CHAPTER IX

THE AUTOMORPHIC TRANSFORMATION OF A BILINEAR FORM

9.01 If the variables of a bilinear form whose matrix is a are transformed cogrediently by a matrix x, the matrix of the new bilinear form is $x'ax$; when this new form is identical with the old, the transformation is said to be *automorphic.* The problem of finding all automorphic transformations of a is therefore equivalent to solving the equation

$$x'ax = a. \tag{1}$$

We shall assume for the present that $|\, a \,| \neq 0$ in which case also $|\, x \,| \neq 0$.

It follows from (1) that $x'a = ax^{-1}$. Hence, if $f(\lambda)$ is a scalar polynomial

$$[f(x)]'a = f(x')a = af(x^{-1}). \tag{1'}$$

In particular, if $f(\lambda) = (1 - \lambda)/(1 + \lambda)$ and $y = f(x^{-1})$, then

$$y = \frac{1 - x^{-1}}{1 + x^{-1}} = \frac{x - 1}{x + 1} = -f(x) \tag{2}$$

provided $|\, x + 1 \,| \neq 0$. Hence from (1') $y'a = -ay$ so that

$$y' = -aya^{-1}. \tag{3}$$

Conversely, if y satisfies (3) and $|\, 1 - y \,| \neq 0$, then $x = (1 + y)/(1 - y)$ is a solution of (1) such that $|\, x + 1 \,| \neq 0$. For from (3) $f(y')a = af(-y)$ so that

$$x'a = \frac{1 + y'}{1 - y'}a = a\frac{1 - y}{1 + y} = ax^{-1}.$$

Similarly, if $|\, x - 1 \,| \neq 0$, we may set $x = (1 - y)/(1 + y)$ and then y is a solution of (3) such that $|\, 1 + y \,| \neq 0$, and conversely. The effect of the transformation (2) is therefore to reduce the solution of (1), which is quadratic in x, to that of (3), which is linear in y, except when both 1 and -1 are roots of x. It is because (3) is linear that it is more convenient than (1); in particular if we regard

$$y'a + ay = 0, \qquad y = ||\, \eta_{pq} \,||, \qquad a = ||\, \alpha_{pq} \,||, \tag{3'}$$

as a system of n^2 linear homogeneous equations in the η's, then the rank of the system gives the number of parameters which enter into the solution when those values of y (or of x) are excluded for which both 1 and -1 are roots.

Since the main problem is thus reduced to the solution of linear equations, it may be regarded as solved; the solution, however, can be given a somewhat more definite form as we shall now show.

9.02 **The equation** $y' = \pm aya^{-1}$. We shall consider in place of (3) the more general equation

$$y' = \delta aya^{-1}, \qquad \delta = \pm 1. \tag{4}$$

Forming the transverse we get $y = \delta a'^{-1}y'a'$ or $y' = \delta a'ya'^{-1}$, whence

$$ya^{-1}a' = a^{-1}a'y, \qquad a'ya'^{-1} = aya^{-1} \tag{5}$$

so that y is commutative with $a^{-1}a'$. Now from (4) we have $y = \delta a^{-1}y'a$ and hence

$$2y = y + \delta a^{-1}y'a.$$

But if b is any matrix commutative with $a^{-1}a'$, then

$$y = b + \delta a^{-1}b'a \tag{6}$$

is a solution of (4); for on substituting this value of y we get

$$y' - \delta aya^{-1} = b' + \delta a'ba'^{-1} - \delta aba^{-1} - b' = 0$$

since, as in (5), $a'ba'^{-1} = aba^{-1}$. It was noted above that y has this form and it therefore follows that the general solution of (3) is obtained by setting

$$y = b - a^{-1}b'a, \qquad ba^{-1}a' = a^{-1}a'b. \tag{7}$$

It should be noted, however, that two different values of b may give rise to the same value of y.

9.03 We are now able to give a solution of (1) under the restriction that either $|x + 1| \neq 0$ or $|x - 1| \neq 0$. Since the first condition is transformed into the second if x is changed into $-x$, it is sufficient for the present to assume that $|x + 1| \neq 0$, and in this case the value of y given by (2) is finite. In terms of y we have

$$x = \frac{1+y}{1-y} = \frac{1 + b - a^{-1}b'a}{1 - b + a^{-1}b'a} = (a - ab + b'a)^{-1}(a + ab - b'a)$$

or, if

$$c = ab - b'a, \tag{8}$$

then

$$\begin{aligned} x &= (a - c)^{-1}(a + c), \qquad |x + 1| \neq 0, \\ x &= (c - a)^{-1}(a + c), \qquad |x - 1| \neq 0. \end{aligned} \tag{9}$$

It follows as in §9.01 that, if x has this form, it is a solution of (1

In place of (8) we may define c by

$$c' = -a'a^{-1}c = -ca^{-1}a'. \tag{10}$$

For from (7) and (8)

$$c' = b'a' - a'b = -a'a^{-1}c = -ca^{-1}a'$$

and, if c is given by (10) and x by (9), then

$$\begin{aligned} x' &= (a' + c')\,(a' - c')^{-1} = (a' - ca^{-1}a')\,(a' + ca^{-1}a')^{-1} \\ &= (1 - ca^{-1})\,(1 + ca^{-1})^{-1} = (1 + ca^{-1})^{-1}(1 - ca^{-1}) = a(a + c)^{-1}(a - c)a^{-1} \\ &= ax^{-1}a^{-1}. \end{aligned}$$

If a is symmetric, (8) or (10) gives $c' = -c$, and c is otherwise arbitrary except that $|\, a - c \,| \neq 0$; in particular if $a = 1$, (9) reduces to the form of an orthogonal matrix already given in 6.03. Similarly if a is skew, (10) shows that c is an arbitrary symmetric matrix subject to the condition that $|\, a - c \,| \neq 0$.

The case in which a is symmetric can also be handled as follows. We can set $a = b^2$ where b is symmetric and, if

$$y = b^{-1}xb,$$

equation (1) gives $yy' = 1$ so that y is orthogonal. Conversely, if y is any orthogonal matrix $x = byb^{-1}$ is a solution of (1).

9.04 **Principal idempotent and nilpotent elements.** Since x is similar to $(x')^{-1}$, the elementary divisors which correspond to roots other than ± 1 occur in pairs with reciprocal roots. If we arrange these roots in pairs g_r, g_r^{-1} and denote the corresponding principal idempotent elements by e_r and e_{-r}, respectively, we may set

$$(11) \qquad x = \Sigma\,[g_r(e_r + \xi_r) + g_r^{-1}e_{-r}(1 + \xi_{-r})^{-1}] + \theta_1(e_1 + \xi_1) - \theta_2(e_{-1} + \xi_{-1})$$

where the ξ's are nilpotent, e_1, e_{-1} are the principal idempotent elements corresponding to 1 and -1, if present as roots, and θ_1, θ_2 are either 0 or 1. The form of x^{-1} is then

$$(12) \quad x^{-1} = \Sigma\,[g_r(e_{-r} + \xi_{-r}) + g_r^{-1}e_r(1 + \xi_r)^{-1}] + \theta_1 e_1(1 + \xi_1)^{-1} - \theta_2 e_{-1}(1 + \xi_{-1})^{-1}$$

and (11) gives

$$(13) \qquad \begin{aligned} e_r' &= ae_{-r}a^{-1}, & e_{-r}' &= ae_ra^{-1} \\ \xi_r' &= a\xi_{-r}a^{-1}, & \xi_{-r}' &= a\xi_ra^{-1} \end{aligned}$$

$$(13') \qquad \begin{aligned} e_1' &= ae_1a^{-1}, & e_{-1}' &= ae_{-1}a^{-1} \\ \xi_1' &= -a\frac{\xi_1}{1 + \xi_1}a^{-1}, & \xi_{-1}' &= -a\frac{\xi_{-1}}{1 + \xi_{-1}}a^{-1}. \end{aligned}$$

We require also the form of $x + x^{-1}$ and $x - x^{-1}$; if

$$(14) \qquad \alpha_r = \xi_r + \xi_{-r}, \qquad \beta_r = \xi_r - \xi_{-r}$$

then

$$\begin{aligned} x + x^{-1} &= \Sigma\,[g_r(e_r + e_{-r} + \alpha_r) + g_r^{-1}(e_r + e_{-r})\,(1 + \alpha_r)^{-1}] \\ &\quad + \theta_1[(e_1 + \xi_1) + e_1(1 + \xi_1)^{-1}] - \theta_2[(e_{-1} + \xi_{-1}) + e_{-1}(1 + \xi_{-1})^{-1}] \\ x - x^{-1} &= \Sigma\,[g_r(e_r - e_{-r} + \beta_r) - g_r(e_r - e_{-r})\,(1 + \beta_r)^{-1}] \\ &\quad + \theta_1[(e_1 + \xi_1) - e_1(1 + \xi_1)^{-1}] - \theta_2[(e_{-1} + \xi_{-1}) - e_{-1}(1 + \xi_{-1})^{-1}] \end{aligned}$$

or if

$$\gamma_r = g_r\alpha_r - g_r^{-1}\frac{\alpha_r}{1+\alpha_r}, \qquad \gamma_1 = \frac{\xi_1^2}{1+\xi_1}, \qquad \gamma_{-1} = \frac{\xi_{-1}^2}{1+\xi_{-1}}$$

$$\delta_r = g_r\xi_r + g_r^{-1}\frac{\xi_r}{1+\xi_r}, \qquad \delta_{-r} = g_r\xi_{-r} + g_r^{-1}\frac{\xi_{-r}}{1+\xi_{-r}}, \tag{15}$$

$$\delta_1 = 2\xi_1 - \gamma_1, \qquad \delta_{-1} = 2\xi_{-1} - \gamma_{-1}$$

we have

$$x + x^{-1} = \Sigma[(g_r + g_r^{-1})(e_r + e_{-r}) + \gamma_r] + \theta_1(2e_1 + \gamma_1) - \theta_2(2e_{-1} + \gamma_{-1}) \tag{16}$$

$$x - x^{-1} = \Sigma[\{(g_r - g_r^{-1})e_r + \delta_r\} - \{(g_r - g_r^{-1})e_{-r} + \delta_{-r}\}] + \theta_1\delta_1 - \theta_2\delta_{-1} \tag{17}$$

where the elements grouped together are principal elements.

The principal idempotent elements of $x - x^{-1}$ are also principal idempotent elements of x except that roots 1 and -1 of x both give the same root 0 of $x - x^{-1}$; no root of x other than ± 1 leads to the coalescing of roots in $x - x^{-1}$. If we put

$$2u = x + x^{-1}, \qquad 2v = x - x^{-1}, \tag{18}$$

then u is a solution of (4) with $\delta = 1$ and v is a solution with $\delta = -1$; also

$$x^2 - 2vx - 1 = 0 \tag{19}$$

which has the formal solution

$$x = v + (v^2 + 1)^{\frac{1}{2}}. \tag{20}$$

Here $v^2 + 1 = u^2$ so that $(v^2 + 1)^{\frac{1}{2}}$ exists whenever x is a solution of (1). Conversely, if v_1 is any solution of (4) with $\delta = -1$ and if u_1 is a determination of $(v_1^2 + 1)^{\frac{1}{2}}$ such that

$$u_1'a = \alpha u_1, \tag{21}$$

then x is a solution of (1); for

$$x'a = v_1'a + u_1'a = -av_1 + au_1 = ax^{-1}$$

since

$$(u_1 + v_1)(u_1 - v_1) = u_1^2 - v_1^2 = 1.$$

If $v^2 + 1$ has no zero root, determinations of $(v^2 + 1)^{\frac{1}{2}}$ always exist which are polynomials in v^2 and therefore satisfy (21); but even in this case this does not give all solutions. The situation is as follows. The general form of v is given by (17) if we replace $g_r - g_r^{-1}$ by, say, $2k_r$ and $\theta_1\delta_1 - \theta_2\delta_{-1}$ by δ_0. When k_r is given, then g_r is determined; from (13) and (14) we have $(\delta_r' + \delta_{-r}')a = a(\delta_r + \delta_{-r})$ and therefore, if $k_r^2 + 1 \neq 0$, the part of $(v^2 + 1)^{\frac{1}{2}}$ corresponding to $e_r + e_{-r}$ exists and satisfies (21); we therefore get all valid expressions for this part of $(v^2 + 1)^{\frac{1}{2}}$ by using the form of the square root given in §8.05 with the restriction

that the only sets of partial units that may be used are those that satisfy (21). However, since $(v^2 + 1)^{\frac{1}{2}} = u$, (16) and (17) show that we need only use the idempotent elements $e_r + e_{-r}$, which are determined by v, in those parts of the square root which do not depend on the zero root of v; e_1 and e_{-1}, however, are not defined by v so that it is necessary in any particular case to consider how e_0 and δ_0 can be broken up into parts which have the required property.

If v has a zero root with the principal idempotent and nilpotent parts e_0, δ_0, then $e^2 = e$ shows that, although $\delta_0' a = -a\delta_0$, we have

$$e_0' a = a e_0. \tag{21'}$$

We therefore seek to divide e_0 into two idempotent parts, e_1 and e_{-1}, which are commutative with v and therefore with δ_0. In forming the square root we then attach the value $+1$ to e_1 and -1 to e_{-1}.

If $k_r^2 + 1 = 0$, then $g_r = i$ and the corresponding part of $(v^2 + 1)^{\frac{1}{2}}$ is $2i(\delta_r + \delta_{-r}) + \delta_r^2 + \delta_{-r}^2$, and it is readily shown from (15) that this has a square root. The details are left to the reader.

If b is a solution of $b'a = -ab$, then so are also $t = \tan b$ and $v = \tan 2b$. A short calculation then gives $x = (1 + t)/(1 - t)$ subject to the restrictions already given; this shows the relations between the rational and irrational solutions.

9.05 **The exponential solution.** Some of the difficulties of the solution in §9.04 can be avoided by setting

$$x = \exp(z) \equiv \epsilon^z, \qquad z = \operatorname{Log} x \tag{22}$$

where a principal determination of $\log x$ is to be used. Since this determination of $\log x$ is a polynomial in x and $x'ax = a$, we have

$$z' = \operatorname{Log} x' = \operatorname{Log} ax^{-1}a^{-1} = a(\operatorname{Log} x^{-1})a^{-1} \tag{23}$$

and therefore

$$1 = x'axa = \epsilon^{z'}\epsilon^{aza^{-1}} = \epsilon^{z' + aza^{-1}}. \tag{24}$$

From (11)

$$\begin{aligned} z &= \Sigma\,[(\log g_r)\, e_r + \eta_r - (\log g_r)\, e_{-r} + \eta_{-r}] + \theta_1\eta_1 + \theta_2(\pi i e_{-1} + \eta_{-1}) \\ \eta_s &= \xi_s - \tfrac{1}{2}\xi_s^2 + \cdots \qquad\qquad (s = r, -r, 1, -1) \end{aligned} \tag{25}$$

and from (13)

$$\begin{aligned} e_s' &= a e_{-s} a^{-1}, \qquad \eta_s' = -a\eta_{-s}a^{-1} \qquad (s = r, -r), \\ e_s' &= a e_s a^{-1}, \qquad \eta_s' = -a\eta_s a^{-1} \qquad (s = 1, -1). \end{aligned} \tag{26}$$

Hence

$$\begin{aligned} z' + aza^{-1} &= \Sigma\,[(\log g_r)\, e_r' + \eta_r' - (\log g_r)\, e_{-r}' + \eta_{-r}'] + \theta_1\eta_1' + \theta_2(\pi i e_{-1}' + \eta_{-1}') \\ &\quad + \Sigma\,[(\log g_r)\, e_{-r}' - \eta_{-r}' - (\log g_r) e_r' - \eta_r'] - \theta_1\eta_1' + \theta_2(\pi i e_{-1}' - \eta_{-1}') \\ &= 2\theta_2\pi i e_{-1}', \end{aligned}$$

and therefore, if we set

$$\zeta = \theta e_{-1}\,(\theta \equiv \theta_2 = 0, 1), \qquad w = z - \pi i\zeta, \tag{27}$$

we have

$$w' + awa^{-1} = 0, \qquad \zeta' - a\zeta a^{-1} = 0, \tag{28}$$

and

$$w = \Sigma\,[(\log g_r)e_r + \eta_r - (\log g_r)e_{-r} + \eta_{-r}] + \theta_1\eta_1 + \theta_2\eta_{-1}. \tag{29}$$

The general value of x can therefore be expressed in terms of the solution of the equation discussed in §9.02.

If we now start with w as a solution of (28) and define x by $x = \epsilon^w$, then

$$x' = \epsilon^{w'} = \epsilon^{-awa^{-1}} = a\epsilon^{-w}a^{-1} = ax^{-1}a^{-1}$$

and therefore x is a solution of (1); to obtain every solution, however, we must add the terms $\pi i\zeta$ to w.

If e_0 is the principal idempotent element corresponding to the root 0, then (29) shows that the presence of the ζ-term depends on the division of e_0 into two parts e_1 and e_{-1} which satisfy the second set of equations in (26); and corresponding to these we have nilpotent parts η_1 and η_{-1} which give rise to 1 and -1, respectively, as roots of x, or 0, πi as roots of z.

A form which gives rational parameters is obtained from the exponential solution as follows. Let

$$t = \tanh(z/2) = \frac{\epsilon^z - 1}{\epsilon^z + 1} = \frac{x - 1}{x + 1} \tag{30}$$

then

$$x = \frac{1 + t}{1 - t}, \tag{31}$$

and

$$ata^{-1} = \frac{axa^{-1} - 1}{axa^{-1} + 1} = \frac{1 - ax^{-1}a^{-1}}{1 + ax^{-1}a^{-1}} = \frac{1 - x'}{1 + x'} = -t'$$

so that (31) gives a solution of (1). If, however, $|\,x + 1\,| = 0$, then t becomes infinite so that (31) cannot give directly any x which has -1 as a root. This difficulty arises from the fact that $\tanh(\theta/2) \to \infty$ when $\theta \to \pi i$; but, since $(t + 1)(t - 1) = \epsilon^z$ for all values of t which do not have an infinite root, that is, one corresponding to a root $(2k + 1)\pi i$ of z, hence x will be a solution of (1) so long as the coordinates of z are continuous functions of the parameters involved and the limiting value of x is finite and determinate.

9.06 **Matrices which admit a given transformation.** In (1) we may regard x as given and a as unknown; the problem then is to find all matrices a such that

$$x'ax = a. \tag{32}$$

If we associate with $a = || a_{ij} ||$ the corresponding tensor of grade 2,

$$u = \Sigma a_{ij} e_i e_j,$$

we see immediately that (23) corresponds to setting (cf. §5.10)

$$\Pi_2(x)u = u. \tag{33}$$

Hence there is a solution if, and only if, x has at least one pair of reciprocal roots; in this case $\Pi_2(x)$ has one or more roots equal to 1 and the various invariant elements corresponding to this root give a linearly independent set of determinations of a.

When it is required that $| a | \neq 0$, another form of solution is preferable. In this case $x' = ax^{-1}a^{-1}$; but, since x and x' are similar, we also have $x' = pxp^{-1}$, where, if p_1 is one determination of p, the general form is

$$p = p_1 b, \qquad bx = xb, \qquad | b | \neq 0. \tag{34}$$

Hence it is necessary that x^{-1} be similar to x, say

$$x^{-1} = q_1^{-1} x q_1, \tag{35}$$

which gives immediately

$$a = p_1 b q_1. \tag{36}$$

Conversely, if p_1, b, and q_1 satisfy the given conditions, it follows immediately that (36) gives a solution of (32).