**Physics 195**
**Supplementary Notes**
**Groups, Lie algebras, and Lie groups**
**020922 F. Porter**

This note defines some mathematical structures which are useful in the discussion of angular momentum in quantum mechanics (among other things).

**Def:** A pair $(G, \circ)$, where $G$ is a non-empty set, and $\circ$ is a binary operation defined on $G$, is called a **group** if:

1. Closure: If $a, b \in G$, then $a \circ b \in G$.
2. Associativity: If $a, b, c \in G$, then $a \circ (b \circ c) = (a \circ b) \circ c$.
3. Existence of right identity: There exists an element $e \in G$ such that $a \circ e = a$ for all $a \in G$.
4. Existence of right inverse: For some right identity $e$, and for any $a \in G$, there exists an element $a^{-1} \in G$ such that $a \circ a^{-1} = e$.

The $\circ$ operation is typically referred to as "multiplication".

The above may be termed a "minimal" definition of a group. It is amusing (and useful) to prove that:

1. The right identity element is unique.
2. The right inverse element of any element is unique.
3. The right identity is also a left identity.
4. The right inverse is also a left inverse.
5. The solution for $x \in G$ to the equation $a \circ x = b$ exists and is unique, for any $a, b \in G$.

We will usually drop the explicit $\circ$ symbol, and merely use juxtaposition to denote group multiplication. Note that both $G$ (the set) and $\circ$ (the "multiplication table") must be specified in order to specify a group. Where the operation is clear, we will usually just refer to "$G$" as a group.

**Def:** An **abelian** (or **commutative**) group is one for which the multiplication is commutative:

$$ab = ba \quad \forall\, a, b \in G. \tag{1}$$

**Def:** The **order** of a group is the number of elements in the set $G$. If this number is infinite, we say it is an "infinite group".

In the discussion of infinite groups of relevance to physics (in particular, Lie groups), it is useful to work in the context of a richer structure called an algebra. For background, we start by giving some mathematical definitions of the underlying structures:

**Def:** A **ring** is a triplet $\langle R, +, \circ \rangle$ consisting of a non-empty set of elements ($R$) with two binary operations ($+$ and $\circ$) such that:

1. $\langle R, + \rangle$ is an abelian group.
2. $(\circ)$ is associative.
3. Distributivity holds: for any $a, b, c \in R$

$$a \circ (b + c) = a \circ b + a \circ c \tag{2}$$

and

$$(b + c) \circ a = b \circ a + c \circ a \tag{3}$$

Conventions:
We use 0 ("zero") to denote the identity of $\langle R, + \rangle$ . We speak of $(+)$ as addition and of $(\circ)$ as multiplication, typically omitting the $(\circ)$ symbol entirely (*i.e.*, $ab \equiv a \circ b$).

**Def:** A ring is called a **field** if the non-zero elements of $R$ form an abelian group under $(\circ)$.

**Def:** An abelian group $\langle V, \oplus \rangle$ is called a **vector space** over a field $\langle F, +, \circ \rangle$ by a scalar multiplication $(*)$ if for all $a, b \in F$ and $v, w \in V$:

1. $a * (v \oplus w) = (a * v) \oplus (a * w)$ distributivity
2. $(a + b) * v = (a * v) \oplus (b * v)$ distributivity
3. $(a \circ b) * v = a * (b * v)$ associativity
4. $1 * v = v$ unit element $(1 \in F)$

Conventions:
We typically refer to elements of $V$ as "vectors" and elements of $F$ as "scalars." We typically use the symbol $+$ for addition both of vectors and scalars. We also generally omit the $*$ and $\circ$ multiplication symbols. Note that this definition is an abstraction of the definition of vector space given in the note on Hilbert spaces, page 1.

**Def:** An **algebra** is a vector space $V$ over a field $F$ on which a multiplication ($\circ$) between vectors has been defined (yielding a vector in $V$) such that for all $u, v, w \in V$ and $a \in F$:

1. $(au) \circ v = a(u \circ v) = u \circ (av)$
2. $(u+v) \circ w = (u \circ w) + (v \circ w) \quad \text{and} \quad w \circ (u+v) = (w \circ u) + (w \circ v)$

(Once again, we often omit the multiplication sign, and hope that it is clear from context which quantities are scalars and which are vectors.)

We are interested in the following types of algebras:

**Def:** An algebra is called **associative** if the multiplication of vectors is associative.

We note that an associative algebra is, in fact, a ring. Note also that the multiplication of vectors is not necessarily commutative. An important non-associative algebra is:

**Def:** A **Lie algebra** is an algebra in which the multiplication of vectors obeys the further properties (letting $u, v, w$ be any vectors in $V$):

1. Anticommutivity: $u \circ v = -v \circ u$.
2. Jacobi Identity: $u \circ (v \circ w) + w \circ (u \circ v) + v \circ (w \circ u) = 0$.

We may construct the idea of a "group algebra": Let $G$ be a group, and $V$ be a vector space over a field $F$, of dimension equal to the order of $G$ (possibily $\infty$). Denote a basis for $V$ by the group elements. We can now define the multiplication of two vectors in $V$ by using the group multiplication table as "structure constants": Thus, if the elements of $G$ are denoted by $g_i$, a vector $u \in V$ may be written:

$$u = \sum a_i g_i$$

We require that, at most, a finite number of coefficients $a_i$ are non-zero. The multiplication of two vectors is then given by:

$$\left(\sum a_i g_i\right)\left(\sum b_j g_j\right) = \sum \left( \sum_{g_i g_j = g_k} a_i b_j \right) g_k$$

[Since only a finite number of the $a_i b_j$ can be non-zero, the sum $\sum_{g_i g_j = g_k} a_i b_j$ presents no problem, and furthermore, we will have closure under multiplication.]

Since group multiplication is associative, our group algebra, as we have constructed it, is an associative algebra.