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# Lectures on Matrices

J. H. M. Wedderburn




American Mathematical Society
Providence, Rhode Island

# PREFACE

This book contains lectures on matrices given at Princeton University at various times since 1920. It was my intention to include full notes on the history of the subject, but this has proved impossible owing to circumstances beyond my control, and I have had to content myself with very brief notes (see Appendix I). A bibliography is given in Appendix II. In compiling it, especially for the period of the last twenty-five years, there was considerable difficulty in deciding whether to include certain papers which, if they had occurred earlier, would probably have found a place there. In the main, I have not included articles which do not use matrices as an algebraic calculus, or whose interest lies in some other part of mathematics rather than in the theory of matrices; but consistency in this has probably not been attained.

Since these lectures have been prepared over a somewhat lengthy period of time, they owe much to the criticism of many friends. In particular, Professor A. A. Albert and Dr. J. L. Dorroh read most of the MS making many suggestions, and the former gave material help in the preparation of the later sections of Chapter X.

J. H. M. WEDDERBURN.

*Princeton, N. J.,*
*July 20, 1934.*

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## APPENDIX II

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# CORRIGENDA

page 4, line 9 from top: on second $\Sigma$ read j for p

page 6, Theorem 1, add: *and conversely, if a matrix is commutative with every other matrix, it is a scalar matrix.*

page 7, line 12 from foot; for first and third A read $|A|$

page 11, lines 10, 11 from foot: for $(Q'Q)$ read $(Q'Q)^{-1}$

page 13, line 9 from foot: for $Sg_ix$ read $S\bar{g}_ix$

page 13, line 6 from foot: for $\bar{g}_{s+1}$ read $\bar{g}'_{s+1}$

page 14, line 3 from top: before $a_{ji}e_j$ read $\Sigma$.

page 18, line 8 from foot: for j read $\gamma_j$

page 20, line 8 from foot: for $r + 1$ read $r - 1$

page 30, line 13 from top: for $=$ read $-$

page 31, line 4 from foot: for second $x_i$ read $xe_i$; add $e_1 + e_2 = 1$

page 42, equation (16): for 1 read $-1$

page 54, line 14 from foot; for (12) read (13)

page 54, line 6 from foot: for (14) read (15)

page 54, line 3 from foot: for (15) read (16)

page 54, line 2 from foot: for (13) read (14)

page 56, line 12 from top: correct term after $\Sigma$ to read $k_{i,\beta+j}z_i$

page 67, lines 4, 5, 6, 7: the exponent n on the second last C should read $\binom{n}{r}$.

page 68, line 11 from foot: before $\Sigma$ read $(-1)^r$

page 68, line 8 from foot: before $|\mathfrak{A}|$ read $(-1)^r$

page 74, line 11 from foot: for $r = 1$ read $r = 3$

page 81, line 4 from foot: for $1/\beta_1!$ read $\beta_1!$ with similar change in last line

page 84, line 13 from foot: interchange i and j.

page 85, line 8 from foot: for $\mathfrak{F}_1$ read $\overline{\mathfrak{F}}_1$

page 86, line 7 from top: for first $e_1$ read $e_i$

page 92, line 11 from foot: delete from "and if" to end of paragraph

page 101, line 6 from foot: after hermitian insert $A = A'$

page 103, line 4 from foot: delete first 0; for $q = t + 1$ read $q = s + 1$

page 112, equation (23): for { } read [ ]

page 116, line 7 from top: add Every power series converges when x is nilpotent.

page 119, line 9 from top: for "at least . . . first" read "the H.C.F. of the t's is relatively prime to m"

page 122, line 4 from foot: multiply bracket by $\epsilon^{\mu_i}$ and delete same inside

page 122, equation (30): for $g_{ij}$ read $p_{ij} = \epsilon^{-\mu_i} g_{ij}$

page 123, lines 2 and 3 from top: for $g_{ij}$ read $p_{ij}$

page 123, equations (32) and (33): for $\pi$ read $2\pi$

page 125, line 4 from top: read $\alpha_1^{r_1}(\lambda)$, $\alpha_2^{r_2}(\lambda)$, $\cdots$, $\alpha_k^{r_k}(\lambda)$

page 126, line 13 from top: for | | read ‖ ‖

page 126, equation (45): for first $\alpha$ read $a$

page 129, equation (63): in first term the bars should be heavy

page 129, line 5 from foot: for |x| read ‖x‖

page 134, line 6 from top: multiply right side of equation by 2

page 136, line 10 from top: for $\xi_r$ read $\xi_{r-1}$

page 137, equation (103): read $\dot{p} = -\partial_q \mathfrak{H}$

page 144, equation (24): read $x'axa^{-1}$

page 156, line 6 from top: for second = read $\leq$ and add "$\leq A$, whence $A = \Sigma A_{ij}$"

page 164, line 8 from top: for primitive read minimal

page 164, line 7 from foot: for invariant read semi-invariant

page 164, last line: before "complete" insert "suitably chosen"

page 166, line 10 from foot: for equivalent read invariant

page 166, line 5 from foot: for first $B_2$ read $B_1$

page 167, Theorem 9: for $j \neq k$ read $i \neq t$

page 171, line 5 from top: delete 80