3 Invariant subspaces

DEFINITIONS 3.1

Subspaces $V_1, \ldots, V_t$ of V are called **independent** *if*

$$v_1 + \cdots + v_t = 0 \Rightarrow v_1 = 0, \ldots, v_t = 0 \qquad \forall v_1 \in V_1, \ldots, v_t \in V_t.$$

*We say that V is the (***internal***) **direct sum** of the subspaces $V_1, \ldots, V_t$ if*

(a) *$V_1, \ldots, V_t$ are independent and*

(b) *$V = V_1 + \cdots + V_t$.*

I.e. every element $v \in V$ is uniquely *expressible as*

$$v = v_1 + \cdots + v_t$$

with $v_i \in V_i$.

Then V is isomorphic to the (external) direct sum $V_1 \oplus \cdots \oplus V_t$ under the isomorphism $v \mapsto (v_1, \ldots, v_t)$ and we write $V = V_1 \oplus \cdots \oplus V_t$.

THEOREM 3.1

If $V = V_1 \oplus \cdots \oplus V_t$ (an internal direct sum) and $\beta_1, \ldots, \beta_t$ are bases for $V_1, \ldots, V_t$ respectively, then

$$\beta = \beta_1 \cup \cdots \cup \beta_t,$$

the sequence formed by juxtaposing the separate bases, is a basis for V. Also

$$\dim V = \dim V_1 + \cdots + \dim V_t.$$

Proof: Left as an exercise.

DEFINITION. Let $T : V \mapsto V$ be a LT and W a subspace of V. Then if

$$w \in W \Rightarrow T(w) \in W,$$

we say W is a ***T*-invariant subspace** of V. We can then consider the linear transformation $T_W : W \to W$ defined by

$$T_W(w) = T(w) \qquad \forall w \in W.$$

If β' is a basis for W, $\{0\} \subset W \subset V$, and β is an extension to a basis of V, then

$$[T]_\beta^\beta = \left[\begin{array}{c|c} [T_W]_{\beta'}^{\beta'} & B_1 \\ \hline 0 & B_2 \end{array}\right].$$

A situation of great interest is when we have T-invariant subspaces $W_1, \ldots, W_t$ and $V = W_1 \oplus \cdots \oplus W_t$. For if $\beta = \beta_1 \cup \cdots \cup \beta_t$, where β_i is a basis for W_i, we see that

$$[T]_\beta^\beta = [T_{W_1}]_{\beta_1}^{\beta_1} \oplus \cdots \oplus [T_{W_t}]_{\beta_t}^{\beta_t}.$$

There are two important examples of T–invariant subspaces that arise in our study of Jordan and rational canonical forms - $\operatorname{Ker} p^t(T)$ and T–cyclic subspaces.

3.1 T–cyclic subspaces

DEFINITION 3.1

The unique monic polynomial f in $F[x]$ of least degree satisfying

$$f(T)(v) = 0$$

is called the **minimum polynomial of the vector** $v \in V$ *relative to the transformation $T : V \mapsto V$ and is denoted $m_{T,v}$.*

Then $f(T)(v) = 0 \Rightarrow m_{T,v} \mid f$, so $m_{T,v} \mid m_T$. Also $m_{T,v} = 1 \Leftrightarrow v = 0$ and so if $v \neq 0$, $\deg m_{T,v} \geq 1$.

EXAMPLE. Let $T = T_A$, where $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Also let

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then $Av_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq c_0 v_1$, so $m_{T,\,v_1} \neq x - c_0$. Next $A^2 v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $m_{T,\,v_1} = x^2$. Also $Av_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so $m_{T,\,v_2} = x$.

DEFINITION 3.2

*(**T-cyclic subspace** generated by v.)*

If $v \in V$, the set of all vectors of the form $f(T)(v)$, $f \in F[x]$, forms a subspace of V called the T-cyclic subspace generated by v. It is denoted by $C_{T,v}$.

PROOF. Exercise.

Also, $C_{T,v}$ is a T-invariant subspace of V. For

$$\begin{aligned} w \in C_{T,v} &\Rightarrow w = f(T)(v) \\ &\Rightarrow T(w) = T(f(T)(v)) = (Tf(T))(v) = ((xf)(T))(v) \in C_{T,v}. \end{aligned}$$

We see that $v = 0$ if and only if $C_{T,v} = \{0\}$.

THEOREM 3.2

Let $v \neq 0,\ v \in V$. Then $C_{T,v}$ has the basis β

$$v, T(v), T^2(v), \ldots, T^{k-1}(v)$$

*where $k = \deg m_{T,v}$. (β is called the T-**cyclic basis** generated by v.) Note that $\dim C_{T,v} = \deg m_{T,v}$.*

Finally,

$$\left[T_{C_{T,v}}\right]_\beta^\beta = C(m_{T,v}),$$

the companion matrix of the minimum polynomial of v.

PROOF.

1. The T-cyclic basis is a basis for $C_{T,v}$:

   Spanning:

   Let $w \in \langle v, T(v), \ldots, T^{k-1}(v)\rangle$, so

   $$\begin{aligned} w &= w_0 v + w_1 T(v) + \cdots + w_{k-1} T^{k-1}(v) \\ &= (w_0 I_V + \cdots + w_{k-1} T^{k-1})(v) \\ &= g(T)(v), \end{aligned}$$

   where $g = w_0 + \cdots + w_{k-1} x^{k-1}$, so $w \in C_{T,v}$. Hence

   $$\langle v, T(v), \ldots, T^{k-1}(v)\rangle \subseteq C_{T,v}.$$

   Conversely, suppose that $w \in C_{T,v}$ so

   $$w = f(T)(v)$$

   and

   $$f = q m_{T,v} + r$$

   where $r = a_0 + a_1 x + \cdots + a_{k-1} x^{k-1}$ and $a_0, \ldots, a_{k-1} \in F$. So

   $$\begin{aligned} f(T)(v) &= q(T) m_{T,v}(T)(v) + r(T)(v) \\ &= q(T) m_{T,v}(T)(v) + a_0 v + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) \\ &= a_0 v + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) \\ &\in \langle v, T(v), \ldots, T^{k-1}(v)\rangle. \end{aligned}$$

Independence:

Assume

$$a_0 v + a_1 T(v) + \cdots + a_{k-1} T^{k-1}(v) = 0,$$

where $a_0, \ldots, a_{k-1} \in F$; that is, $f(T)(v) = 0$ where

$$f = a_0 + a_1 x + \cdots + a_{k-1} x^{k-1}.$$

Hence $m_{T,v} \mid f$ and since

$$\deg f = k - 1 < k = \deg m_{T,v},$$

we have $f = 0$ and thus $a_i = 0 \ \forall i$.

2. $[T_{C_{T,v}}]_\beta^\beta = C(m_{T,v})$:

Let $L = T_{C_{T,v}}$, the restriction of T to $C_{T,v}$.

We want to find $[L]_\beta^\beta$. So

$$\begin{aligned}
L(v) &= T(v) = 0v + 1T(v) + 0T^2(v) + \cdots + 0T^{k-1}(v) \\
L(T(v)) &= T^2(v) = 0v + 0T(v) + 1T^2(v) + \cdots + 0T^{k-1}(v) \\
&\vdots \\
L(T^{k-2}(v)) &= T^{k-1}(v) = 0v + 0T(v) + 0T^2(v) + \cdots + 1T^{k-1}(v)
\end{aligned}$$

Finally, to calculate $L(T^{k-1}(v)) = T^k(v)$, we let

$$m_{T,v} = a_0 + a_1 x + \cdots + a_{k-1} x^{k-1} + x^k.$$

Then $m_{T,v}(T)(v) = 0$ and hence

$$L(T^{k-1}(v)) = T^k(v) = -a_0 v - a_1 T(v) - \cdots - a_{k-1} T^{k-1}(v).$$

Hence

$$[L]_\beta^\beta = \begin{bmatrix} 0 & 0 & \ldots & 0 & -a_0 \\ 1 & 0 & & & -a_1 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & 1 & -a_{k-1} \end{bmatrix} = C(m_{T,v}),$$

as required.

THEOREM 3.3

Suppose that $m_{T,\,v} = (x-c)^k$. *Then the vectors*

$$v,\ (T - cI_V)(v), \ldots, (T - cI_V)^{k-1}(v)$$

form a basis β *for* $W = C_{T,\,v}$ *which we call the* elementary Jordan basis.

Also

$$[T_W]_\beta^\beta = J_k(c).$$

More generally suppose $m_{T,v} = p^k$, *where* p *is a monic irreducible polynomial in* $F[x]$, *with* $n = \deg p$. *Then the vectors*

$$\begin{array}{cccc} v, & T(v), & \ldots, & T^{n-1}(v) \\ p(T)(v), & Tp(T)(v), & \ldots, & T^{n-1}p(T)(v) \\ \vdots & \vdots & \vdots & \vdots \\ p^{k-1}(T)(v), & Tp^{k-1}(T)(v), & \ldots, & T^{n-1}p^{k-1}(T)(v), \end{array}$$

form a basis for $W = C_{T,v}$, *which reduces to the elementary Jordan basis when* $p = x - c$. *Also*

$$[T_W]_\beta^\beta = H(p^k),$$

where $H(p^k)$ *is a* **hypercompanion** *matrix, which reduces to the elementary Jordan matrix* $J_k(c)$ *when* $p = x - c$:

$$H(p^k) = \begin{bmatrix} C(p) & 0 & \cdots & 0 \\ N & C(p) & \cdots & 0 \\ 0 & N & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & N & C(p) \end{bmatrix},$$

where there are k *blocks on the diagonal and* N *is a square matrix of same size as* $C(p)$ *which is everywhere zero, except in the top right–hand corner, where there is a* 1. *The overall effect is an unbroken subdiagonal of* $1's$.

3.1.1 A nice proof of the Cayley-Hamilton theorem

(From Insel, Friedberg and Spence.)

Let $f = \mathrm{ch}_T$, for some $T : V \mapsto V$. We must show that $f(T) = 0_V$—i.e. that $f(T)(v) = 0\ \forall v \in V$.

This is immediate if $v = 0$, so assume $v \neq 0$ and let $W = C_{T,v}$. Let β' be a basis of W and β be an extension of β' to a basis of V. Then

$$[T]_\beta^\beta = \left[\begin{array}{c|c} [T_W]_{\beta'}^{\beta'} & B_1 \\ \hline 0 & B_2 \end{array}\right]$$

and $\mathrm{ch}_T = \mathrm{ch}_{T_W} \cdot \mathrm{ch}_{B_2}$. So $\mathrm{ch}_{T_W} \mid \mathrm{ch}_T$ and since we know that

$$\mathrm{ch}_{T_W} = m_{T,v},$$

we have $m_{T,v} \mid \mathrm{ch}_T$.

Hence $\mathrm{ch}_T = g m_{T,v}$ and

$$\mathrm{ch}_T(T)(v) = (g(T)m_{T,v}(T))(v) = g(T)(m_{T,v}(T)(v)) = g(T)(0) = 0.$$

3.2 An Algorithm for Finding m_T

We use the factorization of ch_T into monic irreducibles.

THEOREM 3.4

Suppose $T : V \mapsto V$,

$$m_T = p_1^{b_1} \dots p_t^{b_t}$$

where $b_1, \dots, b_t \geq 1$, *and* $p_1, \dots, p_t$ *are distinct monic irreducibles.*

Then for $i = 1, \dots, t$ *we have*

(a)
$$V \supset \mathrm{Im}\, p_i(T) \supset \cdots \supset \mathrm{Im}\, p_i^{b_i-1}(T) \supset \mathrm{Im}\, p_i^{b_i}(T) = \cdots$$

(b)
$$\{0\} \subset \mathrm{Ker}\, p_i(T) \subset \cdots \subset \mathrm{Ker}\, p_i^{b_i-1}(T) \subset \mathrm{Ker}\, p_i^{b_i}(T) = \cdots.$$

Note: In terms of nullities, conclusion (b) says that

$$0 < \nu(p_i(T)) < \cdots < \nu(p_i^{b_i-1}(T)) < \nu(p_i^{b_i}(T)) = \cdots$$

so this gives us a method of calculating b_i.

Presently we'll show that if $\mathrm{ch}_T = p_1^{a_1} \dots p_t^{a_t}$, then

$$\mathrm{nullity}\,(p_i^{b_i}(T)) = a_i \deg p_i.$$

Hence b_i is also characterised as the smallest integer h such that

$$\mathrm{nullity}\,(p_i^{h}(T)) = a_i \deg p_i.$$

Also note that (a) and (b) are equivalent, and it is the latter that we prove.

A notational simplification—**the left $F[x]$-module notation**.

If $f \in F[x]$ and $v \in V$, we define

$$fv = f(T)(v).$$

It is easy to verify that

1.
$$(f+g)v = fv + gv \qquad \forall f, g \in F[x], v \in V;$$

2.
$$f(v+w) = fv + fw \qquad \forall f \in F[x], v, w \in V;$$

3.
$$(fg)v = f(gv) \qquad \forall f, g \in F[x], v \in V;$$

4.
$$1v = v \qquad \forall v \in V.$$

These axioms, together with the four axioms for addition on V, turn V into what is called a "left $F[x]$-module". (So there are deeper considerations lurking in the background—ideas of greater generality which make the algorithm we unravel for the rational canonical form also apply to other things such as the theorem that any finite abelian group is a direct product of cyclic prime power subgroups.)

(i) We first prove that $\{0\} \subset \operatorname{Ker} p_i(T)$. We write $p = p_i$, $b = b_i$ for brevity; no confusion should arise since i is fixed.

PROOF. $m_T = pf$, $f \in F[x]$ and $f(T) \neq 0_V$. Hence $\exists v \in V$ such that $fv \neq 0$. Then
$$p(fv) = (pf)v = m_T v = 0,$$
so $fv \in \operatorname{Ker} p(T)$.

(ii) We next prove that
$$\operatorname{Ker} p^b(T) = \operatorname{Ker} p^{b+1}(T).$$

The containment
$$\operatorname{Ker} p^b(T) \subseteq \operatorname{Ker} p^{b+1}(T)$$
is obvious, so we need only show that
$$\operatorname{Ker} p^b(T) \supseteq \operatorname{Ker} p^{b+1}(T).$$

Let $w \in \operatorname{Ker} p^{b+1}(T)$, i.e. $p^{b+1}w = 0$. Now if $m_T = p^b q$, then $\gcd(p^b, q) = 1$. So $\exists u, v \in F[x]$ such that $1 = up^b + vq$. Hence
$$p^b = up^{2b} + vm_T.$$

Hence

$$\begin{aligned} p^b(T) &= (up^{2b})(T) + v(T)m_T(T) \\ &= (up^{2b})(T) \end{aligned}$$

and thus

$$p^b w = (p^{b-1})(p^{b+1}w) = p^{b-1}0 = 0$$

and $w \in \operatorname{Ker} p^b$, as required.

(iii)

$$\begin{aligned} &\operatorname{Ker} p^h(T) = \operatorname{Ker} p^{h+1}(T) \Rightarrow \operatorname{Ker} p^{h+1}(T) = \operatorname{Ker} p^{h+2}(T), \\ \text{i.e.} \quad &\operatorname{Ker} p^h(T) \supseteq \operatorname{Ker} p^{h+1}(T) \Rightarrow \operatorname{Ker} p^{h+1}(T) \supseteq \operatorname{Ker} p^{h+2}(T). \end{aligned}$$

PROOF. Suppose that $\operatorname{Ker} p^h(T) \supseteq \operatorname{Ker} p^{h+1}(T)$. Then

$$\begin{aligned} v \in \operatorname{Ker} p^{h+2}(T) &\Rightarrow p^{h+2}v = 0 \\ &\Rightarrow p^{h+1}(pv) = 0 \Rightarrow pv \in \operatorname{Ker} p^{h+1}(T) \\ &\Rightarrow pv \in \operatorname{Ker} p^h(T) \Rightarrow p^h(pv) = 0 \\ &\Rightarrow p^{h+1}v = 0 \Rightarrow v \in \operatorname{Ker} p^{h+1}(T). \end{aligned}$$

So it follows by induction from (ii) that

$$\operatorname{Ker} p_i^{b_i}(T) = \operatorname{Ker} p_i^{b_{i+1}}(T) = \cdots.$$

(iv)

$$\operatorname{Ker} p^{b-1}(T) \subset \operatorname{Ker} p^b(T)$$

and this forces a chain of proper inclusions:

$$\{0\} \subset \operatorname{Ker} p(T) \subset \cdots \subset \operatorname{Ker} p^{b-1}(T) \subset \operatorname{Ker} p^b(T) = \cdots$$

which is the desired result. For
$p^{b-1}q(T) \neq 0_V$, so $\exists v$ such that $p^{b-1}qv \neq 0$. Then

$$qv \notin \operatorname{Ker} p^{b-1}(T),$$

but $qv \in \operatorname{Ker} p^b(T)$ as

$$p^b qv = m_T v = 0.$$

3.3 Primary Decomposition Theorem

THEOREM 3.5 (Primary Decomposition)
If $T : V \mapsto V$ is a LT with $m_T = p_1^{b_1} \dots p_t^{b_t}$, where $p_1, \dots, p_t$ are monic irreducibles, then

$$V = \operatorname{Ker} p_1^{b_1}(T) \oplus \cdots \oplus \operatorname{Ker} p_t^{b_t}(T),$$

a direct sum of T-invariant subspaces. Moreover for $1 \le i \le t$,

$$\nu(p_i(T)^{b_i}) = a_i \deg p_i,$$

where $ch_T = p_1^{a_1} \cdots p_t^{a_t}$.

REMARK. The same proof gives a slightly more general result:
If $p = p_1^{b_1} \cdots p_t^{b_t}$, then

$$\operatorname{Ker} p(T) = \operatorname{Ker} p_1^{b_1}(T) \oplus \cdots \operatorname{Ker} p_t^{b_t}(T).$$

We subsequently give an application of the decomposition theorem in this form to the solution of the n–th order linear differential equations with constant coefficients. (See Hoffman and Kunze, pages 184–185.)
PROOF. Let $m_T = p_i^{b_i} q_i \ \forall i = 1, \dots, t$. Then

$$(q_i q_j)(T) = 0_V \quad \text{if } i \neq j \text{ as} \quad m_T \mid q_i q_j \quad \text{if } i \neq j.$$

Now $\gcd(q_1, \dots, q_t) = 1$, so $\exists f_1, \dots, f_t \in F[x]$ such that

$$1 = f_1 q_1 + \cdots + f_t q_t$$

and with $T_i = (f_i q_i)(T)$ we have

$$I_V = T_1 + \cdots + T_t. \tag{6}$$

Also

$$\begin{aligned} T_i T_j &= (f_i q_i)(T)(f_j q_j)(T) \\ &= (f_i f_j)(T)(q_i q_j)(T) \\ &= 0_V \qquad \text{if } i \neq j, \end{aligned}$$

Then

$$V = \bigoplus_{i=1}^{t} \operatorname{Im} T_i.$$

For $T_i^2 = T_i(T_1 + \cdots + T_t) = T_i I_V = T_i$. Next, $V = \operatorname{Im} T_1 + \cdots + \operatorname{Im} T_t$. For

$$v \in V \Rightarrow v = I_V(v) = T_1(v) + \cdots + T_t(v) \in \operatorname{Im} T_1 + \cdots + \operatorname{Im} T_t.$$

Next assume $v_1 + \cdots + v_t = 0,\ v_i \in \operatorname{Im} T_i,\ 1 \leq i \leq t$. Then $v_i = T_i(u_i)$ and

$$\begin{aligned} T_1(u_1) + \cdots + T_t(u_t) &= 0 \\ T_i(T_1(u_1) + \cdots + T_t(u_t)) &= T(0) = 0 \\ T_i T_i(u_i) &= 0 \\ v_i = T_i(u_i) &= 0. \end{aligned}$$

We now show that

$$\operatorname{Im} T_i = \operatorname{Ker} p_i^{b_i}(T).$$

"$\subseteq$" Let $v \in \operatorname{Im} T_i$. Then

$$\begin{aligned} v &= f_i q_i w \\ \Rightarrow p_i^{b_i} v &= p_i^{b_i} f_i q_i w \\ &= f_i (p_i^{b_i} q_i) w \\ &= 0. \end{aligned}$$

"$\supseteq$" Suppose $p_i^{b_i} v = 0$.

Now if $j \neq i$, we have $p_i^{b_i} \mid f_j q_j$, so

$$T_j(v) = f_j q_j v = 0.$$

So

$$\begin{aligned} v &= I_V(v) = T_1(v) + \cdots + T_t(v) = T_i(v) \\ &\in \operatorname{Im} T_i, \end{aligned}$$

as required.

Finally, let $V_i = \operatorname{Ker} p_i^{b_i}(T)$ and $L_i = T_{V_i}$. Then because $V_1, \ldots, V_t$ are T–invariant subspaces of V, we have

$$ch_T = ch_{L_1} \cdots ch_{L_t}.$$

Now $p_i^{b_i}(T)(v) = 0$ if $v \in V_i$, so $p_i^{b_i}(L_i) = 0_{V_i}$. Hence m_{L_i} has the form $m_{L_i} = p_i^{e_i}$. Hence ch_{L_i} has the form $ch_{L_i} = p_i^{d_i}$. Hence

$$ch_T = p_1^{a_1} \cdots p_t^{a_t} = p_1^{d_1} \cdots p_t^{d_t}$$

and consequently $d_i = a_i$.

Finally,

$$\dim V_i = \deg ch_{L_i} = \deg p_i^{a_i} = a_i \deg p_i.$$

(Incidentally, we mention that $m_T = \text{lcm}\,(m_{L_1}, \ldots, m_{L_t})$. Hence

$$m_T = p_1^{b_1} \cdots p_t^{b_t} = p_1^{e_1} \cdots p_t^{e_t}$$

and consequently $e_i = b_i$. Hence $m_{L_i} = p_i^{b_i}$.)

THEOREM 3.6 (Commuting diagonable linear transformations)

If $T_1, \ldots, T_n : V \to V$ are commuting diagonable linear transformations, then there exists a basis β for V such that each of $[T_1]_\beta^\beta, \ldots [T_m]_\beta^\beta$ are each diagonal.

(Matrix version) If $A_1, \ldots, A_m$ are commuting diagonable matrices of the same size, then there exists a non-singular matrix P such that the matrices

$$P^{-1}A_1P, \ldots P^{-1}A_mP$$

are simultaneously diagonal.

PROOF. (From Samelson page 158). We prove the result when $m = 2$, the general case follows by an easy iteration. Suppose T_1 and T_2 are commuting diagonable linear transformations on V. Because m_{T_1} splits as a product of distinct linear factors, the primary decomposition theorem gives a direct sum decomposition as a sum of the T_1–eigenspaces:

$$V = U_1 \oplus \cdots \oplus U_t.$$

It turns out that not only are the subspaces U_i T_1–invariant, they are T_2–invariant. For if $U_i = \text{Ker}\,(T_1 - cI_V)$, then

$$\begin{aligned} v \in U_i &\Rightarrow T_1(v) = cv \\ &\Rightarrow T_2(T_1(v)) = cT_2(v) \\ &\Rightarrow T_1(T_2(v)) = cT_2(v) \\ &\Rightarrow T_2(v) \in U_i. \end{aligned}$$

Now because T_2 is diagonable, V has a basis consisting of T_2–eigenvectors and it is an easy exercise to show that in a direct sum of T_2–invariant subspaces, each non-zero "component" of a T_2–eigenvector is itself a T_2–eigenvector; moreover each non–zero component is a T_1– eigenvector. Hence V is spanned by a family of vectors which are simultaneously T_1–eigenvectors and T_2–eigenvectors. If β is a subfamily which forms a basis for V, then $[T_1]_\beta^\beta$ and $[T_2]_\beta^\beta$ are diagonal.

THEOREM 3.7 (Fitting's lemma)

Suppose $T : V \to V$ is a linear transformation over T and

$$\text{Ker} \subset \text{Ker}\, T^2 \subset \cdots \text{Ker}\, T^n = \text{Ker}\, T^{n+1} = \cdots$$

Then $V = \text{Im}\, T^n \oplus \text{Ker}\, T^n$.

COROLLARY 3.1

If $T : V \to V$ is an indecomposable *linear transformation (that is the only T–invariant subspaces of V are $\{0\}$ and V), then T is either* nilpotent *(that is $T^n = 0_V$ for some $n \geq 1$) or T is an isomorphism.*