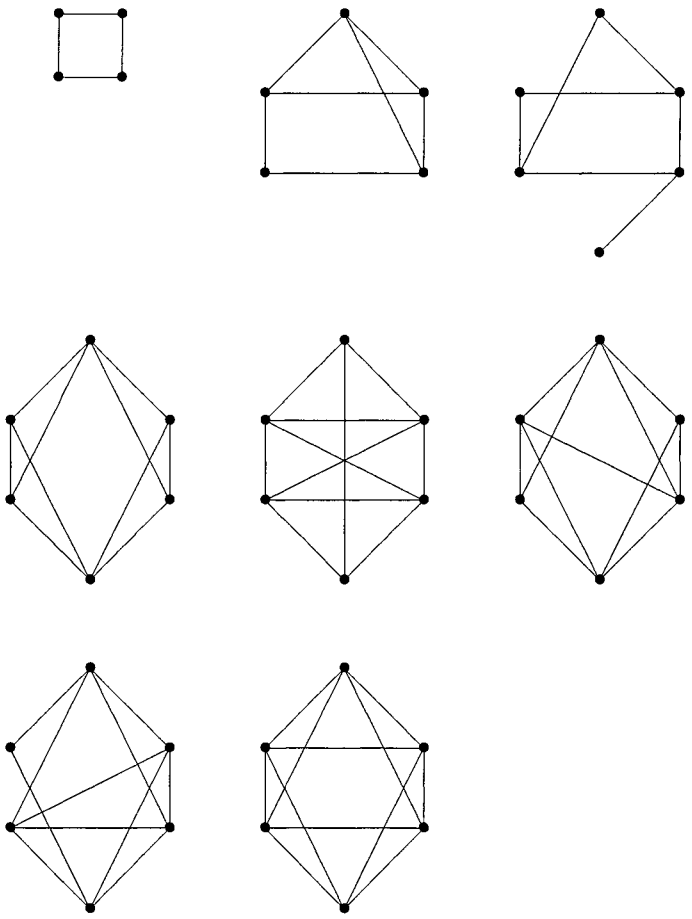






A.2 Unmixed graphs

Next we display the set of all unmixed non Cohen-Macaulay connected graphs with at most six vertices.



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Notation

$(I\colon J^\infty)$, saturation of I, 53
$(N_1 :_R N_2)$, ideal quotient, 4
$A(P)$, Ehrhart ring, 232
$A[F]$, subring generated by F, 23
A_P, normalized Ehrhart ring, 226
$C(G)$, cone over a graph G, 293
$F(M,t)$, Hilbert series, 97
$G_1 * G_2$, join of two graphs, 332
$H(M,i)$, Hilbert function, 14
$H(y,c)$, hyperplane, 210
$H^+(y,c)$, halfspace, 210
$I^{(n)}$, symbolic power, 76
I_Δ, Stanley-Reisner ideal, 142
I_c, ideal of covers, 196
K, a field, 2
$K_\star(\underline{x})$, Koszul complex, 29
$N_G(U)$, neighbor set, 163
$R(-a)$, shift in the graduation, 59
$R^{(k)}$, kth Veronese subring, 203
$R_\mathfrak{p}$, localization at a prime, 3
$S(f,g)$, the S-polynomial, 49
$S^{-1}(M)$, localization, 3
$V(I)$, prime ideals over I, 2
$\operatorname{Ass}(I)$, associated primes, 5
$\operatorname{Ass}_R(M)$, associated primes, 5
Δ^q, q-skeleton of Δ, 145
Δ_I, Stanley-Reisner complex, 142
$\operatorname{Spec}(R)$, spectrum of a ring, 2
$\operatorname{Supp}(M)$, support of a module, 5
$\Upsilon(G)$, minimal covers, 166
$\|x\|$, Euclidean norm, 212
α_+, positive part of a vector, 203
$\alpha_0(G)$, covering number, 165
$\operatorname{ht}(I)$, height of an ideal, 4
$\operatorname{ann}_R(M)$, annihilator of M, 4
$\beta_1(G)$, independence number, 165
$\ell_R(M)$, length of a module, 9
gcd, greatest common divisor, 218
$\langle\ ,\ \rangle$, usual inner product, 210
$\lceil\alpha\rceil$, ceiling of α, 234
lcm, least common multiple, 49
$\operatorname{lk}(F)$, link of a face, 143
$\log(F)$, log set of F, 236
$\log(x^a)$, log of a monomial, 236
$\mathbb{A}^n_k$, affine space, 45
$\mathbb{N}_+$, positive integers, 31
$\mathbb{P}^n_k$, projective space, 56
$\nu(M)$, number of generators, 8
ω_S, canonical module, 106
$\overline{I}$ or I_a, integral closure, 69
$\sqrt{I}$ or $\operatorname{rad}(I)$, radical of I, 4
$|x|$, diamond norm, 217
$e(M)$, multiplicity of M, 101
$e(\mathfrak{q},M)$, relative multiplicity, 85
e_i, unit vector, 156
$f(\Delta)$, the f-vector of Δ, 150
$f \to_F g$, reduction w.r.t F, 49
$h^{\langle j\rangle}$, Macaulay symbol, 119
k, a field, 2
$k[F]$, monomial subring, 201
$k[G]$, edge subring, 282
$k[\Delta]$, Stanley-Reisner ring, 142
$k[\mathbf{x}]$, polynomial ring, 2
$k^* = k \setminus \{0\}$, 48
$r(I)$, reduction number, 339
$x \geq 0$, non negative vector, 218
x^c, monomial, 201
C_n, cycle, 163

$\mathcal{K}_n$, complete graph, 163
$\mathcal{K}_{m,n}$, complete bipartite, 182
$\mathcal{R}(I)$, Rees algebra, 66
$\mathcal{Z}(M)$, zero divisors, 6
$\chi(\Delta)$, Euler characteristic, 141
$\mathbb{R}_+ = \{x \in \mathbb{R} | \, x \geq 0\}$, 211
$\mathbb{R}_+\mathcal{A}$, cone generated by $\mathcal{A}$, 211
$\mathrm{Soc}(M)$, socle of M, 21
$\mathrm{Sym}_R(M)$, symmetric algebra, 65
$\mathrm{adj}(H)$, adjoint of a matrix, 8
$\mathrm{char}(R)$, characteristic of R, 2
$\mathrm{conv}(A)$, convex hull of A, 210
$\deg(v)$, degree of a vertex, 162
$\mathrm{depth}(M)$, depth of a module, 18
$\dim(M)$, Krull dimension, 4
$\mathrm{gr}_I(R)$, associated algebra, 73
iff, if and only if, 2
$\mathrm{in}(I)$, initial ideal, 48
$\mathrm{ri}(X)$, relative interior, 215
s.o.p, system of parameters, 16
$\mathrm{star}(\sigma)$, star of a face, 148
$\mathrm{supp}(\beta)$, support of a vector, 203
$\mathrm{supp}(x^a)$, support of x^a, 131
trdeg, transcendence degree, 34
$\mathrm{type}(M)$, type of a module, 21
w.r.t, with respect to, 14
$\widetilde{H}_q(\Delta; A)$, reduced homology, 139

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