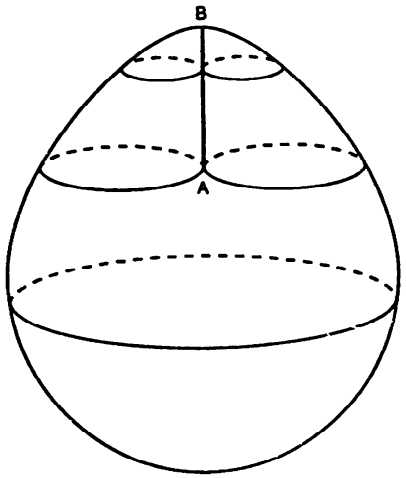


FIGURE 3

By removing the lower hemisphere which is homeomorphic, to a disc, we obtain a surface called a cross-cap (Figure 4).

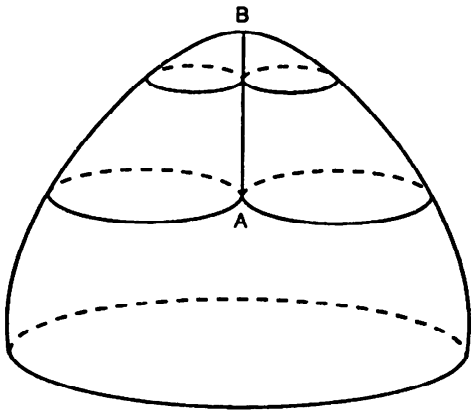


FIGURE 4

Hence,

$$\chi\,[\text{cross-cap}] = \chi\,[\text{real projective plane}] - \chi[\text{disc}] =$$

$$= 1 - 1 = 0.$$

The Euler characteristic of a cross-cap and a Möbius band are the same. Indeed, a Möbius band can be topologically deformed into a cross-cap.

● PROBLEM 9-110

Derive the formula for the Euler characteristic of a sphere with n handles, k holes and l cross-caps.

SOLUTION:

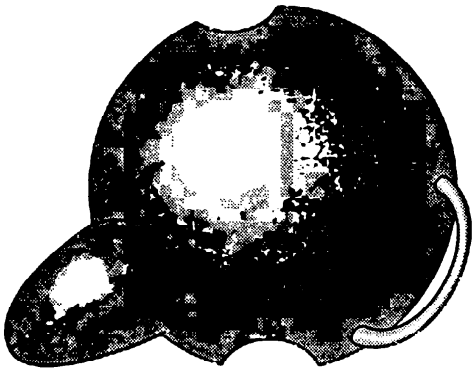


FIGURE 1

A sphere with one handle, one cross-cap and two holes is shown in the figure.

Note that each handle requires two holes and each cross-cap requires one hole. Thus, we have to make

$$2n + l + k$$

holes. The Euler characteristic of a cylinder is zero and of a cross-cap is also zero. Thus, the Euler characteristic of a sphere with n handles, l cross-caps and k holes is

$$\chi = 2 - (2n + l + k) = 2 - 2n - l - k.$$

● PROBLEM 9-111

Present in standard model form disc, sphere, cylinder, real projective plane, Möbius band, torus, two-fold torus and n-fold torus.

SOLUTION:

A surface described as a sphere with n handles, l cross-caps and k holes is said to be represented in standard model form.

	n-handles	l cross-caps	k holes	χ
sphere	0	0	0	2
disc	0	0	1	1
cylinder	0	0	2	0
real projective plane	0	1	0	1
Möbius band	0	1	1	0

torus	1	0	0	0
two-fold torus	2	0	0	-2
n-fold torus	n	0	0	$2-2n$

● PROBLEM 9-112

Determine the rank and Euler characteristic of a torus with a hole.

SOLUTION:

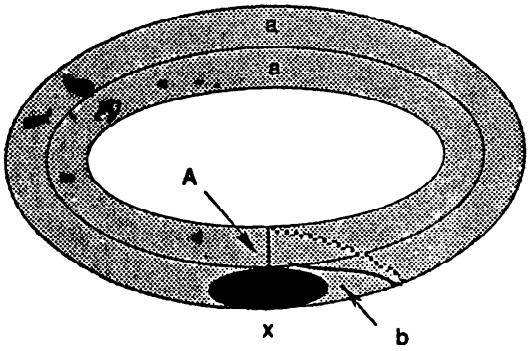


FIGURE 1

The rank of an open surface is defined as requiring the least number of cuts to make the surface homeomorphic to a disc.

From Figure 1, we see that two cuts are required to make a torus with a hole homeomorphic to a disc. The cut surface can be deformed to yield a plane diagram (Figure 2) with the symbolic notation

$$\pm x + a + b - a - b.$$

The torus with a hole has 1 handle, 1 hole and 0 cross-caps. Hence

$$\chi = 2 - (2n + k + l) = 2 - 2 - 1 - 0 = -1.$$

The same value can be obtained from a plane diagram

$$\chi = V - E + F = 1 - 3 + 1 = -1.$$

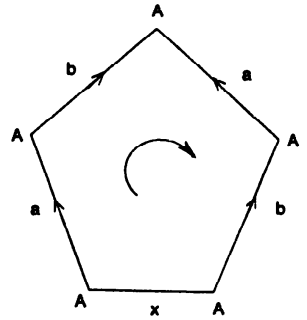


FIGURE 2

● PROBLEM 9-113

Find the rank and Euler characteristic of a sphere with one cross-cap and one hole.

SOLUTION:

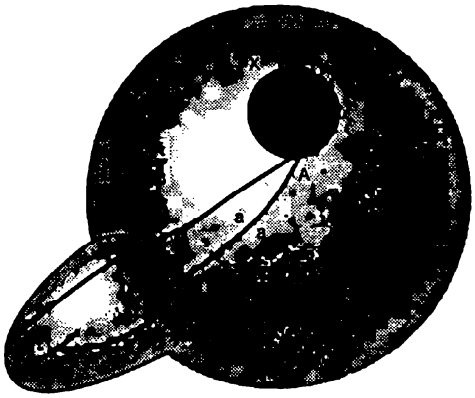


FIGURE 1

This is an open surface. It requires one cut (Figure 1) to become homeomorphic to a disc.

The rank of a sphere with one cross-cap and one hole is one. The surface obtained can be deformed to give a triangle.

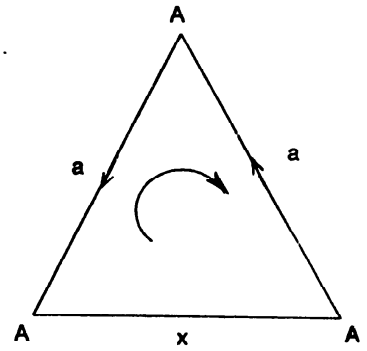


FIGURE 2

Its symbolic representation

$$\pm x - a - a$$

The Euler characteristic

$$\chi = 2 - (2n + k + l) = 0$$

or from the plane diagram

$$\chi = V - E + F = 1 - 2 + 1 = 0.$$

● PROBLEM 9-114

Why is predicting the physical and chemical properties of chemical substances before they are synthesized so important?

SOLUTION:

More than seven million different molecules have been synthesized. At least some of their properties are known with fairly good accuracy. In some sense it forms a basis for future research. The cost of obtaining new molecules can sometimes be enormous. It can happen that the final product turns out to be a disappointment. A new molecule is not exactly what we have been looking for.

Hence, it is important, not only from the scientific point of view, to be able to predict the properties of chemical substances before they are synthesized.

● PROBLEM 9-115

Describe briefly the topological method of making chemical predictions.

SOLUTION:

The heart of this method is the topology of individual molecules. We neglect the nature and lengths of the chemical bonds holding the atoms of a molecule together. We take into account the number of atoms in the molecule and how the atoms are interconnected within the molecule (single straight chain, chain with branches, several chains, a ring, etc.).

As a first step we take a small number of known molecules, for instance, $A_1, A_2, \ldots, A_k$. Then we establish a procedure which converts the topological structure of each molecule A into a single, characteristic number α called index.

$$A_1 \rightarrow \alpha_1$$

$$A_2 \rightarrow \alpha_2$$

$$\vdots$$

$$A_k \rightarrow \alpha_k$$

Suppose we are interested in a certain chemical property, say the boiling point. For each $A_1, A_2, \ldots, A_k$ we determine its boiling point $T_1, T_2, \ldots, T_k$. By using the Cartesian coordinates on one axis, we mark the value of the index

and on the other, the value of the chemical (physical) property.

Thus, we obtain a plot which can be used to predict the future.

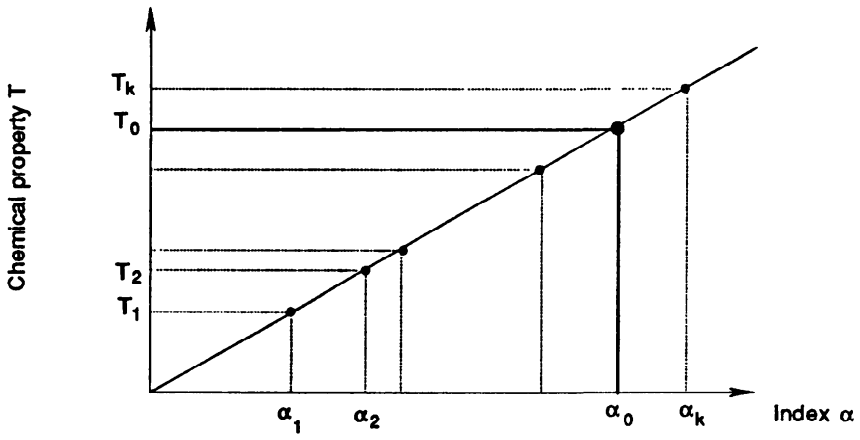


FIGURE 1

Suppose we want to know the boiling point of a molecule whose index is α_0. From the plot, we obtain T_0.

● PROBLEM 9-116

What is a chemical graph?

SOLUTION:

A chemical graph is a drawing in which the atoms of the molecule are depicted as points and the bonds linking the atoms are depicted as straight lines. The length of lines and the angles are immaterial.

Thus, graphs represent structures in an abstract manner. Graph theory, a mathematical discipline, studies graphs in detail.

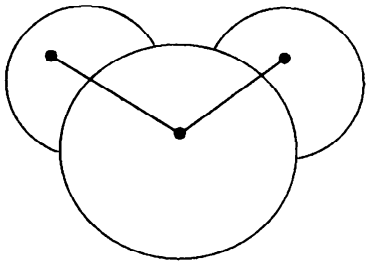


FIGURE 1–Graph of H_2O

Generally, in chemical graphs the hydrogen atoms are omitted.

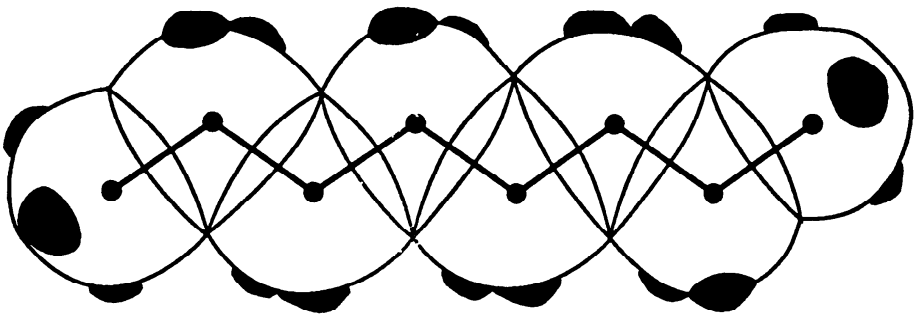


FIGURE 2-Graph of an n-octane.

Large balls depict carbon atoms and small balls depict hydrogen atoms.

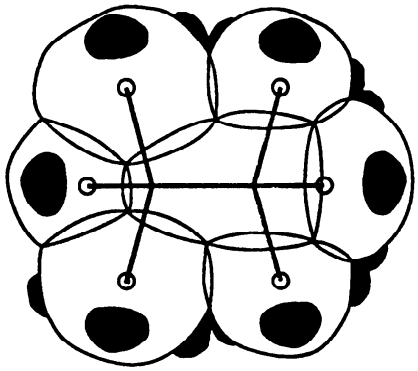


FIGURE 3–The structure of 2, 2, 3, 3,-tetramethylbutane.

The graph is shown inside with the hydrogen atoms omitted.

● PROBLEM 9-117

Describe how the carbon number is determined.

SOLUTION:

Once the chemical graph of a molecule has been drawn, it is fairly easy to compute its characteristic number - index. Such a number is known as a graph invariant. One of the most simple graph invariants is the vertex number, which chemists refer to as the carbon number. It is the number of vertexes in the graph. For hydrocarbon molecules (i.e., molecules consisting only of hydrogen and carbon atoms) it is the number of carbon atoms. It should be remembered that the carbon number has low discriminating power because many different molecules can have the same carbon number.

Describe the Wiener index and its applications.

SOLUTION:

The carbon number is a useful index for analyzing straight-chain molecules for branched molecules. The Wiener index is more appropriate. This index uses the notion of distance between two vertices, which is defined as follows: The distance between any two vertices is equal to the number of edges traversed while taking the shortest route within the graph between these vertices.

The Wiener index of a molecule is equal to the sum of the distances between all pairs of atoms in the molecule.

Consider for example isopentane.

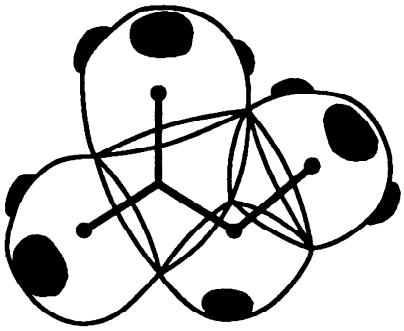
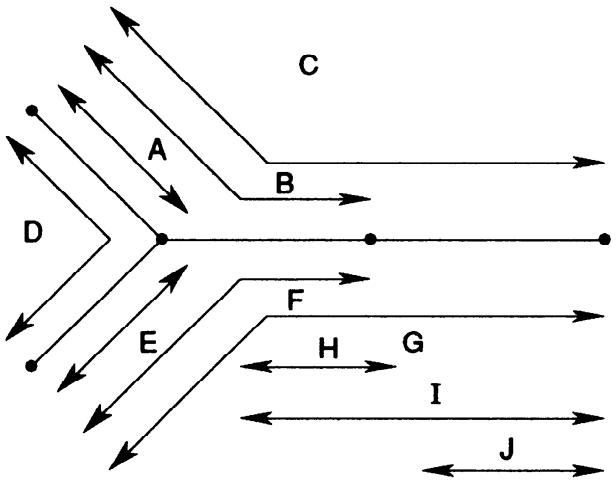


FIGURE 1

Its graph and distances are shown in Figure 2.



A	1
B	2
C	3
D	2
E	1
F	2
G	3
H	1
I	2
J	1
	18

WIENER INDEX

FIGURE 2

Show how the Wiener index can be applied to determine the chemical or physical property of a molecule that has not yet been synthesized.

SOLUTION:

The Wiener index as described in Problem 9-118 converts the topological structure of a molecule into a single number-index. It has been determined that the Wiener index correlates with many physical properties.

First, we establish the molecular structure of some existing molecules and draw their graphs.

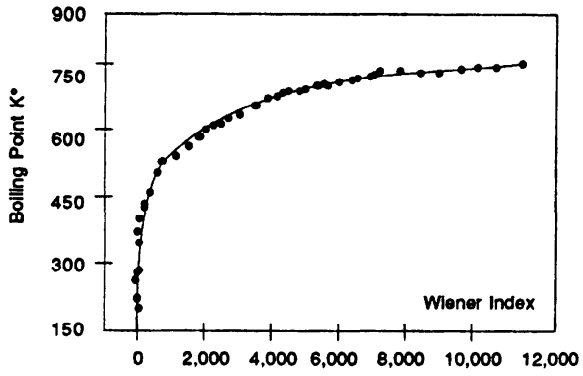


FIGURE 1

Suppose we are interested in the boiling point. For each molecule, we know its Wiener index and its boiling point. Thus, obtained points yield a fairly smooth curve (see Figure 1). The correlation is more evident when we use the logarithms of the Wiener index and boiling point (see Figure 2).

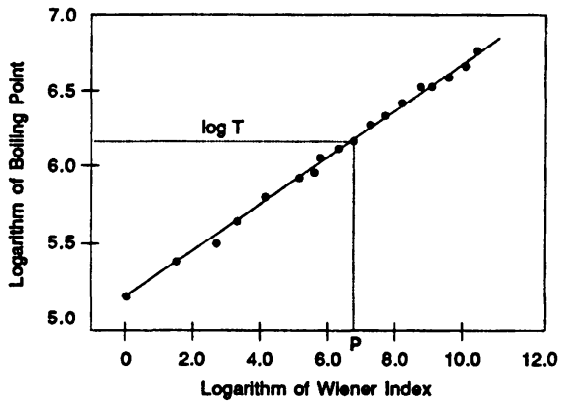
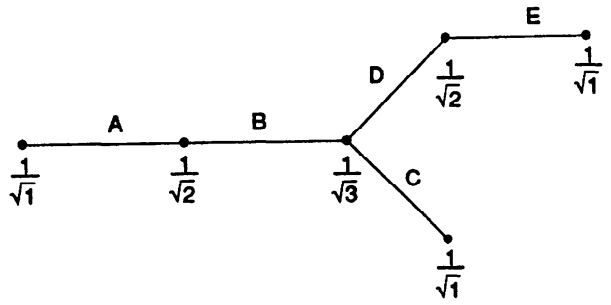


FIGURE 2

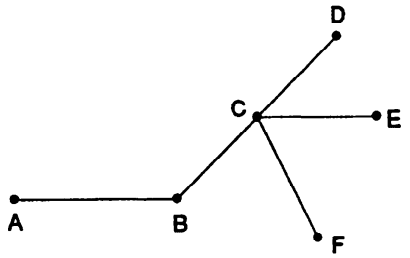
Suppose we want to synthesize a molecule whose logarithm of the Wiener index is P. From the plot we find the logarithm of its boiling point.

Compute the molecular-connectivity index of a molecule whose graph is shown below:



SOLUTION:

The molecular-connectivity index is the most universal index devised so far. It depends on the topological concept of degree. The degree of any vertex is equal to the number of other vertices it is attached to.



Hence the degree of A is 1, of B is 2, of C is 4, of D is 1, of E is 1 and of F is 1.

Each edge has a value: the product of the reciprocals of the square roots of the degrees of the vertices it joins.

$$\text{Molecular - connectivity Index} = \sum \frac{1}{\sqrt{N_i} \times \sqrt{N_j}}$$

where the sum is taken over all the edges.

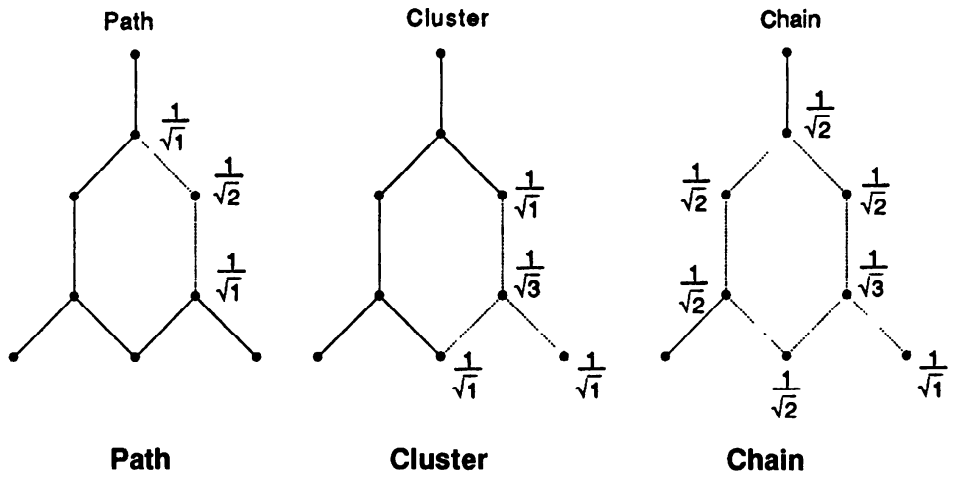
Hence, the molecular-connectivity index for our molecule is

$$\left(\frac{1}{\sqrt{1}} \times \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{1}}\right) + \left(\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{2}}\right) +$$

$$+ \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{1}}\right) = \frac{2}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{6}} = 2.80806$$

● PROBLEM 9-121

In some cases, better correlations are obtained by considering only the fragments of larger molecules. Compute the molecular-connectivity index for the fragments of molecules indicated by dotted lines.



Path Cluster Chain

FIGURE 1

SOLUTION:

From the definition of the molecular-connectivity index, we obtain:

Path

$$\left(\frac{1}{\sqrt{1}} \times \frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{1}}\right) = 1.4142$$

Cluster

$$\left(\frac{1}{\sqrt{1}} \times \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{1}} \times \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{1}} \times \frac{1}{\sqrt{3}}\right) = 1.73205$$

Chain

$$4 \times \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}\right) + 2 \times \left(\frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{1}} \times \frac{1}{\sqrt{3}}\right) = 3.39384$$

● PROBLEM 9-122

Compute the Balaban centric index for a molecule whose graph is shown below.

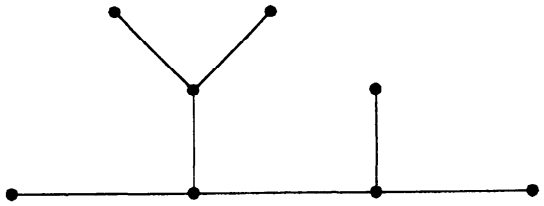


FIGURE 1

SOLUTION:

This index emphasizes the degree of branching in a molecule. All vertices that are connected to only one other vertex are counted and removed from the molecule's graph. The number of vertices removed at each step is squared and added to a running total. The procedure is repeated until all vertices are removed.

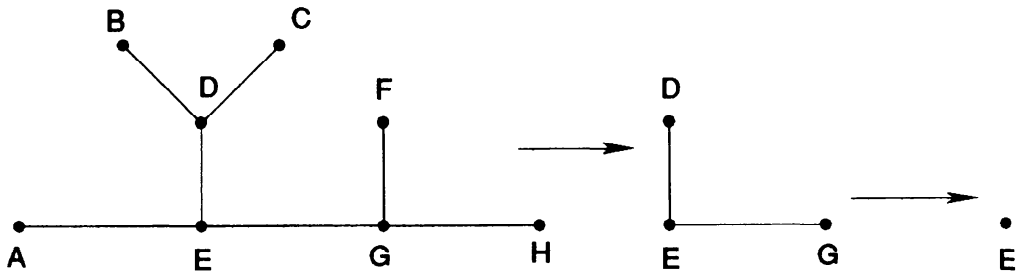


FIGURE 2

As a first step we remove vertices A, *B*, *C*, *F*, and *H* and get 5^2. Then we remove *D* and *G* and obtain 2^2. Finally, we remove *E* and obtain 1^2.

Thus, the index is $5^2 + 2^2 + 1^2$.

● PROBLEM 9-123

List a few applications of molecular-connectivity index.

SOLUTION:

The molecular-connectivity index has numerous and diversified applications. It is used in developing new drugs and in predicting the taste and smell of new substances. This index correlates well with such physical properties as the boiling point, density, heat of vaporization and solubility in water. It is known that many biological responses are launched when an appropriate stimulating molecule docks with a receptor on the surface of a cell. Very

often the specific shape of the molecule is less important than its volume and surface area. Molecular-connectivity indices correlate well with volume and surface area. Thus, the molecular-connectivity indices can predict the ability of molecules to act as anesthetics, hallucinogens or narcotics.

It is also possible to predict the smell of molecules as well as whether a molecule will taste bitter or sweet.

● PROBLEM 9-124

Describe the applications of molecular-connectivity indices in environmental studies.

SOLUTION:

It is a very costly and time-consuming process to determine how fast pollutants spread into the environment. Here, molecular-connectivity indices are very helpful.

The indices correlate well with the ability of many pollutants to spread within the air, water or soil. The relationship exists between the indices and the tendencies of substances to concentrate within living organisms. To test these properties without the indices is extremely difficult and expensive. That is why the U.S. Environmental Protection Agency applies the indices to predict the toxic potentiality of unknown or untested pollutants.

● PROBLEM 9-125

What does the octane number of a fuel describe? What methods are used to predict it?

SOLUTION:

In general, the octane number describes the efficiency with which a fuel burns, that is, its tendency not to "knock." The fuel is mixed with oxygen and then ignited by a spark.

A "knock" appears when during compression, oxygen atoms combine with fuel before it has been ignited. Usually the octane number is determined under standardized conditions in a test engine.

Straight-chain molecules tend to "knock" more than branched molecules.

Attempts to establish the correlation between the tendency of the fuel to "knock" and its Wiener index yielded only fair results.

An index introduced by A. Balaban, called the centric index, emphasized the branching of molecules. It correlated very well with the octane number of

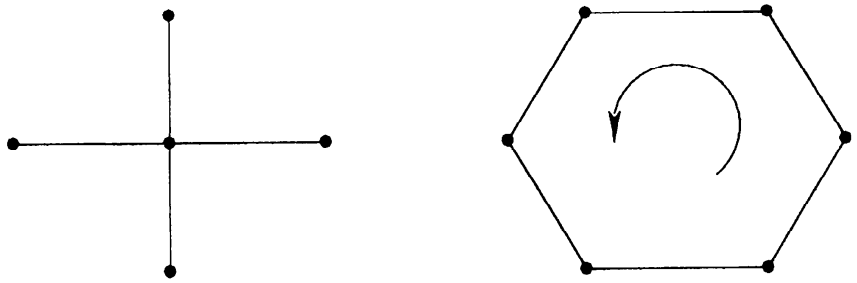
hydrocarbon molecules.

We defined the centric index in Problem 9-123.

● PROBLEM 9-126

An index called hydrogen-deficiency index was devised to predict the amount of soot produced by burning hydrocarbon molecules. Define this index.

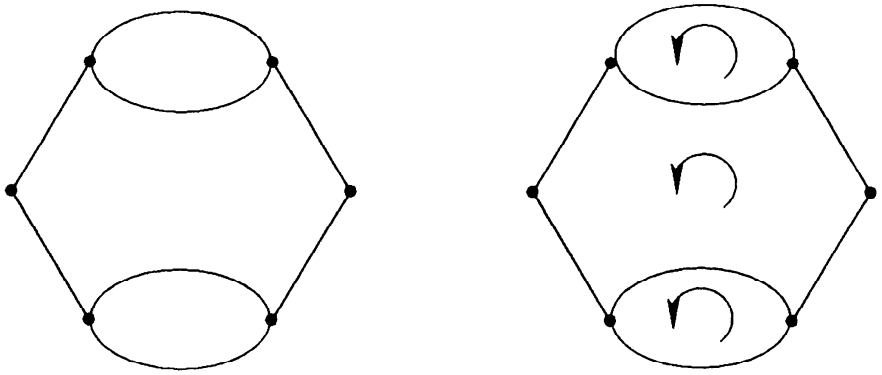
SOLUTION:



0 cycles 1 cycle

FIGURE 1

The hydrogen-deficiency number is equal to the number of independent cycles, or nips, and double bonds in a molecule.



3 cycles

FIGURE 2

The hydrogen-deficiency number combined with the averaged-distance-sum connectivity index yields an index which provides a very good correlation with the soot production for almost 100 hydrocarbon molecules.

Describe how the cancer-causing tendency of a molecule (carcinogenicity) can be predicted by means of topological methods.

SOLUTION:

The task of finding a correlation between carcinogenic behavior of molecules and topological indices is an extremely difficult one.

First of all, the experimental measurements are not necessarily accurate or sufficient. Then the growth of cancer is a process consisting of many stages.

Thus, the index should take into account not only the original molecules but also the molecules produced during successive stages. So far the best results were achieved by William C. Herndon and László van Szentpály. They introduced the combination of simple indices to predict the carcinogenicity of polycyclic aromatic hydrocarbons. Since the molecules, below and above certain size limits, are not carcinogenic, the final index includes the carbon number and the square of the carbon number (which depends on the size of the molecules). Another aspect is that certain regions of hydrocarbons are more important (more carcinogenic) than other regions.

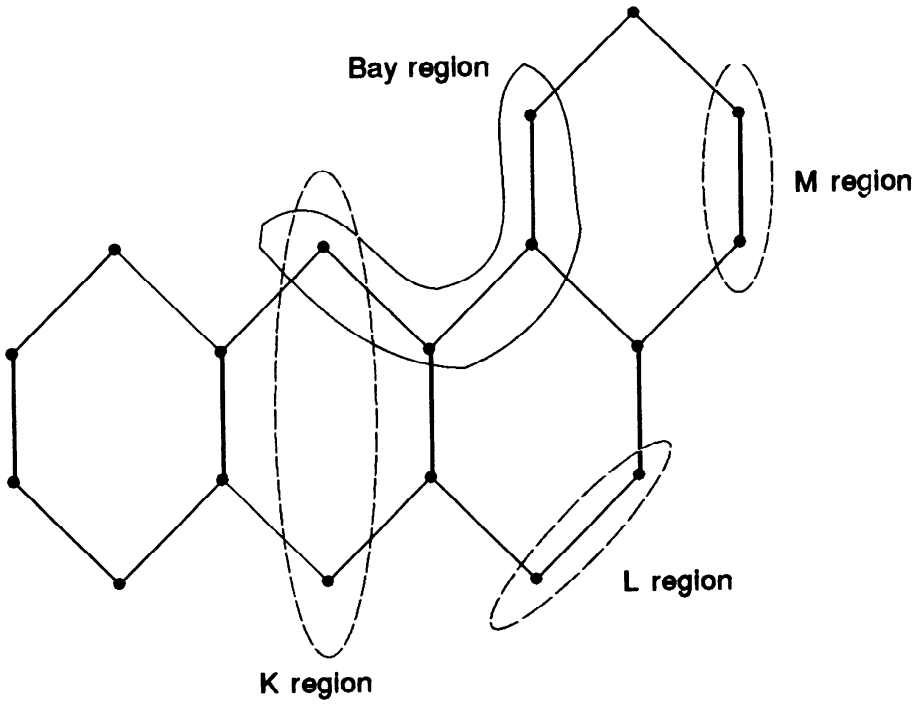


FIGURE 1

The Bay, L and M regions must be fairly active chemically, while the K region must remain fairly inactive.

CHAPTER 10

METRIC SPACES

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Let R represent the set of real numbers. Show that one of the possible metrics for R is the absolute value

$$d(x, y) = |x - y| \tag{1}$$

SOLUTION:

DEFINITION OF A METRIC SPACE

Let X be a set. Function $d(x,y) \in R$

$$d : X \times X \to R \tag{2}$$

is said to be a metric on X if for all $x, y, z \in X$

1. $d(x, y) \geq 0$.
2. $d(x, y) = 0$ iff $x = y$.
3. $d(x, y) = d(y, x)$.
4. $d(x, z) \leq d(x, y) + d(y, z)$.

The set X with a metric d is called a metric space and is denoted by (X, d). The function $d(x, y)$ is called the distance between x and y.

■

For any real numbers x, y, and z we have

$$|x - y| \geq 0$$

$$|x - y| = 0 \quad \text{iff} \quad x - y = 0 \quad \text{iff} \quad x = y$$

$$|x - y| = |y - x|$$

$$|x - z| \leq |x - y| + |y - z|.$$

Hence (1) defines a metric.

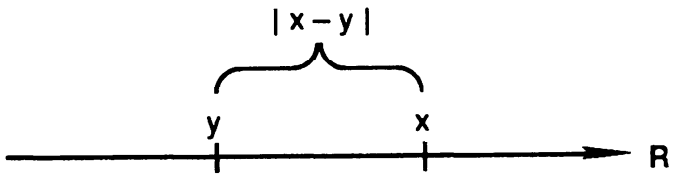


FIGURE 1

If x and y are real numbers on the real axis, then $|x-y|$ is the distance between them.

● PROBLEM 10-2

1. Is it possible to define a metric on any set?

2. Show that the set R^n with the distance defined by

$$d(x, y) = \max \{ |x_i - y_i|, 1 \le i \le n\} \tag{1}$$

is a metric space.

SOLUTION:

1. Let X be any set. Then, function d defined by

$$d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \tag{2}$$

is a metric. Therefore a metric may be defined for any set.

2. We have

$$d(x, y) \ge 0 \quad \text{because } |x_i - y_i| \ge 0$$

$$0 = d(x, y) = \max \{ |x_i - y_i|, 1 \le i \le n\} \quad \text{iff}$$

$$|x_i - y_i| = 0 \quad \text{for} \quad 1 \le i \le n \quad \text{iff} \quad x_i = y_i \quad \text{for}$$

$$1 \le i \le n \quad \text{iff} \quad x = y.$$

We denote

$$x = (x_1, \ldots, x_n)$$

$$y = (y_1, \ldots, y_n) \tag{3}$$

Symmetry is obvious

$$d(x, y) = d(y, x) \tag{4}$$

Triangle inequality.

Suppose for some $1 \le l \le n$

$$d(x, y) = |x_i - y_i|$$

we have

$$d(x, y) = |x_i - y_i| \leq |x_l - z_l| + |z_l - y_l| \leq$$

$$\leq d(x, z) + d(z, y). \tag{5}$$

● PROBLEM 10-3

Show that in the set R^n,

$$d_p(x,y) = \sqrt[p]{\sum_1^n |x_i - y_i|^p} \tag{1}$$

is a metric for each $p \geq 1$.

SOLUTION:

We shall prove the triangle inequality.

For $p \geq 1$, the function

$$f(x) = x^p \tag{2}$$

for $x \in R^1$, $x \geq 0$ satisfies

$$f''(x) \geq 0 \tag{3}$$

and hence it is convex. Thus, $f(x)$ satisfies

$$f(\alpha x + [1 - \alpha]y) \leq \alpha f(x) + (1 - \alpha) f(y) \tag{4}$$

for $x, y \geq 0$.

For any set of real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ let

$$A = \sqrt[p]{\sum_1^n |a_i|^p} \qquad B = \sqrt[p]{\sum_1^n |b_i|^p} \tag{5}$$

Define x, y, and α as; $\forall i \in [1, n]$

$$x_i = \frac{a_i}{A}, \qquad y_i = \frac{b_i}{B}, \qquad \alpha_i = \frac{A}{A + B}. \tag{6}$$

Substituting (6) for each $i = 1, 2, \ldots, n$ into (4) and adding all n inequalities, we obtain

$$\sqrt[p]{\sum_1^n |a_i - b_i|^p} \leq \sqrt[p]{\sum_1^n |a_i|^p} + \sqrt[p]{\sum_1^n |b_i|^p} \tag{7}$$

Equation (7) is called Minkowski's inequality.

Substituting

$$a_i = x_i - y_i$$

$$b_i = y_i - z_i \tag{8}$$

for $n = 1, 2, \ldots, n$ into (7) we obtain

$$d_p(x, z) \leq d_p(x, y) + d_p(y, z). \tag{9}$$

In particular for $n = 2$ and $p = 2$, we obtain

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \tag{10}$$

which is a distance on R^2.

● PROBLEM 10-4

Let d be a metric on a set X. Show that

$$d_0(x,y) = \frac{d(x,y)}{1 + d(x,y)} \tag{1}$$

where $x, y \in X$ is also a metric on X.

SOLUTION:

We have to show that d_0 satisfies the triangle inequality. All remaining conditions for a metric are obviously satisfied.

Since d is a metric we have

$$\frac{d(x,y)}{1 + d(x,y) + d(y,z)} \leq \frac{d(x,y)}{1 + d(x,y)} = d_0(x,y) \tag{2}$$

$$\frac{d(y,z)}{1 + d(x,y) + d(y,z)} \leq \frac{d(y,z)}{1 + d(y,z)} = d_0(y,z) \tag{3}$$

Function d is a metric. Hence $d(x, z) \leq d(x, y) + d(y, z)$ and

$$d_0(x,z) = \frac{d(x,z)}{1 + d(x,z)} \leq \frac{d(x,y) + d(y,z)}{1 + d(x,y) + d(y,z)} =$$

$$= \frac{d(x,y)}{1 + d(x,y) + d(y,z)} + \frac{d(y,z)}{1 + d(x,y) + d(y,z)} \leq d_0(x,y) + d_0(y,z) \tag{4}$$

Thus d_0 is a metric.

Let X represent the set of all functions from the closed interval [0, 1] into itself. Show that for any $f, g \in X$

$$d(f, g) = \text{least upper bound } \{ |f(x) - g(x)| : x \in [0, 1]\} \quad (1)$$

is a metric.

SOLUTION:

Any subset of the real numbers which has an upper bound has a least upper bound and

$$0 \leq |f(x) - g(x)| \leq 1 \quad \text{for all } f, g \in X \text{ and } x \in [0, 1]. \quad (2)$$

Hence $d(f, g)$ is defined for all $f, g \in X$.

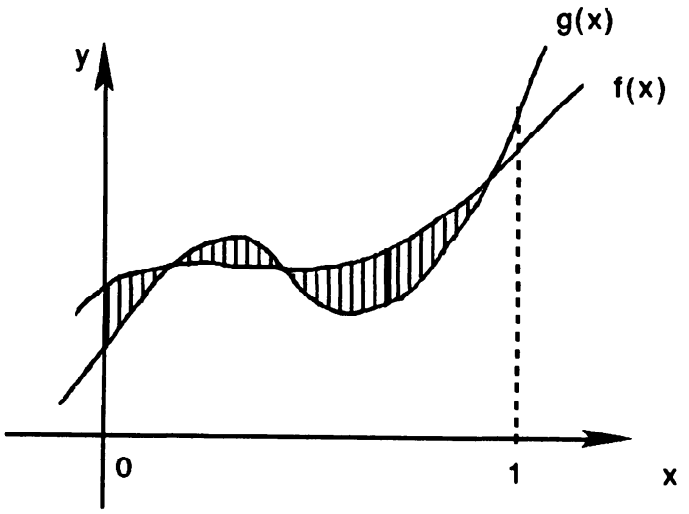


FIGURE 1

1. Since

$$|f(x) - g(x)| \geq 0, x \in [0, 1] \quad (3)$$

the least upper bound of (3) is non-negative, i.e.,

$$d(f, g) \geq 0 \quad \text{for all } f, g \in X. \quad (4)$$

2. $$d(f, g) = 0 \quad \text{iff} \quad f = g. \quad (5)$$

If $d(f, g) = 0$, then a least upper bound

$$\{ |f(x) - g(x)| : x \in [0, 1] \} = 0 \quad (6)$$

and

$$\{\ |f(x) - g(x)| : x \in [0, 1]\ \} = \{\ 0\ \}. \tag{7}$$

Hence

$$|f(x) - g(x)| = 0 : x \in [0, 1] \tag{8}$$

and $f(x) = g(x)$ for all $x \in [0, 1]$.

If $f = g$, then $f(x) = g(x)$ for all $x \in [0, 1]$ and

$$\{\ |f(x) - g(x)| : x \in [0, 1]\ \} = \{\ 0\ \}. \tag{9}$$

Hence

$$d(f, g) = 0. \tag{10}$$

3. $d(f, g) = d(g, f)$ for all $f, g \in X$ because

$$|f(x) - g(x)| = |g(x) - f(x)|. \tag{11}$$

4. $$d(f, h) \leq d(f, g) + d(g, h) \tag{12}$$

because for all $f, g, h \in X$

$$|f(x) - h(x)| \leq |f(x) - g(x)| + |g(x) - h(x)| \tag{13}$$

for all $x \in [0, 1]$.

● PROBLEM 10-6

1. Let (X_1, d_1) and (X_2, d_2) represent metric spaces. Show that $(X_1 \times X_2, d)$ where

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2) \tag{1}$$

for $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$, is a metric space.

2. Applying (1), define a metric on a Cartesian product of n metric spaces $X_1 \times X_2 \times \ldots \times X_n$.

SOLUTION:

1. Since d_1 and d_2 are metrics

$$d_1(x_1, y_1) \geq 0 \text{ and } d_2(x_2, y_2) \geq 0 \tag{2}$$

and

$$d((x_1, x_2), (y_1, y_2)) \geq 0$$

for all $x_1, y_1 \in X_1$ and for all $x_2, y_2 \in X_2$.

Similarly since $d_1(x_1, y_1) = d_1(y_1, x_1)$ and $d_2(x_2, y_2) = d(y_2, x_2)$ we have

$$d((x_1, x_2), (y_1, y_2)) = d((y_1, y_2), (x_1, x_2)) \tag{4}$$

$$d((x_1, x_2), (y_1, y_2)) = 0 \quad \text{iff} \quad (x_1, x_2) = (y_1, y_2). \tag{5}$$

Suppose $d((x_1, x_2), (y_1, y_2)) = 0$ then

$$d_1(x_1, y_1) = 0 \text{ and } d_2(x_2, y_2) = 0.$$

Thus

$$x_1 = y_1 \text{ and } x_2 = y_2 \text{ or } (x_1, x_2) = (y_1, y_2). \tag{6}$$

If $(x_1, x_2) = (y_1, y_2)$, then

$$d_1(x_1, y_1) = 0 \text{ and } d_2(x_2, y_2) = 0. \tag{7}$$

Thus

$$d((x_1, x_2), (y_1, y_2)) = 0. \tag{8}$$

Now, we will show that

$$d((x_1, x_2), (z_1, z_2)) \leq d((x_1, x_2), (y_1, y_2)) + d((y_1, y_2), (z_1, z_2)). \tag{9}$$

Indeed, since d_1 and d_2 are metrics

$$d((x_1, x_2), (z_1, z_2)) = d_1(x_1, z_1) + d_2(x_2, z_2) \leq$$

$$\leq d_1(x_1, y_1) + d_1(y_1, z_1) + d_2(x_2, y_2) + d_2(y_2, z_2) =$$

$$= d((x_1, x_2), (y_1, y_2)) + d((y_1, y_2), (z_1, z_2)). \tag{10}$$

2. If $(X_1, d_1), (X_2, d_2), \ldots, (X_n, d_n)$ are metric spaces, then

$$d(x, y) = d_1(x_1, y_1) + \ldots + d_n(x_n, y_n) \tag{11}$$

is a metric on $X_1 \times X_2 \times \ldots \times X_n$.

● PROBLEM 10-7

The following metrics are defined on R^2

$$d_1((x_1,y_1),(x_2,y_2)) = \begin{cases} 0 & \text{if } (x_1,y_1) = (x_2,y_2) \\ 1 & \text{if } (x_1,y_1) \neq (x_2,y_2) \end{cases}$$

$$d_2((x_1,y_1),(x_2,y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$d_3((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|$$
$$d_4((x_1,y_1),(x_2,y_2)) = \max(|x_1 - x_2|, |y_1 - y_2|).$$

Find the ball of radius 1 and center (0, 0) with respect to metrics d_1, d_2, d_3, and d_4.

SOLUTION:

DEFINITION OF A BALL

The convex set

$$B_d(a, r) = \{x : d(x, a) < r\}$$

is called an open ball of radius r and center a.

■

We have $B_{d_1}((0,0),1) = \{(0,0)\}$.

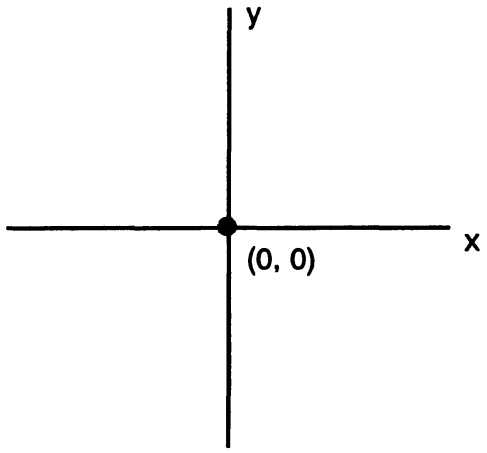


FIGURE 1

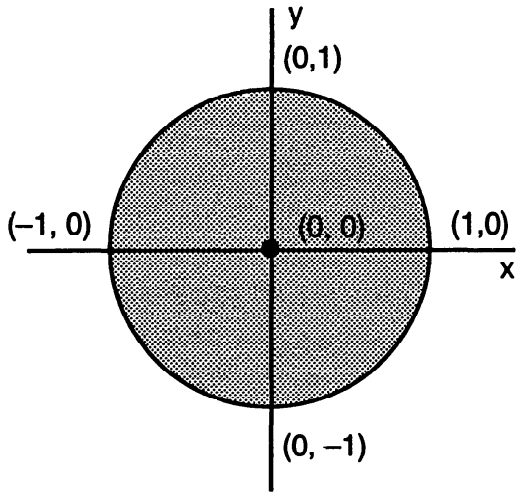


FIGURE 2

Figure 2 illustrates the d_2-ball, $B_{d_2}((0,0), 1)$.

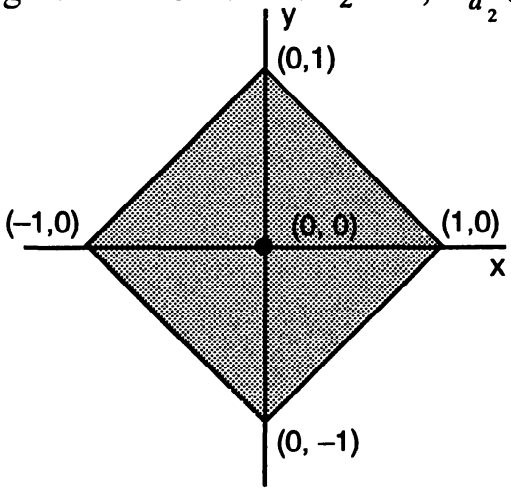


FIGURE 3

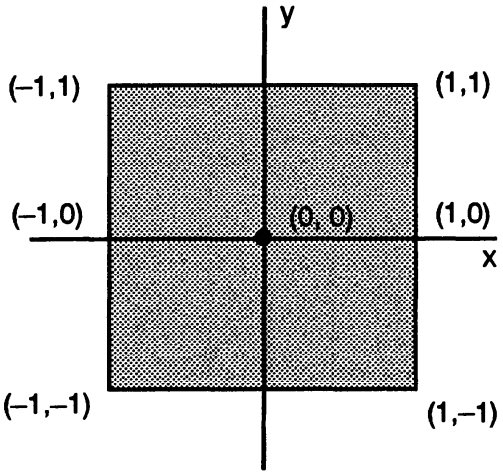


FIGURE 4

B_{d_3} is shown in Figure 3.

B_{d_4} is shown in Figure 4.

Sometimes when it is obvious what metric is used we shall write $B(a, r)$ instead of $B_d(a, r)$.

● PROBLEM 10-8

1. Describe explicitly the ball $B(x_0, r)$ in the metric space (R, d) where $d(x, y) = |x - y|$.

2. Let $C[0, 1]$ be the collection of all continuous functions on $[0, 1]$ with the metric defined by

$$d(f, g) = \sup \{ |f(x) - g(x)| : x \in [0, 1] \}. \quad (1)$$

Find the ball $B(f_0, \varepsilon)$ where f_0 is a continuous function and $\varepsilon > 0$.

SOLUTION:

1. The ball $B(x_0, r)$ is shown in Figure 1.

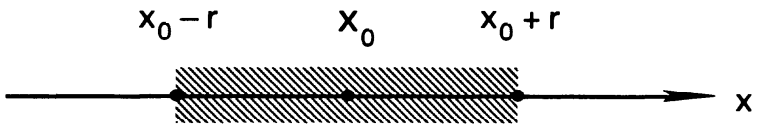


FIGURE 1

The ball consists of all the points x on the real axis such that

$$|x - x_0| < r \quad \text{or} \quad x_0 - r < x < x_0 + r. \quad (2)$$

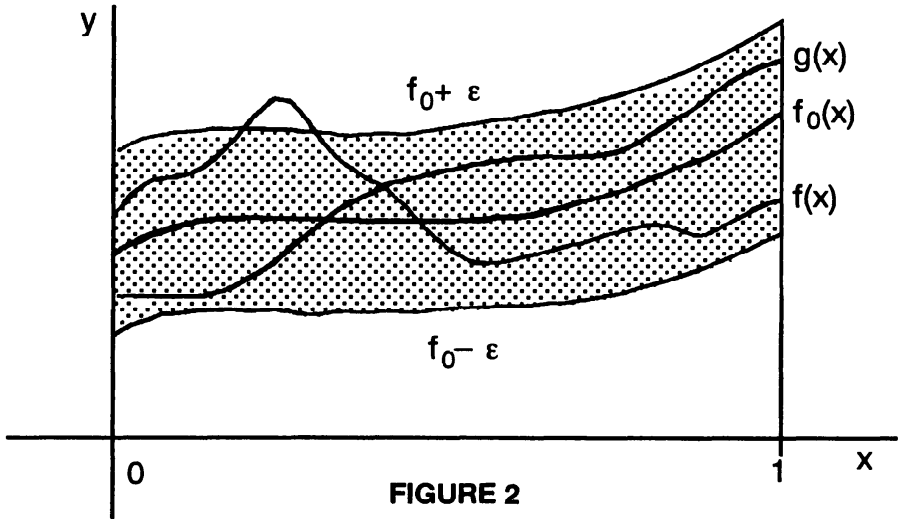


FIGURE 2

2. Let f_0 be a continuous function as given in Figure 2. The open ball $B(f_0, \varepsilon)$ consists of all continuous functions which lie in the area bounded by, (and excluding), the lines at $f_0 - \varepsilon$ and $f_0 + \varepsilon$.

● PROBLEM 10-9

Prove this theorem:

THEOREM

Let $B(x_0, r)$ be a ball. For every point $y \in B(x_0, r)$, there exists a ball $B(y, r')$ such that

$$B(y, r') \subset B(x_0, r). \tag{1}$$

■

SOLUTION:

Point $y \in B(x_0, r)$ hence

$$d(x_0, y) < r. \tag{2}$$

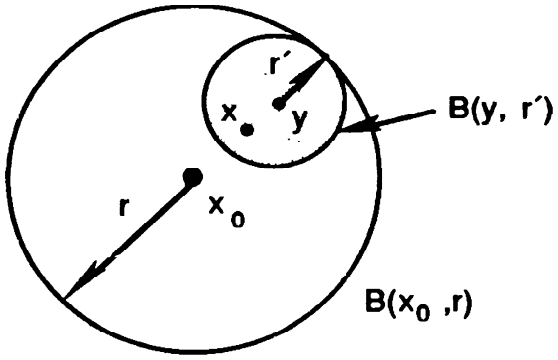


FIGURE 1

Let us define

$$r' = r - d(x_0, y) > 0. \tag{3}$$

We will show that

$$B(y, r') \subset B(x_0, r). \tag{4}$$

Suppose

$$x \in B(y, r') \tag{5}$$

then

$$d(x, y) < r' \tag{6}$$

or

$$d(x, y) < r - d(x_0, y)$$

$$d(x, y) + d(x_0, y) < r. \tag{7}$$

But d is a metric, so

$$d(x, x_0) \leq d(x, y) + d(y, x_0) < r. \tag{8}$$

Hence

$$d(x, x_0) < r$$

and

$$x \in B(x_0, r). \tag{9}$$

● PROBLEM 10-10

1. Prove that if B_1 and B_2 are balls with the same center, then one of them is a subset of the other.

2. Prove this theorem:

THEOREM

Let B_1 and B_2 be balls and let $x \in B_1 \cap B_2$. Then, a ball B exists such that

$$x \in B \subset B_1 \cap B_2 \tag{1}$$

■

SOLUTION:

1. Let $B_1(x_0, r_1)$ and $B_2(x_0, r_2)$. Since r_1 and r_2 are real numbers, we have either $r_1 \leq r_2$ or $r_2 \leq r_1$. Suppose $r_1 \leq r_2$. Then if $x \in B_1(x_0, r_1)$ we obtain $d(x_0, x) < r_1 \leq r_2$.

Thus $d(x_0, x) < r_2$ and $x \in B_2(x^0, r_2)$.

If $r_2 \leq r_1$, then $B_2(x_0, r_2) \subset B_1(x_0, r_1)$.

2. Let x be a point, such that

$$x \in B_1 \cap B_2. \tag{2}$$

Then

$$x \in B_1 \quad \text{and} \quad x \in B_2.$$

By Problem 10–9, if $x \in B_1(x_0, r_1)$, then a ball $B'_1(x, r'_1)$ exists such that

$$x \in B'_1 \subset B_1. \tag{3}$$

Similarly, a ball $B'_2(x, r'_2)$ exists such that

$$x \in B'_2 \subset B_2 \tag{4}$$

as shown in Figure 1.

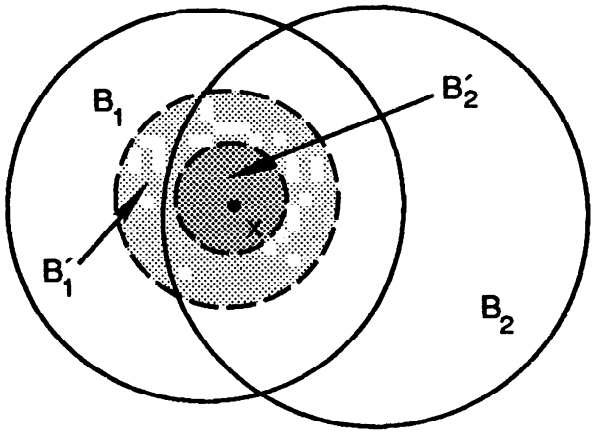


FIGURE 1

The balls B'_1 and B'_2 both have the same center. By part 1 of this problem, one of them is contained in the other. Suppose

$$B'_2 \subset B'_1. \tag{5}$$

Then

$$x \in B'_2 \subset B'_1 \subset B_1. \tag{6}$$

Let $B = B'_2$, then

$$x \in B \subset B_1 \cap B_2. \tag{7}$$

● PROBLEM 10–11

1. Let R be the set of real numbers with the absolute value metric. Show that the open interval (0, 1) is an open set.

2. Show that the ball

$$B(x_0, r) = \{ x : d(x, x_0) < r \} \tag{1}$$

is an open set.

SOLUTION:

DEFINITION OF AN OPEN SET

Let (X, d) be a metric space. A subset $A \subset X$ is said to be open, if for every $x \in A$, a ball $B(x, r)$ exists such that

$$x \in B(x, r) \subset A.$$

■

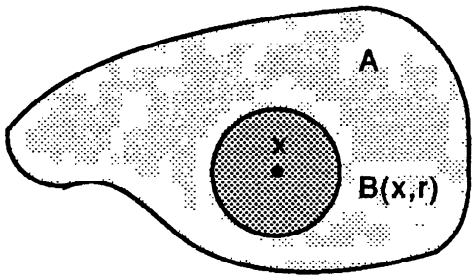


FIGURE 1

1. Let $x \in (0, 1)$, then

$$0 < x < 1. \tag{2}$$

We define

$$r = \min \{ x, 1 - x \}. \tag{3}$$

Then

$$B(x, r) \subset (0, 1) \tag{4}$$

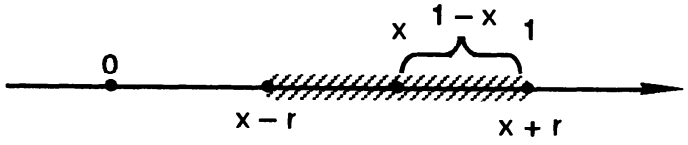


FIGURE 2

Any open interval (a, b) is an open set.

2. Consider ball (1). Let

$$y \in B(x_0, r). \tag{5}$$

Then $d(x_0, y) < r$.

By virtue of Problem 10–9, we conclude that a ball $B'(y, r')$ exists such that

$$y \in B'(y, r') \subset B(x_0, r) \tag{6}$$

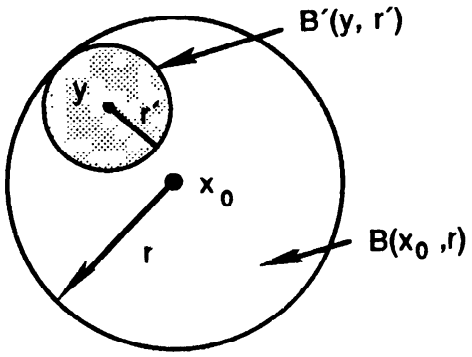


FIGURE 3

● PROBLEM 10-12

Prove the following:
THEOREM

Let (X, d) be a metric space. Then

1. X, ϕ are open sets,

2. the intersection of any two open sets is an open set,

3. the union of any family of open sets is an open set.

■

SOLUTION:

1. If $x \in X$ and $r > 0$, then

$$B(x, r) \subset X. \tag{1}$$

Hence X is an open set.

For each $x \in \phi$ and any $r > 0$,

$$B(x, r) \subset \phi. \tag{2}$$

Hence ϕ is an open set.

2. Suppose A and B are open subsets of (X, d) and $x \in A \cap B$. Since both A and B are open, balls exist such that

$$B_1(x, r_1) \subset A$$

$$B_2(x, r_2) \subset B. \tag{3}$$

Setting

$$r = \min(r_1, r_2) \tag{4}$$

we obtain

$$B(x, r) \subset A \cap B. \tag{5}$$

Hence $A \cap B$ is an open set.

3. Let $\bigcup_\alpha A_\alpha$ be a union of a family of open sets. Suppose

$$x \in \bigcup_\alpha A_\alpha \tag{6}$$

Then an open set A_α exists such that is a member of this family and

$$x \in A_\alpha \tag{7}$$

and since A_α is open

$$x \in B(x,r) \subset A_\alpha \subset \bigcup_\alpha A_a \tag{8}$$

● PROBLEM 10–13

Let (X, d) be a metric space. Show that a subset $A \subset X$ is open if and only if A is the union of a family of balls.

SOLUTION:

We shall prove

$$(A \text{ is open}) \Leftrightarrow (A = \bigcup_{i \in I} B_i) \tag{1}$$

$\Leftarrow$ In Problem 10–12, we proved that the union of any family of open sets is an open set. In Problem 10–11, we proved that a ball is an open set. Hence $A = \bigcup_{i \in I} B_i$ is an open set.

$\Rightarrow$ Suppose A is an open set. Then

$$\forall\, x \in A \quad \exists\, B(x, r) \subset A. \tag{2}$$

Therefore

$$\bigcup_{x \in A} B(x,r) \subset A \tag{3}$$

But the ball $B(x, r) \subset A$ exists for every $x \in A$. Hence

$$A = \bigcup_{x \in A} B(x,r). \tag{4}$$

● PROBLEM 10-14

Show that the closed interval [0, 1] is a closed subset of R with the absolute value metric.

SOLUTION:

DEFINITION OF A CLOSED SET

Let (X, d) be a metric space. A subset B of X is said to be closed in (X, d) when its complement $X - B$ is an open set in (X, d).

■

We shall show that the set $R - [0, 1]$ is open in R.

Suppose

$$x \in R - [0, 1] \tag{1}$$

Then, either $x > 1$ or $x < 0$. We can assume that $x > 1$, (the proof for $x < 0$ is similar).

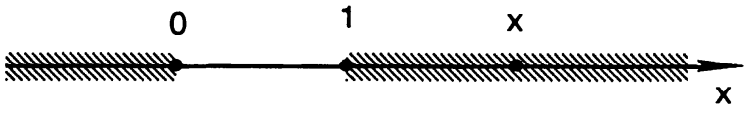


FIGURE 1

We have

$$0 < \delta = x - 1 \tag{2}$$

and

$$B(x, \delta/2) \subset R - [0, 1]. \tag{3}$$

Therefore $R - [0, 1]$ is an open set and [0, 1] is a closed set.

Similarly we can show that a closed ball in the metric space (X, d)

$$B(x, r) = \{ y : d(x, y) \leq r \} \tag{4}$$

is a closed set.

● PROBLEM 10-15

Prove the following:

THEOREM

Let (X, d) be a metric space. Then

1. X and ϕ are closed subsets of X,

2. the union of two closed sets, hence of a finite number of such sets, is a closed set,

3. the intersection of any family of closed sets is a closed set. ■

SOLUTION:

1. ϕ is an open set, therefore $X - \phi = X$ is closed, because it is a complement of an open set. X is an open set, therefore ϕ is a closed set because $\phi = X - X$.

2. Suppose B_1 and B_2 are closed sets. Then $X - B_1$ and $X - B_2$ are open sets. We have

$$X - (B_1 \cup B_2) = (X - B_1) \cap (X - B_2) \tag{1}$$

Since $X - B_1$ and $X - B_2$ are open sets, $(X - B_1) \cap (X - B_2)$ is an open set. Hence $X - (B_1 \cup B_2)$ is a closed set.

3. We shall show that

$$\bigcap_{\alpha} B_{\alpha} \tag{2}$$

is a closed set when all B_α's are closed sets.

$$X - \left(\bigcap_{\alpha} B_{\alpha}\right) = \bigcup_{\alpha} (X - B_{\alpha}) \tag{3}$$

Suppose all sets B_α are closed. Then all sets $X - B_\alpha$ are open sets and their union $\bigcup_{\alpha}(X - B_{\alpha})$ is an open set.

Therefore $X - \left(\bigcap_{\alpha} B_{\alpha}\right)$ is an open set and $\bigcap_{\alpha} B_{\alpha}$ is a closed set.

● PROBLEM 10-16

Which of the following subsets of (R^2, d) where d is the Pythagorean metric are closed?

1. $\{(x, y) : x = 2\}$.

2. $\{(x, y) : x, y$ are integers$\}$.

3. $\{(x, y) : y = x^2\}$.

4. $\{(x, y): x^2 + y^2 \leq 1 \text{ and } x < 1\}$.

5. $\{(x, y): x^2 + y^2 > 2\}$.

SOLUTION:

1.

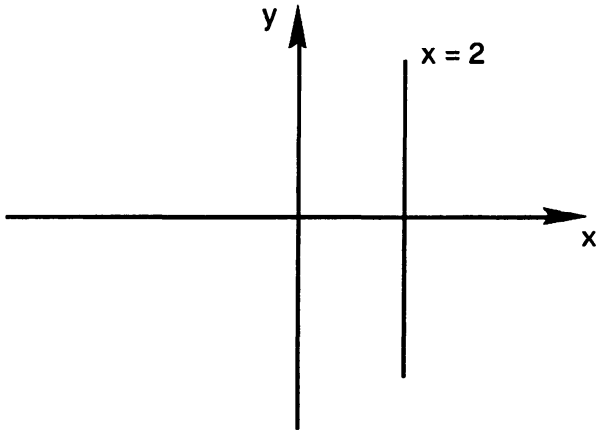


FIGURE 1

This is a closed set because $R^2 - \{(x, y): x = 2\}$ is open.

2.

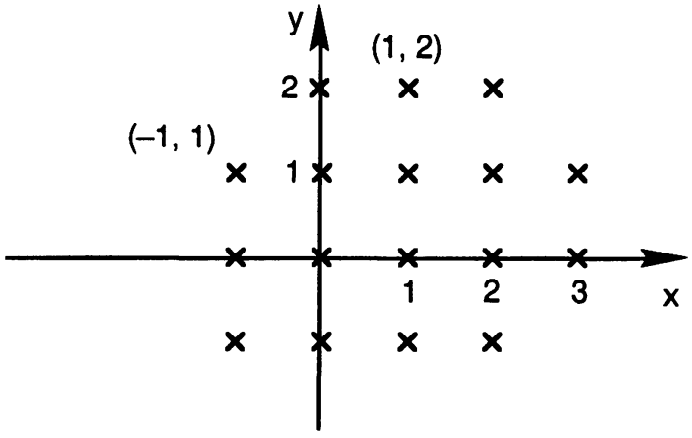


FIGURE 2

This set is closed.

3.

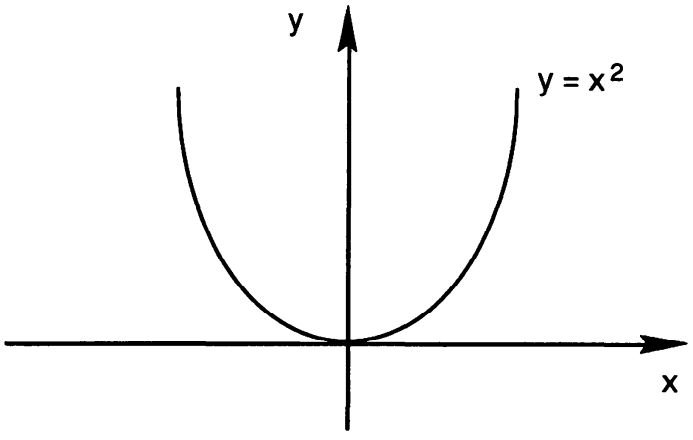


FIGURE 3

This set is closed.

4.

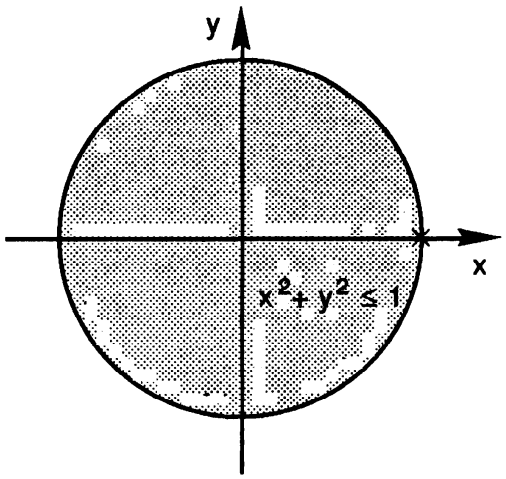


FIGURE 4

This set is neither closed nor open.

5.

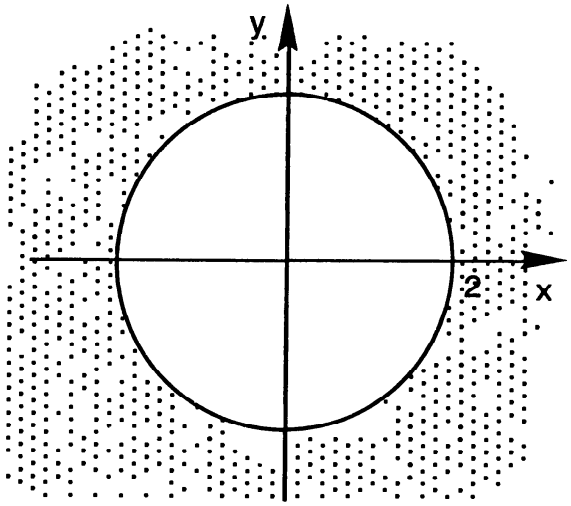


FIGURE 5

Since the circle $x^2 + y^2 = 2$ is not in the set, then the set is open..

● PROBLEM 10–17

DEFINITION OF CONVERGENCE

A sequence of points (x_n) of a metric space (X, d) is convergent to the point x of this space, if the sequence of real numbers $d(x_n, x)$ is convergent to zero. The point $x \in X$ is called the limit of the sequence (x_n). We write

$$x = \lim_{n \to \infty} x_n \tag{1}$$

■

Write down this definition using the symbolism of logic.

Given an example to illustrate that a given sequence can be convergent with respect to one metric and not convergent with respect to another metric.

SOLUTION:

The definition of a limit point may be written in symbolism of logic as follows:

$$\left(\lim_{n\to\infty} x_n = x\right) \equiv \left(\lim_{n\to\infty} d(x_n, x) = 0\right) \equiv$$
$$\equiv \forall \varepsilon > 0 \quad \exists k \in N \quad \forall n \in N$$
$$(n > k) \Rightarrow (d(x_n, x) < \varepsilon) \qquad (2)$$

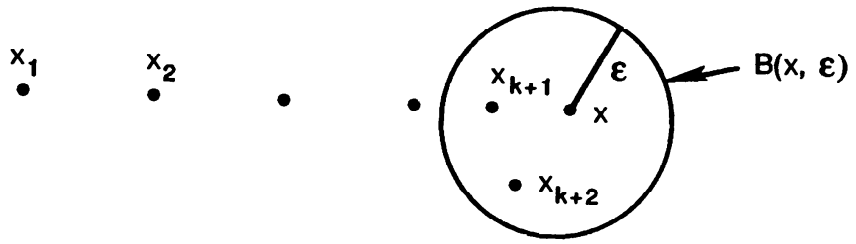


FIGURE 1

If $\lim_{n\to\infty} x_n = x$, then for any ball $B(x, \varepsilon)$ all but a finite number of elements of (x_n) are located inside the ball $B(x, \varepsilon)$.

Suppose the set of real numbers R is equipped with the metric $d_1(x, y) = |x - y|$ and with the metric

$$d_2(x,y) = \begin{cases} 0 & \text{when } x = y \\ 1 & \text{when } x \neq y \end{cases}$$

The sequence $(1/n)$ is given. In the space (R, d_1), this sequence is convergent. In the space (R, d_2), this sequence is not convergent.

● PROBLEM 10-18

Is it possible for a convergent sequence to have two different limits?

SOLUTION:

Let (X, d) be a metric space. Suppose sequence (X_n) has two different limits x_1 and x_2, $x_1 \neq x_2$. Then

$$\lim_{n \to \infty} x_n = x_1 \tag{1}$$

$$\lim_{n \to \infty} x_n = x_2 \tag{2}$$

Since $x_1 \neq x_2$ we have

$$d(x_1, x_2) = \alpha > 0. \tag{3}$$

From the triangle inequality

$$d(x_1, x_2) \leq d(x_n, x_1) + d(x_n, x_2). \tag{4}$$

From (1) and (2) we obtain

$$\lim_{n \to \infty} d(x_n, x_1) = \lim_{n \to \infty} d(x_n, x_2) = 0. \tag{5}$$

This is a contradiction with (3). Hence $x_1 = x_2$. The sequence can have only one limit.

The same result can be reached "graphically." Since $x_1 \neq x_2$

$$d(x_1, x_2) = \alpha > 0. \tag{6}$$

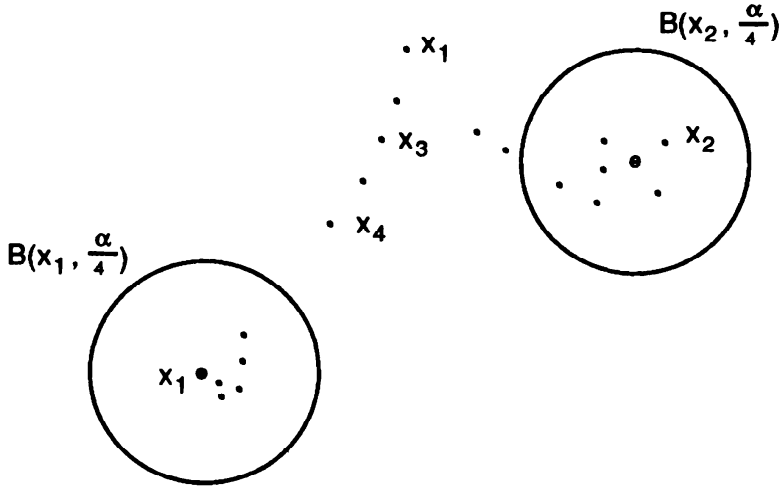


FIGURE 1

Balls $B(x_1, \alpha/u)$ and $B(x_2, \alpha/u)$ are disjoint and each contains all but a finite number of elements of (x_n). Contradiction.

● PROBLEM 10-19

Let X be the set of all functions from [0,1] into itself with the metric d defined as follows

$$d(f, g) = \text{least upper bound } \{ \, | f(x) - g(x) | : x \in [0, 1] \, \}. \tag{1}$$

Check the convergence of the sequence (f_n) where $f_n(x) = x^n$.

SOLUTION:

The elements of the sequence are

$$x, x^2, x^3, x^4, \ldots, x^n, \ldots \tag{2}$$

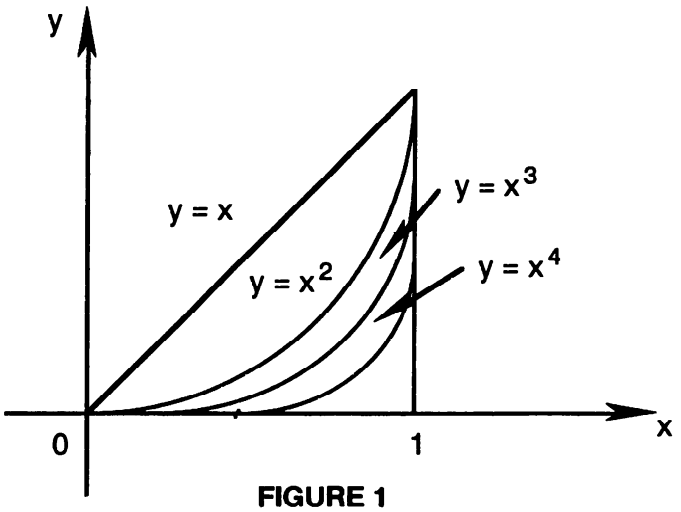


FIGURE 1

Figure 1 shows the first few elements of the sequence. We see that the sequence "converges" to the function

$$f(x) = \begin{cases} 0 & \text{for } x \neq 1 \\ 1 & \text{for } x = 1 \end{cases} \tag{3}$$

Here, the sequence of continuous functions "converges pointwise" to the function which is not continuous.

It converges to f in the sense of metric defined by (1).

$$d(f_n, f) = \text{least upper bound } \{ \, | x^n - f(x) | : x \in [0, 1] \, \} \tag{4}$$

and

$$\lim_{n \to \infty} d(f_n, f) = 0 . \tag{5}$$

● PROBLEM 10-20

Prove the following:

THEOREM

A sequence of points $z_n = (x_n, y_n)$ of the space $Z = X \times Y$ is convergent to the point $z = (x, y)$, if and only if

$$x = \lim_{n \to \infty} x_n \tag{1}$$

$$y = \lim_{n \to \infty} y_n \tag{2}$$

■

SOLUTION:

Suppose $x = \lim_{n \to \infty} x_n$ and $y = \lim_{n \to \infty} y_n$.

Let $\varepsilon > 0$, then $k \in N$ exists such that for $n > k$

$$d_1(x_n, x) < \varepsilon \quad \text{and} \quad d_2(y_n, y) < \varepsilon. \tag{3}$$

Let us define

$$d(z_n, z) = \sqrt{d_1^2(x_n, x) + d_2^2(y_n, y)}. \tag{4}$$

Then

$$d(z_n, z) < \varepsilon\sqrt{2} \tag{5}$$

and

$$\lim_{n \to \infty} z_n = z. \tag{6}$$

Then k exists such that for $n > k$

$$d(z_n, z) < \varepsilon \tag{7}$$

and

$$d(z_n, z) = \sqrt{d_1^2(x_n, x) + d_2^2(y_n, y)} < \varepsilon. \tag{8}$$

Thus

$$d_1(x_n, x) < \varepsilon \quad \text{for} \quad n > k \tag{9}$$

i.e.,

$$\lim_{n \to \infty} x_n = x \tag{10}$$

Similarly, we obtain

$$d_2(y_n, y) < \varepsilon \quad \text{for} \quad n > k \tag{11}$$

i.e.,

$$\lim_{n \to \infty} y_n = y. \tag{12}$$

● PROBLEM 10-21

Show that the function $f(x) = 2x + 1$ is continuous,

$$f : R \to R \tag{1}$$

SOLUTION:

DEFINITION OF A CONTINUOUS FUNCTION

A function $f: X \to Y$, where (X, d) and (Y, d_1) are metric spaces, is said to be continuous, if for every $f(a) \in Y$ and any positive number $\varepsilon > 0$, there is a positive number $\delta > 0$ such that if $x \in B(a, \delta)$, then $f(x) \in B(f(a), \varepsilon)$. ■

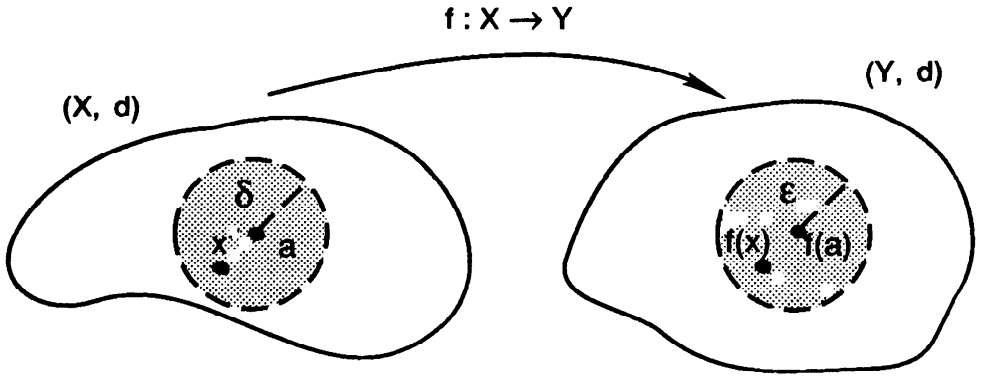


FIGURE 1

$$(x \in B(a, \delta)) \Rightarrow (f(x) \in B(f(a), \varepsilon))$$

In practical applications, this definition is rather difficult to deal with. Some other methods of establishing the continuity of functions will be shown later. Let us rewrite the definition using symbolic notation.

$f: (X, d) \to (Y, d_1)$ is continuous if

$$\forall f(a) \in Y \quad \forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{such that}$$

$$(d(x, a) < \delta) \Rightarrow (d_1(f(a), f(x)) < \varepsilon)$$

Let $f(a) = 2a + 1$ and $\varepsilon > 0$. We can choose $\delta = \varepsilon/2$, if $|x - a| < \delta = \varepsilon/2$, then

$$|f(x) - f(a)| = |2x + 1 - 2a - 1| =$$

$$= |2x - 2a| = 2|x - a| < 2 \cdot \varepsilon/2 = \varepsilon.$$

Hence, the function is continuous.

● PROBLEM 10-22

To determine whether or not a function is continuous, this theorem can be used:

THEOREM

$f: (X, d) \to (Y, d_1)$

$$(f \text{ is continuous}) \Leftrightarrow (\forall B \subset Y, B \text{ is open } f^{-1}(B) \text{ is open}) \quad (1)$$

■

Prove it.

SOLUTION:

$\Rightarrow$ f is continuous and A is an open subset of Y. We need to show that

$$f^{-1}(A) = \{ x \in X : f(x) \in A \} \quad (2)$$

is an open subset of X.

Let $x \in f^{-1}(A)$, then $f(x) \in A$. Since A is open there exists an $\varepsilon > 0$ such that

$$B(f(x), \varepsilon) \subset A. \quad (3)$$

Since f is continuous, then there exists a $\delta > 0$ such that

$$f[B(x, \delta)] \subset B(f(x), \varepsilon) \subset A \quad (4)$$

as shown in Figure 1.

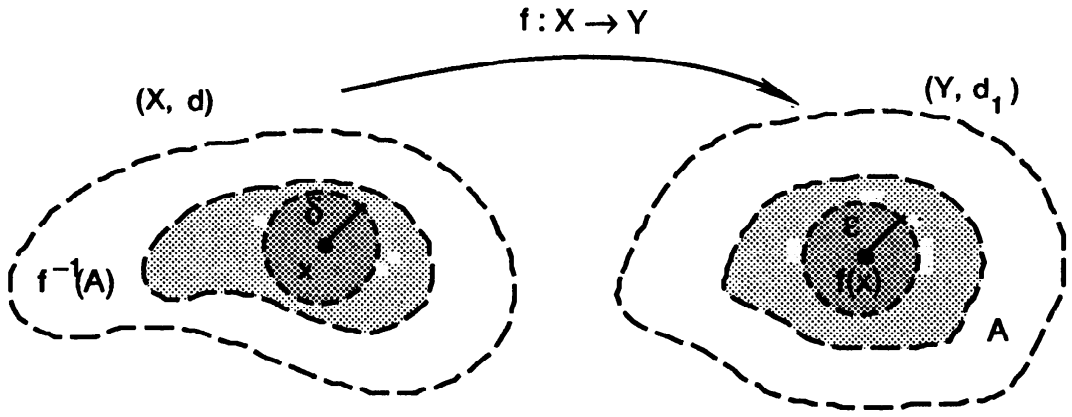


FIGURE 1

Therefore

$$B(x, \delta) \subset f^{-1}(A). \quad (5)$$

For $x \in f^{-1}(A)$, a $\delta > 0$ exists such that (5) is true; hence $f^{-1}(A)$ is an open subset of X.

$\Leftarrow$ Suppose that if A is an open subset of Y, then $f^{-1}(A)$ is an open subset of X. If $f(a) \in Y$ and there exists an $\varepsilon > 0$, then

$$B(f(a), \varepsilon) \quad (6)$$

is an open subset of Y.

Thus,

$$f^{-1}[B(f(a), \varepsilon)] \tag{7}$$

is an open subset of X. There exists a $\delta > 0$ such that

$$B(a, \delta) \subset f^{-1}[B(f(a), \varepsilon)] \tag{8}$$

or

$$f(B(a, \delta)) \subset B(f(a), \varepsilon). \tag{9}$$

Therefore f is continuous.

● PROBLEM 10-23

Prove the following useful:

THEOREM

Let $f: (X, d) \to (Y, d_1)$ then

$$(f \text{ is continuous}) \Leftrightarrow$$

$$(\text{for any } B(y, r) \text{ in } Y, f^{-1}(B) \text{ is an open set of } X) \tag{1}$$

■

SOLUTION:

$\Rightarrow$ Suppose f is continuous and B is any ball in Y. Then B is an open subset of Y. By theorem of Problem 10–22, $f^{-1}(A)$ is an open subset of X.

$\Leftarrow$ Suppose for any ball $B \subset Y$, $f^{-1}(A)$ is an open subset of X.

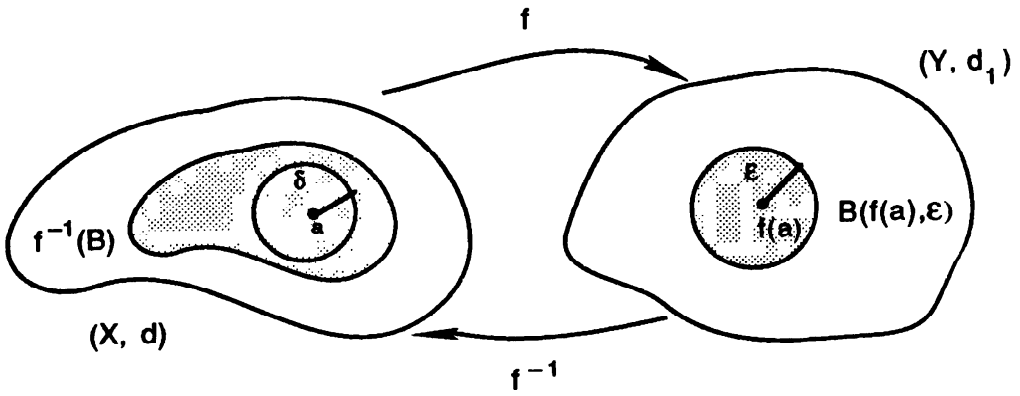


FIGURE 1

Let $f(a)$ be any point in Y and ε be any positive number. Then $B(f(a), \varepsilon)$ is a ball in Y.

The set $f^{-1}[B(f(a), \varepsilon)]$ is an open subset of X. We have

$$a \in f^{-1}[B(f(a), \varepsilon)]. \tag{2}$$

Set $f^{-1}(A)$ is an open subset of X (not necessarily a ball). Hence a ball $B(a, \delta)$ exists such that

$$B(a, \delta) \subset f^{-1}[B(f(a), \varepsilon)]. \tag{3}$$

Thus

$$(x \in B(a, \delta)) \Rightarrow (f(x) \in B(f(a), \varepsilon)).$$

Function f is continuous.

● PROBLEM 10-24

Consider the function

$$f(x, y) = x \tag{1}$$

$$f: (R^2, d_1) \to (R, d_2) \tag{2}$$

where

$$d_1((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \tag{3}$$

$$d_2(x, y) = |x - y| \tag{4}$$

Show that f is continuous.

SOLUTION:

We shall use the theorem proved in Problem 10–23.

Let $B(a, r)$ be any ball in (R, d_2). Then $B(a, r)$ is an open interval $(a - r, a + r)$ in R.

The set $f^{-1}[B(a, r)]$ is

$$f^{-1}[(a - r, a + r)] = \{(x, y) ; a - r < x < a + r\}.$$

This set is shown in Figure 1; it is an open subset of R^2.

Hence, the function defined by (1) and (2) is continuous.

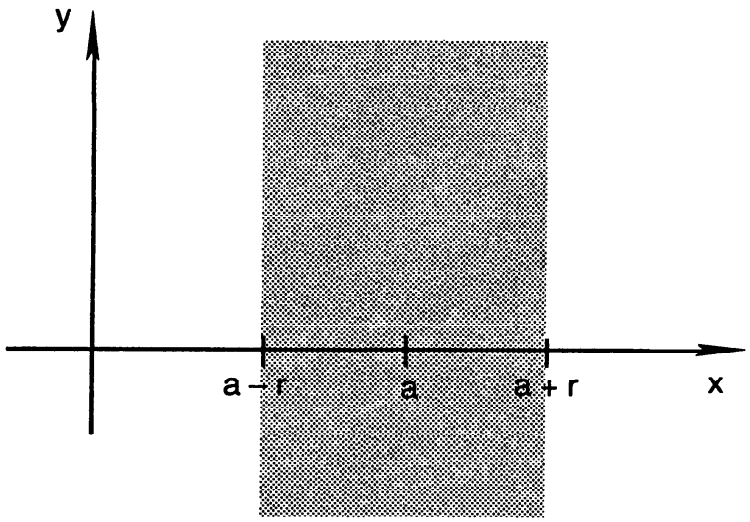


FIGURE 1

● PROBLEM 10-25

Find the limit of the sequence

$$\sin (1 + {}^1/_n)^n \tag{1}$$

SOLUTION:

Consider the sequence $(1 + {}^1/_n)^n$. Its limit is

$$\lim_{n \to \infty} (1 + \tfrac{1}{n})^n = e. \tag{2}$$

Now we shall apply the following:

THEOREM

Let $f : (X, d) \to (Y, d_1)$. Function f is continuous, if and only if given any sequence (x_n), such that

$$\lim_{n \to \infty} x_n = x, \quad \lim_{n \to \infty} f(x_n) = f(x). \qquad \blacksquare$$

A continuous function preserves the convergence of a sequence. Function $f(x) = \sin x$ is continuous. Therefore, since

$$\lim_{n \to \infty} (1 + \tfrac{1}{n})^n \to e \tag{3}$$

we have

$$\lim_{n \to \infty} \{\sin(1 + \tfrac{1}{n})^n\} = e. \tag{4}$$

Sometimes we shall use

$$x_n \underset{n \to \infty}{\to} x \tag{5}$$

to indicate

$$\lim_{n \to \infty} x_n = x.$$

● PROBLEM 10-26

1. Which of the following sets are bounded?

   closed interval [0, 1] in R

   ball $B(0, 1)$ in R^2

   half-line $x < 0$ in R

2. Which of the following functions are bounded?

   $f(x) = x^2$

   $f(x) = \sin x$

SOLUTION:

1. Let A be a subset of (X, d).

DEFINITION

Consider the set A. The diameter of A, denoted by *delta* (A) is the least upper bound of the distances $d(x, y)$ between all pairs of points $x, y \in A$. Sets with finite diameter are said to be bounded. For simplicity, we write "least upper bound" as "lub."

■

Interval [0, 1] is bounded in (R, d) where $d(x, y) = |x - y|$.
The ball $B(0, 1)$ is bounded.
Half-line, $x < 0$ is not bounded.

2. **DEFINITION**

A mapping $f : X \to Y$ where Y is a metric space, is called bounded if the set $f(x)$ is bounded.

■

Function

$$f : R \to R$$

$$f(x) = x^2$$

maps R onto $[0, \infty)$. Hence

$$f(R) = [0, \infty).$$

The set $[0, \infty)$ is not bounded, hence the function is not bounded.

Function $f(x) = \sin x$ maps R onto $[-1, 1]$ hence the function is bounded.

● PROBLEM 10-27

Prove the following:

THEOREM 1

The set $\Phi(X, Y)$ of all bounded mappings $f : X \to Y$, where X is an arbitrary set and (Y, d) a metric space, is a metric space with metric defined by

$$d_1(f, g) = \sup d(f(x), g(x)). \tag{1}$$

■

SOLUTION:

First let us prove:

THEOREM 2

If f and g are bounded mappings to the set X into the metric space (Y, d), then the metric

$$d_1(f, g) = \sup d(f(x), g(x)) \tag{2}$$

is finite.

■

Let a be a given element of X. Then

$$d(f(x), g(x)) \leq d(f(x), f(a)) + d(f(a), g(x)) \leq$$

$$\leq d(f(x), f(a)) + d(f(a), g(a)) + d(g(a), g(x)). \tag{3}$$

Hence

$$d_1(f, g) \leq \delta[f(X)] + d(f(a), g(a)) + \delta[g(X)]. \tag{4}$$

■

We have

$$d_1(f, g) \geq 0 \quad \text{for all } f, g \tag{5}$$

$$d_1(f, g) = 0 \quad \text{iff } f = g \tag{6}$$

Indeed, if $\sup d(f(x), g(x)) = 0$ then $f(x) = g(x)$ for all $x \in X$.

$$d_1(f, g) = d_1(g, f) \text{ for all } f, g \tag{7}$$

$$d_1(f, g) = \sup d(f(x), g(x)) \leq$$

$$\leq \sup [d(f(x), h(x)) + d(f(x), g(x)] \leq$$

$$\leq \sup d(f(x), h(x)) + \sup d(h(x), g(x)) =$$

$$= d_1(f, h) + d_1(h, g). \tag{8}$$

Hence d_1 defined in (1) is a metric.

● PROBLEM 10-28

1. Show that if $A \neq \phi$ and $B \neq \phi$ and $A \subset B$, then

$$\delta(A) \leq \delta(B). \tag{1}$$

2. Show that if $A \cap B \neq \phi$, then

$$\delta(A \cup B) \leq \delta(A) + \delta(B). \tag{2}$$

SOLUTION:

1. We have

$$\delta(A) = \text{lub } d(x, y) \tag{3}$$

where $x, y \in A$.

Note that since $A \subset B$

$$d(x, y) \leq d(x, y) \tag{4}$$

for $x, y \in A$.

Then,

$$\begin{array}{ccc} \text{lub } d(x, y) & \leq & \text{lub } d(x, y) \\ \text{where } x, y \in A & & \text{where } x, y \in B \end{array} \tag{5}$$

and

$$\delta(A) \leq \delta(B). \tag{6}$$

2.

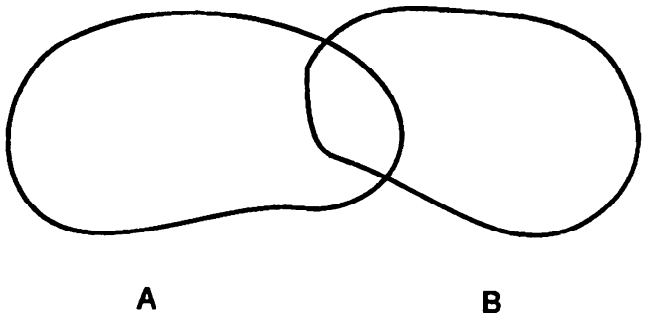


FIGURE 1

Let us define the distance between sets A and B

$$\delta(A, B) = \text{greatest lower bound } \{d(a, b) : a \in A, b \in B\}. \tag{7}$$

Then, for any subsets A and B of (X, d)

$$\delta(A \cup B) \leq \delta(A) + \delta(B) + \delta(A, B). \tag{8}$$

But $A \cap B \neq 0$ therefore

$$\delta(A, B) = 0. \tag{9}$$

Hence

$$\delta(A \cup B) \leq \delta(A) + \delta(B). \tag{10}$$

● PROBLEM 10-29

Let (X, d) be a metric space. Show that a subset A of X is closed, if and only if for any $x \in X - A$,

$$\delta(x, A) \neq 0 \tag{1}$$

where

$$\delta(x, A) =: \text{greatest lower bound or (glb) } \{d(x, a) : a \in A\}. \tag{2}$$

SOLUTION:

Suppose A is a closed set, then $X - A$ is open. Hence for any $x \in X - A$, a ball $B(x, r)$ exists such that

$$B(x, r) \subset X - A \tag{3}$$

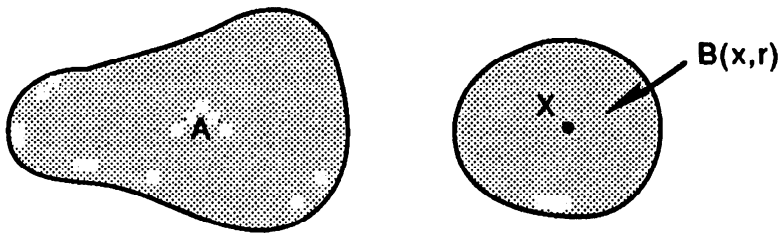


FIGURE 1

Then

$$\delta(x, A) \geq r \tag{4}$$

and

$$\delta(x, A) \neq 0. \tag{5}$$

Now suppose, for any $x \in X - A$, $\delta(x, A) \neq 0$.
Let

$$r = \delta(x, A) \tag{6}$$

then

$$B(x, r) \subset X - A. \tag{7}$$

Hence $X - A$ is open and A is closed

$$A = X - (X - A). \tag{8}$$

● PROBLEM 10-30

Let (X, d) be a metric space. Show that if A and A' are closed subsets of X, such that $A \cap A' = \phi$, then open sets B and B' exist such that

$$A \subset B, \quad A' \subset B', \quad B \cap B' = \phi. \tag{1}$$

SOLUTION:

Let us define for each $x \in A$

$$r_x = {}^1/_2 \, \delta(x, A') \tag{2}$$

and for each $x' \in A'$

$$r'_{x'} = {}^1/_2 \, \delta(x', A) \tag{3}$$

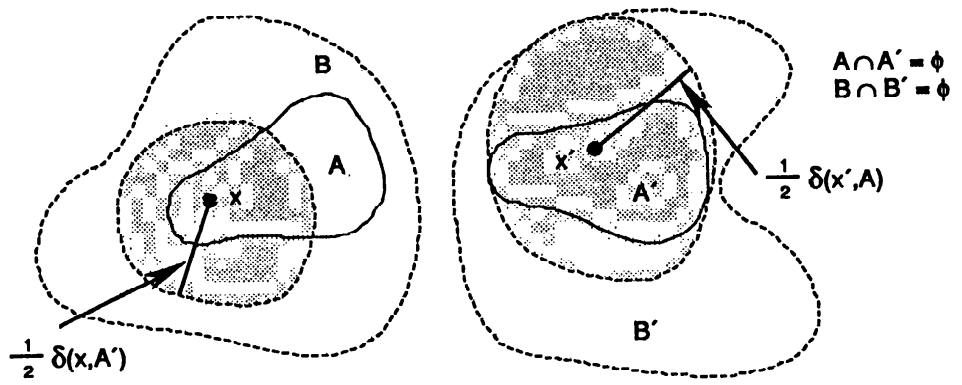


FIGURE 1

We define

$$B = \bigcup_{A} B(x, r_x)$$

$$B' = \bigcup_{A'} B(x', r'_{x'}) \tag{4}$$

Both sets B and B' are open because they are the union of a family of open sets (balls).

Since the union has taken over all the elements of A

$$A \subset B \tag{5}$$

similarly

$$A' \subset B'. \tag{6}$$

We will show that $B \cap B' = \phi$.

Suppose on the contrary

$$x \in B \cap B'. \tag{7}$$

Then for some $y \in A$

$$\delta(y, x) < r_y \tag{8}$$

and for some $y' \in A'$

$$\delta(y', x) < r'_{y'} \tag{9}$$

Suppose $r_y \geq r'_{y'}$, then

$$\delta(y, y') \leq \delta(y, x) + \delta(y', x) < r_y + r'_{y'} \leq$$

$$\leq 2r_y = \delta(y, A') \tag{10}$$

where $\delta(y, y') = d(y, y')$. But

$$\delta(y, y') \geq d(y, A'). \tag{11}$$

Hence a contradiction,

$$B \cap B' = \phi. \tag{12}$$

Two closed disjoint sets are well separable, in the sense that it is possible to put them into two open sets, which again are disjoint.

● PROBLEM 10-31

Consider the metric space (R^2, d) where d is the Pythagorean metric. Let A be the hyperbola $y = {}^1/_x$ and A' the x -axis. Find the open sets B and B' as described in Problem 10–30.

SOLUTION:

Both sets

$$A = \{ (x, y) : y = {}^1/_x, x \neq 0\} \tag{1}$$

and

$$A' = \{ (x, y) : y = 0\} \tag{2}$$

are closed and

$$A \cap A' = \phi. \tag{3}$$

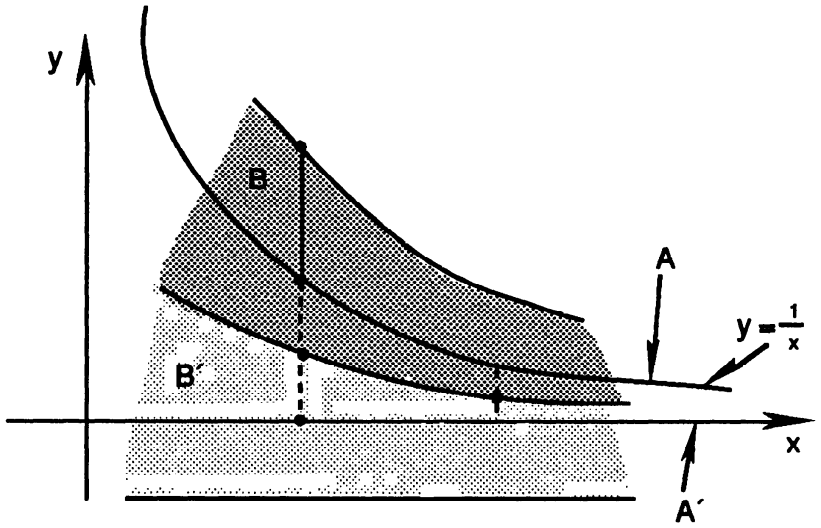


FIGURE 1

At first glance, it seems impossible to find the separating open sets B and B' because

$$\delta(A, A') = 0 \tag{4}$$

i.e., the distance between A and A' is zero. Function $y = 1/x$ has the x-axis for an asymptote. Such sets B and B', however, exist.

$$B = \{ (x, y) : 1/2x < y < 1/x + 1\} \tag{5}$$

$$B' = \{ (x, y) : -1 < y < 1/2x \}. \tag{6}$$

Both B and B' are open. We have

$$A \subset B \text{ and } A' \subset B' \tag{7}$$

$B \cap B' = \phi$.

● PROBLEM 10-32

1. Let (R, d) be the space of real numbers with the absolute value metric and let

$$A = \{1/n : n \in N\}. \tag{1}$$

Find the closure of A.

2. Show that for any subset A of X, the closure of A is a closed subset of X.

SOLUTION:

DEFINITION OF THE CLOSURE OF A

Let (X, d) be the metric space and $A \subset X$. The closure of A, denoted by $\overline{A}$ (sometimes by $C(A)$, is defined by

$$\overline{A} =: \{x \in X : \delta(x, A) = 0\}. \tag{2}$$

■

1. Since $1/n \to 0$, we have

$$\delta(0, A) = 0. \tag{3}$$

For each $x \in A$,

$$\delta(x, A) = 0 \tag{4}$$

Therefore

$$A \subset \overline{A}. \tag{5}$$

From (3) and (5) we obtain

$$A \cup \{0\} \subset \overline{A}. \tag{6}$$

Suppose $x \in \overline{A}$ and $x \notin A \cup \{0\}$. Then

$$\delta(x, A) > 0. \tag{7}$$

We conclude that

$$A \cup \{0\} = \overline{A}. \tag{8}$$

2. Suppose $\overline{A}$ is not closed. Then $X - \overline{A}$ is not open and there is an element $x \in X - \overline{A}$ such that for every $\varepsilon > 0$

$$B(x, \varepsilon) \cap A \neq \phi. \tag{9}$$

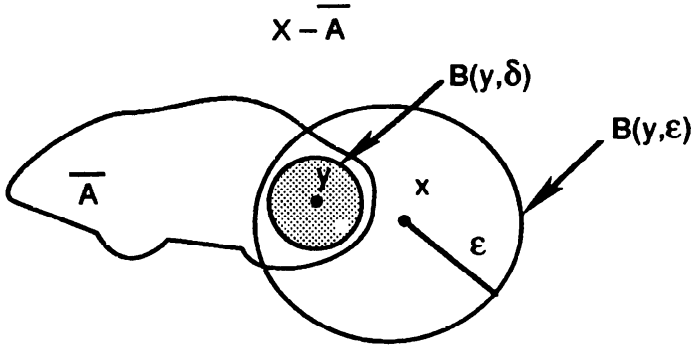


FIGURE 1

Choose $y \in \overline{A} \cap B(x, \varepsilon)$. (10)

Since $B(x, \varepsilon)$ is open, $\delta > 0$ exists such that

$$b(y, \delta) \text{ A } B(x, \varepsilon). \tag{11}$$

On the other hand $y \in \overline{A}$, then

$$\delta(y, A) = 0. \tag{12}$$

Therefore, a exists such that

$$a \in A \cap B(y, \delta) \subset B\,(\text{x}, \varepsilon). \tag{13}$$

Hence, for any $\varepsilon > 0$, there is $a \in A$ such that $\delta(x, a) < \varepsilon$

$$\text{greatest lower bound } \{ \delta(x, a) : a \in A\} = \delta(x, A) = 0. \tag{14}$$

Therefore $x \in \overline{A}$, but we assumed that $x \in X - \overline{A}$. Contradiction.

Prove:
THEOREM

The limit of a uniformly convergent sequence of bounded mappings is bounded.

■

SOLUTION:

DEFINITION OF UNIFORM CONVERGENCE

Let $f_n : X \to Y$, $n \in N$, where X is an arbitrary set and (Y, d) is a metric space. The sequence (f_n) is said to converge uniformly to f of

$$\forall \varepsilon > 0 \quad \exists k \in N \quad \forall x \in X \quad \forall n \geq k \quad d(f_n(x), f(x)) < \varepsilon. \tag{1}$$

■

Consider the space $\Phi(X, Y)$ of all bounded mappings $f : X \to Y$ with the metric defined by

$$d(f, g) =: \sup \{d(f(x), g(x)) : x \in X\}. \tag{2}$$

We have

$$\left(\lim_{n\to\infty} f_n = f\right) \equiv \left(\lim_{n\to\infty} d(f_n, f) = 0\right) \equiv$$
$$\equiv \forall \varepsilon > 0 \, \exists k \in N \, \forall n \geq k \sup\{d(f_n(x), f(x)\} < \varepsilon \equiv$$
$$\equiv \forall \varepsilon > 0 \, \exists k \in N \, \forall n \geq k \forall x \in X \, d(f_n(x), f(x)) < \varepsilon. \tag{3}$$

Thus, in the space $\Phi(X, Y)$ the condition $\lim f_n = f$ means that the sequence (f_n) converges uniformly to f.

Let $\varepsilon > 0$ and let $f_n \to f$, (f_n) be a uniformly convergent sequence of bounded mappings. We choose n such that

$$d(f_n, f) < \varepsilon. \tag{4}$$

We have

$$d(f(x_1), f(x_2)) \leq d(f(x_1), f_n(x_1)) + d(f_n(x_2), f(x_2)) +$$
$$+ d(f_n(x_1), f_n(x_2)). \tag{5}$$

Thus

$$\delta[f(X)] \leq \delta[f_n(X)] + 2\varepsilon \tag{6}$$

and f is bounded.

CHAPTER 11

TOPOLOGICAL SPACES

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1. Let X denote any set and $P(X)$ a family of all subsets of X. Show that $P(X)$ is a topology.

2. Find a topology on X, which contains as few as possible subsets of X.

SOLUTION:

DEFINITION OF TOPOLOGY

T is a topology on X, if and only if

1. $T \subset P(X)$

2. $\phi \in T$ and $X \in T$

3. If $A_1, A_2 \in T$, then $A_1 \cap A_2 \in T$, i.e., each finite intersection of members of T is also a member of T.

4. Each union of members of T is also a member of T.

■

DEFINITION OF TOPOLOGICAL SPACE

A couple (X, T) consisting of a set X and a topology T on X is called a topological space.

■

1. Since $P(X)$ is a family of all subsets of X, $P(X)$ is a topology. This topology is called the discrete topology on X. It contains the maximum possible number of sets.

2. Let X represent any set. The collection $T = \{\phi, X\}$ is a topology on X.
This topology is called the indiscrete topology (or trivial topology) on X. It contains the fewest possible number of sets.

Elements of topological spaces are called points. The members of T are called the open sets of the topological space (X, T).

Let $X = \{a, b, c, d\}$ and $T_0 = \{\{a\}, \{a, b\}, \{a, c, d\}\}$. Find a few possible topologies T on X, such that

$$T_0 \subset T. \tag{1}$$

SOLUTION:

Each union of members of T is also a number of T (see Problem 11–1). Hence

$$\{a, b\} \cup \{a, c, d\} = \{a, b, c, d\} = X. \tag{2}$$

One of the possible topologies on X is

$$T_1 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}\}$$

$$T_0 \subset T_1. \tag{3}$$

Suppose $\{b\} \in T_2$, then

$$T_2 = \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}\}. \tag{4}$$

T_2 is a topology and $T_0 \subset T_2$.

Suppose $\{c\} \in T_3$, then

$$T_3 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}, \{c\}, \{a, c\}, \{a, b, c\}\}. \tag{5}$$

Suppose $\{d\} \in T_4$ then

$$T_4 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}, \{d\}, \{a, d\}, \{a, b, d\}\}. \tag{6}$$

Suppose $\{b\} \in T_5$ and $\{c\} \in T_5$ then

$$T_5 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}, \{b\}, \{c\}, \{a, c\},$$

$$\{a, b, c\}, \{b\, c\}\}. \tag{7}$$

Suppose $\{b\} \in T_6$ and $\{d\} \in T_6$ then

$$T_6 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}, \{b\}, \{d\}, \{a, d\},$$

$$\{b, d\}, \{a, b, d\}\}. \tag{8}$$

Suppose $\{c\} \in T_7$ and $\{d\} \in T_7$ then

$$T_7 = \{\phi, X, \{a\}, \{a, b\}, \{a, c, d\}, \{c\}, \{d\}, \{a, c\},$$

$$\{a, d\}, \{a, b, c\}, \{a, b, d\}, \{c, d\}\}. \tag{9}$$

Now let $T_8 = P(X)$

$$T_8 = \{\phi, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\},$$

$$\{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}. \tag{10}$$

Suppose $\{a, c\} \in T_9$ then

$$T_9 = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{a, c, d\}, \{a, b, c\}\}. \tag{11}$$

● PROBLEM 11-3

Let R represent the set of real numbers. Subset $A \subset R$ is called open if for each $x \in A$, there is an $r > 0$, such that

$$B(x, r) =: \{ y : | y - x | < r \} \subset A \tag{1}$$

(R, d), where $d(x, y) = | x - y |$ is a metric space. Let T denote the family of all open sets (in the sense defined above). Verify that T is a topology on R.

SOLUTION:

Obviously ϕ and R are open sets.

$$\phi \in T, \quad R \in T. \tag{2}$$

Suppose $A_1, \ldots, A_n$ are open sets, then

$$\bigcap_1^n A_k \in T.$$

Indeed

$$\begin{aligned} &(x \in \bigcap_1^n A_k) \Rightarrow (\forall k,\ x \in A_k) \Rightarrow \\ &\Rightarrow (\forall k \exists r_k > 0 : B(x, r_k) \subset A_k) \Rightarrow \\ &\Rightarrow \begin{pmatrix} B(x,r) \subset \bigcap_1^n A_k \\ \text{where } r = \min\{r_1, r_2, \ldots, r_n\} \end{pmatrix} \end{aligned} \tag{3}$$

Suppose each member of $\{A_\omega : \omega \in \Omega\}$ is open, then $\bigcup_\omega A_\omega$ is also open.

$$\begin{aligned} &(x \in \bigcup_\omega A_\omega) \Rightarrow (\exists w : x \in A_\omega) \Rightarrow \\ &\Rightarrow (\exists r > 0 : B(x, r) \subset A_w \subset \bigcup_\omega A_\omega) \end{aligned} \tag{4}$$

Topology defined above is called the Euclidean topology on R'. The topological space (R', T) is called the Euclidean 1-space.

● PROBLEM 11-4

Let (X, d) denote a metric space. Consider the family T of all d-open subsets of X, i.e., subsets which are open in the sense of metric d.

Show that T is a topology on (X, d).

SOLUTION:

Let us recall the definition of an open set A. A is open iff

$$\forall x \in A \quad \exists B(x, r): B(x, r) \subset A. \tag{1}$$

An empty set and the whole space are open sets

$$\phi \in T, \quad X \in T \tag{2}$$

Suppose $A_1, \ldots, A_n$ are open sets. Then $\bigcap_1^n A_k$ is an open set.

$$\begin{aligned} &(x \in \bigcap_1^n A_k) \Rightarrow (\forall k : x \in A_k) \Rightarrow \\ &\Rightarrow (\forall k \exists r_k : B(x, r_k) \subset A_k) \Rightarrow \\ &\Rightarrow \left(\forall k : B(x,r) \subset A_k \text{ where } r = \min\{r_1, \ldots, r_n\}\right) \Rightarrow \\ &\Rightarrow (B(x,r) \subset \bigcap_1^n A_k). \end{aligned} \tag{3}$$

Now, let A_α denote a family of open sets, then $\bigcup_\alpha A_\alpha$ is an open set.

$$\begin{aligned} &(x \in \bigcup_\alpha A_\alpha) \Rightarrow (\exists \alpha : x \in A_\alpha) \Rightarrow \\ &\Rightarrow (\exists B(x,r) : B(x,r) \subset A_\alpha \subset \bigcup_\alpha A_\alpha. \end{aligned} \tag{4}$$

Topology T described here for the metric space (X, d) is called the metric topology induced on X by d. Each metric space (X, d) can become an "automatically" topological space (X, T). Different metrics on X will induce in general different topologies on X.

● PROBLEM 11-5

1. Define the topological space Z and the Sierpinski space.

2. Suppose T is a topology on X, consisting of

$$T = \{\phi, X, A, B\} \tag{1}$$

where A and B are non-empty distinct proper subsets of X. Determine conditions on A and B.

SOLUTION:

1. Let $X = \{0, 1\}$. Then the discrete topology on X, $T = P(X)$, is denoted by Q.

$$T = \{\phi, \{0, 1\}, \{0\}, \{1\}\} \tag{2}$$

The set $\{0, 1\}$ with topology

$$T = \{\phi, \{0, 1\}, \{0\}\} \tag{3}$$

is called the Sierpinski space.

2. Since T defined by (1) is a topology

$$A \cap B \in T = \{\phi, X, A, B\}. \tag{4}$$

We have either

$$A \cap B = \phi \tag{5}$$

or

$$A \cap B = A \tag{6}$$

or

$$A \cap B = B. \tag{7}$$

Suppose $A \cap B = \phi$. Then

$$A \cup B \neq A \quad \text{and} \quad A \cup B \neq B \quad \text{and} \quad A \cup B \neq \phi.$$

Hence

$$A \cup B = X \tag{8}$$

$\{A, B\}$ is a partition of X.

Suppose now $A \cap B = A$ then

$$\phi \subset A \subset B \subset X. \tag{9}$$

If $A \cap B = B$ then

$$\phi \subset B \subset A \subset X. \tag{10}$$

Let $f: X \to Y$ denote a function of $X \neq \phi$ onto a topological space (Y, T_0). Show that the family T of subsets of X, defined by

$$T = \{f^{-1}(B) : B \in T_0\} \tag{1}$$

is a topology on X.

SOLUTION:

T_0 is a topology on Y. Therefore

$$\phi, Y \in T_0 \tag{2}$$

Since f is onto

$$f^{-1}(Y) = X. \tag{3}$$

Also
$$f^{-1}(\phi) = \phi. \tag{4}$$

Thus
$$\phi, X \in T. \tag{5}$$

Let $\{A_\omega : \omega \in \Omega\}$ represent a family of sets in T. Then for each ω

$$f^{-1}(B_\omega) = A_\omega \quad B_\omega \in T_0 \tag{6}$$

We have

$$\bigcup_\omega A_\omega = \bigcup_\omega f^{-1}(B_\omega) = f^{-1}\left(\bigcup_\omega B_\omega\right) \tag{7}$$

Since T_0 is a topology

$$\bigcup_\omega B_\omega \in T_0 \tag{8}$$

therefore $\bigcup_\omega A_\omega \in T$.

Let $A_1, A_2 \in T$. Then sets $B_1, B_2, \in T_0$ exist, such that

$$f^{-1}(B_1) = A_1 \quad \text{and} \quad f^{-1}(B_2) = A_2 \tag{9}$$

Then

$$A_1 \cap A_2 = f^{-1}(B_1) \cap f^{-1}(B_2) = f^{-1}(B_1 \cap B_2). \tag{10}$$

Since T_0 is a topology and $B_1, B_2 \in T_0$, the intersection $B_1 \cap B_2 \in T_0$. Therefore

$$f^{-1}(B_1 \cap B_2) \in T \quad \text{and} \quad A_1 \cap A_2 \in T. \tag{11}$$

Thus (X, T) is a metric space.

● PROBLEM 11-7

Consider set N with the family T of its subsets consisting of ϕ and all sets of the form

$$A_k = \{k, k+1, k+2, \ldots\} \; k = 1, 2, 3, \ldots . \tag{1}$$

Show that T is a topology on N.

SOLUTION:

$$A_1 = \{1, 2, 3, \ldots\} \tag{2}$$

Thus

$$\phi \in T \quad \text{and} \quad N \in T \tag{3}$$

Suppose $A_k, A_l \in T$. Then either $k > l$ or $k < l$ or $k = l$. If $k > l$, then

$$A_k \cap A_l = A_k \in T \tag{4}$$

if $k < l$, then

$$A_k \cap A_l = A_l \in T. \tag{5}$$

Also if $k = l$, then $A_k \cap A_k = A_k \in T$. Thus $A_k \cap A_l = T$.

Let $\{A_i\}$ represent a family of sets of the form (1). Then

$$p = \min \{i_1, i_2, \ldots\} \tag{6}$$

and

$$\bigcup_i A_i = A_p = \{p, p+1, p+2, \ldots\}. \tag{7}$$

Thus T is a topology on N.

● PROBLEM 11-8

1. Show that intersection of any family of topologies on X is also a topology on X.

2. Show that the union of topologies does not have to be a topology.

SOLUTION:

1. Let $\{T_\omega : \omega \in \Omega\}$ be a family of topologies on X.

$$\forall \, \omega \in \Omega : \phi, X \in T_\omega$$

Hence $$\phi, x \in \bigcap_{\omega} T_{\omega}. \tag{1}$$

Suppose $A, B \in \bigcap_{\omega} T_{\omega}$.

$$(A, B \in \bigcap_{\omega} T_{\omega}) \Rightarrow (\forall\ \omega : A, B \in T_{\omega}) \Rightarrow$$

$$\Rightarrow (\forall\ \omega : A \cap B \in T_{\omega}) \Rightarrow (A \cap B \in \bigcap_{\omega} T_{\omega}). \tag{2}$$

Suppose $\{A_i\}$ is a family of sets such that

$$(\forall\ i : A_i \in \bigcap_{\omega} T_{\omega}) \Rightarrow (\forall\ i\ \forall\ \omega\ A_i \in T_{\omega}) \Rightarrow$$

$$\Rightarrow (\forall\ \omega : \bigcup_i A_i \in T_{\omega}) \Rightarrow (\bigcup_i A_i \in \bigcap_{\omega} T_{\omega}). \tag{3}$$

Hence if $\{T_{\omega} : \omega \in \Omega\}$ is a family of topologies on X then $\bigcap_{\omega} T_{\omega}$ is a topology on X.

2. Consider the set

$$X = \{a, b, c\} \tag{4}$$

and two topologies on X

$$T_1 = \{\phi, X, \{a\}\} \tag{5}$$

and

$$T_2 = \{\phi, X, \{b\}\} \tag{6}$$

Then

$$T_1 \cup T_2 = \{\phi, X, \{a\}, \{b\}\} \tag{7}$$

is not a topology because

$$\{a\} \cup \{b\} = \{a, b\} \notin T_1 \cup T_2. \tag{8}$$

● PROBLEM 11-9

How many distinct topologies can a set consisting of three elements have? What is their partial ordering?

SOLUTION:

Let

$$X = \{a, b, c\}. \tag{1}$$

The possible topologies on X are

$$T_1 = \{\phi, X\}$$

$$T_2 = \{\phi, X, \{a\}\}, \qquad T_3 = \{\phi, X, \{b\}\},$$

$$T_4 = \{\phi, X, \{c\}\}, \qquad T_5 = \{\phi, X, \{a, b\}\}$$

$$T_6 = \ldots \tag{2}$$

and so on.

Consider the set $P(X)$

$$P(X) = \{\phi, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}\} \tag{3}$$

1 topology $\{\phi, X\}$
3 topologies of the type $\{\phi, X, \{\cdot\}\}$
where $\{\cdot\}$ indicates the set consisting of one element
3 topologies of the type $\{\phi, X, \{\cdot, \cdot\}\}$
9 topologies of the type $\{\phi, X, \{\cdot\}, \{\cdot, \cdot\}\}$
3 topologies of the type $\{\phi, X, \{\cdot\}, \{\cdot, \cdot\}, \{\cdot, \cdot\}\}$
3 topologies of the type $\{\phi, X, \{\cdot\}, \{\cdot\}, \{\cdot, \cdot\}\}$
6 topologies of the type $\{\phi, X, \{\cdot\}, \{\cdot\}, \{\cdot, \cdot\}, \{\cdot, \cdot\}\}$
1 topology $P(X)$

The numbers must be added up in order to obtain the total number of topologies on $X = \{a, b, c\}$.

We can introduce the relation of inclusion among topologies. For example,

$$\{\phi, X, \{a\}\} \subset \{\phi, X, \{a\}, \{b\}, \{a, b\}\}. \tag{4}$$

This relation defines a partial order. Indeed, Let T_1, T_2, T_3 represent topologies on X, then

$$(T_1 \subset T_2) \wedge (T_2 \subset T_3) \Rightarrow (T_1 \subset T_3)$$

$$\forall T : T \subset T$$

$$(T_1 \subset T_2) \wedge (T_2 \subset T_1) \Rightarrow (T_1 = T_2). \tag{5}$$

● PROBLEM 11-10

Prove that T is the discrete topology on X if and only if every point is an open set.

SOLUTION:

Let $T = P(X)$, then T is called the discrete topology on X. Every set is an open set. We shall prove that

(T is the discrete topology) $\Leftrightarrow$ (every point is an open set). (1)

$\Rightarrow$ Since T consists of all subsets of X

$$\forall x \in X : \{x\} \in T. \tag{2}$$

$\Leftarrow$ Suppose T is a topology on X, such that

$$\forall x \in X : \{x\} \in T. \tag{3}$$

Let A denote any subset of X. Then

$$A = \bigcup_{a \in A} \{a\}. \tag{4}$$

But all $\{a\}$ are open sets, $\{a\} \in T$ and since T is a topology, the union of open sets is an open set.

Hence, $A \in T$, we conclude, that $T = P(X)$.

● PROBLEM 11-11

Let X represent any set and S a family of its subsets, such that

1. $X, \phi \in S$.
2. The union of any two members of S is a member of S.
3. The intersection of any family of members of S is a member of S.

Let T denote a family of subsets of X, such that

$$A \in T \text{ iff } X - A \in S. \tag{1}$$

Show that T is a topology on X.

SOLUTION:

Since $X, \phi \in S$ and $X - X = \phi, X - \phi = X$ $X, \phi \in T$.
Suppose $A_1, A_2 \in T$, then $X - A_1 \in S$ and $X - A_2 \in S$

$$(X - A_1) \cup (X - A_2) = X - (A_1 \cap A_2) \in S. \tag{2}$$

Therefore $A_1 \cap A_2 \in T$.

Suppose $\{A_\alpha\}$ is a family of subsets of T. Then $\{X - A_\alpha\}$ is a family of subsets of S and

$$\bigcap_\alpha (X - A_\alpha) \in S. \tag{3}$$

But

$$\bigcap_\alpha (X - A_\alpha) = X - \bigcup_\alpha A_\alpha. \tag{4}$$

Hence $X - \bigcup_\alpha A_\alpha \in S$ and $\bigcup_\alpha A_\alpha \in T$.

Thus T is a topology on X. Elements of T are called open sets.

DEFINITION OF A CLOSED SET

Subset B of (X, T) is closed, if $X - B$ is an open set, that is, if $X - B \in T$.

● PROBLEM 11-12

Give an example of a basis for the Euclidean topological space (R^n, T).

SOLUTION:

DEFINITION OF A BASIS

Let (X, T) denote a topological space. A family $B \subset T$ is called a basis for T if each open set (i.e., an element of T) is the union of members of B.

■

Note that if T is a topology on X, then T is a basis for T. In (R^n, T), the topology is the family of sets open in the sense of the Euclidean metric.

We shall show that in R^n

$$B = \{B(x, r) : x \in R^n, r > 0\} \tag{1}$$

is a basis for the Euclidean topology.

Let $A \in T$ denote an open subset of R^n. Then

$$A = \bigcup_{x \in A} B(x, r). \tag{2}$$

Each open set can be represented as a union of balls. Note that (1) is not the only possible basis for the Euclidean topology in R^n.

● PROBLEM 11-13

Let $B \subset T$, where T is a topology on X. Show that the following properties of B are equivalent:

1. B is a basis for T.

2. For each $A \in T$ and each $x \in A$, there is $B_\omega \in B$, such that
$$x \in B_\omega \subset A.$$

SOLUTION:

(1) $\Rightarrow$ (2) Let $A \in T$. Since B is a basis for T

$$A = \bigcup_\omega B_\omega \tag{1}$$

where $B = \{B_\omega : \omega \in \Omega\}$. $x \in A$, therefore at least one B_ω exists, such that

$$x \in B_\omega \subset A. \tag{2}$$

(1) $\Leftarrow$ (2) Suppose $A \in T$. Since for each $x \in A$, set B_ω exists, such that

$$B_\omega \in B \tag{3}$$

and

$$x \in B_\omega \subset A \tag{4}$$

we have

$$A = \bigcup_{x \in A} B_\omega, \text{ where } x \in B_\omega. \tag{5}$$

Thus B is a basis for T.

The following theorem is useful in describing open sets.

THEOREM

Let $B \subset T$ represent a basis for T. Then set A is open, i.e., $A \in T$, if and only if, for each $x \in A$, there is a $B_\omega \in B$, such that $x \in B_\omega \subset A$.

● PROBLEM 11-14

Show that R^n with the Euclidean topology has a countable basis.

SOLUTION:

Consider the family of all open balls $B_\omega(x, r)$, such that r is a rational number and all coordinates of $x = (x_1, x_2, \ldots, x_n)$ are rational numbers. This family forms a countable set.

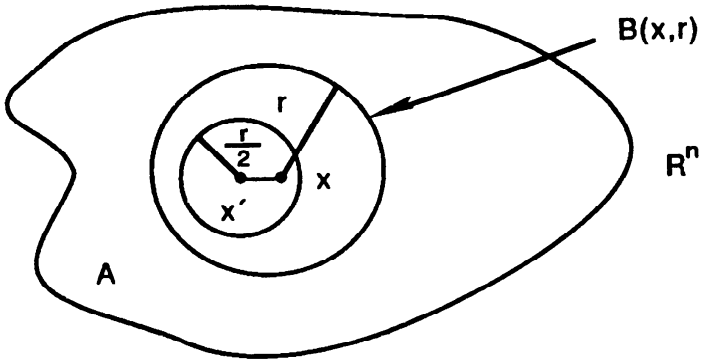


FIGURE 1

Let A denote any open subset of R^n and let $x \in A$. There is a ball $B(x, r)$, such that

$$x \in B(x, r). \tag{1}$$

We can assume that $r > 0$ is a rational number. We can always find a point x', such that

$$x' = (x'_1, x'_2, \ldots, x'_n)$$

where all coordinates of x' are rational and

$$d(x, x') < r/3. \tag{2}$$

Then

$$x \in B(x', r/2) \subset B(x, r). \tag{3}$$

Thus

$$B = \{B_\omega(x, r) : \text{all coordinates of } x \text{ are rational,}$$
$$\text{and } r > 0 \text{ is rational}\} \tag{4}$$

is a countable basis of R^n.

Each open set in R^n is the union of, at the most, countably many balls.

If T is the Euclidean topology of R^n, then

$$\text{card } T = 2^{\text{card } N}. \tag{5}$$

● PROBLEM 11-15

Let B represent a basis for topology T and let B' represent a family of open sets, such that

$$B \subset B' \subset T. \tag{1}$$

Show that B' is also a basis for T.

SOLUTION:

Suppose $A \in T$ is an open subset of X.
Since B is a base for T

$$A = \bigcup_{\omega} B_{\omega} \tag{2}$$

where $B_{\omega} \in B$. Since $B \subset B'$, each $B_{\omega} \in B$ also belongs to B', $B_{\omega} \in B'$.
Therefore

$$A = \bigcup_{\omega} B_{\omega} \text{ where } B_{\omega} \in B'. \tag{3}$$

Hence B' is a base for T.

● PROBLEM 11-16

A set X and a family of all open subsets of X — so called topology T — are given. This pair (X, T) is the topological space. Sometimes we are given the basis B for topology T, then taking all unions of elements of B, we find T.

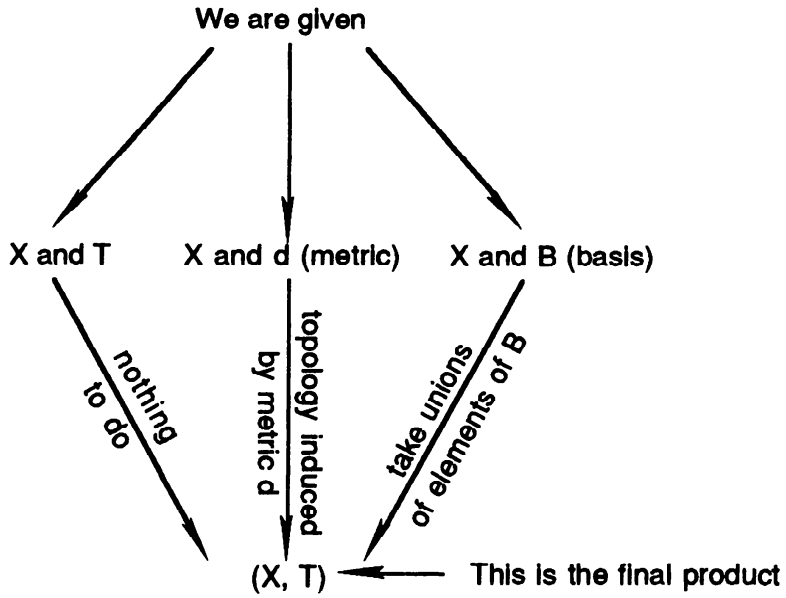


FIGURE 1

Consider this situation. A set X and a family B of its subsets are given. The question is: is this family a basis for any topology T on X? Prove the

following theorem which answers this question:

THEOREM

Let B denote a family of subsets of X, such that

1. X is the union of members of B.

2. The intersection of any two members of B is the union of members of B.

Then B is a basis for topology T on X defined by

$$T = \{A \subset X : A \text{ is the union of members of } B\}. \tag{1}$$

SOLUTION:

1. $X \in T$ because X is the union of members of B. Also $\phi \in T$, as the union of the empty subfamily of B.

2. Let $\{A_\alpha\}$ denote a family of members of T, i.e., for each α, $A_\alpha \in T$. But each A_α is the union of members of B

$$A_\alpha = \bigcup_{\beta(\alpha)} B_{\beta(\alpha)} \tag{2}$$

where $B_{\beta(\alpha)} \in B$. Then

$$\bigcup_\alpha A_\alpha = \bigcup_\alpha \bigcup_{\beta(\alpha)} B_{\beta(\alpha)} \in T. \tag{3}$$

3. Suppose A_1 and $A_2 \in T$. Then

$$A_1 = \bigcup_\alpha B_\alpha \tag{4}$$

$$A_2 = \bigcup_\beta B_\beta \tag{5}$$

where $B_\alpha, B_\beta \in B$ and

$$A_1 \cap A_2 = \bigcup_{\alpha,\beta} (B_\alpha \cap B_\beta). \tag{6}$$

The intersection of any two members of B is the union of members of B. Hence, for each α and β

$$B_\alpha \cap B_\beta = \bigcup_\gamma B_\gamma \tag{7}$$

and

$$A_1 \cap A_2 = \bigcup_{\alpha,\beta} \bigcup_\gamma B_{\gamma(\alpha,\beta)} \in T \tag{8}$$

where each γ depends on α and β. Therefore, T defined by (1) is a topology and B described in the theorem is a basis for T.

● PROBLEM 11-17

Let R represent the set of real numbers.

1. Show that the collection of open intervals is the basis for a topology T on R.

2. (R, d) where $d(x, y) = |x - y|$ is a metric space and hence a topological space (R, T') with topology induced by the metric.

Show that both topologies T and T' are the same.

SOLUTION:

1. R is the union of open intervals. Any intersection of two open intervals is either empty or again an open interval. By virtue of theorem of Problem 11–16, the collection of open intervals in R forms a basis for a topology T on R.

2. Let

$$T = \{A : A \text{ is the union of open intervals}\} \tag{1}$$

$$T' = \{B : B \text{ is open in sense of the absolute value metric}\} \tag{2}$$

We will show that

$$(A \in T) \Rightarrow (A \in T'). \tag{3}$$

If $A \in T$, then A is the union of open interals. Each open interval is a set which is open in the sense of the absolute value metric. A union of open sets is again an open set. Hence, $A \in T'$

$$(A \in T') \Rightarrow (A \in T). \tag{4}$$

Suppose A is open in the sense of the absolute value metric. Then

$$\forall\, x \in A \quad \exists\, B(x, r) : B(x, r) \subset A \tag{5}$$

But

$$B(x, r) = (x - r, x + r). \tag{6}$$

Hence

$$A = \bigcup_{x \in A} B(x, r) = \bigcup_{x \in A} (x - r, x + r). \tag{7}$$

A is the union of open intervals, $A \in T$. We conclude

$$T = T'. \tag{8}$$

Let A represent a family of subsets of X, $A \subset P(X)$. In general, there are many topologies T on X which contain A, for example $A \subset T = P(X)$. We will show that among these topologies a unique, smallest topology $T(A) \supset A$ exists.

THEOREM

Let $A = \{A_\alpha\}$ be a family of subsets of X. Then $T(A)$ is a unique, smallest topology containing A, when

$$T(A) = \{\phi, X, \text{all finite intersections of } A_\alpha, \text{ all unions of finite intersections of } A_\alpha\} \quad (1)$$

$T(A)$ is said to be generated by A, and A is a subbasis for $T(A)$.

■

Prove this theorem.

SOLUTION:

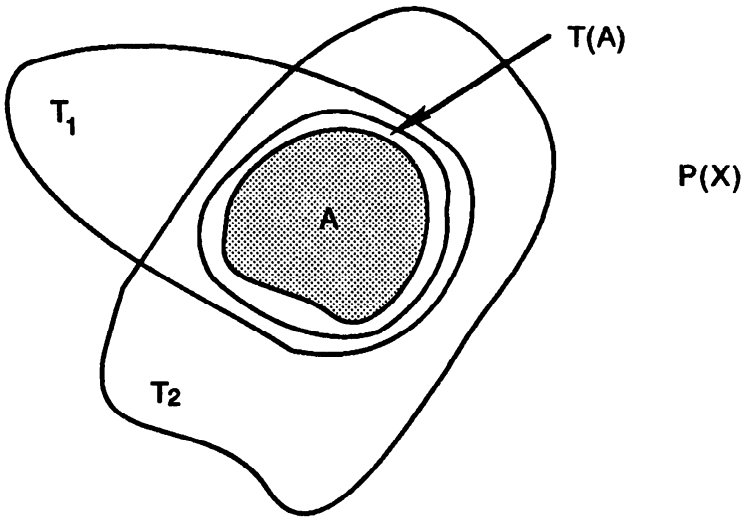


FIGURE 1

Let $T(A)$ be the intersection of all topologies containing A ($P(X)$ is one of such topologies). By Problem 11–8 $T(A)$ is a topology. Since it is the intersection of all topologies containing A, $T(A)$ is the smallest topology, $A \subset T(A)$. By definition $T(A)$ is unique.

We will prove (1). Since $A \subset T(A)$ and $T(A)$ is a topology, $T(A)$ must contain all the sets listed in (1).

On the other hand, the sets listed in (1) form a topology containing A and, therefore, containing $T(A)$.

● PROBLEM 11-19

Let R denote the set of all real numbers and let A denote all the sets of the form

$$\{x : x > a\} \tag{1}$$

$$\{x : x < b\} \tag{2}$$

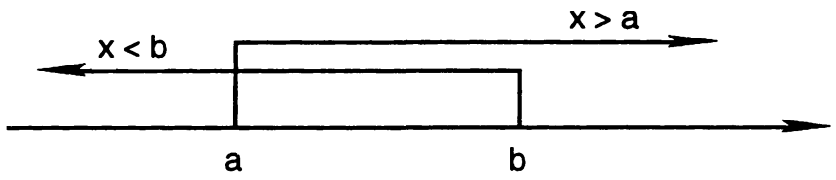


FIGURE 1

Find $T(A)$.

SOLUTION:

By the theorem of Problem 11–18, $T(A)$ consists of

$$T(A) = \{\phi, R, \text{all finite intersections of members of } A,$$

$$\text{all arbitrary unions of finite intersections of members of } A\} \tag{3}$$

All finite intersections of members of A are open intervals (a, b). Hence, we conclude that the

$$\text{set of all open intervals} \subset T(A). \tag{4}$$

The family of all open intervals forms a basis for $T(A)$.

By Problem 11–17 the family of all open intervals forms a basis for the Euclidean topology.

Therefore, $T(A)$ is the Euclidean topology.

● PROBLEM 11-20

R is the set of all real numbers and A consists of all sets of the form

$$\{x : x > a\}$$

$$\{x : x \leq b\} \tag{1}$$

Find $T(A)$ (called the upper limit topology).

SOLUTION:

To find $T(A)$, we must first obtain all finite intersections of elements of A (see Problem 11–18). The sets $(a, b]$

$$(a, b] = \{x : a < x \leq b\} \tag{2}$$

form a basis for topology $T(A)$, which is not Euclidean because $(a, b]$ does not belong to the Euclidean topology.

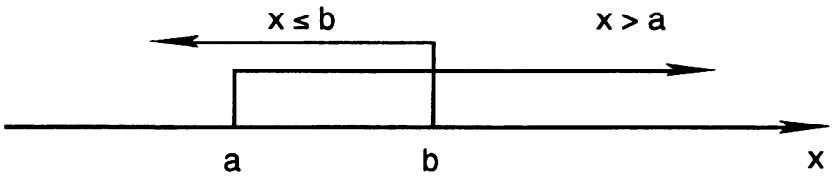


FIGURE 1

Note that we were dealing with the situation, when a family of sets A was given and we had to find topology $T(A)$, such that $A \subset T(A)$. Conversely, for a given topology T, a family of sets $A \subset T$ is called a subbasis for T, if

$$T = T(A).$$

● PROBLEM 11-21

Let $X = \{a, b, c, d, e\}$ and let $A \subset P(X)$ where

$$A = \{\{a, b, c\}, \{a, c, e\}, \{c, d\}\}. \tag{1}$$

Find the topology $T(A)$ on X generated by A.

SOLUTION:

By taking all finite intersections of sets in A and the sets ϕ and X, we find the basis $B(A)$

$$B(A) = \{\phi, X, \{a, b, c\}, \{a, c, e\}, \{c, d\}, \{a, c\}, \{c\}\}. \tag{2}$$

Note that $X \in B$ and $\phi \in B$.

X can be considered the empty intersection of members of A. In order to obtain the topology generated by A, we take all unions of members of $B(A)$.

$$T(A) = \{\phi, X, \{a, b, c\}, \{a, c, e\}, \{c, d\}, \{a, c\}, \{c\}, \{a, b, c, e\},$$
$$\{a, b, c, d\}, \{a, c, e, d\}, \{a, c, d\}\}. \tag{3}$$

● PROBLEM 11-22

Show that a basis determines one and only one topology.

SOLUTION:

Let X represent any set and B a collection of subsets of X. Suppose B is a basis for two topologies T and T', which are different.

Hence, at least one element $A \subset X$ exists, such that $A \in T$ but $A \notin T'$ (or vice versa).

B is a basis for topology T and $A \in T$, therefore

$$A = \bigcup_{\alpha} B_{\alpha} \in T \tag{1}$$

where $B_{\alpha} \in B$ for each α.

Since B is a basis for topology T', any union of elements of B must be an element of T'. Thus,

$$A = \bigcup_{\alpha} B_{\alpha} \in T', \tag{2}$$

which is a contradiction. We conclude that

$$T = T'.$$

A basis as well as a subbasis determines a unique topology.

● PROBLEM 11-23

The following theorem offers a simple method of determining if a given family of sets is a basis.

THEOREM

Let $B = \{B_{\omega} : \omega \in \Omega\}$ denote a family of subsets of X, such that for each $x \in B_{\alpha} \cap B_{\beta}$ and each $\alpha, \beta \in \Omega$ $B_{\gamma} \in B$ exists, such that

$$x \in B_{\gamma} \subset B_{\alpha} \cap B_{\beta} \tag{1}$$

Then $B \cup \{\phi\} \cup \{X\}$ is a basis for some topology $T(B)$ on X. $T(B)$ is unique and is the smallest topology containing B.

■

Prove this theorem.

SOLUTION:

By using B as a subbasis, we obtain a topology $T(B)$. We will show that

B is a basis for topology $T(B)$.

$$T(B) = \{\phi, X, \text{all unions of members of } B\} \tag{2}$$

It is enough to prove that each finite intersection of members of B is a union of members of B. We will prove that for each $x \in B_1 \cap \ldots \cap B_n$, there is a $B_0 \in B$, such that

$$x \in B_0 \subset B_1 \cap \ldots \cap B_n, \tag{3}$$

compare Problem 11–13.

For $n = 2$, (3) is true by the hypothesis. By induction, suppose (3) is true for $n - 1$. Then for n

$$x \in B_1 \cap B_2 \cap \ldots \cap B_{n-1} \cap B_n \tag{4}$$

B_0 exists, such that

$$x \in B_0 \cap B_n \tag{5}$$

$$B_0 \subset B_1 \cap \ldots \cap B_{n-1}. \tag{6}$$

Hence

$$x \in B'_0 \subset B_0 \cap B_n \subset B_1 \cap \ldots B_n \tag{7}$$

for some $B'_0 \in B$.

Thus B is a basis for topology $T(B)$ defined by (2).

● PROBLEM 11-24

Let $C([0, 1])$ represent the set of all continuous functions on $[0, 1]$. For each $f \in C$ and $r > 0$, we define

$$(f,r) = \{g \in C : \int_0^1 |f - g| < r\}. \tag{1}$$

Show that the family

$$\{K(f, r) : f \in C, r > 0\} \tag{2}$$

forms a basis for some topology T on C.

SOLUTION:

We shall apply the theorem of Problem 11–23. Suppose

$$h \in K(f_1, r_1) \cap K(f_2, r_2). \tag{3}$$

Let

$$t_1 = \int_0^1 |f_1 - h|, t_2 = \int_0^1 |f_2 - h| \tag{4}$$

and let

$$R = \min[r_1 - t_1, r_2 - t_2]. \tag{5}$$

Then $R > 0$ and

$$K(h, R) \text{ A } K(f_1, r_1) \cap K(f_2, r_2). \tag{6}$$

Indeed, suppose $F \in K(h, R)$ then

$$\int_0^1 |F - h| < R \tag{7}$$

and

$$\int_0^1 |f_1 - F| \leq \int_0^1 |f_1 - h| + \int_0^1 |h - F| < t_1 + (r_1 - t_1) = r_1. \tag{8}$$

Therefore

$$F \in K(f_1, r_1) \quad \text{and} \quad F \in K(f_2, r_2) \tag{9}$$

because

$$\int_0^1 |f_2 - F| \leq \int_0^1 |f_2 - h| + \int_0^1 |h - F| < t_2 + (r_2 - t_2) = r_2. \tag{10}$$

● PROBLEM 11-25

Each basis in X yields a unique topology. But distinct bases may yield the same topology.

DEFINITION OF EQUIVALENT BASES

Two bases B and B' in X are equivalent if

$$T(B) = T(B') \tag{1}$$

■

Prove this:

THEOREM

Two bases B and B' in X are equivalent if and only if

1. for each $B_\alpha \in B$ and each $x \in B_\alpha$, there is a $B'_\beta \in B'$, such that

$$x \in B'_\beta \subset B_\alpha$$

and

2. for each $B'_\beta \in B'$ and each $x \in B'_\beta$, there is a $B_\alpha \in B$, such that

$$x \in B_\alpha \subset B'_\beta$$

SOLUTION:

Suppose $$T(B) = T(B'). \tag{2}$$

Then for each $B_\alpha \in B \subset T(B)$ and each $x \in B_\alpha$, since B' is a basis for $T(B)$, by Problem 11–13, there is $B'_\beta \in B'$, such that

$$x \in B'_\beta \subset B_\alpha. \tag{3}$$

Similarly we show that condition 2. holds.

Now suppose condition 1 holds. Since each $A \in T(B)$ is a union of $\{B_\alpha\}$ belonging to B, *then* it follows that $A \in T(B')$. yielding $T(B) \subset T(B')$. Similarly we have $T(B') \subset T(B)$ and therefore $T(B) = T(B')$.

● PROBLEM 11-26

Let S represent a subbase for a topology T on X and let A denote any subset of X. Show that the family

$$S_A = \{A \cap S_\alpha : S_\alpha \in S\} \tag{1}$$

is a subbase for the relative topology T_A on A.

SOLUTION:

Let D denote an open subset of A with respect to topology T_A. Then

$$D = A \cap E \tag{2}$$

where E is a T-open subset of X. Family S is a subbase for T, therefore

$$E = \bigcup (S_{n_1} \cap S_{n_2} \cap \ldots \cap S_{n_k}) \tag{3}$$

where $S_{n_1}, S_{n_2}, \ldots, S_{n_k} \in S$.

Therefore

$$D = A \cap E = A \cap [\bigcup (S_{n_1} \cap \ldots \cap S_{n_k})] =$$

$$= \bigcup [(A \cap S_{n_1}) \cap \ldots \cap (A \cap S_{n_k})]. \tag{4}$$

Thus D is the union of finite intersections of elements of S_A, and S_A is a subbase for T_A.

1. Let $A =]0, 1]$ and $B = \{1/n : n \in N\}$ be subsets of R. Find the closure of A and B.

2. Find the closure of 0 and 1 in Sierpinski space, where $X = \{0, 1\}$ and $T = \{\phi, X, 0\}$.

SOLUTION:

DEFINITION OF A NEIGHBORHOOD

Let (X, T) denote a topological space. A neighborhood of an $x \in X$ is any open set N_0 containing x, $x \in N_0$. A neighborhood of x is denoted by $N_0(x)$.

■

DEFINITION OF ADHERENT POINT

Let $A \subset X$. A point $x \in X$ is adherent to A, if for each $N_0(x)$

$$N_0(x) \cap A \neq \phi. \tag{1}$$

■

DEFINITION OF THE CLOSURE OF THE SET

The closure of A, denoted by $\overline{A}$, is the set of all points in X adherent to A.

$$\overline{A} = \{x \in X : \forall N_0(x) : N_0(x) \cap A \neq \phi\}. \tag{2}$$

■

1. The closure of $A =]0, 1]$ is $\overline{A} = [0, 1]$. The closure of $B = \{1/n : n \in N\}$ is

$$\overline{B} = \{0, 1/n\} = \{0\} \cup B \tag{3}$$

2. The open sets are ϕ, X and 0. We have

$$\overline{0} = \{0, 1\} = X$$

$$\overline{1} = 1 \tag{4}$$

● PROBLEM 11-28

Show that:

1. for every set A, $A \subset \overline{A}$.

2. A is closed iff $A = \overline{A}$.

SOLUTION:

DEFINITION

A set $A \subset X$ is called closed if $X - A$ is an open set.

■

1. Obviously, if $x \in A$, then

$$N_0(x) \cap A \neq \phi. \tag{1}$$

Hence
$$x \in \overline{A} \qquad A \subset \overline{A}. \tag{2}$$

2. (A closed) $\Rightarrow (A = \overline{A})$.

If A is closed, then $X - A$ is open. Each $x \notin A$ has a neighborhood $N_0(x)$, such that

$$N_0(x) \cap A \neq \phi. \tag{3}$$

Hence
$$x \notin \overline{A} \text{ and } \overline{A} \subset A. \tag{4}$$

From (4) and (2), we obtain $A = \overline{A}$.

$(A = \overline{A}) \Rightarrow (A$ closed).

Since $A = \overline{A}$, each $x \notin A$ has a $N_0(x)$, such that

$$N_0(x) \cap A \neq \phi. \tag{5}$$

Hence, $X - A$ is open and A is closed.

● PROBLEM 11-29

Show that this is an alternative definition of a closed set:

$\overline{A}$ is the smallest closed set containing A; i.e.,

$$\overline{A} = \cap \{D : (D \text{ is closed}) \wedge (A \subset D)\}. \tag{1}$$

SOLUTION:

$$\overline{A} \subset \bigcap D. \tag{2}$$

Suppose $x \in \overline{A}$ and $x \notin \bigcap D$. Since each set D is closed, $\bigcap D$ is a closed set

$(x \notin \bigcap D) \Rightarrow x \in X - \bigcap D$, where $X - \bigcap D$ is an open set $\Rightarrow$

$\Rightarrow x$ has a nbd G, such that $G \cap \bigcap D = \phi \Rightarrow$

$\Rightarrow G \cap A = \phi$ (because $A \subset \bigcap D$) $\Rightarrow x \notin \overline{A}$. (3)

Now we will prove that $\bigcap D \subset \overline{A}$. Suppose $x \notin \overline{A}$, then there is a neighborhood of x, $N_0(x)$, such that

$$N_0(x) \cap A \neq \phi. \tag{4}$$

Therefore, $X - N_0(x)$ is closed and contains A. Thus $X - N_0(x)$ is one of the sets (denoted by D) in (1).

Since

$$x \notin X - N_0(x) \quad \text{also} \quad x \notin \bigcap D. \tag{5}$$

See Figure 1.

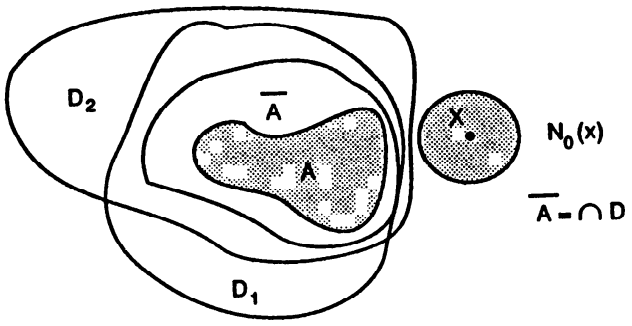


FIGURE 1

● PROBLEM 11-30

Prove that:

1. $\overline{\overline{A}} = \overline{A}$.
2. $A \subset B \Rightarrow \overline{A} \subset \overline{B}$.
3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
4. $\overline{\phi} = \phi$.

SOLUTION:

1. $\overline{A} = \bigcap D$ (where D is closed). Since the intersection of any family of closed sets is a closed set, $\overline{A}$ is a closed set. A is closed if and only if $A = \overline{A}$. Hence

$$\overline{\overline{A}} = \overline{A}. \tag{1}$$

2. $\overline{A} = \bigcap D$ and $\overline{B} = \bigcap D'$, where each D contains A and each D' contains B. Since $A \subset B$ each D' contains B and contains A. Hence,

$$\overline{A} \subset \overline{B}. \tag{2}$$

3. Set $\overline{A \cup B}$ is closed. Since for each A, $A \subset \overline{A}$, we obtain

$$\overline{A} \cup \overline{B} \subset \overline{A \cup B}. \tag{3}$$

On the other hand, set $\overline{A} \cup \overline{B}$ is closed and

$$A \cup B \subset \overline{A} \cup \overline{B}. \tag{4}$$

Hence

$$\overline{A \cup B} \subset \overline{A} \cup \overline{B}. \tag{5}$$

From (3) and (5) we get

$$\overline{A \cup B} = \overline{A} \cup \overline{B}. \tag{6}$$

4. Set ϕ is a closed set, hence

$$\overline{\phi} = \phi. \tag{7}$$

● PROBLEM 11-31

1. Find the derived set of

$A = \{1/n : n \in N\}$

$B =]0, 1]$

$C = \{x : x \in (0,1), x \text{ is a rational number}\}$.

2. Let X denote an indiscrete topological space. Find the derived set A' of any subset $A \subset X$.

SOLUTION:

DEFINITION OF CLUSTER POINT

Let $A \subset X$. A point $x \in X$ is called a cluster point of A if each neighborhood of x, N_0 of x contains at least one point of A distinct from x.

■

It is easy to see that 0 is the only cluster point of $\{1/n\}$.

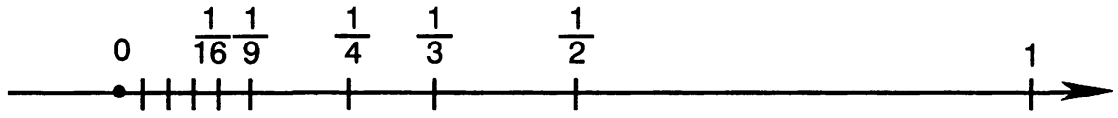


FIGURE 1

Indeed each neighborhood of 0 contains at least one (as a matter of fact, infinitely many) point of $A = \{1/n\}$ distinct from 0.

DEFINITION OF A DERIVED SET

The set

$$A' = \{x \in X : \forall\, 0(x) : 0(x) \cap (A - x) \neq \phi\} \tag{1}$$

of all cluster points of A is called the derived set of A.

■

1. We obtain

$A' = \{0\}$

$B' = [0, 1]$

$C' = [0, 1]$.

2. We have $T = (X, \phi)$. For any point $x \in X$, X is the only open set containing x. We have

$$A' = \begin{cases} \phi & \text{if } A = \phi \\ X & \text{if } A \text{ contains two or more points} \\ X - \{a\} & \text{if } A = \{a\} \end{cases}$$

● PROBLEM 11-32

Show that

$$\overline{A} = A \cup A'. \tag{1}$$

Set A is closed if and only if $A' \subset A$.

SOLUTION:

Suppose $x \in A'$ then $x \in \overline{A}$. Hence $A' \subset \overline{A}$. Also $A \subset \overline{A}$, therefore

$$A' \cup A \subset \overline{A}. \tag{2}$$

To prove the converse inclusion suppose $x \in A$. If $x \in \overline{A}$, then the proof is finished.

If $x \notin A$, then each neighborhood of x intersects A at a point distinct from x, hence $x \in A'$. Thus

$$\overline{A} \subset A \cup A'. \tag{3}$$

We conclude that

$$\overline{A} = A \cup A'. \tag{4}$$

Set A is closed if and only if $A = \overline{A}$.

Thus, A is closed if and only if

$$A' \subset A \tag{5}$$

i.e., if and only if A contains all of its cluster points.

● PROBLEM 11-33

Prove that

$$(A \cup B)' = A' \cup B'. \tag{1}$$

SOLUTION:

Observe that if $A \subset B$, then $A' \subset B'$. Since $A \subset A \cup B$ and $B \subset A \cup B$, we obtain

$$A' \subset (A \cup B)' \quad B' \subset (A \cup B)' \tag{2}$$

Thus

$$A' \cup B' \subset (A \cup B)'. \tag{3}$$

Now we shall prove the inverse inclusion, $(A \cup B)' \subset A' \cup B'$. Suppose, $a \notin A' \cup B'$, then open sets 0_1 and 0_2 exist, such that

$$a \in 0_1 \quad \text{and} \quad a \in 0_2 \tag{4}$$

$$0_1 \cap A \subset \{a\}, \quad 0_2 \cap B \subset \{a\}. \tag{5}$$

Since sets 0_1 and 0_2 are open, $0_1 \cap 0_2$ is open and $a \in 0_1 \cap 0_2$. We obtain

$$(0_1 \cap 0_2) \cap (A \cup B) = (0_1 \cap 0_2 \cap A) \cup (0_1 \cap 0_2 \cap B)$$

$$\subset (0_1 \cap A) \cup (0_2 \cap B) \subset \{a\} \cup \{a\} = \{a\}. \tag{6}$$

Hence

$$a \notin (A \cup B)' \quad \text{and} \quad (A \cup B)' \subset A' \cup B'. \tag{7}$$

● PROBLEM 11-34

1. Find the interior of the sets

   $A = \{1/n : n \in N\}$

   $B = [0, 1]$

2. Prove that

$$Int(A) = X - \overline{(X - A)} \tag{1}$$

SOLUTION:

DEFINITION OF THE INTERIOR

The interior $Int(A)$ of $A \subset X$ is the largest open set contained in A, that is

$$Int(A) = \bigcup \{0 : (0 \text{ open}) \wedge (0 \subset A)\}. \tag{2}$$

■

1. $Int(A) = \phi$.

   $Int(B) = (0, 1)$

2. We shall apply

$$A \subset B \Leftrightarrow X - B \subset X - A. \tag{3}$$

Therefore, if $0 \subset A$, then

$$X - A \subset X - 0 = D. \tag{4}$$

Since 0 is an open set, D is a closed set. We obtain

$$Int(A) = \bigcup \{X - D : (D \text{ closed}) \wedge (X - A \subset D)\} =$$

$$= X - \bigcap \{D : (D \text{ closed}) \wedge (X - A \subset D)\} = X - \overline{(X - A)}. \tag{5}$$

● PROBLEM 11-35

Find the boundary of the sets

$A = \{1/n : n \in N\}$

$B = (0, 1]$

Show that

$$\mathcal{F}r(A) = \overline{A} - Int(A). \tag{1}$$

SOLUTION:

DEFINITION OF BOUNDARY

The boundary of a set $A \subset X$ is denoted by $\mathcal{F}r(A)$ and defined by

$$\mathcal{F}r(A) = \overline{A} \cap \overline{(X - A)}. \tag{2}$$

■

We have

$$\overline{A} = A \cup \{0\} \quad \text{and} \quad \overline{(X - A)} = R.$$

Thus

$$\mathcal{F}r(A) = A \cup \{0\} \quad \overline{B} = [0,1]$$

$$\overline{(X - B)} = (-\infty, 0] \cup [1, \infty) \tag{3}$$

Hence

$$\mathcal{F}r(B) = \overline{B} \cap \overline{(X - B)} = \{0, 1\}. \tag{4}$$

We have

$$\mathcal{F}r(A) = \overline{A} \cap \overline{(X - A)}. \tag{5}$$

But

$$Int(A) = X - \overline{(X - A)}. \tag{6}$$

Also note that

$$X - (X - B) = B \tag{7}$$

Hence

$$\mathcal{F}r(A) = \overline{A} \cap X - [X - \overline{(X - A)}] =$$

$$= \overline{A} \cap [X - Int(A)] = \overline{A} - Int(A) \tag{8}$$

● PROBLEM 11-36

Show that the following statements are equivalent:

1. D is dense in X.
2. If G is a closed set and $D \subset G$, then $G = X$.
3. Each nonempty basic open set in X contains an element of D.
4. The complement of D has empty interior.

SOLUTION:

DEFINITION OF A DENSE SET

Set $D \subset X$ is dense in X if $\overline{D} = X$.

■

1. $\Rightarrow$ 2.

Set D is dense in X, $\overline{D} = X$. $D \subset G$, then $\overline{D} \subset \overline{G}$, but $\overline{G} = G$, hence $X \subset G$ and

$$X = G. \tag{1}$$

2. $\Rightarrow$ 3.

Let $0 \neq \phi$ be open and $0 \cap D = \phi$, then

$$D \subset X - 0 \neq X \tag{2}$$

in contradiction with 2 because $X - 0$ is closed.

3. $\Rightarrow$ 4.

Suppose $Int(X - D) = \phi$. Set $Int(X - D)$ is open, hence, there is a nonempty basic set

$$A \subset Int(X - D). \tag{3}$$

But

$$Int(X - D) \subset X - D. \tag{4}$$

Then $A \subset X - D$ and A contains no points of D.

4. $\Rightarrow$ 1.

$$Int(X - D) = X - \overline{[X - (X - D)]} = X - \overline{D} = \phi. \tag{5}$$

Thus $$\overline{D} = X.$$

● PROBLEM 11-37

Set X is given. To promote it to the topological space (X, T), we have to define a family T of its subsets which satisfies certain conditions. There are many ways of defining (or finding) a topology on X. For example, a function f, which assigns its closure to every subest of X determines a topology

$$f: A \rightarrow \overline{A}. \tag{1}$$

Without getting into details, we want the closure $f(A)$ of the set to satisfy these conditions:

$$A \subset \overline{A} \qquad \text{(See Problem 11–28)}$$

$$\overline{\phi} = \phi, \overline{\overline{A}} = \overline{A}, \overline{A \cup B} = \overline{A} \cup \overline{B} \qquad \text{(See Problem 11–30)}$$

Prove the following:

THEOREM

Let X represent a set and $f: P(X) \rightarrow P(X)$ a function, such that

1. $f(\phi) = \phi$.

2. For each A, $A \subset f(A)$.

3. $f \circ f(A) = f(A)$ for each A.

4. $f(A \cup B) = f(A) \cup f(B)$ for each A, B.

The family

$$T = \{X - f(A) : A \in P(X)\} \tag{2}$$

is a topology and $\overline{A} = f(A)$ for each A. ■

Prove this theorem.

SOLUTION:

Note that

$$(A \subset B) \Rightarrow (f(A) \subset f(B)). \tag{3}$$

Indeed if $A \subset B$, then $A \cup B = B$ and $f(A \cup B) = f(A) \cup f(B) = f(B)$. Now we will show that T is a topology .

I. $\phi, X \in T$.
Since $X \subset f(X)$, we get $X = f(X)$. Hence $\phi \in T$. Since $X - f(\phi) = X$, $X \in T$.

II. Suppose $X - f(A)$ and $X - f(B)$ belong to T. Then

$$[X - f(A)] \cap [X - f(B)] = X - [f(A) \cup f(B)] = X - f(A \cup B) \in T. \tag{4}$$

Thus, intersection of any finite family of sets of T is a member of T.

III. Let $S = \bigcup_{\alpha} X - f(A_\alpha)$.We will show that for some $U \in P(X)$, $S = X - f(U)$. We have

$$S = X - \bigcap_{\alpha} f(A_\alpha) \tag{5}$$

and

$$X - S = \bigcap_{\alpha} f(A_\alpha) \subset f(A_\alpha) \tag{6}$$

for each α.

By (3) and condition 3, we obtain

$$f(X - S) \subset f \circ f(A_\alpha) = f(A_\alpha) \tag{7}$$

for each α. Thus

$$f(X - S) \subset \bigcap_{\alpha} f(A_\alpha) = X - S. \tag{8}$$

from (8) and condition 2, we obtain

Prove the following convenient way of describing F_σ and G_δ sets:

$$X - S = f(X - S). \tag{9}$$

Hence

$$S = X - f(X - S) = X - f(U). \tag{10}$$

The union of any family of members of T is again a member of T.

By using T as the topology on X, we will show that $\overline{A} = f(A)$. Indeed, $\overline{A} \subset f(A)$ because each $f(A)$ is closed in T and $A \subset f(A)$, we obtain $\overline{A} \subset \overline{f(A)} = f(A)$. Similarly $f(A) \subset \overline{A}$. Since $X - \overline{A}$ is open in T, for some B, $f(B) = \overline{A}$. Since

$$A \subset \overline{A} \tag{11}$$

we obtain

$$f(A) \subset f(\overline{A}) = f \circ f(B) = f(B) = \overline{A}. \tag{12}$$

That completes the proof.

● PROBLEM 11-38

Show that in R the closed interval $[a, b]$ is an F_γ set and also a G_δ set. Show that the set of rational numbers in R is F_γ.

SOLUTION:

DEFINITION OF F_σ AND G_δ SETS

Set F is called an F_σ set if it is the union of at most countably many closed sets. A set G is called a G_δ set if it is the intersection of at most countably many open sets.

■

A closed interval is a closed set, hence, it is an F_σ. On the other hand,

$$[a,b] = \bigcap_{n=1} (a - {}^1/_n , b + {}^1/_n) \tag{1}$$

$[a, b]$ is the intersection of countably many open sets. Hence, $[a, b]$ in R is a G_δ.

The set of rational numbers Q is countable and since each point in R is a closed set, Q is the union of countably many closed sets. Thus, Q is an F_γ.

● PROBLEM 11-39

Prove the following convenient way of describing F_σ and G_δ sets:

1. If F is an F_γ, then there is a non-decreasing sequence of closed sets

$$F_1 \subset F_2 \subset F_3 \subset \ldots \text{ where } F = \bigcup_1^\infty F_n. \tag{1}$$

2. If G is a G_δ, then there is a non-increasing sequence of open sets

$$G_1 \supset G_2 \supset G_3 \ldots \text{ where } G = \bigcap_1^\infty G_n. \tag{2}$$

SOLUTION:

1. F is an F_γ. Hence

$$F = \bigcup_1^\infty A_n \text{ , } A_n \text{ closed.} \tag{3}$$

F is the union of at most countably many closed sets. Let

$$F_1 = A_1, F_2 = A_1 \cup A_2, F_3 = A_1 \cup A_2 \cup A_3, \ldots,$$

$$\ldots, F_n = A_1 \cup \ldots \cup A_n. \tag{4}$$

Then sets F_n are closed and

$$\bigcup_1^\infty A_n = \bigcup_1^\infty F_n, \; F_1 \subset F_2 \subset F_3 \ldots. \tag{5}$$

2. Similarly, if G is a G_δ, then

$$G = \bigcap_1^\infty B_n \tag{6}$$

where B_n are open sets. Let

$$G_1 = B_1, G_2 = B_1 \cap B_2, \ldots, G_n = B_1 \cap \ldots \cap B_n. \tag{7}$$

Then each G_n is open and

$$G_1 \supset G_2 \supset G_3 \supset \ldots \bigcap_1^\infty B_n = \bigcap_1^\infty G_n. \tag{8}$$

● PROBLEM 11-40

Prove that:

1. The countable union and finite intersection of F_γ sets is an F_σ.

2. The countable intersection and finite union of G_δ sets is a G_δ.

SOLUTION:

1. Consider the countable union of F_ρ sets F_σ

$$F_i = \bigcup_{k=1}^{\infty} F_{i,k} \tag{1}$$

where $F_{i,k}$ are closed sets. The countable union of F_i's is

$$\bigcup_{i=1}^{\infty}\left[\bigcup_{k=1}^{\infty} F_{i,k}\right] = \bigcup_{i,j}^{\infty}\left\{F_{i,j} : (i,j) \in N \times N\right\}. \tag{2}$$

Set $N \times N$ is countable and $F_{i,j}$ are closed sets, hence, the union is an F_γ. The finite intersection is

$$\bigcap_{i=1}^{n}\bigcup_{k=1}^{\infty} F_{i,k} = \left[\bigcup_{k=1}^{\infty} F_{1,k}\right] \cap \left[\bigcup_{k=1}^{\infty} F_{2,k}\right] \cap \ldots \cap \left[\bigcup_{k=1}^{\infty} F_{n,k}\right] =$$

$$= \bigcup\left\{F_{1,k_1} \cap \ldots \cap F_{n,k_n} : (k_1, \ldots, k_n) \in N \times \ldots \times N\right\}. \tag{3}$$

Each set $F_{1,k_1} \cap \ldots \cap F_{n,k_n}$ is closed and (3) is a countable union.

2. Let

$$G_i = \bigcap_{k=1}^{\infty} G_{i,k} \; ; i = 1, 2, 3, \ldots \tag{4}$$

where $G_{i,k}$ are open sets, be a family of G_δ sets. The countable intersection is

$$\bigcap_{i=1}^{\infty}\bigcap_{k=1}^{\infty} G_{i,k} = \bigcap_{i,k}\left\{G_{i,k} : (i,k) \in N \times N\right\} \tag{5}$$

Sets $G_{i,k}$ are open and $N \times N$ is a countable set. Hence, (5) is a G_δ set. Consider now the finite union of G_i's

$$\bigcup_{i=1}^{n}\bigcap_{k=1}^{\infty} G_{i,k} = \bigcap\{G_{1,k_1} \cup G_{2,k_2} \cup \ldots \cup G_{n,k_n} :$$

$$(k_1, k_2, \ldots, k_n) \in N \times N \times \ldots \times N\}. \tag{6}$$

(6) is a G_δ set.

● PROBLEM 11-41

1. Show that the complement of an F_γ is a G_σ.

2. Show that the complement of a G_δ is an F_γ.

SOLUTION:

1. Let F be an F_γ set, then

$$F = \bigcup_{n=1}^{\infty} F_n \tag{1}$$

where F_n are closed sets. The complement of F is

$$X - F = X - \bigcup_{n=1}^{\infty} F_n. \tag{2}$$

By applying DeMorgan's rule, we find

$$X - F = \bigcap_{n=1}^{\infty} (X - F_n). \tag{3}$$

Since each F_n is closed, $X - F_n$ is open for each n. Thus $X - F$ is the intersection of at most countably many open sets.

2. Let G be a G_δ set. Then

$$G = \bigcap_{n=1}^{\infty} G_n \tag{4}$$

where G_n are open sets. The complement of G is

$$X - G = X - \bigcap_{n=1}^{\infty} G_n = \bigcup_{n=1}^{\infty} (X - G_n) \tag{5}$$

where $X - G_n$ are closed sets. Hence $X - G$ is an F_σ set.

● PROBLEM 11-42

Show that each F_σ and each G_δ is a Borel set.

SOLUTION:

DEFINITION OF A σ-RING

A nonempty family of sets

$$Q \subset P(X) \tag{1}$$

is called a σ-ring if

1. $A \in Q \Rightarrow X - A \in Q$
2. $\forall n : A_n \in Q \Rightarrow \bigcup_{1}^{\infty} A_n \in Q.$

■

DEFINITION OF BOREL SETS

If (X, T) is a topological space, then a unique smallest σ-ring B containing the topology T of X exists, that is $T \subset B$. Family of sets B is called the family of Borel sets in X.

■

Let F denote an F_σ–set, then

$$F = \bigcup_1^\infty F_n \tag{2}$$

where F_n are closed sets. The sets $X - F_n$ are open sets. Since B is a σ-ring such that $T \quad B$, each open set belongs to B. Hence

$$\forall\, n : X - F_n \in B \tag{3}$$

$$(X - F_n \in B) \Rightarrow (X - (X - \mathrm{F_n}) \in B) \tag{4}$$

Thus

$$\forall\, n : F_n \in B \quad \text{and} \quad F = \bigcup_1^\infty F_n \in B \tag{5}$$

If G is a G_δ–set then

$$G = \bigcap_1^\infty G_n \tag{6}$$

where G_n are open sets. But

$$\bigcap_1^\infty G_n = X - \left[\bigcup_1^\infty (X - G_n)\right] \tag{7}$$

Each G_n is an open set, therefore

$$G_n \in B \Rightarrow X - G_n \in B \Rightarrow \bigcup_1^\infty (X - G_n) \in B \Rightarrow$$

$$\Rightarrow X - \left[\bigcup_1^\infty (X - G_n)\right] \in B \Rightarrow G \in B. \tag{8}$$

● PROBLEM 11-43

Show that the countable union, countable intersection, and the difference of Borel sets is a Borel set.

SOLUTION:

Let B denote the family of Borel sets in X. B can be defined as the intersection of all σ–rings containing T.

It is easy to verify that the intersection of any family of σ-rings is a σ-ring. Thus, B is a σ-ring.

$$\left(B_i \in B \text{ for } i = 1, 2, 3, \ldots\right) \Rightarrow \left(\bigcup_1^\infty B_i \in B\right) \tag{1}$$

Because B is a σ-ring.

$$\left(B_i \in B \text{ for } i = 1, 2, \ldots\right) \Rightarrow \left(X - B_i \in B \text{ for } i = 1, 2, \ldots\right) \Rightarrow$$

$$\Rightarrow \left(\bigcup_1^\infty (X - B_i) \in B\right) \Rightarrow \left(X - \bigcup_1^\infty (X - B_i) = \bigcap_1^\infty B_i \in B\right). \quad (2)$$

$$(A_1, A_2 \in B) \Rightarrow (A_1, X - A_2 \in B) \Rightarrow$$

$$\Rightarrow (A_1 \cap (X - A_2) = A_1 - A_2 \in B). \quad (3)$$

● PROBLEM 11-44

Show that in the Euclidean space R^n there are sets that are not Borel sets.

SOLUTION:

We shall apply the following:

THEOREM

Let (X, T) be a topological space and B be the family of Borel sets in X. Then

$$\text{card } (B) \leq \text{card } (T)^{\text{card N}} \quad (1)$$

■

Topological space R^n has a countable basis. Hence, the cardinal number of the Euclidean topology of R^n is $2^{\text{card } N}$

$$\text{card } (T) = 2^{\text{card } N}. \quad (2)$$

From (1) and (2) we find

$$\text{card } (B) \leq 2^{\text{card } N}. \quad (3)$$

Since

$$\text{card } (P(R^n)) = 2^{\text{card } R} \quad (4)$$

we conclude that there are sets in R^n that are not Borel sets.

● PROBLEM 11-45

Let (R, T) denote Euclidean space and X the subset of R

$$X = (0, 1] \cup \{2\}. \tag{1}$$

Find the induced topology T_X.

SOLUTION:

DEFINITION OF SUBSPACE

Let (X, T) denote a topological space and $Y \subset X$. The induced topology T_Y on Y is defined by

$$T_Y =: \{Y \cap T_\alpha : T_\alpha \in T\}. \tag{2}$$

The space (Y, T_Y) is called a subspace of (X, T).

■

It is easy to show that T_Y defined in (2) is a topology on Y. Consider (R, T). T is the family of all open sets in R. By definition

$$T_X = \{[(0, 1] \cup \{2\}] \cap T_\alpha : T_\alpha \in T\} \tag{3}$$

where T_α are open sets.

The family T_X consists of

1. $\{2\}$
2. all open intervals contained in $(0, 1]$
3. all intervals of the form $]a, 1]$ where $a \in (0, 1)$.

Interval $(0, 1]$ belongs to T_X, hence it is an open set. Since $\{2\} \in T_X$, interval $(0, 1]$ is a closed set in X.

● PROBLEM 11-46

Let (X, T) denote a space and (Y, T_Y) a subspace. Show that if $\{B_\omega : \omega \in \Omega\}$ is a basis for T, then $\{Y \cap B_\omega : \omega \in \Omega\}$ is a basis for T_Y.

SOLUTION:

Let (X, T) denote a topological space and $\{B_\omega : \omega \in \Omega\}$ be its basis. Then each open set in X is the union of members of $\{B_\omega\}$. The induced topology T_Y is defined by

$$T_Y = \{Y \cap T_\alpha\}. \tag{1}$$

Suppose A is an open set in Y, $A \in T_Y$. Then

$$A = Y \cap T_\alpha \tag{2}$$

for some α. But $\{B_\omega\}$ is a basis for T,

$$T_\alpha = U B_\omega. \tag{3}$$

Thus

$$A = Y \cap [U B_\omega] = U(Y \cap B_\omega) \tag{4}$$

Therefore $\{Y \cap B_\omega : \omega \in \Omega\}$ is a basis for T_Y.

● PROBLEM 11-47

Let (X, T) denote a topological space and (Y, T_Y) be a subspace. Show that a set $A \subset Y$ is T_Y-closed if and only if $A = Y \cap D$, where D is T-closed.

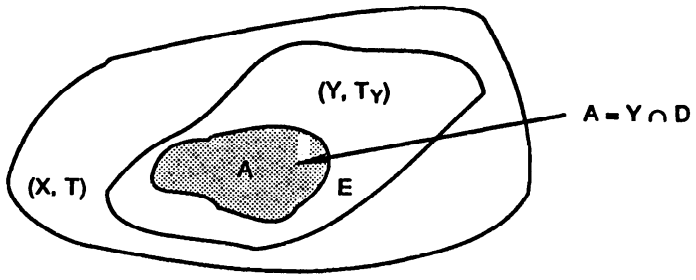


FIGURE 1

SOLUTION:

We will show that the closed sets in Y are the intersections of Y and the closed sets in X. Suppose $A \subset Y$ is closed in Y, then $A = Y - E$, where E is open in Y. Therefore

$$E = Y \cap G \tag{1}$$

where $G \in T$. Hence

$$A = Y - E = Y - Y \cap G = Y \cap (X - G). \tag{2}$$

Since $G \in T$, $X - G$ is T-closed.

Now, suppose

$$A = Y \cap D \tag{3}$$

where D is T-closed. Then

$$Y - A = Y - Y \cap D = Y \cap (X - D) \tag{4}$$

where $X - D$ is T-open and $Y - A$ is T_Y-open. Thus A is T_Y-closed.

● PROBLEM 11-48

1. (X, T) is a topological space and (Y, T_Y) is its subspace. In Problem 11–45, we showed that sets, open in a subspace need not be open in the entire space. Prove the following:

THEOREM

Let (X, T) denote a space and (Y, T_Y) its subspace. If $A \subset Y$ is open (closed) in Y, and Y is open (closed) in X, then A is open (closed) in X.

2. Is a subspace of a subspace a subspace of the entire space?

SOLUTION:

1. Suppose A is open in Y. Then

$$A = Y \cap G \tag{1}$$

where G is open in X. Since Y is open in X, $Y \cap G$ is open in X. The same reasoning holds for closed sets.

2. Yes. Because if

$$Z \subset Y \subset X \tag{2}$$

and T_{YZ} is the topology of Z with respect to Y, then

$$T_{YZ} = T_Z \tag{3}$$

Where T_Z is the topology of Z with respect to X. We shall prove (3). Let

$$U \in T_{YZ} \tag{4}$$

then

$$U = Z \cap V \tag{5}$$

where $V \in T_Y$. But

$$V = Y \cap Q \tag{6}$$

where $Q \in T$. Then

$$U = Z \cap V = Z \cap Y \cap Q = Z \cap Q. \tag{7}$$

Hence $U \in T_Z$ and $T_{YZ} \subset T_Z$.

The converse inclusion is obvious.

CHAPTER 12

CONTINUITY, HOMEOMORPHISMS, AND TOPOLOGICAL EQUIVALENCE

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Let (X,T_X) and (Y, T_Y) represent topological spaces, such that

$$X = \{a, b, c, d\}$$

$$T_X = \{\phi, X, \{a\}, \{a, b\}, \{a, b, c\}\}$$

$$Y = \{x, y, z, w\}$$

$$T_Y = \{\phi, Y, \{y\}, \{y, z, w\}$$

Which of the functions depicted in Figure 1 is continuous?

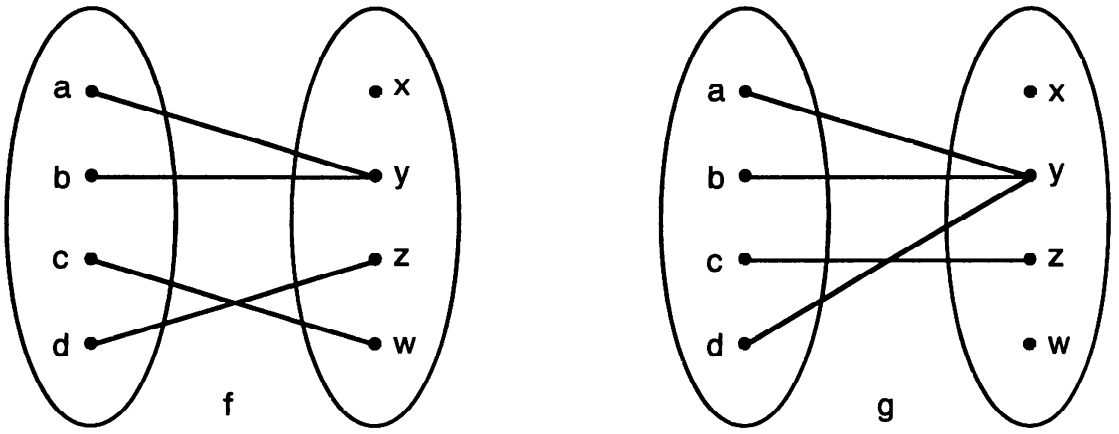


FIGURE 1

SOLUTION:

DEFINITION OF CONTINUOUS FUNCTIONS

Let (X,T_X) and (Y, T_Y) represent topological spaces. A function $f: X \rightarrow Y$ is called continuous if the inverse image of each open set in Y is an open set in X. That is, if $f: X \rightarrow Y$ is continuous, then

$$f^{-1} : T_Y \rightarrow T_X. \tag{1}$$

■

Function f is continuous because f^{-1} maps each open set in T_Y into an open set in T_X.

$$f^{-1}(\phi) = \phi \in T_X$$

$$f^{-1}(Y) = X \in T_X$$

$$f^{-1}(\{y\}) = \{a, b\} \in T_X$$

$$f^{-1}(\{y, z, w\}) = X \in T_X. \tag{2}$$

Function g is not continuous, because

$$\{y\} \in T_Y \text{ but } g^{-1}(\{y\}) = \{a, b, d\} \notin T_X. \tag{3}$$

● PROBLEM 12-2

Show that the identity function

$$f: (X, T) \to (X, T_1) \tag{1}$$

where $\forall x \in X : f(x) = x$, is continuous if and only if T is finer than T_1, that is, if

$$T_1 \subset T. \tag{2}$$

SOLUTION:

Suppose f is continuous then

$$A \in T_1 \Rightarrow f^{-1}(A) \in T \tag{3}$$

But $f^{-1}(A) = A$. Hence $T_1 \subset T$.

Now suppose $f : (X, T) \to (X, T_1)$ is the identity function, $f(x) = x$ and suppose $T_1 \subset T$.

Let $A \in T_1$. Then $f^{-1}(A) = A \in T$ because $T_1 \subset T$. Thus, f is continuous. ■

Consider the sequence of identity functions

$$(X, T_1) \xrightarrow{\text{identity}} (X, T_2) \xrightarrow{\text{identity}} (X, T_3) \xrightarrow{\text{identity}} (X, T_4) \longrightarrow (\ldots$$

To ensure continuity of the functions, we must have

T_1 is finer than T_2
T_2 is finer than T_3
T_3 is finer than T_4.

● PROBLEM 12-3

1. Show that

$$f: (X, P(X)) \to (Y, T_Y) \tag{1}$$

is always continuous.

2. Show that

$$f: (X, T_X) \rightarrow (Y, \{\phi, Y\}) \tag{2}$$

is always continuous.

SOLUTION:

1. Let us take any open subset in T_Y, $A \in T_Y$. Then $f^{-1}(A)$ is an open set in X because $T_X = P(X)$, i.e., each subset of X is an open set.

2. There are only two open sets in Y, ϕ and Y.

$$f^{-1}(\phi) = \phi \tag{3}$$

which is an open set in X, $\phi \in T_x$. Similarly

$$f^{-1}(Y) = X \in T_X. \tag{4}$$

Each topology in X contains ϕ and X. Hence f defined by (2) is always continuous.

● PROBLEM 12-4

Suppose

$$f: (X, T_X) \rightarrow (Y, T_Y) \tag{1}$$

is not a continuous function. Show that the same function

$$f: (X, T'_X) \rightarrow (Y, T'_Y) \tag{2}$$

is also not continuous if

$$T'_X \text{ is coarser than } T_X \text{ (that is } T'_X \subset T_X)$$

and

$$T'_Y \text{ is finer than } T_Y \text{ (that is } T_Y \subset T'_Y).$$

SOLUTION:

Since

$$f: (X, T_X) \rightarrow (Y, T_Y)$$

is not continuous, there is an open set $D \in T_Y$, such that

$$f^{-1}(D) \notin T_X. \tag{3}$$

Consider function (2). Since $D \in T_Y$ and $T_Y \subset T'_Y$, we have

$$D \in T'_Y. \tag{4}$$

Hence $f^{-1}(D) \notin T_X$ and since $T'_X \subset T_X$

$$f^{-1}(D) \notin T'_X. \tag{5}$$

Function defined in (2) is not continuous.

Note that the same will be true if we replace T'_X by T_X in equation (2).

● PROBLEM 12-5

Show that the function

$$f : R \to R$$

defined by

$$f(x) = \begin{cases} x & \text{if } x \leq 1 \\ x + 1 & \text{if } x > 1 \end{cases} \tag{1}$$

is not continuous if R is equipped with the Euclidean topology but becomes continuous when R has the upper limit topology.

SOLUTION:

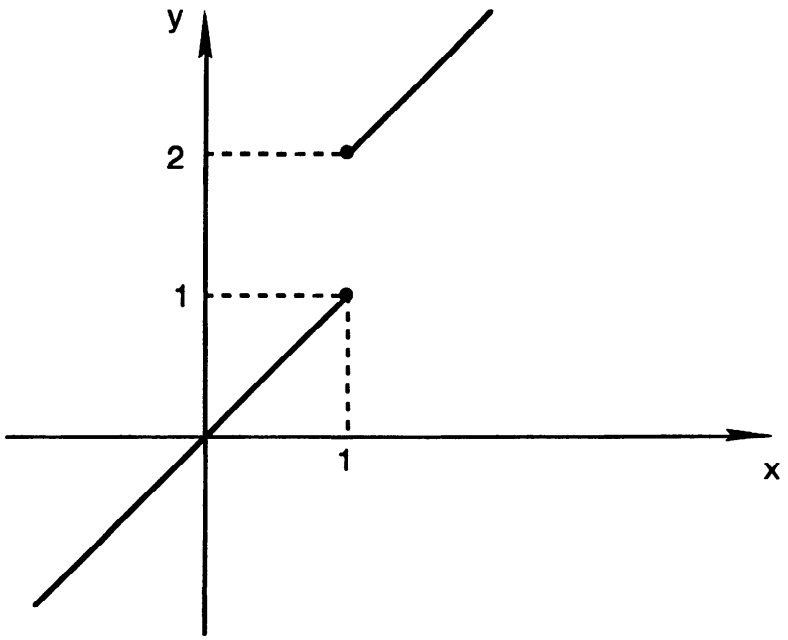


FIGURE 1

The Euclidean topology consists of all open intervals and the unions of open intervals.

Let $A = (0, 3/2)$. Then

$$f^{-1}(A) = f^{-1}(0, 3/2) =]0, 1]. \tag{2}$$

The inverse image of an open set is not an open set. Hence f is not continuous in the Euclidean topology.

The upper limit topology on R consists of unions of open-closed intervals, that is intervals of the form $]a, b]$.

It is easy to verify that for each $]a, b]$

$$f^{-1}[(a, b]] =]c, d]. \tag{3}$$

The inverse image of an open set is an open set, and the function f is continuous.

● PROBLEM 12-6

Function $f : X \to Y$ is continuous if and only if the inverse image of every closed subset of Y is a closed subset of X.

Prove it.

SOLUTION:

$$(f : X \to Y \text{ is continuous}) \Leftrightarrow$$

$$\Leftrightarrow (\forall\, D \subset Y : D \text{ closed in } Y\, f^{-1}(D) \text{ closed in } X)$$

$\Rightarrow$ Suppose $f : X \to Y$ is continuous and $D \subset Y$ is any closed subset of Y. Then $Y - D$ is an open subset of Y and

$$f^{-1}(Y - D) \tag{1}$$

is an open subset of X.

But

$$f^{-1}(Y - D) = X - f^{-1}(D). \tag{2}$$

Hence $f^{-1}(D)$ is closed.

$\Leftarrow$ Suppose for every $D \subset Y$, D is closed in Y, $f^{-1}(D)$ is closed in X. Let A represent an open subset of Y. Then $Y - A$ is closed in Y and $f^{-1}(Y - A)$ is closed in X. Since

$$f^{-1}(Y - A) = X - f^{-1}(A). \tag{3}$$

Hence $f^{-1}(A)$ is a open subset of X and f is a continuous function.

● PROBLEM 12-7

Let

$$f: X \to Y$$

$$g: Y \to Z \tag{1}$$

be continuous functions.

Show that the composition function

$$g \circ f: X \to Z \tag{2}$$

is also continuous.

SOLUTION:

Let A represent an open subset of Z. Then $g^{-1}(A)$ is an open subset of Y because g is continuous.

Also $f^{-1}[g^{-1}(A)]$ is an open subset of X, since f is a continuous function.

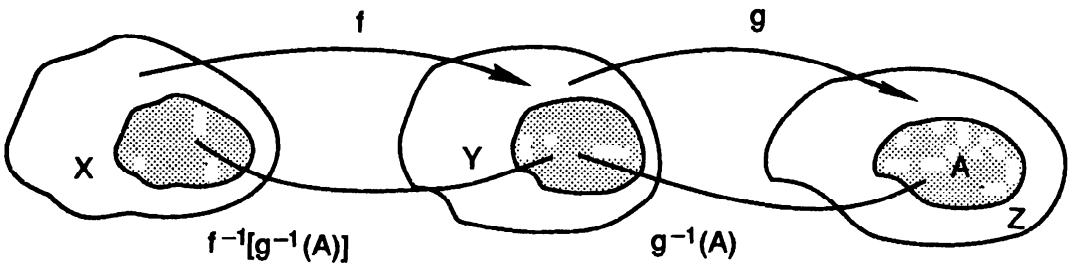


FIGURE 1

But

$$(g \circ f)^{-1}(A) = f^{-1}[g^{-1}(A)]. \tag{3}$$

Thus $(g \circ f)^{-1}(A)$ is an open set in X whenever A is an open set in Z. Hence, $g \circ f$ is a continuous function.

● PROBLEM 12-8

Let

$$f: (X, T_X) \to (Y, T_Y). \tag{1}$$

Prove that

$$(f \text{ continuous}) \Leftrightarrow \begin{pmatrix} \forall f(x) \in Y \quad \forall V \ni f(x) \\ \exists U \ni x \text{ such that } f(U) \subset V \end{pmatrix}$$

SOLUTION:

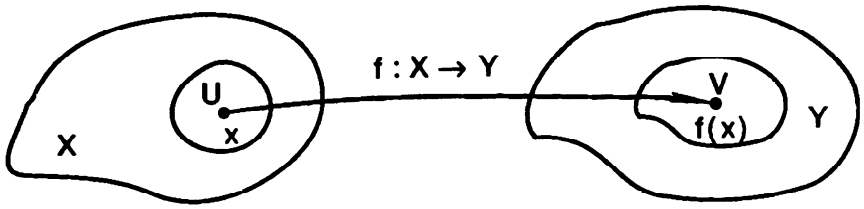


FIGURE 1

$\Rightarrow$ Suppose f is continuous. Take any $f(x) \in Y$ and any neighborhood V of $f(x)$. Then

$$f(x) \in V \text{ and } V \in T_Y. \tag{2}$$

Since f is continuous $f^{-1}(V) \in T_X$.

Also $x \in f^{-1}(V)$. Hence $U = f^{-1}(V)$.

$\Leftarrow$ Let W represent any open subset of Y. Suppose $x \in f^{-1}(W)$, then $f(x) \in W$. Thus W is a neighborhood of $f(x)$. An open set $U \subset X$ exists, such that $x \in U$ and

$$f(U) \subset W. \tag{3}$$

But

$$U \subset f^{-1}(W). \tag{4}$$

Hence, for each $x \in f^{-1}(W)$

$$x \in U \subset f^{-1}(W). \tag{5}$$

where $U \in T_X$.

Therefore $f^{-1}(W)$ is the union of open subsets of X and thus, an open subset.

● PROBLEM 12-9

Let

$$f : (X, T_X) \rightarrow (Y, T_Y) \tag{1}$$

and let $B = \{B_\alpha\}$ denote a basis for T_Y. Function f is continuous if and only if for each $B_\alpha \in B$, $f^{-1}(B_\alpha)$ is an open subset of X.

SOLUTION:

$\Rightarrow$ Suppose f is continuous. Since each $B_\alpha \in B$ is an open subset of Y,

$$f^{-1}(B_\alpha) \in T_X. \tag{2}$$

$\Leftarrow$ Suppose for each $B_\alpha \in B$, $f^{-1}(B_\alpha)$ is an open set in X. Let $A \subset Y$ denote an open subset in Y. Since B is the basis

$$A = \bigcup_\alpha B_\alpha. \tag{3}$$

Then

$$f^{-1}(A) = f^{-1}\left(\bigcup_\alpha B_\alpha\right) = \bigcup_\alpha f^{-1}(B_\alpha). \tag{4}$$

Each $f^{-1}(B_\alpha)$ is an open set. Hence, $f^{-1}(A)$ is an open set as a union of open sets. Function f is continuous.

● PROBLEM 12-10

Let

$$f: (X, T_X) \rightarrow (Y, T_Y) \tag{1}$$

and let $S = \{S_\alpha\}$ denote a subbasis for the topology T_Y.

Prove that f is continuous if and only if the inverse of every $S_\alpha \in S$ is an open subset of X.

SOLUTION:

If (Y, T_Y) is a topological space, then a class S of open subsets $S \subset T_Y$ is a subbasis for T_Y if and only if finite intersections of members of S form a basis for T_Y.

$\Rightarrow$ Suppose f is continuous. Then the inverse of all open sets are open. Hence, since S_α are open, sets $f^{-1}(S_\alpha) \in T_X$.

$\Leftarrow$ Suppose for every $S_\alpha \in S$ $f^{-1}(S_\alpha) \in T_X$. Let $A \in T_Y$. Since $\{S_\alpha\}$ is a subbasis,

$$A = \bigcup_\alpha (S_{\alpha_1} \cap \ldots \cap S_{\alpha_k}). \tag{2}$$

We have

$$f^{-1}(A) = f^{-1}\left[\bigcup_\alpha (S_{\alpha_1} \cap \ldots \cap S_{\alpha_k})\right] =$$

$$= \bigcup_{\alpha} f^{-1}(S_{\alpha_1} \cap \ldots \cap S_{\alpha_k}) =$$

$$= \bigcup_{\alpha} \left[f^{-1}(S_{\alpha_1}) \cap \ldots \cap f^{-1}(S_{\alpha_k}) \right]. \tag{3}$$

Since all sets $f^{-1}(S_{\alpha_1}), \ldots, f^{-1}(S_{\alpha_k})$ are open, so is the set $f^{-1}(A)$. Hence, function f is continuous.

● PROBLEM 12-11

Let $\{T_\alpha\}$ represent a family of topologies on X. Suppose

$$f: X \to Y \tag{1}$$

is continuous with respect to each topology T_α. Show that f is continuous with respect to the topology

$$T = \bigcap_{\alpha} T_\alpha. \tag{2}$$

SOLUTION:

First note that intersection of any family of topologies is a topology.

Let A denote an open subset of Y, $A \subset Y$. Consider set $f^{-1}(A)$. Since f is continuous with respect to each topology T_α, the set $f^{-1}(A)$ belongs to each T_α, $f^{-1}(A) \in T_\alpha$. Hence

$$f^{-1}(A) \in \bigcap_{\alpha} T_\alpha = T. \tag{3}$$

Therefore f is continuous with respect to T.

● PROBLEM 12-12

Prove this theorem

$$(f: X \to Y \text{ is continuous}) \Leftrightarrow (\forall\, A \subset X: f(\overline{A}) \subset \overline{f(A)}).$$

SOLUTION:

$\Rightarrow$ Suppose $f: X \to Y$ is continuous. Since

$$f(A) \subset \overline{f(A)} \tag{1}$$

we obtain

$$A \subset f^{-1}[f(A)] \subset f^{-1}[\overline{f(A)}]. \tag{2}$$

Set $\overline{f(A)}$ is closed and since f is continuous $f^{-1}[\overline{f(A)}]$ is closed as well

$$f^{-1}[\overline{f(A)}] = \overline{f^{-1}[\overline{f(A)}]}. \tag{3}$$

Hence, from (2) and (3), we find

$$A \subset \overline{A} \subset f^{-1}[\overline{f(A)}]. \tag{4}$$

Thus

$$f(\overline{A}) \subset f[f^{-1}[\overline{f(A)}] = \overline{f(A)}. \tag{5}$$

$\Leftarrow$ Suppose for any $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$. Let B denote any closed subset of Y, $B \subset Y$ and let

$$f^{-1}(B) = A. \tag{6}$$

Then

$$f(\overline{A}) = f[\overline{f^{-1}(B)}] \subset \overline{f[f^{-1}(B)]} = \overline{B} = B. \tag{7}$$

Hence

$$\overline{A} \subset f^{-1}[f(\overline{A})] \subset f^{-1}(B) = A \tag{8}$$

Since $A \subset \overline{A}$, we obtain from (8)

$$A = \overline{A}. \tag{9}$$

Thus, the inverse image of any closed subset of Y is a closed subset of X. Therefore, $f: X \to Y$ is continuous, by Problem 12–6.

● PROBLEM 12–13

Let

$$f: (X, T_X) \to (Y, T_Y) \tag{1}$$

represent a continuous function. Show that

$$f_A : (A, T_A) \to (Y, T_Y) \tag{2}$$

where $f_A = f/A$ is the restriction of f to A, is also continuous.

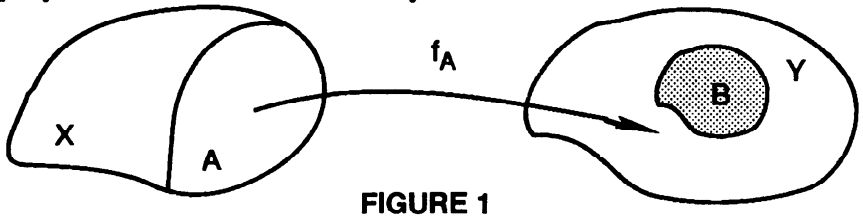


FIGURE 1

SOLUTION:

Note that, for any $B \subset Y$

$$f_A^{-1}(B) = f^{-1}(B) \cap A. \tag{3}$$

Suppose B is an open subset of Y, $B \in T_Y$. Since f is continuous

$$f^{-1}(B) \in T_X. \tag{4}$$

By definition of the induced topology

$$f^{-1}(B) \cap A \in T_A. \tag{5}$$

Hence

$$f^{-1}(B) \cap A = f_A^{-1}(B) \in T_A. \tag{6}$$

and so f_A is continuous.

● PROBLEM 12-14

Prove the following:

THEOREM

If $f: (X, T_X) \rightarrow (Y, T_Y)$ is a function and $X = A \cup B$ and $f|A$ and $f|B$ are both continuous (where A and B are topological subspaces of X), then if A and B are both closed or both open, f is continuous.

■

SOLUTION:

If A and B are not both closed or both open, then f does not have to be continuous.

For example, let

$$A = (0, 1) \quad B = [1, 2) \quad X = (0, 2)$$

and let $f|A = 0$ and $f|B = 1$. Then both $f|A$ and $f|B$ are continuous, but f is not continuous.

Suppose both A and B are closed. Let $D \subset Y$ denote any closed subset of Y. We will show that if $D \in T_Y$ then

$$f^{-1}(D) \in T_X. \tag{1}$$

$(f|A)^{-1}(D)$ is closed in A and $(f|B)^{-1}(D)$ is closed in B since both $f|A$ and $f|B$ are continuous. Since A and B are closed $(f|A)^{-1}(D)$ and $(f|B)^{-1}(D)$ are closed subsets of X.

From

$$A \cup B = X \tag{2}$$

we obtain

$$f^{-1}(D) = (f \mid A)^{-1}(D) \cup (f \mid B)^{-1}(D). \tag{3}$$

Equation (3) is the union of two closed subsets of X, hence $f^{-1}(D)$ is a closed subset of X.

Therefore, f is continuous.

● PROBLEM 12–15

Use the theorem of Problem 12–14 to show that the function

$$f: R \to R$$

$$f(x) = \begin{cases} x & \text{if } x \leq 0 \\ 0 & \text{if } x \geq 0 \end{cases} \tag{1}$$

is continuous. R is equipped with the Euclidean topology.

SOLUTION:

Remember that if $f: X \to Y$ and $A \subset X$, then $f \mid A$ is the restricted function $f \mid A : A \to Y$, where $f \mid A(x) = f(x)$ for each $x \in A$.

Consider the sets

$$A = \{x \in R : x \leq 0\} \quad \text{and} \quad B = \{x \in R : x \geq 0\}. \tag{2}$$

Then

$$R = A \cup B \tag{3}$$

and both A and B are closed.

Both functions $f \mid A$ and $f \mid B$ are continuous. By the theorem of Problem 12–14, function f is continuous.

● PROBLEM 12–16

Let R^2 (coordinate plane) be a topological space with Euclidean topology. Show that any rotation is continuous.

SOLUTION:

Each point of R^2 can be described by its polar coordinates (r, α). Rotation of R^2 about the origin (0, 0) does not change r and transforms α into

$$\alpha_0 + \alpha$$

where α_0 is the angle of rotation. Thus

$$R_{\alpha_0}(r,\alpha) = (r, \alpha + \alpha_0). \tag{1}$$

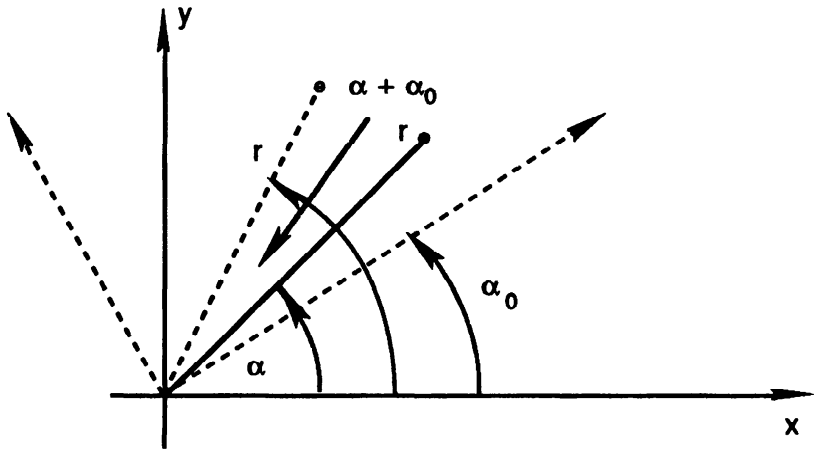


FIGURE 1

Denoting by R_{α_0} rotation through angle α_0, we obtain

$$R^{-1}_{\alpha_0} = R_{-\alpha_0} \tag{2}$$

where $R_{-\alpha_0}$ is the rotation through $-\alpha_0$. Any rotation preserves congruences. Hence, if D is the interior of a square, then $R^{-1}_{\alpha_0}(D)$ is also the interior of a square. The family of interiors of squares forms a basis for R^2 with Euclidean topology.

By Problem 12–9, we conclude that any rotation is continuous.

● PROBLEM 12-17

Show that the projection mappings from the plane R^2 into the line R are continuous with respect to Euclidean topology.

SOLUTION:

Both projections can be defined as follows:

$$P_x, P_y : R^2 \to R$$

$$P_x(x, y) = x$$

$$P_y(x, y) = y \tag{1}$$

Consider $P_y(x, y) = y$. The inverse of any open interval (a, b) is an infinite open strip

$$A = \{(x, y) : a < y < b\}.$$

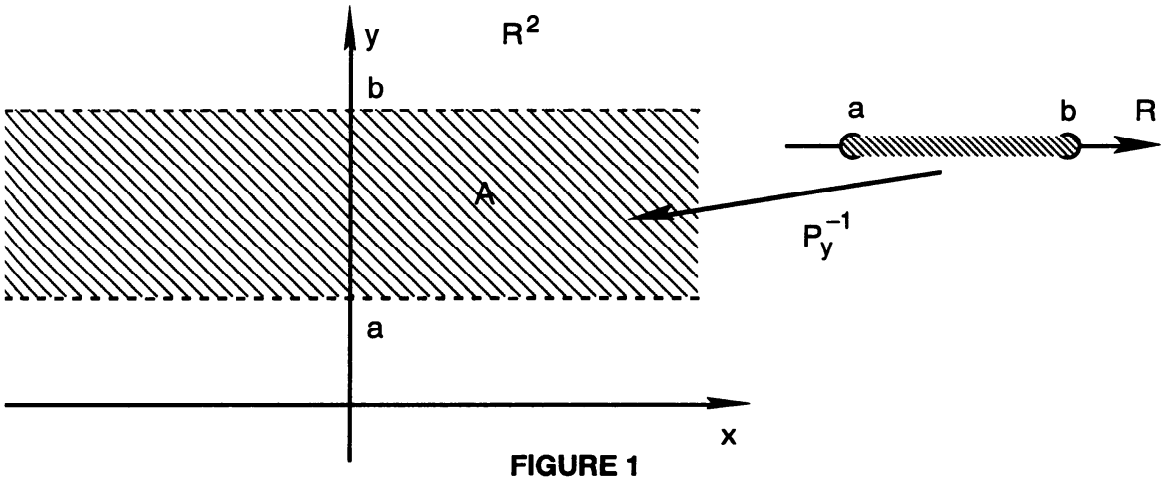


FIGURE 1

The family of open intervals forms a basis for R with the usual topology.

The inverse of any element of the basis is an open set. Hence, by Problem 12–9, the projection P_y is a continuous function. Similarly, we can show that P_x is continuous.

● PROBLEM 12-18

Let (X, T_X) represent a topological space and $\{a\}$ a singleton set, which is an open subset of X, $\{a\} \in T_X$. Show that any function

$$f : X \to Y \tag{1}$$

where (Y, T_Y) is any topological space, is continuous at $a \in X$.

SOLUTION:

DEFINITION OF CONTINUITY AT A POINT

A function $f : X \to Y$ is continuous at $x_0 \in X$ if for each neighborhood $W(f(x_0))$ in Y, a neighborhood $V(x_0)$ in X exists, such that $f(V(x_0)) \subset W(f(x_0))$. ■

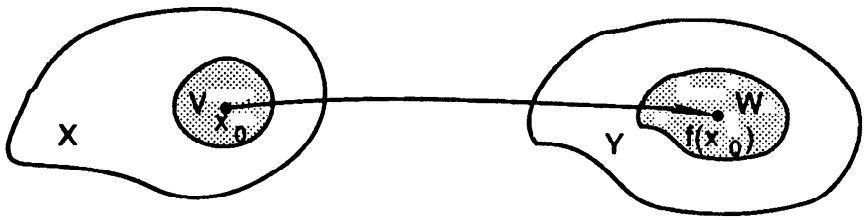


FIGURE 1

Let B represent any open set in Y containing $f(a)$.

$$f(a) \in B. \tag{2}$$

The set $\{a\}$ is an open set in X,

$$\{a\} \in T_X \text{ and } a \in \{a\}. \tag{3}$$

Also, since $f(a) \in B$

$$f(\{a\}) \subset B \tag{4}$$

Thus, f is continuous at $a \in X$.

● PROBLEM 12-19

Consider the topological space (X, T) where

$$X = \{a, b, c, d\} \tag{1}$$

and

$$T = \{\phi, X, \{a\}, \{a, b\}, \{a, b, c\}\}. \tag{2}$$

Show that the function $f: X \to X$, depicted in the diagram, is continuous at $b \in X$ but not continuous at $c \in X$.

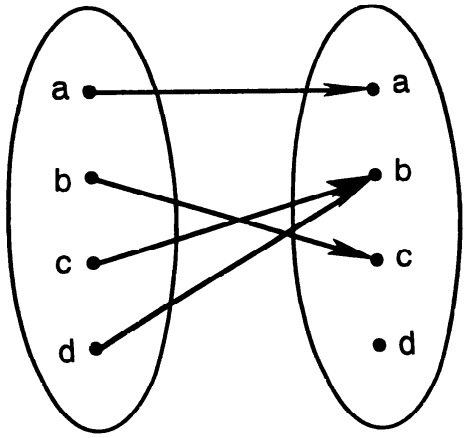


FIGURE 1

SOLUTION:

From the diagram

$$f(b) = c \tag{3}$$

The only open sets containing c are $\{a, b, c\}$ and X. We have

$$f^{-1}(\{a, b, c\}) = X \tag{4}$$

and

$$f^{-1}(X) = X. \tag{5}$$

Thus, the inverse of any open set containing $f(b)$ is an open set containing b, and f is continuous at b.

From the diagram

$$f(c) = b \tag{6}$$

Consider an open set containing $f(c)$

$$\{a, b\} \ni b = f(c), \tag{7}$$

Then

$$f^{-1}(\{a, b\}) = \{a, c, d\} \tag{8}$$

Set $\{a, c, d\}$ does not contain any open set containing c.

Hence, f is not continuous at $c \in X$.

● PROBLEM 12-20

1. Show that a function

$$f: (X, T_X) \rightarrow (Y, T_Y) \tag{1}$$

is continuous if and only if it is continuous at every point $a \in X$.

2. Show that if (1) is continuous at $a \in X$, then the restriction of f to A, where $a \in A \subset X$, is also continuous at a.

SOLUTION:

1. Suppose f is continuous. Let $a \in X$ denote any point, and let $B \subset Y$ denote an open subset of Y, such that $f(a) \in B$, $B \in T_Y$. Then $f^{-1}(B) \in T_X$ and $a \in f^{-1}(B)$. Thus, f is continuous at $a \in X$. Now, suppose f is continuous at every point $a \in X$. Let $A \subset Y$ denote an open set.

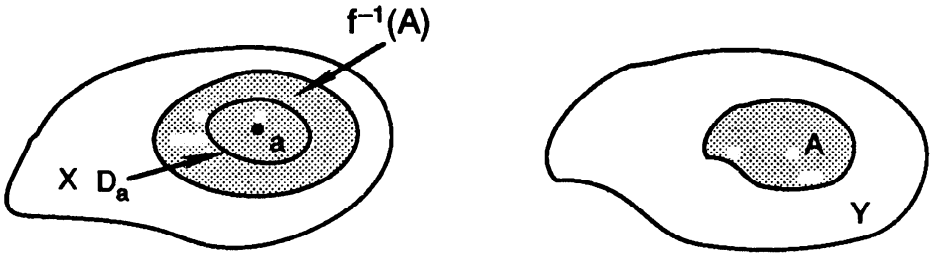


FIGURE 1

For every $a \in f^{-1}(A)$ an open set $D_a \subset X$ exists, such that

$$a \in D_a \subset f^{-1}(A). \tag{2}$$

Thus

$$f^{-1}(A) = \bigcup_a D_a \tag{3}$$

where $a \in f^{-1}(A)$. Set (3) is open as a union of open sets, Hence, f is continuous.

2. Suppose $D \subset Y$ is an open subset containing $f(a)$, see Figure 2.

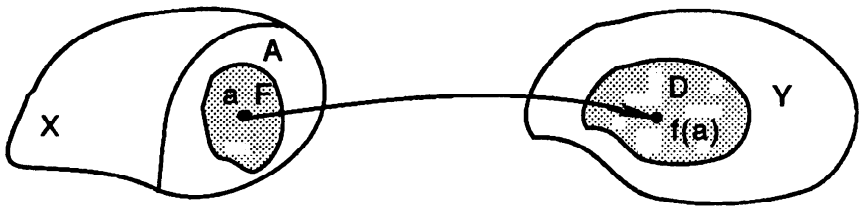


FIGURE 2

Since f is continuous at $a \in X$, there is an open subset F of X, such that

$$a \in F \subset f^{-1}(D). \tag{4}$$

Thus

$$a \in A \cap F \subset A \cap f^{-1}(D) = f_A^{-1}(D) \tag{5}$$

where

$$f_A = f \mid A. \tag{6}$$

But

$$A \cap F \in T_A. \tag{7}$$

Hence f_A is continuous at $a \in X$ with respect to topology T_A.

● PROBLEM 12-21

Show that if a function

$$f: X \to Y \tag{1}$$

is continuous at $a \in X$, then it is sequentially continuous at $a \in X$.

SOLUTION:

DEFINITION OF SEQUENTIAL CONTINUITY

A function $f: X \to Y$ is sequentially continuous at a point $a \in X$ if for every sequence (a_n) in X,

$$(a_n \to a) \Rightarrow (f(a_n) \to f(a)) \tag{2}$$

■

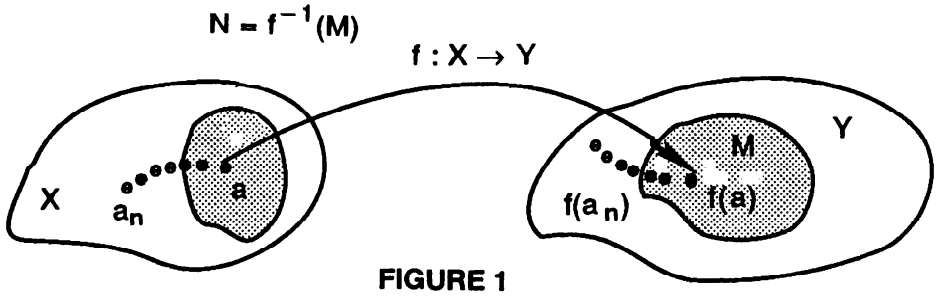


FIGURE 1

Suppose $f: X \to Y$ is continuous at $a \in X$, and let (a_n) denote a sequence convergent to a, $a_n \to a$.

We will show that any neighborhood M of $f(a)$, $f(a) \in M \in T_Y$ contains all but a finite number of the elements of the sequence $(f(a_n))$.

We have

$$f(a) \in M \in T_Y \tag{3}$$

Function f is continuous at $a \in X$. Hence, $f^{-1}(M) = N$ is a neighborhood of a. N contains all but a finite number of elements of (a_n). Then

$$a_n \in N \Rightarrow f(a_n) \in N. \tag{4}$$

Hence, N contains all but a finite number of elements of $f(a_n)$ and

$$f(a_n) \to f(a). \tag{5}$$

● PROBLEM 12-22

Give an example of a neighborhood-finite family in R.

SOLUTION:

DEFINITION OF A NEIGHBORHOOD-FINITE FAMILY

A collection $\{A_\omega : \omega \in \Omega\}$ of sets in a topological space (X, T) is called neighborhood-finite if each point of X has a neighborhood $N_0(x)$, such that

$$N_0(x) \cap A_\omega \neq \emptyset \tag{1}$$

for at most finitely many indices ω.

■

Consider the family $\{A_p\}$,

$$A_p = [p, p + 1\} \tag{2}$$

where p is an integer.

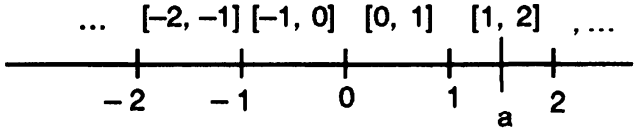


FIGURE 1

Let $a \in R$ denote any point in R. Then, we can always find a sufficiently small neighborhood U of $a \in R$, which intersects with only one element of $\{A_p\}$ (if a is not an integer). If a is an integer, then a sufficiently small neighborhood U of a would intersect with two members of $\{A_p\}$.

Note that a family $\{A_\omega\}$ may be neighborhood-finite, even though each A_α intersects infinitely many other A_β. For example in R_+, we define $\{A_n\}$

$$A_n = \{x : X > n\}. \tag{3}$$

● PROBLEM 12-23

Here is a useful theorem concerning coverings of the space.

THEOREM

Let $\{A_\omega : \omega \in \Omega\}$ represent a family of sets that forms a covering of the space X, that is, $X = \bigcup_\omega A_\omega$. Assume that one of these two conditions holds:

1. all sets A_ω are open, or

2. all sets A_ω are closed, and form an neighborhood-finite family. Then

$$\begin{pmatrix} B \subset X \text{ is open} \\ \text{(or closed)} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \text{all } B \cap A_\omega \text{ are open} \\ \text{(or closed) is the subspace } A_\omega \end{pmatrix}.$$

■

Prove this theorem for the case when condition 2 is true.

SOLUTION:

$\Rightarrow$ This follows immediately from the definition of induced topology.

$\Leftarrow$ Suppose each $B \cap A_\omega$ is closed in the closed A_ω.

The case when each $B \cap A_\omega$ is open in the closed A_ω is similar. Since a subspace of a subspace is a subspace of the entire space X *and each* B $\cap A_\omega$ is closed in the closed A_ω it follows that $B \cap A_\omega$ is closed in X.

$$B \cap A_\omega = \overline{B \cap A_\omega}. \tag{1}$$

Since $\{A_\omega\}$ is neighborhood-finite, so is $\{B \cap A_\omega\}$. Therefore (remember, $\{A_\omega\}$ is a covering)

$$B = \bigcup_\omega \{B \cap A_\omega\} \tag{2}$$

is closed in X.

Here we applied the following:

THEOREM

If $\{A_\omega : \omega \in \Omega\}$ is an neighborhood-finite family in X, then for each $\Lambda \subset \Omega$, the set

$$\bigcup_\lambda \{\overline{A}_\lambda : \lambda \in \Lambda\} \tag{3}$$

is closed in X.

● PROBLEM 12-24

The situation in which a continuous function is defined piecewise appears in analysis. For example, a function which is continuous on a segment $[0, n] \subset R$ is defined for each $[k, k+1]$ separately in such a way, that the adjacent functions agree on the common end points of the segments $[0, 1], [1, 2], \ldots, [n-1, n]$.

Partial definitions of functions are formulated in this theorem.

THEOREM

Let (X, T) represent a topological space and $\{A_\omega : \omega \in \Omega\}$ its cover-

ing. One of two conditions is true:

1. all sets A_ω are open, or

2. all sets A_ω are closed and form a neighborhood-finite family.

For each $\omega \in \Omega$ function

$$f_\omega : A_\omega \to Y \tag{1}$$

is continuous and, such that

$$f_\alpha \mid A_\alpha \cap A_\beta = f_\beta \mid A_\alpha \cap A_\beta \tag{2}$$

for each $\alpha, \beta \in \Omega$.

Then a unique continuous function exists

$$f : X \to Y \tag{3}$$

such that for each $\omega \in \Omega$

$$f \mid A_\omega = f_\omega. \tag{4}$$

■

Prove this theorem.

SOLUTION:

For each $x \in X$, we define

$$f(x) = f_\alpha(x) \tag{5}$$

where $\alpha \in \Omega$ is any index, such that

$$x \in A_\alpha. \tag{6}$$

The definition is unique, because if $x \in A_\alpha$ and $x \in A_\beta$ then

$$f(x) = f_\alpha(x) = f_\beta(x) \tag{7}$$

Since $f_\alpha \mid A_\alpha \cap A_\beta = f_\beta \mid A_\alpha \cap A_\beta$. Hence (5) defines a function which is unique.

Function f is continuous. Let $U \subset Y$ be open. Then

$$f^{-1}(U) \cap A_\omega = f_\omega^{-1}(U) \tag{8}$$

is open in A_ω for each ω.

Thus, by Problem 12–23, $f^{-1}(U)$ is open in X and f is continuous.

● PROBLEM 12-25

Show that the function

$$f: R \to R$$

such that

$$f(x) = x^2 \tag{1}$$

is not open.

SOLUTION:

Observe that if f is a continuous function, then the inverse image of every open set is an open set and the inverse image of every closed set is a closed set. It will be useful to define the following group of functions.

DEFINITION OF AN OPEN (CLOSED) FUNCTION

A function

$$f: X \to Y \tag{2}$$

is called an open (closed) function, if the image of every open (closed) set is open (closed).

■

Consider an open interval $(-1, 1)$. Function $f(x) = x^2$ maps $(-1, 1)$ into

$$f((-1, 1)) = [0, 1) \tag{3}$$

which is not an open set. Hence, f defined by (1) is not open.

● PROBLEM 12-26

Let $f: X \to Y$ denote a function and B represent a basis for a topological space (X, T).

Show that, if for every $B_\alpha \in B$ $f(B_\alpha)$ is open in Y, then f is an open function.

SOLUTION:

Suppose A is an open subset of X. Then

$$A = \bigcup_\alpha B_\alpha \, , \; B_\alpha \in B \tag{1}$$

since B is the basis for X. Hence

$$f(A) = f(\bigcup_{\alpha} B_{\alpha}) = \bigcup_{\alpha} f(B_{\alpha}). \tag{2}$$

By hypothesis, each $f(B_{\alpha})$ is open in Y. Therefore $f(A)$ is open, as a union of open sets, and f is an open function.

As a matter of fact two properties are equivalent:

1. f is an open function.

2. f maps each member of a basis for X to an open set in Y.

● PROBLEM 12-27

Let

$$p : R^2 \to R \tag{1}$$

represent the projection mapping of R^2 into the x-axis, i.e.,

$$p(x, y) = x. \tag{2}$$

1. Show that p is an open function.

2. Show that p is not a closed function.

SOLUTION:

1. The family of all open discs in R^2 forms a basis for R^2. We assume that R^2 is equipped with the Euclidean topology.

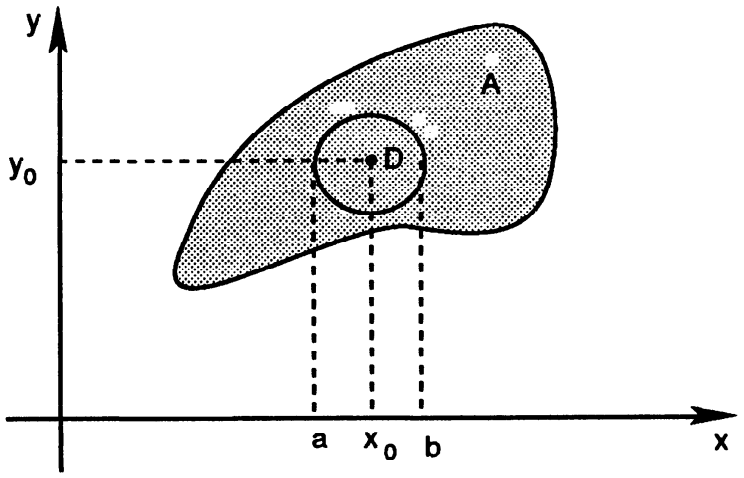


FIGURE 1

Note that the projection of any disc D

$$D = \{(x, y) : (x - x_0)^2 + (y - y_0)^2 < r^2\} \tag{3}$$

is an open interval

$$p(D) = (a, b) \tag{4}$$

where (a, b) is an open interval. According to Problem 12–26, we conclude that p, defined by (2), is an open function.

2. Consider the shaded region shown in Figure 2.

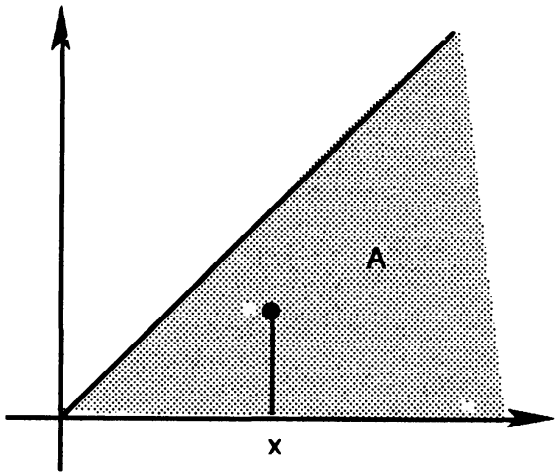


FIGURE 2

$$A = \{(x, y) : x \geq 0, 0 \leq y \leq x\}. \tag{5}$$

Set A is closed, while its projection $p(A) = [0, \infty)$ is not closed. Thus, p is not a closed function.

● PROBLEM 12–28

Prove the following theorem:

THEOREM

Let $f : X \rightarrow Y$ represent a closed function and let D denote any subset of Y. Let $\cup$ denote any open set, such that

$$f^{-1}(D) \subset U \subset X. \tag{1}$$

Then an open V exists, such that $D \subset V$ and $f^{-1}(V) \subset U$.

■

See Figure 1.

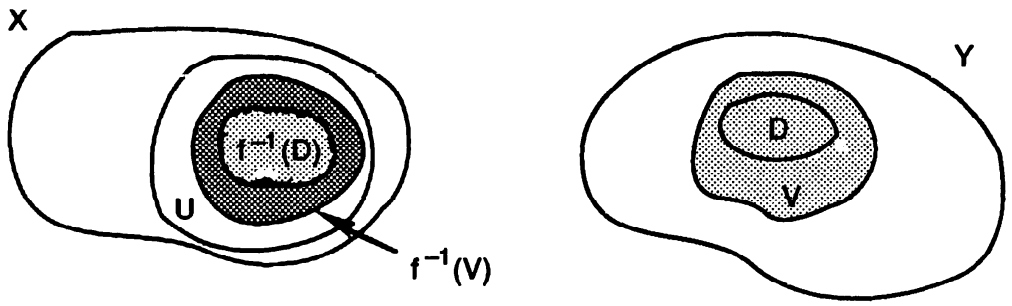


FIGURE 1

SOLUTION:

Let us define

$$V = Y - f(X - U). \tag{2}$$

By hypothesis

$$f^{-1}(D) \subset U. \tag{3}$$

Therefore

$$D \subset V.$$

Indeed, suppose $y \in D$, then $f^{-1}(y) \subset f^{-1}(D)$. Hence, $f^{-1}(y) \subset U$ and $y \in f(U)$. Thus, $y \notin f(X - U)$ and $y \in Y - f(X - U)$. Finally

$$y \in V \tag{4}$$

By hypothesis, f is closed and U is an open subset of X. Then $X - U$ is closed and $f(X - U)$ is closed. Therefore $V = Y - f(X - U)$ is an open subset of Y. Now we prove that $f^{-1}(V) \subset U$.

$$f^{-1}(V) = X - f^{-1}[f(X - U)] \subset X - (X - U) = U \tag{5}$$

A similar theorem exists for open functions.

THEOREM

Let $f : X \to Y$ denote an open function and let D represent any subset of Y. Let W represent any closed set, such that

$$f^{-1}(D) \subset W.$$

Then a closed set P exists, such that $D \subset P$ and

$$f^{-1}(P) \subset W.$$

■

● PROBLEM 12-29

Prove

$$(f : X \to Y \text{ is a closed function}) \Leftrightarrow (\forall A \subset X : \overline{f(A)} \subset f(\overline{A})).$$

SOLUTION:

$\Rightarrow$ Suppose f is a closed function. Let A represent any subset of X. Then $\overline{A}$ is a closed set and $f(\overline{A})$ is closed.

Since

$$f(a) \subset f(\overline{A}) \tag{1}$$

we have

$$\overline{f(A)} \subset \overline{f(\overline{A})} = f(\overline{A}). \tag{2}$$

$\Leftarrow$ Suppose, for each set $A \subset X$, $\overline{f(A)} \subset f(\overline{A})$.

Let A represent a closed set, $\overline{A} = A$. We have

$$f(A) \subset \overline{f(A)} \subset f(\overline{A}) = f(A). \tag{3}$$

Therefore

$$f(A) = \overline{f(A)} \tag{4}$$

so that set $f(A)$ is closed and function $f : X \to Y$ is closed.

● PROBLEM 12-30

Use the function

$$f(x) = \frac{x}{|x| + 1} \tag{1}$$

to show that the spaces R' and $(-1, +1)$ are homeomorphic.

SOLUTION:

DEFINITION OF HOMEOMORPHISM

A continuous bijective (that is, one-to-one and onto) function $f : X \to Y$, such that $f^{-1} : Y \to X$ is also continuous, is called a homeomorphism and denoted by

$$f : X \cong Y. \tag{2}$$

Two spaces X, Y, denoted by $X \cong Y$, are homeomorphic (or of the same topological type), if there is a homeomorphism $f: X \cong Y$.

■

First of all, observe that

$$f(x) = \frac{x}{|x| + 1}$$

where $f: R \to (-1, +1)$ is one-to-one and onto. Both functions f and f^{-1} are continuous. Hence, f, defined by (1), is a homeomorphism.

This result can be generalized to n-dimensional space

$$f: R^n \to B(0, 1) \tag{3}$$

where $B(0, 1)$ is the unit ball. Then $x = (x_1, x_2, \ldots, x_n)$ and

$$|x| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$$

where

$$f(x) = \frac{x}{|x| + 1} \tag{4}$$

Hence, R^n is homeomorphic to its unit ball $B(0, 1)$

$$R^n \cong B(0, 1). \tag{5}$$

● PROBLEM 12-31

Show that the homeomorphism relation is reflexive, symmetric and transitive.

SOLUTION:

Relation is reflexive. For any topological space (X, T), the identity mapping $f(x) = x$

$$f: X \to X \tag{1}$$

is a homeomorphism.

Symmetry. If $X \cong Y$, then a homeomorphism exists, such that

$$f: X \to Y. \tag{2}$$

We define $g = f^{-1}$, where

$$g: Y \to X. \tag{3}$$

Mapping g is a homeomorphism. Hence, $Y \cong X$.

We will show that the homeomorphism relation is transitive.

Suppose $X \cong Y$ and $Y \cong Z$. Then $f: X \to Y$ and $g: Y \to Z$ are homeomorphisms.

Mapping

$$g \circ f: X \to Z \tag{4}$$

where $g \circ f(x) = g[f(x)]$ is a homeomorphism, since both g and f are homeomorphisms.

Relation $\cong$ (X is homeomorphic to Y), defined in any family of topological spaces, is an equivalence relation. Hence, any family of topological spaces can be partitioned into disjoint classes of topologically equivalent spaces.

From now on, we will be more concerned with these classes of topologically equivalent spaces, than with the individual topological spaces.

● PROBLEM 12-32

Show that an area is not a topological property.

SOLUTION:

In Problem 12–31, we defined the classes of topologically equivalent spaces. Now we shall investigate the properties which are common for all members of the same class.

DEFINITION OF TOPOLOGICAL PROPERTIES

A property P of sets is called topological or a topological invariant if, whenever a topological space (X, T) has this property, then every space homeomorphic to (X, T) has property P.

■

That is, property P is common for all members of a class of topologically equivalent spaces.

Consider the function

$$f: R^2 \to R^2 \tag{1}$$

defined by

$$f: (x, y) \to (2x, y), \tag{2}$$

where f is a homeomorphism. It is easy to see that f transforms a unit square into a rectangle, as shown in Figure 1.

Hence, a figure of area one is transformed into a figure of area two.

Area is not a topological property. Similarly, it can be shown that length and volume are not topological properties.

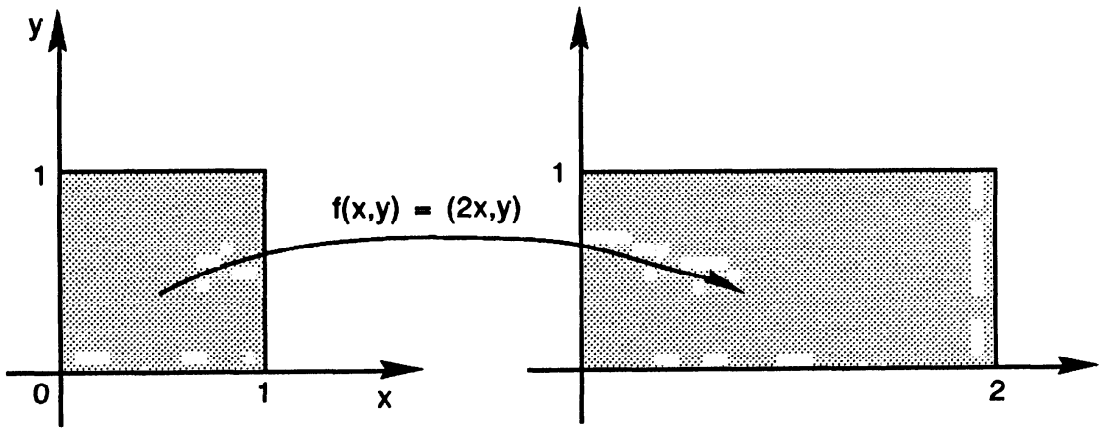


FIGURE 1

● PROBLEM 12-33

Show that each of the following conditions is necessary and sufficient for a one-to-one mapping, f, to be a homeomorphism:

1. $f(\overline{A}) = \overline{f(A)}$ for every $A \subset X$. (1)

2. $f^{-1}(\overline{B}) = \overline{f^{-1}(B)}$ for every $B \subset Y$. (2)

SOLUTION:

We shall apply the following:

THEOREM

$$(f \text{ is continuous}) \Leftrightarrow (f(\overline{A}) \subset \overline{f(A)} \text{ for every } A \subset X) \tag{3}$$

$$(f \text{ is continuous}) \Leftrightarrow \left(\overline{f^{-1}(B)} \subset f^{-1}(\overline{B}) \text{ for every } B \subset Y\right). \tag{4}$$

Suppose f is a homeomorphism, then f is continuous and for every $A \subset X$

$$f(\overline{A}) \subset \overline{f(A)} \tag{5}$$

Also $f^{-1} : Y \to X$ is continuous and by (4), we obtain

$$\overline{f(A)} \subset f(\overline{A}). \tag{6}$$

Note that $(f^{-1})^{-1} = f$. From (5) and (6), we obtain

$$f(\overline{A}) = \overline{f(A)}. \tag{7}$$

Suppose, for every $A \subset X$, $f(\overline{A}) = \overline{f(A)}$. Then

$$f(\overline{A}) \subset \overline{f(A)} \tag{8}$$

and by (3), we conclude that f is continuous.

Also, for every $A \subset X$

$$\overline{f(A)} = f^{-1}[f^{-1}(A)] \subset f^{-1}[f^{-1}(\overline{A})] = f(\overline{A}). \tag{9}$$

Hence, $f^{-1} : Y \to X$ is continuous and f is a homeomorphism.

Similarly, we show that condition (2) is necessary and sufficient for a one-to-one function to be homeomorphic.

● PROBLEM 12-34

Suppose f maps X onto Y, $f : X \to Y$, where X is a T_1–space.

Show that a necessary and sufficient condition for f to be a homeomorphism is:

1. $\overline{A} = f^{-1}(\overline{f(A)})$ for every $A \subset X$ or (1)

2. $(x \in \overline{A}) \equiv (f(x) \in \overline{f(A)})$ (2)

SOLUTION:

DEFINITION OF A T_1-SPACE

A topological space (X, T) is called a T_1-space, if each single element set is closed, that is,

$$\{\overline{a}\} = \{a\} \text{ for each } a \in X.$$

■

For example, each metric space is a T_1-space.

We shall prove : for $f : X \to Y$, where X is a T_1-space

$$(f \text{ is a homeomorphism}) \Leftrightarrow (\overline{A} = f^{-1}(\overline{f(A)}) \text{ for every } A \subset X). \tag{3}$$

$\Rightarrow$ Since f is a homeomorphism, it is thus one-to-one. From Problem 12–33, equation (1),

$$f(\overline{A}) = \overline{f(A)} \tag{4}$$

or

$$f^{-1}[f(\overline{A})] = \overline{A} = f^{-1}[\overline{f(A)}]. \tag{5}$$

$\Leftarrow$ We have to show that f is one-to-one. Suppose $f(a) = f(b)$. Then

$$\overline{\{a\}} = f^{-1}[\overline{f(a)}] = f^{-1}[\overline{f(b)}] = \overline{\{b\}}. \tag{6}$$

Since (X, T) is a T_1-space,

$$a = b. \tag{7}$$

Thus, f is one-to-one. Then from (1),

$$f(\overline{A}) = \overline{f(A)} \tag{8}$$

for every $A \subset X$.

According to Problem 12–33, f is a homeomorphism.

Conditions (1) and (2) are equivalent. From (2), we conclude that every property, expressed in terms of the operation $\overline{A}$ and of operations of set theory and of logics, is topological. We can briefly say that if a point $a \in X$ (or a set, family of sets, etc.) has a given property with respect to the space (X, T), then $f(a)$ has the same property with respect to Y, provided that $f: X \to Y$ is a homeomorphism.

● PROBLEM 12-35

1. Show that two closed intervals, $[a, b]$ and $[c, d]$, are homeomorphic.

2. Show that an open interval $(-1, 1)$ and the real line are homeomorphic.

3. Show that the surface of the sphere with one point removed is homeomorphic to the plane.

SOLUTION:

1. Suppose $a < b$ and $c < d$. Define $f(x)$ by

$$f(x) = \frac{d - c}{b - a} x + \frac{bc - ad}{b - a} \tag{1}$$

f is a homeomorphism, which maps the first interval onto the second. Hence, the two closed intervals are homeomorphic. To show that, one can also use the drawing shown in Figure 1.

2. Function $f: (-1, 1) \to R$

$$f(x) = \tan \frac{\pi x}{2} \tag{2}$$

is a homeomorphism. Hence, $(-1, 1)$ and R are homeomorphic.

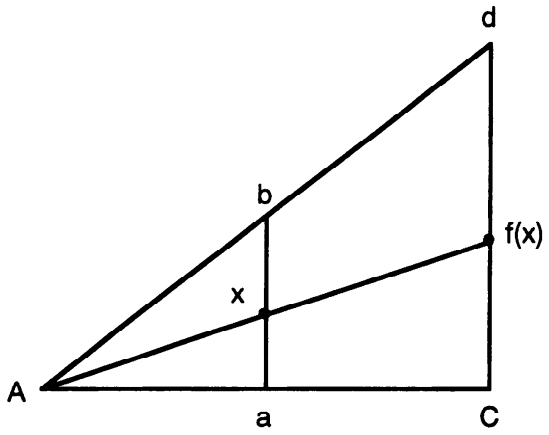


FIGURE 1

3. Consider the surface

$$x^2 + y^2 + (z-1)^2 = 1 \tag{3}$$

with the point (0, 0, 2) removed. (See Figure 2).

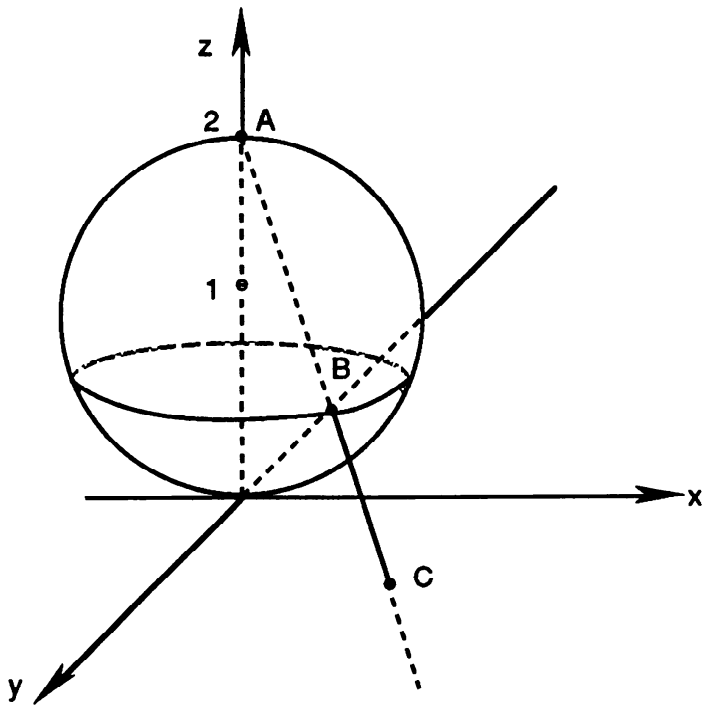


FIGURE 2

Draw a line from A through the point B on the surface of the sphere and through point C on the plane xy.

Each point B on the surface of the sphere is mapped on the plane $z = 0$. This mapping is one-to-one, onto and continuous; the inverse mapping also has the same properties. Thus, a sphere with one point removed and a plane are homeomorphic.

THEOREM

Let $f: X \to Y$ denote a bijection. Then the following properties are equivalent:

1. f is a homeomorphism.
2. $f(\overline{A}) = \overline{f(A)}$ for every $A \subset X$.
3. f is continuous and open.
4. f is continuous and closed.

■

Prove it.

SOLUTION:

1. $\Leftrightarrow$ 2. This was proved in Problem 12-33.

1. $\Leftrightarrow$ 3. f is a homeomorphism, hence $f^{-1}: Y \to X$ is continuous. Therefore, for each open $A \subset X$,

$$f^{-1}[f^{-1}(A)] = f(A) \tag{1}$$

and $f(A)$ is open in Y.

Suppose f is continuous and open, then the image of each open set in X is an open set in Y. Hence, f^{-1} is continuous and f is a homeomorphism.

3. $\Leftrightarrow$ 4. Note, that if $f: X \to Y$ is bijective, then the conditions that f is open and f is closed, are equivalent. Suppose f is open and $A \subset X$ is closed. Then set B is open

$$A = X - \mathrm{B} \tag{2}$$

and

$$f(A) = f(X - B) = f(X) - f(B) = Y - f(B) \tag{3}$$

Hence, since $f(B)$ is open, the set $f(A)$ is closed. Thus, f is closed.

● PROBLEM 12-37

The following theorem helps to determine a given function as being a homeomorphism:

THEOREM 1

Let $f: X \to Y$ and $g: Y \to X$ be continuous and, such that

$$f \circ g = 1_Y \tag{1}$$

and

$$g \circ f = 1_X \tag{2}$$

then f is a homeomorphism and

$$f^{-1} = g. \tag{3}$$

■

Prove this theorem.

SOLUTION:

First let us prove this.

THEOREM 2

Let $f: X \to Y$ and $g: Y \to X$, such that

$$g \circ f = 1_X \tag{4}$$

then f is one-to-one and g is onto.

■

The map of f is one-to-one, since

$$(f(x) = f(y)) \Rightarrow (x = g \circ f(x) = g \circ f(y) = y). \tag{5}$$

Function g is onto, since for any $x \in X$,

$$x = g[f(x)]. \tag{6}$$

By applying Theorem 2 and conditions (1) and (2), we conclude that both f and g are bijective. In addition to that, $f^{-1} = g$. Thus, f and f^{-1} are continuous and f is a homeomorphism.

● PROBLEM 12-38

Let

$$f: (X, T_X) \to (Y, T_Y) \tag{1}$$

denote a homeomorphism and let (A, T_A) represent a subspace of (X, T_X). Let

$$f_A : (A, T_A) \rightarrow (B, T_B) \tag{2}$$

be a restriction of f to A, $f_A = f \mid A$ and let

$$B = f(A)$$

and let T_B be a topology induced on B.

Show that f_A is a homeomorphism.

SOLUTION:

Since f is a bijection (that is, one-to-one and onto), $f_A : A \rightarrow B$ is also a bijection. The restriction of a continuous function is also a continuous function. Hence, f_A is continuous.

Let $U \subset A$ be T_A open, then

$$U = A \cap V \tag{3}$$

where $V \in T_X$. Since f is one-to-one,

$$f(A \cap V) = f(A) \cap f(V) \tag{4}$$

Thus,

$$f_A(U) = f(U) = f(A) \cap f(V) = B \cap f(\mathrm{V}) \tag{5}$$

Since f is open and $V \in T_X$, $f(V) \in T_Y$. Therefore,

$$B \cap f(V) \in T_B \tag{6}$$

and so, f_A is open.

● PROBLEM 12-39

Show that the subsets X and Y of the plane R^2 with the Euclidean topology, shown in Figure 1, are not homeomorphic. The topologies of X and Y are the usual induced topologies.

$$X = \{x : d(x,a_1) = 1 \text{ or } d(x, a_2) = 1\} \tag{1}$$

$$Y = \{y : d(y, a_3) = 1\} \tag{2}$$

$$a_1 = (0, 1) \quad a_2 = (0, -1) \quad a_3 = (3, 0).$$

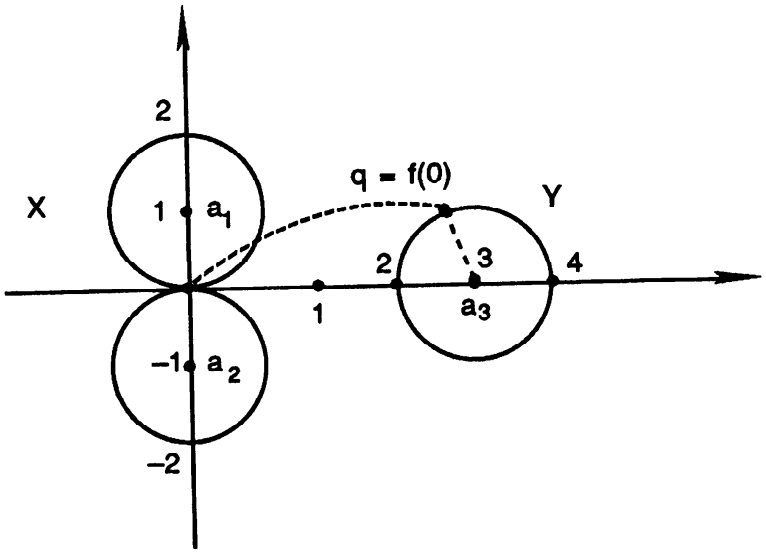


FIGURE 1

SOLUTION:

Suppose a homeomorphism

$$f : X \to Y \tag{3}$$

exists. Let us denote

$$f(0) = q \tag{4}$$

$$X_1 = X - \{0\}$$

$$Y_1 = Y - \{q\}. \tag{5}$$

By Problem 12–38, we conclude that

$$f_1 : X_1 \to Y_1 \tag{6}$$

is also a homeomorphism, with respect to the induced topologies.

Set Y_1 is connected by setting

$$q = (3 + \cos \beta_0, \sin \beta_0), \tag{7}$$

we define

$$g : (0, 2\pi) \to Y_1 \tag{8}$$

by

$$g(\beta) = (3 + \cos(\beta_0 + \beta), \sin(\beta_0 + \beta)). \tag{9}$$

Function g is homeomorphic interval $(0, 2\pi)$ is connected, and thus, Y_1 is connected.

To obtain a contradiction, we will show that X_1 is not connected.

The sets

$$A = \{(x, y) : x > 0\}$$

$$B = \{(x, y) : x < 0\} \tag{10}$$

are both open in R^2. Hence,

$$A_1 = X_1 \cap A$$

$$B_1 = X_1 \cap B \tag{11}$$

are open subsets of X_1, such that

$$A_1, B_1 = \phi, \quad A_1 \cap B_1 = \phi, \quad A_1 \cup B_1 = X_1 \tag{12}$$

space X_1 is not connected. Hence, X_1 and Y_1 are not homeomorphic, because connectedness is a topological property.

● PROBLEM 12-40

A topological space, (X, T), is given, along with an equivalence relation R. Construct the identification topology on $X/_R$ (sometimes called the quotient topology).

SOLUTION:

If X is a set and R is an equivalence relation on X, then R determines a partition of X into equivalence classes.

Two elements, $x, y \in X$, belong to the same class, if and only if, $x R y$ (x is in R-relation to y). The set of equivalence classes is denoted by $X/_R$.

Let us define a mapping

$$f : X \to X/_R \tag{1}$$

by

$$f(x) = [x] \tag{2}$$

where $[x]$ is the equivalence class, such that $x \in [x]$.

f is called the identification mapping (or quotient mapping). See Figure 1.

Let us introduce a topology on $X/_R$.

A subset U of $X/_R$ is open, if and only if, $f^{-1}(U)$ is open in X. In Problem 12-41, we will show that this is, indeed, a topology. This topology is called the identification topology or quotient topology.

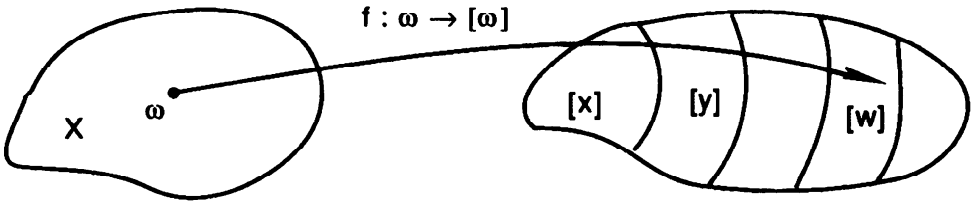


FIGURE 1

● PROBLEM 12-41

Prove that the identification topology, defined in Problem 12–40, is indeed, topology.

SOLUTION:

Let us denote the collection of all open sets of $^{X}/_{R}$ by T_R.

1. $$f^{-1}(\phi) = \phi \text{ and } f^{-1}(^{X}/_{R}) = X \tag{1}$$

(X, T) is a topological space, hence, $\phi, X \in T$. Thus,

$$\phi \in T_R \text{ and } ^{X}/_{R} \in T_R \tag{2}$$

2. Let $E, F \in T_R$, then $f^{-1}(E)$ and $f^{-1}(F)$ are open sets in X. Thus,

$$f^{-1}(E) \cap f^{-1}(F) \in T. \tag{3}$$

But

$$f^{-1}(E) \cap f^{-1}(F) = f^{-1}(E \cap F) \in T. \tag{4}$$

Hence, $E \cap F$ is an open set in $^{X}/_{R}$

$$E \cap F \in T_R \tag{5}$$

3. Let $\{E_\alpha\}$ represent a family of open sets in $^{X}/_{R}$. Then each set

$$f^{-1}(E_\alpha) \in T \tag{6}$$

and

$$\bigcup_{\alpha} f^{-1}(E_\alpha) \in T. \tag{7}$$

We have

$$\bigcup_{\alpha} f^{-1}(E_\alpha) = f^{-1}\left(\bigcup_{\alpha} E_\alpha\right) \in T \tag{8}$$

and $\bigcup_{\alpha} E_{\alpha}$ is an open subset of X/R.

The collection of open sets of X/R forms a topology on X/R.

● PROBLEM 12-42

The closed interval [0, 1] is equipped with the absolute value topology. An equivalence relation is defined by:

1. 0 is equivalent to 1

2. every other element of the segment [0, 1] is equivalent only to itself.

The equivalence classes are $\{0, 1\}$ and $\{x\}$ for $0 < x < 1$.

Show that the identification space defined is homeomorphic to a circle.

SOLUTION:

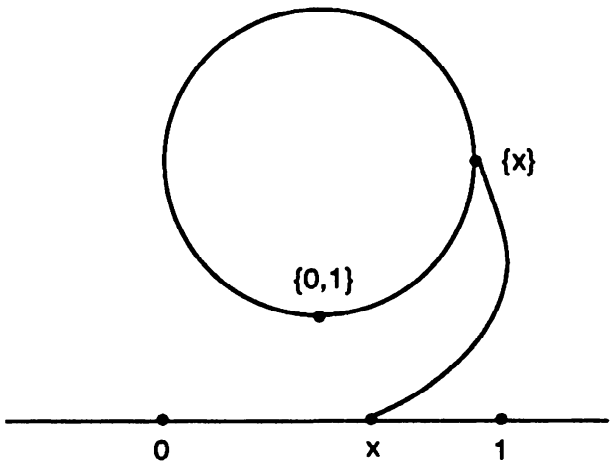


FIGURE 1

The endpoints 0 and 1 of the segment [0, 1] become a single point $\{0, 1\}$ of the new topological space. Hence, we wrap the segment [0, 1] around a circle of a radius of $1/2\pi$ and obtain a continuous function from [0, 1] onto a circle. The inverse function is also continuous.

The identification space, obtained from [0, 1], is homeomorphic to a circle.

CHAPTER 13

SEPARATION AXIOMS

T_0– Spaces	13–1, 13–2
T_1– Spaces	13–3, 13–4, 13–5, 13–6, 13–7
T_2– Spaces	13–8, 13–9, 13–10, 13–11, 13–12, 13–13
Regular Spaces	13–14, 13–15, 13–16, 13–17
T_3– Spaces	13–18, 13–19, 13–20, 13–21
Normal Spaces	13–22, 13–23, 13–24, 13–25
T_4– Spaces	13–26, 13–27, 13–28
Extension of Functions, Urysohn's Lemma	13–29, 13–30, 13–31, 13–32
	13–33, 13–34, 13–35
Functions Separating Points	13–36, 13–37
Completely Regular Spaces	13–38, 13–39
Tychonoff Spaces	13–40, 13–41

Let $X = \{0, 1\}$. Show that X, with the indiscrete topology, is not a T_0–space. Give an example of topology T, for which (X, T) is a T_0–space.

SOLUTION:

DEFINITION OF T_0–SPACE

The space (X, T) is said to be a T_0–space, if for any two distinct $a, b \in X$, there is a neighborhood of at least one, which does not contain the other.

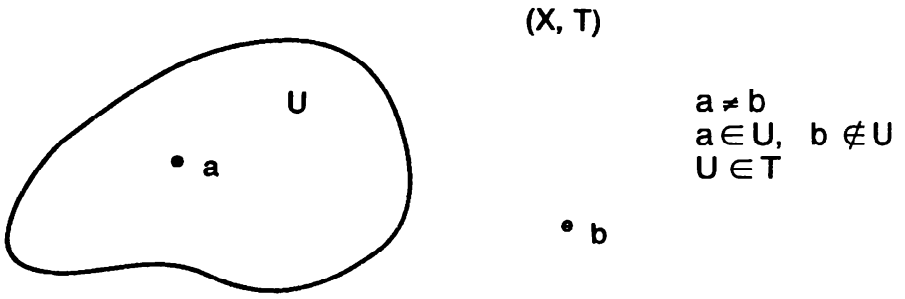


FIGURE 1

■

With the indiscrete topology $T = \{\phi, X\}$ one cannot separate 0 from 1 or 1 from 0. Consider the set $X = \{0, 1\}$ with topology $T = \{\phi, X, \{0\}\}$. Then, for the two distinct points 0 and 1, an open set $\{0\}$ exists, such that

$$0 \in \{0\} \text{ but } 1 \notin \{0\}.$$

Hence, (X, T) is a T_0–space. Note that each metric space (X, d) is a T_0–space.

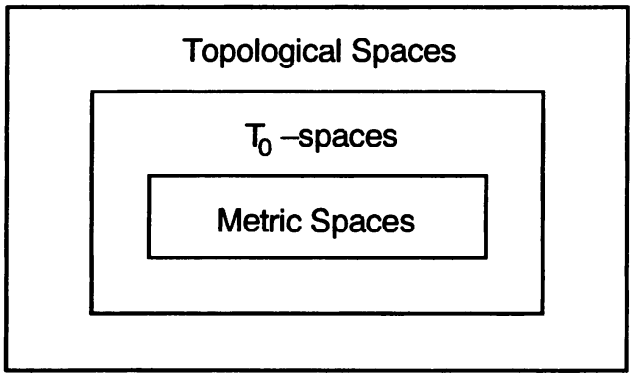


FIGURE 2

● PROBLEM 13-2

Explain why, in general, a pseudometric space is not T_0.

SOLUTION:

A pseudometric on X is a function $D : X \times X \to R_+$, such that

1. $D(x, y) = D(y, x)$

2. $D(x, y) \leq D(x, z) + D(z, y)$

for all $x, y, z, \in X$.

Thus, it is possible that for two distinct points $a, b \in X$,

$$D(a, b) = 0.$$

Any neighborhood of a contains b and any neighborhood of b contains a.

Hence, a pseudometric space is not a T_0–space. Points $a, b \in X$ are distinct, but such that, $D(a, b) = 0$ cannot be separated.

● PROBLEM 13-3

Show that every metric space is a T_1–space.

SOLUTION:

We shall start with the

DEFINITION OF T_1–SPACE

A topological space (X, T) is called a T_1–space, if every single element set is closed, that is,

$$\forall\, a \in X \quad \{a\} = \overline{\{a\}}. \tag{1}$$

■

Note, that in a metric space (X, d), the condition for a set A to be closed, can be expressed by the implication

$$\left(\begin{matrix} \lim\limits_{n \to \infty} x_n = x \\ x_n \in A \end{matrix} \right) \Rightarrow (x \in \overline{A}).$$

Since $\lim a = a$, we have

$$\overline{\{a\}} = \{a\}.$$

Thus, every metric space is a T_1–space (see Figure 1).

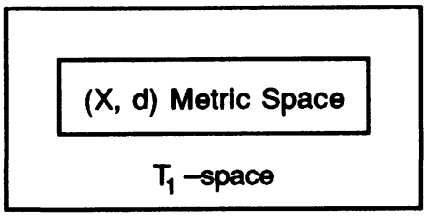


FIGURE 1

There are topological spaces which are not T_1–spaces. For example, space X, containing two points $X = \{a, b\}$ with topology $T = \{\phi, X\}$, is not a T_1–space.

We showed before, that every metric space can be regarded as a topological space. Figure 2 illustrates the results.

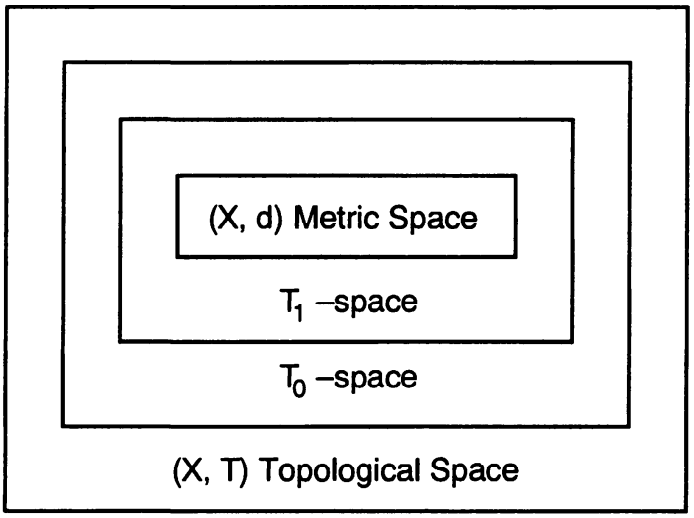


FIGURE 2

From the definitions, it is clear that any T_1–space is also a T_0–space.

● PROBLEM 13-4

Show that a topological space (X, T) is a T_1–space iff for any pair of distinct points $a, b \in X$, the open sets $G, H \in T$ exist, such that

$$a \in G, \quad b \notin G \quad \text{and} \quad b \in H, \quad a \notin H.$$

See Figure 1.

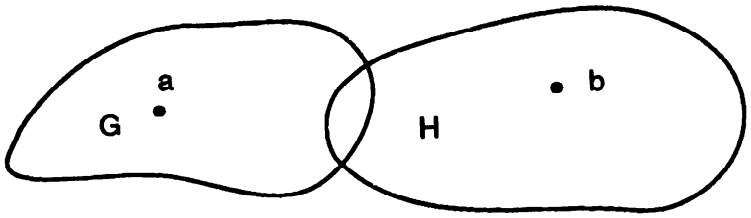


FIGURE 1

SOLUTION:

Suppose (X, T) is a T_1–space. Then, for any $x \in X$, $\{x\}$ is a closed set. Let $a, b \in X$ and $a \neq b$. The sets $X - \{a\}$ and $X - \{b\}$ are open, and

$$a \in X - \{b\} \quad \text{and} \quad b \notin X - \{b\}$$

$$b \in X - \{a\} \quad \text{and} \quad a \notin X - \{a\}.$$

Conversely, suppose $x \in X$. We shall show that $\{x\}$ is closed, i.e., $X - \{x\}$ is open. Let $y \in X - \{x\}$, then $y \neq x$ and an open set H_y exists, such that

$$y \in H_y \quad \text{and} \quad x \in H_y.$$

Thus,

$$y \in H_y \subset X - \{x\} \text{ and } X - \{x\} = \bigcup_{y \neq x} H_y.$$

Since all H_y are open sets, $X - \{x\}$ is open and $\{x\}$ is closed, $\{x\} = \overline{\{x\}}$.

● PROBLEM 13-5

Let X represent a finite set. Prove that the only topology on X, which makes X into a T_1–space, is the discrete topology.

SOLUTION:

Let X represent any finite set and T a topology on X, such that (X, T) is a T_1–space.

A space (X, T) is t_1, iff every one-point subset of X is closed. Since the union of two closed subsets is a closed subset, we conclude that all subsets of X are closed.

Therefore, all subsets of X are open and T is the discrete topology.

● PROBLEM 13-6

Let (X, T) denote a T_1–space. Show that the following conditions are equivalent:

1. $a \in X$ is an accumulation point of A.

2. every open set containing a contains an infinite number of points of A.

SOLUTION:

1. $\Rightarrow$ 2.

Suppose $a \in X$ is an accumulation point of A, and G is an open set $a \in G$, containing only a finite number of points of A different from a. Then

$$B = \{a_1, a_2, \ldots, a_n\} = A \cap [G - \{a\}].$$

B is a finite subset of a T_1–space, hence, it is closed and $X - B$ is open. Let

$$H = (X - \mathrm{B}) \cap G.$$

Then H is open, $a \in H$ and H do not contain any points of A different from a. Hence, a is not an accumulation point of A.

2. $\Rightarrow$ 1.

By definition of an accumulation point.

● PROBLEM 13-7

Prove, that if (X, T) and (Y, T') are homeomorphic and (X, T) is a T_1–space (or T_0–space), then so is (Y, T').

SOLUTION:

Let f denote a homeomorphism

$$f: X \to Y$$

and X be a T_1–space. A space (X, T) is T_1, if and only if every one-point subset of X is closed.

Let y represent any point of Y, $y \in Y$. The set $f^{-1}(y)$ is a one-point subset of X and since X is T_1, the set of $\{f^{-1}(y)\}$ is closed.

Since $f: X \to Y$ is a homeomorphism, it maps closed sets into closed sets. Therefore, for any $y \in Y$

$$\{y\} = \overline{\{y\}}.$$

Thus, (Y, T') is a T_1–space. Similarly, we can show that, if X is T_0, then so is Y.

● PROBLEM 13-8

Prove the following theorem:

THEOREM

Each T_2–space is a T_1–space.

SOLUTION:

DEFINITION OF T_2–SPACE (or a Hausdorff space or a separated space)

A topological space is a Hausdorff space if, for each pair of points $a \neq b$, two disjoint open sets A and B exist, such that

$$a \in A, \quad b \in B \cdot \quad A \cap B = \phi. \tag{1}$$

■

It is easy to see that each metric space is a T_2–space. Now, we shall show that each T_2–space is also a T_1–space. Suppose (X, T) is a T_2–space. Let $a \in X$ represent a given point. For each $x \in X, x \neq a$

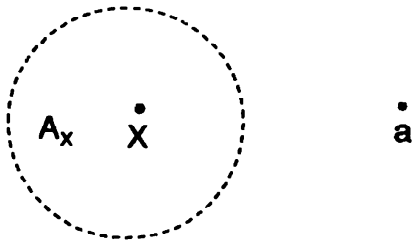


FIGURE 1

there is an open A_x, excluding the dotted circle in Figure 1, such that $x \in A_x$ and $a \notin A_x$. Thus

$$X - \{a\} = \bigcup_{x \neq a} A_x \tag{2}$$

and $X - \{a\}$ is an open set as a union of a family of open sets A_x.

Hence, $\{a\}$ is closed and X is a T_1–space.

● PROBLEM 13-9

Find a T_1–space which is not a T_2–space.

Show that the properties of being a T_1– and a T_2–space are hereditary.

SOLUTION:

Consider the set X, consisting of 0 and all points $1/n$, for $n = 1, 2, 3, \ldots$

$$X = \{0, 1, 1/2, 1/3, 1/4. \ldots\} \tag{1}$$

We define the topology T on X: Sets containing 1 are open, if and only if they are complements of finite sets.

Sets which do not contain the point 1 are open, when they are open in the sense of the usual topology of real numbers.

Hence, each open set containing 0 is infinite.

Therefore, points 0 and 1 cannot be separated by two open disjoint sets. The space is T_1–space, but not T_2.

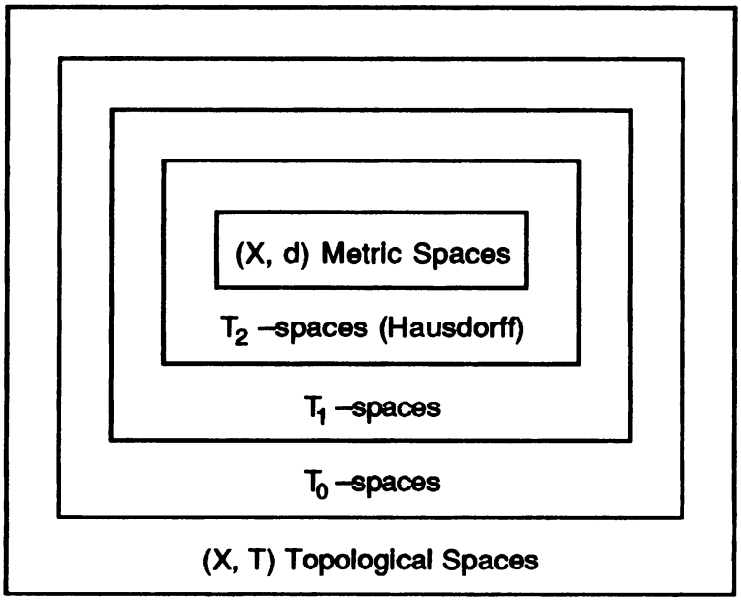


FIGURE 1

● PROBLEM 13-10

Show that the set X with the order topology is a T_2–space.

SOLUTION:

Let X represent any set totally ordered by $<$, $(X, <)$. Let S represent the family of subsets of X of the form

$$\{x : x < a\} \quad \text{or} \quad \{x : a < x\}$$

for all $a \in X$.

The family S forms a subbasis for a topology T on X, called the order topology induced by $<$. We shall show that (X, T) is a T_2–space.

Let $a, b \in X$ represent two distinct points. Since X is totally ordered, either $a < b$ or $b < a$. Suppose $a < b$. There are two possibilities now:

1. An element $c \in X$ exists, such that

$$a < c < b.$$

Then

$$\{x : x < c\} \quad \text{and} \quad \{x : c < x\}$$

are two disjoint neighborhoods of a and b respectively.

2. No $c \in X$ exists, such that

$$a < c < b.$$

Then

$$\{x : x < b\} \quad \text{and} \quad \{x : a < x\}$$

are disjoint neighborhoods of a and b respectively.

● PROBLEM 13–11

If

$$f : (X, T) \rightarrow (Y, T') \tag{1}$$

is onto and one-to-one, f^{-1} is continuous and X is a T_2–space, prove that Y is a T_2–space.

SOLUTION:

Let y_1 and y_2 represent any two distinct points of Y. Since f is one-to-one and onto, two distinct points of X exist, such that

$$x_1, x_2 \in X$$

$$f^{-1}(y_1) = x_1, \quad f^{-1}(y_2) = x_2.$$

(X, T) is a Hausdorff space, therefore, there are two open sets $U_1, U_2 \subset X$, such that

$$x_1 \in U_1, \quad x_2 \in U_2, \quad U_1 \cap U_2 = \phi.$$

Since f is bijective,

$$f(U_1) \subset Y, \quad f(U_2) \subset Y$$

$$f(U_1) \cap f(U_2) = \phi$$

Now, since f^{-1} is continuous, the function $(f^{-1})^{-1} = f$ maps open sets into open sets. Hence, $f(U_1), f(U_2) \in T'$ are open sets.

$$y_1 \in f(U_1), \; y_2 \in f(U_2)$$

We conclude that (Y, T') is a T_2–space.

If two spaces are homeomorphic and one of them is a T_2–space, then so is the other.

● PROBLEM 13-12

Prove the theorems:

1. $\left(\begin{matrix}(X,T) \text{ a Hausdorff space} \\ A \text{ a finite subset of } X\end{matrix}\right) \Rightarrow (A \text{ is closed}).$

2. $\left(\begin{matrix}(X,T) \text{ a Hausdorff space} \\ A \subset X, \; x \text{ is a cluster point of } A \\ U \text{ a neighborhood of } X\end{matrix}\right) \Rightarrow (U \cap A \text{ is infinite}).$

SOLUTION:

1. Let $x \in X$. We shall show that $\{x\}$ is closed. Indeed, let

$$y \in X - \{x\}$$

then a neighborhood U of y exists, such that

$$x \in U.$$

Hence,

$$U \subset X - \{x\}$$

and the set $X - \{x\}$ is open. Thus, $\{x\}$ is closed.

Any subset $\{x_1, \ldots . x_n\}$ of a Hausdorff space is closed.

2. Suppose, on the contrary, that $U \cap A$ is finite. Then

$$U \cap [A - \{x\}]$$

is closed. Hence

$$U - [U \cap (A - \{x\}]$$

is open.

But

$$x \in U - [U \cap (A - \{x\})] = U - (A - \{x\}).$$

Then $U - (A - \{x\})$ is a neighborhood of x.

Since x is a cluster point of A, then

$$\{[U - (A - \{x\})] - \{x\}\} \cap A \neq \phi.$$

Contradiction!

● PROBLEM 13-13

Show, that if (X, T) is a Hausdorff space, then every convergent sequence in X has a unique limit.

SOLUTION:

Let (x_n) denote a convergent sequence with two limits a, b, such that $a \neq b$.

Since (X, T) is a Hausdorff space, the open sets U_1 and U_2 exist, such that

$$a \in U_1, \quad b \in U_2, \quad U_1 \cap U_2 = \phi.$$

But (x_n) converges to a. Thus,

$$\exists k_1 \quad \forall n > k_1 \quad x_n \in U_1.$$

Similarly,

$$\exists k_2 \quad \forall n > k_2 \quad x_n \in U_2.$$

But the sets U_1 and U_2 are disjoint. Contradiction. Hence, $a = b$.

● PROBLEM 13-14

Give an example of a regular space, which is not a T_1–space.

SOLUTION:

DEFINITION OF REGULAR SPACE

A topological space (X, T), is said to be regular if, given any closed subset $F \subset X$ and any point $x \in X$, such that $x \notin F$, there are open sets U and V, such that

$$F \subset U, \quad x \in V, \quad \text{and} \quad U \cap V = \phi \tag{1}$$

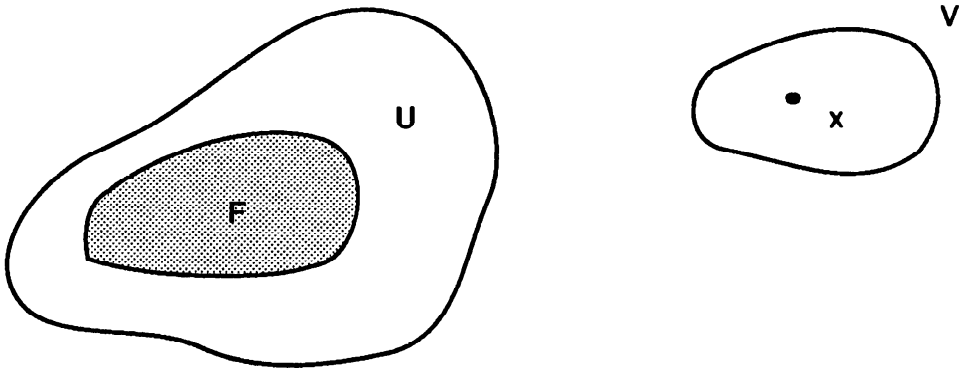


FIGURE 1

■

Consider the set $X = \{a, b, c\}$ with topology $T = \{\phi, X, \{a\}, \{b, c\}\}$. Note that the closed sets are $\phi, X, \{a\}, \{b, c\}$.

The topological space (X, T) is a regular space. But (X, T) is not a T_1–space. Set $\{c\}$ which is finite is not closed $\{c\} \neq \{c\}$.

● PROBLEM 13-15

Let (Y, T_Y) denote the subspace of (X, T). Let $y \in Y$ and $A \subset Y$. Show, that if y does not belong to the T_Y–closure of A, then y does not belong to the T–closure of A.

SOLUTION:

By definition,

$$T_Y\text{–closure of } A = \overline{A} \cap Y \tag{1}$$

where $\overline{A}$ is the T–closure of A.

But

$$y \in Y. \tag{2}$$

Therefore, if

$$y \notin \overline{A} \cap Y \tag{3}$$

then

$$y \notin \overline{A}. \tag{4}$$

Note, that an equivalent definition of regular space is: a topological space (X, T) is called regular if, for every point x and every closed set F, such that $x \notin F$, an open set G exists, such that

$$x \in G \quad \text{and} \quad \overline{G} \cap F = \phi.$$

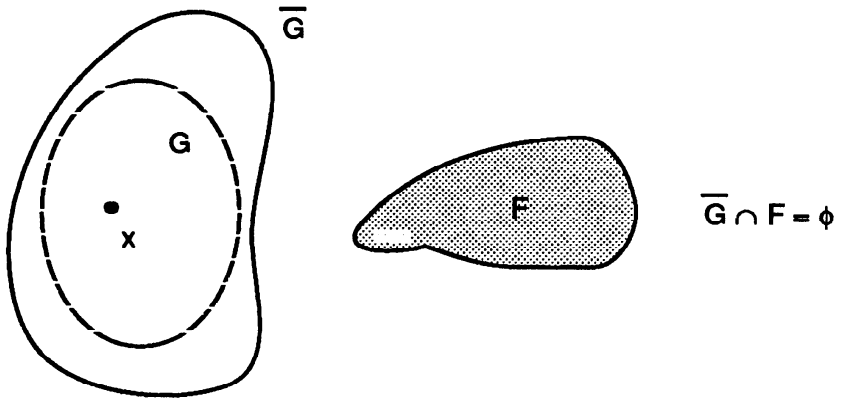


FIGURE 1

● PROBLEM 13–16

Show that every subset of a regular space is regular, i.e., the property of being a regular space is hereditary.

SOLUTION:

Let (X, T) denote a regular space and (A, T_A) its subspace. See Figure 1. Let $y \in A$ and let F represent a T_Y–closed subset of A, such that $y \notin F$. Hence, by Problem 13–15,

$$y \notin \overline{F}$$

where $\overline{F}$ is the T–closure of F. Space (X, T) is regular. Two open sets G and H exist, such that

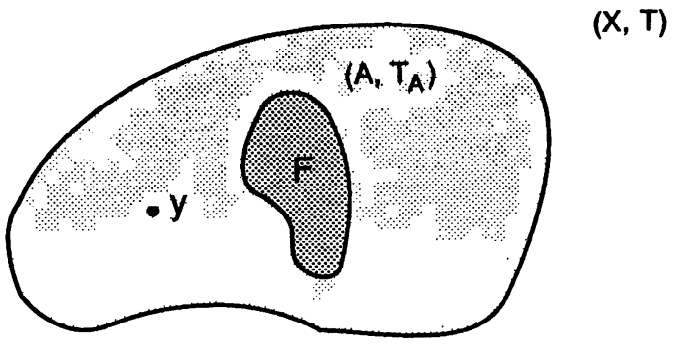


FIGURE 1

$$\overline{F} \subset G, \quad y \in H, G \cap H = \phi .$$

But

$$G \cap A \text{ and } H \cap A \text{ are } T_A\text{–open}$$

subsets of A.

$y \in A \cap H$ because $y \in A$ and $y \in H$. Sets $A \cap G$ and $A \cap H$ are disjoint because $G \cap H = \phi$.

Also, since

$$F \subset A \text{ and } F \subset \overline{F} \subset G \Rightarrow F \subset A \cap G .$$

Thus, (A, T_A) is also regular.

● PROBLEM 13-17

Show that a space (X, T) is regular, if and only if given any $x \in X$ and any neighborhood U of x, $x \in U \in T$, there is a neighborhood V of x, such that

$$\overline{V} \subset U.$$

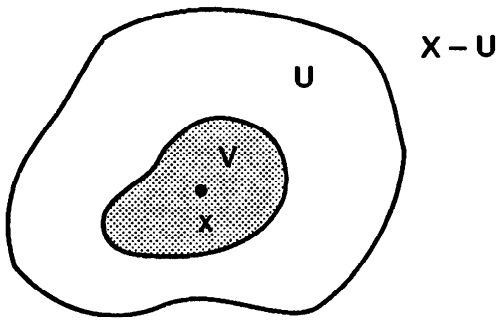


FIGURE 1

SOLUTION:

Suppose (X, T) is regular. Let U represent the neighborhood of x. Set $X - U$

is closed and $x \notin X - U$. Hence, open sets V and Z exist, such that

$$x \in V, \quad X - U \subset Z, \quad V \cap Z = \phi.$$

Since $X - U \subset Z$, we have $X - Z \subset U$. Also, since $V \cap Z = \phi$, we have

$$V \subset X - Z \subset U.$$

Hence,

$$V \subset \overline{V} \subset X - Z \subset U.$$

Now, suppose $x \in X$ and U are any neighborhood of x. Then, a neighborhood V of x exists, such that

$$\overline{V} \subset U.$$

Let $x \in X$ and F represent any closed subset of X, such that $x \notin F$.

$X - F$ is a neighborhood of x. Then V exists, such that, V is open, $x \in V$

$$\overline{V} \subset X - F.$$

Set $X - \overline{V}$ is open and $F \subset X - \overline{V}$, also V is an open subset which contains x. Since $V \subset \overline{V}$

$$V \cap (X - \overline{V}) = \phi.$$

Thus, V and $X - \overline{V}$ are the sets that we are looking for.

(X, T) is regular.

● PROBLEM 13-18

Show that every T_3–space is also a T_2–space.

SOLUTION:

DEFINITION OF A T_3–SPACE

A regular T_1–space is called a T_3–space.

■

Let (X, T) denote a T_3–space. We shall show that (X, T) is also a Hausdorff space.

Let $a, b \in X$ represent distinct points. Space (X, T) is a T_1–space, therefore, $\{a\}$ is a closed set. Since a and b are distinct,

$$b \in \{a\}$$

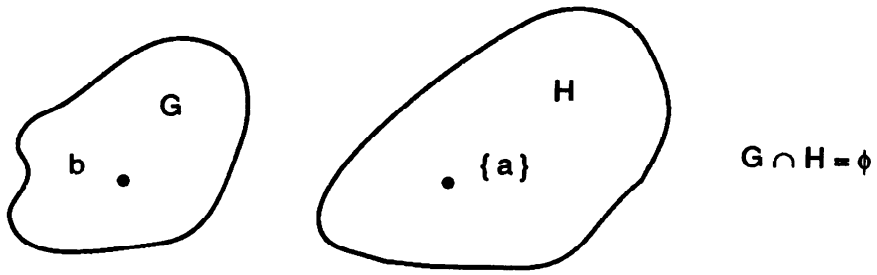


FIGURE 1

Since (X, T) is a regular space, two disjoint open sets G and H exist, such that

$$b \in G \quad \text{and} \quad \{a\} \subset H$$

Hence, a and b belong to two disjoint open sets G and H. (X, T) is a Hausdorff space (T_2–space).

● PROBLEM 13-19

Show that any metric space (X, d) is a T_3–space.

SOLUTION:

Let (X, d) denote a metric space and U represent any neighborhood of $x \in X$. There is a ball $B(x, r)$, such that

$$x \in B(x, r) \subset U. \tag{1}$$

Let us take any number r'

$$0 < r' < r. \tag{2}$$

Then

$$B(x, r') \subset B(x, r) \tag{3}$$

and

$$\overline{B(x, r')} = \{y : d(x, y) \leq r'\} \subset B(x, r) \subset U. \tag{4}$$

Hence, for any $x \in X$ and any neighborhood U of x, there is a neighborhood $B(x, r')$ of x, such that

$$\overline{B(x, r')} \subset U.$$

According to Problem 13–17, we conclude that (X, d) is regular. Since (X, d) is also T_1, any metric space is a T_3–space.

● PROBLEM 13-20

We shall define a topology on the set of real numbers R by giving an open neighborhood system. Let $x \neq 0$, then P_x is the family of all open intervals which contain x. For $x = 0$, we define P_0 as the family of all sets of the form $]-a, a[- \{1/n\}$, where n is a positive integer. Show that (R, T) defined above is T_2, but not T_3.

SOLUTION:

First, we show that (R, T) is T_2. Let $x \in R$ and $y \in R, x \neq y$. If both x and y are different from 0, then

$$U = \left]x - \frac{|x-y|}{2}, x + \frac{|x-y|}{2}\right[$$

$$V = \left]y - \frac{|x-y|}{2}, y + \frac{|x-y|}{2}\right[$$

Bot U and V are open sets and

$$x \in U, \quad y \in V, \quad U \cap V = \phi.$$

If one point is 0, i.e., $x = 0$, then

$$U = \left]-\frac{|y|}{2}, \frac{|y|}{2}\right[- \{\frac{1}{n}\}.$$

Hence, (R, T) is a Hausdorff space (i.e., a T_2–space).

Now, we shall show that (R, T) is not T_3. Let $x = 0$ ad $F = \{1/n\}$, where n is a positive integer. The set F is closed.

Let V represent any neighborhood of $x = 0$

$$V =]-a, a[- \{1/n\}.$$

No open set U exists, such that $F \subset U$ and $V \cap U = \phi$.

Hence, (R, T) is not regular and, therefore, not T_3.

● PROBLEM 13-21

Show that the property of being T_3 is hereditary, that is, that any subspace of a T_3–space is a T_3–space.

SOLUTION:

Let Y denote a subspace of a T_3–space (X, T). Let F be closed in Y and

$x \in Y$ and $x \notin F$.

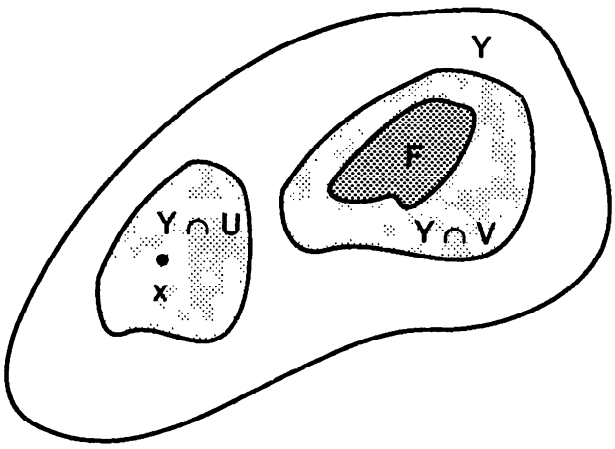


FIGURE 1

Since F is closed in Y,

$$F = Y \cap F' \tag{1}$$

where F' is a closed subset of X. Then $x \notin F'$. Space X is regular, hence, the open sets U and V in X exist, such that

$$x \in U, \quad F' \subset V, \quad U \cap V = \phi. \tag{2}$$

The sets $Y \cap U$ and $Y \cap V$ are open in Y and

$$x \in Y \cap U, \quad F \subset Y \cap V, \quad (Y \cap U) \cap (Y \cap V) = \phi. \tag{3}$$

Hence, space (Y, T_Y) is regular. Since (Y, T_Y) is a subspace of a T_1–space (X, T), it is also T_1. Therefore, (Y, T_Y) is a regular and a T_1–space, i.e., a T_3–space.

● PROBLEM 13-22

Show that any space (X, T), containing more than one point with the indiscrete topology, is normal.

SOLUTION:

DEFINITION OF NORMAL SPACE

A topological space (X, T) is said to be normal if, given any two disjoint closed sets F_1 and F_2 in X, there are disjoint open sets U and V, such that

$$F_1 \subset U \quad \text{and} \quad F_2 \subset V$$

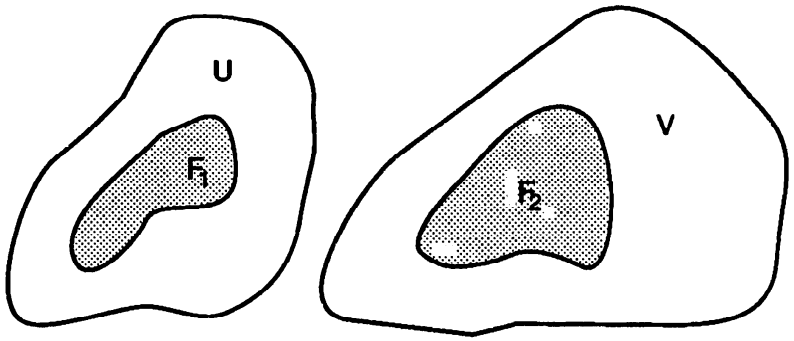


FIGURE 1

■

The indiscrete topology consists of two sets X and ϕ.

$$T = \{X, \phi\}.$$

Hence, the only closed sets are X and ϕ because $X - \phi = X$ and $X - X = \phi$.

Thus, there are no non-empty disjoint closed subsets of X. The space is normal.

It is easy to show that the space with discrete topology is normal. In this topology, each set is closed and open.

● PROBLEM 13-23

Prove this theorem:

THEOREM

A topological space (X, T) is normal iff, for every closed set F and every open set H containing F, an open set U exists, such that

$$F \subset U \subset \overline{U} \subset H. \tag{1}$$

■

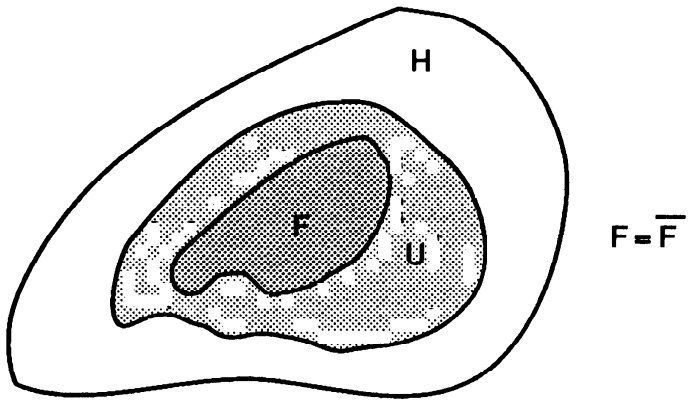


FIGURE 1

SOLUTION:

$\Rightarrow$ Let (X, T) denote a normal space. Let F represent a closed set and H an open set, such that

$$F \subset H \tag{2}$$

The set $X - H$ is closed and

$$F \cap (X - H) = \phi. \tag{3}$$

F and $X - H$ are two disjoint closed sets. Hence, the open sets U and U' exist, such that

$$F \subset U, \quad X - H \subset U', \quad U \cap U' = \phi. \tag{4}$$

Since

$$U \cap U' = \phi, \text{ we have } U \subset X - U' \tag{5}$$

also, since

$$X - H \subset U', \text{ we have } X - U' \subset H. \tag{6}$$

Set $X - U'$ is closed, hence, we conclude

$$F \subset U \subset \overline{U} \subset X - U' \subset H. \tag{7}$$

$\Leftarrow$ Let F_1 and F_2 denote disjoint closed sets. Then

$$F_1 \subset X - F_2 \tag{8}$$

and $X - F_2$ is open. An open set exists, such that

$$F_1 \subset U \subset \overline{U} \subset X - F_2. \tag{9}$$

But, since $\overline{U} \subset X - F_2$, we must have $F_2 \subset X - \overline{U}$. Also since $U \subset \overline{U}$, we have $U \cap (X - \overline{U}) = \phi$.

Thus, since $X - \overline{U}$ is open $F_1 \subset U$, $F_2 \subset X - \overline{U}$ and $U \cap (X - \overline{U}) = \phi$ where U and $X - \overline{U}$ are open sets.

● PROBLEM 13-24

Show that every metric space is normal.

SOLUTION:

Let (X, d) denote a metric space. Metric d induces the topology T on X. To show that (X, T) is normal, we shall apply the separation axiom.

THEOREM (SEPARATION AXIOM)

Let A_1 and A_2 represent closed disjoint subsets of a metric space (X, d). Then the disjoint open sets U_1 and U_2 exist, such that

$$A_1 \subset U_1 \quad \text{and} \quad A_2 \subset U_2$$

■

From this, we conclude that every metric space is normal.

● PROBLEM 13-25

Consider the set $X = \{a, b, c\}$ with the topology

$$T = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}. \tag{1}$$

Is (X, T) a T_1–space, a regular space, or a normal space?

SOLUTION:

The closed sets are

$$X, \phi, \{b, c\}, \{a, c\}, \{c\} \tag{2}$$

Not every singleton subset of X is closed. For example,

$$\{a\} \neq \overline{\{a\}}. \tag{3}$$

Hence, the space (X, T) is not T_1. Also, (X, T) is not a regular space. Take a closed subset $\{c\}$ and $a \notin \{c\}$. Then, the only open set, which contains $\{c\}$ is the whole space X, which contains a. We shall show that (X, T) is a normal space.

Let F_1 and F_2 represent disjoint closed subsets of X. Then one of them, say F_1 is the empty set ϕ.

The sets ϕ and X are disjoint open sets and

$$\phi = F_1 \subset \phi; \quad F_2 \subset X.$$

Thus, (X, T) is a normal space.

● PROBLEM 13-26

Show that every T_4–space is also a T_3–space. See Figure 1.

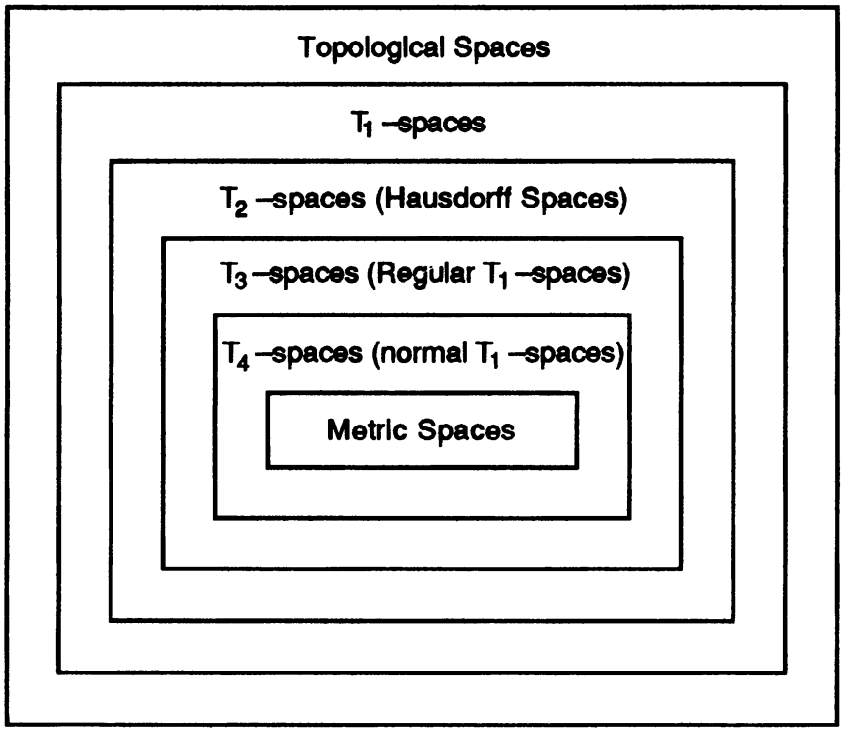


FIGURE 1

SOLUTION:

DEFINITION OF A T_4–SPACE

A normal space which is also a T_1–space is called a T_4–space.

■

Let (X, T) denote a T_4–space. Hence, (X, T) is normal and T_1. Suppose F is a closed subset of X and $a \notin F$. Since (X, T) is T_1, the singleton set $\{a\}$ is closed. Sets F and $\{a\}$ are closed and disjoint. Since (X, T) is normal, the open sets U_1 and U_2 exist, such that

$$\{a\} \subset U_1, \quad F \subset U_2, \quad U_1 \cap U_2 = \phi.$$

Therefore (X, T) is regular and T_3.

The diagram illustrates the relationship between different kinds of topological spaces, defined in this chapter.

● PROBLEM 13-27

Prove that if Y is a closed subset of a T_4–space (X, T), then the subspace (Y, T_Y) is also T_4–space.

SOLUTION:

Since every subspace of a T_1–space is T_1 and (X, T) is T_1 also, Y is a T_1–space. Since Y is closed, a subset F of Y is closed in Y, if and only if F is closed in X. Hence, if F_1 and F_2 are disjoint closed subsets of Y, they are also disjoint closed subsets of X.

Thus, the open sets U_1 and U_2 exist, such that

$$F_1 \subset U_1, \quad F_2 \subset U_2 \text{ and } U_1 \cap U_2 = \phi.$$

Then

$$F_1 \subset U_1 \cap Y, \quad F_2 \subset U_2 \cap Y,$$

and $U_1 \cap Y$ and $U_2 \cap Y$ are disjoint subsets of Y, open in Y. Since (Y, T_Y) is T_1 and normal, it is T_4.

● PROBLEM 13-28

Suppose that the space (X, T) is homeomorphic to the space (Y, T'), and that X is T_4. Prove that Y is T_4.

SOLUTION:

We have already shown that if a space X is homeomorphic to a space Y and X is T_1, then Y is T_1.

Now, suppose X is a normal space and $f: X \to Y$ is a homeomorphism.

Let G_1 and G_2 represent any two disjoint closed subsets of Y. Then $f^{-1}(G_1)$

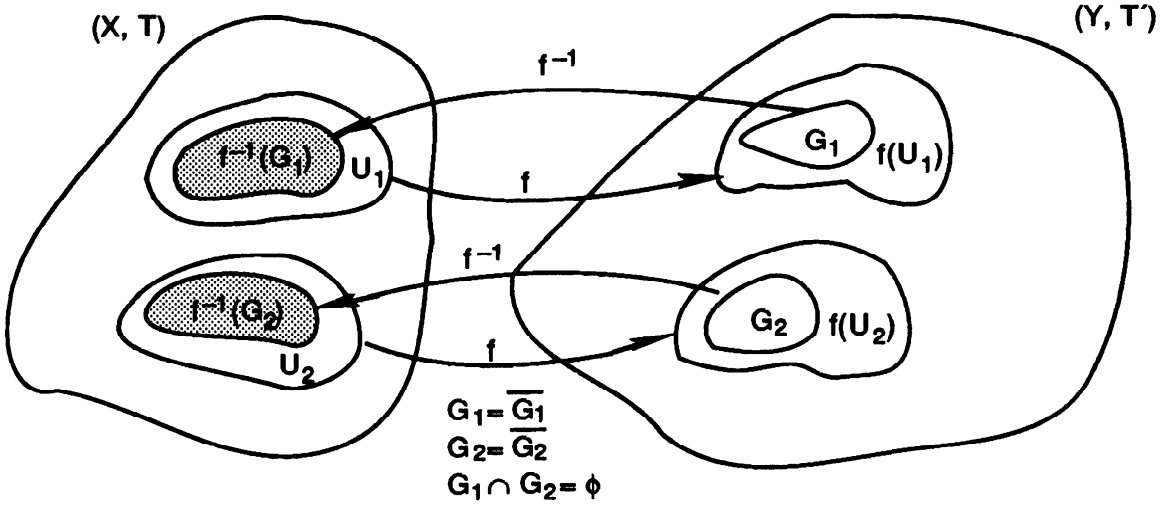


FIGURE 1

and $f^{-1}(G_2)$ are disjoint closed subsets of X. Since X is normal, there are two disjoint open sets U_1 and U_2 in X, such that

$$f^{-1}(G_1) \subset U_1 \quad f^{-1}(G_2) \subset U_2$$

The sets $f(U_1)$ and $f(U_2)$ are open in Y and

$$G_1 \subset f(U_1), \quad G_2 \subset f(U_2)$$

$$f(U_1) \cap f(U_2) = \phi.$$

Hence, (Y, T') is normal and T_4.

● PROBLEM 13-29

Describe the notion of the extension of a continuous function defined on a topological subspace.

SOLUTION:

The problem of extension of a function is one of the central and most difficult questions in all of topology. Suppose (X, T) is a topological space and (Y, T_Y) is its topological subspace.

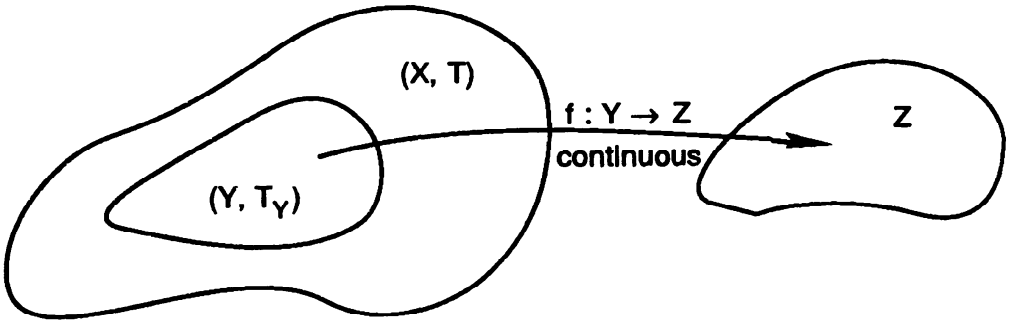


FIGURE 1

Let $f: Y \to Z$ denote a continuous function from Y into some space (Z, T'). The question is: does a continuous function $F: X \to Z$ exist, such that

$$F(y) = f(y) \text{ for each } y \in Y,$$

that is, $F \mid Y = f$?

In most cases the answer to this question is unknown. Only normal spaces have certain extension properties.

● PROBLEM 13-30

Here is one of the most important theorems in topology:

URYSOHN'S LEMMA

If a topological space (X, T) is normal, then, given any disjoint closed non-empty subsets A and B of X, there is a continuous function $f: X \to Z$, $Z = [0, 1]$, (Z has the absolute value topology), such that

$$\text{for every } a \in A, f(a) = 0$$

and

$$\text{for every } b \in B, f(b) = 1.$$

■

Prove that the converse of Urysohn's lemma is true.

SOLUTION:

Suppose (X, T) has the property described in Urysohn's lemma. Let A and B represent closed disjoint non-empty subsets of X. There is a continuous function $f: X \to [0, 1]$, such that for all $a \in A$, $f(a) = 0$ and for all $b \in B$, $f(b) = 1$.

Sets

$$U = \{x : 0 \leq x < {}^1/_2\}$$

$$U' = \{x : {}^1/_2 < x \leq 1\}$$

are disjoint open subsets of $[0, 1]$. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(U')$ are disjoint open subsets of X, such that

$$A \subset f^{-1}(U)$$

$$B \subset f^{-1}(U').$$

Hence, (X, T) is normal.

● PROBLEM 13-31

Let (X, T) denote a T_4–space which contains more than one point. Show that a non-constant continuous function $f: X \to [0, 1]$ exists.

SOLUTION:

Space X contains at least two points, for example, x and y. Since (X, T) is a T_4–space, it must be normal and T_1. For any T_1–space,

$$\{x\} = \{\overline{x}\}$$

$$\{y\} = \{\overline{y}\}$$

That is, any one-point subset is closed. The sets $\{x\}$ and $\{y\}$ are disjoint closed non-empty subsets of X.

Therefore, since X is normal, a continuous function

$$f: X \to [0, 1]$$

exists, such that

$$f(x) = 0 \quad \text{and} \quad f(y) = 1.$$

Function f is a continuous non-constant function from X into $[0, 1]$.

● PROBLEM 13-32

Prove this generalization of Urysohn's lemma:

Let A and B represent disjoint closed non-empty subsets of a normal space (X, T). Then a continuous function

$$f: X \to [a, b\}$$

exists, such that

$$f(x) = a \quad \text{for all} \quad x \in A$$

$$f(x) = b \quad \text{for all} \quad x \in B.$$

SOLUTION:

Since (X, T) is a normal space, we can apply Urysohn's lemma (Problem 13–30).

Function f', which is continuous exists, such that

$$f': X \to [0, 1]$$

and for all $x \in A$, $f(x) = 0$ and for all $x \in B$, $f(x) = 1$.

Consider function f, defined by

$$f(x) = a + (b - a) f'(x).$$

Since f' is continuous, so is f. Furthermore,

$$f(x) = \begin{cases} a & \text{for all } x \in A \\ b & \text{for all } x \in B \end{cases}.$$

● PROBLEM 13-33

Show that Urysohn's lemma is a special case of Tietze's extension theorem.

TIETZE'S EXTENSION THEOREM

Let (X, T) denote a T_4–space and Y any closed subset of X. If f is any continuous function $f: Y \to R$ (where R is the space of real numbers with the absolute value topology), then there is a continuous extension F of f, from X into R

$$F: X \to R.$$

■

SOLUTION:

Let (X, T) denote a T_4–space and let A and B represent disjoint closed non-empty subsets of X. We define

$$Y = A \cup B \quad \text{and} \quad f: Y \to R$$

by $f(x) = 0$ for all $x \in A$ and $f(x) = 1$ for all $x \in B$.

Function f is continuous. Hence, according to Tietze's theorem, it can be extended.

● PROBLEM 13-34

Let (X, T) denote any T_4–space and A represent any closed subset of X. Show that for any continuous function

$$f: A \to R^n$$

a continuous extension F of f exists, such that

$$F: X \to R^n$$

Note that R^n is an n–dimensional Cartesian product of R.

See Figure 1.

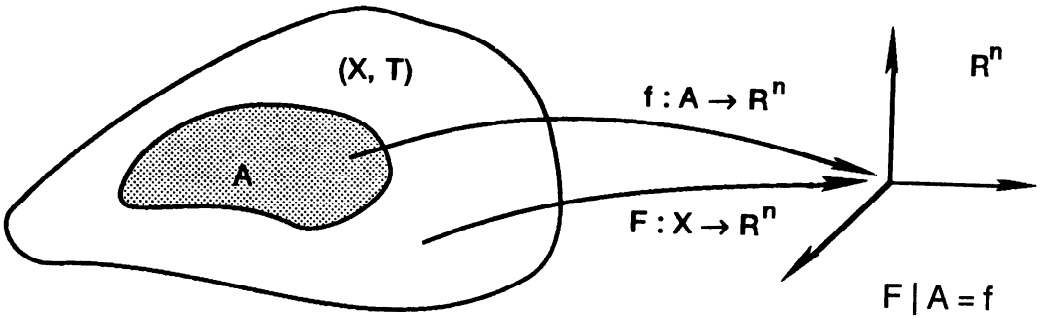


FIGURE 1

SOLUTION:

f is a continuous function of A into R^n. Thus, it can be written as

$$f(x) = (f_1(x), f_2(x), \ldots, f_n(x)).$$

Each of the functions $f_1, \ldots, f_n$ is continuous and maps A into R. According to Tietze's extension theorem, for each of the functions f_i, $i = 1, \ldots, n$, there exists a continuous extension F_i of f_i, $i = 1, 2, \ldots, n$, from X into R. We define

$$F(x) = (F_1(x), \ldots, F_n(x)).$$

Since all functions $F_1, F_2, \ldots, F_n$ are continuous, so is F. Function F is a continuous extension of f from X into R^n.

○ PROBLEM 13-35

Let R denote the space of real numbers with the absolute value topology and Z the set of integers, $Z \subset R$, with subspace topology. Let R^2 represent the plane with product topology and $Z^2 \subset R^2$ the set of all pairs (m, n), where m and n are integers, with subspace topology.

Using this "scenario," prove the existence of a continuous function

$$F : R \to R^2.$$

SOLUTION:

Consider any function

$$f : Z \to Z^2.$$

In both cases, the subspace topology is the discrete topology. Hence, any function from Z into Z^2 is continuous. Sets Z and Z^2 have the same cardinal number. It is possible to find function $f : Z \to Z^2$, which is one-to-one and

continuous.

According to Problem 13–34, f has a continuous extension F from R into R^2.

● PROBLEM 13–36

Consider the class of real-valued functions

$$F = \{f_1(x) = \sin x, f_2(x) = \sin 2x, \ldots, f_n(x) = \sin nx, \ldots\}$$

defined in R. Show that F does not separate points of R.

SOLUTION:

DEFINITION

A class $F = \{f_\omega : \omega \in \Omega\}$ of functions from X into Y is said to separate points if, for any pair of distinct points $a, b \in X$, a function $f \in F$ exists, such that $f(a) \neq f(b)$.

■

Consider a pair of distinct points 0 and π.

$$\sin 0 = \sin 2 \cdot 0 = \sin 3 \cdot 0 = \ldots \quad = 0$$

$$\sin \pi = \sin 2\pi = \sin 3\pi = \ldots \quad = 0$$

Hence class F does not separate points.

● PROBLEM 13–37

Let $C(X, R)$ denote the class of all real-valued continuous functions, defined on a topological space (X, T). Show that, if the class $C(X, R)$ separates points, then (X, T) is a Hausdorff space.

SOLUTION:

Let $a, b \in X$ represent distinct points. Since $C(X, R)$ separates points, a continuous function

$$f: X \to R$$

exists, such that $f(a) \neq f(b)$.

There are two open disjoint subsets U_1 and U_2 of R, such that

$$f(a) \in U_1 \text{ and } f(b) \in U_2.$$

Since f is a continuous function, sets $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open and disjoint.

$$a \in f^{-1}(U_1)$$

$$b \in f^{-1}(U_2).$$

Hence, (X, T) is a Hausdorff space (T_2–space).

● PROBLEM 13-38

Prove that a completely regular space is regular.

SOLUTION:

DEFINITION OF COMPLETELY REGULAR SPACE

A topological space (X, T) is completely regular if, for any closed subset F of X and any $a \in X$, such that $a \notin F$, a continuous function $f: X \to [0, 1]$ exists, such that for every

$$x \in F, \quad f(x) = 1 \quad \text{and} \quad f(a) = 0.$$

■

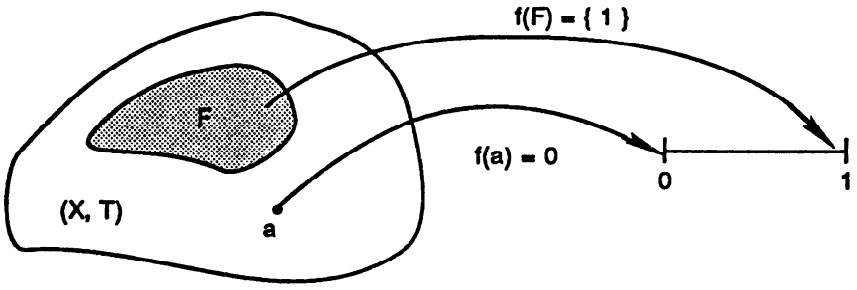


FIGURE 1

Now we show that completely regular space is also regular.

Let F represent a closed subset of X and $a \in X$ a point which does not belong to F.

By hypothesis, a continuous function

$$f: X \to [0, 1]$$

exists, such that $f(F) = \{1\}$ and $f(a) = 0$. An interval $[0, 1]$ is a Hausdorff space. Hence, two open disjoint subsets U_1 and U_2 of $[0, 1]$ exist, such that

$$0 \in U_1 \quad \text{and} \quad 1 \in U_2.$$

Since f is continuous, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open. These subsets are disjoint and such that

$$a \in f^{-1}(U_1), \quad F \subset f^{-1}(U_2).$$

Hence, (X, T) is regular.

● PROBLEM 13-39

Show that the property of being a completely regular space is hereditary, i.e., every subspace of a completely regular space is completely regular.

SOLUTION:

Let (X, T) denote a completely regular space and let (Y, T_Y) be its subspace.

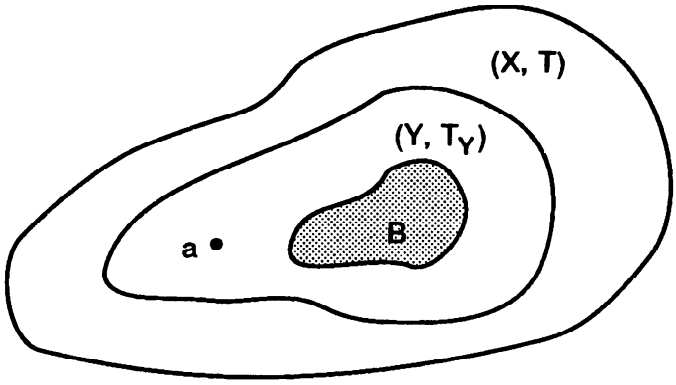


FIGURE 1

Let B be closed in Y and a represent a point in Y, such that $a \notin B$. Then

$$B = Y \cap D$$

where D is a closed set in X. Since $a \in Y$ and $a \notin B$, we have

$$a \notin D.$$

Since (X, T) is completely regular, a continuous function $f: X \to [0, 1\}$ exists, such that

$$f(a) = 0 \quad \text{and for each } x \in D \quad f(x) = 1.$$

Then $f \mid Y$ is a continuous function.

$$f \mid Y : Y \to [0, 1]$$

such that $f \mid Y(a) = 0$ and for each $x \in B, f \mid Y(x) = 1$.

Thus, (Y, T_Y) is a completely regular space.

● PROBLEM 13-40

Consider the drawing of Problem 13–26. Where would you locate on this drawing the Tychonoff spaces?

SOLUTION:

DEFINITION OF TYCHONOFF SPACE

A completely regular T_1–space is called a Tychonoff space.

■

In Problem 13–38, we proved that a completely regular space is also a regular space.

Hence, only a Tychonoff space is a T_3–space.

Any normal T_1–space (i.e., T_4–space) is a Tychonoff space by virtue of Urysohn's lemma.

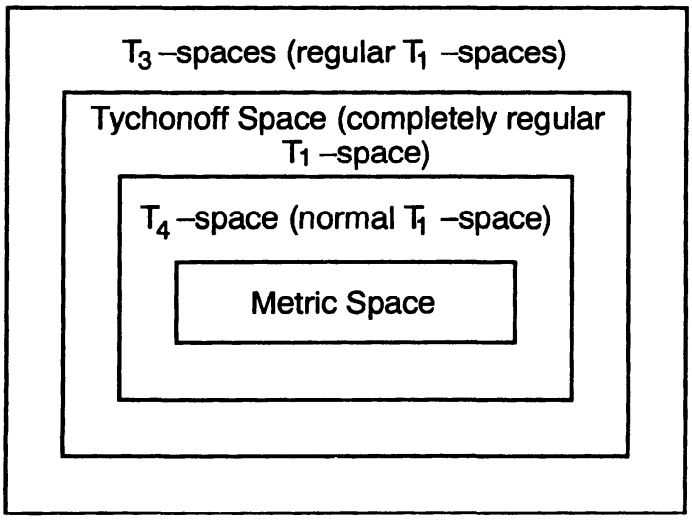


FIGURE 1

● PROBLEM 13-41

Prove the following:

THEOREM

The class $C(X, R)$ of all real-valued continuous functions on a completely regular T_1–space X separates points.

■

SOLUTION:

Let a and b represent any distinct points in X. Since X is T_1, the set $\{a\}$ is closed. Points a and b are distinct, hence,

$$b \notin \{a\}.$$

Space (X, T) is completely regular. Hence, a real-valued continuous function f on X exists, such that

$$f(b) = 0 \quad \text{and} \quad f(a) = 1.$$

This function separates points a and b

$$f(a) \neq f(b).$$

CHAPTER 14

CARTESIAN PRODUCTS

Cartesian Product, Projections	14–1, 14–2, 14–3, 14–4, 14–5, 14–6
Invariants of Cartesian Product	14–7, 14–8
Diagonal	14–9, 14–10, 14–11
Generalized Cartesian Product	14–12, 14–13
Continuity of Maps	14–14, 14–15
Cartesian Product of Metric Spaces	14–16, 14–17

● PROBLEM 14-1

Let (X_1, T_1) and (X_2, T_2) denote topological spaces.

DEFINITION OF OPEN SETS IN $X_1 \times X_2$

A set $A \subset X_1 \times X_2$ is called open in $X_1 \times X_2$ iff it is the union of Cartesian products $G \times H$, where $G \in T_1$, and $H \in T_2$

■

Prove the following:

THEOREM

The Cartesian product of two topological spaces is a topological space.

■

SOLUTION:

From the definition, we conclude that the family of all sets $G \times H$ is a base of $X_1 \times X_2$, where $G \in T_1$ and $H \in T_2$.

From the identity

$$(G_1 \times H_1) \cap (G_2 \times H_2) = (G_1 \cap G_2) \times (H_1 \cap H_2) \tag{1}$$

we see that, if $G_1, G_2 \in T_1$ and $H_1, H_2 \in T_2$, then the intersection $(G_1 \times H_1) \cap (G_2 \times H_2)$ of open sets $G_1 \times H_1$ and $G_2 \times H_2$ is an open set.

Since

$$(\bigcup_{\alpha} G_\alpha) \times (\bigcup_{\beta} H_\beta) = \bigcup_{\alpha,\beta} (G_\alpha \times H_\beta) \tag{2}$$

we conclude that the union of an arbitrary family of open sets in $X_1 \times X_2$ is open.

Thus, the Cartesian product of two topological spaces is a topological space.

● PROBLEM 14-2

Show that in R^2 the usual topology agrees with the definition of Problem 14–1.

SOLUTION:

Let A denote any open set in R^2. It can be easily shown that the collection of open squares with the sides parallel to the x and y axes forms a basis for

R^2 space.

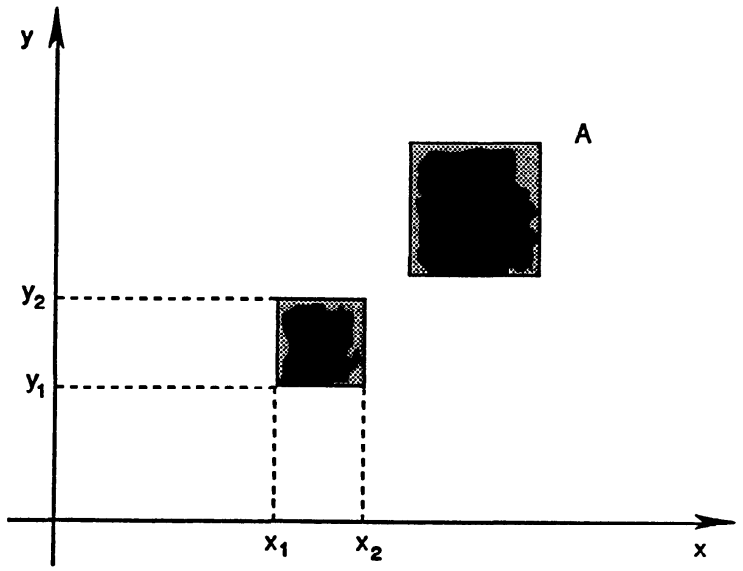


FIGURE 1

Thus

$$A = \bigcup_{\alpha} S_{\alpha}, \tag{1}$$

where each S_α is an open square with the sides parallel to the x and y axes. Since the squares' boundaries are open as indicated by the dashed lines, the boundaries are not included in S_α.

Observe that each square can be represented as the Cartesian product of two open intervals

$$S_\alpha =]\, x_1, x_2\, [\times]\, y_1, y_2\, [\tag{2}$$

Hence, the collection of sets of the type $]\, x_1, x_2\, [\times]\, y_1, y_2\, [$ forms a basis of R^2.

● PROBLEM 14–3

Prove that

THEOREM

If $\{A_\alpha\}$ is a base of X_1, and $\{B_\beta\}$ is a base of X_2, then $\{A_\alpha \times B_\beta\}$ is a base of $X_1 \times X_2$.

■

SOLUTION:

Suppose $W \subset X_1 \times X_2$ is an open set in $X_1 \times X_2$. Then, by Problem 14–1,

W is the union of Cartesian products $E_\gamma \times F_\delta$ where E_γ and F_δ are open subsets of X_1 and X_2 respectively. Thus

$$W = \cup(E_\gamma \times F_\delta) \tag{1}$$

Since $\{A_\alpha\}$ is a base of X_1 and $\{B_\beta\}$ is a base of X_2, we have for each E_γ

$$E_\gamma = \cup A_{\alpha,\gamma} \tag{2}$$

and for each F_δ

$$F_\delta = \cup B_{\beta,\delta} \tag{3}$$

where $A_{\alpha,\gamma} \in \{A_\alpha\}$, $B_{\beta,\delta} \in \{B_\beta\}$. Hence

$$E_\gamma \times F_\delta = (\cup A_{\alpha,\gamma}) \times (\cup B_{\beta,\delta}) = \cup (A_{\alpha,\gamma} \times B_{\beta,\delta}). \tag{4}$$

Therefore

$$W = \cup(E_\gamma \times F_\delta) = \cup (A_\alpha \times B_\beta). \tag{5}$$

The set of $\{A_\alpha \times B_\beta\}$ is the base of $X_1 \times X_2$.

● PROBLEM 14-4

Show that

THEOREM

The projections

$$\pi_1 : X \times Y \to X; \quad \pi_1(x, y) = x \tag{1}$$

$$\pi_2 : X \times Y \to Y; \quad \pi_2(x, y) = y \tag{2}$$

are continuous mappings. ■

SOLUTION:

Let A be an open subset of X. Then

$$\pi_1^{-1}(A) = A \times Y. \tag{3}$$

Since both A and Y are open sets, $\pi_1^{-1}(A)$ is an open set and π_1 is a continuous mapping. Similarly we show that π_2 is a continuous mapping, since for any open subset B of Y

$$\pi_2^{-1}(B) = X \times B \tag{4}$$

$X \times B$ is an open set.

This result holds for the generalized Cartesian product of any family of topological spaces.

● PROBLEM 14–5

Let

$$f: T \to X \times Y \tag{1}$$

where

$$f(t) = (f_1(t), f_2(t)). \tag{2}$$

Show that f is continuous iff f_1 and f_2 are continuous. Also, f is continuous at t_0 iff f_1 and f_2 are continuous at t_0.

SOLUTION:

Suppose that f is continuous at t_0. Since

$$f_1(t) = \pi_1[f(t)] \tag{3}$$

by Problem 14–4, we conclude that f_1 is continuous at t_0.

Now, suppose that f_1 and f_2 are continuous at t_0. Let $A \subset X \times Y$ be open and let

$$t_0 \in f^{-1}(A). \tag{4}$$

We shall show that

$$t_0 \in Int\, f^{-1}(A). \tag{5}$$

We can apply the following property:

If S is a subbase of Y and $f: X \to Y$ and if $f^{-1}(D)$ is open for each $D \in S$, then f is continuous.

Thus, we can assume that A belongs to a subbase of $X \times Y$. Setting

$$A = G \times Y \tag{6}$$

we obtain

$$f^{-1}(A) = f_1^{-1}(G). \tag{7}$$

Therefore

$$t_0 \in f_1^{-1}(G). \tag{8}$$

But f_1 is continuous at t_0, thus

$$t_0 \in Int\, f_1^{-1}(G) = Int\, f^{-1}(A). \tag{9}$$

● PROBLEM 14-6

Let

$$f: X \to Y$$

$$g: W \to Z \tag{1}$$

where X, Y, W and Z are topological spaces.

Show that the product mapping

$$h = f \times g : X \times W \to Y \times Z \tag{2}$$

is continuous iff f and g are continuous.

SOLUTION:

We have

$$h(x, w) = (f(x), g(w)). \tag{3}$$

Let w_0 represent a given element of W. Then

$$h_1(x) =: h(x, w_0) = (f(x), g(w_0)) \tag{4}$$

and we obtain

$$f = \pi_1 \circ h_1. \tag{5}$$

We shall apply the following.

THEOREM

A continuous mapping of two variables is continuous with respect to each variable.

■

Hence, if h is continuous, then so is f. By the same argument g is continuous.

Now, suppose that f and g are continuous. Then, by applying

$$h^{-1}(A \times B) = f^{-1}(A) \times g^{-1}(B) \tag{6}$$

we conclude that h is continuous.

● PROBLEM 14-7

We shall investigate the properties which are invariants of Cartesian multiplication.

Prove:

THEOREM

The product of two closed sets is a closed set. ■

SOLUTION:

Let $A \subset X$ be closed in X and $B \subset Y$ be closed in Y. We have

$$(X \times Y) - (A \times B) = [(X - A) \times Y] \cup [X \times (Y - B)]. \quad (1)$$

Thus, the set $(X \times Y) - (A \times B)$ is the union of two open sets and is therefore an open set.

Similarly it can be shown that if both spaces are T_2–spaces, then their Cartesian product is a T_2–space.

Also regularity is invariant under Cartesian multiplication. But normality is not invariant under Cartesian multiplication.

● PROBLEM 14-8

Show that the Cartesian product of two T_2–spaces is a T_2–space.

SOLUTION:

Suppose that X and Y are T_2–spaces. Let $z_1 \in X \times Y$ and $z_2 \in X \times Y$, such that

$$z_1 \neq z_2 \quad (1)$$

$$z_1 = (x_1, y_1) \quad z_2 = (x_2, y_2) \quad (2)$$

Since $z_1 \neq z_2$, we must have either

$$x_1 \neq x_2 \quad (3)$$

or

$$y_1 \neq y_2 \quad (4)$$

Suppose that $x_1 \neq x_2$. Since X is a T_2–space, two open sets A_1 and A_2 exist, such that,

$$A_1 \subset X, \; A_2 \subset X \quad (5)$$

$$x_1 \in A_1, x_2 \in A_2 \tag{6}$$

$$A_1 \cap A_2 = \phi. \tag{7}$$

Therefore

$$z_1 \in A_1 \times Y \quad z_2 \in A_2 \times Y. \tag{8}$$

The sets $A_1 \times Y$ and $A_2 \times Y$ are open and

$$(A_1 \times Y) \cap (A_2 \times Y) = \phi. \tag{9}$$

Hence the set $X \times Y$ is a T_2–space.

● PROBLEM 14-9

Show that the diagonal $D \subset X \times X$ is homeomorphic to X.

SOLUTION:

DEFINITION OF A DIAGONAL

The set

$$D = \{(x, y) : x = y\} \tag{1}$$

is the diagonal of

$$X^2 = X \times X$$

■

Consider the projection

$$\pi_1 : X \times X \to X$$

$$\pi_1 (x, y) = X \tag{2}$$

In Problem 14–4 we proved that the projection is a continuous mapping. Hence, the required homeomorphism is the projection π_1.

$$X \simeq D. \tag{3}$$

● PROBLEM 14-10

Prove the following:

THEOREM

If X is a T_2–space, then the diagonal is closed in $X \times X = X^2$. ■

SOLUTION:

We shall show that $X^2 - D$ is an open set. That is, we have to show that for any point

$$(x, y) \in X^2 - D \tag{1}$$

there are two open sets A and B, such that

$$x \in A, \quad y \in B \tag{2}$$

and

$$A \times B \subset X^2 - D. \tag{3}$$

Since X is a T_2–space and $x \neq y$, there are two open sets A and B, such that

$$x \in A \quad \text{and} \quad y \in B \tag{4}$$

and

$$A \cap B = \phi. \tag{5}$$

Therefore

$$(A \times B) \cap D = \phi \tag{6}$$

that is

$$A \times B \subset X^2 - D. \tag{7}$$

Note that the following is also true: if the diagonal of $X \times X$ is closed, then X is a T_2–space.

● PROBLEM 14-11

1. Show that if

$$f : X \to Y \tag{1}$$

is continuous, then the graph of f

$$G = \{(x, y) : y = f(x)\} \tag{2}$$

is homeomorphic to X.

2. Also show that if Y is a T_2–space, then G is closed in $X \times Y$.

SOLUTION:

1. Consider the mapping

$$h(x) = (x, f(x)). \tag{3}$$

$h(x)$ is a homeomorphism of X onto G. Thus $X \simeq G$.

2. Let us define a mapping

$$p(x, y) = (f(x), y). \tag{4}$$

Then

$$[p(x, y) \in D] \equiv [f(x) = y] \equiv [(x, y) \in G] \tag{5}$$

Thus

$$G = p^{-1}(D). \tag{6}$$

Since p is continuous and D is closed, we conclude that G is closed.

● PROBLEM 14-12

Construct the subbase of the generalized Cartesian product.

SOLUTION:

Consider the family of topological spaces $\{X_\omega : \omega \in \Omega\}$. The generalized Cartesian product is defined by

$$Z = \underset{\omega \in \Omega}{\times} X_\omega \tag{1}$$

The αth coordinate of $f \in Z$, $\pi_\alpha(f)$ is defined by

$$\pi_\alpha(f) = f(\alpha) \tag{2}$$

then

$$\pi_\alpha : Z \to X_\alpha \tag{3}$$

is the projection of Z on X_α.

The topology in Z (called Tychonow topology) is defined as follows:

DEFINITION

A subbase of Z is defined as the family of sets of the form

$$H_{\alpha,G} = \pi_\alpha^{-1}(G) = \{f : f(\alpha) \in G\} \tag{4}$$

where G is open in X_α.

■

Observe that $H_{\alpha, G}$ is the product of G and of all spaces X_β, where $\beta \neq \alpha$.

● PROBLEM 14-13

Show that if $A_\omega \subset X_\omega$ for each $\omega \in \Omega$, then

$$\overline{\underset{\omega}{\times} A_\omega} = \underset{\omega}{\times} \overline{A}_\omega . \tag{1}$$

That is, the Cartesian product of closed sets is always closed.

SOLUTION:

Let $(x_\omega) \in \underset{\omega}{x} X_\omega$ be, such that

$$(x_\omega) \in \overline{\underset{\omega}{\times} A_\omega}. \tag{2}$$

We shall show that for every ω

$$x_\omega \in \overline{A}_\omega \tag{3}$$

and hence

$$(x_\omega) \in \underset{\omega}{\times} \overline{A}_\omega . \tag{4}$$

Let $x_\omega \in U_\omega$, where U_ω is open in X_ω then

$$(x_\omega) \in < U_\omega >. \tag{5}$$

By $< U_\alpha >$ we denote the "slice" of $\underset{\omega}{\times} X_\omega$ where each factor is X_ω except the αth, which is U_α. We have

$$\phi \neq \underset{\omega}{\times} A_\omega \cap < U_\omega >= (U_\omega \cap A_\omega) \times [X\{A_\beta : \beta \neq \omega\}].$$

Hence

$$U_\omega \cap A_\omega \neq \phi \quad \text{and} \quad \times_\omega \in \overline{A}_\omega .$$

The converse inclusion is proved in the same manner by reversing the steps.

● PROBLEM 14-14

Let $\{X_\omega : \omega \in \Omega\}$ represent any family of topological spaces. Show that for each fixed $\beta \in \Omega$,

$$\pi_\beta : \underset{\omega}{\times} X_\omega \to X_\beta$$

is continuous and open.

SOLUTION:

Let U denote an open set in X_β. Then

$$\pi_\beta^{-1}(U) = < U >$$

is open in $\underset{\omega}{\times} X_\omega$. Hence π_β is continuous.

Since

$$\pi_\beta < U_{\omega_1}, \ldots, U_{\omega_n} > = \begin{cases} X_\beta & \text{if } \beta \neq \omega_1, \ldots, \omega_n \\ U_\beta & \text{if } \beta = \omega_k \end{cases}$$

the image of any basic open set is open. Hence π_β is open.

● PROBLEM 14-15

Let $\{X_\omega : \omega \in \Omega\}$ denote a family of topological spaces and let

$$f : Y \to \underset{\omega}{\times} X_\omega . \tag{1}$$

Prove that

(f is continuous) $\Leftrightarrow$ ($\pi_\omega o f$ is continuous for each $\omega \in \Omega$).

SOLUTION:

$\Rightarrow$ Suppose f is continuous. From 14–14 each π_ω is continuous. Hence

$$\pi_\omega \, o \, f$$

is continuous for each $\omega \in \Omega$.

$\Leftarrow$ Suppose $\pi_\omega \, o \, f$ is continuous. Then for each element $<U_\alpha>$ of subbasis in $\underset{\omega}{x} X_\omega$, we have

$$f^{-1} < U_\alpha > = f^{-1}[\pi_\alpha^{-1}(U_\alpha)] = (\pi_\alpha \, o \, f)^{-1}(U_\alpha).$$

Hence $\times\langle U_\alpha\rangle$ is open and f is continuous.

■

For any

$$f : Y \to \underset{\omega}{\times} X_\omega$$

the map

$$\pi_\alpha \circ f : Y \to X_\alpha$$

is called the αth coordinate function of f.

● PROBLEM 14-16

Let (X_n, d_n) denote a finite or infinite sequence of metric spaces. Define the metric of the product

$$\overset{n}{\underset{k=1}{\times}} X_k \quad \text{and} \quad \overset{\infty}{\underset{n=1}{\times}} X_n .$$

SOLUTION:

Let $X_1, X_2, \ldots, X_n$ denote a finite sequence of metric spaces. Let x and y denote points in $\overset{n}{\underset{k=1}{\times}} X_k$. Then

$$x = (x_1, x_2, \ldots, x_n)$$

$$y = (y_1, y_2, \ldots, y_n). \tag{1}$$

The distance between x and y can be defined as follows:

$$d(x,y) = \sqrt{\sum_{k=1}^{n} d_k(x_k, y_k)}. \tag{2}$$

It is easy to show that (2) defines a metric in the Cartesian product $X_1 \times X_2 \times \ldots \times X_n$.

Consider the infinite Cartesian product $X_1 \times X_2 \times \ldots \times X_n \times \ldots$. The distance between points

$$x = (x_1, x_2, \ldots) \quad \text{and} \quad y = (y_1, y_2, \ldots)$$

can be defined by

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} d_n(x_n, y_n). \tag{3}$$

Note that d_n is the metric in X_n.

In Problem 14–16, we defined the product $\times X_n$ of metric spaces and the distance. Hence $\left(\overset{\infty}{\underset{n=1}{\times}} X_n, d\right)$ is a metric space.

Metric d induces topology T_d on $\overset{\infty}{\underset{n=1}{\times}} X_n$. In Problem 14–12 we defined the Tychonow topology, T, as $\overset{\infty}{\underset{n=1}{\times}} X_n$. What is the relation between these topologies? (i.e., which one is weaker, or are they the same).

SOLUTION:

The following theorem answers this question.

THEOREM

A set $A \subset X_1 \times X_2 \times \ldots$ is open in the metric sense if, and only if, A is open in the Tychonow topology.

■

Hence the topology induced by the metric agrees with the Tychonow topology.

CHAPTER 15

COUNTABILITY PROPERTIES

Open Covers, Refinements	15–1, 15–2
First Countable Spaces	15–3, 15–4, 15–5, 15–6, 15–7, 15–8, 15–9
Second Countable Spaces	15–10, 15–11, 15–13
Separable Spaces	15–12, 15–17, 15–18, 15–19, 15–20, 15–21
Lindelöf Spaces	15–14, 15–15, 15–16
Metric Spaces	15–22, 15–23, 15–24, 15–25, 15–26

Let (R, T) denote the set of real numbers with topology induced by the absolute value metric. Show that

$$\{B(x, 2) : x \in R\}$$

i.e., the collection of open balls

$$B(x, 2) = \{y : y \in R, |x - y| < 2\}$$

is an open cover of R.

Give an example of an open subcover and an improvement of $\{B(x, 2) : x \in R\}$.

SOLUTION:

DEFINITION OF OPEN COVER

An open cover of a topological space (X, T) is a collection

$$\{A_\alpha : \omega \in \Omega, A_\alpha \in T\}$$

of open subsets of X, such that

$$X = \bigcup_\alpha A_\alpha.$$

■

$\{B_\beta\}$ is an open subcover of $\{A_\alpha\}$, if

$$\{B_\beta\} \subset \{A_\alpha\}.$$

The set $\{B(x, 2) : x \in R\}$ is an open cover of R because

$$R = \bigcup_x B(x, 2).$$

The set

$$\{B(n, 2) : n \text{ is an integer}\}$$

is an open subcover of R.

An open cover $\{D_\gamma\}$ is said to be an improvement of $\{A_\alpha\}$, if for each D_γ, there is A_α such that

$$D_\gamma \subset A_\alpha.$$

Hence, we can choose

$$\{B(x, 1) : x \in R\}$$

as an improvement of $\{B(x, 2) : x \in R\}$.

● PROBLEM 15-2

Show that every open cover of a closed and bounded interval $A = [a, b]$ is reducible to a finite cover.

SOLUTION:

We shall apply Heine-Borel Theorem:

If $A = [a, b]$ is a closed and bounded interval and $\{U_n\}$ is a class of open sets, such that

$$A \subset \bigcup_n U_n,$$

then one can choose a finite number of open sets $U_{n_1}, U_{n_2}, \ldots, U_{n_k}$, such that

$$A \subset U_{n_1} \cup U_{n_2} \cup \ldots \cup U_{n_k}.$$

The conclusion from the above theorem is that every open cover of interval $[a, b]$ is reducible to a finite cover.

● PROBLEM 15-3

Show that every metric space is a first countable space.

SOLUTION:

DEFINITION OF FIRST COUNTABLE SPACE

A topological space (X, T) is called a first countable space if, for every point $a \in X$, a countable class B_a of open sets containing a exists, such that every open set A containing a also contains a member of B_a.

■

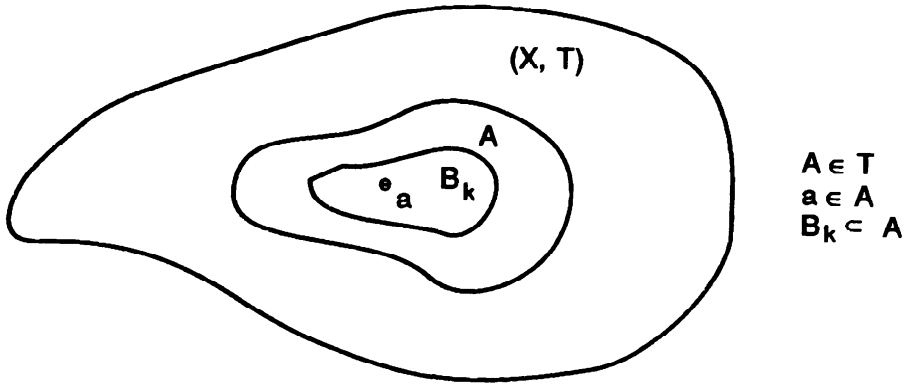


FIGURE 1

That is, a topological space (X, T) is a first countable space iff at every point $a \in X$, a countable local base exists.

Let (X, d) denote a metric space and let $a \in X$ be any point. The set of balls

$$\{B(a, 1), B(a, {}^1/_2), B(a, {}^1/_3), B(a, {}^1/_4), \ldots\}$$

forms a countable local base at $a \in X$.

For every open set A containing a, a ball $B(a, {}^1/_k)$ exists such that

$$a \in B(a, {}^1/_k) \subset A.$$

Hence every metric space is a first countable space.

● PROBLEM 15-4

Show that any subspace (Y, T_Y) of a first countable space (X, T) is also first countable.

SOLUTION:

Let (X, T) be a first countable space and (Y, T_Y) its subspace. Let $a \in Y$.

By hypothesis (X, T) is a first countable space. Hence a countable base B_a in (X, T) exists.

$$B_a = \{B_k : k \in N\}$$

Let us define for each B_k $B'_k = Y \cap B_k$.

Sets B'_k are open in (Y, T_Y) they form a T_Y–local base at $a \in Y$.

The set $\{B'_k : k \in N\}$ is countable. Hence (Y, T_Y) is a first countable space.

● PROBLEM 15-5

Let

$$A_n = \{A_1, A_2, A_3, \ldots\}$$

be a nested local base at $a \in X$. Let $(a_1, a_2, \ldots)$ denote a sequence such that

$$a_1 \in A_1, a_2 \in A_2, \ldots, a_k \in A_k, \ldots.$$

Show that (a_n) converges to a.

SOLUTION:

Let U denote an open set containing a.

Set $\{A_1, A_2, \ldots\}$ is a local base at a. Hence, set A_k exists, such that

$$A_k \subset U.$$

Sets $A_1, A_2, A_3, \ldots$ are nested if

$$A_1 \supset A_2 \supset A_3 \supset A_4 \supset \ldots$$

Hence for any $m > k$ we have

$$A_k \supset A_m$$

and since $a_m \in A_m$ we obtain

$$a_m \in A_k \subset U.$$

Thus

$$a_n \to a.$$

● PROBLEM 15-6

Show that the property of being a first countable space is a topological property.

SOLUTION:

Suppose topological spaces (X, T) and (Y, T_1) are homeomorphic

$$X \simeq Y.$$

We shall show that if (X, T) is a first countable space, then (Y, T_1) is also a first countable space.

Let

$$f: X \to Y$$

be a homeomorphism and let (X, T) be a first countable space. Let $y \in Y$ and V be an open subset of Y, such that $y \in V$. (See Figure 1).

Since f is a homeomorphism, $f^{-1}(y)$ is a point in X and $f^{-1}(V)$ is an open set in X such that

$$f^{-1}(y) \in f^{-1}(V).$$

(X, T) is a first countable space, hence $x = f^{-1}(y)$ has a countable local base B_x. An open set $B \in B_x$ exists, such that

$$x = f^{-1}(y) \in B \subset f^{-1}(V).$$

Thus

$$y \in f(B) \subset V$$

where $f(B)$ is open in (Y, T').

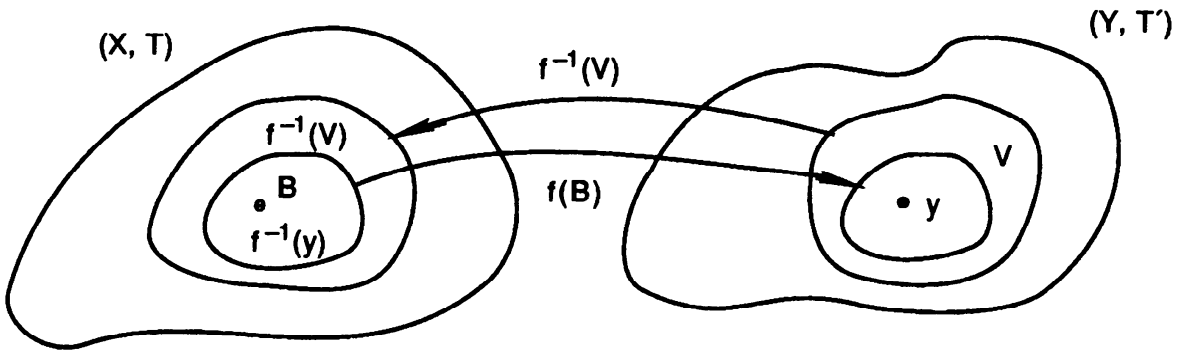


FIGURE 1

● PROBLEM 15-7

Show that if

$$A_x = \{A_1, A_2, A_3, \ldots\}$$

is a countable local base at $x \in X$, then a nested local base exists at x.

SOLUTION:

Let

$$B_1 = A_1, B_2 = A_1 \cap A_2, \ldots, B_n = A_1 \cap \ldots \cap A_n.$$

Each of the sets $B_1, B_2, \ldots$ is open and contains x. Also

$$B_1 \supset B_2 \supset B_3 \supset \ldots$$

Now, let U be an open set containing x, then $k \in N$ exists, such that

$$B_k \subset A_k \subset U.$$

Hence

$$\{B_1, B_2, B_3, \ldots\}$$

is a nested local base at $x \in X$.

● PROBLEM 15-8

Prove this theorem:

THEOREM

A function defined on a first countable space (X, T) is continuous at $x \in X$ iff it is sequentially continuous at x.

■

SOLUTION:

We shall show that if (X, T) is a first countable space, then

$$f: X \to Y$$

is continuous at $x \in X$, if and only if, for every sequence (a_n) converging to $x \in X$, the sequence $(f(a_n))$ converges to $f(x)$. It has been shown that if f is continuous, then f is sequentially continuous. It remains to be shown that

$$(f \text{ is sequentially continuous}) \Rightarrow (f \text{ is continuous})$$

or equivalent

$$(f \text{ is not continuous}) \Rightarrow (f \text{ is not sequentially continuous}).$$

Let $\{A_1, A_2, \ldots\}$ be a nested local base at $x \in X$.

Suppose f is not continuous. Then, an open set $V \subset Y$ exists, such that

$$f(x) \in V, \quad \text{and for every } n \in N, \; A_n \not\subset f^{-1}(V).$$

Thus, for every $n \in N$, an $a_n \in A_n$ exists, such that $a_n \notin f^{-1}(V)$ and therefore $f(a_n) \notin V$.

According to Problem 15–5, (a_n) converges to x, but

$$f(a_n) \not\to f(x).$$

● PROBLEM 15-9

Show that the set of real numbers R with the cofinite topology is not a first countable space.

SOLUTION:

Cofinite topology T contains ϕ and the complements of finite sets. Suppose (R, T) is a first countable space. Let $A(1) = \{A_1, A_2, A_3, \ldots\}$ be a

countable open local base at $1 \in R$. Each A_n is T–open, hence its complement R–A_n is closed and hence finite.

The set

$$A = \bigcup_n (R - A_n)$$

is the countable union of finite sets. Thus A is countable and R is not countable. A point $a \in R$ exists, such that

$$a \neq 1, \quad a \notin A.$$

We have

$$a \in R - A = R - (\bigcup_n (R - A_n)) = \bigcap_n A_n.$$

Hence $a \in A_n$ for every $n \in N$.

The set R–$\{a\}$ is a T–open as a complement of a finite set.

$$1 \in R - \{a\}$$

because $a \neq 1$.

$A(1)$ is a local base at $1 \in R$. $A_k \in A(1)$ exists, such that

$$A_k \subset R - \{a\}.$$

Hence $a \notin A_k$.

That contradicts the fact that for every $n \in N$, $a \in A_n$.

● PROBLEM 15–10

Show that R^2 with the usual topology is a second countable space.

SOLUTION:

DEFINITION OF A SECOND COUNTABLE SPACE

A topological space (X, T) is called a second countable space, if a countable basis B exists for the topology T.

■

Let us define B as a class of open balls in R^2 with rational radii and centers, whose both coordinates are rational numbers. The Cartesian product of three countable sets is a countable set. Hence, B is a countable set. It is easy to see that B is a basis for the usual topology on R^2.

Hence (R^2, T) is a second countable space.

● PROBLEM 15-11

1. Show that (R, T), where R is the set of real numbers and T is the discrete topology, is not the second countable space.

2. Show that: Any second countable space is first countable.

SOLUTION:

1. It is easy to see that if T is a discrete topology, then its base contains all singleton sets.

Set R, and hence, the class of singleton subsets $\{a\}$ of R, is not countable. R with the discrete topology T is not the second countable space.

2. Suppose (X, T) is the second countable space.

Hence, B is a countable base for (X, T). Let B_x consist of all members of B which contain $x \in X$. Then B_x is a countable local base at $x \in X$. Hence, (X, T) is a first countable space.

● PROBLEM 15-12

Show that:

1. Every subspace of a second countable space is second countable.

2. Any second countable space is separable.

SOLUTION:

1. Let B represent a countable base for the second countable space (X, T).

$$B = \{B_n : n \in N\}$$

Then, for any subspace A

$$B_A = \{A \cap B_n : n \in N\}$$

is a countable base for A. Thus, (A, T_A) is a second countable space.

2. DEFINITION OF SEPARABLE SPACE

X is said to be separable if X contains a countable dense subset.

■

Let (X, T) denote a second countable space with a countable base B. For each $B_n : n \in N$ select $x(B_n) \in B_n$.

The set

$$\{x(B_n) : B_n \in B\}$$

is a countable subset of X, which is dense.

Hence, (X, T) is separable.

● PROBLEM 15-13

Prove the following:

THEOREM (Lindelöf)

If A is a subset of a second countable space (X, T), then every open cover of A is reducible to a countable cover. ■

SOLUTION:

Let

$$B = \{B_n : n \in N\}$$

denote a countable base for X and let $H = \{H_\alpha : \alpha \in \Omega\}$ denote an open cover of A.

$$A \subset \cup H_\alpha.$$

Hence, for every $x \in A$, $H_\alpha \in H$ exists, such that $x \in H_\alpha$. Since B is a base for X, for every $x \in A$, $B_x \in B$ exists, such that

$$x \in B_x \subset H_\alpha.$$

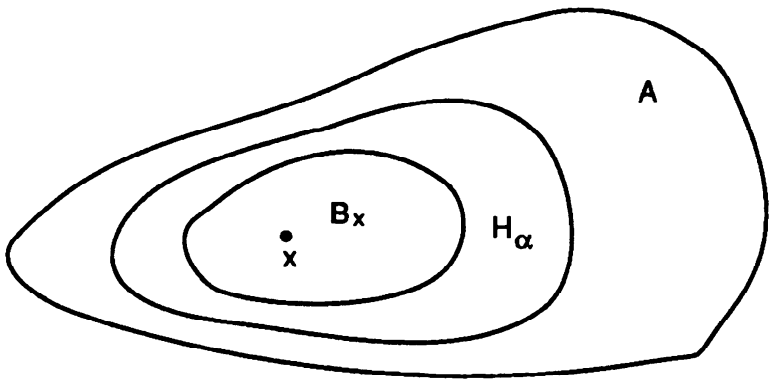


FIGURE 1

Thus

$$A \subset \cup\{B_x : X \in A\}$$

The family of sets $\{B_x : x \in A\}$ is a subset of B, hence it is countable

$$\{B_x : x \in A\} = \{B_k : k \in K\}$$

where K is a countable index set. For each $k \in K$, we can choose one set $H_k \in H$, such that

$$B_k \subset H_k.$$

Thus

$$A \subset \bigcup_k B_k \subset \bigcup_k H_k$$

and $\{H_k : k \in K\}$ is a countable subcover of H.

● PROBLEM 15-14

Show that every second countable space is a Lindelöf space.

SOLUTION:

DEFINITION OF A LINDELÖF SPACE

A topological space (X, T) is called a Lindelöf space, if every open cover of X is reducible to a countable cover.

■

We shall prove the following:

THEOREM

If (X, T) is a second countable space, then every base B for X is reducible to a countable base for X.

■

Let (X, T) represent a second countable space. Then X has a countable base

$$B = \{B_n : n \in N\}.$$

Let $H = \{H_\alpha : \alpha \in \Omega\}$ represent any base for X. Then for each $n \in N$

$$B_n = U\{H_\alpha : H_\alpha \in H_n\}$$

where $H_n \subset H$. Hence H_n is an open cover of B_n.

According to Problem 15–13, H_n is reducible to a countable subcover

H'_n.

$$B_n = U\{H_\alpha : H_\alpha \in H'_n\}, \quad H'_n \subset H_n$$

and H'_n is countable.

Note that

$$H' = \{H_\alpha : H_\alpha \in H'_n, n \in N\}$$

is a base for X, since B is a base. Also

$$H' \subset H$$

and H' are countable.

● PROBLEM 15-15

Show that every closed subspace of a Lindelöf space is Lindelöf.

SOLUTION:

Suppose A is a closed subspace of a Lindelöf space (X, T) and $\{H_k\}$ is an open cover of A, $k \in K$. Each H_k is open in A,

$$H_k = A \cap V_k$$

where V_k is open in X.

Since A is closed, $X - A$ is open in X. Hence,

$$\{X - A\} \cup \{V_k : k \in K\}$$

is an open cover of X.

There is a countable subcover

$$\{X - A\} \cup \{V_{k_n} : n = 1, 2, 3, \ldots\}$$

of

$$\{X - A\} \cup \{V_k : k \in K\}.$$

Then

$$\{V_{k_n} : n = 1, 2, 3, \ldots\}$$

is a countable open cover of A. Therefore

$$\{A \cap V_{k_n} : n = 1, 2, 3, \ldots\}$$

is a countable open subcover of $\{H_k : k \in \mathrm{K}\}$. Hence, A is a Lindelöf space.

Show that a discrete space X is Lindelöf, if and only if, X is a countable set.

SOLUTION:

Let (X, T) denote a discrete topological space, i.e., T consists of all the subsets of X.

Suppose (X, T) is Lindelöf. Then every open cover of X has a countable subcover. Each singleton set $\{x\}$, $x \in X$ is open in discrete topology. The family of sets

$$\{\{x\} : x \in X\}$$

is an open cover of X

$$X = \bigcup_{x \in X} \{x\}.$$

Since X is Lindelöf, a countable subcover exists

$$\{x_1\}, \{x_2\}, \{x_3\}, \ldots.$$

Hence, X is a countable set.

Conversely, if X is a countable set and (X, T) is a discrete space, then X is Lindelöf. Let

$$A = \{A_\alpha : \alpha \in \Omega\}$$

represent an open cover of X.

$$X \subset \bigcup_{\alpha} A_\alpha.$$

For each $x \in X, A_x \in A$ exists, such that

$$x \in A_x$$

since X is countable

$$X = \{x_1, x_2, \ldots\}$$

we can choose

$$x_1 \in A_1, x_2 \in A_2, \ldots$$

Hence, $A_1, A_2, A_3, \ldots$ is an open subcover of A. Thus, (X, T) is Lindelöf.

● PROBLEM 15-17

Show that:

1. The real line R with the usual topology is a separable set.

2. The real line R with the discrete topology is not a separable set.

SOLUTION:

1. Let Q denote the set of rational numbers. We have showed that Q is a countable set. Since Q is dense in R, the set of real numbers R with the usual topology is separable.

2. Let T denote the discrete topology on R. Then every subset of R is both closed and open. Therefore, the only dense set in R is R itself. Set R is not countable. Hence, R with the discrete topology is not a separable space.

● PROBLEM 15-18

Show that if a topological space (X, T) is a second countable space, then (X, T) is separable.

SOLUTION:

Suppose (X, T) is a second countable space. The family of sets

$$B = \{B_n : n \in N\}$$

is a countable base of X. For each $n \in N$, we choose a point a_n, such that

$$a_n \in B_n.$$

The set

$$A = \{a_n : n \in N\}$$

is countable.

We shall show that each point

$$x \in X - A$$

is an accumulation point of A, that is, that $\overline{A} = X$.

Let $x \in X - A$ and let U represent an open set containing x. At least one set $B_k \in B$ exists, such that

$$a_k \in B_k \subset U.$$

Since $a_k \in A$ and $x \in X - A$,

$$a_k \neq x.$$

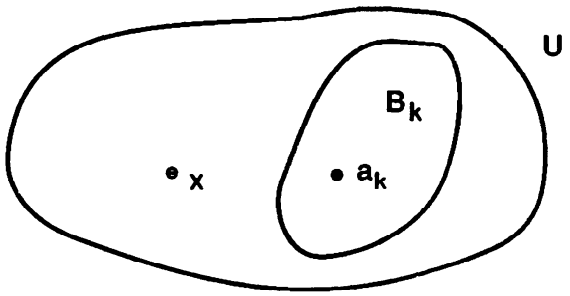


FIGURE 1

We conclude that x is an accumulation point of A, since every open set U containing x also contains a point of A different from x.

● PROBLEM 15-19

1. Show that a discrete space X is separable, if and only if, X is countable.

2. Show that (X, T) with the cofinite topology T is separable.

SOLUTION:

1. Let (X, T) denote a discrete space. Then every subset of X is both open and closed. Therefore, the only dense subset of X is X itself. Hence, X contains a countable dense subset, if and only if, X is countable. Thus, a discrete space X is separable iff X is countable.

2. Suppose X is countable. Then, X is a countable dense subset of (X, T).

Now, suppose, X is not countable. Then X contains non-finite, countable, subset A.

In the cofinite topology, the only closed sets are the finite sets and X. Hence, the closure of A is the entire space X, that is

$$\overline{A} = X.$$

Since A is countable, (X, T) is separable.

● PROBLEM 15–20

Show that (R^2, T) is separable, where R^2 is the plane and T is the topology generated by the half-open rectangles.

$$]a, b] \times]c, d] = \{(x, y) : a < x \leq b, c < y \leq d\}$$

SOLUTION:

For any rectangle $]a, b] \times]c, d]$ a point (x_0, y_0) exists with rational coordinates, such that

$$(x_0, y_0) \in]a, b] \times]c, d] .$$

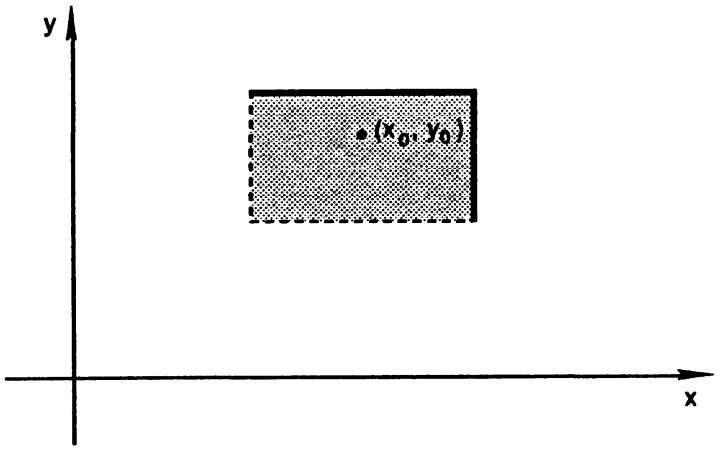


FIGURE 1

Consider the set $A = Q \times Q$ consisting of all points in R^2 with rational coordinates.

Since Q is countable, so is A. The set A is dense in R^2. Hence, (R^2, T) is separable.

● PROBLEM 15–21

Show that any open subspace of a separable space is separable.

SOLUTION:

Let (X, T) represent a separable space and let

$$\{x_n : n \in N\}$$

represent a countable dense subset of (X, T).

We shall show that any open subspace U of X is separable. We define

$$B = U \cap \{x_n : n \in N\}.$$

Let V represent any subset of U which is open in U. Hence V is open in X, since U is open in X. Therefore, V contains some x_n, and hence, a point of B.

We shall apply the following theorem:

THEOREM

A subset A of a topological space (X, T) is dense, if and only if, every open subset of X contains some point of A.

■

Therefore, B is dense in U. Set U contains a countable dense subset B, and hence, U is separable.

● PROBLEM 15-22

Let $B(x, r)$ represent an open sphere in a metric space (X, d), and let

$$d(x, y) < {}^1/_3\, r. \tag{1}$$

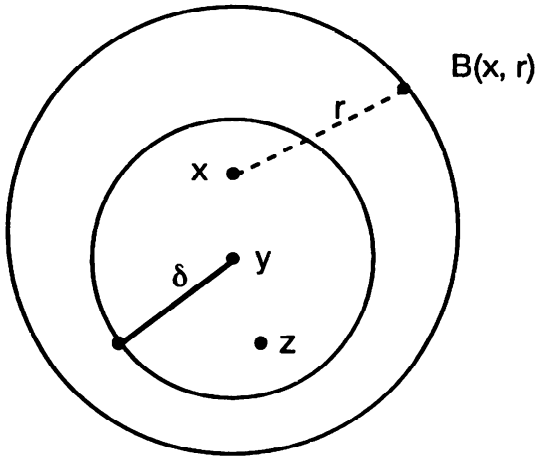


FIGURE 1

Show that if ${}^1/_3\, r < \delta < {}^2/_3\, r$, then

$$x \in B(y, \delta) \subset B(x, r). \tag{2}$$

SOLUTION:

From (1) and condition ${}^1/_3\, r < \delta$ we obtain

$$d(x, y) < {}^1/_3\, r < \delta. \tag{3}$$

Hence

$$x \in B(y, \delta). \tag{4}$$

Suppose

$$z \in B(y, \delta) \tag{5}$$

then

$$d(y, z) < \delta. \tag{6}$$

According to triangle inequality

$$d(z, x) < d(y, z) + d(y, x). \tag{7}$$

From (7), (6), (1) we obtain

$$d(z, x) < d(y, z) + d(y, x) < \delta + {}^1/_3\, r < {}^2/_3\, r + {}^1/_3\, r = r. \tag{8}$$

Hence

$$z \in B(x, r)$$

and

$$B(y, \delta) \subset B(x, r). \tag{9}$$

● PROBLEM 15-23

THEOREM

Every separable metric space is second countable. ■

Prove it.

SOLUTION:

Let (X, d) denote a separable metric space. Then A is a countable dense subset of X

$$\overline{A} = \mathrm{X}.$$

According to S we denote the class of all open spheres with centers in A and with rational radii, that is,

$$S = \{B(a, r) : a \in A, r \in Q\}. \tag{1}$$

Since A and Q are countable sets, S is countable.

We will show that S is a base for the topology on X. That is, that for every open set $U \subset X$ and for every $x \in U$

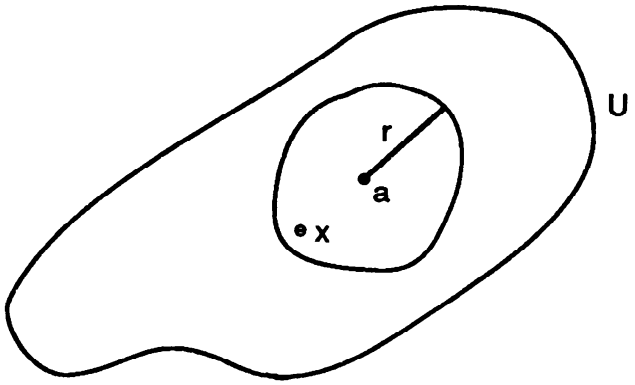


FIGURE 1

there is a sphere $B(a, r) \in S$, such that

$$x \in B(a, r) \subset U. \tag{2}$$

Since $x \in U$, there is an open sphere $B(x, r_0)$, such that

$$x \in B(x, r_0) \subset U. \tag{3}$$

Since A is dense in X, we can always find $a_0 \in A$, such that

$$d(x, a_0) \leq {}^1/_3\, r_0. \tag{4}$$

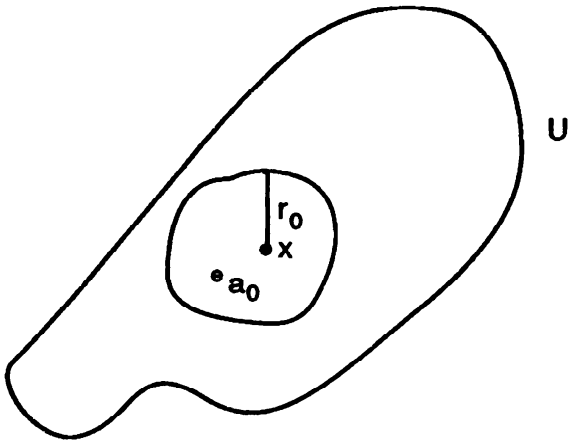


FIGURE 2

Let r_1 denote a rational number, such that

$${}^1/_3\, r_0 < r_1 < {}^2/_3\, r_0. \tag{5}$$

Then, according to Problem 15–22, we have

$$x \in B(a_0, r_1) \subset B(x, r_0) \subset U. \tag{6}$$

Since $B(a_0, r_1) \in S$ and S are countable, S is a countable base for the topology on X.

● PROBLEM 15–24

Show that the linear space $C[0, 1]$ of all continuous functions on the closed interval $[0, 1]$ with the norm defined by

$$\| f \| = \sup\{| f(x) | : x \in [0, 1]\}$$

is second countable.

SOLUTION:

Let $f \in C[0, 1]$. Then, according to the Weierstrass approximation theorem for any $\varepsilon > 0$, a polynomial $P(x)$ exists with rational coefficients, such that

$$\| f(x) - P(x) \| < \varepsilon$$

that is, for all $x \in [0, 1]$

$$| f(x) - P(x) | < \varepsilon.$$

Hence, the collection of polynomials with rational coefficients is dense in $C[0, 1]$.

The set P of all polynomials with rational coefficients is countable. Hence, $C[0, 1]$ contains a countable dense set, i.e., $C[0, 1]$ which is separable. Since each separable metric space (see Problem 15–23) is second countable, $C[0, 1]$ is second countable.

● PROBLEM 15–25

Show that every Lindelöf metric space is separable.

SOLUTION:

Suppose (X, d) is a Lindelöf space. Let $\alpha > 0$ denote any real positive number, and let A denote a maximal subset of X such that

$$d(a, b) \geq \alpha \quad \text{for every } a, b \in A. \tag{1}$$

The existence of such maximal subset is guaranteed according to Zorn's lemma.

For each $a \in A$ consider $B(a, \alpha/2)$ and

$$D = X - U\{\overline{B(a, \alpha/4)} : a \in A\} \tag{2}$$

D is an open set.

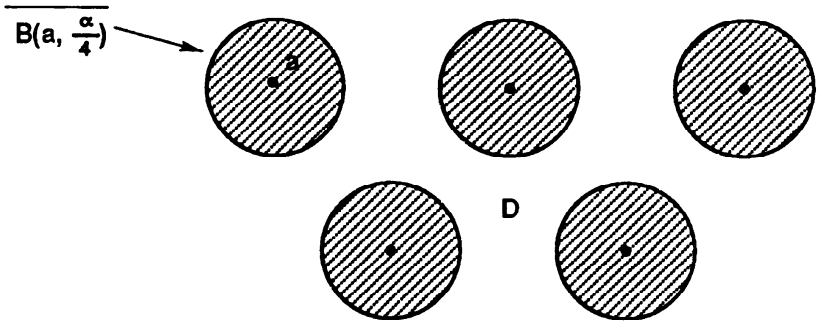


FIGURE 1

Note that

$$\{D\} \cup \{B(a, \alpha/2) : a \in A\} \tag{3}$$

is a covering of X by open sets. Since X is Lindelöf, a countable subcovering exists.

Suppose we remove $B(a, \alpha/2)$ from the original covering (3) for any $a \in A$, then the remaining sets would not cover X (none of them would contain a). Thus, $\{B(a, \alpha/2) : a \in A\}$ is countable, hence, A is countable.

We can repeat the above construction for each $\alpha = 1/n$; $n = 1, 2, 3, \ldots$ and obtain corresponding maximal subsets $\{A_n\}$.

Let

$$P = \bigcup_{n \in N} \{A_n\} \tag{4}$$

P is countable as the union of countably many countable sets. We shall show that P is dense in X.

Suppose $x \in X$ and $\beta > 0$. We have to show that there is $y \in P$, such that $d(y, x) < \beta$.

Set

$$n > 1/\beta. \tag{5}$$

Then there is $y \in A_n$, such that

$$d(y, x) < \beta. \tag{6}$$

Let U represent any nonempty open subset of X. We choose $x \in U$ and $\beta > 0$, such that

$$B(x, \beta) \subset U.$$

Then $B(x, \beta)$, and hence U, contains an element of P. Therefore P is dense in X. Space (X, d) is separable.

● PROBLEM 15-26

Prove the following important theorem:

THEOREM

If (X, d) is a metric space, then the following statements are equivalent:

1. X is separable
2. X is second countable
3. X is Lindelöf.

■

SOLUTION:

In Problem 15–12, we proved that any second countable space is separable, i.e., we proved

$$(2) \Rightarrow (1) \tag{1}$$

In Problem 15–23, we proved that every separable metric space is second countable, i.e.,

$$(1) \Rightarrow (2) \tag{2}$$

Hence

$$(1) \equiv (2).$$

In Problem 15–14, we proved that every second countable space is Lindelöf, i.e.,

$$(2) \Rightarrow (3). \tag{3}$$

In Problem 15–25, we proved that every Lindelöf metric space is separable, i.e.,

$$(3) \Rightarrow (1) \tag{4}$$

From Equations (4) and (2) we obtain

$$(3) \Rightarrow (2). \tag{5}$$

Therefore

$$(1) \equiv (2) \equiv (3). \tag{6}$$

CHAPTER 16

COMPACTNESS

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● PROBLEM 16-1

Why is any compact space also a Lindelöf space?

SOLUTION:

DEFINITION OF COMPACT SPACE

A space (X, T) is said to be compact if given any open cover $\{U_\omega\}$, $\omega \in \Omega$, of X, there is a finite subcover of $\{U_\omega\}$, $\omega \in \Omega$.

■

Let (X, T) represent any space and $A \subset X$. An open cover of A is a collection $\{U_\omega\}$, $\omega \in \Omega$ of open sets, such that

$$A \subset \bigcup_{\omega} U_\omega .$$

Also, $\{U_\omega\}$, $\omega \in \Omega$ is an open cover of A if $\{U_\omega \cap A\}$, $\omega \in \Omega$ is an open cover of the subspace A. A is said to be compact if every open cover of A has a finite subcover. Hence, A is compact if the subspace A is compact.

A space (X, T) is said to be a Lindelöf space if every open cover of X has a countable open subcover.

Therefore, any compact space is also Lindelöf.

● PROBLEM 16-2

Show that (0, 1) is not a compact space with the absolute value topology.

SOLUTION:

The open interval with the absolute value topology is Lindelöf since it is a subspace of a second countable space R. With the absolute value topology, (0, 1) is an example of a space which is Lindelöf, but not compact.

We shall now show that (0, 1) is not compact with the absolute value topology.

Consider the collection of open subsets of (0, 1)

$$\{U_n\},\ n = 1, 2, 3, 4, \ldots$$

where

$$U_n = \left(\frac{1}{n+1}, 1\right).$$

This is an open cover of (0, 1) because

$$(0,1) = \bigcup_n \left(\frac{1}{n+1}, 1\right).$$

No finite number of the U_n will cover (0, 1). Suppose, on the contrary that

$$U_{a_1}, U_{a_2}, \ldots, U_{a_k}$$

cover (0, 1). Let U_{a_l} represent the set with the highest index, then

$$(0,1) = U_{a_1} \cup U_{a_2} \cup \ldots \cup U_{a_k} = U_{a_l} = \left(\frac{1}{a_l + 1}, 1\right) \neq (0,1).$$

Hence, a contradiction.

We conclude that (0, 1) is not a compact space with the absolute value topology.

● PROBLEM 16-3

1. Use the Heine-Borel theorem to show that every closed and bounded interval $[a, b]$ with the absolute value topology is compact.

2. Show that every finite subset of a topological space (X, T) is compact.

SOLUTION:

1. We quote:

HEINE-BOREL THEOREM

Let $[a, b]$ represent a closed and bounded interval and let $\{U_\alpha\}$ represent a class of open sets, such that

$$[a,b] \subset \bigcup_\alpha U_\alpha.$$

Then one can select a finite number of the open sets, say

$$U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}$$

in such a way that

$$[a,b] \subset U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_n}.$$

■

According to the Heine-Borel theorem, every closed and bounded interval $[a, b]$ on the real line with the absolute value topology is compact.

2. Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite subset of a topological space (X, T). Let $\{U_\alpha\}$ be an open cover of A

$$A \subset \bigcup_\alpha U_\alpha .$$

Then each element of A belongs to at least one of the sets $\{U_\alpha\}$. Hence

$$a_1 \in U_{\alpha_1}$$

$$a_2 \in U_{\alpha_2}$$

$$\vdots$$

$$a_n \in U_{\alpha_n}$$

and

$$A \subset U_{\alpha_a} \cup \ldots \cup U_{\alpha_n}.$$

Any finite subset of a topological space is compact.

● PROBLEM 16-4

Let N represent the set of natural numbers with topology T defined as follows: a subset $U \subset N$ is open if it contains all, but mostly finite elements of N. Show that (N, T) is a compact space.

SOLUTION:

Let $\{U_\omega\}$, $\omega \in \Omega$ represent any open cover of N. We choose any element of $\{U_\omega\}$, say U_{ω_1}.

By definition U_{ω_1} contains all but most finitely elements of N. Let us denote these elements by

$$k_2, k_3, \ldots, k_m.$$

Since $\{U_\omega\}$ is an open cover of N, an open set U_{ω_2} exists, such that

$$k_2 \in U_{\omega_2}.$$

Similarly, we can find $U_{\omega_3}, \ldots, U_{\omega_m}$ such that

$$k_3 \in U_{\omega_3}, \ldots, k_m \in U_{\omega_m}.$$

For an open cover $\{U_\omega\}$, $\omega \in \Omega$ we found a finite subcover

$$\{U_{\omega_1}, U_{\omega_2}, \ldots, U_{\omega_m}\} \text{ of } \{U_\omega\}.$$

Since $\{U_\omega\}$, $\omega \in \Omega$ was an arbitrary cover of N, the space (N, T) is compact.

● PROBLEM 16-5

1. Show that any infinite subset A of a discrete topological space X is not compact.

2. Prove that a subset of a discrete space is compact, if and only if it is finite.

SOLUTION:

1. Let (X, T) represent a discrete topological space and A represent an infinite subset of A. We shall find an open cover of A which has no finite subcover. Consider the class of singleton sets

$$\{\{a\} : a \in A\}.$$

It is an open cover of A because

$$A = \bigcup_{a \in A} \{a\}$$

and since all subsets of a discrete space are open, each set $\{a\}$ is open. Note that no proper subclass of $\{\{a\} : a \in A\}$ is a cover of A.

Since A is infinite so is $\{\{a\} : a \in A\}$. Thus, the open cover $\{\{a\} : a \in A\}$ of A contains no finite subcover and A is not compact.

2. In Problem 16–3, we proved that every finite subset of a topological space is compact. Hence, a subset of a discrete space is compact, if and only if it is finite.

● PROBLEM 16-6

Show that a continuous image of a compact set is also compact.

SOLUTION:

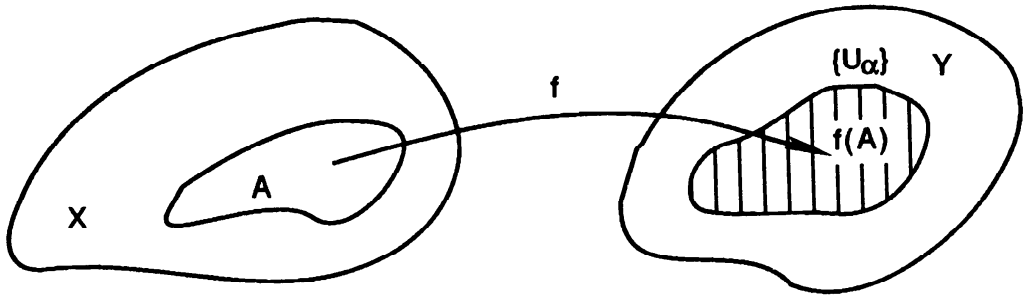


FIGURE 1

Suppose $f : X \to Y$ is continuous and X and Y are topological spaces and A is a compact subset of X.

Let $\{U_\alpha\}$ represent an open cover of $f(A)$

$$f(A) \subset \bigcup_\alpha U_\alpha .$$

Then, we have

$$A \subset f^{-1}[f(A)] \subset f^{-1}[\bigcup_\alpha U_\alpha] = \bigcup_\alpha f^{-1}(U_\alpha).$$

Since f is continuous, all sets $f^{-1}(U_\alpha)$ are open and $\{f^{-1}(U_\alpha)\}$ is an open cover of A. By hypothesis, A is a compact set, hence $\{f^{-1}(U_\alpha)\}$ is reducible to a finite cover, say

$$f^{-1}(U_{\alpha_1}),\ f^{-1}(U_{\alpha_2}),\ \ldots,\ f^{-1}(U_{\alpha_n}).$$

Then

$$A \subset f^{-1}(U_{\alpha_1}) \cup \ldots \cup f^{-1}(U_{\alpha_n})$$

and

$$f(A) \subset f[f^{-1}(U_{\alpha_1}) \cup \ldots \cup f^{-1}(U_{\alpha_n})] \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}.$$

Thus $f(A)$ is compact.

● PROBLEM 16-7

Show that topological space (X, T) with the cofinite topology T is compact.

SOLUTION:

Suppose $\{U_\alpha\}$ is an open cover of X. Choose any

$$U \in \{U_\alpha\}.$$

Since T is the cofinite topology, $X - U$ is a finite set, which we can denote as

$$X - U = \{u_1, u_2, \ldots, u_n\}.$$

$\{U_\alpha\}$ is a cover of X, therefore, for each of the elements of $\{u_1, u_2, \ldots, u_n\}$, at least one set U_{α_i} exists, such that

$$u \in_i U_{\alpha_i} \text{ and } U_{\alpha_i} \in \{U_\alpha\}$$

for $i = 1, 2, \ldots, n$.

Thus

$$X - U \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$$

and

$$X = U \cup (X - U) = U \cup U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}.$$

Hence X is compact.

● PROBLEM 16-8

1. Find an example of a subset of a compact space which is not compact.

2. Prove the following

THEOREM

A closed subset of a compact space is also compact. ■

SOLUTION:

1. Consider the closed interval $[a, b]$ which is compact by the Heine-Borel theorem. Open interval (a, b), which is a subset of $[a, b]$, is not compact (see Problem 16–2).

2. Let $F \subset X$ represent a closed subset of a compact space (X, T) and let

$$\{U_\alpha\}$$

represent an open cover of F,

$$F \subset \bigcup_\alpha U_\alpha.$$

We have

$$X = (X - F) \cup \bigcup_\alpha U_\alpha.$$

Since F is closed, $X - F$ is open and

$$\{U_\alpha, X - F\}$$

is an open cover of X.

By hypothesis, X is compact and the cover $\{U_\alpha, X - F\}$ is reducible to a finite cover, say

$$X = U_1 \cup \ldots \cup U_n \cup (X - F)$$

where $U_1, \ldots, U_n \in \{U_\alpha\}$.

Since F and $X - F$ are disjoint we obtain

$$F \subset U_1 \cup \ldots \cup U_n.$$

Therefore, an open cover $\{U_\alpha\}$ of F is reducible to a finite subcover $\{U_1, U_2, \ldots, U_n\}$.

Set F is compact.

● PROBLEM 16-9

Show that the following conditions are equivalent:

1. X is compact

2. For every family $\{F_\alpha\}$ of closed subsets of X,

$$\left(\bigcap_\alpha F_\alpha = \phi\right) \Rightarrow \left(\begin{array}{c}\{F_\alpha\} \text{ contains a finite subclass} \\ \{F_{\alpha_1}, \ldots, F_{\alpha_k}\} \text{ such that} \\ F_{\alpha_1} \cap \ldots \cap F_{\alpha_k} = \phi\end{array}\right).$$

SOLUTION:

1. $\Rightarrow$ 2.

Suppose $\bigcap_\alpha F_\alpha = \phi$, then according to de Morgan's law

$$X = X - \left(\bigcap_\alpha F_\alpha\right) = \bigcup_\alpha (X - F_\alpha)$$

$\{X - F_\alpha\}$ is an open cover of X, because all F_α are closed. But, by hypothesis X is compact, hence a finite subcover exists, say

$$X = (X - F_{\alpha_1}) \cup \ldots \cup (X - F_{\alpha_k})$$

where

$$F_{\alpha_1}, \ldots, F_{\alpha_k} \in \{F_\alpha\}.$$

Again, according to de Morgan's law

$$\phi = X - X = X - [(X - F_{\alpha_a}) \cup \ldots \cup (X - F_{\alpha_k})] = F_{\alpha_1} \cap \ldots \cap F_{\alpha_n}.$$

2. $\Rightarrow$ 1.

Let $\{U_\alpha\}$ represent an open cover of X.

$$X = \bigcup_\alpha U_\alpha.$$

According to de Morgan's law

$$\phi = X - X = \bigcap_{\alpha} (X - U_{\alpha}).$$

All sets $X - U_{\alpha}$ are closed and have an empty intersection. By hypothesis, a subclass of $\{X - U_{\alpha}\}$ exists, say $(X - U_{\alpha_1}), \ldots, (X - U_{\alpha_k})$, such that

$$(X - U_{\alpha_1}) \cap \ldots \cap (X - U_{\alpha_k}) = \phi.$$

According to de Morgan's law

$$X - [(X - U_{\alpha_1}) \cap \ldots \cap (X - U_{\alpha_k})] = U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_k} = X.$$

Thus, X is compact.

● PROBLEM 16-10

Show that the family of open intervals

$$\{(0, 1), (0, 1/2), (0, 1/3), (0, 1/4), \ldots\}$$

has the finite intersection property.

SOLUTION:

DEFINITION

A family $\{A_{\alpha}\}$ of sets is said to have the finite intersection property, if every finite subfamily

$$\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$$

has a non-empty intersection, that is

$$A_{\alpha_1} \cap \ldots \cap A_{\alpha_n} = \phi.$$

■

Let $\{(0, a_1), \ldots, (0, a_n)\}$ represent any subfamily of open intervals, then

$$(0, a_1) \cap \ldots \cap (0, a_n) = (0, a) \neq \phi$$

where

$$a = min\ (a_1, a_2, \ldots, a_n).$$

Observe that the intersection of all the members of the family

$$(0, 1) \cap (0, 1/2) \cap \ldots = \phi$$

is empty.

● PROBLEM 16-11

Prove this theorem:

THEOREM

A topological space (X, T) is compact, if and only if every class $\{F\alpha\}$ of closed subsets of X, which has the finite intersection property (see Problem 16–10), has a non-empty intersection, i.e.,

$$\bigcap_{\alpha} F_{\alpha} \neq \phi.$$

■

SOLUTION:

The implication of Problem 16–9 can be written in the form

$$\left(\bigcap_{\alpha} F_{\alpha} \neq \phi\right) \Rightarrow \left(\begin{array}{l}\exists\ \alpha_1, \alpha_2, \ldots, \alpha_k \text{ such that} \\ F_{\alpha_1} \cap F_{\alpha_2} \cap \ldots \cap F_{\alpha_k} = \phi\end{array}\right) \tag{1}$$

Let a and b represent sentences, then

$$(a \Rightarrow b) \equiv (a' \vee b)$$

and

$$(b' \Rightarrow a') \equiv (b \vee a').$$

Let us take the negation of (1)

$$\left(\begin{array}{l}\forall\ \alpha_1, \alpha_2, \ldots, \alpha_k \text{ such that} \\ F_{\alpha_1} \cap F_{\alpha_2} \cap \ldots \cap F_{\alpha_k} = \phi\end{array}\right) \Rightarrow \left(\bigcap_{\alpha} F_{\alpha} \neq \phi\right). \tag{2}$$

Since (1) implies (2), the proof is completed.

● PROBLEM 16-12

Let A represent a compact subset of a Hausdorff space (X, T). Show that, if

$$q \in X - A, \tag{1}$$

then the open sets U and V exist, such that

$$q \in U, \quad A \subset V, \quad U \cap V = \phi. \tag{2}$$

See Figure 1.

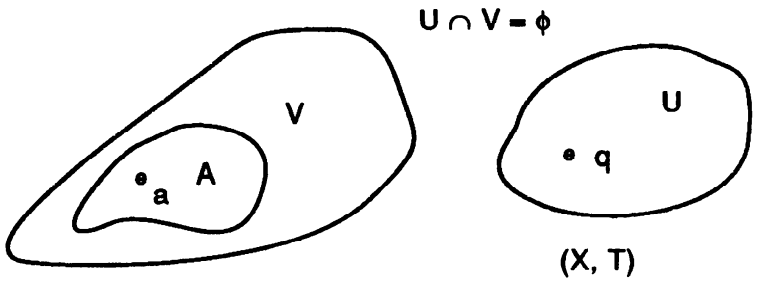


FIGURE 1

SOLUTION:

Choose $a \in A$, since $q \in X - A$, $a \neq q$ and since (X, T) is a Hausdorff space, open sets U_a and V_a exist such that

$$q \in U_a, \quad a \in V_a, \quad U_a \cap V_a = \phi. \tag{3}$$

The family of sets $\{V_a : a \in A\}$ forms an open cover of A

$$A \subset \bigcup_{a \in A} V_a. \tag{4}$$

Since A is compact, a finite subcover $V_{a_1}, V_{a_2}, \ldots, V_{a_k}$ exists, such that

$$A \subset V_{a_1} \cup V_{a_2} \cup \ldots \cup V_{a_k}. \tag{5}$$

Let us define

$$V = V_{a_1} \cup V_{a_2} \cup \ldots \cup V_{a_k} \tag{6}$$

and

$$U = U_{a_1} \cap \ldots \cap U_{a_k}. \tag{7}$$

Obviously U and V are open as the union and finite intersection of open sets.

From (5) and (6) we have

$$A \subset V,$$

from (3) and (7) we have

$$q \in U.$$

We have to show that $U \cap V = \phi$.

$$U \cap V = (U_{a_1} \cap \ldots \cap U_{a_k}) \cap (V_{a_1} \cup V_{a_2} \cup \ldots \cup V_{a_k}). \tag{8}$$

Since

$$U_{a_l} \cap V_{a_l} = \phi \tag{9}$$

for $l = 1, 2, \ldots, k$, we have

$$U \cap V = \phi.$$

● PROBLEM 16-13

Show that any compact subset of a Hausdorff space is closed.

SOLUTION:

Let A represent a compact subset of a Hausdorff space (X, T) and let $q \notin A$.

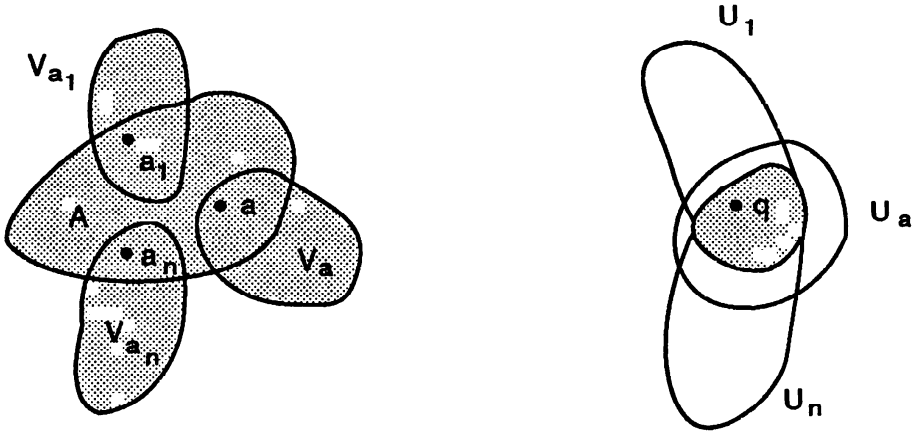


FIGURE 1

We shall show that if A is compact, then $X - A$ is open.

Let $a \in A$; then since (X, T) is Hausdorff, there are neighborhoods U_a and V_a of a and q respectively, such that

$$a \in U_a, \quad q \in V_a, \quad U_a \cap V_a = \phi. \tag{1}$$

The family of sets

$$\{V_a : a \in A\} \tag{2}$$

forms an open cover of A. Since A is compact, there are finitely many

$$a_1, a_2, \ldots, a_n \text{ such that } \{V_{a_1}, \ldots, V_{a_n}\} \tag{3}$$

forms on an open cover of A. Let

$$U = U_{q_1} \cap \ldots \cap U_{q_n} \text{ and } V = V_{a_1} \cup \ldots \cup V_{a_n}. \tag{4}$$

Then $q \in U$ and $U \cap V = \phi$. Since $q \in X - A$ and $q \in U$ and $U \cap A = \phi$, then $X - A$ is open and A is closed.

● PROBLEM 16-14

Show that a subset of a compact T_2–space (Hausdorff space) is compact, if and only if it is closed.

SOLUTION:

Let (X, T) represent a Hausdorff compact space and $A \subset X$, then

$$(A \text{ is compact}) \Leftrightarrow (A \text{ is closed}).$$

$\Leftarrow$ In Problem 16–8 we proved that any closed subset of a compact space is compact.

$\Rightarrow$ In Problem 16–13 we proved that any compact subset of a Hausdorff space is closed.

● PROBLEM 16-15

Prove

THEOREM

Let A and B represent disjoint compact subsets of a Hausdorff space (X, T). Then open sets U and V exist, such that

$$A \subset U, \quad B \subset V, \quad U \cap V = \phi. \tag{1}$$

■

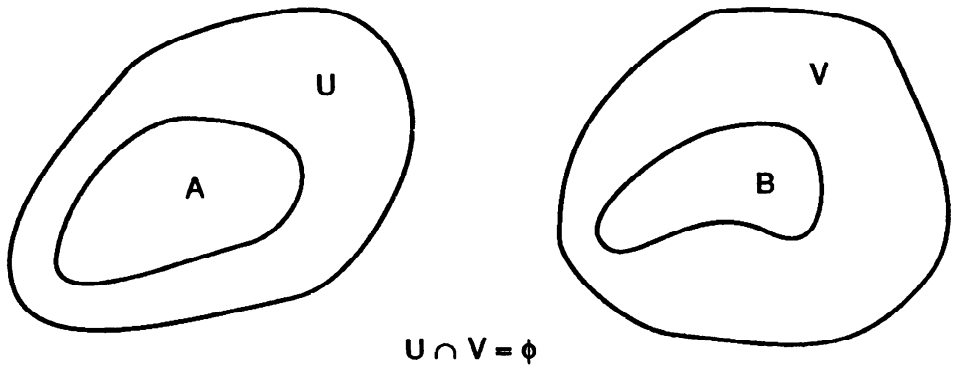


FIGURE 1

SOLUTION:

Let $a \in A$: then, since $A \cap B = \phi$, $a \notin B$. By hypothesis, B is compact and open sets U_a and V_a exist, such that

$$a \in U_a, \quad B \subset V_a, \quad U_a \cap V_a = \phi. \tag{2}$$

The family of sets $\{U_a : a \in A\}$ forms an open cover of A and since A is compact we can select $\{U_{a_1}, U_{a_2}, \ldots, U_{a_n}\}$ a finite subcover of A.

Let us denote

$$U = U_{a_1} \cup \ldots \cup U_{a_n}$$

$$V = V_{a_1} \cap \ldots \cap V_{a_n}. \tag{3}$$

Then

$$A \subset U \quad \text{and} \quad B \subset V. \tag{4}$$

Both U and V are open sets. For each $k = 1, 2, \ldots, n$

$$U_{a_k} \cap V_{a_k} = \phi. \tag{5}$$

Thus

$$U_{a_k} \cap V = \phi$$

and

$$U \cap V = (U_{a_1} \cup \ldots \cup U_{a_n}) \cap V = \phi. \tag{6}$$

● PROBLEM 16–16

Show that every compact Hausdorff space is normal.

SOLUTION:

Suppose (X, T) is a compact Hausdorff space and F_1 and F_2 are disjoint closed subsets of X.

By Problem 16–8, both sets F_1 and F_2 are also compact.

By Problem 16–15, open subsets U_1 and U_2 exist such that

$$F_1 \subset U_1, \quad F_2 \subset U_2, \quad U_1 \cap U_2 = \phi.$$

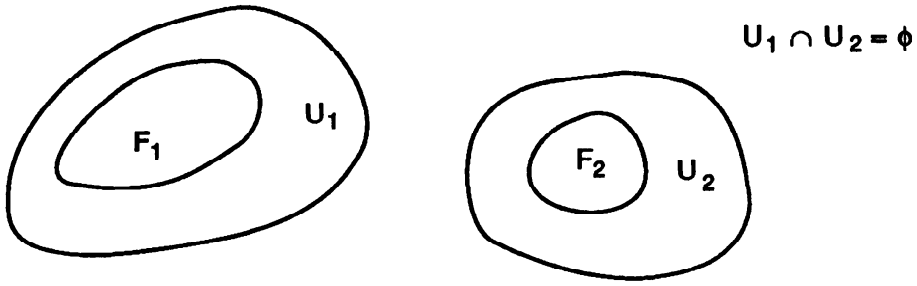


FIGURE 1

We conclude that (X, T) is normal (see definition in Problem 13–22). See Figure 2.

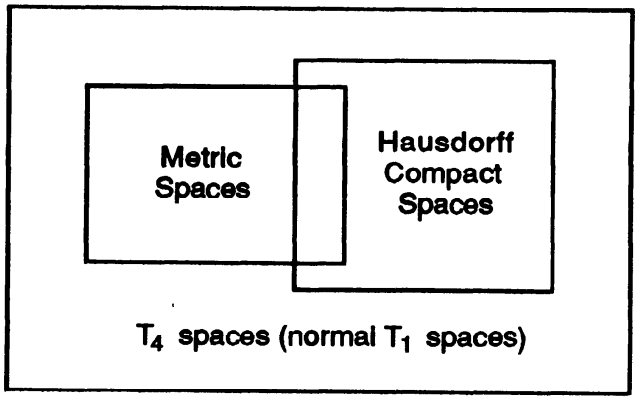


FIGURE 2

● PROBLEM 16-17

1. Let A represent a compact subset of a Hausdorff space (X, T).

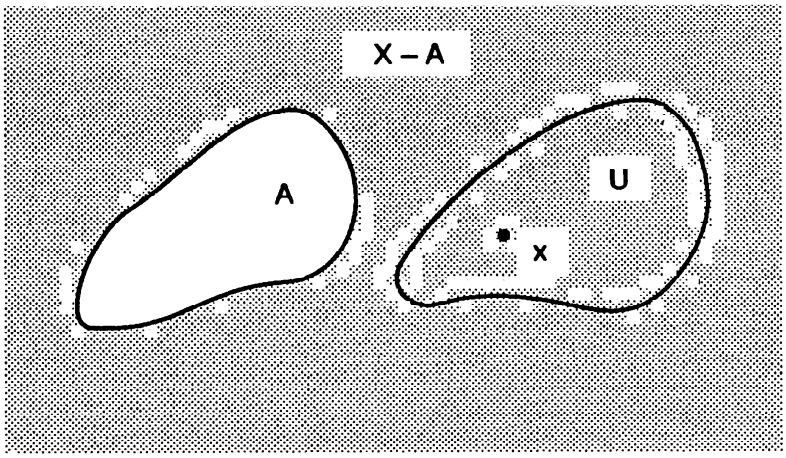


FIGURE 1

Show that if $x \notin A$, then there is an open set U, such that

$$x \in U \subset X - A. \tag{1}$$

2. Prove that a compact subset A of a Hausdorff space (X, T) is closed.

SOLUTION:

1. Open sets U and V exist such that

$$x \in U, \quad A \subset V \quad \text{and} \quad U \cap V = \phi. \tag{2}$$

Therefore,

$$U \cap A = \phi \quad \text{and} \quad x \in U \subset X - A. \tag{3}$$

2. We shall prove that $X - A$ is open. Let

$$x \in X - A$$

that is,

$$x \notin A.$$

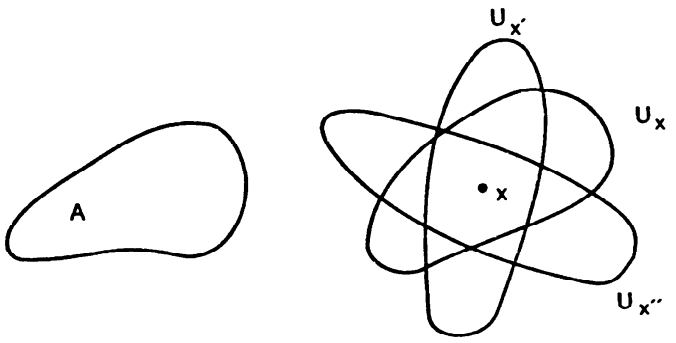


FIGURE 2

Since A is compact, an open set U_x exists, such that

$$x \in U_x \subset X - A. \tag{4}$$

Therefore,

$$X - A = \bigcup_{x \in X - A} U_x \tag{5}$$

and $X - A$ is open, while the union of open sets and A is closed. Compare with Problem 16–13.

● PROBLEM 16–18

How does an equivalence relation lead to the new compact spaces? (Hint: use results of Problem 16–17).

Show that the circle is compact.

SOLUTION:

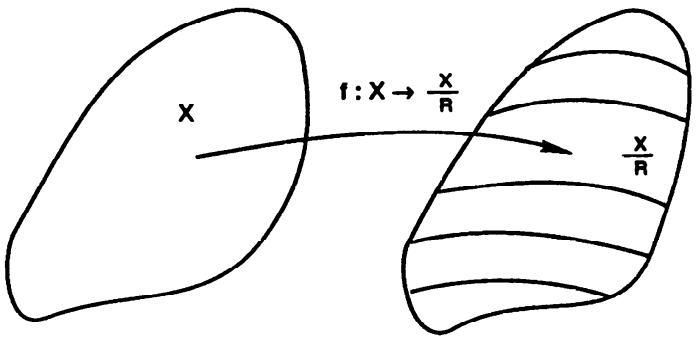


FIGURE 1

Suppose X is a compact space and R is an equivalence relation on X.

The space $X/_R$ is called the identification space. Since the identification mapping f from X onto $X/_R$,

$$f: X \to X/_R$$

is continuous and the space $X/_R$ is compact.

We have already shown that the closed interval [0, 1] is compact. An equivalence relation between [0, 1] and the circle can be established as follows:

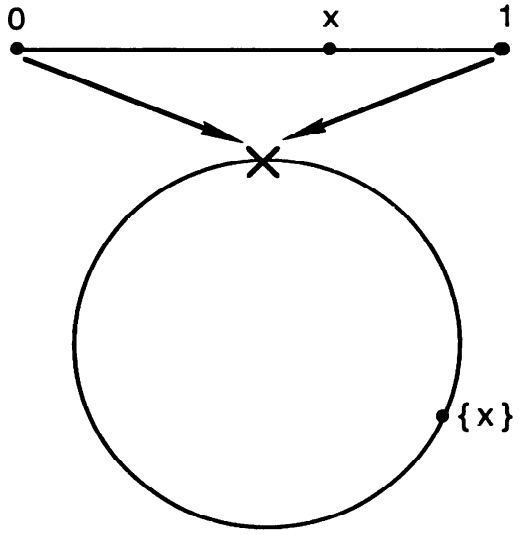


FIGURE 2

Let 0 be equivalent to 1, and every other element of [0, 1] be equivalent only to itself.

By identifying 0 and 1, we obtain a circle. The circle is an identification space derived from [0, 1], hence the circle is compact.

● PROBLEM 16-19

Prove:

THEOREM

Let f represent a one-to-one continuous function

$$f: X \to Y$$

from a compact space X into a Hausdorff space Y. Then X and $f(X)$ are homeomorphic.

■

See Figure 1.

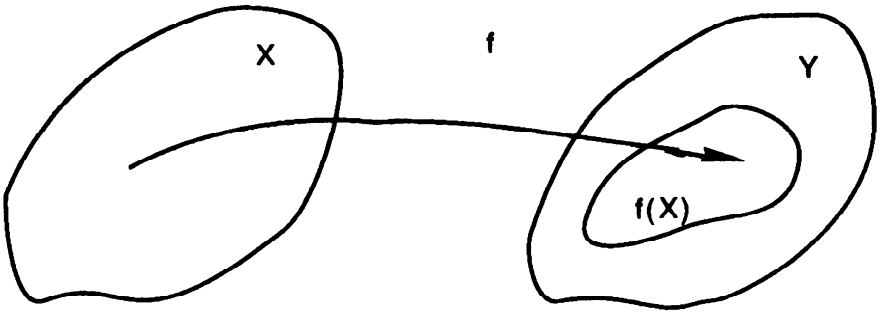


FIGURE 1

SOLUTION:

Function

$$f: X \to f(X) \tag{1}$$

is a continuous bijection (i.e., one-to-one and onto). Hence,

$$f^{-1}: f(X) \to X \tag{2}$$

exists. By hypothesis, f is continuous, so we only have to show that f^{-1} is continuous.

Function f^{-1} is continuous if, for every closed subset F of X,

$$[f^{-1}]^{-1}(F) = f(F) \tag{3}$$

it is a closed subset of $f(X)$.

Let F represent a closed subset of a compact space, then (by Problem 16–8) F is compact. Since F is continuous, $f(F)$ is a compact subset of $f(X)$. $f(X)$ is the Hausdorff space as a subspace of the Hausdorff space Y. Thus, $f(F)$ is closed and f^{-1} is continuous. Therefore,

$$f: X \to f(X) \tag{4}$$

is a homeomorphism.

● PROBLEM 16–20

Let (X, T) represent compact and let (X, T') represent the Hausdorff space.

Prove that

$$(T' \subset T) \Rightarrow (T' = T).$$

SOLUTION:

Consider the identity function defined on X:

$$f(x) = x$$

$$f: (X, T) \to (X, T').$$

Function f is one-to-one and onto. Since

$$T' \subset T,$$

f is continuous.

Since (X, T) is compact and (X, T') is the Hausdorff space, we conclude that (X, T) and (X, T') are homeomorphic. Therefore

$$T' = T.$$

● PROBLEM 16-21

Here is an important theorem which enables us to find more compact spaces.

TYCHONOFF THEOREM

Let $\underset{k}{\times} X_k \; k \in K$ represent the product space of the countable family of non-empty spaces $\{(X_\kappa, T_k) : k \in K\}$. Then

$$\underset{k}{\times} X_k$$

is compact, if and only if each component space is compact. ■

Prove it.

SOLUTION:

Suppose $\underset{k}{\times} X_k$ is compact. For each $l \in K$, the projection map

$$p_l : \underset{k}{\times} X_k \to X_l$$

is continuous and onto.

Therefore X_l is compact for each $l \in K$. Now, suppose each X_k is compact. Let $\{t_j\}, j \in J$ represent any ultranet in $\underset{k}{\times} X_k$ with the kth coordinate of t_j denoted by $t_j(k)$. Then

$$\{p_k(t_j)\} = \{t_j(k)\}, j \in J$$

is an ultranet in X_k and $\{t_j(k)\}$ converges in X_k. Therefore, $\{t_k\}$, $k \in K$ converges in $\underset{k}{\times} X_k$.

A space (X, T) is compact, if and only if every ultranet in X converges. Hence,

$$\underset{k}{\times} X_k$$

is compact.

● PROBLEM 16-22

Show that the cylinder, the torus and the cube are compact spaces.

SOLUTION:

We have already proved that the closed interval $I = [0, 1]$ and the circle C are compact.

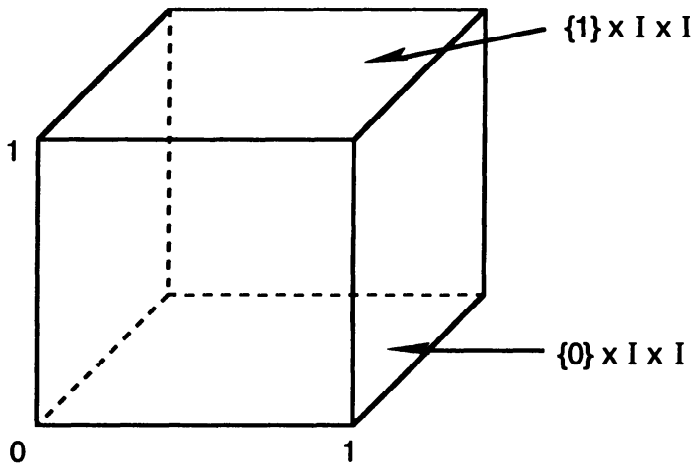


FIGURE 1

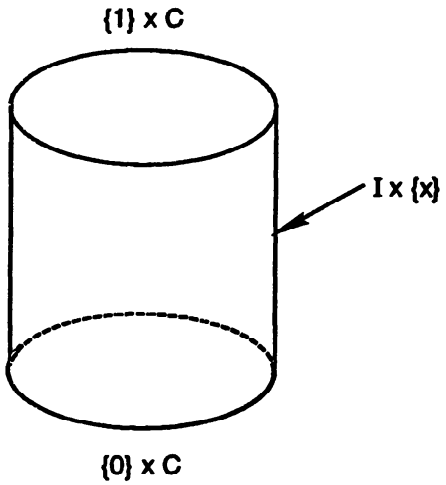


FIGURE 2

According to the Tychonoff theorem, the cube $[0, 1] \times [0, 1] \times [0, 1]$ is compact because $[0, 1]$ is compact.

Similarly, we show that the cylinder is compact, see Figure 2.

Since C and $[0, 1]$ are compact, the cylinder $C \times [0, 1]$ is compact.

The torus $C \times C$ is also compact, see Figure 3.

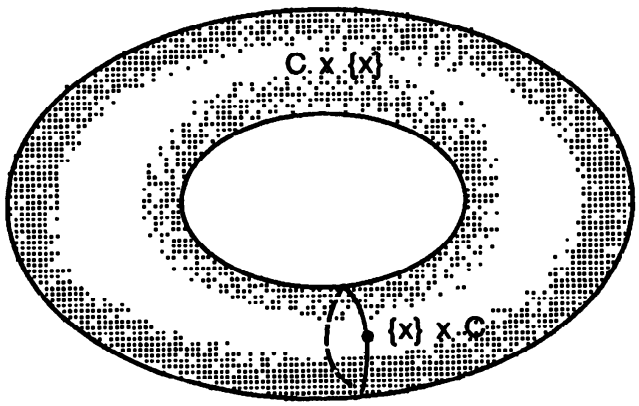


FIGURE 3

1. Show that if A is compact and F is closed, then $A \cap F$ is compact.

2. Prove that if $A_1, A_2, \ldots, A_k$ are compact subsets of a topological space X, then $A_1 \cup \ldots \cup A_k$ is also compact.

SOLUTION:

1.

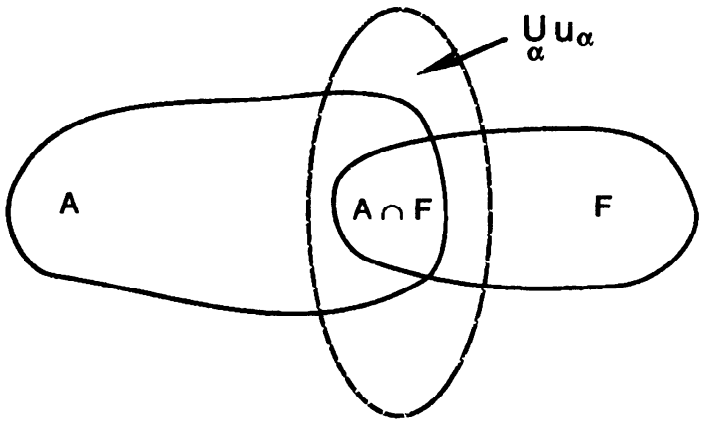


FIGURE 1

Let $\{U_\alpha\}$ represent an open cover of $A \cap F$

$$A \cap F \subset \bigcup_\alpha U_\alpha. \tag{1}$$

Then

$$A \subset \bigcup_\alpha U_\alpha \cup (X - F). \tag{2}$$

and since F is closed $\{\{U_\alpha\}, X - F\}$ is an open cover of A. Set A is compact so we can choose a finite subcover, say

$$A \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_n} \cup (X - F). \tag{3}$$

Hence

$$A \cap F \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_n} \tag{4}$$

and $A \cap F$ is compact.

2. Let A and B represent compact subsets of (X, T) and let $\{U_\alpha\}$ represent an open cover of $A \cup B$

$$A \cup B \subset \bigcup_\alpha U_\alpha. \tag{5}$$

Then $\{U_\alpha\}$ is an open cover of A and B, which are compact.

$$A \subset \bigcup_\alpha U_\alpha. \tag{6}$$

We can choose finite subcover

$$A \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_n} \tag{7}$$

and

$$B \subset U_{\alpha_{n+1}} \cup \ldots \cup U_{\alpha_{n+k}}. \tag{8}$$

Hence

$$A \cup B \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_{n+k}} \tag{9}$$

and $A \cup B$ is compact.

● PROBLEM 16-24

Show that a finite subset A of a topological space (X, T) is sequentially compact.

SOLUTION:

DEFINITION OF SEQUENTIALLY COMPACT SETS

A subset A of a topological space (X, T) is sequentially compact if every sequence in A contains a subsequence which converges to a point in A.

■

Let A represent a finite subset of (X, T), and let

$$(x_1, x_2, x_3, \ldots) \tag{1}$$

represent a sequence in A.

Since A is finite, at least one of the elements in A, say x_0, appears an infinite number of times in the sequence. We can choose the subsequence of (1) to be

$$(x_0, x_0, x_0, \ldots) \tag{2}$$

which converges to the point x_0 belonging to A.

● PROBLEM 16-25

Show that a continuous image of a sequentially compact set is sequentially compact.

SOLUTION:

Let

$$f: X \to Y \tag{1}$$

be a continuous function and let A represent a subset of X which is sequentially compact. We will show that $f(A)$ is sequentially compact in Y.

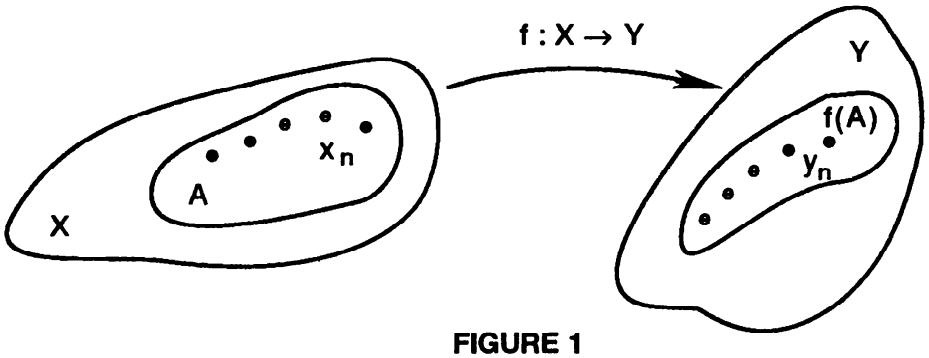


FIGURE 1

Let

$$(y_1, y_2, \ldots) \tag{1}$$

represent a sequence in $f(A)$. Then there exists a sequence in A

$$(x_1, x_2, \ldots) \tag{2}$$

such that

$$f(x_1) = y_1, f(x_2) = y_2, \ldots, f(x_n) = y_n, \ldots.$$

Since A is sequentially compact, sequence (2) contains a subsequence, say

$$(x_{k_1}, x_{k_2}, \ldots) \tag{3}$$

which converges to a point $x_0 \in A$.

Since f is continuous, we have

$$\left((x_{k_1}, x_{k_2}, \ldots) \text{ converges to } x_0\right) \Rightarrow \begin{pmatrix} (f(x_{k_1}), f(x_{k_2}), \ldots) \\ \text{converges to } f(x_0) \end{pmatrix}.$$

We have $f(x_0) \in f(A)$.

Thus, $f(A)$ is sequentially compact.

● PROBLEM 16-26

1. Show that every bounded closed interval $[a, b]$ is countably compact.
2. Show that the open interval (a, b) is not countably compact.

SOLUTION:

DEFINITION OF COUNTABLY COMPACT SETS

A subset A of a topological space (X, T) is countably compact, if every infinite subset $B \subset A$ has an accumulation point in A.

■

1. Let B represent an infinite subset of a closed interval $[a, b]$.

$$B \subset [a, b].$$

We shall apply:

BOLZANO-WEIERSTRASS THEOREM

Every bounded infinite set of real numbers has an accumulation point.

■

Thus B has an accumulation point x. Since $[a, b]$ is closed, and $B \subset [a, b]$, the accumulation point x of B belongs to $[a, b]$. Hence $[a, b]$ is countably compact.

2. We shall show that the open interval (0, 1) is not countably compact. Consider the infinite subset of (0, 1)

$$\{1/2, 1/3, 1/4, 1/5, \ldots\}.$$

This subset has only one accumulation point 0 which does not belong to (0, 1).

Hence (0, 1) is not countably compact.

● PROBLEM 16-27

Show that a closed subset of a countably compact space is countably compact.

SOLUTION:

Let (X, T) represent a countably compact space and F be a closed subset of X.

Let A represent any infinite subset of F. A is also an infinite subset of a countably compact space (X, T). Therefore an accumulation point x of A exists, such that

$$x \in X$$

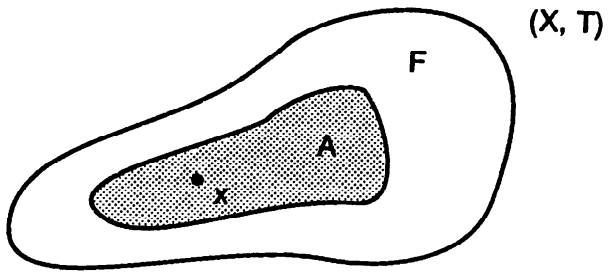


FIGURE 1

Since $A \subset F$, x is also an accumulation point of F. By hypothesis, F is closed, so it contains all its accumulation points. Thus

$$x \in F.$$

Hence, any infinite subset A of a closed set F has an accumulation point $x \in F$. Therefore, F is countably compact.

● PROBLEM 16-28

Prove:

$$(X \text{ is compact}) \Rightarrow (X \text{ is countably compact}).$$

SOLUTION:

Suppose (X, T) is compact.
Let A represent a subset of X

$$A \subset X$$

with no accumulation points in X.

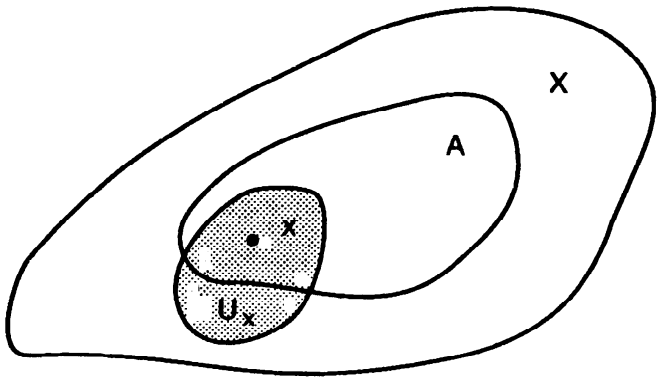


FIGURE 1

Each point $x \in X$ belongs to an open set U_x which contains, at most, one point of A.

Now, consider the family of sets

$$\{U_x : x \in X\} \tag{1}$$

Family (1) consists of open sets and

$$X \subset \bigcup_{x \in X} U_x. \tag{2}$$

Hence (1) is an open cover of X. Since X is compact (by hypothesis), we can select a finite subcover

$$U_{x_1}, U_{x_1}, \ldots, U_{x_n}. \tag{3}$$

Then

$$A \subset X \subset U_{x_1} \cup \ldots \cup U_{x_n}. \tag{4}$$

Since each of the sets (3) contains, at most, one point of A, A is finite. Therefore, every infinite subset of X contains an accumulation point in X, that is, X is countably compact.

● PROBLEM 16-29

In Problem 16–28 we showed that if the space is compact, then it is countably compact.

$$((X, T) \text{ compact}) \Rightarrow ((X, T) \text{ countably compact}).$$

Give an example showing that this implication cannot be reversed.

SOLUTION:

We shall give an example of a space which is countably compact, but not compact. Let N represent the set of positive integers with the topology T generated by the sets

$$\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots$$

and let $A \neq \phi$ represent a non-empty subset of N.

Let $k \in A$, then, if k is even, then $k - 1$ is a limit point of A and if k is odd, then $k + 1$ is a limit point of A.

Hence A has an accumulation point and (N, T) is countably compact. Since

$$\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \ldots\}$$

is an open cover of N which has no finite subcover, (N, T) is not compact.

● PROBLEM 16-30

1. Show that sequential compactness is a topological property.

2. Show that countable compactness is a topological property.

SOLUTION:

1. Let (X, T) and (Y, T') represent homeomorphic topological spaces. Then a homeomorphism exists

$$f: X \to Y.$$

Suppose X is sequentially compact and let

$$(y_n) \tag{1}$$

be a sequence in Y.

Then, since f is onto and one-to-one,

$$f^{-1}(y_n) \tag{2}$$

is a sequence in X. Since X is sequentially compact, (2) contains a subsequence, say

$$f^{-1}(y_{k_n}) \tag{3}$$

which converges to a point in X,

$$f^{-1}(y_{k_n}) \to x \in X\,. \tag{4}$$

Since f is continuous,

$$f(f^{-1}(y_{k_n})) = (y_{k_n}) \to f(x) \in Y. \tag{5}$$

Thus, (1) contains a subsequence (y_{k_n}), which converges to a point $f(x)$ in Y. (Y, T') is sequentially compact.

2. Let B represent an infinite subset of Y. Then $f^{-1}(B)$ is an infinite subset of X. X is countably compact. Hence $f^{-1}(B)$ has an accumulation point x in X. Thus B has an accumulation point $f(x)$ in Y and Y is countably compact.

● PROBLEM 16-31

Prove that if a space (X, T) is sequentially compact, then it is countably compact.

(X sequentially compact) $\Rightarrow$ (X countably compact).

SOLUTION:

Suppose (X, T) is sequentially compact and A is any infinite subset of X, $A \subset X$. We shall show that A has an accumulation point in A. We can find a sequence

$$(a_n) \quad n \in N$$

in A with distinct terms, i.e., for any $k, l \in N$, $k \neq l$

$$a_k \neq a_l.$$

Since X is sequentially compact, (a_n) contains a subsequence which converges to a point in X.

$$(a_{k_n}) \to x \in X.$$

Then $x \in X$ is an accumulation point of A and (X, T) is countably compact.

There are some other properties related to compactness, for example, paracompactness, metacompactness, and pseudocompactness.

● PROBLEM 16-32

Show that the coordinate plane R^2 with the Pythagorean metric topology is locally compact.

SOLUTION:

DEFINITION OF LOCALLY COMPACT SPACES

A space (X, T) is said to be locally compact if for any $x \in X$ and any neighborhood U of x, there is a compact set A such that

$$x \in Int(A) \subset A \subset U \tag{1}$$

■

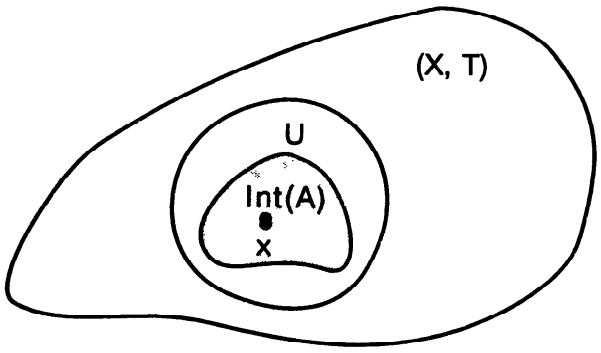


FIGURE 1

Plane R^2 is not compact, since it is not bounded.

Let

$$x \in R^2$$

and let U represent any neighborhood of x. Then there exists $r > 0$ such that

$$x \in B(x, r) \subset U. \tag{2}$$

We have

$$B(x, r/2) \subset \overline{B(x, r/2)} \subset B(x, r) \subset U. \tag{3}$$

The set

$$\overline{B(x, r/2)}$$

is a closed and bounded subset of R^2, hence it is compact.

$$x \in B(x, r/2) \subset \overline{B(x, r/2)} \subset U. \tag{4}$$

Hence R^2 with the Pythagorean metric topology is locally compact.

● PROBLEM 16-33

Prove that any compact T_2–space (X, T) is locally compact.

SOLUTION:

The criterion for local compactness given in Problem 16–32 is simpler for the T_2–spaces.

DEFINITION

Let (X, T) represent a T_2–space. Then X is locally compact if and only if, given any

$$x \in X,$$

there is a compact set A, such that

$$x \in Int(A).$$

■

One can prove that for T_2–spaces, the existence of one compact subset A of X such that $x \in Int(A)$ guarantees that for any neighborhood U of x, there is a compact set B, such that

$$x \in Int(B) \subset B \subset U.$$

Let (X, T) be a compact T_2–space. Then X is a compact neighborhood of any $x \in X$.

● PROBLEM 16-34

Let Q represent the subspace of rational numbers in the space R of real numbers with the absolute value topology. Show that Q is not locally compact.

SOLUTION:

Space R is obviously locally compact.

Take any $x \in Q$ and a compact set A such that

$$x \in int(A). \tag{1}$$

Remember that a subset A of R (or R^n) is compact, if and only if A is bounded and closed.

Then A contains infinitely many elements of Q.

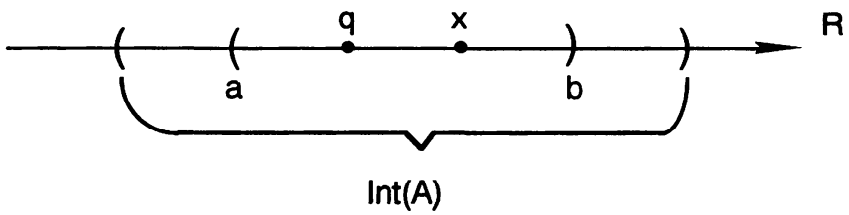


FIGURE 1

We can find rational numbers a and b, such that $(a, b) \subset R$

$$x \in (a, b) \cap Q \subset Int(A). \tag{2}$$

Let q represent an irrational number, such that

$$q \in (a, b). \tag{3}$$

We define an open cover of A as follows:

$$\text{for each } a \in A,$$

$$U(a) = \begin{cases} if \ \ q < a \ \ \{s \in R : a < s\} \\ if \ \ q > a \ \ \{t \in R : t < a\} \end{cases} \tag{4}$$

$\{U(a) \cap A : a \in A\}$ is an open cover of A which has not finite subcover. Hence Q is not locally compact.

● PROBLEM 16-35

Prove the following

THEOREM

If (X, T) is a Hausdorff space (i.e., a T_2–space) locally compact, then (X, T) is a regular space

■

SOLUTION:

We shall apply the results of Problem 13–17. A space (X, T) is regular iff given any $x \in X$ and any neighborhood U of x, there is a neighborhood V of x such that

$$\overline{V} \subset U.$$

Let (X, T) represent a locally compact T_2–space. If $x \in X$, and U is any neighborhood of x, then there is a compact set A, such that

$$x \in Int(A) \subset A \subset U \tag{1}$$

(see Problem 16–32).

Since A is compact, A is closed, hence

$$\overline{Int(A)} \subset A. \tag{2}$$

We define V as

$$V = Int(A) \tag{3}$$

and obtain

$$x \in V \subset \overline{V} \subset U. \tag{4}$$

Since V is a neighborhood of x, (X, T) is regular.

● PROBLEM 16-36

1. Show that an open subspace of a locally compact space is locally compact.

2. Show that a closed subspace of a locally compact T_2–space is locally compact.

SOLUTION:

1. Let U represent an open subspace of a locally compact space (X, T) and $x \in U$.

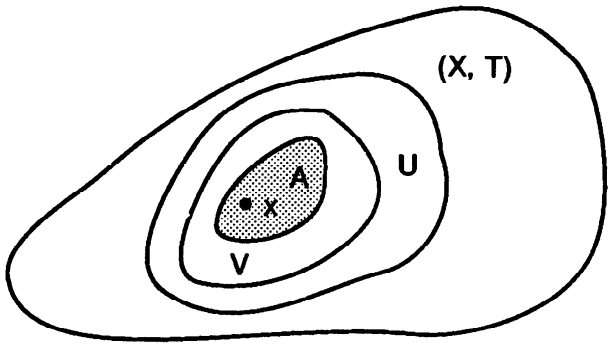


FIGURE 1

Let V represent any neighborhood of x, such that

$$x \in V \subset U. \tag{1}$$

Since (X, T) is locally compact, a compact set A exists such that

$$x \in Int(A) \subset A \subset V. \tag{2}$$

Hence, U is locally compact.

2. Let F represent a closed subspace of X and $x \in F$ and let A represent any compact subset of X, such that

$$x \in Int(A) \subset A. \tag{3}$$

A is closed because (X, T) is T_2. The set

$$F \cap A$$

is a closed subset of the compact set A and thus, is compact. We have

$$x \in Int_F(F \cap A) \subset F \cap A \subset F \tag{4}$$

Where $Int_F(F \cap A)$ denotes $Int(F \cap A)$ in F.

By definition in Problem 16–33, we conclude that since F is T_2, F is locally compact.

● PROBLEM 16–37

In Problem 16–6, we showed that compactness is preserved by continuous functions. One can find examples showing that local compactness is not preserved by continuous functions. Prove the following:

THEOREM

If $f : (X, T) \to (Y, T')$ is a continuous, open function from a locally compact space (X, T) onto Y, then (Y, T') is also locally compact. ■

SOLUTION:

A function $f : X \to Y$ is said to be open, if for any open subset $U \subset X$, $f(U)$ is open in Y.

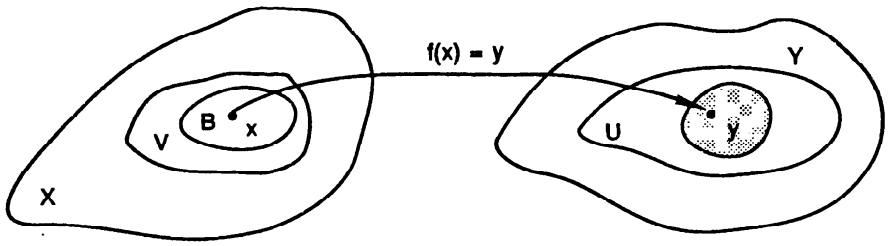


FIGURE 1

Let $y \in Y$ and U represent a neighborhood of y. For some $x \in X$

$$f(x) = y.$$

Since f is continuous, an open set V exists, such that

$$x \in V \in T \quad \text{and} \quad f(V) \subset U.$$

Since (X, T) is locally compact, there is a compact set B such that

$$x \in Int(B) \subset B \subset V.$$

Then

$$y = f(x) \in f(Int\ B) \subset f(B) \subset U.$$

Since f is open, $f(Int\ B)$ is open. Since B is compact and f is continuous, $f(B)$ is compact. Thus, Y is locally compact.

● PROBLEM 16-38

Prove the existence of a Lebesque number described in this theorem.

THEOREM

If (X, d) is a compact metric space and $\{U_\alpha\}$ is an open cover of X, then there is a positive number $r > 0$, such that

$$B(x, r) \subset U_\alpha \text{ for some } \alpha \text{ for any } x \in X.$$

■

That is, number $r > 0$ exists such that the r–neighborhood $B(x, r)$ of any point $x \in X$ is a subset of at least one of U_α. Number r is called a Lebesque number of the cover.

SOLUTION:

Since $\{U_\alpha\}$ is a cover of X, each $x \in X$ belongs to at least one U_α. Each U_α is open, hence for each $x \in X$

$$x \in B(x, r_x) \subset U_\alpha. \tag{1}$$

The collection of sets

$$\left\{B\left(x, \frac{r_x}{2}\right) : x \in X\right\} \tag{2}$$

forms an open cover of X and since X is compact, a finite subcover of (2) exists

$$\left\{B\left(x_1, \frac{r_{x_1}}{2}\right), \ldots, B\left(x_n, \frac{r_{x_n}}{2}\right)\right\}. \tag{3}$$

The Lebesque number is defined by

$$r = \min\left(\frac{r_{x_1}}{2}, \ldots, \frac{r_{x_n}}{2}\right). \tag{4}$$

Indeed, let $x \in X$ then

$$x \in B\left(x_k, \frac{r_{x_k}}{2}\right)$$

for some $1 \leq k \leq n$. Let

$$y \in B(x, r) \tag{5}$$

then

$$d(y, x_k) \leq d(y, x) + d(x, x_k) < r + \frac{r_{x_k}}{2} \leq r_{x_k}. \tag{6}$$

Thus

$$B(x, r) \subset B(x_k, r_{x_k}) \subset U_\alpha \tag{7}$$

for some U_α.

● PROBLEM 16-39

Show that the function $f: R \to R$

$$f(x) = 2x \tag{1}$$

is uniformly continuous.

SOLUTION:

The function

$$f: X \to Y \tag{2}$$

from a metric space (X, d) into a metric space (Y, d') is said to be uniformly continuous if for every $\varepsilon > 0$, there is $\delta > 0$ such that for every $x \in X$

$$f(B(x, \delta)) \subset B(f(x), \varepsilon). \tag{3}$$

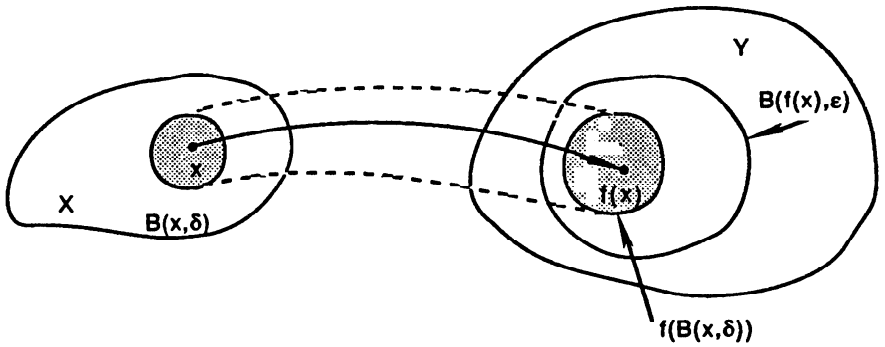


FIGURE 1

Let us choose any $\varepsilon > 0$ and set $\delta = \varepsilon/2$, then for any $x \in X$

$$f(B(x, \delta)) \subset B(f(x), \varepsilon). \tag{4}$$

Indeed let

$$y \in B(x, \delta)$$

then

$$|x - y| < \varepsilon/2$$

and

$$|f(x) - f(y)| = |2x - 2y| < 2 \cdot \varepsilon/2 = \varepsilon. \tag{5}$$

Thus, f is uniformly continuous.

● PROBLEM 16-40

Let $f : X \to Y$ represent a continuous function from a compact metric space (X, d) into a metric space (Y, d'). Show that f is uniformly continuous.

SOLUTION:

Let ε represent any positive number, $\varepsilon > 0$. We can define an open cover of Y as

$$\{B(y, \varepsilon/2) : y \in Y\}. \tag{1}$$

Since f is a continuous function, each $f^{-1}(B(y, \varepsilon/2))$ is an open set in X and

$$\{f^{-1}(B(y, \varepsilon/2))\} \tag{2}$$

is an open cover of X. But X is a compact metric space. Hence, by Problem 16–38, a Lebesque number r exists for cover (2).

Then for each $x \in X$,

$$f(B(x, q)) \subset B(f(x), \varepsilon). \tag{3}$$

Indeed, from the definition of the Lebesque number ($r = q$)

$$B(x, r) \subset f^{-1}(B(y, \varepsilon/2)) \tag{4}$$

where $f(x) = y$

● PROBLEM 16–41

Let (R^2, d) represent R^2 space with the Pythagorean distance. Show that the subset $B(x, 1) \subset R^2$, where $x = (1, 1)$, is totally bounded.

SOLUTION:

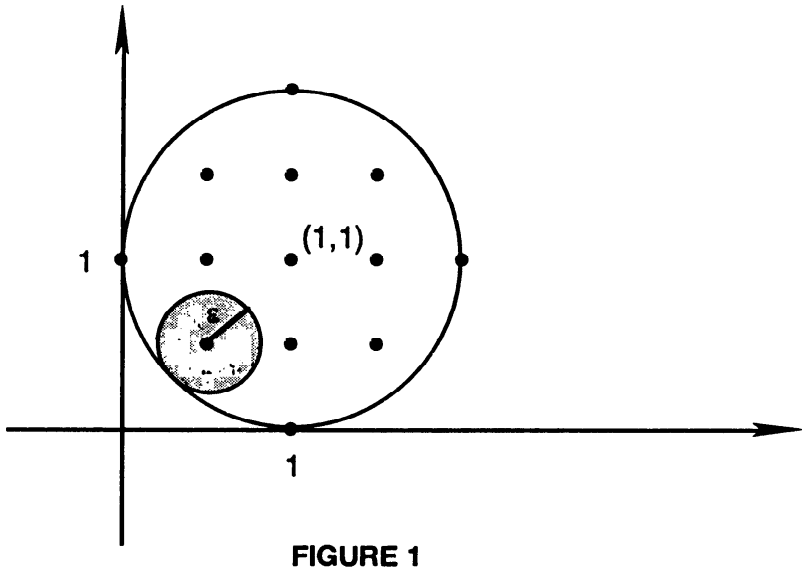


FIGURE 1

Let A represent a subset of a metric space (X, d) and let $\varepsilon > 0$. An ε–net for A is a finite set of points in X

$$N = \{x_1, x_2, \ldots, x_n\}$$

such that for every $x \in A$, $x_k \in N$ exists, such that

$$d(x_1, x_k) < \varepsilon.$$

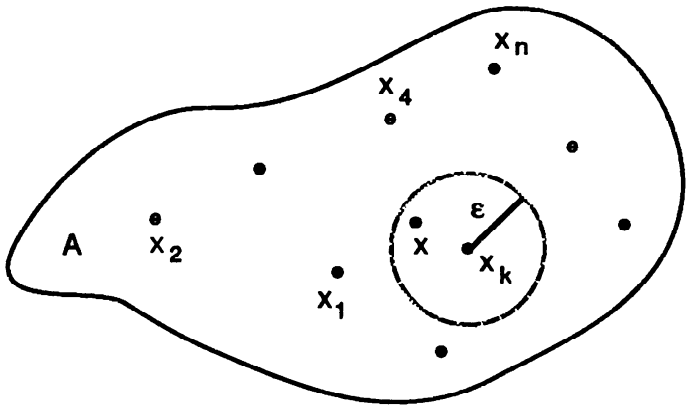


FIGURE 2

DEFINITION

A subset A of a metric space (X, d) is totally bounded if A has an ε-net for every $\varepsilon > 0$.

■

Let $\varepsilon > 0$. As an ε-net for a circle $B(x, 1)$ with the center at $x = (1, 1)$ and radius 1, we can choose all points inside the circle $B(x, 1)$ with coordinates

$$(1, 1), (1, 1 \pm \varepsilon), (1, 1 \pm 2\varepsilon), \ldots, (1 \pm \varepsilon, 1), (1 \pm 2\varepsilon, 1), \ldots,$$

$$\ldots, (1 \pm k\varepsilon, 1 \pm n\varepsilon)$$

where n and k are positive integers. Note that we are not looking for the smallest number of elements to form an ε-net.

● PROBLEM 16-42

Prove:

THEOREM

Totally bounded sets are bounded. ■

SOLUTION:

The diameter of A, $\delta(A)$, is defined by

$$\delta(A) = \sup \{d(x, x') : x, x' \in A\}. \quad (1)$$

Set A is bounded if

$$\delta(A) < \infty \tag{2}$$

Suppose A is a totally bounded set and let

$$N = \{x_1, x_2, \ldots, x_n\} \tag{3}$$

be ε-net for A.

Let $x, x' \in A$ be any elements of A. Then two elements of N exist, x_l and x_m, such that

$$d(x_1, x_l) < \varepsilon \quad d(x', x_m) < \varepsilon. \tag{4}$$

Also for any $x_l, x_m \in N$

$$d(x_l, x_m) < p < \infty \tag{5}$$

where p is some positive finite number.

We have

$$d(x, x') \leq d(x, x_l) + d(x_l, x') \leq$$

$$\leq d(x, x_l) + d(x', x_m) + d(x_m, x_l) <$$

$$< \varepsilon + \varepsilon + p < \infty. \tag{6}$$

Hence

$$\delta(A) = \sup \{d(x, x') : x, x' \in A\} < \infty \tag{7}$$

and A is bounded.

● PROBLEM 16-43

Give an example of a bounded set which is not totally bounded.

SOLUTION:

We shall first define Hilbert space H. By R^∞ we denote the class of all infinite sequences

$$(a_1, a_2, \ldots) \tag{1}$$

such that

$$\sum_{n=1}^{\infty} a_n^2 < \infty. \tag{2}$$

Let $a, b \in R^\infty$ and

$$a = (a_1, a_2, \ldots) \quad b = (b_1, b_2, \ldots) \tag{3}$$

then the l_2–metric on R^∞ is defined by

$$d(a,b) = \sqrt{\sum_{n=1}^{\infty} |a_n - b_n|^2}. \tag{4}$$

The metric space (R^∞, d) with the l_2–metric is called Hilbert space, or l_2–space, and denoted by H. Let A represent a subset of H consisting of elements

$$e_1 = (1, 0, 0, \ldots)$$

$$e_2 = (0, 1, 0, \ldots)$$

$$e_3 = (0, 0, 1, \ldots)$$

Then

$$d(e_j, e_k) = \sqrt{1 + 1} = \sqrt{2} \tag{5}$$

and A is bounded because

$$\delta(A) = \sup\{d(e_j, e_k) : e_j, e_k \in A\} = \sqrt{2} < \infty. \tag{6}$$

But A is not totally bounded. Let $\varepsilon = 1/2$ then, ε–net of A consists of all elements of A. The infinite set A cannot be separated into a finite number of subsets, each with diameter less than $\varepsilon = 1/2$.

● PROBLEM 16-44

Show that sequentially compact subsets of a metric space are totally bounded.

SOLUTION:

Let A represent a subset of a metric space (X, d). We shall prove that if A is not totally bounded, then A is not sequentially compact. Suppose A is not totally bounded, then $\varepsilon > 0$ exists, such that no finite ε-net of A exists. Let $a_1 \in A$. See Figure 1.

Then $a_2 \in A$ exists, such that

$$d(a_1, a_2) \geq \varepsilon.$$

Otherwise $\{a_1\}$ would be ε-net. A point $a_3 \in A$ exists, such that

$$d(a_1, a_3) \geq \varepsilon$$

$$d(a_1, a_2) \geq \varepsilon$$

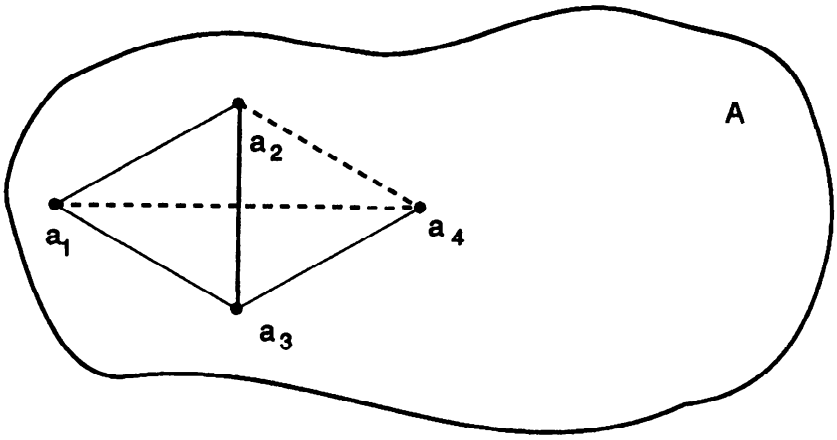


FIGURE 1

Otherwise $\{a_1, a_2\}$ would be ε-net. Following this procedure, we obtain a sequence of points

$$(a_1, a_2, a_3, \ldots),$$

such that

$$d(a_l, a_k) \geq \varepsilon \quad \text{for } l \neq k.$$

This sequence $(a_1, a_2, \ldots)$ does not have any convergent subsequence. Thus, A is not sequentially compact.

● PROBLEM 16-45

Show that if A is a countably compact subset of a metric space (X, d), then A is also sequentially compact.

SOLUTION:

Let

$$(a_1, a_2, \ldots) \tag{1}$$

represent a sequence in A. We shall show that a subsequence of (1) exists which converges to a point in A. If the set

$$\{a_1, a_2, \ldots\} \tag{2}$$

is finite then one of the elements, say a_l, of (2) appears infinitely many times in (1). Hence the subsequence

$$(a_l, a_l, a_l, \ldots) \tag{3}$$

of (1) converges to $a_l \in A$.

Suppose (2) is infinite. By hypothesis, A is countably compact. Hence, the infinite subset of A

$$\{a_1, a_2, a_3, \ldots\}$$

has an accumulation point a in A. Since (X, d) is a metric space, we can choose a subsequence

$$(a_{n_1}, a_{n_2}, a_{n_3}, \ldots) \tag{4}$$

of (1), which converges to $a \in A$. Thus, A is sequentially compact.

● PROBLEM 16-46

Prove this theorem:

THEOREM

If (X, d) is a metric space, then the following statements are equivalent:

1. X is compact.
2. X is countably compact
3. X is sequentially compact.

■

SOLUTION:

In Problem 16–28, we proved that any compact space is countably compact. Hence

$$1 \Rightarrow 2$$

for any metric space.

In Problem 16–45, we proved that any countably compact metric space is sequentially compact. Hence

$$2 \Rightarrow 3.$$

We shall prove that if a subset A of a metric space (X, d) is sequentially compact, then it is compact, i.e., $3 \Rightarrow 1$.

Let A be sequentially compact and let $\{U_\alpha\}$ be an open cover of A. Set $\{U_\alpha\}$ has a Lebesque number $r > 0$, since every open cover of a sequentially compact subset of a metric space has a Lebesque number. Since A is totally bounded, there is a decomposition of A into a finite number of subsets

$$B_1, B_2, \ldots, B_k$$

such that $d(B_1), d(B_2), \ldots, d(B_k) < r$. But r is a Lebesque number of A, hence open sets exist such that

$$B_1 \subset U_{\alpha_1}, \ldots, B_k \subset U_{\alpha_k}.$$

Thus

$$A \subset B_1 \cup \ldots \cup B_k \subset U_{\alpha_1} \cup \ldots \cup U_{\alpha_k}$$

and $\{U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ is a finite subcover of $\{U_\alpha\}$. A is compact and

$$1 \Leftrightarrow 2 \Leftrightarrow 3.$$

● PROBLEM 16-47

Show that any countably compact metric space (X, d) is separable.

SOLUTION:

Let s represent any positive number. For any s, there is a maximal subset A_s of X, such that for any $a, b \in A_s$

$$d(a, b) \geq s. \tag{1}$$

Suppose A_s is infinite for some $s > 0$, then A_s has an accumulation point y. Then $B(y, s/2)$ contains infinitely many points of A_s. But any of these points are closer than s. Thus, a contradiction and A_s is finite for each s. But then for any $x \in X$

$$A_s \cap B(x, s) \neq \phi. \tag{2}$$

Otherwise we would obtain a contradiction to the maximality of A_s. For each positive integer n we find $A_{\frac{1}{n}}$. Then

$$\bigcup_{n \in N} A_{\frac{1}{n}}$$

is a countable dense subset of X. Hence X is separable.

● PROBLEM 16-48

Find an example of a compactification of the open interval (0, 1) with the absolute value topology.

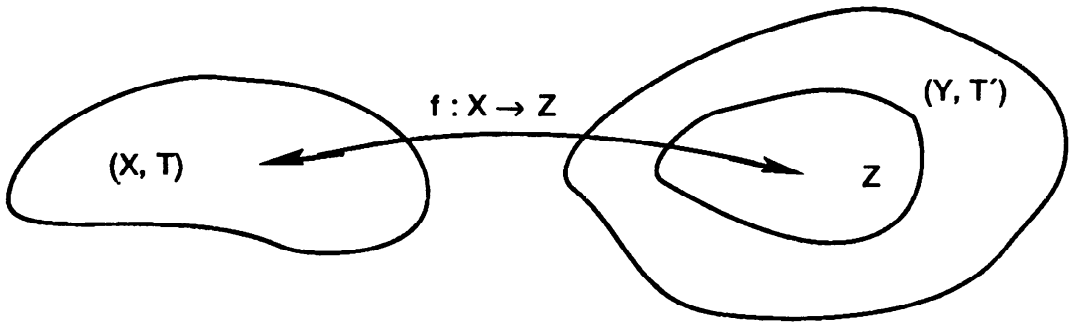
SOLUTION:

DEFINITION OF COMPACTIFICATION

Let (X, T) represent any topological space. A compactification of X is a compact space (Y, T'), such that X is homeomorphic to a dense subspace of Y.

■

Note that by compactification of X, we understand a space Y and a homeomorphism from X onto a dense subspace of Y.



f is homeomorphism
Z is dense in Y

FIGURE 1

Thus, for a given X and Y, it is possible to find more than one compactification (possibly infinitely many).

Two possible compactifications of the open interval (0, 1) are the circle and the closed interval [0, 1].

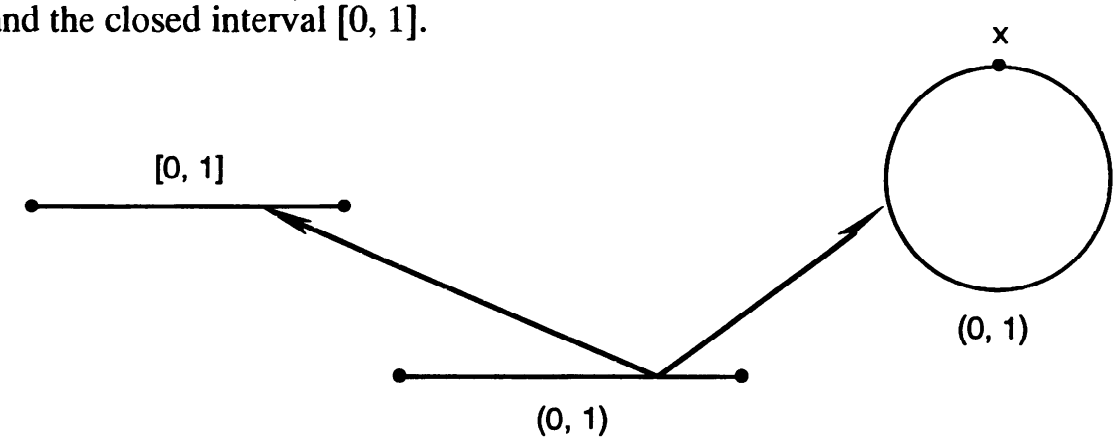


FIGURE 2

Obviously (0, 1) is homeomorphic to a dense subset of [0, 1]. Also (0, 1) is homeomorphic to a circle with one point removed.

● PROBLEM 16-49

Let P denote the (x, y) plane in the R^3 space with the Euclidean topology. Let S be the sphere of radius 1 and center at (0, 0, 1).

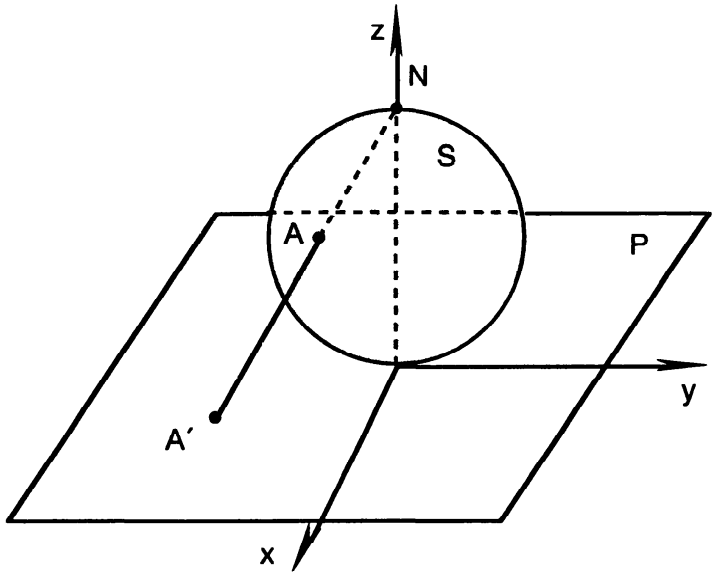


FIGURE 1

Show that S is a compactification of P.

SOLUTION:

Consider a line passing through the point $N = (0, 0, 2)$ on the sphere S and any point A on the sphere as shown in the figure.

Take any point A' on the plane, $A' \in P$. Then a line from A' to N determines the image of A' on the sphere.

$$f: P \to S$$

is defined by

$$f(A') = A.$$

f is one-to-one and onto and both f and f^{-1} are continuous. Hence, f is a homeomorphism from the plane P onto the subset $S - \{N\}$ of S. Obviously P is not compact, while S is compact.

The set $S - \{N\}$ is dense in S. Thus, S is a compactification of P.

● PROBLEM 16-50

Define the Alexandroff (sometimes called one-point) compactification of a T_2–space.

SOLUTION:

Let (X, T) represent any T_2–space. If X is compact then we define the Alexandroff compactification of X to be X itself.

Now suppose X is not compact. Let A be any point which does not belong to X. We define

$$Y = X \cup \{A\}$$

and the topology on Y as follows: U is open in Y if U is open in X, i.e., $U \in T$; or if $A \in U$, then $Y - U$ is a compact subset of X.

The Alexandroff, or one-point, compactification of X is defined as the set Y with the topology defined above.

Point A is called an ideal point. It can be shown that

1. Y is a topological space

2. Y is compact

3. X is dense in Y.

It can be also proved that:

THEOREM

The Alexandroff compactification Y of any topological T_2–space, (X, T) is a topological space and it is a compactification of X in the sense of the definition of Problem 16–48.

● PROBLEM 16-51

Prove the following:

THEOREM

The Alexandroff compactification of a space (X, T) is T_2 if and only if X is T_2 and locally compact. ■

SOLUTION:

Suppose (X, T) is compact. Then the Alexandroff compactification of X is X itself.

Then if X is T_2, X is T_2 and also locally compact. Also if X is T_2 and locally compact, then X is T_2.

Suppose X is not compact and Y is its Alexandroff compactification. If Y

is T_2, then X is T_2, since X is a subspace of Y.

Since Y is T_2 and compact, Y is also locally compact. X is locally compact as an open subspace of Y.

Now suppose X is T_2 and locally compact. Let x and y represent elements of Y, $x \neq y$. If $x, y \in X$, then open sets U and V exist such that

$$x \in U \quad \text{and} \quad y \in V$$

$$U \cap V = \phi$$

because X is T_2.

Suppose $x = A$ (A is an ideal point). Then $y \in X$ and a compact subset B of X exists, such that

$$y \in Int\, B \subset B.$$

Hence $Y - B$ is a neighborhood of $x = A$ and $Int\, B$ is a neighborhood of y and

$$(Y - B) \cap Int\, B = \phi.$$

Thus Y is T_2.

CHAPTER 17

CONNECTEDNESS

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Show that any set X, consisting of two or more points, with discrete topology is disconnected.

SOLUTION:

DEFINITION OF MUTUAL SEPARATION

In the space (X, T), two nonempty, disjoint sets $U \subset X$, $V \subset X$ are said to be mutually separated if neither U, nor V contains a boundary point of the other.

■

DEFINITION OF CONNECTED SPACES

A space (X, T) is said to be disconnected if the set X can be expressed as the union of multiple, mutually separated, nonempty subsets of X. The set X is connected if it is not disconnected, (see Figure 1).

■

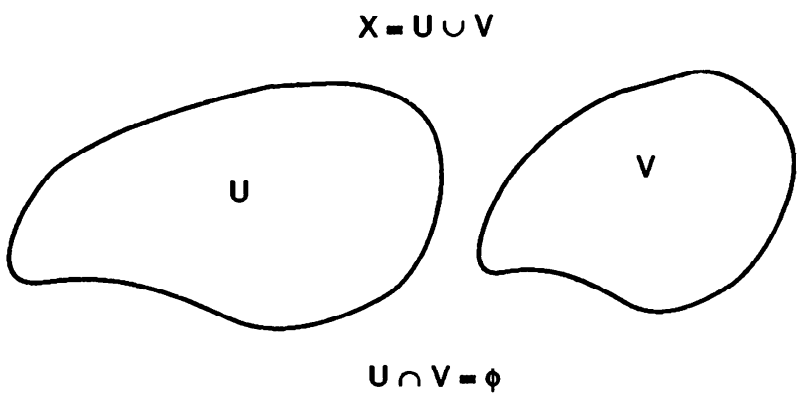


FIGURE 1

Space (X, T) with the discrete topology consisting of more than one point is disconnected. Indeed take any

$$x \in X \tag{1}$$

and consider the sets

$$\{x\} \quad \text{and} \quad X - \{x\}. \tag{2}$$

Both sets (2) are open, nonempty, and disjoint. Their union is X

$$\{x\} \cup [X - \{x\}] = X. \tag{3}$$

Any set with the trivial topology is connected since the only nonempty open subset of X is X itself.

Show that connectedness is a topological property.

SOLUTION:

We shall prove that if $f: (X, T) \to (Y, T')$ is a continuous function onto Y and X is connected, then Y is connected.

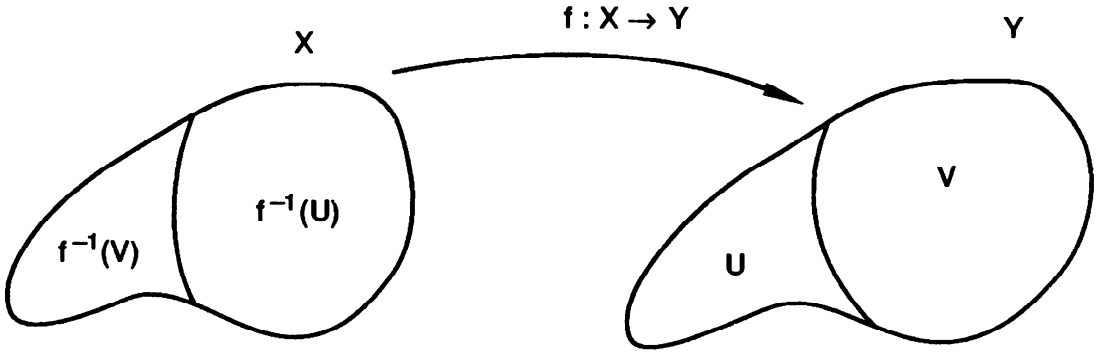


FIGURE 1

Suppose Y is not connected. Then there are open sets U and V such that

$$Y = U \cup V, \quad U \cap V = \phi. \tag{1}$$

Since f is continuous, then both $f^{-1}(U)$, and $f^{-1}(V)$ are open subsets of X. Since f is a bijection then

$$f^{-1}(U) \cap f^{-1}(V) = \phi \tag{2}$$

and

$$f^{-1}(U) \cup f^{-1}(V) = X. \tag{3}$$

Hence X is disconnected which is a contradiction. Therefore, Y is connected.

For example, if X is a connected space and R is an equivalence relation on R, then the identification space X/R is connected.

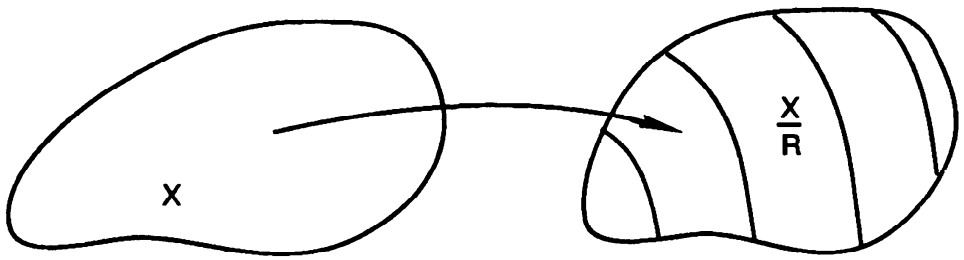


FIGURE 2

The identification mapping $f: X \to X/R$ is continuous.

● PROBLEM 17-3

Show that if (X, T) is any topological space, then the following statements are equivalent:

1. X is connected.

2. X cannot be expressed as the union of two nonempty, disjoint, closed subsets.

SOLUTION:

1. $\Rightarrow$ 2.

Suppose X is connected and two sets F and G exist such that

$$X = F \cup G \text{ and } F \cap G = \phi \tag{1}$$

and F, G are nonempty and closed in X. Then

$$G = X - F$$

$$F = X - G \tag{2}$$

as complements of closed sets are open. Hence,

$$X = F \cup G$$

is the union of disjoint, nonempty, open subsets of X. Thus, X is not connected.

2. $\Rightarrow$ 1.

Suppose X is disconnected. Then open sets U, V exist such that

$$X = U \cup V, \quad U \cap V = \phi \tag{3}$$

where U, V are nonempty. Then

$$U = X - V \quad \text{and} \quad V = X - U.$$

Hence, U and V are closed, a contradiction with 2.

● PROBLEM 17-4

Let (X, T) be any topological space. Show that the following statements are equivalent:

1. X is connected.

2. The only subsets of X which are open and closed are X and ϕ.

3. No continuous

$$f: X \to Y$$

is onto, where $Y = \{0, 1\}$ with the discrete topology.

SOLUTION:

1. $\Rightarrow$ 2.

Suppose $A \subset X$ is both open and closed, then

$$A \text{ and } X - A$$

are both open and

$$A \cup (X - A) = X, \quad A \cap (X - A) = \phi.$$

Hence, X is disconnected, a contradiction.

2. $\Rightarrow$ 3.

Suppose

$$f: X \to \{0, 1\}$$

is a continuous and onto. Then

$$f^{-1}(0) \neq \phi, \; f^{-1}(0) \neq X.$$

Since the subset $\{0\}$ of $\{0, 1\}$ is open and closed,

$$f^{-1}(0)$$

is open and closed in X. Contradiction.

3. $\Rightarrow$ 1.

Let U and V be disjoint, nonempty, open sets, such that

$$X = U \cup V.$$

The U and V are also closed. The characteristic function

$$\chi_A : X \to \{0, 1\}$$

is continuous and onto.

● PROBLEM 17-5

Suppose (X, T) is a space such that

$$X = U \cup V$$

$$U \cap V = \phi \tag{1}$$

where U and V are open, nonempty subsets of X. Show that if A is any connected subset of X, then either

$$A \subset U \quad \text{or} \quad A \subset V. \tag{2}$$

SOLUTION:

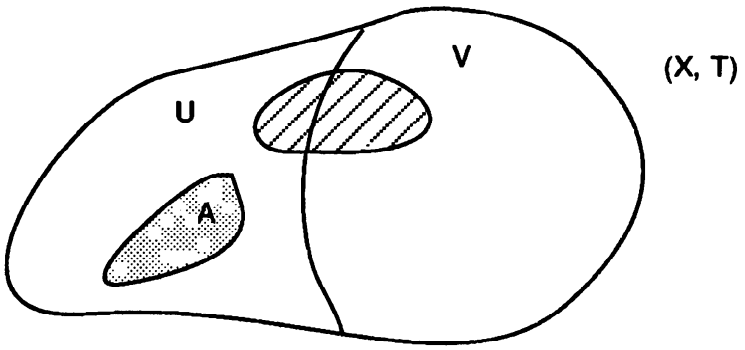


FIGURE 1

Suppose

$$A \cap U \neq \phi \quad \text{and} \quad A \cap V \neq \phi. \tag{3}$$

Then sets $A \cap U$ and $A \cap V$ are nonempty, disjoint subsets of A which are open in A. Then

$$A = (A \cap U) \cup (A \cap V). \tag{4}$$

Hence, A is not connected.

Therefore, either $A \cap U = \phi$ or $A \cap V = \phi$. Hence, we have

$$A \subset V \quad \text{or} \quad A \subset U.$$

Note that a subset A of X is said to be connected if the subspace A is connected.

● PROBLEM 17-6

Let A and B be connected sets which are not mutually separated. Show that

$A \cup B$

is connected.

SOLUTION:

For working convenience write the definition of mutually separated sets, as given in Problem 17–1, in terms of set theory.

DEFINITION OF MUTUALLY SEPARATED SETS

In the topological space (X, T), $A \subset X$ and $B \subset X$ are said to be mutually separated if

$$A \cap B = \phi$$

$$\overline{A} \cap B = \phi$$

$$A \cap \overline{B} = \phi$$

■

Suppose $A \cup B$ is disconnected. Then

$$A \cup B = U \cup V.$$

Since A is a connected subset of $A \cup B$, we have either

$$A \subset U \quad \text{or} \quad A \subset V$$

by Problem 17–5.

Also, either

$$B \subset U \quad \text{or} \quad B \subset V.$$

If $A \subset U$ and $B \subset V$ (or $A \subset V$ and $B \subset U$), then

$$A = (A \cup B) \cap U$$

$$B = (A \cup B) \cap V$$

are separated sets, a contradiction with the hypothesis. Thus, either

$$A \cup B \subset U \quad \text{or} \quad A \cup B \subset V.$$

Therefore, $U \cup V$ is not a disconnection of $A \cup B$ and $A \cup B$ is connected.

● PROBLEM 17-7

Prove the following:

THEOREM

The union of any family of connected subsets of any space (X, T) having at least one point in common is also connected.

■

SOLUTION:

Let $\{A_\alpha\}$ be a family of connected subsets of X such that for each α

$$x_0 \in A_\alpha$$

$$A = \bigcup_\alpha A_\alpha$$

$$x_0 \in \bigcap_\alpha A_\alpha$$

Let (Y, T') be the space $Y = \{0, 1\}$ with discrete topology. Let

$$f : A \to Y$$

be continuous. Since each A_α is connected and each $f \mid A_\alpha$ is continuous, no $f \mid A_\alpha$ is onto. Because for each α

$$x_0 \in A_\alpha$$

we obtain

$$f(x) = f(x_0)$$

for each $x \in A_\alpha$ for all α. Hence, f is not onto and $\bigcup_\alpha A_\alpha$ is connected.

Note that the intersection of two connected sets need not be connected.

● PROBLEM 17-8

Let A be a connected subset of (X, T) and let

$$A \subset B \subset \overline{A} \tag{1}$$

Show that B is connected.

SOLUTION:

Suppose B is disconnected, then

$$B = U \cup V.$$

Since A is a connected subset of B, we have either

$$A \cap U = \phi \quad \text{or} \quad A \cap V = \phi.$$

Suppose $A \cap U = \phi$. Then $X - U$ is closed and

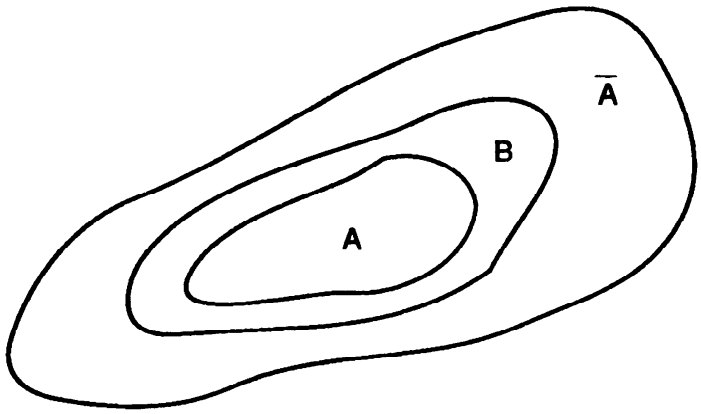


FIGURE 1

$$A \subset B \subset \overline{A} \subset X - U.$$

Thus,

$$B \cap U = \phi.$$

This is in contradiction with $B = U \cup V$. Hence, B is connected.

● PROBLEM 17-9

Show that the only connected subsets of R with the absolute value topology, having more than one point, are R and the intervals (open, closed, half-open).

SOLUTION:

We shall show that

$$(Y \text{ connected subset of } R) \Leftrightarrow (Y \text{ is an interval}).$$

$\Rightarrow$ Suppose Y is connected and is not an interval. Then there are

$$a, b \in Y \quad \text{and} \quad c \notin Y \tag{1}$$

such that $a < c < b$. Then

$$Y \cap \{x : x > c\}$$

$$Y \cap \{x : x < c\} \tag{2}$$

is a decomposition of Y, since both sets (2) are open, disjoint, and nonempty.

$\Leftarrow$ Suppose Y is an interval and is not connected. Then disjoint, nonempty, open sets A, B exist such that

$$Y = A \cup B \tag{3}$$

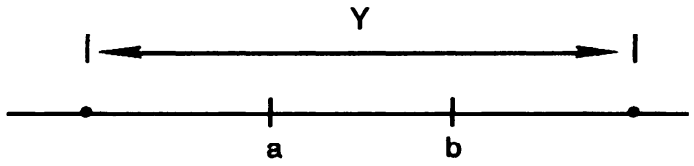


FIGURE 1

we can find $a \in A$, $b \in B$, such that

$$a < b. \tag{4}$$

Let us define

$$\omega = \sup \{x : [a, x) \subset A\} \tag{5}$$

and since Y is an interval,

$$\omega \in Y \quad \omega \leq b. \tag{6}$$

We have

$$\omega \in \overline{A_Y} \tag{7}$$

and since

$$A = Y - B$$

is closed in Y, we have

$$\omega \in A. \tag{8}$$

But, A is also open in Y, hence, $\varepsilon > 0$ exists such that

$$(\omega - \varepsilon, \omega + \varepsilon) \subset A \tag{9}$$

which contradicts the definition of ω.

Prove the following variation of the fixed-point theorem:

THEOREM 1

If $f: [0, 1] \to [0, 1]$ is continuous, then $x_0 \in [0, 1]$ exists such that

$$f(x_0) = x_0.$$

■

SOLUTION:

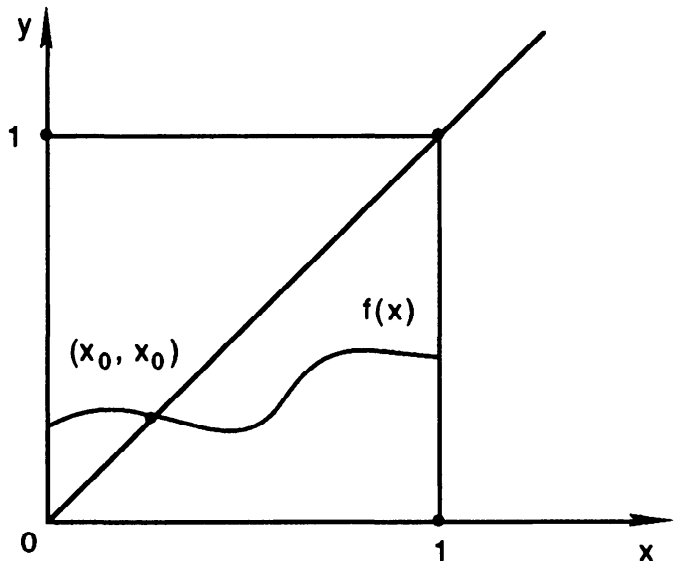


FIGURE 1

In Problem 17–2, we proved that a continuous image of a connected set is connected. The only connected subsets (consisting of more than one point) of R are intervals. Hence, the conclusion in the form of a theorem.

THEOREM 2

If $f: X \to R$ is a continuous function defined on a connected set X, then f assumes all values between any two of its values.

■

Theorem 1 is a conclusion of Theorem 2. Also note that the graph of the continuous function

$$f: [0, 1] \to [0, 1]$$

is located in the square

$$[0, 1] \times [0, 1].$$

Hence, it must intersect the diagonal joining (0, 0) with (1, 1). Therefore, point $x_0 \in [0, 1]$ exists such that

$$f(x_0) = x_0.$$

● PROBLEM 17-11

Consider Theorem 1 of Problem 17–10. Prove this theorem using the sets

$$U = \{(x, y) : x < y\}$$

$$V = \{(x, y) : x > y\}. \tag{1}$$

Observe that the graph of f is connected.

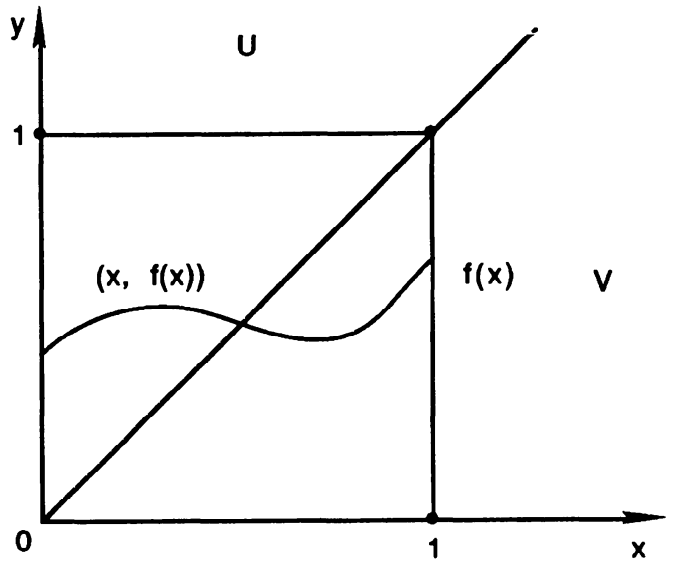


FIGURE 1

SOLUTION:

If $f(0) = 0$ or $f(1) = 1$, the theorem is proved. Otherwise

$$f(0) > 0 \quad \text{and} \quad f(1) < 1 \tag{2}$$

then

$$(0, f(0)) \in U \quad (1, f(1)) \in V. \tag{3}$$

Suppose the graph of f

$$F = \{(x, y) : x \in [0, 1], y = f(x)\} \tag{4}$$

does not contain any points of the diagonal

$$D = \{(x, y) : x = y\}. \tag{5}$$

Then $U \cup V$ is a disconnection of F. But F is connected as a continuous image of a connected set. Contradiction. Thus, F contains a point (x_0, x_0) of the diagonal, hence

$$f(x_0) = x_0. \tag{6}$$

● PROBLEM 17-12

Let (X, T) be a space such that for any two elements

$$x, y \in X \tag{1}$$

a connected set $A \subset X$ exists such that

$$x, y \in A. \tag{2}$$

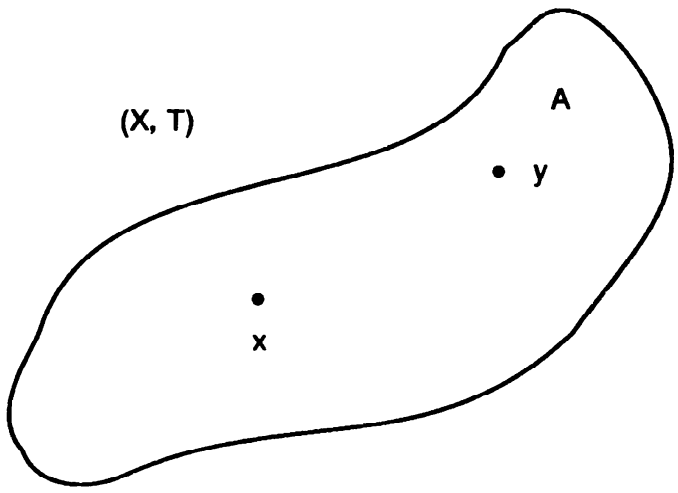


FIGURE 1

Show that (X, T) is connected.

SOLUTION:

Let x be a fixed element of X. Then, for any $y \in X$ there exists a connected subspace $A(x, y)$ of X such that

$$x, y \in A(x, y).$$

Consider the family of sets

$$\{A(x, y) : y \in X\} \tag{3}$$

which consists of connected subspaces of X and

$$X = \bigcup_{y \in X} A(x, y). \tag{4}$$

The intersection of (3) is nonempty

$$X \in \bigcap_{y \in X} A(x, y). \tag{5}$$

By Problem 17–7, we conclude that (X, T) is connected.

● PROBLEM 17–13

Apply Problem 17–12 to show that the Euclidean n-space R^n $(n > 1)$ with one point S removed

$$R^n - \{S\}$$

is connected.

SOLUTION:

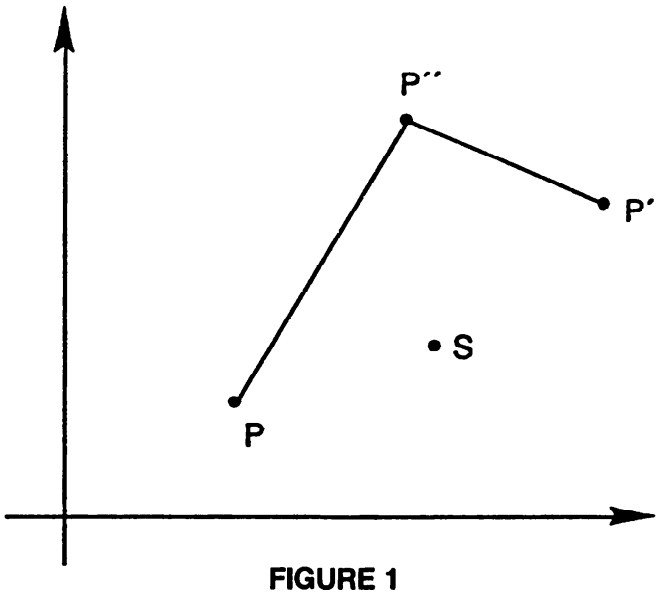


FIGURE 1

Suppose P and P' are any two points of $R^n - \{S\}$. Any closed line segment in the Euclidean space R^n is homeomorphic to the closed interval [0, 1]. Hence, it is a connected subspace of R^n. We choose point $P'' \in R^n - \{S\}$ such that

$$\overline{S \notin PP''}$$

and

$$S \notin \overline{P'' P'}.$$

The sets $\overline{PP''}$ and $\overline{P'' P'}$ are connected and their intersection is nonempty

$$\overline{PP''} \cap \overline{P''P'} = \{P''\} \neq \phi.$$

Hence,

$$\overline{PP''} \cup \overline{P''P'}$$

is connected.

Points P and P' belong to the connected subspace of $R^n - \{S\}$. Hence, $R^n - \{S\}$ is connected. Note that this is not true for $n = 1$.

● PROBLEM 17-14

Which of the subsets of R^2 are polygonally connected?

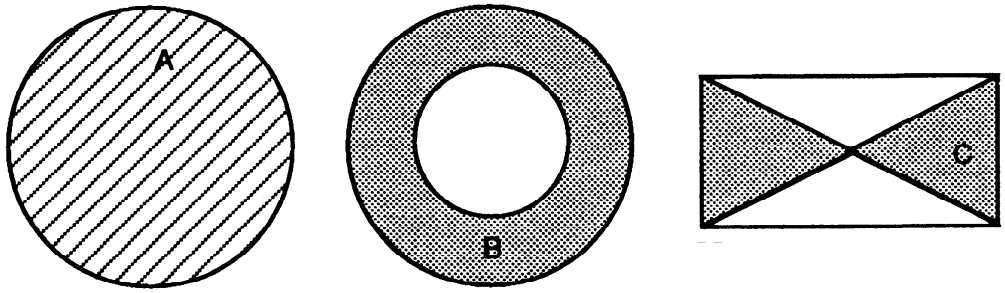


FIGURE 1

1. A
2. B
3. C
4. $A \cup B$

SOLUTION:

DEFINITION OF A PATH AND PATH-CONNECTED SPACES

A subspace Y of any space (X, T) is said to be a path in X if there is a continuous function from $[0, 1]$ (with the absolute value topology)

$$f: [0, 1] \rightarrow Y$$

onto Y.

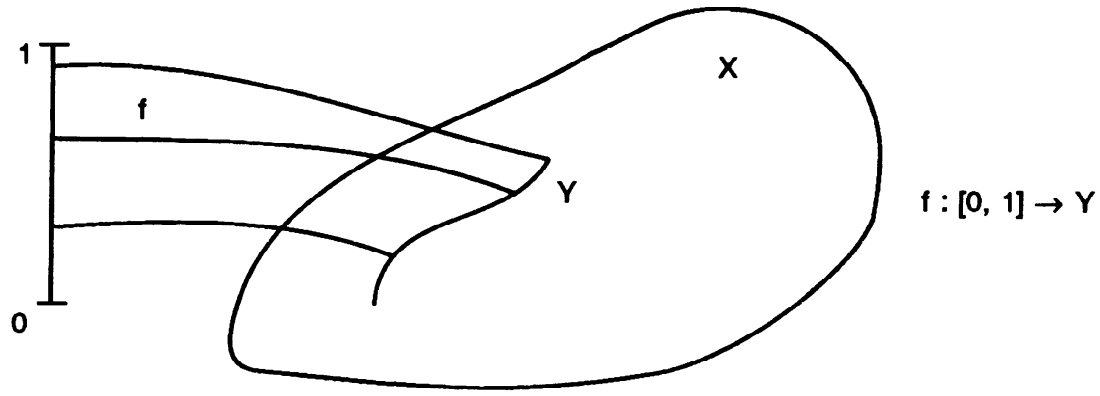


FIGURE 2

X is said to be path-connected if any two points x, y of X belong to some path in X.

■

Consider the Euclidean space R^n. A subset $A \subset R^n$ is said to be polygonally connected if for any $x, y \in A$ there are points

$$x = x_0, x_1, \ldots, x_{k-1}, x_k = y$$

such that

$$\bigcup_{i=0}^{k-1} \overline{x_i x_{i+1}} \subset A$$

where $\overline{x_i x_{i+1}}$ is a closed segment joining x_i and x_{i+1}.

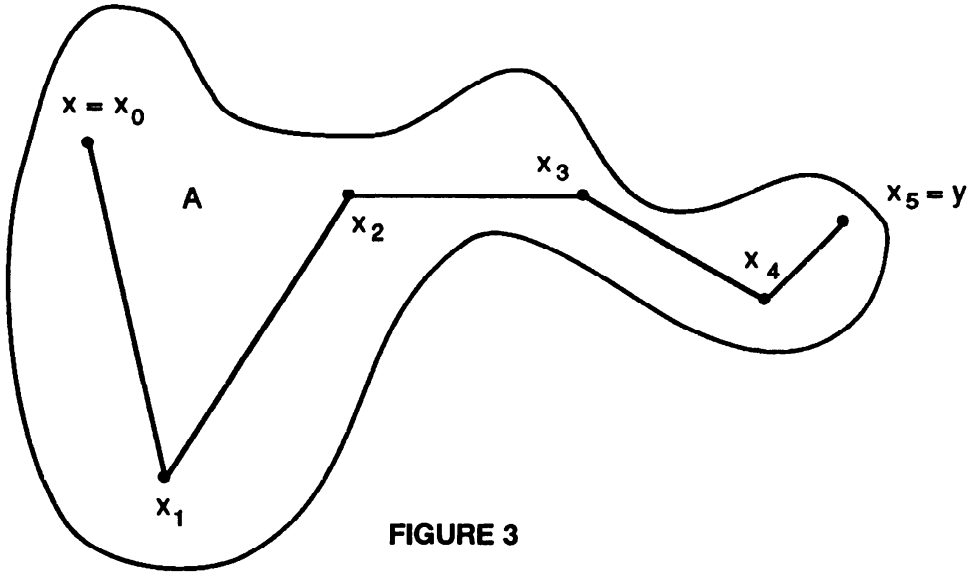


FIGURE 3

Hence, A, B, and C are polygonally connected, while $A \cup B$ is not.

1. Show that a subset of R^n, which is path-connected, does not have to be polygonally connected.

2. Give an example of a connected subspace of R^2 which is not path-connected.

SOLUTION:

1. The circle

$$\{(x, y) : x^2 + y^2 = 1\} \qquad R^2$$

is path-connected, but is not polygonally connected.

2. Consider the set

$$A = \{(x, y) : y = \sin {}^1/_x, x > 0\} \cup \{0, 0\}$$

which is a subset of R^2.

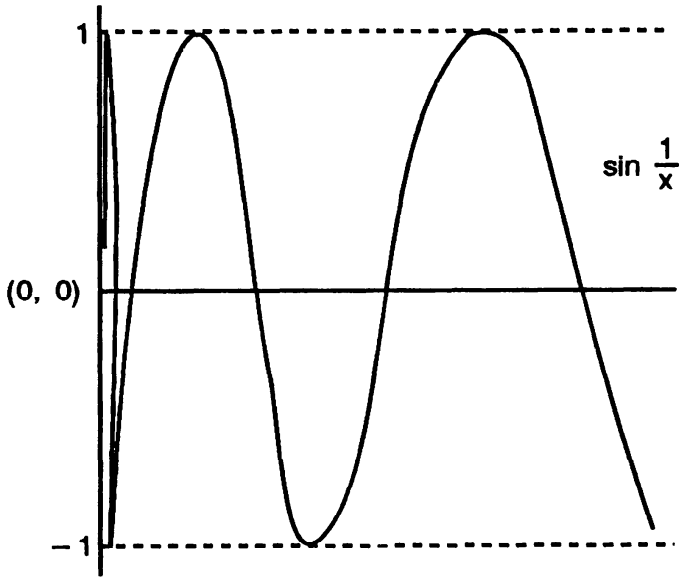


FIGURE 1

From Problem 17–8 we see that A is connected.

But A is not path-connected. There is no path in A which contains (0, 0) and any point $P \neq (0, 0)$ of A. Otherwise the function $f(x) = \sin {}^1/_x$, $x > 0$ would be continuous.

● PROBLEM 17-16

Prove that any polygonally connected subspace of R^n is path-connected.

SOLUTION:

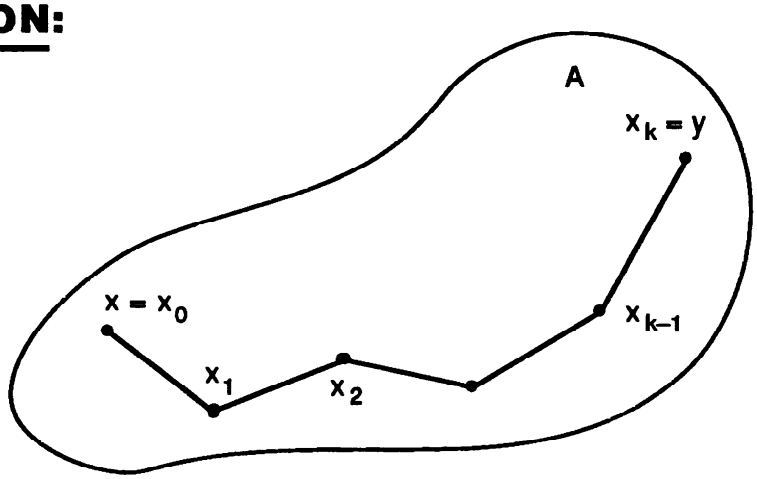


FIGURE 1

Let $A \subset R^n$ be polygonally connected and let x, y be any points in A.

Since A is polygonally connected, there are points $x = x_0, x_1, \ldots, x_k = y$ such that

$$\bigcup_{i=0}^{k-1} \overline{x_i x_{i+1}} \subset A$$

Set $\bigcup_{i=0}^{k-1} \overline{x_i x_{i+1}}$ is a path. A continuous function exists

$$f : [0,1] \to \bigcup_{i=0}^{k-1} \overline{x_i x_{i+1}}.$$

Hence, A is path-connected.

● PROBLEM 17-17

Let

$$f : [0, 1] \to X, f(0) = a, f(1) = b$$

be a path from $a \in X$ to $b \in X$ and

$$g : [0, 1] \to X, g(0) = b, g(1) = c$$

be a path from $b \in X$ to $c \in X$.

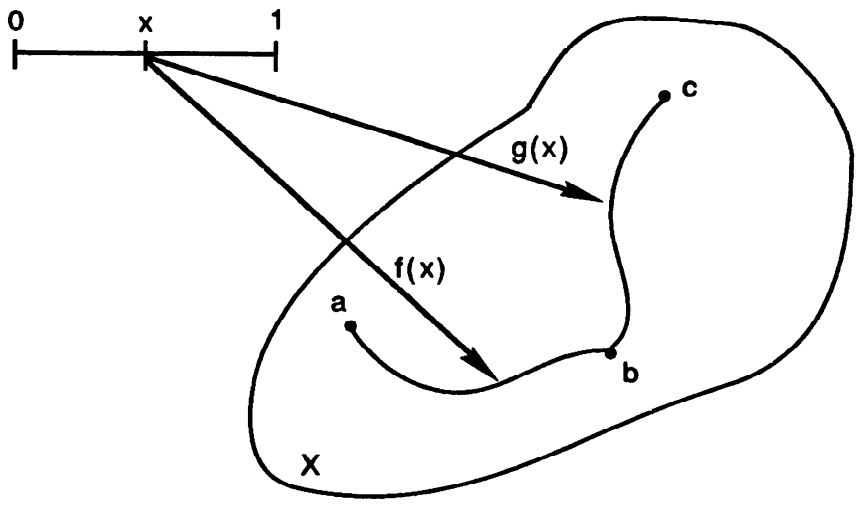


FIGURE 1

Show that the juxtaposition of the two paths f and g is a path from a to c.

SOLUTION:

The juxtaposition of the two paths f and g, denoted by f o g is defined as follows:

$$f \circ g : [0, 1] \to X$$

$$(f \circ g)(t) = \begin{cases} f(2t) & \text{for} \quad 0 \leq t \leq \frac{1}{2} \\ g(2t - 1) & \text{for} \quad \frac{1}{2} \leq t \leq 1 \end{cases}.$$

Note that $$(f \circ g)(0) = f(0) = a$$

$$(f \circ g)(1/2) = f(1) = b = g(0)$$

and $$(f \circ g)(1) = g(1) = c.$$

Thus, f o g is a path from a to c.

● PROBLEM 17-18

1. Show that if (X, T) is a path-connected space, then X is connected.

2. Explain why any convex subset of R^n is connected.

SOLUTION:

1. In Problem 17–7 we proved that if (X, T) is a space and

$$X = \bigcup_{\alpha} A_\alpha$$

where $\{A_\alpha\}$ is a collection of connected subspaces of X and

$$\bigcap_{\alpha} A_\alpha \neq \phi$$

contained in some connected subspace of X, then X is connected.

Suppose X is path-connected and $x \in X$. Then for each $y \in X$, $P(x, y)$ denotes a path from x to y.

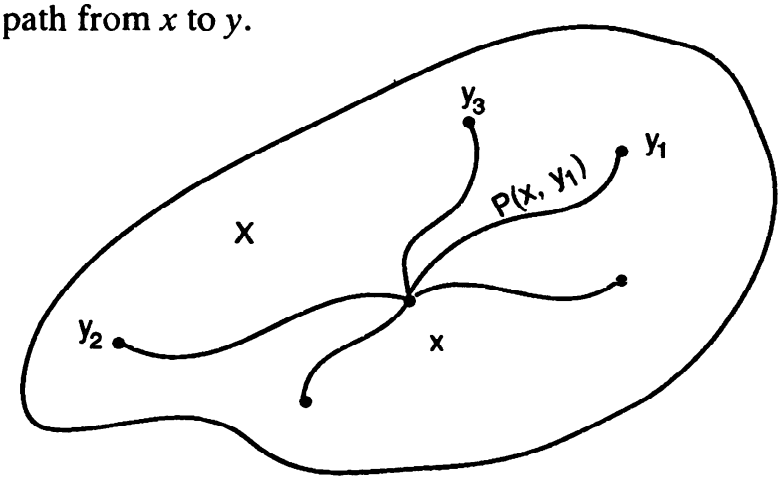


FIGURE 1

Then

$$X = \bigcup_{y \in X} P(x,y)$$

and

$$X \in \bigcap_{y \in X} P(x,y) \neq \phi.$$

Since each $P(x, y)$ is connected (as a continuous image of a connected set $[0, 1]$), X is connected.

2.

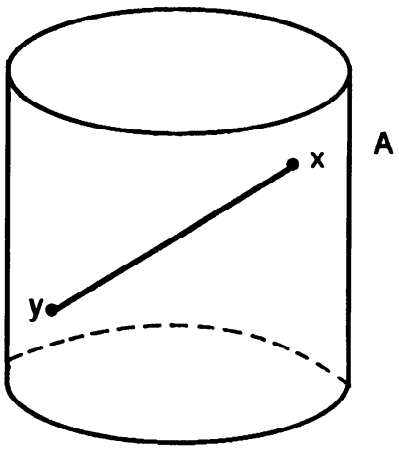


FIGURE 2

A subset A of R^n is convex if, for any points x and y of A, the closed segment $\overline{xy}$ is a subset of A.

Any convex subset of R^n is polygonally connected and, hence, by Problem 17–16, is path-connected and, hence, is connected.

● PROBLEM 17-19

Prove the following theorem concerning the Euclidean R^n spaces.

THEOREM

If U is an open connected subset of R^n, then U is polygonally connected. ■

SOLUTION:

Suppose $U \subset R^n$ is open and connected and $u \in U$.

Let X be all points of U which can be polygonally connected with u

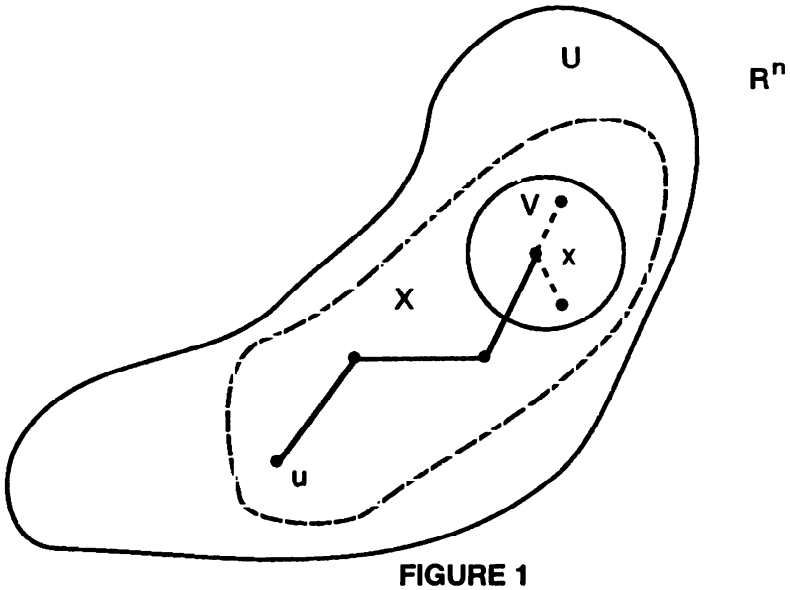


FIGURE 1

$$X = \{x \in U : x \text{ can be polygonally connected to } u \in U\}. \tag{1}$$

Let

$$Y = U - X. \tag{2}$$

Since U is open for any $x \in X$, there is an open neighborhood V of x such that u can be polygonally connected with any point of V; hence,

$$V \subset X. \tag{3}$$

Therefore, X is open.

The set $Y = U - X$ consists of elements which cannot be polygonally connected to u. By the same argument Y is open. Thus,

$$U = X \cup Y, X \cap Y = \phi \tag{4}$$

where X and Y are disjoint open subsets of U. Since U is connected, we have two options:

either $X = \phi, \quad Y = U$

or $X = U, \quad Y = \phi.$ (5)

Since $u \in X, X \neq \phi$

and $X = U, \quad Y = \phi$

and U is polygonally connected.

● PROBLEM 17-20

Show that for any $n > 1$, R^n is not homeomorphic to R.

SOLUTION:

Consider the space R (with the absolute value topology) with one point removed $R - \{a\}$.

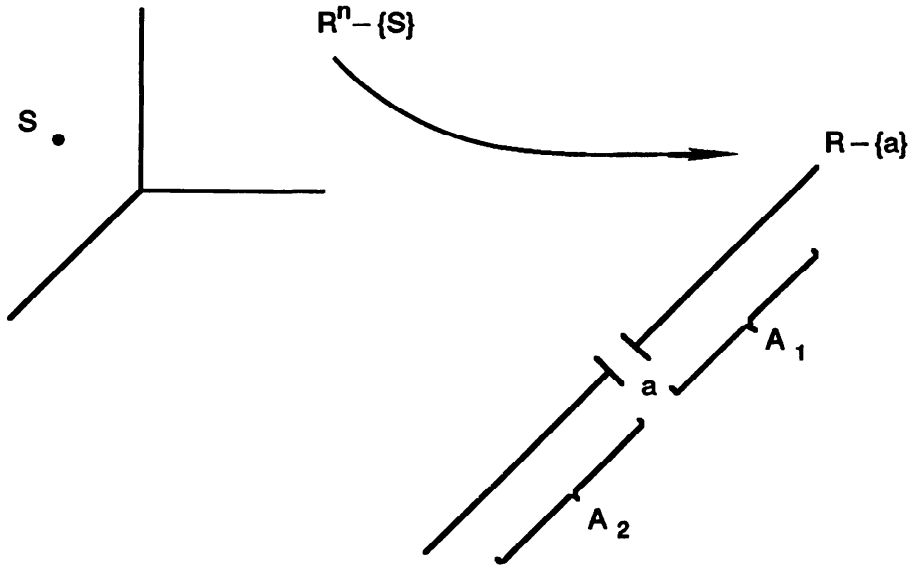


FIGURE 1

Then R is disconnected into two disjoint open sets

$$A_1 = \{x \in R : x > a\}$$

$$A_2 = \{x \in R : x < a\}$$

Hence, $R - \{a\}$ is disconnected. Suppose R^n $(n > 1)$ is homeomorphic to R, then for any point $S \in R^n$, $R^n - \{S\}$ is not connected.

In Problem 17–13, we showed that $R^n - \{S\}$ is connected. Therefore, R^n is not homeomorphic to R.

● PROBLEM 17–21

In Problem 17–18 we proved that

$$((X, T) \text{ is path-connected}) \Rightarrow ((X, T) \text{ is connected}).$$

Show that the converse of this theorem is not true.

SOLUTION:

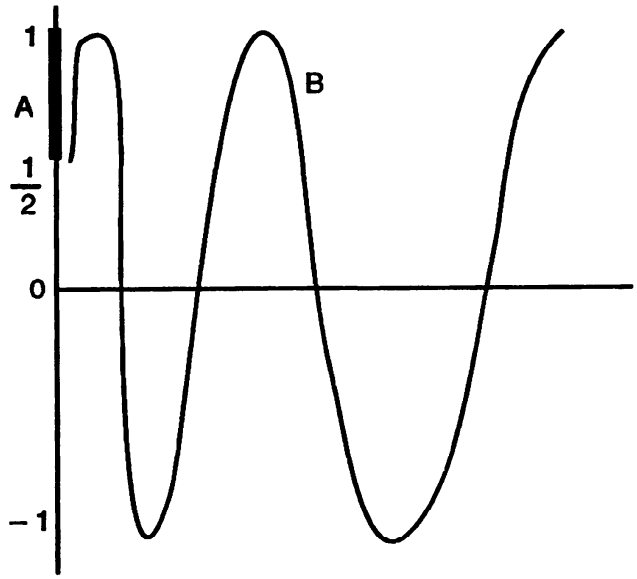


FIGURE 1

Let A and B be the subsets of R^2

$$A = \{(0, y) : {}^1/_2 \leq y \leq 1\}$$

$$B = \{(x, \sin {}^1/_x) : 0 < x \leq 1\}.$$

Set A is a closed interval and B is a continuous image of a closed interval; hence, both are connected. Each point of A is an accumulation point of B, hence A and B are not separated. Thus,

$A \cup B$

is connected.

But $A \cup B$ is not path-connected. There is no path connecting any point of A with any point of B.

The converse implication is not true.

● PROBLEM 17-22

Show that the continuous image of a path-connected set is path-connected.

SOLUTION:

Let $A \subset X$ be path-connected and let $f : X \to Y$ be continuous. We shall show that $f(A)$ is path-connected. Let $a, b \in f(A)$, then $a', b' \in A$ (not necessarily uniquely defined) exist that

$$f(a') = a \quad \text{and} \quad f(b') = b.$$

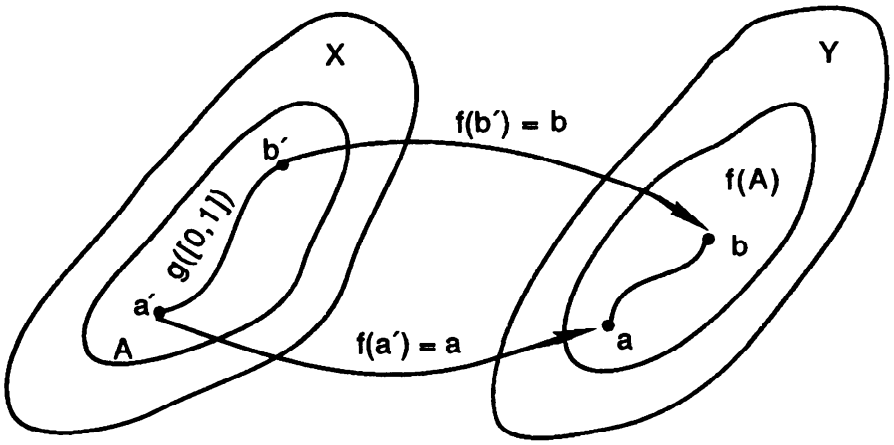


FIGURE 1

Since A is path-connected, there is a continuous function

$$g : [0, 1] \to A$$

such that $g(0) = a'$, $g(1) = b'$.

Since both f and g are continuous,

$$f \circ g : [0, 1] \to f(A)$$

is continuous and

$$f(g(0)) = f(a') = a$$

$$f(g(1)) = f(b') = b.$$

Thus, $f(A)$ is path-connected.

● PROBLEM 17-23

1. Show that an open disc in the Euclidean R^2 plane is path-connected (do not use results of Problem 17–19).

2. Let P be a class of path-connected subsets of X with a nonempty intersection

$$P = \{A_\alpha : \alpha \in A\}.$$

Show that

$$A = \bigcup_\alpha A_\alpha$$

is path-connected.

SOLUTION:

1. Let p and q be any points of an open disc in the R^2 plane.

$$p = (x_1, y_1)$$

$$q = (x_2, y_2)$$

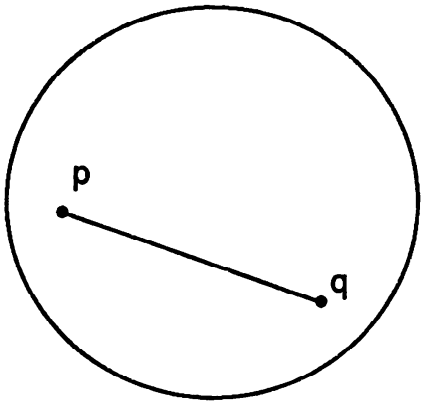


FIGURE 1

A path from p to q may be defined as

$$f : [0, 1] \to R^2$$

$$f(t) = (x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$$

for $t \in [0, 1]$. Then $f(t)$ is the line segment between p and q. For all points p and q in the open disc, $f(t)$ is a path contained completely in the disc. Therefore the open disc in R^2 is path-connected.

2. Let $x, y \in A$. Path-connected sets exist such that

$$x \in A_x, \quad y \in A_y$$

$$A_x, A_y \in P.$$

Nonempty intersection $\bigcap_\alpha A_\alpha$ contains at least one point, say q

$$q \in \bigcap_\alpha A_\alpha.$$

Then $x, q \in A_x$ and there is a path

$$f : [0, 1] \to A_x$$

$$f(0) = x, f(1) = q.$$

Similarly, there is a path

$$g : [0, 1] \to A_y$$

$$g(0) = q, \quad q(1) = y.$$

The juxtaposition of f and g is a path from x to y (see Problem 17–17). Hence, A is path-connected.

● PROBLEM 17–24

Let X be connected and let $P = \{A_\alpha\}$ be an open cover of X. Show that any pair of points of X can be joined by a simple chain consisting of elements of P.

SOLUTION:

DEFINITION OF A SIMPLE CHAIN

Subsets $A_1, A_2, \ldots, A_K$ of X are said to form a simple chain joining points x and y in X if

1. only A_1 contains x

2. only A_K contains y

3. $A_i \cap A_j = \phi$ iff $|i-j| > 1$

■

Let x be any point in X and let B be the set consisting of all points of X which can be joined to x by a simple chain consisting of elements of P.

Since $x \in B$, $B \neq \phi$. We shall prove that B is both open and closed, i.e., $B = X$ because X is connected.

B is open.

Let $b \in B$, then there is a simple chain $A_1, A_2, \ldots, A_l$ from x to b.

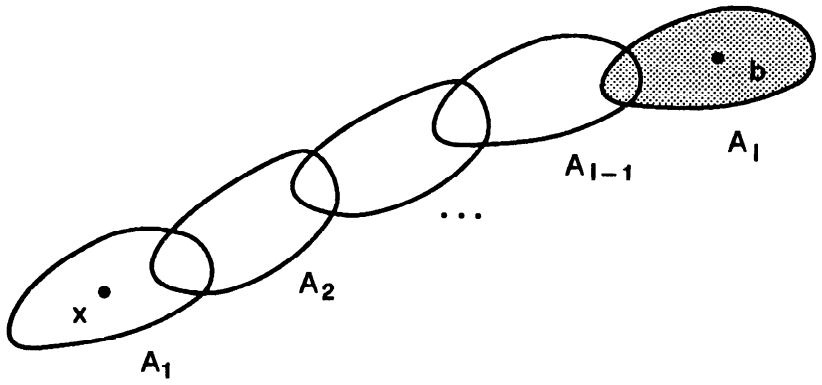


FIGURE 1

Note that all elements of A_l belong to B (chain $A_1, \ldots, A_l$ connects any element of A_l with x). Hence,

$$b \in A_l \subset B.$$

Since A_l is open, B is open.

B is closed.

Suppose

$$y \notin B.$$

Since P is a cover of X, there is an open $A_t \in P$ such that

$$y \in A_t$$

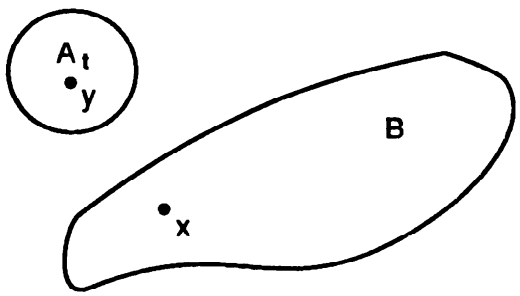


FIGURE 2

Suppose

$$A_t \cap B \neq \phi$$

then there is some chain from y to x and $y \in B$, contradiction.

Thus,

$$A_t \cap B = \phi$$

and

$$y \in A_t \subset X - B.$$

Hence, $X - B$ is open and B is closed. We conclude

$$X = B.$$

● PROBLEM 17-25

Let C_1 and C_2 be subsets of R^2

$$C_1 = \{(x, y) : x^2 + y^2 < 1\}$$

$$C_2 = \{(x, y) : (x-2)^2 + y^2 < 1\}$$

Are C_1 and C_2 mutually separated?

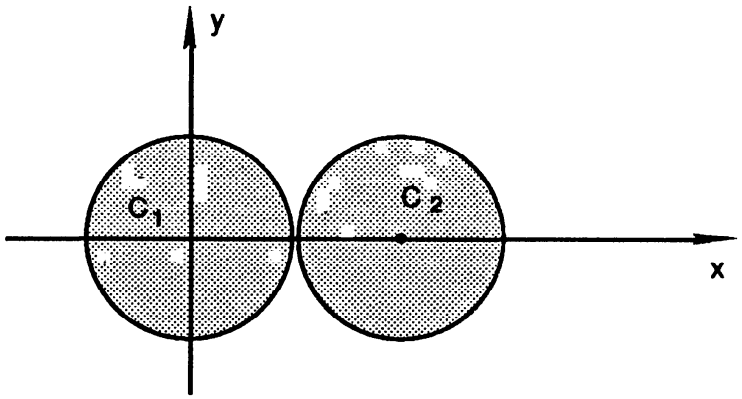


FIGURE 1

SOLUTION:

DEFINITION OF MUTUALLY SEPARATED SETS

Two sets A_1 and A_2 of a space (X, T) are said to be mutually separated if

$$\overline{A_1} \cap A_2 = A_1 \cap \overline{A_2} = \phi.$$

■

Where $\overline{A_1}$ indicates the closure of A_1. Hence,

$$\overline{C_1} = \{(x, y) : x^2 + y^2 \leq 1\}$$

$$\overline{C_2} = \{(x, y) : (x-2)^2 + y^2 \leq 1\}$$

and

$$\overline{C_1} \cap C_2 = C_1 \cap \overline{C_2} = \phi.$$

The sets C_1 and C_2 are mutually separated.

● PROBLEM 17-26

Not every subspace of a connected space is connected. The following theorem enables us to determine whether or not a subspace of a given space is connected.

THEOREM

A subspace A of (X, T) is connected iff A cannot be expressed as the union $C \cup D$, where C and D are nonempty, mutually separated subsets of X.

■

Prove it.

SOLUTION:

If A is not connected, then

$$A = C \cup D \quad \text{and} \quad C \cap D = \phi \tag{1}$$

where C and D are nonempty subsets of A, which are open and closed in A. Suppose

$$x \in C \cap \overline{D} \text{ then } x \in C \text{ and } x \in \overline{D}. \tag{2}$$

Since

$$C \subset A$$

$$x \in A \cap \overline{D} = \overline{D_A} = D. \tag{3}$$

Then

$$x \in C \cap D = \phi \tag{4}$$

a contradiction.

Hence,

$$C \cap \overline{D} = \phi. \tag{5}$$

Similarly, we show that

$$D \cap \overline{C} = \phi. \tag{6}$$

Now, suppose

$$A = C \cup D$$

$$C \cap \overline{D} = D \cap \overline{C} = \phi \quad \text{and} \quad C, D \neq \phi. \tag{7}$$

Then

$$\overline{C_A} = \overline{C} \cap A = (C \cup D) \cap \overline{C} =$$

$$= (C \cap \overline{C}) \cup (D \cap \overline{C}) = C \cup \phi = C. \tag{8}$$

Similarly,

$$\overline{D_A} = D \tag{9}$$

where $\overline{D_A}$ indicates the closure of D in A, i.e., $\overline{D_A} = \overline{D} \cap A$.

Hence, C and D are closed in A, and A is disconnected.

● PROBLEM 17-27

1. Show that if a space (X, T) contains a connected dense subspace, then X is connected.

2. Show that if (X, T) is connected then any compactification Y of X is connected.

SOLUTION:

1. Let $A \subset X$ be a connected dense subspace of X.

$$\overline{A} = X.$$

In Problem 17–8 we proved that if A is a connected subset of (X, T) and

$$A \subset B \subset \overline{A}$$

then B is connected.

Thus,

$$A \subset X \subset \overline{A}$$

and since A is connected, X is also connected.

2. Remember that a compactification of (X, T) is a compact space (Y, T') such that X is homeomorphic to a dense subspace of Y.

In other words, X is a dense connected subspace of Y; hence, Y is connected.

● PROBLEM 17-28

Let

$$X = \{(x, y) : 0 < x, y = \sin {}^1/_x \} \cup \{(0, 0)\} \subset R^2.$$

Apply Problem 17–8 to show that X is connected. (See following figure.)

SOLUTION:

Consider the set

$$A = X - \{(0, 0)\}.$$

We define a function

$$f : R \to R^2$$

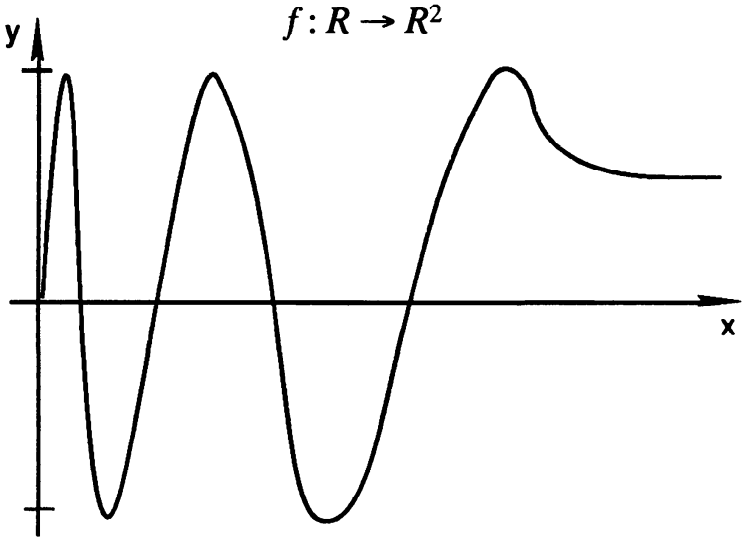


FIGURE 1

by

$$R \supset \{x : x > 0\} \ni x \to f(x) = (x, \sin {}^1/_x) \in R^2.$$

Note that f is continuous. Hence, since $\{x : x > 0\}$ is connected, $A = X - \{(0, 0)\}$ is connected as a continuous image of a connected set.

Since every neighborhood of (0, 0) contains an infinite amount of points of A,

$$(0, 0) \in \overline{A}.$$

We have

$$A \subset X \subset \overline{A}.$$

Hence, since A is connected, by Problem 17–8, we conclude that X is connected.

● PROBLEM 17-29

Show that a component A of a space (X, T) is closed.

SOLUTION:

A disconnected space has connected subspaces.

DEFINITION OF COMPONENT

A component A of (X, T) is a maximal connected subset of X.

■

We see that A is connected and A is not a proper subset of any connected subset of X. By Problem 17–8 since

$$A \subset \overline{A} \subset \overline{A}$$

$\overline{A}$ is connected.

But

$$A \subset \overline{A}$$

and A is a maximal connected subspace of X; hence,

$$A = \overline{A}.$$

Therefore, A is closed.

● PROBLEM 17-30

1. Let R be the space of real numbers with the absolute value topology and let

$$A_+ = \{x \in R : x > a\}$$

$$A_- = \{x \in R : a > x\}$$

where $a \in R$. Find the components of $R - \{a\}$.

2. Let R^2 be the plane

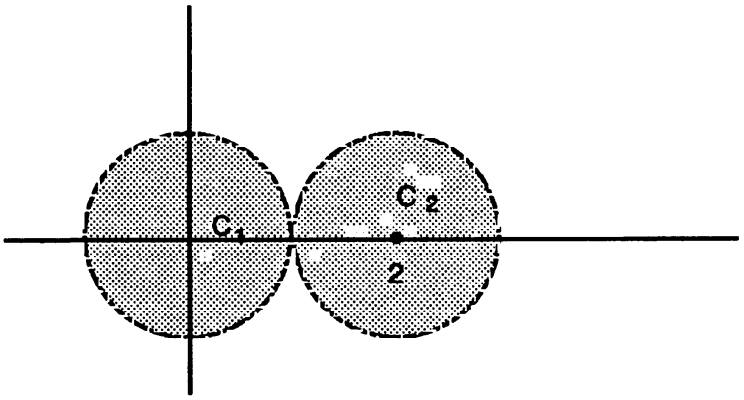


FIGURE 1

with the Pythagorean topology and

$$C_1 = \{(x, y) : x^2 + y^2 < 1\}$$

$$C_2 = \{(x, y) : (x-2)^2 + y^2 < 1\}$$

Find the components of the space $C_1 \cup C_2 \subset R^2$.

SOLUTION:

1. The components of

$$R - \{a\} = A_+ \cup A_-$$

are A_+ and A_-. Each is the maximal connected subspace of $R - \{a\}$.

2. Here the components of $C_1 \cup C_2$ are C_1 and C_2.

● PROBLEM 17-31

Show that each element of (X, T) is contained in a component of X.

SOLUTION:

Obviously each element x of X is contained in at least one connected subspace of X, namely $\{x\} \subset X$. We will show that there is a maximal connected subspace of X which contains x.

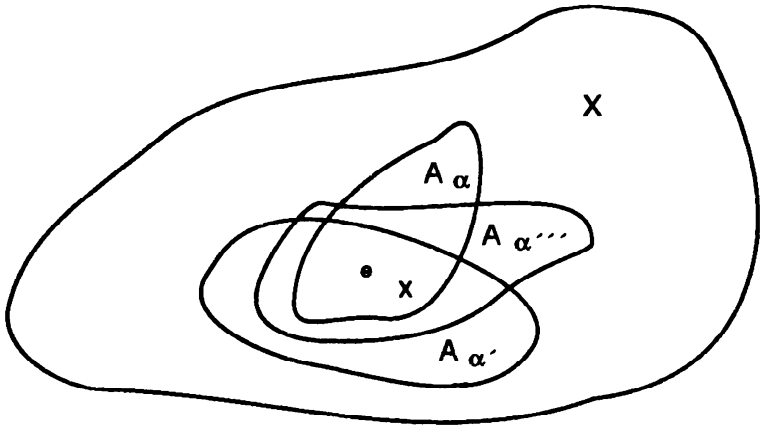


FIGURE 1

Let

$$\{A_\alpha : \alpha \in A\}$$

be the family of all connected subspaces of X which contains x,

$$x \in A_\alpha \quad X$$

and A_α is connected for each $\alpha \in A$. Then

$$\bigcup_\alpha A_\alpha$$

is a connected subspace of X which contains x. Indeed, since all A_α are connected and

$$x \in \bigcap_\alpha A_\alpha \neq \phi$$

then by Problem 17–7, $\bigcup_\alpha A_\alpha$ is connected.

$\bigcup_\alpha A_\alpha$ is a maximal connected subspace of X which contains x and is unique.

● PROBLEM 17–32

Let $\{A_\alpha\}$, $\alpha \in A$ be the set of components of a space (X, T). Show that the components of X form a partition of X, i.e., (see Figure 1 on following page)

$$X = \bigcup_\alpha A_\alpha$$

if $\alpha \neq \beta$, then $A_\alpha \cap A_\beta = \phi$.

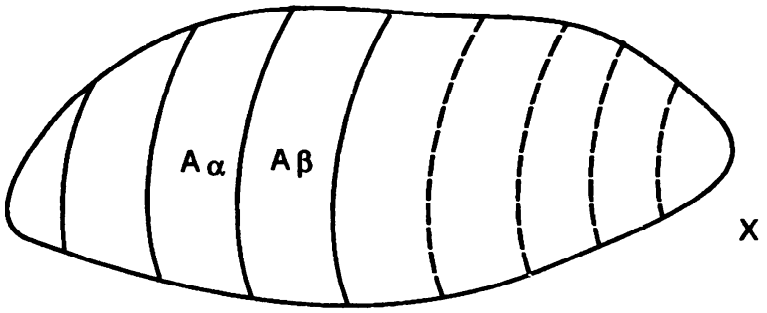


FIGURE 1

SOLUTION:

Any element x of X belongs to at least one connected subspace of X (for example, $x \in \{x\} \subset X$). Thus, a maximal connected subspace of X contains x. We have

$$X = \bigcup_{\alpha} A_{\alpha}.$$

We will show that if $\alpha \neq \beta$, then $A_\alpha \cap A_\beta = 0$. Suppose $\alpha \neq \beta$ and $A_\alpha \cap A_\beta \neq 0$, then $A_\alpha \cup A_\beta$ is a connected subspace of X which contains both A_α and A_β. A contradiction arises because A_α and A_β are maximal.

● PROBLEM 17-33

Prove that if a space has only a finite number of components, then each component is open.

SOLUTION:

Let

$$A_1, A_2, \ldots, A_K$$

be components of X.

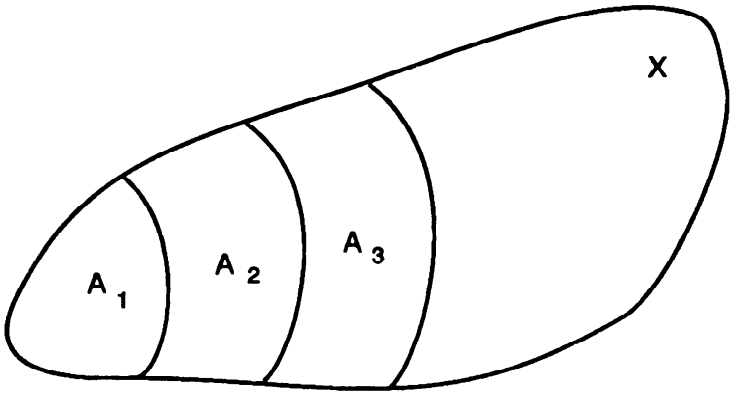


FIGURE 1

Then

$$X = A_1 \cup \ldots \cup A_K.$$

Since each pair of components are disjoint sets

$$A_i \cap A_j = \phi \quad \text{for } i \neq j$$

we have

$$A_1 = X - (A_2 \cup \ldots \cup A_K).$$

In Problem 17–29 we proved that a component of a space is closed. Therefore, since there are a finite number of components,

$$A_2 \cup \ldots \cup A_K$$

is a closed set. Hence, A_1 is an open set. Similarly, we show that all sets $A_1, A_2, \ldots, A_K$ are open.

● PROBLEM 17–34

Show that a finite product of connected spaces is connected.

SOLUTION:

Let X and Y be connected spaces and let

$$p = (x_1, y_1) \in X \times Y$$

$$q = (x_2, y_2) \in X \times Y. \tag{1}$$

The set

$$\{x_1\} \times Y$$

is homeomorphic to Y and since Y is connected, so is $\{x_1\} \times Y$. Similarly, we show that

$$X \times \{y_2\}$$

is connected since it is homeomorphic to X.

Therefore,

$$\{x_1\} \times Y \cup X \times \{y_2\}$$

is connected because

$$\{(x_1, y_2)\} = \{x_1\} \times Y \cap X \times \{y_2\} \neq \phi.$$

Also,

$$p, q \in \{x_1\} \times Y \cup X \times \{y_2\}.$$

Hence, p and q belong to the same component. But p and q were chosen arbitrarily; hence, $X \times Y$ has only one component — itself.

We conclude that a finite product of connected spaces is connected.

● PROBLEM 17-35

Show that a product of connected spaces is connected.

SOLUTION:

Let

$$\{X_\alpha : \alpha \in A\}$$

be a collection of connected spaces and let

$$X = \underset{\alpha \in A}{\times} X_\alpha$$

be the product space.

Let

$$x \in X$$

$$x = (X_\alpha : \alpha \in A)$$

and let $P \quad X$ be the component to which x belongs, $x \in P$.

We will show that an arbitrary point $y \in X$ belongs to P.

Let

$$F = X\{X_\alpha : \alpha \neq \alpha_1, \ldots, \alpha_m\} \times \bigcup_{\alpha_1} \times \ldots \times \bigcup_{\alpha_m}$$

be any basic open set containing $y \in X$. The set

$$G = X\{\{X_\alpha\} : \alpha \neq \alpha_1, \ldots, \alpha_m\} \times X_{\alpha_1} \times \ldots \times X_{\alpha_m}$$

is homeomorphic to

$$X_{\alpha_1} \times \ldots \times X_{\alpha_m}$$

hence, it is connected.

Since $x \in G$

$$G \subset P$$

where P is the component of x.

The set

$$F \cap G \neq \phi$$

and $$P \cap P \neq \phi.$$

Thus,

$$y \in P = P.$$

X has one component; hence, it is connected.

● PROBLEM 17-36

Show that the space of real numbers R with the topology T generated by the closed-open intervals $[a, b) \subset R$ is totally disconnected.

SOLUTION:

DEFINITION OF TOTALLY DISCONNECTED SPACES

A topological space (X, T) is said to be totally disconnected if, for each

$$x, y \in X$$

open nonempty sets exist, U and V such that

$$X = U \cup V$$

$$U \cap V = \phi$$

and $$x \in U, \quad y \in V.$$

■

Let $x, y \in R$ and $x < y$. We define

$$U = (-\infty, y)$$

$$V = [y, +\infty).$$

Sets U and V are disjoint, open, nonempty sets and $R = U \cup V$. Also,

$$x \in U \quad \text{and} \quad y \in V.$$

Thus, R with the topology generated by the closed-open intervals is totally disconnected.

● PROBLEM 17-37

1. Show that the set Q of rational numbers with the absolute value topology is totally disconnected.

2. Show that a totally disconnected space is T_2 (i.e., Hausdorff).

SOLUTION:

1. Let $x, y \in Q$ and

$$x < y.$$

An irrational number a exists such that

$$x < a < y.$$

Define sets U and V

$$U = \{z \in Q : z < a\}$$

$$V = \{z \in Q : z > a\}.$$

Thus, $U \cup V$ is a disconnection of Q and

$$x \in U, y \in V.$$

Q is totally disconnected.

2. Let (X, T) be totally disconnected and let

$$x, y \in X.$$

Open, nonempty, disjoint sets U, V exist such that

$$x \in U \quad \text{and} \quad y \in V$$

$$U \cap V = \phi.$$

Hence, X is a Hausdorff space.

● PROBLEM 17-38

Prove that the components of a totally disconnected space are the singleton sets.

That statement leads to an equivalent definition of totally disconnected spaces. A space (X, T) is said to be totally disconnected if X is not connected and the only connected subspaces of X are the singleton subsets of X and ϕ.

SOLUTION:

Suppose $A \subset X$ is a component of X and

$$x, y \in A, \quad x \neq y.$$

Since X is totally disconnected, there are sets U and V such that

$$X = U \cup V, \quad U \cap V = \phi$$

$$x \in U, \quad y \in V.$$

U and V are open.

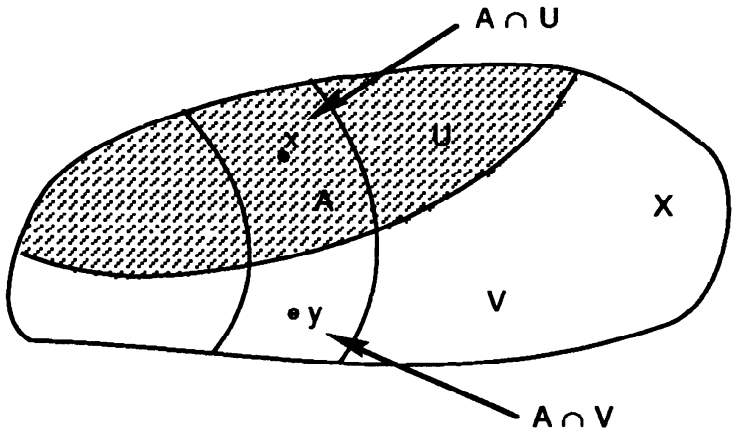


FIGURE 1

Hence, the sets

$$A \cap U \quad \text{and} \quad A \cap V$$

are nonempty, and $U \cup V$ is a disconnection of A. But as a component, A has to be connected.

Contradiction. Thus, A consists of one point.

Show that a space X with the discrete topology is totally disconnected and locally connected.

SOLUTION:

The only connected subsets of a discrete space are singleton sets and ϕ. Hence, X is totally disconnected.

DEFINITION OF LOCALLY CONNECTED SPACE

A space (X, T) is locally connected, if for any $x \in X$ and any neighborhood U of x, there is a connected neighborhood V of x such that

$$V \subset U.$$

■

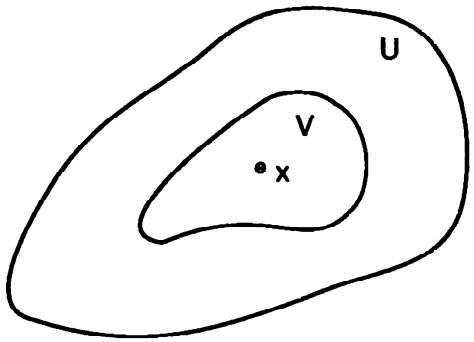


FIGURE 1

We can use this equivalent definition:

DEFINITION

Space (X, T) is said to be locally connected if there is an open neighborhood system for T such that for each $x \in X$, N_x consists of connected subspaces.

■

Let (X, T) be the space with discrete topology. For each

$$x \in X$$

we define

$$N_x = \{\{x\}\}$$

which establishes an open neighborhood system. Each subspace $\{x\}$ is connected.

● PROBLEM 17-40

Consider the space

$$X = \{(x, y) : 0 < x, y = \sin {}^1/_x\} \cup \{(0, 0)\}.$$

Show that X is not locally connected. Use this space to show that local connectedness is not preserved by a continuous function.

SOLUTION:

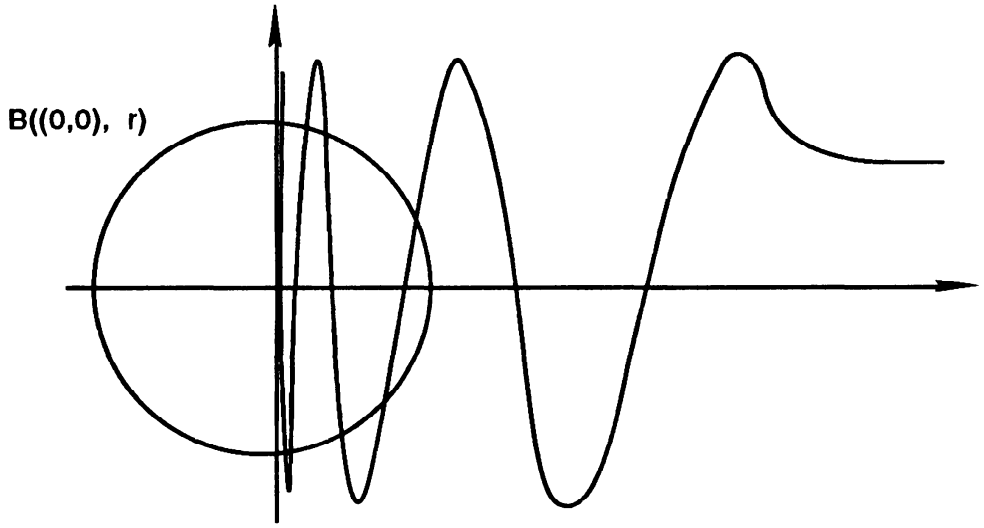


FIGURE 1

For any $0 < r < 1$, $B((0, 0), r)$ has an infinite number of components. X is connected but not locally connected.

On the other hand, X is a continuous image of a locally connected space.

Hence, the local connectedness is not generally perserved by continuous functions.

● PROBLEM 17-41

1. Let X and Y be locally connected. Show that $X \times Y$ is locally connected.

2. Show that a component of a locally connected space X is open.

SOLUTION:

1. Space X is locally connected if and only if X has a base B consisting of connected sets. Also, Y has a base B' consisting of connected sets. Since $X \times Y$ is a finite product, a base for $X \times Y$ is defined by

$$\{U \times V : U \in B, V \in B'\}.$$

Each of the sets $U \times V$ is connected, since both U and V are connected. The product $X \times Y$ has a base consisting of connected sets; hence, $X \times Y$ is locally connected.

2. Let A be a component of a locally connected space X and

$$a \in A.$$

Since X is locally connected, a belongs to at least one open connected set $U_a \subset A$.

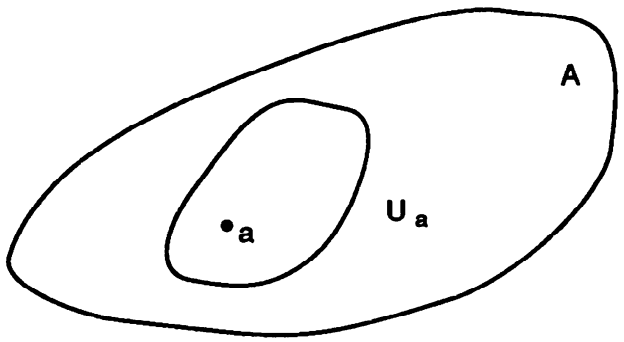


FIGURE 1

But A is a component of a; hence,

$$a \in U_a \subset A$$

and

$$A = \bigcup_{a \in A} U_a.$$

Thus A is open, as the union of open sets is.

● PROBLEM 17-42

Show that a compact, locally connected space (X, T) has a finite number of components.

SOLUTION:

Let

$$(A_\alpha : \alpha \in A\}$$

be the family of components of X. Hence, $\{A_\alpha\}$ forms an open cover of X.

Since X is compact, a finite subcover exists, say

$$A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_k}.$$

On the other hand, since $\{A_\alpha\}$ is the family of components, for $\alpha \neq \beta$ and α and $\beta \in A$, we have

$$A_\alpha \cap A_\beta = \phi.$$

Therefore, no A_α can be omitted from $\{A_\alpha\}$ and the remaining components still form a cover of X.

Hence,

$$A_{\alpha_1}, A_{\alpha_2}, \ldots, A_{\alpha_k}$$

are all of the components of X.

● PROBLEM 17-43

Continuous functions do not preserve local connectedness. But continuous open functions do. Prove:

THEOREM

If f is a continuous open function from a locally connected space (X, T) onto (Y, T').

$$f : X \to Y$$

then Y is locally connected. ■

SOLUTION:

f is a continuous open and onto function

$$f : X \to Y.$$

Let U be any neighborhood of $y \in Y$.

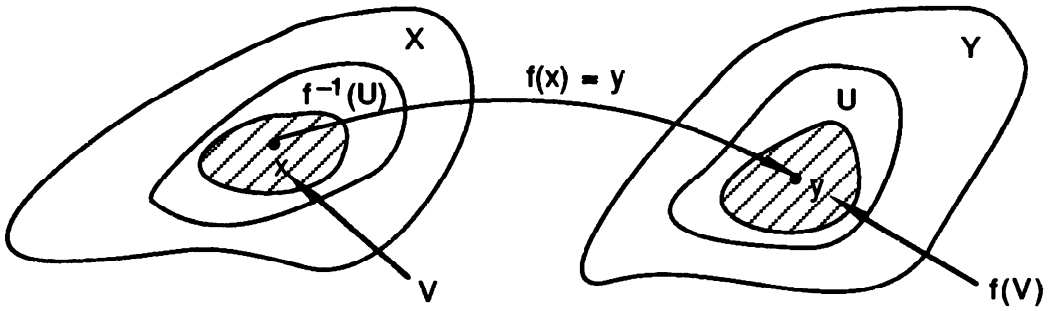


FIGURE 1

Since f is onto, $x \in X$ exists such that

$$f(x) = y.$$

f is continuous; hence, $f^{-1}(U)$ is an open set $x \in f^{-1}(U)$. But X is locally connected; thus, there is a connected neighborhood V of x such that

$$x \in V \subset f^{-1}(U).$$

Since f is continuous and open, $f(V)$ is a connected open set and

$$y \in f(V) \subset U.$$

Hence, Y is locally connected.

● PROBLEM 17-44

Prove the following:

THEOREM

If $\{X_\alpha : \alpha \in A\}$ is a family of connected locally connected spaces, then $\underset{\alpha}{\times} X_\alpha$ is locally connected. ■

SOLUTION:

Let

$$x = (x_\alpha) \in \underset{\alpha}{\times} X_\alpha$$

and let $U \subset \underset{\alpha}{\times} X_\alpha$ be the neighborhood of x. A member of the defining base exists

$$B = U_{\alpha_1} \times \ldots \times U_{\alpha_k} \times \left(\underset{\alpha \neq \alpha_1, \ldots, \alpha_k}{x} X_\alpha\right)$$

such that

$$x \in B \subset U$$

and

$$x_{\alpha_l} \in U_{\alpha_l}.$$

Each coordinate space is locally connected; hence. connected open subsets V_{α_l} exist such that

$$V_{\alpha_l} \subset X_{\alpha_l}$$

and

$$x_{\alpha_l} \in V_{\alpha_l} \subset U_{\alpha_l}$$

for $l = 1, 2, \ldots, k$.

Let

$$V = V_{\alpha_1} \times \ldots \times V_{\alpha_k} \times \left(\underset{\alpha \neq \alpha_1, \ldots, \alpha_k}{x} X_\alpha\right).$$

V is connected because each X_α is connected and each V_{α_l} is connected. Since V is open and

$$x \in V \subset B \subset U,$$

X is locally connected.

● PROBLEM 17-45

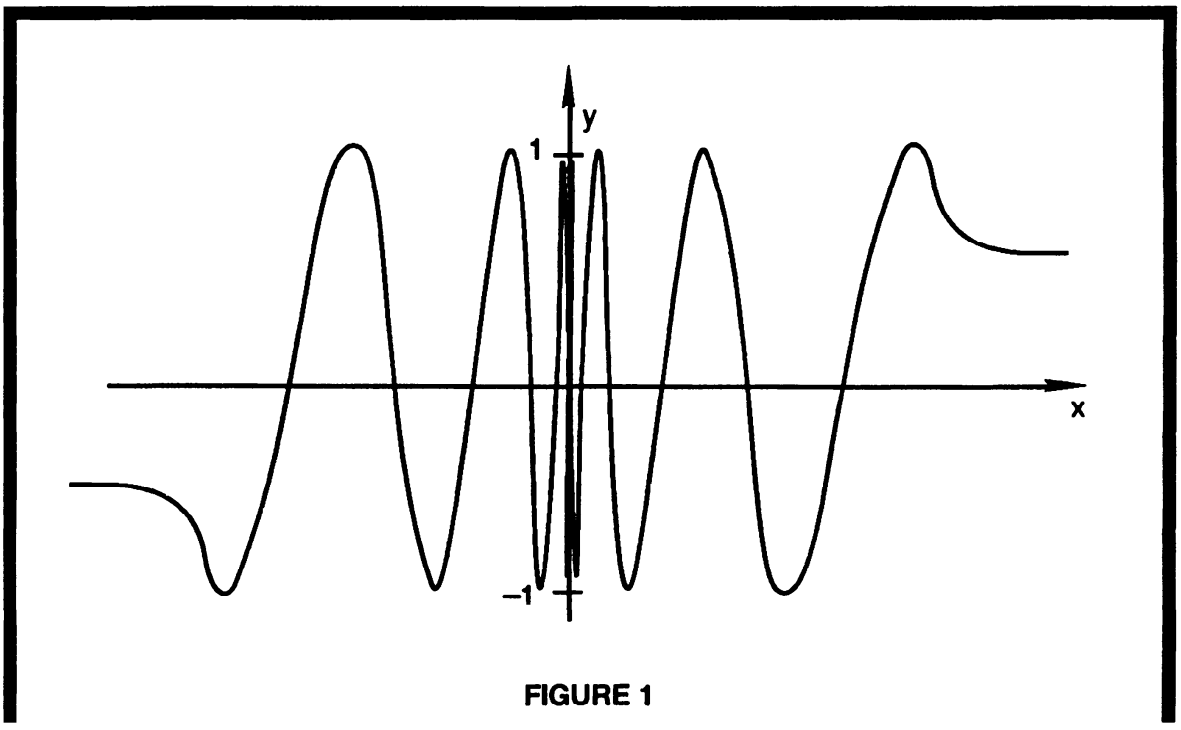


FIGURE 1

Which of the following sets is a continuum:

1. closed interval

2. closed n-dimensional cube

3. the set P of points defined by

$$y = \sin 1/x \text{ for } 0 < |x| \leq 1$$

$$|y| \leq 1 \quad \text{for} \quad x = 0$$

SOLUTION:

A compact connected T_2–space (X, T) is called a continuum.

■

Note that any closed, bounded, and connected subset of R^n for any n is a continuum. This is due to the fact that the closure of any bounded subset of R^n is compact.

Also, any path in a T_2–space is a continuum.

We see that each of the spaces described in 1, 2, and 3 is a continuum.

● PROBLEM 17-46

Prove this:

THEOREM

The union of two continuua which have a common point is a continuum.

■

SOLUTION:

Let (X, T) and (Y, T') be compact connected spaces. We proved that a topological space which is the union of two compact sets is compact.

Hence, $X \cup Y$ is compact. We also proved the following:

If the sets A and B are connected and are not separated, then their union is connected.

Hence, $X \cup Y$ is connected.

Therefore, $X \cup Y$ is a continuum.

Let X and Y be continua such that

$$Y \text{ A } X$$

and let $X - Y$ be the union of two disjoint open sets A and B. Show that

$$Y \cup A \quad \text{and} \quad Y \cup B$$

are continua.

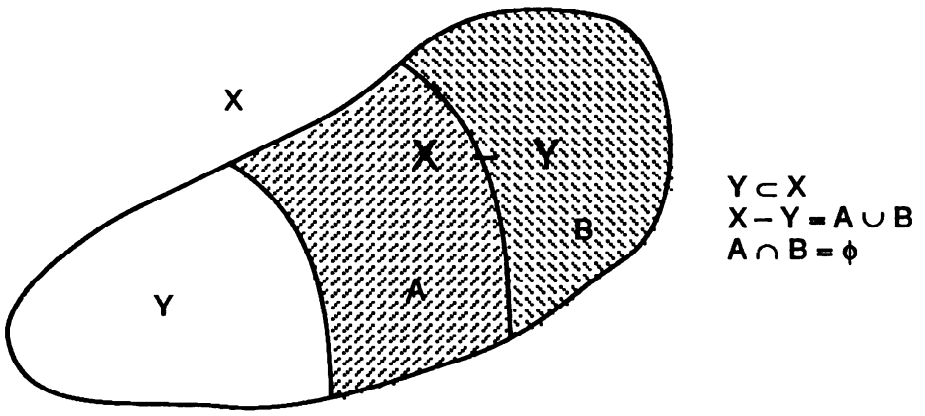


FIGURE 1

SOLUTION:

We shall apply the following:

THEOREM

If Y is a connected subset of the connected space X and

$$X - Y = A \cup B$$

where A and B are separated, then the sets

$$Y \cup A \quad \text{and} \quad Y \cup B$$

are connected.

Note that sets A and B are separated, if

$$\overline{A} \cap B = \phi \quad \text{and} \quad A \cap \overline{B} = \phi.$$

We conclude that $Y \cup A$ and $Y \cup B$ are continua.

Prove the following:

THEOREM

If $f: X \to Y$ is a homeomorphism and X is a continuum, then Y is a continuum.

■

SOLUTION:

We shall apply two theorems.

THEOREM

The continuous image of a compact set is compact.

■

THEOREM

The image under a continuous mapping of a connected space is a connected space.

■

Also, if $f: X \to Y$ is a homeomorphism and X is T_2, then Y is T_2.

Hence, if the spaces are homeomorphic and one is a continuum, then the other one is also a continuum.

The above proved theorem is a generalization of the theorem from analysis. Suppose X is a continuum and f is a homeomorphic real-valued function, then $f(X)$ is either a single point or a closed interval.

CHAPTER 18

METRIZABLE SPACES

Metrizable Space	18–1, 18–5, 18–31
Induced Metric	18–2, 18–3
Countable Metric Space	18–4, 18–19
Cauchy Space, Cauchy Sequence	18–6, 18–7, 18–8, 18–9, 18–10 18–11, 18–12, 18–13, 18–14 18–27, 18–28
Complete Metric Space	18–15, 18–16, 18–17, 18–20, 18–21, 18–22, 18–23, 18–26, 18–29, 18–30, 18–35
Cantor Theorem	18–18
Contract Mapping	18–24
Fixed Point Theorem	18–25
Limit Point	18–32
Dense and isometric	18–33, 18–34, 18–36, 18–37, 18–38
Baire's Category Theorem	18–39

● PROBLEM 18-1

Show that if X is metrizable, then $X \times X$ is also metrizable.

SOLUTION:

Suppose (X, T) is metrizable.

DEFINITION OF METRIZABLE SPACE

A space (X, T) is said to be metrizable if a metric d can be defined on X such that the topology induced by d is T. Otherwise, the space X is called non-metrizable.

■

Metric d exists such that d induces T. We define topology on $X \times X$ in the usual way. It can be proved that this topology is induced by the metric

$$d'((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + d(x_2, y_2).$$

Hence, if X is metrizable, so is $X \times X$.

● PROBLEM 18-2

R is the space of real numbers with the topology T generated by the basis consisting of open intervals. Show that (R, T) is metrizable.

SOLUTION:

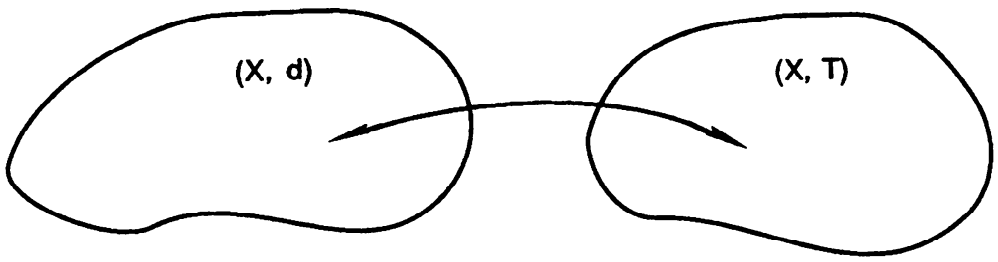


FIGURE 1

Suppose a metric space (X, d) is given. The metric d induces a certain topology T_d on X.

$$(X, d) \rightarrow (X, T_d)$$

In the metric space (X, d) a d-ball of radius r and center a is defined by

$$B(a, r) = \{x \in X : d(x, a) < r\}.$$

The family

$$\{B(a, r) : a \in X, r > 0\}$$

of all d-balls in X can serve as the basis for topology T.

Hence, we obtain a topology T on X determined (or induced) by the metric d.

Let R be the space of real numbers with the absolute value metric. Then this metric induces a topology with the open intervals as the basis. Hence, R with the topology generated by the basis consisting of open intervals is metrizable. The metric we are looking for is the absolute value metric.

● PROBLEM 18-3

Let $\{(X_n, d_n) : n \in N\}$ be a countable family of second countable metric spaces.

1. Show that the product space

$$\underset{n \in N}{\times} X_n$$

is second countable.

2. Show that

$$d(x, y) = \sum_{n \in N} \frac{\min(d_n(x_n, y_n), 1)}{2^n}.$$

is a metric on xX_n, where

$$x = (x_1, x_2, \ldots, x_n, \ldots); x_n, y_n \in X_n$$

$$y = (y_1, y_2, \ldots, y_n, \ldots)$$

SOLUTION:

1. Since each X_n is second countable, $\times X_n$ is second countable. We proved earlier that the product space of a countable family of nonempty spaces is second countable, if and only if each component space is second countable.

2. We have

$$d(x, y) = \sum_n \frac{\min(d_n(x_n, y_n), 1)}{2^n} \leq \sum_n \frac{1}{2^n}.$$

Hence, $d(x, y)$ is defined for all

$$x, y \in \underset{n}{\times} X_n.$$

Also,

$$d(x, y) \geq 0$$

$d(x, y) = 0$ iff $x_n = y_n$ for all $n \in N$, hence, $x = y$.

$$d(x,y) = d(y,x)$$

$$d(x,y) = \sum_n \frac{\min(d_n(x_n, y_n), 1)}{2^n} \leq 1$$

$$\leq \sum_n \left[\frac{\min(d_n(x_n, z_n), 1)}{2^n} + \frac{\min(d_n(z_n, y_n), 1)}{2^n} \right] =$$

$$= d(x,z) + d(z,y).$$

Hence, d is a metric on $\underset{n}{\times} X_n$.

● PROBLEM 18–4

Using the metric defined in Problem 18–3, show that the space $\underset{n \in N}{\times} X_n$ is metrizable, where $\{(X_n, d_n) : n \in N\}$ is a countable family of second countable metric spaces.

SOLUTION:

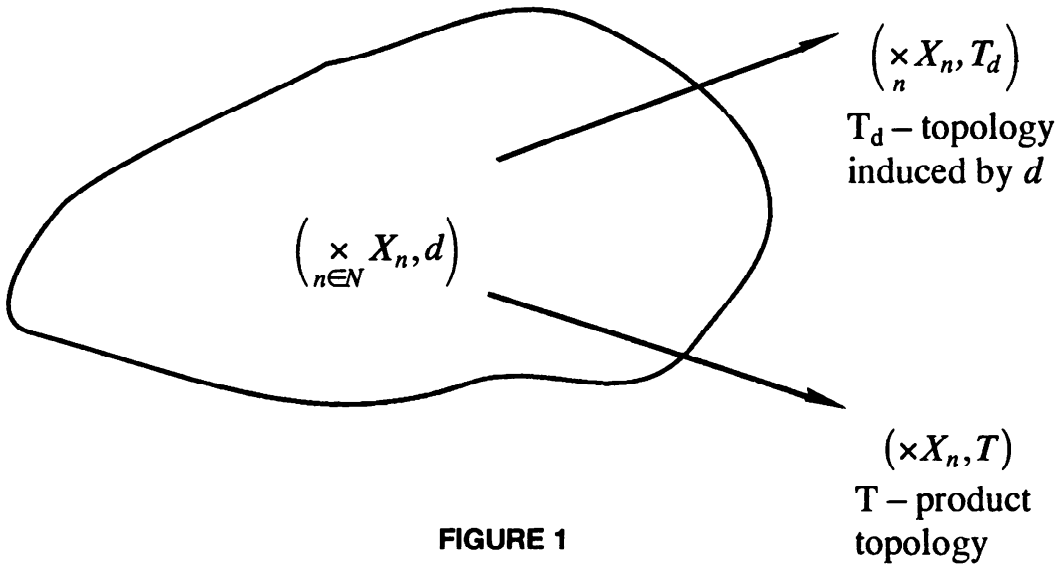


FIGURE 1

We have to show that

$$T_d = T.$$

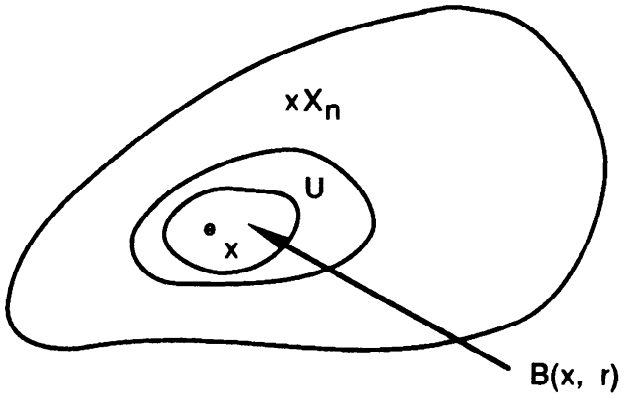


FIGURE 2

Let $x \in xX_n$ and let U be the basic neighborhood of x in the product topology. Then

$$U = \underset{n \in N}{\times} V_n$$

where V_n is open in X_n and for all but at most finitely many n, say

$$n_1, \ldots, n_k$$

$$V_n = X_n$$

The sets $V_{n_1}, \ldots, V_{n_k}$ are open in $X_{n_1}, \ldots, X_{n_k}$ respectively. Hence, positive numbers $r_1, \ldots, r_k$ exist such that

$$\begin{array}{ccccc} x_{n_1} & \in & B(x_{n_1}, r_1) & \subset & V_{n_1} \\ \vdots & & \vdots & & \vdots \\ x_{n_k} & \in & B(x_{n_k}, r_k) & \subset & V_{n_k} \end{array}$$

Choose

$$r = \min(r_1, \ldots, r_k).$$

Then if $y \in B(x, r)$

$$y_n \in V_n$$

thus

$$y \in U \quad \text{and} \quad B(x, r) \subset U.$$

To complete the proof, we have to show that for any $B(x, r)$ there is U such that $U \subset B(x, r)$. Let $B(x, r)$ be a d-neighborhood of x. Choose $m \in N$ such that

$$\sum_{n=m}^{\infty} \frac{1}{2^n} < \frac{r}{2}$$

and let $r_1, \ldots, r_{m-1}$ be such that

$$r_1 + \ldots + r_{m-1} < r/2.$$

We define

$$U = \underset{n \in N}{x} B_n$$

where $B_n = B(x_n, r_n)$ for $n = 1, 2, \ldots, m-1$ and $B_n = X_n$ for other n's.

U is a basic neighborhood of x in the product topology and

$$U \subset B(x, r).$$

Hence, the product and metric topologies on xX_n are equivalent and $\underset{N}{\times} X_n$ is metrizable.

● PROBLEM 18–5

Explain why the Hilbert cube $\underset{n \in N}{\times} [0,1]$ is second countable and metrizable.

SOLUTION:

Since the closed interval [0, 1] is second countable, the Hilbert cube $\underset{N}{\times} [0,1]$ is second countable.

The closed interval [0, 1] with the absolute value topology is metrizable. By Problem 18–4 the product of the countable family of second countable metric spaces is metrizable.

● PROBLEM 18–6

Show that the sequence

$$a_n = \frac{1}{n}, \; n \in N$$

in the space of real numbers R with the absolute value metric is a Cauchy sequence.

SOLUTION:

We shall define Cauchy sequences.

DEFINITION OF CAUCHY SEQUENCES

A sequence (a_n) in (X, d) is a Cauchy sequence iff

$\forall \varepsilon > 0, \exists N_0 \in N$ such that

$$n, m > N_0 \Rightarrow d(a_n, a_m) < \varepsilon.$$

■

We take an arbitrary fixed $\varepsilon > 0$. The solution of the problem lies in finding $N_0 = N^0(\varepsilon)$ such that for every $n, m > N_0$

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \varepsilon.$$

Let N_0 be such that

$$\frac{1}{N_0} < \frac{\varepsilon}{2}.$$

Note that we do not have to find the smallest possible N_0.

For any $n, m > N_0$

$$\frac{1}{n} < \frac{1}{N_0} \quad \text{and} \quad \frac{1}{m} < \frac{1}{N_0}$$

and

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \frac{1}{n} + \frac{1}{m} < \frac{1}{N_0} + \frac{1}{N_0} <$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, $(1/n)$ is a Cauchy sequence in $(R, 1 \cdot 1)$.

● PROBLEM 18-7

Show that every convergent sequence in a metric space (X, d) is a Cauchy sequence.

SOLUTION:

Let (a_n) be a convergent sequence in (X, d)

$$a_n \underset{n \to \infty}{\to} a.$$

In other words,

$$d(a_n, a) \to 0.$$

Then for every $\varepsilon > 0$ there exists N_0 such that

$$n > N_0 \Rightarrow d(a_n, a) < \varepsilon/2.$$

Hence, by the triangle inequality

$$n,m > N_0 \Rightarrow d(a_n, a_m) \leq d(a_n, a) + d(a_m, a) <$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus, (a_n) is a Cauchy sequence.

● PROBLEM 18–8

Find an example which illustrates that the converse of the statement in Problem 18–7 is not true; that is, not every Cauchy sequence is convergent.

SOLUTION:

Consider the space Q of rational numbers with the absolute value metric $(Q, 1 \cdot 1)$. The sequence is given as follows:

$$a_1 = 1$$

$$a_2 = 1.4$$

$$a_3 = 1.41$$

$$a_4 = 1.414$$

$$\vdots$$

Sequence (a_n) described above is the decimal expansion of $\sqrt{2}$. It is easy to verify that (a_n) is a Cauchy sequence. But (a_n) does not converge in Q (it converges in R). Hence, (a_n) is a sequence which is Cauchy but not convergent.

● PROBLEM 18–9

Show that every Cauchy sequence (a_n) in a metric space (X, d) is bounded and totally bounded.

SOLUTION:

Let (a_n) be a Cauchy sequence in (X, d). We shall prove that (a_n) is totally bounded (and thus bounded).

A subset A of a metric space (X, d) is totally bounded if A has an ε-net for every $\varepsilon > 0$. An ε-net for A is a finite set of points $\{a_1, \ldots, a_n\}$, such that for every $a \in A$ there is $a_k \in \{a_1, \ldots, a_n\}$ such that

$$d(a, a_k) < \varepsilon.$$

Let $\varepsilon > 0$. Since (a_n) is a Cauchy sequence,

$\exists N_0 \in N$ such that for every

$$n, m > N_0, d(a_n, a_m) < \varepsilon.$$

Hence, the diameter of the set

$$\{a_{N_0+1}, a_{N_0+2}, \ldots\}$$

is at most ε.

The ε-net for the sequence (a_n) is

$$\{a_1, a_2, \ldots, a_{N_0}, a_{N_0+1}\}.$$

● PROBLEM 18-10

Let (X, d) be a space with the trivial metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}.$$

Find a general formula for a Cauchy sequence in this space.

SOLUTION:

Suppose (a_n) is a Cauchy sequence in the trivial space.

Let $\varepsilon = 1/2$. Then there exists N_0 such that

$$n, m > N_0 \Rightarrow d(a_n, a_m) < 1/2 \text{ that is}$$

$$a_n = a_m.$$

Hence, a Cauchy sequence in the space with the trivial metric is of the form

$$(a_1, a_2, \ldots, a_k, a, a, a \ldots)$$

that is constant from a certain term on.

● PROBLEM 18-11

Let (a_n) be a Cauchy sequence in (X, d) and let (a_{K_n}) be its subsequence. Show that

$$d(a_n, a_{K_n}) \underset{n\to\infty}{\to} 0.$$

SOLUTION:

Let $\varepsilon > 0$ be an arbitrary positive number. Since (a_n) is a Cauchy sequence $\exists\, N_0 \in N$ such that,

$$n, m > N_0 - 1 \Rightarrow d(a_n, a_m) < \varepsilon.$$

Hence,

$$K_{N_0} \geq N_0 > N_0 - 1 \quad \text{and} \quad d(a_{N_0}, a_{K_{N_0}}) < \varepsilon.$$

That is

$$\lim_{n\to\infty} d(a_n, a_{K_n}) = 0.$$

● PROBLEM 18-12

Let (a_n) be a sequence in a metric space (X, d). Let

$$A_1 = \{a_1, a_2, \ldots\}$$

$$A_2 = \{a_2, a_3, \ldots\}$$

$$A_3 = \{a_3, a_4, \ldots\}$$

$$\vdots$$

Show that

$$((a_n) \text{ is a Cauchy sequence}) \Leftrightarrow (\lim_{n\to\infty} \delta(A_n) = 0)$$

where $\delta(A_n)$ denotes the diameter of A_n.

SOLUTION:

$\Rightarrow$ Suppose (a_n) is a Cauchy sequence. For $\varepsilon > 0$ there exists $N_0 \in N$ such that for $n, m > N_0$

$$d(a_n, a_m) < \varepsilon.$$

Thus, for $n > N_0$

$$\delta(A_n) < \varepsilon.$$

Since ε was chosen arbitrarily,

$$\lim_{n \to \infty} \delta(A_n) = 0.$$

$\Leftarrow$ $$\lim_{n \to \infty} \delta(A_n) = 0.$$

For any $\varepsilon > 0$ there exists N_0 such that for any $n > N_0$

$$\delta(A_n) < \varepsilon.$$

In other words, for any $n, m > N_0$

$$d(a_n, a_m) < \varepsilon$$

and (a_n) there is a Cauchy sequence.

● PROBLEM 18-13

Let (a_n) be a Cauchy sequence in (X, d) and (b_n) be a sequence in (X, d) such that

$$d(a_n, b_n) < \frac{1}{n}$$

for every $n \in N$. Show that:

1. (b_n) is also a Cauchy sequence.

2. (b_n) converges to $a \in X$ iff
   (a_n) converges to $a \in X$.

SOLUTION:

1. Applying the triangle inequality,

$$d(b_m, b_n) \leq d(b_m, a_m) + d(a_m, b_n) \leq$$

$$\leq d(b_m, a_m) + d(b_n, a_n) + d(a_n, a_m).$$

Let $\varepsilon > 0$. We can find $N_1 \in N$ such that for

$$n, m > N_1$$

we have

$$d(b_m, b_n) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + d(a_m, a_n).$$

Since (a_n) is a Cauchy sequence,

$\exists N_2 \in N$ such that

$$n, m > N_2 \Rightarrow d(a_m, a_n) \leq \frac{\varepsilon}{3}.$$

Setting

$$N_3 = \max\{N_1, N_2\}$$

we get

$$n, m > N_3 \Rightarrow d(b_m, b_n) < \varepsilon.$$

Hence, (b_n) is a Cauchy sequence.

2. Similarly,

$$d(a_n, a) \leq d(a_n, b_n) + d(b_n, a).$$

Hence,

$$\lim_{n \to \infty} d(a_n, a) \leq \lim_{n \to \infty} d(a_n, b_n) + \lim_{n \to \infty} d(b_n, a).$$

But

$$\lim d(a_n, b_n) = 0.$$

If $b_n \to a$, then

$$\lim_{n \to \infty} d(a_n, a) \leq \lim_{n \to \infty} d(b_n, a) = 0$$

and

$$\lim_{n \to \infty} a_n = a$$

Similarly, if $a_n \underset{n \to \infty}{\to} a$ then $b_n \underset{n \to \infty}{\to} a$.

● PROBLEM 18-14

Show that if (x_n) is a Cauchy sequence in Euclidean m-space R^m

$$x_1 = (x_1^{(1)}, x_1^{(2)}, \ldots, x_1^{(m)})$$

$$\vdots$$

$$x_n = (x_n^{(1)}, x_n^{(2)}, \ldots, x_n^{(m)})$$

$$\vdots$$

then each of the sequences

$$\left(x_K^{(1)}\right), \left(x_K^{(2)}\right), \ldots, \left(x_K^{(m)}\right)$$

$k = 1, \ldots, n$ is a Cauchy sequence in R.

SOLUTION:

Let $\varepsilon > 0$. Since (x_n) is a Cauchy sequence,

$\exists\, N_0 \in N$ such that for $j,\ l > N_0$

$$d(x_j, x_l) = \sqrt{\left|x_j^{(1)} - x_l^{(1)}\right|^2 + \ldots + \left|x_j^{(m)} - x_l^{(m)}\right|^2} < \varepsilon.$$

Thus, for each term we obtain

$$\left|x_j^{(1)} - x_l^{(1)}\right| < \varepsilon$$

$$\vdots$$

$$\left|x_j^{(m)} - x_l^{(m)}\right| < \varepsilon.$$

Each of the m sequences

$$\left(x_k^{(1)}\right), \ldots, \left(x_k^{(m)}\right)$$

is a Cauchy sequence.

● PROBLEM 18-15

1. Show that the real line R with the usual metric is complete.

2. Give an example of a subspace of R which is not complete.

SOLUTION:

DEFINITION OF COMPLETE METRIC SPACE

A metric space (X, d) is complete if every Cauchy sequence (a_n) in X converges to a point $a \in X$.

$$\lim_{n \to \infty} a_n = a \in X.$$

The real line R is complete. This is the result of the Cauchy theorem: Every Cauchy sequence of real numbers converges to a real number.

2. Consider the open interval]0, 1[with the absolute metric. The sequence $(1/2, 1/3, 1/4, 1/5, \ldots)$ is a Cauchy sequence which converges, in the limit as

$$\lim_{n \to \infty} \frac{1}{n} = 0 .$$

However, 0 does not belong to]0, 1[, hence,]0, 1[is not complete.

● PROBLEM 18-16

1. Is completeness a topological property?

2. Find the diameters of A, B, and C

$$A = \{(x, y) : x^2 + y^2 \leq 1 \} \subset R^2$$

$$B = \{(x, y) : x^2 + y^2 = 1 \} \subset R^2$$

$$C = \{(x, y) : |x| < 1, |y| \leq 2 \} \subset R^2$$

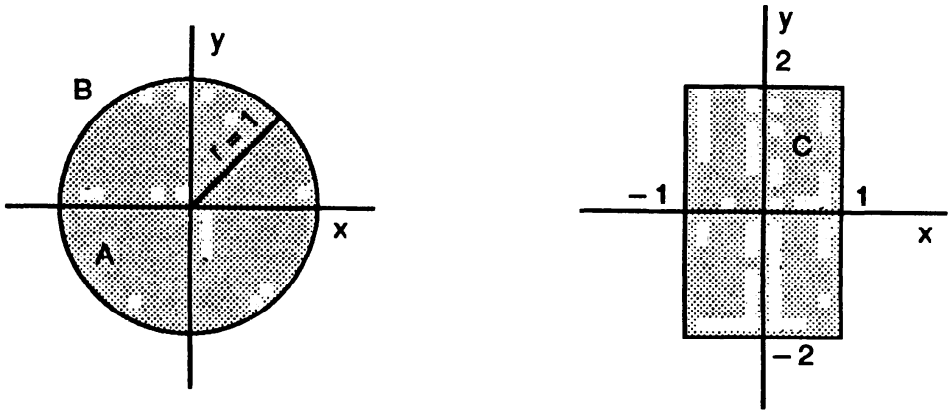


FIGURE 1

SOLUTION:

1. The real line R with the absolute metric is complete. As demonstrated in Problem 18-15, the open interval]0, 1[with the absolute metric is not complete. But R is homeomorphic to]0, 1[, then completeness is not a topological property.

2. The diameter of a subset A of a metric space (X, d) is defined to be the least upper bound of

$$\{d(x, y) : x, y \in A\}.$$

We denote the diameter of A by $\delta(A)$ or $d(A)$.

The diameter of A, B, and C are

$$\delta(A) = 1$$

$$\delta(B) = 1$$

$$\delta(C) = \sqrt{20}.$$

● PROBLEM 18-17

Show that for any subset A of a metric space (X, d)

$$\delta(A) = \delta(\overline{A}). \tag{1}$$

SOLUTION:

We shall prove that for any positive number $r > 0$

$$\delta(\overline{A}) < \delta(A) + r \tag{2}$$

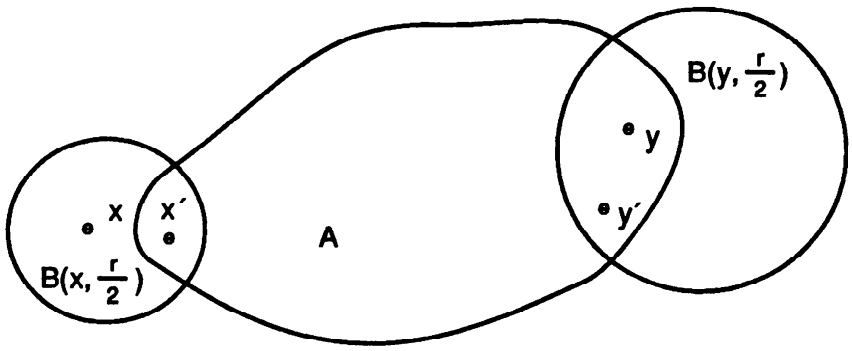


FIGURE 1

Suppose

$$x, y \in \overline{A}. \tag{3}$$

Then

$$B(x, r/2) \cap A \neq \phi$$

$$B(y, r/2) \cap A \neq \phi. \tag{4}$$

We choose x' and y' such that

$$x' \in B(x, r/2) \cap A$$

$$y' \in B(y, r/2) \cap A \tag{5}$$

Then

$$d(x, y) \leq d(x, x') + d(y, x') \leq$$

$$\leq d(x, x') + d(y, y') + d(x', y') < r/2 + r/2 + \delta(A) =$$

$$= r + \delta(A). \tag{6}$$

Therefore,

$$\delta(\overline{A}) < \delta(A) + r \tag{7}$$

and

$$\delta(\overline{A}) \leq \delta(A). \tag{8}$$

Since $A \subset \overline{A}$,

$$\delta(A) \leq d(\overline{A}). \tag{9}$$

Thus,

$$\delta(A) = \delta(\overline{A}). \tag{10}$$

● PROBLEM 18-18

Prove the Cantor theorem.

CANTOR THEOREM

Let $\{A_n\}$, $n \in N$ be a countable family of closed nonempty subsets of complete space X such that

$$A_1 \supset A_2 \supset A_3 \ldots \text{ and } \delta(A_n) \underset{n \to \infty}{\to} 0 \tag{1}$$

Then

$$\bigcap_{n \in N} A_n \neq \phi. \tag{2}$$

■

SOLUTION:

For every $n \in N$ choose $a_n \in A_n$. We shall show that (a_n) is a Cauchy sequence. Let $\varepsilon > 0$. There exists

$$N_0 \in N$$

such that for $n > N_0$

$$\delta(A_n) < \varepsilon/2. \tag{3}$$

For $k,l > N_0$, a_k, $a_l \in A_{N_0+1}$.

Hence,

$$d(a_k, a_l) < \varepsilon. \tag{4}$$

Thus, (a_n) is a Cauchy sequence in (X, d). Since, by hypothesis, X is complete

$$\lim_{n\to\infty} a_n = a \in X. \tag{5}$$

Note that for any n

$$(a_n, a_{n+1}, \ldots)$$

is also a sequence which converges to a.

$$\{a_n, a_{n+1}, \ldots\} \subset A_n. \tag{6}$$

Since for each n, A_n is closed,

$$a \in \overline{A_n} = A_n. \tag{7}$$

Thus,

$$a \in \bigcap_{n\in N} A_n \quad \text{and} \quad \bigcap_{n\in N} A_n \neq \phi. \tag{8}$$

● PROBLEM 18-19

Prove the following:

THEOREM

A metric space (X, d) is complete if and only if every nested sequence (i.e., $A_1 \supset A_2 \supset A_3 \ldots$) of nonempty closed sets such that $\delta(A_n) \to 0$ has a nonempty intersection, $\bigcap_n A_n \neq \phi$.

■

SOLUTION:

We have to show that

$$((X, d) \text{ complete}) \Leftrightarrow (\bigcap_n A_n \neq \phi).$$

For any countable family $\{A_n\}$ of closed, nonempty subsets of X such that

$$A_1 \supset A_2 \supset A_3 \ldots$$

and

$$\delta(A_n) \to 0.$$

In Problem 18–18 we proved $\Rightarrow$.

Now we shall show $\Leftarrow$.

Let $A_1 \supset A_2 \supset A_3 \ldots$ be any sequence of closed, nonempty subsets of X such that

$$\delta(A_n) \to 0, \cap A_n \neq \phi.$$

Let $\{X_n\}$, $n \in N$ be a Cauchy sequence in X. We define

$$B_1 = \{x_1, x_2, x_3, \ldots\}, A_1 = \overline{B}_1$$

$$B_2 = \{x_2, x_3, \ldots\}, A_2 = \overline{B}_2$$

$$B_3 = \{x_3, x_4, \ldots\}, A_3 = \overline{B}_3$$

$$\vdots$$

Then $A_1 \supset A_2 \supset A_3 \supset \ldots$ and $\delta(A_n) \to 0$ and $\{A_n\}$ are closed, nonempty subsets of X. Hence,

$$\bigcap_{n \in N} A_n \neq \phi.$$

Let

$$x \in \bigcap_n A_n.$$

We show that

$$x_n \to x \in X.$$

Let $\varepsilon > 0$. Then

$\exists N_0 \in N$ such that for $n > N_0$

$$\delta(B_n) < \varepsilon.$$

Hence (by Problem 18–17),

$$\delta(B_n) = \delta(A_n) < \varepsilon.$$

For all $n > N_0$

$$d(x_n, x) < \varepsilon.$$

Hence,

$$x_n \to x \in X.$$

● PROBLEM 18–20

Show that any compact metric space (X, d) is complete.

SOLUTION:

A space X is said to be compact if given any open cover $\{U_i\}$, $i \in I$, of X, there is a finite subcover of $\{U_i\}$, $i \in I$.

Suppose $(A_n\}$, $n \in N$, are closed, nonempty subsets of X such that

$$A_1 \supset A_2 \supset A_3 \supset \ldots \tag{1}$$

We shall apply the following: X is compact if and only if for any family $\{F_i\}$, $i \in I$, of closed subsets of X such that the intersection of any finite number of the F_i is nonempty,

$$\bigcap_{i \in I} F_i \neq \phi. \tag{2}$$

Then

$$\bigcap_{n \in N} A_n \neq \phi. \tag{3}$$

Hence, by Problem 18–19, the space (X, d) is complete.

● PROBLEM 18–21

1. Show that a subspace of a complete metric space can be incomplete.

2. Show that any separable metric space (X, d) is homeomorphic to a dense subspace of a complete metric space.

SOLUTION:

1. The set of rational numbers forms an incomplete subspace of the complete set of real numbers.

2. Space X is said to be separable if X contains a countable dense subset.

Any separable metric space (X, d) has a metrizable compactification, say (Y, d'). If a metric space (Y, d') is compact, then (Y, d') is complete.

● PROBLEM 18-22

Show that any closed subspace A of a complete metric space (X, d) is a complete metric space.

SOLUTION:

Suppose

$$(a_n),\ n \in N$$

is a Cauchy sequence in A. Then (a_n) is also a Cauchy sequence in X. Since X is complete

$$a_n \to a \in X.$$

But

$$a \in \overline{A} = A.$$

Hence, (a_n) converges in A and A is complete.

● PROBLEM 18-23

Show that

$$((\underset{n}{x} X_n, d) \text{ is complete}) \Leftrightarrow (\text{each } (X_n, d_n) \text{ is complete}).$$

SOLUTION:

Let us define d as follows:

$$d(x,y) = \sum_{n \in N} \frac{\min(d_n(x_n, y_n), 1)}{2^n}.$$

where $$x = (x_1, x_2, x_3, \ldots)$$

$$y = (y_1, y_2, y_3, \ldots).$$

It can be shown that d is a metric in $\underset{n}{\times} X_n$.

$\Leftarrow$ Suppose each (X_n, d_n) is complete. Let

$$(a_k),\ k \in N$$

be a Cauchy sequence in $\underset{n}{x} X_n$. The l-th coordinate of a_k is denoted by

$$a_k(l).$$

The sequence $(a_k(l))$, $k \in N$ is a Cauchy sequence in X_l. Hence, $(a_k(l))$ converges in X_l for each $l \in N$. Thus, (a_k) converges in $\underset{n}{x} X_n$.

$\Rightarrow$ Suppose (X_1, d_1) is not complete. There is a Cauchy sequence

$$(a_k(1)), k \in N$$

in X_1, which does not converge. Choose from each X_n, $n > 1$ a point

$$x_n \in X_n.$$

We define a sequence as

$$(a_K(l)) = \begin{cases} a_K(1) & \text{for } l = 1 \\ x_l & \text{for } l \neq 1 \end{cases}$$

which is a Cauchy sequence but does not converge in $(\underset{n}{x} X_n, d)$.

Contradiction.

● PROBLEM 18-24

1. Show that

$$f: R^2 \to R^2$$

$$f(x) = \frac{x}{2}$$

defined on Euclidean R^2 space is a contracting mapping.

2. Show that a contracting mapping is continuous.

SOLUTION:

DEFINITION OF CONTRACTING MAPPING

A function $f: X \to X$, defined as (X, d) is called a contracting mapping if a real number $a \in R$, $0 \leq a \leq 1$ exists there, such that for every $x, y \in X$

$$d(f(x), f(y)) \leq a\, d(x, y) < d(x, y).$$

■

1. $$d(f(x), f(y)) = \|f(x) - f(y)\| = \left\|\frac{x}{2} - \frac{y}{2}\right\| =$$

$$= \frac{1}{2} \|x - y\| = \frac{1}{2}\, d(x, y).$$

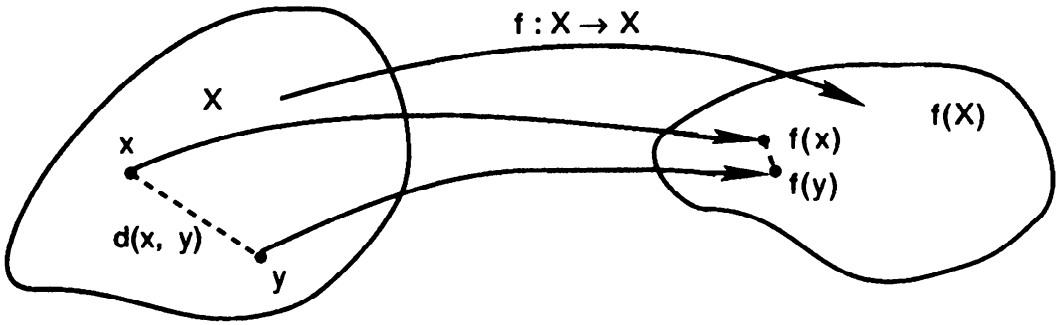


FIGURE 1

2. Let $f : X \to X$ be a contracting mapping on a metric space (X, d). A real number $0 \leq a < 1$ exists such that for every $x, y \in X$

$$d(f(x), f(y)) \leq a\, d(x, y).$$

Let $x_1 \in X$ and $\varepsilon > 0$.

We have

$$d(x, x_1) < \varepsilon \Rightarrow d(f(x), f(x_1)) \leq a\, d(x, x_1) \leq \varepsilon\, a < \varepsilon.$$

Hence, f is continuous.

● PROBLEM 18-25

Show that if $f : X \to X$ is a contracting mapping on a complete metric space (X, d) then there exists one and only one point $x_0 \in X$ such that

$$f(x_0) = x_0. \tag{1}$$

SOLUTION:

Let $a_0 \in X$. We define

$$f(a_0) = a_1, f(a_1) = a_2 = f^2(a_0), \ldots$$

$$f(a_{n-1}) = a_n = f^n(a_0), \ldots \tag{2}$$

and show that

$$a_0, a_1, a_2, \ldots$$

is a Cauchy sequence.

$$d(f^{k+l}(a_0), f^l(a_0)) \leq a\, d(f^{k+l-1}(a_0), f^{l-1}(a_0)) \leq \ldots$$

$$\ldots \leq a^l\, d(f^k(a_0), a_0) \leq$$

$$\leq a^l[d(a_0, f(a_0)) + d(f(a_0), f^2(a_0)) + \ldots + d(f^{k-1}(a_0), f^k(a_0))]. \quad (3)$$

But

$$d(f^{m+1}(a_0), f^m(a_0)) \leq a^m\, d(f(a_0), a_0). \quad (4)$$

Hence, from (3) and (4) we obtain

$$d(f^{k+l}(a_0), f^l(a_0)) \leq a^l\, d(f(a_0), a_0)\,[1 + a + a^2 + \ldots + a^{k-1}]. \quad (5)$$

Since $a < 1$,

$$1 + a + a^2 + \ldots + a^{k-1} \leq \frac{1}{1-a}. \quad (6)$$

From (5) and (6) we find

$$d(f^{k+l}(a_0), f^l(a_0)) \leq \frac{a^l}{1-a}\, d(f(a_0), a_0). \quad (7)$$

For $\varepsilon > 0$, we define

$$\lambda = \begin{cases} \varepsilon(1-a) & \text{if} \quad d(f(a_0), a_0) = 0 \\ \dfrac{\varepsilon(1-a)}{d(f(a_0), a_0)} & \text{if} \quad d(f(a_0), a_0) \neq 0 \end{cases} \quad (8)$$

Since $a < 1$, $M \in N$ exists such that

$$a^M < \lambda. \quad (9)$$

For $k \geq l > M$,

$$d(a_k, a_l) \leq a^l \cdot \frac{1}{1-a} \cdot d(f(a_0), a_0) < \frac{\lambda}{1-a} \cdot d(f(a_0), a_0) \leq \varepsilon. \quad (10)$$

Thus, $(a_1, a_2, a_3, \ldots)$ is a Cauchy sequence and since X is complete $x_0 \in X$ exists such that

$$\lim_{n \to \infty} a_n = x_0. \quad (11)$$

Function f is continuous, thus

$$f(x_0) = f(\lim a_n) = \lim f(a_n) = \lim a_n = x_0. \quad (12)$$

Point x_0 is unique. For, suppose $f(x_0) = x_0$ and $f(y_0) = y_0$, then

$$d(x_0, y_0) = d(f(x_0), f(y_0)) \leq a\, d(x_0, y_0). \quad (13)$$

But $a < 1$. Therefore,

$$d(x_0, y_0) = 0$$

and $x_0 = y_0$.

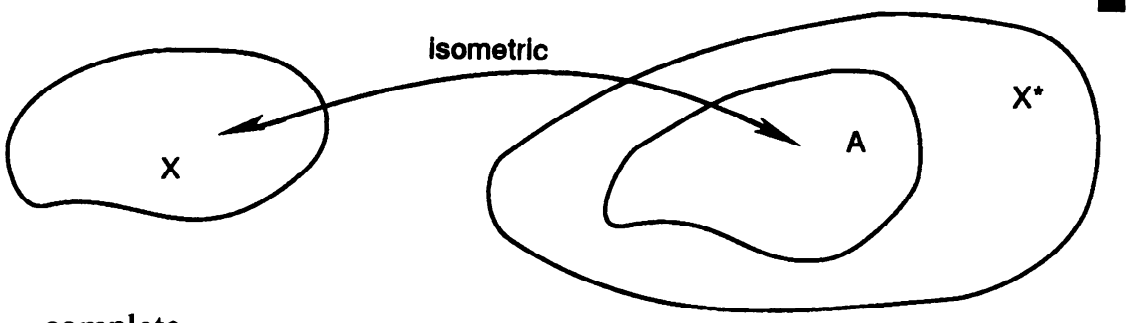
● PROBLEM 18-26

Show that the set of real numbers R is a completion of the set of rational numbers Q.

SOLUTION:

DEFINITION OF COMPLETION OF METRIC SPACE

A metric space X^* is called a completion of a metric space X if X^* is complete and X is isometric to a dense subset of X^*. ■



X^* – complete.
X and A are isometric.
$\overline{A} = X^*$.

FIGURE 1

Consider the set Q of rational numbers. The closure of Q is the entire set R of real numbers, i.e.,

$$\overline{Q} = R.$$

The set R is complete. Hence, the set R of real numbers is a completion of the set Q of rational numbers.

● PROBLEM 18-27

Let $C[X]$ denote the collection of all Cauchy sequences on X. The relation $\sim$ on $C[X]$ is defined by

$$((a_n) \sim (b_n)) \Leftrightarrow (\lim_{n \to \infty} d(a_n, b_n) = 0). \quad (1)$$

Show that $\sim$ is an equivalence relation on $C[X]$.

SOLUTION:

$$(a_n) \sim (a_n)$$

is obvious, since $d(x, x) = 0$

$$((a_n) \sim (b_n)) \Rightarrow ((b_n) \sim (a_n))$$

because

$$d(x, y) = d(y, x).$$

Suppose $(a_n) \sim (b_n)$ and $(b_n) \sim (c_n)$. Then

$$\lim d(a_n, b_n) = \lim d(b_n, c_n) = 0.$$

From the definition of a metric,

$$d(a_n, c_n) \leq d(a_n, b_n) + d(b_n, c_n).$$

Taking the limit

$$\lim d(a_n, c_n) \leq \lim d(a_n, b_n) + \lim d(b_n, c_n) = 0.$$

Thus,

$$(a_n) \sim (c_n).$$

The relation $\sim$ is an equivalence relation on $C[X]$.

● PROBLEM 18-28

Show that

$$((a_n) \sim (b_n)) \Leftrightarrow \begin{pmatrix} (a_n) \text{ and } (b_n) \text{ are subsequences} \\ \text{of some Cauchy sequence } (c_n) \end{pmatrix}.$$

SOLUTION:

$\Leftarrow$ A Cauchy sequence (c_n) exists such that

$$(a_n) = (c_{k_n}) \text{ and } (b_n) = (c_{l_n}).$$

Then

$$\lim_{n \to \infty} d(a_n, b_n) = \lim_{n \to \infty} d(c_{k_n}, c_{l_n}) = 0$$

since (c_n) is a Cauchy sequence.

$\Rightarrow$ Suppose

$$(a_n) \sim (b_n)$$

that is

$$\lim_{n \to \infty} d(a_n, b_n) = 0.$$

We define a sequence

$$(c_n) = (a_1, b_1, a_2, b_2, a_3, b_3, \ldots)$$

that is

$$(c_n) = \begin{cases} a_{\frac{n+1}{2}} & \text{if } n \text{ is odd} \\ b_{\frac{n}{2}} & \text{if } n \text{ is even} \end{cases}.$$

and show that (c_n) is a Cauchy sequence. Let $\varepsilon > 0$, then

$\exists\, n_1 \in N$ such that

$$n > n_1 \Rightarrow d(a_n, b_n) < \varepsilon/2$$

$\exists\, n_2 \in N$ such that

$$m, n > n_2 \Rightarrow d(a_m, a_n) < \varepsilon/2$$

$\exists\, n_3 \in N$ such that

$$m, n > n_3 \Rightarrow d(b_m, b_n) < \varepsilon/2$$

Set $n_0 = \max(n_1, n_2, n_3)$. Then for $m > 2n_0$

$$\frac{1}{2} m > n_1, \frac{1}{2} m > n_3$$

$$\frac{m+1}{2} > n_1, \frac{m+1}{2} > n_2.$$

We shall show that

$$m, n > 2n_0 \Rightarrow d(c_m, c_n) < \varepsilon.$$

We have

m, n even $\Rightarrow c_m = b_{\frac{m}{2}}, c_n = b_{\frac{n}{2}}$

then

$$d(c_m, c_n) < \varepsilon/2 < \varepsilon$$

m, n odd $\Rightarrow c_m = a_{\frac{m+1}{2}}, c_n = a_{\frac{n+1}{2}}$

then

$$d(c_m, c_n) < \varepsilon/2 < \varepsilon$$

$$m,n \text{ one odd and one even} \Rightarrow c_m = b_{\frac{m}{2}}\,,\ c_n = a_{\frac{n+1}{2}}$$

then

$$d(c_m,c_n) \le d(a_{\frac{m}{2}},b_{\frac{m}{2}}) + d(a_{\frac{m}{2}},a_{\frac{n+1}{2}}) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, (c_n) is a Cauchy sequence.

● PROBLEM 18–29

Describe the construction of a completion X^* of a metric space (X, d).

SOLUTION:

A metric space (X, d) is given. By $C[X]$ we denote the collection of all Cauchy sequences in X. Among Cauchy sequences the relation $\sim$ is introduced as follows:

$$(a_n) \sim (b_n) \text{ iff } \lim_{n\to\infty} d(a_n,b_n) = 0.$$

In Problem 18–27 we showed that $\sim$ is an equivalence relation. We define X^*

$$X^* = \frac{C[X]}{\sim}$$

as the quotient space. Elements of X^* are the equivalence classes $[(a_n)]$ of Cauchy sequences (a_n) of $C[X]$. Thus, if $(a_n) \in [(a_n)]$ and $(b_n) \in [(a_n)]$ we have $(a_n) \sim (b_n)$ and $\lim_{n\to\infty} d(a_n,b_n) = 0$.

Later we shall show that the function δ defined by

$$\delta([(a_n)], [(b_n)]) = \lim_{n\to\infty} d(a_n,b_n)$$

is a metric on X^*.

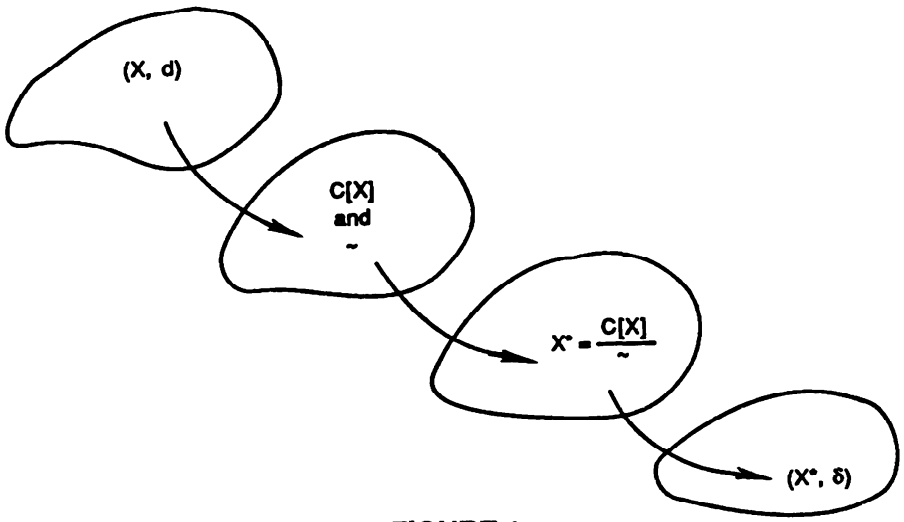


FIGURE 1

Show that the function δ defined in Problem 18–29 by

$$\delta([(a_n)], [(b_n)]) = \lim_{n \to \infty} d(a_n, b_n)$$

is well-defined, i.e., that

$$\begin{pmatrix} (a'_n) \sim (a_n) \\ (b'_n) \sim (b_n) \end{pmatrix} \Rightarrow (\lim d(a'_n, b'_n) = \lim d(a_n, b_n).$$

SOLUTION:

Suppose $(a'_n) \sim (a_n)$ and $(b'_n) \sim (b_n)$ then

$$\lim_{n \to \infty} d(a'_n, a_n) = 0$$

and

$$\lim_{n \to \infty} d(b'_n, b_n) = 0.$$

Set

$$a = \lim d(a_n, b_n)$$

$$a' = \lim d(a'_n, b'_n).$$

We have

$$d(a_n, b_n) \leq d(a_n, a'_n) + d(b_n, b'_n) + d(a'_n, b'_n).$$

Let $\varepsilon > 0$, then

$\exists\, n_1 \in N$ such that

$$n > n_1 \Rightarrow d(a_n, a'_n) < \varepsilon/3$$

$\exists\, n_2 \in N$ such that

$$n > n_2, \Rightarrow d(b_n, b'_n) < \varepsilon/3$$

$\exists\, n_3 \in N$ such that

$$n > n_3 \Rightarrow d(a'_n, b'_n) - a' < \varepsilon/3$$

For

$$n > \max(n_1, n_2, n_3)$$

we get

$$d(a_n,b_n) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + a' = a' + \varepsilon$$

and

$$\lim_{n\to\infty} d(a_n,b_n) = a \leq a' + \varepsilon .$$

Hence, since $\varepsilon > 0$ is arbitrary

$$a \leq a'.$$

In the same way we show that

$$a' \leq a$$

and conclude that

$$a' = a.$$

● PROBLEM 18-31

Show that

$$\delta([(a_n)], [(b_n)]) = \lim_{n\to\infty} d(a_n,b_n)$$

is metric on

$$X^* = \frac{C[X]}{\sim}$$

(compare Problem 18–29).

SOLUTION:

We have already shown that δ is well-defined (Problem 18–30).

1. $$\delta([a_n)], [(b_n)] \geq 0$$

because $d(a_n, b_n) \geq 0$.

$$\delta([a_n)], [(a_n)] = 0$$

because $d(a_n, b_n) = 0$.

2. $$\delta([(a_n)], [(b_n)]) = \delta([(b_n)], [(a_n)])$$

because $d(a_n, b_n) = d(b_n, a_n)$.

3. $$\delta([(a_n)], [(b_n)]) = \lim_{n\to\infty} d(a_n,b_n) \leq$$

$$\leq \lim_{n \to \infty} d(a_n, c_n) + \lim_{n \to \infty} d(b_n, c_n) =$$

$$= \delta([(a_n)], [(c_n)]) + \delta([(b_n)], [(c_n)]).$$

4. If $[(a_n)] \neq [(b_n)]$ then

$$\delta([(a_n)], [(b_n)]) > 0.$$

Indeed

$$\delta([(a_n)], [(b_n)]) = \lim_{n \to \infty} d(a_n, b_n) > 0$$

because

$$(a_n) \text{ is not in relation } \sim \text{ with } (b_n) \Rightarrow \lim d(a_n, b_n) > 0.$$

Hence,

$$\delta([(a_n)], [(b_n)]) > 0.$$

● PROBLEM 18-32

Suppose $a \in X$ then $(a, a, a, \ldots)$ is a Cauchy sequence

$$(a, a, a, \ldots) \in C[X].$$

We define space $\hat{X}$ as follows:

$$\hat{a} \in \hat{X} = \{\hat{a} : a \in X\}$$

$\hat{a} = [(a, a, a, \ldots)]$. Space $\hat{X}$ is a subspace of X^*.

Now, suppose $(a_n) \in C[X]$. Show that

$$a = [(a_n)] \in X^*$$

is the limit of the sequence

$$(\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots)$$

in $\hat{X}$.

SOLUTION:

Sequence (a_n) is a Cauchy sequence in X. We have

$$\lim_{n \to \infty} \delta(a, \hat{a}_n) = \lim_{n \to \infty} \left(\lim_{n \to \infty} d(a_m, a_n)\right) =$$

$$= \lim_{\substack{n \to \infty \\ m \to \infty}} d(a_m, a_n) = 0.$$

Thus,

$$(\hat{a}_n) \to a.$$

● PROBLEM 18–33

1. Show that X is isometric to space $\hat{X}$ defined in Problem 18–32.
2. Show that $\hat{X}$ is dense in X^*.

SOLUTION:

1. Suppose $a, b \in X$. Then

$$\hat{a} = [(a, a, a, \ldots)] \in \hat{X}$$

$$\hat{b} = [(b, b, b, \ldots)] \in \hat{X}$$

and

$$\delta(\hat{a}, \hat{b}) = \lim_{n \to \infty} d(a, b) = d(a, b).$$

Hence, X is isometric to $\hat{X}$.

2. We shall show that every point in X^* is the limit of a sequence in $\hat{X}$. Let

$$a = [(a_1, a_2, a_3, \ldots)] \in X^*.$$

Then $(a_1, a_2, a_3, \ldots)$ is a Cauchy sequence in X. Hence, a is the limit of the sequence

$$(\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots)$$

in $\hat{X}$. Thus, $\hat{X}$ is dense in X^*.

● PROBLEM 18–34

Show that (X^*, d) is a completion of X, i.e., that every Cauchy sequence in (X^*, δ) converges.

SOLUTION:

Let

$$(a_1, a_2, a_3, \ldots)$$

be a Cauchy sequence in (X^*, δ). In Problem 18–33 we proved that

$$\hat{X} \text{ is dense in } X^*.$$

Hence,

$$\forall\, n \in N \quad \exists\, \hat{a}_n \in \hat{X} \text{ such that}$$

$$\delta(\hat{a}_n, a_n) < \frac{1}{n}.$$

Hence, $(\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots)$ is also a Cauchy sequence.

By Problem 18–33 $(\hat{a}_1, \hat{a}_2, \hat{a}_3, \ldots)$ converges to

$$a = [(a_1, a_2, \ldots)] \in X^*.$$

Thus, (a_n) also converges to a. We conclude that (X^*, δ) is complete.

● PROBLEM 18–35

Show that

$$(Y^* \text{ is a completion of } X) \Rightarrow (Y^* \text{ is isometric to } X^*).$$

SOLUTION:

X is a subspace of Y^*. Thus for every

$$y \in Y^*$$

there exists a Cauchy sequence $(y_1, y_2, y_3, \ldots)$ in X converging to y. Define a mapping

$$f : Y^* \to X^*$$

by

$$f(y) = [(y_1, y_2, \ldots)].$$

We shall show that f is an isometry between Y^* and X^*. Mapping f is well-defined. Indeed, suppose

$$(y'_1, y'_2, y'_3, \ldots) \to y$$

where (y'_n) is a sequence in X.

Then

$$\lim_{n \to \infty} d(y_n, y'_n) = 0$$

and

$$[(y_n)] = [(y'_n)].$$

Also f is subjective. Suppose

$$[(x_1, x_2, \ldots)] \in X^*.$$

Then

$$(x_1, x_2, \ldots)$$

is a Cauchy sequence in $X \subset Y^*$. But Y^* is complete. Hence, (x_n) converges to $x \in Y^*$ and

$$f(x) = [(x_n)].$$

Now suppose $x, y \in Y^*$. Then there are sequences (x_n) and (y_n) in X such that

$$x_n \to x$$

$$y_n \to y.$$

We have

$$\delta(f(x), f(y)) = \delta([(x_n)], [(y_n)]) =$$

$$= \lim_{n \to \infty} d(x_n, y_n) = d(x, y).$$

Hence, f is an isometry between Y^* and X^*.

● PROBLEM 18-36

1. Show that the set Z of integers is a nowhere dense subset of the real line R.

2. Show that the set Q of rational numbers is not nowhere dense in R.

SOLUTION:

1. A subset A of a topological space (X, T) is nowhere dense in X iff the interior of the closure of A is empty

$$int(\overline{A}) = \phi.$$

The set Z is closed and its interior is empty. Hence,

$$int(\overline{Z}) = int(Z) = \phi.$$

2. The closure of Q is R

$$\overline{Q} = R$$

and

$$int(\overline{Q}) = int(R) = R \neq \phi.$$

Thus, the set Q of rational numbers is not nowhere dense in R.

● PROBLEM 18-37

Let U be an open subset of the metric space (X, d) and let M be nowhere dense in X. Show that $x \in X$ exists and $\varepsilon > 0$ such that

$$B(x, \varepsilon) \subset U$$

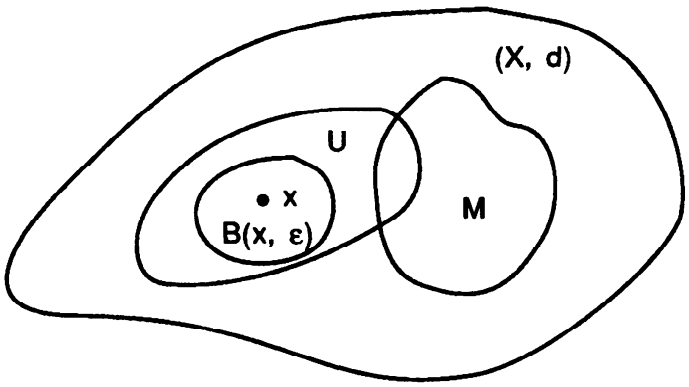


FIGURE 1

SOLUTION:

Let us denote

$$U \cap \overline{M}^c = V.$$

Then

$$V \subset U \quad \text{and} \quad V \cap M = \phi.$$

Set V is nonempty because U is open and $\overline{M}^c$ is dense in X.

$$V \neq \phi.$$

Hence,

$$x \in V.$$

V is open because U is open and $\overline{M^c}$ is open.

Thus, $\varepsilon > 0$ exists such that

$$B(x, \varepsilon) \subset V$$

and

$$B(x, \varepsilon) \cap M = \phi.$$

● PROBLEM 18-38

Show that if M is a nowhere dense subset of X then $\overline{M^c}$ is dense in X. (We denote $M^c = X - M$.)

SOLUTION:

Suppose on the contrary that $\overline{M^c}$ is not dense in X. Then $x \in X$ exists and an open set U such that

$$x \in U$$

$$U \cap \overline{M^c} = \phi.$$

Hence,

$$x \in U \subset \overline{M}$$

and

$$x \in int(\overline{M}).$$

It is a contradiction, because M is nowhere dense in X, $int(\overline{M}) = \phi$. Thus, $\overline{M^c}$ is dense in X.

● PROBLEM 18-39

Prove Baire's Category Theorem.

THEOREM

Every complete metric space (X, d) is of second category. ■

SOLUTION:

DEFINITION OF FIRST CATEGORY SPACES

A topological space (X, T) is said to be of first category if X is the countable union of nowhere dense subsets of X. All other spaces are of second category.

■

Let $A \subset X$ be of first category. We will show that $A \neq X$. Since A is of first category, by definition

$$A = M_1 \cup M_2 \cup \ldots \tag{1}$$

where each M_n is a nowhere dense subset of X.

Since M_1 is nowhere dense in X, there exists $x_1 \in X$ and $\varepsilon_1 > 0$ such that

$$B(x_1, \varepsilon_1) \cap M_1 = \phi. \tag{2}$$

Then

$$\overline{B(x_1, \tfrac{\varepsilon_1}{2})} \cap M_1 = \phi. \tag{3}$$

Since $B(x_1, \frac{\varepsilon_1}{2})$ is open and M_2 is nowhere dense in X, by Problem 18–38 we conclude that

$\exists\, x_2 \in X, \exists\, \varepsilon_2 > 0$ such that

$$B(x_2, \varepsilon_2) \subset B(x_1, \tfrac{\varepsilon_1}{2}) \subset \overline{B(x_1, \tfrac{\varepsilon_1}{2})} \tag{4}$$

and

$$B(x_2, \varepsilon_2) \cap M_2 = \phi. \tag{5}$$

Set

$$\varepsilon_2 \leq \frac{\varepsilon_1}{2}. \tag{6}$$

Then

$$\overline{B(x_2, \tfrac{\varepsilon_2}{2})} \subset \overline{B(x_1, \tfrac{\varepsilon_1}{2})} \tag{7}$$

and

$$\overline{B(x_2, \tfrac{\varepsilon_2}{2})} \cap M_2 = \phi. \tag{8}$$

In this way we get a nested sequence of closed sets

$$\overline{B(x_1, \tfrac{\varepsilon_1}{2})} \supset \overline{B(x_2, \tfrac{\varepsilon_2}{2})} \supset \overline{B(x_3, \tfrac{\varepsilon_3}{2})} \supset \ldots \tag{9}$$

such that for every $n \in N$

$$\overline{B(x_n, \frac{\varepsilon_n}{2})} \cap M_n = \phi \tag{10}$$

and $$\frac{\varepsilon_n}{2} \leq \frac{\varepsilon_1}{2^n}.$$

Hence, $$\lim_{n\to\infty} \varepsilon_n \leq \lim_{n\to\infty} \frac{\varepsilon_1}{2^{n-1}} = 0$$

and $x \in X$ exists such that $$x \in \bigcap_{n-1}^{\infty} \overline{B(x_n, \frac{\varepsilon_n}{2})}. \tag{11}$$

Also, $x \notin M_n$ for every $n \in N$ and

$$x \notin A.$$

Thus, (X, d) is of second category.

CHAPTER 19

ELEMENTS OF HOMOTOPY THEORY

Homotopic Functions	19–1, 19–2, 19–3, 19–17
Homotopy is an Equivalence Relation	19–4, 19–5, 19–12
Homotopy Class	19–6
Paths, Arcs, and Loops are Homotopic	19–7, 19–8, 19–10, 19–11 19–12, 19–13, 19–21
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Equivalence Class	19–14, 19–15, 19–16, 19–18, 19–20
Suitable Homotopy	19–19
Fundamental Group	19–22, 19–23, 19–24

Show that the two continuous functions,

$$f : X \to R$$

$$g : X \to R$$

where R is the space of real numbers with standard topology, are always homotopic.

SOLUTION:

DEFINITION OF HOMOTOPIC FUNCTIONS

Two continuous functions

$$f : X \to Y$$

$$g : X \to Y$$

are given. Functions f and g are said to be homotopic, if a continuous function h

$$h : X \times [0, 1] \to Y$$

exists, such that

$$h(x, t) \in Y$$

$$h(x, 0) = f(x) \quad \text{and} \quad h(x, 1) = g(x).$$

■

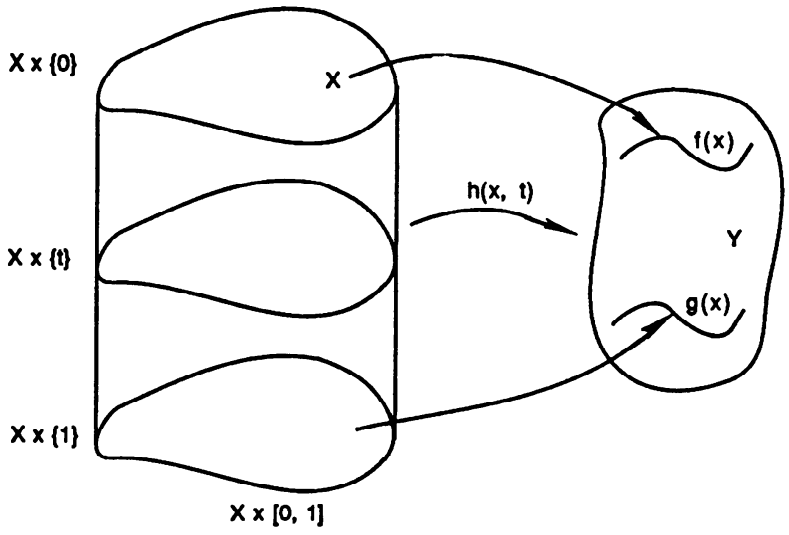


FIGURE 1

If $Y = R$ (or more generally, $Y = R^n$), then the functions f and g are always homotopic. Indeed,

$$h(x, t) = f(x) + t[g(x) - f(x)]$$

and $h(x, t)$ is continuous.

● PROBLEM 19-2

Show that the identity function on the unit disk is homotopic to a constant function which maps the whole disk into (0, 0).

SOLUTION:

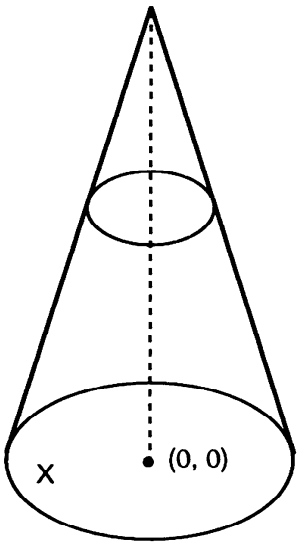


FIGURE 1

Suppose X and Y are unit disks and f is the identity function

$$f(x) = x \quad \text{for} \quad x \in X.$$

Let $g(x)$ denote a constant function defined on the unit disk

$$g(x) = (0, 0) \quad \text{for} \quad x \in X.$$

We shall define function $h(x, t)$.

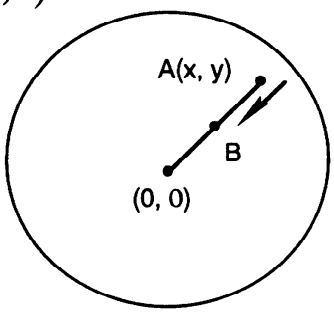


FIGURE 2

Let $A = (x, y)$ represent a point in the disk. Point $h(A, t) = B$ is the point on $(0, 0)$ (x, y) that is t times the distance from (x, y) to $(0, 0)$. Hence

$$h(A, 0) = A$$

$$h(A, 1) = (0, 0) = \text{const.}$$

● PROBLEM 19-3

Let Y^X denote the set of all continuous functions from X into Y, and let $f \sim g$ dneote that f is homotopic to g. Show that if $f \in Y^X$, then

$$f \sim f.$$

SOLUTION:

The homotopy

$$h : X \times [0, 1] \to Y$$

between f and g is given by

$$h(x, t) = f(x)$$

for all $x \in X$ and all $t \in [0, 1]$.

We must show that $h(x, t)$ is continuous and that

$$h(x, 0) = h(x, 1) = f(x).$$

Suppose U is an open subset of Y. Then

$$h^{-1}(U) = f^{-1}(U) \times [0, 1].$$

Since f is continuous $f^{-1}(U)$ is an open subset of X and $f^{-1}(U) \times [0, 1]$ is an open subset of $X \times [0, 1]$. Hence, $h(x, t)$ is continuous.

● PROBLEM 19-4

Show that if $f \sim g$, then $g \sim f$.

SOLUTION:

Since $f \sim g$, there is a homotopy

$$h : X \times [0, 1] \to Y$$

such that

$$h(x, 1) = f \quad \text{and} \quad h(x, 0) = g.$$

Let us define H by

$$H : X \times [0, 1] \to Y$$

$$H(x, t) = h(x, 1 - t)$$

for all $x \in X$ and all $t \in [0, 1]$.

We have

$$H(x, 0) = h(x, 1) = f$$

$$H(x, 1) = h(x, 0) = g.$$

Since $h(x, t)$ is continuous, so is $H(x, t)$.

Hence, $H(x, t)$ is a homotopy between g and f.

● PROBLEM 19–5

Show that the relation, $f \sim g$

f is homotopic to g

is an equivalence relation.

SOLUTION

In Problem 19–3, we have shown that

$$f \sim f$$

and in Problem 19–4, we have shown that

$$\text{if } f \sim g, \text{ then } g \sim f.$$

It remains to be shown that, if $f \sim g$ and $g \sim k$, then $f \sim k$. Since $f \sim g$, there is a homotopy

$$h_1 : X \times [0, 1] \to Y$$

such that

$$h_1(x, 0) = g(x) \quad \text{and}$$

$$h_1(x, 1) = f(x) \text{ for all } x \in X.$$

Similarly, since $g \sim k$, there is a homotopy

$$h_2 : X \times [0, 1] \to Y$$

such that

$$h_2 (x, 0) = \mathrm{k}(x) \quad \text{and}$$

$$h_2 (x, 1) = g(x) \text{ for all } x \in X.$$

Let us define mapping

$$h : X \times [0, 1] \to Y$$

by

$$h(x,t) = \begin{cases} h_2(x, 2t) & \text{for} \quad 0 \le t \le \frac{1}{2} \\ h_1(x, 2t - 1) & \text{for} \quad \frac{1}{2} \le t \le 1 \end{cases}$$

for all $x \in X$.

Note that

$$h(x, 0) = h_2(x, 0) = k(x)$$

$$h(x, 1) = h_1(x, 1) = f(x)$$

$$h(x, {}^1/_2) = h_1(x, 0) = h_2(x, 1) = g(x).$$

Mapping $h(x, t)$ is well-defined and since both h_1 and h_2 are continuous, so is $h(x, t)$.

Thus, $h(x, t)$ is a homotopy between f and k.

The relation $\sim$ is an equivalence relation on Y^X.

● PROBLEM 19-6

1. Why do the homotopy classes of Y^X form a partition of Y^X?

2. Can you find any relationship between the extension of functions and functions being homotopic?

SOLUTION:

DEFINITION OF A HOMOTOPY CLASS

The family of all continuous functions from X into Y, which are homotopic to a continuous function f, is called the homotopy class of f.

■

The homotopy classes of Y^X form a partition of Y^X because $\sim$ is an equivalence relation on Y^X (See Problem 19–5).

2. Suppose two continuous functions f and g are given, $f, g \in Y^X$. We define function h as follows:

$$h : (X \times \{0\}) \cup (X \times \{1\}) \rightarrow Y$$

$$h(x, 0) = f(x) \quad \text{and} \quad h(x, 1) = g(x)$$

for all $x \in X$.

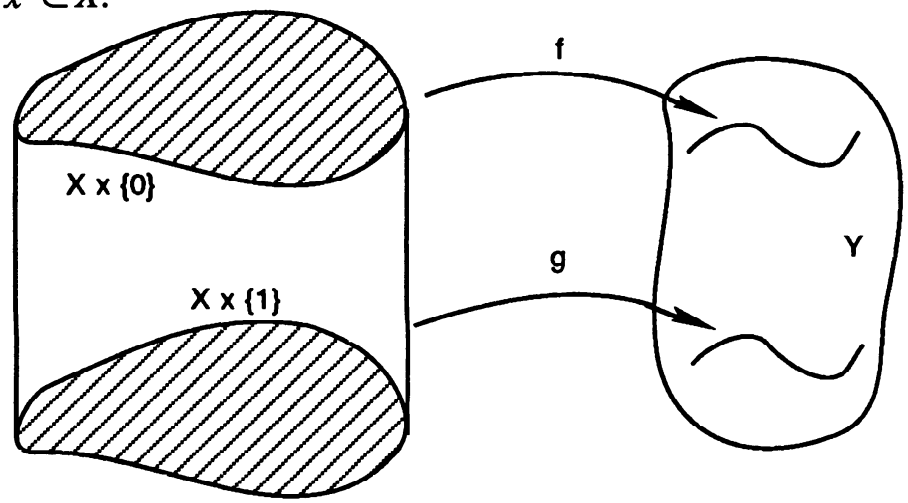


FIGURE 1

It is easy to see that f and g are homotopic, if and only if function h can be extended to a continuous function H

$$H : X \times [0, 1] \rightarrow Y$$

such that

$$H \mid (X \times \{0\}) \cup (X \times \{1\}) = h.$$

● PROBLEM 19–7

Why are any two paths in R^2 homotopic?

SOLUTION:

A continuous function from the interval [0, 1] into any space (X, T) is called a path in X. (See Figure 1.)

The space

$$[0, 1] \times [0, 1]$$

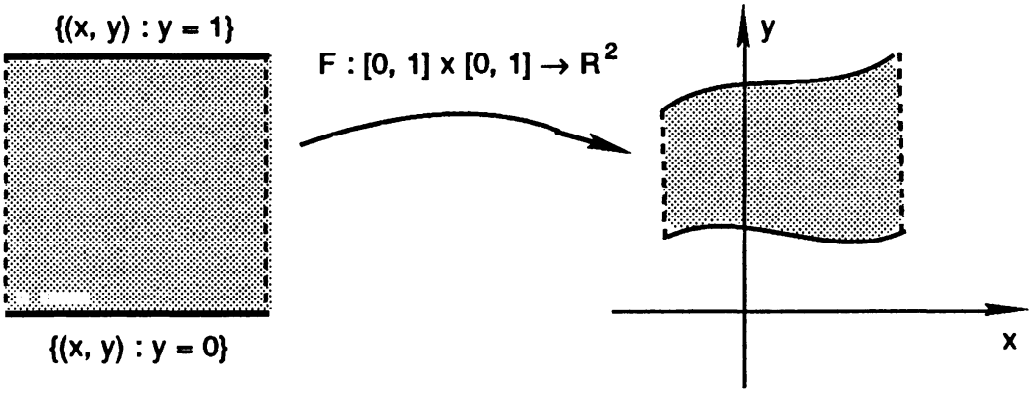
with the product topology is a normal space and its subset

$$A = \{(x, y) : y = 1\} \cup \{(x, y) : y = 0\}$$

is a closed set.

If f is any continuous function from A into R^2, then, by Tietze's Extension Theorem, f has a continuous extension F.

Thus, any two paths in R^2 are homotopic.



● PROBLEM 19–8

In Problem 19–2, we have shown that the identity function $f_1 = i$ on the unit disk Y is homotopic to the function f_0, which maps the whole disk into (0, 0). Show that $f_1 = i$ is homotopic to the function

$$g : Y \to Y$$

defined by

$$g(x, y) = (0, 1/2)$$

for all $(x, y) \in Y$.

SOLUTION:

Let h denote the homotopy between $f_1 = i$ and f_0. We define

$$H((x,y),t) = \begin{cases} h((x,y),1-2t) & \text{for } 0 \le t \le \frac{1}{2} \\ (0,t-\frac{1}{2}) & \text{for } \frac{1}{2} \le t \le 1 \end{cases}.$$

Note, that $H((x, y), t)$ on the segment $[0, 1/2]$ contracts Y to (0, 0) and then on the segment $[1/2, 1]$, slides it from (0, 0) to $(0, 1/2)$ along the straight line between these two points.

Since $H \mid (Y \times [0, 1/2])$ and $H \mid (Y \times [1/2, 1])$ are continuous and

$$H((x, y), 1/2) = h((x, y), 0) = (0, 0)$$

is well-defined, H is continuous. Thus H is a homotopy, since

$$H \mid (Y \times \{0\}) = g$$

$$H \mid (Y \times \{1\}) = f_1 = i.$$

● PROBLEM 19-9

Show, that if the homotopic mappings f_0 and f_1 have values in the space Y and if g_0 and g_1 are defined on Y and homotopic, then the mappings $g_0 \circ f_0$ and $g_1 \circ f_1$ are homotopic, i.e., prove that

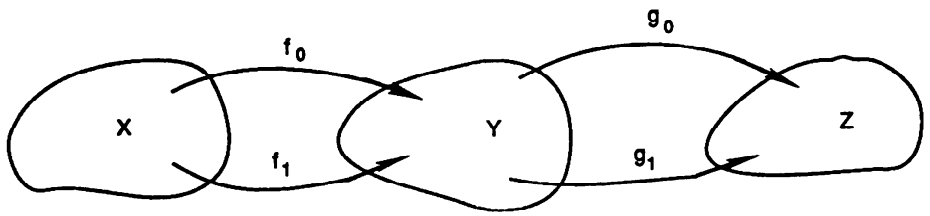


FIGURE 1

$$\begin{pmatrix} f_0 \sim f_1 \\ g_0 \sim g_1 \end{pmatrix} \Rightarrow (g_0 \circ f_0 \sim g_1 \circ f_1).$$

SOLUTION:

Since $f_0 \sim f_1$, a continuous function

$$h_1(x, t) \in Y$$

exists, such that

$$h_1(x, 0) = f_0(x) \; ; \; h_1(x, 1) = f_1(x).$$

Since $g_0 \sim g_1$, a continuous function

$$h_2(y, t) \in Z$$

exists, such that

$$h_2(y, 0) = g_0(y) \; ; \; h_2(y, 1) = g_1(y).$$

To prove that

$$g_0 \circ f_0 \sim g_1 \circ f_1$$

we have to show that a continuous function

$$H(x, t) \in Z$$

exists, such that

$$H(x, 0) = g_0 \circ f_0 \quad \text{and} \quad H(x, 1) = g_1 \circ f_1.$$

Let us define

$$H(x, t) = h_2(h_1(x, t), t)$$

then

$$H(x, 0) = h_2(h_1(x, 0), 0) = h_2(f_0(x), 0) =$$

$$= g_0(f_0(x)) = g_0 \circ f_0(x)$$

and

$$H(x, 1) = h_2(h_1(x, 1), 1) = h_2(f_1(x), 1) =$$

$$= g_1(f_1(x)) = g_1 \circ f_1(x).$$

Since h_1 and h_2 are continuous, H is also continuous. Thus, H is a homotopy between $g_0 \circ f_0$ and $g_1 \circ f_1$.

● PROBLEM 19-10

A homeomorphism of [0, 1] into any space (X, T) is called an arc. A continuous function of [0, 1] into any space is called a path. In Problem 19–7, we showed that any two paths in R^2 are homotopic. Thus, we can state that any two arcs in R^2 are homotopic. Let A represent any point in R^2. Show that two arcs in $R^2 - \{A\}$ are not necessarily homotopic.

SOLUTION:

Let a_1 and a_2 denote distinct arcs in $R^2 - \{A\}$, such that

$$a_1(0) = a_2(0) \quad \text{and} \quad a_1(1) = a_2(1).$$

Obviously a_1 and a_2 are homotopic in R^2. We choose a_1 and a_2 in such a way, that A is a point in the area bounded by a_1 and a_2. (See Figure 1.)

Then a_1 and a_2 are not homotopic in $R^2 - \{A\}$ because a_1 cannot be continuously transformed into a_2.

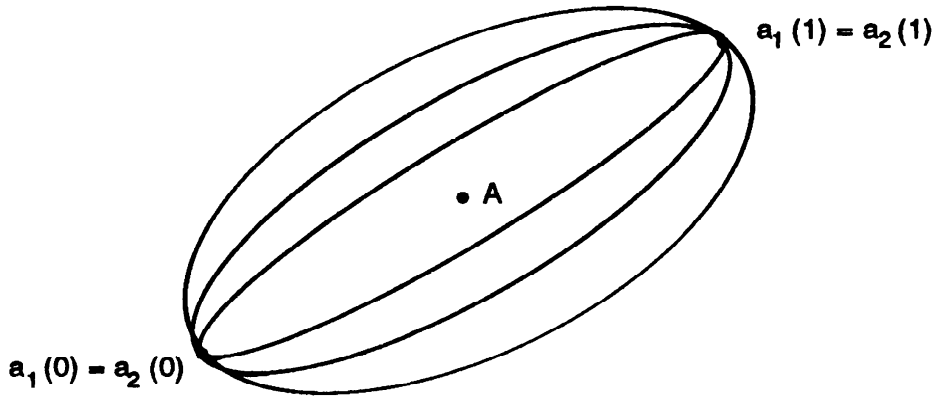


FIGURE 1

● PROBLEM 19-11

When are two loops a_0 and a_1 in X with base point x_0 homotopic relative to x_0?

SOLUTION:

A continuous function a

$$a : [0, 1] \to X$$

such that

$$a(0) = a(1) = x_0$$

is called a loop in X with base point y_0.

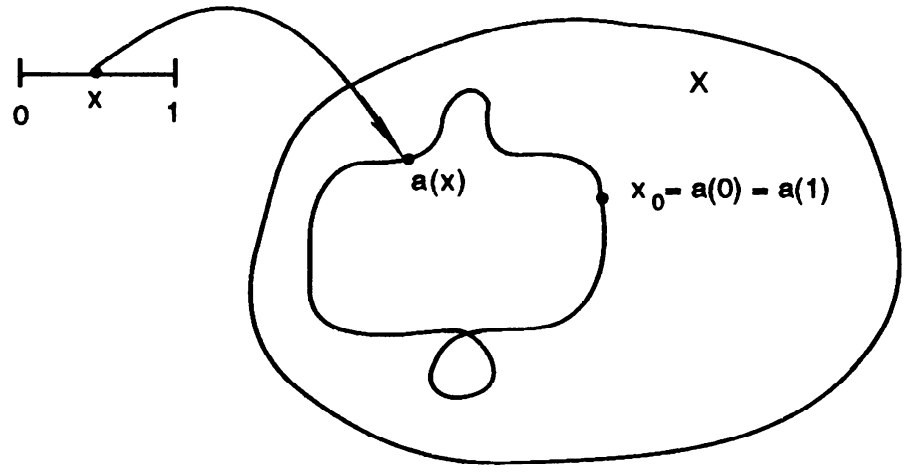


FIGURE 1

Two loops a_0 and a_1 in X with base point x_0 are said to be homotopic

relative to x_0, if a homotopy H between a_0 and a_1 exists, i.e.,

$$H(x, t) \in X;\ x, t \in [0, 1]$$

$$H(x, 0) = a_0\ ;\ H(x, 1) = a_1$$

such that, for each $t \in [0, 1]$

$$H(0, t) = H(1, t) = x_0.$$

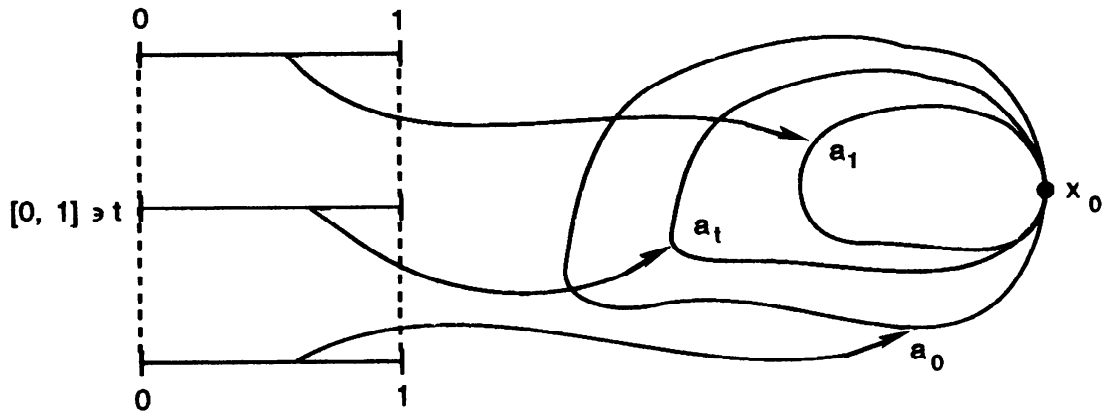


FIGURE 2

$$a_t(0) = a_t(1) = x_0$$

That is, for each $t \in [0, 1]$

$$H \mid [0, 1] \times \{t\}$$

is a loop in X with base point x_0. See Figure 2.

● PROBLEM 19–12

Let $L(X, x_0)$ denote the set of all loops in X with base point x_0. Show that the relation "homotopic relative to x_0 to" (denoted by ~) defined on $L(X, x_0)$, is an equivalence relation.

SOLUTION:

In Problem 19–5, we showed that "f, homotopic to g", is an equivalence relation on Y^X.

For any

$$a \in L(X, x_0)$$

a is homotopic relative to x_0 to a

$$a \sim a.$$

Let

$$a \sim b.$$

Then, the homotopy H exists between a and b, such that

$$H(0, t) = H(1, t) = x_0.$$

Setting

$$H'(x, t) = H(x, 1 - t)$$

we obtain homotopy between b and a, such that

$$H'(0, t) = H'(1, t) = x_0.$$

Hence,

$$b \sim a.$$

Suppose $a \sim b$ and $b \sim c$.

Homotopies exist, such that

$$h_1(0, t) = h_1(1, t) = x_0$$

$$h_1(0, t) = h_2(1, t) = x_0.$$

Then

$$h(x,t) = \begin{cases} h_2(x, 2t) & \text{for} \quad 0 \le t \le \frac{1}{2} \\ h_1(x, 2t - 1) & \text{for} \quad \frac{1}{2} \le t \le 1 \end{cases}$$

is a homotopy between a and c, such that

$$h(0, t) = h(1, t) = x_0.$$

● PROBLEM 19-13

Let $L(X, x_0)$ denote the set of all loops in X with base point x_0. For any a_1, $a_2 \in L(X, x_0)$, we define an operation $\oplus$ as follows:

$$(a_1 \oplus a_2)(t) = \begin{cases} a_1(2t), & 0 \le t \le \frac{1}{2} \\ a_2(2t - 1), & \frac{1}{2} \le t \le 1 \end{cases}.$$

Show that

$$a_1 \oplus a_2 \in L(X, x_0).$$

See Figure 1.

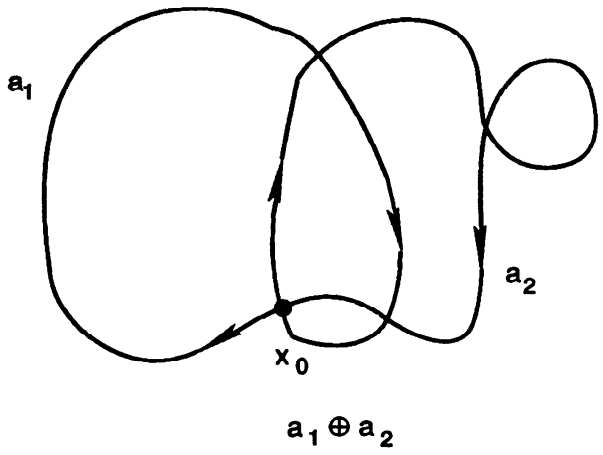


FIGURE 1

SOLUTION:

Loop $a_1 \oplus a_2$ is formed by moving arround a_1 once and then around a_2 once.

$$a_1 \oplus a_2 : [0, 1] \to X.$$

$a_1 \oplus a_2$ is continuous on $[0, {}^1/_2]$ and on $[{}^1/_2, 1]$ because both a_1 and a_2 are continuous and at $t = {}^1/_2$

$$(a_1 \oplus a_2)\,({}^1/_2) = a_1(1) = x_0 = a_2(0).$$

Also

$$(a_1 \oplus a_2)\,(0) = a_1(0) = x_0$$

$$(a_1 \oplus a_2)\,(1) = a_2(1) = x_0.$$

Hence,

$$a_1 \oplus a_2 \in L(X, x_0).$$

● PROBLEM 19-14

We proved that ~ (is homotopic relative y_0 to) is an equivalence relation on $L(X, x_0)$. We denote

$$\frac{L(X,x_0)}{\sim} = \pi(X,x_0)$$

the set of homotopy relative to x_0 classes of $L(X,x_0)$. If $a \in L(X, x_0)$, then by

$$[a] \in \pi(X, x_0)$$

we denote the equivalence class of a.

Justify the following definition:

$$[a_1] \oplus [a_2] = [a_1 \oplus a_2].$$

SOLUTION:

We define $[a_1] \oplus [a_2]$ to be the equivalence class of the loop $a_1 \oplus a_2$. In Problem 19–13, we proved that $a_1 \oplus a_2$ is a loop.

$$a_1 \oplus a_2 \in L(X, x_0).$$

Taking its homotopy equivalence class $[a_1 \oplus a_2]$, we obtain an element of $\pi(X, x_0)$.

To make sure that $[a_1] \oplus [a_2] = [a_1 \oplus a_2]$ is a well-defined operation on $\pi(X, x_0)$, it must be shown that this definition is independent of the representatives a_1 and a_2 that we choose. This will be done in the next problem.

● PROBLEM 19–15

Let

$$a_1, a_2, a_3 \in L(X, x_0).$$

Show, that if $a_1 \sim a_3$, then

$$a_1 \oplus a_2 \sim a_3 \oplus a_2.$$

SOLUTION:

Since $a_1 \sim a_3$, a homotopy relative to x_0 exists between a_1 and a_3, such that

$$h : [0, 1] \times [0, 1] \to X.$$

We define

$$H : [0, 1] \times [0, 1] \to X$$

by

$$H(r,t) = \begin{cases} h(2r,t), & 0 \le r \le \frac{1}{2} \\ a_2(2r-1), & \frac{1}{2} \le r \le 1 \end{cases}$$

Thus,

$$H(r,0) = \begin{cases} h(2r,0) = a_1(2r), & 0 \le r \le \frac{1}{2} \\ a_2(2r-1), & \frac{1}{2} \le r \le 1 \end{cases}$$

and

$$H(r, 0) = (a_1 \oplus a_2)\,(r) \text{ for } r \in [0, 1].$$

Also

$$H(r, 1) = (a_3 \oplus a_2)\,(r) \text{ for } r \in [0, 1].$$

We have

$$H(0, t) = h(0, t) = x_0$$

$$H(1, t) = a_2(1) = x_0.$$

Continuity of H can easily be shown. Hence,

$$a_1 \oplus a_2 \sim a_3 \oplus a_2.$$

● PROBLEM 19–16

Show that the operation

$$[a_1] \oplus [a_2] = [a_1 \oplus a_2] \tag{1}$$

defined in Problem 19–14 is well-defined.

SOLUTION:

Suppose $a_1 \sim a_3$ and $a_2 \sim a_4$. In Problem 19–15, we proved that

$$a_1 \oplus a_2 \sim a_3 \oplus a_2. \tag{2}$$

Similary, we can show that

$$a_1 \oplus a_2 \sim a_1 \oplus a_4. \tag{3}$$

Thus,

$$[a_1 \oplus a_2] = [a_1] \oplus [a_2] = [a_3 \oplus a_2] =$$

$$= [a_3] \oplus [a_2] \tag{4}$$

and

$$[a_1] \oplus [a_2] = [a_1] \oplus [a_4]. \tag{5}$$

We have

$$a_1 \oplus a_2 \sim a_3 \oplus a_2 \sim a_3 \oplus a_4. \tag{6}$$

Hence, if $[a_1] = [a_3]$ and $[a_2] = [a_4]$, then

$$[a_1] \oplus [a_2] = [a_3] \oplus [a_4]. \tag{7}$$

● PROBLEM 19–17

Draw the homotopy $H(r, t)$ described in Problem 19–15. Explain why $H(r, t)$ is continuous on $[0, 1] \times [0, 1]$.

SOLUTION:

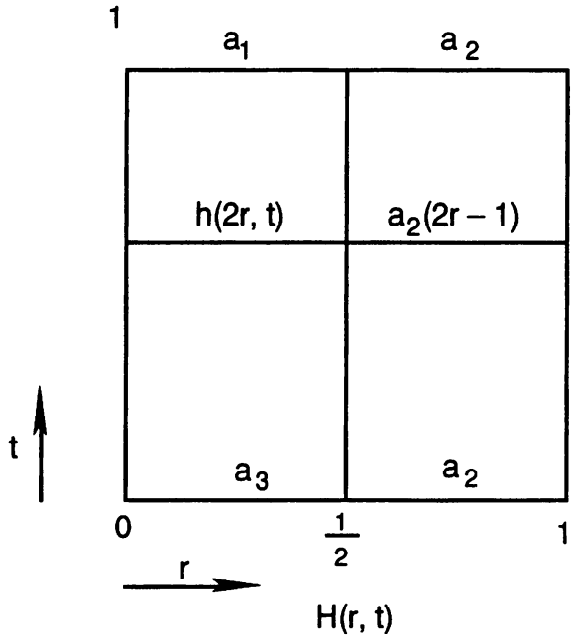


FIGURE 1

On $[0, 1/2] \times [0, 1]$, we continuously deform a_1 into a_3, while a_2 remains fixed on $[1/2, 1] \times [0, 1]$.

For $r = 1/2$, we have

$$H(1/2, t) = h(1, t) = a_2(1) = x_0$$

$$H(1/2, t) = a_2(0) = x_0.$$

The homotopy $H(r, t)$ is continuous on $[0, 1/2] \times [0, 1]$ and on $[1/2, 1] \times [0, 1]$, hence, it is continuous on $[0, 1] \times [0, 1]$.

Show that operation $\oplus$, defined on $\pi(X, x_0)$, is an associative operation; that is, show that

$$([a_1] \oplus [a_2]) \oplus [a_3] = [a_1] + ([a_2] \oplus [a_3]).$$

SOLUTION:

Let $[a_1]$, $[a_2]$, $[a_3]$ represent any elements of $\pi(X, x_0)$. Since operation $\oplus$ defined on $\pi(X, x_0)$ does not depend on the choice of representatives, it suffices to show, that

$$(a_1 \oplus a_2) \oplus a_3 \sim a_1 \oplus (a_2 \oplus a_3).$$

By definition

$$((a_1 \oplus a_2) \oplus a_3)(r) = \begin{cases} (a_1 \oplus a_2)(2r); & 0 \le r \le \frac{1}{2} \\ a_3(2r-1); & \frac{1}{2} \le r \le 1 \end{cases}.$$

Thus,

$$((a_1 \oplus a_2) \oplus a_3)(r) = \begin{cases} a_1(4r); & 0 \le r \le \frac{1}{4} \\ a_2(4r-1); & \frac{1}{4} \le r \le \frac{1}{2} \\ a_3(2r-1); & \frac{1}{2} \le r \le 1 \end{cases}.$$

Similarly,

$$(a_1 \oplus (a_2 \oplus a_3)(r) = \begin{cases} a_1(2r); & 0 \le r \le \frac{1}{2} \\ (a_2 \oplus a_3)(2r-1); & \frac{1}{2} \le r \le 1 \end{cases} =$$

$$= \begin{cases} a_1(2r); & 0 \le r \le \frac{1}{2} \\ a_2(4r-2); & \frac{1}{2} \le r \le \frac{3}{4} \\ a_3(4r-3); & \frac{3}{4} \le r \le 1 \end{cases}.$$

The homotopy between $(a_1 \oplus a_2) \oplus a_3$ and $a_1 \oplus (a_2 \oplus a_3)$ is defined as follows:

$$H(r,t) = \begin{cases} a_1(\frac{4r}{1+t}); & 0 \le r \le \frac{1}{4}(1+t) \\ a_2(4r-1-t); & \frac{1}{4}(1+t) \le r \le \frac{1}{4}(2+t). \\ a_3(1-\frac{4(1-r)}{2-t}); & \frac{1}{4}(2+t) \le r \le 1 \end{cases}$$

One can directly compute that $H(r, t)$ is indeed the homotopy between $(a_1 \oplus a_2) \oplus a_3$ and $a_1 \oplus (a_2 \oplus a_3)$.

● PROBLEM 19–19

Draw the homotopy defined in Problem 19–18. Explain why it is a suitable homotopy.

SOLUTION:

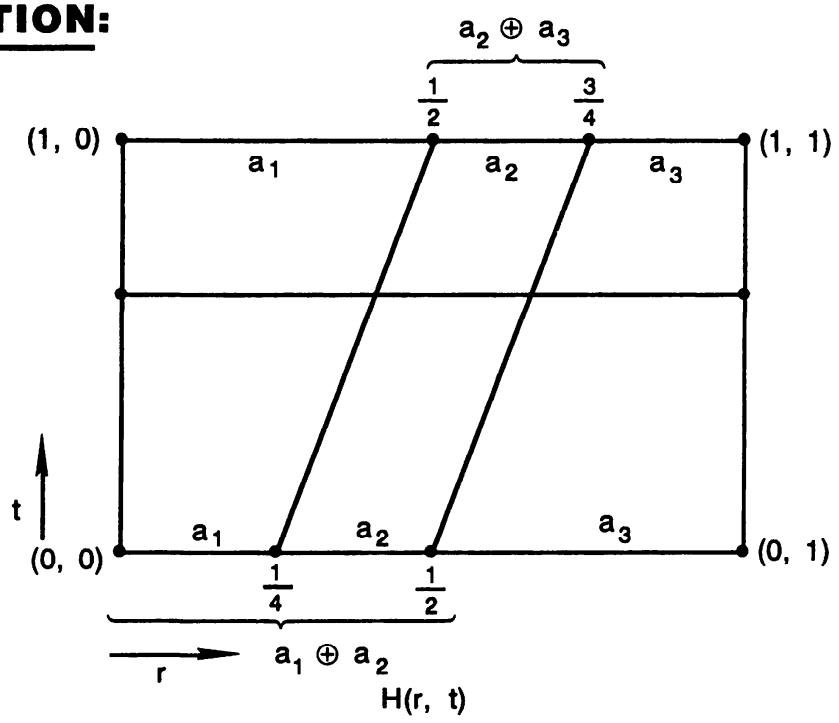


FIGURE 1

The homotopy $H(r, t)$ was defined as follows:

$$H(r,t) = \begin{cases} a_1(\frac{4r}{1+t}); & 0 \le r \le \frac{1}{4}(1+t) \\ a_2(4r-1-t); & \frac{1}{4}(1+t) \le r \le \frac{1}{4}(2+t). \\ a_3(1-\frac{4(1-r)}{2-t}); & \frac{1}{4}(2+t) \le r \le 1 \end{cases}$$

Note that for $t = 0$

$$H(r,0) = \begin{cases} a_1(4r);\ 0 \le r \le \frac{1}{4} \\ a_2(4r-1); \quad \frac{1}{4} \le r \le \frac{1}{2} \\ a_3(2r-1); \quad \frac{1}{2} \le r \le 1 \end{cases}$$

which is $((a_1 \oplus a_2) \oplus a_3)\,(r)$.

Similarly, for $t = 1$, we have

$$H(r,1) = \begin{cases} a_1(2r);\ 0 \le r \le \frac{1}{2} \\ a_2(4r-2); \quad \frac{1}{2} \le r \le \frac{3}{4} \\ a_3(4r-3); \quad \frac{3}{4} \le r \le 1 \end{cases}$$

which is $(a_1 \oplus (a_2 \oplus a_3))\,(r)$. It is evident that $H(r, t)$ is continuous.

● PROBLEM 19-20

Let

$$f: [0, 1] \to x_0 \in X$$

denote a constant function that maps $[0, 1]$ onto $x_0 \in X$. Show that $[f]$ is an identity for $\pi(X, x_0)$ with respect to $\oplus$.

SOLUTION:

We shall prove, that for any $[a] \in \pi(X, x_0)$,

$$[a] \oplus [f] = [f] \oplus [a] = [a].$$

By definition

$$[a] \oplus [f] = [a \oplus f]$$

and

$$(a \oplus f)(r) = \begin{cases} a(2r); & 0 \le r \le \frac{1}{2} \\ f(2r-1); & \frac{1}{2} \le r \le 1 \end{cases}.$$

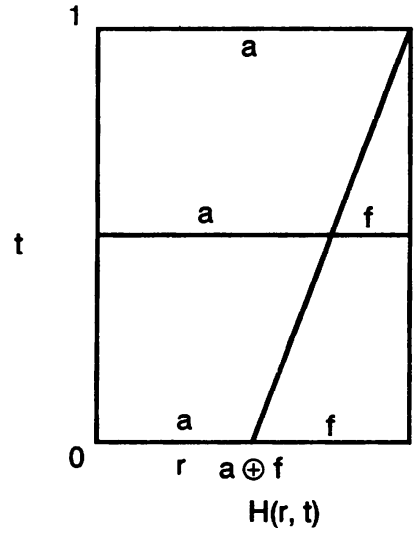


FIGURE 1

The figure illustrates the homotopy between $a \oplus f$ and a. Hence,

$$[a] \oplus [f] = [a].$$

Similarly, we show that

$$[f] \oplus [a] = [a].$$

● PROBLEM 19-21

Let a denote a loop in X with base point x_0, $a \in L(X, x_0)$. Define an inverse loop a^{-1}.

SOLUTION:

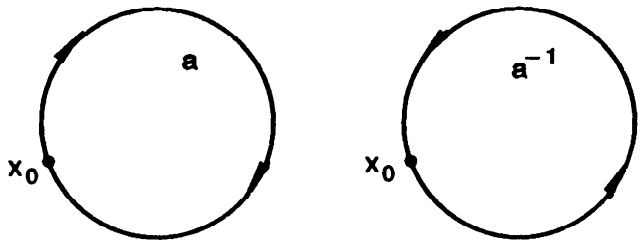


FIGURE 1

We define a^{-1} by

$$a^{-1}(r) = a(1 - r)$$

for each $r \in [0, 1]$.

Then

$$a^{-1}(0) = a(1) = x_0$$

and

$$a^{-1}(1) = a(0) = x_0.$$

Also, since a is continuous, so is a^{-1}, hence,

$$a^{-1} \in L(X, x_0).$$

Note that a^{-1} has the shape of a, but is drawn in the opposite direction.

● PROBLEM 19-22

Define the fundamental group of the space X, given that $x_0 \in X$.

SOLUTION:

A topological space (X, T) is given and $x_0 \in X$. $L(X, x_0)$ is the set of all loops in X with base point x_0.

On the set of homotopy, the classes of $L(X, x_0)$ denoted by $\pi(X, x_0)$, we defined operation $\oplus$, which is associative and has an identity $[f]$ with respect to x.

It is easy to prove, that

$$a \oplus a^{-1} \sim a^{-1} \oplus a \sim f$$

by using the homotopy between $a \oplus a^{-1}$ and f

$$H(r,t) = \begin{cases} a(2r(1-t)); & 0 \le r \le \frac{1}{2} \\ a(2(1-r)(1-t)); & \frac{1}{2} \le r \le 1 \end{cases}.$$

Similarly, we can define the homotopy between $a^{-1} \oplus a$ and f. Thus, the inverse of $[a] \in \pi(X, x_0)$ is $[a^{-1}]$.

The space $\pi(X, x_0)$ with operation $\oplus$ is a group with identity $[f]$. Each $[a] \in \pi(X, x_0)$ has its inverse $[a^{-1}] \in \pi(X, x_0)$.

DEFINITION

$(\pi(X, x_0), \oplus)$ is called the fundamental group of the space X based on $x_0 \in X$.

■

Consider the space

$$X = \{(x, y) : x^2 + y^2 = 1\} \subset R^2.$$

Let x_0 represent any point of X and a the loop $a \subset X$, which goes once around the circle in the clockwise direction. For each integer $n \in Z$ and each loop $a \in L(X, x_0)$, we define

$$na = \begin{cases} \overbrace{a \oplus a \oplus \ldots \oplus a}^{n \text{ times}} & \text{for } n \text{ positive} \\ f & \text{for } n = 0 \\ \underbrace{a^{-1} \oplus a^{-1} \oplus \ldots a^{-1}}_{-n \text{ times}} & \text{for } n \text{ negative} \end{cases}$$

where

$$f : [0, 1] \to x_0.$$

Find the fundamental group of X based on x_0.

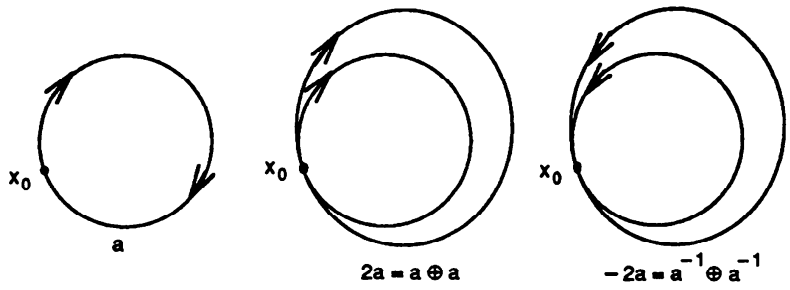


FIGURE 1

SOLUTION:

Note, that

$$m\,a \sim n\,a$$

if and only if $m = n$. For every loop $a_1 \in L(X, x_0)$, there is $n \in Z$, such that

$$a_1 \in [n\,a].$$

Thus, there is an isomorphism h between

$$(\pi(X, x_0), \oplus) \text{ and } (Z, +)$$

given by

$$h([n\,a]) = n.$$

Group $(\pi(X, x_0), \oplus)$ is generated by one element $[a]$.

Find the fundamental group based on x_0 of the space X

$$X = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \subset R^3$$

$x_0 \in X$.

SOLUTION:

Let A represent any point of X. Then $X - \{A\}$ is homeomorphic to R^2 and therefore, a contractible space. The plane λ is tangent to X at A'. A' is the point of X antipodal to A.

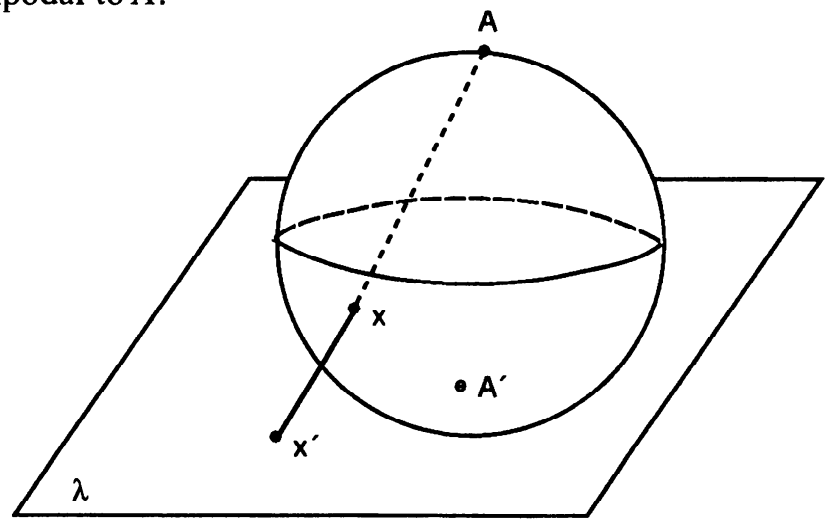


FIGURE 1

The projection of a point $x \in X - \{A\}$ is a point $x' \in \lambda$ determined by the intersection of a straight line $\overline{Ax}$ with the plane λ. Thus, $X - \{A\}$ and λ are homeomorphic.

If a is any loop in X and A is any point in $X - a$, then a is a loop in R^2 (remember R^2 and $X - \{A\}$ are homeomorphic).

Thus, a is homotopic to the identity function f of $\pi(X, x_0)$. Hence, any loop in X based on $x_0 \in X$ is homotopic to f.

Therefore, $\pi(X, x_0)$ consists only of $[f]$.

GLOSSARY OF SYMBOLS AND ABBREVIATIONS

Symbol	Meaning	See Problem where symbol appears
$\alpha, \beta, \gamma, \delta, \ldots$	sentences	1-1
0,1	logical values	1-1
$\equiv$	equivalent	1-1
$\vee$	or	1-1
$\wedge$	and	1-1
$'$	not	1-1
$\Rightarrow$	implies	1-5
$\Leftrightarrow$	if and only if	1-5
iff	if and only if	1-5
$\mp$	symmetric difference	1-11
■	end of a proof, theorem, or definition	1-6
$a \in A$	a is an element of A	2-1
$a \notin A$	a is not an element of A	2-1
$\{a, b, c, \ldots\}$	set consisting of elements $a, b, c, \ldots$	2-1
$\{x : p(x)\}$	set of all x such that $p(x) \equiv 1$	2-1
N	set of all natural numbers	
Z	set of all integers	
Q	set of all rational numbers	
R	set of all real numbers	
C	set of all imaginary numbers	
$A \subset B$	A is a subset of B	2-3
$A \cup B$	union of A and B	2-4
$A \cap B$	intersection of A and B	2-4
$A - B$	difference of A and B	2-4
ϕ	empty set	2-7
$P(A)$ (or 2^A)	power set of A	2-8
A^C (or $X - A$)	complement of A	2-12

$A \div B$	symmetric difference of A and B	2-18
$\exists$	there exists	3-2
$\forall$	for each	3-2
(x_n)	sequence consisting of elements $x_1, x_2, \ldots$	3-4
$\lim_{n \to \infty} x_n$	limit of (x_n)	3-4
(a, b)	ordered pair	3-5
$A \times B$	Cartesian product	3-5
$]a, b[$ (or (a, b))	open interval	3-9
$\bigcup_\alpha A_\alpha$	union of a family of sets	3-10
$\bigcap_\alpha A_\alpha$	intersection of a family of sets	3-10
$[a, b]$	closed interval	3-11
$a \rho b$	a is in relation ρ with b	3-13
ρ^{-1}	inverse relation	3-14
Δ	diagonal of a set	3-14
$\rho \circ \lambda$	composition of relations	3-15
R	equivalence relation	3-17
$\frac{X}{R}$	quotient space	3-23
$f : X \to Y$	function defined on X	4-1
$f \mid A$	restriction of f to A	4-3
$f \circ g$	composition of functions	4-5
1_A	identity function on A	4-8
A^B	set of all functions $f : B \to A$	4-22
χ_A	characteristic function of A	4-25
$\underset{\alpha}{X} A_\alpha$ (or $\prod_\alpha A_\alpha$)	Cartesian product of a family of sets	4-28
$A \sim B$	sets A and B are equivalent	5-1
card$A$ (or $\overline{\overline{A}}$)	cardinal number of set A	5-4
$\alpha + \beta$	sum of cardinal numbers	6-14
$\alpha \cdot \beta$	product of cardinal numbers	6-16

β^{α}	α^{th} power of β	6-23
H	Hilbert space	6-37
$(A, <)$	preordered set	7-4
$(A, \leq)$	partially ordered set	7-7
$=:$	equals by definition	7-10
$(A, \leq)$	totally ordered set	7-8
lub	the least upper bound	7-18
glb	the greatest lower bound	7-18
$A \simeq B$	order – isomorphic sets	8-2
ord $(A, \leq)$	ordinal number of a well-ordered set A	8-4
$d(x, y)$	metric	10-1
(X, d)	metric space	10-1
R^n	n-dimensional Euclidean space	10-3
$B_d(a, r)$	d-ball of radius r and center a	10-7
$\overline{B}(a, r)$	closed ball	10-14
(x_n)	sequence of points	10-17
$\lim_{n \to \infty} x_n$	limit of a sequence	10-17
$\delta(A)$	diameter of A	10-26
$\overline{A}$	closure of A	10-32
(X, T)	topological space	11-1
$T(A)$	topology generated by A	11-18
$N_0(x)$	neighborhood of x	11-27
$\overline{A}$	closure of A	11-27
A^1	derived set of A	11-31
Int(A)	interior of A	11-34
Fr(A)	boundary of A	11-35
T_Y	induced topology	11-45
$X \cong Y$	homeomorphic spaces	12-30
$<U_\alpha>$	"slice" of Cartesian product $\underset{\alpha}{\times} X_\alpha$	14-13

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