Course 311: 2001–02. Supplementary Notes
The Resolvent Cubic of a Quartic Polynomial

Let α, β, γ and δ be the roots of the quartic polynomial

$$x^4 - px^2 - qx - r,$$

so that

$$(x-\alpha)(x-\beta)(x-\gamma)(x-\delta) = x^4 - px^2 - qx - r.$$

Equating coefficients of x, we find that

$$\alpha + \beta + \gamma + \delta = 0,$$

and

$$\begin{aligned} p &= -(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta), \\ q &= \beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma, \\ r &= -\alpha\beta\gamma\delta. \end{aligned}$$

Let

$$\begin{aligned} \lambda &= (\alpha+\beta)(\gamma+\delta) = -(\alpha+\beta)^2 = -(\gamma+\delta)^2, \\ \mu &= (\alpha+\gamma)(\beta+\delta) = -(\alpha+\gamma)^2 = -(\beta+\delta)^2, \\ \nu &= (\alpha+\delta)(\beta+\gamma) = -(\alpha+\delta)^2 = -(\beta+\gamma)^2. \end{aligned}$$

We wish to express $\lambda+\mu+\nu$, $\mu\nu+\lambda\nu+\lambda\mu$ and $\lambda\mu\nu$ in terms of p, q and r.

To do this we eliminate α from the above expressions using the identity $\alpha+\beta+\gamma+\delta = 0$. We find

$$\begin{aligned} p &= (\beta+\gamma+\delta)(\beta+\gamma+\delta) - \gamma\delta - \beta\delta - \beta\gamma \\ &= \beta^2 + \gamma^2 + \delta^2 + \gamma\delta + \beta\delta + \beta\gamma, \\ q &= \beta\gamma\delta - (\beta+\gamma+\delta)(\gamma\delta + \beta\delta + \beta\gamma) \\ &= -(\beta^2\gamma + \beta^2\delta + \gamma^2\beta + \gamma^2\delta + \delta^2\beta + \delta^2\gamma) - 2\beta\gamma\delta, \\ r &= \beta^2\gamma\delta + \gamma^2\beta\delta + \delta^2\beta\gamma. \end{aligned}$$

Then

$$\begin{aligned} \lambda+\mu+\nu &= -\Big((\gamma+\delta)^2 + (\beta+\delta)^2 + (\beta+\gamma)^2\Big) \\ &= -2\Big(\beta^2 + \gamma^2 + \delta^2 + \gamma\delta + \beta\delta + \beta\gamma\Big) \\ &= -2p, \end{aligned}$$

$$\begin{aligned}
\lambda^2+\mu^2+\nu^2 &= (\gamma+\delta)^4+(\beta+\delta)^4+(\beta+\gamma)^4\\
&= \gamma^4+4\gamma^3\delta+6\gamma^2\delta^2+4\gamma\delta^3+\delta^4\\
&\quad +\beta^4+4\beta^3\delta+6\beta^2\delta^2+4\beta\delta^3+\delta^4\\
&\quad +\beta^4+4\beta^3\gamma+6\beta^2\gamma^2+4\beta\gamma^3+\gamma^4\\
&= 2(\beta^4+\gamma^4+\delta^4)+4(\beta^3\gamma+\beta^3\delta+\gamma^3\beta+\gamma^3\delta+\delta^3\beta+\delta^3\gamma)\\
&\quad +6(\gamma^2\delta^2+\beta^2\delta^2+\beta^2\gamma^2),\\
p^2 &= \beta^4+\gamma^4+\delta^4+3(\gamma^2\delta^2+\beta^2\delta^2+\beta^2\gamma^2)\\
&\quad +4(\beta^2\gamma\delta+\gamma^2\beta\delta+\delta^2\beta\gamma)\\
&\quad +2(\beta^3\gamma+\beta^3\delta+\gamma^3\beta+\gamma^3\delta+\delta^3\beta+\delta^3\gamma).
\end{aligned}$$

Therefore

$$\begin{aligned}
\lambda^2+\mu^2+\nu^2 &= 2p^2-8(\beta^2\gamma\delta+\gamma^2\beta\delta+\delta^2\beta\gamma)\\
&= 2p^2-8r.
\end{aligned}$$

But

$$4p^2=(\lambda+\mu+\nu)^2=\lambda^2+\mu^2+\nu^2+2(\mu\nu+\lambda\nu+\lambda\mu)$$

Therefore

$$\begin{aligned}
\mu\nu+\lambda\nu+\lambda\mu &= 2p^2-\tfrac{1}{2}(\lambda^2+\mu^2+\nu^2)\\
&= p^2+4r.
\end{aligned}$$

Finally, we note that

$$\lambda\mu\nu=-\Big((\gamma+\delta)(\beta+\delta)(\beta+\gamma)\Big)^2.$$

Now

$$\begin{aligned}
(\gamma+\delta)(\beta+\delta)(\beta+\gamma) &= \beta^2\gamma+\beta^2\delta+\gamma^2\beta+\gamma^2\delta+\delta^2\beta+\delta^2\gamma+2\beta\gamma\delta\\
&= -q.
\end{aligned}$$

Therefore

$$\lambda\mu\nu=-(-q)^2=-q^2.$$

Thus λ, μ and ν are the roots of the *resolvent cubic*

$$x^3+2px^2+(p^2+4r)x+q^2.$$