HAAGERUP PROPERTY FOR ALGEBRAIC GROUPS OVER LOCAL FIELDS

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Abstract. We classify, among the linear algebraic groups over a local field of characteristic zero, those that have the Haagerup property (also called a-(T)-menability). Our method relies essentially on a discussion on the existence of a subgroup isomorphic, up to a finite covering, to the semidirect product of SL_2 by an irreducible representation, or a one-dimensional central extension of an even-dimensional irreducible representation (that is, a Heisenberg group). We also obtain a new proof for the analogous case of Lie groups.

We give an application by giving examples of discrete groups not having the Haagerup property, but all of whose infinite subgroups are not relatively Kazhdan.

1. Introduction

Let G be a locally compact, second countable group. One says that G has the *Haagerup property* if G has a proper isometric action on some affine Hilbert space.

If H is a subgroup of G, one says that the pair (G, H) has *relative Kazhdan property (T)* if every isometric action of G on any affine Hilbert space has a fixed point in restriction to H.

As an immediate consequence of these definitions, if (G, H) has relative property (T) and H is not relatively compact in G, then G does not have the Haagerup property (we shall show, section 4, that this is not the only obstruction to Haagerup property).

The class of groups with the Haagerup property generalizes the class of amenable groups as a strong negation of Kazhdan's property (T). For other motivations for the Haagerup property and equivalent definitions, see [CCJJV].

Lie groups with the Haagerup property are classified in [CCJJV], chap. 4: they are all obtained from simple Lie groups of rank one and amenable Lie groups, by taking direct products and coverings. We extend this classification to algebraic groups over a local field of characteristic zero. More precisely, let us state our main result.

If $\mathbf{K}$ is local field of characteristic zero, we denote by $V_n = V_n(\mathbf{K})$ the irreducible $(n+1)$-dimensional representation of $SL(2, \mathbf{K})$, and $H_n = H_n(\mathbf{K})$ is the $(2n+1)$-dimensional Heisenberg group, with $SL(2, \mathbf{K})$ acting trivially on its center, and irreducibly modulo its center, as we will describe more precisely in section 2.

Theorem 1.1. *Let G be a linear algebraic group over a local field $\mathbf{K}$ of characteristic zero. The following properties are equivalent:*

- *(1) G has the Haagerup property.*
- *(2a) If H is a subgroup of G such that (G, H) has Kazhdan's property (T), then H is relatively compact.*
- *(2b) If H is an Zariski closed subgroup of G such that (G, H) has property (T), then H is compact.*
- *(3a) No simple factor of G of rank 1 has property (T), and G has no Zariski closed subgroup isomorphic to a semidirect product $SL(2, \mathbf{K}) \ltimes V_n$ (or perhaps $PSL(2, \mathbf{K}) \ltimes V_n$ if n is even), or to $SL(2, \mathbf{K}) \ltimes H_n$, for any $n \geq 1$.*
- *(3b) No simple noncompact factor of G has property (T), and G has no Zariski closed subgroup H isomorphic to $(P)SL(2, \mathbf{K}) \ltimes V_n$, nor $SL(2, \mathbf{K}) \ltimes H_n$, such that the radical $\mathscr{R}H$ be included in the unipotent radical $\mathscr{R}_u G$, for any $n \geq 1$.*
- *(4) G is isogeneous to the direct product of a semisimple group, none of whose factors has property (T), and an amenable group (that is, a solvable extension of a compact group).*

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(Recall that two algebraic groups are said to be isogeneous if they have isomorphic algebraic coverings).

This theorem should remind the following one, proved in [CCJJV, chap. 4], for which we propose a new proof (throughout this article, Lie groups mean *real* Lie groups, and Lie subgroups mean analytic subgroups, hence need not be closed).

Theorem 1.2. *If G is a Lie group with a finite number of connected components, then the following properties are equivalent:*

- *(1') G has the Haagerup property.*
- *(2') If H is a subgroup of G such that (G, H) has the property (T), then H is relatively compact in G.*
- *(4') G is locally isomorphic to the direct product of a semisimple group, none of whose factors has property (T), and an amenable group.*

We shall also show that these conditions are equivalent to the following conditions (3'a) and (3'b):

- *(3'a) No simple rank one factor of G has property (T), and G has no Lie subgroup locally isomorphic to $SL(2,\mathbf{R}) \ltimes V_n$ or $SL(2,\mathbf{R}) \ltimes H_n$, for any $n \geq 1$.*
- *(3'b) No simple noncompact factor of G has property (T), and G has no Lie subgroup H locally isomorphic to $SL(2,\mathbf{R}) \ltimes V_n$ or $SL(2,\mathbf{R}) \ltimes H_n$, such that $\mathrm{rad}(H) \subset \mathrm{rad}(G)$, for any $n \geq 1$.*

The latter theorem remains true in the context of *complex* Lie groups, replacing $\mathbf{R}$ by $\mathbf{C}$, and is proved in the same lines.

• First note that the implications (1)⇒(2a)⇒(2b) and (1')⇒(2') are obvious.

• To prove the implication (2b)⇒(3a), we shall show that, if $n \geq 1$, the pairs $((P)SL(2,\mathbf{K}) \ltimes V_n, V_n)$ and $(SL(2,\mathbf{K}) \ltimes H_n, H_n)$ have property (T). This is what we shall do in section 2 (corollaries 2.4 and 2.8).

The implication (2')⇒(3'a) is obtained in an analogous way (corollaries 2.5 and 2.9), but we must also prove that the radical of the Lie subgroup we have constructed is not relatively compact (proposition 2.11).

• To obtain the implication (3a)⇒(3b), it suffices to observe (see [MARG, Chapter I, (1.6.2)]) that every simple linear algebraic group of $\mathbf{K}$-rank ≥ 2 contains a subgroup isomorphic to $SL(3,\mathbf{K})$ or $(P)Sp(2,\mathbf{K})$. It turns out that $SL(3,\mathbf{K})$ contains an algebraic subgroup isomorphic to $SL(2,\mathbf{K}) \ltimes V_1$, and $Sp(2,\mathbf{K})$ contains an algebraic subgroup isomorphic to $SL(2,\mathbf{K}) \ltimes V_2$ (see [BHV], section 1.4). An analogous argument leads to (3'a)⇒(3'b). Note that the condition on rank one factors is empty if $\mathbf{K} \neq \mathbf{R}$.

• Section 3 is devoted to the implications (3b)⇒(4) (theorem 3.6) and (3'b)⇒(4') (theorem 3.2). Both will result from the analogous statement on Lie algebras (theorem 3.1).

• We shall not prove again the implications (4)⇒(1) and (4')⇒(1'). In the algebraic case, it comes down to state that amenable groups, and simple algebraic groups not having property (T) have the Haagerup property.

As regards amenable groups, this is a general fact (see chapter one in [CCJJV]).

If $\mathbf{K} = \mathbf{R}$, simple algebraic groups not having property (T) are some simple Lie groups of rank 1, namely, up to isogeny, the $SO(n,1)$ ($n \geq 2$) and $SU(n,1)$ ($n \geq 1$), see section 2.5 in [BHV]. This includes the $\mathbf{K} = \mathbf{C}$ case, for which the only possibility is $SL(2,\mathbf{C})$, which is locally isomorphic to $SO(3,1)$ as a real Lie group.

In the nonarchimedean case, the simple algebraic groups not having property (T) are exactly those of rank one, and they act properly on a tree (see [BT]); this implies (see [CCJJV], chapter 1) that they have the Haagerup property.

In the case of Lie groups, if the semisimple part has a finite center, then, up to a finite covering, we come down again to the case of a product of an amenable group and a product of simple Lie groups as in the $\mathbf{R}$-algebraic case, and we are done.

Greater difficulty occurs when the semisimple part has infinite center, which is possible since the groups $SU(n,1)$ have an infinite fundamental group. In this case, the proof is considerably more involved, see [CCJJV], chapter 4.

2. Relative property (T)

In this section, we denote by $\mathbf{K}$ a local field of characteristic zero (that is, $\mathbf{R}$, $\mathbf{C}$, or a finite extension of the p-adics $\mathbf{Q}_p$).

Recall the lemma (see [BHV], section 1.4).

Lemma 2.1. *Let G be a locally compact group, N a closed normal abelian subgroup. Assume that the only mean on the Borel subsets of the Pontryagin dual $\widehat{N} = \mathrm{Hom}(N, \mathbf{R}/\mathbf{Z})$, invariant under the action of G (induced by conjugation), is the Dirac measure at zero. Then the pair (G,N) has relative property (T).*

Recall that we denote by V_n the unique irreducible $(n+1)$-dimensional representation of $SL(2,\mathbf{K})$. In order to apply lemma 2.1 to $SL(2,\mathbf{K}) \ltimes V_n$, we need the following lemma:

Lemma 2.2. *Fix $|\cdot|$ a norm on $\mathbf{K}$. Let N be the operator of $\mathbf{K}^{n+1}$ ($n \geq 1$) given, on the canonical basis (e_i), by the Jordan matrix ($Ne_i = e_{i-1}$ if $1 \leq i \leq n$ and $Ne_0 = 0$). Set, for $t \in \mathbf{K}$, $T^t = \exp(tN)$ (well-defined since N is nilpotent). Let $\Omega \subset \mathbf{K}^{n+1}$ be the cone $\{(x_0,\dots,x_n),\ x_n \neq 0, |x_{n-1}| \leq a|x_n|\}$, for some fixed real $a > 0$. Then there exists a sequence (t_k) in $\mathbf{K}$ such that the subsets $T^{t_k}\Omega$ of $\mathbf{K}^n$ are pairwise disjoint.*

Proof: Let $(x_0,\dots,x_n) \in \Omega$, and set $(y_0,\dots,y_n) = T^t(x_0,\dots,x_n)$. Then $y_n = x_n$ and $y_{n-1} = x_{n-1} + tx_n$. Thus, since $|x_{n-1}| \leq a|x_n| = a|y_n|$, we obtain

$$(|t| - a)|y_n| \leq |y_{n-1}| \leq (|t| + a)|y_n|.$$

Then we choose a sequence (t_k) in $\mathbf{K}$ such that $|t_k| + a \leq |t_{k+1}| - a$, so that the subsets $T^{t_k}\Omega$ are pairwise disjoint. ■

Proposition 2.3. *The only $SL(2,\mathbf{K})$-invariant mean on V_n ($n \geq 1$) is the Dirac measure at zero.*

Proof: Let $\rho : SL(2,\mathbf{K}) \to GL(V_n)$ be the representation. One can choose a basis $(e_0,\dots,e_n)$ of V_n, such that the operator N of lemma 2.2 is in $d\rho(\mathfrak{sl}(2,\mathbf{K}))$, so that the operators T^t of lemma 2.2 are in $\rho(\mathfrak{sl}(2,\mathbf{K}))$.

Assume by contradiction that there exists an invariant mean m on $V_n\backslash\{0\}$. As the projective space $P^n(\mathbf{K})$ is compact, there exists a line D, all of whose conic neighborhoods have positive m-measure. The set of all lines D satisfying this hypothesis is $SL(2,\mathbf{K})$-invariant, hence, by irreducibility, spans linearly V_n. So we can suppose D is not contained in the hyperplane $\mathrm{Ker}(N^n) = \{x_n = 0\}$. Then D is included in the interior of a cone Ω as in the lemma 2.2, having null m-measure by the conclusion of the same lemma, which contradicts the hypothesis on D. ■

Corollary 2.4. *The pair $(SL(2,\mathbf{K}) \ltimes V_n, V_n)$ has property (T) for all $n \geq 1$.*

Proof: By the lemma 2.1, and observing that V_n is isomorphic to its Pontryagin dual as a $SL(2,\mathbf{K})$-module, it suffices to check that the only $SL(2,\mathbf{K})$-invariant mean on Borel subsets is the Dirac measure at 0; this is the result of proposition 2.3. ■

Corollary 2.5. *If G is a Lie group locally isomorphic to $SL(2,\mathbf{R}) \ltimes V_n$ ($n \geq 1$), then the pair $(G, \mathrm{rad}(G))$ has property (T).*

Proof: The argument is exactly the same as that of the preceding corollary. ■

We next have to consider invariant symplectic forms on V_m for odd $m = 2n-1$. The material here is borrowed from [CDSW]. Fix a field K with $\mathrm{char}(K) = 0$. Denote by $\pi : \mathfrak{sl}(2,K) \to \mathrm{End}(V_m)$ the unique irreducible $(m+1)$-dimensional representation. Take a basis $(e_0,\dots,e_m)$ of V_m, so that π is in a standard form:

$$\pi(H)e_i = (m-2i)e_i, \qquad \pi(X)e_i = (m-i+1)e_{i-1}\ (i \geq 1),$$
$$\pi(Y)e_i = (i+1)e_{i+1} \quad (i \leq m-1), \qquad \pi(X)e_0 = \pi(Y)e_m = 0.$$

We also denote by π the corresponding representation of $SL(2,K)$ on V_m.

Lemma 2.6. *Let $\mathcal{B}_n$ be the space of bilinear forms on V_{2n-1} ($n \geq 1$) preserved by $SL(2,K)$. Then $\mathcal{B}_n$ is one-dimensional, and consists of symplectic forms. Namely, $\mathcal{B}_n$ is generated by the form φ defined by, with $m = 2n-1$*

$$\varphi(e_i, e_{m-i}) = (-1)^i \binom{i}{m}; \quad \varphi(e_i, e_j) = 0 \text{ if } i+j \neq m.$$

Proof: Let φ be a bilinear form on V_m preserved by $\pi(SL(2,K))$, that is, making $\mathfrak{sl}(2,K)$ act by skew-adjoint operators. Then, using that $\pi(H)$ is skew-adjoint, we obtain $\varphi(e_i, e_j) = 0$ if $i+j \neq m$. Write $\varphi(e_i, e_{-i}) = \lambda_i$. Then, using that $\pi(X)$ is skew-adjoint, we obtain

$$\lambda_{i+1} = -\frac{m-i}{i+1}\lambda_i, \quad 0 \leq i \leq n-1$$

hence $\lambda_i = (-1)^i \binom{i}{m} \lambda_0$ for all i. It is straightforward that, conversely, this defines a nondegenerate bilinear form that makes $\pi(\mathfrak{sl}(2,K))$ skew-adjoint, which is symplectic (it would be symmetric if m were even). ■

The algebraic K-group H_n is defined as follows. Let Z be a one-dimensional vector space identified with $\mathrm{Hom}_K(\mathcal{B}_n, K)$. As a K-variety, H_n is the vector space $V_{2n-1} \oplus Z$. Define, for $x, x' \in V_{2n-1}$, $e_{(x,x')} \in Z$ by $e_{(x,x')}(\varphi) = \varphi(x, x')$. We equip H_n with the product $(x,z)\cdot(x',z') = (x+x', z+z'+e_{(x,x')})$. It is immediately checked that this is an algebraic group structure. Moreover, the action of $SL(2,K)$ on V_n extends naturally to H_n by setting $\pi(g)(x,z) = (\pi(g), z)$, and this action is by group automorphisms: indeed, for all $g \in SL(2,K), (x,z) \in H_n$,

$$\begin{aligned}\pi(g)(x,z)\pi(g)(x',z') &= (\pi(g)x, z)(\pi(g)x', z') \\ &= (\pi(g)x + \pi(g)x', z + z' + e_{\pi(g)x, \pi(g)x'}) \\ &= (\pi(g)(x+x'), z + z' + e_{x,x'}) \\ &= \pi(g)((x,z)(x',z'))\end{aligned}$$

Note that Z is both the center and the derived subgroup of H_n. We denote by $\mathfrak{h}_n$ the Lie algebra of H_n.

Now we have to prove that the pair $(SL(2,\mathbf{K}) \ltimes H_n, H_n)$ has property (T) for all $n \geq 1$.

Proposition 2.7. *Let G be a locally compact, σ-compact, group and N a closed normal subgroup. Suppose that $\overline{[N,N]}$ is contained in the center of G, and that the pair $(G/\overline{[N,N]}, N/\overline{[N,N]})$ has property (T). Then (G,N) has property (T).*

Proof: This proof is inspired by that of Proposition 4.1.4 in [CCJJV]. First we prove that $(G, \overline{[N,N]})$ has property (T).

Let π_i be a net of irreducible representations of G converging to the trivial representation. Let $\overline{\pi}_i$ be the contragredient representation of π_i. Then $\pi_i \otimes \overline{\pi_i}$ converges to the trivial representation. By irreducibility, π_i is scalar in restriction to $[N,N]$, hence $\pi_i \otimes \overline{\pi_i}$ is trivial on $\overline{[N,N]}$, so factors through $G/\overline{[N,N]}$. Since $(G/\overline{[N,N]}, N/\overline{[N,N]})$ has property (T), the restriction to N of $\pi_i \otimes \overline{\pi_i}$ eventually contains the trivial representation. By a standard argument ([BHV], Appendice 1), this means that $\pi_i|_N$ eventually contains a finite-dimensional representation. By Lie's theorem, since N is solvable, it preserves a one-dimensional subspace. Then this subspace is pointwise fixed by $[N,N]$, hence by $\overline{[N,N]}$. This shows that, eventually, the restriction to $\overline{[N,N]}$ of π_i contains the trivial representation, hence $(G, \overline{[N,N]})$ has property (T).

Now, since the pairs $(G, \overline{[N,N]})$ and $(G/\overline{[N,N]}, N/\overline{[N,N]})$ have property (T), (G,N) has property (T). ■

Corollary 2.8. *The pair $(SL(2,\mathbf{K}) \ltimes H_n, H_n)$ has property (T).*

Proof: This is an immediate consequence of proposition 2.7, the fact that $SL(2,\mathbf{K}) \ltimes H_n$ modulo its center is isomorphic to $SL(2,\mathbf{K}) \ltimes V_{2n-1}$, and corollary 2.4. ■

Corollary 2.9. *If G is a Lie group locally isomorphic to $SL(2,\mathbf{R}) \ltimes H_n$, then $(G, \mathrm{rad}(G))$ has property (T).*

Proof: The same argument as in the proof of corollary 2.8 works with proposition 2.7 and corollary 2.5. Note that $\mathrm{rad}(G)$ is either isomorphic to H_n, or H_n modulo a cyclic subgroup of its center, even if G is not always a semidirect product. ■

Now we have proved $((P)SL(2,\mathbf{K}) \ltimes V_n, V_n)$ and $(SL(2,\mathbf{K}) \ltimes H_n, H_n)$ have property (T); since the subgroups V_n and H_n are closed and not compact, we have proved the implication (2b)⇒(3a).

In the Lie framework, it is not supposed that the subgroups be closed, so that we also need the following observation.

Lemma 2.10. *Let G be a Lie group, and V an abelian connected Lie subgroup. Suppose that V contains no nontrivial central element of G. Then V is closed in G.*

Proof: Note that the normalizer $N_G(V)$ of V is closed, since it coincides with the normalizer of the Lie algebra of V for the adjoint representation. So, upon replacing G by the connected component of the unit of $N_G(V)$, we can suppose that V is a normal subgroup and that G is connected.

Let W be the closure of V. Then G acts on W by conjugation. Let K be the maximal torus of the abelian connected Lie group W. By connectedness, G acts trivially on K, i.e. K is central in G. This implies that $V \cap K = \{1\}$, so that, in the universal covering $\widetilde{W}$, V and $\widetilde{K}$ are supplementary subspaces. Thus, V is closed modulo the kernel Γ of the natural morphism $\widetilde{W} \to W$. Since V is also dense modulo Γ, this proves $W = V$. ■

Proposition 2.11. *Let G be a Lie group, and H a Lie subgroup locally isomorphic to $SL(2,\mathbf{R}) \ltimes V_n$ or $SL(2,\mathbf{R}) \ltimes H_n$ ($n \geq 1$). Then $\mathrm{rad}(H)$ is not relatively compact in G.*

Proof: Case 1: H is locally isomorphic to $SL(2,\mathbf{R}) \ltimes V_n$. Lemma 2.10 implies that $\mathrm{rad}(H)$ is closed. Since $\mathrm{rad}(H)$ is homeomorphic to $\mathbf{R}^{n+1}$, we are done.

Case 2: H is locally isomorphic to $SL(2,\mathbf{R}) \ltimes H_n$. Upon replacing G by its closed subgroup $N_G(H)$, we can assume that H is normal in G. So its center is also normal. Let Z be the closure in G of the center of H. Note that $H/(Z \cap H)$ is isomorphic to $SL(2,\mathbf{R}) \ltimes V_{2n-1}$. In the quotient by Z, the subgroup corresponding to V_{2n-1} is not relatively compact (by case 1), and it is the image of $\mathrm{rad}(H)$. So neither $\mathrm{rad}(H)$ is relatively compact. ■

3. Action on the radical

The key result is the following one, and is a generalization of the real case, worked out in [CDSW].

Theorem 3.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field of characteristic zero, and $\mathfrak{g} = \mathfrak{r}\mathfrak{s}$ a Levi decomposition ($\mathfrak{r} = \mathrm{rad}(\mathfrak{g})$ and $\mathfrak{s}$ is a semisimple factor). Then we have exactly one of the two alternatives:*

- *The simple factors of positive rank of the semisimple part $\mathfrak{s}$ centralize the radical $\mathfrak{r}$, or*
- *$\mathfrak{g}$ has a Lie subalgebra $\mathfrak{h}$ isomorphic to $\mathfrak{sl}(2,K) \ltimes V_n$ or $\mathfrak{sl}(2,K) \ltimes \mathfrak{h}_n$ for some $n \geq 1$, with $\mathrm{rad}(\mathfrak{h}) \subset \mathrm{rad}(\mathfrak{g})$.*

Proof: One sees easily that the two possibilities are incompatible: indeed, let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,K)$. Then $\mathfrak{h}$ is contained in a Levi factor $\mathfrak{s}'$, and, upon replacing $\mathfrak{h}$ by a conjugate, we can suppose $\mathfrak{h} \subset \mathfrak{s}$. Then the projection of $\mathfrak{h}$ on the rank zero factors of $\mathfrak{s}$ is trivial, so that $\mathfrak{h}$ is contained in the direct sum of simple factors of positive rank of $\mathfrak{s}$. Thus, if $\mathfrak{h}$ acts nontrivially on the radical, so does a simple factor of positive rank of $\mathfrak{s}$.

Suppose now that a simple factor of positive rank acts nontrivially on the radical. Since it is simple, it acts faithfully on the radical, and, since it has positive rank, it has a subalgebra $\mathfrak{h}$ isomorphic to $\mathfrak{sl}(2,K)$, which acts faithfully, hence nontrivially on the radical. Replacing $\mathfrak{g} = \mathfrak{r}\mathfrak{s}$ by $\mathfrak{r}\mathfrak{h}$, we can now suppose that $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}(2,K)$.

The adjoint representation of $\mathfrak{s}$ preserves all the derived algebras $D^n\mathfrak{r}$. Let n be such that $[\mathfrak{s}, D^n\mathfrak{r}] \neq 0$ and $[\mathfrak{s}, D^{n+1}\mathfrak{r}] = 0$. Upon replacing $\mathfrak{r}$ by $D^n\mathfrak{r}$, one can suppose that $\mathfrak{s}$ acts trivially on $[\mathfrak{r}, \mathfrak{r}]$.

Let $\mathfrak{m}$ be an irreducible submodule of $\mathfrak{r}$ on which $\mathfrak{s}$ acts nontrivially. Then, since $[\mathfrak{s}, \mathfrak{m}] = \mathfrak{m}$, by Jacobi identity we obtain (in the following lines, the bracket $[\mathfrak{a}, \mathfrak{b}]$ means the *vector space* generated

by the brackets $[a,b]$, $(a,b) \in \mathfrak{a} \times \mathfrak{b}$)

$$[\mathfrak{m}, D\mathfrak{r}] = [[\mathfrak{s}, \mathfrak{m}], D\mathfrak{r}] \subset [[D\mathfrak{r}, \mathfrak{s}], \mathfrak{m}] + [[\mathfrak{m}, D\mathfrak{r}], \mathfrak{s}] = 0.$$

In particular, $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] = 0$, so that $\mathfrak{r}' = \mathfrak{m} \oplus [\mathfrak{m}, \mathfrak{m}]$ is a Lie subalgebra, normalized my $\mathfrak{s}$.

Let u be any linear form on $[\mathfrak{m}, \mathfrak{m}]$. Then $(x,y) \mapsto u([x,y])$ defines a alternate form b_u on $\mathfrak{m}$, for which $\mathfrak{s}$ acts as skew-adjoint operators. Thus, by lemma 2.6, the image of $u \mapsto b_u$ is either null, or perhaps one-dimensional if $\mathfrak{m}$ has even dimension. Moreover, the mapping $u \mapsto b_u$ is injective.

Then, either $[\mathfrak{m}, \mathfrak{m}] = 0$ and $\mathfrak{r}'$ is abelian, and $\mathfrak{sr}'$ is isomorphic to $\mathfrak{sl}(2,K) \ltimes V_n$, or $[\mathfrak{m}, \mathfrak{m}]$ is one-dimensional, $\mathfrak{r}'$ is isomorphic to the Heisenberg Lie algebra $\mathfrak{h}_n$ of dimension $2n+1$, and $\mathfrak{sr}'$ is isomorphic to $\mathfrak{sl}(2,K) \ltimes \mathfrak{h}_n$. ■

Theorem 3.2. *Let G be a connected Lie group. Suppose the noncompact simple factors of its semisimple part do not all centralize the radical. Then G has a Lie subgroup H, locally isomorphic to $SL(2,\mathbf{R}) \ltimes V_n$ or $SL(2,\mathbf{R}) \ltimes H_n$ ($n \geq 1$), where the subgroup isomorphic to V_n (or H_n) is included in* $\mathrm{rad}(G)$.

Proof: This is an immediate consequence of theorem 3.1. ■

Now, we turn to the algebraic case.

Lemma 3.3. *Let G be a linear algebraic group over a field K of characteristic zero, and $\mathfrak{g}$ its Lie algebra. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Suppose $\mathfrak{h}$ is perfect, that is, $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$. Then $\mathfrak{h}$ is the Lie algebra of an unique connected algebraic subgroup H of $\mathfrak{g}$. Moreover, every linear algebraic K-group with Lie algebra isomorphic to $\mathfrak{h}$ is isogeneous to H.*

Proof: See [BOREL] p. 109, and [SPR] pp. 87-88. ■

Lemma 3.4. *Let K be a field of characteristic zero. The Lie algebras $\mathfrak{sl}(2,K) \ltimes V_n$ and $\mathfrak{sl}(2,K) \ltimes \mathfrak{h}_n$ are perfect for all $n \geq 1$.*

Proof: This is immediate, since $[\mathfrak{sl}(2,K), \mathfrak{sl}(2,K)] = \mathfrak{sl}(2,K)$ and $[\mathfrak{sl}(2,K), V_n] = V_n$ (resp. $[\mathfrak{sl}(2,K), \mathfrak{h}_n] = \mathfrak{h}_n$). ■

Lemma 3.5. *Let G be a linear algebraic group over a field K of characteristic zero. If its Lie algebra is isomorphic to $\mathfrak{sl}(2,K) \ltimes V_n$ ($n \geq 1$), then G is isomorphic to $SL(2,K) \ltimes V_n$, or possibly $PSL(2,K) \ltimes V_n$ if n is even. If its Lie algebra is isomorphic to $\mathfrak{sl}(2,K) \ltimes \mathfrak{h}_n$ ($n \geq 1$), then G is isomorphic to $SL(2,K) \ltimes H_n$.*

Proof: Apply lemmas 3.3 and 3.4. Note that $SL(2,K) \ltimes V_n$ and $SL(2,K) \ltimes H_n$ are simply connected as algebraic groups (that is, they have no nontrivial connected algebraic covering). The former has trivial center for odd n, and center isomorphic to $\mathbf{Z}/2\mathbf{Z}$ for even n (this gives the quotient $PSL(2,K) \ltimes V_n$), whereas $SL(2,K) \ltimes H_n$ has torsionless center, so has not finite quotient. ■

Theorem 3.6. *Let G be a linear algebraic group over a field K of characteristic zero, and $G = \mathscr{R}_u G.S$ a Levi decomposition ($\mathscr{R}_u G$ is the unipotent radical and S a reductive factor). Then we have exactly one of the two following possibilities:*

- *The simple factors of positive rank of the reductive factor S act trivially on the radical $\mathscr{R}G$, or*
- *G has an algebraic subgroup H isomorphic to $(P)SL(2,K) \ltimes V_n$ or $SL(2,K) \ltimes H_n$ for some $n \geq 1$, where $\mathscr{R}H$ is included in the unipotent radical $\mathscr{R}_u G$.*

Proof: Suppose that the first condition is not satisfied. Let $\mathfrak{g}$ be the Lie algebra of G. Then proposition 3.1 states that there exists a Lie subalgebra $\mathfrak{h}$, isomorphic to $\mathfrak{sl}(2,K) \ltimes V_n$ or $\mathfrak{sl}(2,K) \ltimes \mathfrak{h}_n$, $n \geq 1$, such that $\mathrm{rad}(\mathfrak{h}) \subset \mathrm{rad}(\mathfrak{g})$. Then lemmas 3.3 and 3.4 tell us that $\mathfrak{h}$ is the Lie algebra of a connected algebraic subgroup H of $\mathfrak{g}$. By lemma 3.5, H is isomorphic to $(P)SL(2,K) \ltimes V_n$ or $SL(2,K) \ltimes H_n$. Since $\mathrm{rad}(\mathfrak{h})$ is included in $\mathrm{rad}(\mathfrak{g})$, the radical of H is included in the radical $\mathscr{R}G$. Besides, we know that $\mathscr{R}G \cap [G,G]$ is always included in the unipotent radical $\mathscr{R}_u G$. Hence $\mathscr{R}H \subset \mathscr{R}_u G$. ■

4. An application

The following theorem solves an open question in [CCJJV, section 7.1].

Theorem 4.1. *Let $\Gamma = SO(3, \mathbf{Z}[1/p]) \ltimes \mathbf{Z}[1/p]^3$, p an odd prime. Then*

- *(1) Γ does not have the Haagerup property.*
- *(2) For every infinite subgroup $\Sigma \subset \Gamma$, the pair (Γ, Σ) does not have Kazhdan's relative property (T).*

Proof of (1) in theorem 4.1. Recall that $\mathbf{Z}[1/p]$ embeds (diagonally) as a lattice in $\mathbf{R} \times \mathbf{Q}_p$. Let $G(K) = SO(3, K)$ and $W(K) = SO(3, K) \ltimes K^3$. Then $\Gamma = W(\mathbf{Z}[1/p])$ is a lattice in $W(\mathbf{R}) \times W(\mathbf{Q}_p)$. Since $p \neq 2$, $SO(3, \mathbf{Q}_p)$ is not compact (if $p \equiv 1 \,[\mathrm{mod}\, 4]$, this is clear since then -1 is a square in $\mathbf{Q}_p$). By theorem 1.1, $W(\mathbf{Q}_p)$ does not have the Haagerup property, hence neither does Γ (Haagerup property is inherited from lattices, proposition 6.1.5 in [CCJJV]). ■

For the second statement of the theorem, we need the following obvious proposition.

Proposition 4.2. *Let G_1, G_2 be topological groups, $H \subset G$ a subgroup. Let $f : G_1 \to G_2$ be a continuous homomorphism, and suppose that (G_1, H) has relative property (T). Then so does $(G_2, f(H))$.*

Proof of (2) in theorem 4.1. Let us take a subgroup $\Sigma \subset \Gamma$ such that (Γ, Σ) has relative property (T). We want to prove that Σ is finite.

First take the embedding i of Γ in $SO(3, \mathbf{R}) \ltimes \mathbf{R}^3$. This group is amenable, hence has the Haagerup property. Then, using proposition 4.2, $\overline{i(\Sigma)}$ must be a compact subgroup of $SO(3, \mathbf{R}) \ltimes \mathbf{R}^3$. This implies $i(\Sigma) \cap \mathbf{R}^3 = \{1\}$, so that the obvious morphism $\pi : \Gamma \to SO(3, \mathbf{Z}[1/p])$ is injective in restriction to Σ.

Now, $SO(3, \mathbf{Z}[1/p])$ embeds as a discrete subgroup (in fact, as a lattice) in $SO(3, \mathbf{R}) \times SO(3, \mathbf{Q}_p)$, which has the Haagerup property ($SO(3, \mathbf{Q}_p)$ has rank one). So $SO(3, \mathbf{Z}[1/p])$ has the Haagerup property. On the other hand, by proposition 4.2, $(SO(3, \mathbf{Q}_p), \pi(\Sigma))$ has relative property (T). Hence $\pi(\Sigma)$ is finite.

Thus, we conclude that Σ is finite. ■

There are many analogous ways to find groups Γ satisfying the conclusion of theorem 4.1, the proofs being similar. We give an example.

Let q be the quadratic form $\sqrt{p}\, x_0^2 + x_1^2 + x_2^2 + \cdots + x_n^2$. Then $\Gamma = SO(q)(\mathbf{Z}[\sqrt{p}]) \ltimes \mathbf{Z}[\sqrt{p}]^{n+1}$ satisfies the conclusion of theorem 4.1. Note that Γ is finitely presented: it is an extension of $SO(q)(\mathbf{Z}[\sqrt{p}])$ by the finitely generated group $\mathbf{Z}[\sqrt{p}]^{n+1}$, and the former is finitely presented: it is a cocompact lattice in the semisimple Lie group $SO(1, n)$.

5. Conclusion

We finish by a remark on generalization of theorem 1.1 to positive characteristic. Several new difficulties arise, among others:

- Most of the results to recognize or construct an algebraic group from its Lie algebra fail in positive characteristic, so that we must work within the algebraic group.
- Levi factors do not always exist.
- It is not clear whether algebraic actions of $SL(2, K)$ on a unipotent algebraic group behave like a linear representation.
- Representations of $SL(2, K)$ are not always completely reducible.

Thus, richness of positive characteristic might provide interesting examples of algebraic groups with the Haagerup property, or at least without noncompact closed subgroup with relative property (T) (which are not, up to finite covering, direct products of simple groups of rank one and amenable groups). We leave it as an open problem.

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