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THE TOPOLOGY PROBLEM SOLVER®

REGISTERED TRADEMARK

A Complete Solution Guide to Any Textbook

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WHAT THIS BOOK IS FOR

Students have generally found topology a difficult subject to understand and learn. Despite the publication of hundreds of textbooks in this field, each one intended to provide an improvement over previous textbooks, students continue to remain perplexed as a result of the numerous conditions that must often be remembered and correlated in solving a problem. Various possible interpretations of terms used in topology have also contributed to much of the difficulties experienced by students.

In a study of the problem, REA found the following basic reasons underlying students' difficulties with topology taught in schools:

(a) No systematic rules of analysis have been developed which students may follow in a step-by-step manner to solve the usual problems encountered. This results from the fact that the numerous different conditions and principles which may be involved in a problem, lead to many possible different methods of solution. To prescribe a set of rules to be followed for each of the possible variations, would involve an enormous number of rules and steps to be searched through by students, and this task would perhaps be more burdensome than solving the problem directly with some accompanying trial and error to find the correct solution route.

(b) Textbooks currently available will usually explain a given principle in a few pages written by a professional who has an insight in the subject matter that is not shared by students. The explanations are often written in an abstract manner which leaves the students confused as to the application of the principle. The explanations given are not sufficiently detailed and extensive to make the student aware of the wide range of applications and different aspects of the principle being studied. The numerous possible variations of principles and their applications are usually not discussed, and it is left for the students to discover these for themselves while doing exercises. Accordingly, the average student is expected to rediscover that which has been long known and practiced, but not published or explained extensively.

(c) The examples usually following the explanation of a topic are too few in number and too simple to enable the student to obtain a thorough grasp of the principles involved. The explanations do not provide sufficient basis to enable a student to solve problems that may be subsequently assigned for homework or given on examinations.

The examples are presented in abbreviated form which leaves out much material between steps, and requires that students derive the omitted material

themselves. As a result, students find the examples difficult to understand—contrary to the purpose of the examples.

Examples are, furthermore, often worded in a confusing manner. They do not state the problem and then present the solution. Instead, they pass through a general discussion, never revealing what is to be solved for.

Examples, also, do not always include diagrams/graphs, wherever appropriate, and students do not obtain the training to draw diagrams or graphs to simplify and organize their thinking.

(d) Students can learn the subject only by doing the exercises themselves and reviewing them in class, to obtain experience in applying the principles with their different ramifications.

In doing the exercises by themselves, students find that they are required to devote considerably more time to topology than to other subjects of comparable credits, because they are uncertain with regard to the selection and application of the theorems and principles involved. It is also often necessary for students to discover those "tricks" not revealed in their texts (or review books), that make it possible to solve problems easily. Students must usually resort to methods of trial-and-error to discover these "tricks," and as a result they find that they may sometimes spend several hours to solve a single problem.

(e) When reviewing the exercises in classrooms, instructors usually request students to take turns in writing solutions on the boards and explaining them to the class. Students often find it difficult to explain in a manner that holds the interest of the class, and enables the remaining students to follow the material written on the boards. The remaining students seated in the class are, furthermore, too occupied with copying the material from the boards, to listen to the oral explanations and concentrate on the methods of solution.

This book is intended to aid students in topology overcoming the difficulties described, by supplying detailed illustrations of the solution methods which are usually not apparent to students. The solution methods are illustrated by problems selected from those that are most often assigned for class work and given on examinations. The problems are arranged in order of complexity to enable students to learn and understand a particular topic by reviewing the problems in sequence. The problems are illustrated with detailed step-by-step explanations, to save the students the large amount of time that is often needed to fill in the gaps that are usually found between steps of illustrations in textbooks or review/outline books.

The staff of REA considers topology a subject that is best learned by allowing students to view the methods of analysis and solution techniques themselves. This approach to learning the subject matter is similar to that practiced in various scientific laboratories, particularly in the medical fields.

In using this book, students may review and study the illustrated problems at their own pace; they are not limited to the time allowed for explaining problems on the board in class.

When students want to look up a particular type of problem and solution, they can readily locate it in the book by referring to the index which has been extensively prepared. It is also possible to locate a particular type of problem by glancing at just the material within the boxed portions. To facilitate rapid scanning of the problems, each problem has a heavy border around it. Furthermore, each problem is identified with a number immediately above the problem at the right-hand margin.

To obtain maximum benefit from the book, students should familiarize themselves with the section, "How To Use This Book," located in the front pages.

Special thanks are due to Dr. Nathan Busch for his technical editing of the book.

MAX FOGIEL, Ph.D.
Program Director

HOW TO USE THIS BOOK

This book can be an invaluable aid to students in topology as a supplement to their textbooks. The book is subdivided into 19 chapters, each dealing with a separate topic. The subject matter is developed beginning with the fundamental concepts of sentence calculus, sets, mappings, and extending through metric and topological spaces, continuity, homeomorphisms, axioms, compactness, connectedness, and homotopy theory.

TO LEARN AND UNDERSTAND A TOPIC THOROUGHLY

1. Refer to your class text and read the section pertaining to the topic. You should become acquainted with the principles discussed there. These principles, however, may not be clear to you at that time.

2. Then locate the topic you are looking for by referring to the "Table of Contents" in front of this book, "The Topology Problem Solver."

3. Turn to the page where the topic begins and review the problems under each topic, in the order given. For each topic, the problems are arranged in order of complexity, from the simplest to the more difficult. Some problems may appear similar to others, but each problem has been selected to illustrate a different point or solution method.

To learn and understand a topic thoroughly and retain its contents, it will be generally necessary for students to review the problems several times. Repeated review is essential in order to gain experience in recognizing the principles that should be applied, and in selecting the best solution technique.

TO FIND A PARTICULAR PROBLEM

To locate one or more problems related to a particular subject matter, refer to either the index, located at the back of the book or the indexes found at the beginning of each chapter. In using the indexes, be certain to note that the numbers given refer to problem numbers, not page numbers. This arrangement of the indexes is intended to facilitate finding a problem more rapidly, since two or more problems may appear on a page.

If a particular type of problem cannot be found readily, it is recommended that the student refer to the "Table of Contents" in the front pages,

and then turn to the chapter which is applicable to the problem being sought. By scanning or glancing at the material that is boxed, it will generally be possible to find problems related to the one being sought, without consuming considerable time. After the problems have been located, the solutions can be reviewed and studied in detail. For this purpose of locating problems rapidly, students should acquaint themselves with the organization of the book as found in the "Table of Contents."

In preparing for an exam, locate the topics to be covered on the exam in the "Table of Contents," and then review the problems under those topics several times. This should equip the student with what might be needed for the exam.

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CHAPTER 1

INTRODUCTION TO SENTENCE CALCULUS

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● PROBLEM 1-1

Let α, β, γ, ... denote sentences, each of which has one of two logical values 0 or 1. We assign the value 0 to a false sentence and 1 to a true sentence. To express the fact that α is a true sentence we write

$$\alpha \equiv 1 \text{ or } \beta \equiv 0 \tag{1}$$

if β is false.

We consider here only sentences of mathematical nature, i.e., sentences which take values either 0 or 1. For example, the sentence: "two times two is four" is (no doubt) a true one. On the other hand, the statement: "Goethe is a great poet" is not a sentence even though most people would agree that he indeed was a great poet.

Define the sum, the product and negation of sentences and establish the truth tables for each case.

SOLUTION:

If α and β are two sentences, then the sentence "α or β," denoted by $\alpha \vee \beta$, is called the sum of α and β. The sentence "α and β," denoted by $\alpha \wedge \beta$, is called the product of α and β. Clearly, the sentence $\alpha \vee \beta$ is true if at least one of the components is a true proposition. It has $2 \times 2 = 4$ logical possibilities.

Table 1

α	β	$\alpha \vee \beta$
1	1	1
1	0	1
0	1	1
0	0	0

Sentence $\alpha \wedge \beta$ is true if both factors are true sentences.

Table 2

α	β	$\alpha \wedge \beta$
1	1	1
1	0	0
0	1	0
0	0	0

The sum and product of sentences are commutative and associative, i.e.

$$\alpha \vee \beta \equiv \beta \vee \alpha, \quad \alpha \wedge \beta \equiv \beta \wedge \alpha \tag{2}$$

$$\alpha \vee (\beta \vee \gamma) \equiv (\alpha \vee \beta) \vee \gamma, \quad \alpha \wedge (\beta \wedge \gamma) \equiv (\alpha \wedge \beta) \wedge \gamma. \tag{3}$$

The distributive law holds

$$\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma). \tag{4}$$

In the above formulas we used the equivalence sign $\equiv$. The equivalence $\alpha \equiv \beta$ holds, if and only if α and β have the same logical value. The operation of negation of a sentence α denoted by α' (or by $\rceil \alpha$ or $\sim \alpha$) is defined in such a way that if α is true, then α' is false, and if α is false, then α' is true.

Table 3

α	α'
0	1
1	0

From this we obtain the law of double negation

$$\alpha'' \equiv \alpha. \tag{5}$$

● PROBLEM 1-2

Using the logical notation introduced in Problem 1-1, write down two fundamental theorems of Aristotelian logic. Prove that $\alpha \vee 1 \equiv 1$, $\alpha \wedge 0 \equiv 0$.

SOLUTION:

The first theorem is called the law of the excluded middle (principium tertii exclusi) and, in classical logic, is formulated in the following manner:

From two contradictory sentences, one is true.

Let α represent a sentence, then α' is its contradiction. The law can be written as

$$\alpha \vee \alpha' \equiv 1. \tag{1}$$

The second theorem, called the law of contradiction, states: no sentence can

be true simultaneously with its negation. It can be written briefly as follows:

$$\alpha \wedge \alpha' \equiv 0. \tag{2}$$

Let us examine the truth table for the statement $\alpha \vee 1$:

α	$\alpha \vee 1$
1	1
0	1

Hence, $\alpha \vee 1$ is a statement which is always true.

$$\alpha \vee 1 \equiv 1. \tag{3}$$

Similarly, the truth table for $\alpha \wedge 0$ is

α	$\alpha \wedge 0$
1	0
0	0

and

$$\alpha \wedge 0 \equiv 0. \tag{4}$$

● PROBLEM 1-3

So far, we defined three operations on sentences: product, sum and negation. By applying DeMorgan's laws we see that the number of these fundamental operations can be reduced to two.

SOLUTION:

DeMorgan's laws play an important role in mathematical logic. They can be formulated as follows:

$$(\alpha \vee \beta)' \equiv \alpha' \wedge \beta' \tag{1}$$

$$(\alpha \wedge \beta)' \equiv \alpha' \vee \beta'. \tag{2}$$

The first law (1) asserts that, if it is not true that one of the sentences α and β is true, then both of these sentences are false and conversely.

Taking the negation of (1) we obtain:

$$(\alpha \vee \beta)'' \equiv \alpha \vee \beta \equiv (\alpha' \wedge \beta')'. \tag{3}$$

Equation (3) can be treated as a definition of the sum of sentences.

Here, the sum is defined in terms of two operations: product and negation. Hence, we limit the number of fundamental operations to two, product and negation. Similarly, taking the negation of (2) we obtain

$$\alpha \wedge \beta \equiv (\alpha' \vee \beta')'. \tag{4}$$

Now, the product is defined in terms of sum and negation. Again the number of fundamental operations is reduced to two. Product is defined with the aid of sum and negation. It is easy to verify DeMorgan's laws using the truth tables.

Table 1
for $(\alpha \vee \beta)' \equiv \alpha' \wedge \beta'$

α	β	$(\alpha \vee \beta)'$		$\alpha' \wedge \beta'$
1	1	0		0
1	0	0	$\equiv$	0
0	1	0		0
0	0	1		1

Table 2
for $(\alpha \wedge \beta)' \equiv \alpha' \vee \beta'$

α	β	$(\alpha \wedge \beta)'$		$\alpha' \vee \beta'$
1	1	0		0
1	0	1	$\equiv$	1
0	1	1		1
0	0	1		1

● PROBLEM 1-4

Generalize DeMorgan's laws for *n* components.

SOLUTION:

DeMorgan's laws state that

$$(\alpha_1 \vee \alpha_2)' \equiv \alpha_1' \wedge \alpha_2' \tag{1}$$

$$(\alpha_1 \wedge \alpha_2)' \equiv \alpha_1' \vee \alpha_2'. \tag{2}$$

We shall generalize (1) for the system of n sentences: $\alpha_1, \alpha_2, \ldots, \alpha_n$.

$$(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-1} \vee \alpha_n)' \equiv [(\alpha_1 \vee \alpha_2 \ldots \vee \alpha_{n-1}) \vee \alpha_n]' \equiv \quad (3)$$

The sum of the sentences is associative. Applying (1) to (3) we find

$$\equiv (\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-1})' \wedge \alpha_n' \equiv$$

$$\equiv [(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-2}) \vee \alpha_{n-1}]' \wedge \alpha_n' \equiv \quad (4)$$

Applying (1) again we obtain

$$(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-1})' \wedge \alpha_n' \equiv (\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_{n-2})' \wedge \alpha'_{n-1} \wedge \alpha'_n . \quad (5)$$

Observe that applying the above procedure to the sentence $(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n)'$, $(n-1)$ times we find

$$(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n)' \equiv \alpha_1' \wedge \alpha_2' \wedge \ldots \wedge \alpha_n'. \quad (6)$$

Now, take the negation of the product, $(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n)'$. Since the product is associative, we have

$$(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n)' \equiv [(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_{n-1}) \wedge \alpha_n]' \equiv \quad (7)$$

Applying (2) to (7) we obtain

$$\equiv (\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_{n-1})' \vee \alpha_n' \equiv \quad (8)$$

This procedure can be repeated again:

$$\equiv [(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_{n-2}) \wedge \alpha_{n-1}]' \vee \alpha_n' \equiv$$

$$\equiv (\alpha_1 \wedge \ldots \wedge \alpha_{n-2})' \vee \alpha_{n-1} \vee \alpha_n' \equiv \quad (9)$$

After $(n-1)$ steps we obtain

$$(\alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_n)' \equiv \alpha_1' \vee \alpha_2' \vee \ldots \vee \alpha_n'. \quad (10)$$

● PROBLEM 1-5

Another important operation on sentences is implication. It is defined by

$$(\alpha \Rightarrow \beta) \equiv (\alpha' \vee \beta) \quad (1)$$

The sign "$\Rightarrow$" is read "implies" or "if ..., then ...".

1. Write the truth table for the implication $\alpha \Rightarrow \beta$.

2. Explain the difference between "if ..., then ..." as defined in a strictly mathematical way by (1) and everyday "if ..., then ...."

3. Show that

$$\text{if } \alpha \Rightarrow \beta \text{ and } \beta \Rightarrow \alpha, \text{ then } \alpha \equiv \beta. \qquad (2)$$

SOLUTION:

Let us start with the truth table for $\alpha \Rightarrow \beta$.

Table 1

α	β	$(\alpha \Rightarrow \beta) \equiv (\alpha' \vee \beta)$
1	1	1
1	0	0
0	1	1
0	0	1

1. The implication $\alpha \Rightarrow \beta$ is always true, except when α is true ($\alpha \equiv 1$) and β is false ($\beta \equiv 0$).
2. Implication has properties similar to deduction. There is, however, a small but significant difference. Let α denote the statement "The sun is shining" and let β denote the statement "I am swimming." The statement "If the sun is shining, then I am swimming" makes sense. When is such a statement false?

When the sun is shining, I am not swimming, i.e., $\alpha \wedge \beta' \equiv 1$. But, that is equivalent to

$$1' \equiv 0 \equiv (\alpha \wedge \beta')' \equiv \alpha' \vee \beta \qquad (3)$$

in agreement with Table 1. In everyday language a sentence of the form "If α, then β" means that β is true whenever α is true. In such a case, the statement "If two times two is five, then Paris is the capital of Germany" is regarded as nonsense, since both components are false.

The same statement, in terms of formal language, not only makes sense but is also true. To each of the four possibilities shown in Table 1 (for $\alpha \Rightarrow \beta$), the system of formal language assigns either 0 or 1, even though two of the cases ($\alpha \equiv 0$ and $\beta \equiv 0$, $\alpha \equiv 0$ and $\beta \equiv 1$) appear to be pure nonsense in ordinary language.

3. To prove (2), note that

$$(\alpha \Rightarrow \beta) \text{ and } (\beta \Rightarrow \alpha) \equiv (\alpha' \vee \beta) \wedge (\beta' \vee \alpha) \equiv$$

$$\equiv (\alpha' \wedge \beta') \vee (\alpha' \wedge \alpha) \vee (\beta \wedge \beta') \vee (\beta \wedge \alpha) \equiv$$

$$\equiv (\alpha' \wedge \beta') \vee (\alpha \wedge \beta). \qquad (4)$$

The sentence $(\alpha' \wedge \beta') \vee (\alpha \wedge \beta)$ is true when both components α and β have the same logical value; i.e., they are both true or both false.

Hence,

$$\text{if } \alpha \Rightarrow \beta \text{ and } \beta \Rightarrow \alpha, \text{ then } \alpha \equiv \beta. \tag{5}$$

Symbols $\Leftrightarrow$ and $\equiv$ may be used interchangeably.

The symbol $\Leftrightarrow$ is read as: "if, and only if," and indicates implication in both directions. Sometimes, instead of if, and only if, we write iff.

Table 2 lists the symbols used so far.

Table 2

Symbol	Meaning
$\alpha, \beta, \gamma, \delta, \ldots$	logical sentences
$'$ or $\sim$ or $\rceil$	negation (not)
$\vee$	sum (or)
$\wedge$	product (and)
$\Rightarrow$	implication (if ..., then ...)
$\Leftrightarrow$	iff (if, and only if)

● PROBLEM 1-6

1. Prove the syllogism law (sometimes called the law of implication):

$$\text{If } \alpha \Rightarrow \beta \text{ and } \beta \Rightarrow \gamma, \text{ then } \alpha \Rightarrow \gamma. \tag{1}$$

2. Prove the law of contraposition, which is the basis of indirect proof (proof by contradiction, reductio ad absurdum):

$$(\alpha \Rightarrow \beta) \Leftrightarrow (\beta' \Rightarrow \alpha'). \tag{2}$$

3. Prove the law of Duns Scotus:

$$\text{If } \alpha \equiv 0, \text{ then } \alpha \Rightarrow \beta. \tag{3}$$

4. Prove the law of Clausius:

$$\text{If } \alpha' \Rightarrow \beta \text{ for each } \beta, \text{ then } \alpha \equiv 1. \tag{4}$$

SOLUTION:

1. Let us write (1) in the form

$$[(\alpha' \vee \beta) \wedge (\beta' \vee \gamma)] \Rightarrow (\alpha' \vee \gamma). \tag{5}$$

Note that the implication $\alpha \Rightarrow \beta$ is always true, except in the case when $\alpha \equiv 1$ and $\beta \equiv 0$. Suppose

$$(\alpha' \vee \beta) \wedge (\beta' \vee \gamma) \equiv 1 \quad \text{and} \quad \alpha' \vee \gamma \equiv 0. \tag{6}$$

From $\alpha' \vee \gamma \equiv 0$, we conclude that $\alpha \equiv 1$ and $\gamma \equiv 0$. Substituting into (6) we obtain

$$(0 \vee \beta) \wedge (\beta' \vee 0) \equiv 1 \quad \text{or} \quad \beta \wedge \beta' \equiv 1 \tag{7}$$

which is a contradiction.

2. $\Rightarrow$ We shall prove that

$$(\alpha \Rightarrow \beta) \Rightarrow (\beta' \Rightarrow \alpha') \tag{8}$$

which is equivalent to

$$(\alpha \Rightarrow \beta)' \vee (\beta' \Rightarrow \alpha') \tag{9}$$

and to

$$(\alpha' \vee \beta)' \vee (\beta \vee \alpha') \tag{10}$$

and to

$$(\alpha \wedge \beta') \vee (\alpha' \vee \beta). \tag{11}$$

The last statement is true for any combintation of α and β.

$\Leftarrow$

$$(\beta' \Rightarrow \alpha') \Rightarrow (\alpha \Rightarrow \beta) \tag{12}$$

or

$$(\beta \vee \alpha')' \vee (\alpha' \vee \beta) \tag{13}$$

or

$$(\beta' \wedge \alpha) \vee (\alpha' \vee \beta) \equiv 1. \tag{14}$$

■

3. Suppose $\alpha \equiv 0$, then $\alpha' \equiv 1$. Therefore

$$(\alpha \Rightarrow \beta) \equiv (\alpha' \vee \beta) \equiv 1. \tag{15}$$

4. If for $\beta \equiv 0$ and $\beta \equiv 1$ we have

$$(\alpha' \Rightarrow \beta) \equiv (\alpha \vee \beta) \equiv 1 \tag{16}$$

then

$$\alpha \equiv 1. \tag{17}$$

Throughout this book we will be using a box (■) to indicate the end of a proof, theorem, or definition.

Note that there is a shorter way to prove (2)

$$(\alpha \Rightarrow \beta) \equiv (\beta' \vee \alpha'). \tag{18}$$

Indeed,

$$(\alpha' \vee \beta) \equiv (\beta'' \vee \alpha'). \tag{19}$$

● PROBLEM 1-7

Prove that:

1. $\alpha \Rightarrow \alpha \vee \beta$ (1)

2. $\alpha \Rightarrow \alpha$ (2)

3. $\alpha \wedge \beta \Rightarrow \alpha \wedge \beta$ (3)

4. $\alpha \wedge \beta \Rightarrow \beta$ (4)

SOLUTION:

1. We shall apply the definition of implication (see Problem 1–5, (1))

$$(\alpha \Rightarrow \beta) \equiv \alpha' \vee \beta\,. \tag{5}$$

$$(\alpha \Rightarrow \alpha \vee \beta) \equiv \alpha' \vee \alpha \vee \beta \equiv (\alpha' \vee \alpha) \vee \beta \equiv$$

$$\equiv 1 \vee \beta \equiv 1. \tag{6}$$

Here, we applied the law of excluded middle (see Problem 1–2, (1)).

2. $$(\alpha \Rightarrow \alpha) \equiv \alpha' \vee \alpha \equiv 1. \tag{7}$$

3. In (2) we can replace α by $\alpha \wedge \beta$ to obtain (3).

4. $$(\alpha \wedge \beta \Rightarrow \beta) \equiv (\alpha \wedge \beta)' \vee \beta \equiv$$

$$\equiv \alpha' \vee \beta' \vee \beta \equiv 1. \tag{8}$$

In (8) we applied DeMorgan's law (See Problem 1–3, (2)).

Note that each of the above problems can be solved "automatically" using the truth tables.

Table

α	β	$\alpha \vee \beta$	$\alpha \wedge \beta$	$\alpha \Rightarrow \alpha \vee \beta$	$\alpha \Rightarrow \alpha$	$\alpha \wedge \beta \Rightarrow \alpha \wedge \beta$	$\alpha \wedge \beta \Rightarrow \beta$
1	1	1	1	1	1	1	1
1	0	1	0	1	1	1	1
0	1	1	0	1	1	1	1
0	0	0	0	1	1	1	1

Now it should be clear why this system is called a formal language.

● PROBLEM 1-8

1. Prove that:

$$(\alpha \Rightarrow \beta) \Rightarrow (\alpha \wedge \gamma \Rightarrow \beta \wedge \gamma). \quad (1)$$

2. Prove that:

$$(\alpha \equiv \beta) \equiv \quad (\alpha \wedge \beta) \vee (\alpha' \wedge \beta'). \quad (2)$$

SOLUTION:

1. From the definition of implication, we conclude that statement (1) is equivalent to

$$(\alpha \Rightarrow \beta)' \vee (\alpha \wedge \gamma \Rightarrow \beta \wedge \gamma) \quad (3)$$

which is equivalent to

$$(\alpha' \vee \beta)' \vee [(\alpha \wedge \gamma)' \vee (\beta \wedge \gamma)]. \quad (4)$$

From DeMorgan's law, we find that (4) can be replaced by

$$(\alpha \wedge \beta') \vee (\alpha' \vee \gamma') \vee (\beta \wedge \gamma). \quad (5)$$

Statement (5) is always true. Indeed, it is true when $\alpha' \equiv 1$ or $\gamma' \equiv 1$. Suppose $\alpha' \equiv 0$ and $\gamma' \equiv 0$, then $\alpha \equiv 1$ and $\gamma \equiv 1$ and for any β, $(1 \wedge \beta') \vee (\beta \wedge 1) \equiv 1$. Thus

$$(\alpha \wedge \beta') \vee (\alpha' \vee \gamma') \vee (\beta \wedge \gamma) \equiv 1. \quad (6)$$

2. Remember that $\equiv$ can be replaced by $\Leftrightarrow$ (See Problem 1-5). Then

$$(\alpha \equiv \beta) \equiv (\alpha \Leftrightarrow \beta) \equiv (\alpha \Rightarrow \beta) \wedge (\beta \Rightarrow \alpha) \equiv$$

$$\equiv (\alpha' \vee \beta) \wedge (\beta' \vee \alpha) \equiv (\alpha' \wedge \beta') \vee (\alpha' \wedge \alpha) \vee (\beta \wedge \beta') \vee (\alpha \wedge \beta) \equiv$$

$$\equiv (\alpha' \wedge \beta') \vee (\alpha \wedge \beta). \tag{7}$$

As always, the proofs can be carried out by means of the truth tables.

● PROBLEM 1-9

1. Prove that if $\alpha \Rightarrow \beta$, then

$$\alpha \wedge \beta \equiv \alpha \tag{1}$$

and

$$\alpha \vee \beta \equiv \beta. \tag{2}$$

2. Prove that if $\alpha \Rightarrow \beta$ and $\gamma \Rightarrow \delta$, then

$$\alpha \wedge \gamma \Rightarrow \beta \wedge \delta \tag{3}$$

$$\alpha \vee \gamma \Rightarrow \beta \vee \delta. \tag{4}$$

SOLUTION:

1. Since $\alpha \Rightarrow \beta$, we have

$$\alpha' \vee \beta \equiv 1 \tag{5}$$

and

$$\alpha \equiv \alpha \wedge 1 \equiv \alpha \wedge (\alpha' \vee \beta) \equiv (\alpha \wedge \alpha') \vee (\alpha \wedge \beta) \equiv$$

$$\equiv \alpha \wedge \beta. \tag{6}$$

Similarly, since $\alpha' \vee \beta \equiv 1$ we obtain

$$(\alpha' \vee \beta)' \equiv \alpha \wedge \beta' \equiv 1' \equiv 0 \tag{7}$$

and

$$\beta \equiv \beta \vee 0 \equiv \beta \vee (\alpha \wedge \beta') \equiv \alpha \vee \beta. \tag{8}$$

2. Since $\alpha \Rightarrow \beta$, then for any sentence γ we have

$$\alpha \wedge \gamma \Rightarrow \beta \wedge \gamma. \tag{9}$$

See Problem 1–8, (1).

Similarly, since $\gamma \Rightarrow \delta$, then

$$\beta \wedge \gamma \Rightarrow \beta \wedge \delta. \tag{10}$$

Combining (9) and (10) we obtain, by virture of Problem 1–6,

$$\alpha \wedge \gamma \Rightarrow \beta \wedge \delta. \tag{11}$$

To prove (4), note that

$$(\alpha \Rightarrow \beta) \Rightarrow (\beta' \Rightarrow \alpha') \tag{12}$$

$$(\gamma \Rightarrow \delta) \Rightarrow (\delta' \Rightarrow \gamma'). \tag{13}$$

Applying (11) to (12) and (13), we find

$$\beta' \wedge \delta' \Rightarrow \alpha' \wedge \gamma'. \tag{14}$$

Applying the law of contraposition (Problem 1–6), (2)) to (14) we obtain from (14)

$$(\alpha' \wedge \gamma')' \Rightarrow (\beta' \wedge \delta')' \tag{15}$$

or

$$(\alpha \vee \gamma) \Rightarrow (\beta \vee \delta). \tag{16}$$

● PROBLEM 1-10

Prove the law of absorption:

$$\alpha \equiv \alpha \vee (\alpha \wedge \beta) \equiv \alpha \wedge (\alpha \vee \beta) \tag{1}$$

and its more general version

$$\alpha \vee (\beta \wedge \gamma) \equiv (\alpha \vee \beta) \wedge (\alpha \vee \gamma). \tag{2}$$

SOLUTION:

We have

$$\alpha \wedge (\alpha \vee \beta) \equiv (\alpha \wedge \alpha) \vee (\alpha \wedge \beta) \equiv \alpha \vee (\alpha \wedge \beta). \tag{3}$$

Now we shall show that

$$\alpha \equiv \alpha \vee (\alpha \wedge \beta). \tag{4}$$

Indeed, if $\alpha \equiv 1$, then

$$\alpha \vee (\alpha \wedge \beta) \equiv 1 \vee (1 \wedge \beta) \equiv 1 \equiv \alpha. \tag{5}$$

If $\alpha \equiv 0$, then

$$\alpha \vee (\alpha \wedge \beta) \equiv 0 \vee (0 \wedge \beta) \equiv 0 \equiv \alpha. \tag{6}$$

That completes (1).

To prove (2), note that

$$(\alpha \vee \beta) \wedge (\alpha \vee \gamma) \equiv [\alpha \wedge (\alpha \vee \gamma)] \vee [\beta \wedge (\alpha \vee \gamma)] \equiv \quad (7)$$

By applying (1) we obtain

$$\equiv \alpha \vee [\beta \wedge (\alpha \vee \gamma)] \equiv \alpha \vee [(\alpha \wedge \beta) \vee (\beta \wedge \gamma)] \equiv$$

$$\equiv [\alpha \vee (\alpha \wedge \beta)] \vee (\beta \wedge \gamma) \equiv \quad (8)$$

Again, by applying (1), we obtain from (8)

$$(\alpha \vee \beta) \wedge (\alpha \vee \gamma) \equiv \alpha \vee (\beta \wedge \gamma). \quad (9)$$

● PROBLEM 1-11

The symmetric difference of the sentences α and β is defined by

$$(\alpha \div \beta) \equiv [(\alpha \wedge \beta') \vee (\alpha' \wedge \beta)]. \quad (1)$$

Prove that:

$$\alpha \vee \beta \equiv [(\alpha \div \beta) \div (\alpha \wedge \beta)]. \quad (2)$$

SOLUTION:

The easiest way to prove (2) is to establish the truth table and to compare the corresponding values. The truth table for $\alpha \div \beta$ is:

Table 1

α	β	$\alpha \wedge \beta'$	$\alpha' \wedge \beta$	$\alpha \div \beta$
1	1	0	0	0
1	0	1	0	1
0	1	0	1	1
0	0	0	0	0

Utilizing Table 1, we obtain

Table 2

α	β	$\alpha \vee \beta$	$(\alpha \div \beta) \div (\alpha \wedge \beta)$
1	1	1	1
1	0	1	1
0	1	1	1
0	0	0	0

By comparing the results of the last two columns of Table 2, we find that (2) is true.

We can also carry out the proof of (2) without the use of the truth table.

$$(\alpha \div \beta) \div (\alpha \wedge \beta) \equiv [(\alpha \div \beta) \wedge (\alpha \wedge \beta)'] \vee [(\alpha \div \beta)' \wedge (\alpha \wedge \beta)] \equiv$$

$$\equiv \{[(\alpha \wedge \beta') \vee (\alpha' \wedge \beta)] \wedge (\alpha' \vee \beta')\} \vee$$

$$\vee \{[(\alpha \wedge \beta') \vee (\alpha' \wedge \beta)]' \wedge (\alpha \wedge \beta)\} \equiv$$

$$\equiv [(\alpha' \vee \beta') \wedge (\alpha \wedge \beta')] \vee [(\alpha' \vee \beta') \wedge (\alpha' \vee \beta)] \vee$$

$$\vee [(\alpha \wedge \beta')' \wedge (\alpha' \wedge \beta)' \wedge (\alpha \wedge \beta)] \equiv$$

$$\equiv (\alpha \wedge \beta') \vee (\alpha' \wedge \beta) \vee [(\alpha' \vee \beta) \wedge (\alpha \vee \beta') \wedge (\alpha \wedge \beta)] \equiv$$

$$\equiv (\alpha \wedge \beta') \vee (\alpha' \wedge \beta) \vee (\alpha \wedge \beta) \equiv$$

$$\equiv (\alpha \wedge \beta') \vee [\beta \wedge (\alpha \vee \alpha')] \equiv$$

$$\equiv \beta \vee (\alpha \wedge \beta') \equiv \alpha \vee \beta. \qquad (3)$$

● PROBLEM 1-12

The digital (logic) circuits operate in the binary mode where each input and output voltage is either 1 or 0. This enables us to use Boolean algebra, which differs from ordinary algebra in that Boolean variables and constants are only allowed to have two possible values, 1 or 0. The voltage values in the circuit are predetermined, in the sense that any voltage in the range of 0V to 0.5V corresponds to "0" and any voltage in the range of 2V to 6V corresponds to "1." The values between 0.5V and 2V should not occur in the circuit. Table 1 lists some commonly used terms.

Table 1

Logic 1	Logic 0
True	False
On	Off
High	Low
Yes	No
Closed switch	Open switch

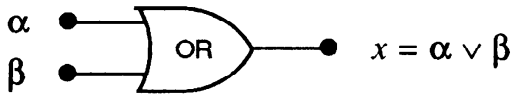
1. Describe the OR gate.

2. Describe the AND gate.

3. Describe the NOT gate (inverter).

SOLUTION:

1. In a digital circuit, an OR gate has two or more inputs, and one output is equal to the sum of the inputs.



α	β	α ∨ β = x
1	1	1
1	0	1
0	1	1
0	0	0

FIGURE 1

Figure 1 shows the symbol of the OR gate with two inputs and its truth table.

An OR gate with 4 inputs is shown in Figure 2.

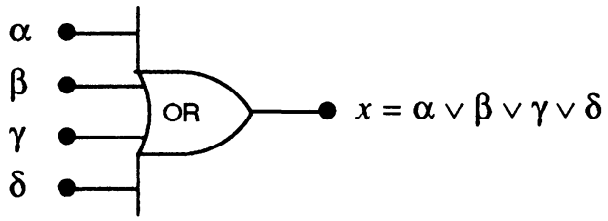
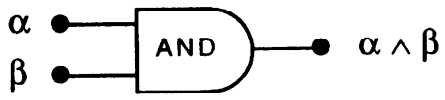


FIGURE 2

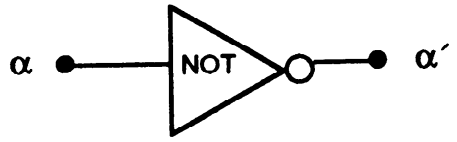
2. A two-input AND gate and its truth table is shown in Figure 3.



α	β	α ∧ β
1	1	1
1	0	0
0	1	0
0	0	0

FIGURE 3

3. The NOT gate (inverter) has one input and one output.



α	α′
1	0
0	1

FIGURE 4

● PROBLEM 1-13

1. Replace the following digital circuit with the simpler one:

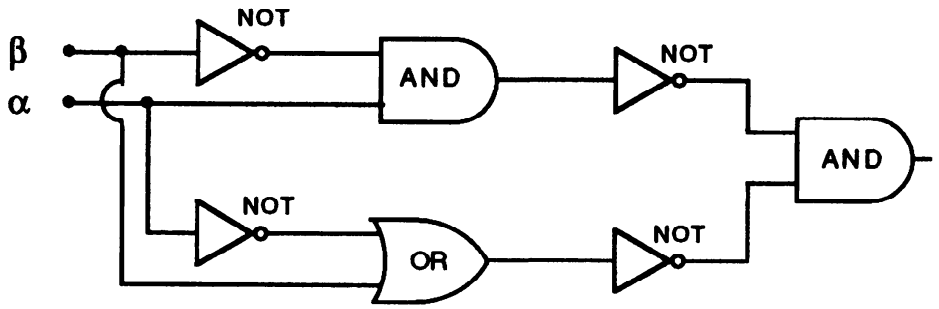


FIGURE 1

2. Illustrate DeMorgan's law using logic gates.

SOLUTION:

1. The circuit shown in Figure 1 has two inputs, one output, and it performs the function described by

$$(\beta' \wedge \alpha)' \wedge (\alpha' \vee \beta)'. \qquad (1)$$

By applying DeMorgan's laws and some basic properties of sentence calculus, we obtain

$$(\beta' \wedge \alpha)' \wedge (\alpha' \vee \beta)' \equiv (\beta \vee \alpha') \wedge (\alpha \wedge \beta') \equiv$$

$$\equiv [(\alpha \wedge \beta') \wedge \beta] \vee [(\alpha \wedge \beta') \wedge \alpha'] \equiv$$

$$\equiv (\alpha \wedge \beta' \wedge \beta) \vee (\alpha \wedge \alpha' \wedge \beta') \equiv$$

$$\equiv (\alpha \wedge 0) \vee (0 \wedge \beta') \equiv 0 \vee 0 \equiv 0. \quad (2)$$

No matter what the input, the output is always 0. To construct a circuit equivalent to the one shown in Figure 1, all that is needed is wire. See Figure 2.

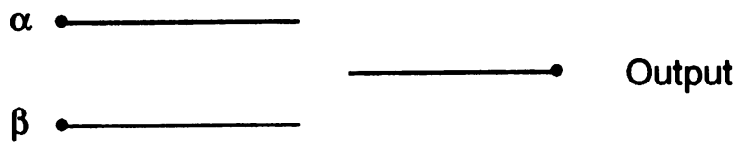


FIGURE 2

Luckily, not all of the computer logic circuits can be simplified to such an extent.

2. We will build the circuit for

$$(\alpha \wedge \beta)' \equiv \alpha' \vee \beta'. \quad (3)$$

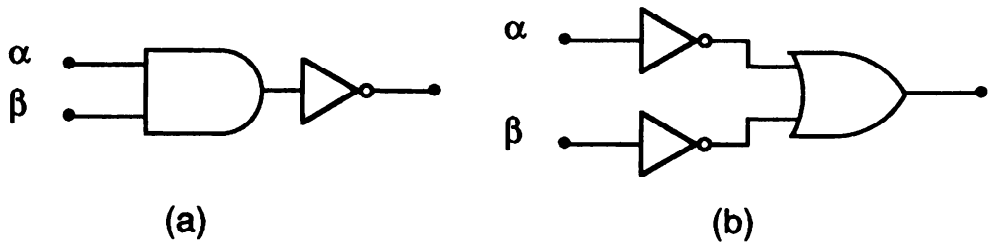


FIGURE 3

Circuits (a) and (b) shown in Figure 3 are equivalent.

Now we will build the circuit for

$$(\alpha \vee \beta)' \equiv \alpha' \wedge \beta'. \quad (4)$$

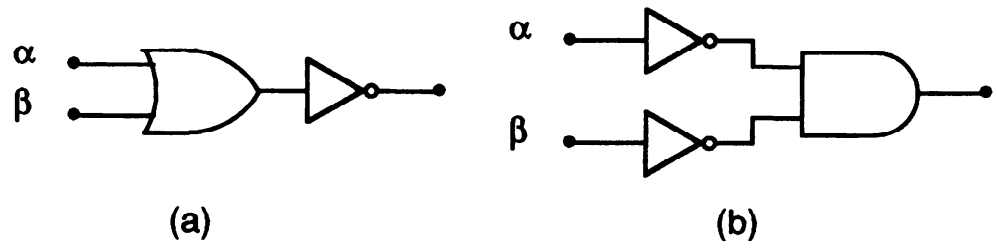


FIGURE 4

Circuits (a) and (b) of Figure 4 are equivalent.

● PROBLEM 1-14

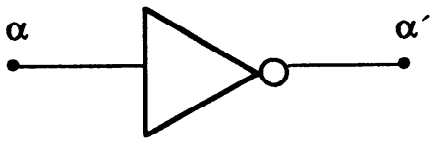
Design the circuit that implements the expression

1. $[(\alpha' \wedge \beta) \vee (\beta' \wedge \gamma)]'$. (1)

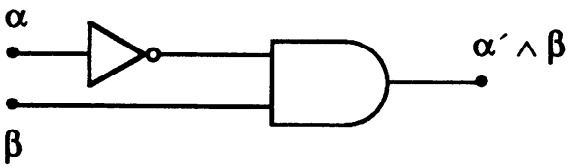
2. $[(\alpha \wedge \beta \wedge \gamma) \vee (\alpha' \wedge \beta')] \wedge \gamma'$. (2)

SOLUTION:

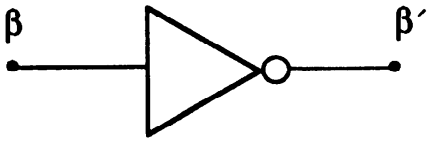
1. Step by step we shall move α', which is realized by



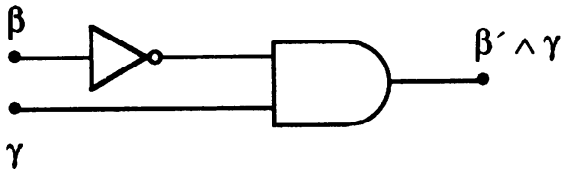
$\alpha' \wedge \beta$ is realized by



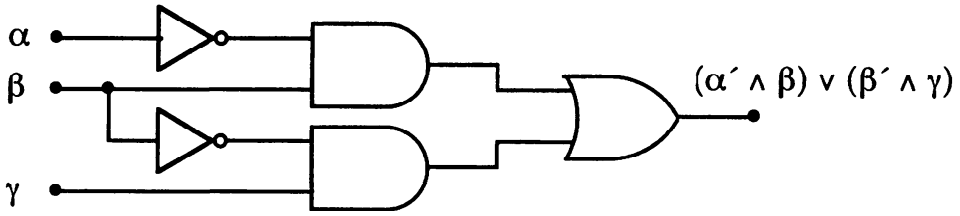
β' is realized by



$\beta' \wedge \gamma$ is realized by



$(\alpha' \wedge \beta) \vee (\beta' \wedge \gamma)$ is realized by



$[(\alpha' \wedge \beta) \vee (\beta' \wedge \gamma)]'$ is realized by

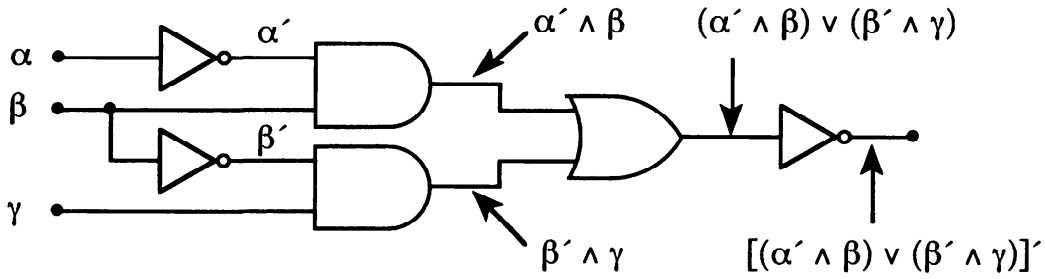
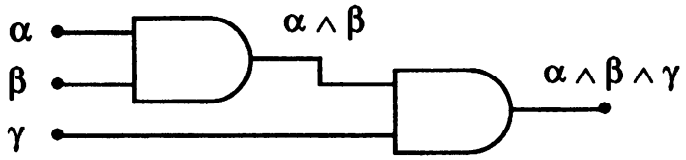
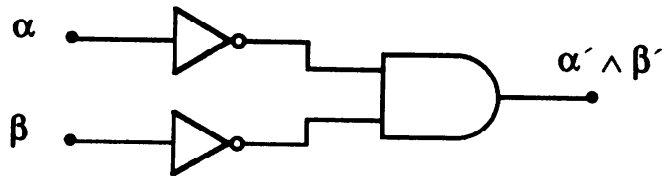


FIGURE 1

2. $\alpha \wedge \beta \wedge \gamma$ is implemented by



$\alpha' \wedge \beta'$ is implemented by



$[(\alpha \wedge \beta \wedge \gamma) \vee (\alpha' \wedge \beta')] \wedge \gamma'$ is implemented by

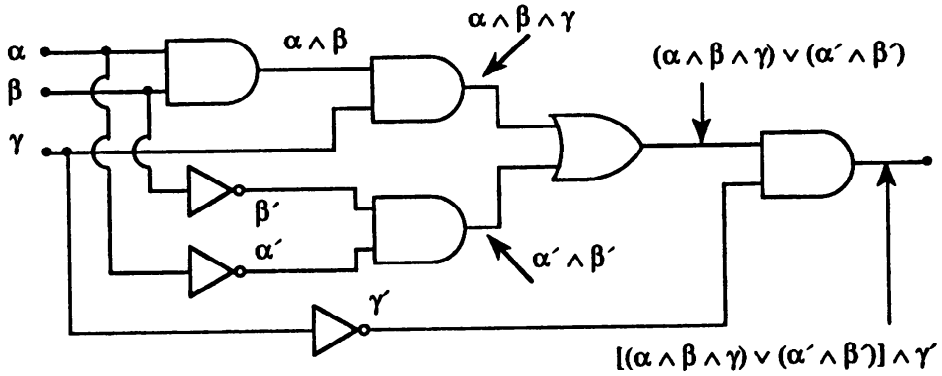


FIGURE 2

Note that since

$$[(\alpha \wedge \beta \wedge \gamma) \vee (\alpha' \wedge \beta')] \wedge \gamma' \equiv$$

$$\equiv (\alpha \wedge \beta \wedge \gamma \wedge \gamma') \vee (\alpha' \wedge \beta' \wedge \gamma') \equiv (\alpha' \wedge \beta') \wedge \gamma' \qquad (3)$$

the circuit in the last part of Figure 2 can be replaced by

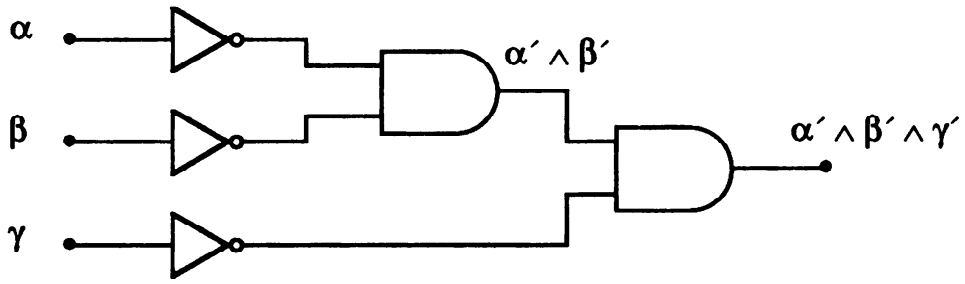


FIGURE 3

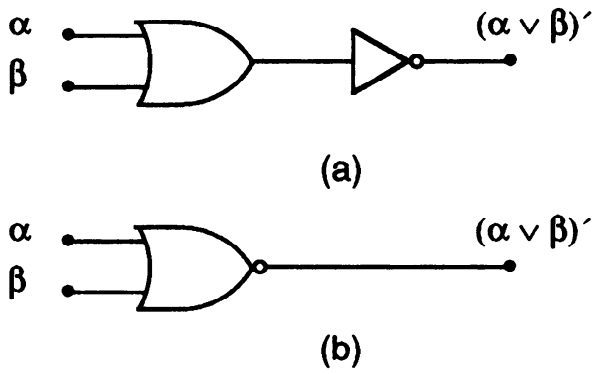
● PROBLEM 1-15

1. Define NOR and NAND gates.

2. Show that any logic circuit can be built exclusively with the NAND gates.

SOLUTION:

1. NOR gate consists of OR gate and an INVERTER.

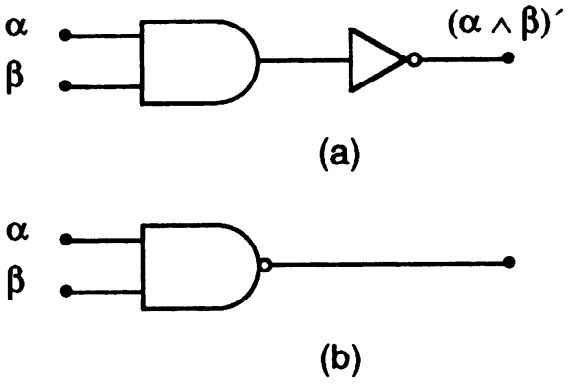


α	β	α ∨ β	(α ∨ β)′
1	1	1	0
0	1	1	0
1	0	1	0
0	0	0	1

FIGURE 1—NOR Gate

Both figures represent the same logic circuit. The symbol in Figure (b) is used for convenience.

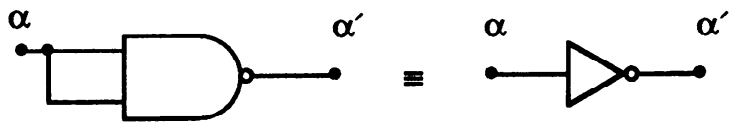
NAND gate and its symbol are shown in Figure 2.



α	β	α ∧ β	(α ∧ β)′
1	1	1	0
1	0	0	1
0	1	0	1
0	0	0	1

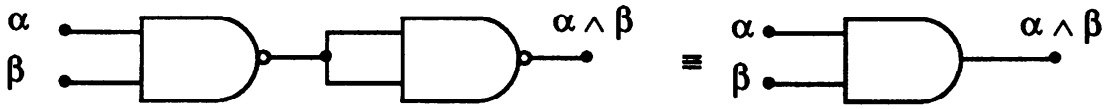
FIGURE 2—NAND Gate

2. All Boolean expressions consist of various combinations of the basic operations of OR, AND, and INVERT. Thus, any circiut can be implemented using OR, AND, and INVERT gates. It is possible, however, to build any logic circuit using only NAND gates. Using NAND gates, we can design any logic operations OR, AND, and INVERT.



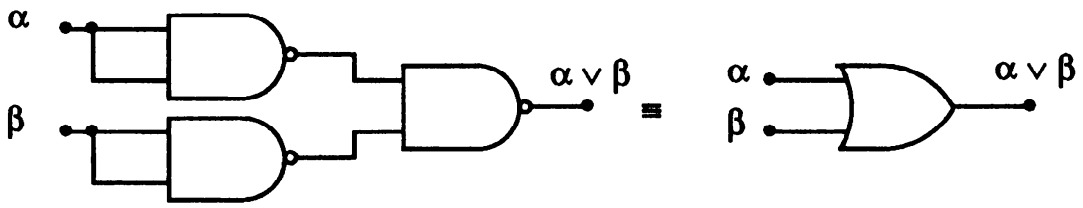
(a) INVERTER

Indeed $(\alpha \wedge \alpha)' \equiv \alpha'$



(b) AND Gate

Indeed $[(\alpha \wedge \beta)' \wedge (\alpha \wedge \beta)']' \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \beta) \equiv \alpha \wedge \beta$.



(c) OR Gate

Indeed $[(\alpha \wedge \alpha)' \wedge (\beta \wedge \beta)']' \equiv (\alpha' \wedge \beta')' \equiv \alpha \vee \beta$.

FIGURE 3

NAND gates can be used to perform any Boolean function.

1. Design a logic circuit that implements the function

$$(\alpha \wedge \beta) \vee (\gamma \wedge \delta). \qquad (1)$$

2. Suppose you have the TTL IC's shown in Figure 1 at your disposal. Each IC is a quad; that is, each chip contains four identical two-input gates. Using the minimum number of IC's, design the circuit which performs function (1).

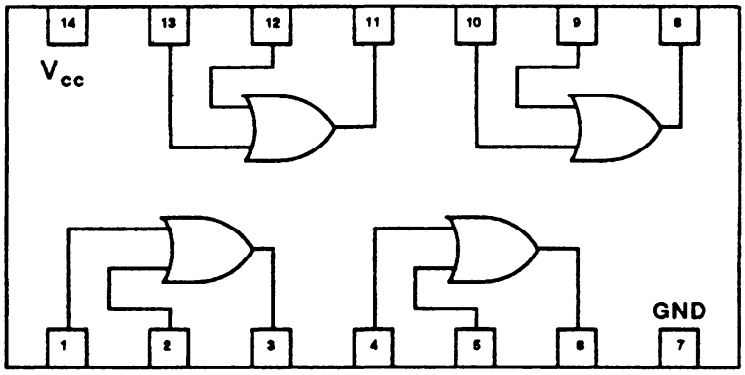


FIGURE 1a — IC 7432

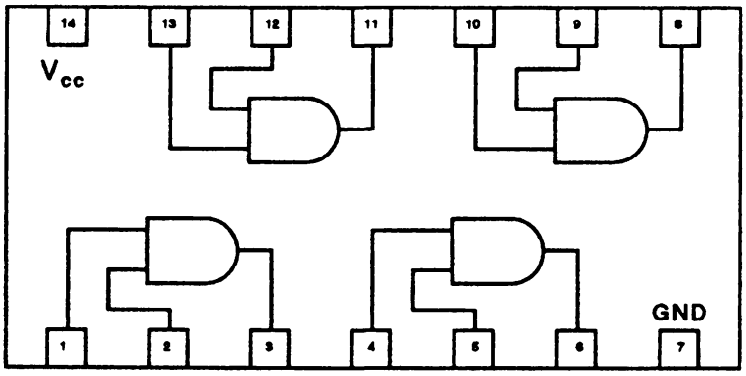


FIGURE 1b — IC 7408

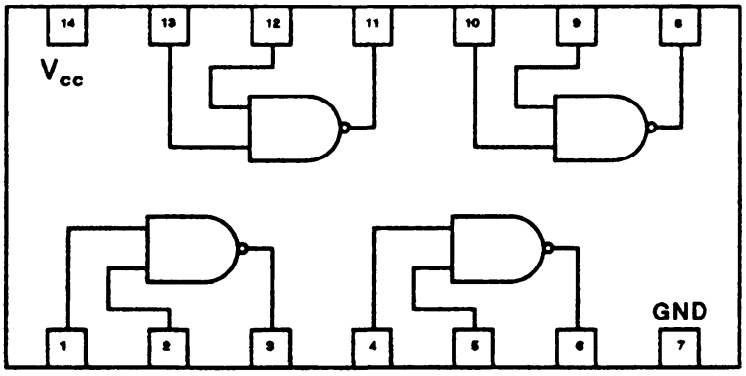


FIGURE 1c — IC 7400

SOLUTION:

1.

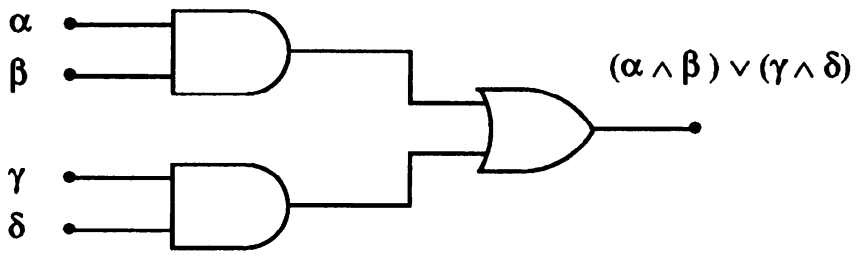


FIGURE 2

Figure 2 shows the circuit implementating operation $(\alpha \wedge \beta) \vee (\gamma \wedge \delta)$.

2. From Figure 2 we conclude that in order to build the circuit (1), we have to use two gates from IC 7408 and one gate from IC 7432, as shown in Figure 3.

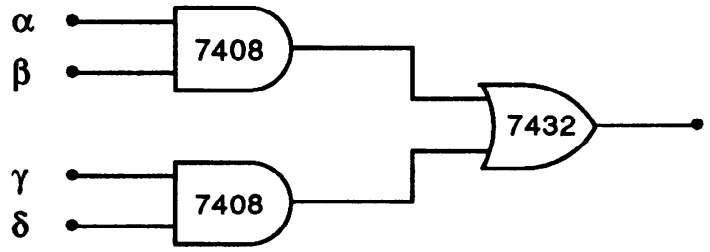


FIGURE 3

Each AND gate and OR gate can be replaced by a combination of NAND gates (See Problem 1–15).

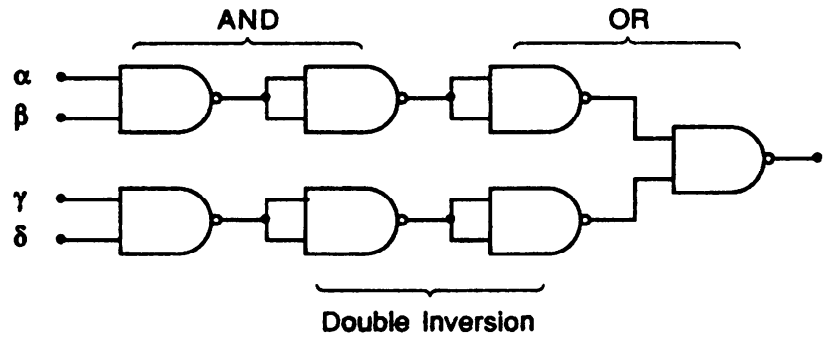


FIGURE 4

By eliminating double inversion, we obtain

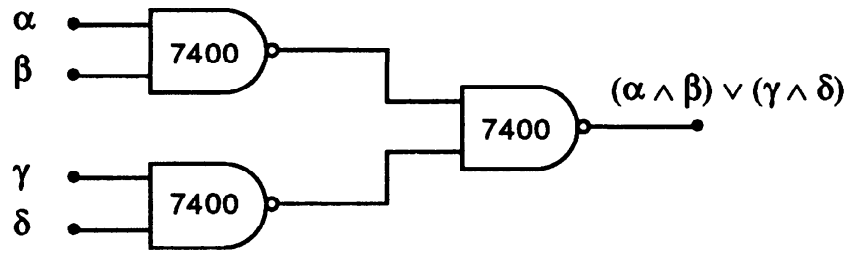
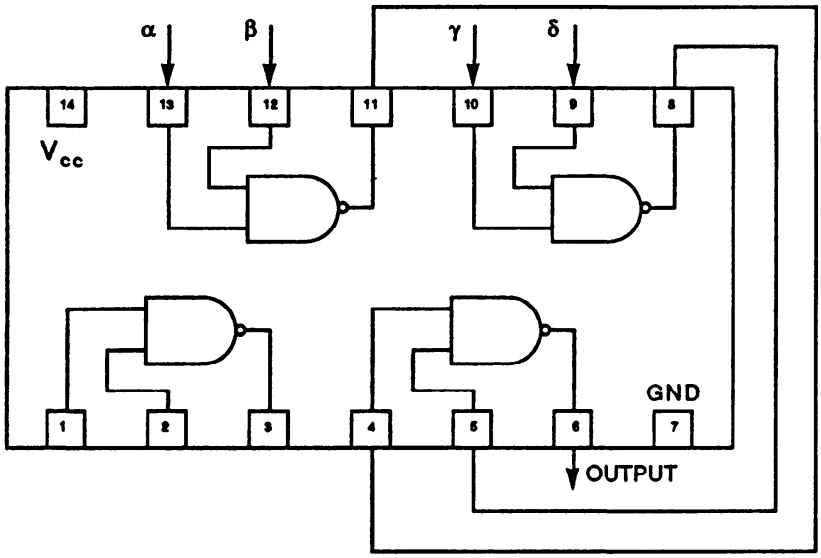


FIGURE 5

Now to implement the operation, we only need three NAND gates. Therefore, instead of using two IC's as in Figure 3, we can use one IC 7400, as shown in Figure 5 and Figure 6.



IC 7400

$$\text{OUTPUT} = (\alpha \wedge \beta) \vee (\gamma \wedge \delta)$$

FIGURE 6

CHAPTER 2

ALGEBRA OF SETS

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● PROBLEM 2-1

Which of the following statements is true?

1. $7 \in A$, where

$$A = \{x : x \in N \text{ and } 5 < x < 8\}$$

and N is the set of natural numbers.

2. $B = 4$, where

$$B = \{x : x + 1 = 5\}.$$

SOLUTION:

The theory of sets was founded and developed into a mathematical system by G. Cantor (1845-1918). According to him, we are to understand by a set — a collection into a whole, of definite, well-distinguished objects (called the elements of A) of our perception or of our thought."

For example, the set of vertices of a square consists of four elements. The set can be defined by listing its elements.

For example:

$$A = \{\text{John, seven}\}$$

is the set consisting of two elements: John and seven. The statement "x is an element of A" is written

$$x \in A.$$

The negation of the statement $x \in A$ is

$$x \notin A.$$

The other way to define a set is to characterize the properties (or property) of its elements.

$$A = \{x : p(x)\}.$$

If $x \in A$, then $p(x)$ is true ($p(x) \equiv 1$).

For example:

$$N = \{n : n \text{ is a natural number}\}.$$

1. The set A consists of two elements

$$A = \{6, 7\}.$$

Thus

$$7 \in A.$$

2. The set B consists of a single element

$$B = \{4\}.$$

The number 4 belongs to the set B, $4 \in B$, but it does not equal B. The statement $B = 4$ is not true.

● PROBLEM 2-2

Show that equality of sets is reflexive, symmetric, and transitive, i.e., show that

1. $A = A$ for all sets A.

2. If $A = B$, then $B = A$ for all sets A, B.

3. If $A = B$ and $B = C$, then A = C for all sets A, B, C.

SOLUTION:

Two sets, A and B, are said to be equal, $A = B$, provided that they contain the same elements. From this definition we conclude that:

1. $x \in A$ if and only if $x \in A$, thus

$$A = A.$$

Frequently, instead of writing if, and only if, we shall use iff.

"iff" = if and only if.

2. The statement $x \in A$ if and only if (iff) $x \in B$ is equivalent to $x \in B$, iff $x \in A$.

3. ($x \in A$ if and only if $x \in B$) and ($x \in B$ if and only if $x \in C$) imply ($x \in A$ if and only if $x \in C$). Using the logic notation, we can write this statement in the form

$$[(x \in A \text{ iff } x \in B) \wedge (x \in B \text{ iff } x \in C)] \Rightarrow (x \in A \text{ iff } x \in C)$$

or

$$(A = B) \wedge (B = C) \Rightarrow (A = C).$$

1. Show that inclusion of sets is reflexive, anti-symmetric, and transitive; i.e.

   a. $A \subset A$ for all sets A.

   b. $A \subset B$ and $B \subset A$ imply $A = B$ for all A, B.

   c. If $A \subset B$ and $B \subset C$, then $A \subset C$ for all sets A, B, and C.

2. Prove

$$[(A \subset B) \wedge (B \subset C) \wedge (C \subset A)] \Rightarrow (A = B = C). \quad (1)$$

SOLUTION:

DEFINITION: INCLUSION

The set A is included in B; i.e., A is a subset of B, which is written as $A \subset B$ or $B \subset A$, when

$$x \in A \Rightarrow x \in B \quad (2)$$

■

1. Obviously

$$(x \in A) \Rightarrow (x \in A) \text{ for all } A. \quad (3)$$

Hence

$$A \subset A.$$

$$(x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in A) \Rightarrow$$

$$(x \in A \text{ iff } x \in B). \quad (4)$$

Hence

$$(A \subset B) \wedge (B \subset A) \Rightarrow A = B. \quad (5)$$

$$(x \in A \Rightarrow x \in B) \wedge (x \in B \Rightarrow x \in C) \Rightarrow$$

$$(x \in A \Rightarrow x \in C). \quad (6)$$

Thus

$$(A \subset B) \wedge (B \subset C) \Rightarrow A \subset C. \quad (7)$$

2. To prove that $A = B$, we must prove that $x \in A$ iff $x \in B$. Since $A \subset B$, it is

enough to show that $B \subset A$. But $B \subset C$ and $C \subset A$, thus $B \subset A$. And

$$A \subset B \text{ and } B \subset A \Rightarrow A = B. \tag{8}$$

Similarly, we show that $B = C$.

$$(C \subset A \text{ and } A \subset B) \Rightarrow C \subset B. \tag{9}$$

Thus

$$(B \subset C \text{ and } C \subset B) \Rightarrow C = B.$$

● PROBLEM 2-4

Using Venn diagrams, illustrate the union, intersection and difference of two sets A and B.

Show that intersection is associative; i.e., that

$$A \cap (B \cap C) = (A \cap B) \cap C \tag{1}$$

for all sets A, B, and C.

SOLUTION:

The union of two sets A and B is the set whose elements are all the elements of the set A and all the elements of the set B and which does not contain any other elements. The union of A and B is denoted by $A \cup B$; see Figure 1.

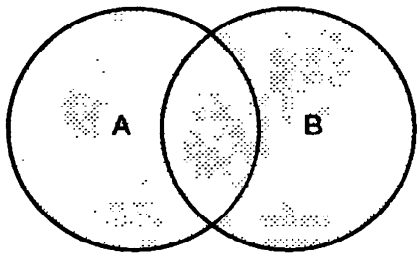


FIGURE 1. A ∪ B is shaded.

The common part of two sets is called the intersection. The intersection of A and B contains those, and only those, elements which belong to A and to B. The intersection is denoted by $A \cap B$.

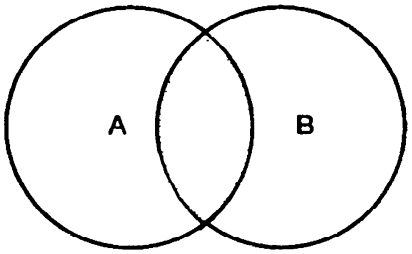


FIGURE 2. A ∩ B is shaded.

The difference of two sets A and B is the set consisting of those, and only those, elements which belong to A, but do not belong to B. The difference is denoted by $A - B$ (sometimes $A \setminus B$).

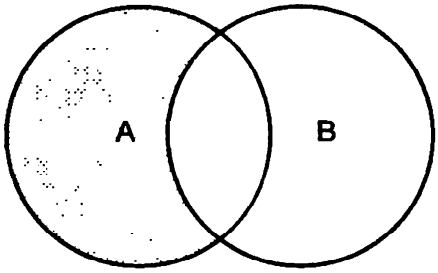


FIGURE 3. A − B is shaded.

Now we will use Venn diagrams to prove (1).

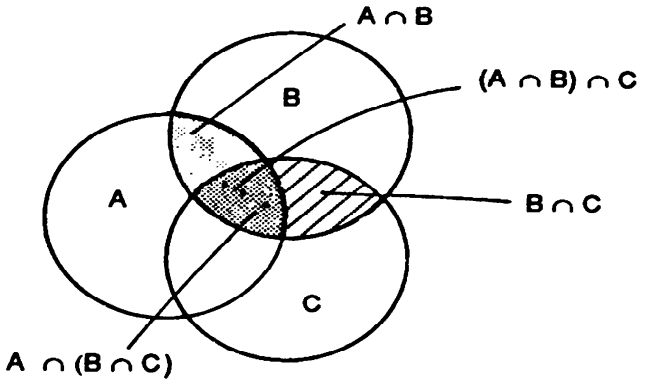


FIGURE 4.

From Figure 4 we conclude that

$$A \cap (B \cap C) = (A \cap B) \cap C.$$

By applying the sentence calculus, prove the following formula:

$$A \cup (A \cap B) = A = A \cap (A \cup B). \tag{1}$$

SOLUTION:

Operations on sets are related to operations on sentences. To denote that x is an element of A we write

$$x \in A. \tag{2}$$

Similarly

$$(x \notin A) \equiv (x \in A)' \tag{3}$$

where $x \notin A$ means x is not an element of A.

The following equivalences hold for all x:

$$(x \in A \cup B) \equiv (x \in A) \vee (x \in B) \tag{4}$$

$$(x \in A \cap B) \equiv (x \in A) \wedge (x \in B) \tag{5}$$

$$(x \in A - B) \equiv (x \in A) \wedge (x \in B)' \tag{6}$$

Using formulas (3) - (6) we can deduce theorems on the calculus of sets from analogous theorems of sentence calculus. Furthermore, let us note that

$$\text{if } x \in A \equiv x \in B \text{ holds for all } x, \text{ then } A = B. \tag{7}$$

We shall now prove (1).

$$[x \in A \cup (A \cap B)] \equiv (x \in A) \vee (x \in A \cap B) \equiv$$

$$\equiv (x \in A) \vee [(x \in A) \wedge (x \in B)] \equiv (x \in A). \tag{8}$$

Thus

$$A \cup (A \cap B) = A. \tag{9}$$

Similarly

$$[x \in A \cap (A \cup B)] \equiv (x \in A) \wedge (x \in A \cup B) \equiv$$

$$\equiv (x \in A) \wedge [(x \in A) \vee (x \in B)] \equiv (x \in A). \tag{10}$$

Hence

$$A \cap (A \cup B) = A. \tag{11}$$

Note that in (8) and (10) we applied the law of absorption of sentence calculus.

$$\alpha \vee (\alpha \wedge \beta) \equiv \alpha \equiv \alpha \wedge (\alpha \vee \beta) \tag{12}$$

● PROBLEM 2-6

Prove these basic formulas for unions and intersections:

1. $A \cup A = A$
   $A \cap A = A$ } indempotency (1)

2. $A \cup B = B \cup A$
   $A \cap B = B \cap A$ } commutativity (2)

3. $A \cup (B \cup C) = (A \cup B) \cup C$
   $A \cap (B \cap C) = (A \cap B) \cap C$ } associativity (3)

4. $A \cup (A \cap B) = A$
   $A \cap (A \cup B) = A$ } adjunction (4)

5. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
   $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ } distributivity (5)

SOLUTION:

1. $$(x \in A \cup A) \equiv (x \in A) \vee (x \in A) \equiv (x \in A) \tag{6}$$

Hence

$$A \cup A = A. \tag{7}$$

$$(x \in A \cap A) \equiv (x \in A) \wedge (x \in A) \equiv (x \in A) \tag{8}$$

Hence

$$A \cap A = A. \tag{9}$$

From (7) and (9) we see that, in contrast to arithmetic, neither multiples nor exponents arise in set theory.

2. $$(x \in A \cup B) \equiv (x \in A) \vee (x \in B) \equiv$$

$$\equiv (x \in B) \vee (x \in A) \equiv (x \in B \cup A). \tag{10}$$

Thus $$A \cup B = B \cup A. \tag{11}$$

$$(x \in A \cap B) \equiv (x \in A) \wedge (x \in B) \equiv$$

$$\equiv (x \in B) \wedge (x \in A) \equiv (x \in B \cap A). \tag{12}$$

Hence $$A \cap B = B \cap A. \tag{13}$$

3. $$[x \in A \cup (B \cup C)] \equiv [(x \in A) \vee (x \in B \cup C)] \equiv$$

$$\equiv [(x \in A) \vee (x \in B) \vee (x \in C)] \equiv [(x \in A \cup B) \vee (x \in C)] \equiv \tag{14}$$

$$\equiv [x \in (A \cup B) \cup C]$$

Thus $$A \cup (B \cup C) = (A \cup B) \cup C. \tag{15}$$

Replacing "v" by "∧" in (14) we prove

$$A \cap (B \cap C) = (A \cap B) \cap C. \tag{16}$$

4. See Problem 2–5.

5. $$[x \in A \cup (B \cap C)] \equiv [(x \in A) \vee (x \in B \cap C)] \equiv$$

$$\equiv \{(x \in A) \vee [(x \in B) \wedge (x \in C)]\} \equiv$$

$$\equiv \{[(x \in A) \vee (x \in B)] \wedge [(x \in A) \vee (x \in C)]\} \equiv$$

$$\equiv (x \in A \cup B) \wedge (x \in A \cup C) \equiv$$

$$\equiv [x \in (A \cup B) \cap (A \cup C)]. \tag{17}$$

Hence $$A \cup (B \cap C) = (A \cup B) \cap (A \cup C). \tag{18}$$

$$[x \in A \cap (B \cup C)] \equiv [(x \in A) \wedge (x \in B \cup C)] \equiv$$

$$\equiv \{(x \in A) \wedge [(x \in B) \vee (x \in C)]\} \equiv$$

$$\equiv \{[(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \in C)]\} \equiv$$

$$\equiv [(x \in A \cap B) \vee (x \in A \cap C)] \equiv$$

$$\equiv [x \in (A \cap B) \cup (A \cap C)] . \tag{19}$$

Hence $$A \cap (B \cup C) = (A \cap B) \cup (A \cap C). \tag{20}$$

● PROBLEM 2-7

1. Find the elements of the set

$$A = \{x : x + 1 = 3, x^2 = 1\}. \tag{1}$$

2. Prove that:

If A is a subset of the null set ϕ, then $A = \phi$.

3. Prove that:

$$(A - B) \cap B = \phi. \tag{2}$$

SOLUTION:

DEFINITION OF THE NULL SET

The null set (or the empty set or the void set) is the set which contains no elements. The null set is denoted by ϕ. Often the null set is defined as

$$\phi = \{x : x \neq x\}. \tag{3}$$

1. Number x, such that $x + 1 = 3$ and $x^2 = 1$ does not exist, hence

$$A = \phi. \tag{4}$$

2. The null set ϕ is a subset of every set, thus $\phi \subset A$. But, by hypothesis, $A \subset \phi$, therefore

$$A = \phi. \tag{5}$$

3. $$[x \in (A - B) \cap B] \equiv$$

$$\equiv [(x \in A - B) \wedge (x \in B)] \equiv$$

$$\equiv [(x \in A) \wedge (x \in B)' \wedge (x \in B)] \equiv 0. \tag{6}$$

Thus, the set $(A - B) \cap B$ contains no elements

$$(A - B) \cap B = \phi. \tag{7}$$

● PROBLEM 2-8

1. Find the power set $P(A)$ of the set

$$A = \{1, 2, 3\}. \tag{1}$$

2. Find the power set $P(B)$ of the set

$$B = \{1, \{2, 3\}\}. \tag{2}$$

3. Set A consists of 10 elements. How many elements does the power set $P(A)$ have?

SOLUTION:

DEFINITION OF POWER SET

The power set $P(A)$ of A is the class of all subsets of A.

■

1. The subsets of A are:

0 elements: ϕ
1 element: $\{1\}, \{2\}, \{3\}$ (3)
2 elements: $\{1, 2\}, \{1, 3\}, \{2, 3\}$
3 elements: $\{1, 2, 3\}$

Hence

$$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1, 2, 3\}\} \tag{4}$$

$P(A)$ contains $2^3 = 8$ elements.

2. Note that B contains only two elements, 1 and $\{2, 3\}$. Thus, its power set $P(B)$ is

$$P(B) = \{\phi, \{1\}, \{2,3\}, \{1, \{2, 3\}\}\}. \tag{5}$$

3. From the theory of permutations we have the equation for the number of different combinations that can be formed from n different elements, using m elements at a time.

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \tag{6}$$

where

$$n! = 1 \cdot 2 \cdot \ldots \cdot (n-1) \cdot n \tag{7}$$

and

$$0! = 1.$$

Suppose set A contains n elements; then there are

$\binom{n}{0} = 1$ subsets with 0 elements

$\binom{n}{1} = \frac{n!}{1!(n-1)!} = n$ subsets with 1 element

$\binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$ subsets with 2 elements.

The total number of subsets of a set A consisting of n elements is

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = (1+1)^n = 2^n. \tag{8}$$

Set A consists of 10 elements, hence its power set $P(A)$ consists of $2^{10} = 1024$.

● PROBLEM 2-9

If A is any non-empty set, then $P(A)$ (or 2^A) is the power set of A. Prove that:

1. $2^A \cap 2^B = 2^{A \cap B}$ (1)

2. $2^A \cup 2^B \subset 2^{A \cup B}$ (2)

3. Show that $2^A \cup 2^B \neq 2^{A \cup B}$ (3)

SOLUTION:

1. Let α represent an element of $2^A \cap 2^B$. Recall that the power set 2^A of A is the class of all subsets of A. Suppose α consists of elements $p_1, p_2, \ldots$

$$\alpha = \{p_1, p_2, \ldots\} \tag{4}$$

Since $\alpha \in 2^A$ and $\alpha \in 2^B$, each of $p_1, p_2, \ldots$ belongs to $A \cap B$. Thus, α is a subset of $A \cap B$ and

$$\alpha \in 2^{A \cap B} \tag{5}$$

Hence,

$$2^A \cap 2^B \subset 2^{A \cap B.} \tag{6}$$

Now, suppose $\alpha \in 2^{A \cap B}$; i.e., α is a subset of $A \cap B$, and each element of $\alpha = \{p_1, p_2, \ldots\}$ belongs to $A \cap B$. Hence, each of $p_1, p_2, \ldots$ belongs to A and to B.

Therefore, α is a subset of A, $\alpha \in 2^A$; and α is a subset of B, $\alpha \in 2^B$.

$$\alpha \in 2^A \cap 2^B. \tag{7}$$

Hence,

$$2^{A \cap B} \subset 2^A \cap 2^B. \tag{8}$$

From (6) and (8) we conclude that

$$2^{A \cap B} = 2^A \cap 2^B. \tag{9}$$

2. Suppose $\alpha \in 2^A \cup 2^B$; that is, $\alpha \in 2^A$ or $\alpha \in 2^B$. Assume $\alpha \in 2^A$, then α is a subset of A. If α is a subset of A, then α is a subset of $A \cup B$ and

$$\alpha \in 2^{A \cup B}. \tag{10}$$

That proves

$$2^A \cup 2^B \subset 2^{A \cup B.} \tag{11}$$

3. Now we shall show that

$$2^A \cup 2^B \neq 2^{A \cup B}. \tag{12}$$

Let $A = \{a\}$, $B = \{b\}$, then

$$2^A = \{\phi, \{a\}\} \tag{13}$$

$$2^B = \{\phi, \{b\}\} \tag{14}$$

and $A \cup B = \{a, b\}$

$$2^{A \cup B} = \{\phi, \{a\}, \{b\}, \{a, b\}\}. \tag{14}$$

Hence,

$$2^A \cup 2^B = \{\phi, \{a\}, \{b\}\} \tag{15}$$

and

$$2^A \cup 2^B \neq 2^{A \cup B}. \tag{16}$$

● PROBLEM 2-10

Prove the following formulas:

1. $A \cap B \subset A \subset A \cup B$ (1)

2. $A - B \subset A$ (2)

3. If $A \subset B$ and $C \subset D$, then

$$A \cup C \subset B \cup D \tag{3}$$

and

$$A \cap C \subset B \cap D \tag{4}$$

4. $(A \subset B) \equiv (A \cup B = B) \equiv (A \cap B = A)$ (5)

5. $(A \cup B) \cap (A \cup C) = A \cup (B \cap C)$ (6)

SOLUTION:

1. $[x \in A \cap B] \equiv [(x \in A) \wedge (x \in B)] \Rightarrow (x \in A)$ (7)

Thus,

$$A \cap B \subset A \tag{8}$$

$$(x \in A)\,[(x \in A) \vee (x \in B)] \equiv (x \in A \cup B) \tag{9}$$

Hence,

$$A \subset A \cup B. \tag{10}$$

2. $[x \in A - B] \equiv [(x \in A) \vee (x \in B)'] \Rightarrow (x \in A)$ (11)

Hence,

$$A - B \subset A. \tag{12}$$

3. We have to show that

$$[(A \subset B) \wedge (C \subset D)] \Rightarrow \begin{pmatrix} A \cup C \subset B \cup D \\ \text{and} \\ A \cap C \subset B \cap D \end{pmatrix} \tag{13}$$

$$[x \in A \cup C] \equiv [(x \in A) \vee (x \in C)] \Rightarrow$$

$$\Rightarrow [(x \in B) \vee (x \in D)] \equiv (x \in B \cup D) \tag{14}$$

Hence,

$$A \cup C \subset B \cup D. \tag{15}$$

$$(x \in A \cap C) \equiv [(x \in A) \wedge (x \in C)] \Rightarrow$$

$$\Rightarrow [(x \in B) \wedge (x \in D)] \equiv (x \in B \cap D) \tag{16}$$

Hence,

$$A \cap C \subset B \cap D. \tag{17}$$

4. Suppose $A \subset B$; then from $(A \subset B)$ and $(B \subset B)$ we conclude that

$$(A \cup B) \subset (B \cup B) = B \tag{18}$$

by virtue of (13). But since $B \subset (A \cup B)$, we have

$$A \cup B = B. \tag{19}$$

Conversely from $A \cup B = B$, it follows that $A \subset B$. Hence, the relations $A \subset B$ and $A \cup B = B$ are equivalent.

$$(A \subset B) \equiv (A \cup B = B). \tag{20}$$

Similarly, combining $A \subset B$ with $A \subset A$, we obtain $A \subset (A \cap B)$. Since $A \cap B \subset A$,

$$A = A \cap B. \tag{21}$$

Conversely from $A = A \cap B$ we conclude that $A \subset B$.

5. $(A \cup B) \cap (A \cup C) = (A \cap A) \cup (A \cap C) \cup (B \cap A) \cup (B \cap C) =$

$$= A \cup (A \cap C) \cup (A \cap B) \cup (B \cap C) \tag{22}$$

But $A \cap B \subset A$, hence, $A \cup (A \cap B) = A$ and $A \cap C \subset A$, hence, $A \cup (A \cap C) = A$. Thus,

$$(A \cup B) \cap (A \cup C) = A \cup (B \cap C). \tag{23}$$

● PROBLEM 2-11

Prove the following formulas:

1. $A \cap B = A - (A - B)$ (1)

2. $A \cup (B - A) = A \cup B$ (2)

3. $A - (A \cap B) = A - B$ (3)

4. $A \cap (B - C) = (A \cap B) - (A \cap C)$ (4)

5. $A \cup (B - C) \neq (A \cup B) - (A \cup C)$ (5)

SOLUTION:

1. $$[x \in A - (A - B)] \equiv [(x \in A) \wedge (x \in A - B)'] \equiv$$
$$\equiv \{(x \in A) \wedge [(x \in A) \wedge (x \in B)']'\} \equiv$$
$$\equiv \{(x \in A) \wedge [(x \in A)' \vee (x \in B)]\} \equiv$$
$$\equiv [(x \in A) \wedge (x \in B)] \equiv (x \in A \cap B). \quad (6)$$

2. $$[x \in A \cup (B - A)] \equiv [(x \in A) \vee (x \in B - A)] \equiv$$
$$\equiv \{(x \in A) \vee [(x \in B) \wedge (x \in A)']\} \equiv$$
$$\equiv \{[(x \in A) \vee (x \in B)] \wedge [(x \in A) \vee (x \in A)']\} \equiv$$
$$\equiv [(x \in A) \vee (x \in B)] \equiv (x \in A \cup B). \quad (7)$$

Hence,

$$A \cup (B - A) = A \cup B. \quad (8)$$

3. $$[x \in A - (A \cap B)] \equiv [(x \in A) \wedge (x \in A \cap B)'] \equiv$$
$$\equiv \{(x \in A) \wedge [(x \in A) \wedge (x \in B)]'\} \equiv$$
$$\equiv \{(x \in A) \wedge [(x \in A)' \vee (x \in B)']\} \equiv$$

$$\equiv \{[(x \in A) \wedge (x \in A)'] \vee [(x \in A) \wedge (x \in B)']\} \equiv$$

$$\equiv [(x \in A) \wedge (x \in B)'] \equiv (x \in A - B). \tag{9}$$

Hence,

$$A - (A \cap B) = A - B. \tag{10}$$

4. $$[x \in (A \cap B) - (A \cap C)] \equiv [(x \in A \cap B) \wedge (x \in A \cap C)'] \equiv$$

$$\equiv \{(x \in A) \wedge (x \in B) \wedge [(x \in A) \wedge (x \in C)]'\} \equiv$$

$$\equiv \{(x \in A) \wedge (x \in B) \wedge [(x \in A)' \vee (x \in C)']\} \equiv$$

$$\equiv [(x \in A) \wedge (x \in B) \wedge (x \in C)'] \equiv$$

$$\equiv [(x \in A) \wedge (x \in B - C)] \equiv [x \in A \cap (B - C)]. \tag{11}$$

5. Suppose A, B, C are three disjoint sets as shown in the Venn diagram:

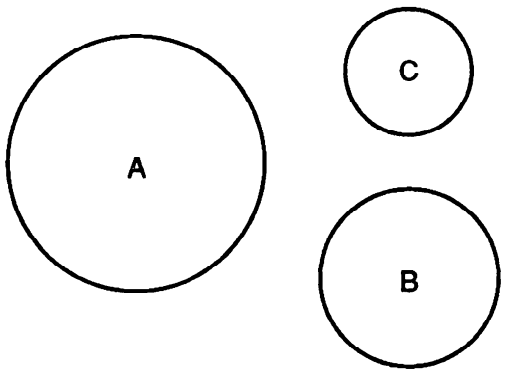


FIGURE 1

Then

$$A \cup (B - C) = A \cup B \tag{12}$$

and

$$(A \cup B) - (A \cup C) = B \tag{13}$$

Hence,

$$A \cup (B - C) \neq (A \cup B) - (A \cup C). \tag{14}$$

In the theory of sets, we assume that all the sets under consideration are subsets of some fixed set, called the space (or universal set). For example, in geometry the space is the Euclidean space, and in analysis the space is the set of real or complex numbers.

Throughout this book, all the sets considered will belong to the space which we shall denote by (1). Hence

$$A \subset 1 \tag{1}$$

for each of the sets considered.

DEFINITION OF COMPLEMENT OF THE SET

The complement of the set A with respect to the given space 1 is denoted by A^C (or CA or $\sim A$) and defined by

$$A^C = 1 - A \tag{2}$$

See Figure 1.

■

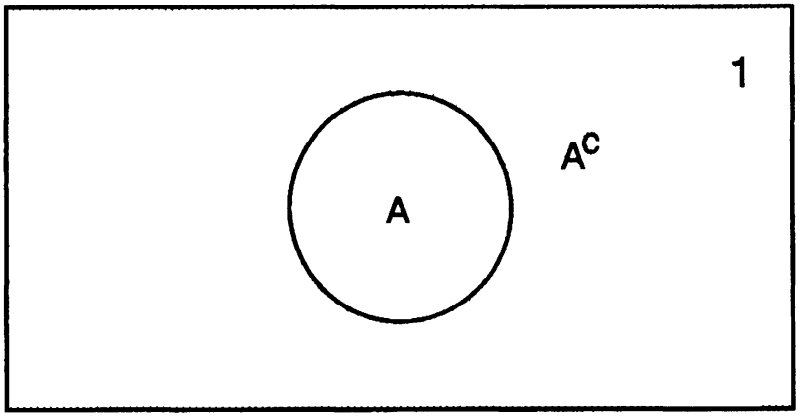


FIGURE 1.

We nave

$$(x \in A^C) \equiv (x \in A)' \equiv (x \notin A) \tag{3}$$

Prove that:

$$1^C = \phi, \quad \phi^C = 1 \tag{4}$$

$$A^{CC} = A \tag{5}$$

$$A \cup A^C = 1, \quad A \cap A^C = \phi \tag{6}$$

SOLUTION:

From definition (2) we have

$$1^C = 1 - 1 = \phi \tag{7}$$

$$\phi^C = 1 - \phi = 1 \tag{8}$$

$$A^{CC} = (A^C)^C = (1 - A)^C =$$

$$= 1 - (1 - A) = A \tag{9}$$

$$A \cup A^C = A \cup (1 - A) = 1 \tag{10}$$

$$A \cap A^C = A \cap (1 - A) = \phi \tag{11}$$

● PROBLEM 2-13

Prove DeMorgan's Theorem:

1. $(A \cup B)^C = A^C \cap B^C$ (1)

2. $(A \cap B)^C = A^C \cup B^C$ (2)

SOLUTION:

1. $$[x \in (A \cup B)^C] \equiv [x \in 1 - (A \cup B)] \equiv$$

$$\equiv [x \in A \cup B]' \equiv [(x \in A) \vee (x \in B)]' \equiv$$

$$\equiv [(x \in A)' \wedge (x \in B)'] \equiv [(x \in A^C) \wedge (x \in B^C)] \equiv$$

$$\equiv [x \in A^C \cap B^C]. \tag{3}$$

Hence,

$$(A \cup B)^C = A^C \cap B^C. \tag{4}$$

2. Similarly

$$[x \in (A \cap B)^C] \equiv [x \in 1 - (A \cap B)] \equiv$$

$$\equiv [x \in A \cap B]' \equiv [(x \in A) \wedge (x \in B)]' \equiv$$

$$\equiv [(x \in A)' \vee (x \in B)'] \equiv [(x \in A^C) \vee (x \in B^C)] \equiv$$

$$\equiv [x \in A^C \cup B^C]. \tag{5}$$

Hence,

$$(A \cap B)^C = A^C \cup B^C. \tag{6}$$

DeMorgan's theorem can be generalized to any finite number of sets.

$$(A_1 \cup A_2 \cup \ldots \cup A_n)^C = A_1^C \cap A_2^C \cap \ldots \cap A_n^C \tag{7}$$

$$(A_1 \cap A_2 \cap \ldots \cap A_n)^C = A_1^C \cup A_2^C \cup \ldots \cup A_n^C. \tag{8}$$

● PROBLEM 2-14

Prove that:

1. $A - B = A \cap B^C$ (1)

2. $A \subset B$ if and only if $B^C \subset A^C$ (2)

3. $A^C - B^C = B - A$ (3)

4. Prove Equation (4) of Problem 2–11 using the notion of complement.

SOLUTION:

1. $$(x \in A \cap B^C) \equiv [(x \in A) \wedge (x \in B^C)] \equiv$$

$$\equiv [(x \in A) \wedge (x \in 1 - B)] \equiv [(x \in A) \wedge (x \in B)'] \equiv$$

$$\equiv (x \in A - B). \tag{4}$$

2. Here, we shall apply the law of contraposition:

$$\text{if } \alpha \Rightarrow \beta, \text{ then } \beta' \Rightarrow \alpha' \tag{5}$$

$$(A \subset B) \equiv [(x \in A) \Rightarrow (x \in B)] \equiv$$

$$\equiv [(x \in B)' \Rightarrow (x \in A)'] \equiv [(x \notin B) \Rightarrow (x \notin A)] \equiv$$

$$\equiv [(x \in 1 - B) \Rightarrow (x \in 1 - A)] \equiv [(x \in B^C) \Rightarrow (x \in A^C)] \equiv$$

$$\equiv (B^C \subset A^C). \tag{6}$$

Thus,

$$A \subset B \text{ iff } B^C \subset A^C. \tag{7}$$

3. $$(x \in A^C - B^C) \equiv [(x \in A^C) \wedge (x \in B^C)'] \equiv$$

$$\equiv [(x \in 1 - A) \wedge (x \in 1 - B)'] \equiv [(x \in A)' \wedge (x \in B)''] \equiv$$

$$\equiv [(x \in B) \wedge (x \notin A)] \equiv (x \in B - A). \tag{8}$$

4. We have to prove that

$$(A \cap B) - (A \cap C) = A \cap (B - C) \tag{9}$$

By applying (1), we obtain

$$(A \cap B) - (A \cap C) = (A \cap B) \cap (A \cap C)^C =$$

$$= (A \cap B) \cap [1 - (A \cap C)] = (A \cap B) \cap (A^C \cup C^C) =$$

$$= (A \cap B \cap A^C) \cup (A \cap B \cap C^C) =$$

$$= (A \cap A^C \cap B) \cup (A \cap B \cap C^C) =$$

$$= A \cap B \cap C^C = A \cap (B - C). \tag{10}$$

In (10), we used DeMorgan's theorem.

● PROBLEM 2-15

Let $A_1, A_2, \ldots, A_n$ represent fixed subsets of the space 1. Let us denote

$$A_i^1 = 1 - A_i = A_i^C$$

$$A_i^0 = A_i. \tag{1}$$

A constituent of the space with respect to the sets $A_1, A_2, \ldots, A_n$ is any set of the form

$$A_1^{i_1} \cap A_2^{i_2} \cap \ldots \cap A_n^{i_n}$$

where $i_1, \ldots, i_n = 1$ or 0. (2)

1. Show that constituents are disjoint.

2. Prove that the union of all constituents is equal to 1.

SOLUTION:

1. Let $A_1^{i_1} \cap A_2^{i_2} \cap \ldots \cap A_n^{i_n}$ and $A_1^{k_1} \cap A_2^{k_2} \cap \ldots \cap A_n^{k_n}$ denote two constituents. Then at least one index i_l exists, such that

$$i_l \neq k_l. \tag{3}$$

Then the intersection of these two constituents is

$$A_1^{i_1} \cap A_2^{i_2} \ldots \cap A_n^{i_n} \cap A_1^{k_1} \cap A_2^{k_2} \cap \ldots \cap A_n^{k_n} \cap A_l^{i_l} \cap A_l^{k_l} =$$

$$= \ldots \cap A_l \cap A_l^C = \phi. \tag{4}$$

2. Now we shall show that the union of constituents is equal to 1.

Let $x \in 1$ represent an element of the space. We shall show that a constituent exists, such that x belongs to this constituent. Consider the sets $A_1, A_2, \ldots, A_n$. Two possibilities exist:

$$x \in A_1 \quad \text{or} \quad x \notin A_1.$$

If $x \in A_1 = A_1^0$, we set $i_1 = 0$. If $x \notin A_1$, then $x \in 1 - A_1 = A_1^1$ and we set $i_1 = 1$. We repeat the above procedure for each set $A_1, A_2, \ldots, A_n$ and obtain $i_1, i_2, \ldots, i_n$. The element x belongs to

$$\begin{aligned} x &\in A_1^{i_1} \\ x &\in A_2^{i_2} \\ &\vdots \\ x &\in A_n^{i_n} \end{aligned} \tag{5}$$

Hence,

$$x \in A_1^{i_1} \cap A_2^{i_2} \cap \ldots \cap A_n^{i_n} \tag{6}$$

Since x is an arbitrary element of 1, we conclude that the union of constituents is equal to 1.

● PROBLEM 2-16

Let A, B, and C represent the subsets of the space.

1. Find the constituents of the space with respect to the sets A, B, and C.

2. Represent the set

$$A - (B \cup C) \tag{1}$$

as the union of the constituents.

SOLUTION:

1. We shall consider all possible sets of the form

$$A^{i_1} \cap B^{i_2} \cap C^{i_3} \tag{2}$$

where

$$i_1, i_2, i_3 = 0 \text{ or } 1 \text{ and}$$

$$A^0 = A,\ B^0 = B,\ C^0 = C$$

$$A^1 = A^C,\ B^1 = B^C,\ C^1 = C^C. \tag{3}$$

The constituents are

$$A \cap B \cap C$$

$$A \cap B \cap C^C$$

$$A \cap B^C \cap C$$

$$A^C \cap B \cap C$$

$$A \cap B^C \cap C^C$$

$$A^C \cap B^C \cap C$$

$$A^C \cap B \cap C^C$$

$$A^C \cap B^C \cap C^C. \tag{4}$$

2. Let us write $A - (B \cup C)$ in the form

$$A - (B \cup C) = (A - B) - C = (A \cap B^C) - C =$$

$$= A \cap B^C \cap C^C. \tag{5}$$

Note that $A \cap B^C \cap C^C$ is itself a constituent.

● PROBLEM 2-17

The sets $A_1, A_2, \ldots, A_n$ are called independent if all the constituents are non-empty. Independent sets play an important role in the probability theory. Find the number of constituents of the independent sets $A_1, A_2, \ldots, A_n$.

SOLUTION:

The constituents are

$$A_1^{i_1} \cap A_2^{i_2} \cap \ldots \cap A_n^{i_n} \tag{1}$$

where $i_1, i_2, \ldots, i_n$ take one of two values, 0 or 1.

$$A_i^0 = A_i \text{ and } A_i^1 = 1 - A_i \tag{2}$$

The sets $A_1, A_2, \ldots, A_n$ are independent, therefore the constituents are non-empty sets. Setting

$$i_1 = i_2 = \ldots = i_n = 0 \tag{3}$$

in (1) we obtain

$$A_1 \cap A_2 \cap \ldots \cap A_n \neq \phi . \tag{4}$$

Thus, none of the sets $A_1, A_2, \ldots, A_n$ are empty sets. Similarly, setting $i_1 = i_2 = \ldots i_n = 1$, we find that

$$A_1^c \cap A_2^c \cap \ldots \cap A_n^c \neq \phi . \tag{5}$$

That is, none of the sets $A_1, A_2, \ldots, A_n$ are whole spaces.

Each of the superscripts $i_1, i_2, \ldots, i_n$ in

$$A_1^{i_1} \cap A_2^{i_2} \cap \ldots \cap A_n^{i_n}$$

takes up one of two values. Thus, the total number of combinations is

$$2 \times 2 \times \ldots \times 2 = 2^n. \tag{6}$$

The number of constituents is 2^n.

● PROBLEM 2-18

The symmetric difference of the sets A and B is defined by

$$A \div B = (A - B) \cup (B - A). \tag{1}$$

Prove the following formulas:

1. $A \div B = B \div A$ (2)

2. $A \div (B \div C) = (A \div B) \div C$ (3)
   The symmetric difference is associative.

3. $A \cup B = A \div B \div A \cap B$ (4)

SOLUTION:

We shall use

$$A - B = A \cap B^C \tag{5}$$

to write (1) in the form

$$A \div B = (A \cap B^C) \cup (A^C \cap B). \tag{6}$$

From (6) we obtain

$$(A \div B)^C = (A \cap B^C)^C \cap (A^C \cap B)^C =$$

$$= (A \cap B) \cup (A^C \cap B^C). \tag{7}$$

1. $$A \div B = (A - B) \cup (B - A) = (B - A) \cup (A - B) =$$

$$= B \div A. \tag{8}$$

2. $$A \div (B \div C) = [A \cap (B \div C)^C] \cup [(B \div C) \cap A^C] = \tag{9}$$

$$= \{A \cap [(C \cap B) \cup (C^C \cap B^C)]\} \cup \{[(B \cap C^C) \cup (B^C \cap C)] \cap A^C\} =$$

$$= (A \cap B \cap C) \cup (A \cap B^C \cap C^C) \cup (A^C \cap B \cap C^C) \cup (A^C \cap B^C \cap C).$$

Similarly,

$$(A \div B) \div C = [(A \cap B^C) \cup (A^C \cap B)] \div C = \tag{10}$$

$$= \{[(A \cap B^C) \cup (A^C \cap B)] \cap C^C\} \cup [(A \div B)^C \cap C] =$$

$$= (A \cap B^C \cap C^C) \cup (A^C \cap B \cap C^C) \cup \{[(A \cap B) \cup (A^C \cap B^C)] \cap C\} =$$

$$= (A \cap B^C \cap C^C) \cup (A^C \cap B \cap C^C) \cup (A \cap B \cap C) \cup (A^C \cap B^C \cap C).$$

Comparing (9) and (10) we find

$$(A \div B) \div C = A \div (B \div C) = A \div B \div C. \tag{11}$$

3. $$A \div B \div A \cap B = (A \div B) \div (A \cap B) =$$

$$= [(A \cap B^c) \cup (A^c \cap B)] \div (A \cap B) =$$

$$= \{[(A \cap B^c) \cup (A^c \cap B)] \cap (A^c \cup B^c)\} \cup$$

$$\cup \{[(A^c \cup B) \cap (A \cup B^c)] \cap (A \cap B)\} =$$

$$= [(A \cap B^c) \cap (A^c \cup B^c)] \cup [(A^c \cap B) \cap (A^c \cup B^c)] \cup$$

$$\cup [(A^c \cap B^c) \cap (A \cap B)] \cup [(A \cap B) \cap (A \cap B)] =$$

$$= (A \cap B^c) \cup (A^c \cap B) \cup (A \cap B) =$$

$$= (A - B) \cup (B - A) \cup (A \cap B) = A \cup B. \tag{12}$$

Similarly, we can show that

$$A - B = A \div A \cap B. \tag{13}$$

● PROBLEM 2-19

Show that

$$(A_1 \cup \ldots \cup A_n) \div (B_1 \cup \ldots \cup B_n) \subset (A_1 \div B_1) \cup \ldots \cup (A_n \div B_n) \tag{1}$$

where

$$A \div B = (A - B) \cup (B - A). \tag{2}$$

SOLUTION:

Let us denote

$$A = A_1 \cup \ldots \cup A_n$$

$$B = B_1 \cup \ldots \cup B_n. \tag{3}$$

Suppose

$$x \in A \div B = (A - B) \cup (B - A). \tag{4}$$

The sets $A - B$ and $B - A$ are disjoint sets (i.e., they have no common elements). We can assume that

$$x \in A - B, \tag{5}$$

then $x \in A$ and $(x \notin B) \equiv (x \in B)'$. We have

$$x \in A_1 \cup \ldots \cup A_n \tag{6}$$

and

$$x \notin B_1 \cup \ldots \cup B_n. \tag{7}$$

Therefore a set A_l exists, such that

$$x \in A_l \tag{8}$$

and

$$(x \in B)' \equiv (x \in B_1 \cup \ldots \cup B_n)' \equiv$$

$$\equiv [(x \in B_1) \vee (x \in B_2) \vee \ldots \vee (x \in B_n)]' \equiv$$

$$\equiv (x \in B_1)' \wedge (x \in B_2)' \wedge \ldots \wedge (x \in B_n)' \equiv$$

$$\equiv (x \notin B_1) \wedge (x \notin B_2) \wedge \ldots \wedge (x \notin B_n). \tag{9}$$

Thus,

$$x \in A_l \text{ and } x \notin B_l \tag{10}$$

and

$$x \in A_l \div B_l = (A_l - B_l) \cup (B_l - A_l). \tag{11}$$

Hence,

$$(A_1 \cap \ldots \cap A_n) \div (B_1 \cap \ldots \cap B_n) \subset (A_1 \div B_1) \cup \ldots \cup (A_n \div B_n). \tag{12}$$

● PROBLEM 2-20

DEFINITION OF A RING

The Operations + and · form a commutative ring, if they satisfy the conditions:

1. $x + y = y + x$, $x \cdot y = y \cdot x$ (1)

2. $x + (y + z) = (x + y) + z$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (2)

3. $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ (3)

4. An element 0 exists, such that

$$x + 0 = x \tag{4}$$

5. For every pair x, y, an element z exists, such that

$$x = y + z. \tag{5}$$

■

Show that the sets form a ring with respect to the operations $\div$ and $\cap$ but don't form a ring with respect to the operations $\cup$ and $\cap$.

SOLUTION:

Let us replace in (1) - (5) operation + by $\div$, and operation $\cdot$ by $\cap$. We obtain

$$A \div B = B \div A,\ A \cap B = B \cap A \tag{6}$$

which are obviously true. Then

$$A \div (B \div C) = (A \div B) \div C,\ A \cap (B \cap C) = (A \cap B) \cap C. \tag{7}$$

The relationship

$$A \cap (B \div C) = (A \cap B) \div (A \cap C) \tag{8}$$

is true.

For an empty set ϕ, we obtain

$$A \div \phi = A. \tag{9}$$

To prove condition 5, we have to show that

$$(A = B \div C) \Rightarrow (C = B \div A) \tag{10}$$

which is easy to prove because

$$A = B \div C = B \div B \div A = (B \div B) \div A = A. \tag{11}$$

Hence, operations $\div$ and $\cap$ form a ring.

Consider operations $\cup$ and $\cap$. We have

$$A \cup B = B \cup A,\ A \cap B = B \cap A \tag{12}$$

$$A \cup (B \cup C) = (A \cup B) \cup C,\ A \cap (B \cap C) = (A \cap= B) \cap C. \tag{13}$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \tag{14}$$

$$A \cup \{\phi\} = A \tag{15}$$

It is not true that for every pair A, B, a set C exists, such that

$$A = B \cup C. \tag{16}$$

For example, let $A = \{\phi\}$ and $C \neq \{\phi\}$. Thus, $\cup$ and $\cap$ do not form a ring.

● PROBLEM 2-21

1. Show that the family of all subsets of a given set A is an ideal.

2. Show that the family of all sets B, such that $A \subset B \subset 1$, is a filter.

SOLUTION:

DEFINITION OF IDEAL

A family is a collection of sets. A non-empty family I of subsets of 1 is called ideal if

$$(A \in I) \wedge (B \subset A) \Rightarrow B \in I \tag{1}$$

$$(A \in I) \wedge (B \in I) \Rightarrow (A \cup B \in I). \tag{2}$$

1. Let P denote the family of all subsets of a given set A. If $D \in P$, then $D \subset A$. If $G \subset D$, then $G \subset A$ and $G \in P$. Hence, condition (1) is fulfilled.

If $D \in P$ and $G \in P$, then $D \subset A$ and $G \subset A$. Hence, $D \cup G \subset A$ and $D \cup G \in P$. Therefore, the family of all subsets of a given set is an ideal.

DEFINITION OF A FILTER

A non-empty family F is called a filter if

$$(A \in F) \wedge (A \subset B) \Rightarrow (B \in F) \tag{3}$$

$$(A \in F) \wedge (B \in F) \Rightarrow (A \cap B \in F) \tag{4}$$

■

2. Let A represent a given set and R the family of all sets B, such that $A \subset B \subset 1$. Suppose $D \in R$ and $D \subset G$, then $A \subset D \subset G \subset 1$ and $G \in R$. Condition (3) is fulfilled.

If $D \in R$ and $G \in R$, then $A \subset D$ and $A \subset G$. Therefore, $A \subset D \cap G$ and $D \cap G \in R$. Hence, R is a filter.

● PROBLEM 2-22

Show that a family of sets is a filter if and only if the family of the complements of these sets is an ideal.

SOLUTION:

Let $S = \{A, B, C, \ldots\}$ represent a family of sets. We shall prove that

$$\begin{pmatrix} S = \{A, B, C, \ldots\} \text{ is} \\ \text{a filter} \end{pmatrix} \Rightarrow \begin{pmatrix} P = \{A^c, B^c, C^c, \ldots\} \text{ is} \\ \text{an ideal} \end{pmatrix}$$

Let $A^C \in P$ and $B^C \subset A^C$, then $A \subset B$. But $A \in S$ (which is a filter); hence, $B \in S$. Therefore $B^C \in P$.

Let $A^C \in P$ and $B^C \in P$, then $A \in S$ and $B \in S$. Since S is a filter, $A \cap B \in S$.

But $A^C \cup B^C = (A \cap B)^C \in P$.

Similarly, we can show that

$$\begin{pmatrix} P = \{A^c, B^c, \ldots\} \text{is} \\ \text{an ideal} \end{pmatrix} \Rightarrow \begin{pmatrix} S = \{A, B, C, \ldots\} \text{ is} \\ \text{a filter} \end{pmatrix}$$

Let $A \in S$ and $A \subset B$. Then $A^C \in P$ and $B^C \subset A^C$. Hence, $B^C \in P$ and $B \in S$.

Let $A \in S$ and $B \in S$. Then $A^C \in P$ and $B^C \in P$.

Since P is an ideal, $A^C \cup B^C \in P$ and $A \cap B \ (A^C \cup B^C)^C \in S$.

● PROBLEM 2-23

In 1895, Georg Cantor created a theory of sets. At that time, it was accepted that a universal set (that is, "the set of all sets") exists. In 1902, Bertrand Russell showed in his famous paradox that the admission of a set of all sets leads to a contradiction. Explain Russell's paradox.

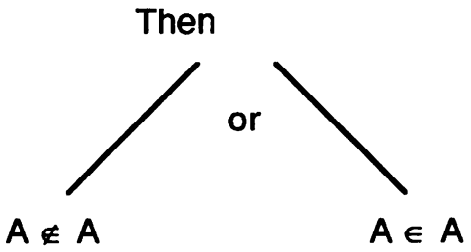
SOLUTION:

We shall show that the assumption that the universal set exists leads to contradictory statements.

THEOREM 1

Suppose that the set of all sets, denoted by W, exists. Let

$$A = \{x \in W : x \notin x\}$$



Indeed,

$$(x \in A) \equiv (x \notin x); \text{ therefore } (A \in A) \equiv (A \notin A).$$

That leads to:

THEOREM 2

A set of all sets does not exist.

This fact was briefly described by Paul Halmos in this statement: "Nothing contains everything." New paradoxes of set theory appeared shortly after Russell's paradox. To overcome this difficulty, mathematicians created several variations of axiomatic set theory. None of these theories was completely satisfactory, nevertheless they eliminated most of the antinomies.

● PROBLEM 2-24

Consider the axiomatic formulation of the algebra of sets. It consists of the concept of element, of set and of the relation of an element belonging to a set ($x \in A$) and four fundamental axioms. Using this system, show that

1. the null set exists.

2. for any two sets A and B, only one set $A \cup B$ exists.

SOLUTION:

We shall list the four fundamental axioms.

I. UNIQUENESS AXIOM (Axiom of Extension).

If the sets A and B have the same elements, then A and B are identical.

II. UNION AXIOM.

For two arbitrary sets A and B, a set exists which contains all the elements of the set A, and all the elements of the set B and which does not contain any other element.

III. DIFFERENCE AXIOM.

For two arbitrary sets A and B, a set exists which contains those and only those elements of the set A which are not elements of the set B.

IV. EXISTENCE AXIOM.

At least one set exists.

1. Let us define the null set by the formula

$$\phi = A - A.$$

The existence of at least one set is guaranteed by Axiom IV.

2. For given sets A and B, one and only one set satisfying Axiom II exists. Hence, the operation $\cup$ is unique.

Since

$$A \cap B = A - (A - B),$$

it is not necessary to include an axiom on the existence of an intersection. The intersection can be defined in terms of the difference.

● PROBLEM 2-25

Explain the system of axioms called Boolean algebra.

SOLUTION:

Note that in most of the theorems of the algebra of sets the symbol $\in$ does not appear, although it occurs in their proofs. This suggests the possibility of establishing the system of axioms, which will enable us to prove the theorems without referring to the relation $\in$.

BOOLEAN ALGEBRA

The fundamental concepts are:

The set ϕ and the operations $\cup$, $\cap$, $-$.

We assume the following axioms:

1. $A \cup B = B \cup A$

2. $A \cap B = B \cap A$

3. $A \cup (B \cup C) = (A \cup B) \cup C$

4. $A \cap (B \cap C) = (A \cap B) \cap C$

5. $A \cup \phi = A$

6. $A \cap (A \cup B) = A$

7. $A \cup (A \cap B) = A$

8. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

9. $(A - B) \cup B = A \cup B$

10. $B \cap (A - B) = \phi$

11. $A \cap 1 = A$

12. $(A \subset B) \equiv (A \cup B = B)$

From the above axioms we can deduce all the theorems of the algebra of sets. Axiom 12 defines the Inclusion Theory of sets as only one of the applications of Boolean algebra. Replacing in axioms the notion of a set by sentences, we obtain the algebra of sentences. This explains the close relationship between the algebra of sentences and the algebra of sets.

Sentence		Sets
$\vee$	corresponds to	$\cup$
$\wedge$	corresponds to	$\cap$
$'$ (negation)	corresponds to	^{c} (complement)

By omitting axioms 8 - 12, we obtain the axiomatic formulation of the lattice theory, applied, for example, in electrical networks.

CHAPTER 3

EQUIVALENCE RELATIONS, AXIOMATIC SET THEORY, QUANTIFIERS

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Axiomatic Formulation, Axiom of Choice	3-26

1. The set of all values of the variable x, for which $\varphi(x)$ is a true sentence, is denoted by the symbol

$$\{x : \varphi(x)\}. \tag{1}$$

The following equivalence holds:

$$\text{For every } a\text{: } [a \in \{x : \varphi(x)\}] \equiv \varphi(a). \tag{2}$$

Using (1) and (2), describe the set of the real numbers which are larger than zero and smaller than or equal to 5.

2. Prove the following formulas:

$$\{x : \varphi(x) \vee \zeta(x)\} = \{x : \varphi(x)\} \cup \{x : \zeta(x)\} \tag{3}$$

$$\{x : \zeta(x) \wedge \varphi(x)\} = \{x : \zeta(x)\} \cap \{x : \varphi(x)\} \tag{4}$$

$$\{x : \varphi(x) \wedge [\zeta(x)]'\} = \{x : \varphi(x)\} - \{x : \zeta(x)\}$$

$$\text{and } \{x : [\varphi(x)]'\} = \{x : \varphi(x)\}^C. \tag{5}$$

SOLUTION:

1. The set can be described as follows:

$$\{x : (x \in R) \wedge (x > 0) \wedge (x \leq 5)\}. \tag{6}$$

2.
$$a \in \{x : \varphi(x) \vee \zeta(x)\} \equiv \varphi(a) \vee \zeta(a) \equiv$$

$$\equiv [a \in \{x : \varphi(x)\}] \vee [a \in \{x : \zeta(x)\}] \equiv$$

$$\equiv a \in \{x : \varphi(x)\} \cup \{x : \zeta(x)\}. \tag{7}$$

We employ the strategy used in the solution of (3) to prove (4).

$$a \in \{x : \zeta(x) \wedge \varphi(x)\} \equiv \zeta(a) \wedge \varphi(a) \equiv$$

$$\equiv [a \in \{x : \zeta(x)\}] \wedge [a \in \{x : \varphi(x)\}] \equiv$$

$$\equiv a \in \{x : \zeta(x)\} \cap \{x : \varphi(x)\}. \tag{8}$$

Proving (5),

$$a \in \{x : \varphi(x) \wedge [\zeta(x)]'\} \equiv \varphi(a) \wedge [\zeta(a)]' \equiv$$

$$\equiv [a \in \{x : \varphi(x)\}] \wedge [a \notin \{x : \zeta(x)\}] \equiv$$

$$\equiv a \in \{x : \varphi(x)\} - \{x : \zeta(x)\}. \tag{9}$$

Proving (6),

$$a \in \{x : [\varphi(x)]'\} \equiv [\varphi(a)]' \equiv$$

$$\equiv (a \in 1) \wedge [a \notin \{x : \varphi(x)\}] \equiv$$

$$\equiv a \in \{x : \varphi(x)\}^C. \tag{10}$$

The symbol $\{x : \varphi(x)\}$ will be used very often, sometimes with some modifications, throughout this book.

● PROBLEM 3-2

1. Using qualifiers, write the sentence: For every natural number, a larger natural number exists.

2. Explain the quantifiers by using the calculus of sentences.

SOLUTION:

1. The quantifier "there exists" is denoted by $\exists$ and the quantifier "for each" is denoted by $\forall$.

Let A represent the domain of a general statement $p(x)$, i.e., $x \in A$. Then

$$\forall x \in A : \ p(x) \tag{1}$$

asserts that for all $x \in A$, the statement $p(x)$ is true. Similarly,

$$\exists x \in A : \ p(x) \tag{2}$$

means that at least one $x \in A$ exists, such that $p(x)$ is true.

By denoting the set of natural numbers by N, we write

$$\forall n \ \exists m : n < m \tag{3}$$

where $m, n \in N$.

2. Suppose the elements of the set A can be written in the form $a_1, a_2, a_3, \ldots$. The sentence $\forall x : p(x)$ means that $p(x)$ is true for each $a_1, a_2, a_3, \ldots$. Thus

$$\forall x : p(x) \text{ is equivalent to } p(a_1) \wedge p(a_2) \wedge \ldots \tag{4}$$

Similarly, the sentence $\exists x : p(x)$ means that $p(x)$ is true for at least one of

the elements $a_1, a_2, a_3, \ldots$. Thus

$$\exists x : p(x) \text{ is equivalent to } p(a_1) \vee p(a_2) \vee p(a_3) \ldots \tag{5}$$

Note that a universal quantifier $\forall$ is "stronger" than existential quantifier $\exists$.

$$[\forall x : p(x)] \Rightarrow [\exists x : p(x)]. \tag{6}$$

● PROBLEM 3-3

1. Using quantifiers write the statement: For each x there exists a y such that for all z the statement $p(x, y, z)$ is true.

2. Write the negation of this statement.

SOLUTION:

We shall apply the following rule of quantifier negation:

$$[\forall x : p(x)]' \equiv \exists x : [p(x)]' \tag{1}$$

$$[\exists x : p(x)]' \equiv \forall x : [p(x)]' \tag{2}$$

Equations (1) and (2) are generalized DeMorgan formulas.

Note that by taking negation of (1), we can define a universal quantifier in terms of the existential quantifier and negation

$$\forall x : p(x) \equiv [\exists x : [p(x)]']'. \tag{3}$$

Similarly, existential quantifiers can be defined in terms of the universal quantifier and negation

$$\exists x : p(x) \equiv [\forall x : [p(x)]']'. \tag{4}$$

1. This statement can be written as

$$\forall x \, \exists y \, \forall z : p(x, y, z). \tag{5}$$

2. The negation of (5) is

$$[\forall x \, \exists y \, \forall z : p(x, y, z)]' \equiv$$

$$\equiv [\forall x \, [\exists y \, \forall z : p(x, y, z)]]' \equiv$$

$$\equiv \exists x \, [\exists y \, \forall z : p(x, y, z)]'. \tag{6}$$

Here we used (1).

$$\equiv \exists x\,[\exists y\,[\forall z : p(x, y, z)]]' \equiv$$

$$\equiv \exists x\,\forall y\,[\forall z : p(x, y, z)]' \equiv$$

$$\equiv \exists x\,\forall y\,\exists z : [p(x, y, z)]' \equiv \tag{7}$$

● PROBLEM 3-4

Using quantifiers, write down the definitions of:

1. Limit of a sequence of real numbers.
2. Continuous functions.
3. Uniformly continuous functions.

SOLUTION:

1. Let (x_n) denote a sequence of real numbers. This sequence is said to have a limit x. We write

$$\lim_{n \to \infty} x_n = x \tag{1}$$

when

$$\forall\, \varepsilon > 0 \quad \exists\, m \in N \quad \forall\, n \geq m \quad |x_n - x| < \varepsilon. \tag{2}$$

2. Let $f : R^1 \to R^1$ be a function continuous at x_0. Then

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x \in R^1 \quad |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon. \tag{3}$$

Function $f(x)$ is continuous at x_0 if, for every $\varepsilon > 0$, $\delta > 0$ exists such that for every $x \in R^1$, if $|x - x_0| < \delta$, implies $|f(x) - f(x_0)| < \varepsilon$.

3. Function $f : R^1 \to R^1$ is said to be uniformly continuous on R^1 if

$$\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad \forall\, x, x' \in R^1 \quad |x - x'| < \delta \Rightarrow |f(x) - f(x')| < \varepsilon. \tag{4}$$

From (3) and (4) we conclude that if a function is uniformly continuous on R^1, then it is continuous on R^1. The opposite is not true. For example, function $f(x) = x^2$ is continuous on R^1 but not uniformly continuous.

1. Is the following definition of an ordered pair correct?

$$(a, b) = \{\{a\}, \{a, b\}\}. \tag{1}$$

2. Find the Cartesian product $A \times B$, where $A = [1, 2]$ and $B = [2, 3]$.

3. Let Z denote the set of all positive and negative integers and zero. Find $A \times B$, where $A = B = Z$.

SOLUTION:

1. To each of two elements a, b there corresponds their ordered pair (a, b), which satisfies the condition

$$(a, b) = (c, d) \text{ if, and only if } a = c \text{ and } b = d. \tag{2}$$

Hence, in general $(a, b) \neq (b, a)$. But

$$(a, b) = (b, a) \text{ iff } a = b. \tag{3}$$

Definition (1) satisfies condition (2) because

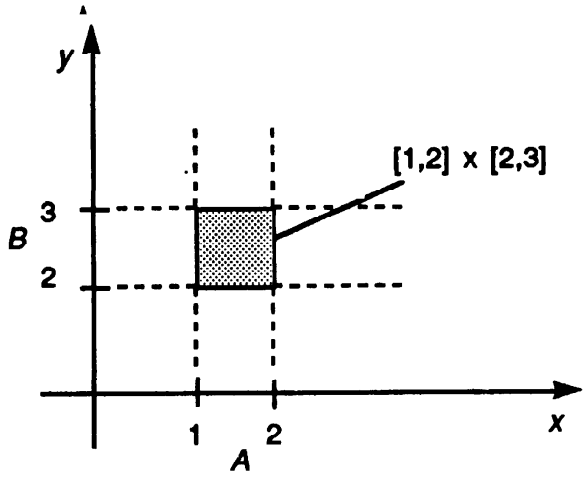
$$\{a, \{a, b\}\} = \{c, \{c, d\}\} \text{ if, and only if } a = c \text{ and } b = d.$$

Thus, definition (1) is correct.

2. **DEFINITION**

Let A and B represent two sets. Their Cartesian product, $A \times B$, is the set of all ordered couples

$$A \times B = \{(a, b) : a \in A,\ b \in B\}. \tag{4}$$



The Cartesian product is represented by the shaded area.

FIGURE 1.

3. The set $A \times B$, where $A = B = Z$ represents the set of lattice points in R^2.

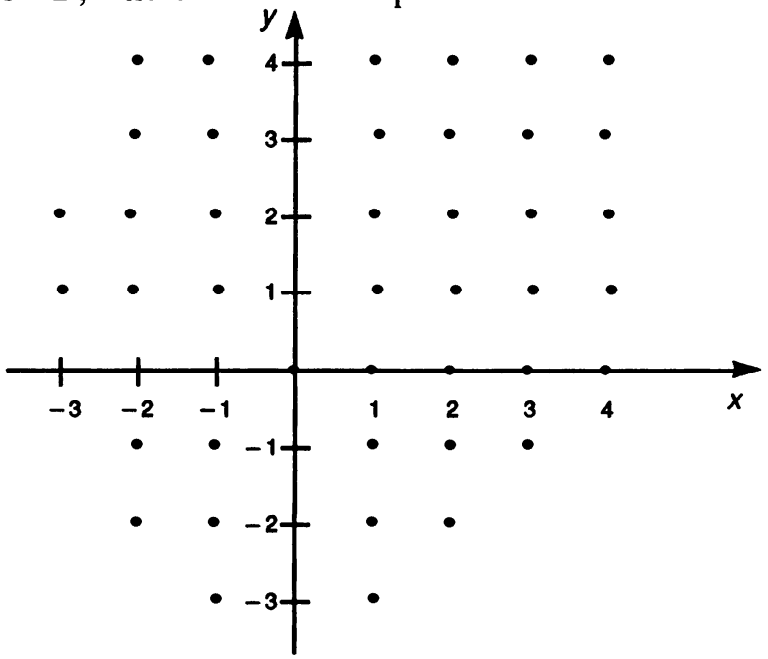


FIGURE 2.

Dots represent the elements of the set $Z \times Z$.

● PROBLEM 3-6

1. Let A represent the circle in the xy-plane, $x^2 + y^2 \leq 1$ and let B represent the set of points along the z coordinate, such that $0 \leq z \leq 1$. Find $A \times B$.

2. Show that,

$$(A \times B = \phi) \Leftrightarrow [(A = \phi) \vee (B = \phi)]. \quad (1)$$

3. Show that,

if $A \times B \neq \phi$, then $A \times B \subset C \times D$ iff

$$(A \subset C) \wedge (B \subset D). \quad (2)$$

SOLUTION:

1. The set $A \times B$ is a cylinder of altitude 1, shown in Figure 1.

2. If $A \times B \neq \phi$, then an element exists $(a, b) \in A \times B$. Thus, $a \in A$ and $b \in B$ and $A \neq \phi$ and $B \neq \phi$.

Similarly, if $A \neq \phi$ and $B \neq \phi$, then elements a, b exist, such that $a \in A$ and $b \in B$. Therefore, $(a, b) \in A \times B$ and $A \times B \neq \phi$.

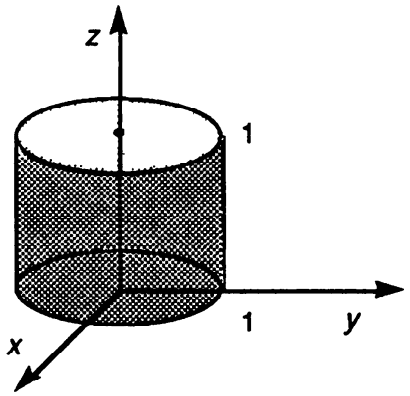


FIGURE 1.

3. We shall prove that

$$(A \times B \subset C \times D) \Leftrightarrow (A \subset C) \wedge (B \subset D). \tag{3}$$

$\Rightarrow$

Let $(a, b) \in A \times B$. Then $(a, b) \in C \times D$. Therefore, if $a \in A$, then $a \in C$. Similarly, if $b \in B$, then $b \in D$.

$\Leftarrow$

Let $(a, b) \in A \times B$. Then $a \in A$ and $a \in C$. Also $b \in B$ and $b \in D$. Therefore $(a, b) \in C \times D$ and

$$A \times B \subset C \times D. \tag{4}$$

● PROBLEM 3-7

Show that

$$A \times (B \cup C) = A \times B \cup A \times C \tag{1}$$

$$A \times (B \cap C) = A \times B \cap A \times C \tag{2}$$

$$A \times (B - C) = A \times B - A \times C \tag{3}$$

SOLUTION:

$$A \times (B \cup C) = \{(x, y) : (x \in A) \wedge (y \in B \cup C)\} =$$

$$= \{(x, y) : (x \in A) \wedge [(y \in B) \vee (y \in C)]\} =$$

$$= \{(x, y) : [(x \in A) \wedge (y \in B)] \vee [(x \in A) \wedge (y \in C)]\} =$$

$$= \{(x, y) : [(x, y) \in A \times B] \vee [(x, y) \in A \times C]\} =$$

$$= A \times B \cup A \times C. \tag{4}$$

Similarly, we prove (2).

$$A \times (B \cap C) = \{(x, y) : (x \in A) \wedge (y \in B \cap C)\} =$$

$$= \{(x, y) : (x \in A) \wedge [(y \in B) \wedge (y \in C)]\} =$$

$$= \{(x, y) : [(x, y) \in A \times B] \wedge [(x, y) \in A \times C]\} =$$

$$= A \times B \cap A \times C. \tag{5}$$

And (3).

$$A \times (B - C) = \{(x, y) : (x \in A) \wedge (y \in B - C)\} =$$

$$= \{(x, y) : (x \in A) \wedge [(y \in B) \wedge (y \notin C)]\} =$$

$$= \{(x, y) : [(x, y) \in A \times B] \wedge [(x, y) \notin A \times C]\} =$$

$$= A \times B - A \times C. \tag{6}$$

● PROBLEM 3-8

1. Prove that:

$$(A \times C) \cap (B \times D) = (A \cap B) \times (C \cap D) \tag{1}$$

$$(A \times C) \cup (B \times D) \subset (A \cup B) \times (C \cup D) \tag{2}$$

where $A, B \subset X$ and $C, D \subset Y$.

2. $$A \times B = (A \times Y) \cap (X \times B) \tag{3}$$

$$(A \times B)^C = (A^C \times Y) \cup (X \times B^C) \tag{4}$$

where $A \subset X$ and $B \subset Y$. The symbols A^c and B^c denote the complements with respect to X and Y, and $(A \times B)^c$ denotes the complement with respect to $X \times Y$.

SOLUTION:

1. $$[(x, y) \in (A \times C) \cap (B \times D)] \equiv$$

$$\equiv [(x, y) \in A \times C] \wedge [(x, y) \in B \times D] \equiv$$

$$\equiv (x \in A) \wedge (y \in C) \wedge (x \in B) \wedge (y \in D) \equiv$$

$$\equiv [(x \in A) \wedge (x \in B)] \wedge [(y \in C) \wedge (y \in D)] \equiv$$

$$\equiv (x \in A \cap B) \wedge (y \in C \cap D) \equiv$$

$$\equiv (x, y) \in (A \cap B) \times (C \cap D). \tag{5}$$

Applying Equation (1) of Problem 3-7 we find:

$$(A \cup B) \times (C \cup D) = [A \times (C \cup D)] \cup [B \times (C \cup D)] =$$

$$= (A \times C) \cup (A \times D) \cup (B \times C) \cup (B \times D). \tag{6}$$

Equation (2) follows from (6):

$$(A \times C) \cup (B \times D) \subset (A \cup B) \times (C \cup D). \tag{7}$$

2. Let us write (1) in the form:

$$(A \times Y) \cap (X \times B) = (A \cap X) \times (Y \cap B). \tag{8}$$

Since $A \subset X$, $A \cap X = A$. Similarly, since $B \subset Y$, $Y \cap B = B$.
Therefore,

$$A \times B = (A \times Y) \cap (X \times B) . \tag{9}$$

To prove (4), we shall apply DeMorgan's law. Since

$$A \times B = (A \cap X) \times (Y \cap B) = (A \times Y) \cap (X \times B) \tag{10}$$

we obtain

$$(A \times B)^c = (A \times Y)^c \cup (X \times B)^c. \tag{11}$$

But

$$(A \times Y)^c = A^c \times Y \quad \text{and} \tag{12}$$

$$(X \times B)^c = X \times B^c. \tag{13}$$

Therefore

$$(A \times B)^C = (A^C \times Y) \cup (X \times B^C). \tag{14}$$

● PROBLEM 3-9

1. Give a definition of a family of sets and show that

$$\{A_n : n \in N\} \tag{1}$$

where $A_n =]-n, n[$ is a family of sets.

2. Is it true, that any set of sets can be converted to a family of sets?

SOLUTION:

DEFINITION OF A FAMILY OF SETS

The collection of sets

$$\{A_\lambda : \lambda \in \Lambda\} \tag{2}$$

is called a family of sets if, to each element λ of some set $\Lambda \neq \phi$, a set A_λ corresponds. Set Λ is called an indexing set for the family.

Unlike the definition of a set, where all elements have to be different, in a family of sets it is not required that sets with distinct indices be different. Equation (1) defines a family of sets, where N is the indexing set. Set A_n is an open interval

$$]-n, n[= \{x \in R' : -n < x < n\}. \tag{3}$$

2. Let G represent any set of sets. Then we can use G itself as an indexing set and assign the set it represents to each member of the set G.

● PROBLEM 3-10

Let

$$A_n =]-\frac{1}{n}, \frac{1}{n}[\tag{1}$$

where the indexing set is the set of all positive integers Z^+

$$Z^+ = \{1, 2, 3, \dots\}. \tag{2}$$

Find

$$\bigcup_n A_n$$

and

$$\bigcap_n A_n .$$

SOLUTION:

Let X represent a given set, and $\{A_\lambda : \lambda \in \Lambda\}$ a family of subsets of X. The union, denoted by $\bigcup_\lambda A_\lambda$, is the set

$$\{x \in X : \exists \lambda \in \Lambda; x \in A_\lambda\}. \tag{3}$$

Hence, for the family of sets defined by (1), we obtain

$$\bigcup_n A_n = \bigcup_n \left]-\frac{1}{n}, \frac{1}{n}\right[=]-1,1[. \tag{4}$$

The intersection, denoted by $\bigcap_\lambda A_\lambda$, is the set

$$\{x \in X : \forall \lambda \in \Lambda; x \in A_\lambda\}. \tag{5}$$

Thus

$$\bigcap_n A_n = \bigcap_n \left]-\frac{1}{n}, \frac{1}{n}\right[= \{0\} \tag{6}$$

since 0 is the only common point of all intervals $\left]-\frac{1}{n}, \frac{1}{n}\right[$.

Obviously, the union and intersection of a family of sets does not depend on how the family is indexed.

● PROBLEM 3-11

1. Show that $\bigcap_\lambda$ is distributive over $\cup$ and $\bigcup_\lambda$ is distributive over $\cap$. That is, show that:

$$[\cap \{A_\lambda : \lambda \in \Lambda\}] \cup [\cap \{B_\omega : \omega \in \Omega\}] =$$

$$= \cap \{A_\lambda \cup B_\omega : (\lambda, \omega) \in \Lambda \times \Omega\} \tag{1}$$

$$[\cup \{A_\lambda : \lambda \in \Lambda\}] \cap [\cup \{B_\omega : \omega \in \Omega\}] =$$

$$= \cup \{A_\lambda \cap B_\omega : (\lambda, \omega) \in \Lambda \times \Omega\} \tag{2}$$

2. Show that

$$\left(\bigcup_{\lambda} A_{\lambda}\right)^{C} = \bigcap_{\lambda} A_{\lambda}^{C} \tag{3}$$

and

$$\left(\bigcap_{\lambda} A_{\lambda}\right)^{C} = \bigcup_{\lambda} A_{\lambda}^{C} \tag{4}$$

where complements are taken with respect to X.

SOLUTION:

1.
$$x \in [\cap \{A_{\lambda} : \lambda \in \Lambda\}] \cup [\cap \{B_{\omega} : \omega \in \Omega\}] \equiv$$

$$\equiv [\forall \lambda \in \Lambda; x \in A_{\lambda}] \vee [\forall \omega \in \Omega; x \in B_{\omega}] \equiv$$

$$\equiv [\forall (\lambda, \omega) \in \Lambda \times \Omega : x \in A_{\lambda} \cup B_{\omega}] \equiv$$

$$\equiv x \in \cap \{A_{\lambda} \cup B_{\omega} : (\lambda, \omega) \in \Lambda \times \Omega\}. \tag{5}$$

In the same manner, we prove (2)

$$x \in [\cup \{A_{\lambda} : \lambda \in \Lambda\}] \cap [\cup \{B_{\omega} : \omega \in \Omega\}] \equiv$$

$$\equiv [\, x \in \cup A_{\lambda}] \wedge [x \in \cup B_{\omega}] \equiv$$

$$\equiv (\exists \lambda \in \Lambda : x \in A_{\lambda}) \wedge (\exists \omega \in \Omega : x \in B_{\omega})$$

$$\equiv [\exists (\lambda, \omega) \in \Lambda \times \Omega : x \in A_{\lambda} \cap B_{\omega}] \equiv$$

$$\equiv x \in \cup \{A_{\lambda} \cap B_{\omega} : (\lambda, \omega) \in \Lambda \times \Omega\}. \tag{6}$$

2.
$$x \in \left(\bigcup_{\lambda} A_{\lambda}\right)^{C} \equiv x \in X - \left(\bigcup_{\lambda} A_{\lambda}\right) \equiv$$

$$\equiv x \notin \left(\bigcup_{\lambda} A_{\lambda}\right) \equiv [\, \exists \lambda \in \Lambda : x \in A_{\lambda}]' \equiv$$

$$\equiv \forall \lambda \in \Lambda : x \notin A_{\lambda} \equiv \forall \lambda \in \Lambda : x \in A_{\lambda}^{C} \equiv$$

$$\equiv x \in \bigcap_{\lambda} A_{\lambda}^{C}. \tag{7}$$

Similarly,
$$x \in \left(\bigcap_{\lambda} A_{\lambda}\right)^{C} \equiv x \notin \left(\bigcap_{\lambda} A_{\lambda}\right) \equiv$$

$$\equiv [\, \forall \lambda \in \Lambda : x \in A_{\lambda}]' \equiv [\, \exists \lambda \in \Lambda : x \notin A_{\lambda}] \equiv$$

$$\equiv [\, \exists \lambda \in \Lambda : x \in A_{\lambda}^{C}] \equiv x \in \bigcup_{\lambda} A_{\lambda}^{C}. \tag{8}$$

Let $\{A_n : n \in N\}$ be a family of sets and let

$$B_k = A_0 \cup A_1 \cup \ldots \cup A_k, \quad k = 0, 1, 2, \ldots \tag{1}$$

Show that the union

$$\bigcup_0^\infty A_n = A_0 \cup (A_1 - B_0) \cup \ldots \cup (A_n - B_{n-1}) \cup \ldots \tag{2}$$

is pairwise disjoint.

SOLUTION:

First, we shall prove (2). Suppose

$$x \in \bigcup_0^\infty A_n \tag{3}$$

and $x \in A_0$. Then $x \in A_0 \cup (A_1 - B_0) \cup \ldots$. Now, suppose A_m is the first set in the sequence

$$A_0, A_1, \ldots, A_m, \ldots \tag{4}$$

such that

$$x \notin A_0, x \notin A_1, \ldots, x \notin A_{m-1}, x \in A_m, \ldots \tag{5}$$

Then

$$x \in \bigcup_0^\infty A_n \text{ and}$$

$$x \in A_m - B_{m-1} = A_m - (A_0 \cup \ldots \cup A_{m-1}). \tag{6}$$

That proves (2).

To show that the union

$$A_0 \cup (A_1 - B_0) \cup \ldots \cup (A_m - B_{m-1}) \cup \ldots \tag{7}$$

consists of pairwise disjoint sets, suppose

$$x \notin A_0, x \notin A_1, \ldots, x \notin A_{m-1}, x \in A_m, \tag{8}$$

Then

$$x \notin A_0, x \notin A_1 - B_0, x \notin A_2 - B_1, \ldots \tag{9}$$

but

$$x \in A_m - B_{m-1}. \tag{10}$$

Again

$$x \notin A_{m+1} - B_m = A_{m+1} - (A_1 \cup \ldots \cup A_m). \tag{11}$$

Hence the sets $A_0, A_1 - B_0, \ldots, A_n - B_{n-1}, \ldots$ in (7) are pairwise disjoint.

1. Let ρ denote the relation $<$ on $A \times B$, where $A = \{1, 3, 4, 5\}$ and $B = \{3, 4, 5\}$. Write all elements of ρ, its domain and range. Plot ρ on a co-ordinate diagram.

2. Let R be the set of real numbers and ρ the relation defined by

$$x \rho y \text{ iff both } x \in [n, n+1] \text{ and } y \in [n, n+1]$$

for some integer n.
Plot the relation ρ.

SOLUTION:

We shall begin with a definition:

DEFINITION

A relation ρ is a subset $\rho \subset A \times B$. Often, $(a, b) \in \rho$ is written $a \rho b$. ■

1. A relation ρ consists of ordered pairs $(a, b) \in A \times B$, such that $a < b$. Therefore,

$$\rho = \{(1, 3), (1, 4), (1, 5), (3, 4), (3, 5), (4, 5)\}. \quad (1)$$

Note that the domain of ρ is the set of the first coordinates of the pairs in ρ. From (1) we have the

$$\text{domain of } \rho = \{1, 3, 4\}. \quad (2)$$

Similarly, the range of a relation is the set of the second coordinates of the pairs in this relation.
Hence,

$$\text{range of } \rho = \{3, 4, 5\}. \quad (3)$$

Figure 1 depicts the sets A and B and the relation $<$.

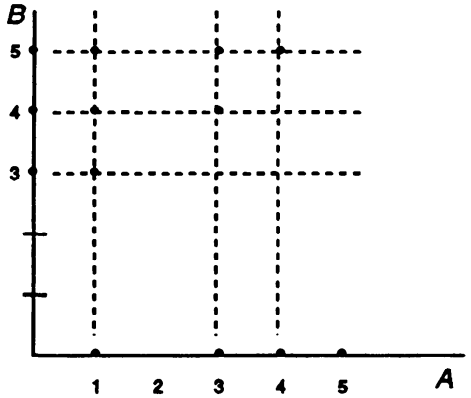


FIGURE 1.

2. Suppose $n = 0$, then

$$x \rho y \text{ if } x \in [0, 1] \quad \text{and} \quad y \in [0, 1],$$

which is a square shown in Figure 2.

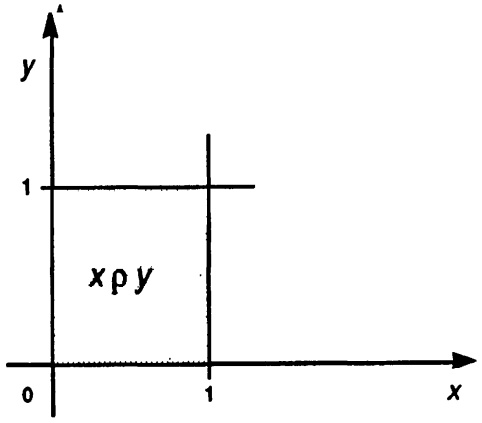


FIGURE 2.

For each integer n we obtain a square as shown in Figure 3.

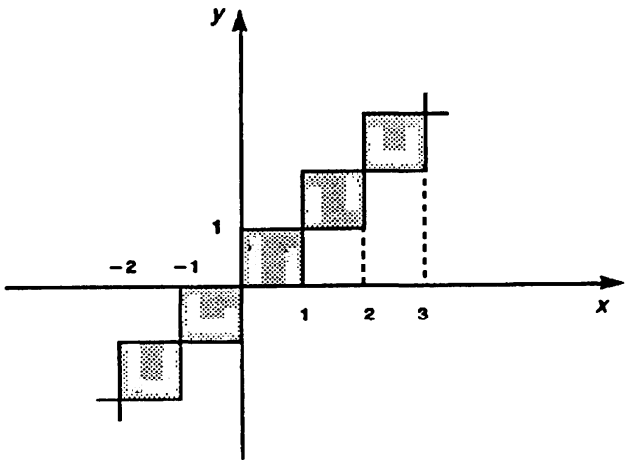


FIGURE 3.

● PROBLEM 3-14

1. Let Z denote the set of all integers. Find the diagonal of Z.

2. Let N denote the set of natural numbers and ρ, a relation defined by

$$n \rho m \text{ iff } n = 3m + 1. \qquad (1)$$

Find the inverse of ρ, denoted by ρ^{-1}.

SOLUTION:

1. For any set A, its diagonal Δ is the relation of equality defined by

$$\Delta = \{(a, a) : a \in A\} \subset A \times A. \tag{2}$$

Thus, if Z is the set of all integers, its diagonal is the set

$$\Delta = \{(n, n) : n \in Z\} =$$

$$= \{\ldots (-2, -2), (-1, -1), (0, 0), (1, 1), (2, 2), \ldots\}. \tag{3}$$

2. Let ρ denote a relation defined as $A \times B$. The inverse of ρ, denoted by ρ^{-1}, is the subset of $B \times A$, defined by

$$\rho^{-1} = \{(b, a) : (a, b) \in \rho\}. \tag{4}$$

Relation ρ, defined by (1), consists of all the pairs

$$\rho = \{(1, 0), (4, 1), (7, 2), (10, 3), (-2, -1), (-5, -2) \ldots\} \tag{5}$$

such that

$$(n, m) \in \rho \text{ if, and only if } n = 3m + 1. \tag{6}$$

The inverse ρ^{-1} consists of all the pairs $(m, n) \in \rho^{-1}$, such that

$$(m,n) \in \rho^{-1} \text{ iff } m = \frac{n-1}{3}. \tag{7}$$

That leads to

$$\rho^{-1} = \{(0, 1), (1, 4), (2, 7), (3, 10), (-1, -2), (-2, -5) \ldots\} \tag{8}$$

● PROBLEM 3-15

Consider three sets A, B, and C.

$$A = \{1, 2, 3, 4\}$$

$$B = \{5, 6, 7, 8\}$$

$$C = \{x, y, z\}. \tag{1}$$

Let ρ be a relation on $A \times B$

$$\rho = \{(1, 6), (1, 7), (3, 6), (3, 7), (3, 8), (4, 5)\} \tag{2}$$

and λ be a relation on $B \times C$

$$\lambda = \{(5, x), (6, y), (7, y), (7, z), (8, x)\}. \tag{3}$$

Find the composition of ρ and λ denoted by $\lambda \circ \rho$ (sometimes the symbol $\rho \circ \lambda$ is used instead).

SOLUTION:

We shall start with a definition of a composition of relations. Let A, B, C represent the sets, and $\rho \subset A \times B$ and $\lambda \subset B \times C$ represent two relations. The composition of ρ and λ, denoted by $\lambda \circ \rho$, is the relation on $A \times C$, such that

$$\lambda \circ \rho = \{(a, c) : a \in A,\ c \in C,\ \exists b \in B \text{ such that}$$

$$(a, b) \in \rho \wedge (b, c) \in \lambda\}. \tag{4}$$

Therefore, from (2), (3), and definition (4), we find

$(1, y) \in \lambda \circ \rho$ because $(1, 6) \in \rho$ and $(6, y) \in \lambda$

$(4, x) \in \lambda \circ \rho$ because $(4, 5) \in \rho$ and $(5, x) \in \lambda$

$(1, z) \in \lambda \circ \rho$ because $(1, 7) \in \rho$ and $(7, z) \in \lambda$

$(3, y) \in \lambda \circ \rho$ because $(3, 6) \in \rho$ and $(6, y) \in \lambda$

$(3, z) \in \lambda \circ \rho$ because $(3, 7) \in \rho$ and $(7, z) \in \lambda$

$(3, x) \in \lambda \circ \rho$ because $(3, 8) \in \rho$ and $(8, x) \in \lambda$

$(4, x) \in \lambda \circ \rho$ because $(4, 5) \in \rho$ and $(5, x) \in \lambda$

The composition $\lambda \circ \rho$ is the set consisting of

$$\lambda \circ \rho = \{(1, y), (4, x), (1, z), (3, y), (3, z), (3, x), (4, x)\}. \tag{5}$$

The relation $\lambda \circ \rho$ is illustrated in Figure 1.

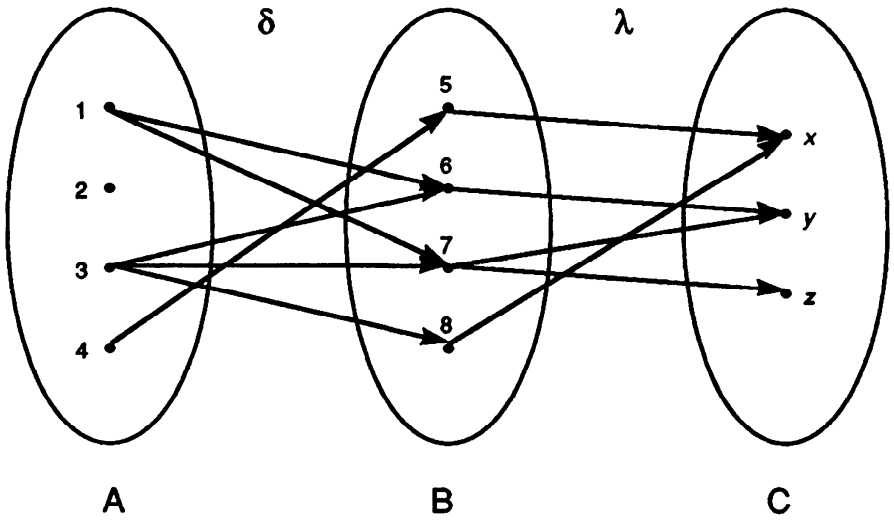


FIGURE 1.

For example $(3, x) \in \lambda \circ \rho$ because there is a "route" from 3 to x via 8.

● PROBLEM 3-16

Let A, B, C, and D represent the sets and $\rho \subset A \times B$, $\lambda \subset B \times C$, and $\delta \subset C \times D$ represent the relations. Prove that:

1. $(\delta \circ \lambda) \circ \rho = \delta \circ (\lambda \circ \rho)$ (1)

2. $(\lambda \circ \rho)^{-1} = \rho^{-1} \circ \lambda^{-1}$ (2)

SOLUTION:

1.

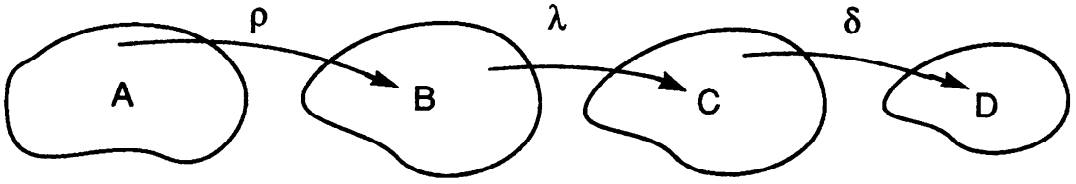


FIGURE 1.

$$(\delta \circ \lambda) \circ \rho = \{(a, d) : a \in A \wedge d \in D, \exists b \in B \text{ such that}$$

$$(a, b) \in \rho \wedge (b, d) \in \delta \circ \lambda\} =$$

$$= \{(a, d) : \exists b \in B, \exists c \in C \text{ such that}$$

$$(a, b) \in \rho, (b, c) \in \lambda, (c, d) \in \delta\} =$$

$$= \{(a, d) : \exists c \in C \text{ such that } (a, c) \in \lambda \circ \rho, (c, d) \in \delta\} =$$

$$= \delta \circ (\lambda \circ \rho). \quad (3)$$

Hence, we can write

$$(\delta \circ \lambda) \circ \rho = \delta \circ (\lambda \circ \rho) = \delta \circ \lambda \circ \delta. \quad (4)$$

2. $$(\lambda \circ \rho)^{-1} = \{(c, a) : (a, c) \in \lambda \circ \rho\} =$$

$$= \{(c, a) : \exists b \in B, \text{ such that } (a, b) \in \rho, (b, c) \in \lambda\} =$$

$$= \{(c, a) : \exists b \in B, \text{ such that } (b, a) \in \rho^{-1}, (c, b) \in \lambda^{-1}\} =$$

$$= \{(c, a) : \exists b \in B, \text{ such that } (c, b) \in \lambda^{-1}, (b, a) \in \rho^{-1}\} =$$

$$= \rho^{-1} \circ \lambda^{-1}. \tag{5}$$

Similarly,

$$(\delta \circ \lambda \circ \rho)^{-1} = [\delta \circ (\lambda \circ \rho)]^{-1} =$$

$$= (\lambda \circ \rho)^{-1} \circ \delta^{-1} = \rho^{-1} \circ \lambda^{-1} \circ \delta^{-1}. \tag{6}$$

● PROBLEM 3-17

Let A represent the set of all people and R represent a relation defined as $A \times A$, such that for $a, b \in A$ and $a\ R\ b$ iff a knows b; that is, a is in relation to b if and only if a knows b. Is this relation an equivalence relation?

SOLUTION:

An equivalence relation is a very useful and frequently applied concept in mathematics. It enables us to divide the set into subsets (classes) according to some preassigned characteristics.

DEFINITION OF EQUIVALENCE RELATION

A relation R in A is called an equivalence relation if:

1. It is reflexive: $\forall a \in A,\ a\ R\ a$

2. It is symmetric: $(a\ R\ b) \Rightarrow (b\ R\ a)$

3. It is transitive: $(a\ R\ b) \wedge (b\ R\ c) \Rightarrow (a\ R\ c)$

If $a\ R\ b$, we say that a and b are equivalent.

■

The relation "knows" is reflexive, since obviously a knows a.

It is symmetric, because if a knows b, then b knows a.

It is not transitive, because if a knows b and b knows c, it does not guarantee that a knows c.

Therefore, the relation "knows" is not an equivalence relation.

Prove the following:

Let R denote a relation in A, that is, $R \subset A \times A$.

Then:

1. R is reflexive iff $\Delta \subset R$ (where Δ is the diagonal of A).

2. R is symmetric iff $R = R^{-1}$.

3. R is transitive iff $R \circ R \subset R$.

4. (R is symmetric) $\Rightarrow$ ($R \circ R^{-1} = R^{-1} \circ R$)

5. (R is reflexive) $\Rightarrow$ $\begin{pmatrix} 1.\ R \subset R \circ R \\ 2.\ R \circ R \text{ is reflexive} \end{pmatrix}$

6. (R is transitive) $\Rightarrow$ ($R \circ R$ is transitive)

SOLUTION:

1. The diagonal of A is

$$\Delta = \{(a, a) : a \in A\}. \tag{1}$$

Thus R is reflexive iff,

$$\forall a \in A, \quad (a, a) \in R$$

iff $\Delta \subset R$.

2. (R – symmetric) $\Leftrightarrow$ ($R = R^{-1}$)
   $\Rightarrow$ If $(a, b) \in R$, then $(b, a) \in R$, therefore $R = R^{-1}$.
   $\Leftarrow$ Obvious.

3. Let $(a, c) \in R \circ R$. Then $\exists b \in A$, such that $(a, b) \in R$ and $(b, c) \in R$. Since R is transitive, and $(a, b), (b, c) \in R$ implies that $(a, c) \in R$, hence $R \circ R \subset R$.
   Conversely, if $R \circ R \subset R$, then $(a, b) \in R$ and $(b, c) \in R$ imply that $(a, c) \in R \circ R \subset R$. That is, R is transitive.

4. $R \circ R^{-1} = \{(a, c) : \exists b \in A$, such that $(a, b) \in R^{-1} \wedge (b, c) \in R\} =$

$$= \{(a, c) : \exists b \in A, (a, b) \in R \wedge (b, c) \in R^{-1}\} =$$

$$= R^{-1} \circ R. \tag{2}$$

5. Let $(a, b) \in R$. Then

$$R \circ R = \{(a, c) : \exists b \in A, (a, b) \in R \wedge (b, c) \in R\}. \tag{3}$$

But, $(a, b) \in R$ and since R is reflexive, $(b, b) \in R$. Hence $(a, b) \in R \circ R$, that is, $R \subset R \circ R$.

Also,

$$\Delta \subset R \subset R \circ R \text{ implies that } R \circ R \text{ is reflexive.}$$

6. Let $(a, b) \in R \circ R$ and $(b, c) \in R \circ R$. Then from (3), $R \circ R \subset R$. Thus, $(a, b), (b, c) \in R$. So $(a, c) \in R \circ R$, and $R \circ R$ is transitive.

● PROBLEM 3-19

The set A consists of

$$A = \{1, 2, 3, 4, \alpha, \beta\}. \tag{1}$$

Find the smallest equivalence relation R, such that

$$(1, 2) \in R, \quad (2, \alpha) \in R, \quad (4, 2) \in R. \tag{2}$$

SOLUTION:

Relation R has to be reflexive, therefore,

$$(1, 1) \in R, \ (2, 2) \in R, \ (3, 3) \in R, \ (4, 4) \in R,$$

$$(\alpha, \alpha) \in R, \ (\beta, \beta) \in R. \tag{3}$$

It has to be symmetric, hence

$$(2, 1) \in R, \ (\alpha, 2) \in R, \ (2, 4) \in R. \tag{4}$$

Finally, R has to be transitive, thus,

$$(1, 2) \in R \wedge (2, \alpha) \in R \Rightarrow (1, \alpha) \in R$$

$$(1, 2) \in R \wedge (4, 2) \in R \Rightarrow (1, 4) \in R$$

$$(2, \alpha) \in R \wedge (4, 2) \in R \Rightarrow (4, \alpha) \in R. \tag{5}$$

The smallest equivalence relation is

$$R = \{(1, 1), (2, 2), (3, 3), (4, 4), (\alpha, \alpha), (\beta, \beta), (1, 2), (2, \alpha), (4, 2),$$

$$(2, 1), (\alpha, 2), (2, 4), (1, \alpha), (\alpha, 1), (1, 4), (4, 1), (4, \alpha), (\alpha, 4)\}. \tag{6}$$

The largest equivalence relation consists of all 36 elements of $A \times A$.

● PROBLEM 3-20

Let Z^+ denote the set of positive integers,

$$Z^+ = \{1, 2, \ldots\} \tag{1}$$

and let ρ represent the relation in $Z^+ \times Z^+$ defined by

$$(a, b)\ \rho\ (c, d) \text{ iff } a + d = b + c. \tag{2}$$

Show that ρ is an equivalence relation.

SOLUTION:

Element (a, b) is in relation ρ with itself. Indeed

$$(a, b)\ \rho\ (a, b) \text{ because } a + b = b + a. \tag{3}$$

Thus, ρ is reflexive.

Suppose:

$$(a, b)\ \rho\ (c, d), \text{ then } a + d = b + c \text{ or}$$

$$c + b = d + a. \tag{4}$$

Hence, $(c, d)\ \rho\ (a, b)$ and ρ are symmetric.

Suppose:

$$(a, b)\ \rho\ (c, d) \quad \text{and} \quad (c, d)\ \rho\ (e, f) \tag{5}$$

then

$$a + d = b + c \quad c + f = d + e. \tag{6}$$

Hence,

$$a + d + c + f = b + c + d + e \tag{7}$$

and

$$a + f = b + e. \tag{8}$$

From (8) we conclude that

$$(a, b)\ \rho\ (e, f). \tag{9}$$

Relation ρ is transitive. Thus, ρ is an equivalence relation.

● PROBLEM 3-21

Let $Z^+ \times Z^+$ denote the set of ordered pairs of positive integers. Let ρ denote the relation in $Z^+ \times Z^+$ defined by

$$(a, b)\ \rho\ (c, d) \text{ iff } ad = bc. \tag{1}$$

Show that ρ is an equivalence relation. Find the equivalence class of (1,1).

SOLUTION:

Relation ρ is reflexive, because for every $(a, b) \in Z^+ \times Z^+$

$$(a, b)\ \rho\ (a, b), \text{ that is, } ab = ba. \tag{2}$$

Relation ρ is symmetric.

Suppose

$$(a, b)\ \rho\ (c, d), \text{ that is, } ad = bc, \tag{3}$$

and $cb = da$. Therefore,

$$(c, d)\ \rho\ (a, b) \tag{4}$$

and ρ is symmetric.

Suppose

$$(a, b)\ \rho\ (c, d) \quad \text{and} \quad (c, d)\ \rho\ (e, f). \tag{5}$$

That is,

$$ad = cb \text{ and } cf = de. \tag{6}$$

Hence,

$$adcf = cbde \tag{7}$$

and

$$af = be. \tag{8}$$

Therefore, $(a, b)\ \rho\ (e, f)$ and ρ is transitive.

Relation ρ is reflexive, symmetric and transitive, thus it is an equivalence relation.

Suppose (a, b) is in relation ρ with $(1, 1)$

$$(a, b)\ \rho\ (1, 1) \quad \text{iff} \quad a = b. \tag{9}$$

The equivalence class of $(1, 1)$ consists of all pairs (a, a), $a \in Z^+$.

● PROBLEM 3-22

Consider a set X and its partition into disjoint classes

$$X = \bigcup_{i \in I} A_i \,. \tag{1}$$

This partition defines a relation $R \subset X \times X$, in such a way that $(x, y) \in R$ when $\exists i : x, y \in A_i$, that is, when the elements x, y belong to the same Class A_i. Show that R is an equivalence relation.

SOLUTION:

R is reflexive, because $(x, x) \in R$ means that x belongs to a certain class A_i, which is true, since

$$X = \bigcup_i A_i$$

That R is symmetric is obvious, because if $x, y \in A_i$, then $y, x \in A_i$. Hence

$$(x, y) \in R \Rightarrow (y, x) \in R. \tag{2}$$

Suppose $(x, y) \in R \wedge (y, z) \in R$. That means

$$\exists i\ \exists j\ (x, y \in A_i) \wedge (y, z \in A_j). \tag{3}$$

Thus

$$y \in A_i \cap A_j \neq 0 \tag{4}$$

which is a contradiction, because the classes are disjoint.

We proved that relation R is an equivalence.

In the next problem we shall prove the converse.

● PROBLEM 3-23

Prove the following:

THEOREM

Any equivalence relation in X defines a partition of X into classes of the form

$$A_x = \{y \in X : (x, y) \in R\} \tag{1}$$

■

SOLUTION:

Suppose R is an equivalence relation in X. Therefore R is reflexive

$$\forall x \in X \ (x, x) \in R \quad \text{and} \quad x \in A_x. \tag{2}$$

Hence

$$X = \bigcup_x A_x \ . \tag{3}$$

The classes cover the whole set X. We shall show that the classes are disjoint.

Suppose

$$A_x \cap A_y \neq \phi. \tag{4}$$

Then z exists, such that $z \in A_x \wedge z \in A_y$

$$[(z, x) \in R \wedge (z, y) \in R] \Rightarrow (x, y) \in R. \tag{5}$$

Therefore,

$$A_x = A_y. \tag{6}$$

That completes the proof.

The set of classes

$$\{A_x : x \in X\} \tag{7}$$

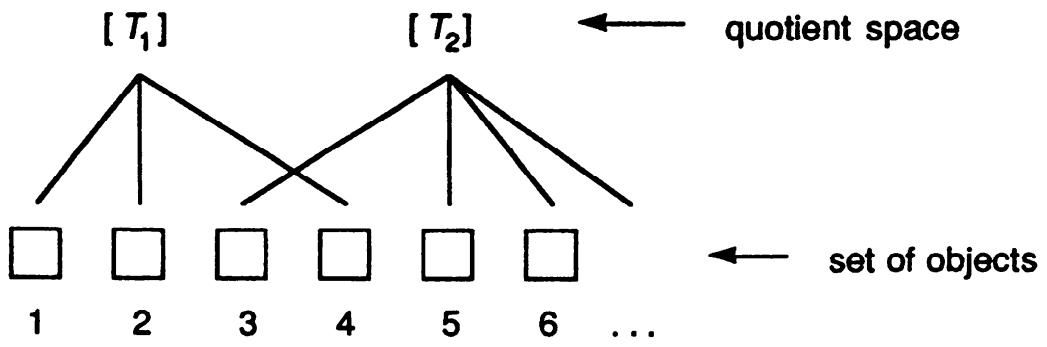
is called a quotient space and is denoted by $X/_R$. Mapping

$$\zeta : X \to \frac{X}{R} \tag{8}$$

where $\zeta(x) = A_x$, is called a canonical mapping or projection. The class A_x is denoted by $[x]$.

The principle of partition of a set into classes forms a basis of abstract thinking – one of the most interesting features of the human mind.

For example, abstract thinking leads us to the concept of temperature. Let X represent the set of all objects.



By using a thermometer, we establish an equivalence relation.

Two objects x and y are in relation T with each other, if they have the same temperature,

$$x\,T\,y. \tag{9}$$

It is easy to see that this is an equivalence relation. Neglecting all other properties like color, weight, age, etc., we classify objects into classes of equal temperature.

● PROBLEM 3-24

Let $\zeta(x, y)$ represent a given sentence function of two variables. Thus,

$$\forall y : \zeta(x, y) \text{ and } \exists y : \zeta(x, y) \tag{1}$$

are sentence functions of one variable. Show that operation $\forall$ is commutative with respect to $\forall$, and $\exists$ is commutative with respect to $\exists$. That is, show that

$$\forall x\, \forall y\, \zeta(x, y) \equiv \forall y\, \forall x\, \zeta(x, y) \tag{2}$$

$$\exists x\, \exists y\, \zeta(x, y) \equiv \exists y\, \exists x\, \zeta(x, y) \tag{3}$$

Hence, the sequence of quantifiers is immaterial in (2) and (3).

On the other hand, the sequence of quantifiers $\forall$ and $\exists$ is significant. Prove the formula:

$$\exists x\, \forall y\, \zeta(x, y) \Rightarrow \forall y\, \exists x\, \zeta(x, y). \tag{4}$$

SOLUTION:

The sentence $\zeta(x, y)$ is true for all x and y, therefore, it is true for all y and x.

Similarly, if x and y exist, such that $\zeta(x, y)$ is true, then y and x exist, such that $\zeta(x, y)$ is true.

The left-hand side of (4) indicates that x_0 exists such that, for every y, $\zeta(x_0, y)$ is true. Therefore, for every y, an x (namely $x = x_0$) exists, such that $\zeta(x, y)$ is true. Hence, the left-hand side implies the right-hand side.

1. Show that the implication (4) of Problem 3–24 is not valid in the opposite direction.

2. Prove the formulas:

$$\exists x\,[\zeta(x) \wedge \varphi(x)] \Rightarrow \exists x\,\exists y\,[\zeta(x) \wedge \varphi(y)] \equiv$$

$$\equiv \exists x : \zeta(x) \wedge \exists y : \varphi(y) \equiv \exists x : \zeta(x) \wedge \exists x : \varphi(x) \qquad (1)$$

and

$$\forall x : \zeta(x) \vee \forall x : \varphi(x) \equiv \forall x, y\,[\zeta(x) \vee \varphi(y)] \Rightarrow$$

$$\Rightarrow \forall x\,[\zeta(x) \vee \varphi(x)]. \qquad (2)$$

SOLUTION:

1. Take, for instance, the set of real numbers. It is true that

$$\forall y\,\exists x : y < x. \qquad (3)$$

On the other hand, the statement

$$\exists x\,\forall y : y < x \qquad (4)$$

is not true. Hence, the implication in the opposite direction is not true.

2. Obviously the implication

$$\forall x : \zeta(x) \Rightarrow \exists x : \zeta(x) \qquad (5)$$

is true. Setting $X = Y$, we can replace (5) by

$$\forall x\,\forall y : \zeta(x, y) \Rightarrow \forall x : \zeta(x, x) \Rightarrow \exists x : \zeta(x, x) \Rightarrow \exists x\,\exists y: \zeta(x, y). \qquad (6)$$

Similarly, from

$$[\exists x : \zeta(x) \wedge \varphi(x)] \Rightarrow \exists x : \zeta(x) \wedge \exists x : \varphi(x) \qquad (7)$$

we obtain

$$\exists x\,[\zeta(x) \wedge \varphi(x)] \Rightarrow \exists x\,\exists y\,[\zeta(x) \wedge \varphi(y)]\,. \qquad (8)$$

The right-hand side is equivalent to

$$\exists x\,\exists y\,[\zeta(x) \wedge \varphi(y)] \equiv \exists x : \zeta(x) \wedge \exists y : \varphi(y) \equiv$$

$$\equiv \exists x : \zeta(x) \wedge \exists x : \varphi(x). \qquad (9)$$

Similarly, from

$$[\forall x : \zeta(x) \vee \forall x : \varphi(x)] \Rightarrow \forall x\, [\zeta(x) \vee \varphi(x)] \tag{10}$$

we obtain (2).

● PROBLEM 3-26

In Chapter 2, we discussed the axiomatic formulation of the algebra of sets. Then, four axioms:

1. Uniqueness Axiom
2. Union Axiom
3. Difference Axiom
4. Existence Axiom

were sufficient. What additional axioms are necessary for the purposes of Chapter 3?

SOLUTION:

We should add the following axioms:

5. For every sentence function $\zeta(x)$ and for every set A, a set exists consisting of those, and only those elements of A for which $\zeta(x)$ is true.

This set is denoted by

$$\{x : \zeta(x), x \in A\}.$$

Without realizing it, we used this axiom frequently.

For example, Axiom 5 guarantees the existence of the set $\{a, b\}$

$$\{a, b\} = \{x : x = a \vee x = b, x \in A\}.$$

6. For every set A, a set, whose elements are all the subsets of A, exists. This set is denoted by 2^A.

7. **AXIOM OF CHOICE.**

For every family R of non-empty disjoint sets, a set exists which has one, and only one, element in common with each of the sets of the family R.

The last axiom is an existential axiom. In general, a set obtained by application of this axiom is not uniquely determined. In 1938, K. Gödel showed that if the axiomatic set theory without axiom of choice is consistent, then it is also consistent *with* the axiom of choice.

For some time, some mathematicians suspected that axiom of choice can be derived from the other axioms. In 1963, P.J. Cohen proved that this is not the case.

The axiom of choice is an independent axiom.

CHAPTER 4

MAPPINGS

Function , Domain	4-1
Surjection, Injection, Bijection	4-2
Graph, Equal Functions	4-3
Restriction, Extension	4-3
Inverse Function	4-4
Composition of Functions	4-5, 4-6, 4-7, 4-8
Image, Inverse Image	4-9, 4-10, 4-11
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Properties of f^{-1}	4-13, 4-14
Coverings, Partitions	4-15, 4-16, 4-17, 4-18
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Real-valued Functions	4-24
Characteristic Function	4-25
Relation-preserving Functions	4-26, 4-27
General Cartesian Products	4-28, 4-29, 4-30, 4-31

Which of the diagrams depicts a function from $X = \{a, b, c\}$ to $Y = \{x, y, z\}$?

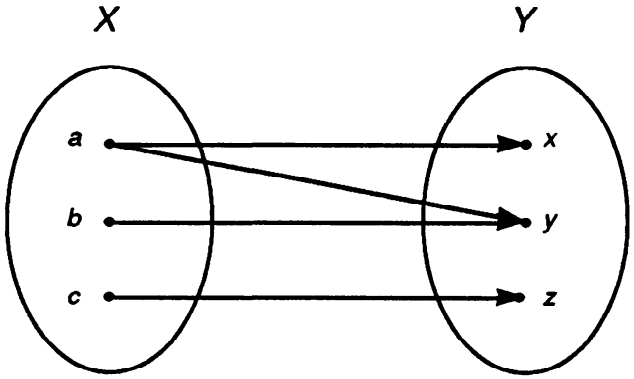


FIGURE 1

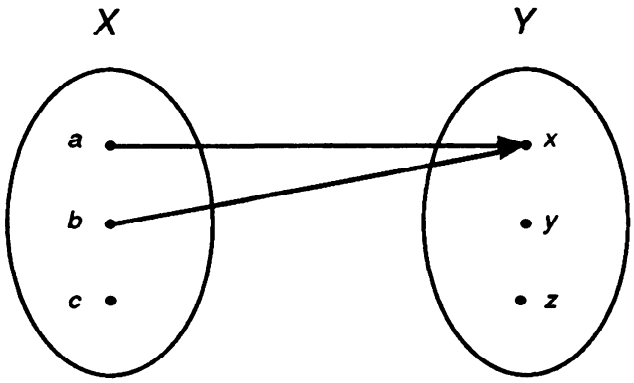


FIGURE 2

SOLUTION:

DEFINITION

f is a function from X to Y iff $f \subseteq X \times Y$, for all $x \in X$, there is a $y \in Y$, such that $(x, y) \in f$; $(x, y_1) \in f$ and $(x, y_2) \in f$ imply $y_1 = y_2$.

Function f is denoted

$$f: X \to Y. \tag{1}$$

The domain of f is X, the co-domain is Y.

The element $f(x) = y \in Y$ is called the image of x under f. Figure 1 illustrates the relation consisting of the pairs

$$(a, x), (a, y), (b, y), (c, z).$$

This relation assigns two elements, $x \in Y$ and $y \in Y$, to element $a \in X$. Hence, it is not a function.

Figure 2 also does not define a function. This relation is not defined for all elements of X. Element c is left "alone." It is easy to upgrade the relation shown in Figure 2 to obtain a function as shown in Figure 3.

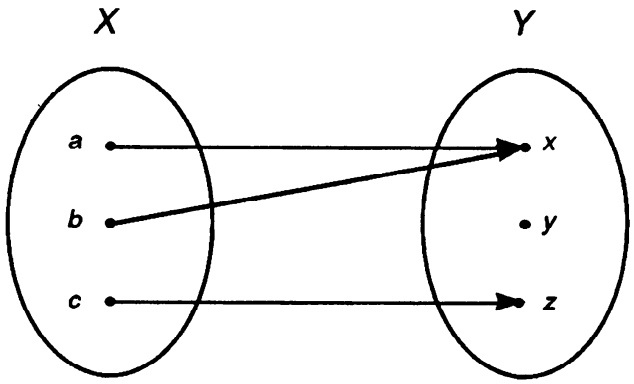


FIGURE 3

Note that a function $f: X \rightarrow Y$ has to be defined for all elements of X, but it does not have to "cover" all elements of Y, as shown in Figure 3. In the extreme case, a function f can assign all elements of X to one element of Y.

Sometimes a word mapping or map is used instead of a function.

● PROBLEM 4-2

Prove that not every injection of a set into itself is a bijection.

SOLUTION:

We shall start with some definitions.

DEFINITION (SURJECTION (ONTO) FUNCTION)

A function f from X to Y, $f: X \rightarrow Y$ is an onto function (also called surjection) if, and only if, for all $y \in Y$, an $x \in X$ exists such that $y = f(x)$.

For example, function $f: R \rightarrow R$, where R is the set of real numbers, is a surjection when $f(x) = 3x + 1$ and is not a surjection when $f(x) = x^2$.

DEFINITION (INJECTION – ONE-TO-ONE FUNCTION)

A function $f: X \rightarrow Y$ is a one-to-one function (also called injection) iff $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

DEFINITION (BIJECTION)

A function $f: X \rightarrow Y$ is a bijection iff f is both a surjection and injection.

For example, the function shown in Figure 1

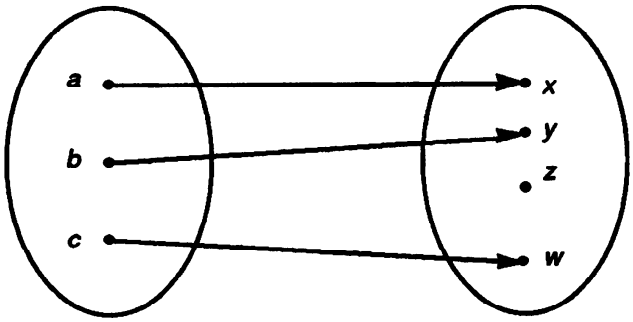


FIGURE 1

is one-to-one, but it is not onto, hence it is not a bijection.

We shall find function $f : X \to X$ which is an injection, but not a surjection. Let Z^+ denote the set of positive integers, then

$$f : Z^+ \to Z^+$$

such that

$$f(n) = n + 1$$

is one-to-one but not onto. Indeed no positive integer n exists such that

$$f(n) = 1.$$

Hence, $f(n) = n + 1$ is not a bijection.

● PROBLEM 4-3

1. Find the graph of the function shown in Figure 1.

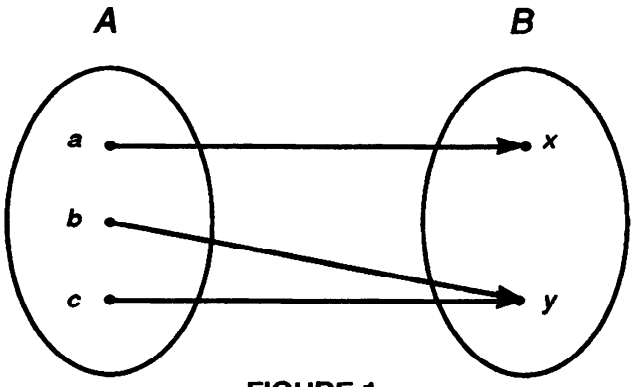


FIGURE 1

2. Let $f : R \to R$ and $g : R \to R$, where $f(x) = (x + 1)^2$, and $g(x) = x^2 + 2x + 1$ and R is the set of real numbers..Are these functions equal?

3. Define the restriction and extension of a function.

SOLUTION:

1. The set G

$$G = \{(x, y) : f(x) = y\} \subset X \times Y \tag{1}$$

is called the graph of f. From Figure 1, we find the graph of the function to be

$$\{(a, x), (b, y), (c, y)\}. \tag{2}$$

2. **DEFINITION**

Two functions $f : A \to B$ and $g : A \to B$ are equal, $f = g$ iff

$$f(a) = g(a) \text{ for every } a \in A.$$

Obviously, for all real numbers

$$(x + 1)^2 = x^2 + 2x + 1. \tag{3}$$

Hence $f(x) = g(x)$.

3. **DEFINITION**

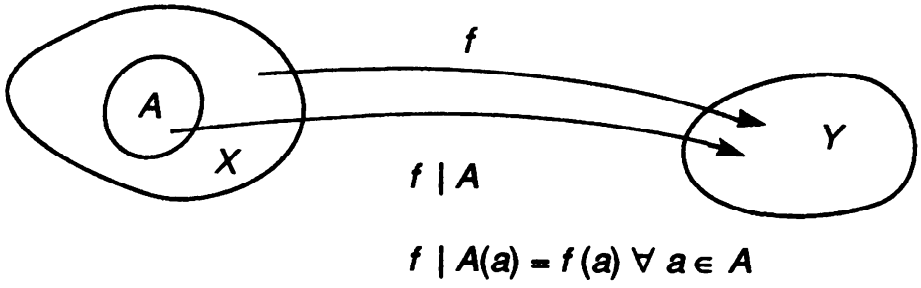
Let $f : X \to Y$ and $A \subset X$; the function f considered only on A is called the restriction of f to A, written $f \mid A$, if

$$f \mid A = f \cap (A \times Y). \tag{4}$$

Similarly, if $A \subset X$ and $g : A \to Y$ is a given function, then

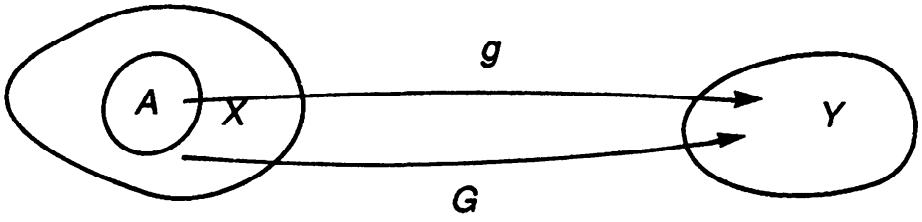
$$G : X \to Y$$

such that $G \mid A = g$ is called an extension of g over X.



$f \mid A$ is the restriction of f to A

FIGURE 2



G is an extension of g.

FIGURE 3

● PROBLEM 4-4

Let f^{-1} be the inverse of the function f. In general, f^{-1} is only a relation. Find the conditions for f^{-1} to be a function.

SOLUTION:

We defined a function $f: X \to Y$ as a subset of $X \times Y$, $f \subset X \times Y$, such that

$$[(x, y_1) \in f \wedge (x, y_2) \in f] \Rightarrow [y_1 = y_2]. \tag{1}$$

Since f is a relation, we can always find f^{-1}.

Function f must be a surjection, that is, $f(X) = Y$, then f^{-1} is defined for all $y \in Y$. In order for f^{-1} to be a function, function f must be an injection. Indeed, if

$$[(x_1, y) \in f \wedge (x_2, y) \in f] \Rightarrow [x_1 = x_2]$$

then we have

$$[(y, x_1) \in f' \wedge (y, x_2) \in f'] \Rightarrow [x_1 = x_2].$$

We conclude that only a function, which is a bijection, has the inverse function. It is easy to show that

$$(f^{-1})^{-1} = f.$$

● PROBLEM 4-5

Let the functions

$$f: R \to R \text{ and } g: R \to R$$

be defined by

$$f(x) = 3x - 1 \quad g(x) = x^3 + 1.$$

Find their compositions

$$f \circ g \quad \text{and} \quad g \circ f.$$

SOLUTION:

DEFINITION OF A COMPOSITION

Let $f: A \to B$ and $g: B \to C$. The composition of f and g is the function $g \circ f$ defined by:

$$(g \circ f)(a) = g(f(a)). \tag{1}$$

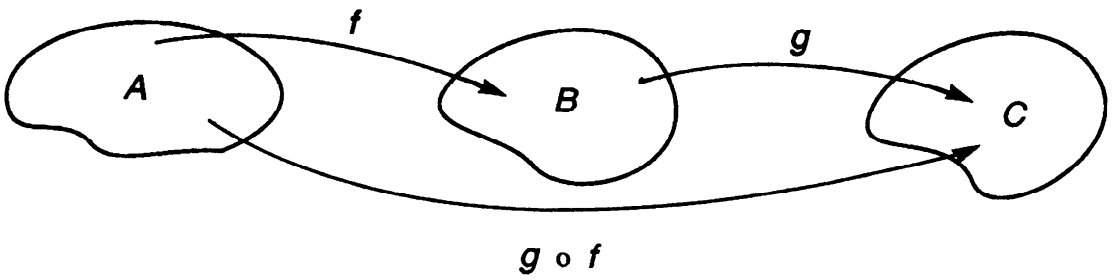


FIGURE 1

Let us find the composition $g \circ f$

$$(g \circ f)(x) = g(f(x)) = g(3x - 1) = (3x - 1)^3 + 1. \tag{2}$$

Similarly,

$$(f \circ g)(x) = f(g(x)) = f(x^3 + 1) = 3(x^3 + 1) - 1. \tag{3}$$

From (2) and (3) we conclude that

$$f \circ g \neq g \circ f. \tag{4}$$

● PROBLEM 4-6

Prove that the composition of functions is associative. That is, prove that if

$$f: A \to B, \quad g: B \to C, \quad h: C \to D$$

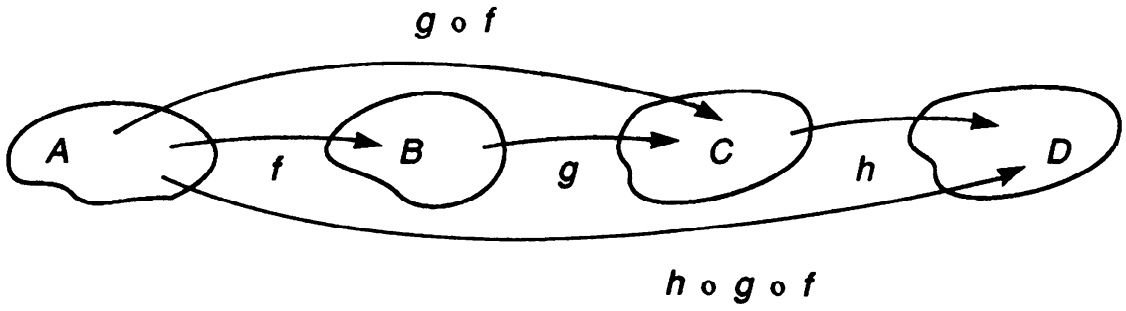


FIGURE 1

then

$$(h \circ g) \circ f = h \circ (g \circ f). \tag{1}$$

SOLUTION:

In Chapter 3 we proved that relations are associative and since functions are relations, this result holds for functions as well. Here is another way of showing it:

$$[(h \circ g) \circ f\,](a) = (h \circ g)\,[f(a)] =$$

$$= h[g(f(a))] \quad \text{for every } a \in A. \tag{2}$$

Similarly,

$$[h \circ (g \circ f)\,](a) = h\,[(g \circ f)\,(a)] =$$

$$= h[g(f(a))] \quad \text{for every } a \in A. \tag{3}$$

Thus

$$(h \circ g) \circ f = h \circ (g \circ f) = h \circ g \circ f. \tag{4}$$

● PROBLEM 4-7

1. Let $f : A \to B$, $g : B \to C$. Prove that

   a) If f and g are onto, then

   $$g \circ f : A \to C \text{ is onto;}$$

   b) If f and g are one-to-one, then

   $$g \circ f \text{ is one-to-one.}$$

2. Determine which of the following functions are onto, one-to-one or bijection.

$$f : A \to A, \quad A = [-1, 1]$$

$$f(x) = x^2, \quad f(x) = \sin x$$

$$f(x) = x^3, \quad f(x) = \sin \pi x \tag{1}$$

SOLUTION:

1. We shall show that $g \circ f$ is onto when both g and f are onto.
   Let $c \in C$, since g is onto

$$\exists b \in B, \text{ such that } g(b) = c.$$

Also, f is onto, therefore,

$$\exists a \in A, \text{ such that } f(a) = b.$$

We conclude

$$\forall c \in C \quad \exists a \in A, \text{ such that } (g \circ f)(a) = c \tag{2}$$

hence $g \circ f$ is onto.

Now we show that if f and g are one-to-one, then $g \circ f$ is one-to-one. Suppose it is not the case

$$(g \circ f)(a) = (g \circ f)(a') \tag{3}$$

or

$$g(f(a)) = g(f(a')). \tag{4}$$

But g is one-to-one. Hence

$$f(a) = f(a'). \tag{5}$$

Also f is one-to-one, hence

$$a = a'.$$

We conclude that $g \circ f$ is one-to-one.

2. Function $f(x) = x^2$ (Figure 1) is

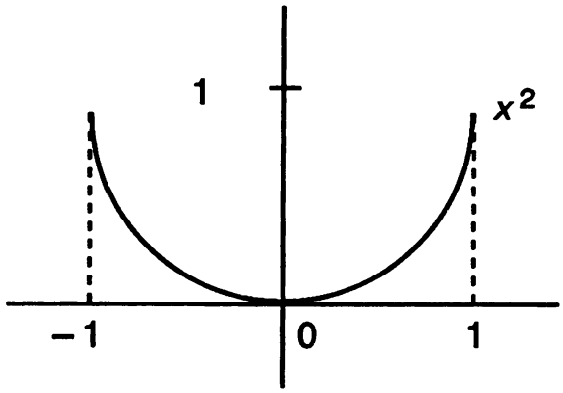


FIGURE 1

not onto because $f([-1, 1]) = [0,1]$. It is not one-to-one because, for example,

$$(-1)^2 = (1)^2.$$

Function $f(x) = \sin x$ (Figure 2) is one-to-one, but is not onto.

Function $f(x) = x^3$ (Figure 3) is one-to-one and onto, hence it is a bijection.

Function $f(x) = \sin \pi x$ (Figure 4) is onto but not one-to-one.

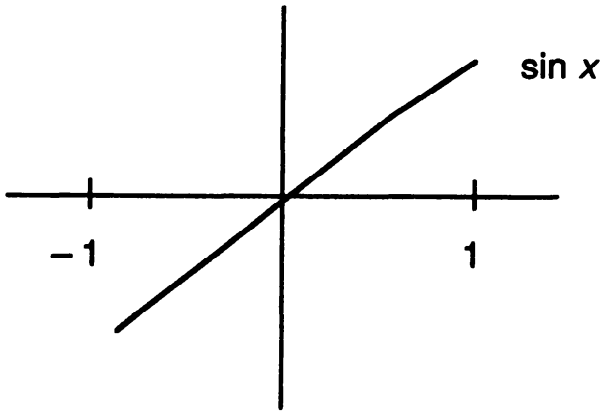


FIGURE 2

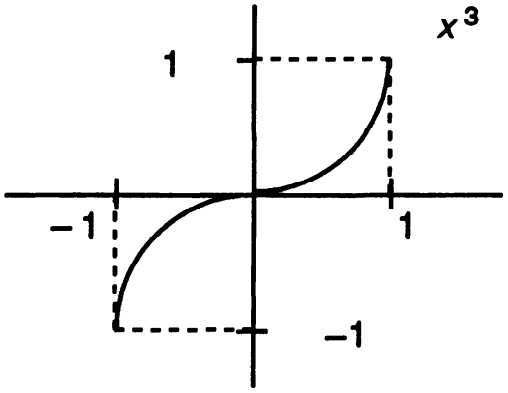


FIGURE 3

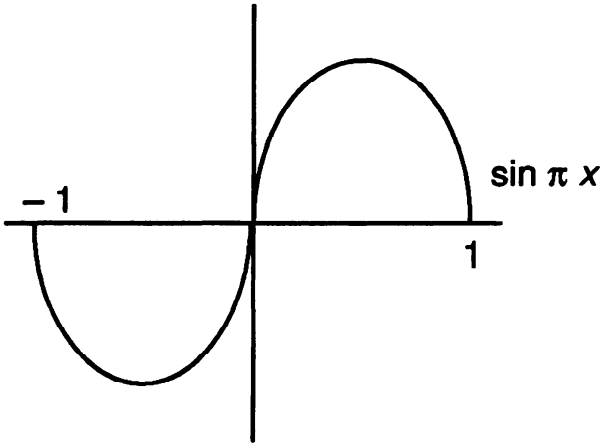


FIGURE 4

● PROBLEM 4–8

Prove that:
If $f : A \to B$ and $g : B \to C$ are one-to-one and onto, then

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}. \qquad (1)$$

SOLUTION:

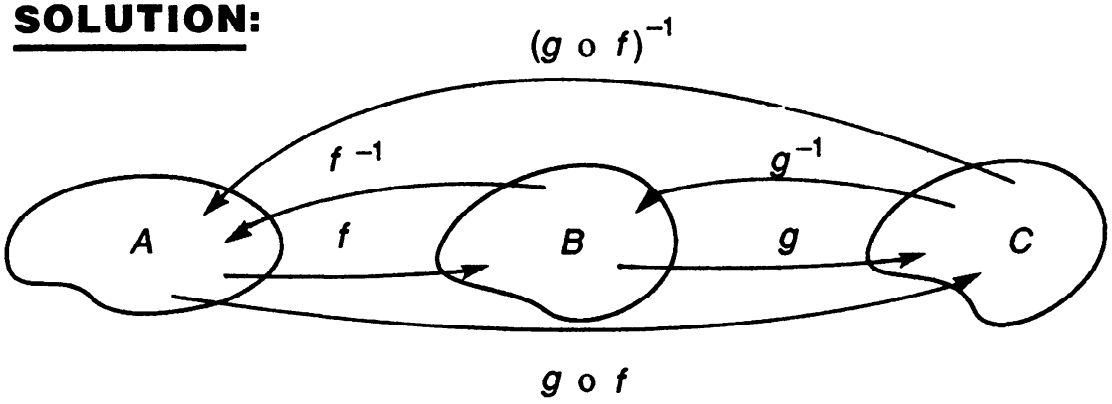


FIGURE 1

Let 1_A be the identity function on A

$$1_A(a) = a, \forall a \in A. \tag{2}$$

By applying the associative law of composition of functions, we find

$$(f^{-1} \circ g^{-1}) \circ (g \circ f) = f^{-1} \circ [g^{-1} \circ (g \circ f)] =$$

$$= f^{-1} \circ [(g^{-1} \circ g) \circ f] = f^{-1} \circ (1 \circ f) =$$

$$= f^{-1} \circ f = 1_A \tag{3}$$

since $g^{-1} \circ g = 1_A$ and $1_A \circ f = f \circ 1_A = f$.

Similarly,

$$(g \circ f) \circ (f^{-1} \circ g^{-1}) = g \circ [f \circ (f^{-1} \circ g^{-1})] =$$

$$= g \circ [(f \circ f^{-1}) \circ g^{-1}] =$$

$$= g \circ (1 \circ g^{-1}) = g \circ g^{-1} = 1_A. \tag{4}$$

Therefore,

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}. \tag{5}$$

● PROBLEM 4-9

1. Let $X = [-1, 1]$ and $Y = R$, and

$$f : X \to R \text{ be } f(x) = x^2. \tag{1}$$

Find the image of X in R under f.

2. Find the inverse image of $[1, 4]$ in X under f, $f(x) = x^2$.

3. Prove that:

If $f: X \to Y$ and $A \subset X$, $B \subset X$, then

$$f(A \cup B) = f(A) \cup f(B) \tag{2}$$

$$f(A) - f(B) \subset f(A - B). \tag{3}$$

SOLUTION:

1. Let $f: X \to Y$. The image of $A \subset X$, denoted by $f(A)$, is the set of images of points of A

$$f(A) = \{f(x) : x \in A\}. \tag{4}$$

Hence, the image of $[-1, 1]$ is

$$f([-1, 1]) = [0, 1]. \tag{5}$$

2. The inverse image $f^{-1}(B)$ of any subset $B \subset Y$, is the set

$$f^{-1}(B) = \{x : x \in X, f(x) \in B\}. \tag{6}$$

The inverse image of $[1, 4]$ is

$$f^{-1}([1, 4]) = [1, 2] \tag{7}$$

or

$$f^{-1}([1, 4]) = [-1, -2]. \tag{8}$$

3. First, we shall show that

$$f(A \cup B) \subset f(A) \cup f(B). \tag{9}$$

Let $y \in f(A \cup B)$, that is, $x \in A \cup B$ exists such that $f(x) = y$. We have

$$x \in A \quad \text{or} \quad x \in B \quad \text{then}$$

$$x \in A \Rightarrow f(x) = y \in f(A) \text{ or } x \in B \Rightarrow f(x) = y \in f(B). \tag{10}$$

Hence

$$y = f(x) \quad \text{and} \quad y \in f(A) \cup f(B). \tag{11}$$

Now we prove

$$f(A) \cup f(B) \subset f(A \cup B). \tag{12}$$

Let $y \in f(A) \cup f(B)$ then

$$y \in f(A) \quad \text{or} \quad y \in f(B)$$

$$y \in f(A) \Rightarrow \exists x \in A, \text{ such that } f(x) = y$$

$$y \in f(B) \Rightarrow \exists x \in B, \text{ such that } f(x) = y. \tag{13}$$

Hence

$$y = f(x) \quad \text{and} \quad x \in A \cup B, \text{ that is,}$$

$$y \in f(A \cup B).$$

Let $y \in f(A) - f(B)$. Then $\exists x \in A$, such that $f(x) = y$ but $y \notin \{f(x) : x \in B\}$. Hence $x \notin B$ and $x \in A - B$. Then, $y \in f(A - B)$.

● PROBLEM 4-10

Prove the following:
Let $f : X \to Y$, then for any subsets A and B of X,

1. $f(A \cap B) \subset f(A) \cap f(B)$ (1)

2. $A \subset B \Rightarrow f(A) \subset f(B)$ (2)

3. For any family A_α of subsets of X

$$f(\bigcup_\alpha A_\alpha) = \bigcup_\alpha f(A_\alpha) \tag{3}$$

$$f(\bigcap_\alpha A_\alpha) \subset \bigcap_\alpha f(A_\alpha) \tag{4}$$

4. Show that inclusion in (1) cannot be replaced, in general, by equality.

SOLUTION:

1. Suppose $y \in f(A \cap B)$. Hence $x \in A \cap B$ exists, such that $f(x) = y$. But $x \in A$ and $x \in B$, therefore

$$y = f(x) \in f(A) \quad \text{and} \quad y = f(x) \in f(B). \tag{5}$$

Thus

$$y \in f(A) \cap f(B) \tag{6}$$

and

$$f(A \cap B) \subset f(A) \cap f(B). \tag{7}$$

2. Suppose $A \subset B$ and $y \in f(A)$. Then an x exists such that $f(x) = y$ and $x \in$

A. Hence $x \in B$ and $f(x) = y \in f(B)$. Therefore,

$$A \quad B \Rightarrow f(A) \quad f(B). \tag{8}$$

3. In Problem 4-9 we proved that

$$f(A \cup B) = f(A) \cup f(B). \tag{9}$$

Similarly,

$$y \in \bigcup_{\alpha} f(A_\alpha) \equiv \exists\, \alpha : y \in f(A_\alpha) \equiv$$
$$\equiv \exists\, x \in A_\alpha : f(x) = y \equiv \exists\, x \in \bigcup_{\alpha} A_\alpha : f(x) = y \equiv$$
$$\equiv y = f(x) \in f(\bigcup_{\alpha} A_\alpha). \tag{10}$$

That proves (3).

Suppose $y \in f(\bigcap_{\alpha} A_\alpha)$, then

$$y \in f(\bigcap_{\alpha} A_\alpha) \equiv \exists x \in \bigcap_{\alpha} A_\alpha : f(x) = y \equiv$$
$$\equiv \exists\, x : \forall\, \alpha : x \in A_\alpha : f(x) = y \Rightarrow$$
$$\exists\, x : \forall\, \alpha : y \in f(A_\alpha) \equiv y \in \bigcap_{\alpha} f(A_\alpha). \tag{11}$$

4. Consider the function $f(x) = x^2$. Let $A = [-1, 0]$ and $B = [0, 1]$ then

$$A \cap B = [0] \quad \text{and} \quad f(A \cap B) = f(0) = [0]$$

$$f(A) = f([-1, 0]) = [0, 1].$$

$$f(B) = [0, 1].$$

$$f(A) \cap f(B) = [0, 1] \neq f(A \cap B) = [0]. \tag{12}$$

● PROBLEM 4-11

Prove that:

$$\begin{pmatrix} f : X \to Y \\ \text{is one - to - one} \end{pmatrix} \Leftrightarrow \begin{pmatrix} \forall A, \forall B : A, B \subset X \\ f(A \cap B) = f(A) \cap f(B) \end{pmatrix} \tag{1}$$

SOLUTION:

First we shall prove $\Rightarrow$

Suppose $f : X \to Y$ is one-to-one. Let A and B be any subsets of X.

Then, for any $y \in f(A \cap B)$

$$y \in f(A \cap B) \equiv \exists x : x \in A \cap B, f(x) = y \equiv \tag{2}$$

(Note that there is only one $x \in A \cap B$, such that $f(x) = y$.)

$$\equiv x \in A \wedge x \in B \wedge f(x) = y \equiv$$

$$\equiv y \in f(A) \wedge y \in f(B) \equiv y \in f(A \cap B). \tag{3}$$

$\Leftarrow$ Now, suppose for any sets $A \subset X$ and $B \subset X$

$$f(A \cap B) = f(A) \cap f(B) \tag{4}$$

and suppose $f : X \rightarrow Y$ is not one-to-one. Then x_1 and x_2, $x_1 \neq x_2$, exist such that

$$y = f(x_1) = f(x_2). \tag{5}$$

Let $A = x_1$ and $B = x_2$; we have

$$f(A) \cap f(B) = f(x_1) \cap f(x_2) = y \neq f(A \cap B) =$$

$$= f(\phi). \tag{6}$$

● PROBLEM 4-12

Let $f : X \rightarrow Y$ and let $P(X)$ and $P(Y)$ denote power sets of X and Y respectively. Function f induces functions

$$f : P(X) \rightarrow P(Y) \tag{1}$$

by $X \supset A \rightarrow f(A)$ and

$$f^{-1} : P(Y) \rightarrow P(X) \tag{2}$$

by

$$Y \supset B \rightarrow f^{-1}(B).$$

Prove the following:

THEOREM

Let $f : X \rightarrow Y$ denote one-to-one. Then the induced function

$$f : P(X) \rightarrow P(Y)$$

is also one-to-one.

SOLUTION:

If $X = \phi$, then $P(X) = \{\phi\}$, and

$$f: P(X) \rightarrow P(Y)$$

is one-to-one, because no two different elements of $P(X)$ can have the same image, as the set $P(X)$ consists of only one element.

Suppose $X \neq \phi$, then $P(X)$ has at least two elements. Let $A \in P(X)$ and $B \in P(X)$, but $A \neq B$. Then $x \in X$ exists, such that $x \in A$ and $x \notin B$. Hence,

$$f(x) \in f(A) \quad \text{and} \quad f(x) \notin f(B)$$

because f is one-to-one. We find

$$f(A) \neq f(B)$$

therefore, the induced function is also one-to-one.

● PROBLEM 4-13

Prove the following theorem:

THEOREM

Let $f: X \rightarrow Y$, then the induced function $f^{-1}: P(Y) \rightarrow P(X)$ preserves the elementary set operations

1. $f^{-1}(\bigcup_{\alpha} B_\alpha) = \bigcup_{\alpha} f^{-1}(B_\alpha)$ (1)

2. $f^{-1}(\bigcap_{\alpha} B_\alpha) = \bigcap_{\alpha} f^{-1}(B_\alpha)$ (2)

3. $f^{-1}(B_1 - B_2) = f^{-1}(B_1) - f^{-1}(B_2)$ (3)

SOLUTION:

1. $$x \in f^{-1}(\bigcup_{\alpha} B_\alpha) \equiv f(x) \in \bigcup_{\alpha} B_\alpha \equiv$$

$$\equiv \exists\, \alpha : f(x) \in B_\alpha \equiv \exists\, \alpha : x \in f^{-1}(B_\alpha) \equiv$$

$$\equiv x \in \bigcup_{\alpha} f^{-1}(B_\alpha). \qquad (4)$$

2. $$x \in f^{-1}(\bigcap_{\alpha} B_\alpha) \equiv f(x) \in \bigcap_{\alpha} B_\alpha \equiv$$

$$\equiv \forall\, \alpha : f(x) \in B_\alpha \equiv \forall\, \alpha : x \in f^{-1}(B_\alpha) \equiv$$

$$\equiv x \in \bigcap_{\alpha} f^{-1}(B_\alpha). \qquad (5)$$

3. $$x \in f^{-1}(B_1 - B_2) \equiv f(x) \in B_1 - B_2 \equiv$$

$$\equiv f(x) \in B_1 \wedge f(x) \notin B_2 \equiv$$

$$\equiv x \in f^{-1}(B_1) \wedge x \notin f^{-1}(B_2) \equiv$$

$$\equiv x \in f^{-1}(B_1) - f^{-1}(B_2). \tag{6}$$

We conclude that the induced function f^{-1} preserves the elementary set operations. The induced function $f : P(X) \rightarrow P(Y)$ preserves unions, but it does not preserve intersections in general.

● PROBLEM 4-14

Show that:
If $f : X \rightarrow Y$, then:

1. For each $A \subset X$,

$$A \subset f^{-1}[f(A)]. \tag{1}$$

2. For each $A \subset X$ and $B \subset Y$,

$$B \cap f(A) = f[f^{-1}(B) \cap A] \tag{2}$$

in particular,

$$B \cap f(X) = f[f^{-1}(B)]. \tag{3}$$

SOLUTION:

1. Let A represent any subset of X and $x \in A$. Function f maps x into $f(x) \in Y$. Since function f is not necessarily one-to-one, it is possible that elements $x, x_1, x_2, \ldots$ exist, such that

$$f(x) = f(x_1) = f(x_2) \ldots \text{ (See Figure 1).} \tag{4}$$

Therefore,

$$x \in f^{-1}[f(x)] \tag{5}$$

Hence

$$A \subset f^{-1}[f(A)]. \tag{6}$$

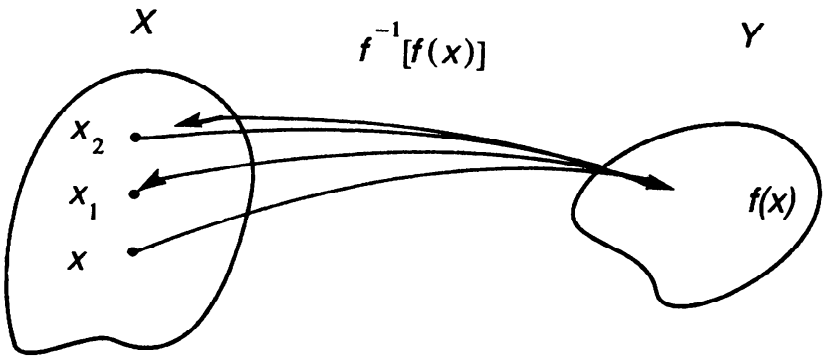


FIGURE 1

2. Suppose $y \in Y$ and

$$y \in f[f^{-1}(B) \cap A] \equiv \exists x : [x \in f^{-1}(B) \cap A] \wedge [f(x) = y] \equiv$$

$$\equiv \exists x : x \in A \wedge x \in f^{-1}(B) \wedge f(x) = y \equiv$$

$$\equiv \exists x : f(x) \in f(A) \wedge f(x) \in B \wedge f(x) = y \equiv$$

$$\equiv y \in f(A) \wedge y \in B \equiv y \in f(A) \cap B. \tag{7}$$

Therefore,

$$f[f^{-1}(B) \cap A] = f(A) \cap B. \tag{8}$$

In particular, setting $A = X$ we obtain

$$f[f^{-1}(B)] = f(X) \cap B \tag{9}$$

because

$$f^{-1}(B) \cap X = f^{-1}(B). \tag{10}$$

● PROBLEM 4-15

Prove this theorem:

THEOREM

Let X represent any set and $\{A_\alpha : \alpha \in \Omega\}$ its covering, i.e.

$$X = \bigcup_\alpha A_\alpha \tag{1}$$

Furthermore, let

$$\forall \alpha \quad \exists f_\alpha : A_\alpha \to Y \tag{2}$$

such that

$$\forall \alpha, \beta \quad f_\alpha | A_\alpha \cap A_\beta = f_\beta | A_\alpha \cap A_\beta. \tag{3}$$

Then one, and only one, $f : X \to Y$ exists which is an extension of each f_α, i.e. $\forall \alpha : f | A_\alpha = f_\alpha$.

SOLUTION:

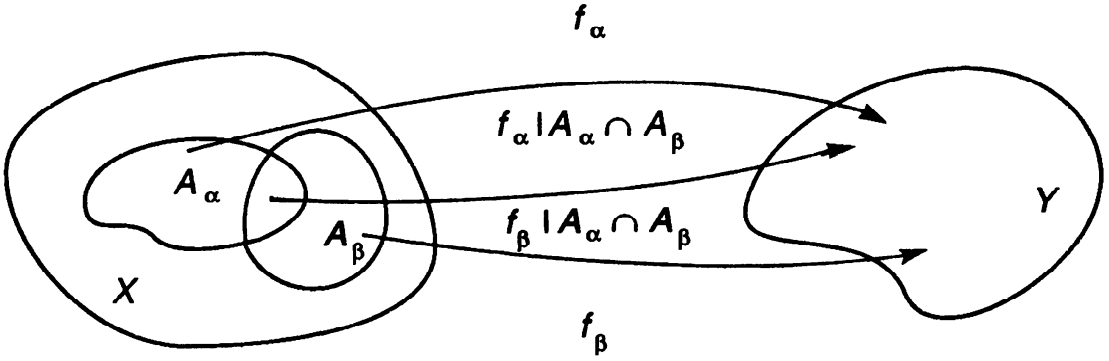


FIGURE 1

Let us define for each $x \in X$, $f(x) = f_\alpha(x)$, where $x \in A_\alpha$. If also $x \in A_\beta$, then $f(x) = f_\beta(x)$ but

$$f_\alpha | A_\alpha \cap A_\beta(x) = f_\beta | A_\alpha \cap A_\beta(x). \tag{4}$$

Hence, this is a correct definition of a function: it assigns to an element of X only one element of Y.

$$f(x) = f_\alpha(x) = f_\beta(x). \tag{5}$$

The function

$$f : X \to Y \tag{6}$$

is an extension of each f_α over X. Function f is unique, each x belongs to some A_α (A_α is a covering of X) and the value $f_\alpha(x)$ is uniquely defined.

● PROBLEM 4-16

Let

$$\{A_\alpha : \alpha \in \Omega\}$$

be a covering of X, i.e.

$$X = \bigcup_\alpha A_\alpha$$

such that if $\alpha \neq \beta$, then $A_\alpha \cap A_\beta = \phi$; then the family $\{A_\alpha : \alpha \in \Omega\}$ is called a partition of X.

Show that:
If $\{A_\alpha : \alpha \in \Omega\}$ is a partition of X and for each A_α, f_α is given, such that $f_\alpha : A_\alpha \to Y$, then $f : X \to Y$, which is an extension of each f_α, is uniquely defined by

$$f(x) = f_\alpha(x) \quad \text{for} \quad x \in A_\alpha. \tag{1}$$

SOLUTION:

We shall use the theorem proved in Problem 4-15. Since $\{A_\alpha : \alpha \in \Omega\}$ is a partition of X, the condition

$$f_\alpha \mid A_\alpha \cap A_\beta = f_\beta \mid A_\alpha \cap A_\beta \tag{2}$$

is satisfied. Therefore, (1) defines a function on X which is unique.

● PROBLEM 4-17

Let

$$\{A_\alpha : \alpha \in \Omega\} \text{ and } \{B_\beta : \beta \in \Lambda\} \tag{1}$$

denote two coverings (partitions) of X. Show that

$$\{A_\alpha \cap B_\beta : (\alpha, \beta) \in \Omega \times \Lambda\} \tag{2}$$

is also a covering (partition) of X.

SOLUTION:

Both $\{A_\alpha\}$ and $\{B_\beta\}$ are coverings of X. Let $x \in X$ represent any element of X. Since

$$X = \bigcup_\alpha A_\alpha$$

α' exists, such that $x \in A_{\alpha'}$. Also, β' exists, such that $x \in A_{\beta'}$. Therefore, $x \in A_{\alpha'} \cap B_{\beta'}$.

Hence,

$$\bigcup_{(\alpha,\beta)} A_\alpha \cap A_\beta = X \tag{3}$$

and (2) is a covering of X. Now we shall show that if both $\{A_\alpha\}$ and $\{B_\beta\}$ are partitions, then $\{A_\alpha \cap B_\beta\}$ is also a partition.

Consider,

$$(A_\alpha \cap B_\beta) \cap (A_{\alpha'} \cap B_{\beta'}) =$$

$$= (A_\alpha \cap A_{\alpha'}) \cap (B_\beta \cap B_{\beta'}). \tag{4}$$

Then

$$A_\alpha \cap A_{\alpha'} = \phi \quad \text{for } \alpha \neq \alpha'$$

and

$$B_\beta \cap B_{\beta'} = \phi \quad \text{for } \beta \neq \beta'.$$

Therefore, $\{A_\alpha \cap B_\beta\}$ is a partition.

● PROBLEM 4-18

Let $\{A_\alpha : \alpha \in \Omega\}$ and $\{B_\beta : \beta \in \Lambda\}$ denote coverings (partitions) of sets X and Y, respectively. Show that

$$\{A_\alpha \times B_\beta : (\alpha, \beta) \in \Omega \times \Lambda\} \tag{1}$$

is a covering (partition) of $X \times Y$.

SOLUTION:

Let $(x, y) \in X \times Y$. Then $x \in X$ and $y \in Y$. $\{A_\alpha\}$ is a covering of X. Hence

$$X = \bigcup_\alpha A_\alpha \tag{2}$$

and

$$\exists \alpha' \in \Omega : x \in A_{\alpha'}. \tag{3}$$

$\{B_\beta\}$ is a covering of Y. Hence

$$Y = \bigcup_\beta B_\beta \tag{4}$$

and

$$\exists \beta' \in \Lambda : y \in B_{\beta'}. \tag{5}$$

Thus,

$$(x, y) \in A_{\alpha'} \times B_{\beta'} \tag{6}$$

and (1) is a covering of $X \times Y$

$$X \times Y = \bigcup_{(\alpha,\beta)} A_\alpha \times B_\beta. \tag{7}$$

Suppose $\{A_\alpha\}$ and $\{B_\beta\}$ are partitions of X and Y, respectively; then

$$(A_\alpha \times B_\beta) \cap (A_{\alpha'} \times B_{\beta'}) = (A_\alpha \cap A_{\alpha'}) \times (B_\beta \cap B_{\beta'}) =$$

$$= \phi \times \phi = \phi. \tag{8}$$

Hence, $\{A_\alpha \times B_\beta\}$ is also a partition.

Let T be a family of subsets of X. Show that the smallest additive family U exists, such that $T \subset U$.

SOLUTION:

DEFINITION (ADDITIVE FAMILY)

The family of sets U is additive if

$$(A \in U, B \in U) \Rightarrow (A \cup B \in U) \tag{1}$$

Let N represent the class of all additive families S, such that

$$T \subset S. \tag{2}$$

Since the family of all subsets of X is an element of N, $N \neq \phi$. Let

$$U = \cap N. \tag{3}$$

We shall show that U defined by (3) is additive.

Let $A \in U$ and $B \in U$, then for every $S \in N$, $A \in S$ and $B \in S$. The set N consists of additive families, therefore,

$$A \cup B \in S. \tag{4}$$

Equation (4) holds for every $S \in N$, hence

$$A \cup B \in U. \tag{5}$$

U is additive.

Now, we shall show $T \subset U$. By assumption:

$$\text{for every } S \in N, T \subset S.$$

Hence, if $A \in T$, then $A \in S$ and

$$A \in U. \tag{6}$$

Therefore,

$$A \in T \Rightarrow A \in U \tag{7}$$

or

$$T \subset U.$$

The family $U = \cap N$ is the smallest additive family containing the family T, because U is the intersection of all families with this property.

We define:

The family S of sets is multiplicative, if

$$(A \in S \wedge B \in S) \Rightarrow (A \cap B \in S). \tag{8}$$

For multiplicative families, a similar theorem exists.

THEOREM

For every family T of subsets of X, the smallest multiplicative family U exists, such that $T \subset U$.

● PROBLEM 4-20

Consider the set X and the family T of all its one-element subsets. Find the smallest additive family U, such that $T \subset U$.

SOLUTION:

Since T is the family of all one-element subsets of X, we have

$$\forall\{a\}\ \forall\{b\},\ \{a\} \in T,\ \{b\} \in T$$

$$\{a\} \cup \{b\} = \{a, b\} \in U \tag{1}$$

Because U is additive and $T \subset U$, U contains all one- and two-element subsets. By the same token, we can show that U consists of all finite subsets of X.

From the above considerations, it follows that a necessary and sufficient condition for a set X to be finite is that the family of all its non-empty subsets must be identical with U.

Note that by using this property, we can define a finite set without referring to the concept of natural numbers.

● PROBLEM 4-21

Let S denote a family of sets and S' denote the family of all sets of the form $D' = A - B$, where $A, B \in S$. Prove that

$$S' \subset S''. \tag{1}$$

SOLUTION:

The set S' consists of

$$S' = \{D' : D' = A - B, A \in S \wedge B \in S\} \tag{2}$$

while S'' consists of

$$S'' = \{D'' : D'' = E' - F', E' \in S' \wedge F' \in S'\}. \tag{3}$$

We understand that

$$S'' = (S')' \tag{4}$$

From (2) and (3), we obtain

$$S'' = \{D'' : D'' = (A - B) - (G - H), A, B, G, H \in S\} \tag{5}$$

where $E' = A - B$ and $F' = G - H$.

Suppose $A' \in S'$, then

$$A' = A - B \tag{6}$$

where $A, B \in S$.

Since S' consists of all sets of the form $A - B$, we conclude that

$$\phi \in S' \tag{7}$$

Therefore,

$$(A - B) - (A - A) = (A - B) - \phi = A - B \in S'' \tag{8}$$

and

$$S' \subset S''. \tag{9}$$

The inverse inclusion, in general, is false.

● PROBLEM 4-22

Prove that:

$$(A^B = \phi) \Leftrightarrow (A = \phi \wedge B \neq \phi). \tag{1}$$

Remember that $A^B = \{f : f : B \rightarrow A\}$.

SOLUTION:

$\Leftarrow$ Suppose $A = \phi$ and $B \neq \phi$. Then $f \in A^B$ implies $f \subset B \times A = B \times \phi = \phi$. But ϕ is not a function $f : B \rightarrow A$, $B \neq \phi$ and $A = \phi$. Therefore,

$$A^B = \phi. \tag{2}$$

$\Rightarrow$ Suppose $A \neq \phi$. Then $\exists a : a \in A$ and

$$\{(x, a) : x \in B\}$$

is a function from B to A. Hence

$$A^B \neq \phi. \tag{3}$$

Now, suppose $B = \phi$, then ϕ is a function from B to A and

$$A^B \neq \phi.$$

● PROBLEM 4-23

Prove that:

$$(A^B = B^A) \Rightarrow (A = B). \tag{1}$$

SOLUTION:

Suppose $A^B \neq \phi$, then a function f exists, such that

$$f : B \rightarrow A$$

$F \in A^B = B^A$. Since $f \in A^B$, the domain of f is B.

On the other hand, since $F \in B^A$, the domain of f is A. Thus,

$$A = B. \tag{2}$$

Now, suppose $A^B = \phi$ then

$$(A^B = \phi) \Rightarrow (A = \phi \wedge B \neq \phi). \tag{3}$$

Hence $A = \phi$.

But $A^B = B^A = \phi$ and $B = \phi$. Therefore,

$$A = B. \tag{4}$$

● PROBLEM 4-24

Let $F(X, R)$ denote the set of all real-valued functions $f \in F(X, R)$

$$f : X \rightarrow R. \tag{1}$$

Show that $F(X, R)$, with the usual algebraic operations, forms a real linear vector space.

SOLUTION:

We shall define algebraic operations on the elements $F(X, R)$. Let $f : X \rightarrow R$ and $g : X \rightarrow R$ and $a \in R$, then we define

$$(f + g)(x) = f(x) + g(x) \tag{2}$$

$$(a \cdot f)(x) = a \cdot f(x) \tag{3}$$

$$(f \cdot g)(x) = f(x) \cdot g(x) \tag{4}$$

$$(f + a)(x) = f(x) + a \tag{5}$$

We shall verify that $F(X, R)$ satisfies the axioms of the real linear vector space:

1. $(f + g) + (h) = f + (g + h)$
2. $f + g = g + f$
3. $\exists 0 \in F(X, R)$, such that $\forall f \in F(X, R), f + 0 = f$
4. $\forall f \in F(X, R)$ $\exists -f \in F(X, R)$, such that $f + (-f) = 0$
5. $a \cdot (b \cdot f) = (ab) \cdot f$ for any real numbers a and b
6. $1 \cdot f = f$
7. $a \cdot (f + g) = af + ag$
8. $(a + b) f = af + bf$

● PROBLEM 4-25

Let X represent the space (universal set), and A any subset of X. The characteristic function of A is defined by

$$X_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{1}$$

Show that

1. $X_{A \cap B} = X_A X_B$ (2)
2. $X_{A \cup B} = X_A + X_B - X_{A \cap B}$ (3)
3. $X_{A - B} = X_A - X_{A \cap B}$ (4)

SOLUTION:

1. We shall prove that

$$X_{A \cap B} = X_A \cdot X_B \tag{5}$$

Suppose $x \in A \cap B$, then

$$X_{A \cap B} = 1 \text{ and } X_A = X_B = 1.$$

If, on the other hand, $x \notin A \cap B$, then $X_{A \cap B} = 0$ and at least one of the functions X_A or X_B is equal to zero, $X_A \cdot X_B = 0$.

2. $X_{A \cup B} = 1$ when

   (a) $x \in A \wedge x \notin B$

   (b) $x \notin A \wedge x \in B$

   (c) $x \in A \wedge x \in B$

For (a) we have $X_A + X_B - X_{A \cap B} = 1 + 0 - 0 = 1$; for (b) we have $X_A + X_B - X_{A \cap B} = 0 + 1 - 0 = 1$; for (c) we have $X_A + X_B - X_{A \cap B} = 1 + 1 - 1 = 1$.

If $X_{A \cup B} = 0$, then $x \notin A$ and $x \notin B$ hence $X_A + X_B - X_{A \cap B} = 0$.

3. Suppose $X_{A-B} = 1$, then $x \in A$ and $x \notin B$,

$$X_A - X_{A \cap B} = 1 - 0 = 1$$

If $X_{A-B} = 0$, then

   (a) $x \notin A$, or

   (b) $x \in A \wedge x \in B$

For (a) $X_A - X_{A \cap B} = 0 - 0 = 0$; for (b) $X_A - X_{A \cap B} = 1 - 1 = 0$.

● PROBLEM 4-26

Prove the following:

THEOREM

Let $f: A \to B$ represent relation-preserving, and let R represent relation in A, and S relation in B. Then, there is one, and only one function

such that

$$F: \frac{A}{R} \to \frac{B}{S}, \tag{1}$$

$$p_B \circ f = F \circ p_A \tag{2}$$

F is called the function induced by f and $p_A : A \to {}^{A}/_{R}$ and $p_B : B \to {}^{B}/_{R}$.

SOLUTION:

Let us start with the definition.

DEFINITION (RELATION-PRESERVING FUNCTION)

Let A, B denote two sets with equivalence relations R and S respectively. A function $f : A \to B$ is called relation-preserving if

$$(a, b) \in R \Rightarrow [f(a), f(b)] \in S. \tag{3}$$

Now we shall show that the diagram

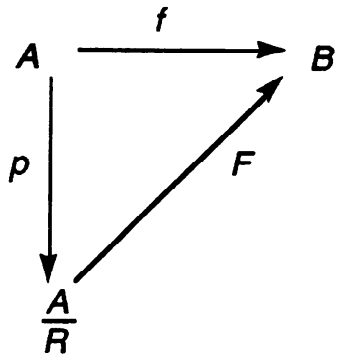


FIGURE 1

commutes. An equivalence class (see Problem 3–23) is denoted by $[a]$. Let us define function F by

$$F : [a] \to [f(a)] \tag{4}$$

where $[a] \in {}^{A}/_{R}$ and $[f(a)] \in {}^{B}/_{S}$. Note that the equivalence class, $[f(a)] \in {}^{B}/_{S}$ is independent of the representative $a \in [a]$.

Take for example, $a' \in [a]$, then $(a, a') \in R$; but since f is relation preserving,

$$(a, a') \in R \Rightarrow [f(a), f(a')] \in S. \tag{5}$$

Therefore,

$$f(a) \in [f(a)] \quad \text{and} \quad f(a') \in [f(a)]$$

and F defined by (4) is uniquely defined.

Function F is unique because p_A is a surjective. Suppose F^* is another function for which the diagram commutes, then for at least one $[a] \in {}^{A}/_{R}$,

$$F^*([a]) \neq f([a]). \tag{6}$$

But since p_A is a surjective, (6) leads to

$$F^* \circ p_A(a) \neq F \circ p_A(a) \tag{7}$$

which is impossible, because of commutativity.

That the diagram commutes follows from

$$p_B \circ f(a) = [f(a)] = F([a]) = F \circ p_A(a). \tag{8}$$

● PROBLEM 4-27

Let $f : A \to B$. Show that

$$(a, b) \in R \Leftrightarrow f(a) = f(b) \tag{1}$$

is an equivalence relation and show that a function $F : {}^{A}/_{R} \to B$ exists, such that

$$f = F \circ p \tag{2}$$

i.e. show that the diagram (Figure 1) commutes.

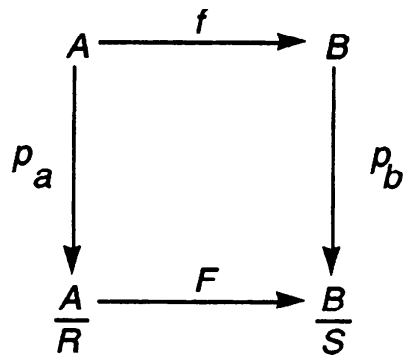


FIGURE 1

SOLUTION:

We shall show that (1) defines an equivalence relation.

1. $\forall a \in A : f(a) = f(a) \Rightarrow (a, a) \in R.$

2. $\forall a, b \in A : (a, b) \in R \Leftrightarrow f(a) = f(b) \Leftrightarrow (b, a) \in R.$

3. $\forall a, b, c \in A : (a, b) \in R \wedge (b, c) \in R \Leftrightarrow f(a) = f(b) \wedge f(b) = f(c) \Rightarrow$
   $\Rightarrow f(a) = f(c) \Leftrightarrow (a, c) \in R$

Hence, R is an equivalence relation.

Let us define $F : {}^{A}/_{R} \to B$ by

$$F([a]) = f(a). \tag{3}$$

Function F is uniquely defined. Indeed, suppose $b \in [a]$, then

$$(a, b) \in R \Rightarrow f(a) = f(b). \tag{4}$$

The diagram commutes because

$$(F \circ p)(a) = F(p(a)) = F([a]) = f(a). \tag{5}$$

● PROBLEM 4-28

Let $\{A_\alpha : \alpha \in \Omega\}$ be a family of sets. Define the Cartesian product of $\{A_\alpha\}$.

SOLUTION:

The Cartesian product of two sets A_1 and A_2, denoted $A_1 \times A_2$, was defined as a set of all ordered pairs

$$A_1 \times A_2 = \{(a, b) : a \in A_1 \wedge b \in A_2\} \tag{1}$$

On the other hand, $A_1 \times A_2$ can be considered to be the set of functions f of the index set $\{1, 2\}$ into $A_1 \cup A_2$, such that

$$f : \{1, 2\} \rightarrow A_1 \cup A_2$$

$$f(1) \in A_1 \quad \text{and} \quad f(2) \in A_2. \tag{2}$$

DEFINITION

Let $\{A_\alpha : \alpha \in \Omega\}$ represent the family of sets. The Cartesian product, denoted $\underset{\alpha}{X} A_\alpha$, is the set of all maps

$$F : \Omega \rightarrow \bigcup_\alpha A_\alpha \tag{3}$$

such that

$$\forall \alpha \in \Omega : F(\alpha) \in A_\alpha.$$

An element $f \in \underset{\alpha}{X} A_\alpha$ is usually denoted as $\{a_\alpha\}$, where $\forall \alpha \in \Omega : f(\alpha) \in a_\alpha$.

The element $a_\alpha \in A_\alpha$ is called the α^{th} coordinate of $\{a_\alpha\}$.

The set A_α is called the α^{th} factor of $\underset{\alpha}{X} A_\alpha$.

When the index set of Ω is the set of natural numbers and all A_n are $A_n = R$, where R is the set of real numbers, we obtain

$$\underset{m=1}{\overset{n}{X}} R = R^n$$

the n-dimensional Euclidean space; and the product,

$$\prod_{n=1}^{\infty} R_n$$

is the extension of the n-dimensional space R^n to an infinite number of dimensions.

● PROBLEM 4-29

Prove that the following properties are equivalent:

1. If each $A_\alpha \neq \phi$ and $\{A_\alpha : \alpha \in \Omega\}$ is a non-empty family, then $\prod_\alpha A_\alpha \neq \phi$

2. The axiom of choice.

3. If each $A_\alpha \neq \phi$ and $\{A_\alpha : \alpha \in \Omega\}$ is a non-empty family of sets, then a function $F: \Omega \rightarrow \bigcup_\alpha A_\alpha$ exists such that $\forall \alpha \in \Omega : F(\alpha) \in A_\alpha$. F is called the choice function.

SOLUTION:

(1) $\Rightarrow$ (2).

Suppose $\{A_\alpha : \alpha \in \Omega\}$ is a non-empty family of sets. Since $\prod_\alpha A_\alpha \neq \phi$, an element $F = \{a_\alpha\}$ exists and

$$F(\Omega)$$

is a set which satisfies the axiom of choice.

(2) $\Rightarrow$ (3).

Let us define for each $\alpha \in \Omega$

$$A'_\alpha = \{\alpha\} \times A_\alpha$$

Each A'_α is a non-empty set.

According to the axiom of choice, a set T exists consisting of exactly one element from each A'_α. Hence, for each α, there is a unique $(\alpha, a_\alpha) \in T$, $a_\alpha \in A_\alpha$

$$T \subset \bigcup_\alpha (\{\alpha\} \times A_\alpha) \subset \bigcup_\alpha (\Omega \times A_\alpha) = \Omega \times \bigcup_\alpha A_\alpha$$

and T is a function

$$T: \Omega \rightarrow \bigcup_\alpha A_\alpha .$$

(3) $\Rightarrow$ (1).

If $F: \Omega \rightarrow \bigcup_\alpha A_\alpha$, then F is an element of $\prod_\alpha A_\alpha$.

Let $\{A_\alpha : \alpha \in \Omega\}$ represent a family of non-empty sets, and let

$$\Lambda \subset \Omega.$$

Show that the function

$$P : \underset{\alpha}{X} \{A_\alpha : \alpha \in \Omega\} \rightarrow \underset{\alpha}{X} \{A_\alpha : \alpha \in \Lambda\} \tag{1}$$

defined by

$$P(F) = F \mid \Lambda \tag{2}$$

is onto and that each projection

$$p_\beta : \underset{\alpha}{X} A_\alpha \rightarrow A_\beta \tag{3}$$

is onto.

SOLUTION:

Let f be any element

$$f \in X \{A_\alpha : \alpha \in \Lambda\}. \tag{4}$$

We shall show that

$$F \in X \{A_\alpha : \alpha \in \Omega\} \tag{5}$$

exists, such that

$$P(F) = f. \tag{6}$$

By Problem 4-29 part 3, a choice function exists such that

$$G : \Omega - \Lambda \rightarrow \cup \{A_\alpha : \alpha \in \Omega - \Lambda\} \tag{7}$$

Then

$$F : \Omega \rightarrow \cup \{A_\alpha : \alpha \in \Omega\} \tag{8}$$

defined by

$$F \mid \Lambda = f$$

$$F \mid \Omega - \Lambda = G \tag{9}$$

is an element

$$F \in X \{A_\alpha : \alpha \in \Omega\} \tag{10}$$

and

$$P(F) = F \mid \Lambda = f.$$

Setting $\Lambda = \{\beta\}$, $\beta \in \Omega$ we obtain $p_B = P$, which completes the proof.

Two families are given $\{A_\alpha : \alpha \in \Omega\}$ and $\{B_\beta : \beta \in \Lambda\}$. Prove that

$$\left(\bigcup_{\alpha\in\Omega} A_\alpha\right) \times \left(\bigcup_{\beta\in\Lambda} B_\beta\right) = \bigcup_{(\alpha,\beta)\in\Omega\times\Lambda} (A_\alpha \times B_\beta). \tag{1}$$

SOLUTION:

Suppose

$$(a,b) \in (\bigcup_{\alpha} A_\alpha) \times (\bigcup_{\beta} B_\beta) \tag{2}$$

then

$$a \in \bigcup_{\alpha\in\Omega} A_\alpha \quad \text{and} \quad b \in \bigcup_{\beta\in\Lambda} B_\beta. \tag{3}$$

Hence, for some $\alpha \in \Omega$, $a \in A_\alpha$ and for some $\beta \in \Lambda$, $b \in B_\beta$. Therefore,

$$a \in A_\alpha \text{ and } b \in B_\beta \text{ for some } (\alpha, \beta) \in \Omega \times \Lambda.$$

Therefore,

$$(a, b) \in A_\alpha \times B_\beta \text{ for some } (\alpha, \beta) \in \Omega \times \Lambda \tag{4}$$

and finally

$$(a,b) \in \bigcup_{(\alpha,\beta)\in\Omega\times\Lambda} A_\alpha \times B_\beta. \tag{5}$$

CHAPTER 5

POWER OF A SET, EQUIVALENT SETS

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● PROBLEM 5-1

Let $N = \{1, 2, 3, \ldots\}$ and $P = \{2, 4, 6, \ldots\}$. Show that N and P are equivalent sets. Are they infinite?

SOLUTION:

We shall start with a very important definition.

DEFINITION OF EQUIVALENT SETS

Two sets A and B are called equivalent, written $A \sim B$, if a function $f: A \to B$ exists, which is one-to-one and onto. Function f defines a one-to-one correspondence between the sets A and B.

Let us define

$$f: n \to 2n, \quad f: N \to P. \tag{1}$$

Function (1) is one-to-one and onto. Therefore, sets N and P are equivalent.

Here is another definition.

DEFINITION OF FINITE SETS

A set is finite iff it is empty or equivalent to $\{1, 2, \ldots, n\}$ for some $n \in N$. Otherwise, the set is called infinite.

The set N is infinite because it is not empty and not equivalent to any set $\{1, 2, \ldots, n\}$. Since N and P are equivalent, P is also infinite.

● PROBLEM 5-2

Determine which of the following pairs are equivalent sets.

1. $\{1, 2, 3\}$ and $\{a, b, 4\}$.

2. Points of two circles.

SOLUTION:

1. There is a one-to-one and onto correspondence between the sets $\{1, 2, 3\}$ and $\{a, b, 4\}$.

$f: 1 \to a$

$2 \to b$

$3 \rightarrow 4$

Two finite sets are equivalent if, and only if, they contain the same number of elements.

2. We can assume that circles are concentric.

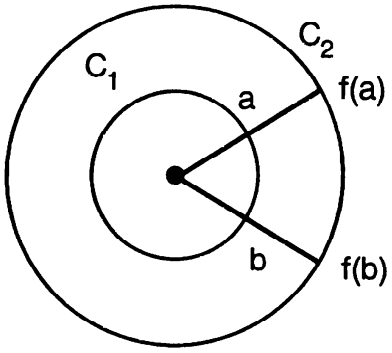


FIGURE 1

If they have equal radii, then equivalence is obvious. If $r_1 \neq r_2$, we can establish a one-to-one and onto correspondence in the following way:

Let a represent any point on the circle C_1. Then $f(a)$ is the point of intersection of the radius from the center, through a, to C_2, as shown in Figure 1. Function f defined above is one-to-one and onto. Hence the sets consisting of the points of C_1 and C_2 are equivalent.

● PROBLEM 5-3

Show that the open interval $]-1, 1[$ and the real axis R are equivalent.

SOLUTION:

Consider the "structure" shown in Figure 1.

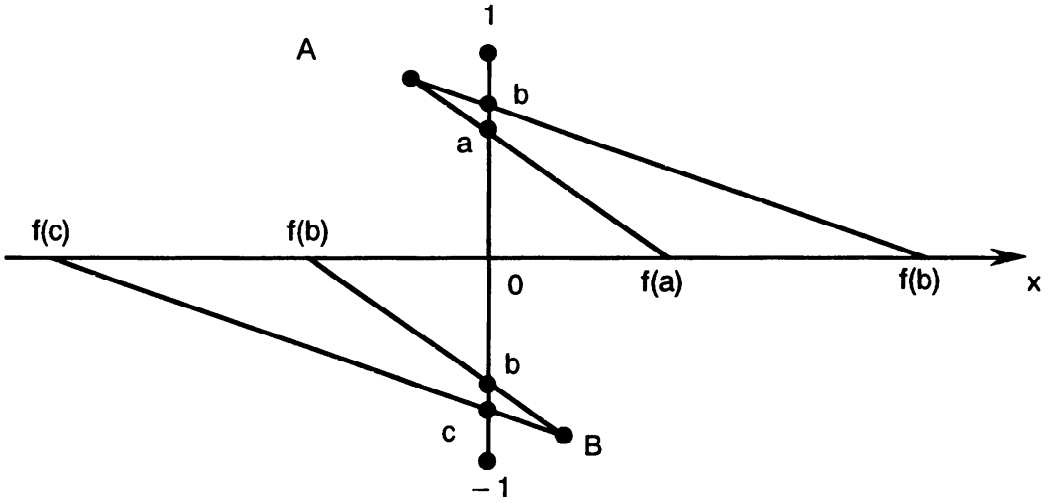


FIGURE 1

The "upper" half of the interval is mapped on the positive part of the real axis in the following manner:

Let a represent any point, such that $a \in]1, 0]$. We will draw a line from point A (which is at the same level as point 1 of the interval, and to the left of it) through point a to the x-axis. The point of intersection is $f(a)$. In the same manner, we assign a unique point of the negative axis ($x < 0$) to each point of $]0, -1[$. The determined function is one-to-one and onto. Hence, the open interval $]-1, 1[$ is equivalent to R.

Another way to show it is to define function f, such that

$$f:]-1, 1[\rightarrow R \text{ and } f(x) = \frac{x}{1 - |x|}.$$

Function f is onto and one-to-one. The desired result follows.

● PROBLEM 5-4

1. Show that the set of elements of any infinite sequence of distinct terms is denumerable.

2. Prove that the relation $A \sim B$ (A is equivalent to B) is an equivalence relation.

SOLUTION:

1. We shall start with a definition.

DEFINITION OF DENUMERABLE SETS

Set A is called denumerable if it is equivalent to the set of positive integers $N = \{1, 2, 3, \ldots\}$. Such a set is said to have cardinality aleph-null, denoted $\aleph_0$ or card N.

Let us define the function $f: N \rightarrow \{a_n\}$ by

$$f: n \rightarrow a_n.$$

Its domain is the set N and its range is the set of all elements of the sequence (a_n).

Since the elements of (a_n) are distinct, the function is one-to-one and onto. Hence, the set of elements of an infinite sequence of distinct terms is denumerable.

DEFINITION OF COUNTABLE SETS

A set is called countable if it is finite or denumerable.

Any sequence is a countable set.

2. We shall show that $A \sim B$ is an equivalence relation. The identity function , i.e.

$$1 : x \to x$$

is a one-to-one mapping of the set A onto itself. Since the inverse of a one-to-one mapping is one-to-one, it follows that

$$(A \sim B) \Rightarrow (B \sim A).$$

The composition of two one-to-one and onto mappings is one-to-one and onto. Therefore,

$$(A \sim B) \wedge (B \sim C) \Rightarrow A \sim C.$$

Remark:

Relation ~ is an equivalence relation. Hence we can classify the sets with respect to their power.

To each set A we assign a cardinal number (denoted sometimes by $\overline{\overline{A}}$ or card A) in such a way that the same cardinal number is assigned to two distinct sets if these sets are equivalent (have the same power).

● PROBLEM 5-5

1. Show that the lattice points (n, m) in a plane form a denumerable set, where n and m are integers.

2. Show that the set of positive rational numbers is denumerable.

SOLUTION:

1. Let us order the lattice points in the manner shown in Figure 1.

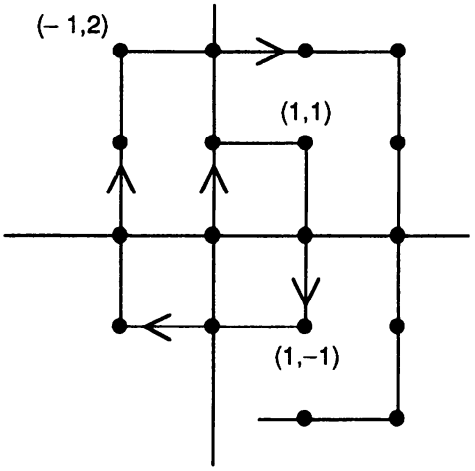


FIGURE 1

We order the set of lattice points into a sequence. That is, we establish a one-to-one and onto correspondence between the set of natural numbers, and the set of lattice points.

The lattice points are ordered as follows:

(0,0) – 1
(0,1) – 2
(1,1) – 3
(1,0) – 4
(1, –1) – 5 etc.

The ordering starts at the origin.

2. The set of rational numbers is arranged as shown in Figure 2.

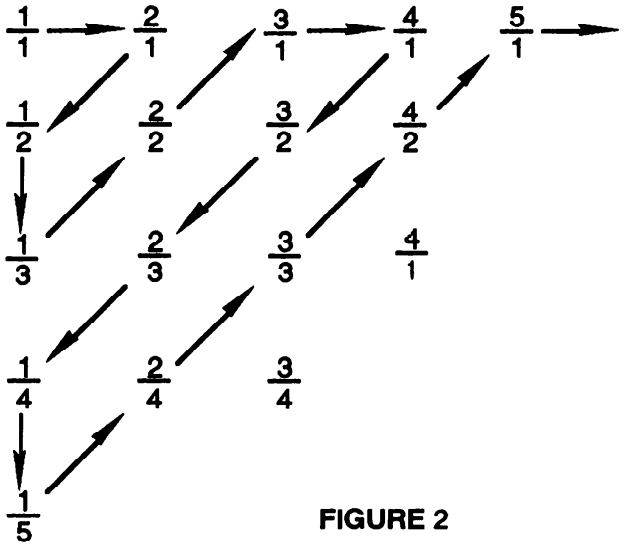


FIGURE 2

The diagram is traversed in the diagonally arrowed path. Every positive rational number will be included. This procedure establishes a one-to-one and onto mapping between the set of positive integers N and the set of rational positive numbers

1 – $^1/_1$
2 – $^2/_1$
3 – $^1/_2$
4 – $^1/_3$
5 – $^2/_3$
⋮

Hence, the set of rational numbers is denumerable.

● PROBLEM 5–6

Prove the following:

THEOREM

The set of all real numbers is non-countable.

SOLUTION:

To prove this theorem, it suffices to prove that for every sequence of real numbers $a_1, a_2, \ldots, a_n, \ldots$ a real number, a, which does not belong to the sequence, exists.

We define a sequence of closed intervals

$$\alpha_1 \beta_1, \alpha_2 \beta_2, \ldots, \alpha_n \beta_n, \ldots \tag{1}$$

such that

$$\beta_n - \alpha_n = \frac{1}{3^n} \quad \text{and} \quad \alpha_n \beta_n \subset \alpha_{n-1}\beta_{n-1}, \; a_n \notin \alpha_n \beta_n \tag{2}$$

We divide the closed interval [0, 1] into three $[0, 1/3]$, $[1/3, 2/3]$, $[2/3, 1]$ and choose one which does not contain the point a_1. That interval is $\alpha_1\beta_1$. Similarly, we divide $\alpha_1\beta_1$ into three intervals and choose the one which does not contain a_2. That interval is $\alpha_2\beta_2$. So, in the closed interval $\alpha_{n-1}\beta_{n-1}$ we determine a closed interval $\alpha_n\beta_n$ of length $1/3^n$, which does not contain the point a_n. Let a denote the common point of all the closed intervals $\alpha_n\beta_n$

$$a = \bigcap_{n=1}^{\infty} \alpha_n \beta_n \tag{3}$$

Then

$$a = \lim_{n \to \infty} \alpha_n = \lim_{n \to \infty} \beta_n. \tag{4}$$

For every n, $a \in \alpha_n\beta_n$ while $a \neq a_n$, because $a_n \notin \alpha_n\beta_n$.

We have a sequence of real numbers $(a_1, a_2, a_3, \ldots)$. A real number, a, exists, which does not belong to the sequence.

Hence, the set of real numbers is not countable.

● PROBLEM 5-7

Show that the union $A \cup B$ of two countable sets A and B is countable.

SOLUTION:

The set A is countable, therefore its elements can be written in the form of an infinite sequence

$$a_1, a_2, a_3, \ldots, a_n, \ldots \tag{1}$$

Similarly, the elements of B can be written in the form of a sequence

$$b_1, b_2, \ldots, b_n, \ldots \tag{2}$$

We form the sequence

$$a_1, b_1, a_2, b_2, a_3, b_3, \ldots, a_n, b_n, \ldots \tag{3}$$

The terms of sequence (3) form the set $A \cup B$. Hence $A \cup B$ is a countable set.

From this, we conclude that the set of all integers is countable.

The set of all positive integers is countable and the set of all negative integers is countable.

● PROBLEM 5-8

Prove that the set of all rational numbers is denumerable.

SOLUTION:

Let Q^+ denote the set of positive rational numbers and let Q^- be the set of negative rational numbers. Then the set of all rational numbers Q is

$$Q = Q^+ \cup Q^- \cup \{0\}. \tag{1}$$

In Problem 5-5 we proved that the set of positive rational numbers Q^+ is denumerable. By the same token, we can show that the set of negative rational numbers Q^- is denumerable. The sets Q^+, Q^- and $\{0\}$ are countable.

From Problem 5-7, we conclude that the set Q is denumerable as the union of Q^+, Q^- and $\{0\}$.

● PROBLEM 5-9

Prove the following:

THEOREM

The Cartesian product of two (or, more generally, of a finite number of) countable sets is a countable set.

SOLUTION:

First we shall prove that the set of pairs (m, n), where m and n are natural numbers, is countable.

We will arrange the pairs in a sequence, in such a way that if

$$m + n < m' + n' \tag{1}$$

then (m, n) comes before (m', n'). If $m + n = m' + n'$, then the pair with the smaller antecedent comes first. We obtain the sequence of pairs

$$(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), (1, 4), (2, 3), \ldots \tag{2}$$

Similarly, using the indexes of a_n and b_m, we write

$$(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_1, b_3), (a_2, b_2), (a_3, b_1), (a_1, b_4), \ldots \quad (3)$$

Suppose A and B are countable sets, then

$$A = \{a_1, a_2, a_3, a_4, \ldots\} \text{ and } B = \{b_1, b_2, b_3, \ldots\} \quad (4)$$

The Cartesian product of A and B is

$$A \times B = \{(a_n, b_m) : a_n \in A \wedge b_m \in B\}. \quad (5)$$

The elements of $A \times B$ can be arranged in a sequence as shown in (3).

If the sets A, B, and C are countable, then $A \times B$ is countable, and

$$(A \times B) \times C = A \times B \times C$$

is a countable set. Hence, the Cartesian product of a finite number of countable sets is a countable set.

It is easy to conclude from the above theorem that the set of all rational numbers is countable.

● PROBLEM 5–10

Prove:
The collection P of all polynomials with integer coefficients

$$p(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n \quad (1)$$

is a denumerable set.

SOLUTION:

Let us define for each pair

$$(m, n) \in N \times N \quad (2)$$

the set P_{mn} of polynomials $p(x)$ of degree n, such that

$$|a_0| + |a_1| + \ldots + |a_n| = m. \quad (3)$$

For each pair (m, n), the set P_{mn} is finite.

The set of all polynomials P can be represented as

$$P = \bigcup_{m,n} \{P_{mn} : (m, n) \in N \times N\}. \quad (4)$$

The set P is a countable union of countable sets, therefore, P is a countable set. Since P is not finite, P is denumerable.

● PROBLEM 5-11

Prove that the set Ω of algebraic numbers is denumerable.

SOLUTION:

A real number q is called an algebraic number if q is a solution to a polynomial equation

$$a_m x^m + \ldots + a_1 x + a_0 = 0 \tag{1}$$

with integral coefficients.

The set P of all polynomials with integer coefficients is a denumerable set (see Problem 5-10). Hence

$$P = \{f_1(x) = 0, f_2(x) = 0, f_3(x) = 0, \ldots\} \tag{2}$$

where $f_1, f_2, f_3, \ldots$ are polynomials.

Let us denote by Q_i

$$Q_i = \{x : f_i(x) = 0\} \tag{3}$$

the set of the solutions of $f_i(x) = 0$. Since a polynomial of degree m has, at most, m solutions, each Q_i is finite. Therefore

$$\Omega = \bigcup_{i \in N} Q_i \tag{4}$$

is denumerable.

The set of all real numbers is non-countable. Each real number is either algebraic or transcendental (i.e. non-algebraic),

$$R = \Omega \cup T$$

where T is the set of transcendental numbers. We conclude that transcendental numbers exist and that the set T of transcendental numbers is non-countable. The members e and π are transcendental numbers.

● PROBLEM 5-12

Use the axiom of choice to prove:

THEOREM

Every infinite set A contains a subset B which is denumerable.

SOLUTION:

Let $P(A)$ represent the power set of A, that is, the class of all subsets of A. We define the choice function F, such that

$$F : P(A) \rightarrow A. \tag{1}$$

For each non-empty subset D of A, $D \subset A$, $f(D) \in A$. By virtue of the axiom of choice, such a function exists. We shall start from the "top" and move "down":

$$F(A) = a_1$$

$$F(A - \{a_1\}) = a_2$$

$$F(A - \{a_1, a_2\}) = a_3$$

$$\vdots$$

$$F(A - \{a_1, a_2, \ldots, a_n\}) = a_{n+1}. \tag{2}$$

The set A is infinite, hence for every $n \in N$, the set

$$A - \{a_1, a_2, \ldots, a_n\} \neq \phi \tag{3}$$

is non-empty.

Since F is a choice function

$$a_n \neq a_k \quad \text{for} \quad n \neq k \tag{4}$$

Hence all a_n are distinct and the set

$$B = \{a_1, a_2, a_3, \ldots\} \tag{5}$$

is a denumerable subset of A, $B \subset A$. In the first step, the choice function F chooses one element from A, $F(A) = a_1$. Then from the set $A - \{a_1\}$, the choice function F chooses a_2, $F(A - \{a_1\}) = a_2$. The remaining set $A - \{a_1, a_2, \ldots, a_n\}$ is non-empty because the set A is infinite.

● PROBLEM 5-13

Prove the theorems:

1. A subset of a denumerable set is countable.

2. Every subset of a countable set is countable.

SOLUTION:

1. Let A denote any denumerable set

$$A = \{a_1, a_2, a_3, \ldots\} \tag{1}$$

and let B denote a subset of A, $B \subset A$. When $B = \phi$, then of course B is finite. If $B \neq \phi$, then we can denote the least positive integer as k, such that

$$a_{k_1} \in B. \tag{2}$$

By k_2 we denote the least positive integer, such that

$$k_2 > k_1 \quad \text{and} \quad a_{k_2} \in B. \tag{3}$$

By repeating the procedure, we obtain

$$B = \{a_{k_1}, a_{k_2}, a_{k_3}, \ldots\}. \tag{4}$$

If the set of integers $\{k_1, k_2, \ldots\}$ is finite, then B is finite, otherwise B is denumerable.

2. Suppose A is a countable set, then A is either finite or denumerable. In either case, its subsets are countable.

● PROBLEM 5-14

Let A and B denote disjoint sets, $A \cap B = \phi$, and let A denote an infinite set and B denote a denumerable set. Show that

$$A \cup B \sim A. \tag{1}$$

SOLUTION:

Since A is an infinite set, it must contain a denumerable subset $P = \{p_1, p_2, p_3, \ldots\}$ (See Problem 5-12).

Let us write

$$A \cup B = (A - P) \cup (P \cup B) \quad \text{and} \quad A = (A - P) \cup P. \tag{2}$$

We shall establish an equivalence relation between $A \cup B$ and A in the following manner:

$$f(x) = \begin{cases} x & \text{for } x \in A - P \\ p_{2n-1} & \text{for } x = p_n \\ p_{2n} & \text{for } x = b_n \end{cases} \tag{3}$$

where

$$B = \{b_1, b_2, b_3, \ldots\} \tag{4}$$

Figure 1 illustrates function $f(x)$.

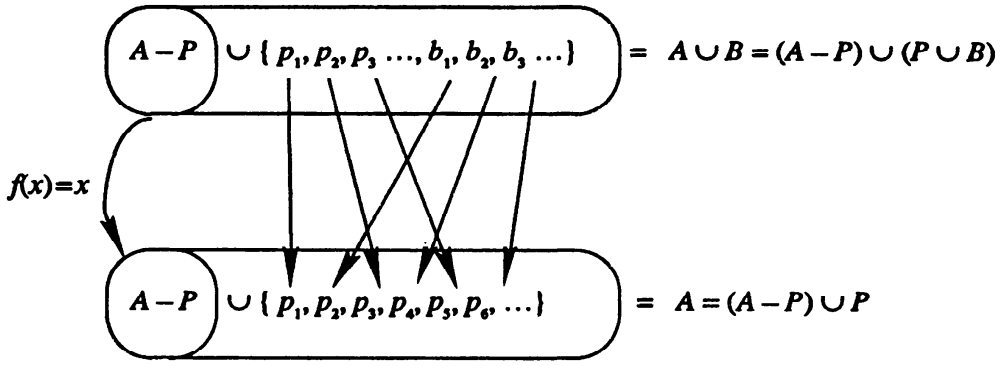


FIGURE 1

Function $f: A \cup B \rightarrow A$ is one-to-one and onto. Hence, the sets $A \cup B$ and A are equivalent

$$A \cup B \sim A. \tag{5}$$

● PROBLEM 5-15

Prove the following important theorem:

THEOREM

Let $\{A_1, A_2, A_3, \ldots\}$ denote a countable sequence of countable sets, then the union

$$A = A_1 \cup A_2 \cup A_3 \cup \ldots \tag{1}$$

is a countable set.

SOLUTION:

We proved (see Problem 5-9) that a Cartesian product of two countable sets is a countable set. From that, it follows that every double sequence can be transformed into a simple sequence. The elements of the array

$$\begin{array}{l} a_{11}, a_{12}, \ldots, a_{1n}, \ldots \\ a_{21}, a_{22}, \ldots, a_{2n}, \ldots \\ \vdots \\ a_{m1}, a_{m2}, \ldots, a_{mn}, \ldots \end{array} \tag{2}$$

can be written in the form of the infinite sequence. Each of the sets A_m is a countable set. Therefore,

$$A_m = \{a_{m1}, a_{m2}, a_{m3}, \ldots\}. \tag{3}$$

We obtain a double sequence as in (2), because all A_m are countable. A double sequence (a_{mn}) can be transformed into a simple sequence with possible repetitions

$$a_{11}, a_{12}, a_{21}, a_{31}, \ldots \tag{4}$$

We apply the axiom of choice, because the set of sequences consisting of the elements of the set A_m contains more than one element. We showed here that the elements of the set

$$A = A_1 \cup A_2 \cup A_3 \cup \ldots = \bigcup_{n=1}^{\infty} A_n \tag{5}$$

can be arranged in a sequence. Therefore, A is a countable set.

● PROBLEM 5-16

Prove that the set of all intervals on the real axis, with both endpoints rational numbers, is countable.

Is the set of all oriented intervals on the real axis, with both endpoints rational numbers, countable as well?

SOLUTION:

The set of rational numbers Q is countable. Hence the Cartesian product $Q \times Q$ is a countable set. Let P denote the set of intervals

$$P = \{[a, b] : a \in Q \wedge b \in Q\}$$

with endpoints rational numbers. We define mapping f, such that

$$f: P \to Q \times Q$$

$$f: [a, b] \to (a, b) \in Q \times Q$$

When P is the set of oriented intervals, then f is one-to-one and onto. Indeed, for example

$$f([0,1]) = (0, 1)$$

$$f([1, 0]) = (1, 0).$$

Therefore, the set of oriented intervals is countable. From that, we conclude that the set of intervals is countable.

● PROBLEM 5-17

Show that the power set 2^A of A is equivalent to the collection of characteristic functions on A, $C(A)$, i.e. prove that

$$2^A \sim C(A). \tag{1}$$

SOLUTION:

Let B represent any subset of A,

$$B \in 2^A. \tag{2}$$

We define function

$$f: 2^A \to C(A) \tag{3}$$

in such a way that

$$f(B) = X_B = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}. \tag{4}$$

Then function f is one-to-one and onto. Therefore, the sets 2^A and $C(A)$ are equivalent.

$$2^A \sim C(A). \tag{5}$$

● PROBLEM 5-18

Prove that the set of spheres in a three-dimensional space, which have the rational coordinates of the center and rational radii, is countable.

SOLUTION:

The set Q of all rational numbers is countable.

Now, consider the set P of all points in a three-dimensional space, such that $p \in P$, if all three coordinates of p are rational numbers

$$P = \{p : p \in R^3; p = (p_1, p_2, p_3); p_1 \in Q \wedge p_2 \in Q \wedge p_3 \in Q\}.$$

It is easy to see that

$$P \sim Q \times Q \times Q.$$

Hence, the set of centers of the spheres is countable. The radii of the spheres are rational numbers.

Let Q^+ represent the set of all non-negative rational numbers. Then the set of all spheres, which have the rational coordinates of the center and

rational radii, is equivalent to the set $Q \times Q \times Q \times Q^+$. The Cartesian product of a finite number of countable sets is a countable set.

Therefore, $Q \times Q \times Q \times Q^+$ is a countable set and the set of spheres is countable.

● PROBLEM 5-19

Show that every set of disjoint intervals is countable.

SOLUTION:

Let P denote the set of disjoint intervals, and let $p \in P$ represent an interval

$$p = [\alpha, \beta] \tag{1}$$

where α and β are real numbers. We can always find an interval p', such that

$$p' = [a, b] \tag{2}$$

and $p' \subset p$, that is

$$[a, b] \subset [\alpha, \beta] \tag{3}$$

where a and b are rational numbers.

The set of rational numbers is countable. Hence, the set of all intervals with rational endpoints is countable.

Let $F : P \rightarrow P'$ represent a one-to-one and onto function, such that to every interval $[\alpha, \beta] \in P$, F assigns an interval $[a, b]$, such that $[a, b] \subset [\alpha, \beta]$ and a and b are rational numbers. Here P' is the set of disjoint intervals with rational endpoints. Since P' is a subset of a countable set (the set of all intervals with rational endpoints), P' is countable.

Therefore, P is also countable.

● PROBLEM 5-20

Prove that the set of proper maxima of the function is countable.

SOLUTION:

Let f represent a function with real arguments and values.

DEFINITION

Function f has a proper maximum at the point a, if an interval $[b, c]$ exists

such that $a \in]b, c[$, and such that for each $x \in]b, c[$ and $x \neq a$,

$$f(x) < f(a). \tag{1}$$

For each point x', which is the proper maximum of function $f(x)$, an interval $[b, c]$ with the properties described in the definition exists.

It is possible that to the proper maximum x' there corresponds an interval $[b', c']$ which contains another proper maximum (or maxima) as shown in Figure 1.

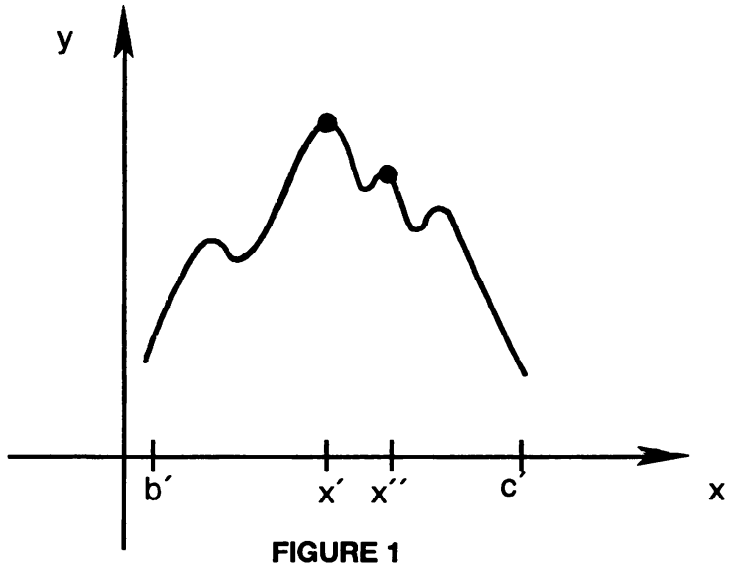


FIGURE 1

For each proper maximum, we can find an interval $[\alpha, \beta]$ with the rational endpoints which does not contain other proper maxima. Hence, we obtain a family of disjoint intervals, which is countable (see Problem 5-19).

Therefore, the set of proper maxima of the function is countable.

● PROBLEM 5-21

Prove that the set of points of discontinuity of a monotonic function in an interval $[a, b]$ is countable.

SOLUTION:

Suppose $f(x)$ is a monotonically increasing function. Function $f(x)$ is discontinuous at α, if and only if

$$p(\alpha) = f(\alpha + 0) - f(\alpha - 0) > 0 \tag{1}$$

where $f(\alpha + 0)$ denotes the right-hand limit, and $f(\alpha - 0)$ the left-hand limit. Of course, we should set

$$f(a - 0) = f(a) \quad \text{and} \quad f(b + 0) = f(b). \tag{2}$$

If

$$a < \alpha_1 < \alpha_2 < \ldots < \alpha_s < b \tag{3}$$

and

$$\alpha_\mu < x_\mu < \alpha_{\mu+1} \tag{4}$$

then

$$f(x_\mu) - f(x_{\mu-1}) \geq f(\alpha_\mu + 0) - f(\alpha_\mu - 0) = p(\alpha_\mu). \tag{5}$$

Hence

$$f(b) - f(a) = \sum_{\mu=1}^{S} [f(x_\mu) - f(x_{\mu-1})] \geq \sum_{\mu=1}^{S} p(\alpha_\mu) \tag{6}$$

where

$$x_0 = a \quad \text{and} \quad x_s = b. \tag{7}$$

When α_μ are numbers with $p(\alpha_\mu) > 1/n$, it follows that

$$M < n\,[f(b) - f(a)]. \tag{8}$$

Therefore, the number of points of discontinuity α with $p(\alpha) > 1/n$ has a fixed upper bound. Thus, a finite number of points of discontinuity with $p(\alpha) > 1/n$ belong to the interval $[a, b]$.

Now we can write the finite number of points of discontinuity with $p(\alpha) > 1$, then $p(\alpha) > 1/2$, then $p(\alpha) > 1/3$, etc.

For each point of discontinuity α, n exists, such that $p(\alpha) > 1/n$. Therefore, each point of discontinuity appears in the sequence. The set of points of discontinuity is countable.

● PROBLEM 5-22

Prove this rather amazing result: that the set of points of a straight line and the set of points of a space are equivalent, that is, that R^1 and R^3 are equivalent.

SOLUTION:

We shall first prove that the set of points of a cube of unit length and the set of points of an interval [0, 1] are equivalent. The cube is shown in Figure 1.

The coordinates of a point P interior to the cube are x, y z. Each of these real numbers can be written as a decimal fraction

$$x = 0.a_1a_2a_3 \ldots$$

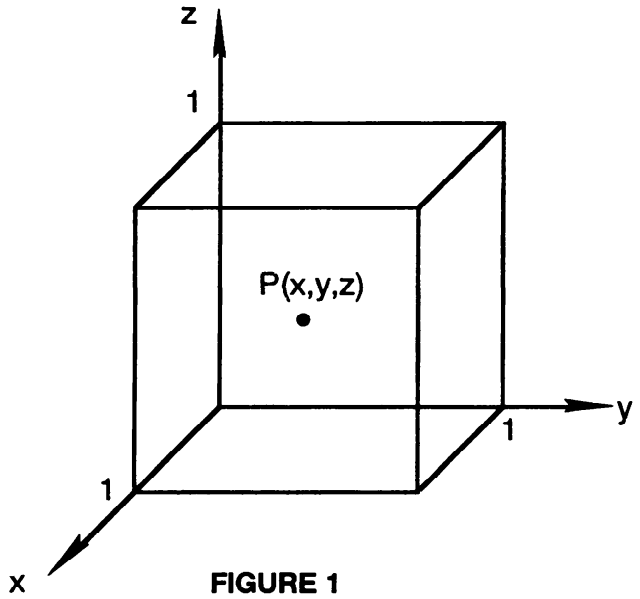


FIGURE 1

$$y = 0.b_1b_2b_3 \ldots$$

$$z = 0.c_1c_2c_3 \ldots$$

Let us form a real number p in the following manner:

$$p = 0.a_1b_1c_1a_2b_2c_2a_3b_3c_3\ldots$$

Hence, to every point P interior to the cube a unique point p can be assigned, where $0 < p < 1$.

This is a one-to-one and onto mapping. Therefore, the sets of points of an interval and of a cube are equivalent. Since the interval and the whole real axis are equivalent, the result follows.

It is surprising that the sets of different dimensions are equivalent. A one-to-one and onto correspondence between them exists.

Hilbert, Peano and Brouwer proved that this correspondence cannot be continuous.

THEOREM (HILBERT, PEANO, BROUWER)

No one-to-one correspondence that maintains continuity exists between two continuums of different order.

Correspondence is continuous when the neighboring points of one continuum can be mapped to the neighboring points of the other continuum.

● PROBLEM 5-23

Show that, if $A \sim B$, then

$$A^C \sim B^C. \qquad (1)$$

SOLUTION:

Sets A and B are equivalent, hence a function

$$g : A \to B \tag{2}$$

exists, which is a bijection (one-to-one and onto).

Let $f \in A^C$. We must define a function

$$G : A^C \to B^C \tag{3}$$

which is a bijection, to show that the sets A^C and B^C are equivalent.

Let us set (see Figure 1)

$$G(f) = g \circ f. \tag{4}$$

Function G is one-to-one, since

$$g \circ f = g \circ h \text{ implies } f = h.$$

Function G is also onto, because for $h \in B^C$, we obtain

$$G(g^{-1} \circ h) = g \circ g^{-1} \circ h = h \tag{5}$$

Since G is one-to-one and onto, it is bijective. Hence $A^C \sim B^C$.

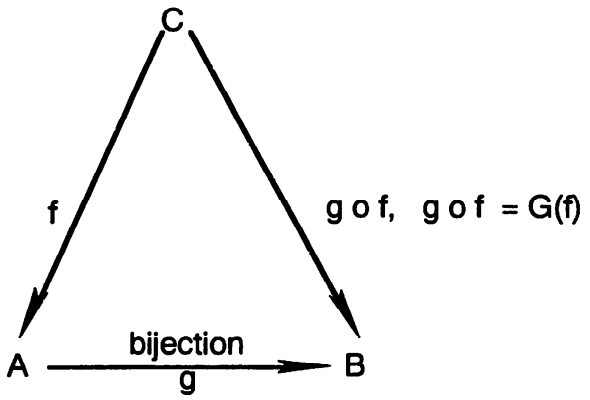


FIGURE 1

● PROBLEM 5-24

Show that

$$A^C \times B^C \sim (A \times B)^C. \tag{1}$$

SOLUTION:

We repeat

$$A^C = \{f : f : C \to A\}. \tag{2}$$

We define

$$G : A^C \times B^C \to (A \times B)^C \tag{3}$$

in such a way that

$$G(f, g)(x) = (f(x), g(x)) \text{ for all } x \in C. \tag{4}$$

Function G is one-to-one. Indeed let

$$G(f_1, g_1) = G(f_2, g_2). \tag{5}$$

Then, for all $x \in C$

$$G(f_1, g_1)(x) = G(f_2, g_2)(x) \tag{6}$$

and

$$(f_1(x), g_1(x)) = (f_2(x), g_2(x)). \tag{7}$$

Hence

$$f_1(x) = f_2(x) \text{ for all } x \in C$$

and

$$g_1(x) = g_2(x) \text{ for all } x \in C.$$

Thus

$$f_1 = f_2 \quad \text{and} \quad g_1 = g_2,$$

$$(f_1, g_1) = (f_2, g_2). \tag{8}$$

Now we shall prove that G is onto. Let $h \in (A \times B)^C$,

$$h(x) \in A \times B \text{ for all } x \in C.$$

Let

$$(h_1(x), h_2(x)) = h(x). \tag{9}$$

Then

$$h_1 \in A^C \quad \text{and} \quad h_2 \in B^C$$

$$(h_1, h_2) \in A^C \times B^C. \tag{10}$$

Thus

$$G(h_1, h_2) = h \quad \text{since} \tag{11}$$

$$G(h_1, h_2)(x) = (h_1(x), h_2(x)) = h(x) \tag{12}$$

for all $x \in C$.

Function $G : A^C \times B^C \to (A \times B)^C$ is bijective, hence, the sets $A^C \times B^C$ and $(A \times B)^C$ are equivalent.

CHAPTER 6

CARDINAL NUMBERS, CARDINAL ARITHMETIC

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● PROBLEM 6-1

The power of the set of natural numbers is denoted by card N (or $\aleph$) and the power of the set of real numbers is denoted by card R or by C (continuum). These are the most important of the infinite cardinal numbers. The following important inequality holds:

$$\text{card } R \neq \text{card } N. \tag{1}$$

Prove (1) by applying the results of Chapter 5.

SOLUTION:

In Chapter 5, we showed that for every sequence of real numbers a_1, a_2, a_3, ..., a_n, ..., we can define a real number b, which does not belong to this sequence. This can be formulated in the form of a theorem.

THEOREM

The set of all real numbers is non-countable.

Therefore, a cardinal number assigned to the set of natural numbers, card N, is different from the cardinal number assigned to the set of real numbers, card R.

$$\text{card } R \neq \text{card } N$$

The cardinal number assigned to the set A is denoted by card A.

● PROBLEM 6-2

In Problem 3-26, we discussed the axiomatic formulation of set theory. Describe how the concept of cardinal numbers can be established axiomatically.

SOLUTION:

The cardinal numbers describe the "size" of the sets. In this sense, they can be considered a generalization of natural numbers. To the set of, say, five books, we assign the natural number five, which describes how large the set is.

The axiomatic formulation given here does not define a cardinal number and is not related to counting. The concept of a cardinal number is related to "size." We want to determine if one of two given sets has more members than the other. Thus, we do not have to count; just pair off each member of

one set with a member of the other and see if any elements are left over.

The following axioms introduce the concept of cardinal numbers:

I. Each set X is associated with a cardinal number, denoted by card X, and for each cardinal number p, a set X exists such that

$$\text{card } X = p.$$

II. Card $X = 0$, if and only if $X = \phi$.

III. If X is a non-empty finite set, that is, $X \sim \{1, 2, \ldots, n\}$ for some $n \in N$, then card $X = n$.

IV. For any two sets X and Y, card X = card Y, if and only if $X \sim Y$.

Axioms I and IV are usually called the axiom of cardinality.

From II and III, we see that the cardinal number of a finite set is the number of elements of that set.

● PROBLEM 6-3

Explain why

$$\text{card } N < \text{card } R \qquad (1)$$

where N is the set of positive integers and R is the set of real numbers.

SOLUTION:

The cardinal number of a finite set is called a finite cardinal number. Finite cardinal numbers are actually non-negative integers. The cardinal number of an infinite set is called a transfinite cardinal number. Finite cardinal numbers have the inherited order of natural numbers

$$0 < 1 < 2 < \ldots < n < n + 1 < \ldots \qquad (2)$$

The question is how to compare two transfinite cardinal numbers. By applying axiom IV (see Problem 6-2), we can determine whether two cardinal numbers are equal or not. To establish which of the two cardinal numbers is "larger," we shall use the following:

DEFINITION

Let A and B represent two sets and card A and card B denote their cardinal numbers, respectively. Then card A is said to be less than card B; we write

$$\text{card } A < \text{card } B \tag{3}$$

when the set A is equipotent to a subset of B, but the set B is not equipotent to any subset of A.

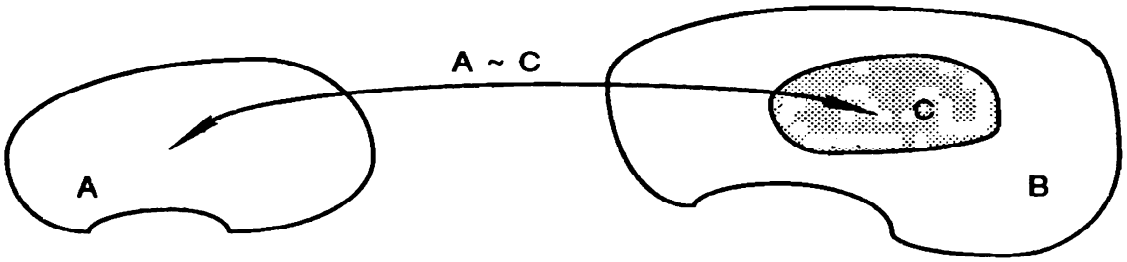


FIGURE 1

D does not exist, such that $D \subset A$ and $D \sim B$.

The set N is a subset of R, hence N is equipotent to a subset of R

$$N \sim N \subset R. \tag{4}$$

The infinite set R is nondenumerable. Thus, R is not equipotent to any subset of N. Therefore,

$$\text{card } N < \text{card } R. \tag{5}$$

● PROBLEM 6-4

Let m denote any finite cardinal number. Prove that

$$m < \text{card } N. \tag{1}$$

SOLUTION:

Let A represent the subset of the set of natural numbers N.

$$A = \{1, 2, \ldots, m\} \tag{2}$$

Then

$$\text{card } A = m \quad \text{and} \quad A \subset N. \tag{3}$$

We have

$$m = \text{card } A < \text{card } N \tag{4}$$

because the set A is equipotent to a subset of N, but N is not equipotent to any subset of A (see Problem 6-3).

Prove the following:

LEMMA

If B is a subset of A, and if an injection (one-to-one) $f : A \to B$ exists, then there is a bijection (one-to-one and onto) $g : A \to B$.

SOLUTION:

If $B = A$, then the identity function is one-to-one and onto (bijection). Suppose B is a proper subset of A. Let us denote the set

$$C = \bigcup_{n=0}^{\infty} f^n(A - B) \tag{1}$$

by C where f^0 is the identify function on A and

$$f^n(x) = f(f^{n-1}(x)) \tag{2}$$

for each $x \in A$ and for each positive integer n. Note that $A - B \subset C$ and $f(C) \subset C$. For any two distinct non-negative integers m,n, the sets $f^m(A - B)$ and $f^n(A - B)$ are disjoint.

Indeed, suppose $f^m(A - B) \cap f^n(A - B) \neq \phi$, then $x_1, x_2 \in A - B$ exists, such that

$$f^m(x_1) = f^n(x_2) \tag{3}$$

and

$$f^m(x_1) = f^{n-m}(f^m(x_2)) = f^m(f^{n-m}(x_2)). \tag{4}$$

Function f is one-to-one, hence

$$f^{n-m}(x_2) = x_1 \in (A - B) \cap B \tag{5}$$

is a contradiction.

We define function $g : A \to B$

$$g(x) = \begin{cases} f(x) & \text{for } x \in C \\ x & \text{for } x \in A - C \end{cases} \tag{6}$$

Function g is one-to-one, also:

$$\begin{aligned} g(A) &= f(C) \cup (A - C) = \\ &= f(\bigcup_{n=0}^{\infty} f^n(A - B)) \cup (A - \bigcup_{n=0}^{\infty} f^n(A - B)) = \\ &= (\bigcup_{n=1}^{\infty} f^n(A - B)) \cup (A - \bigcup_{n=0}^{\infty} f^n(A - B)) = \\ &= A - (A - B) = B. \end{aligned} \tag{7}$$

Remember f^0 is the identity function. Thus, function g is a bijection. The diagram illustrates the proof.

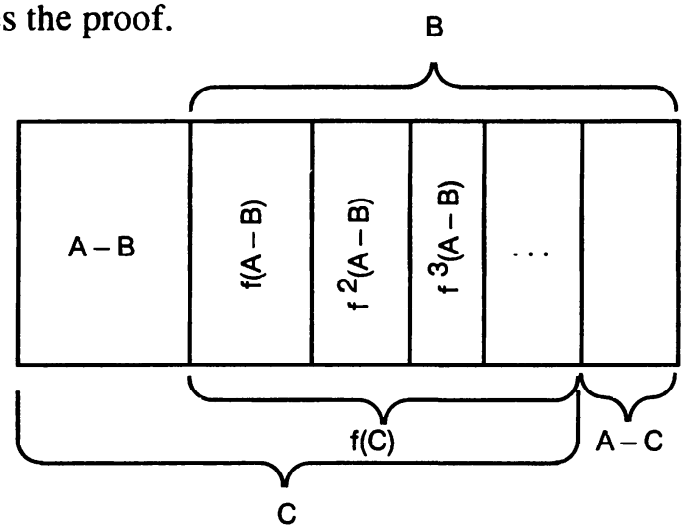


FIGURE 1

The set A is represented by the whole rectangle.

● PROBLEM 6–6

So far, we have considered the following cases:

1. If two sets A and B are equipotent, then

$$\text{card } A = \text{card } B. \tag{1}$$

2. Set A is equipotent to a subset of B, but the set B is not equipotent to any subset of A, then

$$\text{card } A < \text{card } B. \tag{2}$$

The question is how two cardinal numbers, card A and card B, compare when A is equipotent to a subset of B and B is equipotent to a subset of A. Georg Cantor conjectured that in such a case

$$\text{card } A = \text{card } B. \tag{3}$$

It was later independently proven by both E. Schröder and F. Bernstein. Prove:

SCHRÖDER-BERNSTEIN THEOREM

If A and B are sets, such that A is equipotent to a subset of B and B is equipotent to a subset of A, then A and B are equipotent, i.e.

$$\text{card } A = \text{card } B.$$

SOLUTION:

Suppose $A_1 \subset A$ and $B_1 \subset B$ and

$$A_1 \sim B, \; B_1 \sim A. \tag{4}$$

Then two bijections exist

$$f_1 : A \to B_1 \quad \text{and} \quad g_1 : B \to A_1. \tag{5}$$

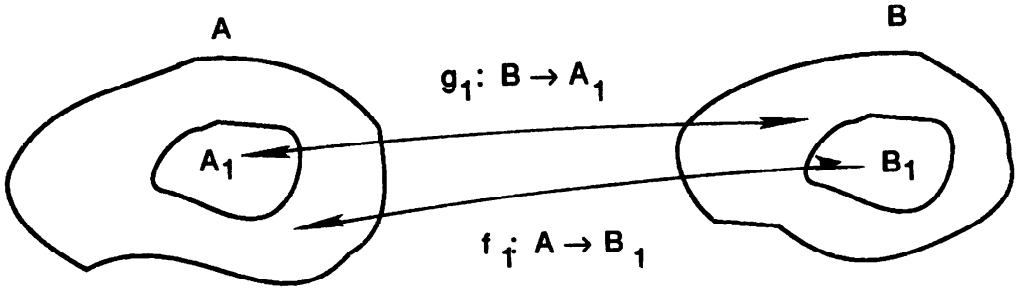


FIGURE 1

Let us define

$$f : A \to A_1 \text{ by } f(x) = g_1(f_1(x)). \tag{6}$$

Then f is one-to-one. By the lemma proved in Problem 6-5, a bijection $g : A \to A_1$ exists.

Therefore, since $g : A \to A_1$ is a bijection and $g_1^{-1} : A_1 \to B$ is a bijection, their composition

$$g_1^{-1} \circ g : A \to B \tag{7}$$

is a bijection. Hence, $A \sim B$ and

$$\text{card } A = \text{card } B. \tag{8}$$

● PROBLEM 6-7

Prove the following version of the Schröder-Bernstein theorem:

THEOREM

If card $A \leq$ card B and card $B \leq$ card A, then card A = card B.

SOLUTION:

We shall write card $A \leq$ card B to mean card $A <$ card B or card A = card B.

Suppose two sets are given, A and B. By applying the Schröder-Bernstein theorem, we conclude that one of three possibilities takes place:

$$\text{card}\, A < \text{card}\, B$$

or

$$\text{card}\, A > \text{card}\, B$$

or

$$\text{card}\, A = \text{card}\, B.$$

Hence, an immediate consequence of the Schröder-Bernstein theorem is

$$\begin{pmatrix}\text{card}\, A \leq \text{card}\, B \\ \text{card}\, B \leq \text{card}\, A\end{pmatrix} \Rightarrow \begin{pmatrix}\text{card}\, A = \text{card}\, B\end{pmatrix}$$

● PROBLEM 6-8

Prove that card N is the smallest transfinite cardinal number.

SOLUTION:

The cardinal number assigned to a finite set is called a finite cardinal number and the cardinal number assigned to an infinite set is called a transfinite cardinal number. We already know two transfinite cardinal numbers, card N and card R, where

$$\text{card}\, N < \text{card}\, R. \tag{1}$$

Thus

$$\underbrace{0 < 1 < 2 < \ldots}_{\text{Finite Cardinal Numbers}} \quad \underbrace{< \ldots ? \ldots < \text{card}\, N < \ldots ? \ldots < \text{card}\, R \ldots ? \ldots}_{\text{Transfinite Cardinal Numbers}}$$

FIGURE 1

Figure 1 illustrates the state of our knowledge about cardinal numbers. Each question mark indicates that we are not sure if the respective cardinal number exists.

Let A represent an infinite set. We shall apply the following:

THEOREM

Every infinite set contains a denumerable subset.

Therefore, a set B exists, such that

$$B \subset A \text{ and } B \sim N. \tag{2}$$

Thus

$$\text{card } B = \text{card } N \tag{3}$$

and

$$\text{card } B \leq \text{card } A. \tag{4}$$

We conclude, that for every infinite set A,

$$\text{card } N \leq \text{card } A. \tag{5}$$

The number, card N, is the smallest transfinite cardinal number. We can eliminate one of the question marks; see Figure 2.

$$\underbrace{0 < 1 < 2 < \ \ldots} \qquad \underbrace{< \text{card } N < \ \ldots ?\ldots < \text{card } R < \ldots ?\ldots}$$

Finite Cardinal Numbers Transfinite Cardinal Numbers

FIGURE 2

We shall eliminate the remaining question marks later.

● PROBLEM 6-9

Prove the following:

$$\begin{pmatrix} A \subset B \subset C \\ A \sim C \end{pmatrix} \Rightarrow (A \sim B).$$

SOLUTION:

We shall apply the Schröder-Bernstein theorem. Since $A \subset B \subset C$, we have

$$B \sim B \subset C \tag{1}$$

and

$$A \subset B. \tag{2}$$

But $A \sim C$, therefore

$$B \sim B \subset C \tag{3}$$

$$C \sim A \subset B \tag{4}$$

as shown in Figure 1.

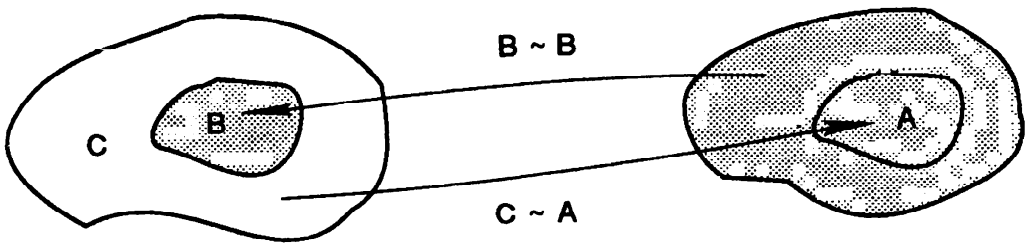


FIGURE 1

By the Schröder-Bernstein theorem we conclude that $C \sim B$. Therefore, since $A \sim C$ and $C \sim B$, we obtain $A \sim B$.

● PROBLEM 6-10

Show that the largest cardinal number does not exist.

SOLUTION:

Georg Cantor proved the following theorem:

CANTOR'S THEOREM

For any set X,

$$\text{card } X < \text{card } P(X). \tag{1}$$

Remember that the power set $P(X)$ of X is the set of all subsets of X.

Let X represent a set and $P(X)$ its power set. Then, by applying Cantor's theorem

$$\text{card } X < \text{card } P(X). \tag{2}$$

Since $P(X)$ is again the set, we have

$$\text{card } P(X) < \text{card } P(P(X)). \tag{3}$$

By applying Cantor's theorem, we obtain

$$\text{card } X < \text{card } P(X) < \text{card } P(P(X)) < \ldots \tag{4}$$

In such a way, we can obtain a sequence of cardinal numbers.

By setting $X = R$, we obtain a sequence of transfinite cardinal numbers

$$\text{card } R < \text{card } P(R) < \text{card } P(P(R)) < \ldots \tag{5}$$

Thus, there is no largest cardinal number.

● PROBLEM 6-11

Let A and B represent sets. Prove that if $A \sim B$, then

$$\text{card } P(A) = \text{card } P(B). \tag{1}$$

SOLUTION:

Since $A \sim B$, a bijection exists

$$f: A \to B. \tag{2}$$

We shall show that a bijection exists

$$F: P(A) \to P(B). \tag{3}$$

Let us define

$$F(X) = f(X) \text{ for all } X \in P(A). \tag{4}$$

Since f is a bijective function, F is also. Hence,

$$P(A) \sim P(B). \tag{5}$$

● PROBLEM 6-12

Let X represent a denumerable set. Prove that the power set $P(X)$ of X is nondenumerable.

SOLUTION:

We shall apply Cantor's theorem. Suppose, on the contrary, that a denumerable set X exists, such that its power set $P(X)$ is also denumerable. Then

$$\text{card } X = \text{card } P(X) \tag{1}$$

but this contradicts Cantor's theorem.

That completes the proof.

● PROBLEM 6-13

Prove the following:

CANTOR'S THEOREM

If A is a set, then

$$\text{card } A < \text{card } P(A). \tag{1}$$

SOLUTION:

If $A = \phi$, then card $\phi = 0$ and card $P(\phi) = 1$.

Suppose $A \neq \phi$. Let us define a function

$$f : A \to P(A) \tag{2}$$

such that

$$f(x) = \{x\} \in P(A) \tag{3}$$

for all $x \in A$.

Function f is one-to-one. Therefore, the sets A and $\{\{x\} : x \in A\}$ are equipotent. But

$$\{\{x\} : x \in A\} \subset P(A). \tag{4}$$

Therefore,

$$\text{card } A \leq \text{card } P(A). \tag{5}$$

Now we must show that the sets A and $P(A)$ are not equipotent, i.e. that card $A \neq$ card $P(A)$. Assume, on the contrary, that the sets A and $P(A)$ are equipotent, hence a bijection exists

$$g : A \to P(A). \tag{6}$$

Let us define the set

$$T = \{x \in A : x \notin g(x)\}. \tag{7}$$

Since $T \in P(A)$ and $g : A \to P(A)$, an element $a \in A$ exists, such that

$$g(a) = T. \tag{8}$$

The existence of such an element leads to a contradiction. Because if $a \in T$, then $a \notin g(a)$, according to the definition of T; on the other hand, $g(a) = T$ and $a \in T$.

If $a \notin T$, then since $g(a) = T$, we have $a \notin g(a)$ and $a \in T$ and consequently, $a \in g(a)$.

That completes the proof of Cantor's theorem.

● PROBLEM 6-14

Let α and β denote two cardinal numbers. Define their sum and show that the sum is uniquely defined, i.e. is independent of the choice of sets.

SOLUTION:

The sum $\alpha + \beta$ of two cardinal numbers α and β is defined to be the power of the union of two disjoint sets, which have the powers α and β, respectively.

We have

$$\text{card}\, A + \text{card}\, B = \text{card}\, (A \cup B), \text{ if } A \cap B = \phi. \tag{1}$$

The cardinal number assigned to set X we denote by card X.

Sometimes the other notations are used, like $\overline{\overline{X}}$, $\#X$, etc.

In this definition, we employ two disjoint sets. Suppose A_1 and B_1 have powers, card A_1 and card B_1, respectively. Then, the sets A and B exist, which are disjoint and such that

$$\text{card}\, A_1 = \text{card}\, A \text{ and } \text{card}\, B_1 = \text{card}\, B. \tag{2}$$

Indeed, suppose $a \neq b$ are two distinct elements, then

$$A = \{a\} \times A_1 \quad B = \{b\} \times B_1$$

$$\text{and} \quad A \cap B = \phi. \tag{3}$$

Since

$$\left.\begin{array}{c} A_1 \sim B_1 \\ A_2 \sim B_2 \\ A_1 \cap A_2 = B_1 \cap B_2 = \phi \end{array}\right) \Rightarrow (A_1 \cup A_2 \sim B_1 \cup B_2),$$

we conclude that for every two cardinal numbers, their sum is uniquely defined, that is, independent of the choice of sets A and B.

Remember that $A \sim B$ indicates that sets A and B have the same power (are equipotent).

● PROBLEM 6-15

Find the sum of cardinal numbers

$$\text{card}\, N + \text{card}\, N. \tag{1}$$

SOLUTION:

Let N_O and N_E denote the set of odd natural numbers and the set of even natural numbers. We have

$$N_O \cap N_E = \phi \tag{2}$$

$$N_O \subset N, \quad N_E \subset N \tag{3}$$

$$N_O \cup N_E = N. \tag{4}$$

The sets N_O and N_E are denumerable. Hence,

$$\text{card } N + \text{card } N = \text{card } N_O + \text{card } N_E =$$

$$= \text{card } (N_O \cup N_E) = \text{card } N. \tag{5}$$

Similarly, card R + card R = card R. It is easy to show that card N + card R = card R. Indeed

$$(0, 1) \sim R \quad \text{and} \quad \text{card } (0,1) = \text{card } R. \tag{6}$$

Let $P = (0, 1) \cup N$. Since N and (0,1) are disjoint sets,

$$\text{card } P = \text{card } (0,1) + \text{card } N = \text{card } N + \text{card } R\,. \tag{7}$$

On the other hand,

$$R \sim (0,1) \subset P \tag{8}$$

and

$$P \sim P \subset R. \tag{9}$$

According to the Schröder-Bernstein theorem,

$$P \sim R \quad \text{and} \quad \text{card } P = \text{card } N + \text{card } R = \text{card } R. \tag{10}$$

The power of the set of natural numbers will be denoted throughout this book by card N or by $\aleph_0$ (aleph-null).

● PROBLEM 6-16

Define the product of two cardinal numbers and show that the product is uniquely defined.

SOLUTION:

Let α and β denote two cardinal numbers and let A and B represent two sets, such that

$$\text{card } A = \alpha \quad \text{and} \quad \text{card } B = \beta. \tag{1}$$

Then, we define the product $\alpha \cdot \beta$ of α and β to be the power of the Cartesian product of sets A and B:

$$\text{card } A \cdot \text{card } B = \text{card } (A \times B). \tag{2}$$

This definition is unique, in the sense that it does not depend on the choice of sets A and B. Suppose A_1 and B_1 are two sets, such that

$$A \sim A_1 \quad \text{and} \quad B \sim B_1 \tag{3}$$

Since the following holds

$$\begin{pmatrix} A \sim A_1 \\ B \sim B_1 \end{pmatrix} \Rightarrow (A \times B \sim A_1 \times B_1) \tag{4}$$

we conclude that the definition of the product of two cardinal numbers is unique.

● PROBLEM 6-17

Show that

$$\text{card } N + \text{card } N = \text{card } N \tag{1}$$

$$\text{card } N \cdot \text{card } N = \text{card } N \tag{2}$$

$$\text{card } N + n = \text{card } N \tag{3}$$

$$\text{card } N \cdot n = \text{card } N \tag{4}$$

where n is a natural number and card N is the power of the set of natural numbers.

SOLUTION:

In Chapter 5, we proved the following useful theorem:

THEOREM

The union $A \cup B$ of two countable sets A and B is countable.

Therefore, since the set of natural numbers is countable, we conclude that

$$\text{card } N + \text{card } N = \text{card } N \tag{5}$$

From this theorem, we conclude as well that

$$\text{card } N + n = \text{card } N \tag{6}$$

To prove (2) we shall apply:

THEOREM

The Cartesian product of two countable sets is a countable set.

Therefore,

$$\text{card } N \cdot \text{card } N = \text{card } N \tag{7}$$

and similarly,

$$\text{card } N \cdot n = \text{card } N. \tag{8}$$

● PROBLEM 6-18

Show that addition and multiplication satisfy the associative, commutative and distributive laws.

SOLUTION:

Since

$$A \cup (B \cup C) = (A \cup B) \cup C \quad \text{and} \quad A \cup B = B \cup A$$

addition is associative and commutative.

$$\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$$

$$\alpha + \beta = \beta + \alpha \tag{1}$$

Similarly, since

$$A \times (B \times C) = (A \times B) \times C \tag{2}$$

multiplication is associative.

By applying the following formula

$$A \times B \sim B \times A \tag{3}$$

we conclude that multiplication is commutative

$$\alpha \cdot \beta = \beta \cdot \alpha. \tag{4}$$

To prove the distributive law

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma \tag{5}$$

let us choose three sets A, B and C, such that

$$\alpha = \text{card } A, \beta = \text{card } B, \gamma = \text{card } C \tag{6}$$

where $B \cap C = \phi$. Then

$$A \times (B \cup C) = A \times B \cup A \times C \tag{7}$$

$$(A \times B) \cap (A \times C) = A \times (B \cap C) = \phi \tag{8}$$

and

$$\text{card}\,[A \times (B \cup C)] = \text{card}\,A \times B + \text{card}\,A \times C \tag{9}$$

which proves (5).

● PROBLEM 6-19

Let x denote an arbitrary cardinal number. Compute the following cardinal numbers:

1. $0x$

2. $1x$.

SOLUTION:

1. Let X represent a set, such that

$$\text{card}\,X = x. \tag{1}$$

Then

$$X \times \phi = \phi. \tag{2}$$

Hence

$$0x = \text{card}\,\phi \cdot \text{card}\,X =$$

$$= \text{card}\,(\phi \times X) = \text{card}\,\phi = 0. \tag{3}$$

2. Let $\{a\}$ represent any set consisting of one element. Then the sets X and $X \times \{a\}$ are equipotent

$$X \sim X \times \{a\}. \tag{4}$$

We obtain

$$1x = \text{card}\,\{a\} \cdot \text{card}\,X =$$

$$= \text{card}\,\{a\} \times X = \text{card}\,X = x. \tag{5}$$

● PROBLEM 6-20

Which of the following theorems is true?

1. If x, y, z are cardinal numbers, such that $x \leq y$, then $xz \leq yz$.

2. If x, y and z are cardinal numbers, such that $x < y$, $z \neq 0$, then $xz < yz$.

SOLUTION:

1. Let A, B, and C represent the sets, such that

$$\text{card } A = x, \text{ card } B = y, \text{ card } C = z. \tag{1}$$

Since

$$\text{card } A \leq \text{card } B \tag{2}$$

a one-to-one function exists, such that

$$f : A \to B. \tag{3}$$

Let $g : C \to C$, $g(c) = c$ for every $c \in C$. Then function

$$F : A \times C \to B \times C \tag{4}$$

defined by

$$F(a, c) = (f(a), g(c)) \tag{5}$$

is also one-to-one. Therefore,

$$\text{card } (A \times C) \leq \text{card } (B \times C) \tag{6}$$

or

$$\text{card } A \cdot \text{card } C = xz \leq \text{card } B \cdot \text{card } C = yz. \tag{7}$$

2. This theorem is not true. The following example proves why:
Let

$$x = 1 \quad y = z = \text{card } N \tag{8}$$

then $x < y$ but

$$xz = 1 \cdot \text{card } N = \text{card } N \quad \text{and} \quad yz = \text{card } N \text{ card } N = \text{card } N.$$

Hence $xz = yz$.

● PROBLEM 6-21

Prove that

$$\text{card } R \cdot \text{card } R = \text{card } R. \tag{1}$$

SOLUTION:

In Chapter 5 we proved that the set of real numbers R and the open interval (0, 1) are equipotent

$$R \sim (0, 1). \tag{2}$$

To evaluate card $R \cdot$ card R, let us note that

$$\text{card } R = \text{card } (0, 1) \tag{3}$$

$$\text{card } R \cdot \text{card } R =$$

$$= \text{card } (0,1) \cdot \text{card } (0,1) = \text{card } (0,1) \times (0,1). \tag{4}$$

Hence, we have to compute card (0,1) × (0,1). To do it, a function f will be defined, such that

$$f: (0,1) \times (0,1) \rightarrow (0,1) \tag{5}$$

and f is bijective (i.e. one-to-one and onto).

Each $x \in (0,1)$ can be expressed by its infinite decimal expansion. For example

$$^2/_7 = 0.28571428\ldots$$

$$^1/_2 = 0.4999999\ldots \tag{6}$$

We define f in the following manner:

$$x, y \in (0,1)$$

$$f(x, y) = f(.x_1x_2x_3\ldots, .y_1y_2y_3\ldots) =$$

$$= .x_1y_1x_2y_2x_3y_3\ldots \tag{7}$$

It is easy to verify that such a function is one-to-one and onto. Thus

$$(0,1) \times (0,1) \sim (0,1) \tag{8}$$

and

$$\text{card } (0,1) \times (0,1) = \text{card } (0,1). \tag{9}$$

From (4) and (9), we obtain

$$\text{card } R \cdot \text{card } R = \text{card } (0,1) = \text{card } R. \tag{10}$$

● PROBLEM 6-22

Let x and y denote cardinal numbers. Prove that

1. If xy = 0 then $x = 0$ or $y = 0$.

2. If $xy = 1$ then $x = 1$ or $y = 1$.

SOLUTION:

1. Let A and B represent two sets, such that

$$\text{card}\, A = x$$

$$\text{card}\, B = y \tag{1}$$

then

$$xy = \text{card}\,(A \times B) = 0. \tag{2}$$

Hence $A \times B$ is an empty set

$$A \times B = \phi \tag{3}$$

From (3), we conclude that

$$A = \phi \quad \text{and} \quad B = \phi. \tag{4}$$

Thus

$$\text{card}\, A = x = 0$$

$$\text{card}\, B = y = 0. \tag{5}$$

2. $$xy = \text{card}\, A \cdot \text{card}\, B =$$

$$= \text{card}\,(A \times B) = 1. \tag{6}$$

Set $A \times B$ consists of one element. Thus, each of the sets A and B consists of one element. We have

$$\text{card}\, A = x = 1$$

$$\text{card}\, B = y = 1. \tag{7}$$

● PROBLEM 6-23

Define b^a (a^{th} power of b), where a and b are cardinal numbers, and justify this definition using an analogy with natural numbers.

SOLUTION:

Let us start with a definition:

DEFINITION

Let $a \neq 0$ and b denote cardinal numbers and A and B represent two sets such that

$$\text{card}\, A = a, \quad \text{card}\, B = b \tag{1}$$

Then, we define b^a by

$$b^a = card^{B^A} \tag{2}$$

where B^A denotes the set of all functions from A to B.

Consider the finite case of two natural finite numbers m and n. Then

$$n^m = n \cdot n \cdot n \cdot \ldots \cdot n, \; m \text{ times.} \tag{3}$$

Let A represent a set with m elements and B a set with n elements.

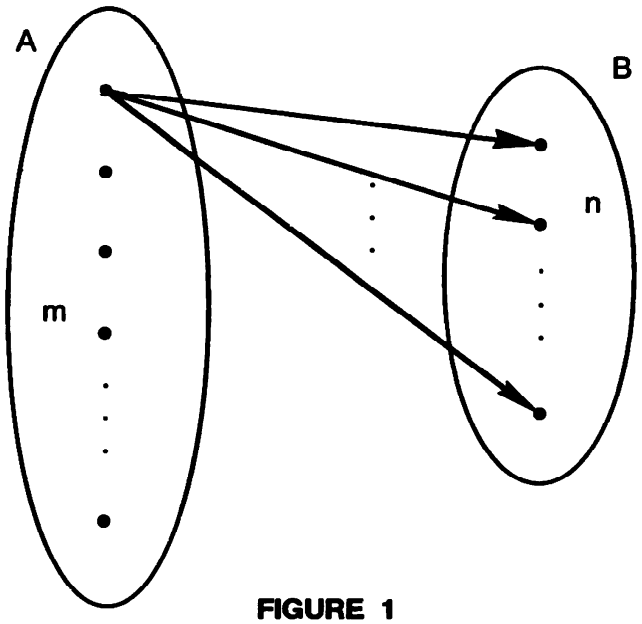


FIGURE 1

Each of the m elements of A has n choices for its image. The choices for each element of A are independently made and, since A consists of m elements, there are

$$n \cdot n \cdot n \cdot \ldots \cdot n = n^m$$

functions from A to B.

The definition of b^a is a generalization of this concept.

● PROBLEM 6-24

Show that the definition of b^a (see Problem 6-23) is independent of the choice of representatives, A and B.

SOLUTION:

We shall prove the following:

THEOREM

Let A, B, A', B' represent sets, such that $A \sim A'$ and $B \sim B'$. Then

$$B^A \sim B'^{A'} \tag{1}$$

Proof:

Since $A \sim A'$ and $B \sim B'$ are bijective functions, g and h exist

$$g : A \to A', \quad h : B \to B'. \tag{2}$$

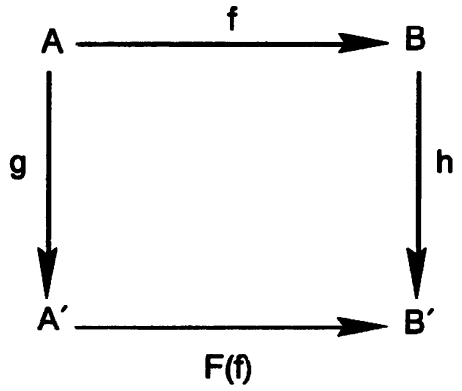


FIGURE 1

Suppose $f : A \to B$ is a function, $f \in B^A$. Let us define

$$F : B^A \to B'^{A'} \tag{3}$$

by $F(f) : A' \to B'$

$$F(f)(x) = h(f(g^{-1}(x))) = h \circ f \circ g^{-1}(x). \tag{4}$$

Functions g and h are bijective, hence $F(f)$ is bijective. Therefore,

$$F : B^A \to B'^{A'} \tag{5}$$

is bijective and

$$B^A \sim B'^{A'} \tag{6}$$

● PROBLEM 6–25

Show that for any set A

$$\text{card } P(A) = 2^{\text{card } A}. \tag{1}$$

SOLUTION:

Let $B = \{0, 1\}$. To each subset X of A, $X \subset A$, we assign the characteristic function

$$X_x : A \to B \tag{2}$$

as follows:

$$X_x(x) = \begin{cases} 1 & \text{if } x \in X \\ 0 & \text{if } x \in A - X \end{cases} \tag{3}$$

and a one-to-one and onto mapping exists between the sets $P(A)$ and B^A.

Remember that $P(A)$ is the set of all subsets of A, and B^A is the set of all functions from A to $\{0, 1\}$. To each subset $X \subset A$, $X \in P(A)$, we assign its characteristic function X_x, $X_x \in B^A$. Two different subsets cannot have the same characteristic function, hence mapping is one-to-one. Any characteristic function uniquely defines a subset, hence mapping is onto.

Bijection exists between $P(A)$ and B^A. Thus

$$\text{card } P(A) = \text{card } B^A \tag{4}$$

Since $B = \{0, 1\}$, card $B = 2$ and

$$\text{card } P(A) = 2^{\text{card } A}. \tag{5}$$

● PROBLEM 6-26

Let a, x, and y denote cardinal numbers. Show that

$$a^x a^y = a^{x+y}. \tag{1}$$

SOLUTION:

First of all, let us choose the sets A, X, Y, such that

$$\text{card } A = a$$

$$\text{card } X = x$$

$$\text{card } Y = y$$

$$X \cap Y = \phi. \tag{2}$$

From the definition of the sum of cardinal numbers, we have

$$\text{card } (X \cup Y) = x + y. \tag{3}$$

Now we shall show that the sets

$$A^X \times A^Y \text{ and } A^{X \cup Y}$$

are equipotent.

Suppose $f \in A^X$ and $g \in A^Y$, then $(f, g) \in A^X \times A^Y$. Then the function $f \cup g$ belongs to $A^{X \cup Y}$.

The union $f \cup g$ is the function

$$f \cup g : X \cup Y \to A \tag{4}$$

defined by

$$f \cup g(x) = \begin{cases} f(x) & \text{if } x \in X \\ g(x) & \text{if } x \in Y \end{cases} \tag{5}$$

Thus, we have established a one-to-one and onto relationship between the sets $A^X \times A^Y$ and $A^{X \cup Y}$.

Hence,

$$\text{card } A^X \times A^Y = a^x a^y =$$

$$= \text{card } A^{X \cup Y} = a^{x+y}. \tag{6}$$

● PROBLEM 6-27

Let a, x, and y denote cardinal numbers. Show that

$$(a^x)^y = a^{xy}. \tag{1}$$

SOLUTION:

Let A, X, and Y represent sets, such that

$$\text{card } A = a, \text{ card } X = x, \text{ card } Y = y. \tag{2}$$

We shall prove that the sets $A^{X \times Y}$ and $(A^X)^Y$ are equipotent. Let $f : X \times Y \to A$ represent a given function and b a given element of Y, $b \in Y$.

Then a function exists

$$f^b : X \to A \tag{3}$$

defined by

$$f^b(a) = f(a, b) \tag{4}$$

for all $a \in X$.

Now, let us define the function

$$F : A^{X \times Y} \rightarrow (A^X)^Y \tag{5}$$

such that to each $f \in A^{X \times Y}$, we assign the function

$$p_f \in (A^X)^Y \tag{6}$$

defined by

$$p_f(b) = f^b \tag{7}$$

for all $b \in Y$.

Function F is one-to-one and onto, hence the sets $A^{X \times Y}$ and $(A^X)^Y$ are equivalent. Thus

$$(a^x)^y = a^{xy}. \tag{8}$$

● PROBLEM 6-28

Prove the following:

THEOREM

If a, b, and x are cardinal numbers, then

$$(ab)^x = a^x b^x. \tag{1}$$

SOLUTION:

Let A, B, and X represent sets, such that

$$\text{card } A = a, \text{ card } B = b, \text{ card } X = x. \tag{2}$$

We must prove that the sets $(A \times B)^X$ and $A^X \times B^X$ are equipotent.

Recall that the A-projection

$$p_A : A \times B \rightarrow A \tag{3}$$

is a function which assigns element $a \in A$ to each ordered pair $(a, b) \in A \times B$. Let us define a bijection

$$F : (A \times B)^X \rightarrow A^X \times B^X. \tag{4}$$

Function F assigns to each

$$f \in (A \times B)^X, f : X \rightarrow A \times B \tag{5}$$

a function defined by

$$(p_A \circ f, p_B \circ f) \in A^X \times B^X. \tag{6}$$

This assignment is one-to-one and onto (it can be easily verified), hence

$$(A \times B)^X \sim A^X \times B^X \tag{7}$$

and

$$(ab)^x = a^x b^x. \tag{8}$$

● PROBLEM 6-29

Prove

$$2^{\text{card } N} = \text{card } R. \tag{1}$$

SOLUTION:

Recall that the symbols, card N and card R, denote the cardinal numbers of the sets N and R, respectively. Equation (1) will be proven in two steps: first we shall show that

$$2^{\text{card } N} \geq \text{card } R \tag{2}$$

and then we shall prove

$$\text{card } R \geq 2^{\text{card } N}. \tag{3}$$

Let Q represent the set of rational numbers; then

$$N \sim Q. \tag{4}$$

The function

$$f: R \rightarrow P(Q) \tag{5}$$

defined by

$$f(x) = \{y \in Q : y < x\} \tag{6}$$

is one-to-one. Indeed, if $x, x' \in R$ and $x < x'$, then a rational number $y \in Q$ exists, such that

$$x < y < x'. \tag{7}$$

The rational numbers are a dense subset of the set of real numbers.

Then $y \in f(x')$, but $y \notin f(x)$, thus, $f(x) \neq f(x')$ and f is one-to-one.

For any two sets, A and B, card $A \leq$ card B, if and only if an injection $f : A \rightarrow B$ exists. Hence,

$$\text{card } R \leq \text{card } P(Q) = 2^{\text{card } Q} = 2^{\text{card } N}. \tag{8}$$

To prove card $R \geq 2^{\text{card } N}$, define

$$F : \{0, 1\}^N \rightarrow R \tag{9}$$

such that

$$F(f) = 0.f(1)\,f(2)\,f(3)\,\ldots \tag{10}$$

where $f : N \to \{0, 1\}$. $F(f)$ is a decimal number consisting of 0's and 1's. Function F is one-to-one. Suppose $f, g \in \{0, 1\}^N$ and $f \neq g$, then $F(f) \neq F(g)$ because decimals defining $F(f)$ and $F(g)$ are different. Therefore,

$$\text{card } \{0, 1\}^N \leq \text{card } R \tag{11}$$

or

$$2^{\text{card } N} \leq \text{card } R \tag{12}$$

Q.E.D.

● PROBLEM 6-30

Prove

$$\text{card } N < \text{card } R. \tag{1}$$

SOLUTION:

According to Cantor's theorem (If X is a set, then card $X <$ card $P(X)$) we have

$$\text{card } N < \text{card } P(N). \tag{2}$$

For any set A, the following equality holds:

$$\text{card } P(A) = 2^{\text{card } A} \tag{3}$$

(see Problem 6-25). Thus

$$\text{card } N < \text{card } P(N) = 2^{\text{card } N} =$$

$$= \text{card } R. \tag{4}$$

● PROBLEM 6-31

Show that for any finite $n \geq 2$

$$n^{\text{card } N} = (\text{card } N)^{\text{card } N} = \text{card } R. \tag{1}$$

SOLUTION:

We proved (see Problem 6-29)

$$\text{card } R = 2^{\text{card } N}. \tag{2}$$

Since $n \geq 2$,

$$2^{\text{card } N} \leq n^{\text{card } N} \tag{3}$$

and

$$n^{\text{card } N} \leq (\text{card } N)^{\text{ card } N}. \tag{4}$$

In Problem 6-30, we proved

$$\text{card } N < 2^{\text{card } N}. \tag{5}$$

Combining (2), (3), (4), and (5)

$$\text{card } R = 2^{\text{card } N} \leq n^{\text{card } N} \leq (\text{card } N)^{\text{card } N} \leq (2^{\text{card } N})^{\text{card } N} =$$

$$= 2^{\text{card } N \cdot \text{ card } N} = \tag{6}$$

but card $N \cdot$ card N = card N (see Problem 6-19). Hence

$$2^{\text{card } N \cdot \text{ card } N} = 2^{\text{card } N} = \text{card } R. \tag{7}$$

From (7), we conclude that

$$\text{card } R = n^{\text{card } N} = (\text{card } N)^{\text{card } N}. \tag{8}$$

● PROBLEM 6-32

Prove that:

1. card C = card R (1)
   where C is the set of complex numbers.

2. card $R \cdot$ card N = card R. (2)

SOLUTION:

1. A complex number

$$z = x + iy \tag{3}$$

can be represented by an ordered pair

$$(x, y)$$

where x and y are real numbers. Thus,

$$C \sim R \times R \tag{4}$$

and

$$\text{card } C = \text{card } R \cdot \text{card R}. \tag{5}$$

From Problem 6-21

$$\text{card}\, R \cdot \text{card}\, R = \text{card}\, R. \tag{6}$$

Hence

$$\text{card}\, C = \text{card}\, R. \tag{7}$$

2. It is easy to verify that

$$\text{card}\, R \leq \text{card}\, R \cdot \text{card}\, N \leq \text{card}\, R \cdot \text{card}\, R = \text{card}\, R. \tag{8}$$

Therefore

$$\text{card}\, R \cdot \text{card}\, N = \text{card}\, R. \tag{9}$$

● PROBLEM 6-33

In Problem 6-21, we proved

$$\text{card}\, R \cdot \text{card}\, R = \text{card}\, R. \tag{1}$$

Using

$$2^{\text{card}\, N} = \text{card}\, R \tag{2}$$

prove (1).

SOLUTION:

For three cardinal numbers a, b, and c

$$a^b a^c = a^{b+c} \tag{3}$$

(see Problem 6-26).

Thus, from (2) and (3), we obtain

$$\text{card}\, R \cdot \text{card}\, R = 2^{\text{card}\, N} 2^{\text{card}\, N} = 2^{\text{card}\, N + \text{card}\, N} =$$

$$= 2^{\text{card}\, N} = \text{card}\, R. \tag{4}$$

Note that in Problem 6-15 we proved

$$\text{card}\, N + \text{card}\, N = \text{card}\, N. \tag{5}$$

● PROBLEM 6-34

Let

$$F = \{f : f : R \to R\} \tag{1}$$

represent a set of all functions from R to R. Compare the cardinal numbers, card F and card R.

SOLUTION:

According to the definition of exponentiation of cardinal numbers, we have

$$\text{card } F = \text{card } R^R = (\text{card } R)^{\text{card } R} \tag{2}$$

(see Problem 6-23).

Since

$$2^{\text{card } N} = \text{card } R \tag{3}$$

(2) leads to

$$\text{card } F = (\text{card } R)^{\text{card } R} = (2^{\text{card } N})^{\text{card } R} =$$

$$= 2^{\text{card } N \cdot \text{card } R}. \tag{4}$$

From Problem 6-32, we obtain

$$\text{card } N \cdot \text{card } R = \text{card } R \tag{5}$$

therefore

$$\text{card } F = 2^{\text{card } R}. \tag{6}$$

According to Cantor's theorem (for any set X, card $X <$ card $P(X)$) and

$$\text{card } P(X) = 2^{\text{card } X} \tag{7}$$

we find setting $X = R$

$$2^{\text{card } R} = \text{card } P(R) > \text{card } R. \tag{8}$$

From (6) and (8), we obtain

$$\text{card } F = 2^{\text{card } R} > \text{card } R. \tag{9}$$

● PROBLEM 6-35

Prove that

$$\text{card } C(R, R) = \text{card } C(Q, R) =$$

$$= \text{card } K(R, R) = \text{card } R \tag{1}$$

where $C(R, R)$ and $C(Q, R)$ are the sets of continuous real-valued functions with domains R and Q, respectively. The set of all real-valued constant functions with the domain R is denoted by $K(R, R)$.

SOLUTION:

Note that the each function $f: R \to R$ a function corresponds $f \mid Q : Q \to R$, such that for all $x \in Q$, $f(x) = f \mid Q(x)$. Function $f \mid Q$ is the restriction of f to Q. We denote this correspondence by

$$F : C(R, R) \to C(Q, R). \tag{2}$$

The restriction of a continuous function is a continuous function. We show that F is one-to-one. Indeed, let $f, g \in C(R, R)$ denote such that for all $x \in Q$

$$f(x) = g(x). \tag{3}$$

Let x' be any real number, $x' \in R$. Then a sequence $(x_n) \in Q$ of rational numbers exists, such that

$$\lim_{n \to \infty} x_n = x' \tag{4}$$

Since both f and g are continuous

$$f(x') = g(x'). \tag{5}$$

Therefore, $f = g$. Function F is injective, thus

$$\text{card } C(R, R) \leq \text{card } C(Q, R). \tag{6}$$

On the other hand,

$$\text{card } C(Q, R) \leq \text{card } R^Q = (\text{card } R)^{\text{card } N} =$$

$$= (2^{\text{card } N})^{\text{card } N} = 2^{\text{card } N} = \text{card } R. \tag{7}$$

From (6) and (7),

$$\text{card } C(R, R) \leq \text{card } C(Q, R) \leq \text{card } R. \tag{8}$$

Consider the set $K(R, R)$. To each real number $p \in R$, a constant function corresponds

$$f_p : R \to R \tag{9}$$

such that for all $x \in R$, $f_p(x) = p$. Thus

$$\text{card } K(R, R) = \text{card } R. \tag{10}$$

Each constant function is continuous

$$K(R, R) \subset C(R, R). \tag{11}$$

Hence

$$\text{card } R = \text{card } K(R, R) \leq \text{card } C(R, R). \tag{12}$$

We conclude from (8) and (12) that

$$\text{card } R = \text{card } C(R, R) = \text{card } C(Q, R) = \text{card } K(R, R). \tag{13}$$

Surprisingly, there are as many constant functions as continuous functions.

● PROBLEM 6-36

Let $D(R, R)$ represent the set of all differentiable real-valued functions. Show that

$$\text{card } D(R, R) = \text{card } R. \tag{1}$$

SOLUTION:

Each differentiable function is continuous

$$D(R, R) \subset C(R, R). \tag{2}$$

Hence

$$\text{card } D(R, R) \leq \text{card } C(R, R) = \text{card } R \tag{3}$$

(see Problem 6-35).

Each constant function is differentiable

$$K(R, R) \subset D(R, R) \tag{4}$$

hence

$$\text{card } R = \text{card } K(R, R) \leq \text{card } D(R, R) \tag{5}$$

(see Problem 6-35). Thus

$$\text{card } D(R, R) = \text{card } R. \tag{6}$$

● PROBLEM 6-37

The so-called classic Hilbert space, H, consists of all infinite sequences

$$(x_1, x_2, x_3, \ldots) ; x_k \in R ; k \in N \tag{1}$$

such that the series

$$x_1^2 + x_2^2 + x_3^2 + \ldots \tag{2}$$

converges. Find card H.

SOLUTION:

Define function

$$f: R \to H \tag{3}$$

such that for every $x \in R$

$$f(x) = (x, 0, 0, 0, \ldots). \tag{4}$$

The infinite sequence belongs to the Hilbert space, H,

$$(x, 0, 0, \ldots) \in H. \tag{5}$$

Function f is an injection, hence

$$\text{card } R \leq \text{card } H. \tag{6}$$

On the other hand,

$$\text{card } H \leq (\text{card } R)^{\text{card } N} = (2^{\text{card } N})^{\text{card } N} =$$

$$= 2^{\text{card } N \cdot \text{ card } N} = 2^{\text{card } N} = \text{card } R. \tag{7}$$

From (6) and (7), we conclude that

$$\text{card } H = \text{card } R. \tag{8}$$

● PROBLEM 6-38

Compare the cardinal numbers of these two sets:

1. The set of all infinite sequences

$$(x_1, x_2, x_3, \ldots) \tag{1}$$

of real numbers, denoted by $R^{\text{card } N}$.

2. The set of all infinite sequences of integers.

SOLUTION:

The cardinal number of the set of all infinite sequences of integers is

$$(\text{card } N)^{\text{card } N}. \tag{2}$$

From Problem 6-31, we obtain

$$(\text{card } N)^{\text{card } N} = \text{card } R. \tag{3}$$

The cardinal number of the set $R^{\text{card } N}$ is

$$\text{card } R^{\text{card } N} = (\text{card } R)^{\text{card } N} \tag{4}$$

Equation (4) leads to

$$(\text{card } R)^{\text{card } N} = (2^{\text{card } N})^{\text{card } N} = 2^{\text{card } N \cdot \text{card } N} =$$

$$= 2^{\text{card } N} = \text{card } R. \tag{5}$$

Therefore, both sets have the same cardinal numbers. Sometimes, points

$$(x_1, x_2, x_3, \ldots)$$

where all x_k's are integers are called lattice points.

● PROBLEM 6-39

Compare the cardinal numbers of the following sets:

1. The set of all functions of one real variable, which assume values of 0 or 1.

2. The set of all real-valued functions of n real variables.

SOLUTION:

Consider the set of all functions

$$f(x) = \begin{cases} 0 \\ 1 \end{cases} \qquad f : R \to \{0,1\}.$$

The cardinal number of this set is

$$\text{card } 2^R = 2^{\text{card } R}. \tag{1}$$

The cardinal number of the set of real-valued functions of n real variables

$$f : R^n \to R \tag{2}$$

is

$$\text{card } R^{R^n} = \text{card } R^{\text{card } R^n} \tag{3}$$

Since

$$(\text{card } R)^n = \text{card } R \tag{4}$$

we have

$$\text{card } R^{\text{card } R^n} = \text{card } R^{\text{card } R} = (2^{\text{card } N})^{\text{card } R} \tag{5}$$

but

$$\text{card } N \text{ card } R = \text{card } R \tag{6}$$

hence

$$\text{card } R^{\text{card } R^n} = 2^{\text{card } R}. \tag{7}$$

Both sets have the same cardinal number.

● PROBLEM 6-40

Explain the continuum hypothesis and the generalized continuum hypothesis.

SOLUTION:

We recall briefly the major facts concerning cardinal numbers. There are finite cardinal numbers (which are non-negative integers) and transfinite cardinal numbers. The smallest transfinite cardinal number is card N (denoted $\aleph_0$, aleph-null). There is no largest cardinal number. Let us denote card $R = c$

$$\underbrace{0 < 1 < 2 < \ldots}_{\text{finite cardinal numbers}} \qquad \underbrace{< \aleph_0 < \ldots < c < \ldots < 2^c < \ldots}_{\text{transfinite cardinal numbers}}$$

We proved that $\aleph_0 < 2^{\aleph_0} = c$.

The following question posed by Cantor is known as the continuum problem: Is there a cardinal number that lies between $\aleph_0$ and c? Many mathematicians have spent many hours trying to solve this problem. Nobody has found a set A, such that card $A = a$ and $\aleph_0 < a < c$. Hence, the hypothesis is that such a set does not exist.

CONTINUUM HYPOTHESIS

There is no cardinal number p, such that

$$\aleph_0 < p < c = 2^{\aleph_0}.$$

A similar question exists for any transfinite cardinal number: Is there a cardinal number that lies between a transfinite cardinal number p and 2^p? This question has not been answered.

For any transfinite cardinal number p, no cardinal number r has been found, such that $p < r < 2^p$. Thus:

GENERALIZED CONTINUUM HYPOTHESIS

For any transfinite cardinal number p, there is no cardinal number r, such that

$$p < r < 2^p.$$

● PROBLEM 6-41

Do we know the solution of the continuum problem?

SOLUTION:

The continuum problem was formulated by Cantor in 1880. It became one of the most famous unresolved mathematical problems. On Hilbert's list of 23 mathematical questions with no answers, it occupied first place. (This list was announced in 1900 at the International Congress of Mathematicians in Paris.)

In 1938, Kurt Gödel proved that the generalized continuum hypothesis is consistent with the axioms of set theory. What this means can be explained as follows: Consider two systems of axioms

A – axioms for set theory, and

B – all axioms of A with the generalized continuum hypothesis added.

Any contradiction that might be implied by system B could be formulated as a contradiction implied by system A.

Later, P.J. Cohen proved that the negation of the continuum hypothesis is also consistent with the axioms for set theory. Thus, both the continuum and generalized continuum hypotheses are independent of the other axioms for set theory. They cannot be proved or disproved on the basis of these axioms.

W. Sierpinski has shown that, if the generalized continuum hypothesis is added as an axiom to the axioms of set theory, then the axiom of choice becomes redundant. It can be derived from the generalized continuum hypothesis and remaining axioms.

The situation is that the generalized continuum hypothesis is not provable on the basis of the axioms of set theory. We may postulate the generalized continuum hypothesis or deny it and, in either case, obtain a consistent mathematical theory.

CHAPTER 7

AXIOM OF CHOICE AND ITS EQUIVALENT FORMS

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● PROBLEM 7-1

Quote the axiom of choice and explain it for a finite family of sets.

SOLUTION:

AXIOM OF CHOICE

Let $\mathcal{F}$ be any non-empty family of non-empty sets. Then a function called a choice function exists

$$f:\mathcal{F} \rightarrow \bigcup_{A \in \mathcal{F}} A \tag{1}$$

such that for all $A \in \mathcal{F}$,

$$f(A) \in A \tag{2}$$

Suppose $\mathcal{F}$ is a set of phone books from different cities. It is possible to choose one and only one name from each book. The axiom of choice is trivial for $\mathcal{F}$ finite.

In this chapter we shall examine the axiom of choice and its three equivalent principles (the Hausdorff maximality principle, Zorn's lemma, and the well-ordering principle of Zermelo).

The axiom of choice is, indeed, an axiom that cannot be proven as a theorem using the classical axioms of mathematics.

● PROBLEM 7-2

Let

$$f: A \rightarrow B \tag{1}$$

denote onto mapping. Apply the axiom of choice to prove that a subset C of A exists, such that C and B are equipotent and hence card $A \geq$ card B = card C.

SOLUTION:

Suppose $B \neq \phi$ (if $B = \phi$, then $C = \phi$). Consider the family of sets

$$\{f^{-1}(y) : y \in B\} \tag{2}$$

which forms a partition of A because sets (2) are disjoint and

$$\bigcup_{y \in B} \{f^{-1}(y) : y \in B\} = A. \tag{3}$$

Note that f is onto.

According to the axiom of choice, the family of sets $\{f^{-1}(y) : y \in B\}$ has a set of representatives C, such that for each $y \in B$

$$\{f^{-1}(y) : y \in B\} \cap C \tag{4}$$

is a single element.

Restriction of f to C

$$f \mid C : C \to B \tag{5}$$

is one-to-one and onto. Therefore, sets C and B are equipotent. Hence

$$\text{card } A \geq \text{card } B = \text{card } C. \tag{6}$$

● PROBLEM 7-3

Let $\{A_i : i \in I\}$ represent a family of non-empty sets. The generalized Cartesian product $\underset{i \in I}{X} A_i$ of the family $\{A_i\}$ is the set of all functions

$$f : I \to \bigcup_{i \in I} A_i \tag{1}$$

such that

$$f(i) \in A_i \text{ for all } i \in I. \tag{2}$$

Prove that, if $I \neq \phi$, then $\underset{i \in I}{X} A_i$ is non-empty.

SOLUTION:

Set $\{A_i : i \in I\}$ is a non-empty set, whose elements are non-empty sets (i.e., for each $i \in I$, $A_i \neq \phi$). According to the axiom of choice, a function g, called a choice function, exists such that

$$g : \{A_i : i \in I\} \to \bigcup_{i \in I} A_i. \tag{3}$$

Then, we can define function f by

$$f(i) = g(A_i) \in A_i \tag{4}$$

for all $i \in I$, where

$$f : I \to \bigcup_{i \in I} A_i \tag{5}$$

Hence, if $I \neq \phi$, then

$$\underset{i \in I}{X} A_i \neq \phi. \tag{6}$$

1. Let A represent any set. Show that the relation $\{(a, b) : a = b\}$ is a preordering.

2. Show that in R, $\leq$ is a preordering and $<$ is not.

3. Is inclusion a preordering?

SOLUTION:

We shall start with

DEFINITION OF PREORDER

A relation ρ in a set A, $\rho \subset A \times A$, is called a preorder if it is reflexive and transitive, that is, if

1. $\forall a \in A : (a, a) \in \rho$

2. $\forall a, b \in A : (a, b) \in \rho \wedge (b, c) \in \rho \Rightarrow (a, c) \in \rho$

A set A with preorder is called a preordered set $(A, <)$. Preorder is usually denoted by $<$ ($a < b$ is read "a precedes b").

1. Relation $=$ is a preorder. It is reflexive because for each $a \in A$, $a = a$. It is transitive:

$$(a = b) \wedge (b = c) \Rightarrow a = c.$$

Hence $(A, =)$ is a preordered set.

2. For each $a \in R$, $a \leq a$. Also

$$(a \leq b) \wedge (b \leq c) \Rightarrow (a \leq c).$$

Therefore, $\leq$ is a preordering and $(R, \leq)$ is a preordered set. Of course, $a < a$ is not true and $<$ is not a preorder.

3. Consider the power set $P(X)$ (i.e., the set of all subsets of X). The relation $\subset$ is a preorder because for each $A, B, C \in P(X)$

$$A \subset A \text{ and } (A \subset B) \wedge (B \subset C) \Rightarrow A \subset C.$$

Preordering by inclusion is defined by

$$A < B \quad \text{iff} \quad A \subset B.$$

A family of sets preordered in this manner is called preordered by inclusion.

● PROBLEM 7-5

Let C represent the set of complex numbers. Show that $|z_1| \leq |z_1|$, $z_1, z_2 \in C$ defines a preordering.

SOLUTION:

Any complex number $z \in C$ can be represented by

$$z = x + iy$$

where x, y are real numbers. Then

$$|z| = \sqrt{x^2 + y^2}.$$

For every $z \in C$ $|z| \leq |z|$ because

$$\sqrt{x^2 + y^2} \leq \sqrt{x^2 + y^2}.$$

Hence relation $|\ldots| \leq |\ldots|$ is reflexive. Let $z_1, z_2, z_3 \in C$ and $|z_1| \leq |z_2|$ and $|z_2| \leq |z_3|$. Then

$$\sqrt{x_1^2 + y_1^2} \leq \sqrt{x_2^2 + y_2^2} \text{ and } \sqrt{x_2^2 + y_2^2} \leq \sqrt{x_3^2 + y_3^2}$$

where $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$, $z_3 = x_3 + iy_3$. Hence

$$\sqrt{x_1^2 + y_1^2} \leq \sqrt{x_3^2 + y_3^2}$$

or

$$|z_1| \leq |z_3|.$$

Relation $|\ldots| \leq |\ldots|$ is also transitive. Thus $|\ldots| \leq |\ldots|$ is a preorder and $(C, |\ldots| \leq |\ldots|)$ is a preordered set.

● PROBLEM 7-6

Let D denote the diagonal, $D \subset A \times A$. Show that $\rho \subset A \times A$ is a preorder, if and only if $D \subset \rho$ and $\rho \circ \rho = \rho$. The diagonal is defined by

$$D = \{(x, x) : x \in A\}.$$

SOLUTION:

We have to prove that

$$(\rho \text{ is a preorder}) \Leftrightarrow \begin{pmatrix} D \subset \rho \\ \rho \circ \rho = \rho \end{pmatrix}.$$

$\Rightarrow$ ρ is a preorder. For each $a \in A$, $(a, a) \in \rho$. Hence $D \subset \rho$. Furthermore

$$(a, b) \in \rho \wedge (b, c) \in \rho \Rightarrow (a, c) \in \rho$$

which is equivalent to $\rho \circ \rho = \rho$.

$\Leftarrow$ Since $D \subset \rho$ we have

$$\forall a \in A : (a, a) \in \rho.$$

The set $\rho \circ \rho$ is

$$\rho \circ \rho = \{(a, c) : a, c \in A, \exists b \in A \text{ such that } (a, b) \in \rho \wedge (b, c) \in \rho\}.$$

By assumption

$$\rho \circ \rho = \rho.$$

Hence

$$(a, b) \in \rho \wedge (b, c) \in \rho \Rightarrow (a, c) \in \rho.$$

● PROBLEM 7-7

Show that the relation $\leq$ defined on R^2 by

$$(a_1, a_2) \leq (b_1, b_2) \tag{1}$$

if and only if

$$a_1 \leq b_1 \text{ and } a_2 \leq b_2 \tag{2}$$

is a partial order relation on R^2.

SOLUTION:

We shall apply the following:

DEFINITION OF PARTIAL ORDER

A relation $\leq$ on a set A is called a partial order, if and only if $\leq$ is reflexive, transitive and antisymmetric on A, that is, if $a \leq b$ and $b \leq a$, then $a = b$. Set A with a partial order relation on A is denoted by $(A, \leq)$ and is called a partially ordered set.

Relation $\leq$ defined in (1) is reflexive,

$$(a, b) \leq (a, b) \text{ because } a \leq a \text{ and } b \leq b. \tag{3}$$

It is also transitive.

Suppose

$$(a_1, b_1) \leq (a_2, b_2) \quad \text{and} \quad (a_2, b_2) \leq (a_3, b_3) \tag{4}$$

then

$$a_1 \leq a_2 \quad \text{and} \quad a_2 \leq a_3$$

$$b_1 \leq b_2 \quad \text{and} \quad b_2 \leq b_3$$

Hence

$$a_1 \leq a_3 \quad \text{and} \quad b_1 \leq b_3$$

or

$$(a_1, b_1) \leq (a_3, b_3). \tag{5}$$

Finally we show that $\leq$ is antisymmetric. Indeed, if

$$(a_1, b_1) \leq (a_2, b_2) \quad \text{and} \quad (a_2, b_2) \leq (a_1, b_1) \tag{6}$$

then

$$a_1 \leq a_2 \quad \text{and} \quad a_2 \leq a_1$$

$$b_1 \leq b_2 \quad \text{and} \quad b_2 \leq b_1.$$

Therefore,

$$a_1 = a_2 \quad \text{and} \quad b_1 = b_2 \quad \text{or} \quad (a_1, b_1) = (a_2, b_2). \tag{7}$$

● PROBLEM 7-8

Show that the relation defined in Problem 7-7 is not a total order.

SOLUTION:

We shall use

DEFINITION OF TOTAL ORDER

A total order relation $\leq$ on a set A is a partial order relation, such that for any pair $a, b \in A$, either $a \leq b$ or $b \leq a$. Set A with a total order relation, denoted by $(A, \leq)$, is called a totally ordered set.

Total order relation is sometimes called linear order relation. According

to the above definition, the relation described in Problem 7-7 is not a total order relation. Neither $(0, 1) \leq (1, 0)$, nor $(1, 0) \leq (0, 1)$.

This example shows that a partially ordered set need not be a totally ordered set.

● PROBLEM 7-9

Set $A = \{a, b, c\}$ is ordered as shown in the diagram.

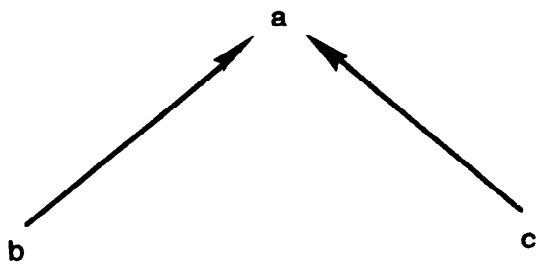


FIGURE 1

The order can be defined as follows: $x \leq y$ iff $x = y$ or one can move from x to y in the direction of arrows. Let B denote the collection of all non-empty totally ordered subsets of A. Draw the diagram of the order of B, when B is partially ordered by set inclusion.

SOLUTION:

The set of all subsets of A consists of

$$\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}.$$

Only the subsets $\{a, b, c\}$ and $\{b, c\}$ are not totally ordered, because neither $b \leq c$ nor $c \leq b$.

Set B consists of

$$\{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\},$$

and is ordered by set inclusion. Hence

$$\{a\} \leq \{a, b\}$$

$$\{b\} \leq \{a, b\}$$

$$\{a\} \leq \{a, c\}$$

$$\{c\} \leq \{a, c\}$$

Set B is partially ordered as shown in the diagram

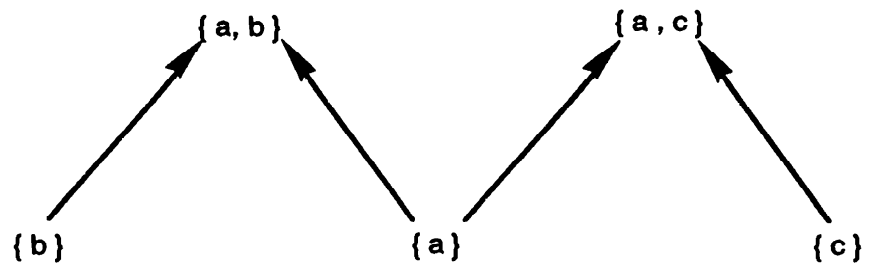


FIGURE 2

● PROBLEM 7-10

Consider the relation defined in Problem 7-7. Show that the diagonal of the plane R^2 is a chain.

SOLUTION:

Let A denote a partially ordered set $(A, \leq)$; then

$$P =: \{(x, y) : x, y \in A, x \leq y\}.$$

The sign =: will be used to denote "equals by definition."

For any subset B of A, $B \subset A$, we define $\leq'$ to be the intersection

$$P \cap (B \times B).$$

The set $(B, \leq')$ is a partially ordered set. It may happen that $(B, \leq')$ is a totally ordered subset of a partially ordered set.

DEFINITION OF A CHAIN

A totally ordered subset of a partially ordered set is called a chain.

In Problem 7-7, we showed that a relation defined by (1) is a partial order. The diagonal

$$D = \{(x, x) : x \in R\}$$

of the plane R^2 is a chain because on D, a relation becomes a total order.

● PROBLEM 7-11

Prove that the relation "m divides n" defined on the set of natural numbers N is a partial order, but not a total order.

SOLUTION:

A relation is reflexive, because for any natural number n, n divides n is true.

This relation is also transitive. Suppose m divides n and n divides k, that is,

$$\frac{n}{m} \in N \, , \, \frac{k}{n} \in N$$

then

$$\frac{n}{m} \cdot \frac{k}{n} = \frac{k}{m} \in N .$$

That is, m divides k.

A relation is antisymmetric because, if m divides n and n divides m, then $m = n$. Hence, the relation is a partial order.

It is not a total order because

3 does not divide 2
and
2 does not divide 3.

● PROBLEM 7-12

Show that

1. The identity relation "=" is a partial order relation on any set.

2. The set $(R, \leq)$ is a totally ordered set, where R is the set of real numbers and "$\leq$" is understood in the ordinary sense.

SOLUTION:

1. Let A represent a set. Relation "=" is reflexive because for all $x \in A$

$$x = x. \tag{1}$$

It is also transitive. If

$$x = y \text{ and } y = z, \text{ then } x = z. \tag{2}$$

Furthermore, relation "=" is antisymmetric. Hence, $(A, =)$ is a partially ordered set.

If set A contains only one element, then $(\{a\}, =)$ is a totally ordered set.

2. Relation "$\leq$", understood in the ordinary sense and defined on the set of real numbers R, is a total order. For each $x \in R$

$$x \leq x.$$

If $x \leq y$ and $y \leq z$, then

$$x \leq z.$$

If $x \leq y$ and $y \leq x$, then

$$x = y.$$

For any pair of real numbers x, y, either $x \leq y$ or $y \leq x$.

● PROBLEM 7-13

Using the notion of temperature, define a total order relation on the set of all physical objects.

SOLUTION:

Temperature can be introduced in the following manner. Let a and b represent two objects. If there is a heat transfer from a to b, we say that the temperature of a is higher than the temperature of b. If there is no heat transfer, we say that a and b are of the same temperature. Here, we neglected all physical details.

Let us define the relationship $a \leq b$ iff the temperature of b is higher than or equal to the temperature of a. This relation is reflexive:

$$a \leq b.$$

It is transitive:

If $a \leq b$ and $b \leq c$, then

$$a \leq c.$$

It is antisymmetric:

If $a \leq b$ and $b \leq a$, then $a = b$ (that is, a and b have the same temperature).

Note that "temperature" defines the total order, since, for any two a and b, we have either $a \leq b$ or $b \leq a$.

● PROBLEM 7-14

Consider the set F of all functions

$$f: R \to R \tag{1}$$

with the relation S defined by

$$S = \{(f, g) \in F \times F : \forall x \in R, f(x) \leq g(x)\}. \tag{2}$$

Prove that (F, S) is a partially ordered set.

SOLUTION:

Relation S is reflexive since

$$\forall f \in F \quad \forall x \in R \quad f(x) \leq g(x). \tag{3}$$

S is transitive,

$$\begin{pmatrix} \forall f,g,h \in F, \forall x \in R \\ f(x) \leq g(x) \\ g(x) \leq h(x) \end{pmatrix} \Rightarrow \begin{pmatrix} f(x) \leq h(x) \\ \forall x \in R \end{pmatrix}$$

S is antisymmetric,

$$\begin{pmatrix} f(x) \leq g(x), \forall x \in R \\ g(x) \leq f(x) \end{pmatrix} \Rightarrow \begin{pmatrix} f(x) = g(x) \\ \forall x \in R \end{pmatrix}$$

S is a partial order.

It is not a total order. For example, if $f(x) = x$ and $g(x) = x^2$, then neither $f(x) \leq g(x)$ nor $g(x) \leq f(x)$ for all $x \in R$.

● PROBLEM 7-15

Let A and B denote totally ordered sets $(A, \leq')$, $(B, \leq'')$. Show that $A \times B$ can be totally ordered.

SOLUTION:

We can define the order on $A \times B$ in this manner:

$$(a,b) \leq (c,d) \begin{cases} \text{if } a \leq' c \\ \text{or, if } a = c \text{ and } b \leq'' d \end{cases}$$

Since both $\leq'$ and $\leq''$ are reflexive, $\leq$ is reflexive. Relation $\leq$ is also transitive.

Suppose $(a, b) \leq (c, d)$ and $(c, d) \leq (a, b)$. Then $a \leq' c$ and $c \leq' a$ imply $a = c$. Also $b \leq'' d$ and $d \leq'' b$ imply $b = d$.

Thus, $(a, b) = (c, d)$ and relation $\leq$ is antisymmetric.

Since $\leq'$ and $\leq''$ are total orders, $\leq$ is a total order.

The order defined above is called the lexicographical order on $A \times B$ because it is similar to the arrangement of the words in a dictionary.

● PROBLEM 7-16

Give an example of the partially ordered set that has no upper bound.

SOLUTION:

We shall start with

DEFINITION OF UPPER (LOWER) BOUND

Let $(A, \leq)$ denote a partially ordered set and B its subset, $B \subset A$. An element, $x \in A$, is an upper bound (lower bound) of B, if and only if

$$x \geq a \ (x \leq a) \text{ for all } a \in B.$$

For example, let $(N, \leq)$ denote a set of natural numbers $N = \{1, 2, 3, 4, \ldots\}$ and $B = \{2, 4, 8\}$ its subset. Relation "$\leq$" is an ordinary relation. Element $10 \in N$ is an upper bound of B. Any element of the set $\{8, 9, 10, 11, \ldots\}$ is an upper bound of B.

Elements 1 and 2 are lower bounds of B. Now consider the set of even numbers

$$\{2, 4, 6, 8, \ldots\}.$$

This set has no upper bound.

● PROBLEM 7-17

Let S represent a family of sets partially ordered by the relation of inclusion, and let R represent a subfamily of S.

1. Show that

$$\cup \{A : A \in R\}$$

does not have to be an upper bound of R.

2. Show that if $B \in S$ is an upper bound of R, then

$$\cup \{A : A \in R\} \subset B. \qquad (1)$$

SOLUTION:

1. Set S is a family of sets and R is its subfamily. It does not guarantee that the union of elements of R

$$\cup \{A : A \in R\}$$

is an element of S.

Hence, $\cup \{A : A \in R\}$ is an upper bound of R, if and only if it is an element of S.

2. Let

$$x \in \cup \{A : A \in R\}. \tag{2}$$

Then set A_0 exists, such that $x \in A_0$ and $A_0 \in R$.

Set B is an upper bound of R, therefore

$$A_0 \subset B \quad \text{and} \quad x \in B$$

which proves (1).

● PROBLEM 7-18

The set

$$A = \{a, b, c, d, e, f, g, h\} \tag{1}$$

is partially ordered as shown in the diagram.

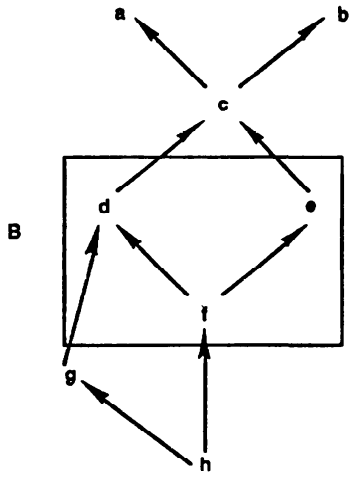


FIGURE 1

Two elements, x and y, are related, $x \leq y$, iff $x = y$ or one can move along the arrows from x to y. Consider the subset B of A

$$B = \{d, e, f\}.$$

Find the least upper bound and the greatest lower bound of B.

SOLUTION:

The elements a, b, and c are upper bounds of B.

DEFINITION OF LEAST UPPER BOUND (GREATEST LOWER BOUND)

Let $(A, \leq)$ represent a partially ordered set and B its subset. An upper bound (lower bound) x_0 of B is the least upper bound (the greatest lower bound) of B, if and only if $x_0 \leq x$ ($x_0 \geq x$) for every upper (lower) bound x of B.

Element c is the least upper bound of B because

$$c \leq a \quad \text{and} \quad c \leq b.$$

Elements f and h are lower bounds of B.

The greatest lower bound of B is f.

Sometimes, we denote the least upper bound of A by

$$\sup(A) \quad (\text{or lub } A)$$

and the greatest lower bound of A by

$$\inf(A) \quad (\text{or glb } A).$$

● PROBLEM 7-19

Let B denote a subset of the partially ordered set $(A, \leq)$. Prove that if the least upper bound (greatest lower bound) of B exists, then it is unique.

SOLUTION:

Proof (reductio ad absurdum or r.a.a.)

Suppose there are two least upper bounds of B, x and x', such that $x \neq x'$. Since x is the least upper bound of B and x' is an upper bound of B

$$x \leq x'. \tag{1}$$

Since x' is the least upper bound of B and x is an upper bound of B

$$x' \leq x. \tag{2}$$

Relation $\leq$ defined as A is a partial order. Hence, it is antisymmetric

$$\begin{pmatrix} x \leq x' \\ x' \leq x \end{pmatrix} \Rightarrow (x = x').$$

This is in contradiction with the assumption that $x \neq x'$. Thus, the least upper bound is unique.

In the same way, it can be shown that the greatest lower bound is unique.

● PROBLEM 7-20

Let X denote a non-empty set. The set $(P(X), \subset)$ is a partially ordered set. Show that, for any $Q \subset P(X)$, the least upper bound of Q is

$$\sup(Q) = \bigcup_{A \in Q} A \tag{1}$$

and the greatest lower bound of Q is

$$\inf(Q) = \bigcap_{A \in Q} A. \tag{2}$$

SOLUTION:

Let B represent any element of Q, $B \in Q$, then

$$B \subset \bigcup_{A \in Q} A. \tag{3}$$

Hence, $\bigcup_{A \in Q} A$ is the upper bound of Q. Suppose D is the upper bound of Q, then for any $B \in Q$

$$B \subset D. \tag{4}$$

Hence

$$\bigcup_{A \in Q} A \subset D \tag{5}$$

and $\bigcup_{A \in Q} A$ is the least upper bound.

The set $\bigcap_{A \in Q} A$ is the lower bound of Q because for any $B \in Q$

$$\bigcap_{A \in Q} A \subset B. \tag{6}$$

The set $\bigcap_{A \in Q} A$ is the greatest lower bound of Q. Indeed let D be the lower bound, then

$$D \subset B \text{ for all } B \in Q$$

and

$$D \subset \bigcap_{A \in Q} A. \tag{7}$$

● PROBLEM 7-21

The set

$$A = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 14\}$$

is ordered by

$$x \leq y \text{ iff } x \text{ is a multiple of } y.$$

Find the maximal and minimal elements of A.

SOLUTION:

The order of A is depicted in the diagram

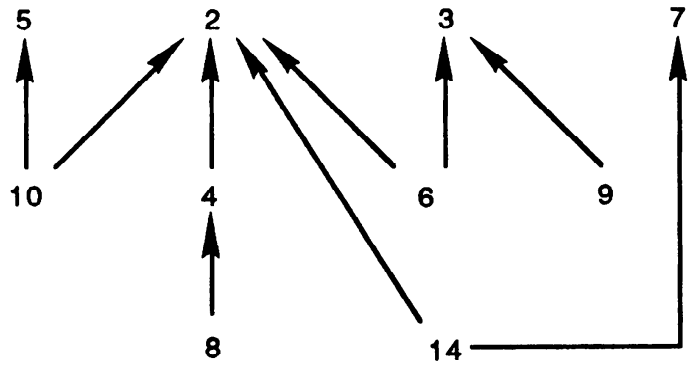


FIGURE 1

DEFINITION OF MAXIMAL (MINIMAL) ELEMENT

An element $x_0 \in A$ of a partially ordered set $(A, \leq)$ is said to be maximal (minimal), if and only if $x_0 \leq x$ ($x \leq x_0$) implies $x = x_0$ for all $x \in A$.

According to the definition, the maximal elements of A are 2, 3, 5 and 7. The minimal elements are 6, 8, 9, 10, and 14.

● PROBLEM 7-22

Consider the set

$$A = \{2, 3, 4, 5, \ldots\} = N - \{1\}$$

ordered by $m \leq n$, iff m divides n.

Find maximal and minimal elements of A.

SOLUTION:

All prime numbers are minimal elements. If $p \in A$ is a prime number, then only p divides p, because $1 \notin A$.

Only prime numbers are minimal elements. Suppose $a \in A$ is not a prime number, then $b \in A$ exists, such that

$$b \text{ divides } a \ (b \leq a).$$

Hence, a is not a minimal element.

Set A has no maximal elements since, for every $m \in A$, $2m$ also belongs to A and

$$m \text{ divides } 2m \ (m \leq 2m).$$

● PROBLEM 7-23

Give an example of a partially ordered set that has more than one maximal element and more than one minimal element.

SOLUTION:

Consider set A consisting of elements

$$A = \{a, b, c, d, e\}$$

which are partially ordered as shown in the diagram.

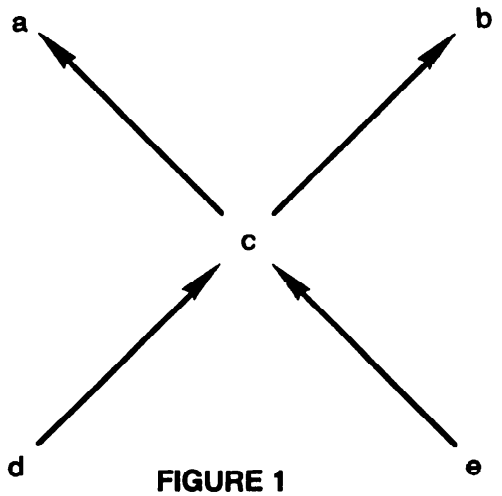


FIGURE 1

The maximal elements of A are a and b.

The minimal elements of A are d and e.

From this example, it is clear that maximal and minimal elements of a partially ordered set are not unique.

The situation changes for totally ordered sets (see Problem 7-24).

● PROBLEM 7-24

Show that if a totally ordered set has a maximal (or minimal) element, then this element is unique.

SOLUTION:

Set A is a totally ordered set $(A, \leq)$.

Suppose the set has two maximal elements, x_1 and x_2. We have $x_1 \in A$ and $x_2 \in A$. Since A is a totally ordered set for any pair $x_1, x_2 \in A$, either $x_1 \leq x_2$ or $x_2 \leq x_1$. Suppose $x_1 \leq x_2$. Then, since x_1 is a maximal element,

$$(x_1 \leq x_2) \Rightarrow (x_1 = x_2).$$

Hence, $x_1 = x_2$. The maximal element of a totally ordered set is unique.

In the same way, we can show that the minimal element of a totally ordered set is unique.

● PROBLEM 7-25

Draw the diagram for the ordered set (A, S) where

$$A = \{a, b, c, d, e, f, g\}$$

$$S = \{(b, c), (d, e), (e, f), (d, f), (g, a), (g, b), (g, c),$$

$$(g, d), (g, e), (g, f)\} \cup I_A.$$

Find the elements which are maximal, minimal, maximum, and minimum.

SOLUTION:

The diagram for the set (A, S) is shown in Figure 1.

The maximal elements are a, c, and f.

The minimal element is g.

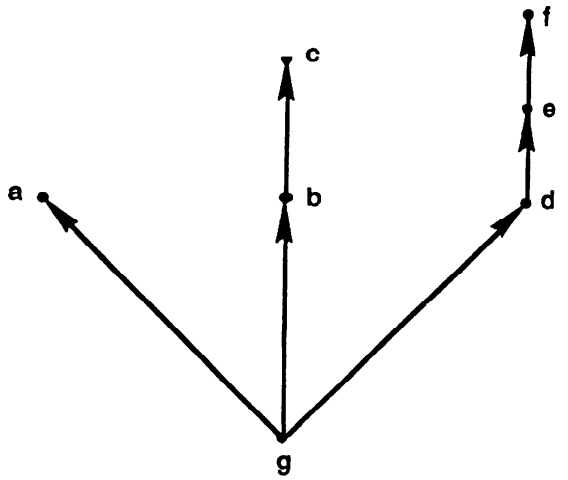


FIGURE 1

DEFINITION OF MAXIMUM (MINIMUM)

Element $a \in X$ of an ordered set X is a maximum (minimum) element of X or greatest (least) element of X, if and only if $x \leq a$ ($a \leq x$) for all $x \in X$.

Set (A, S) has no maximum. Element g is a minimum element of A.

● PROBLEM 7-26

Show that if an ordered set A has two distinct minimal elements, then it has no minimum element.

SOLUTION:

Let x_1 and x_2 denote two distinct minimal elements of A. Suppose a is a minimum element of A. Therefore

$$a \leq x_1 \quad a \leq x_2 \tag{1}$$

because a is a minimum element of A and $x_1, x_2 \in A$.

Since x_1 is a minimal element of A, there is no $x \in A$, such that

$$x_1 \neq x \leq x_1. \tag{2}$$

From (1) and (2), we conclude

$$x_1 = a. \tag{3}$$

Similarly, since x_2 is a minimal element of A

$$x_2 = a. \tag{4}$$

Hence

$$x_1 = x_2 \tag{5}$$

which is a contradiction. Set A with two distinct minimal elements has no minimum element.

● PROBLEM 7-27

Let X represent a linearly ordered set and A a finite totally ordered subset of X. Show that if

$$a = \sup A$$

then

$$a = \text{maximum } A$$

Show that the conclusion can be false when A is not totally ordered.

SOLUTION:

Let

$$A = \{a_1, a_2, \ldots, a_n\} \tag{1}$$

and let

$$b_1 = a_1 = \text{maximum } \{a_1\}. \tag{2}$$

Moving step by step, we define

$$b_2 = \text{maximum } \{b_1, a_2\}$$

$$b_3 = \text{maximum } \{b_2, a_3\} \tag{3}$$

In general

$$b_i = \text{maximum } \{b_{i-1}, a_i\}$$

and

$$b_n = \text{maximum } \{b_{n-1}, a_n\}.$$

Set A is totally ordered, hence we can always find the maximum. Since the maximum exists,

$$a = \sup A = \text{maximum } A. \tag{4}$$

Suppose $x = \{a, b, c\}$ and $A = \{a, b\}$. Set A is not totally ordered; see Figure 1.

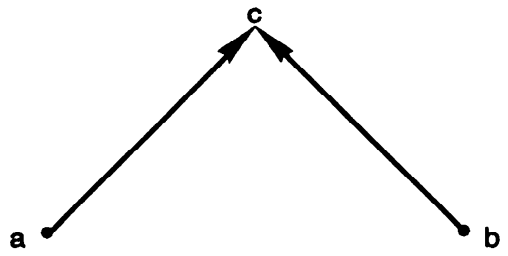


FIGURE 1

Then

$$c = \sup A \tag{5}$$

but c is not a maximum of A.

● PROBLEM 7-28

Prove that not every antisymmetric relation may be extended to an order.

SOLUTION:

Let X represent a set and T an antisymmetric relation on X. We have to choose T in such a way that no partial order Q on X exists, such that

$$T \subset Q. \tag{1}$$

Let

$$T = \{(a, b), (b, c), (c, d), (d, a)\} \tag{2}$$

denote an antisymmetric relation on X, where X is any set containing elements a, b, c, d.

Suppose a partial order Q exists, such that $T \subset Q$. Then

$$(a, b) \in Q \quad \text{and} \quad (b, c) \in Q. \tag{3}$$

Hence, since Q is transitive

$$(a, c) \in Q. \tag{4}$$

Furthermore

$$(a, d) \in Q \quad \text{and} \quad (d, a) \in Q \tag{5}$$

and, since Q is antisymmetric

$$a = d. \tag{6}$$

Similarly, we show that

$$a = b = c = d. \tag{7}$$

● PROBLEM 7-29

Let A and B represent ordered sets and let

$$f : A \rightarrow B \tag{1}$$

be onto. Prove that if for all $x, y \in A$

$$(x \leq y) \Leftrightarrow (f(x) \leq f(y)) \tag{2}$$

then f is one-to-one.

SOLUTION:

Suppose

$$f(x) = f(y) \tag{3}$$

Then

$$f(x) \geq f(y) \Rightarrow x \geq y$$

$$f(x) \leq f(y) \Rightarrow x \leq y. \tag{4}$$

Since order is a relation which is antisymmetric we conclude from (4) that

$$x = y. \tag{5}$$

● PROBLEM 7-30

We shall accept the following theorem without a proof.

THEOREM 1

Let $(A, \leq)$ represent a non-empty partially ordered set, such that every totally ordered subset of A has a least upper bound in A. If function $f : A \rightarrow A$ is such that for every $a \in A, f(a) \geq a$, then element $b \in A$ exists such that

$$f(b) = b.$$

Use Theorem 1 to prove:

THEOREM 2

Let $(A, \leq)$ represent a non-empty partially ordered set, such that every totally ordered subset of A has a greatest lower bound. If function $f : A \rightarrow A$ is such that for every $a \in A, f(a) \leq a$, then element $b \in A$ exists such that

$$f(b) = b.$$

SOLUTION:

Suppose (A, ρ) is a partially ordered set and ρ^{-1} is a relation inverse to ρ. Then (A, ρ^{-1}) is also a partially ordered set, where

$$(a \; \rho^{-1} \; b) \quad \text{iff} \quad (b \; \rho \; a). \tag{1}$$

Assume that (A, ρ) is such that every totally ordered subset of A has a least upper bound in A. Note that, if B is a totally ordered subset of (A, ρ), then (B, ρ^{-1}) is a totally ordered subset of (A, ρ^{-1}). If x_0 is a least upper bound of (B, ρ), then x_0 is a greatest lower bound of (B, ρ^{-1}).

Suppose function $f: A \to A$ is such that for every $a \in A$, $f(a) \geq a$. Then by applying Theorem 1, we conclude that element $b \in A$ exists, such that

$$f(b) = b.$$

If function $f: (A, \rho^{-1}) \to (A, \rho^{-1})$ is such that for every $a \in A$, $a \rho^{-1} f(a)$, then by virtue of Theorem 1, element $b \in A$ exists, such that

$$f(b) = b.$$

● PROBLEM 7-31

Show that the Hausdorff maximality principle and the axiom of choice are equivalent. Apply Theorem 1 of Problem 7-30.

SOLUTION:

HAUSDORFF MAXIMALITY PRINCIPLE

Let $(A, \leq)$ represent a partially ordered set and let S represent the set of all totally ordered subsets of A partially ordered by inclusion $(S, \subset)$. Then $(S, \subset)$ has a maximal element.

We shall use the reductio ad absurdum method.

Assume that S has no maximal element. Then, for every element s, $s \in S$, we can find a non-empty set

$$P = \{s' \in S : s \subset s', s \neq s'\}. \tag{1}$$

Elements s and s' of S are themselves ordered sets.

The axiom of choice guarantees the existence of a function f defined on $\{P : s \in S\}$ and such that

$$f(P) \in P. \tag{2}$$

Hence we can define a function

$$g : S \to S \tag{3}$$

such that

$$g(s) = f(P). \tag{4}$$

Note that since $g(s) = f(P)$, we have

$$s \subset g(s) = f(P), \quad g(s) \neq s \tag{5}$$

because $f(P) \in P$.

By applying Theorem 1 of Problem 7-30 [set $(S, \subset)$ and function g satisfy the hypotheses of the theorem], we conclude that an element $q \in S$ exists, such that

$$g(q) = q \tag{6}$$

which is a contradiction because for all $s \in S$

$$s \subset g(s) \quad (s \neq g(s)). \tag{7}$$

We showed that

(Axiom of Choice) $\Rightarrow$ (Hausdorff Maximality Principle).

Later we will carry out the proof in the opposite direction.

● PROBLEM 7-32

Use the Hausdorff maximality principle to prove the following:

Let $(A, \leq)$ represent a partially ordered set and B represent a totally ordered subset of A, $B \subset A$. Then A has a maximal totally ordered subset C such that

$$B \subset C. \tag{1}$$

SOLUTION:

Let S represent the set of all totally ordered subsets of A that contain B.

Note that the set $(S, \subset)$ is partially ordered.

The restriction that S consists of only these sets that contain B does not change the structure of the proof of theorem in Problem 7-31. Hence, we conclude that S has a maximal element, say C, such that

$$B \subset C.$$

● PROBLEM 7-33

Show that

(Axiom of Choice) $\Rightarrow$ (Zorn's Lemma).

SOLUTION:

ZORN'S LEMMA

If $(A, \leq)$ is a partially ordered set, such that every chain in A has an upper bound in A, then A contains a maximal element.

According to the Hausdorff maximality principle, the non-empty partially ordered set $(A, \leq)$ has a totally ordered subset $B \subset A$, which is maximal with respect to the relation of inclusion. By hypothesis, set B has an upper bound $x \in A$.

We shall show that element x is a maximal element of A.

Suppose, on the contrary, that an element $y \in A$ exists, such that

$$x \leq y.$$

Then $B \cup \{y\}$ is a totally ordered subset of A that contains B

$$B \subset B \cup \{y\}$$

where B is the maximal totally ordered subset of A. Hence

$$B \cup \{y\} = B$$

and

$$y \leq x.$$

Thus, x is a maximal element of $(A, \leq)$.

That completes the proof of

(Hausdorff Maximality Principle) $\Rightarrow$ (Zorn's Lemma).

From Problem 7-31, we obtain

(Axiom of Choice) $\Rightarrow$ (Hausdorff Maximality Principle) $\Rightarrow$

$\Rightarrow$ (Zorn's Lemma).

● PROBLEM 7-34

Prove that if set J of subsets of X has finite character, then $(J, \subset)$ has a maximal element.

SOLUTION:

DEFINITION OF FINITE CHARACTER

Set J of subsets of X has finite character, provided that $A \in J$, if and only if every finite subset of A belongs to J.

Set J is partially ordered by inclusion $(J, \subset)$.

Let $\mathcal{H}$ denote a chain in $(J, \subset)$. We shall show that $\mathcal{H}$ has an upper bound

$$B = \bigcup_{H \in \mathcal{H}} H$$

Let $\{x_1, x_2, \ldots, x_n\}$ denote any finite subset of B,

$$\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{H \in \mathcal{H}} H.$$

Then

$$x_l \in H_l \text{ for some } H_l \in \mathcal{H}$$

$$l = 1, 2, \ldots, n.$$

Set $\mathcal{H}$ is a chain, therefore $H \in \mathcal{H}$ exists, such that for all $l = 1, 2, \ldots, n$

$$H_l \subset H.$$

Hence

$$\{x_1, x_2, \ldots, x_n\} \subset H$$

and

$$\{x_1, x_2, \ldots, x_n\} \in J.$$

Therefore $B \in J$ is an upper bound for $\mathcal{H}$.

According to Zorn's lemma, $(J, \subset)$ has a maximal element.

● PROBLEM 7-35

Apply Zorn's lemma to show that every vector space has a basis.

SOLUTION:

Let V denote a vector space, and let S denote the set of all linearly independent subsets of vectors in V.

Set S can be partially ordered by inclusion $(S, \subset)$. If H is a chain in $(S, \subset)$, then

$$T = \bigcup_{h \in H} h$$

is linearly independent. Hence T is an upper bound for H, $T \in S$.

According to Zorn's lemma, set $(S, \subset)$ has a maximal element. It can be easily shown that this element forms a basis for V.

● PROBLEM 7-36

Prove that every ring with an identity has a proper maximal ideal.

SOLUTION:

Let K denote a ring and let $1 \in K$ denote the identity.

Set $\mathcal{A}$ of all proper ideals of K is partially ordered by inclusion, $(\mathcal{A}, \subset)$. Let $(H, \subset)$ denote a chain in $(\mathcal{A}, \subset)$. Then

$$\bigcup_{h \in H} H$$

is a proper ideal of K.

Indeed

$$1 \notin \bigcup_{h \in H} h$$

Hence $\bigcup_{h \in H} h \in \mathcal{A}$ is an upper bound of H.

According to Zorn's lemma, we conclude that $(\mathcal{A}, \subset)$ has a maximal element. Hence, ring K with an identity has a proper maximal ideal.

● PROBLEM 7-37

Prove

(Zorn's lemma) $\Rightarrow$ (Kuratowski's lemma).

SOLUTION:

KURATOWSKI'S LEMMA

Each chain in a partially ordered set $(X, \leq)$ is included in a maximal chain.

Let $(X, \leq)$ denote a partially ordered set and h a given chain in X. Let H represent a set of all chains in X which include h, $h \in H$. Set H is ordered by inclusion $(H, \subset)$. Let a non-empty set $\mathcal{F} \neq \phi$ represent an inclusion chain of elements of H; then

$$\cup \mathcal{F} \in H. \tag{1}$$

Suppose $a, b \in \cup \mathcal{F}$; then for some $A, B \in \mathcal{F}$

$$a \in A \in \mathcal{F}$$

$$b \in B \in \mathcal{F}. \tag{2}$$

Since A and B are members of $\mathcal{F}$, we must have

$$A \subset B \quad \text{or} \quad B \subset A. \tag{3}$$

Hence

$$a \leq b \quad \text{or} \quad b \leq a$$

because $\cup \mathcal{F}$ is a chain. We have

$$h \subset A \subset \cup \mathcal{F} \in H. \tag{4}$$

Set $\mathcal{F}$ has an upper bound in H.

Therefore, according to Zorn's lemma, H has a maximal element.

Chain h is included in a maximal element.

One can prove that

$$\text{(Kuratowski's Lemma)} \Rightarrow \text{(Zorn's Lemma)}.$$

Therefore, Zorn's lemma and Kuratowski's lemma are equivalent.

● PROBLEM 7-38

Show that the set of natural numbers $(N, \leq)$ is well ordered when $\leq$ is understood in the usual sense.

SOLUTION:

DEFINITION OF A WELL ORDERED SET

A totally ordered set $(A, \leq)$ is a well ordered set iff every non-empty subset B of A contains a minimal element; that is, if

$$\underset{\phi \neq B \subset A}{\forall} \quad \underset{b \in B}{\exists} \quad \underset{x \in B}{\forall} \quad b \leq x$$

Element b is called the least element of B and relation $\leq$ is called a well ordered relation.

The set of natural numbers N is well ordered because every non-empty subset of N contains the least element.

On the other hand, the set of rational numbers with the ordinary "less than or equal" $\leq$ relation is not well ordered. Not all subsets of R (for example, $\{-1, -2, -3, -4, \ldots\}$, contain the least element.

● PROBLEM 7-39

Show that the set Q of rational numbers can be well ordered.

SOLUTION:

Set $(N, \leq)$ is well ordered. Set Q is denumerable, hence a bijection

$$f: Q \to N \tag{1}$$

exists.

Relation $\leq$ on Q can be introduced in the following manner:

$$\left(\begin{matrix} \forall \\ a,b \in Q \end{matrix} \quad a \leq b \right) \Leftrightarrow \left(\begin{matrix} f(a) \leq f(b) \\ f(a), f(b) \in N \end{matrix} \right).$$

Now, since N is well ordered, so is Q.

In this proof, Q can be replaced by any denumerable set.

● PROBLEM 7-40

Let (X, T) denote a totally ordered set and $Y \subset X$. Prove that a is the minimum element of Y, if and only if

$$\text{Im}(T \cap (\{a\} \times Y)) = Y. \tag{1}$$

SOLUTION:

Let $y \in Y$, then

$$y \in \text{Im}(T \cap (\{a\} \times Y)). \tag{2}$$

Hence

$$(x, y) \in T \cap (\{a\} \times Y) \tag{3}$$

for some $x \in X$ if $(a, y) \in T$. Furthermore

$$(x, y) \in (T \cap (\{a\} \times Y)) = Y \tag{4}$$

if and only if

$$(a, y) \in T$$

for all $y \in Y$ and if and only if a is the minimum element of Y.

In 1904 Ernst Zermelo used the axiom of choice to prove the amazing well ordering principle.

WELL ORDERING PRINCIPLE

Every set can be well ordered.

Prove

(Zorn's Lemma) ⇒ (Well Ordering Principle).

SOLUTION:

Let A represent any set and $\tilde{A}$ a family of all well ordered sets $(A_0, \leq_0)$ which are subsets of A. Set $\tilde{A}$ can be partially ordered by the relation $\tilde{\leq}$ defined by:

$$(A_1, \leq_1) \ \tilde{\leq} (A_2, \leq_2) \tag{1}$$

iff

1. $A_1 \subset A_2$, and

2. $x, y \in A_1$ and $x \leq_1 y$ imply $x \leq_2 y$, and

3. $x \in A_2 - A_1$ implies $y \leq_2 x$ for all $y \in A_1$. Set $(\tilde{A}, \tilde{\leq})$ is partially ordered.

Now, let $\tilde{B}$ represent a totally ordered subset of $\tilde{A}$. To apply Zorn's lemma, we have to show that $\tilde{B}$ has an upper bound. Consider $\left(\bigcup_{A \in \tilde{B}} A, \leq^*\right)$ where $\leq^*$ is defined by:

$$x \leq^* y \text{ iff both } x, y \text{ belong to some } (A_0, \leq_0)$$

and $(A_0, \leq_0) \in \tilde{B}$, and $x \leq_0 y$.

It is easy to verify that $\leq^*$ is a total order relation on $\bigcup_{A \in \tilde{B}} A$. Furthermore, we will show that

$$\left(\bigcup_{A \in \tilde{B}} A, \leq^*\right) \tag{2}$$

is well ordered.

Let

$$\phi \neq C \subset \bigcup_{A \in \tilde{B}} A, \tag{3}$$

then $(A_0, \leq_0) \in \tilde{B}$ exists, such that

$$A_0 \cap C \neq \phi.$$

Set $(A_0, \leq_0)$ is well ordered, so $A_0 \cap C$ contains the unique least element of $A_0 \cap C$ denoted by a_0.

For any $x \in C$, $(A_1, \leq_1) \in \tilde{B}$ exists, such that

$$(A_0, \leq_0) \tilde{\leq} (A_1, \leq_1) \tag{4}$$

and $a_0, x \in A_1$.

We have $a_0 \leq x$ and $a_0 \leq^* x$. Hence a_0 is the least element for $(C, \leq^*)$.

Thus $\left(\bigcup_{A \in \tilde{B}} A, \leq^* \right)$ is well ordered.

According to Zorn's lemma, $(\tilde{A}, \tilde{\leq})$ has a maximal element $(A_1, \leq_1)$. We will show that $A = A_1$.

Indeed, suppose $A \neq A_1$; then $y \in A - A_1$ and relation $\leq_1$ on A_1 can be extended to $A_1 \cup \{y\}$ by

$$x \leq_1 y \text{ for all } x \in A_1.$$

We conclude that

$$(A_1, \leq_1) \tilde{\leq} (A_1 \cup \{y\}, \leq_1). \tag{5}$$

But $(A_1, \leq_1)$ is a maximal element, hence, a contradiction.

● PROBLEM 7-42

Prove that

(Well Ordering Principle) $\Rightarrow$ (Axiom of Choice).

SOLUTION:

Let $\mathcal{A}$ represent a non-empty family of non-empty sets. According to the well ordering principle (see Problem 7-41) a total order relation $\leq$ exists, such that

$$\left(\bigcup_{A \in \mathcal{A}} A, \leq \right) \tag{1}$$

is a well ordered set.

Since (1) is well ordered, each $A \in \mathcal{A}$ contains a least element. Thus, we can define function f

$$f : \mathcal{A} \rightarrow \bigcup_{A \in \mathcal{A}} A \tag{2}$$

by

$$f(A) = \text{least element of } A. \tag{3}$$

Function f is the choice function of the axiom of choice.

Show that the following statements are equivalent:

1. The axiom of choice
2. Hausdorff maximality principle
3. Zorn's lemma
4. Kuratowski's lemma
5. The well ordering principle.

SOLUTION:

There is nothing left to prove. We will merely summarize the results of this chapter.
In Problem 7-31, we showed that

$$\text{(Axiom of Choice)} \Rightarrow \text{(Hausdorff Maximality Principle)}.$$

In Problem 7-33 we proved

$$\text{(Hausdorff Maximality Principle)} \Rightarrow \text{(Zorn's Lemma)}.$$

In Problem 7-41 we proved

$$\text{(Zorn's Lemma)} \Rightarrow \text{(Well Ordering Principle)}.$$

To make the circle complete, we proved in Problem 7-42

$$\text{(Well Ordering Principle)} \Rightarrow \text{(Axiom of Choice)}.$$

Hence

$$\text{(Axiom of Choice)} \equiv \text{(Zorn's Lemma)} \equiv$$

$$\equiv \text{(Well Ordering Principle)} \equiv \text{(Hausdorff Maximality Principle)} \equiv$$

$$\equiv \text{(Kuratowski's Lemma)}.$$

For Kuratowski's lemma, see Problem 7-37.

Note that all these propositions are purely existential. They state the fact that "something" exists but do not offer the means of finding "it." For example, the well ordering principle states that the set of real numbers can be well ordered. So far it is not known *how* the set of real numbers can be well ordered.

● PROBLEM 7-44

Give an example of a segment of (N, ≤), where ≤ is the usual "less than or equal."

SOLUTION:

DEFINITION OF A SEGMENT

A segment of a totally ordered set $(A, \leq)$ is a subset S of A such that, if

$$x \in S, \text{ and } y \in A, \text{ and } y \leq x,$$

then $y \in S$. A proper segment of A is a segment which is a proper subset of A.

Set $\{1\}$ is a segment of N, where $N = \{1, 2, 3, 4, \ldots\}$ is the set of natural numbers.

Each set of the form

$$\{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \ldots$$

is a segment.

Also the set of natural numbers is a segment.

● PROBLEM 7-45

Show that any union or intersection of segments is a segment.

SOLUTION:

Let $(A, \leq)$ represent a totally ordered set, and let $\mathcal{F}$ represent a family of segments of A.

Suppose

$$x \in \bigcup_{S \in \mathcal{F}} S \tag{1}$$

and $y \in A$ where

$$y \leq x. \tag{2}$$

Element x belongs to some segment, say S_0 of A, $x \in S_0$. Since S_0 is a segment and $x \in S_0$, $y \in A$, $y \leq x$, we have

$$y \in S_0. \tag{3}$$

Hence,

$$y \in \bigcup_{S \in \mathcal{F}} S \tag{4}$$

and $\bigcup_{S \in \mathcal{F}} S$ is a segment of A.

Suppose

$$x \in \bigcap_{S \in \mathcal{F}} S \tag{5}$$

and $y \in A$, and $y \leq x$. Then for every $S \in \mathcal{F}$, $x \in S$ and since S is a segment, $y \in S$.

Hence, $y \in \bigcap_{S \in \mathcal{F}} S$.

Therefore, $\bigcap_{S \in \mathcal{F}} S$ is a segment.

● PROBLEM 7-46

Prove that if $(A, \leq)$ is a well ordered set, then:

1. All segments of a segment are again segments.

2. For each segment S of A, $S \neq A$, an element $x \in A$ exists, such that $S = A_x$, where

$$A_x = \{a \in A : a < x\}. \tag{1}$$

We use $a < x$ to indicate $a \leq x$ and $a \neq x$.

SOLUTION:

1. Let S represent a segment of A and P represent a segment of S. Suppose $x \in P$ and $y \in A$, and

$$y \leq x. \tag{2}$$

Since x belongs to the segment S, we have

$$y \in S.$$

Now from $x \in P$, $y \in S$ and $y \leq x$ we conclude that $y \in P$ because P is a segment of S. Thus, P is a segment of A.

2. Let S represent a segment of A, such that $S \neq A$. Then set $A - S$ is non-empty. This set has a least element, which we will denote by x.

Thus, as can be easily verified

$$A_x = S \quad \text{where} \quad A_x = \{a \in A : a < x\}. \tag{3}$$

Prove that if $(A, \leq)$ is a well ordered set and $\mathcal{F}$ a family of segments of A, such that

1. any union of elements of $\mathcal{F}$ belongs to $\mathcal{F}$.

2. if $A_x \in \mathcal{F}$, then $A_x \cup \{x\} \in \mathcal{F}$,

$$A_x = \{a \in A : a < x\}$$

then $\mathcal{F}$ contains all segments of A.

SOLUTION:

Suppose $x \in A$ exists, such that $A_x \notin \mathcal{F}$. Then, non-empty set

$$B = \{x \in A : A_x \notin \mathcal{F}\} \subset A \tag{1}$$

has a least element b, because $(A, \leq)$ is well ordered. We have

$$b \in B \quad \text{and} \quad A_b \notin \mathcal{F}. \tag{2}$$

If $y \in A$ and $y < b$, then

$$y \notin B \text{ and hence } A_y \notin \mathcal{F}.$$

By number 1, of the hypothesis,

$$\bigcup_{y<b} A_y \in \mathcal{F} \tag{3}$$

An element $a \in A$ exists (see Problem 7-46), such that

$$\bigcup_{y<b} A_y = A_a \in \mathcal{F} \tag{4}$$

hence $a < b$ and

$$A_a \cup \{a\} = A_c \text{ for some } c \in A. \tag{5}$$

Thus

$$a < c < b \quad \text{and}$$

$$a \in A_c \subset \bigcup_{y<b} A_y = A_a. \tag{6}$$

But $a \notin A_a$, hence a contradiction. Thus

$$\text{for all } x \in A, A_x \in \mathcal{F}. \tag{7}$$

Now, we shall prove that $A \in \mathcal{F}$. Indeed

$$\bigcup_{x \in A} A_x \in \mathcal{F} \tag{8}$$

if

$$A = \bigcup_{x \in A} A_x \; ; \tag{9}$$

then everything is all right. Suppose

$$A \neq \bigcup_{x \in A} A_x . \tag{10}$$

Then $y \in A$ exists, such that

$$\bigcup_{x \in A} A_x = A_y . \tag{11}$$

Furthermore

$$A_y \cup \{y\} \in \mathcal{F}. \tag{12}$$

Then

$$x \leq y \quad \text{for all} \quad x \in A.$$

Thus

$$A = A_y \cup \{y\} \in \mathcal{F}. \tag{13}$$

● PROBLEM 7-48

Prove the following important principle:

PRINCIPLE OF TRANSFINITE INDUCTION

Let $(A, \leq)$ denote a well ordered set and let $p(x)$ represent a sentence for each $x \in A$.

If, for each $x \in A$,

$$(p(y) \equiv 1 \text{ for every } y < x) \Rightarrow (p(x) \equiv 1) \tag{1}$$

then for every $x \in A$, $p(x) \equiv 1$.

Note that $p(x) \equiv 1$ denotes that the sentence $p(x)$ is true and $p(x) \equiv 0$ denotes that the sentence $p(x)$ is false.

SOLUTION:

Suppose $x \in A$ exists, such that

$$p(x) \equiv 0. \tag{2}$$

Let B denote a non-empty subset of A, such that

$$B = \{x \in A : p(x) \equiv 0\}, \tag{3}$$

and let b denote the least element of B. Remember that $(A, \leq)$ is well ordered.

Since $b \in B$, $p(b) \equiv 0$. Suppose $y \in A$, such that $y < b$; then $y \notin B$ and $p(y) \equiv 1$. Hence for every $y < b$

$$p(y) \equiv 1. \tag{4}$$

From (1) and (4), we conclude that

$$p(b) \equiv 1$$

which is a contradiction.

Thus

$$p(x) \equiv 1 \tag{5}$$

for every $x \in A$.

● PROBLEM 7-49

Apply the principle of transfinite induction to prove the following:

THEOREM

Let $(A, \leq)$ and $(A_0, \leq_0)$ denote well ordered sets. If $f : A \to A_0$ is increasing, $f(A)$ is a segment of A_0 and $F : A \to A_0$ is strictly increasing, then

$$\text{for all } x \in A, \quad f(x) \leq_0 F(x). \tag{1}$$

SOLUTION:

We shall start with

DEFINITION OF INCREASING FUNCTION

Let $(A, \leq)$ and $(A_0, \leq_0)$ denote well ordered sets. A function $f : A \to A_0$ is increasing iff

$$\begin{pmatrix} a \leq b \\ a, b \in A \end{pmatrix} \Rightarrow \begin{pmatrix} f(a) \leq_0 f(b) \\ f(a), f(b) \in A_0 \end{pmatrix}.$$

Function $f : A \to A_0$ is strictly increasing iff

$$\begin{pmatrix} a < b \\ a, b \in A \end{pmatrix} \Rightarrow \begin{pmatrix} f(a) < f(b) \\ f(a), f(b) \in A_0 \end{pmatrix}.$$

Let $p(x)$ represent the sentence

$$f(x) \leq_0 F(x).$$

Suppose an element $a \in A$ exists, such that for all $x < a$, $p(x) \equiv 1$ but $p(a) \equiv 0$. That is, for all $x < a$,

$$f(x) \leq_0 F(x) \quad \text{and} \quad F(a) <_0 f(a). \tag{2}$$

Function f is increasing and F is strictly increasing, thus for all $x < a$ and for all $a \leq y$

$$f(x) \leq_0 F(x) <_0 F(a) <_0 f(a) \leq_0 f(y). \tag{3}$$

Hence

$$F(a) <_0 f(a) \quad \text{and} \quad F(a) \notin f(A),$$

which is a contradiction because $f(A)$ is a segment of A_0. Hence, for each $x \in A$

$$\text{if } f(y) \leq_0 F(y) \text{ for all } y < x, \text{ then } f(x) \leq_0 F(x).$$

Now we can apply the principle of transfinite induction:

$$f(x) \leq_0 F(x) \text{ for all } x \in A.$$

CHAPTER 8

ORDINAL NUMBERS AND ORDINAL ARITHMETIC

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● PROBLEM 8-1

Show that if the functions

$$f: A \to B$$

$$g: B \to C \tag{1}$$

are order isomorphic, then f^{-1} and $g \circ f$ are also order isomorphic.

SOLUTION:

DEFINITION OF ORDER ISOMORPHISM

Two well ordered sets $(A, \leq)$ and $(B, \leq_0)$ are order isomorphic if a bijection $f: A \to B$ exists, such that

$$\begin{pmatrix} a_1, a_2 \in A \\ a_1 \leq a_2 \end{pmatrix} \Rightarrow (f(a_1) \leq_0 f(a_2)). \tag{2}$$

Function $f: A \to B$ is called an order isomorphism.

■

If $f: A \to B$ is an order isomorphism, then so is $f^{-1} : B \to A$. Indeed, both sets A and B are well ordered. If f is a bijection, then f^{-1} is a bijection. Furthermore, if f has property (2), so does f^{-1}.

If functions f and g of (1) are bijections, then $g \circ f: A \to C$ is a bijection. Also if f and g have property (2), then $g \circ f$ has property (2).

● PROBLEM 8-2

Let $(A, \leq)$ and $(B, \leq_0)$ represent well ordered sets each consisting of the same finite number of elements. Show that $(A, \leq)$ and $(B, \leq_0)$ are order-isomorphic.

SOLUTION:

Sets $(A, \leq)$ and $(B, \leq_0)$ are equivalent, finite, and well ordered. Hence, their elements can be arranged as follows:

$$A = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

$$B = \{\beta_1, \beta_2, \ldots, \beta_n\} \tag{1}$$

The arrangement $\alpha_1, \alpha_2, \ldots, \alpha_n$ is such that we denote the least element of

the set A by α_1; then we denote the least element of $A - \{\alpha_1\}$ by α_2, etc. The same procedure is repeated for the set B.

Bijection

$$f: A \to B$$

is defined by

$$f(\alpha_k) = \beta_k$$

for $k = 1, \ldots, n$.

Function f is an order isomorphism because if $\alpha_l \leq \alpha_k$, then $f(\alpha_l) \leq f(\alpha_k)$, where $\alpha_l \leq \alpha_k$ indicates that α_k appears after α_l in the sequence $\alpha_1, \ldots, \alpha_n$. We shall write

$$(A, \leq) \approx (B, \leq_0) \quad \text{or} \quad A \approx B$$

to indicate that the sets $(A, \leq)$ and $(B, \leq_0)$ are order isomorphic.

● PROBLEM 8-3

The set of natural numbers N is well ordered as follows.

$$(N, \leq) = \{1, 2, 3, 4, \ldots\} \tag{1}$$

and as

$$(N, \leq_0) = (1, 3, 5, \ldots, 2, 4, 6, \ldots\}. \tag{2}$$

Show that both sets are not order isomorphic.

SOLUTION:

It is easy to verify that both sets are well ordered. Suppose

$$f: (N, \leq) \to (N, \leq_0) \tag{3}$$

exists, such that f is order isomorphic. Then, to ensure that

$$\begin{pmatrix} a_1, a_2 \in (N, \leq) \\ a_1 \leq a_2 \end{pmatrix} \Rightarrow (f(a_1) \leq f(a_2))$$

we must have

$$f(1) = 1, f(2) = 3, f(3) = 5, \ldots$$

or in general

$$f(n) = 2n - 1. \tag{4}$$

Function (4) maps set N onto the set of odd numbers. Hence, (3) is not a bijection.

Thus, $(N, \leq)$ and $(N, \leq_0)$ are not order isomorphic.

● PROBLEM 8-4

Describe the concept of ordinal numbers.

SOLUTION:

1. To each well ordered set $(A, \leq)$, an ordinal number denoted by $\text{ord}(A, \leq)$, is assigned. Also, if α is an ordinal number, then a well ordered set $(A, \leq)$ exists, such that

$$\alpha = \text{ord}(A, \leq). \tag{1}$$

2. $\text{ord}(A, \leq) = \text{ord}(B, \leq_0)$ iff $(A, \leq) \approx (B, \leq_0)$

i.e., iff the sets $(A, \leq)$ and $(B, \leq_0)$ are order isomorphic.

3. $\text{ord}\,(A, \leq) = 0$ iff $A = \phi$.

4. If $(A, \leq)$ is a well ordered set consisting of n elements, then

$$\text{ord}\,(A, \leq) = n.$$

The ordinal number of the set N with the usual 1, 2, 3, ... order is denoted by ω. Thus

$$\text{ord}(N, \leq) = \omega.$$

A given set can have only one cardinal number, but depending on its order set, it can have many different ordinal numbers (for example see Problem 8-3).

● PROBLEM 8-5

Show that if $(A, \leq)$ is a well ordered set, then the only order isomorphism of $(A, \leq)$ onto a segment of $(A, \leq)$ is the identity function of A onto A.

SOLUTION:

Suppose, on the contrary, that an order isomorphism exists

$$f: A \to A_a \tag{1}$$

where A_a is a segment

$$A_a = \{x \in A : x < a\} \tag{2}$$

then $f(a) \in A_a$ and $f(a) < a$.

Let

$$B = \{x \in A : f(x) < x\} \neq \phi \tag{3}$$

and let b denote the least element of B. Then, since $b \in B$

$$f(b) < b \tag{4}$$

hence $f(b) \in B$, which is a contradiction because $f(b) < b$ and b is the least element of B. We conclude that a well ordered set cannot be order isomorphic to any of its proper segments. Now, suppose

$$f : A \rightarrow A \tag{5}$$

is order isomorphism.

Both functions f and the identity function $1_A : A \rightarrow A$ are strictly increasing. Hence,

$$1_A(x) \leq f(x) \leq 1_A(x) \tag{6}$$

for all $x \in A$ and $f = 1_A$.

● PROBLEM 8-6

Let α and β denote two ordinal numbers. Show that the relation $\alpha \leq \beta$ (α is less than or equal to β) is reflexive and transitive.

SOLUTION:

We shall begin with the following:

DEFINITION

Let α and β denote ordinal numbers and let $(A, \leq)$ and $(B, \leq_0)$ denote well ordered sets, such that

$$\alpha = \text{ord}(A, \leq), \quad \beta = \text{ord}(B, \leq_0). \tag{1}$$

Number α is less than or equal to β we denote it by $\alpha \leq \beta$, if and only if $(A, \leq)$ is order isomorphic to a segment of $(B, \leq_0)$. If $\alpha \leq \beta$ and $\alpha \neq \beta$, then we write $\alpha < \beta$.

■

Relation is reflexive if for every ordinal number α, $\alpha \leq \alpha$. Let $(A, \leq)$ represent a well ordered set, such that

$$\alpha = \text{ord}(A, \leq). \tag{2}$$

Set A is a segment of $(A, \leq)$. Identity function on A is an order isomorphism. Hence, $\alpha \leq \alpha$.

To prove that relation $\leq$ is transitive, we must show that for any ordinal numbers

$$(\alpha \leq \beta, \beta \leq \rho) \Rightarrow (\alpha \leq \rho). \tag{3}$$

Three well ordered sets exist, such that

$$\alpha = \text{ord}(A, \leq), \beta = \text{ord}(B, \leq_1), \rho = \text{ord}(C, \leq_2).$$

If $(A, \leq)$ is order isomorphic to a segment of $(B, \leq_1)$ and $(B, \leq_1)$ is order isomorphic to a segment of $(C, \leq_2)$, then $(A, \leq)$ is order isomorphic to a segment of $(C, \leq_2)$.

● PROBLEM 8-7

Show that the relation $\leq$ for the ordinal numbers is a partial order relation.

SOLUTION:

A partial order relation is reflexive, transitive, and antisymmetric. We proved in Problem 8-5, that $\leq$ is reflexive and transitive. Now, we shall prove

THEOREM

$$\left(\begin{array}{c}\alpha, \beta \text{ are ordinal numbers} \\ \alpha \leq \beta \wedge \beta \leq \alpha\end{array}\right) \Rightarrow (\alpha = \beta). \tag{1}$$

■

Let $(A, \leq)$ and $(B, \leq_0)$ represent well ordered sets, such that

$$\alpha = \text{ord}(A, \leq), \quad \beta = \text{ord}(B \leq_0). \tag{2}$$

Since $\alpha \leq \beta$ and $\beta \leq \alpha$, two order isomorphisms exist, such that

$$f : A \rightarrow B_b \quad g : B \rightarrow A_a \tag{3}$$

where A_a and B_b are segments of A and B respectively. Function

$$h : A \rightarrow A_c \tag{4}$$

defined by

$$h(x) = g(f(x)), \text{ for all } x \in A \tag{5}$$

is an order isomorphism of A to a segment A_c of A_a. Since A_a is a segment of A, so is A_c. By Problem 8-5, $A_c = A$.

Thus, $A_a = A$ and since g is the order isomorphism $g : B \to A$, we conclude that

$$\alpha = \beta. \tag{6}$$

● PROBLEM 8-8

In Problem 8-3 we proved that

$$\text{ord}(N, \leq) \neq \text{ord}(N, \leq_0) \tag{1}$$

where

$$(N, \leq) = \{1, 2, 3, 4, \ldots\}$$

$$(N, \leq_0) = \{1, 3, 5, \ldots, 2, 4, 6, \ldots\}. \tag{2}$$

Show that

$$\text{ord}(N, \leq) < \text{ord}(N, \leq_0). \tag{3}$$

SOLUTION:

Consider the function

$$f(n) = 2n - 1$$

$$f : (N, \leq) \to (N, \leq_0). \tag{4}$$

Let A denote the segment of $(N, \leq_0)$

$$A = \{1, 3, 5, 7, \ldots\}$$

Function $f(n) = 2n - 1$

$$f : (N, \leq) \to A \tag{5}$$

is a bijection, which preserves order. Thus, f defined by (5) is an order isomorphism. Therefore,

$$\text{ord}(N, \leq) \leq \text{ord}(N, \leq_0). \tag{6}$$

From (6) and (1), we obtain

$$\text{ord}(N, \leq) < \text{ord}(N, \leq_0). \tag{7}$$

● PROBLEM 8-9

Is the relation $\leq$, defined on ordinal numbers, a total order relation?

SOLUTION:

In Problem 8-7 we proved that relation $\leq$ is reflexive, transitive, and antisymmetric. That is, $\leq$ is a partial order.

We shall quote the following without a proof.

THEOREM

For any two cardinal numbers α and β, either

$$\alpha \leq \beta \quad \text{or} \quad \beta \leq \alpha.$$

■

Thus, for any two ordinal numbers, their order is determined.

We conclude that $\leq$ is a total order and ordinal numbers are totally ordered.

The above results can be summarized in the following theorem.

THEOREM

If α and β are ordinal numbers, then one and only one of the following is true:

1. $\alpha < \beta$
2. $\alpha = \beta$
3. $\beta < \alpha$.

■

● PROBLEM 8-10

Let $(A, \leq)$ and $(B, \leq_0)$ represent well ordered sets. Prove that

$$\left(\text{ord}(A,\leq) < \text{ord}(B,\leq_0)\right) \Leftrightarrow \begin{pmatrix} A \text{ is order isomorphic} \\ \text{to a proper segment of } B \end{pmatrix}.$$

SOLUTION:

$\Leftarrow$ If A is order isomorphic to a segment of B, then

$$\text{ord}(A, \leq) \leq \text{ord}(B, \leq_0). \tag{1}$$

But A is order isomorphic to a proper segment of B. From (1) and the theorem of Problem 8-5, we conclude that

$$\text{ord}(A, \leq) < \text{ord}(B, \leq_0). \tag{2}$$

$\Rightarrow$ Condition (2) can be written as

$$\text{ord}(A, \leq) \leq \text{ord}(B, \leq_0) \tag{3}$$

and

$$\text{ord}(A, \leq) \neq \text{ord}(B, \leq_0). \tag{4}$$

From (3), we conclude that A must be order isomorphic to a segment of B; and from (4) and theorem of Problem 8-5, we conclude that A is order isomorphic to a proper segment of B.

● PROBLEM 8-11

Show that

$$\text{ord } \{k, k+1, k+2, \ldots\} = \text{ord } \{1, 2, \ldots\} \tag{1}$$

where k is any natural number.

SOLUTION:

Let us define function

$$f: \{1, 2, 3, \ldots\} \rightarrow \{k, k+1, k+2, \ldots\} \tag{2}$$

by

$$f(n) = n + k - 1 \quad \text{for} \quad n = 1, 2, 3, \ldots \tag{3}$$

Function f is a bijection and such that if $n_1 \leq n_2$ where $n_1, n_2 \in \{1, 2, 3, \ldots\}$, then

$$f(n_1) \leq f(n_2) \text{ where } f(n_1), f(n_2) \in \{k, k+1, \ldots\}.$$

Thus, f is an order isomorphism and

$$\{k, k+1, k+2, \ldots\} \approx \{1, 2, 3, 4, \ldots\} \tag{4}$$

which proves (1).

● PROBLEM 8-12

Prove that

$$\text{ord } \{1, 2, 3, \ldots\} < \text{ord } \{k, k+1, k+2, \ldots, 1, 2, 3, \ldots, k-1\}. \quad (1)$$

SOLUTION:

In other words, we have to prove that $\{1, 2, 3, \ldots\}$ is order isomorphic to a proper segment of $\{k, k+1, \ldots, 1, 2, 3, \ldots, k-1\}$.

The subset

$$\{k, k+1, k+2, \ldots\} \quad (2)$$

is a proper segment of

$$\{k, k+1, k+2, \ldots, 1, 2, \ldots, k-1\}. \quad (3)$$

In Problem 8-11, we proved that (2) is order isomorphic to

$$\{1, 2, 3, 4, \ldots\}.$$

That completes proof of (1).

● PROBLEM 8-13

What is the smallest transfinite ordinal number?

SOLUTION:

The ordinal number of the set consisting of n elements is n, assuming the set is well ordered. The infinite set with the lowest cardinal number is the set of natural numbers or any set equivalent to it.

A given set has only one cardinal number, but a set may have different ordinal numbers under distinct well orderings. The ordinal number of the set of natural numbers ordered as

$$\{1, 2, 3, 4, \ldots\}$$

is denoted by ω. Hence,

$$\omega = \text{ord } (N, \leq).$$

Any other ordering of N leads to the same or larger ordinal number. See, for example, Problem 8-12.

Hence, the smallest transfinite ordinal number is ω.

● PROBLEM 8-14

Find the ordinal sum

$$3 + 4$$

of the two finite ordinal numbers 3 and 4.

SOLUTION:

DEFINITION OF SUM

The ordinal sum $\alpha + \beta$ of ordinal numbers α and β is the ordinal number of the set $(A \cup B, \preceq')$, where $(A, \preceq)$ and $(B, \preceq_0)$ are disjoint well ordered sets, such that

$$\alpha = \text{ord}(A, \preceq) \quad \text{and} \quad \beta = \text{ord}(B, \preceq_0).$$

Relation $\preceq'$ is defined by

1. If $a, b \in A$ (or $a, b \in B$), then we write $a \preceq' b$ iff $a \preceq b$ (or $a \preceq_0 b$).

2. If $a \in A$ and $b \in B$, we write $a \preceq' b$.

■

Since

$$3 = \text{ord}\{1, 2, 3\}$$

and

$$4 = \text{ord}\ \{4, 5, 6, 7\}$$

we have

$$3 + 4 = \text{ord}\ \{1, 2, 3, 4, 5, 6, 7\} = 7.$$

● PROBLEM 8-15

Consider the definition of the sum of ordinal numbers given in Problem 8-14. Explain why the sets

$$(A \cup B, \preceq') \text{ and } (B \cup A, \preceq')$$

should be considered distinct.

SOLUTION:

To find the sum of ordinal numbers α and β, we take two well ordered

disjoint sets A and B, such that

$$\alpha = \text{ord}(A, \leq), \quad \beta = \text{ord}(B, \leq_0).$$

If the sets are not disjoint we can replace them by $A \times \{x\}$ and $B \times \{y\}$. Then

$$A \approx A \times \{x\} \quad \text{and} \quad B \approx B \times \{y\}.$$

In the next step, we form the sum of the sets A and B and define well ordering, $(A \cup B, \leq')$, such that:

$$\text{if } a \in A \quad \text{and} \quad b \in B, \text{ we write } a \leq' b.$$

Hence, any element $a \in A$ is less than any element $b \in B$. While, for the sum $(B \cup A, \leq')$, any element $b \in B$ is less than any element $a \in A$, $b \leq' a$.

To avoid this whole problem, one can use different notation, such as $(A \cup B, \leq'_{A,B})$ meaning "A is less than B." Then

$$(A \cup B, \leq'_{A,B}) = (B \cup A, \leq'_{A,B}).$$

● PROBLEM 8-16

Let $k \neq 0$ denote any finite ordinal number. Show that

$$k + \omega = \omega \tag{1}$$

$$\omega + k \neq \omega. \tag{2}$$

SOLUTION:

We have

$$k = \text{ord } \{1, 2, \ldots, k\} \tag{3}$$

$$\omega = \text{ord } \{k + 1, k + 2, \ldots\} \tag{4}$$

(see Problem 8-11.) Thus,

$$k + \omega = \text{ord}\{1, 2, \ldots, k, k + 1, \ldots\} = \omega \tag{5}$$

Similarly, we obtain

$$\omega + k = \text{ord } \{k + 1, k + 2, \ldots, 1, 2, 3, \ldots\} > \omega \tag{6}$$

(see Problem 8-12.) Thus,

$$\omega + k \neq \omega. \tag{7}$$

In general, addition of ordinal numbers is not communicative.
It is rather amazing that

$$\omega + k \neq k + \omega. \tag{8}$$

Let (A1, $\leq_1$), (A_2, $\leq_2$), (B_1, $\leq_1'$), and (B_2, $\leq_2'$) represent well ordered sets, such that

$$A_1 \cap B_1 = A_2 \cap B_2 = \phi \text{ and}$$

$$A_1 \approx A_2, \quad B_1 \approx B_2. \tag{1}$$

Prove that

$$(A_1 \cup B_1, \leq_1^*) \approx (A_2 \cup B_2, \leq_2^*). \tag{2}$$

SOLUTION:

Since $A_1 \approx A_2$ and $B_1 \approx B_2$, we have: bijection $f: A_1 \to A_2$ exists, such that

$$(a_1 \leq_1 a_2) \Rightarrow (f(a_1) \leq_2 f(a_2)) \tag{3}$$

and bijection $g: B_1 \to B_2$ exists, such that

$$(b_1 \leq_1' b_2) \Rightarrow (g(b_1) \leq_2' g(b_2)). \tag{4}$$

Functions f and g are order isomorphisms. We define

$$F: A_1 \cup B_1 \to A_2 \cup B_2 \tag{5}$$

by

$$F(x) = \begin{cases} f(x) \text{ if } x \in A_1 \\ g(x) \text{ if } x \in B_1. \end{cases} \tag{6}$$

Remember that $A_1 \cap B_1 = A_2 \cap B_2 = \phi$. It is easy to verify that F is a bijection. We define well orderings $\leq_1^*$ and $\leq_2^*$ according to the definition of Problem 8-14.

$$\text{if } x_1 \leq_1^* x_2, \text{ then} \begin{cases} x_1, x_2 \in A_1, \ f(x_1) = F(x_1), \ f(x_2) = F(x_2) \\ \text{or} \downarrow \\ x_1, x_2 \in B_1, \ g(x_1) = F(x_1), \ g(x_2) = F(x_2) \\ \text{or} \downarrow \\ x_1 \in A_1, x_2 \in B_1, \ F(x_1) = f(x_1) \in A_2, \ F(x_2) = g(x_2) \in B_2 \end{cases} \Rightarrow F(x_1) \leq_2^* F(x_2) \tag{7}$$

Thus, F defined in (4) and (5) is an order isomorphism and

$$(A_1 \cup B_1, \leq_1^*) \approx (A_2 \cup B_2, \leq_2^*). \tag{8}$$

Prove that ordinal sum is associative; that is, for any ordinals α, β, and ρ

$$(\alpha + \beta) + \rho = \alpha + (\beta + \rho). \tag{1}$$

SOLUTION:

It is possible to find three pair-wise disjoint sets A, B, and C, such that

$$\alpha = \text{ord}(A, \leq_1)$$

$$\beta = \text{ord}(B, \leq_2)$$

$$\rho = \text{ord}(C, \leq_3). \tag{2}$$

According to the definition of Problem 8-14, we obtain

$$\alpha + \beta = \text{ord}(A \cup B, \leq_1^*) \tag{3}$$

and

$$\beta + \rho = \text{ord}(B \cup C, \leq_2^*) \tag{4}$$

where well ordering $\leq_1^*$ means "A is less than B" and well ordering $\leq_2^*$ means "B is less than C." We obtain

$$(\alpha + \beta) + \rho = \text{ord}(A \cup B \cup C, \leq_3^*)$$

$$\alpha + (\beta + \rho) = \text{ord}(A \cup B \cup C, \leq_4^*) \tag{5}$$

but $\leq_3^*$ and $\leq_4^*$ are exactly the same orders; i.e., "A is less than B" and "B is less than C." Thus,

$$(\alpha + \beta) + \rho = \alpha + (\beta + \rho) = \alpha + \beta + \rho =$$

$$= \text{ord}(A \cup B \cup C, \leq^*). \tag{6}$$

If $x_1, x_2 \in A$, we write $x_1 \leq^* x_2$ iff $x_1 \leq_1 x_2$.

If $x_1, x_2 \in B$, we write $x_1 \leq^* x_2$ iff $x_1 \leq_2 x_2$.

If $x_1, x_2 \in C$, we write $x_1 \leq^* x_2$ iff $x_1 \leq_3 x_2$.

We write $x_1 \leq^* x_2$ if $x_1 \in A$ and $x_2 \in B \cup C$, or if $x_1 \in B$ and $x_2 \in C$.

● PROBLEM 8-19

Prove that

$$(\beta < \rho) \Rightarrow (\alpha + \beta < \alpha + \rho) \tag{1}$$

for any ordinal numbers α, β, and ρ.

SOLUTION:

Let $(A, \leq)$, $(B, \leq_1)$, and $(C, \leq_2)$ represent well ordered sets, such that

$$\alpha = \text{ord}(A, \leq)$$

$$\beta = \text{ord}(B, \leq_1)$$

$$\rho = \text{ord}(C, \leq_2) \tag{2}$$

and

$$A \cap B = \phi$$

$$A \cap C = \phi. \tag{3}$$

Since $\beta < \rho$, the set $(B, \leq_1)$ is order isomorphic to some segment of $(C, \leq_2)$, which we will denote by C_t. Hence

$$f : B \rightarrow C_t. \tag{4}$$

We form the sets $(A \cup B, \leq_1^*)$ and $(A \cup C, \leq_2^*)$ with the orders defined in Problem 8-14. Then

$$F : A \cup B \rightarrow A \cup C_t \tag{5}$$

where

$$F(x) = \begin{cases} x & \text{if } x \in A \\ f(x) & \text{if } x \in B \end{cases} \tag{6}$$

is an order isomorphism to $A \cup B$ onto a proper segment $A \cup C_t$ of $A \cup C$. Therefore,

$$\alpha + \beta < \alpha + \rho. \tag{7}$$

● PROBLEM 8-20

Prove that for any ordinal numbers α, β, and ρ

$$(\alpha + \beta = \alpha + \rho) \Rightarrow (\beta = \rho). \tag{1}$$

SOLUTION:

r.a.a. (reductio ad absurdum)

Suppose ordinal numbers exist, such that

$$\alpha + \beta = \alpha + \rho \quad \text{and} \quad \beta \neq \rho.$$

When $\beta \neq \rho$, then either $\beta < \rho$ or $\rho < \beta$.

Suppose $\beta < \rho$. From Problem 8-19, we have

$$(\beta < \rho) \Rightarrow (\alpha + \beta < \alpha + \rho), \tag{2}$$

which is a contradiction.

Rule (1) is called left cancellation. In Problem 8-21, we will show that right cancellation is not true for ordinal numbers.

● PROBLEM 8-21

Give a counter-example to show that the right cancellation rule is not true for ordinal numbers.

SOLUTION:

We have to find three ordinal numbers, such that

$$\beta + \alpha = \rho + \alpha \quad \text{and} \quad \beta \neq \rho.$$

In Problem 8-16, we proved

$$n + \omega = \omega. \tag{1}$$

Hence,

$$1 + \omega = \omega$$

and

$$2 + \omega = \omega. \tag{2}$$

We obtain

$$1 + \omega = 2 + \omega \quad \text{but} \quad 1 \neq 2.$$

● PROBLEM 8-22

Prove that for any ordinal numbers α, β, and ρ

$$(\beta < \rho) \Rightarrow (\beta + \alpha \leq \rho + \alpha). \tag{1}$$

SOLUTION:

Let $(A, \leq)$, $(B, \leq')$, and $(C, \leq'')$ represent well ordered sets, such that

$$\alpha = \text{ord}(A, \leq), \quad \beta = \text{ord}(B, \leq'), \quad \rho = \text{ord}(C, \leq'') \tag{2}$$

and

$$B \cap A = \phi \quad C \cap A = \phi. \tag{3}$$

Since $\beta < \rho$, the set B is order isomorphic to some proper segment C' of C. Hence,

$$f : B \rightarrow C' \tag{4}$$

is an order isomorphism.

We shall use the reductio ad absurdum method.

Suppose that

$$\beta + \alpha > \rho + \alpha. \tag{5}$$

Hence, $C \cup A$ is order isomorphic to some proper segment $(B \cup A)'$ of $B \cup A$. Hence,

$$g : C \cup A \rightarrow (B \cup A)' \tag{6}$$

is order-isomorphic, which leads to contradiction.

● PROBLEM 8-23

Show that relation $\leq$ cannot be replaced by $<$ in Problem 8-22.

SOLUTION:

We have to find three ordinal numbers α, β, and ρ, such that

$$\beta < \rho \tag{1}$$

and

$$\beta + \alpha < \rho + \alpha. \tag{2}$$

We shall use the helpful formula,

$$k + \omega = \omega. \tag{3}$$

Let us substitute $\beta = 1$, $\rho = 2$, and $\alpha = \omega$ to obtain

$$1 < 2 \quad \text{and} \quad 1 + \omega = \omega = 2 + \omega. \tag{4}$$

Hence, (2) is not true.

● PROBLEM 8-24

Prove that for any ordinal number α, $\alpha + 1$ is an ordinal number and

$$\alpha < \alpha + 1. \tag{1}$$

SOLUTION:

It is easy to show that for any ordinal number α,

$$\alpha + 0 = 0 + \alpha = \alpha. \tag{2}$$

Substituting $\beta = 0$ and $\rho = 1$ in equation (1) of Problem 8-22, we obtain

$$(0 < 1) \Rightarrow (\alpha + 0 < \alpha + 1). \tag{3}$$

From (2) and (3), we conclude that for any ordinal number α,

$$\alpha < \alpha + 1. \tag{4}$$

● PROBLEM 8-25

Show that the greatest ordinal number does not exist.

SOLUTION:

Suppose, on the contrary, that the greatest ordinal number exists, and denote it by δ. From Problem 8-24, we know that for any ordinal number

$$\alpha < \alpha + 1.$$

Hence,

$$\delta < \delta + 1.$$

Therefore, $\delta + 1$ is an ordinal number which is larger than δ.

Contradiction. Conclusion: the greatest ordinal number does not exist.

● PROBLEM 8-26

Let α and β denote ordinal numbers, such that $\alpha \leq \beta$. Prove that a unique ordinal number ρ exists, such that

$$\alpha + \rho = \beta. \tag{1}$$

Number ρ is sometimes denoted by $(-\alpha) + \beta$.

SOLUTION:

First we shall show that if ρ exists, then it is unique.

Suppose, on the contrary, that ρ and ρ' exist, such that

$$\alpha + \rho = \beta \quad \text{and} \quad \alpha + \rho' = \beta. \tag{2}$$

Then

$$\alpha + \rho = \alpha + \rho' \tag{3}$$

and

$$\rho = \rho'. \tag{4}$$

If $\alpha = \beta$, then $\rho = 0$. Suppose $\alpha < \beta$; then sets A, B, B_t exist, such that $A \cap B = \phi$ and

$$\alpha = \text{ord}(A, \leq), \quad \beta = \text{ord}(B, \leq_1)$$

and an order isomorphism exists

$$f: A \rightarrow B_t$$

where B_t is a proper segment of B.

Let

$$\rho = \text{ord}(B - B_t, \leq_1). \tag{5}$$

Then

$$\alpha + \rho = \beta \tag{6}$$

where $\alpha + \gamma = \text{ord}(A \cup (B - B_t), \leq^*)$.

Order isomorphism can be defined as follows:

$$F(x) = \begin{cases} f(x) & \text{if } x \in A \\ x & \text{if } x \in B - B_t \end{cases} \tag{7}$$

● PROBLEM 8-27

Prove that for any ordinal numbers α, β, and ρ

$$(\beta < \rho) \Leftrightarrow (\alpha + \beta < \alpha + \rho). \tag{1}$$

SOLUTION:

$\Rightarrow$ Demonstrated in Problem 8-19.

$\Leftarrow$ Since $\alpha + \beta < \alpha + \rho$, an ordinal number δ exists, such that (see Problem 8-26)

$$\alpha + \beta + \delta = \alpha + \rho. \tag{2}$$

By applying left cancellation, we obtain

$$\beta + \delta = \rho$$

which leads to

$$\beta < \rho \tag{3}$$

because $\delta \neq 0$.

● PROBLEM 8-28

Let

$$A = \{a, b, c, d\} \tag{1}$$

and

$$B = \{1, 2, 3\} \tag{2}$$

represent well ordered sets with orderings defined by:

for A $\quad a \leq b \leq c \leq d$

and for B $\quad 1 \leq 2 \leq 3$.

Find any well ordering of the Cartesian product of A and B.

SOLUTION:

Let us denote the sets by

$$(A, \leq) \text{ and } (B, \leq_1).$$

We have to determine when

$$(a, b) \leq^* (c, d) \tag{3}$$

where $a, c \in A$ and $(b, d) \in B$.

The elements of $A \times B$ can be well ordered as follows:

$$(a, 1) \leq^* (a, 2) \leq^* (a, 3) \leq^* (b, 1) \leq^* (b, 2) \leq^*$$

$$\leq^* (b, 3) \leq^* (c, 1) \leq^* (c, 2) \leq^* (c, 3) \leq^* (d, 1) \leq^*$$

$$\leq^* (d, 2) \leq^* (d, 3).$$

It is obvious that the set $(A \times B, \leq^*)$ is well ordered.

There are many ways (exactly 12!) of well ordering the set $A \times B$. We have just chosen one possibility.

● PROBLEM 8-29

Let $(A, \leq)$ and $(B, \leq_1)$ denote well ordered sets. Define the lexicographic (dictionary type) ordering of $A \times B$.

SOLUTION:

We construct the Cartesian product of the sets A and B. The lexicographic ordering is defined as follows:

$$(a,b) \leq^* (c,d) \text{ iff } \begin{cases} a < c \\ \text{or} \\ a = c \text{ and } b \leq_1 d \end{cases}$$

where $a, c \in A \quad b, d \in B.$ (1)

If both sets $(A, \leq)$ and $(B, \leq_1)$ are well ordered, then $(A \times B, \leq^*)$ is a well ordered set.

In a similar manner, we can impose the lexicographic ordering on the Cartesian product of n well ordered sets $(A_1, \leq_1), (A_2, \leq_2), \ldots, (A_n, \leq_n)$.

$$(x_1, x_2, \ldots x_n) \leq^* (y_1, \ldots, y_n) \text{ iff } \begin{cases} x_1 <_1 y_1 \\ x_1 = y_1 \text{ and } x_2 <_2 y_2 \\ x_1 = y_1 \text{ and } x_2 = y_2 \text{ and } x_3 <_3 y_3 \\ \vdots \\ x_1 = y_1 \text{ and } \ldots x_{n-1} = y_{n-1} \text{ and } x_n <_n y_n \end{cases}$$

where $x_1, y_1, \in A_1, x_2, y_2 \in A_2, \ldots, x_n, y_n \in A_n$.

● PROBLEM 8-30

Prove the following:

THEOREM

If $(A, \leq)$ and $(B, \leq_1)$ are well ordered sets, then the set $(A \times B, \leq^*)$ where $\leq^*$ is the lexicographic ordering is a well ordered set.

■

SOLUTION:

Since both sets A and B are totally ordered for any two pairs $(a,b) \in A \times B$ and $(c, d) \in A \times B$, we have either $(a, b) \leq^* (c, d)$ or $(c, d) \leq^* (a, b)$.

The set $(A \times B, \leq^*)$ is totally ordered.

To show that $(A \times B, \leq^*)$ is well ordered, we must show that every subset $\phi \neq T \subset A \times B$ contains the least element. Consider the following projection:

$$p_A(T) = \{x \in A : \text{for some } b \in B, (x, b) \in T\}. \tag{1}$$

The set $p_A(T)$ is a non-empty subset of a well ordered set $(A, \leq)$. Hence, it contains the least element; i.e., t_1. By the same token,

$$p_B(T) = \{y \in B : \text{for some } a \in A, (a, y) \in T\}$$

contains the least element; i.e., t_2. The element $(t_1, t_2) \in T$ is the least element of T.

Therefore, $(A \times B, \leq^*)$ is a well ordered set.

● PROBLEM 8-31

Prove the following:

THEOREM

Let $(A_1, \leq_1)$, $(A_2, \leq_2)$, $(B_1, \leq_1')$, and $(B_2, \leq_2')$ denote well ordered sets, such that

$$(A_1, \leq_1) \approx (A_2, \leq_2) \tag{1}$$

and

$$(B_1, \leq_1') \approx (B_2, \leq_2') \tag{2}$$

then

$$(A_1 \times B_1, \leq_1^*) \approx (A_2 \times B_2, \leq_2^*) \tag{3}$$

where $\leq_1^*$ and $\leq_2^*$ are lexicographic orderings.

■

SOLUTION:

Since $A_1 \approx A_2$ and $B_1 \approx B_2$, order isomorphisms exist, such that

$$f : A_1 \rightarrow A_2 \text{ and } g : B_1 \rightarrow B_2. \tag{4}$$

We will show that the function

$$F : A_1 \times B_1 \rightarrow A_2 \times B_2 \tag{5}$$

defined by

$$F(x, y) = (f(x), g(y)) \text{ for all } (x, y) \in A_1 \times B_1$$

is an order isomorphism.

Both functions f and g are bijections, thus F is a bijection. To complete the proof, we must show that

$$((a, b) \leq_1^* (c, d)) \Rightarrow (F(a, b) \leq_2^* F(c, d)) \tag{6}$$

where $(a, b), (c, d) \in A_1 \times B_1$.

If

$$(a, b) \leq_1^* (c, d) \tag{7}$$

then two possibilities exist:

1. $a <_1 c$ and hence, f is an order isomorphism

$$f(a) <_2 f(c). \tag{8}$$

Then

$$F(a, b) = (f(a), g(b)) \leq_2^* (f(c), g(d)) = F(c, d). \tag{9}$$

2. $a = c$ and $b \leq_1' d$. Functions f and g are order isomorphisms. Thus $f(a) = f(c)$ and $g(b) \leq_2' g(d)$, and

$$F(a, b) = (f(a), g(b)) \leq_2^* (f(a), g(d)) = F(c, d). \tag{10}$$

Thus, (6) is proven and F is an order isomorphism.

● PROBLEM 8-32

Define the ordinal product $\alpha\ \beta$ of ordinal numbers α and β, and explain why the ordinal product is not commutative.

SOLUTION:

DEFINITION

Let α and β denote ordinal numbers, the ordinal product $\alpha\ \beta$ is defined by

$$\alpha\ \beta = \text{ord}(B \times A, \leq^*) \tag{1}$$

where $(A, \leq)$ and $(B, \leq_1)$ are well ordered sets, such that

$$\alpha = \text{ord}(A, \leq) \quad \beta = \text{ord}(B, \leq_1)$$

and $\leq^*$ is the lexicographic ordering of $B \times A$.

■

According to the definition above, the product $\alpha\,\beta$ is different from $\beta\,\alpha$. In general, the Cartesian product

$$A \times B \neq B \times A. \tag{2}$$

Furthermore, the lexicographic ordering of $A \times B$ is different from the lexicographic ordering of $B \times A$.

● PROBLEM 8-33

Show that

$$2\omega \neq \omega 2. \tag{1}$$

SOLUTION:

Let $(N, \leq) = \{1, 2, 3, \ldots\}$ and $(B, \leq_1) = \{a, b\}$, so that

$$\text{ord}(N, \leq) = \omega \quad \text{ord}(B, \leq_1) = 2 \tag{2}$$

Hence,

$$2\omega = \text{ord}(N \times B, \leq^*) \tag{3}$$

where $\leq^*$ is the lexicographic ordering.

Let us define the function

$$f : N \times B \rightarrow N \tag{4}$$

$$f(n, a) = 2n - 1$$

$$f(n, b) = 2n.$$

It is easy to see that f is an order isomorphism. Thus

$$2\omega = \omega. \tag{5}$$

Now we shall form the set $(B \times N, \leq^*)$ with the lexicographic ordering.

$$\text{ord}(B \times N, \leq^*) = \omega 2 \tag{6}$$

$$(B \times N, \leq^*) = \{(a, 1), (a, 2), \ldots, (b, 1), (b, 2), \ldots\} \tag{7}$$

We have

$$\text{ord}(B \times N, \leq^*) = \text{ord}((\{a\} \times N) \cup (\{b\} \times N), \leq^*) =$$

$$= \text{ord}(\{a\} \times N, \leq_1) + \text{ord}(\{b\} \times N, \leq_2) =$$

$$= \omega + \omega. \tag{8}$$

Thus,

$$2\omega \neq \omega 2. \tag{9}$$

● PROBLEM 8-34

Prove that for any finite ordinal k

$$k\omega = \omega. \tag{1}$$

SOLUTION:

Let $(A, \leq_1)$ denote a well ordered set

$$(A, \leq_1) = \{a_1, a_2, \ldots, a_k\}$$

and $(N, \leq_2) = \{1, 2, 3, \ldots\}$

$$\omega = \text{ord}(N, \leq_2) \tag{2}$$

Then

$$k\omega = \text{ord}(N \times A, \leq^*) \tag{3}$$

where $\leq^*$ is the lexicographic ordering. Consider the function

$$f : N \times A \rightarrow N \tag{4}$$

defined by

$$f(m, a_n) = k(m - 1) + n \tag{5}$$

where m is any natural number and $n = 1, 2, \ldots, k$.

Function f, defined by (4) and (5), is one-to-one and onto, hence it is a bijection.

Furthermore,

$$\left((n_1, a_{l_1}) \leq^* (n_2, a_{l_2})\right) \Rightarrow \left(f(n_1, a_{l_1}) \leq f(n_2, a_{l_2})\right). \tag{6}$$

Function f is order isomorphic. Hence,

$$k\omega = \omega. \tag{7}$$

Show that

1. $\alpha 0 = 0\alpha = 0$ for any ordinal number α.

2. $\alpha 1 = 1\alpha = \alpha$ for any ordinal number α.

3. $\alpha\beta = 0$, if and only if either $\alpha = 0$ or $\beta = 0$.

SOLUTION:

1. $\text{ord}(A, \leq_1) = 0$, if and only if $A = \phi$. Let

$$\alpha = \text{ord}(B, \leq_2). \tag{1}$$

Then

$$\alpha 0 = \text{ord}(A \times B, \leq^*) = \text{ord}(\phi \times B, \leq^*) =$$

$$= \text{ord}(\phi, \leq^*) = 0. \tag{2}$$

Similarly,

$$\alpha 0 = \text{ord}(B \times A, \leq_1{}^*) = \text{ord}(\phi, \leq_1{}^*) = 0 \tag{3}$$

2. Let

$$1 = \text{ord}(\{a\}, \leq_1) \quad \alpha = \text{ord}(B, \leq_2). \tag{4}$$

Then

$$\alpha 1 = \text{ord}(\{a\} \times B, \leq_1{}^*) = \text{ord}(B, \leq_2) = \alpha. \tag{5}$$

Similarly,

$$1\alpha = \text{ord}(B \times \{a\}, \leq_2{}^*) = \text{ord}(B, \leq_2) = \alpha. \tag{6}$$

3. We shall prove that

$$(\alpha\beta = 0) \Leftrightarrow (\alpha = 0 \text{ or } \beta = 0). \tag{7}$$

$\Leftarrow$ was proven in part 1 of this problem.

$\Rightarrow$ Suppose $\alpha\beta = 0$. But

$$\text{ord}(A, \leq) = 0, \text{ if and only if } A = \phi.$$

On the other hand,

$$\alpha\beta = \text{ord}(B \times A, \leq^*). \tag{8}$$

Hence,

$$B \times A = \phi.$$

Therefore, either $A = \phi$ or $B = \phi$.

● PROBLEM 8-36

Prove the following:

THEOREM

Let α, β, and ρ denote ordinal numbers, such that $0 < \rho$. Then

$$(\alpha < \beta) \Rightarrow (\rho\alpha < \rho\beta). \tag{1}$$

■

SOLUTION:

Let $(A, \leq_1)$, $(B, \leq_2)$, and $(C, \leq_3)$ denote well ordered sets, such that

$$\alpha = \text{ord}(A, \leq_1), \quad \beta = \text{ord}(B, \leq_2), \quad \rho = \text{ord}(C, \leq_3). \tag{2}$$

Since $\alpha < \beta$, $b \in B$ exists, such that

$$A \approx B_b. \tag{3}$$

We form the sets $(A \times C, \leq_1^*)$ and $(B \times C, \leq_2^*)$ with the lexicographic orderings. Hence,

$$(A \times C, \leq_1^*) \approx (B_b \times C, \leq_2') \tag{4}$$

where $\leq_2'$ is ordering inherited from $(B \times C, \leq_2^*)$. Let $c \in C$ denote the least element of C.

The set $B_b \times C$ is the segment of $B \times C$

$$B_b \times C = \{(x, y) \in B \times C : (x, y) \leq_2'' (b, c)\}. \tag{5}$$

Hence,

$$\rho\alpha < \rho\beta. \tag{6}$$

● PROBLEM 8-37

Show that, for any ordinal numbers α, β, and ρ, such that $0 < \rho$

$$(\rho\alpha = \rho\beta) \Rightarrow (\alpha = \beta). \tag{1}$$

SOLUTION:

Assume, on the contrary, that the ordinal numbers α, β, and ρ exist, such that $\rho > 0$ and $\rho\alpha = \rho\beta$ and $\alpha \neq \beta$.

Suppose $\alpha < \beta$.

Then by Problem 8-36, we obtain

$$\rho\alpha < \rho\beta \tag{2}$$

which is a contradiction.

It should be noted that ordinal multiplication is not right cancellative. The ordinal numbers α, β, and $0 < \rho$ exist, such that

$$\alpha\rho = \beta\rho, \quad \text{but} \quad \alpha \neq \beta.$$

For example,

$$2\omega = 1\omega \quad \text{but} \quad 1 \neq 2.$$

● PROBLEM 8-38

Prove the following important theorem.

THEOREM

Let α denote any ordinal number. Then the set

$$\{\beta : \beta \text{ is an ordinal}, \beta < \alpha\} \tag{1}$$

is a well ordered set whose ordinal number is α. ■

SOLUTION:

Let

$$\text{ord}(A, \leq) = \alpha. \tag{2}$$

Then for any ordinal number $\beta < \alpha$ and any well ordered set $(B, \leq_1)$, such that

$$\beta = \text{ord}(B, \leq_1) \tag{3}$$

the set $(B, \leq_1)$ is order isomorphic to a proper segment A_b, $b \in A$, of A. Element $b \in A$ is uniquely determined by the ordinal number β. This defines a function.

$$f: \{\beta : \beta < \alpha\} \rightarrow A \tag{4}$$

defined by

$$f(\beta) = b \tag{5}$$

where

$$\beta = \text{ord}(B, \leq_1) \text{ and } A_b \approx B. \tag{6}$$

Function f is a bijection. Furthermore,

$$(\beta_1 < \beta_2) \Rightarrow (f(\beta_1) \leq f(\beta_2)). \tag{7}$$

Hence, f is an order isomorphism and the set $\{\beta : \beta < \alpha\}$ is well ordered.

$$\text{ord}\{\beta : \beta < \alpha\} = \text{ord}\, A = \alpha. \tag{8}$$

● PROBLEM 8-39

Prove the following theorem:

THEOREM

Any set of ordinal numbers is well ordered. ■

SOLUTION:

r.a.a. (reductio ad absurdum)

Suppose a set of ordinal numbers A exists which is not well ordered. Then, at least one subset $B \subset A$ exists that does not have a least element. Hence, set B must contain a strictly decreasing infinite sequence of ordinal numbers

$$\beta_1 > \beta_2 > \beta_3 > \beta_4 \ldots$$

This sequence is contained in

$$\{\beta : \beta < \beta_1\}.$$

Hence, the set $\{\beta : \beta < \beta_1\}$ is not well ordered because it contains a strictly decreasing infinite sequence. That is a contradiction.

By theorem of Problem 8-38, this set is well ordered.

● PROBLEM 8-40

Use the theorem of Problem 8-38 to explain how each ordinal number α can be identified with the set

$$\{\beta : \beta < \alpha\}.$$

SOLUTION:

Let α denote an arbitrary ordinal number. Then the set

$$\{\beta : \beta < \alpha\} \tag{1}$$

is well ordered and

$$\alpha = \text{ord}\{\beta : \beta < \alpha\}. \tag{2}$$

Hence, we can identify the ordinal number α with the set $\{\beta : \beta < \alpha\}$. Each ordinal number can be regarded as a well ordered set of ordinal numbers. We have

$0 \equiv \phi$
$1 \equiv \{0\}$
$2 \equiv \{0, 1\}$
$3 \equiv \{0, 1, 2\}$
$4 \equiv \{0, 1, 2, 3\}$
$\vdots$
$\omega \equiv \{0, 1, 2, 3, \ldots\}$
$\omega + 1 \equiv \{0, 1, 2, \ldots, \omega\}$
$\omega + 2 \equiv \{0, 1, 2, \ldots, \omega, \omega + 1\}$
$\omega 2 \equiv \{0, 1, 2, \ldots, \omega, \omega + 1, \ldots\}$
$\omega 2 + 1 \equiv \{0, 1, 2, \ldots, \omega, \omega + 1, \ldots, \omega 2\}$
$\vdots$

● PROBLEM 8-41

Russell's paradox leads to the conclusion that a set of all sets does not exist. Similar reasoning for the ordinal numbers is called the Burali-Forti paradox. Prove that:

THEOREM

A set of all ordinal numbers does not exist.

■

SOLUTION:

Suppose a set S of all ordinal numbers exists. By theorem of Problem 8-39, S is a well ordered set. Let s denote the ordinal number of S, then s

must be an element of S. We have

$$S = \text{ord}\,\{\alpha \in S : \alpha < s\} = \text{ord}\,S_s$$

$$< \text{ord}\,S = s$$

which is a contradiction.

CHAPTER 9

FUNDAMENTAL CONCEPTS OF TOPOLOGY

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1. Which of the figures are congruent?

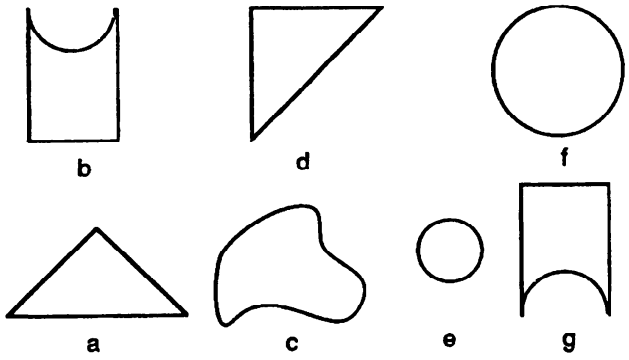


FIGURE 1

2. List a few properties of the triangle shown in Figure 2 which are geometric and a few which are not.

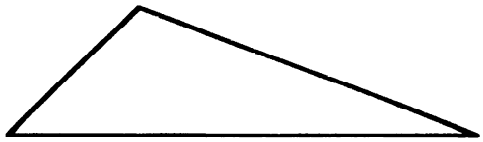


FIGURE 2

SOLUTION:

Geometry deals with figures and certain properties of figures in Euclidean space. To determine which properties are geometric and which are not, we shall introduce the notion of geometric equivalence, often called congruence.

Two figures are congruent if and only if one can be placed on top of the other, so as to fit perfectly.

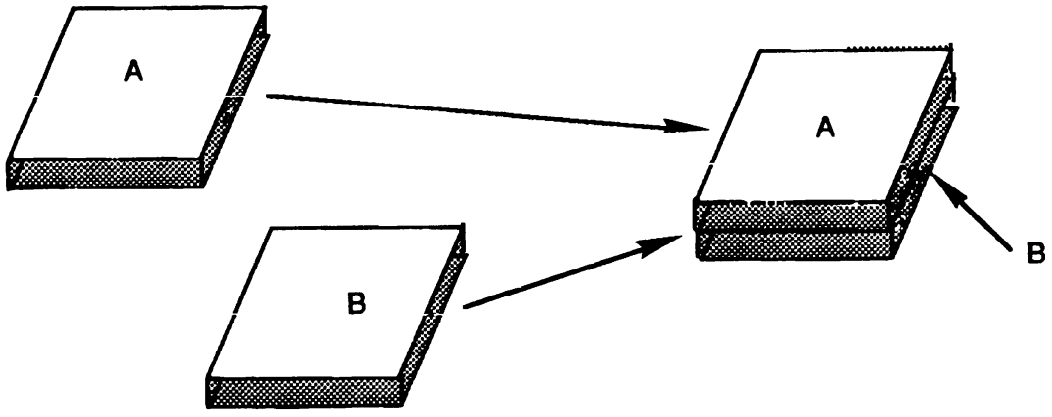


FIGURE 3

A and *B* shown in Figure 3 are congruent. The properties which are common for the congruent class of figures are called geometric properties.

1. From Figure 1, we see that *a* and *d* are congruent, *b* and *g* are congruent.

2. The geometric properties of the triangle are:
   the number of sides
   the length of its sides
   the number of angles
   the value of its angles
   the area enclosed by its perimeter.

The properties which are not geometric are:
   its color
   its orientation with respect to some given axes in the plane.

● PROBLEM 9-2

By using a rigid transformation (also called isometry), show that the triangles in Figure 1 are congruent.

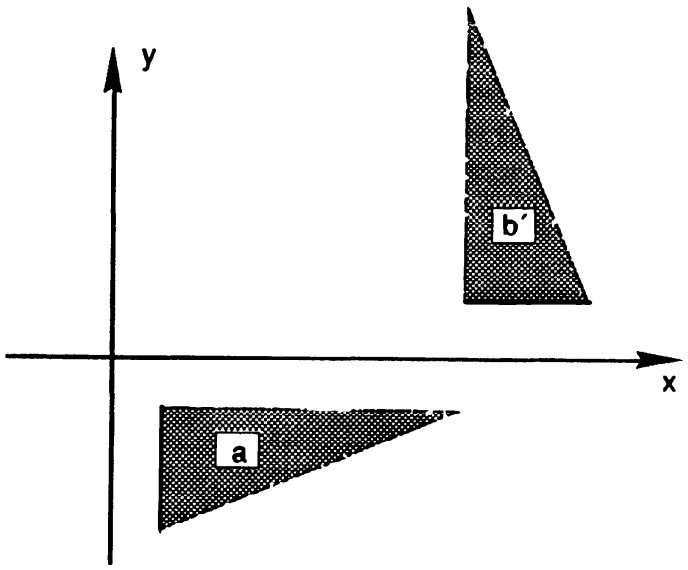


FIGURE 1

SOLUTION:

There are three fundamental rigid transformations: translation, rotation, and reflection. Figure 2 shows translation of a point P into P'.

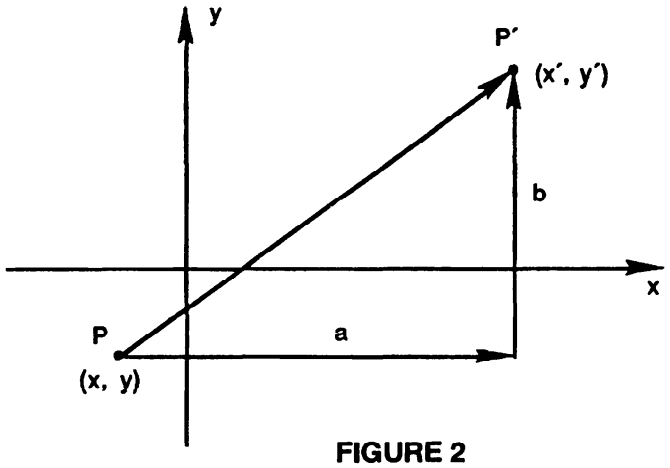


FIGURE 2

$$x' = x + a \qquad y' = y + b \tag{1}$$

or

$$(x', y') = (x, y) + (a, b) \tag{2}$$

Rotation of a point P about the origin is illustrated in Figure 3. The coordinates of P are (x, y).

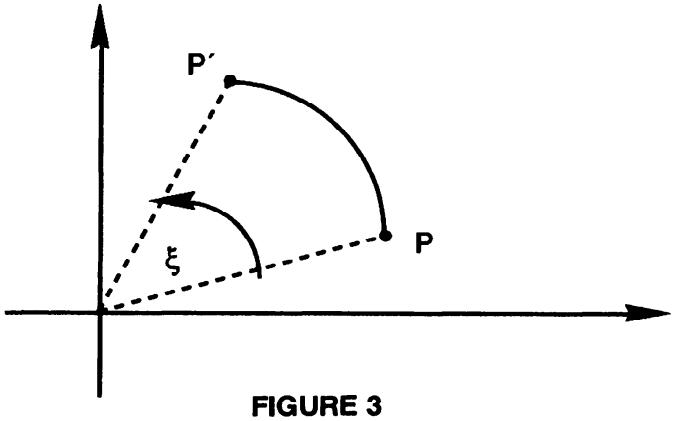


FIGURE 3

The coordinates of P' are (x', y') where

$$x' = x \cos \zeta - y \sin \zeta \qquad y' = x \sin \zeta + y \cos \zeta \tag{3}$$

Suppose reflection takes place with respect to the x axis, then

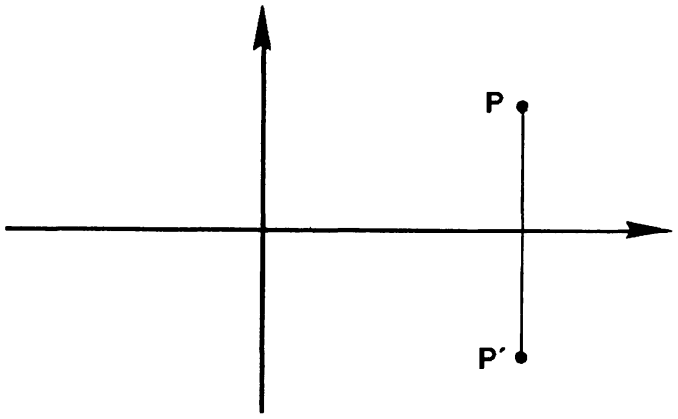


FIGURE 4

$$x' = x \qquad y' = -y$$

Now, back to the main problem (Figure 1).

Transformation yields a'. Rotation yields a'' and reflection yields a''', which is triangle b.

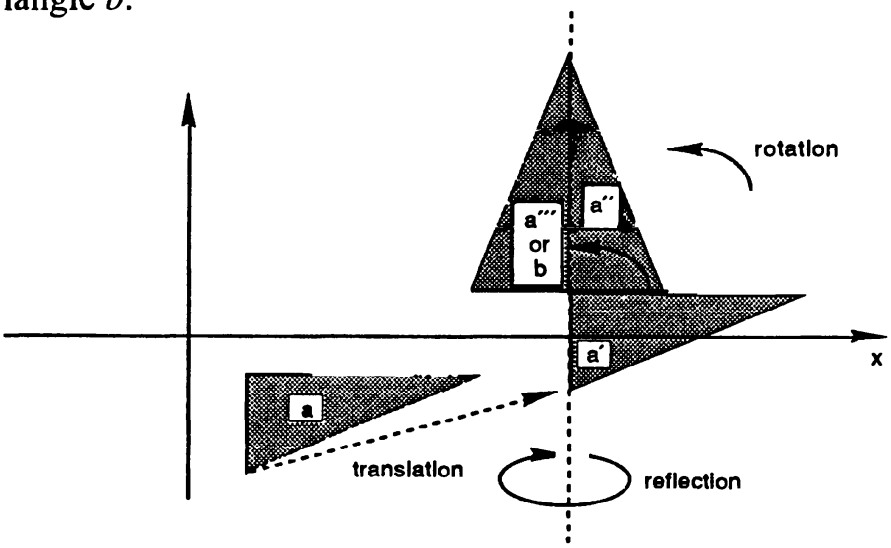


FIGURE 5

Thus, triangles a and b are congruent.

● PROBLEM 9-3

Why are rigid transformations isometric?

SOLUTION:

Isometric means preserving the distance.

Consider translation by a vector (a, b). Points A and B become A' and B'. The distance between A and B is

$$d_{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

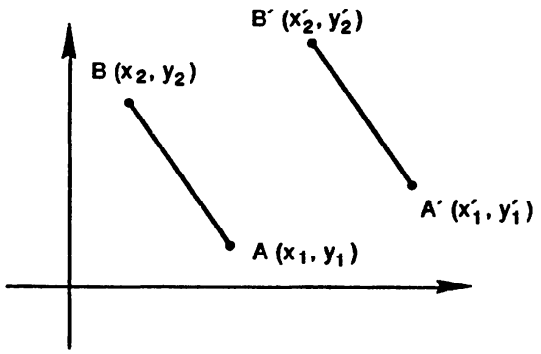


FIGURE 1

while the distance between A' and B' is

$$d_{A'B'} = \sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2} =$$

$$= \sqrt{[(x_2 + a) - (x_1 + a)]^2 + [(y_2 + b) - (y_1 + b)]^2} =$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = d_{AB} \tag{2}$$

Translation preserves the distances. Similarly, rotation preserves the distances. Consider rotation by angle ζ. From Problem 9-2, Equation (3), we obtain:

$$\sqrt{\begin{matrix}[(x_2\cos\zeta - y_2\sin\zeta) - (x_1\cos\zeta - y_1\sin\zeta)]^2 + \\ + [(x_2\sin\zeta - y_2\cos\zeta) - (x_1\sin\zeta - y_1\cos\zeta)]^2\end{matrix}} =$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{3}$$

As for reflection, we can always assume that reflection takes place with respect to the x axis. Then

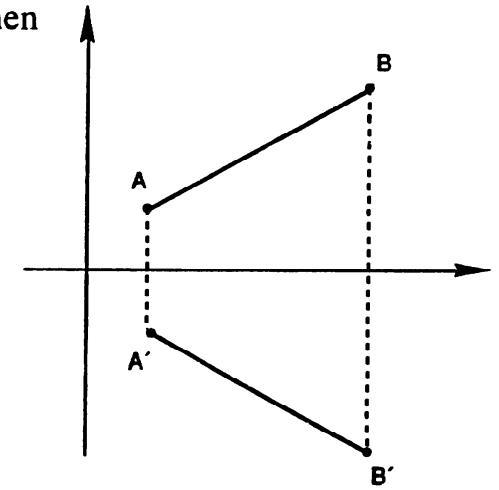


FIGURE 2

$$\sqrt{(x_2' - x_1')^2 + (y_2' - y_1')^2} = \sqrt{(x_2 - x_1)^2 + ((-y_2) - (-y_1))^2} =$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{4}$$

Conclusion: rigid transformations are isometric.

● PROBLEM 9-4

Show that the relation of congruence divides all geometric figures into disjoint classes. Explain the role that these classes play in Euclidean geometry.

SOLUTION:

Instead of writing

$$a \text{ is congruent to } b$$

we shall write

$$a \approx b. \tag{1}$$

Note that since

$$a \approx a \tag{2}$$

each geometric figure belongs to some class. Furthermore, if

$$a \approx b \text{ and } b \approx c, \text{ then } a \approx c. \tag{3}$$

Also, if $a \approx b$, then $b \approx a$.

Congruence is an equivalence relation. It separates the set of figures in Euclidean space into disjoint equivalence classes. Geometry deals with the equivalence classes and not with the particular elements of some class. Within each equivalence (congruence) class, all elements share the same geometric properties.

Suppose we are given two figures a and b. If we can point out one geometric property which these figures do not have in common, then a and b belong to two different congruence classes.

It is quite possible that these figures share many other geometric properties.

● PROBLEM 9-5

Suppose the congruence classes are given as described in Problem 9-1. Can you give an example (or two) of how to subdivide further each class?

SOLUTION:

We can take into account the location of figures in the space. For example, a class of squares of a certain size can be subdivided according to the

distance of their centers from the origin of the coordinate system. All the squares (of the same size) whose centers are located at distance p from the origin, belong to the same subclass as shown in Figure 1.

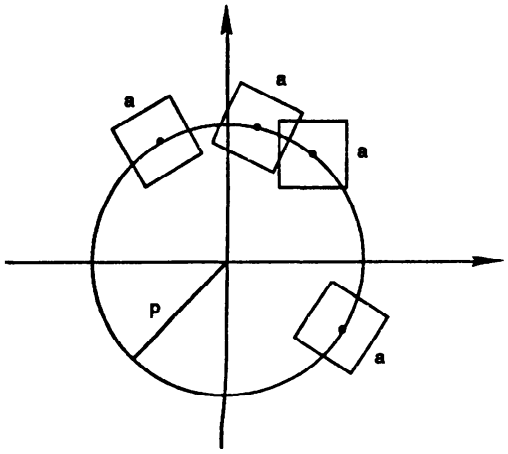


FIGURE 1

Here is another example. Suppose a line is given. The figures of some congruence class belong to the same subclass when their sides make the same angles with the line (see Figure 2).

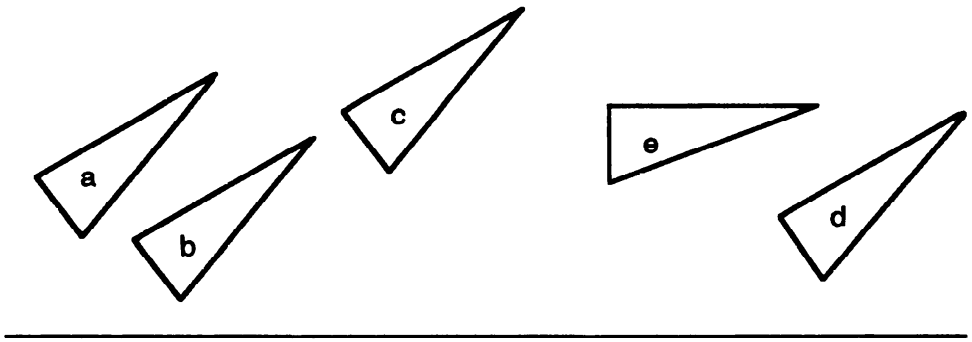


FIGURE 2

Note that triangles *a, b, c,* and *d* belong to the same subclass, while *e* does not. Often, vectors are subdivided into the classes in this manner; that is, vectors with equal length and orientation belong to the same class.

● PROBLEM 9-6

Remember the geometry described in Problem 9-4. We shall move a step further and define what might be called geometry of the magnifying glass or similarity geometry. Now, within each equivalence class, we permit right transformations and proportional magnification or contraction.

1. What are the conditions for two rectangles to belong to the same equivalence class?

2. Name a few geometric properties which remain invariant under the permitted transformations.

SOLUTION:

1. Note that all straight line segments belong to the same equivalence class in similarity geometry. Also, all squares are equivalent, and all circles are equivalent. Obviously, area is no longer an invariant.

The rectangles with the same ratio of side lengths are equivalent.

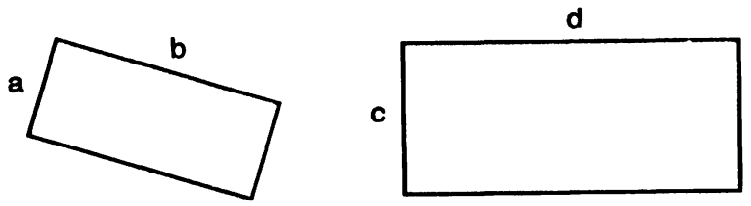


FIGURE 1

Rectangles of Figure 1 are equivalent, if and only if $^{b}/_{a} = {}^{d}/_{c}$.

2. The values of angles remain the same under the new transformations. Also, straight line segments continue to be straight line segments. In similarity geometry, the overall shape of the figures is preserved. Certain congruence classes of ordinary geometry have been combined to form equivalence classes of similarity geometry (see Figure 2).

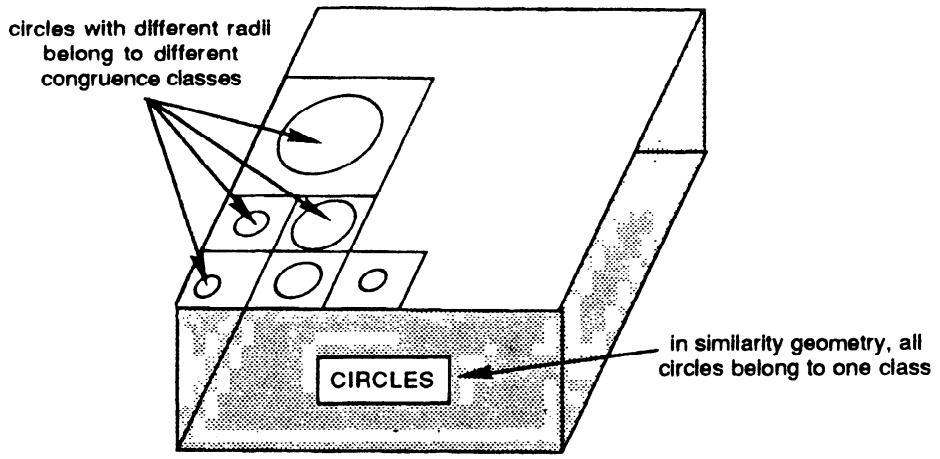


FIGURE 2

As a next step on our way to more and more general equivalence classes, we shall define affine geometry. The permitted transformations in the plane are defined now by

$$(x, y) \rightarrow (x', y') \tag{1}$$

where

$$x' = ax + by + c \quad y' = dx + ey + f. \tag{2}$$

The numbers a, b, c, d, e, and f are real and, such that

$$ae - bd \neq 0. \tag{3}$$

Show that in affine geometry the lines which were originally parallel, remain parallel although the angles are not invariant.

SOLUTION:

Suppose two parallel lines AB and CD are given, as shown in Figure 1 below:

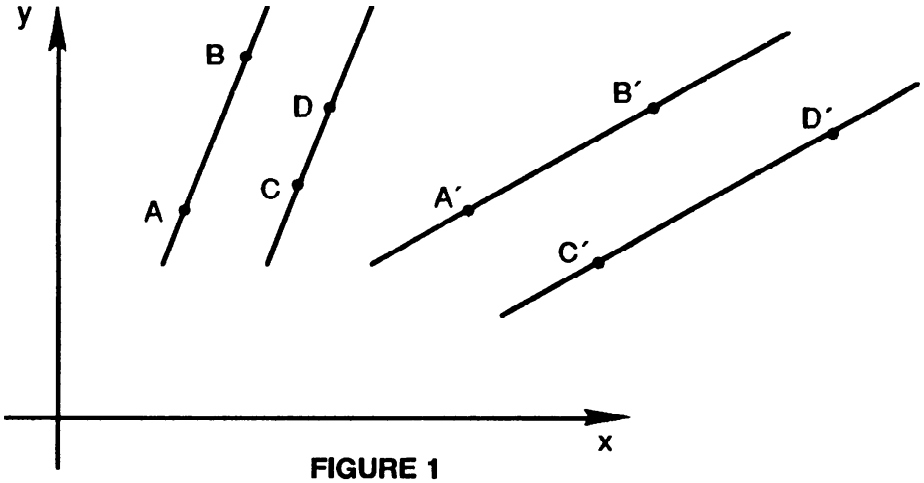


FIGURE 1

We denote the coordinates of the points A, B, C, and D by (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4). Since the lines are parallel, we have

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}. \tag{4}$$

Under transformation (2), the points A, B, C, D are transformed to A', B', C', and D' to correspond with coordinates

$$A' \ (ax_1 + by_1 + c, dx_1 + ey_1 + f) = (x_1', y_1')$$

$$B' \ (ax_2 + by_2 + c, dx_2 + ey_2 + f) = (x_2', y_2')$$

$$C' \ (ax_3 + by_3 + c, dx_3 + ey_3 + f) = (x_3', y_3')$$

$$D' \ (ax_4 + by_4 + c, dx_4 + ey_4 + f) = (x_4', y_4'). \tag{5}$$

To show that the lines $A'B'$ and $C'D'$ are parallel, we must prove that

$$\frac{y_2' - y_1'}{x_2' - x_1'} = \frac{y_4' - y_3'}{x_4' - x_3'}. \tag{6}$$

Indeed,

$$\frac{d(x_2 - x_1) + e(y_2 - y_1)}{a(x_2 - x_1) + b(y_2 - y_1)} = \frac{d + e\dfrac{y_2 - y_1}{x_2 - x_1}}{a + b\dfrac{y_2 - y_1}{x_2 - x_1}} =$$

$$= \frac{d + e\dfrac{y_4 - y_3}{x_4 - x_3}}{a + b\dfrac{y_4 - y_3}{x_4 - x_3}} = \frac{d(x_4 - x_3) + e(y_4 - y_3)}{a(x_4 - x_3) + b(y_4 - y_3)}. \tag{7}$$

Hence, lines $A'B'$ and $C'D'$ are parallel.

● PROBLEM 9-8

1. Using drawings, explain shear and strain.

2. Show that all triangles are equivalent in affine geometry and not all four-sided polygons are equivalent.

SOLUTION:

1. Shear is illustrated in Figure 1.

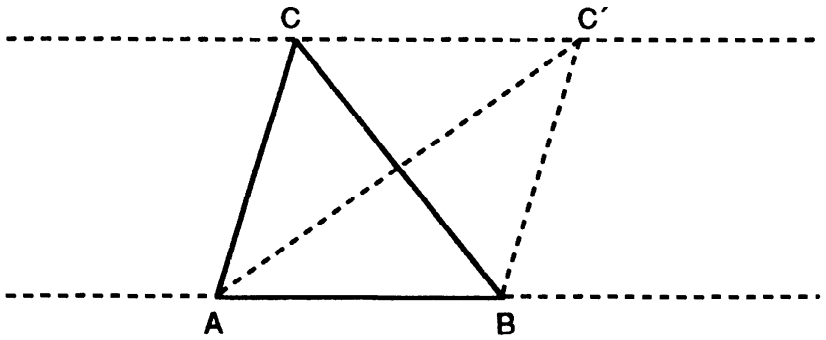


FIGURE 1

Here, the base of the triangles remains the same, while the vertex C is moved

along a line parallel to the base.

Another transformation allowed in affine geometry is strain. Here the vertex is moved along a line which is not parallel to the base. See Figure 2.

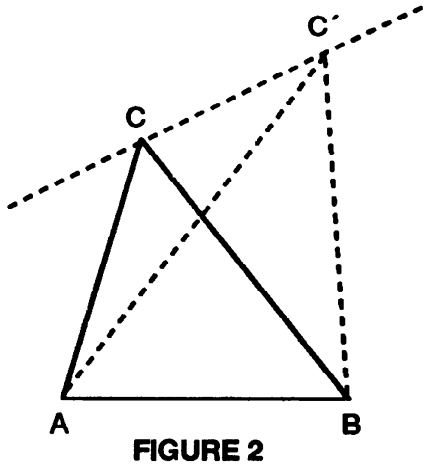


FIGURE 2

2. Suppose we are given any two triangles. Then by using only affine transformations, we can transform one triangle into another. Hence, all triangles belong to one equivalence class in affine geometry. Consider the two four-sided polygons, shown in Figure 3.



FIGURE 3

Since affine transformations transform parallel lines into parallel lines, *A* cannot be transformed into *B*. Hence, *A* and *B* belong to two different equivalence classes in affine geometry.

● PROBLEM 9-9

Explain why the equivalence classes of affine geometry are combinations of equivalence classes of similarity geometry which are, in fact, combinations of equivalence (congruence) classes of ordinary geometry.

SOLUTION:

Each geometry class is determined by the set of transformations it allows.

Ordinary Geometry	Similarity Geometry	Affine Geometry
↓	↓	↓
Rigid Transformations	Rigid Transformations and Contraction and Magnification	Rigid Transformations (Shear, Strain)

FIGURE 1

Each transformation allowed in ordinary geometry is allowed in similarity geometry, and each transformation allowed in similarity geometry is also allowed in affine geometry. The situation is illustrated in Figure 2.

Equivalence classes of

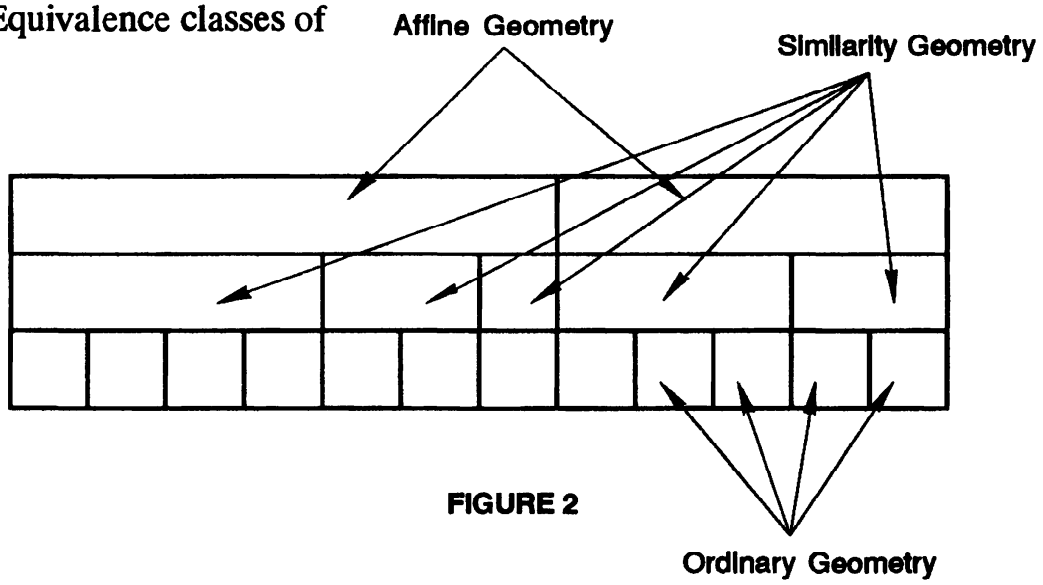


FIGURE 2

● PROBLEM 9-10

One of the invariants of affine geometry is that the parallel lines are transformed into parallel lines. Show that this is not true for projective geometry.

SOLUTION:

In projective geometry, the projective transformations are permitted. Such transformations are perspective projections of a figure.

In Figure 1, the triangle ABC in one plane is transformed into $A'B'C'$ in the other plane by projection from an exterior point. We shall also consider parallel projections, as shown in Figure 2.

Obviously, two parallel lines can be projected into two lines which are not parallel.

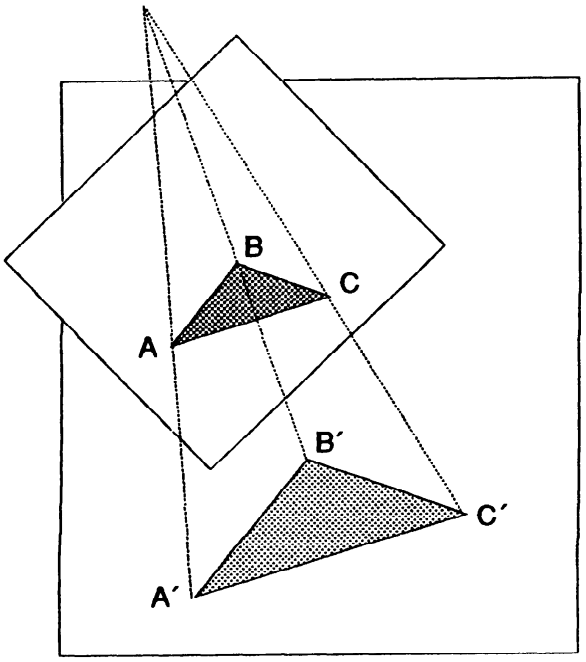


FIGURE 1

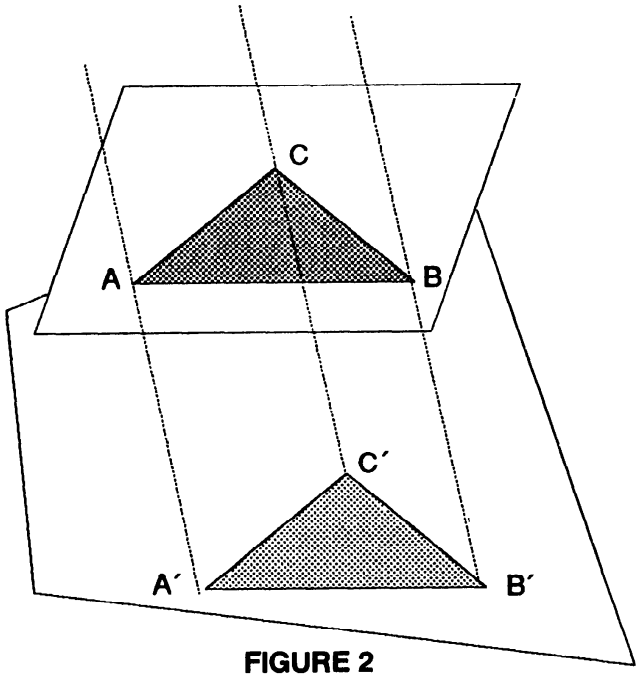


FIGURE 2

● PROBLEM 9-11

In projective geometry, straight lines remain straight lines. Show that cross-ratio is also an invariant; that is, prove that

$$\frac{AC:BC}{AD:BD} = \frac{A'C':B'C'}{A'D':B'D'}. \tag{1}$$

SOLUTION:

Here we shall prove (1) for projection from an exterior point. The proof

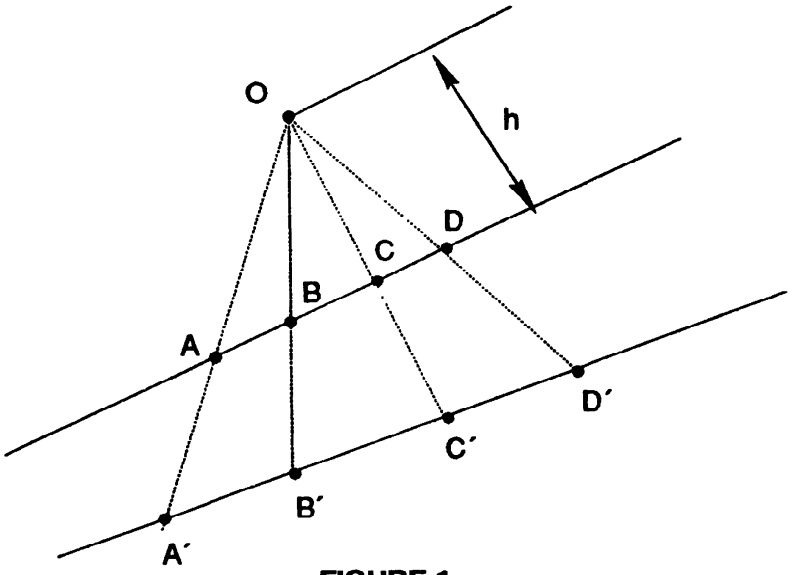


FIGURE 1

for the parallel projection is similar. Consider the triangle AOC. Its area is equal to

$$S_{AOC} = {}^1\!/_2\, h{\cdot}AC = AO{\cdot}OC \sin AOC. \tag{2}$$

We obtain similar equations for all involved triangles. Thus,

$$\frac{AC:BC}{AD:BD} = \frac{S_{AOC}:S_{BOC}}{S_{AOD}:S_{BOD}} =$$

$$= \frac{AO\cdot OC\cdot \sin AOC}{BO\cdot OC\cdot \sin BOC} \cdot \frac{BO\cdot OD\cdot \sin BOD}{AO\cdot OD\cdot \sin AOD} =$$

$$= \frac{\sin AOC\cdot \sin BOD}{\sin BOC\cdot \sin AOD}. \tag{3}$$

The final result in (3) does not depend on the choice of points A', B', C', and D'; hence, cross-ratio is an invariant.

● PROBLEM 9–12

All the geometries considered, so far, have had one important invariant in common: a straight line remained a straight line. This is not true in topology, where a square can be transformed into a circle, since we allow all one-to-one bi-continuous transformations. The new transformations, now permitted, are called elastic deformations.

Bending, stretching, and twisting is now allowed. Cutting is not allowed, unless the cuts are repaired in such a way that the points which were "near" remain "near."

Which figures shown in Figure 2 and Figure 3 belong to the same topological equivalence classes?

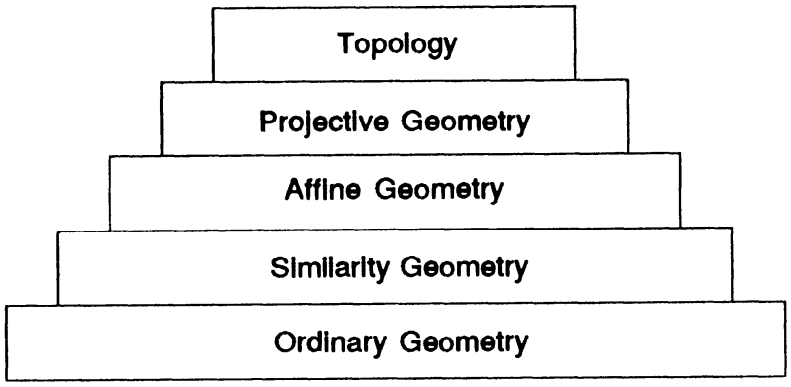


FIGURE 1

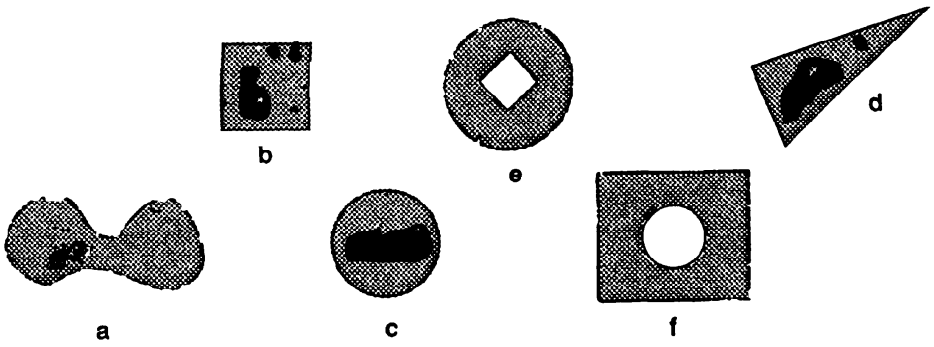


FIGURE 2

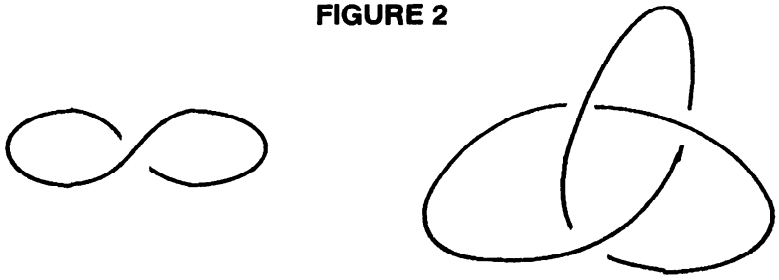


FIGURE 3

SOLUTION:

Figures *a, b, c,* and *d* of Figure 2 belong to the same topological equivalence class. Each may be transformed into any of the others by elastic deformations.

Figures *e* and *f* belong to the same class.

Figure 3 shows two plane curves which belong to the same topological class. In three-dimensional space, *a* cannot be transformed into *b* without cutting and subsequent re-connecting.

● PROBLEM 9-13

The plane closed curve C is deformed into C^{I}, then into C^{II}, then into C^{III} and finally into C^{IV}, see Figure 1. Which of the transformations are

topological?

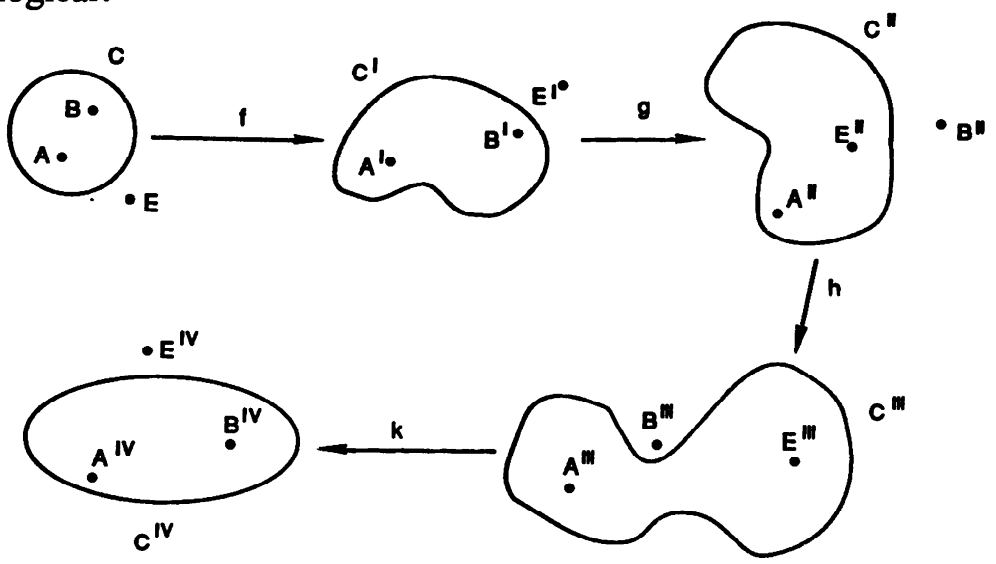


FIGURE 1

SOLUTION:

In topology the distances are much less important than in the geometries defined so far. Here we investigate non-metric spatial relationships. What is important is the preservation of the "nearness" of the points. We understand "nearness" in the topological sense. The property of separating a plane surface into a region outside and a region inside the plane closed curve is a topological invariant.

We conclude that f(of Figure 1) is a topological transformation. Points A and B inside C are transformed into A^{I} and B^{I} inside C^{1} while the point E outside C is transformed into E^{I} outside C^{I}. Transformations g and k are not topological while h is a topological transformation.

● PROBLEM 9-14

Consider the curves C_1 and C_2.

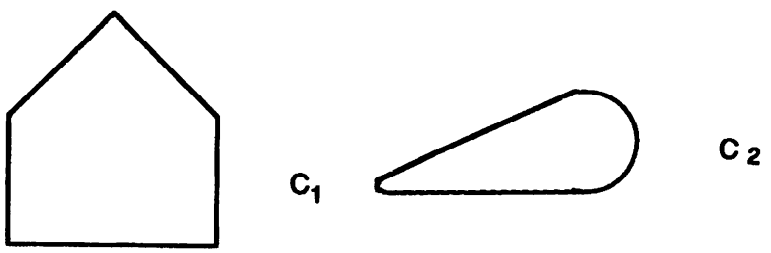


FIGURE 1

Which of the following properties are topological, which are geometric, and which are neither?

1. C_1 is above C_2.

2. The area enclosed by C_1 is larger than the area enclosed by C_2.

3. C_1 and C_2 have no common points.

4. C_1 consists of straight lines, while C_2 consists of straight lines and a semicircle.

5. C_1 has more vertices than C_2.

SOLUTION:

1. This property is neither geometric nor topological. Rotation can place C_2 above C_1.

2. The area is an invariant in ordinary geometry. But under affine transformations, the area is not preserved. This is not a topological property.

3. This is a geometric and topological property.

4. This is not a topological property. It is a geometric property in ordinary geometry.

5. It is a geometric but not topological property.

● PROBLEM 9-15

There is a sphere with a continuous closed non-self-intersecting curve C drawn on its surface. The curve C divides the surface into two disjoint regions A_1 and A_2. One cannot move from any point in A_1 to any point in A_2 without crossing C, see Figure 1.

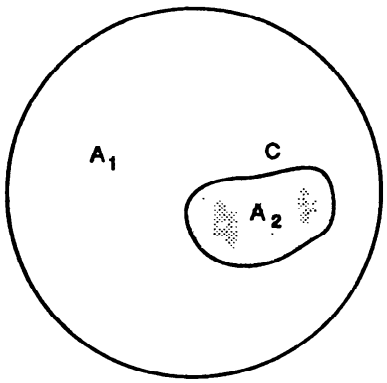


FIGURE 1

1. How many times does a path which joins a point belonging to one region with a point in the other region cross the curve C? What happens when both points belong to the same region?

2. Is it possible, by continuous transformations, to contract C to a point?

SOLUTION:

1. Figure 2 depicts a sphere with a curve C.

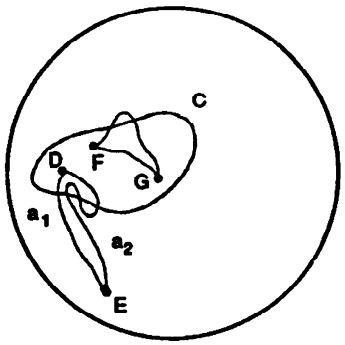


FIGURE 2

Points D and E belong to different regions; path a, crosses C once, a_2 three times. Hence, a point belonging to one region may be joined to a point belonging to the other by paths which cross C an odd number of times.

When both points belong to the same region (points F and G), the path from F to G will cross C an even number of times.

2. Curve C, which is closed, non-self-intersecting, and continuous may gradually be contracted on the surface of a sphere to a point. This is an important property, which we shall discuss later in detail.

● PROBLEM 9-16

The surface of a one-fold torus is shown in Figure 1. It has the form of a well-known doughnut. (If you are on a diet, compare it to a bicycle tire).

1. Curve C separates the surface into two disjoint regions, and it may be continuously contracted on the surface into a point. Is this true for any continuous non-self-intersecting closed curve?

2. What is the maximum number of continuous non-self-intersecting closed curves which may be drawn on the surface of a torus without dividing it?

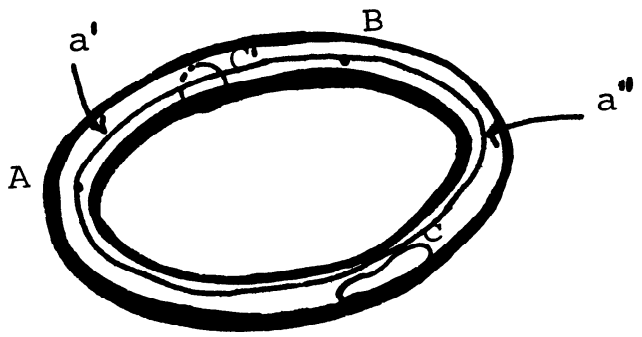


FIGURE 1

SOLUTION:

1. Consider curve C' of Figure 1. It cannot be continuously contracted into a single point. Furthermore, C' does not divide the surface of the torus into two regions. Points A and B can be joined by path a'' which does not cross C'.

The sphere and the torus belong to different topological equivalence classes. Topologically equivalent surfaces are called homeomorphic.

2. One. If we draw any other closed curve besides C (as in Figure 2), the surface of the torus will be divided. Any path AB crosses either C or C' at least once.

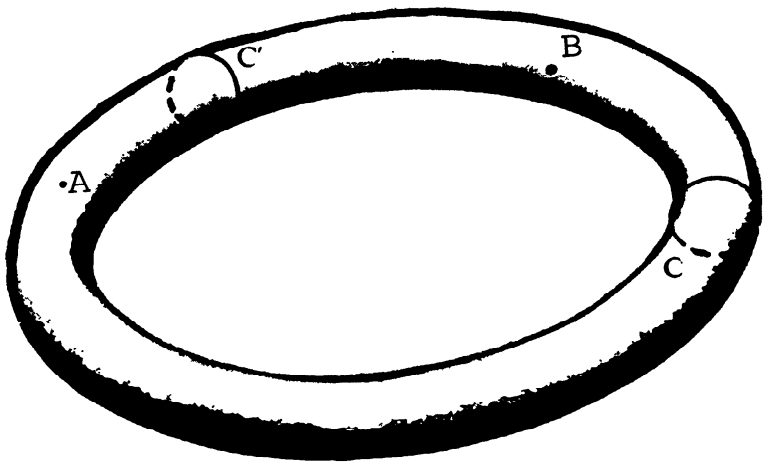


FIGURE 2

● PROBLEM 9-17

We shall move a step further and investigate the curves on the surface of a two-fold torus, see Figure 1.

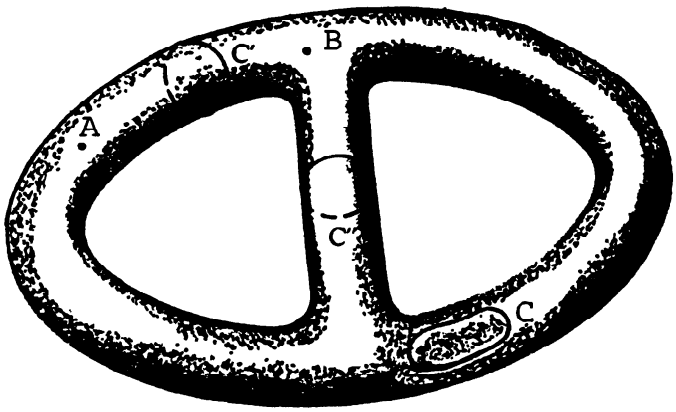


FIGURE 1

How many continuous, closed curves can be drawn on the surface of this torus without dividing it into distinct regions?

SOLUTION:

Curve C, which can be continuously contracted into a point, divides the torus into two regions. Hence, we shall investigate only the curves which cannot be contracted into a point. Draw curves C' and C''. Any two points A and B can be joined without crossing C' or C''. Adding one more curve C''' (see Figure 2) changes the situation. Any path AB has to cross at least one of the curves at least once. The maximum number of curves which do not divide a two-fold torus is two.

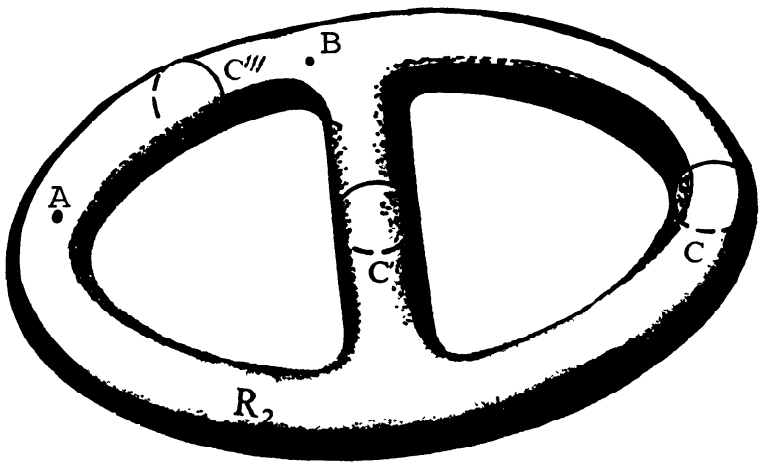


FIGURE 2

What is the genus of the surface of:

1. a sphere
2. a one-fold torus
3. an n-fold torus
4. a fork with a hot-dog on it
5. a ladder with five rungs.

SOLUTION:

The greatest number of continuous non-self-intersecting closed curves which may be drawn on a surface without dividing it into distinct regions defines the genus of the surface.

1. The genus of the surface of a sphere is 0.

2. The genus of the surface of a one-fold torus is 1.

3. Generally, the genus of an n-fold torus is n.

4. The genus of a fork with a hot dog on it is three, as shown in Figure 1.

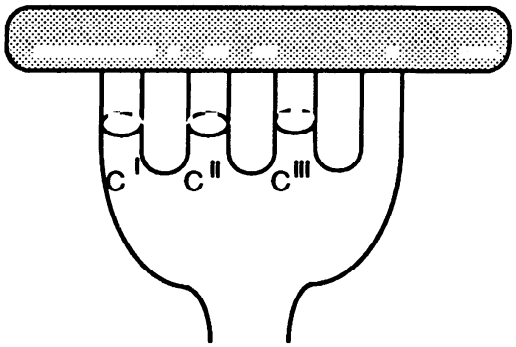


FIGURE 1

5. A ladder with five rungs has a genus of 4.

The genus of a surface is invariant under all one-to-one bicontinuous transformations; hence, it is a topological property. All surfaces belonging to the same topological equivalence class have the same genus.

● PROBLEM 9-19

Which of the following surfaces are closed and which are open?

1. a sphere
2. an n-fold torus
3. a hollow cylinder
4. a cube.

SOLUTION:

A surface is closed if it has no boundary curves. Otherwise, the surface is called open. A sphere, an n-fold torus, and a cube are closed surfaces.

The edges of a cube are not boundary curves. A hollow cylinder is an open surface. Similarly, a disc is open. Boundary curves of two-sided surfaces are curves which separate one side of a surface from the other.

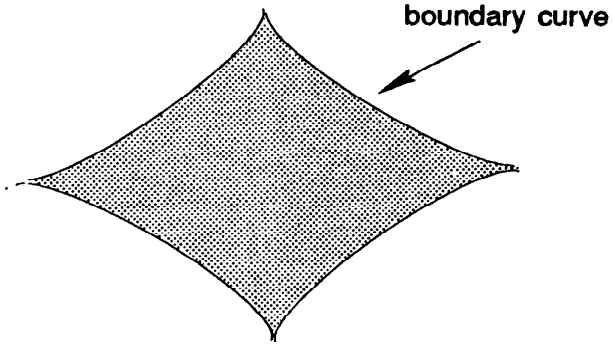


FIGURE 1

The edges of a piece of infinitely thin paper are boundary curves of this surface.

● PROBLEM 9-20

Describe the Möbius band, which is a one-sided surface. What is the boundary curve of the Möbius band? (See following figure.)

SOLUTION:

To obtain the Möbius band, take a rectangular strip of paper $ABCD$, make a 180° twist with the end DC and join up the ends. The obtained surface does not have two sides; it cannot be painted in different colors. An ant walking

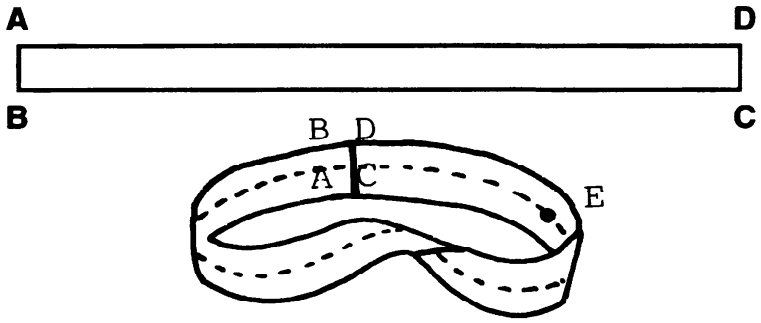


FIGURE 1

along a dotted line from a point E after making one round will end up on the other side of E without crossing the boundary line.

The Möbius band has only one boundary curve, which is topologically equivalent to a circle.

● PROBLEM 9-21

Why is the surface of a Klein bottle closed and one-sided?

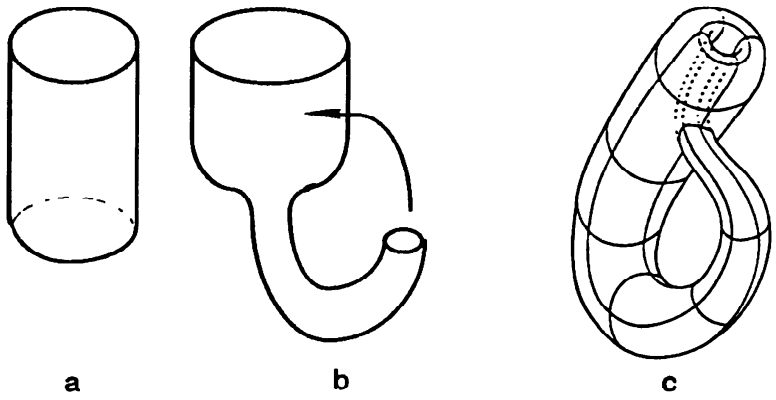


FIGURE 1

SOLUTION:

We start with an open cylinder, stretch out and bend one end, then pass through the side surface without intersecting or breaking it. At the final stage, we join up the ends.

This bottle cannot be physically constructed in the three-dimensional space. Similarly, the Möbius band, which is a two-dimensional structure, cannot be constructed in two dimensions, because of the twist. Any two points on the surface of a Klein bottle may be joined by a continuous path not crossing any boundary curve. The surface of a Klein bottle has no bound-

ary curve. Note that two open ends of the original cylinder were joined together in such a way that the outside of one was connected to the inside of the other. Hence, the outside and inside became impossible to distinguish.

The surface of a Klein bottle is one-sided and closed.

● PROBLEM 9-22

Why is a Möbius band a non-orientable surface?

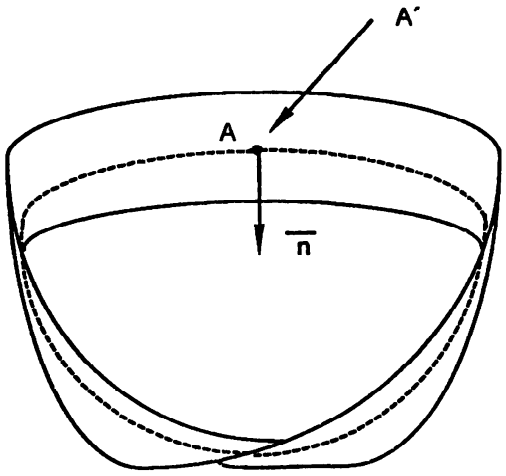


FIGURE 1

SOLUTION:

Consider a point A on the surface of a Möbius band. Let $\bar{n}$ be a normal to the surface at A. Suppose $\bar{n}$ moves along a dotted line from A to A'. The normal of A' has the opposite direction of that at A. Note that $\bar{n}$ is continuously defined as it moves from A to A'. Similarly, the surface of a Klein bottle is nonorientable.

● PROBLEM 9-23

A one-sided surface is connected if it is possible to join any two points of this surface with a continuous path. A two-sided surface is connected if its sides, taken separately, are both connected. Which of the surfaces depicted in Figure 1 is connected?

SOLUTION:

It is important to keep in mind the distinction between a solid and the surface of a solid. In most cases of a sphere, a torus, etc., we understand a surface is not a solid.

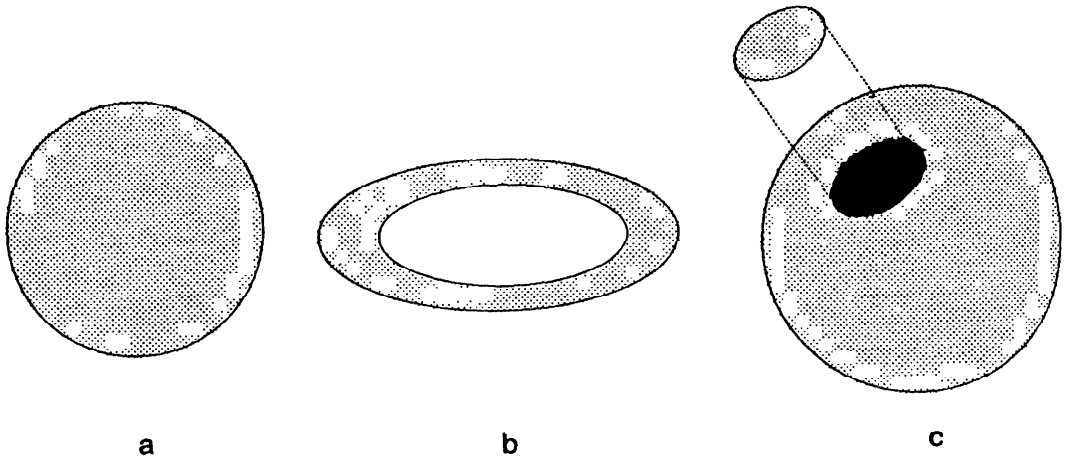


FIGURE 1

Hence, a sphere and a torus (Figure 1) are two-sided connected surfaces.

In Figure 1 *c* a disc is separated from a sphere. The total surface is disconnected. Now the total surface consists of two surfaces (each connected in itself) which are homeomorphic to a disc.

● PROBLEM 9-24

Which of these surfaces are simply connected?

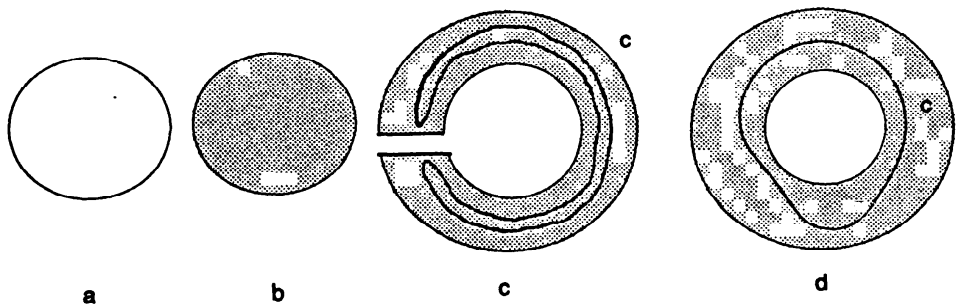


FIGURE 1

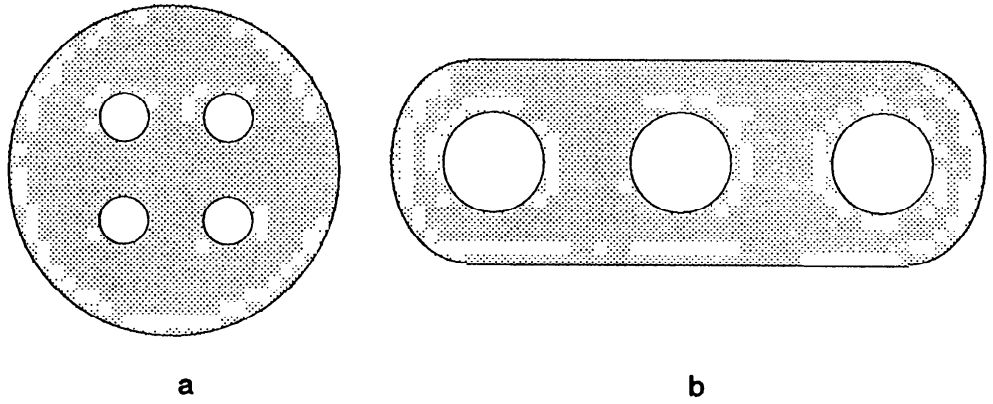
SOLUTION:

A surface is said to be simply connected if every continuous non-self-intersecting closed curve drawn on it may be continuously contracted on the surface into a point. The disc (a) and the sphere (b) are simply connected.

The surface shown in (c) is topologically equivalent to (a); hence, it is simply connected. The annulus (d) is not simply connected. Curve *C* cannot be continuously contracted to a point.

Annulus is doubly connected, since it requires only one cut to make it homeomorphic to a disc.

Which of the surfaces is quadruply connected?



a b

FIGURE 1

SOLUTION:

The surface is n-tuply connected if it requires $n - 1$ cuts in order to make it homeomorphic to a disc. Thus, annulus is doubly connected; it requires one cut to render it homeomorphic to a disc. Surface (a) of Figure 1 requires four cuts; hence, it is quintuply connected ($n = 5$).

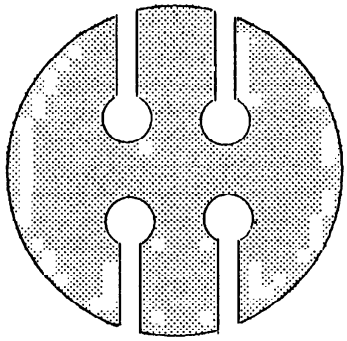
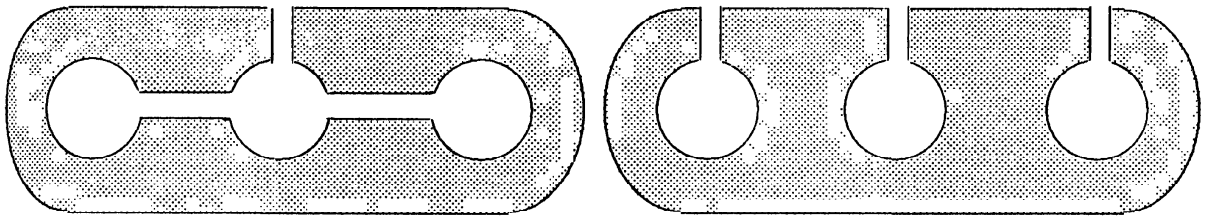


FIGURE 2

Surface (b) of Figure 1 is quadruply connected. It requires three cuts which can be made in many different ways (See Figure 3 (a) and (b)).



a b

FIGURE 3

Note that each new cut forms a boundary of the surface, and the cuts cannot intersect.

● PROBLEM 9–26

Determine n for these n-tuply connected surfaces:

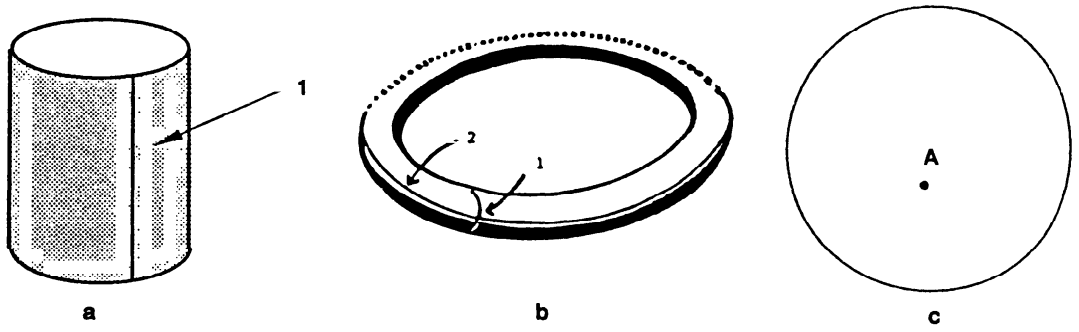


FIGURE 1

where (a) is the curved surface of a cylinder, (b) is the torus, and (c) is the sphere with one point removed.

SOLUTION:

We can cut the cylinder along line 1 and obtain a rectangle, which is homeomorphic to a disc; hence, $n = 2$.

We can cut the torus along line 1 to obtain a cylinder and then along 2. Thus, $n = 3$ and the surface of the torus is triply connected.

A sphere with a single hole is homeomorphic to a disc, $n = 1$.

● PROBLEM 9–27

Show that any two continuous non-self-intersecting closed curves on the sphere are homotopic. (Figures shown on following page.)

How many homotopy classes does the torus have?

SOLUTION:

Any continuous closed non-self-intersecting figure on a simply connected surface can be continuously contracted to a point. It follows that, on a simply connected surface, any curve may be continuously deformed into any other curve. C in Figure 1 can be continuously deformed into C'. All continuous

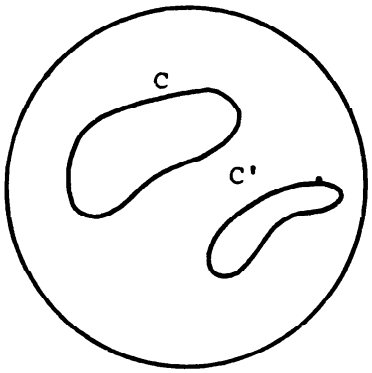


FIGURE 1

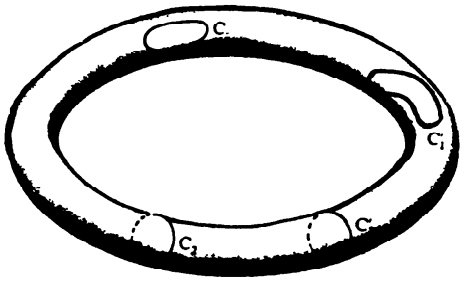


FIGURE 2

closed non-self-intersecting curves on a simply connected surface are homotopic to each other.

On the torus, C_1 can be transformed to C_1' and C_2 can be transformed to C_2'. But it is impossible to continuously transform C_1 into C_2. Curves C_1 and C_1' are homotopic, and curves C_2 and C_2' are homotopic. There are two homotopy classes for the surface of the torus.

Curves like C_1 and C_1', which may be continuously contracted to a point, belong to the null homotopy class.

● PROBLEM 9-28

What is the rank of a two-fold torus with a removed disc?

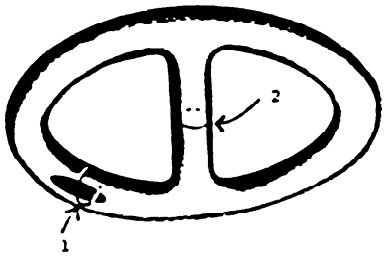


FIGURE 1

SOLUTION:

Let us start with a definition: The rank of an open surface is the least number of cuts required to make the surface homeomorphic to a disc. The rank of an n-tuply connected surface is $n - 1$.

The rank of a closed surface is the rank of the open surface obtained from the closed surface by the removal of a single disc. A disc and a sphere have rank zero. An annulus, the curved surface of a cylinder, and a Möbius band have rank 1.

Three cuts are required to make the surface shown in the figure homeomorphic to a disc.

We can also use an alternative definition of rank. Rank is the greatest number of non-intersecting cuts which can be made without making the surface disconnected.

● PROBLEM 9-29

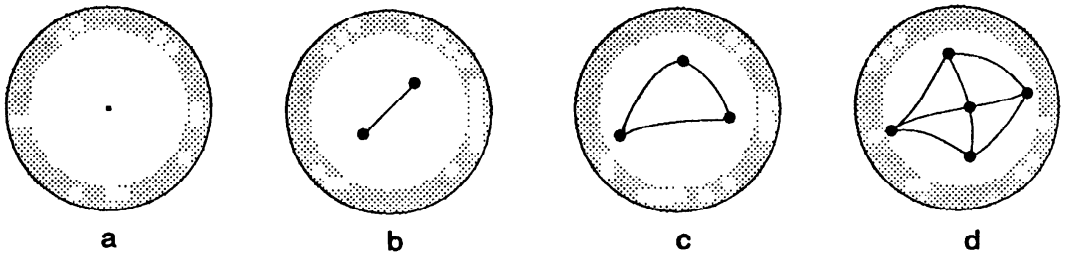


FIGURE 1

Verify the Euler rule for the figures *a, b, c,* and *d* drawn on a sphere.

SOLUTION:

Let us draw a figure having the number of vertices V of regions (or faces) F and of edges (or arcs) E, on a sphere.

Then, any figure drawn on a surface of a sphere satisfies the following rule:

$$V + F - E = 2 \tag{1}$$

discovered by Euler.

For *a*, we have:

$$V = 1, \quad F = 1, \quad E = 0 \tag{2}$$

hence $1 + 1 - 0 = 2$.

For b, we have

$$V = 2, \quad F = 1, \quad E = 1 \tag{3}$$

hence $2 + 1 - 1 = 2$.

For c, we have

$$V = 3, \quad F = 2, \quad E = 3 \tag{4}$$

hence $3 + 2 - 3 = 2$.

For d, we have

$$V = 5, \quad F = 5, \quad E = 8 \tag{5}$$

hence $5 + 5 - 8 = 2$.

In the next problem, we shall prove Euler's formula.

● PROBLEM 9-30

Prove Euler's formula:

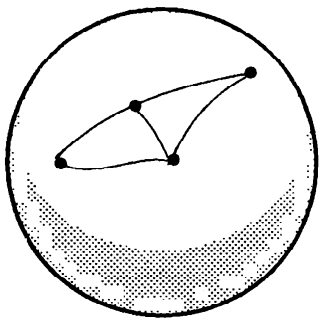
$$V + F - E = 2 \tag{1}$$

for the figures drawn on a sphere.

SOLUTION:

Suppose (1) is true for some figure depicted in Figure 1.

Let us draw a line connecting a vertex to another vertex (which does not cross any existing edges), as shown in Figure 2 a and b:



a

FIGURE 1

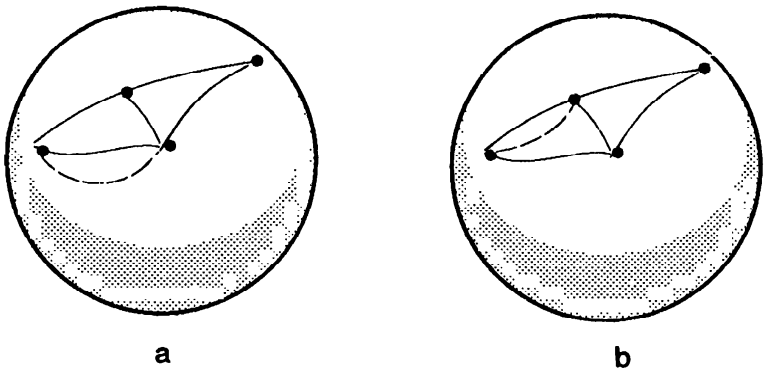


FIGURE 2

Let Δ denote the increase, then

$$\Delta V = 0, \quad \Delta F = 1, \quad \Delta E = 1. \tag{2}$$

Then

$$V + \Delta V + F + \Delta F - E - \Delta E = V + F - E = 2 \tag{3}$$

and (1) holds.

Suppose we draw a line which ends with a new vertex (Figure 3).

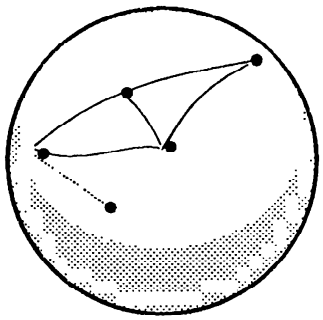


FIGURE 3

Then, $\Delta V = 1$, $\Delta F = 0$, $\Delta E = 1$ and (1) holds again.

Any figure on the sphere can be drawn, beginning with a vertex and then adding lines one after another. Hence, Euler's rule is proved.

Find one or two simple regular divisions of a sphere. A division is called regular when every face has the same number of lines f as a boundary and the same number of lines v meet at every vertex.

SOLUTION:

Euler's formula states that

$$V + F - E = 2. \tag{1}$$

Since there are F faces on the sphere, the total number of lines (edges) is

$$\frac{Ff}{2} = E \tag{2}$$

On the other hand,

$$\frac{Vv}{2} = E \tag{3}$$

From (1), (2), and (3), we obtain:

$$\frac{1}{v} + \frac{1}{f} = \frac{1}{E} + \frac{1}{2} \tag{4}$$

The simple solutions of (4) are

	v	f	E	V	F
I	$v = E$	2		2	$F = E$
II	2	$f = E$		$V = E$	2

Division I has 2 vertices and as many faces as edges. Each face is bounded by two lines, $f = 2$. The number of edges is arbitrary $E = 1, 2, 3, \ldots$ (see Figure 1).

SOLUTION I

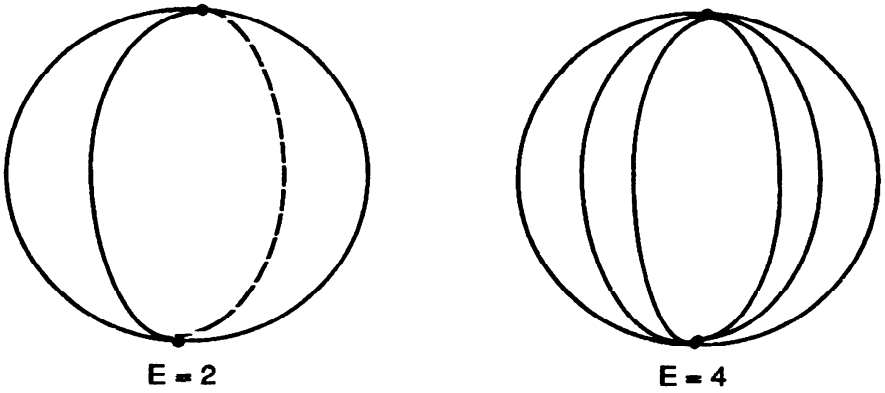


FIGURE 1

Division II has two faces and as many vertices as edges, see Figure 2. The number of edges is arbitrary $E = 1, 2, 3, \ldots$

SOLUTION II

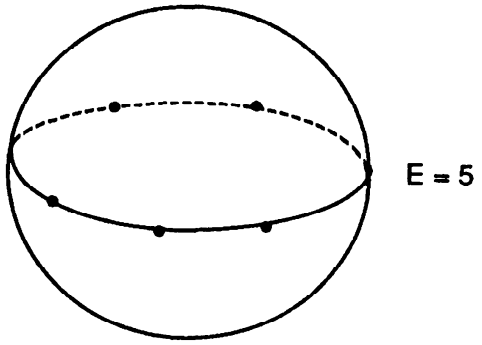


FIGURE 2

Here we have a sphere divided into two hemispheres separated by an E-gon.

● PROBLEM 9-32

Consider the equation:

$$\frac{1}{v} + \frac{1}{f} = \frac{1}{E} + \frac{1}{2} \tag{1}$$

which describes the regular divisions of a sphere. Find all possible non-trivial solutions of (1).

SOLUTION:

For $E = 1$, we obtain trivial solutions

$$v = 1, \quad f = 2 \quad \text{or} \quad v = 2, \quad f = 1.$$

For $E = 2$, the only solution is $f = v = 2$, discussed in Problem 9-31. We are left with $E = 3, 4, 5, \ldots$ Hence,

$$\frac{1}{2} < \frac{1}{v} + \frac{1}{f} < 1 \tag{2}$$

(2) excludes $f = 1$. Case $f = 2$ was already discussed.

The possible values of f are $f = 3, 4, 5$. Note that $f = 6, 7, 8, \ldots$ are incompatible with (2).

By the same token, the admissible values of v are $v = 3, 4, 5$.

f \ v	3	4	5
3	OK	OK	OK
4	OK		
5	OK		

The crossed-out values do not satisfy (2). Hence, we are left with five cases.

● PROBLEM 9-33

Find the number of faces F, edges E and vertices V for each of five regular polyhedra found in Problem 9-32.

SOLUTION:

In Problem 9-32, we found possible values of f and v:

v	f	E	F	V	Name
3	3	6	4	4	tetrahedron
4	3	12	8	6	octahedron
3	4	12	6	8	cube
5	3	30	20	12	icosahedron
3	5	30	12	20	dodecahedron

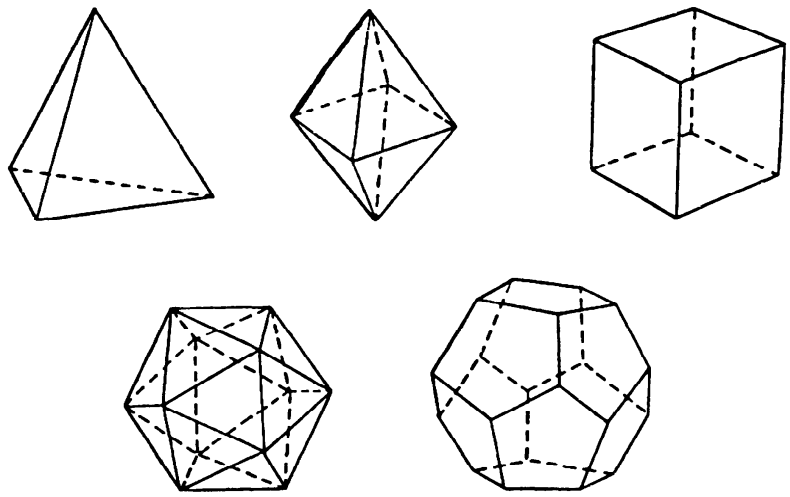
The values of E are computed from

$$\frac{1}{E} = \frac{1}{f} + \frac{1}{v} - \frac{1}{2} \tag{1}$$

The values of F and V are obtained from

$$F = \frac{2E}{f}, \qquad V = \frac{2E}{v} \tag{2}$$

Such five regular polyhedra, known as Platonic polyhedra, are depicted in the following figure.



● PROBLEM 9-34

A separation of a surface consisting of vertices linked together by edges is called a map. For any given surface, the expression

$$V - E + F = \chi \tag{1}$$

is invariant. Hence, it remains invariant for all surfaces homeomorphic to the given surface.

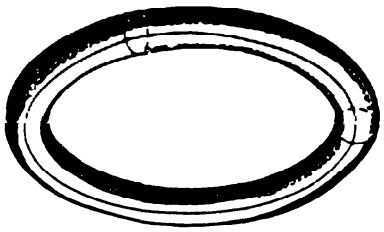


FIGURE 1

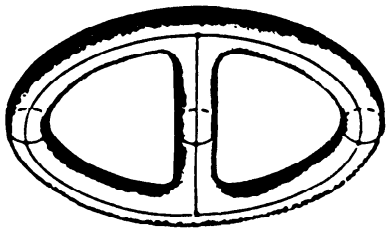


FIGURE 2

This invariant number χ is called the Euler characteristic of the surface, and it is a topological property of the surface independent of the actual values of V, E, and F. For the sphere $\chi = 2$, find χ for a torus and for a two-fold torus, see Figures 1 and 2.

SOLUTION:

The Euler characteristic remains the same for any map on a torus, so we can use the map of Figure 1.

$$V = 2 \quad E = 4 \quad F = 2. \tag{2}$$

Thus,

$$V - E + F = 2 - 4 + 2 = 0. \tag{3}$$

For torus, $\chi = 0$.

From Figure 2, for a two-fold torus

$$V = 5 \quad E = 9 \quad F = 2. \tag{4}$$

Hence, for a two-fold torus

$$\chi = 5 - 9 + 2 = -2. \tag{5}$$

● PROBLEM 9-35

1. Figure 1 depicts regular polyhedra. Find the Euler characteristic for each one.

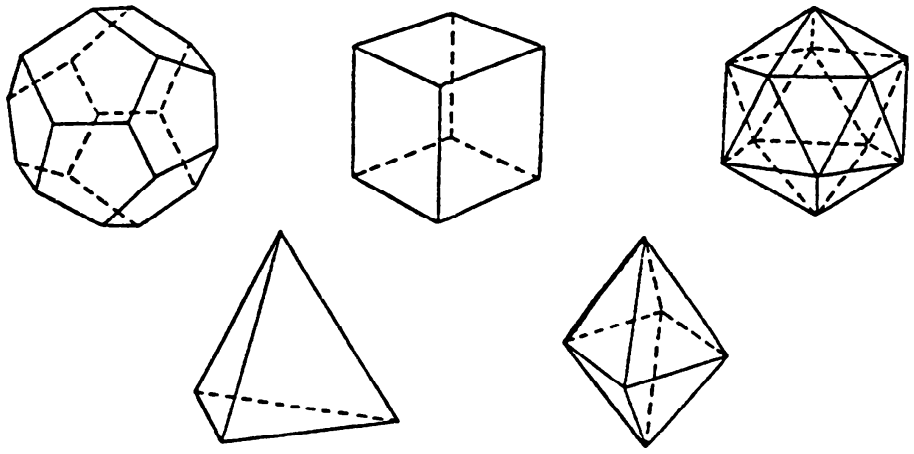


FIGURE 1

2. Is it possible

a) to draw a map on a sphere having seven vertices linked by ten arcs and defining four regions?

b) to draw a map on a two-fold torus having five vertices linked by eight arcs and defining two regions?

SOLUTION:

1. Consider the tetrahedron

$$V = 4 \quad E = 6 \quad F = 4.$$

Hence,

$$V - E + F = 4 - 6 + 4 = 2. \tag{1}$$

We can perform calculations for all polyhedra and in each case, obtain $\chi = 2$. This result is obvious and does not require any computing. Any simple polyhedron can be continuously deformed into a sphere; hence, $\chi = 2$.

2. a) For a sphere

$$V - E + F = 2$$

$$7 - 10 + 4 = 1 \neq 2 \tag{2}$$

Not possible.

b) For a two-fold torus

$$V - E + F = -2$$

$$5 - 8 + 2 = -1 \neq -2 \tag{3}$$

It is not possible to draw such a map.

● PROBLEM 9-36

Use the triangulation method to prove Euler's formula

$$V - E + F = 2 \tag{1}$$

for a cube.

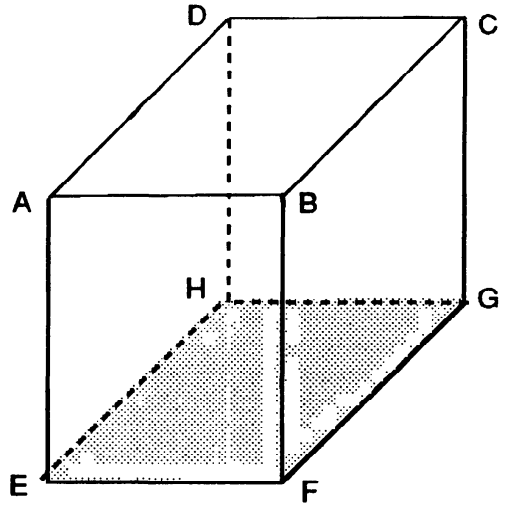


FIGURE 1

SOLUTION:

From the cube *ABCDEFGH*, we have to remove any face, say *EFGH*. The Euler characteristic $\chi = V - E + F$ for a cube is now decreased by 1 since V

and E remain constant and F decreases by 1. The cube with the removed face is transformed until all vertices lie in a plane (see Figure 2).

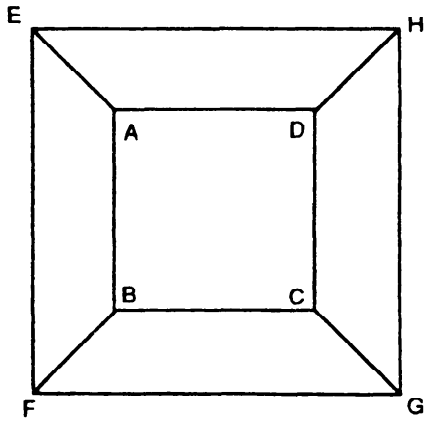


FIGURE 2

This deformation does not change χ. Now we shall carry out the triangulation as shown in Figure 3.

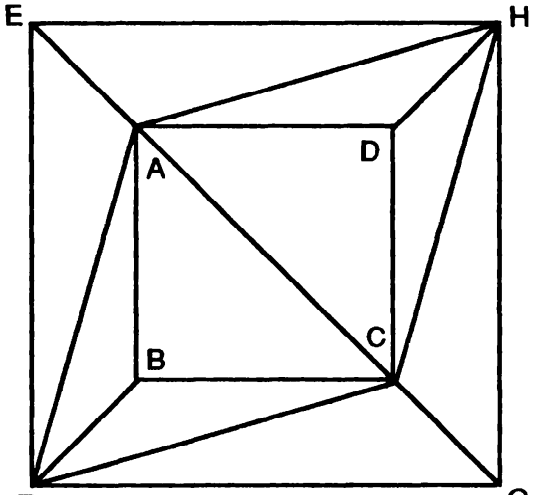


FIGURE 3

We have $V = 8, E = 17, F = 10$.

Note that whenever we add a triangle, V remains constant, whereby E and F increase by 1. Hence, χ remains the same.

Triangles are now removed until one remains; e.g., ABC, for which

$$V = 3, \quad E = 3, \quad F = 1$$

Hence,

$$V - E + F = 1$$

for a triangle and

$$V - E + F = 1 + 1 = 2$$

for a cube.

The value of $V - E + F$ remains invariant within any topological equivalence class.

Describe the method of triangulation applied to prove Euler's formula

$$V - E + F = 2$$

for polyhedra.

SOLUTION:

From a polyhedron with V vertices, E edges, and F faces we remove one face. Removal of one face decreases $V - E + F$ by one.

The new surface is deformed until all vertices and edges lie in a plane. We obtain something that is topologically equivalent to a disc. Now we perform the triangulation. Each face is divided into a triangle in such a way that no new vertices are created (Figure 1).

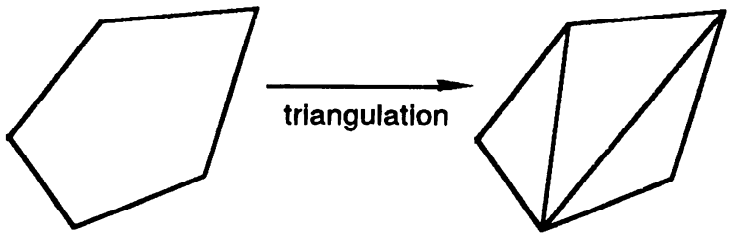


FIGURE 1

Note that $V - E + F$ remains unchanged.

We shall remove the triangles in such a way that the triangle removed has at least one edge on the boundary. Two situations are possible (see Figure 2).

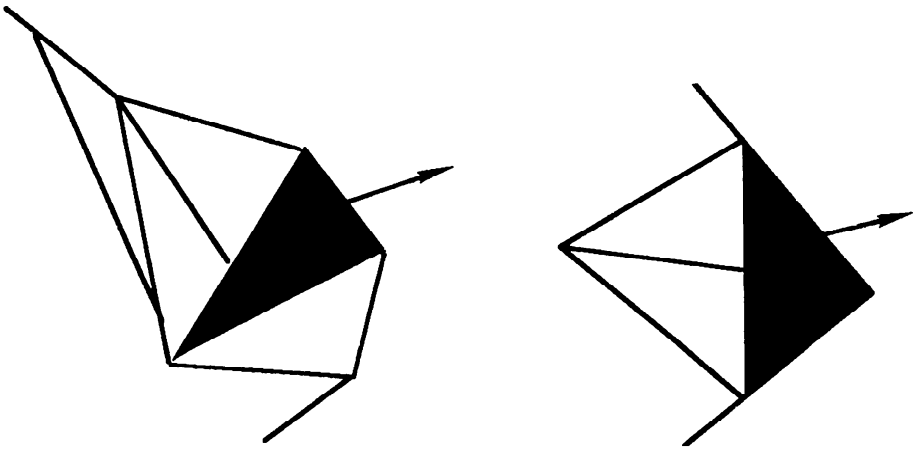


FIGURE 2

a b

For a) V remains the same, E and F are decreased by one; hence, $V - E + F$ is invariant.

Similarly, for b) $V-E+F$ is invariant because V and F decrease by one and E decreases by two. Finally we are left with one triangle for which

$$V-E+F=3-3+1=1.$$

Therefore, for a polyhedron, $\chi = 2$.

● PROBLEM 9-38

Verify the expression

$$V-E+F=\chi \tag{1}$$

for the drawings depicted in Figure 1 and Figure 2.

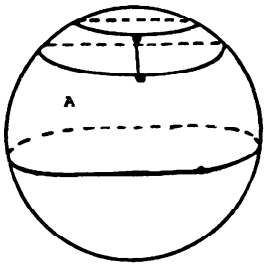


FIGURE 1

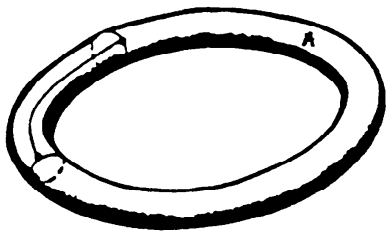


FIGURE 2

Explain any discrepancies.

SOLUTION:

For a sphere

$$V-E+F=2. \tag{2}$$

From Figure 1

$$V=3, \quad E=4, \quad F=4. \tag{3}$$

Hence,

$$V-E+F=3-4+4=3\neq 2. \tag{4}$$

All regions of a map should be simply connected. Region A is not simply connected. Thus rule (1) cannot be applied.

For a torus $\chi = 0$. From Figure 2

$$V=2, \quad E=3, \quad F=2. \tag{5}$$

and

$$V - E + F = 1 \neq 0 \tag{6}$$

because A is not simply connected.

● PROBLEM 9-39

Each surface is characterized by two numbers

1. the genus, g

2. the Euler characteristic, χ.

What is the relationship, if any, between these numbers?

SOLUTION:

The genus of the surface is the greatest number of distinct continuous non-self-intersecting closed curves which may be drawn on a surface without separating it into distinct regions. Each division of a surface to become a map, has to include some edges so that all regions are simply connected. That is related to the genus of the surface.

	Genus	Euler characteristic
Sphere	0	2
Torus	1	0
Two-fold Torus	2	-2
n-fold Torus	n	$2-2n$

The relation between the genus and the Euler characteristic of any given surface is

$$V - E + F = \chi = 2 - 2g.$$

● PROBLEM 9-40

Suppose you have a "hairy" sphere and a "hairy" torus. What happens when you comb down the hair on a sphere? Is it possible to comb a torus nicely?

SOLUTION:

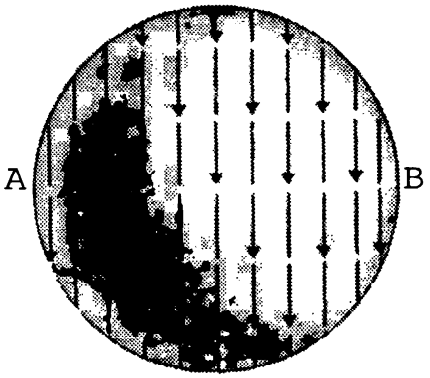


FIGURE 1

Figure 1 depicts a sphere with its hair combed. This "hairdo" has two points of discontinuity, A and B, as illustrated in Figure 2.

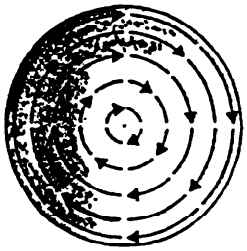


FIGURE 2

Another way of brushing a sphere is shown in Figure 3, which has two points of discontinuity C and D.

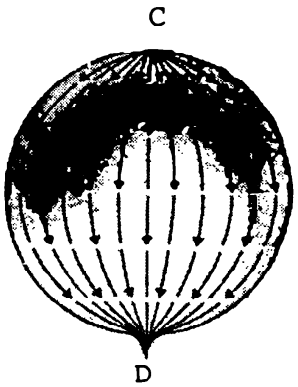


FIGURE 3

It is possible to brush the hair of a torus in a nice way; that is, in such a way that continuity is preserved everywhere (see Figure 4).

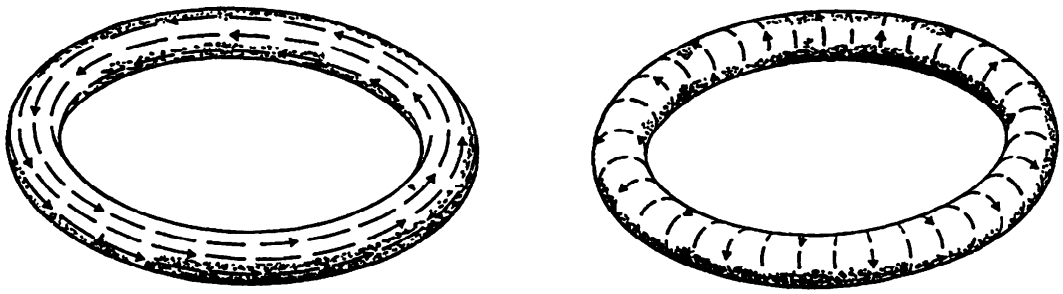


FIGURE 4

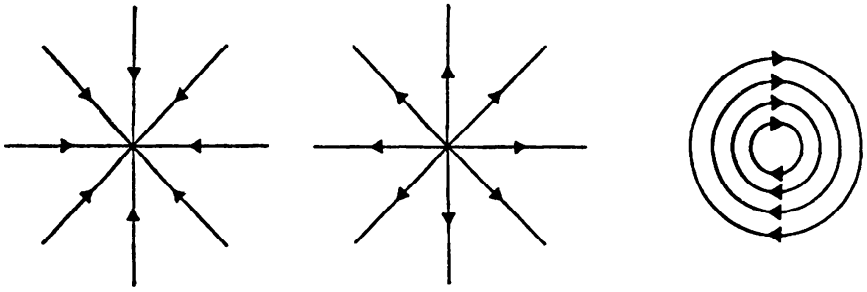
Note that all this "brushing" is equivalent to assigning a direction to each point of a surface.

This problem belongs to differential topology.

● PROBLEM 9-41

Instead of combing hair, we can visualize a flow of fluid on the surface. The points of discontinuity are called singular points. Make a sketch of a few singular points.

SOLUTION:

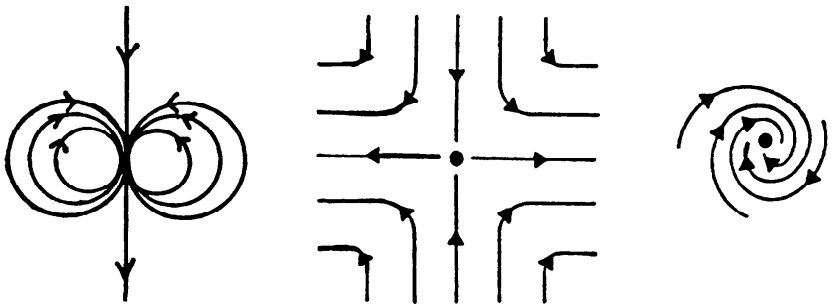


Vortex Source Sink

FIGURE 1

Figure 1 depicts some often encountered singular points. Some other singular points are illustrated in Figure 2.

In most cases, the name of a singular point explains its nature.



Dipole Crosspoint Focus

FIGURE 2

● PROBLEM 9-42

For each singular point described in Problem 9-41, find its index.

SOLUTION:

To each singular point, we assign an integer called its index, which is obtained by travelling around this point along a circle in a counter-clockwise direction and counting the number of counter-clockwise revolutions made by an arrow with its base on the path and pointing in the direction of the flow on the surface.

Let us compute the index of a source.

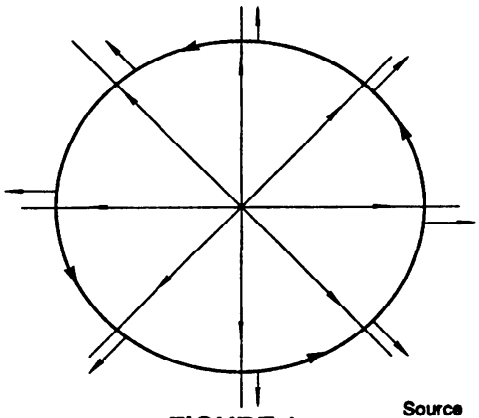


FIGURE 1

The little arrow makes one counter-clockwise revolution, hence the index of a source is one. It is easy to see that for a sink, a vortex and a focus, the index is one.

For a dipole (see Figure 2), the index is 2.

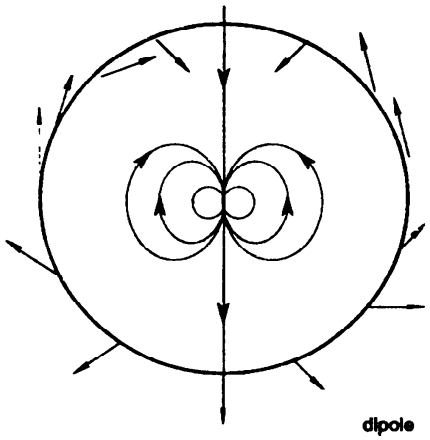


FIGURE 2

For the crosspoint (see Problem 9-41) the index is – 1. It is easy to see that the little arrow makes one clockwise revolution.

● PROBLEM 9-43

Brush a hairy sphere so that it has only one singular point. Make a sketch and determine the index of this singularity.

SOLUTION:

It is possible to brush a sphere in such a way, that it would have only one singularity, namely a dipole.

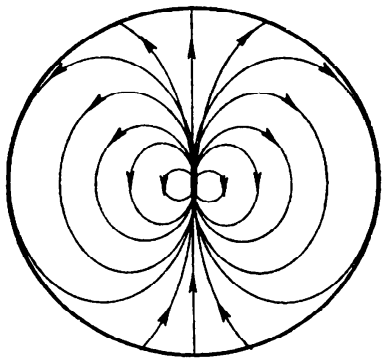


FIGURE 1

The index of a dipole is two.

● PROBLEM 9-44

THEOREM

The Euler characteristic of the surface is equal to the sum of the indices

of the singular points of this surface.

■

Verify this theorem for the surfaces shown in Problems 9-40 and 9-43.

SOLUTION:

Consider the surface of a sphere depicted in Figure 1 of Problem 9-40. It has two vortices. The index of a vortex is one. Hence,

$$1 + 1 = 2$$

which is the Euler characteristic of a sphere.

Similarly, Figure 3 of a sphere of Problem 9-40 has two singular points: a sink with index one and a source with index one. Again,

$$1 + 1 = 2.$$

Both toruses of Figure 4 of Problem 9-40 have no singular points. Hence, the sum of indices is 0 which is the Euler characteristic of a torus.

The sphere depicted in Problem 9-43 has one dipole, whose index is 2.

● PROBLEM 9-45

There is some fluid flow on the sphere which contains a crosspoint. Complete the description of this flow. What other singularities contain a flow with a vortex on a torus?

SOLUTION:

The sum of the indices of the singularities on a sphere is two. The index of a crosspoint is –1. Thus,

$$-1 + \Sigma \text{ indices of other singular points} = 2.$$

Here are some possible combinations:

sink and two sources

source and two sinks

dipole and a sink.

The Euler characteristic of a torus is 0. So, if there is a vortex with index 1, there must be a crosspoint with index – 1.

We obtain

$$1 + (-1) = 0.$$

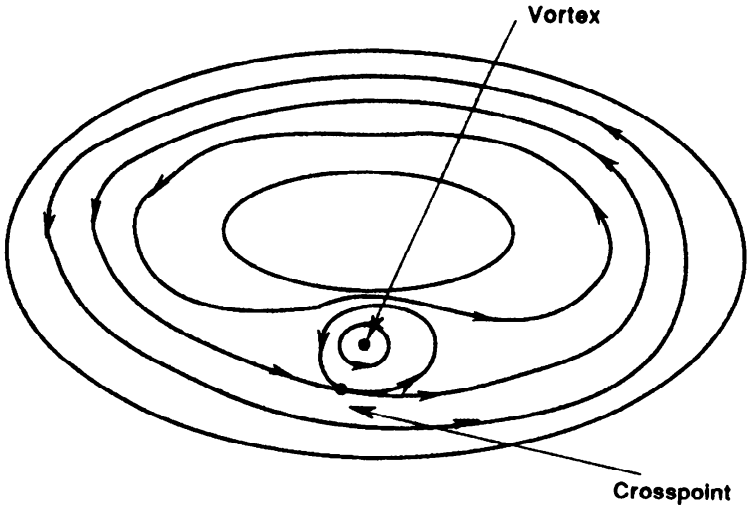


FIGURE 1

● PROBLEM 9-46

Show that the sum of the indices of singular points of a surface is equal to its Euler characteristic.

SOLUTION:

For any map on a given surface the expression

$$V - E + F$$

is invariant and equal to its Euler characteristic. Thus

$$V - E + F = \chi.$$

Suppose there is a map drawn on some surface. This map can be replaced by a flow as follows:

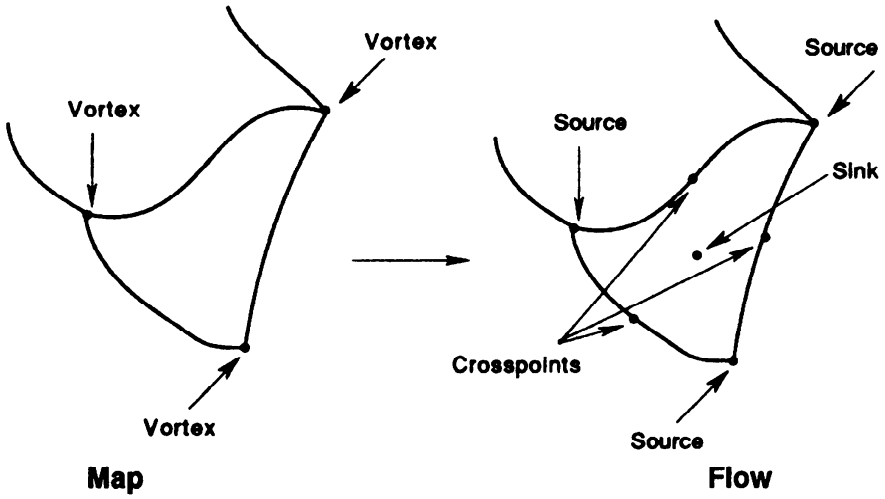


FIGURE 1

1. Replace vertices by sources.

2. Put crosspoints at the center of each arc.

3. Put a sink at the center of each region.

We obtain:

V sources of index 1

E crosspoints of index -1

F sinks of index 1

The sum of indices is

$$V - E + F$$

■

● PROBLEM 9-47

Can you draw these figures without lifting the pencil and passing once and only once through each line. If not, explain why.

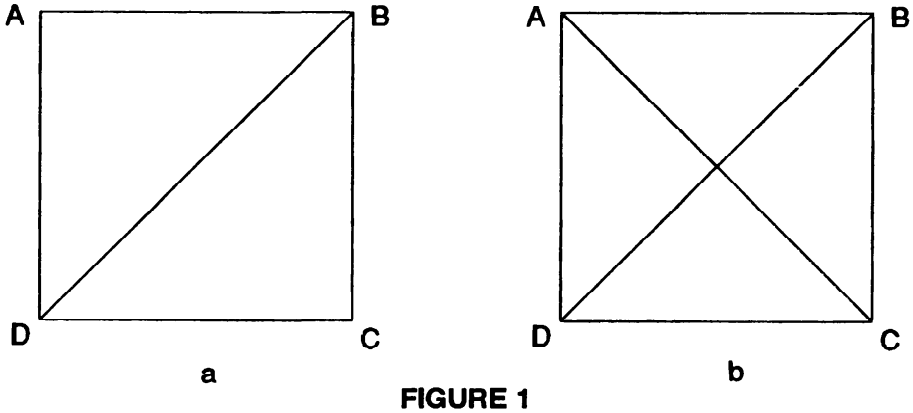


FIGURE 1

SOLUTION:

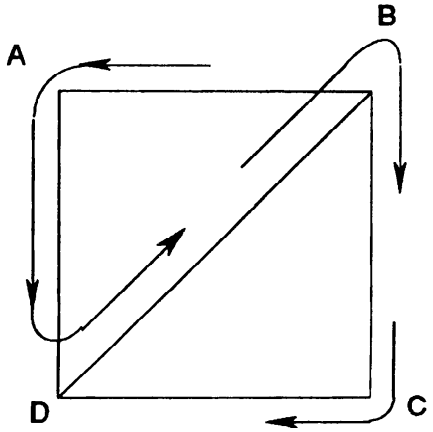


FIGURE 2

Figure 1a can be easily drawn according to the requirements as shown in Figure 2.

We can start from *B* and then move to *A, D, B, C, D.*

It is not possible to draw Figure 2b. Let us return to Figure 2. There are two points *B* and *D* where three lines merge. Then, to make a drawing we have to start with one such point and end with the other. In case of Figure 1b, we have four points, *A, B, C, D* where three lines merge. One can be used to start a drawing, the second to end it; but there is no way we can "get rid" of the remaining two points.

● PROBLEM 9-48

Determine the order of each vertex.

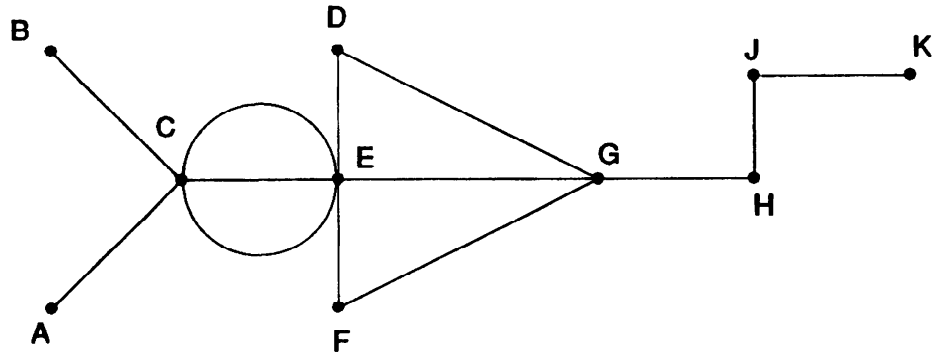


FIGURE 1

Which vertices are odd and which are even?
Why must the total number of arc-ends in any network be even?
Show that the total number of odd vertices must be even.

SOLUTION:

A network consists of a finite number of vertices linked by arcs. The arcs must be non-intersecting. Such a network is sometimes called a linear graph. The number of arc-ends meeting at the vertex is called an order.

A vertex whose order is an even number is called an even vertex. A vertex with an odd order is called an odd vertex.

From the drawing we have

A	1	odd
B	1	odd
C	5	odd
D	2	even

E	6	even
F	2	odd
G	4	even
H	2	even
J	2	even
K	1	odd

The number of arc-ends is twice the number of arcs; hence, it is even.

The number of arc-ends is equal to the sum of the orders of all the vertices. Hence, the total number of odd vertices must be even.

● PROBLEM 9-49

Show that the network

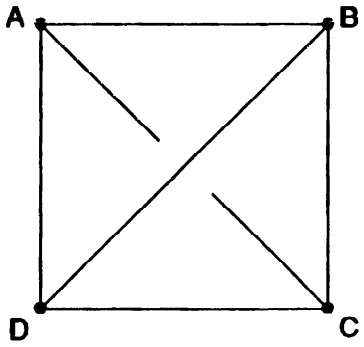


FIGURE 1

is planar.

SOLUTION:

A network that can be mapped onto a simply connected surface in such a way that the arcs don't intersect is called planar.

The network of Figure 1 can be mapped on the plane (see Figure 2).

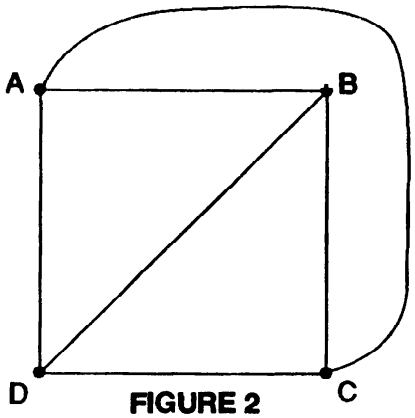


FIGURE 2

This network can also be mapped onto a sphere as shown in Figure 3.

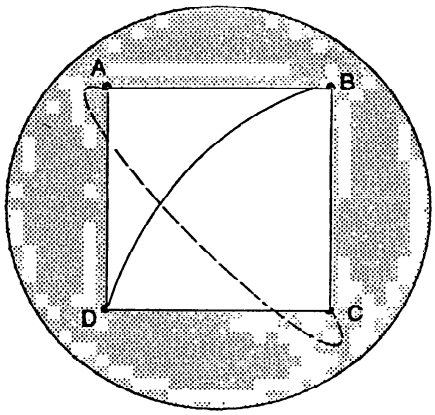


FIGURE 3

● PROBLEM 9–50

Suppose there are three houses, *A*, *B*, and *C*. Each house has to be connected with water, electricity and gas in such a way that the pipes do not intersect. Can it be done?

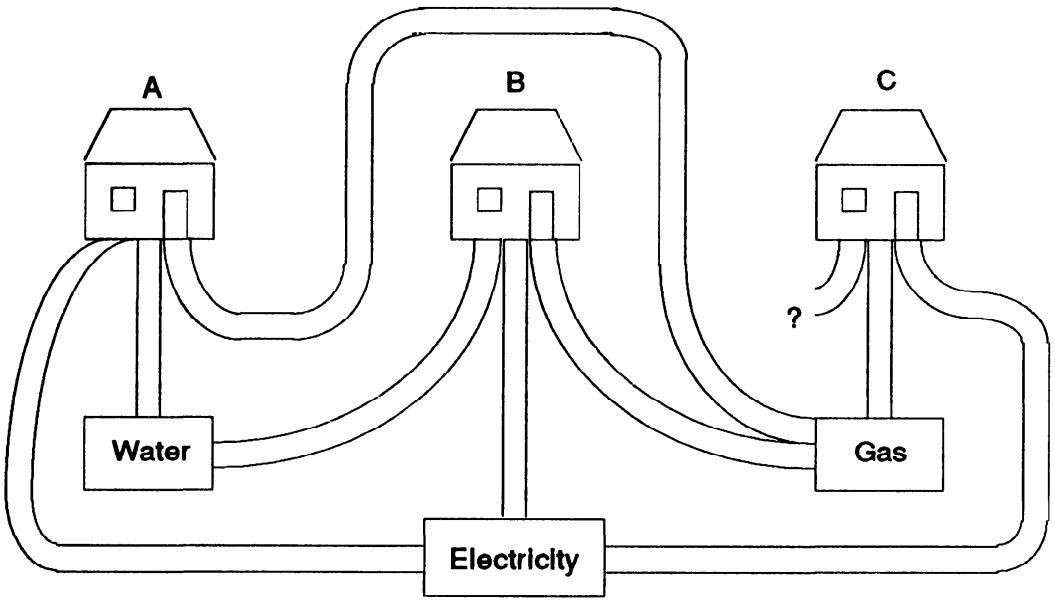


FIGURE 1

SOLUTION:

It cannot be done. Here is why.

Let us connect the first house *A* with water, electricity, and gas, and then proceed through these points to the second house *B*, see Figure 2.

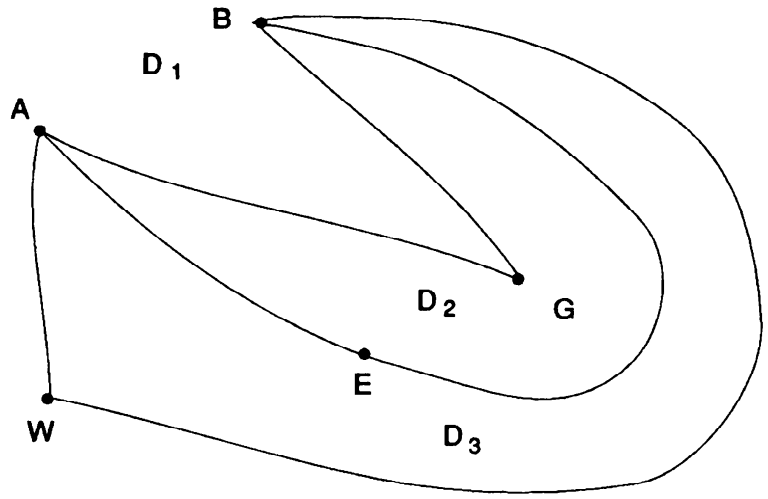


FIGURE 2

We have three lines from A to B which do not cross. These lines divide the whole plane into three areas: D_1, D_2, D_3. The third house C lies in one of the areas.

If it lies in D_1, then E cannot be connected; if in D_2, then W cannot be connected; and if in D_3, then G cannot be connected.

● PROBLEM 9-51

Give an example of a non-planar network.

SOLUTION:

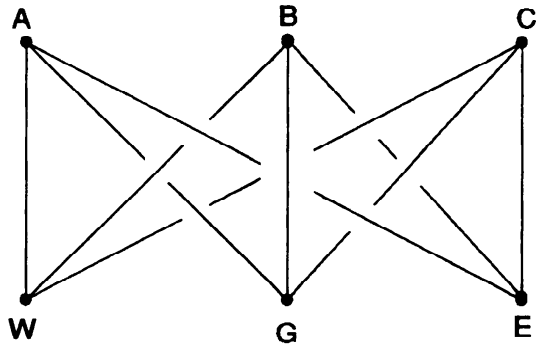


FIGURE 1

Note that the network depicted in Figure 1 illustrates Problem 9-50.

Another example is the complete network on five vertices shown in Figure 2.

It can be proven that every non-planar network must contain either a network of Figure 1 or network of Figure 2 as a sub-network.

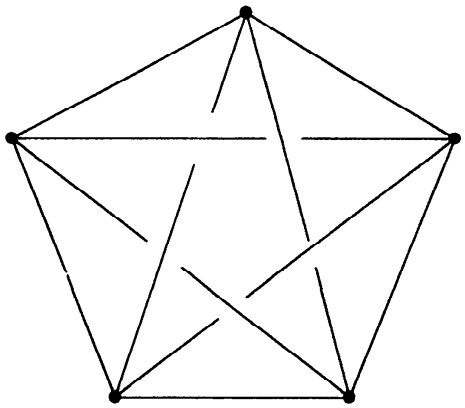


FIGURE 2

A network is complete when all vertices are directly linked to each other by the minimum number of arcs.

● PROBLEM 9-52

Give an example of a network that cannot be traversed by a single path.

SOLUTION:

A sequence of arcs which can be followed continuously without any arc being passed more than once is called a path. Figure 1 depicts a path, *ABCD*.

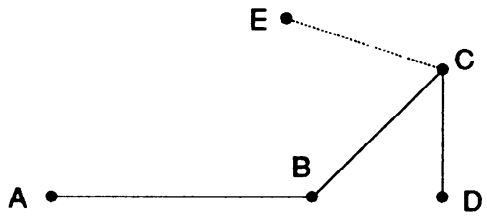


FIGURE 1

A path traverses a network if every arc of the network is included in the path.

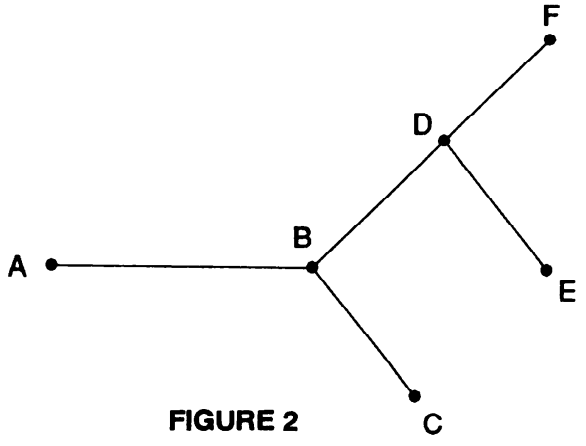


FIGURE 2

No path traverses a network shown in Figure 2.

A closed path (that is such, which starts and finishes at the same vertex)

is called a circuit.

A path that is not closed is called open.

● PROBLEM 9-53

Show that a network which has more than two odd vertices cannot be traversed by a single path.

SOLUTION:

Figure 1 depicts a network, which

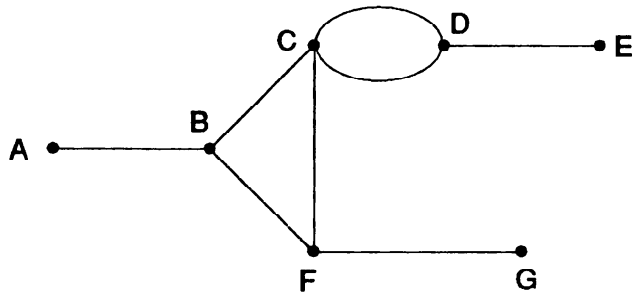


FIGURE 1

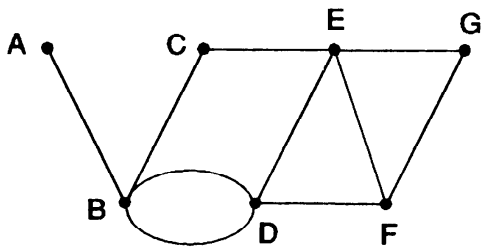
cannot be traversed by a single path.

This network has three vertices of the order of one, three vertices of the order of three and one vertex of the order of four. If a path traversing a network is closed, then an even vertex may be the starting and terminating point of the "journey." If a path traversing a network is open, then it starts and terminates at odd vertices. Hence, we conclude that if a network has more than two odd vertices, it cannot be traversed by a single path.

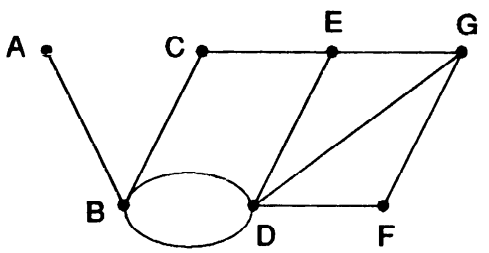
The network in Figure 1 has six odd vertices and cannot be traversed by a single path.

● PROBLEM 9-54

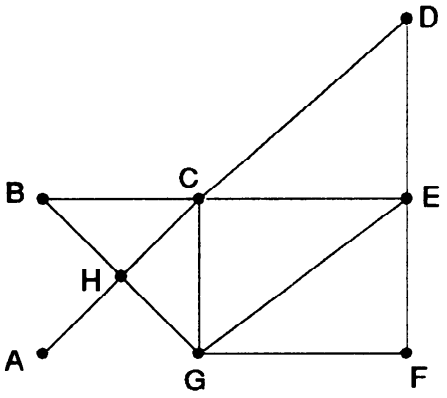
Which of the following networks can be traversed by a single path?



(a)



(b)



(c)

FIGURE 1

SOLUTION:

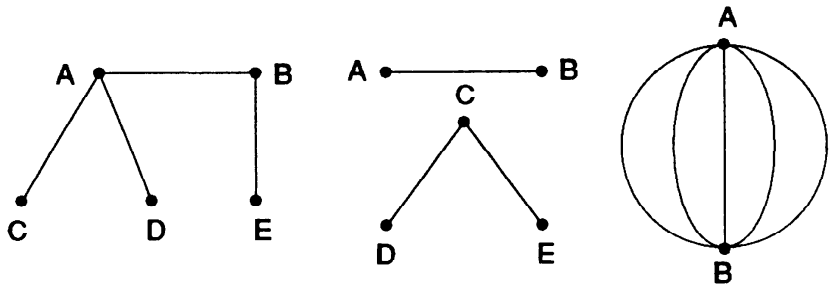
A connected network can be traversed by a single open path, if and only if it has exactly two odd vertices.

Network (a) of Figure 1 has two odd vertices A and F. It can be traversed $ABDBCEDFGEF$. Network (b) of Figure 1 has four odd vertices A, E, G, D. It cannot be traversed by a single path.

Network (c) of Figure 1 has two odd vertices A and C, and it can be traversed by a single path $AHCDEFGHBCGEC$.

● PROBLEM 9-55

Which of the networks is connected and which is disconnected?



(a) (b) (c)

FIGURE 1

Find a necessary condition for a network to be connected.

SOLUTION:

A network is said to be connected if every pair of vertices belongs to some path; otherwise the network is disconnected.

Networks (a) and (c) are connected.

Network (b) is disconnected, vertices A and C don't belong to any path. Let a be the number of arcs and n the number of vertices. Then a necessary condition for a network to be connected is

$$n - 1 \leq a.$$

This condition is necessary but not sufficient.

● PROBLEM 9-56

Reduce a network depicted in Figure 1 to a tree.

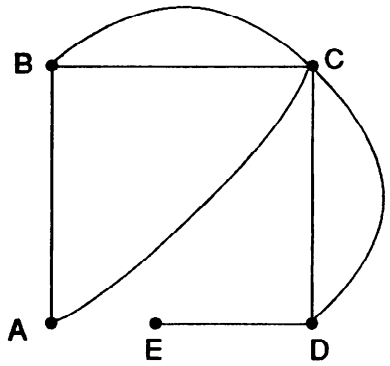


FIGURE 1

SOLUTION:

Any connected network, such that the number of its vertices is one more than the number of its arcs is called a tree. Any connected network can be reduced to a tree by the removal of appropriate arcs. We can remove BC, AC, and CD to obtain a network (see Figure 2) which is a tree.

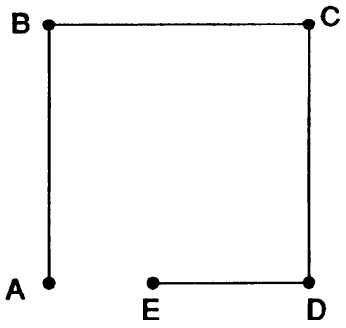


FIGURE 2

● PROBLEM 9-57

A planar network is mapped onto a simply connected surface, so that no arcs intersect.

The number of arcs and vertices is given. Determine the number of bounded regions into which the surface is separated.

SOLUTION:

Let v be the number of vertices and a the number of arcs. Then the number of bounded regions is

$$a - n + 1.$$

Indeed, for a triangle

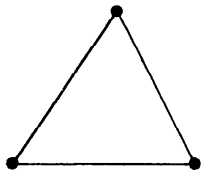


FIGURE 1

we have $3 - 3 + 1 = 1$. For an arc with two vertices $1 - 2 + 1 = 0$.

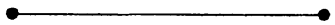


FIGURE 2

According to the network depicted in Figure 2, by adding new arcs and vertices, we can obtain any network.

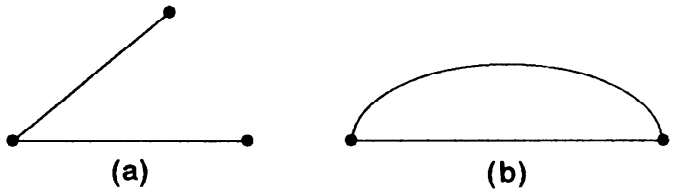


FIGURE 3

In Figure 3(a) we added one arc and one vertex, hence the number $a - n + 1$ remains the same. In Figure 3(b) one arc was added and the number of vertices is constant, hence $a - n + 1$ increases by 1.

In such a way, we can obtain any network. That completes the proof.

● PROBLEM 9-58

List the tie-sets, which specify the network in Figure 1.

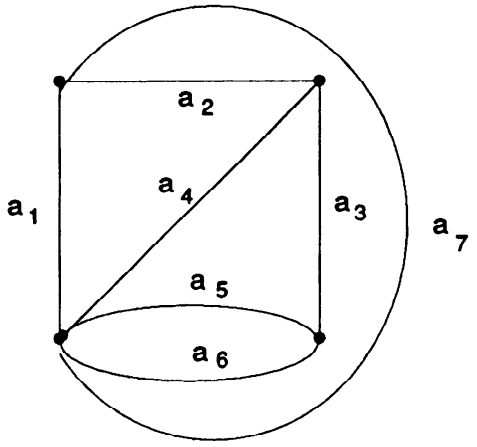


FIGURE 1

SOLUTION:

Any planar network can be completely determined in terms of its independent bounded regions or in terms of paths between its independent pairs of vertices. There are $\frac{V(V-1)}{2}$ pairs but only $v - 1$ independent pairs (v is the number of vertices). A network is defined by tie-sets when its structure is determined by paths between its independent pairs of vertices. A tie-set is a single closed path, such that exactly two arcs meet at each vertex. Figure 2 depicts some tie-sets.

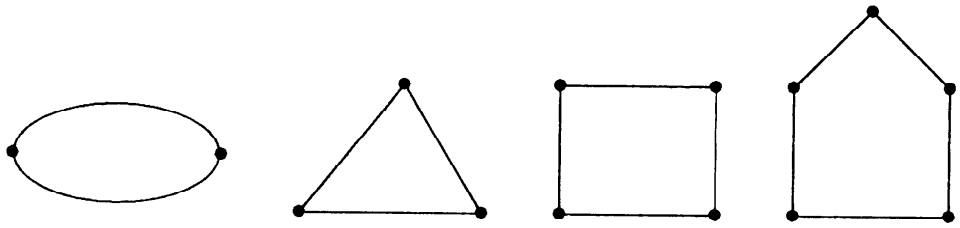


FIGURE 2

Here is one of of possible lists of tie-sets of the network of Figure 1

$$a_5\, a_6,\ a_1\, a_2\, a_4,\ a_3\, a_5\, a_4,\ a_2\, a_3\, a_6\, a_7.$$

Another possibility is

$$a_1\, a_2\, a_3\, a_6,\ a_1\, a_4\, a_3\, a_6\, a_7,\ a_3\, a_4\, a_6,\ a_1\, a_2\, a_3\, a_5$$

or any other combination of four independent closed paths.

● PROBLEM 9-59

Is it possible to traverse a network depicted in Figure 1 by a single path?

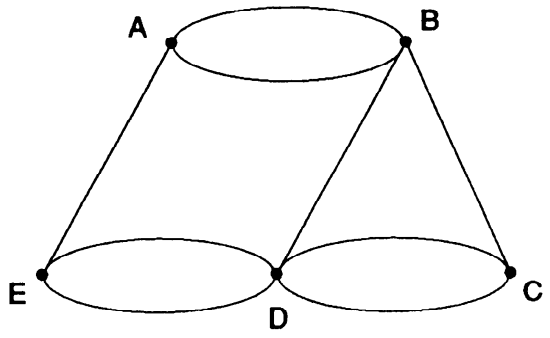


FIGURE 1

SOLUTION:

To be traversed by a single path, a network has to be connected (it is a necessary condition but not sufficient).

If a network has two odd vertices, then these vertices have to be the initial and final vertices of the path. Now suppose a network has an even number $2n > 2$ of odd vertices. Then at least n paths are necessary to traverse it.

The network of Figure 1 has four odd vertices, hence two paths are needed to traverse this network. We can choose for example

ABAEDBCDC and *ED*.

● PROBLEM 9-60

Here is the famous problem of the Königsberg bridges solved by Euler. Is it possible to cross them all in turn, passing no more than once over each?

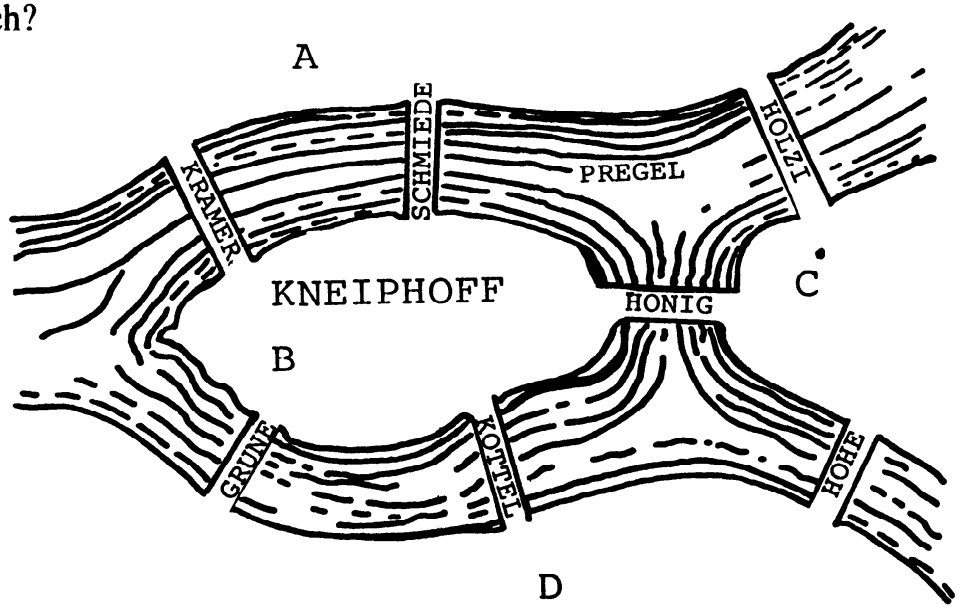


FIGURE 1

SOLUTION:

Let us denote the regions as shown in Figure 1.

Then we have to traverse a network shown in Figure 2 with one single path.

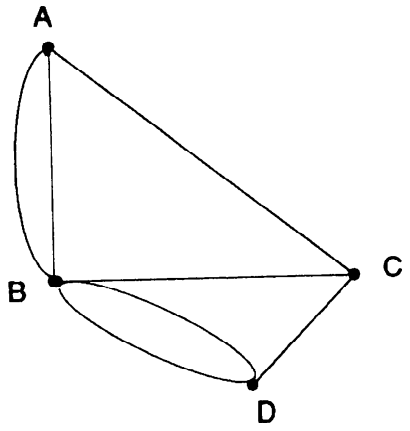


FIGURE 2

But this is impossible, because *A, B, C,* and *D* are odd vertices.

● PROBLEM 9-61

Consider again the Königsberg bridges. Where would the eighth bridge be built in order to make the walk possible? Again, we have to cross all bridges by passing only once over each.

SOLUTION:

From Figure 2 of Problem 9-60, we see that there are four odd vertices.

Hence the eighth bridge has to eliminate two odd vertices, for example, *C* and *D* (see Figure 1)

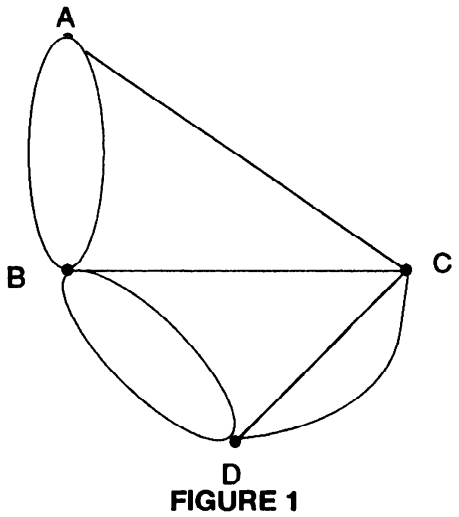


FIGURE 1

We have to build a bridge between *C* and *D* as shown in Figure 2.

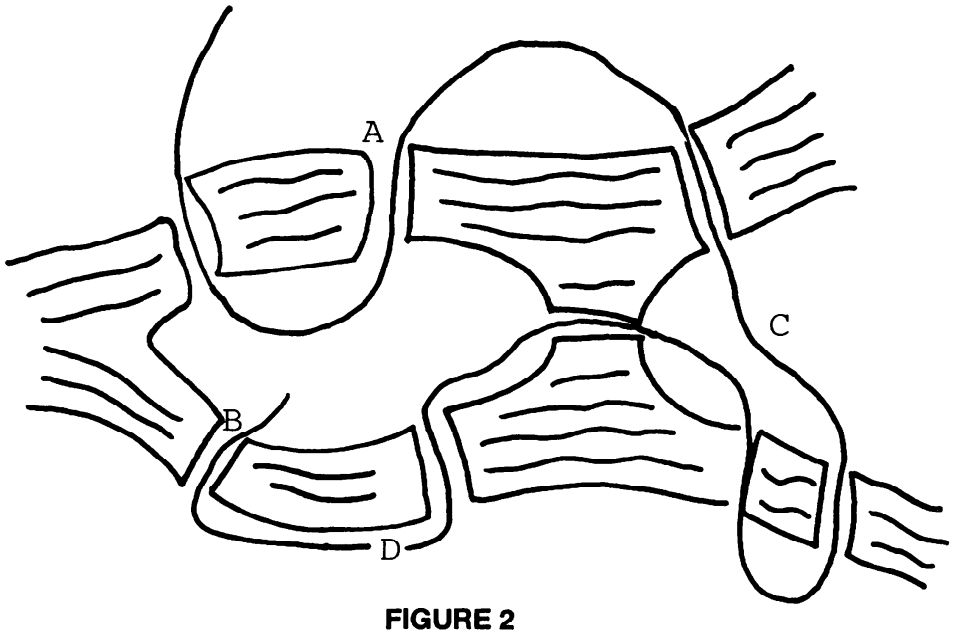


FIGURE 2

The only two odd vertices are now A and B. Hence a walk has to start at A (or B) and end at B (or A).

● PROBLEM 9-62

Is it possible to draw the figure shown in Figure 1 without lifting the pencil and passing once and only once through each point?

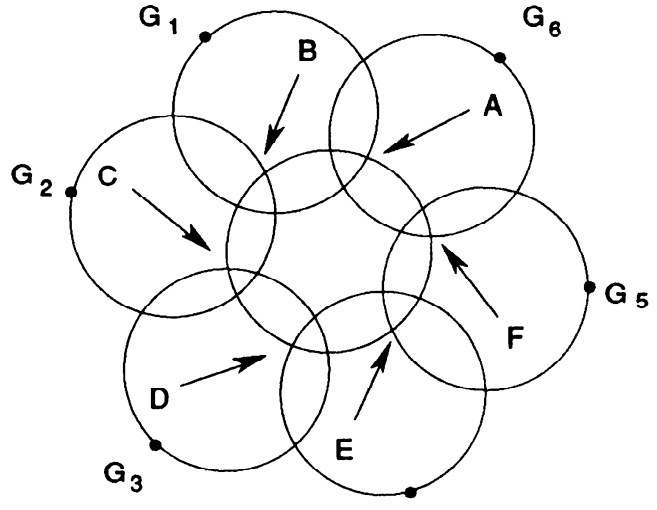


FIGURE 1

SOLUTION:

Yes. Note that all the vertices are of the order of four.

We can start from A and draw a circle $A\ G_1\ A$, then move to B and draw a circle $B\ G_2\ B$, then move to C and so on, until we draw the last surrounding circle $F\ G_6\ F$. To complete the drawing we sketch FA, which closes the inner circle.

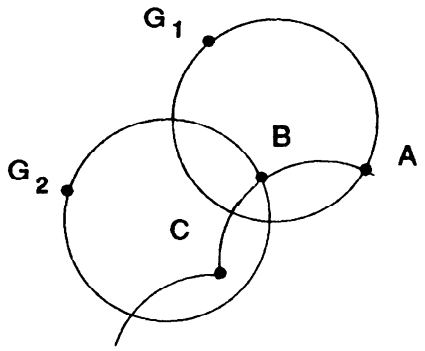


FIGURE 2

It is also possible to draw the Olympic symbol (see Figure 3) without lifting the pencil and passing only once through each point.

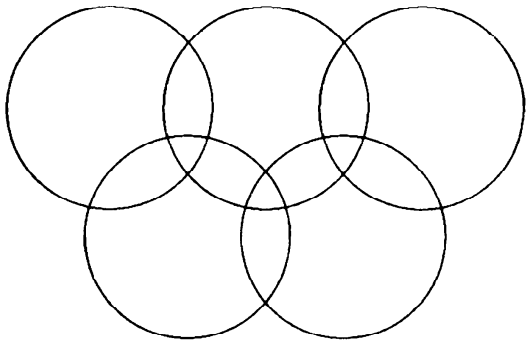


FIGURE 3

● PROBLEM 9–63

Here is an odd puzzle. It's quite possible that you already know the solution. If you don't, try to find it.

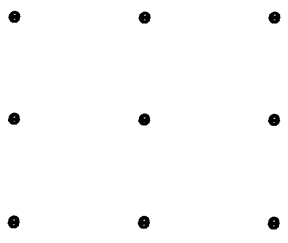


FIGURE 1

The network consists of 9 dots (see Figure 1). Draw 4 lines without lifting the pencil from the paper so as to cross out every dot.

SOLUTION:

To solve the problem, we have to leave the frame of the dots.

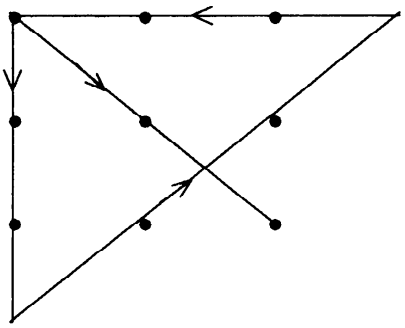


FIGURE 2

Another solution is shown in Figure 3. Here the line crosses each dot once and only once.

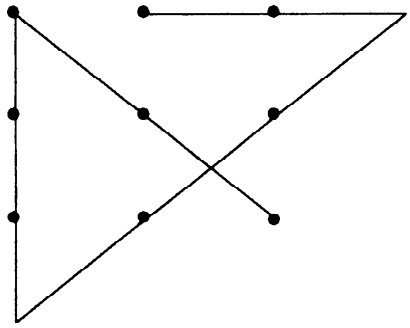


FIGURE 3

● PROBLEM 9-64

Suppose you own a mansion with the floor-plan depicted in Figure 1.

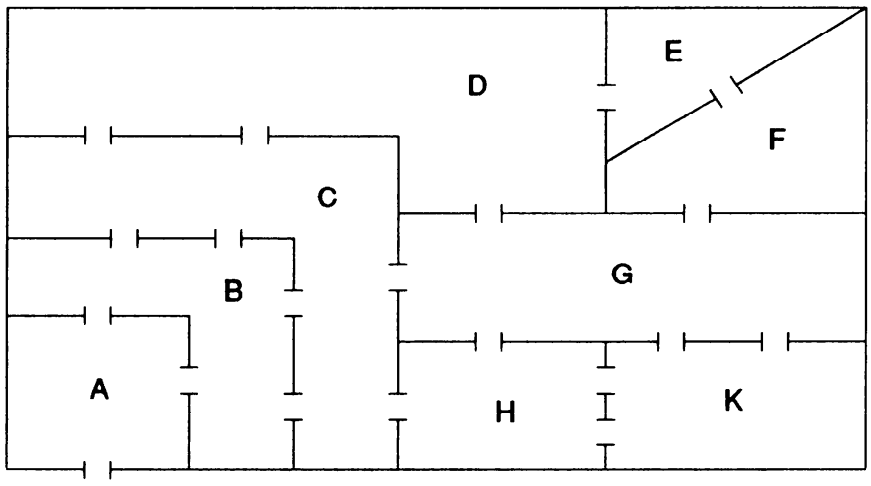


FIGURE 1

To support yourself, you have to guide tourists through each room. Is is possible to traverse the mansion starting at the entrance and passing each door once and only once?

SOLUTION:

Let us denote the rooms A, B, C, D, E, F, G, H, and K.

Each room represents a vertex of a network. The number of doors is the order of the vertex, hence A has order 3, $B - 4$, $C - 4$, $D - 4$, $E - 2$, $F - 2$, $G - 6$, $H - 4$, $k - 4$. There is only one odd vertex. It is possible to traverse the network by a single path. One possible route is shown in Figure 2.

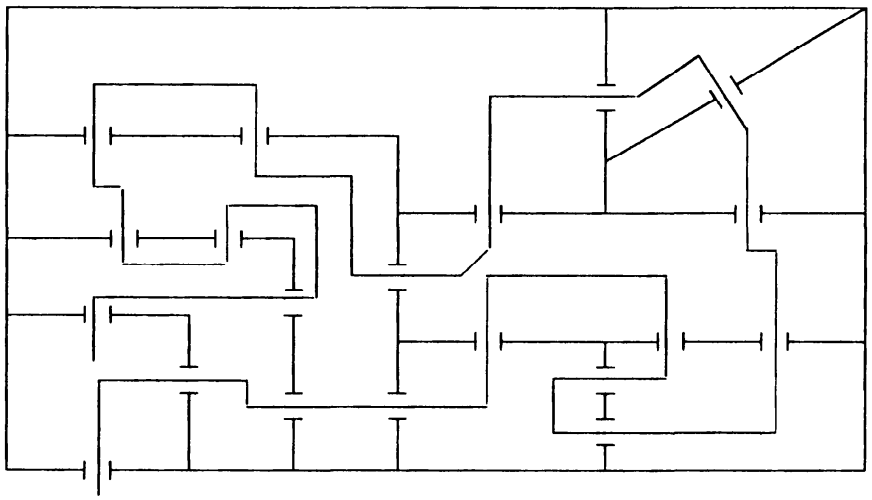


FIGURE 2

● PROBLEM 9-65

Prove the following theorem:

THEOREM

Every network with all even vertices, or with, at the most, two odd vertices can be traversed by a single path.

SOLUTION:

Consider first, the case when all vertices are even.

We can start at an arbitrary vertex A. Since A is even, the path must end at A. If parts of the network not already visited remain and the order of A is higher than 2, we can draw another closed path.

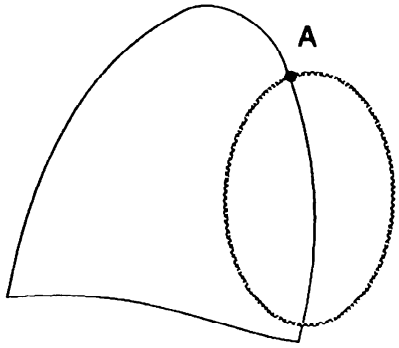


FIGURE 1

Two closed paths with a common point A can be considered as one closed path.

If the task is not yet accomplished, the new path must be linked at B with the rest of the network.

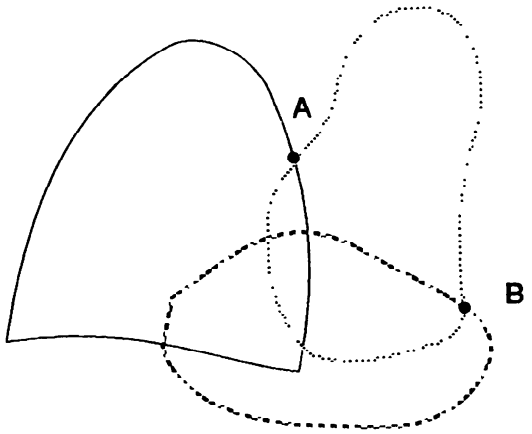


FIGURE 2

We can repeat this procedure until we reach the solution, that is when the path traverses the whole network.

The reasoning is almost the same for the case of a network with two odd vertices A and B.

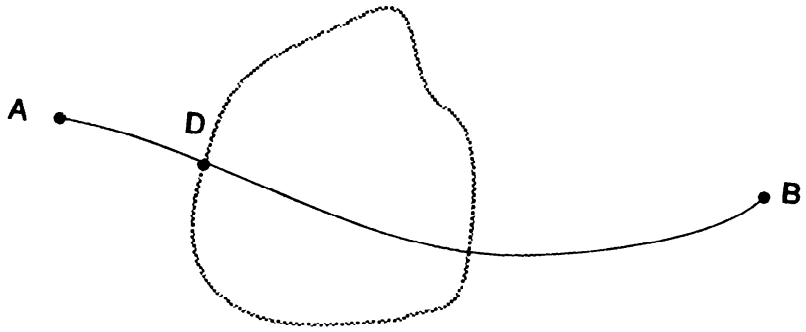


FIGURE 3

We start at A (or B) and end at B (or A). The rest of the network contains only even vertices previously discussed and is linked to AB at some point D.

What regular polygons with all diagonals can you draw with one stroke, that is, without lifting a pen and passing once and only once through each one?

SOLUTION:

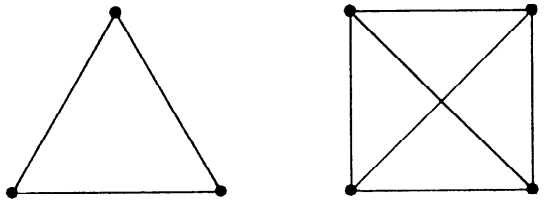


FIGURE 1

It is possible to draw a triangle but not a square, which has four odd vertices.

For a corresponding network to have only even vertices, the polygon must have an odd number of sides. Figure 2 depicts 5-gon and 7-gon.

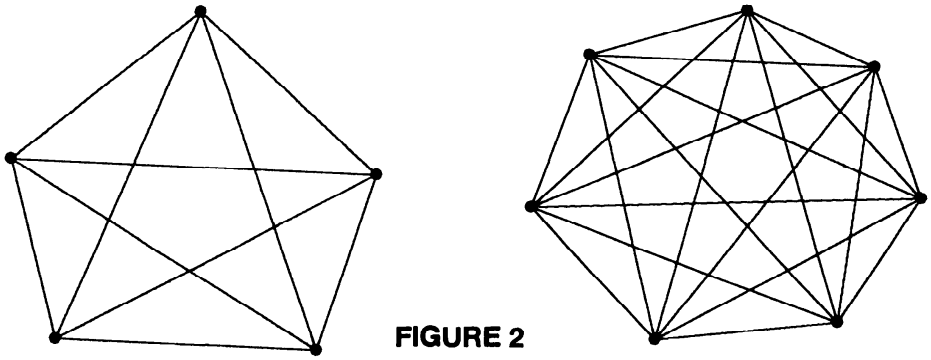


FIGURE 2

If you have enough patience you can draw polygons of higher order 19-, 21-, 23-, 25-gons etc. Figure 3 illustrates 23-gon with all of its diagonals.

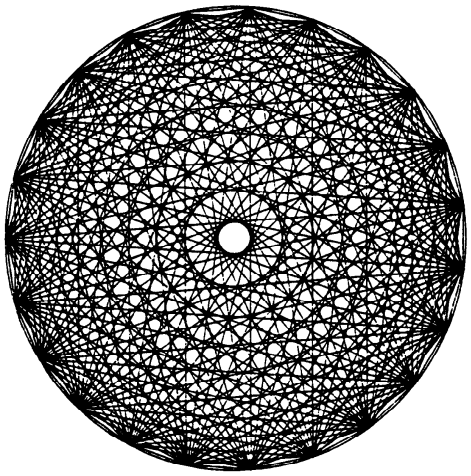


FIGURE 3

● PROBLEM 9-67

Suppose you have three chains with clips. Arrange them in such a way that they interlock as a whole but individual pairs do not interlock. That is, if you open any one chain, they all become separated.

SOLUTION:

Figure 1 does not offer a solution.

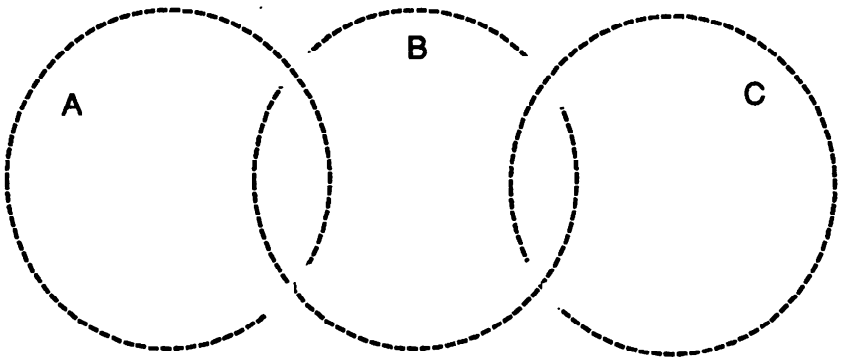


FIGURE 1

Opening *B* separates all of the chains. But opening *A or* C does not. Figure 2 illustrates the solution.

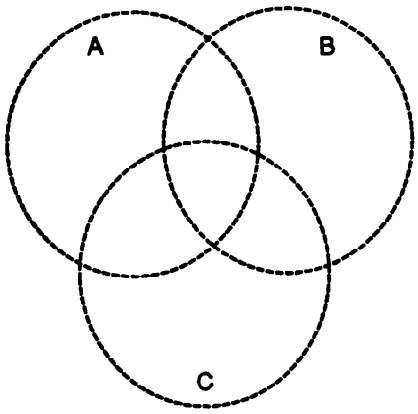


FIGURE 2

Opening any chain separates the remaining two.

The knowledge of knots is part of topology. It has been proven that it is impossible to tie two knots at two ends of a rope in such a way that when brought together they may cancel each other.

● PROBLEM 9-68

Prove that five (or more) countries are never neighbors of each other.

SOLUTION:

It is easy to show that four countries can be neighbors of each other, see Figure 1.

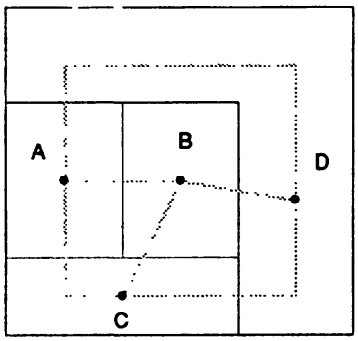


FIGURE 1

Let *A, B, C,* and *D* be the capitals or respective countries. Their capitals can be connected by roads that pass through only two countries (Figure 1). Connecting the capitals *A, B,* and *C* we get the triangle shown in Figure 2.

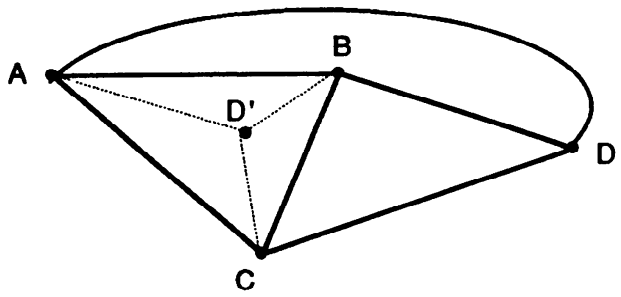


FIGURE 2

D can be interior or exterior of this triangle. In both cases, we obtain a large triangle composed of three small adjacent triangles. *E* is the capital of the fifth country. It lies in one of the small triangles or outside the large triangle, Figure 3.

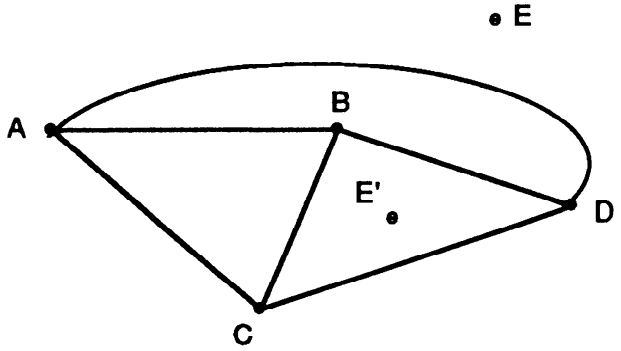


FIGURE 3

In either case, *E* is separated from one capital by a triangle of roads. To get from *E* to *B* (in our case) one has to cross at least one existing road.

Hence, countries with capitals *B* and *E* are not neighbors. That completes the proof.

● PROBLEM 9-69

Show that the chromatic number of a plane surface is larger than three.

SOLUTION:

Here we deal with the famous four-color problem. In 1976 it was proven that:

every map drawn on a sheet of paper can be colored with only four colors in such a way that countries having a common border receive different colors.

This problem was formulated by Francis Guthrie in 1852.

The least number of colors required to color a map on any given surface is called the chromatic number of that surface. A chromatic number is a topological property of the surface.

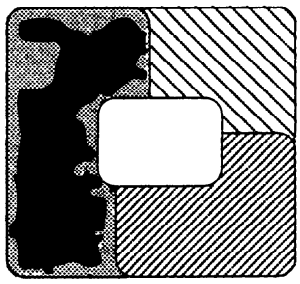


FIGURE 1

The drawing shows that three colors are insufficient to color a map on the plane surface. We will prove later that five colors are sufficient to color any map. So far we know that:

$$\text{chromatic number of a plane surface} \geq 4.$$

● PROBLEM 9-70

Derive the regular map with the least number of additional vertices and arcs from the non-regular map depicted in Figure 1.

SOLUTION:

A map is a network consisting of a number of vertices and non-intersecting arcs linked together in a such a way that an area is separated into simply connected regions. Figures 1 depicts a map.

A map is called regular, if

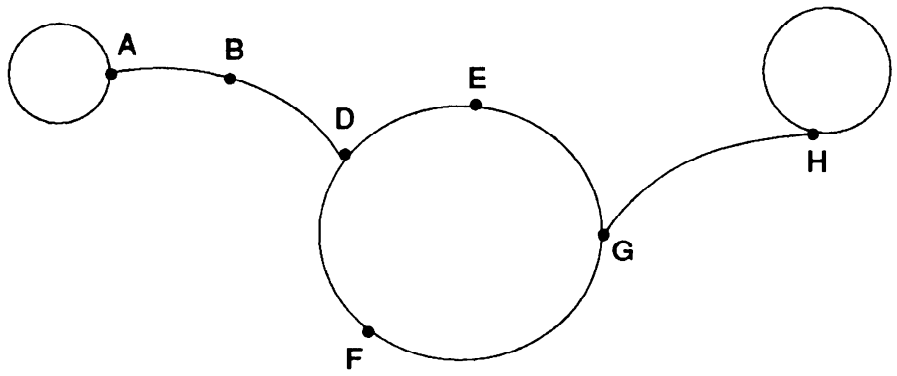


FIGURE 1

1. its vertices are of order three or more.

2. each arc separates two distinct regions.

3. each arc joins two distinct vertices.

The map of Figure 1 is not regular. We shall add some vertices and arcs to make it regular.

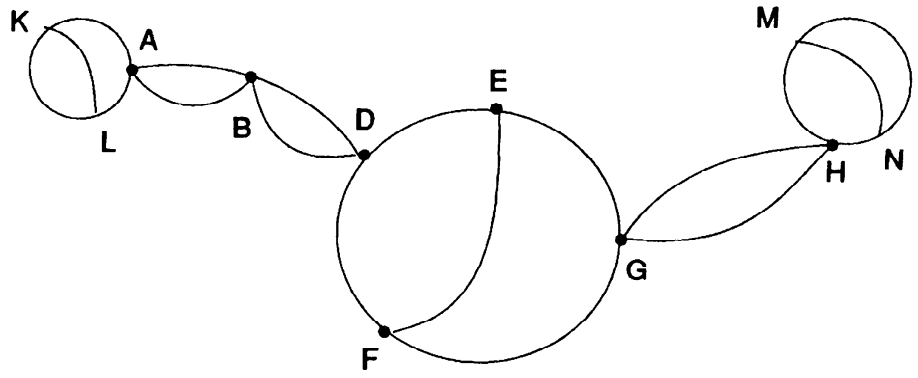


FIGURE 2

All vertices of the map of Figure 2 are of an order higher than or equal to three. The new map is regular.

● PROBLEM 9-71

In problem 9-68, we proved that it is not possible for five countries to be located in such a way that each of them is adjacent to each of the other four. This result led Augustus DeMorgan (who was the first one to prove it) to believe that he solved the four-color problem. Why was he wrong?

SOLUTION:

The fact that five mutually adjacent countries do not exist on a map does not constitute a proof of the four-color conjecture. Quite a few mathematicians have made this mistake. Consider a map of six countries.

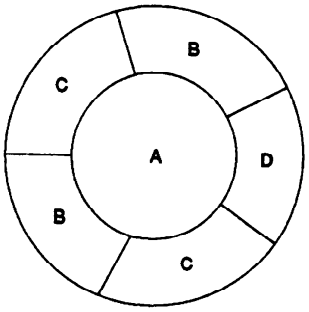


FIGURE 1

Among the six countries, there is no collection of four in which each member is adjacent to the other three. Nevertheless it takes four colors to color this map. From the figure we see that the number of colors required for a map (here 4) is not the same as the maximum number of mutually adjacent countries (here 3).

● PROBLEM 9–72

Prove the following:

THEOREM

Any regular map on the plane surface or the surface of a sphere must have at least one region bounded by fewer than six arcs.

SOLUTION:

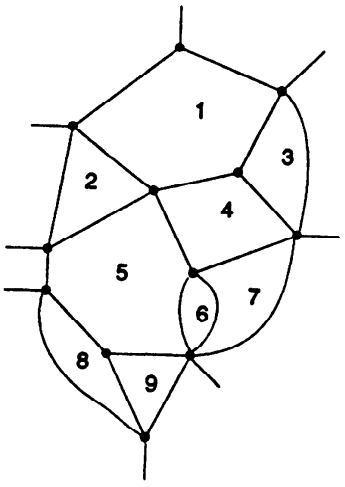


FIGURE 1

Let χ be the Euler characteristic, then

$$V - E + F = \chi \tag{1}$$

where V is the number of vertices, E, arcs, and F regions. Hence, for any surface of the Euler characteristic $\chi > 0$, we have

$$V - E + F > 0 \quad \text{or} \quad 6V - 6E + 6F > 0. \tag{2}$$

Since the map is regular, (see Problem 9-70) all vertices are of the order of 3 or higher.

$$2E \geq 3V \tag{3}$$

From (2) and (3), we obtain

$$6F - 2E > 0 \tag{4}$$

Let F_n denote the number of regions, each bounded by exactly n arcs. In Figure 1, regions 2, 3, 7, 8, and 9 are bounded by 3 arcs, hence $F_3 = 5$. While $F_2 = 1$, $F_4 = 1$, $F_5 = 1$, $F_6 = 1$. The map is regular, therefore

$$F_1 = 0 \tag{5}$$

The total number of regions is

$$F = \sum_{n=2,3,4\ldots} F_n \tag{6}$$

From (4), (5), and (6), we find

$$6 \sum_{n=2,3,\ldots} F_n - \sum_{n} nF_n = \sum_{n=2,3,\ldots} (6-n)F_n > 0 \tag{7}$$

Then

$$4F_2 + 3F_3 + 2F_4 + F_5 - F_7 - \ldots > 0 \tag{8}$$

Hence some $n < 6$ must exist for (8) to hold. Thus, any regular map on a surface of the Euler characteristic > 0 must have at least one region bounded by less than six arcs.

● PROBLEM 9-73

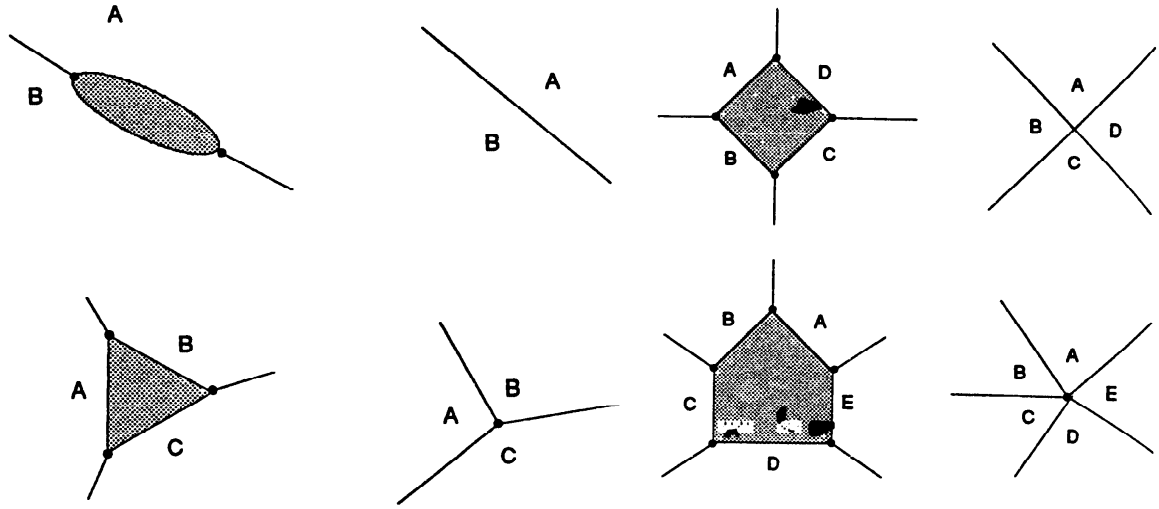
Using the results of Problem 9-72, prove the six color theorem.

SIX COLOR THEOREM

Any regular map on a surface of the Euler characteristic > 0 requires six colors at the most, if no neighboring regions are to be colored the same.

SOLUTION:

Obviously the theorem is true for $F \leq 6$. We shall prove that, if the theorem holds for F', then it holds for $F' + 1$. We proved in Problem 9-72 that at least one region exists which is bounded by less than six arcs, that is by two, three, four, or five arcs. Suppose this region shrinks to a point. The four possibilities are illustrated below.



For all cases with a contracted map, which contains F' regions the theorem holds by assumption. That is each of the contracted maps requires six colors. From the drawing, we see that when the region is restored there is a color available for it without the total of six colors being exceeded. Hence, if the theorem is true for F, then it is true for $F + 1$. Since it is true for $F \leq 6$, the proof is completed.

● PROBLEM 9-74

Any regular map on a surface of the Euler characteristic χ can be colored by at most β colors, where

$$\beta F > 6(F - \chi) \tag{1}$$

Show that at least one region must have a boundary consisting of less than β arcs.

SOLUTION:

For any regular map

$$V - E + F = \chi \tag{2}$$

$$2E \geq 3V$$

Thus

$$6(E - V) = 6(F - \chi) \geq 2E \tag{3}$$

Combining (1) and (3) we find

$$\beta F > 2E \tag{4}$$

Since the map is regular

$$\beta \sum_n F_n > \sum_n nF_n \tag{5}$$

$$\sum_n F_n(\beta - n) > 0\,,\ n \geq 2 \tag{6}$$

From (6) we conclude that at least one region must have a boundary consisting of less than β arcs.

● PROBLEM 9-75

Determine the smallest integer β_0 satisfying

$$F\beta > 6(F - \chi) \tag{1}$$

when $F > \beta$. Consider the following values of χ : 0, 1, 2, –1, –2, ..., –10.

SOLUTION:

For $\chi = 0$, we have

$$\beta > 6. \tag{2}$$

The smallest integer β_0 satisfying (2) is $\beta_0 = 7$.

Let us write (1) in the form

$$\beta > 6(1 - \tfrac{\chi}{F}) \tag{3}$$

Then for $\chi = 1$, we obtain $\beta_0 = 6$. Similarly for $\chi = 2$, we find

$$\beta > 6 - \tfrac{12}{F} \tag{4}$$

The smallest integer β_0 satisfying (4) is $\beta_0 = 6$.

Note that the same values of β_0 were obtained from the six-color theorem.

We find:

$$\text{for } \chi = -1,\ \beta > 6 + {}^{6}/_{F},\ \beta_0 = 7 \tag{5}$$

$$\text{for } \chi = -2,\ \beta > 6 + {}^{12}/_{F},\ \beta_0 = 8$$

$$\text{for } \chi = -3,\ \beta > 6 + {}^{18}/_{F},\ \beta_0 = 9$$

$$\text{for } \chi = -4,\ \beta > 6 + {}^{24}/_{F},\ \beta_0 = 9$$

$$\text{for } \chi = -5\ \ \beta > 6 + {}^{30}/_{F},\ \beta_0 = 10$$

$$\text{for } \chi = -6\ \ \beta > 6 + {}^{36}/_{F},\ \beta_0 = 10$$

$$\text{for } \chi = -7\ \ \beta > 6 + {}^{42}/_{F},\ \beta_0 = 10$$

$$\text{for } \chi = -8\ \ \beta > 6 + {}^{48}/_{F},\ \beta_0 = 11$$

$$\text{for } \chi = -9 \text{ we get } \beta_0 = 11$$

$$\text{for } \chi = -10 \text{ we get } \beta_0 = 12 \qquad (6)$$

We can summarize the results:

χ	β_0
2	6
1	6
0	7
−1	7
−2	8
−3	9
−4	9
−5	10
−6	10
−7	10
−8	11
−9	11
−10	12

● PROBLEM 9-76

From the inequality

$$\beta F > 6(F - \chi) \text{ for all } F > \beta \qquad (1)$$

derive the formula for β_0 for negative values of χ (see Problem 9-75).

SOLUTION:

From (1) we have

$$\beta > 6(1 - \chi/F) \tag{2}$$

Since $\chi < 0$, we can substitute for F the smallest admissible $\beta + 1$. Thus

$$\beta > 6(1 - \frac{x}{\beta+1}) \tag{3}$$

and

$$(\beta + 1)\,\beta > 6\beta + 6 - 6\chi \tag{4}$$

or

$$(\beta - \tfrac{5}{2})^2 > \tfrac{49}{4} - 6x \tag{5}$$

$$\beta > \sqrt{\tfrac{49}{4} - 6x} + \frac{5}{2} \tag{6}$$

The smallest integer β_0 satisfying (6) is

$$\beta_0 = \text{int}\left[\sqrt{\tfrac{49}{4} - 6x} + \frac{5}{2}\right] + 1 \tag{7}$$

where int $x = a$ means a is an integer and $x \geq a$.

For negative values of χ, we obtain:

χ	β_0
−1	7
−2	8
−3	9
−4	9
−5	10
−6	10
−7	10
−8	11
−9	11
−10	12
−11	12
−12	12
−13	13
−14	13
−15	13

● PROBLEM 9-77

Show that seven colors are necessary to paint a torus.

SOLUTION:

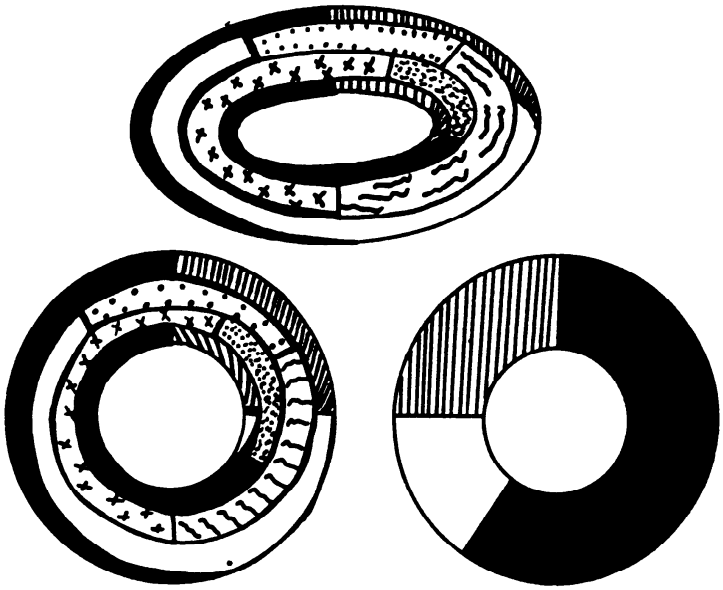


FIGURE 1

Figure 1 shows a torus divided into seven regions. Each country is a neighbor to the remaining six countries. We see that to paint a map on the torus, one needs at least seven colors. Hence the chromatic number β of the torus is

$$\beta \geq 7. \tag{1}$$

The Euler characteristic χ of the torus is $\chi = 0$. From Problem 9-75, we obtain

$$\beta \leq 7. \tag{2}$$

Hence, the chromatic number of the torus is 7.

● PROBLEM 9-78

Discuss the results of Problems 9-69, 9-71, 9-73, 9-75, 9-77.

SOLUTION:

A chromatic number, defined as the least number of colors required to color a map on any given surface, is a topological property of the surface. We tried to determine the chromatic number as a function of the Euler characteristic of the surface. Here are the results:

from Problem 9-69	for $\chi = 2$	$\beta \geq 4$
from Problem 9-73	for $\chi > 0$	$\beta \leq 6$
from Problem 9-75	for $\chi = 2$ or 1	$\beta \leq 6$
	for $\chi = 0$	$\beta \leq 7$
	for $\chi = -1$	$\beta \leq 7$
	for $\chi = -2$	$\beta \leq 8$
	for $\chi = -3$	$\beta \leq 9$
	for $\chi = -4$	$\beta \leq 9$
	for $\chi = -5$	$\beta \leq 10$
	for $\chi = -6$	$\beta \leq 10$
	for $\chi = -7$	$\beta \leq 10$
	for $\chi = -8$	$\beta \leq 11$
from Problem 9-77	for $\chi = 0$	$\beta \geq 7$

Note that the table in Problem 9-75 gives only the sufficient numbers of colors. Another problem is to determine the necessary number of colors.

It has been proven that for the surfaces of the Euler characteristic $\chi = 1$ or $\chi = 0$ or $\chi =$ an even negative integer, the values obtained in Problem 9-75 are necessary as well as sufficient. For those surfaces, the calculated values of β are the chromatic numbers. Hence

χ	Chromatic Number
1	6
0	7
-2	8
-4	9
-6	10
-8	11
-10	12

● PROBLEM 9-79

Suppose the world has the shape of a two-fold torus. What is the least number of colors needed to depict on the model of the world political situation?

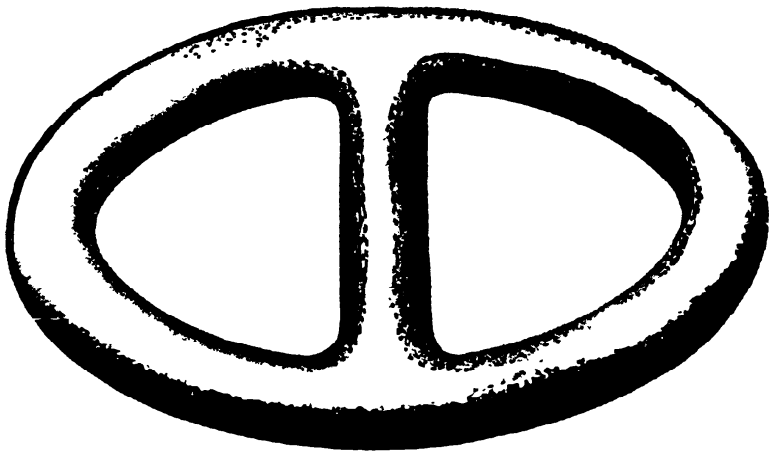


FIGURE 1

SOLUTION:

The Euler characteristic of a two-fold torus is $\chi = -2$. From the table (Problem 9-78), we find that chromatic number of a two-fold torus is $\beta = 8$. Hence, eight colors are needed to paint any map on the surface of a two-fold torus.

● PROBLEM 9-80

It has been proven that the maximum number of colors required for regular maps on a sphere or on a plane is six. Using Heawood's argument, show that this number can be reduced to five.

SOLUTION:

Any regular map on a sphere contains a region bounded by less than six arcs.

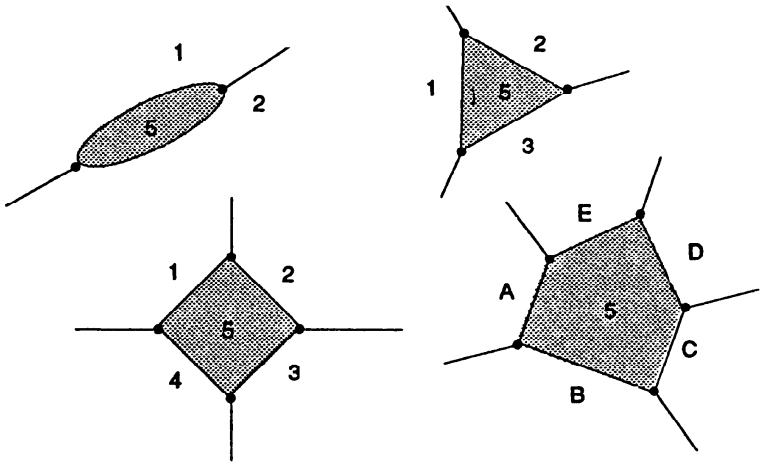


FIGURE 1

This region can always be painted with the color number five. It's obvious for the regions bounded by two, three, or four arcs.

For the region having five arcs as a boundary, some pair of the regions *A, B, C, D, E* must have no common boundary. Indeed, suppose *A* and *C* have a common boundary, then *B* has no common boundary with *D* (or *E*). Thus we need four colors to paint *A, B, C, D*, and *E* and the fifth one is available.

Suppose one arc is removed from the region bounded by two, three, or four arcs.

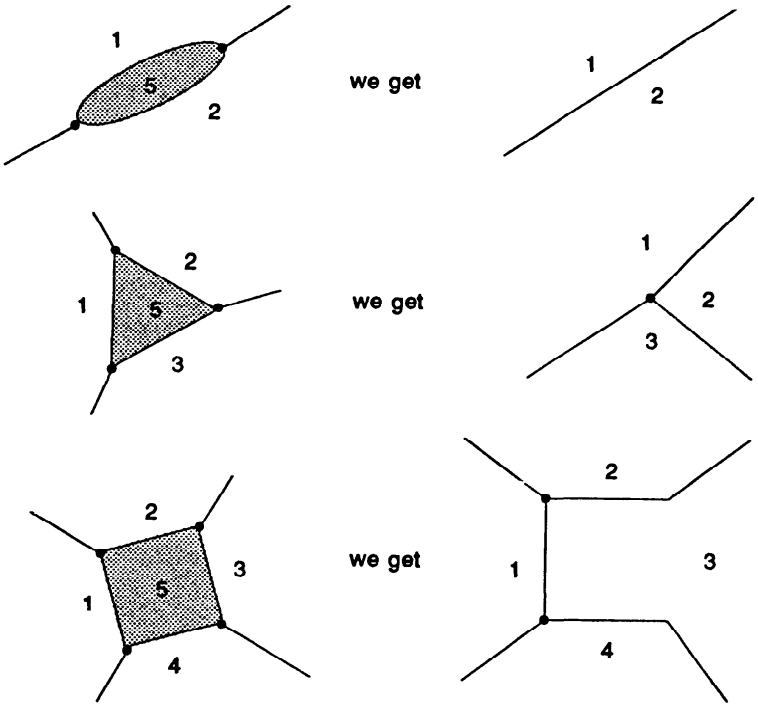


FIGURE 2

The number of regions decreases by one. If the maps in Figure 2 to the right can be colored with five colors, so can the original maps to the left.

If, for instance, *A* and *C* are of the same color, then by removing two arcs (Figure 3) we decrease the number of regions by two.

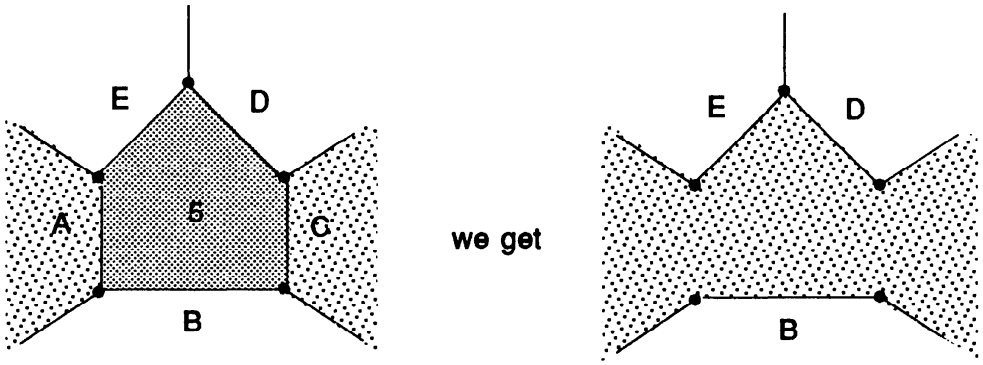


FIGURE 3

Again, if the map to the right (Figure 3) can be colored with five colors, then so can the original map. Gradually we reduce the number of regions until it is less than six. Since that can obviously be colored with less than six colors, the original map requires five colors or less. This result is known as Heawood's five color theorem.

● PROBLEM 9-81

How many colors are required to color every map on the plane surface?

SOLUTION:

Four! In 1976 the four-color problem was solved. For over a hundred years many mathematicians have tried in vain to prove (or disprove) this simple statement:

Four colors are required to color any map on the plane surface.

Attempting to find the solution, mathematicians like Heawood, Kempe and Birkhoff developed the new branch-graph theory, now used in arranging the airline routes and wiring diagrams.

In 1975 a group of scientists at the University of Illinois, using an IBM 360 computer, proved the four-color conjecture. The method used involved the reducibility of over 1,500 configurations.

It is amazing that the complete proof of this simple statement consists of hundreds of pages, miles of calculations and is impossible to carry out without the use of the computer.

Chances are that one day some bright teenager will find a short and elegant proof. It is also conceivable that no such proof is possible.

● PROBLEM 9-82

1. Which of the points *A, B, C,* and *D* lie inside the curve and which lie outside the curve?

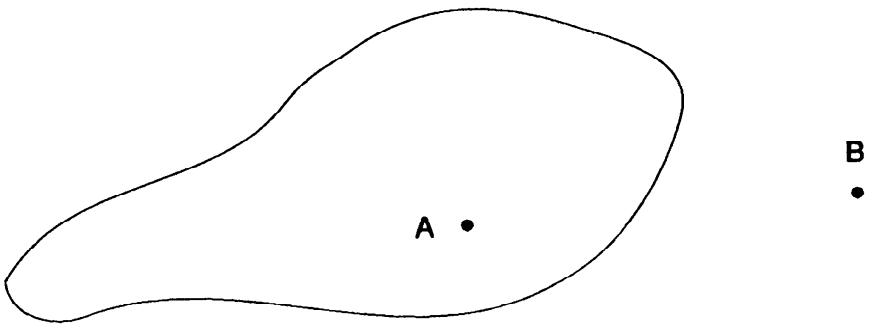


FIGURE 1

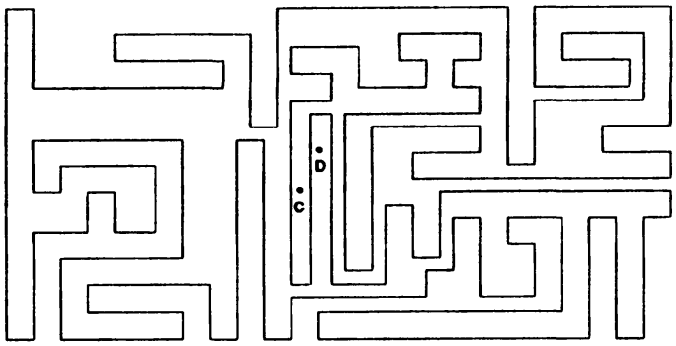


FIGURE 2

2. Why are both curves Jordan curves?

SOLUTION:

1. It is obvious that a triangle (or circle) separates a plane surface into an area inside and outside its perimeter. The same property is assumed for any continuous non-self-intersecting closed curve. For the curve of Figure 1, it is easy to see that A lies inside its perimeter and B outside its perimeter.

For more complicated curves, such as in Figure 2, it takes a while to determine what is inside and what is outside. In this case, C is outside and D is inside.

Even though the terms "inside" and "outside" are intuitively simple, they require a rigorous mathematical treatment.

2. By definition, any curve homeomorphic to a circle is called a Jordan curve.

● PROBLEM 9-83

Define and give an example of a polygonal path.

SOLUTION:

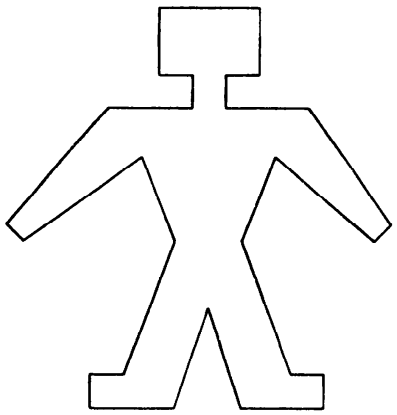


FIGURE 1

A polygonal path consists of n straight line segments joining n distinct points in a plane. Line segments can intersect only at their end points. Each line segment joins two points uniquely. The straight line segments are called the sides of the polygon. Polygonal paths determine the perimeter of the polygon. The figure depicts a polygonal path.

● PROBLEM 9-84

JORDAN CURVE THEOREM

On a plane or on the surface of a sphere, a Jordan curve separates the surface into two disjoint regions having the curve as a common boundary.

■

It is surprising that this theorem is obvious to almost everybody but it requires a fairly complex proof.

Prove the Jordan curve theorem for a polygonal path.

SOLUTION:

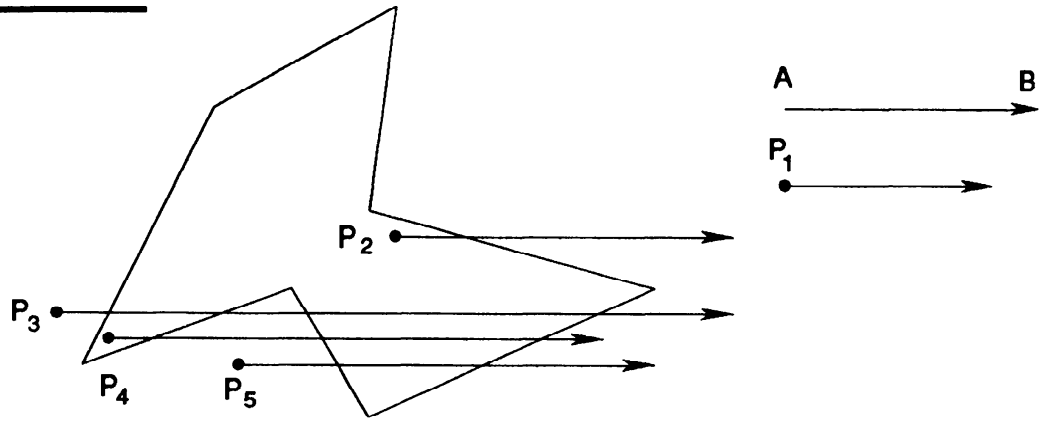


FIGURE 1

For a given polygonal path, it is always possible to find a line segment AB which is not parallel to any of the sides. This separates the plane (with the exception of the polygonal path) into two disjoint sets:

α - the set of points such that a ray from the point in the direction parallel to AB intersects the polygonal path an even number of times;

β - an odd number of times.

$$P_1 \in \alpha, \quad P_3 \in \alpha, \quad P_5 \in \alpha,$$
$$P_2 \in \beta, \quad P_2 \in \beta$$

from Figure 1.

Suppose a point moves along a line segment not intersecting a polygonal path and not parallel to AB, (Figure 2).

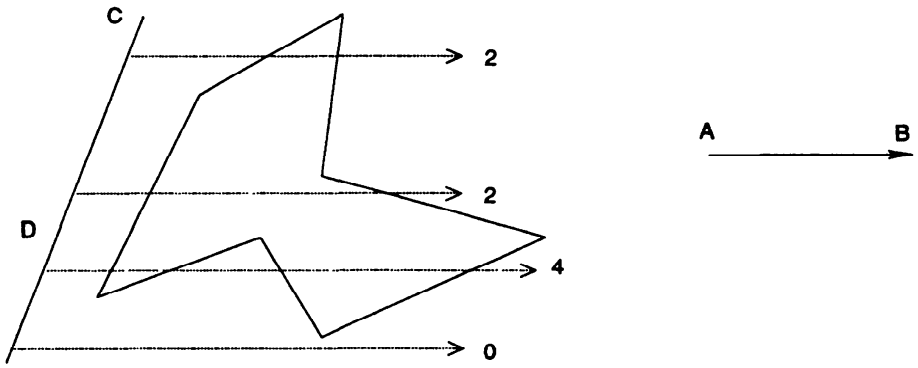


FIGURE 2

The number of intersections of the ray with the polygonal path always changes by a multiple of two. Thus, if any point belonging to α is joined to any point of β by a polygonal path, this new path must cross the old one an odd number of times. See Figure 3.

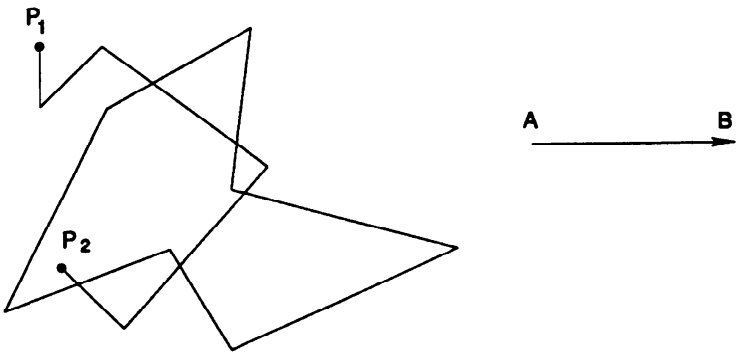


FIGURE 3

If both points P_1 and P_2 belong to the same subset (α or β), they can be joined by a polygonal path not intersecting the original polygonal path.

Thus it is possible to identify α as the set of points outside the polygonal path and β as the set of points inside.

● PROBLEM 9-85

Look again at the proof of the Jordan curve theorem for a polygonal path (Problem 9-84). Why does the same method applied to any Jordan curve lead to difficulties?

SOLUTION:

Since a polygonal path contains the finite number of sides of the polygon, it is easy to find a direction not parallel to any side.

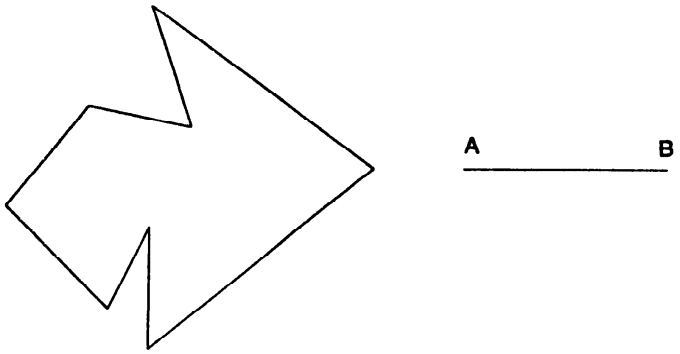


FIGURE 1

AB is not parallel to any side. For a closed curve it may no longer be possible to find a direction which is nowhere tangential to a curve. Obviously for a circle, no straight line in the plane exists, which is not parallel to some tangent to the circle. See Figure 2 where *CD* is parallel to *AB*.

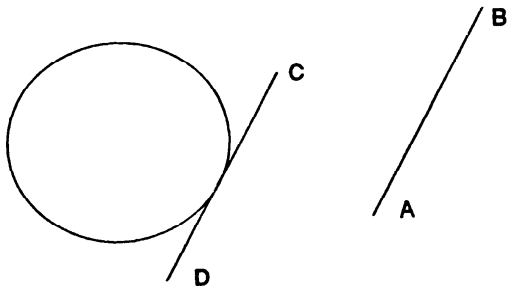


FIGURE 2

● PROBLEM 9-86

Another amazing statement in topology is that in a plane there are infinitely many simply connected bounded regions which all have the same boundary. Consider this puzzle for three regions:



FIGURE 1

There is an island. Within the first day you have to dig three canals:

from the sea, from the cold lake, and from the warm lake in such a way that the waters are separated and any point of the island is less than 1 mile from any water by the end of the day.

In the next half day the digging continues according to the same principles and each point has to be less than a $^1/_2$ mile from each kind of water. The work is carried out in intervals 1 day, $^1/_2$, $^1/_4$, $^1/_8$, $^1/_{16}$, ... of the day.

What happens at the end of the second day?

SOLUTION:

The island (or whatever is left of it) forms a closed set A nowhere dense. Each point of A is arbitrarily near to each kind of water.

Note that A is the common boundary of three regions: the sea, the warm lake, and the cold lake.

● PROBLEM 9-87

Rotate a disc and an annulus in its own plane about its own center. What point (or points) is always mapped into itself?

SOLUTION:

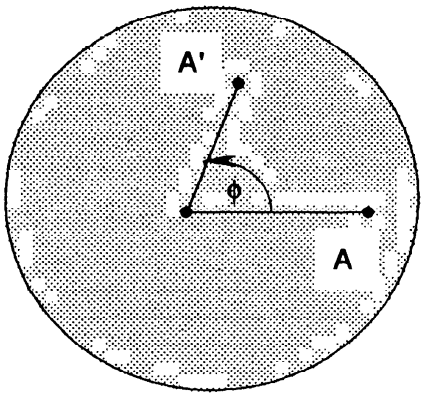


FIGURE 1

Let ϕ be the angle of rotation. The transformation is rigid, one-to-one and continuous. Each point A of the disc is mapped into A'. For ϕ not an integer multiple of 2π, there is one point only, which maps into itself, namely the center of the disc.

For the same rotation of an annulus there is no point which maps into itself.

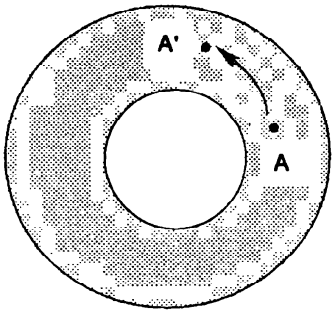


FIGURE 2

● PROBLEM 9-88

Explain the result of Problem 9-87 in terms of Brouwer's fixed point theorem.

SOLUTION:

BROUWER'S FIXED POINT THEOREM

For any continuous transformation of a disc to itself, there is at least one point which is mapped to itself.

■

Since rotation is a continuous transformation, at least one point of a disc is mapped into itself.

There is no continuous transformation f of a disc into itself

$$f: D \rightarrow D$$

such that, for every $x \in D$, $f(x) \neq X$.

Brouwer's fixed point theorem deals with the topological property, hence it holds for any region homeomorphic to a disc. It can be applied to various situations in life.

Consider for example a pool of oil spilled on the road.

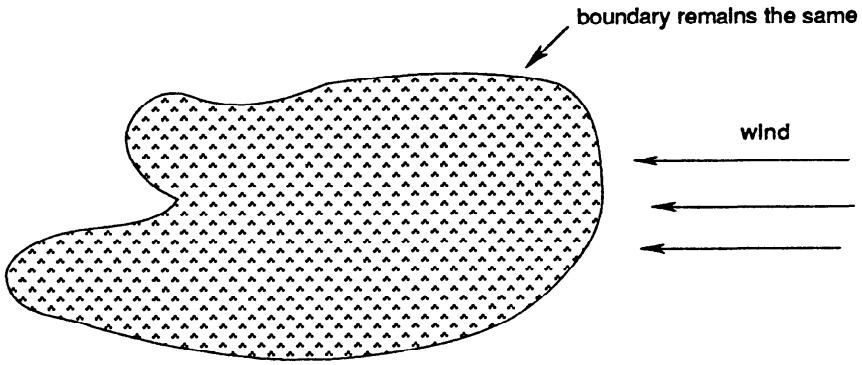


FIGURE 1

When the breeze blows, the surface of oil is moved. Assuming the boundary of oil remains the same and the oil is not "broken" in any way, there is at least one point where the oil remains in exactly the same place as it was before the wind began to blow.

● PROBLEM 9-89

A thin sheet of an elastic material is stretched as shown in Figure 1a.

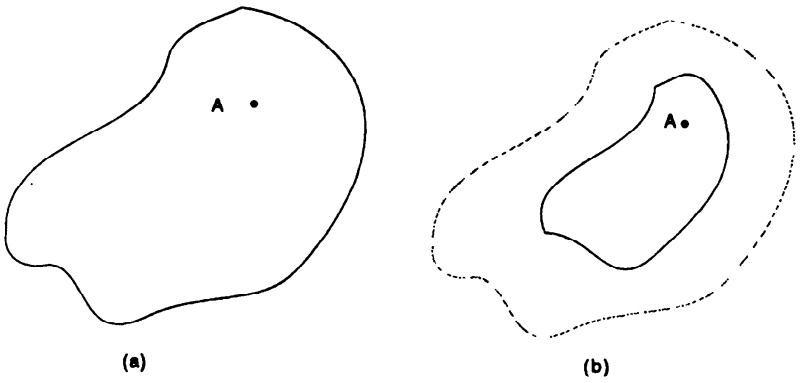


FIGURE 1

Show that when the material contracts so as to occupy only a part of its original area (Figure 1b), then there will always be a point A that occupies the same location before and after contraction.

SOLUTION:

Contraction is a continuous transformation, thus we can apply Brouwer's fixed point theorem.

We conclude that at least one point occupies the same place after contraction as before. Note that the same is true if we stretch out an elastic material.

● PROBLEM 9-90

Prove Problem 9-89 without direct reference to Brouwer's fixed point theorem.

SOLUTION:

Suppose the original region is *colored in the form of a chessboard* and

point P occupies P' after contraction. All squares can be grouped into three categories:

I. For every point belonging to the square, P' lies nearer the right side than this point.

II. For every point belonging to the square, P' lies nearer the left side than this point.

III. Squares that are neither I nor II.

It is easy to see that the squares of type I and II are never neighbors. Thus, square III goes from top to the bottom and a line can be drawn passing only through the squares of type III, which join the upper edge with the base.

Let us draw an arrow PP' at each point of this line. Since not all P' can lie higher than P not all arrows are directed upward. By the same token not all arrows are directed downward. Hence at least one point Q exists with a horizontal QQ'.

Since Q belongs to square III, there must be point R in this square with a vertical RR'.

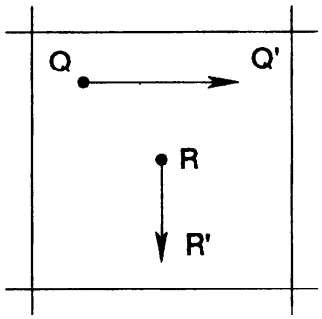


FIGURE 1

Since the square is small, such an abrupt change of direction is impossible (note that transformation is continuous) for a large PP'.

Hence, the arrow PP' for all points of the square must be small. When the number of squares increases, we reach the limit of point P_0, such that

$$P_0P_0' = 0 \quad \text{that is,} \quad P_0 = P_0'$$

and P_0 is the fixed point of the transformation.

● PROBLEM 9-91

Here is a curious consequence of the Brouwer's fixed point theorem:

At any given time there are two points on the earth which are antipodes and which have the same temperature and the same air pressure.

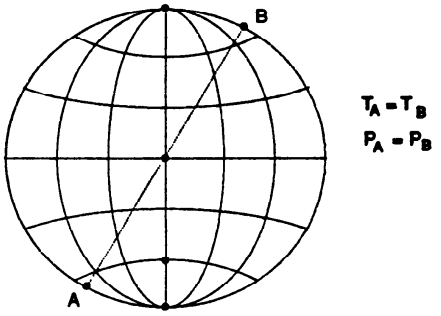


FIGURE 1

Explain why.

SOLUTION:

Suppose you have a sphere made of elastic material, for example, an inflatable ball. If the ball is folded and deflated so as to become flat, then there are two antipodes, A and B, which will lie one upon another.

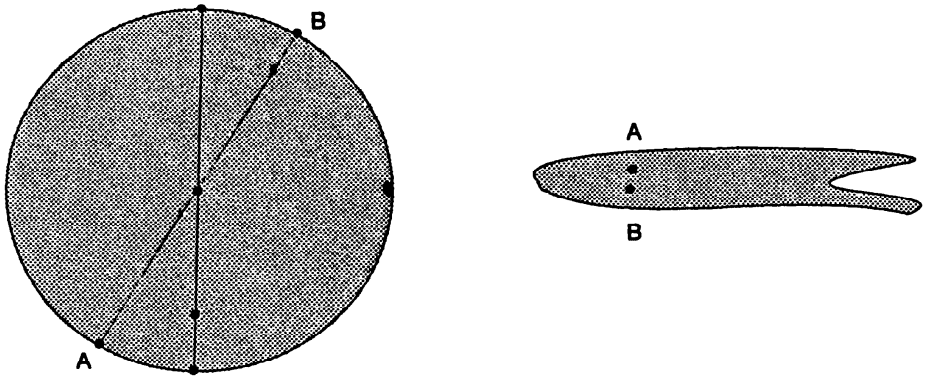


FIGURE 2

It can be proven by using Brouwer's fixed point theorem. Since both temperature and pressure are continuous, the conclusion follows.

● PROBLEM 9-92

Here is a fixed point theorem for one dimension:

If an interval is continuously transformed to itself, then there is at least one point of the interval which remains fixed.

■

One way to prove it is to divide the interval into small segments. List all complete segments for each division of the interval AB shown below.

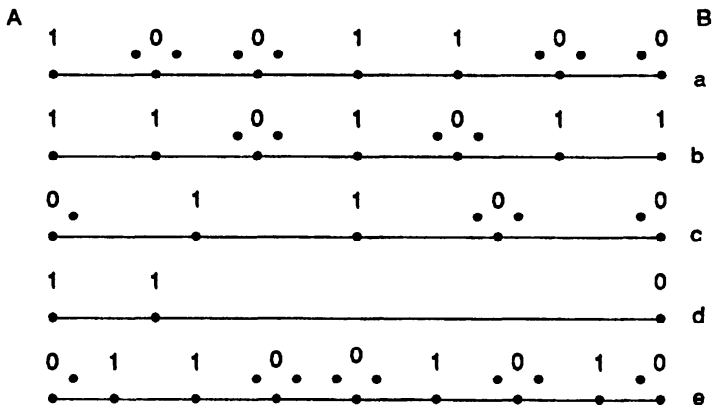


FIGURE 1

SOLUTION:

On the original segment AB we arbitrarily choose any number of points. All these points and the end-points A and B are labeled at random, 0 or 1.

On both sides of every 0, we put a dot. When a zero is at the end-point A or B, it gets one dot.

A small segment (i.e., one that does not include any subsegments) is called a complete segment when it has a 0 at one end and a 1 at the other.

For each division a to e, we list all small segments and circle the complete segments.

a. (10), 00 , (01), 11 , (10), 00

b. 11 , (10), (01), (10), (01), 11

c. (01), 11 , (10), 00

d. 11 , (10)

e. (01), 11 , (10), 00 , (01), (10), (01), (10)

Each complete segment contains one dot, other segments contain no dots or two dots.

● PROBLEM 9–93

Show that if the end-points of the original line segment are labeled 0 and 1, then any division of this segment contains at least one complete segment.

SOLUTION:

Let n be the number of complete segments. Then the total number of dots is n plus some even positive integer

$$\#\text{ dots} = n + 2k.$$

On the other hand,

$$\#\text{ dots} = \#\text{ of 0's at the end-points} + 2 \cdot l$$

where l is the number of interval 0's.

We conclude that if the end-points of the original segment are labeled 0 and 1, then the number of complete segments must be odd. Thus, an original segment 01 (or 10) contains at least one complete segment.

● PROBLEM 9-94

By applying Problems 9-92 and 9-93, prove Brouwer's fixed point theorem for one dimension.

SOLUTION:

Suppose the original line segment, labeled 01, is continuously transformed into itself.

We label the points of the transformed segment 0, if their distance from the end-point 0 has not decreased and 1, if their distance from end-point 1 has not decreased.

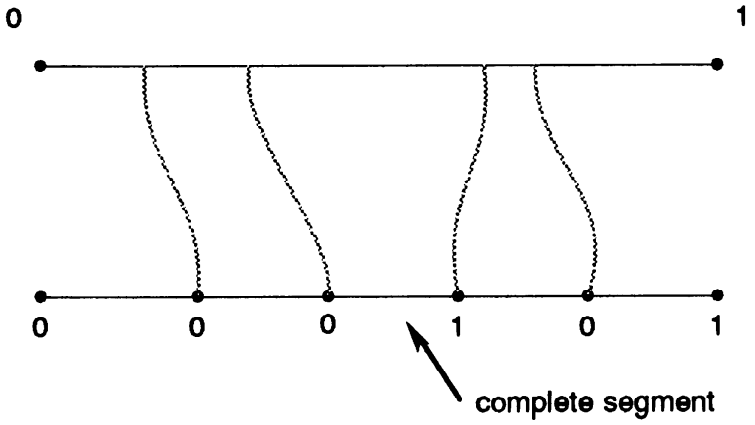


FIGURE 1

There must be at least one complete segment and since the labeling can go on infinitely, this segment can be made arbitrarily small.

At the limit, this segment tends to a single point, which is the fixed point we are seeking.

● PROBLEM 9-95

Prove Brouwer's fixed point theorem for two dimensions.

SOLUTION:

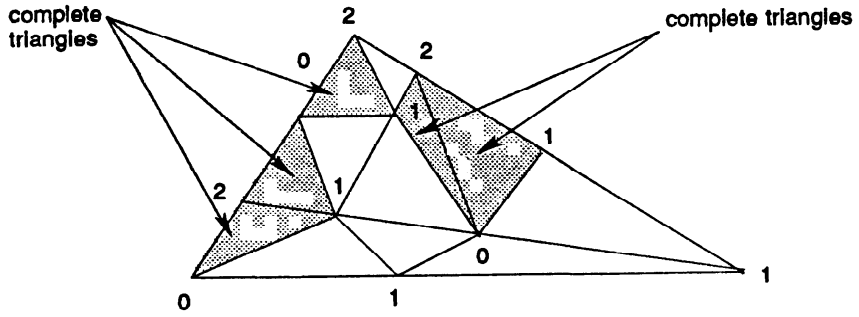


FIGURE 1

We shall use a method similar to the one-dimensional case. Consider a triangle arbitrarily subdivided into smaller triangles. The original triangle is labeled 012. The vertices lying on the side 01 are labeled 0 or 1 only, vertices on 02 are labeled 0 or 2 only and vertices on 12 are labeled 1 or 2 only.

A complete triangle is defined as a small triangle with vertices labeled 012.

This division of an original triangle contains at least one complete triangle. Next we continuously transform and divide the original triangle. Diminishing the complete triangle becomes the fixed point.

The above theorem is sometimes called Sperner's lemma.

In the same manner, we can prove Brouwer's fixed point theorem for any dimension.

● PROBLEM 9-96

Make a necessary cut of the cylinder (Figure 1) and draw the corresponding plane diagram.

SOLUTION:

Curves x and y are the boundaries of the cylinder. By cutting the cylinder along the line AB and then opening it, we obtain a rectangle. To ensure that the rectangle can be folded back, so as to give the cylinder we described its vertices; and use arrows to indicate that there is no twisting.

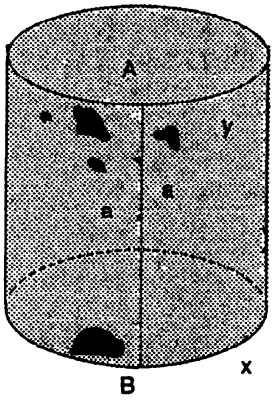


FIGURE 1

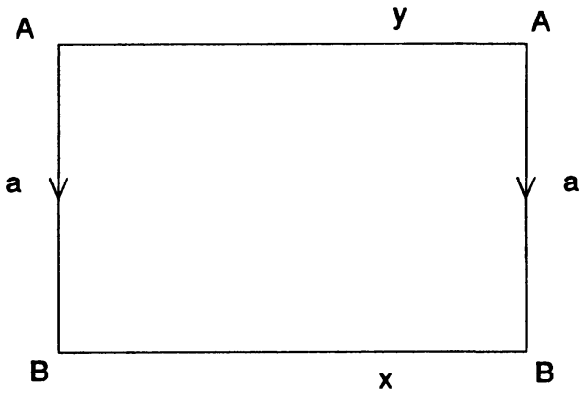


FIGURE 2

The diagram obtained as such is called a plane diagram.

● PROBLEM 9-97

For the torus depicted in Figure 1, make the needed cuts and sketch its corresponding plane diagram.

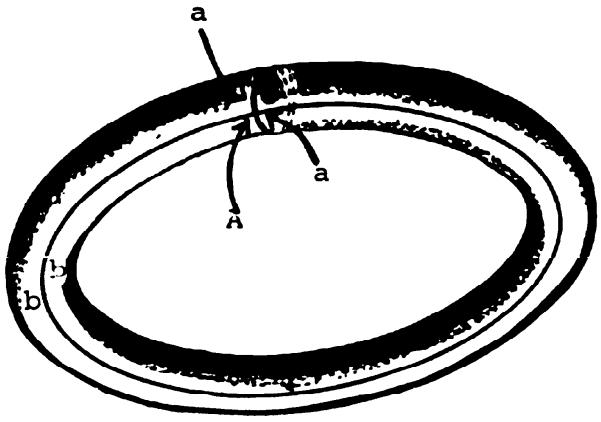


FIGURE 1

SOLUTION:

Two cuts, *a, b* are shown in Figure 1. The first cut, *a*, enables us to straighten the torus so as to obtain a cylinder.

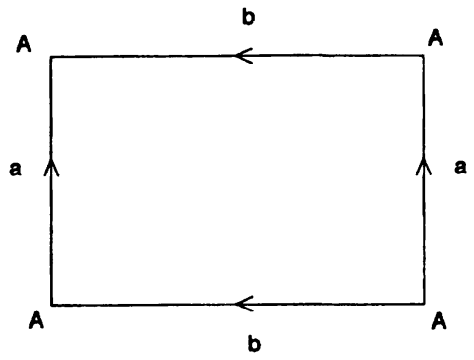


FIGURE 2

Here again arrows indicate that there is no twisting. Also note that all four vertices are labeled with the same letter.

● PROBLEM 9-98

Make the necessary cuts and draw a plane diagram (rectangle) of a sphere.

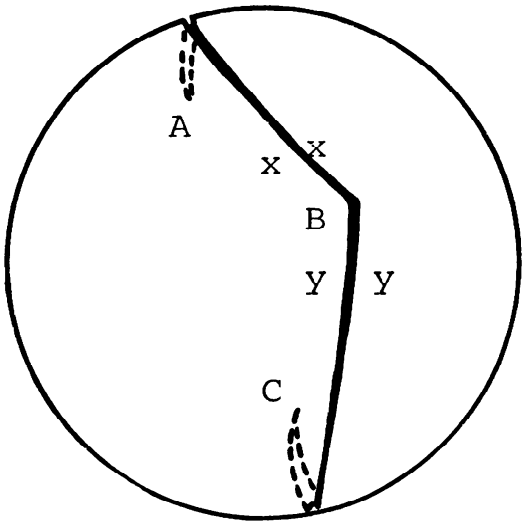


FIGURE 1

SOLUTION:

To obtain a plane diagram, we have to cut the sphere twice as shown in Figure 1. To obtain a rectangle, we have to deform the cuts. The plane diagram of the sphere is shown in Figure 2.

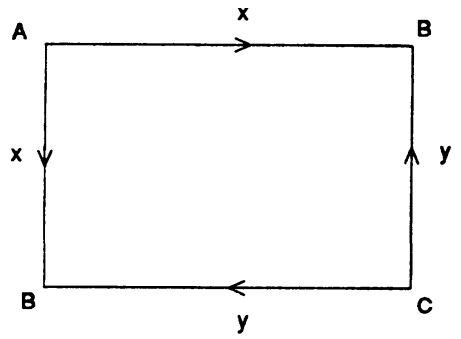


FIGURE 2

● PROBLEM 9-99

Sketch the plane diagrams representing a Möbius band and a Klein bottle.

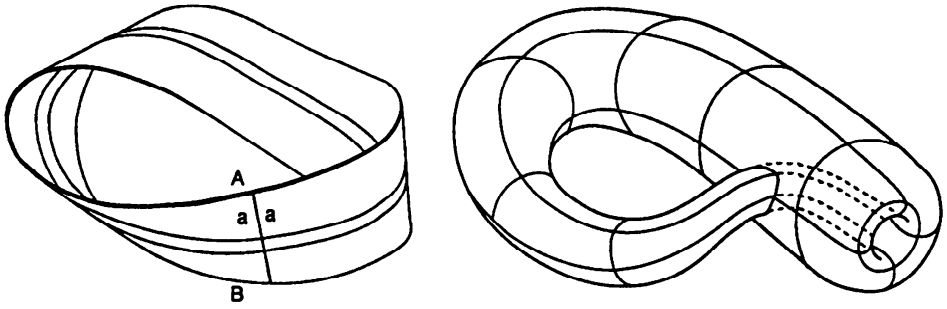


FIGURE 1

SOLUTION:

We cut a Möbius band along the line AB and then twist it (Figure 2). Note that we used two different letters, x and y, since the edges are not joined together.

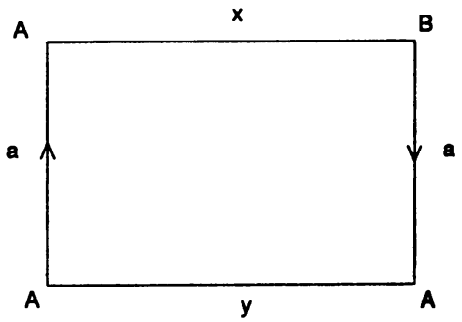


FIGURE 2

Note that from the plane diagram of a Klein bottle it is impossible to obtain a Klein bottle in three-dimensional space.

We have to make two cuts in order to obtain a plane diagram of a Klein bottle (Figure 3).

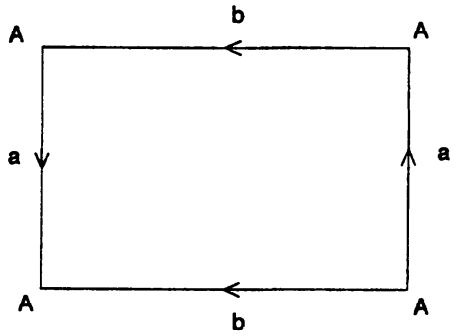


FIGURE 3

● PROBLEM 9-100

Draw the plane diagram of the real projective plane. Compare the plane diagrams of a Möbius band, Klein bottle and the real projective plane.

SOLUTION:

Compare these three diagrams:

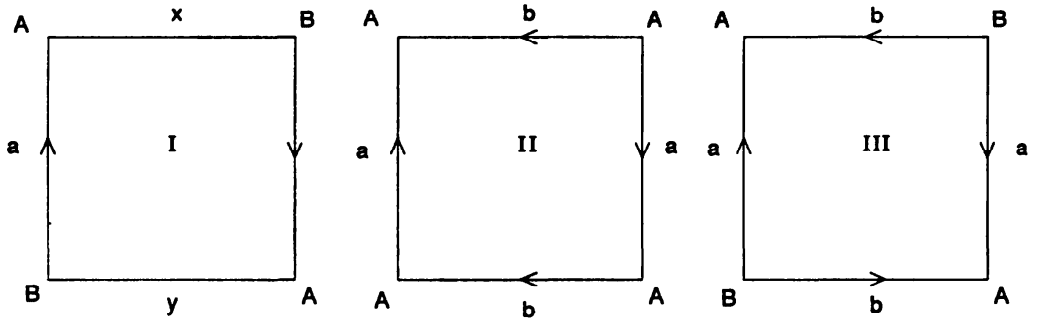


FIGURE 1

I is the plane diagram of a Möbius band. By adding arrows to avoid twisting, we obtain a diagram of a Klein bottle. If the opposite sides of the rectangle are to be joined together, then we should have to place the arrows in opposite directions as in diagram III. III is the plane diagram of the real projective plane, where each pair of opposite sides is joined in the opposite direction (with a twist).

The real projective plane can be represented as follows:

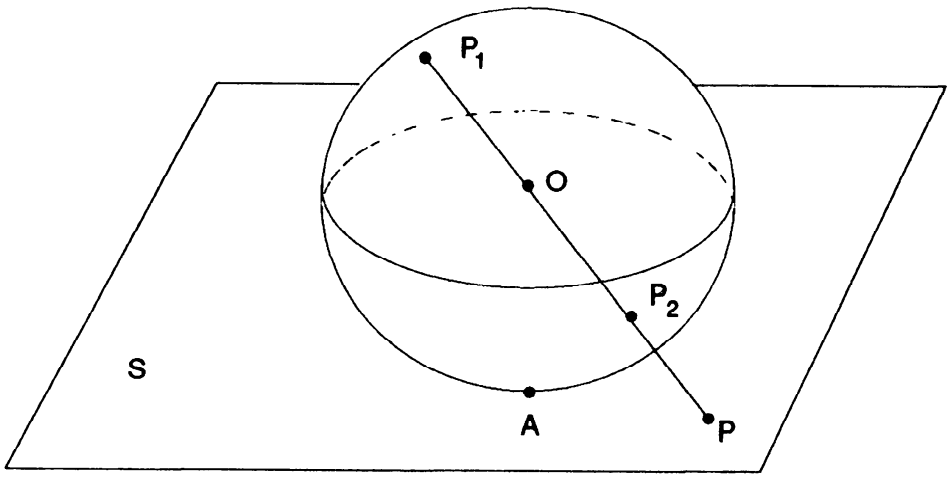


FIGURE 2

There is a sphere and a plane, which is tangential to the sphere at point A. Points P_1 and P_2 on the sphere are mapped to the single point P on the plane. Every great circle of the sphere is mapped to a line, with the exception of the great circle parallel to S. This circle is mapped into a line at infinity. The Euclidean plane, with the line of infinity added, is called the real projective plane.

● PROBLEM 9-101

From the plane diagrams, obtain the Euler characteristic χ of

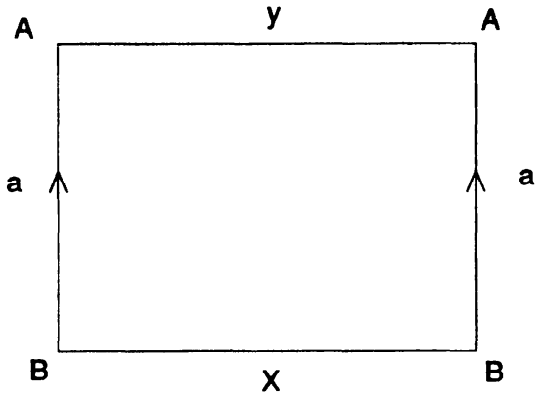
1. a cylinder
2. a one-fold torus
3. a sphere
4. a Möbius band
5. a Klein bottle
6. a real projective plane.

SOLUTION:

We shall apply the expression

$$\chi = V - E + F. \tag{1}$$

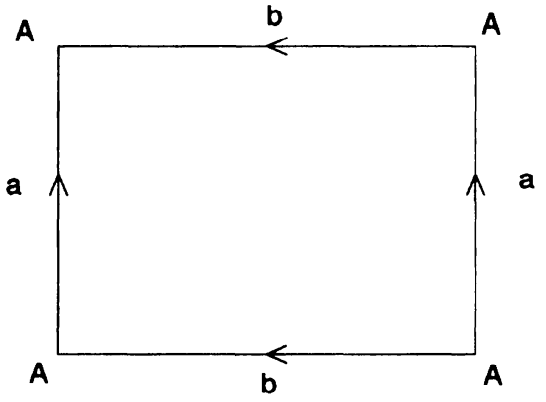
For a cylinder



there are two distinct vertices, $V = 2$, three distinct sides, a, x, y, $E = 3$, and one face $F = 1$. Thus

$$\chi = 2 - 3 + 1 = 0.$$

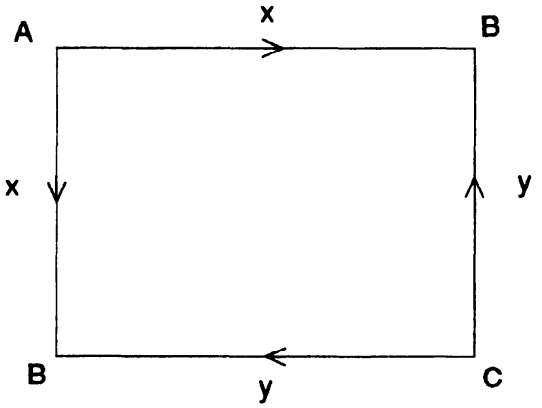
For a one-fold torus



$V = 1, E = 2, F = 1$, hence

$$\chi = 1 - 2 + 1 = 0.$$

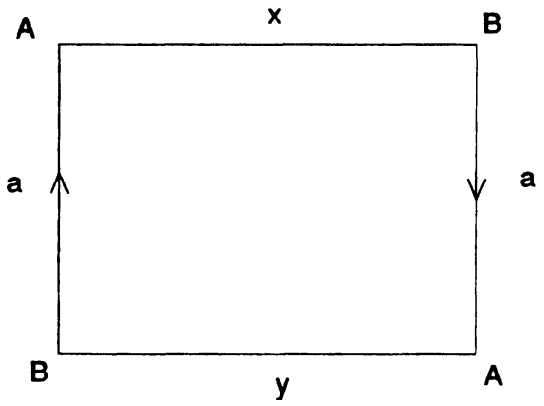
For a sphere



$V = 3, E = 2, F = 1$, hence

$$\chi = 3 - 2 + 1 = 2.$$

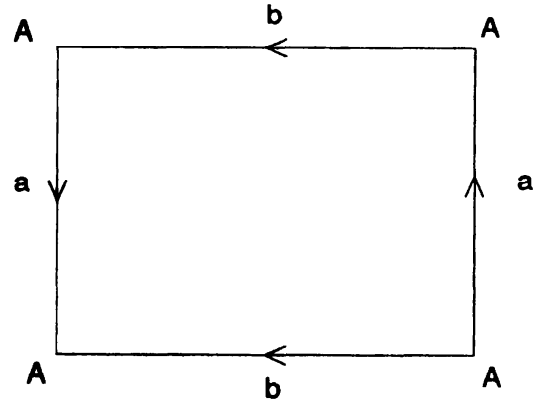
For a Möbius band



$V = 2, E = 3, F = 1$, hence

$$\chi = 2 - 3 + 1 = 0.$$

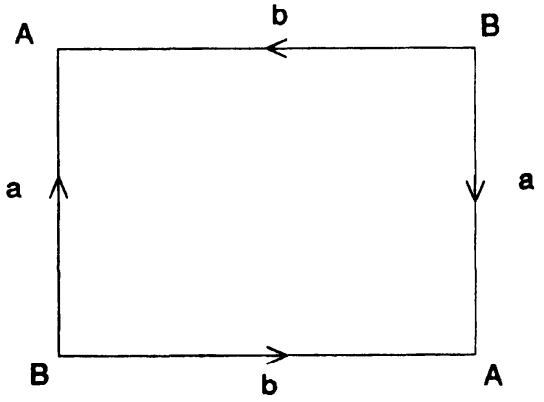
For a Klein bottle



$V = 1, E = 2, F = 1$, hence

$$\chi = 1 - 2 + 1 = 0.$$

For a real projective plane



$V = 2, E = 2, F = 1$, hence

$$\chi = 1.$$

● PROBLEM 9-102

We already proved that the maximum number of colors required for any map on the surface of a torus is seven.

By using a plane diagram of a torus, show that seven is a necessary number. Thus, prove that the chromatic number of a torus is seven.

SOLUTION:

By remembering that a plane diagram will be folded into a torus, we can color the rectangle as follows:

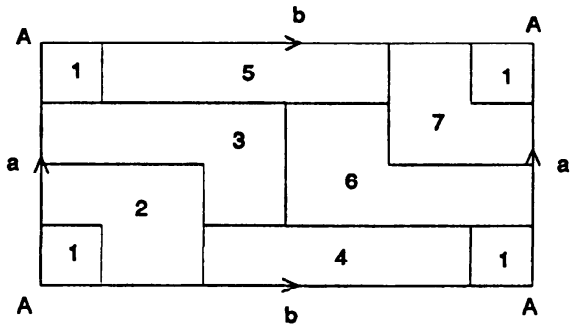


FIGURE 1

Note that when folded, each color borders with the remaining 6. This result is known as the seven color theorem for a one-folded torus.

● PROBLEM 9-103

Find the symbolic representation of a torus and a sphere.

SOLUTION:

The plane diagrams are

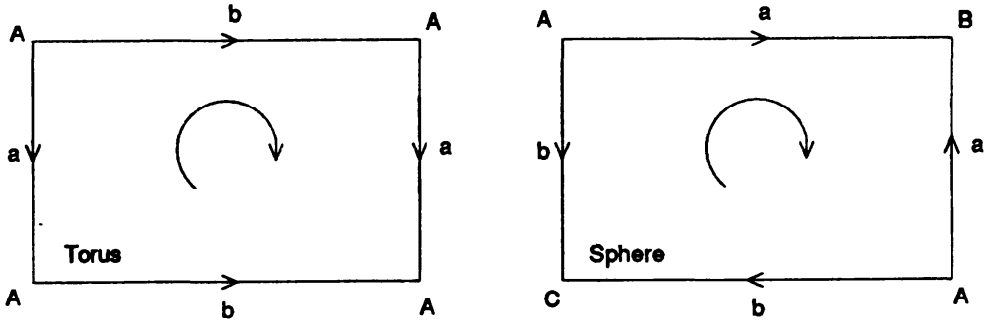


FIGURE 1

Let us establish the reference orientation as clockwise (we can choose counter-clockwise as well).

Starting from, for instance, the left upper corner and assigning + to each edge when the arrow moves clockwise or – when counter-clockwise we obtain

$$+b+a-b-a$$

for a torus. And

$$+a-a+b-b$$

for a sphere. Note that we can start from any vertex, hence, for a sphere equivalent representations are

$$-a+b-b+a$$

$$+b-b+a-a$$

$$-b+a-a+b$$

● PROBLEM 9-104

What is the Euler characteristic of a sphere with s holes?

SOLUTION:

The Euler characteristic of a sphere is $\chi = 2$.

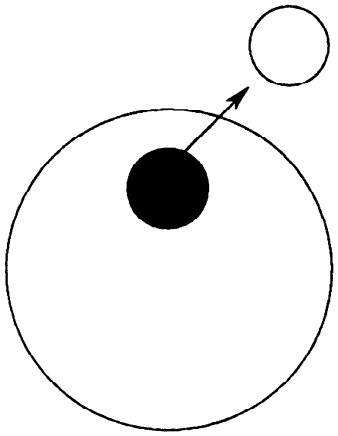


FIGURE 1

Suppose a hole is removed from a sphere (Figure 1). Then the number of the arc increases by one. The Euler characteristic

$$2 = \chi = V - E + F \tag{1}$$

decreases by one. Hence for a sphere with one disc removed

$$\chi = 2 - 1 = 1. \tag{2}$$

When two discs are removed

$$\chi = 2 - 2 = 0. \tag{3}$$

The sphere with two holes is homeomorphic to an open cylinder.

In general, a sphere with s holes has the Euler characteristic

$$\chi = 2 - s. \tag{4}$$

● PROBLEM 9-105

What is the Euler characteristic of a sphere with n handles and no holes?

SOLUTION:

A sphere with two handles and no holes (see Figure 1) is homeomorphic to a two-fold torus.

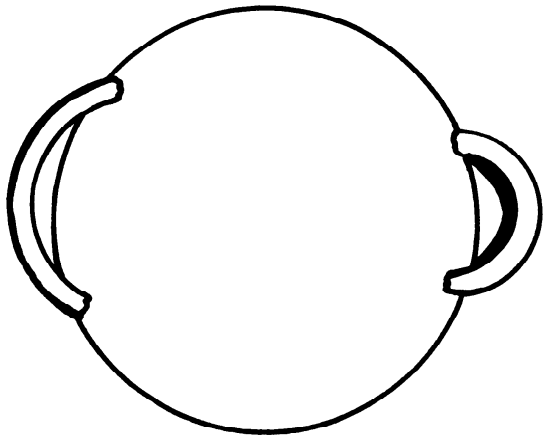


FIGURE 1

The Euler characteristic of a two-fold torus is – 2. Hence, the Euler characteristic of a sphere with two handles is – 2.

In general, a sphere with n handles and no holes left open is homeomorphic to an n-fold torus. The genus of the surface of an n-fold torus is n. But

$$\chi = 2 - 2g$$

Then

$$\chi = 2 - 2n$$

is the Euler characteristic of a sphere with n handles.

The Euler characteristic of a sphere with no handles is

$$\chi = 2 - 2 \cdot 0 = 2$$

with one handle

$$\chi = 2 - 2 = 0$$

with two handles

$$\chi = 2 - 4 = -2$$

etc.

● PROBLEM 9-106

Suppose a sphere with no holes and n handles is given with a map drawn on it. Since every region of the map must be simply connected, at least one arc must be drawn along every handle (Figure 1).

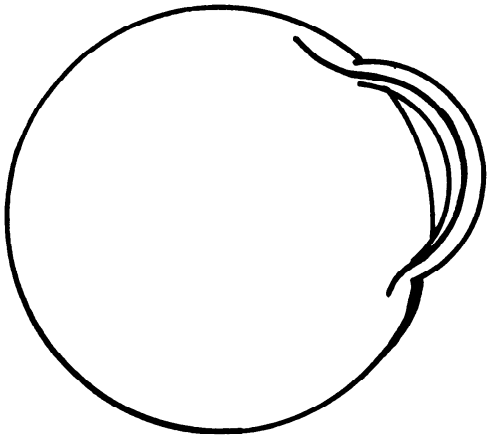


FIGURE 1

By using the procedure described below, determine the Euler characteristic of a sphere with n handles from

$$\chi = V - E + F.$$

SOLUTION:

We shall detach one end of each handle in order to obtain a sphere with n holes and n pipes, called cuffs.

I. **First Stage**

We have a map on a sphere with n handles and no holes, such that $\chi = V - E + F$.

II. Additional vertices are added to the map, one at every intersection of an arc and a boundary where a handle is to be disengaged. Such a boundary is now treated as an arc. Hence

V becomes $V + k$

E becomes $E + 2k$

F becomes $F + k$

where k is the number of additional vertices to be added.

III. **Detachment**

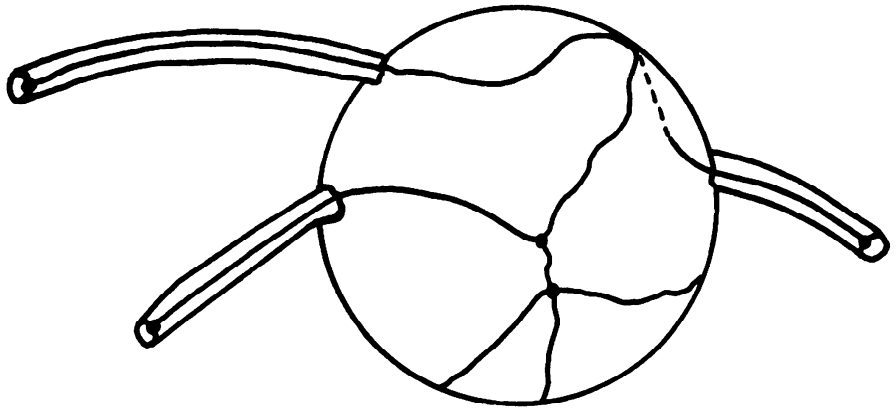


FIGURE 2

Figure 2 shows a sphere with 3 handles detached. In order to make a surface closed, we have to cover n holes and n open cuffs with $2n$ discs. When the handles are detached, k arcs and k vertices are added.

Now we have

$V = 2k$ vertices, $E + 3k$ arcs, and F + K + $2n$ regions.

The thereby obtained surface is homeomorphic to a sphere. Hence,

$$\chi = 2 = (V + 2k) - (E + 3k) + (F + k + 2n) \tag{2}$$

and

$$V - E + F = 2 - 2n \tag{3}$$

$$\chi = 2 - 2n. \tag{4}$$

Cf. Problem 9-106.

● PROBLEM 9-107

The Euler characteristic of a cylinder is zero and of a disc is one. By attaching and removing various surfaces, find the Euler characteristic of a sphere with 7 handles and 4 holes.

SOLUTION:

It is easy to calculate the Euler characteristic from the formula

$$\chi = 2 - 2n - k \tag{1}$$

where n is the number of handles and k is the number of holes

$$\chi = 2 - 2 \cdot 7 - 4 = -16. \tag{2}$$

The other way is to start with a brand new sphere $\chi = 2$. Then make $2 \cdot 7 = 14$ holes for handles $\chi = 2 - 2 \cdot 7$, then 4 holes $\chi = 2 - 2 \cdot 7 - 4$, then put 7 cylinders to make handles, $\chi = 2 - 2 \cdot 7 - 4 + 7 \cdot 0$. Thus,

$$\chi = \chi\,[\text{sphere}] - \chi\,[\text{disc}] \cdot (2n + k) + \chi\,[\text{cylinder}] \cdot n =$$

$$= 2 - 14 - 4 = -16. \tag{3}$$

● PROBLEM 9-108

Generally, if n open surfaces $P_1, P_2, \ldots, P_n$ are joined together along the boundaries, then the Euler characteristic χ of the resulting surface is

$$\chi = \chi[P_1] + \chi[P_2] + \ldots + \chi[P_n]\,. \tag{1}$$

By applying (1), obtain the Euler characteristic of a sphere, putting together

1. n cylinders and two discs.

2. a sphere with k holds and k discs.

SOLUTION:

1. By putting n open cylinders, end to end, we obtain a single cylinder. By adding two discs, we obtain a surface homeomorphic to the surface of a sphere.

$$\chi\,[\text{sphere}] = \chi\,[\text{cylinder with two discs}] =$$

$$= n\chi\,[\text{cylinder}] + 2\chi\,[\text{disc}] = n \cdot 0 + 2 \cdot 1 = 2. \tag{2}$$

2. The Euler characteristic of a sphere with k holes is

$$\chi = 2 - k.$$

By covering k holes with k discs, we obtain a sphere. Hence

$$\chi\,[\text{sphere}] = \chi\,[\text{sphere with } k \text{ holes}] +$$

$$+ k\chi[\text{disc}] = 2 - k + k = 2. \tag{3}$$

● PROBLEM 9-109

Determine the Euler characteristic of a cross-cap.

SOLUTION:

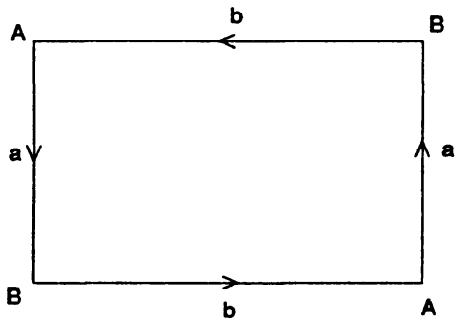


FIGURE 1

The plane diagram (Figure 1) of the real projective plane (see Problem 9-100) can be topologically deformed into a sphere with the appropriate hole shown in Figure 2.

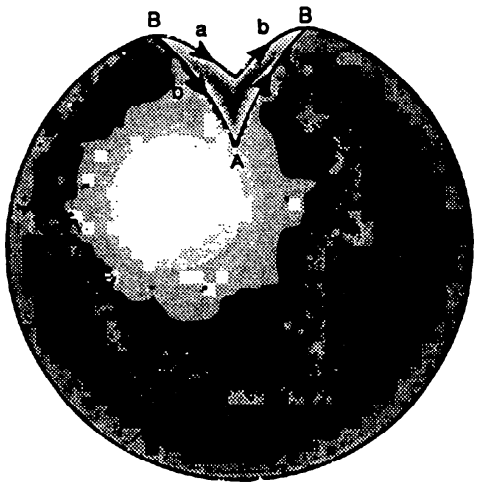


FIGURE 2

By closing the hole in such a way that the sides are joined according to the arrows, we obtain a closed surface intersecting itself (see Figure 3). The surface is one-sided. Points A and B are single; all other points of the segment AB are double points.