COHERENT RINGS AND *FP*-INJECTIVE MODULES

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1. *Introduction*

There are many results of a homological nature which may be generalized from Noetherian rings to coherent rings. In this process, finitely generated modules should in general be replaced by finitely presented modules. In this note we will in particular study the consequences of extending the notion of injective module to that of "*FP*-injective module", where a module M is called *FP-injective* if

$$\mathrm{Ext}^1(F, M) = 0,$$

for every finitely presented module F.

The properties of *FP*-injective modules over arbitrary rings are considered in §2. In §3 we specialize to coherent rings, and show that these rings are characterized by the property that direct limits of *FP*-injective modules are *FP*-injective. The observation (*cf.* D. G. McRae [**11**]) that the weak dimension of a coherent ring may be obtained as the global *FP*-injective dimension is used for generalizing a dimension result proved for Noetherian rings by Osofsky [**12**]. In §4 we consider self-*FP*-injective coherent rings and their significance for duality theory. In particular it is proved that a left self-*FP*-injective left coherent and right perfect ring is a quasi-Frobenius ring.

Throughout this paper, A denotes a ring with 1 and all modules are unitary A-modules.

2. *FP-injective modules*

For a finitely presented right module F and a direct system $(M_i)_I$ of right modules, we consider the canonical homomorphism

$$\xi_n : \varinjlim \mathrm{Ext}^n(F, M_i) \to \mathrm{Ext}^n(F, \varinjlim M_i).$$

Recall that finite presentation of F is equivalent to ξ_0 being an isomorphism for every $(M_i)_I$.

PORPOSITION 2.1. *ξ_1 is a monomorphism.*

Proof. We employ the Yoneda description of Ext, so that the proof works in more general categories where there are not necessarily enough projectives. ξ_n is then induced by the pushout construction for long exact sequences. Suppose we have the exact commutative diagram

$$\begin{array}{ccccccc} 0 \to & M_i & \to N_i & \to F & \to 0 \\ & \downarrow & \downarrow & \| & \\ 0 \to & \varinjlim M_i & \to N & \to F & \to 0 \end{array}$$

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and the lower row splits. For $j \geqslant i$ we may choose $0 \to M_j \to N_j \to F \to 0$ as pushouts of the upper row. The lower row then becomes

$$0 \to \varinjlim M_j \to \varinjlim N_j \to F \to 0.$$

Since F is finitely presented, the splitting homomorphism $F \to \varinjlim N_j$ factors over some N_j and thereby defines a splitting of $0 \to M_j \to N_j \to F \to 0$.

PROPOSITION 2.2. *ξ_1 is an isomorphism whenever $(M_i)_I$ is a direct system of submodules of some module.*

Proof. In order to prove that ξ_1 is surjective, let there be given an exact sequence

$$0 \to \bigcup_I M_i \to N \to F \to 0.$$

There exists a finitely generated submodule N' of N which maps onto F. We obtain

$$\begin{array}{ccccccc} 0 \to & K & \to N' & \to F & \to 0 \\ & \alpha\downarrow & \downarrow & \| & \\ 0 \to & \bigcup M_i & \to N & F & \to 0 \end{array}$$

Since K is finitely generated, α factors over some M_i. The given sequence is then the pushout of an exact sequence $0 \to M_i \to N'' \to F \to 0$.

A module M will be called *FP-injective* if $\operatorname{Ext}^1(F, M) = 0$ for every finitely presented F. From the preceding proposition we obtain:

COROLLARY 2.3. *If $(M_i)_I$ is a direct family of FP-injective submodules of some module, then also $\bigcup_I M_i$ is FP-injective.*

COROLLARY 2.4. *Any direct sum of FP-injective modules is FP-injective.*

If M is a right module, let $A(M)$ denote the family of subsets of M of the form $\{x \in M \mid xI = 0\}$, I a right ideal of A. M will be called *semi-compact* if every finitely solvable set of congruences

$$x \equiv x_\alpha \pmod{M_\alpha},$$

where $x_\alpha \in M$ and $M_\alpha \in A(M)$, has a solution in M[**9**].

PROPOSITION 2.5. *An FP-injective module is injective if and only if it is semi-compact.*

Proof. Essentially as in [**9**; Prop. 2 and 3].

The concept of FP-injectivity is closely related to that of pure exact sequences. A short exact sequence of right modules is called *pure* if every finitely presented right module is a relative projective for it, or equivalently, if it remains exact when tensoring it with any left module [**5** and **13**; Th. 2]. One easily verifies:

PROPOSITION 2.6. *The following statements are equivalent for a right module M:*

(a) *M is FP-injective.*

(b) *If G is a finitely generated submodule of a free module F, then every homomorphism $G \to M$ may be extended to $F \to M$.*

(c) *Every exact sequence $0 \to M \to M' \to M'' \to 0$ is pure.*

(d) *There exists a pure exact sequence $0 \to M \to M' \to M'' \to 0$ with M' FP-injective.*

Definitions. (1) For any module M, let FP-inj.dim M denote the smallest integer $n \geqslant 0$ such that $\operatorname{Ext}^{n+1}(F, M) = 0$ for every finitely presented module F.

(2) r.FP-dim $A = \sup\{FP\text{-inj.dim } M \mid M \text{ a right module}\}$.

The duality formula [3; p. 120]:

$$\operatorname{Ext}^n(F, L^*) = \operatorname{Tor}_n(F, L)^*, \tag{1}$$

where F is a right module, L a left module and $(\cdot)^* = \operatorname{Hom}_Z(\cdot, Q/Z)$, gives:

PROPOSITION 2.7. gl.w.dim $A \leqslant$ r.FP-dim $A \leqslant \sup\{\text{proj.dim } F \mid F$ *a finitely presented right module*$\}$.

3. *Coherent rings*

We recall that the ring A is said to be *right coherent* if every finitely generated submodule of a finitely presented right module also is finitely presented [4, 11].

LEMMA 3.1. *Let A be right coherent. For a right module M, the following assertions are equivalent*:

(a) FP-inj.dim $M \leqslant n$.

(b) $\operatorname{Ext}^{n+1}(F, M) = 0$ *for all finitely presented modules F.*

(c) $\operatorname{Ext}^{n+1}(A/I, M) = 0$ *for all finitely generated right ideals I.*

(d) *If the sequence* $0 \to M \to E_0 \ldots \to E_n \to 0$ *is exact with* $E_0, \ldots, E_{n-1}$ FP-*injective, then also E_n is FP-injective.*

Proof. (a) $\Rightarrow$ (b): Use induction on n. Clear if FP-inj.dim $M = n$. If FP-inj.dim $M \leqslant n-1$, resolve F by $0 \to K \to P \to F \to 0$ with P finitely generated free. K is finitely presented since A is coherent, and $\operatorname{Ext}^{n+1}(F, M) \cong \operatorname{Ext}^n(K, M) = 0$ by induction hypothesis.

(c) $\Rightarrow$ (b): We use induction on the number of generators of F. If F is finitely presented on m generators, let F' be the submodule generated by one of these generators. Since A is coherent, both F' and F/F' are finitely presented on less than m generators, so we get an exact sequence

$$\operatorname{Ext}^{n+1}(F/F', M) \to \operatorname{Ext}^{n+1}(F, M) \to \operatorname{Ext}^{n+1}(F', M),$$

where both end terms are zero by induction, and hence $\operatorname{Ext}^{n+1}(F, M) = 0$. (b) $\Rightarrow$ (a) and (b) $\Rightarrow$ (c) are trivial, and (b) $\Leftrightarrow$ (d) follows from the isomorphism $\operatorname{Ext}^1(F, E_n) \cong \operatorname{Ext}^{n+1}(F, M)$.

It is well-known that Noetherian rings are characterized by the fact that direct limits of injective modules are injective. For coherent rings we have similarly:

THEOREM 3.2. *The following assertions are equivalent for the ring A*:

(a) *A is right coherent.*

(b) $\xi_1 : \varinjlim \operatorname{Ext}^1(F, M_i) \to \operatorname{Ext}^1(F, \varinjlim M_i)$ *is an isomorphism for every finitely presented module F and direct system (M_i).*

(c) $\xi_n : \varinjlim \operatorname{Ext}^n(F, M_i) \to \operatorname{Ext}^n(F, \varinjlim M_i)$ *are isomorphisms for all $n \geqslant 0$, for every finitely presented right module F and direct system $(M_i)_I$.*

(d) *Every direct limit of FP-injective right modules is FP-injective.*

(e) *Every direct limit of right modules of FP-injective dimension $\leqslant n$ has FP-injective dimension $\leqslant n$.*

Proof. (a) $\Rightarrow$ (c): Every finitely presented right module has a projective resolution consisting of finitely presented modules.

(c) $\Rightarrow$ (b) $\Rightarrow$ (d) and (c) $\Rightarrow$ (e) $\Rightarrow$ (d) are obvious, in view of the preceding lemma.

(d) $\Rightarrow$ (a): Let L be a finitely generated submodule of a finitely presented module F, and consider any homomorphism $\alpha : L \to \varinjlim M_i$. Choose a direct family of injective modules $E(M_i) \supset M_i$. Since $\varinjlim E(M_i)$ is FP-injective, α extends to a homomorphism $\beta : F \to \varinjlim E(M_i)$. But F is finitely presented, so β factors over some $E(M_j)$. The composed homomorphism $L \to F \to E(M_j) \to E(M_j)/M_j$ goes to zero in the direct limit, so we may assume it to be the zero map itself by taking j big enough. $L \to F \to E(M_j)$ then factors over M_j, and this gives the desired factorization of α over M_j. Hence L is finitely presented.

The following theorem extends results by D. G. McRae [**11**] and B. Osofsky [**12**]:

THEOREM 3.3. *The following integers are identical when A is right coherent*:

(a) gl.w.dim A.

(b) r.FP-dim A.

(c) sup {proj.dim F | F *finitely presented right module*}.

(d) sup {FP-inj.dim A/I | I *finitely generated right ideal*}.

Proof. When A is right coherent and F is a finitely presented right module, one has the duality formula ([**3**; p. 120], slightly generalized):

$$\operatorname{Ext}^n(F, M)^* = \operatorname{Tor}_n(F, M^*),$$

where $(\cdot)^* = \operatorname{Hom}_Z(\cdot, Q/Z)$ as before. Together with Proposition 2.7 this gives the equality of (a), (b) and (c). Using the same type of argument as in the proof of (c) $\Rightarrow$ (b) in Lemma 3.1, one obtains

sup {FP-inj.dim A/I | I finitely generated} = sup {FP-inj.dim M | M finitely presented}.
Since every module is a direct limit of finitely presented modules, the equality of (b) and (d) is then obtained with the aid of Theorem 3.2 (e).

Let us denote the right finitistic global weak (resp. FP-injective) dimension of A by r.f.w.dim A and r.f.FP-dim A. Similarly to the above theorem on has (*cf.* [**10**; Th. 1]):

PROPOSITION 3.4. l.f.w.dim A = r.f.FP-dim A *when A is right coherent.*

PROPOSITION 3.5. *Suppose A is right coherent. If* gl.w.dim $A < \infty$, *then it coincides with the right self-FP-injective dimension of A.*

Proof. Clearly it suffices to show that if gl.w.dim $A = n$, then FP-inj.dim $A \geqslant n$. There exists some module M and a finitely presented module F such that

$$\operatorname{Ext}^n(F, M) \neq 0.$$

Choose a resolution $0 \to K \to P \to M \to 0$ with P free. There results an exact sequence

$$\operatorname{Ext}^n(F, P) \to \operatorname{Ext}^n(F, M) \to \operatorname{Ext}^{n+1}(F, K),$$

where the last term is zero. It follows that the first term cannot be zero, and hence $\operatorname{Ext}^n(F, A) \neq 0$ by Th. 3.2.

As a simple example we mention:

PROPOSITION 3.6. *The following assertions are equivalent for* A:

(a) *A is von Neumann regular,.*

(b) *A is right coherent, right self-FP-injective and has finite global weak dimension.*

(c) $\mathrm{r.}FP\text{-dim}\, A = 0$.

(d) *Every cyclic module is FP-injective.*

Proof. (a) $\Rightarrow$ (b) is clear, (b) $\Rightarrow$ (c) follows from Proposition 3.5, (c) $\Rightarrow$ (d) is trivial, and (d) $\Rightarrow$ (a) has been proved by Ikeda and Nakayama [7].

4. *Self-FP-injective coherent rings.*

LEMMA 4.1. *If A is right coherent and right self-FP-injective, then every flat right module is FP-injective.*

Proof. If L is flat, let $0 \to K \to P \to L \to 0$ be exact with P free. P is FP-injective by Corollary 2.3, and the sequence is pure since L is flat. Hence K is FP-injective by Proposition 2.6. For any finitely presented F we get an exact sequence

$$0 \to \mathrm{Ext}^1(F, L) \to \mathrm{Ext}^2(F, K),$$

where also the last term is zero by Lemma 3.1. Hence L is FP-injective.

PROPOSITION 4.2. *Suppose A is both right and left coherent. The following assertions are equivalent*:

(a) *A is left self-FP-injective.*

(b) *Every FP-injective right module is flat.*

(c) *Every injective right module is flat.*

Proof. (a) $\Rightarrow$ (b): Let M be an FP-injective right module. The duality formula (2) shows that M^* is a flat left module. M^* is then FP-injective by the preceding Lemma, and the duality formula (1) gives that M is flat.

(b) $\Rightarrow$ (c) is trivial.

(c) $\Rightarrow$ (a): For every finitely presented left module F and injective right module M one has a canonical homomorphism [3; p. 120]

$$\mathrm{Tor}_1(M, F) = \mathrm{Tor}_1\big(\mathrm{Hom}\,(A, M), F\big) \to \mathrm{Hom}\big(\mathrm{Ext}^1(F, A), M\big), \tag{3}$$

which is an isomorphism since A is left coherent. It now suffices to take M to be an injective cogenerator.

We notice here that even if A is not coherent, the homomorphism (3) is at any rate surjective, as is easily verified. This implies e.g. that if A is an injective cogenerator for the category of right modules, then A is left self-FP-injective.

If S is any subset of A, denote by $l(S)$ (resp. $r(S)$) the set $\{a \in A \mid aS = 0\}$ (resp. $\{a \in A \mid Sa = 0\}$). We recall from [7] that A is left self-FP-injective if and only if $r(I \cap J) = r(I) + r(J)$ for all finitely generated left ideals I, J and $r\big(l(a)\big) = aA$ for all $a \in A$. From this one obtains:

LEMMA 4.3. *If A is left self-FP-injective and left coherent, then $r\big(l(I)\big) = I$ for all finitely generated right ideals I.*

We may now prove the main result of this section.

THEOREM 4.4. *If A is left self-FP-injective, left coherent and right perfect, then A is a quasi-Frobenius ring.*

Proof. Consider any ascending chain of finitely generated right ideals (I_α). Since A is left coherent, $(l(I_\alpha))$ is a descending chain of finitely generated left ideals, which is stationary by a theorem of Björk [2]. The original chain (I_α) is then stationary by Lemma 4.3, so A is right Noetherian. In particular A is right coherent, so every injective right module is flat by Proposition 4.2. But since A is right perfect, this means that every injective right module is projective. It follows that A is quasi-Frobenius [6; Th. 5.3].

There are three other combinations of sides for self-FP-injectiveness, coherence and perfectness. It remains an open problem if any of these combinations imply A quasi-Frobenius. Our only result in that direction is:

PROPOSITION 4.5. *If A is left self-FP-injective, left coherent and left perfect, then every left module has projective dimension* 0 *or* ∞.

Proof. Let F be a finitely presented left module of finite projective dimension n. There exists a module M with $\mathrm{Ext}^n(F, M) \neq 0$. Exactly as in the proof of Proposition 3.5 we find that $\mathrm{Ext}^n(F, A) \neq 0$. Hence $n = 0$, and thus every finitely presented module has projective dimension 0 or ∞. Since A is left perfect, this holds also for arbitrary modules [1; Lemma 6.1].

We conclude with a generalization of the well-known characterization of quasi-Frobenius rings as Noetherian rings over which all finitely generated modules are reflexive [8]. From now on we set $M^* = \mathrm{Hom}_A(M, A)$ for any left or right A-module M. We recall that M is *torsionless* if the canonical homomorphism $\sigma : M \to M^{**}$ is a monomorphism, and is *reflexive* if σ is an isomorphism.

LEMMA 4.6 (Jans, McRae). *Let A be right and left coherent. For any finitely presented torsionless right module M there exists a finitely presented left module L and exact sequences*

$$0 \to M \to M^{**} \to \mathrm{Ext}^1(L, A) \to 0$$

$$0 \to L \to L^{**} \to \mathrm{Ext}^1(M, A) \to 0$$

with canonical monomorphisms.

Proof. [11; Ch. III, Prop. 2.4] (*cf.* [8]).

LEMMA 4.7. *If A is right coherent and L is a finitely presented left module, then L^* is a finitely presented right module.*

Proof. Easy.

THEOREM 4.8. *If A is left self-FP-injective and both right and left coherent, then every finitely presented right module is reflexive*

Proof. It follows from Lemma 4.6 that every finitely presented torsionless right module is reflexive. Let M be an arbitrary finitely presented right module, and let $0 \to K \to P \to M \to 0$ be exact with P finitely generated free. K is then finitely presented and torsionless. By dualizing we get an exact sequence

$$0 \to M^* \to P^* \to K^*.$$

Since A is left self-FP-injective we get, using Lemma 4.7, a commutative diagram

$$\begin{array}{ccccccc} 0 \to & K & \to & P & \to & M & \to 0 \\ & \| & & \| & & \downarrow & \\ & K^{**} & \to & P^{**} & \to & M^{**} & \to 0 \end{array}$$

with exact rows. Hence $M \to M^{**}$ is an isomorphism.

THEOREM 4.9. *Let A be right and left coherent. The following assertions are equivalent:*

(a) *A is right and left self-FP-injective.*

(b) *All finitely presented right or left modules are reflexive*

Proof. (a) $\Rightarrow$ (b) by the preceding theorem.

(b) $\Rightarrow$ (a) follows immediately from Lemma 4.6.

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