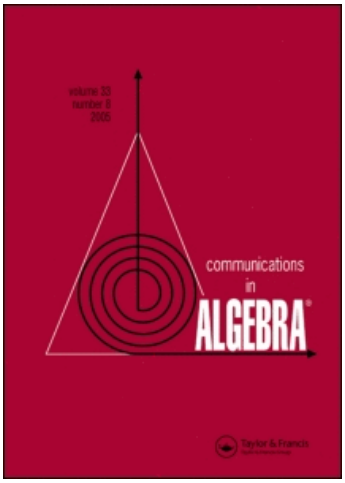
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Communications in Algebra
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713597239

Weak-Injective Modules

Sang Bum Lee [a]
[a] Department of Mathematics Education, Sangmyung University, Seoul, South Korea

To cite this Article Lee, Sang Bum(2006) 'Weak-Injective Modules', Communications in Algebra, 34: 1, 361 — 370
To link to this Article: DOI: 10.1080/00927870500346339
URL: http://dx.doi.org/10.1080/00927870500346339

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Communications in Algebra®, 34: 361–370, 2006
Copyright © Taylor & Francis, LLC
ISSN: 0092-7872 print/1532-4125 online
DOI: 10.1080/00927870500346339

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WEAK-INJECTIVE MODULES

Sang Bum Lee
Department of Mathematics Education, Sangmyung University, Seoul, South Korea

Pure-injective and RD-injective R-modules over domains R have been investigated by many authors. We introduce another class of R-modules, called weak-injective modules, which turn out to be useful in addressing several unanswered questions between the two classes of modules. We also find that this class is an envelope class over any domain, giving a partial answer to the existence of envelope classes in the hierarchy of injective and divisible modules.

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Key Words: Cover and envelope class; Flat, Projective and weak dimension; Prüfer domains; Pure-injective, *RD*-injective, and weak-injective modules; Torsion-free modules and modules of weak dimension ≤ 1.

Mathematics Subject Classification: 13C11; 13G05.

1. INTRODUCTION

In this note, R will denote a commutative domain with 1 and $Q(\neq R)$ its field of quotients. The R-module Q/R will be denoted by K. Various generalizations of injective R-modules were investigated in the literature. In Section 2, we consider two classes of R-modules, Pure-injective and RD-injective R-modules. Over Prüfer domains, they are identical and there are no other domains over which these two concepts coincide. We address two open problems concerning these modules in view of character modules of flat modules and modules of weak dimension ≤ 1. In Section 3, we introduce a class of R-modules between injective and h-divisible R-modules, called weak-injective R-modules. Recall that an h-divisible R-module is one which is an epic image of an injective R-module. We find that weak-injective R-modules are identical to injective R-modules if and only if R is Prüfer. Recall that R is called a *Matlis* domain if the projective dimension of Q (or equivalently K) is 1. We find that there exists an h-divisible but not weak-injective R-module over non-Matlis domains R. Among other properties, we note a seemingly anticipated one that injective is to flat as weak-injective is to torsion-free. Recall that R is called a *semi-Dedekind* domain if every h-divisible R-module is pure-injective. For details of these domains, we refer to Lee (2003). As an application of these

Received September 22, 2004; Revised February 9, 2005. Communicated by I. Swanson.
Address correspondence to Sang Bum Lee, Department of Mathematics Education, Sangmyung University, Seoul 110-743, South Korea, Fax: 02-3217-6019; E-mail: sblee@smu.ac.kr

modules, we provide characterizations of Prüfer and semi-Dedekind domains. In Section 4, we consider cover and envelope class. For details of these concepts, we refer to Trlifaj (2000) and Lee (2003). We drop the condition on the domain in Lee (2003, Theorem 4.10) and prove that over any domain R, every R-module has a $\mathscr{W}_1$-cover and a weak-injective envelope. Here $\mathscr{W}_1$ denotes the class of all R-modules of weak dimension ≤ 1. In Section 5, we consider the case when K is pure-injective and see how far it is from R being Prüfer. For unexplained definitions and terminologies, we refer to Fuchs and Salce (2001) and Trlifaj (2000).

2. PRELIMINARIES

Recall that R is called *Prüfer* if every finitely generated ideal of R is projective. There are numerous characterizations of Prüfer domains. Classical results can be found in Gilmer (1972).

Here we first consider two classes of R-modules which provide a characterization of Prüfer domains. Recall that an R-submodule N of an R-module M is called *pure* in M if, for all (finitely presented) R-modules F, the map $F \otimes_R N \to F \otimes_R M$ induced by the inclusion map $N \to M$ is injective. An exact sequence $0 \to N \xrightarrow{\phi} M \to L \to 0$ is called a *pure-exact* sequence if $\operatorname{Im}\phi$ is pure in M. An R-module M is called *pure-injective* if it has the injectivity property relative to all pure-exact sequences. An R-submodule N of an R-module M is called *relatively divisible* (or briefly *RD*) in M if $rN = N \cap rM$ for each $r \in R$. Accordingly, an R-module M is called *RD-injective* if it has the injectivity property relative to all *RD*-exact sequences (the definition of which is self-evident). The relations between these concepts are cummulated in the following lemma.

Lemma 2.1 (Naudé et al., 1986; Warfield, 1969). *For a domain R, the following are equivalent*:

(a) *R is a Prüfer domain*;
(b) *Rlative divisibility and purity are equivalent for R-modules*;
(c) *RD-injectivity and pure-injectivity are equivalent for R-modules.*

For more details on these modules, we refer to Fuchs and Salce (2001). A paticular result that we are interested in is the following lemma. The *character module* $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of an R-module M will be denoted by $M^{\flat}$.

Lemma 2.2 (Fuchs and Salce, 2001, p. 430; Lambek, 1964).

(a) *If E is pure-injective, then so is* $\operatorname{Hom}_R(M, E)$ *for every R-module M.*
(b) *For every R-module M,* $M^{\flat}$ *is pure-injective.*
(c) *An R-module F is flat if and only if* $F^{\flat}$ *is injective.*

We first try to answer the following question: What are then the R-modules M for which $M^{\flat}$ is *RD*-injective?

Recall that an R-module M is called *Warfield-cotorsion* if $\operatorname{Ext}^1_R(A, M) = 0$ for all torsion-free R-modules A. It is known that *RD*-injective R-modules are always

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Warfield-cotorsion, and torsion-free Warfield-cotorsion R-modules are RD-injective. The following lemma is crucial to this note.

Lemma 2.3. *For an R-module A, the following are equivalent*:

(a) *A is torsion-free*;
(b) $\mathrm{Tor}_1^R(M, A) = 0$ *for all R-modules M of projective dimension* ≤ 1;
(c) $\mathrm{Tor}_1^R(M, A) = 0$ *for all R-modules M of weak dimension* ≤ 1.

Proof. (a) $\Rightarrow$ (c) Let A be a torsion-free R-module and M an R-module of weak dimension ≤ 1. Consider a flat resolution $0 \to F_1 \to F_2 \to M \to 0$ of M, where F_1 and F_2 are flat R-modules. From the induced exact sequence $0 \to M^\flat \to F_2^\flat \to F_1^\flat \to 0$, we derive that the injective dimension of $M^\flat$ is ≤ 1 since $F_2^\flat$ and $F_1^\flat$ are injective by Lemma 2.2(c). The isomorphism

$$\mathrm{Ext}_R^1(Q, \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}_1^R(Q, M), \mathbb{Q}/\mathbb{Z}) = 0,$$

implies that $M^\flat$ is Warfield-cotorsion (see Fuchs and Salce, 2001, p. 458). Thus we obtain $0 = \mathrm{Ext}_R^1(A, M^\flat) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}_1^R(A, M), \mathbb{Q}/\mathbb{Z})$ and therefore conclude that $\mathrm{Tor}_1^R(A, M) = 0$.

(c) $\Rightarrow$ (b) Trivial.

(b) $\Rightarrow$ (a) It is easy to prove that $\mathrm{Tor}_1^R(R/rR, A) \cong \{a \in A \mid ra = 0\}$ for every $r \in R$ (see Fuchs and Salce, 2001, p. 34). Since the R-module R/rR is of projective dimension ≤ 1, $\{a \in A \mid ra = 0\} = 0$ by hypothesis. This implies A is torsion-free. □

From this, we can obtain

Corollary 2.4. *An R-module M has weak dimension* ≤ 1 *if and only if* $\mathrm{Tor}_1^R(A, M) = 0$ *for all torsion-free R-modules A.*

Proof. $\Rightarrow$) This follows from Lemma 2.3.

$\Leftarrow$) Let N be an R-module and $0 \to H \to F \to N \to 0$ be a presentation of N, where F is free and H torsion-free. From the induced exact sequence $0 = \mathrm{Tor}_1^R(M, H) \to \mathrm{Tor}_2^R(M, N) \to \mathrm{Tor}_2^R(M, F) = 0$, we have $\mathrm{Tor}_2^R(M, N) = 0$. Since N was arbitrary, we conclude M has weak dimension ≤ 1. □

This in turn gives us an analog of Lemma 2.2(c).

Corollary 2.5. *An R-module M has weak dimension* ≤ 1 *if and only if $M^\flat$ is Warfield-cotorsion.*

Proof. $\Rightarrow$) see the proof of Lemma 2.3.

$\Leftarrow$) see the proof of Corollary 2.4. □

Combining Corollary 2.5 with the simple fact that an R-module D is divisible if and only if $D^\flat$ is torsion-free, we obtain a partial answer to the above mentioned question.

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Theorem 2.6. *An R-module D is divisible of weak dimension* ≤ 1 *if and only if* $D^\flat$ *is torsion-free RD-injective.*

Recall that an R-module F is flat if and only if $\mathrm{Ext}^1_R(F, E) = 0$ for all pure-injective R-modules E. The following corollaries are analogs of this for torsion-free modules and modules of weak dimension ≤ 1.

Corollary 2.7. *For an R-module A, the following are equivalent*:

(a) *A is torsion-free*;
(b) $\mathrm{Ext}^1_R(A, E) = 0$ *for all RD-injective R-modules E*;
(c) $\mathrm{Ext}^1_R(A, K^\flat) = 0$.

Proof. (a) $\Rightarrow$ (b) Trivial since *RD*-injectives are Warfield-cotorsion.

(b) $\Rightarrow$ (c) Trivial since $K^\flat$ is *RD*-injective by Theorem 2.6.

(c) $\Rightarrow$ (a) The isomorphism $\mathrm{Ext}^1_R(A, K^\flat) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}^R_1(A, K), \mathbb{Q}/\mathbb{Z})$ implies $\mathrm{Tor}^R_1(A, K) = 0$. Hence A is torsion-free. □

Note that in Corollary 2.7, $K^\flat$ plays the role of a single test module for the torsion-freeness of A.

Corollary 2.8. *An R-module M has weak dimension* ≤ 1 *if and only if* $\mathrm{Ext}^2_R(M, E) = 0$ *for all pure-injective R-modules E.*

Proof. This follows from the fact that every pure-injective R-module is a summand of $N^\flat$ for some R-module N, and the isomorphism $\mathrm{Ext}^2_R(M, N^\flat) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}^R_2(M, N), \mathbb{Q}/\mathbb{Z})$. □

3. WEAK-INJECTIVE MODULES

An R-module D is called *weak-injective* if $\mathrm{Ext}^1_R(M, D) = 0$ for all R-modules M of weak dimension ≤ 1. Evidently, direct products and summands of weak-injective R-modules are again weak-injective. Since $\mathrm{Ext}^1_R(K, D) = 0$ for a weak-injective R-module D, we conclude that weak-injective modules are h-divisible by Lee (2003). Injective R-modules are trivially weak-injective. Over a Prüfer domain R, weak-injective R-modules are injective since the weak global dimension of a Prüfer domain is 1. We first prove that this in fact characterizes Prüfer domains. The following lemma is an analog of Lemma 2.2(c).

Lemma 3.1. *An R-module A is torsion-free if and only if* $A^\flat$ *is weak-injective.*

Proof. The isomorphism $\mathrm{Ext}^1_R(M, A^\flat) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Tor}^R_1(M, A), \mathbb{Q}/\mathbb{Z})$, together with Lemma 2.3, proves the result. □

Recall that an R-module M is called *FP-injective* if $\mathrm{Ext}^1_R(N, M) = 0$ for all finitely presented R-modules N. Note that direct sums of *FP*-injective R-modules are again *FP*-injective.

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Theorem 3.2. *For a domain R, the following are equivalent*:

(a) *R is Prüfer*;
(b) *All weak-injectives are FP-injective*;
(c) *All weak-injectives are injective.*

Proof. Since (a) $\Rightarrow$ (c) $\Rightarrow$ (b) is trivial, it remains to show (b) $\Rightarrow$ (a). Let A be a torsion-free R-module. By Lemma 3.1, $A^\flat$ is weak-injective and thus is FP-injective by hypothesis. It is wellknown that FP-injective R-modules, which are pure-injective, are injective. Hence $A^\flat$ is injective and thus A is flat by Lemma 2.2(c). Since A was arbitrary, we conclude that R is Prüfer. □

In view of the fact that divisible R-modules (and thus h-divisible R-modules) which are RD-injective are injective, we address the following question: What are then divisible R-modules that are pure-injective? This was already mentioned in Lee (2003) but can be improved as follows.

Theorem 3.3. *For a pure-injective R-module D, the following are equivalent*:

(a) *D is divisible*;
(b) *D is h-divisible*;
(c) *D is weak-injective.*

Proof. We have only to prove (a) $\Rightarrow$ (c). Let D be a divisible R-module. Then $D^\flat$ is torsion-free and hence $D^{\flat\flat}$ is weak-injective by Lemma 3.1. The pure-imbedding $D \to D^{\flat\flat}$ together with the hypothesis implies that D is a summand of $D^{\flat\flat}$. Hence D is weak-injective. □

As a special case of Lemma 2.2(a), $\mathrm{Hom}_R(K, E)$ is pure-injective for every pure-injective R-module E. Its converse is considered in the following lemma.

Lemma 3.4 (Lee, 2003)**.** *Let R be a Matlis domain and D an h-divisible R-module. Then D is pure-injective if and only if* $\mathrm{Hom}_R(K, D)$ *is pure-injective.*

Proof. We can assume, without loss of generality, that D is torsion. Let D^* be the pure-injective hull of D. The pure-exact sequence

$$0 \to \mathrm{Hom}_R(K, D) \to \mathrm{Hom}_R(K, D^*) \to \mathrm{Hom}_R(K, D^*/D) \to \mathrm{Ext}^1_R(K, D) = 0$$

induced by the pure-exact sequence $0 \to D \to D^* \to D^*/D \to 0$ is splitting by hypothesis. Note that $\mathrm{Ext}^1_R(K, D) = 0$, since the projective dimension of K is 1 for a Matlis domain R. Hence we have

$$\mathrm{Hom}_R(K, D^*) \cong \mathrm{Hom}_R(K, D) \oplus \mathrm{Hom}_R(K, D^*/D).$$

By Matlis duality (see Fuchs and Salce, 2001, p. 280), we have $tD^* \cong D \oplus t(D^*/D)$, where t denotes "torsion part". As D is pure-essential in tD^*, which is pure in D^*, splitting can be ruled out i.e., D^*/D is torsion-free. Hence the divisible R-module D^*/D is isomorphic to a direct sum of copies of Q. Since the projective dimension

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of D^*/D is 1 and D is divisible, we have $\mathrm{Ext}^1_R(D^*/D, D) = 0$. Thus D is a summand of D^* and thus $D^*/D = 0$, establishing the result. □

Looking at the proof of Lemma 3.4, we note that the above lemma can be modified as follows: Let D be an R-module satisfying $\mathrm{Ext}^1_R(K, D) = 0$. Then D is pure-injective if and only if $\mathrm{Hom}_R(K, D)$ is pure-injective. As a consequence of this, we can conclude that a weak-injective R-module D is pure-injective if and only if $\mathrm{Hom}_R(K, D)$ is pure-injective. It is easy to prove that every weak-injective R-module D is Warfield-cotorsion if and only if every torsion-free R-module has weak dimension ≤ 1. The next lemma weakens the hypothesis in Theorem 3.3.

Lemma 3.5. *For an R-module D such that $\mathrm{Hom}_R(K, D)$ is pure-injective, the following are equivalent*:

(a) *D is h-divisible*;
(b) $\mathrm{Ext}^1_R(K, D) = 0$;
(c) *D is weak-injective.*

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Proof. In Lee (2003, Lemma 2.3), it is shown that an R-module D which satisfies (b) is h-divisible. Hence we have only to show that (a) $\Rightarrow$ (c). Since the torsion submodule of an h-divisible R-module D is a summand of D, we can assume, without loss of generality, that D is torsion. Write $A = \mathrm{Hom}_R(K, D)$. From the Matlis category equivalence between h-divisible torsion R-modules and R-complete torsion-free R-modules, we have $D \cong E(A)/A$, where $E(A)$ is the injective hull of A (see Fuchs and Salce, 2001, p. 280). Consider the exact sequence $0 \to A \to E(A) \to D \to 0$. Let N be an R-module of weak dimension ≤ 1. In the induced exact sequence $\mathrm{Ext}^1_R(N, E(A)) \to \mathrm{Ext}^1_R(N, D) \to \mathrm{Ext}^2_R(N, A)$, both ends are zero by Corollary 2.8. Hence we have $\mathrm{Ext}^1_R(N, D) = 0$. Since N was arbitrary, we conclude D is weak-injective. □

We can prove that if $\mathrm{Hom}_R(K, D)$ is *RD*-injective in Lemma 3.5, then D in (c) is injective.

Lemma 3.6. *For a domain R, the following are equivalent*:

(a) *Every R-module of weak dimension ≤ 1 has projective dimension ≤ 1*;
(b) *All h-divisible R-modules are weak-injective*;
(c) *All divisible R-modules are weak-injective.*
(d) *Epic images of weak-injective R-modules are weak-injective.*

Proof. (a) $\Leftrightarrow$ (b) This follows from Fuchs and Salce (2001, Proposition 2.5, p. 252).

(d) $\Rightarrow$ (b) Trivial since injectives are weak-injective.

(b) $\Rightarrow$ (d) Trivial since weak-injectives are h-divisible.

(c) $\Rightarrow$ (b) Trivial since h-divisibles are divisible.

(a) $\Rightarrow$ (c) Since (a) implies that R is a Matlis domain, divisible R-modules are h-divisible. □

The following is an analog of Fuchs and Salce (2001, Theorem 6.4, p. 452).

Corollary 3.7. *For a domain R, the following are equivalent*:

(a) *R is semi-Dedekind*;
(b) $\mathrm{Hom}_R(K, D)$ *is pure-injective for all h-divisible R-modules D*;
(c) $\mathrm{Hom}_R(D', D)$ *is pure-injective for all h-divisible R-modules D′, D.*

Proof. By Lemma 2.2(a), (a) $\Rightarrow$ (c) $\Rightarrow$ (b) is immediate. We show (b) $\Rightarrow$ (a). By Lemma 3.5, every h-divisible R-module D is weak-injective. Then by Lemma 3.6, R is a Matlis domain. By Lemma 3.4, D is pure-injective. □

Recall the famous Baer's criterion for injectivity: an R-module E is injective if and only if $\mathrm{Ext}^1_R(R/I, E) = 0$ for all ideals I of R. Noting that an R-module D is divisible if and only if $\mathrm{Ext}^1_R(R/I, D) = 0$ for all projective ideals I of R, we raise the following question: Is it true that an R-module D is weak-injective if and only if $\mathrm{Ext}^1_R(R/I, D) = 0$ for all flat ideals I of R?

We can also give a characterization of injectives in view of divisibles.

Lemma 3.8. *An R-module E is injective if and only if* $\mathrm{Ext}^1_R(D, E) = 0$ *for all divisible R-modules D.*

Proof. We have only to prove sufficiency. Let M be an R-module. By Fuchs and Salce (2001, Proposition 1.4, p. 249), there is an exact sequence $0 \to M \to D(M) \to D(M)/M \to 0$, where $D(M)$ is a divisible hull of M and $D(M)/M$ has projective dimension ≤ 1. In the induced exact sequence $\mathrm{Ext}^1_R(D(M), E) \to \mathrm{Ext}^1_R(M, E) \to \mathrm{Ext}^2_R(D(M)/M, E)$, both ends are zero. Hence we have $\mathrm{Ext}^1_R(M, E) = 0$. Since M was arbitrary, we conclude E is injective. □

This provides another characterization of Prüfer domains.

Corollary 3.9. *R is a Prüfer domain if and only if every divisible R-module has weak dimension* ≤ 1.

Proof. We have only to prove sufficiency. By hypothesis, $\mathrm{Ext}^1_R(D, D') = 0$ for all divisible R-modules D and weak-injective R-modules D'. By Lemma 3.8, D' is injective. Hence, by Theorem 3.2, R is Prüfer. □

Recall that R is a Dedekind domain if and only if every cyclic torsion R-module has projective dimension ≤ 1. Next corollary is another characterization.

Corollary 3.10. *R is a Dedekind domain if and only if every h-divisible torsion R-module has projective dimension* ≤ 1.

Proof. We have only to prove suffiency. Let D be a divisible R-module and D' an h-divisible R-module. Since R is a Matlis domain by hypothesis, D is h-divisible and thus has projective dimension ≤ 1. Hece we have $\mathrm{Ext}^1_R(D, D') = 0$. Since D, D' were arbitrary, we conclude D' is injective and R is Dedekind. □

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4. $\mathscr{W}_1$-COVER AND WEAK-INJECTIVE ENVELOPE

Let $\mathscr{C}$ be a class of R-modules and M an R-module. A homomorphism $\phi \in \mathrm{Hom}_R(N, M)$ with $N \in \mathscr{C}$ is called a *$\mathscr{C}$-precover* of M if the induced map

$$\mathrm{Hom}_R(1_{N'}, \phi) : \mathrm{Hom}_R(N', N) \to \mathrm{Hom}_R(N', M)$$

is surjective for all $N' \in \mathscr{C}$. A $\mathscr{C}$-precover $\phi \in \mathrm{Hom}_R(N, M)$ is called a *$\mathscr{C}$-cover* if each $\gamma \in \mathrm{Hom}_R(N, N)$ satisfying $\phi = \phi\gamma$ is an automorphism of N. $\mathscr{C}$ is called a *precover* (*cover*) *class* if every R-module has a $\mathscr{C}$-precover ($\mathscr{C}$-cover).

$\mathscr{C}$-preenvelope, *$\mathscr{C}$-envelope, preenvelope class* and *envelope class* are defined dually (see Trlifaj, 2000). In particular, if $\mathscr{C}$ is the class of weak-injective R-modules, a $\mathscr{C}$-envelope is called a *weak-injective envelope*. As usual, Mod-R denotes the category of all R-modules.

We will use the following notations:

$$\mathscr{C}^{\top} = \{X \in \text{Mod-}R \mid \mathrm{Tor}_1^R(Y, X) = 0 \text{ for all } Y \in \mathscr{C}\},$$
$$\mathscr{C}^{\perp} = \{X \in \text{Mod-}R \mid \mathrm{Ext}_R^1(Y, X) = 0 \text{ for all } Y \in \mathscr{C}\},$$
$${}^{\perp}\mathscr{C} = \{X \in \text{Mod-}R \mid \mathrm{Ext}_R^1(X, Y) = 0 \text{ for all } Y \in \mathscr{C}\}.$$

Let $\mathscr{C}, \mathscr{D}$ be classes of R-modules. The pair $(\mathscr{C}, \mathscr{D})$ is called a *cotorsion theory* if $\mathscr{C} = {}^{\perp}\mathscr{D}$ and $\mathscr{D} = \mathscr{C}^{\perp}$. It is easy to show that $({}^{\perp}\mathscr{C}, ({}^{\perp}\mathscr{C})^{\perp})$ and $({}^{\perp}(\mathscr{C}^{\perp}), \mathscr{C}^{\perp})$ are cotorsion theories, called the cotorsion theory *generated* and *cogenerated*, respectively, by the class $\mathscr{C}$.

In this note, we are particularly interested in the hierarchy of injective and divisible R-modules. It is well known that every R-module over any ring R has an injective envelope. Trlifaj (2000) proved that every free R-module has a divisible envelope if and only if R is a Matlis domain. He also proved that if R is a non-Matlis Prüfer domain, then no free R-module has a divisible envelope. Lee (2003) proved that over a semi-Dedekind domain R, every R-module has a divisible envelope. We pursue as wide an envelope class as possible over any domain.

Theorem 4.1. *Over a domain R, every R-module has a $\mathscr{W}_1$-cover and a weak-injective envelope.*

Proof. Let $\mathscr{A}$ be the class of all torsion-free R-modules. Lemma 2.3 and Corollary 2.4 imply that $\mathscr{W}_1 = \mathscr{A}^{\top}$ and $\mathscr{A} = \mathscr{W}_1^{\top}$. By Trlifaj (2000, Lemma 1.11), $(\mathscr{W}_1, \mathscr{W}_1^{\perp})$ is a cotorsion theory generated by $\{A^{\flat} \mid A \in \mathscr{A}\}$. Hence , by Trlifaj (2000, Theorem 2.8), $\mathscr{W}_1$ is a cover class and $\mathscr{W}_1^{\perp}$ (the class of all weak-injective R-modules) is an envelope class. □

Note that in the proof of Theorem 3.11, $(\mathscr{A}, \mathscr{A}^{\perp})$ is also a cotorsion theory, which yields the torsion-free cover class $\mathscr{A}$ and the Warfield-cotorsion envelope class $\mathscr{A}^{\perp}$. This was proved independently by Enochs (1963), and more generally by Trlifaj (2000) over any rings. Theorem 3.11 says that both $\mathscr{A}$ and $\mathscr{W}_1$ are cover classes coarser than the flat cover class. Let $\mathscr{P}_1$ denote the class of all R-modules of projective dimension ≤ 1. Note that the following corollary improves Lee (2003, Theorem 4.10).

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Corollary 4.2. *Let R be a domain over which every h-divisible R-module is weak-injective. Then every R-module has a $\mathcal{P}_1$-cover and a divisible envelope.*

Proof. This follows from Lemma 3.6. □

5. WHEN K IS FP-INJECTIVE

In this section, we consider the following well-known result.

Lemma 5.1 (Fuchs and Salce, 2001, p. 314)**.** *For a domain R, the following are equivalent*:

(a) *R is Prüfer*;
(b) *All h-divisible R-modules are FP-injective*;
(c) *All torsion-free R-modules are flat.*

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In case K is FP-injective, it is easy to prove that every h-divisible R-module of projective dimension ≤ 1 is FP-injective. This follows from the fact that the torsion submodule of an h-divisible R-module is always a summand of the module and every h-divisible torsion R-module of projective dimension ≤ 1 is a summand of a direct sum of copies of K. Now we consider this case with regard to (c). In Lee (2003), it is shown that over a Matlis domain R with global dimension 2, every torsion-free R-module has projective dimension ≤ 1. We consider a similar condition.

Corollary 5.2. *Suppose R is a domain with global dimension 2 and K is FP-injective. Then R is Prüfer.*

Proof. Let D be an h-divisible R-module. Without loss of generality, we can assume D is torsion. There is an exact sequence $0 \to H \to \bigoplus K \to D \to 0$, where H is h-divisible torsion. By hypothesis, injective dimension of H is ≤ 1. Note that $\bigoplus K$ is FP-injective. Let F be a finitely presented R-module. In the induced exact sequence $\mathrm{Ext}^1_R(F, \bigoplus K) \to \mathrm{Ext}^1_R(F, D) \to \mathrm{Ext}^2_R(F, H)$, both ends are zero. Hence we have $\mathrm{Ext}^1_R(F, D) = 0$. Since D, F were arbitrary, we conclude D is FP-injective and thus R is Prüfer by Lemma 5.1(b). □

Now we prove the main result of this section.

Theorem 5.3. *If K is FP-injective, then every torsion-free R-module of projective dimension ≤ 1 is flat.*

Proof. By Fuchs and Salce (2001, Theorem 2.1, p. 278), K is pure in its injective hull $E(K)$. From the Matlis category equivalence, we can conclude that $\mathrm{Hom}_R(K, K)$ is pure in $\mathrm{Hom}_R(K, E(K))$. It is known that $\mathrm{Hom}_R(K, K) \cong \widetilde{R}$, the R-completion of R and $\mathrm{Hom}_R(K, E(K)) \cong \widehat{R}$, the RD-injective hull of R (see Fuchs and Salce, 2001, p. 426). Since the R-module $\widetilde{R}/R$ is torsion-free and divisible, we can conclude that R is pure in $\widetilde{R}$, and therefore in $\widehat{R}$. Let $0 \to R \to M \to M/R \to 0$ be an RD-exact sequence of R-modules. Using the pure imbedding $\mu : R \to \widetilde{R}$, form the pushout

diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \xrightarrow{a} & M & \longrightarrow & M/R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \nu} & & \| & & \\ 0 & \longrightarrow & \widehat{R} & \xrightarrow{a'} & M' & \longrightarrow & M/R & \longrightarrow & 0. \end{array}$$

The bottom row is also RD-exact, and since $\widehat{R}$ is RD-injective, we can conclude that the bottom row splits. Thus $\alpha'\mu = \nu\alpha$ imbedds R as a pure submodule of M', whence the purity of $\alpha(R)$ in M is obvious. Since any direct sum $\bigoplus K$ of copies of K is again FP-injective, $\bigoplus K$ is pure in its injective hull $E(\bigoplus K)$. Again, by Matlis category equivalence, $\mathrm{Hom}_R(K, \bigoplus K)$ is pure in $\mathrm{Hom}_R(K, E(\bigoplus K))$. We also have $\mathrm{Hom}_R(K, \bigoplus K) \cong \widetilde{\bigoplus} R$, the R-completion of $\bigoplus R$ and $\mathrm{Hom}_R(K, E(\bigoplus K)) \cong \widehat{\bigoplus} R$, the RD-injective hull of $\bigoplus R$ (see Fuchs and Salce, 2001, p. 280). Applying the same argument to the pure imbedding $\mu : \bigoplus R \to \widehat{\bigoplus} R$, we can conclude that any free R-module $\bigoplus R$, which is contained in an R-module M as an RD-submodule, is in fact pure in M. We can also apply this argument to projective R-modules. Now let A be a torsion-free R-module of projective dimension ≤ 1 and $0 \to H \to F \to A \to 0$ a presentation of A, where F is free and H is projective. By the above argument, H is pure in F, which implies that A is flat. □

Note that under the hypothesis of Theorem 5.3, we can in fact prove that if A is a torsion-free R-module of projective dimension $\leq n$, then it has weak dimension $\leq n - 1$.

ACKNOWLEDGMENTS

The article was completed while the author was visiting Tulane University. He expresses his thanks to the Department of Mathematics of Tulane University for the hospitality, and to Professor Laszlo Fuchs for several discussions on the subject. He also thanks the referee for valuable comments.

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