GENERAL TOPOLOGY

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1. Sets, functions and relations

1.1. **Sets.** A set is a collection of mathematical objects. We write $a \in S$ if the set S contains the object a.

1.1. **Example.** The natural numbers $1, 2, 3, \ldots$ can be collected to form the set $\mathbf{Z}_+ = \{1, 2, 3, \ldots\}$.

This naïve form of set theory unfortunately leads to paradoxes. *Russel's paradox*[1] concerns the formula $S \notin S$. First note that it may well happen that a set is a member of itself. The set of all infinite sets is an example. The Russel set

$$R = \{S \mid S \notin S\}$$

is the set of all sets that are not a member of itself. Is $R \in R$ or is $R \notin R$?

How can we remove this contradiction?

1.2. **Definition.** *The universe of mathematical objects is stratified. Level* 0 *of the universe consists of (possibly) some* atomic *objects. Level $i > 0$ consists of collections of objects from lower levels. A set is a mathematical object that is not atomic.*

No object of the universe can satisfy $S \in S$ for atoms do not have elements and a set and an element from that set can not be in the same level. Thus R consists of everything in the universe. Since the elements of R occupy all levels of the universe there is no level left for R to be in. Therefore R is outside the universe, R is not a set. The contradiction has evaporated!

Axiomatic set theory is an attempt to make this precise formulating a theory based on axioms, the ZFC-axioms, for set theory. (Z stands for Zermelo, F for Fraenkel, and C for Axiom of Choice.) It is not possible to prove or disprove the statement "ZFC is consistent" within ZFC – that is within mathematics [12].

If A and B are sets then

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\} \qquad A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$
$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\} \quad A \amalg B = \{(1, a) \mid a \in A\} \cup \{(2, b) \mid b \in B\}$$

and

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

are also sets. These operations satisfy

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \qquad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A - (B \cup C) = (A - B) \cap (A - C) \qquad A - (B \cap C) = (A - B) \cup (A - C)$$

as well as several other rules.

We say that A is a subset of B, or B a superset of A, if all elements of A are elements of B. The sets A and B are equal if A and B have the same elements. In mathematical symbols,

$$A \subset B \iff \forall x \in A \colon x \in B$$
$$A = B \iff (\forall x \in A \colon x \in B \text{ and } \forall x \in B \colon x \in A) \iff A \subset B \text{ and } B \subset A$$

The *power set* of A,

$$\mathcal{P}(A) = \{B \mid B \subset A\}$$

is the set of all subsets of A.

1.2. **Functions.** Functions or maps are fundamental to all of mathematics. So what is a function?

1.3. **Definition.** *A function from A to B is a subset f of $A \times B$ such that for all a in A there is exactly one b in B such that $(a, b) \in f$.*

We write $f \colon A \to B$ for the function $f \subset A \times B$ and think of f as a rule that to any element $a \in A$ associates a unique object $f(a) \in B$. The set A is the domain of f, the set B is the codomain of f; $\mathrm{dom}(f) = A$, $\mathrm{cod}(f) = B$.

The function f is

- injective or one-to-one if distinct elements of A have distinct images in B,
- surjective or onto if all elements in B are images of elements in A,

[1] If a person says "I am lying" – is he lying?

- bijective if is both injective and surjective, if any element of B is the image of precisely one element of A.

In other words, the map f is injective, surjective, bijective iff the equation $f(a) = b$ has at most one solution, at least one solution precisely one solution, for all $b \in B$.

If $f\colon A \to B$ and $g\colon B \to C$ are maps such that $\mathrm{cod}(f) = \mathrm{dom}(g)$, then the composition is the map $g \circ f\colon A \to C$ defined by $g \circ f(a) = g(f(a))$.

1.4. **Proposition.** *Let A and B be two sets.*

(1) *Let $f\colon A \to B$ be any map. Then*

$$\begin{aligned} f \text{ is injective} &\iff f \text{ has a left inverse} \\ f \text{ is surjective} &\overset{AC}{\iff} f \text{ has a right inverse} \\ f \text{ is bijective} &\iff f \text{ has an inverse} \end{aligned}$$

(2) *There exists a surjective map $A \twoheadrightarrow B$ $\overset{AC}{\iff}$ There exits an injective map $B \rightarrowtail A$*

Any left inverse is surjective and any right inverse is injective.

If $f\colon A \to B$ is bijective then the inverse $f^{-1}\colon B \to A$ is the map that to $b \in B$ associates the unique solution to the equation $f(a) = b$, ie

$$a = f^{-1}(b) \iff f(a) = b$$

for all $a \in A$, $b \in B$.

Let $\mathrm{map}(A, B)$ denote the set of all maps from A to B. Then

$$\mathrm{map}(X, A \times B) = \mathrm{map}(X, A) \times \mathrm{map}(X, B), \qquad \mathrm{map}(A \amalg B, X) = \mathrm{map}(A, X) \times \mathrm{map}(B, X)$$

for all sets X, A, and B. Some people like rewrite this as

$$\mathrm{map}(X, A \times B) = \mathrm{map}(\Delta X, (A, B)), \qquad \mathrm{map}(A \amalg B, X) = \mathrm{map}((A, B), \Delta X)$$

where $\Delta X = (X, X)$. Here, (A, B) is a pair of spaces and maps $(f, g)\colon (X, Y) \to (A, B)$ between pairs of spaces are defined to be pairs of maps $f\colon X \to A$, $g\colon Y \to B$. These people say that the product is right adjoint to the diagonal and the coproduct is left adjoint to the diagonal.

1.5. **Relations.** There are many types of relations. We shall here concentrate on equivalence relations and order relations.

1.6. **Definition.** *A relation R on the set A is a subset $R \subset A \times A$.*

1.7. **Example.** We may define a relation D on $\mathbf{Z}_+$ by aDb if a divides b. The relation $D \subset \mathbf{Z}_+ \times \mathbf{Z}_+$ has the properties that aDa for all a and aDb and $bDc \implies aDc$ for all a, b, c. We say that D is reflexive and transitive.

1.5.1. *Equivalence relations.* Equality is a typical equivalence relation. Here is the general definition.

1.9. **Definition.** *An equivalence relation on a set A is a relation $\sim \subset A \times A$ that is*

Reflexive: *$a \sim a$ for all $a \in A$*
Symmetric: *$a \sim b \Rightarrow b \sim a$ for all $a, b \in A$*
Transitive: *$a \sim b \sim c \Rightarrow a \sim c$ for all $a, b, c \in A$*

The *equivalence class* containing $a \in A$ is the subset

$$[a] = \{b \in A \mid a \sim b\}$$

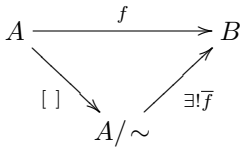
of all elements of A that are equivalent to a. There is a canonical map $[\,]\colon A \to A/\!\sim$ onto the set

$$A/\!\sim = \{[a] \mid a \in A\} \subset \mathcal{P}(A)$$

of equivalence classes that takes the element $a \in A$ to the equivalence class $[a] \in A/\!\sim$ containing a.

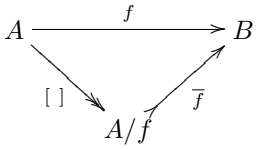
A map $f\colon A \to B$ is said to respect the equivalence relation $\sim$ if $a_1 \sim a_2 \implies f(a_1) = f(a_2)$ for all $a_1, a_2 \in A$ (f is constant on each equivalence class). The canonical map $[\,]\colon A \to A/\!\sim$ respects the equivalence relation and it is the universal example of such a map: Any map $f\colon A \to B$ that

respects the equivalence relation factors uniquely through $A/\sim$ in the sense that there is a unique map $\overline{f}$ such that the diagram



commutes. How would you define $\overline{f}$?

1.10. **Example.** (1) Equality is an equivalence relation. The equivalence class $[a] = \{a\}$ contains just one element.
(2) $a \bmod b \mod n$ is an equivalence relation on $\mathbf{Z}$. The equivalence class $[a] = a + n\mathbf{Z}$ consists of all integers congruent to $a \mod n$ and the set of equivalence classes is $\mathbf{Z}/n\mathbf{Z} = \{[0], [1], \ldots, [n-1]\}$.
(3) $x \sim y \overset{\text{def}}{\iff} |x| = |y|$ is an equivalence relation in the plane $\mathbf{R}^2$. The equivalence class $[x]$ is a circle centered at the origin and $\mathbf{R}^2/\sim$ is the collection of all circles centered at the origin. The canonical map $\mathbf{R}^2 \to \mathbf{R}^2/\sim$ takes a point to the circle on which it lies.
(4) If $f\colon A \to B$ is any function, $a_1 \sim a_2 \overset{\text{def}}{\iff} f(a_1) = f(a_2)$ is an equivalence relation on A. The equivalence class $[a] = f^{-1}(f(a)) \subset A$ is the fibre over $f(a) \in B$. we write A/f for the set of equivalence classes. The canonical map $A \to A/f$ takes a point to the fibre in which it lies. Any map $f\colon A \to B$ can be factored



as the composition of a surjection followed by an injection. The corestriction $\overline{f}\colon A/f \to f(A)$ of $\overline{f}$ is a bijection between the set of fibres A/f and the image $f(A)$.
(5) [Ex 3.2] (Restriction) Let X be a set and $A \subset X$ a subset. Declare any two elements of A to be equivalent and any element outside A to be equivalent only to itself. This is an equivalence relation. The equivalence classes are A and $\{x\}$ for $x \in X - A$. One writes X/A for the set of equivalence classes.
(6) [Ex 3.5] (Equivalence relation generated by a relation) The intersection of any family of equivalence relations is an equivalence relation. The intersection of all equivalence relations containing a given relation R is called the equivalence relation generated by R.

1.11. **Lemma.** *Let $\sim$ be an equivalence relation on a set A. Then*

(1) $a \in [a]$
(2) $[a] = [b] \iff a \sim b$
(3) *If* $[a] \cap [b] \neq \emptyset$ *then* $[a] = [b]$

Proof. (1) is reflexivity, (2) is symmetry, (3) is transitivity: If $c \in [a] \cap [b]$, then $a \sim c \sim b$ so $a \sim b$ and $[a] = [b]$ by (2). □

This lemma implies that the collection $A/\sim$ is a *partition* of A, a collection of nonempty, disjoint subsets of A whose union is all of A. Conversely, given any partition of A we define an equivalence relation by declaring a and b to be equivalent if they lie in the same subset of the partition. We conclude that an equivalence relation is essentially the same thing as a partition.

1.5.2. *Linear Orders.* The usual order relation $<$ on $\mathbf{Z}$ or $\mathbf{R}$ is an example of a linear order. Here is the general definition.

1.12. **Definition.** *A linear order on the set A is a relation $<\subset A \times A$ that is*

Comparable: *If $a \neq b$ then $a < b$ or $b < a$ for all $a, b \in A$*
Nonreflexive: *$a < a$ for no $a \in A$*
Transitive: *$a < b < c \Rightarrow a < c$ for all $a, b, c \in A$*

What are the right maps between ordered sets?

1.13. **Definition.** *Let $(A,<)$ and $(B,<)$ be linearly ordered sets. An* order preserving *map is a map $f\colon A\to B$ such that $a_1<a_2 \Longrightarrow f(a_1)<f(a_2)$ for all $a_1,a_2\in A$. An* order isomorphism *is a bijective order preserving map.*

An order preserving map $f\colon A\to B$ is always injective. If there exists an order isomorphism $f\colon A\to B$, then we say that $(A,<)$ and $(B,<)$ have the same *order type.*

How can we make new ordered sets out of old ordered sets? Well, any *subset* of a linearly ordered set is a linearly ordered set in the obvious way using the restriction of the order relation. Also the *product* of two linearly ordered set is a linearly ordered set.

1.14. **Definition.** *Let $(A,<)$ and $(B,<)$ be linearly ordered sets. The dictionary order on $A\times B$ is the linear order given by*

$$(a_1,b_1)<(a_2,b_2) \overset{def}{\iff} (a_1<a_2) \text{ or } (a_1=a_2 \text{ and } b_1<b_2)$$

The restriction of a dictionary order to a product subspace is the dictionary order of the restricted linear orders. (Hey, what did that sentence mean?)

What about orders on $A\amalg B$, $A\cup B$, $\mathrm{map}(A,B)$ or $\mathcal{P}(A)$?

What are the invariant properties of ordered sets? In a linearly ordered set $(A,<)$ it makes sense to define intervals such as

$$(a,b)=\{x\in A\mid a<x<b\},\qquad (-\infty,b]=\{x\in A\mid x\leq b\}$$

and similarly for other types of intervals, $[a,b]$, $(a,b]$, $(-\infty,b]$ etc.

If $(a,b)=\emptyset$ then a is the *immediate predecessor* of b, and b the *immediate successor* of a.

Let $(A,<)$ be an ordered set and $B\subset A$ a subset.

- M is a *largest element* of B if $M\in B$ and $b\leq M$ for all $b\in B$. The element m is a *smallest element* of B if $m\in B$ and $m\leq b$ for all $b\in B$. We denote the largest element (if it exists) by $\max B$ and the smallest element (if it exists) by $\min B$.
- M is an *upper bound* for B if $M\in A$ and $b\leq M$ for all $b\in B$. The element m is a *lower bound* for B if $m\in A$ and $m\leq b$ for all $b\in B$. The set of upper bounds is $\bigcap_{b\in B}[b,\infty)$ and the set of lower bounds is $\bigcap_{b\in B}(-\infty,b]$.
- If the set of upper bounds has a smallest element, $\min\bigcap_{b\in B}[b,\infty)$, it is called the *least upper bound* for B and denoted $\sup B$. If the set of lower bounds has a largest element, $\max\bigcap_{b\in B}(-\infty,b]$, it is called the *greatest lower bound* for B and denoted $\inf B$.

1.15. **Definition.** *An ordered set $(A,<)$ has the* least upper bound property *if any nonempty subset of A that has an upper bound has a least upper bound. If also $(x,y)\neq\emptyset$ for all $x<y$, then $(A,<)$ is a* linear continuum.

1.16. **Example.** (1) $\mathbf{R}$ and $(0,1)$ have the same order type. $[0,1)$ and $(0,1)$ have distinct order types for $[0,1)$ has a smallest element and $(0,1)$ doesn't. $\{-1\}\cup(0,1)$ and $[0,1)$ have the same order type as we all can find an explicit order isomorphism between them.

(2) $\mathbf{R}\times\mathbf{R}$ has a linear dictionary order. What are the intervals $(1\times 2,1\times 3)$, $[1\times 2,3\times 2]$ and $(1\times 2,3\times 4]$? Is $\mathbf{R}\times\mathbf{R}$ a linear continuum? Is $[0,1]\times[0,1]$?

(3) We now consider two subsets of $\mathbf{R}\times\mathbf{R}$. The dictionary order on $\mathbf{Z}_+\times[0,1)$ has the same order type as $[1,\infty)$ so it is a linear continuum. In the dictionary order on $[0,1)\times\mathbf{Z}_+$ each element (a,n) has $(a,n+1)$ as its immediate successor so it is not a linear continuum. Thus $\mathbf{Z}_+\times[0,1)$ and $[0,1)\times\mathbf{Z}_+$ do not have the same order type. (So, in general, $(A,<)\times(B,<)$ and $(B,<)\times(A,<)$ represent different order types. This is no surprise since the dictionary order is not symmetric in the two variables.)

(4) $(\mathbf{R},<)$ is a linear continuum as we all learn in kindergarten. The sub-ordered set $(\mathbf{Z}_+,<)$ has the least upper bound property but it is not a linear continuum as $(1,2)=\emptyset$.

(5) $(-1,1)$ has the least upper bound property: Let B be any bounded from above subset of $(-1,1)$ and let $M\in(-1,1)$ be an upper bound. Then B is also bounded from above in $\mathbf{R}$, of course, so there is a least upper bound, $\sup B$, in $\mathbf{R}$. Now $\sup B$ is the smallest upper bound so that $\sup B\leq M<-1$. We conclude that $\sup B$ lies in $(-1,1)$ and so it is also a least upper bound in $(-1,1)$. In fact, any convex subset of a linear continuum is a linear continuum.

(6) $\mathbf{R}-\{0\}$ does not have the least upper bound property as the subset $B=\{-1,-\frac{1}{2},-\frac{1}{3},\ldots\}$ is bounded from above (by say 100) but the set of upper bounds $(0,\infty)$ has no smallest element.

2. The integers and the real numbers

We shall **assume** that the real numbers $\mathbf{R}$ exists with all the usual properties: $(\mathbf{R}, +, \cdot)$ is a field, $(\mathbf{R}, +, \cdot, <)$ is an ordered field, $(\mathbf{R}, <)$ is a linear continuum (1.15).

What about $\mathbf{Z}_+$?

2.1. **Definition.** *A subset $A \subset \mathbf{R}$ is inductive if $1 \in A$ and $a \in A \Longrightarrow a + 1 \in A$.*

There are inductive subsets of $\mathbf{R}$, for instance $\mathbf{R}$ itself and $[1, \infty)$.

2.2. **Definition.** *$\mathbf{Z}_+$ is the intersection of all inductive subsets of $\mathbf{R}$.*

We have that $1 \in \mathbf{Z}_+$ and $\mathbf{Z}_+ \subset [1, \infty)$ because $[1, \infty)$ is inductive so $1 = \min \mathbf{Z}_+$ is the smallest element of $\mathbf{Z}_+$.

Theorem 2.3. *(Induction Principle) Let J be a subset of $\mathbf{Z}_+$ such that*

$$1 \in J \text{ and } \forall n \in \mathbf{Z}_+ \colon n \in J \Longrightarrow n + 1 \in J$$

Then $J = \mathbf{Z}_+$.

Proof. J is inductive so J contains the smallest inductive set, $\mathbf{Z}_+$. □

Theorem 2.4. *Any nonempty subset of $\mathbf{Z}_+$ has a smallest element.*

Before the proof, we need a lemma.

For each $n \in \mathbf{Z}_+$, write

$$S_n = \{x \in \mathbf{Z}_+ \mid x < n\}$$

for the set of positive integers smaller than n (the *section* below n). Note that $S_1 = \emptyset$ and $S_{n+1} = S_n \cup \{n\}$.

2.5. **Lemma.** *For any $n \in \mathbf{Z}_+$, any nonempty subset of S_n has a smallest element.*

Proof. Let $J \subset \mathbf{Z}_+$ be the set of integers for which the lemma is true. It is enough (2.3) to show that J is inductive. $1 \in \mathbf{J}$ for the trivial reason that there are no nonempty subsets of $S_1 = \emptyset$. Suppose that $n \in J$. Consider a nonempty subset A of S_{n+1}. If A consists of n alone, then $n = \min A$ is the smallest element of A. If not, A contains integers $< n$, and then $\min(A \cap S_n)$ is the smallest element of A. Thus $n + 1 \in J$. □

Proof of Theorem 2.4. Let $A \subset \mathbf{Z}_+$ be any nonempty subset. The intersection $A \cap S_n$ is nonempty for some n, so it has a smallest element (2.5). This is also the smallest element of A. □

Theorem 2.6 (General Induction Principle)**.** *Let J be a subset of $\mathbf{Z}_+$ such that*

$$\forall n \in \mathbf{Z}_+ \colon S_n \subset J \Longrightarrow n \in J$$

Then $J = \mathbf{Z}_+$.

Proof. We show the contrapositive. Let J be a proper subset of $\mathbf{Z}_+$. Consider the smallest element $n = \min(\mathbf{Z}_+ - J)$ outside J. Then $n \notin J$ and $S_n \subset J$ (for n is the smallest element not in J meaning that all elements smaller than n are in J). Thus J does not satisfy the hypothesis of the theorem. □

Theorem 2.7 (Archimedean Principle)**.** *$\mathbf{Z}_+$ has no upper bound in $\mathbf{R}$: For any real number there is a natural number which is greater.*

Proof. We assume the opposite and derive a contradiction. Suppose that $\mathbf{Z}_+$ is bounded from above. Let $b = \sup \mathbf{Z}_+$ be the least upper bound ($\mathbf{R}$ has the least upper bound property). Since $b - 1$ is not an upper bound (it is smaller than the least upper bound), there is a positive integer $n \in \mathbf{Z}_+$ such that $n > b - 1$. Then $n + 1$ is also an integer ($\mathbf{Z}_+$ is inductive) and $n + 1 > b$. This contradicts that b is an upper bound for $\mathbf{Z}_+$. □

Theorem 2.8 (Principle of Recursive Definitions)**.** *For any set B and any function*

$$\rho \colon \operatorname{map}(\{S_n \mid n \in \mathbf{Z}_+\}, B) \to B$$

there exists a unique function $h \colon \mathbf{Z}_+ \to B$ such that $h(n) = \rho(h|S_n)$ for all $n \in \mathbf{Z}_+$.

Proof. See [9, Ex 8.8]. □

This follows from the Induction Principle, but we shall not go into details. It is usually considered bad taste to define h in terms of h but the Principle of Recursive Definition is a permit to do exactly that in certain situations. Here is an example of a recursive definition from computer programing

```
fibo := func< n | n le 2 select 1 else $$(n-1) + $$(n-2) >;
```

of the Fibonacci function. Mathematicians (sometimes) prefer instead to apply the Principle of Recursive Definitions to the map

$$\rho(S_n \xrightarrow{f} \mathbf{Z}_+) = \begin{cases} 1 & n < 2 \\ f(n-2) - f(n-1) & n > 2 \end{cases}$$

Recursive functions can be computed by Turing machines.

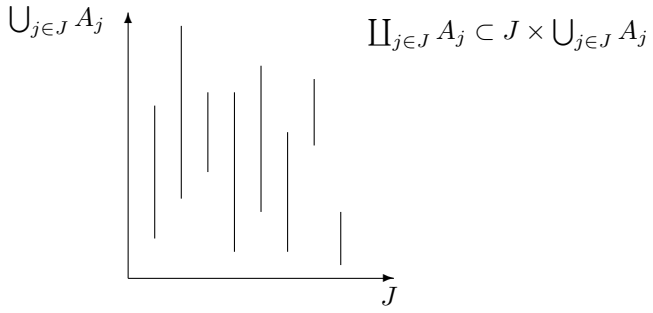


FIGURE 1. The coproduct

3. Products and coproducts

3.1. **Definition.** *An indexed family of sets consists of a collection $\mathcal{A}$ of sets, an index set J, and a surjective function $f\colon J \to \mathcal{A}$.*

We often denote the set $f(j)$ by A_j and the whole indexed family by $\{A_j\}_{j\in J}$. Any collection $\mathcal{A}$ can be made into an indexed family by using the identity map $\mathcal{A} \to \mathcal{A}$ as the indexing function.

We define the union, the intersection, the product, and the coproduct of the indexed family as

$$\bigcap_{j\in j} A_j = \{a \mid a \in A_j \text{ for all } j \in J\}, \qquad \bigcup_{j\in j} A_j = \{a \mid a \in A_j \text{ for at least one } j \in J\}$$

$$\prod_{j\in J} A_j = \{x \in \operatorname{map}(J, \bigcup A_j) \mid \forall j \in J\colon x(j) \in A_j\} \qquad \coprod_{j\in J} A_j = \bigcup_{j\in J} \{(j,a) \in J \times \bigcup_{j\in J} A_j \mid a \in A_j\}$$

There are natural maps

$$\pi_j\colon \prod_{j\in J} A_j \to A_j \quad \text{(projection)} \qquad \iota_j\colon A_j \to \coprod_{j\in J} A_j \quad \text{(injection)}$$

given by $\pi_j(x) = x(j)$ and $\iota_j(a) = (j,a)$ for all $j \in J$. These maps are used in establishing the identities

$$\operatorname{map}(X, \prod_{j\in J} A_j) = \prod_{j\in J} \operatorname{map}(X, A_j), \qquad \operatorname{map}(\coprod_{j\in J} A_j, Y) = \prod_{j\in J} \operatorname{map}(A_j, Y)$$

for any sets X and Y. This gives in particular maps

$$\Delta\colon \bigcap_{j\in J} A_j \to \prod_{j\in J} A_j \quad \text{(diagonal)}, \qquad \nabla\colon \coprod_{j\in J} A_j \to \bigcup_{j\in J} A_j \quad \text{(codiagonal)}$$

If the index set $J = S_{n+1} = \{1, \dots, n\}$ then we also write

$$A_1 \cup \cdots \cup A_n, \quad A_1 \cap \cdots \cap A_n, \quad A_1 \times \cdots \times A_n \quad A_1 \amalg \cdots \amalg A_n$$

for $\bigcup_{j\in S_{n+1}} A_j$, $\bigcap_{j\in S_{n+1}} A_j$, $\prod_{j\in S_{n+1}} A_j$ $\coprod_{j\in S_{n+1}} A_j$, respectively. If also and $A_j = A$ for all $j \in S_{n+1}$ we write A^n for the product $\prod_{j\in S_{n+1}} A$. The elements of A^n are all n-tuples $(a_1, \dots, a_n)$ of elements from A.

If the index set $J = \mathbf{Z}_+$ then we also write

$$A_1 \cup \cdots \cup A_n \cup \cdots, \quad A_1 \cap \cdots \cap A_n \cap \cdots, \quad A_1 \times \cdots \times A_n \times \cdots \quad A_1 \amalg \cdots \amalg A_n \times \cdots$$

for $\bigcup_{j\in \mathbf{Z}_+} A_j$, $\bigcap_{j\in \mathbf{Z}_+} A_j$, $\prod_{j\in \mathbf{Z}_+} A_j$, $\coprod_{j\in \mathbf{Z}_+} A_j$, respectively. If also $A_j = A$ for all j we write A^ω for the product $\prod_{j\in \mathbf{Z}_+} A$, the set of all functions $x\colon \mathbf{Z}_+ \to A$, i.e. all sequences $(x_1, \dots, x_n, \dots)$ of elements from A. [2]

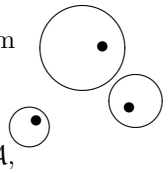
3.2. **Example.** (1) $S_1 \cup S_2 \cup \cdots \cup S_n \cdots = \bigcup_{n\in \mathbf{Z}_+} S_n = \mathbf{Z}_+$.

(2) If the collection $\mathcal{A} = \{A\}$ consists of just one set A then $\bigcap_{j\in J} A = A = \bigcup_{j\in J} A$, $\prod_{j\in J} A = \operatorname{map}(J, A)$, and $\coprod_{j\in J} A = J \times A$.

(3) There is a bijection (which one?) between $\{0,1\}^\omega = \operatorname{map}(\mathbf{Z}_+, \{0,1\})$ and $\mathcal{P}(\mathbf{Z}_+)$. More generally, there is a bijection (which one) between the product $\prod_{j\in J}\{0,1\} = \operatorname{map}(J, \{0,1\})$ and the power set $\mathcal{P}(J)$.

[2] ω is the formal set within set theory corresponding to the naïve set $\mathbf{Z}_+$ [12, V.1.5]

Even though we shall not specify our (ZF) axioms for set theory, let us mention just one axiom which has a kind of contended status since some of its consequences are counter-intuitive.



3.3. **Axiom** (Axiom of Choice (AC))**.** For any nonempty colllection of nonempty disjoint sets, $\mathcal{A}$, there exists a set $C \subset \bigcup_{A\in\mathcal{A}} A$ such that $C \cap A$ contains exactly one element for all $A \in \mathcal{A}$.

If the ZF axioms of set theory are consistent, then both ZF+AC (Gödel 1938) and ZF+$\neg$AC (Fraenkel and Mostowsski, Cohen) are consistent theories [12, IV.2.8]. You may take or leave AC without penalty. (Just like you may take or leave Euclid's axiom on parallels depending on what kind of geometry you like to do.) We shall here include AC and work within ZFC (ZF + AC).

Unlike the other axioms of set theory, the AC does not determine the set C uniquely.

Theorem 3.4. [3, Thm B.18] *The following statements are equivalent:*

(1) *The Axiom of Choice*
(2) *Any surjective map has a right inverse.*
(3) *For any nonempty indexed family of (not necessarily disjoint) nonempty sets, $\{A_j\}_{j\in J}$, there exists a function $c\colon J \to \bigcup_{j\in J} A_j$ (a choice function) such that $c(j) \in A_j$ for all $j \in J$.*
(4) $\prod_{j\in J} A_j \neq \emptyset$ *for any nonempty indexed family of nonempty sets.*

Proof. (1) $\Longrightarrow$ (2): Let $f\colon A \to B$ be a surjective map. Define the right inverse $g\colon B \to A$ by $\{g(b)\} = C \cap f^{-1}(b)$ where $C \subset A = \bigcup_{b\in B} f^{-1}(b)$ is a set such that $C \cap f^{-1}(b)$ contains exactly one point for each $b \in B$.
(2) $\Longrightarrow$ (3): Define c to be $J \to \coprod A_j \xrightarrow{\nabla} \bigcup A_j$ where the first map is a right inverse to the function $\coprod_{j\in J} A_j \to J$ taking A_j to j for all $j \in J$.
(3) $\Longleftrightarrow$ (4): The product is defined to be the set of choice functions.
(3) $\Longrightarrow$ (1): Let $\mathcal{A}$ be a nonempty collection of nonempty sets. Put $C = c(\mathcal{A})$ where $c\colon \mathcal{A} \to \bigcup_{A\in\mathcal{A}} A$ is a choice function. □

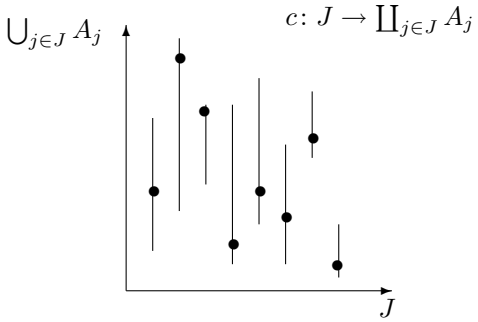


FIGURE 2. A choice function

Here is a special, but often used, case. Let A be any nonempty set and $\mathcal{P}'(A) = \mathcal{P}(A) - \{\emptyset\}$ the collection of nonempty subsets of A. Then there exists (3.4.(3)) a choice function $c\colon \mathcal{P}'(A) \to A$ such that $c(B) \in B$ for any nonempty $B \subset A$. (The choice function selects an element in each nonempty subset of A.)

4. FINITE AND INFINITE SETS

4.1. **Definition.** *A set A is* finite *if $S_{n+1} \sim A$ for some $n \in \mathbf{Z}_+$. A set is* infinite *if it is not finite.*

We write $X \sim Y$ if there is a bijection between the two sets X and Y.

4.2. **Lemma.** *Let $n \in \mathbf{Z}_+$ and let B be a proper subset of S_{n+1}.*

(1) *It is impossible to map B onto S_{n+1}.*
(2) *$S_{m+1} \sim B$ for some $m < n$.*

Proof. Both statements are proved by induction.
(1) If $n = 1$, then $S_2 = \{1\}$ and $B = \emptyset$ so the assertion is true in this case. Assume it is true for some $n \in \mathbf{Z}_+$. Consider a proper subset B of S_{n+1+1}. Suppose that there exists a surjection $f\colon B \to S_{n+1+1}$. By permuting the elements of B and S_{n+1+1} if necessary, we can assume that $f(n+1) = n+1$. Then $B - f^{-1}(n+1)$ is a proper subset of S_{n+1} which is mapped onto S_{n+1} by f. But that is impossible by induction hypothesis.
(2) If $n = 1$, then $S_2 = \{1\}$ and $B = \emptyset$ so $S_1 \sim B$. Assume the assertion is true for some $n \in \mathbf{Z}_+$. Consider a proper subset B of S_{n+1+1}. By permuting the elements of S_{n+1+1} if necessary, we can assume that $n+1 \notin B$ so that B is a subset of S_{n+1}. If $B = S_{n+1}$, then $B \sim S_{n+1}$, of course. Otherwise, B is a proper subset of S_{n+1} and then $S_{m+1} \sim B$ for some $m < n < n+1$ by induction hypothesis. □

4.3. **Corollary.** *Let A be a finite set.*

(1) *If $S_{m+1} \sim A \sim S_{n+1}$, then $m = n$.*
(2) *Any subset of A is finite.*

Proof. (1) Suppose that $m \neq n$. We may then assume that $m < n$. But then S_{m+1} is a proper subset of S_{n+1} which can be mapped onto S_{n+1}. That is not possible.
(2) Since this is true for the model finite set S_{n+1}, it is true for all finite sets. □

We have just learned that if A is finite then $S_{n+1} \sim A$ for a *unique* $n \in \mathbf{Z}_+$. This n is called the *cardinality* of A and it is denoted card$A$ or $|A|$. We also learned that if $B \subset A$ then B is finite, $|B| \leq |A|$, and

$$|B| < |A| \iff B \subsetneq A \tag{4.4}$$

which is often called the 'pidgeon-hole principle'.

Theorem 4.5 (Characterization of finite sets)**.** *Let A be a set. The following statements are equivalent*

(1) *A is finite*
(2) *There exists a surjection $S_{n+1} \to A$ for some $n \in \mathbf{Z}_+$*
(3) *There exists an injection $A \to S_{n+1}$ for some $n \in \mathbf{Z}_+$*

Proof. (1) $\Longrightarrow$ (2): There even exists a bijection $S_{n+1} \to A$.
(2) $\iff$ (3): 1.4.(2)
(3) $\Longrightarrow$ (1): If there exists an injection $A \to S_{n+1}$, then there exists a bijection between A and a subset of S_{n+1}. But we have just seen that all subsets finite sets are finite. □

4.6. **Corollary** (Hereditary properties of finite sets)**.**

(1) *Subsets of finite sets are finite.*
(2) *Images of finite sets are finite.*
(3) *Finite unions of finite sets are finite.*
(4) *Finite Cartesian products of finite sets are finite.*

Proof. (1) Proved already in 4.3.
(2) $S_{n+1} \twoheadrightarrow A \twoheadrightarrow B$.
(3) To see that the union of two finite sets is finite, it is enough to show $S_{m+1} \amalg S_{n+1}$ is finite (for the union of any two finite sets is an image of this set). But it is immediate that $S_{m+n+1} \sim S_{m+1} \amalg S_{n+1}$. Induction now shows that $A_1 \cup \cdots \cup A_n$ is finite when $A_1, \ldots, A_n$ are finitely many finite sets.
(4) Let A and B be finite. Since $A \times B = \coprod_{a \in A} B$ is the union of finitely many finite sets, it is finite. Induction now shows that $A_1 \times \cdots \times A_n$ is finite when $A_1, \ldots, A_n$ are finitely many finite sets. □

Are all sets finite? No!

4.7. **Corollary.** $\mathbf{Z}_+$ *is infinite.*

Proof. There is a surjective map of the proper subset $\mathbf{Z}_+ - \{1\}$ onto $\mathbf{Z}_+$. □

Theorem 4.8 (Characterization of infinite sets)**.** *Let A be a set. The following are equivalent:*

(1) *A is infinite*

(2) *There exists an injective map* $\mathbf{Z}_+ \to A$
(3) *There exists a surjective map* $A \to \mathbf{Z}_+$
(4) *A is in bijection with a proper subset of itself*

Proof. (1) $\Longrightarrow$ (2): Let $c\colon \mathcal{P}'(A) \to A$ be a choice function. Define $h\colon \mathbf{Z}_+ \to A$ recursively by

$$h(1) = c(A)$$
$$h(i) = c(A - \{h(1), \dots, h(i-1)\}), \qquad i > 1$$

Then h is injective (if $i < j$ then $h(j) \in A - \{h(1), \dots, h(i), \dots, h(j-1)\}$ so $h(i) \neq h(j)$).
(2) $\iff$ (3): 1.4.(2)
(2) $\Longrightarrow$ (4): We view $\mathbf{Z}_+$ as a subset of A. Then $A = (A - \mathbf{Z}_+) \cup \mathbf{Z}_+$ is in bijection with the proper subset $A - \{1\} = (A - \mathbf{Z}_+) \cup (\mathbf{Z}_+ - \{1\})$.
(4) $\Longrightarrow$ (1): This is 4.2. □

Here we applied the Principle of Recursive Definitions (2.8) to $\rho(S_n \xrightarrow{f} \mathbf{Z}_+) = c(A - f(S_n))$.

5. Countable and uncountable sets

5.1. **Definition.** *A set C is* countably infinite *if $\mathbf{Z}_+ \sim C$. It is* countable *if it is finite or countably infinite. It is* uncountable *if it is not countable.*

5.2. **Lemma.** *Any subset of $\mathbf{Z}_+$ is either finite or countably infinite (in bijection with $\mathbf{Z}_+$).*

Proof. Let $C \subset \mathbf{Z}_+$ be an infinite set of positive integers. We show (the stronger statement) that C has the order type of $\mathbf{Z}_+$. Define a function $h\colon \mathbf{Z}_+ \to C$ recursively (2.8) by

$$h(1) = \min C$$
$$h(i) = \min(C - \{h(1), \dots, h(i-1)\}), \quad i > 1$$

using 2.4. Note that $C - \{h(1), \dots, h(i-1)\}$ is nonempty since C is infinite (4.5). We claim that h is bijective.
h is order preserving: If $i < j$, then

$$h(i) = \min(C - \{h(1), \dots, h(i-1)\}) < \min(C - \{h(1), \dots, h(i-1), \dots, h(j-1)\}) = h(j)$$

because $C - \{h(1), \dots, h(i-1)\} \supsetneq C - \{h(1), \dots, h(i-1), \dots, h(j-1)\}$.
h is surjective: Let $c \in C$. We must find a positive integer m such that $c = h(m)$. Our only hope is

$$m = \min\{n \in \mathbf{Z}_+ \mid h(n) \geq c\}$$

(Note that this has a meaning since the set $\{n \in \mathbf{Z}_+ \mid h(n) \geq c\}$ is nonempty as we can not inject the infinite set $\mathbf{Z}_+$ into the finite set $\{1, \dots, c-1\} = S_c$ (4.3). Note also that again we use 2.4.) By definition of m,

$$h(m) \geq c \quad \text{and} \quad h(n) \geq c \Rightarrow n \geq m$$

The last of these two properties is equivalent to $n < m \Rightarrow h(n) < c$, so $c \notin \{h(1), \dots, h(m-1)\}$, or $c \in C - \{h(1), \dots, h(m-1)\}$, and therefore

$$h(m) = \min(C - \{h(1), \dots, h(m-1)\}) \leq c$$

by definition of h. Thus $h(m) = c$. □

Here we applied the Principle of Recursive Definitions (2.8) to $\rho(S_n \xrightarrow{f} C) = \min(C - f(S_n))$.

Theorem 5.3 (Characterization of countable sets)**.** *Let A be a set. The following statements are equivalent*

(1) *A is countable*
(2) *There exists a surjection $\mathbf{Z}_+ \to A$*
(3) *There exists an injection $A \to \mathbf{Z}_+$*

Proof. If A is finite, the theorem is true, so we assume that A is countably infinite.
(1) $\Longrightarrow$ (2): Clear.
(2) $\iff$ (3): 1.4.(2)
(3) $\Longrightarrow$ (1): We may as well assume that $A \subset \mathbf{Z}_+$. Since A is assumed to infinite, A is countably infinite by Lemma 5.2. □

5.4. **Example.** $\mathbf{Z}_+$ is obviously infinitely countable. The map $f\colon \mathbf{Z}_+ \times \mathbf{Z}_+ \to \mathbf{Z}_+$ given by $f(m,n) = 2^m 3^n$ is injective by uniqueness of prime factorizations. The map $g\colon \mathbf{Z}_+ \times \mathbf{Z}_+ \to \mathbf{Q}_+$ given by $g(m,n) = \frac{m}{n}$, is surjective. Thus $\mathbf{Z}_+ \times \mathbf{Z}_+$ and $\mathbf{Q}_+$ are infinitely countable.

5.5. **Corollary** (Hereditary properties of countable sets)**.**

(1) *A subset of a countable set is countable*
(2) *The image of a countable set is countable.*
(3) *A countable union of countable sets is countable (assuming AC).*[3]
(4) *A finite product of countable sets is countable.*

Proof. (1) $B \rightarrowtail A \rightarrowtail \mathbf{Z}_+$.
(2) $\mathbf{Z}_+ \twoheadrightarrow A \twoheadrightarrow B$.
(3) Let $\{A_j\}_{j\in J}$ be an indexed family of sets where J is countable and each set A_j is countable. It is enough to show that $\coprod A_j$ is countable. We leave the case where the index set J is finite as an exercise and consider only the case where J is infinite. Then we may as well assume that $J = \mathbf{Z}_+$. Choose (!) for each $n \in \mathbf{Z}_+$ an injective map $f_n\colon A_n \to \mathbf{Z}_+$. Then we have injective maps

$$\coprod A_n \xrightarrow{\coprod f_n} \coprod \mathbf{Z}_+ = \mathbf{Z}_+ \times \mathbf{Z}_+ \xrightarrow{(5.4)} \mathbf{Z}_+$$

so $\coprod A_n$ is countable.
(4) If A and B are countable, so is $A \times B = \coprod_{a\in A} B$ as we have just seen. Now use induction to show that if $A_1, \ldots, A_n$ are countable, so is $A_1 \times \cdots \times A_n$. □

You may think that a countable product of countable sets is countable or indeed that all sets are finite or countable – but that's false.

Theorem 5.6. *Let A be any set.*

(1) *There is no injective map $\mathcal{P}(A) \to A$*
(2) *There is no surjective map $A \to \mathcal{P}(A)$*

Proof. (Cantor's diagonal argument.) Is is a general fact (1.4.(2)) that (1) $\iff$ (2). Thus it suffices to show (2). Let $g\colon A \to \mathcal{P}(A)$ be any function. Then

$$\{a \in A \mid a \notin g(a)\} \in \mathcal{P}(A)$$

is not in the image of g. Because if this set were of the form $g(b)$ for some $b \in A$, then we'd have

$$b \in g(b) \iff b \notin g(b)$$

□

5.7. **Corollary.** *The set $\mathcal{P}(\mathbf{Z}_+) = \mathrm{map}(\mathbf{Z}_+, \{0,1\}) = \prod_{n\in\mathbf{Z}_+}\{0,1\} = \{0,1\}^\omega$ is uncountable.*

Russel's paradox also exploits Cantor's diagonal argument.

We have seen (5.2) that any subset of $\mathbf{Z}_+$ is either finite or in bijection with $\mathbf{Z}_+$. What about subsets of $\mathbf{R}$?

5.8. **Conjecture** (Cantor's Continuum Hypothesis, CH)**.** Any subset of $\mathbf{R}$ is either countable or in bijection with $\mathbf{R}$.

CH is independent of the ZFC axioms for set theory in that if ZFC is consistent then both ZFC+CH (Gödel 1950) and ZFC+¬CH (Cohen 1963) are consistent theories [12, VII.4.26] [4]. Our axioms are not adequate to settle the CH.

Look up the generalized continuum hypothesis (GCH) [Ex 11.8] (due to Hausdorff) somewhere [15, 16]. It is not customary to assume the GCH; if you do, the AC becomes a theorem.

[3] In set theory without AC, $\mathbf{R}$ is a countable union of countable sets [12, p 228]

6. Well-ordered sets

We have seen that all nonempty subsets of $(\mathbf{Z}_+, <)$ have a smallest element and we have used this property in quite a few places so there is reason to suspect that this is an important property in general. This is the reason for the following definition. You may think of well-ordered sets as some kind of generalized versions of $\mathbf{Z}_+$.

6.1. **Definition.** *A set A with a linear order $<$ is* well-ordered *if any nonempty subset has a smallest element.*

Any well-ordered set has a smallest element. Any element (but the largest) in a well-ordered set has an immediate successor, the smallest successor [Ex 10.2]. (And any element (but the smallest) has an immediate predecessor?) A well-ordered set can not contain an infinite descending chain $x_1 > x_2 > \cdots$, in fact, a linearly ordered set is well-ordered if and only if it does not contain a copy of the negative integers $\mathbf{Z}_-$ [Ex 10.4].

Let $(A, <)$ be a well-ordered set and α an element of A. The subset

$$S_\alpha(A) = S_\alpha = (-\infty, \alpha) = \{a \in A \mid a < \alpha\}$$

is called the *section* of A by α.

The induction principle and the principle of recursive defintions apply not only to $\mathbf{Z}_+$ but to any well-ordered set.

Theorem 6.2 (Principle of Transfinite Induction)**.** *[Ex 10.7] (Cf 2.6) Let $(A, <)$ be a well-ordered set and $J \subset A$ a subset such that*

$$\forall \alpha \in A \colon S_\alpha \subset J \Longrightarrow \alpha \in J$$

Then $J = A$.

Proof. Formally identical to the proof of 2.6. □

Theorem 6.3 (Principle of Transfinite Recursive Definitions)**.** *Let $(A, <)$ be a well-ordered set. For any set B and any function*

$$\rho \colon \operatorname{map}(\{S_\alpha \mid \alpha \in A\}, B) \to B$$

there exists a unique function $h \colon A \to B$ such that $h(\alpha) = \rho(h|S_\alpha)$ for all $\alpha \in A$.

6.4. **Proposition** (Hereditary properties of well-ordered sets)**.**

(1) *A subset of a well-ordered set is well-ordered.*
(2) *The coproduct of any well-ordered family of well-ordered sets is well-ordered [Ex 10.8].*
(3) *The product of any finite family of well-ordered sets is well-ordered.*

Proof. (1) Clear.

(2) Let J be a well-ordered set and $\{A_j\}_{j \in J}$ a family of well-ordered sets indexed by J. For $i, j \in J$ and $x \in A_i$, $y \in A_j$, define

$$(i, x) < (j, y) \overset{\text{def}}{\iff} i < j \text{ or } (i = j \text{ and } x < y)$$

and convince yourself that this is a well-ordering.

(3) If $(A, <)$ and $(B, <)$ well-ordered then $A \times B = \coprod_{a \in A} B$ is well-ordered. Now use induction to show that the product $A_1 \times \cdots \times A_n$ of *finitely* many well-ordered sets $A_1, \ldots, A_n$ is well-ordered.

□

If C a nonempty subset of $A \times B$ then $\min C = (c_1, \min \pi_2(C \cap \pi_1^{-1}(c_1)))$ where $c_1 = \min \pi_1(C)$ is the smallest element of C.

6.5. **Example.** (1) The positive integers $(\mathbf{Z}_+, <)$ is a well-ordered set.
(2) $\mathbf{Z}$ and $\mathbf{R}$ are not well-ordered sets.
(3) Any section $S_{n+1} = \{1, 2, \ldots, n\}$ of $\mathbf{Z}_+$ is well-ordered (6.4.(1)).
(4) $S_{n+1} = \{1, 2, \ldots, n\}$ of $\mathbf{Z}_+$ is well-ordered. The product $S_{n+1} \times \mathbf{Z}_+$ is well-ordered (6.4.(3)). The finite products $\mathbf{Z}_+^n = \mathbf{Z}_+ \times \mathbf{Z}_+ \cdots \times \mathbf{Z}_+$ are well-ordered (6.4.(3)).
(5) The infinite product $\{0, 1\}^\omega$ is not well-ordered for it contains the infinite descending chain $(1, 0, 0, 0, \ldots) > (0, 1, 0, 0, \ldots) > (0, 0, 1, 0, \ldots) > \cdots$.

(6) The set $\overline{S}_\omega = [1,\omega] = \mathbf{Z}_+ \amalg \{\omega\}$ is well-ordered (6.4.(2)). It has ω as its largest element. The section $S_\omega = [1,\omega) = \mathbf{Z}_+$ is countably infinite but any other section is finite. Any finite subset A of $[1,\omega)$ has an upper bound because the set of non-upper bounds

$$\{x \in [1,\omega) \mid \exists a \in A\colon x < a\} = \bigcup_{a\in A} S_a$$

is finite (4.6.(3)) but $[1,\omega)$ is infinite. $\overline{S}_\omega$ has the same order type as the interval $[1\times 1, 2\times 1]$ in $\mathbf{Z}_+ \times \mathbf{Z}_+$.

Which of these well-ordereds have the same order type [Ex 10.3]? Draw pictures of examples of well-ordered sets.

We can classify completely all finte well-ordered sets.

Theorem 6.6 (Finite order types)**.** *[Ex 6.4] Any finite linearly ordered set A of cardinality n has the order type of $(S_{n+1}, <)$; in particular, it is well-ordered and it has a largest element.*

Proof. Define $h\colon S_{n+1} \to A$ recursively by $h(1) = \min A$ and

$$h(i) = \min(A - \{h(1), \dots, h(i-1)\}, \qquad i > 1$$

Then h is order preserving. In particular, h is injective and hence bijective (by the pidgeon hole principle (4.4)) since the two sets have the same cadinality. □

Can you find an explicit order preserving bijection $S_{m+1} \times S_{n+1} \to S_{m+n+1}$?

So there is just one order type of a given finite cardinality n. There are many countably infinite well-ordered sets (6.5). Is there an uncountable well-ordered set? Our examples, $\mathbf{R}$ and $\{0,1\}^\omega$, of uncountable sets are not well-ordered (6.5.(2), 6.5.(5)).

Theorem 6.7 (Well-ordering theorem)**.** *(Zermelo 1904) Any set can be well-ordered.*

We focus on the minimal criminal, the *minimal uncountable well-ordered set*. (It may help to look at 6.5.(6) again.)

6.8**. Lemma.** *There exists a well-ordered set $\overline{S}_\Omega = [0,\Omega]$ with a smallest element, 0, and a largest element, Ω, such that:*

1. *The section $S_\Omega = [0,\Omega)$ by Ω is uncountable but any other section, $S_\alpha = [0,\alpha)$ for $\alpha < \Omega$, is countable.*
2. *Any countable subset of $S_\Omega = [0,\Omega)$ has an upper bound in $S_\Omega = [0,\Omega)$*

Proof. (Cf 6.5.(6)) Take any uncountable well-ordered set A. Append a greatest element to A. Call the result A again. Now A has at least one uncountable section. Let Ω be the smallest element of A such that the section by this element is uncountable, that is $\Omega = \min\{\alpha \in A \mid S_\alpha \text{ is uncountable}\}$. Put $\overline{S}_\Omega = [0,\Omega]$ where 0 is the smallest element of A. This well-ordered set satisfies (1) and (2). Let C be a countable subset of $S_\Omega = [0,\Omega)$. We want to show that it has an upper bound. We consider the set of elements of S_Ω that are *not* upper bounds, i.e

$$\{x \in S_\Omega \mid x \text{ is not an upper bound for } C\} = \{x \in S_\Omega \mid \exists c \in C\colon x < c\} = \overbrace{\bigcup_{c\in C} S_c}^{\text{countable}} \subsetneq \overbrace{S_\Omega}^{\text{uncountable}}$$

This set of not upper bounds is countable for it is a countable union of countable sets (5.5.(4)). But S_Ω is uncountable, so the set of not upper bounds is a proper subset. □

See [SupplExI : 8] for an explicit construction of a well-ordered uncountable set. $\mathbf{Z}_+$ is the well-ordered set of all finite (nonzero) order types and S_Ω is the well-ordered set of all countable (nonzero) order types. See [Ex 10.6] for further properties of S_Ω.

Recall that the ordered set $\mathbf{Z}_+ \times [0,1)$ is a linear continuum of the same order type as $[1,\infty) \subset \mathbf{R}$. What happens if we replace $\mathbf{Z}_+$ by S_Ω [Ex 24.6, 24.12]?

7. Partially ordered sets, The Maximum Principle and Zorn's lemma

If we do not insist on comparability in our order relation we obtain a partially ordered set (poset):

7.1. **Definition.** *A strict partial order on a set A is a relation $\prec$ on A which is non-reflexive and transitive:*

(1) *$a \prec a$ holds for no $a \in A$*
(2) *$a \prec b$ and $b \prec c$ implies $a \prec c$.*

We do not require that any two nonidentical element can be compared. For instance, power sets are strictly partially ordered by proper inclusion $\subsetneq$ (in fact, it is good idea to read $\prec$ as "is contained in").

Theorem 7.2 (Hausdorff's Maximum Principle)**.** *Any linearly ordered subset of a poset is contained in a maximal linearly ordered subset.*

Proof. We prove that any poset contains maximal linearly ordered subsets and leave general form of the theorem to the reader. As a special case, suppose that the poset is infinitely countable. We may as well assume that the poset is $\mathbf{Z}_+$ with some partial order $\prec$. Define $h\colon \mathbf{Z}_+ \to \{0,1\}$ recursively by $h(1) = 0$ and

$$h(i) = \begin{cases} 1 & \{j < i | h(j) = 0\} \cup \{i\} \text{ is linearly ordered wrt } \prec \\ 0 & \text{otherwise} \end{cases}$$

for $i > 0$. Then $H = h^{-1}(0)$ is a maximal linearly ordered subset.

□

For the proof in the general case, let $(A, <)$ be a poset, well-order A (6.7), apply the Principle of Transfinite Recursion (6.3) to the function

$$\rho(S_\alpha \xrightarrow{f} \{0,1\}) = \begin{cases} 0 & f^{-1}(0) \cup \{\alpha\} \text{ is a linearly ordered subset of } A \\ 1 & \text{otherwise} \end{cases}$$

and get a function $h\colon A \to \{0,1\}$.

7.3. **Definition.** *Let $(A, \prec)$ be a set with a strict partial order. An element m of A is* maximal *if no elements of A are greater than m.*

Let B be a subset of A. If $b \preceq c$ for all $b \in B$, then c is an upper bound *on B.*

Maximal elements are often of great interest.

Theorem 7.4 (Zorn's lemma)**.** *Let $(A, <)$ be a poset. Suppose that any linearly ordered subset of A has an upper bound. Then A contains maximal elements.*

Proof. Any upper bound on a maximal linearly ordered subset is maximal: Let $H \subset A$ be a maximal linearly ordered subset. By hypothesis, H has an upper bound m, i.e. $x \preceq m$ for all $x \in H$. By maximality of H, m must be in H, and there can be no element greater than m. (Suppose that $m \prec d$ for some d. Then $x \preceq m \prec d$ for all elements of H so $H \cup \{d\}$ is linearly ordered, contradicting maximality of H.) □

We shall later use Zorn's lemma to prove Tychonoff's theorem (18.16) that a product of compact spaces is compact. In fact, The Axiom of Choice, Zermelo's Well-ordering theorem, Hausdorff's maximum principle, Zorn's lemma, and Tychonoff's theorem are equivalent.

Here are two typical applications. Recall that a basis for a vector space over a field is a maximal independent subset.

Theorem 7.5. *[Ex 11.8] Any linearly independent subset of a vector space is contained in a basis.*

Proof. Let $M \subset V$ be a linearly independent subset. Apply Zorn's lemma to the set, strictly partially ordered by proper inclusion $\subsetneq$, of independent subsets containing M. A maximal element is a basis. □

As a corollary we see that $\mathbf{R}$ and $\mathbf{R}^2$ are isomorphic as vector spaces over $\mathbf{Q}$.

A maximal ideal in a ring is a maximal proper ideal.

Theorem 7.6. *Any proper ideal of a ring is contained in a maximal ideal.*

Proof. Let I be a proper ideal. Consider the set, strictly partially ordered by proper inclusion, of proper ideals containing the given ideal I. □

Other authors prefer to work with partial orders instead of strict partial orders.

7.7. **Definition.** *[Ex 11.2] Let A be a set. A relation $\preceq$ on A is said to be a partial order precisely when it is symmetric (that is $a \preceq a$ for all a in A), transitive (that is $a \preceq b$ and $b \preceq c$ implies $a \preceq c$), and anti-symmetric (that is $a \preceq b$ and $b \preceq a$ implies $a = b$).*

8. Topological spaces

What does it mean that a map $f\colon X \to Y$ between two sets is continuous? To answer this question, and much more, we equip sets with topologies.

8.1. **Definition.** *Let X be a set. A* topology *on X is a collection $\mathcal{T}$ of subsets of X, called* open *sets, such that*

(1) *$\emptyset$ and X are open*
(2) *The intersection of finitely many (two) open sets is open*
(3) *The union of an arbitrary collection of open sets is open*

A topological space *is a set X together with a topology $\mathcal{T}$ on X.*

8.2. **Example.** (1) In he *trivial topology* $\mathcal{T} = \{\emptyset, X\}$, only two subsets are open.
(2) In the *discrete topology* $\mathcal{T} = \mathcal{P}(X)$, all subsets are open.
(3) In the *particular point topology*, the open sets are $\emptyset$, X and all subsets containing a particular point $x \in X$. For instance, the Sierpinski space is the set $X = \{0, 1\}$ with the particular point topology for the point 0. The open sets are $\mathcal{T} = \{\emptyset, \{0\}, X\}$.
(4) In the *finite complement topology* (or *cofinite topology*), the open sets are $\emptyset$ and X and all subsets with a finite complement. .
(5) The *standard topology* on the real line $\mathbf{R}$ is $\mathcal{T} = \{\text{unions of open intervals}\}$.
(6) More generally, suppose that (X, d) is a metric space. The open r-ball centered at $x \in X$ is the set $B_d(x, r) = \{y \in X \mid d(x, y) < r\}$ of points within distance $r > 0$ from x. The *metric topology* on X is the collection $\mathcal{T}_d = \{\text{unions of open balls}\}$. The open sets of the topological space $(X, \mathcal{T}_d)$ and the open sets in the metric space (X, d) are the same. (See §§20–21 for more on metric topologies.)

The Sierpinski topology and the finite complement topology on an infinite set are not metric topologies.

Topologies on X are partially ordered by inclusion. For instance, the finite complement topology (8.2.(4)) on $\mathbf{R}$ is contained in the standard topology (8.2.(5), and the indiscrete topology (8.2.(1)) on $\{0, 1\}$ is contained in the Sierpinski topology (8.2.(3)) is contained in the discrete topology (8.2.(2)).

8.3. **Definition** (Comparison of topologies). *Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on the same set X.*

$$\left.\begin{array}{r} \mathcal{T} \text{ is finer than } \mathcal{T}' \\ \mathcal{T}' \text{ is coarser than } \mathcal{T} \end{array}\right\} \overset{def}{\Leftrightarrow} \mathcal{T} \supset \mathcal{T}'$$

The finest topology is the topology with the most opens sets, the coarsest topology is the one with fewest open sets. (Think of sandpaper!) The discrete topology is finer and the indiscrete topology coarser than any other topology: $\mathcal{P}(X) \supset \mathcal{T} \supset \{\emptyset, X\}$. Of course, two topologies may also be *incomparable*.

8.4. **Subbasis and basis for a topology.** Any intersection of topologies is a topology [Ex 13.4]. A standard way of defining a topology is to specify some collection of subsets and look at the coarsest topology containing this collection. In praxis, we restrict ourselves to collections of subsets that cover X.

8.5. **Definition.** *A* subbasis *is a collection $\mathcal{S}$ of subsets of X, that cover X (whose union is X). The subbasis $\mathcal{S}$ is a basis if the intersection of any two $\mathcal{S}$-sets is a union of $\mathcal{S}$-sets.*

If $\mathcal{S} \subset \mathcal{P}(X)$ is a subbasis then

$$\mathcal{T}_{\mathcal{S}} = \{\text{unions of finite intersections of subbasis sets}\}$$

is a topology, called the topology *generated by* the subbasis $\mathcal{S}$. It is the coarsests topology containing $\mathcal{S}$ [Ex 13.5]. (Use the distributive laws [§1] for $\cup$ and $\cap$.)

If $\mathcal{B} \subset \mathcal{P}(X)$ is a basis then

$$\mathcal{T}_{\mathcal{B}} = \{\text{unions of basis sets}\}$$

is a topology, called the topology *generated by* the basis $\mathcal{S}$. It is the coarsests topology containing $\mathcal{T}$ [Ex 13.5]. See 8.2.(5) and 8.2.(3) for examples.

If $\mathcal{S}$ is a subbasis, then

$$\mathcal{B}_{\mathcal{S}} = \{\text{Finite intersections of } \mathcal{S}\text{-sets}\} \tag{8.6}$$

is a basis generating the same topology as $\mathcal{S}$, $\mathcal{T}_{\mathcal{B}_{\mathcal{S}}} = \mathcal{T}_{\mathcal{S}}$.

A topology is a basis is a subbasis.

How can we compare topologies given by bases? How can we tell if two bases, or a subbasis and a basis, generate the same topology? (Two topologies, bases or subbases are said to be *equivalent* if they generate the same topology.)

8.7. **Lemma** (Comparison)**.** *Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases and $\mathcal{S}$ a subbasis.*

(1)

$$\mathcal{T}_{\mathcal{B}} \subset \mathcal{T}_{\mathcal{B}'} \iff \mathcal{B} \subset \mathcal{T}_{\mathcal{B}'} \iff \textit{All } \mathcal{B}\textit{-sets are open in } \mathcal{T}_{\mathcal{B}'}$$

(2)

$$\mathcal{T}_{\mathcal{B}} = \mathcal{T}_{\mathcal{B}'} \iff \left\{ \begin{array}{l} \textit{All } \mathcal{B}\textit{-sets are open in } \mathcal{T}_{\mathcal{B}'} \\ \textit{All } \mathcal{B}'\textit{-sets are open in } \mathcal{T}_{\mathcal{B}} \end{array} \right\}$$

(3)

$$T_{\mathcal{B}} = T_{\mathcal{S}} \iff \left\{ \begin{array}{l} \textit{All } \mathcal{B}\textit{-sets are open in } \mathcal{T}_{\mathcal{S}} \\ \textit{All } \mathcal{S}\textit{-sets are open in } \mathcal{T}_{\mathcal{B}} \end{array} \right\}$$

Proof. (1) is obvious since $\mathcal{T}_{\mathcal{B}}$ is the coarsest topology containing $\mathcal{B}$. Item (2) is immediate from (1). Item (3) is proved in the same way since $\mathcal{T}_{\mathcal{B}} = T_{\mathcal{S}}$ iff $\mathcal{T}_{\mathcal{B}} \subset T_{\mathcal{S}}$ and $\mathcal{T}_{\mathcal{B}} \supset T_{\mathcal{S}}$ iff $\mathcal{B} \subset T_{\mathcal{S}}$ and $\mathcal{T}_{\mathcal{B}} \supset \mathcal{S}$. □

8.8. **Example.** (1) In a metric space, the set $\mathcal{B} = \{B(x, r) \mid x \in X, r > 0\}$ of open balls is (by definition) a basis for the metric topology $\mathcal{T}_d$. The collection of open balls of radius $\frac{1}{n}$, $n \in \mathbf{Z}_+$, is an equivalent topology basis for $\mathcal{T}_d$.

(2) The collection of rectangular regions $(a_1, b_1) \times (a_2, b_2)$ in the plane $\mathbf{R}^2$ is a topology basis equivalent to the standard basis of open balls $B(a, r) = \{x \in \mathbf{R}^2 \mid |x - a| < r\}$. you can always put a ball inside a rectangle and a rectangle inside a ball.

(3) Let $f\colon X \to Y$ be any map. If $\mathcal{T}$ is a topology on Y with basis $\mathcal{B}$ or subbasis $\mathcal{S}$, then the pull-back $f^{-1}(\mathcal{T})$ is a topology, the *initial topology* for f, on X with basis $f^{-1}(\mathcal{B})$ and subbasis $f^{-1}(\mathcal{S})$.

(4) More generally, let X be a set, $\{Y_j\}$ a collection of topological spaces, and $f_j\colon X \to Y_j$, $j \in J$, a set of maps. Let $\mathcal{T}_j$ be the topology on Y_j, $\mathcal{B}_j$ a basis and $\mathcal{S}_j$, $j \in J$ a subbasis. Then $\bigcup f_j^{-1}(\mathcal{T}_j)$, $\bigcup f_j^{-1}(\mathcal{B}_j)$, $\bigcup f_j^{-1}(\mathcal{S}_j)$ are equivalent subbases on X. The topology they generate is called the *initial topology* for the maps f_j, $j \in J$.

8.9. **Example.** [Topologies on $\mathbf{R}$] We consider three topologies on $\mathbf{R}$:

$\mathbf{R}$: The *standard* topology with basis the open intervals (a, b).
$\mathbf{R}_\ell$: The *right half-open interval* topology with basis the right half-open intervals $[a, b)$.
$\mathbf{R}_K$: The K-topology with basis $\{(a, b)\} \cup \{(a, b) - K\}$ where $K = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$.

The right half-open interval topology is strictly finer than the standard topology because any open interval is a union of half-open intervals but not conversely (8.7) ($(a, b) = \bigcup_{a<x<b}[x, b)$, and $[0, 1)$ is open in $\mathbf{R}_\ell$ but not in $\mathbf{R}$; an open interval containing 0 is not a subset of $[0, 1)$). The K-topology is strictly finer than the standard topology because its basis contains the standard basis and $\mathbf{R} - K$ is open in $\mathbf{R}_K$ but not in $\mathbf{R}$ (an open interval containing 0 is not a subset of $\mathbf{R} - K$). The topologies $\mathbf{R}_\ell$ and $\mathbf{R}_K$ are not comparable [Ex 13.6].

8.10. **Example.** The collection of all *open rays*

$$\mathcal{S} = \{(-\infty, b)\} \cup \{(a, +\infty)\}$$

is a subbasis and the collection $\mathcal{B} = \{(a, b)\}$ of all open intervals is a basis for the standard topology on $\mathbf{R}$ (8.9) by 8.7.(3).

9. Order topologies

We associate a topological space to any linearly ordered set and obtain a large supply of examples of topological spaces. You may view the topological space as a means to study the ordered set, to find invariants, or you may view this construction as a provider of interesting examples of topological spaces.

Let $(X, <)$ be a linearly ordered set containing at least two points. The *open rays* in X are the subsets

$$(-\infty, b) = \{x \in X \mid x < b\}, \quad (a, +\infty) = \{x \in X \mid a < x\}$$

of X. The collection of all open rays is clearly a subbasis (just as in 8.10).

9.1. **Definition.** *The order topology $\mathcal{T}_<$ on the linearly ordered set X is the topology with all open rays as subbasis. A linearly ordered space is a linearly ordered set with the order topology.*

The *open intervals* in X are the subsets of the form

$$(a, b) = (-\infty, b) \cap (a, +\infty) = \{x \in X \mid a < x < b\}, \quad a, b \in X, a < b.$$

9.2. **Lemma.** *The collection of all open rays together with all open intervals is a basis for the order topology $\mathcal{T}_<$.*

Proof. $\mathcal{B}_\mathcal{S} \overset{(8.6)}{=} \{\text{Finite intersections of } \mathcal{S}\text{-sets}\} = \mathcal{S} \cup \{(a, b)\}$. □

If X has a smallest element a_0 then $(-\infty, b) = [a_0, b)$ is open. If X has no smallest element, then the open ray $(-\infty, b) = \bigcup_{a<c}(a, c)$ is a union of open intervals and we do not need this open ray in the basis. Similar remarks apply to the greatest element when it exists.

9.3. **Lemma.** *(Cf 8.10) If X has no smallest and no largest element, then the set $\{(a, b)\}$ of open intervals is a basis for the order topology.*

9.4. **Example.**
(1) The order topology on the ordered set $(\mathbf{R}, <)$ is the standard topology (8.9).
(2) The order topology on the ordered set $\mathbf{R}^2$ has as basis the collection of all open intervals $(a_1 \times a_2, b_1 \times b_2)$. An equivalent basis (8.7) consists of the open intervals $(a \times b_1, a \times b_2)$. $\mathbf{R}^2_<$ is strictly finer than $\mathbf{R}^2_d$.
(3) The order topology on $\mathbf{Z}_+$ is the discrete topology because $(-\infty, n) \cap (\infty, n + 1) = \{n\}$ is open.
(4) The order topology on $\mathbf{Z}_+ \times \mathbf{Z}_+$ is not discrete. Any open set that contains the element 2×1 also contains elements from $\{1\} \times \mathbf{Z}_+$. Thus the set $\{1 \times 2\}$ is not open.
(5) The order topology on $\mathbf{Z} \times \mathbf{Z}$ is discrete.
(6) Is the order topology on S_Ω discrete?
(7) $I^2 = [0, 1]^2$ with the order topology is denoted I^2_o and called the *ordered square*. The open sets containing the point $x \times y \in I^2_o$ look quite different depending on whether $y \in \{0, 1\}$ or $0 < y < 1$.

10. The product topology

Let $(X_j)_{j\in J}$ be an indexed family of topological spaces. Let $\pi_k \colon \prod_{j\in J} X_j \to X_k$ be the projection map. An *open cylinder* is a subset of the product space of the form

$$\pi_k^{-1}(U_k), \quad U_k \subset X_k \text{ open}, \quad k \in J.$$

The set $\pi_k^{-1}(U_k)$ consists of the points $(x_j) \in \prod X_j$ with kth coordinate in U_k. Alternatively, $\pi_k^{-1}(U_k)$ consists of all choice functions $c \colon J \to \bigcup_{j\in J} U_j$ such that $c(k) \in U_k$ (Figure 2).

10.1. **Definition.** *The* product *topology on $\prod_{j\in J} X_j$ is the topology with subbasis*

$$\mathcal{S}_\prod = \bigcup_{j\in j} \{\pi_j^{-1}(U) \mid U \subset X_j \text{ open}\}$$

consisting of all open cylinders or, equivalently, with basis (8.6)

$$\mathcal{B}_\prod = \{\prod_{j\in J} U_j \mid U_j \subset X_j \text{ open and } U_j = X_j \text{ for all but finitely many } j \in J\}$$

The product topology is the coarsest topology making all the projection maps $\pi_j\colon \prod X_j \to X_j$, $j \in J$, continuous.

This becomes particularly simple when we consider finite products.

10.2. **Lemma.** *Let $X = X_1 \times X_2 \times \cdots X_k$ be a finite Cartesian product. The collection*

$$\mathcal{B} = \{U_1 \times U_2 \times \cdots \times U_k \mid U_1 \text{ open in } X_1, U_2 \text{ open in } X_2, \ldots, U_k \text{ open in } X_k\}$$

of all products of open sets is a basis for the product topology.

10.3. **Corollary.** *Suppose that $\mathcal{B}_j$ is a basis for the topology on X_j, $j = 1, \ldots, k$. Then $\mathcal{B}_1 \times \cdots \times \mathcal{B}_k$ is a basis for the product topology on $X_1 \times \cdots \times X_k$.*

Proof. Note that $\mathcal{B}_1 \times \cdots \times \mathcal{B}_k$ is indeed a basis. Compare it to the basis of 10.2 using 8.7. □

10.4. **Products of linearly ordered spaces.** When $(X, <)$ and $(Y, <)$ are linearly ordered sets, we now have two topologies on the Cartesian product $X \times Y$: The product topology of the order topologies, $(X, \mathcal{T}_<) \times (Y, \mathcal{T}_<)$, and the order topology of the product dictionary order, $(X \times Y, \mathcal{T}_<)$. These two topologies are in general *not* identical or even comparable. It is difficult to imagine any general relation between them since $X_< \times Y_<$ is essentially symmetric in X and Y whereas the dictionary order has no such symmetry. But even when $X = Y$ there does not seem to be a general pattern: The order topology $(\mathbf{Z}_+ \times \mathbf{Z}_+, \mathcal{T}_<)$ is coarser than the product topology $(\mathbf{Z}_+, \mathcal{T}_<) \times (\mathbf{Z}_+, \mathcal{T}_<)$ (which is discrete) (9.4.(4)–(5)). On the other hand, the order topology $(\mathbf{R} \times \mathbf{R}, \mathcal{T}_<)$ is finer than the product topology $(\mathbf{R}, \mathcal{T}_<) \times (\mathbf{R}, \mathcal{T}_<)$ which is the standard topology, see 9.4.(1)–(2) and [Ex 16.9].

10.5. **Corollary** (Cf [Ex 16.9]). *Let $(X, <)$ and $(Y, <)$ be two linearly ordered sets. Suppose that Y does not have a largest or a smallest element. Then the order topology $(X \times Y)_< = X_d \times Y_<$ where X_d is X with the discrete topology. This topology is finer than $X_< \times Y_<$.*

Proof. The equations

$$(-\infty, a \times b) = \bigcup_{x<a} \{x\} \times Y \cup \{a\} \times (-\infty, b), \quad (a \times b, +\infty) = \{a\} \times (b, +\infty) \cup \bigcup_{x>a} \{x\} \times Y$$

show that the order topology is coarser than the product of the discrete and the order topology. On the other hand, the sets $\{x\} \times (c, d)$ is a basis for $= X_d \times Y_<$ (9.3, 10.3) and $\{x\} \times (c, d) = (x \times c, x \times d)$ is open in the order topology since it is an open interval. □

The corollary shows that the order topology on $X \times Y$ does not yield anything new in case the second factor is unbounded in both directions. As an example where this is not the case we have already seen $\mathbf{Z}_+ \times \mathbf{Z}_+$ and I_o^2 (9.4.(4), 9.4.(7)).

10.6. **The coproduct topology.** The coproduct topology on the coproduct $\coprod X_j$ (3) is the finest topology making all the inclusion maps $\iota_j\colon X_j \to \coprod X_j$, $j \in J$, continuous.

This means that

$$U \subset \coprod_{j \in J} X_j \text{ is open } \iff \iota_j^{-1}(U) \text{ is open for all } j \in J$$

Alternatively, the open sets of the coproduct $\coprod X_j$ are the coproducts $\coprod U_j$ of open set $U_j \subset X_j$.

11. The subspace topology

Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$ a subset. We make Y into a topological space.

11.1. **Definition.** *The* subspace *topology on Y is the topology $\mathcal{T}_\subset = Y \cap \mathcal{T} = \{Y \cap U \mid U \in \mathcal{T}\}$. A subset $V \subset Y$ is open relative to Y if it is open in the subspace topology.*

It is immediate that $\mathcal{T}_\subset$ is indeed a topology. The subset $V \subset Y$ is open relative to Y if and only if $V = Y \cap U$ for some open set $U \subset X$.

11.2. **Lemma.** *If $\mathcal{B}$ is a basis for $\mathcal{T}$, then $Y \cap \mathcal{B} = \{Y \cap U \mid U \in \mathcal{B}\}$ is a basis for the subspace topology $Y \cap \mathcal{T}$. If $\mathcal{S}$ is a subbasis for $\mathcal{T}$, then $Y \cap \mathcal{S} = \{Y \cap U \mid U \in \mathcal{S}\}$ is a subbasis for the subspace topology $Y \cap \mathcal{T}$.*

Proof. This is 8.8.(3) applied to the inclusion map $A \hookrightarrow X$. □

If $V \subset Y$ is open, then V is also relatively open. The converse holds if Y is open.

11.3. **Lemma.** *Assume that $A \subset Y \subset X$. Then*

(1) *A is open in $Y \iff A = Y \cap U$ for some open set U in X*
(2) *If Y is open then: A is open in $Y \iff A$ is open in X*

Proof. (1) This is the definition of the subspace topology.
(2) Suppose that Y is open and that $A \subset Y$. Then

$$A \text{ open in } Y \iff A = Y \cap U \text{ for some open } U \subset X \iff A \text{ open in } X$$

in that $A = A \cap Y$. □

The lemma says that an open subset of an open subset is open.

The next theorem says that the subspace and the product space operations commute.

Theorem 11.4. *Let $Y_j \subset X_j$, $j \in J$. The subspace topology that $\prod Y_j$ inherits from $\prod X_j$ is the product topology of the subspace topologies on Y_j.*

Proof. The subspace topology on $\prod Y_j$ has subbasis

$$\prod Y_j \cap \mathcal{S}_{\prod X_j} = \prod Y_j \cap \bigcup_{k \in j} \{\pi_k^{-1}(U_k)\} = \bigcup_{k \in J} \{\prod Y_j \cap \pi_k^{-1}(U_k)\}, \quad U_k \subset X_k \text{ open},$$

and the product topology on $\prod Y_j$ has subbasis

$$\mathcal{S}_{\prod Y_j} = \bigcup_{k \in J} \{\pi_k^{-1}(Y_k \cap U_k)\}, \quad U_k \subset X_k \text{ open}$$

These two subbases are identical for $\pi_k^{-1}(Y_k \cap U_k) \overset{\text{[Ex 2.2]}}{=} \pi_k^{-1}(Y_k) \cap \pi_k^{-1}(U_k) = (\prod Y_j) \cap \pi_k^{-1}(U_k)$. □

11.5. **Subspaces of linearly ordered spaces.** When $(X, <)$ is a linearly ordered set and $Y \subset X$ a subset we now have two topologies on Y. We can view Y as a subspace of the topological space $X_<$ with the order topology or we can view Y as a sub-ordered set of X and give Y the order topology. Note that these two topologies are *not* the same when $X = \mathbf{R}$ and $Y = [0, 1) \cup \{2\} \subset \mathbf{R}$: In the subspace topology $\{2\}$ is open in Y but $\{2\}$ is not open in the order topology because Y has the order type of $[0, 1]$. See 11.7 for more examples. The point is that

- Any open ray in Y is the intersection of Y with an open ray in X
- The intersection of Y with an open ray in X need not be an open ray in Y

However, if Y happens to be convex then open rays in Y are precisely Y intersected with open rays in X.

11.6. **Lemma** ($Y_< \subset Y \cap X_<$). *Let $(X, <)$ be a linearly ordered set and $Y \subset X$ a subset. The order topology on Y is coarser than the subspace topology Y in general. If Y is convex, the two topologies on Y are identical.*

Proof. The order topology on Y has subbasis

$$\mathcal{S}_< = \{Y \cap (-\infty, b) \mid b \in Y\} \cup \{Y \cap (a, \infty) \mid a \in Y\}$$

and the subspace topology on Y has subbasis

$$\mathcal{S}_\subset = \{Y \cap (-\infty, b) \mid b \in X\} \cup \{Y \cap (a, \infty) \mid a \in X\}$$

Clearly, $\mathcal{S}_< \subset \mathcal{S}_\subset$, so the order topology on Y is coarser than the subspace topology in general.

If Y is convex and $b \in X - Y$ then b is either a lower or an upper bound for Y as we cannot have $y_1 < b < y_2$ for two points y_1 and y_2 of Y. If b is a lower bound for Y, then $Y \cap (-\infty, b) = \emptyset$, and if b is an upper bound for Y, then $Y \cap (-\infty, b) = Y$. Therefore also $\mathcal{S}_\subset \subset \mathcal{T}_<$ so that in fact $\mathcal{T}_< = \mathcal{T}_\subset$.

□

11.7. **Example.** (1) $\overline{S}_\omega = [1 \times 1, 2 \times 1]$ (6.5) is a convex subset of $\mathbf{Z}_+ \times \mathbf{Z}_+$ so (11.6) the subspace topology (9.4.(4)) is the same as the order topology.

(2) The subset $\mathbf{Z}_+ \times \mathbf{Z}_+$ is not convex in $\mathbf{Z} \times \mathbf{Z}$ so we expect the subspace topology to be strictly finer than the order topology. Indeed, the subspace topology that $\mathbf{Z}_+ \times \mathbf{Z}_+$ inherits from the discrete space $\mathbf{Z} \times \mathbf{Z}$ is discrete but the order topology is not discrete (9.4.(4)–(5)).

(3) Consider $X = \mathbf{R}^2$ and $Y = [0,1]^2$ with order topologies. Y is not convex so we expect the subspace topology to be strictly finer than the order topology. Indeed, the subspace topology on $[0,1]^2$, which is $[0,1]_d \times [0,1]$, is strictly finer than I_o^2 (9.4.(7)): The set

$$[0, \frac{1}{2}] \times [0,1] = ([0,1] \times [0,1]) \cap (-\infty, \frac{1}{2} \times 2)$$

is open in the subspace topology on Y but it is not open in the order topology on Y as any basis open set (9.2) containing $\frac{1}{2} \times 1$ also contains points with first coordinate $> \frac{1}{2}$.

(4) Consider $X = \mathbf{R}^2$ and $Y = (0,1)^2$ with order topologies. Y is not convex so we expect the subspace topology to be strictly finer than the order topology. But it isn't! The reason is that $(0,1)$ does not have a greatest nor a smallest element (10.5).

(5) The subset $\mathbf{Q}$ of the linearly ordered set $\mathbf{R}$ is not convex but nevertheless the subspace topology inherited from $\mathbf{R}$ is the order topology. Again, the reason seems to be that $\mathbf{Q}$ does not have a greatest nor a smallest element.

12. Closed sets and limit points

Let $(X, \mathcal{T})$ be a topological space and A a subset of X.

12.1. **Definition.** *We say that the set A is* closed *if its complement $X - A$ is open.*

Note that X and $\emptyset$ are closed (and open), a finite union of closed sets is closed, an arbitrary intersection of closed sets is closed. The finer the topology, the more closed sets.

12.2. **Example.** (1) $[a, b]$ is closed in $\mathbf{R}$, $[a, b)$ is neither closed nor open, $\mathbf{R}$ is both closed and open.
(2) Let $X = [0, 1] \cup (2, 3)$. The subsets $[0, 1]$ and $(2, 3)$ of X are both closed and open (they are *clopen*) as subsets of X. The interval $[0, 1]$ is closed in $\mathbf{R}$ while $(2, 3)$ is open in $\mathbf{R}$.
(3) $K = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is not closed in $\mathbf{R}$ but it is closed in $\mathbf{R}_K$ (8.9).
(4) $\mathbf{R}_\ell$ is (8.9) the set of real numbers equipped with the topology generated by the basis sets $[a, b)$ of right half-open intervals. All sets that are (open) closed in the standard topology $\mathbf{R}$ are also (open) closed in the finer topology $\mathbf{R}_\ell$. Sets of the form $(-\infty, a) = \bigcup_{x<a}[x, a) = \mathbf{R} - [a, \infty)$, $[a, b) = (-\infty, b) \cap [a, \infty)$, and $[a, \infty) = \bigcup_{a<x}[a, x) = \mathbf{R} - (-\infty, a)$ are both open and closed. Sets of the form $(-\infty, b]$ or $[a, b]$ are closed (since they are closed in the standard topology) and not open since they are not unions of basis sets. Sets of the form (a, ∞) are open (since they are open in standard topology) and not closed. Sets of the form $(a, b]$ are neither open nor closed.
(5) Let X be a well-ordered set (§6). In the order topology (§9), sets of the form $(a, \infty) = [a^+, \infty) = X - (-\infty, a]$, $(-\infty, b] = (\infty, b^+) = X - (b, \infty)$ and $(a, b] = (a, \infty) \cap (-\infty, b]$ are closed and open. (Here, b^+ denotes b if b is the largest element and the immediate successor of b if b is not the largest element.)

12.3. **Closure and interior.** We consider the largest open set contained in A and the smallest closed set containing A.

12.4. **Definition.** *The* interior *of A is the union of all open sets contained in A:*

$$\operatorname{Int} A = \bigcup \{U \subset A \mid U \text{ open}\} = A^\circ$$

The closure *of A is the intersection of all closed sets containing A:*

$$\operatorname{Cl} A = \bigcap \{C \supset A \mid C \text{ closed}\} = \overline{A}$$

Note [9, Ex 17.6, 17.7, 17.8, 17.9] [3, p 9] [6, 7.1.7–9] that

- $X - \overline{A} = (X - A)^\circ$ and $X - A^\circ = \overline{X - A}$
- $A^\circ \subset A \subset \overline{A}$
- A° is open, it is the largest open set contained in A, and A is open iff $A^\circ = A$
- $\overline{A}$ is closed, it is the smallest closed set containing A, and A is closed iff $A = \overline{A}$
- $(A \cap B)^\circ = A^\circ \cap B^\circ$, $\overline{A \cup B} = \overline{A} \cup \overline{B}$, $(A \cup B)^\circ \supset A^\circ \cap B^\circ$, $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$,
- $A \subset B \Rightarrow \begin{cases} \operatorname{Int} A \subset \operatorname{Int} B \\ \operatorname{Cl} A \subset \operatorname{Cl} B \end{cases}$

The first point, for instance, is the computation

$$\begin{aligned} X - \overline{A} = X - \bigcap \{C \supset A \mid C \text{ closed}\} &= \bigcup \{X - C \subset X - A \mid X - C \text{ open}\} \\ &= \bigcup \{U \subset X - A \mid U \text{ open}\} = (X - A)^\circ \end{aligned}$$

or one can simply say that $X - \overline{A}$ is the largest open subset of $X - A$ since $\overline{A}$ is the smallest closed superset of A.

12.5. **Definition.** *A* neighborhood *of the point $x \in X$ is an open set containing x. A* neighborhood *of the set $A \subset X$ is an open set containing A.*

12.6. **Proposition.** *Let $A \subset X$ and let x be a point in X. Then*

$$x \in A^\circ \iff \text{There exists a neighborhood } U \text{ of } x \text{ such that } U \subset A$$
$$x \in \overline{A} \iff U \cap A \neq \emptyset \text{ for all neighborhoods } U \text{ of } x$$

Proof. The first assertion is clear as A° is the union of the open sets contained in A. The second assertion is just a reformulation of the first one,

$$x \notin \overline{A} \iff x \in X - \overline{A} \iff x \in (X - A)^\circ \iff x \text{ has a neighborhood disjoint from } A,$$

since $X - \overline{A} = (X - A)^\circ$. □

12.7. **Proposition** (Closure with respect to subspace)**.** *Let $A \subset Y \subset X$. Then*

(1) *A is closed in $Y \iff A = Y \cap C$ for some closed set C in X*
(2) *If Y is closed then: A is closed in $Y \iff A$ is closed in X*
(3) $\mathrm{Cl}_Y(A) = Y \cap \overline{A}$.

Proof. (1) Let A be a subset of Y. Then:

$$\begin{aligned} A \text{ is closed in } Y &\iff Y - A \text{ is open in } Y \\ &\overset{11.3.(1)}{\iff} Y - A = Y \cap U \text{ for some open } U \subset X \\ &\iff A = Y \cap (X - U) \text{ for some open } U \subset X \\ &\iff A = Y \cap C \text{ for some closed } C \subset X \end{aligned}$$

(2) If $A \subset Y$ and Y is closed then

$$\begin{aligned} A \text{ is closed in } Y &\iff A = Y \cap C \text{ for some closed } C \subset X \\ &\iff A \text{ is closed in } X \end{aligned}$$

(3) The set $\mathrm{Cl}_Y(A)$ is the intersection of all relatively closed sets containing A. The relatively closed sets are the sets of the form $Y \cap C$ where C is closed in X. This means that

$$\mathrm{Cl}_Y(A) = \bigcap \{Y \cap C \mid C \supset A, C \text{ closed}\} = Y \bigcap \{C \mid C \supset A, C \text{ closed}\} = Y \cap \overline{A}$$

by a direct computation. □

12.8. **Definition.** *A subset $A \subset X$ is said to be dense if $\overline{A} = X$, or, equivalently, if every open subset of X contains a point of A.*

12.9. **Proposition.** *Let A be a dense and U an open subset of X. Then $\overline{A \cap U} = \overline{U}$.*

Proof. The inclusion $\overline{A \cap U} \subset \overline{U}$ is general. For the other inclusion, consider a point $x \in \overline{U}$. Let V be any neighborhood of x. Then $V \cap (A \cap U) = (V \cap U) \cap A$ is not empty since $V \cap U$ is a neighborhood of x and A is dense. But this says that x is in the closure of $A \cap U$. □

12.10. **Limit points and isolated points.** Let X be a topological space and A a subset of X.

12.11. **Definition** (Limit points, isolated points)**.** *A point $x \in X$ is a* limit point *of A if $U \cap (A - \{x\}) \neq \emptyset$ for all neighborhoods U of x. The set of limit points*[1] *of A is denoted A'. A point $a \in A$ is an* isolated point *if a has a neighborhood that intersects A only in $\{a\}$.*

Equivalently, $x \in X$ is a limit point of A iff $x \in \overline{A - \{x\}}$, and $a \in A$ is an isolated point of A iff $\{a\}$ is open in A. These two concepts are *almost* each others opposite:

$$\begin{aligned} x \text{ is } \textit{not} \text{ a limit point of } A &\iff x \text{ has a neighborhood } U \text{ such that } U \cap (A \cup \{x\}) = \{x\} \\ &\iff x \text{ is an isolated point of } A \cup \{x\} \end{aligned}$$

12.12. **Proposition.** *Let A be a subset of X and A' the set of limit points of A. Then $A \cup A' = \overline{A}$ and $A \cap A' = \{a \in A \mid a \text{ is not an isolated point of } A\}$ so that*

$$\begin{aligned} A \supset A' &\iff A \textit{ is closed} \\ A \subset A' &\iff A \textit{ has no isolated points} \\ A \cap A' = \emptyset &\iff A \textit{ is discrete} \\ A' = \emptyset &\iff A \textit{ is closed and discrete} \end{aligned}$$

If $B \subset A$ then $B' \subset A'$.

[1] The set of limit points of A is sometimes denotes A^d and called the derived set of A

Proof. It is clear that all limit points of A are in $\overline{A}$ and that all points in the closure of A that are not in A are limit points,

$$\overline{A} - A = \{x \in X - A \mid \text{all neighborhoods of } x \text{ meet } A\} \subset A' \subset \overline{A},$$

which implies that $A \cup A' = \overline{A}$. From the above discusion we have that the set of limit points in A, $A \cap A' = A - (A - A')$, is the set of non-isolated points of A. If $A \cap A' = \emptyset$ then all all points of A are isolated so that the subspace A has the discrete topology. We have $A' = \emptyset \iff A \supset A'$, $A \cap A' = \emptyset \iff A$ is closed and discrete. □

12.13. **Example.** (1) $\overline{\mathbf{Q}} = \mathbf{R} = \overline{\mathbf{R} - \mathbf{Q}}$ ($\mathbf{R} - \mathbf{Q}$ contains $(\mathbf{Q} - \{0\})\sqrt{2}$). $\operatorname{Cl}\operatorname{Int}\mathbf{Q} = \emptyset$, $\operatorname{Int}\operatorname{Cl}\mathbf{Q} = \mathbf{R}$.

(2) Let $C_r \subset \mathbf{R}^2$ be the circle with center $(0, r)$ and radius $r > 0$. Then

$$\overline{\bigcup_{n\in\mathbf{Z}_+} C_{1/n}} = \bigcup_{n\in\mathbf{Z}_+} C_{1/n}, \qquad \overline{\bigcup_{n\in\mathbf{Z}_+} C_n} = (\mathbf{R} \times \{0\}) \cup \bigcup_{n\in\mathbf{Z}_+} C_n$$

The first set of decreasing circles (known as the Hawaiian Earring [9, Exmp 1 p 436]) is closed because it is also the intersection of a collection (which collection?) of closed sets.

(3) The set of limit points of $K = \{\frac{1}{n} \mid n \in \mathbf{Z}_+\}$ is $\{0\}$ in $\mathbf{R}$ and $\mathbf{R}_\ell$ and $\emptyset$ in $\mathbf{R}_K$.

(4) Let X be a linearly ordered space (9.1). Closed intervals $[a, b]$ are closed because their complements $X - [a, b] = (-\infty, a) \cup (b, \infty)$ are open. Therefore the closure of an interval of the form $[a, b)$ is either $[a, b)$ or $[a, b]$. If b has an immediate predecessor b^- then $[a, b) = [a, b^-]$ is closed so that $\overline{[a, b)} = [a, b)$. Otherwise, $\overline{[a, b)} = [a, b]$ because any neighborhood of b contains an interval of the form $(c, b]$ for some $c < b$ and $[a, b) \cap (c, b]$, which equals $[a, b]$ or $(c, b]$, is not empty.

12.14. **Convergence, the Hausdorff property, and the T_1-axiom.** Let x_n, $n \in \mathbf{Z}_+$, be a sequence of points in X, ie a map $\mathbf{Z}_+ \to X$.

12.15. **Definition.** *The sequence x_n converges to the point $x \in X$ if for any neighborhood U of x there exists some N such that $x_n \in U$ for all $n > N$.*

If a sequence converges in some topology on X it also converges in any coarser topology but not necessarily in a finer topology.

12.16. **Example.** In the Sierpinski space $X = \{0, 1\}$ (8.2.(3)), the sequence $0, 0, \cdots$ converges to 0 and to 1. In $\mathbf{R}$ with the finite complement topology (8.2.(4)), the sequence $1, 2, 3, \cdots$ converges to any point. In $\mathbf{R}_K$, the sequence $\frac{1}{n}$ does not converge; in $\mathbf{R}$ and $\mathbf{R}_\ell$ it converges to 0 (and to no other point). The sequence $-\frac{1}{n}$ converges to 0 in $\mathbf{R}$ and $\mathbf{R}_K$, but does not converge in $\mathbf{R}_\ell$. In the ordered space $[1, \omega] = \mathbf{Z}_+ \amalg \{\omega\}$ (6.5.(6)), the sequence n converges to ω (and to no other point). Can you find a sequence in the ordered space $[0, \Omega]$ (6.8) that converges to Ω?

12.17. **Definition** (Separation Axioms). *A topological space is T_1-space if points are closed: For any two distinct points $x_1 \neq x_2$ in X there exists an open set U such that $x_1 \in U$ and $x_2 \notin U$.*

A topological space X is a T_2-space (or a Hausdorff space) if there are enough open sets to separate points: For any two distinct points $x_1 \neq x_2$ in X there exist disjoint open sets, U_1 and U_2, such that $x_1 \in U_1$ and $x_2 \in U_2$.

All Hausdorff spaces are T_1. X is T_1 iff all finite subsets are closed. Cofinite topologies are T_1 by construction and not T_2 when the space has infinitely many points. Particular point topologies are not T_1 (on spaces with more than one point).

All linearly ordered [Ex 17.10] or metric spaces are Hausdorff; in particular, $\mathbf{R}$ is Hausdorff. $\mathbf{R}_\ell$ and $\mathbf{R}_K$ are Hausdorff because the topologies are finer than the standard Hausdorff topology $\mathbf{R}$.

Theorem 12.18. *A sequence in a Hausdorff space can not converge to two distinct points.*

A property of a topological space is said to be (weakly) hereditary if any (closed) subspace of a space with the property also has the property. Hausdorffness is hereditary and also passes to product spaces.

Theorem 12.19 (Hereditary properties of Hausdorff spaces). *[Ex 17.11, 17.12] Any subset of a Hausdorff space is Hausdorff. Any product of Hausdorff spaces is Hausdorff.*

Theorem 12.20. *Suppose that X is T_1. Let A be a subset of and x a point in X. Then*

$$x \text{ is a limit point} \iff \text{All neighborhoods of } x \text{ intersect } A \text{ in infinitely many points}$$

Proof. $\Longleftarrow$: If all neighborhoods of x intersect A in infinitely many points, then, clearly, they also intersect A in a point that is not x.
$\Longrightarrow$: Assume that x is a limit point and let U be a neighborhood of x. Then U contains a point a_1 of A different from x. Remove this point from U. Since points are closed, $U - \{a_1\}$ is a new neighborhood of x. This new neighborhood of x contains a point a_2 of A different from x and a_1. In this way we recursively find a whole sequence of distinct points in $U \cap A$. □

13. Continuous functions

Let $f\colon X \to Y$ be a map between two topological spaces.

13.1. **Definition.** *The map $f\colon X \to Y$ is continuous if all open subsets of Y have open preimages in X: V open in $X \Longrightarrow f^{-1}(V)$ open in Y*

If $\mathcal{T}_X$ is the topology on X and $\mathcal{T}_Y$ is the topology on Y then

$$f \text{ is continuous} \iff f^{-1}(\mathcal{T}_Y) \subset \mathcal{T}_X \iff f^{-1}(\mathcal{B}_Y) \subset \mathcal{T}_X \iff f^{-1}(\mathcal{S}_Y) \subset \mathcal{T}_X \tag{13.2}$$

where $\mathcal{B}_Y$ is a basis and $\mathcal{S}_Y$ a subbasis for $\mathcal{T}_Y$. The finer the topology in Y and the coarser the topology in X are, the more difficult is it for $f\colon X \to Y$ to be continuous.

Theorem 13.3. *Let $f\colon X \to Y$ be a map between two topological spaces. The following are equivalent:*

1. *f is continuous*
2. *The preimage of any open set in Y is open in X: V open in $Y \Longrightarrow f^{-1}(V)$ open in X*
3. *The preimage of any closed set in Y is closed in X: C closed in $Y \Longrightarrow f^{-1}(C)$ closed in X*
4. *$f^{-1}(B^\circ) \subset (f^{-1}(B))^\circ$ for any $B \subset Y$*
5. *$\overline{f^{-1}(B)} \subset f^{-1}(\overline{B})$ for any $B \subset Y$*
6. *$f(\overline{A}) \subset \overline{f(A)}$ for any $A \subset X$*
7. *For any point $x \in X$ and any neighborhood $V \subset Y$ of $f(x)$ there is a neighborhood $U \subset X$ of x such that $f(U) \subset V$.*

Proof. It is easy to see that (7),(1),(2), and (3) are equivalent.
(2) $\Longrightarrow$ (4): $\left.\begin{array}{l} f^{-1}(B^\circ) \subset f^{-1}(B) \\ f^{-1}(B^\circ) \text{ open} \end{array}\right\} \Longrightarrow f^{-1}(B^\circ) \subset (f^{-1}(B))^\circ$.
(4) $\Longrightarrow$ (2): Let B be any open set in Y. Then $f^{-1}(B) \overset{B=B^\circ}{=} f^{-1}(B^\circ) \overset{(4)}{\subset} (f^{-1}(B))^\circ \subset f^{-1}(B)$ so $f^{-1}(B)$ is open since it equals its own interior.
(3) $\iff$ (5): Similar to (2) $\iff$ (4).
(3) $\Longrightarrow$ (6): $\left.\begin{array}{l} A \subset f^{-1}(\overline{f(A)}) \\ f^{-1}(\overline{f(A)}) \text{ is closed} \end{array}\right\} \Longrightarrow \overline{A} \subset f^{-1}(\overline{f(A)}) \iff f(\overline{A}) \subset \overline{f(A)}$.
(6) $\Longrightarrow$ (5): $f(\overline{f^{-1}(B)}) \overset{(6)}{\subset} \overline{f(f^{-1}(B))} \overset{f(f^{-1}(B)) \subset B}{\subset} \overline{B}$ □

13.4. **Example.** The function $f(x) = -x$ is continuous $\mathbf{R} \to \mathbf{R}$, but not continuous $\mathbf{R}_K \to \mathbf{R}_K$ (K is closed in $\mathbf{R}_K$ but $f^{-1}(K)$ is not closed (12.16)) and not continuous $\mathbf{R}_\ell \to \mathbf{R}_\ell$ ($[0,\infty)$ is open in $\mathbf{R}_\ell$ but $f^{-1}([0,\infty)) = (-\infty, 0]$ is not open (12.2.(4))). Since $[a,b)$ is closed and open in $\mathbf{R}_\ell$ (12.2.(4)) the map $1_{[a,b)}\colon \mathbf{R}_\ell \to \{0,1\}$, with value 1 on $[a,b)$ and value 0 outside $[a,b)$, is continuous.

Theorem 13.5. (1) *The identity function is continuous.*
(2) *The composition of two continuous functions is continuous.*
(3) *Let B be a subspace of Y, A a subspace of X, and $f\colon X \to Y$ a map taking values in B. Then*

$$f\colon X \to Y \text{ is continuous} \Longrightarrow f|A\colon A \to Y \text{ is continuous} \qquad (\textit{restriction})$$
$$f\colon X \to Y \text{ is continuous} \iff B|f\colon X \to B \text{ is continuous} \qquad (\textit{corestriction})$$

(4) *Let $f_j\colon X_j \to Y_j$, $j \in J$, be an indexed set of maps. Then*

$$\prod f_j\colon \prod X_j \to \prod Y_j \text{ is continuous} \iff f_j\colon X_j \to Y_j \text{ is continuous for all } j \in J$$

(5) *Suppose that $X = \bigcup U_\alpha$ is a union of open sets. Then*

$$f\colon X \to Y \text{ is continuous} \iff f|U_\alpha \text{ is continuous for all } \alpha$$

for any map $f\colon X \to Y$.

(6) *(Glueing lemma) Suppose that $X = C_1 \cup \cdots \cup C_n$ is a finite union of closed sets. Then*

$$f\colon X \to Y \text{ is continuous} \iff f|C_i \text{ is continuous for all } i = 1, \ldots, n$$

for any map $f\colon X \to Y$.

Proof. This is easy to check. The $\Longrightarrow$-part of (4) uses 13.14 below. □

13.6. **Homeomorphisms and embeddings.** One of the central problems in topology is to decide if two given spaces are homeomorphic.

13.7. **Definition** (Homeomorphism). *A bijective continuous map $f\colon X \to Y$ is a* homeomorphism *if its inverse is continuous.*

A bijection $f\colon X \to Y$ is a homeomorphism iff

$$U \text{ is open in } X \iff f(U) \text{ is open in } Y \quad \text{or} \quad f^{-1}(V) \text{ is open in } X \iff V \text{ is open in } Y$$

holds for all $U \subset X$ or $V \subset Y$.

We now extend the subspace topology (11.1) to a slightly more general situation.

13.8. **Definition** (Subspace topology). *Let X be a set, Y a topological space, and $f\colon X \to Y$ an injective map. The* subspace topology *on X (wrt to the map f) is the collection*

$$f^{-1}(\mathcal{T}_Y) = \{f^{-1}(V) \mid V \subset Y \text{ open}\}$$

of subsets of X.

The subspace topology is the coarsest topology on X such that $f\colon X \to Y$ is continuous. There are as few open sets in X as possible without destroying continuity of $f\colon X \to Y$.

13.9. **Proposition** (Characterization of the subspace topology). *Suppose that X has the subspace topology wrt to the map $f\colon X \to Y$. Then*

(1) *$X \to Y$ is continuous and,*
(2) *for any map $A \to X$ into X,*

$$A \to X \text{ is continuous} \iff A \to X \xrightarrow{f} Y \text{ is continuous}$$

The subspace topology is the only topology on X with these properties.

Proof. This is because

$$A \xrightarrow{g} X \text{ is continuous} \overset{(13.2)}{\iff} g^{-1}(\mathcal{T}_X) \subset \mathcal{T}_A \iff g^{-1}(f^{-1}\mathcal{T}_Y) \subset \mathcal{T}_A \iff$$
$$(fg)^{-1}\mathcal{T}_Y) \subset \mathcal{T}_A \iff A \xrightarrow{g} X \xrightarrow{f} Y \text{ is continuous}$$

by definition of the subspace topology. The identity map of X is a homeomorphism whenever X is eqiopped with a topology with these two properties. □

13.10. **Definition** (Embedding). *An injective continuous map $f\colon X \to Y$ is an* embedding *if X has the subspace topology.*

This means that f is an embedding if and only if for all $U \subset X$:

$$U \text{ is open in } X \iff U = f^{-1}(V) \text{ for some open } V \subset Y$$
$$\iff f(U) = f(X) \cap V \text{ for some open } V \subset Y \iff f(U) \text{ is open in } f(X)$$

Alternatively, the injective map $f\colon X \to Y$ is an embedding if and only if the bijective map $f(X)|f\colon X \to f(X)$ is a homeomorphism. An embedding is a homeomorphism followed by an inclusion.

13.11. **Example.** (1) The map $f(x) = 3x + 1$ is a homeomorphism $\mathbf{R} \to \mathbf{R}$.
(2) The identity map $\mathbf{R}_\ell \to \mathbf{R}$ is bijective and continuous but not a homeomorphism.
(3) The map $[0, 1) \to S^1 \colon t \mapsto (\cos(2\pi t), \sin(2\pi t))$ is continuous and bijective but not a homeomorphism. The image of the open set $[0, \frac{1}{2})$ is not open in S^1.
(4) Find an example of an injective continuous map $\mathbf{R} \to \mathbf{R}^2$ that is not an embedding.
(5) The obvious bijection $[0, 1) \cup \{2\} \to [0, 1]$ is continuous but not a homeomorphism (the domain has an isolated point, the codomain has no isolated points). There does not exist any continuous surjection in the other direction.
(6) The spaces $[1 \times 1, 2 \times 1] \subset \mathbf{Z}_+ \times \mathbf{Z}_+$ and $\overline{K} = \overline{\{\frac{1}{n} \mid n \in \mathbf{Z}_+\}} \subset \mathbf{R}$ are homeomorphic.
(7) The map $\mathbf{R} \to \mathbf{R} \times \mathbf{R} \colon t \mapsto (t, t)$ is an embedding. For any continuous map $f \colon X \to Y$, the map $X \to X \times Y \colon x \mapsto (x, f(x))$ is an embedding; see (25.4) for a generalization.
(8) $\mathbf{R}^n$ embeds into S^n via stereographic projection.
(9) $\mathbf{R}^2$ embeds in $\mathbf{R}^3$. Does $\mathbf{R}^3$ imbed in $\mathbf{R}^2$? (See [8] for the answer.)
(10) Are the spaces $\bigcup C_n$ and $\bigcup C_{1/n}$ of 12.13.(2) homeomorphic?
(11) A knot is in embedding of S^1 in $\mathbf{R}^3$ (or S^3). Two knots, $K_0 \colon S^1 \hookrightarrow \mathbf{R}^3$ and $K_1 \colon S^1 \hookrightarrow \mathbf{R}^3$, are equivalent if there exists a homeomorphism h of $\mathbf{R}^3$ such that $h(K_0) = K_1$. The fundamental problem of knot theory [1] is to classify knots up to equivalence.

13.12. **Lemma.** *If $f \colon X \to Y$ is a homeomorphism (embedding) then the corestriction of the restriction $f(A)|f|A \colon A \to f(A)$ ($B|f|A \colon A \to B$) is a homeomorphism (embedding) for any subset A of X (and any subset B of Y containing $f(A)$). If the maps $f_j \colon X_j \to Y_j$ are homeomorphisms (embeddings) then the product map $\prod f_j \colon \prod X_j \to Y_j$ is a homeomorphism (embedding).*

Proof. In case of homeomorphisms there is a continuous inverse in both cases. In case of embeddings, use that an embedding is a homeomorphism followed by an inclusion map. □

13.13. **Maps into products.** There is an easy test for when a map *into* a product space is continuous.

Theorem 13.14 (Characterization of the product topology)**.** *Give $\prod Y_j$ the product topology. Then*

(1) *the projections $\pi_j \colon \prod Y_j \to Y_j$ are continuous, and,*
(2) *for any map $f \colon X \to \prod_{j \in J} Y_j$ into the product space we have*

$$X \xrightarrow{f} \prod_{j \in J} Y_j \text{ is continuous} \iff \forall j \in J \colon X \xrightarrow{f} \prod_{j \in J} Y_j \xrightarrow{\pi_j} Y_j \text{ is continuous}$$

The product topology is the only topology on the product set with thes two properties.

Proof. Let $\mathcal{T}_X$ be the topology on X and $\mathcal{T}_j$ the topology on Y_j. Then $\mathcal{S}_\prod = \bigcup_{j \in J} \pi_j^{-1}(\mathcal{T}_j)$ is a subbasis for the product topology on $\prod_{j \in J} Y_j$ (10.1). Therefore

$$\begin{aligned} f \colon X \to \prod_{j \in J} Y_j \text{ is continuous} &\overset{(13.2)}{\iff} f^{-1}\Big(\bigcup_{j \in J} \pi_j^{-1}(\mathcal{T}_j)\Big) \subset \mathcal{T}_X \\ &\iff \bigcup_{j \in J} f^{-1}(\pi_j^{-1}(\mathcal{T}_j)) \subset \mathcal{T}_X \\ &\iff \forall j \in J \colon (\pi_j \circ f)^{-1}(\mathcal{T}_j) \subset \mathcal{T}_X \\ &\iff \forall j \in J \colon \pi_j \circ f \text{ is continuous} \end{aligned}$$

by definition of continuity (13.2).

We now show that the product topology is the unique topology with these properties. Take two copies of the product set $\prod_{j \in J} X_j$. Equip one copy with the product topology and the other copy with some topology that has the two properties of the theorem. Then the identity map between these two copies is a homeomorphism. □

The reason for the great similarity between 13.14 and 13.9 is that in both cases we use an initial topology.

13.15. **Example.** Suppose that J and K are sets and that $(X_j)_{j\in J}$ and $(Y_k)_{k\in k}$ are indexed families of topological spaces. Given a map $g\colon J \to K$ between index sets and an indexed family of continuous maps $(f_j\colon Y_{g(j)} \to X_j)_{j\in J}$. Then there is a unique map between product spaces such that

$$\begin{array}{ccc} \prod_{k\in K} Y_k & \longrightarrow & \prod_{j\in J} X_j \\ {\scriptstyle \pi_{g(j)}}\downarrow & & \downarrow{\scriptstyle \pi_j} \\ Y_{g(j)} & \xrightarrow[f_j]{} & X_j \end{array}$$

commutes and this map of product spaces is continuous by 13.14.

Theorem 13.16. *Let $(X_j)_{j\in J}$ be an indexed family of topological spaces with subspaces $A_j \subset X_j$. Then $\prod A_j$ is a subspace of $\prod X_j$.*

(1) $\overline{\prod A_j} = \prod \overline{A_j}$.
(2) $\left(\prod A_j\right)^\circ \subset \prod A_j^\circ$ *and equality holds if $A_j = X_j$ for all but finitely many $j \in J$.*

Proof. (1) Let (x_j) be a point of $\prod X_j$. Since $\mathcal{B}_\prod = \bigcup_{j\in J} \pi_j^{-1}(U_j)$ is a subbasis for the product topology we have:

$$\begin{aligned} (x_j) \in \overline{\prod A_j} &\iff \forall k \in J\colon \pi_k^{-1}(U_k) \cap \prod A_j \neq \emptyset \text{ for all neighborhoods } U_k \text{ of } x_k \\ &\iff \forall k \in J\colon U_k \cap A_k \neq \emptyset \text{ for all neighborhoods } U_k \text{ of } x_k \\ &\iff \forall k \in J\colon x_k \in \overline{A_k} \\ &\iff (x_j) \in \prod \overline{A_j} \end{aligned}$$

(2) $\left(\prod A_j\right)^\circ \subset \prod A_j^\circ$ because π_j is an open map (14.3) so that $\pi_j\left(\left(\prod A_j\right)^\circ\right) \subset A_j^\circ$ for all $j \in J$. If $A_j = X_j$ for all but finitely many $j \in J$ then $\prod A_j^\circ \subset \left(\prod A_j\right)^\circ$ because $\prod A_j^\circ$ is open and contained in $\prod A_j$. □

It follows that a product of closed sets is closed. (Whereas a product of open sets need not be open in the product topology.)

13.17. **Maps out of coproducts.**

Theorem 13.18. *Let $f\colon \coprod_{j\in J} X_j \to Y$ be a map out of a coproduct space. Then*

$$f\colon \coprod_{j\in J} X_j \to Y \text{ is continuous} \iff f \circ \iota_j\colon X_j \to Y \text{ is continuous for all } j \in J$$

where $\iota_j\colon X_j \to \coprod_{j\in J} X_j$ is the inclusion map.

Let $f_j\colon X_j \to Y_j$, $j \in J$, be an indexed set of maps. Then

$$\coprod f_j\colon \coprod X_j \to \coprod Y_j \text{ is continuous} \iff f_j\colon X_j \to Y_j \text{ is continuous for all } j \in J$$

14. The quotient topology

In this section we will look at the quotient space construction. But first we consider open and closed maps.

14.1. **Open and closed maps.** [2, I.§5] Let X and Y be topological spaces and $f\colon X \to Y$ a map.

14.1. **Definition.** *The map* $f\colon X \to Y$ *is* $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ *if for all* $U \subset X$ *we have:*

$$U \left\{\begin{matrix}\textit{open}\\ \textit{closed}\end{matrix}\right\} \textit{ in } X \Longrightarrow f(U) \left\{\begin{matrix}\textit{open}\\ \textit{closed}\end{matrix}\right\} \textit{ in } Y$$

The restriction (14.12) of an open (or closed) map to an arbitrary subspace need not be open (closed). However,

14.2. **Proposition.** [9, Ex 22.5] *The restriction of an open (or closed) map* $f\colon X \to Y$ *to an open (closed) subspace* $A \subset X$ *is an open (closed) map* $f(A)|f|A\colon A \to f(A)$ *or* $f|A\colon A \to Y$.

Proof. Suppose that $f\colon X \to Y$ is an open map and $A \subset X$ an open subset. Let U be an open subset of A. The implications

$$\begin{aligned} &U \text{ is open in } A && (A \text{ is open in } X)\\ &\Longrightarrow U \text{ open in } X && (f \text{ is open})\\ &\Longrightarrow f(U) \text{ open in } Y\\ &\Longrightarrow f(U) \text{ open in } f(A) \end{aligned}$$

show that $f|A\colon A \to f(A)$ is open. □

14.3. **Proposition** (Projections are open)**.** *The projection map* $\pi_j\colon \prod X_j \to X_j$ *is open.*

Proof. The map π_j takes the basis $\mathcal{B}_{\prod}$ (10.1) for the product topology into the topology on X_j. □

A bijective continuous and open or closed map is a homeomorphism.

14.4. **Example.** (1) The projection map $\pi_1\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is continuous and open. It is not a closed map for $H = \{(x, y) \in \mathbf{R} \times \mathbf{R} \mid xy = 1\}$ is closed but $\pi_1(H) = \mathbf{R} - \{0\}$ is not closed.
(2) The map $f\colon [-1, 2] \to [0, 1]$ given by

$$f(x) = \begin{cases} 0 & -1 \le x \le 0\\ x & 0 \le x \le 1\\ 1 & 1 \le x \le 2 \end{cases}$$

is continuous and closed (18.8.(1)). It is not open for $f([-1, -1/2)) = \{0\}$ is not open.
(3) The map $\mathbf{R}_\ell \to \{0, 1\}$ with value 1 on $[0, 1)$ and value 0 elsewhere is continuous because (12.2.(4)) the half-open interval $[0, 1)$ is both closed and open. This map is closed and open because all subsets of the discrete space $\{0, 1\}$ are closed and open. See 14.14 for another example of a closed and open map.

14.2. **Quotient topologies and quotient maps.** Quotient maps are continuous surjective maps that generalize both continuous, open surjective maps and and continuous, closed surjective maps.

14.5. **Definition** (Quotient topology)**.** *Let* X *be a topological space,* Y *a set, and* $p\colon X \to Y$ *a surjective map. The* quotient topology *on* Y *consists of those subsets of* Y *that have open preimages in* X*; ie the quotient topology on* Y *is the collection*

$$\{V \subset Y \mid p^{-1}(V) \textit{ is open in } X\}$$

of subsets of Y*.*

The quotient topology[2] is the finest topology on Y such that $p\colon X \to Y$ continuous. There are as many open sets in Y as possible without destroying continiuty of $f\colon X \to Y$. The concepts of quotient topology and subspace topology (13.8) are 'dual'.

[2]The quotient topology is sometimes called the *final topology* [2, I.§2.4] with respect to the map f

14.6. **Lemma** (Characterization of the quotient topology). *Suppose that Y has the quotient topology with respect to the map $p\colon X \to Y$. Then*

(1) *$p\colon X \to Y$ is continuous, and,*
(2) *for any map $g\colon Y \to Z$ out of Y*

$$Y \xrightarrow{g} Z \text{ is continuous} \iff X \xrightarrow{p} Y \xrightarrow{g} Z \text{ is continuous}$$

The quotient topology is the only topology on Y with these properties.

Proof. This is because

$$\begin{aligned} Y \xrightarrow{g} Z \text{ is continuous} &\iff g^{-1}(\mathcal{T}_Z) \subset \mathcal{T}_Y \iff p^{-1}g^{-1}(\mathcal{T}_Z) \subset \mathcal{T}_X \\ &\iff (pg)^{-1}(\mathcal{T}_Z) \subset \mathcal{T}_X \iff X \xrightarrow{p} Y \xrightarrow{g} Z \text{ is continuous} \end{aligned}$$

by definition of the quotient topology.

If we give Y some topology with these two properties then the identity map between the two topologies is a homeomorphism. □

14.7. **Definition** (Quotient map). *A surjective continuous map $p\colon X \to Y$ is a* quotient map *if Y has the quotient topology.*

A surjective map $p\colon X \to Y$ is a quotient map when the topology on Y, $\mathcal{T}_Y = \{V \subset Y \mid p^{-1}(V) \text{ is open in } X\}$, is the finest topology on Y such that p is continuous. This means that p is a quotient map if and only if

$$p^{-1}(V) \text{ is open in } X \iff V \text{ is open in } Y$$

holds for all $V \subset Y$.

Sets of the form $p^{-1}(V) = \bigcup_{y\in V} p^{-1}(y)$, consisting of unions of fibres, are called *saturated* sets. The *saturation* of $A \subset X$ is the union $f^{-1}f(A) = \bigcup_{y\in f(A)} f^{-1}(y)$ of all fibres that meet A.

14.8. **Proposition.** *For a surjective map $p\colon X \to Y$ the following are equivalent:*

(1) *$p\colon X \to Y$ is a quotient map*
(2) *For all $V \subset Y$ we have: $p^{-1}(V)$ is $\left\{\begin{smallmatrix}\text{open}\\\text{closed}\end{smallmatrix}\right\}$ in $X \iff V$ is $\left\{\begin{smallmatrix}\text{open}\\\text{closed}\end{smallmatrix}\right\}$ in Y.*
(3) *$p\colon X \to Y$ is continuous and maps saturated $\left\{\begin{smallmatrix}\text{open}\\\text{closed}\end{smallmatrix}\right\}$ sets to $\left\{\begin{smallmatrix}\text{open}\\\text{closed}\end{smallmatrix}\right\}$ open sets*

Proof. Condition (1) and condition (2) with the word "open" are clearly equivalent. Suppose now that: $p^{-1}(V)$ is open $\iff V$ is open. Then we get

$$\begin{aligned} p^{-1}(C) \text{ is closed} &\iff X - p^{-1}(C) \text{ is open} \iff p^{-1}(Y - C) \text{ is open} \\ &\iff Y - C \text{ is open} \iff C \text{ is closed} \end{aligned}$$

for all $C \subset Y$. This shows that the two conditions of (2) are equivalent. The content of (3) is just a reformulation of (2). □

Thus a quotient map $p\colon X \to Y$ induces a bijective correspondence

$$\left\{\begin{smallmatrix}\text{Saturated open (closed) sets}\\\text{in } X\end{smallmatrix}\right\} \underset{p^{-1}(V)\leftarrow V}{\overset{U\to p(U)}{\longleftrightarrow}} \left\{\begin{smallmatrix}\text{Open (closed) sets}\\\text{in } Y\end{smallmatrix}\right\}$$

Just as for open (closed) continuous bijective maps we evidently have:

14.9. **Corollary.** *A bijective continuous map is a quotient map if and only if it is a homeomorphism.*

Quotient maps behave nicely with respect to composition:

14.10. **Corollary.** *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be continuous maps. Then*

$$f \text{ and } g \text{ are quotient} \Longrightarrow g \circ f \text{ is quotient} \Longrightarrow g \text{ is quotient}$$

Proof. The first assertion is a tautology: Assume that f and g are quotient. Then

$$(g\circ f)^{-1}(V) \text{ open in } X \iff f^{-1}g^{-1}(V) \text{ open in } X \overset{f \text{ is quotient}}{\Longrightarrow} g^{-1}(V) \text{ open in } Y \overset{g \text{ is quotient}}{\Longrightarrow} V \text{ open in } Z$$

for any set $V\subset Y$. Next, suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a quotient map. Then the last map g is surjective and

$$g^{-1}(V) \text{ is open in } Y \overset{f \text{ continuous}}{\Longrightarrow} (g\circ f)^{-1}(V) \text{ is open in } X \overset{g\circ f \text{ quotient}}{\Longrightarrow} V \text{ is open in } Z$$

for any set $V\subset Z$. □

14.11. **Corollary.** *Any* $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ *continuous surjective map is a quotient map. Conversely, a quotient map* $f\colon X\to Y$ *is* $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ *if and only if all* $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ *sets* $A\subset X$ *have* $\left\{\begin{matrix}\text{open}\\ \text{closed}\end{matrix}\right\}$ *saturations* $f^{-1}f(A)$.

Proof. Let $f\colon X\to Y$ be an open map. Then

$$f^{-1}(V) \text{ is open} \overset{f \text{ open}}{\Longrightarrow} ff^{-1}(V) \text{ is open} \iff V \text{ is open} \overset{f \text{ is continuous}}{\Longrightarrow} f^{-1}(V) \text{ is open}$$

for all $V\subset Y$. This shows that f is quotient. Suppose next that $f\colon X\to Y$ is quotient. Then

$$\begin{aligned} f \text{ is open} &\overset{(14.1)}{\iff} f(A) \text{ is open in } Y \text{ for all open sets } A\subset X \\ &\overset{(14.7)}{\iff} f^{-1}f(A) \text{ is open in } X \text{ for all open sets } A\subset X \end{aligned}$$

which is the claim. □

There are quotient maps that are neither open nor closed [9, Ex 22.3] [2, Ex 10 p 135] and there are (non-identity) quotient maps that are both open and closed (14.14) [2, Ex 3 p 128].

14.12. **Example.** (1) The projection map $\pi_1\colon \mathbf{R}\times\mathbf{R}\to\mathbf{R}$ is (14.4.(1)) open, continuous, and surjective so it is a quotient map. The restriction $\pi_1|H\cup\{(0,0)\}$ is continuous and surjective, even bijective, but it is not a quotient map (14.8) for it is not a homeomorphism: $\{(0,0)\}$ is open and saturated in $H\cup\{(0,0)\}$ but $\pi_1(\{(0,0)\}=\{0\}$ is not open.

Thus the restriction of a quotient map need not be a quotient map in general. On the positive side we have

14.13. **Proposition.** *The restriction of a quotient map* $p\colon X\to Y$ *to an open (or closed) saturated subspace* $A\subset X$ *is a quotient map* $p|A\colon A\to p(A)$.

Proof. (Similar to the proof of 14.2.) Let $p\colon X\to Y$ be a quotient map and $B\subset Y$ an open set. (The case where B is closed is similar.) The claim is that $p|p^{-1}(B)\colon p^{-1}(B)\to B$ is quotient. For any $U\subset B$ the implications

$$\begin{aligned} &p^{-1}(U) \text{ open in } p^{-1}(B) && (p^{-1}(B) \text{ is open}) \\ &\Longrightarrow p^{-1}(U) \text{ open in } X && (p \text{ is quotient}) \\ &\Longrightarrow U \text{ is open in } Y \\ &\Longrightarrow U \text{ is open in } B \end{aligned}$$

show that $p|p^{-1}(B)\colon p^{-1}(B)\to B$ is quotient. □

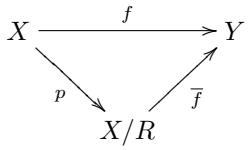
A typical situation is when R is an equivalence relation on the space X and $X\to X/R$ is the map that takes a point to its equivalence class. We call X/R with the quotient topology for the *quotient space of the equivalence relation* R. A set of equivalence classes is an open subset of X/R if and only if the union of equivalence classes is an open subset of X: $V\subset X/R$ is open $\iff \bigcup_{[x]\in V}[x]\subset X$ is open. We shall often say that X/R is the space obtained by *identifying* equivalent points of X.

14.14. **Example** (Orbit spaces). (1) The real projective n-space $\mathbf{R}P^n$ is the quotient space of S^n by the equivalence relation with equivalence classes $\{\pm x\}$, $x \in S^n$. The quotient map $p\colon S^n \to \mathbf{R}P^n$ is both open and closed since (14.11) the saturation $\pm U$ of an open (closed) set $U \subset S^n$ is open (closed) because $x \to -x$ is a homeomorphism. Elements of $\mathbf{R}P^n$ can be thought of as lines through the origin of $\mathbf{R}^{n+1}$. A set of lines is open if the set of intersection points with the unit sphere is open.
(2) More generally, let $G \times X \to X$ be the action of a discrete group G on a space X. Give the orbit space $G\backslash X$ the quotient topology and let $p_G\colon X \to G\backslash X$ be the quotient map. The points in the orbit space are orbits of points in X and the open subsets are orbits of open subsets of X. The saturation of any subset A of X is the orbit $GA = \bigcup_{g in G} gA$ of A. If A is open, GA is open; if A is closed and G is finite, GA is closed. Thus the quotient map p_G is always open (14.11); if G is finite, it is also closed.

A continuous map $f\colon X \to Y$ *respects the equivalence relation* R if equivalent points have identical images, that is if $x_1 R x_2 \implies f(x_1) = f(x_2)$. The quotient map $X \to X/R$ respects the equivalence relation R and it is the *universal example* of such a map.

Theorem 14.15. *Let $f\colon X \to Y$ be a continuous map. Then*

(1) *The map $p\colon X \to X/R$ respects the equivalence relation R.*
(2) *If the continuous map $f\colon X \to Y$ respects R then there exists a unique continuous map $\overline{f}\colon X \to Y$ such that*



commutes. (We say that f factors uniquely through X/R.) Conversely, if f factors through X/R then f respects R.

(3) *If $\overline{f}$ exists then: $\overline{f}$ is quotient $\iff$ f is quotient*

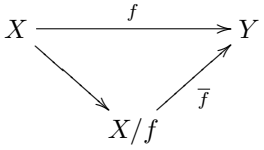
Proof. If $\overline{f}$ exists then clearly f respects R. Conversely, if f respects R then we can define $\overline{f}[x] = f(x)$ and this map is continuous by 14.6 and it is the only possibility. The rest follows from 14.10. □

The theorem says that there is a bijective correspondence

$$\left\{\begin{matrix}\text{Continuous maps } X \to Y \\ \text{that respect } R\end{matrix}\right\} \underset{g\circ p \leftarrow g}{\overset{f \to \overline{f}}{\longleftrightarrow}} \left\{\begin{matrix}\text{Continuous maps} \\ X/R \to Y\end{matrix}\right\}$$

taking quotient maps to quotient maps.

14.16. **Example.** (1) Let $f\colon X \to Y$ be any *surjective* continuous map. Consider the equivalence relation corresponding to the partition $X = \bigcup_{y \in Y} f^{-1}(y)$ of X into fibres fibres $f^{-1}(y)$, $y \in Y$. Let X/f denote the quotient space. Thus X/f is the set of fibres equipped with the quotient topology. By constuction, the map f respects this equivalence relation so there is a unique continuous map $\overline{f}$ such that the diagram



commutes. Note that $\overline{f}$ is bijective. The bijective continuous map $\overline{f}\colon X/f \to Y$ is a homeomorphism if and only if f is quotient (14.9, 14.15). In particular, all quotient maps have (up to homeomorphism) the form $X \to X/R$ for some equivalence relation R on X.

(2) Let $f\colon X \to Y$ be any surjective continuous map. The induces map $\overline{f}\colon X/f \to Y$ is a continuous bijection but in general not a homeomorphism. Instead, the quotient space X/f is homeomorphic to Y with the quotient topology which is finer than the given topology. For instance, X/f is Hausdorff if Y is Hausdorff.

(3) Let $f\colon X \to Y$ be any continuous map. Then f has a canonical decomposition

$$X \xrightarrow{p} X/f \xrightarrow{\overline{f}} f(X) \xrightarrow{\iota} Y$$

where p is a quotient map, $\overline{f}$ is a continuous bijection, and ι is an inclusion map.

(4) Let X be a topological space and $A_1, A_2, \ldots, A_k$ a finite collection of closed subsets. Consider the equivalence relation where the equivalence classes are the sets $A_1, A_2, \ldots, A_k$ together with the sets $\{x\}$ for $x \notin A_1 \cup A_2 \cup \cdots \cup A_k$. The quotient space $X/(A_1, \ldots, A_k)$ is obtained from X by identifying each of the sets A_i to the point $p(A_i)$. The quotient map $p\colon X \to X/(A_1, \ldots, A_k)$ is closed because (14.11) closed sets $A \subset X$ have closed saturations $A \cup \bigcup_{A_i \cap A \neq \emptyset} A_i$. A continuous map $f\colon X \to Y$ factors through the quotient space $X/(A_1, \ldots, A_k)$ if and only if it sends each of the sets $A_i \subset X$ to a point in Y (14.15). The restriction $p|X - (A_1 \cup \cdots \cup A_k)\colon X - (A_1 \cup \cdots \cup A_k) \to X/(A_1, \ldots, A_k) - \{p(A_1), \ldots, p(A_k)\}$ to the complement of $A_1 \cup \cdots \cup A_k$ is a homeomorphism (14.2). In case of just one closed subspace $A \subset X$, the quotient space is denoted X/A.

(5) The standard map $f\colon [0,1] \to S^1$ that takes $t \in [0,1]$ to $(\cos(2\pi t), \sin(2\pi t))$ is quotient because it is continuous and closed. (If you can't see this now, we will prove it later (18.7.(1)).) The induced map $[0,1]/\{0,1\} \to S^1$ is a homeomorphism. More generally, the standard map $D^n/S^{n-1} \to S^n$ is a homeomorphism where $D^n \subset \mathbf{R}^n$, the unit disc, is the set of vectors of length ≤ 1.

(6) Let R be the equivalence relation "zero or not zero" on $\mathbf{R}$. The quotient space $\mathbf{R}/R$ is homeomorphic to Sierpinski space $\{0,1\}$.

(7) There is an obvious continuous surjective map $f\colon \mathbf{Z}_+ \times S^1 \to \bigcup C_n$ (12.13.(2)) that takes $\mathbf{Z}_+ \times \{1\}$ to the point common to all the circles. This map is continuous because its restriction to each of the open sets $\{n\} \times S^1$ is continuous (13.5.(5). However, f is not a quotient map (14.8) for the image of the closed saturated set consisting of the points $n \times (\cos(\frac{\pi}{2}), \sin(\frac{\pi}{2}))$ is not closed as it does not contain all its limit points. The induced bijective continuous map $\overline{f}\colon \mathbf{Z}_+ \times S^1/\mathbf{Z}_+ \times \{1\} \to \bigcup C_n$ is therefore *not* a homeomorphism. There is an obvious continuous surjective map $g\colon \mathbf{Z}_+ \times S^1 \to \bigcup C_{1/n}$ (12.13.(2)) that takes $\mathbf{Z}_+ \times \{1\}$ to the point common to all the circles. This map is continuous because its restriction to each of the open sets $\{n\} \times S^1$ is continuous (13.5.(5)). However, g is not a quotient map for the image of the closed saturated set $\mathbf{Z}_+ \times \{-1\}$ is not closed as it does not contain all its limit points. The induced bijective continuous map $\overline{g}\colon \mathbf{Z}_+ \times S^1/\mathbf{Z}_+ \times \{1\} \to \bigcup C_{1/n}$ is therefore *not* a homeomorphism either. (Actually (15.10.(7)), the quotient space $\mathbf{Z}_+ \times S^1/\mathbf{Z}_+ \times \{1\}$, known as the countable wedge of circles $\bigvee_{n \in \mathbf{Z}_+} S^1$ [9, Lemma 71.4], is not homeomorphic to *any* subspace of the plane.)

(8) [9, p 451] Let P_{4g} be a regular $4g$-gon and with edges labeled $a_1, b_1, a_1, b_1, \ldots, a_g, b_g, a_g, b_g$ in counter-clockwise direction. The closed orientable surface M_g of genus $g \geq 1$ is (homeomorphic to) the quotient space P_{4g}/R where R is the equivalence relation that makes the identifications $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ on the perimeter and no identifications in the interior of the polygon. See [9, p 452] for the case $g = 2$. Are any of these surfaces homeomorphic to each other? [9, Thm 77.5]

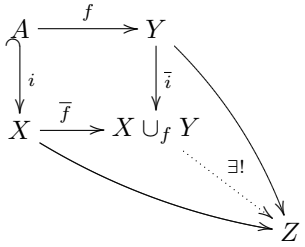
(9) [9, p 452] Let P_{2g} be a regular $2g$-gon with edges labeled $a_1, a_1, \ldots, a_g, a_g$ in counter-clockwise direction. The closed non-orientable surface N_g of genus $g \geq 1$ is the quotient space P_{2g}/R where R is the equivalence relation that makes the identifications $a_1^2 \cdots a_g^2$ on the perimeter and no identifications in the interior of the polygon. For $g = 1$ we get the projective plane $\mathbf{R}P^2$ and for $g = 2$ we get the Klein Bottle [9, Ex 74.3].

14.17. **Example.** (The adjunction space) [9, Ex 35.8] [5, p 93] [10, Chp 1, Exercise B p 56] Consider the set-up $X \xleftarrow{i} A \xrightarrow{f} Y$ consisting of a space X and a continuous map $f\colon A \to Y$ defined on the closed subspace $A \subset X$. Let R be the smallest equivalence relation on $X \amalg Y$ such that $aRf(a)$ for all $a \in A$; the equivalence classes of R are $\{a\} \amalg f(a)$ for $a \in A$, $\{x\}$ for $x \in X - A$, anf $f^{-1}(y) \amalg \{y\}$ for $y \in Y$. The adjunction space is the quotient space

$$X \cup_f Y = X \amalg Y/R$$

for the equivalence relation R. Let $\overline{f}\colon X \to X \cup_f Y$ be the map $X \to X \amalg Y \to X \cup_f Y$ and let $\overline{i}\colon Y \to X \cup_f Y$ be the map $Y \to X \amalg Y \to X \cup_f Y$. These two continuous maps agree on A in the

sense that $\overline{f} \circ i = \overline{i} \circ f$ and the adjunction space is the universal space with this property. For any other space Z receiving maps $X \to Z \leftarrow Y$ that agree on A there exists a unique continuous map $X \cup_f Y$ such that the diagram



commutes. The map $\overline{i}\colon Y \to X \cup_f Y$ is closed for closed sets $B \subset Y \subset X \amalg Y$ have closed saturations $f^{-1}(B) \amalg B$. Since $\overline{i}$ is injective it is an embedding; its image is a closed subspace of $X \cup_f Y$ homeomorphic to Y. The map $\overline{f}|X - A\colon X - A \to X \cup_f Y$ is open for open sets $U \subset X - A \subset X \amalg Y$ have open saturations $U \amalg \emptyset$. Since $\overline{f}|X - A$ is injective it is an embedding; its image is an open subspace of $X \cup_f Y$ homeomorphic to $X - A$. The quotient map $X \amalg Y \to X \cup_f Y$ is closed if the map f is closed for then also closed subsets $B \subset X \subset X \amalg Y$ have closed saturations $B \cup f^{-1}f(B \cap A) \amalg f(A \cap B)$. We shall later see [9, Ex 35.8] that $X \cup_f Y$ is normal when X and Y are normal.

Only few topological properties are preserved by quotient maps. The reason is that surjective open maps and surjective closed maps are quotient maps so that any property invariant under quotient maps must also be invariant under both open and closed maps.

As we saw in 14.16.(6) the quotient space of a Hausdorff space need not be Hausdorff, not even T_1. In general, the quotient space X/R is T_1 if and only if all equivalence classes are closed sets. (For instance, if X and Y are T_1 then also the adjunction space $X \cup_f Y$ is T_1.) The quotient space X/R is Hausdorff if and only if any two distinct equivalence classes are contained in disjoint open saturated sets. We record an easy criterion for Hausdorfness even though you may not yet know the meaning of all the terms.

14.18. **Proposition.** *If X is regular and $A \subset X$ is closed then the quotient space X/A is Hausdorff.*

The product of two quotient maps need not be a quotient map [2, I.§5.3]. See [9, Ex 29.11] for an important example where the product of two quotient maps is quotient, though.

15. Metric topologies

If X is a set with a metric $d\colon X \times X \to [0,\infty)$ the collection $\{B_d(x,\varepsilon) \mid x \in X, \varepsilon > 0\}$ of balls $B_d(x,\varepsilon) = \{y \in X \mid d(x,y) < \varepsilon\}$ is a basis for the *metric topology* $\mathcal{T}_d$ induced by d.

15.1. **Definition.** *A metric space is the topological space associated to a metric set. A topological space is* metrizable *if the topology is induced by some metric on X.*

Hausdorff dimension, fractals, or chaos are examples of metric, rather than topological, concepts.

Theorem 15.2 (Continuity in the metric world)**.** *Let $f\colon X \to Y$ be a map between metric spaces with metrics d_X and d_Y, respectively. The following conditions are equivalent:*

(1) *f is continuous*
(2) $\forall x \in X \forall \varepsilon > 0 \exists \delta > 0 \forall y \in X\colon d_X(x,y) < \delta \Rightarrow d_Y(f(x),f(y)) < \varepsilon$
(3) $\forall x \in X \forall \varepsilon > 0 \exists \delta > 0\colon f(B_X(x,\delta)) \subset B_Y(f(x),\varepsilon)$

Proof. Essentially [9, Ex 18.1]. □

15.3. **Proposition** (Comparison of metric topologies)**.** *Let d and d' be metrics on X and $\mathcal{T}_d$, $\mathcal{T}_{d'}$ the associated metric topologies. Then*

$$\mathcal{T}_d \subset \mathcal{T}_{d'} \Leftrightarrow \forall x \in X \forall \varepsilon > 0 \exists \delta > 0\colon B_{d'}(x,\delta) \subset B_d(x,\varepsilon)$$

Proof. $\mathcal{T}_d \subset \mathcal{T}_{d'}$ if and only if the identity map $(X,\mathcal{T}_{d'}) \to (X,\mathcal{T}_d)$ is continuous [9, Ex 18.3] . □

15.4. **Lemma** (Standard bounded metric)**.** *Let d be a metric on X. Then $d'(x,y) = \min\{d(x,y),1\}$ is a bounded metric on X (called the standard bounded metric corresponding to d) that induce the same topology on X as d.*

Proof. Either use the above proposition or use that for any metric the collection of balls of radius < 1 is a basis for the metric topology. These bases are the same for the two metrics. □

Theorem 15.5 (Hereditary properties of metrizable spaces)**.** *Any subspace of a metrizable space is a metrizable. Any countable product of metrizable spaces is metrizable.*

Proof. See [9, Ex 21.1] for the first assertion. To prove the second assertion, let X_n, $n \in \mathbf{Z}_+$, be a countable collection of metric spaces. We may assume that each X_n has diameter at most 1 (15.4). Put

$$d((x_n),(y_n)) = \sup\{\frac{1}{n} d_n(x_n,y_n) \mid n \in \mathbf{Z}_+\}$$

for points (x_n) and (y_n) of $\prod X_n$ and convince yourself that d is a metric. The idea here is that $\frac{1}{n} d_n(x_n,y_n) \leq \frac{1}{n}$ becomes small when n becomes large. For any $\varepsilon > 0$

$$d((x_n),(y_n)) \leq \varepsilon \iff \forall n \leq N\colon \frac{1}{n} d_n(x_n,y_n) \leq \varepsilon$$

where N is such that $N\varepsilon > 1$.

The claim is that the metric toplogy coincides with the product topology on $\prod X_n$. We need to show that the metric topology enjoys the two properties that characterizes the product topology (13.14). First, the projection maps $\pi_n\colon \prod X_n \to X_n$ are continuous because $d(x,y) < \varepsilon \Longrightarrow d_n(x,y) < n\varepsilon$ (15.2). Second, let $f\colon X \to \prod X_n$ be a map such that $X \xrightarrow{f} \prod X_n \xrightarrow{\pi_n} X_n$ is continuous for all n. Given $x \in X$ and $\varepsilon > 0$, there exist neighborhoods U_n of x such that $d_n(\pi_n f(x), \pi_n f(y)) < n\varepsilon$ for all $y \in U_n$. Then $d(f(x),f(y)) < \varepsilon$ for all $y \in U_1 \cap \ldots \cap U_N$ where $N\varepsilon > 1$ (remember that all the spaces X_n have diameter at most 1). This shows that $f\colon X \to \prod X_n$ is continuous. □

15.6. **The first countability axiom.** Which topological spaces are metrizable? To address this question we need to build up an arsenal of metrizable and non-metrizable spaces and to identify properties that are common to all metrizable spaces. Here are the first such properties: All metric spaces are Hausdorff and first countable.

15.7. **Definition** (Neighborhood basis)**.** *A neighborhood basis at $x \in X$ is a collection of neighborhoods of x such that any neighborhood of x contains a member of the collection.*

15.8. **Definition** (First countable spaces). *Let X be a space and x a point in X. We say that X has a* countable basis at x *if there is a countable neighborhood basis at x. X is* first countable *if all points of X have a countable neighborhood basis.*

All metrizable spaces are first countable since $\{B(x, 1/n) \mid n \in \mathbf{Z}_+\}$ is a countable neighborhood basis at x.

15.9. **Proposition** (Hereditary properties of first countable spaces). [9, Thm 30.2] *Any subspace of a first countable space is first countable. Any countable product of first countable spaces is first countable.*

Proof. The first assertion is immediate. Let $\prod X_n$ be a countable product of first countable spaces. Let (x_n) be a point of $\prod X_n$. Let $\mathcal{B}_n$ be countable basis at $x_n \in X_n$. The collection of all products $\prod U_n$ where $U_n \in \mathcal{B}_n$ for finitely many n and $U_n = X_n$ for all other n is then a countable (5.5) basis at (x_n). □

15.10. **Example.** (1) $\mathbf{R}$ is first countable because it is a metric space.

(2) $\mathbf{R}_\ell$ is first countable. The collection of half-open intervals $[a, b)$ where $b > a$ is rational is a countable basis of neighborhoods at the point a. Is $\mathbf{R}_\ell$ metrizable? (22.3) [9, Ex 30.6]

(3) Is $\mathbf{R}_K$ first countable? Is it metrizable?

(4) The ordered space $\overline{S}_\Omega = [0, \Omega]$ (6.8) is not first countable at Ω but it is first countable at any other point. Let $\{U_n\}$ be any countable collection of neighborhoods of Ω. Choose $a_n < \Omega$ such that $(a_n, \Omega] \subset U_n$ and chose a such that $a_n < a < \Omega$ for all n. This is possible because the countable set $\{a_n\} \subset [0, \Omega)$ of left end-points has an upper bound in $[0, \Omega)$ (6.8.(2)). Then $(a, \Omega]$ is a neighborhood of Ω that does not contain any of the U_n because it does not even contain any of the intervals $(a_n, \Omega]$. The section S_Ω, however, *is* first countable: $\{1\} = [1, 2)$ is open so S_Ω has a finite basis at the first element 1. For any other element, $\alpha > 1$, we can use the countable collection of neighborhoods of the form $(\beta, \alpha]$ for $\beta < \alpha$ (12.2.(5)). Is S_Ω metrizable? (20.3)[9, Exmp 3 p 181, Ex 30.7].

(5) $\mathbf{R}^J$ is not first countable at any point when J is uncountable: Let $\{U_n\}_{n\in\mathbf{Z}_+}$ be any countable collection of neighborhoods of, say, the point $(0)_{j\in J}$ of $\mathbf{R}^J$. I claim that there is a neighborhood that does not contain any of the U_n. This is because there is an index $j_0 \in J$ such that $\pi_{j_0}(U_n) = \mathbf{R}$ for all n. Indeed, the set of js for which this is not true

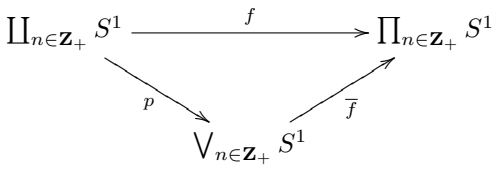
$$\{j \in J \mid \exists n \in \mathbf{Z}_+ \colon \pi_j(U_n) \neq \mathbf{R}\} = \bigcup_{n\in\mathbf{Z}_+} \{j \in J \mid \pi_j(U_n) \neq \mathbf{R}\}$$

is a countable union of finite sets, hence countable (5.5.(3)). Then the neighborhood $\pi_{j_0}^{-1}(\mathbf{R} - \{1\})$ of $(0)_{j\in J}$ does not contain any of the neighborhoods U_n in the countable collection. ($\mathbf{R}^J$ does not even satisfy the sequence lemma (15.12.(1)) [9, Exmp 2 p 133].)

(6) The closed (14.16.(4)) quotient map $\mathbf{R} \to \mathbf{R}/\mathbf{Z}$ takes the first countable space $\mathbf{R}$ to a space that is not first countable [5, 1.4.17] at the point corresponding to $\mathbf{Z}$. This can be seen by a kind of diagonal argument: Let $\{U_n\}_{n\in\mathbf{Z}}$ be any countable collection of open neighborhoods of $\mathbf{Z} \subset \mathbf{R}$. Let U be the open neighborhood of $\mathbf{Z}$ such that $U \cap (n, n + 1)$, $n \in \mathbf{Z}$, equals $U_n \cap (n, n + 1)$ with one point deleted. Then U does not contain any of the U_n.

(7) The quotient map $p\colon \coprod_{n\in\mathbf{Z}_+} S^1 \to \bigvee_{n\in\mathbf{Z}_+} S^1$ is closed (14.16.(4)). The domain $\coprod_{n\in\mathbf{Z}_+} S^1 = \mathbf{Z}_+ \times S^1 \subset \mathbf{R} \times \mathbf{R}^2$ is first countable (15.9) but the image $\bigvee_{n\in\mathbf{Z}_+} S^1$ (14.16.(7)) is not: Let $\{U_n\}_{n\in\mathbf{Z}_+}$ be any collection of saturated neighborhoods of $\mathbf{Z}_+ \times \{1\}$. The saturated neighborhood U which at level n equals U_n with one point deleted (cf Cantor's diagonal argument (5.7)) does not contain any of the U_n. It follows (15.9) that $\bigvee_{n\in\mathbf{Z}_+} S^1$ does not embed in $\mathbf{R}^2$ nor in any other first countable space. For instance, the universal property of quotient spaces (14.15) gives a factorization

(15.11)



of the continuous map f such that $\pi_m(f|\{n\} \times S^1)$ is the identity function when $m = n$ and the constant function when $m \neq n$. The induced map $\overline{f}$ is an injective continuous map. It can not be an embedding for the countable product $\prod S^1$ of circles is first countable (15.9). The topology on $\bigvee S^1$ is finer than the subspace topology inherited from $\prod S^1$.

These examples show that the uncountable product of first countable (even metric) spaces and the quotient of a first countable space may fail to be first countable (15.10.(5), 15.10.(7)). Some linearly ordered spaces are first countable, some are not (15.10.(1), 15.10.(4)).

In a first countable (eg metric) space X, the points of the closure of any $A \subset X$ can be approached by sequences from A in the sense that they are precisely the limit points of convergent sequences in A. This is not true in general (15.10.(4)).

15.12. **Lemma.** *Let X be a topological space.*

(1) *(The sequence lemma) Let $A \subset X$ be a subspace and x a point of X. Then*

$$x \text{ is the limit of a sequence of points from } A \Longrightarrow x \in \overline{A}$$

The converse holds if X is first countable.

(2) *(Continuous map preserve convergent sequences) Let $f\colon X \to Y$ be a map of X into a space Y. Then*

$$f \text{ is continuous} \Longrightarrow f(x_n) \to f(x) \text{ whenever } x_n \to x \text{ for any sequence } x_n \text{ in } X$$

The converse holds if X is first countable.

Proof. (1) The direction $\Longrightarrow$ is clear. Conversely, suppose that $x \in \overline{A}$. Let U_n be a countable basis at x. We may assume that $U_1 \supset U_2 \supset \cdots \supset U_n \supset U_{n+1} \supset \cdots$ as we may replace U_n by $U_1 \cap \ldots \cap U_n$. For each n choose a point $x_n \in A \cap U_n$. We claim that the sequence (x_n) converges to x. Let U be any neighborhood of x. Then $U_n \subset U$ for some n so that $x_m \in U_m \subset U_n \subset U$ for $m \geq n$.
(2) The direction $\Longrightarrow$ is clear. Conversely, suppose that $f(x_n) \to f(x)$ whenever $x_n \to x$. We want to show that f is continuous, ie (13.3.(6)) that $f(\overline{A}) \subset \overline{f(A)}$ for any $A \subset X$. Let $x \in \overline{A}$. Since X is first countable, there is by (1) a sequence of points $a_n \in A$ converging to x. By hypothesis, the sequence $f(a_n) \in f(A)$ converges to $f(x)$. Thus $f(x) \in \overline{f(A)}$ by (1) again. □

We say that X satisfies the sequence lemma[1] if for any $A \subset X$ and for any $x \in \overline{A}$ there is a sequence of points in A converging to x.

To summarize:

$$X \text{ is metrizable} \Longrightarrow X \text{ is 1st countable} \Longrightarrow X \text{ satisfies the sequence lemma (is Frechét)}$$

Examples show that neither of these arrows reverse.

The largest element Ω is a limit point of $[0, \Omega)$ but it is not the limit of any sequence in $[0, \Omega)$ as any such such sequence has an upper bound in $[0, \Omega)$ (6.8.(2)). Thus $[0, \Omega]$ does not satisfy the sequence lemma. Hnece it is not 1st countable and not metrizable.

15.13. **The uniform metric.** Let Y be a metric space with a *bounded* metric d and let J be a set. We shall discuss uniform convergence which is a metric, not a topological, concept.

15.14. **Definition.** [9, pp 124, 266] *The uniform metric on $Y^J = \prod_{j \in J} Y = \mathrm{map}(J, Y)$ is the metric given by $\overline{d}(f, g) = \sup\{j \in J \mid d(f(j), g(j))\}$.*

It is easy to see that this is indeed a metric.

Theorem 15.15. *On $\mathbf{R}^J$ we have: product topology $\subset$ uniform topology $\subset$ box topology*

Proof. Omitted. □

The elements of Y^J are functions $f\colon J \to Y$. Note that a sequence of functions $f_n\colon J \to Y$ converges in the uniform metric to the function $f\colon J \to Y$ if and only if

$$\forall \varepsilon > 0 \exists N > 0 \forall j \in J \forall n \in \mathbf{Z}_+ \colon n > N \Longrightarrow d(f_n(j), f(j)) < \varepsilon$$

We say that the sequence of functions $f_n\colon J \to Y$ converges uniformly to the function $f\colon J \to Y$.

[1] Aka a Frechét space

Theorem 15.16 (Uniform limit theorem)**.** *(Cf* [9, Thm 43.6]*) Suppose that J is a topological space and that Y is a metric space. The uniform limit of any sequence (f_n) of continuous functions $f_n\colon J \to Y$ is continuous.*

Proof. Well-known. □

16. Connected spaces

16.1**. Definition.** *The topological space X is* connected *if it is not the union $X = X_0 \cup X_1$ of two disjoint open non-empty subsets X_0 and X_1.*

Two subsets A and B of a space X are *separated* if $A \cap \overline{B} = \emptyset = \overline{A} \cap B$. This means that the two sets are disjoint and neither contains a limit point of the other. Two disjoint open (closed) sets are separated. If $C \subset A$ and $D \subset B$ and A and B are separated, then C and D are separated. A *separation* of X consists of two separated non-empty subsets A and B with union $X = A \cup B$.

Theorem 16.2. *The following are equivalent:*

(1) *X is connected*
(2) *The only clopen (closed and open) subsets of X are $\emptyset$ and X*
(3) *X has no separations*
(4) *Every continuous map $X \to \{0,1\}$ to the discrete space $\{0,1\}$ is constant.*

Proof. (1) $\Longrightarrow$ (2): If C is a closed and open subset of X then also the complement $X - C$ is closed and open. Thus $X = C \cup (X - C)$ is the union of two disjoint open subsets and therefore one of these sets must be empty.
(2) $\Longrightarrow$ (3): Suppose that $X = A \cup B$ where A and B are separated. Then $\overline{A} \subset A$ for $\overline{A}$ does not meet B. Thus A is closed. Similarly, B is closed. Therefore A is both closed and open. By hypothesis, A is either the empty set or all of X.
(3) $\Longrightarrow$ (4): Suppose that there exists a surjective continuous map $f\colon X \to \{0,1\}$. Then $X = f^{-1}(0) \cup f^{-1}(1)$ is a separation of X.
(4) $\Longrightarrow$ (1): We prove the contrapositive. Suppose that X is non-connected. Then $X = U_1 \cup U_2$ where U_1 and U_1 are disjoint, open, and non-empty. The map $f\colon X \to \{0,1\}$ given by $f(U_1) = 0$ and $f(U_1) = 1$ is a non-constant continuous map. □

Theorem 16.3. *If $f\colon X \to Y$ is continuous and X is connected, then $f(X)$ is connected.*

Proof. We use the equivalence of (1) and (4) in Theorem 16.2. Let $f\colon X \to Y$ be a continuous map. If $f(X)$ is not connected, there is a non-constant continuous map $f(X) \to \{0,1\}$ and hence a non-constant continuous map $X \to \{0,1\}$. So X is not connected. □

16.4**. Example.** (1) $\mathbf{R} - \{0\} = \mathbf{R}_- \cup \mathbf{R}_+$ is not connected for it is the union of two disjoint open non-empty subsets.
(2) We shall later prove that $\mathbf{R}$ is connected (17.3) and that the connected subsets of $\mathbf{R}$ are precisely the intervals, rays, $\mathbf{R}$, and $\emptyset$.
(3) $\mathbf{R}_\ell$ is not connected for $[a,b)$ is a closed and open subset whenever $a < b$ (12.2.(4)). In fact, any subset Y of $\mathbf{R}_\ell$ containing at least two points $a < b$ is disconnected as $Y \cap [a,b)$ is closed and open but not equal to $\emptyset$ or Y. ($\mathbf{R}_\ell$ is *totally disconnected*.)
(4) $\mathbf{R}_K$ is connected [9, Ex 27.3].
(5) $\mathbf{Q}$ is totally disconnected (and not discrete): Let Y be any subspace of $\mathbf{Q}$ containing at least two points $a < b$. Choose an irrational number t between a and b. Then $Y \cap (t,\infty) = Y \cap [t,\infty)$ closed and open but not equal to $\emptyset$ or Y.
(6) Particular point topologies (8.2.(3)) are connected.

A subspace of X is said to be connected if it is connected in the subspace topology. A subspace of a connected space need obviously not be connected. So how can we tell if a subspace is connected?

16.5**. Lemma.** *Let $Y \subset X$ be a subspace. Then*

$$Y \text{ is connected} \iff Y \text{ is not the union of two separated non-empty subsets of } X$$

Proof. Suppose that $Y = Y_1 \cup Y_2$ is the union of two subspaces. Observe that

$$Y_1 \text{ and } Y_2 \text{ are separated in } Y \iff Y_1 \text{ and } Y_2 \text{ are separated in } X \tag{16.6}$$

because $\mathrm{Cl}_Y(Y_1) \cap Y_2 \overset{12.7.(3)}{=} \overline{Y_1} \cap Y \cap Y_2 = \overline{Y_1} \cap Y_2$. The lemma now follows immediately from 16.2, the equivalence of (1) and (3). □

16.7. **Corollary.** *Suppose that $Y \subset X$ is a connected subspace. For every pair A and B of separated subsets of X such that $Y \subset A \cup B$ we have either $Y \subset A$ or $Y \subset B$.*

Proof. The subsets $Y \cap A$ and $Y \cap B$ are separated (since the bigger sets A and B are separated) with union Y. By 16.5, one of them must be empty, $Y \cap A = \emptyset$, say, so that $Y \subset B$. □

Theorem 16.8. *Let $(Y_j)_{j\in J}$ be an indexed family of connected subspaces of the space X. Suppose that there is an index $j_0 \in J$ such that Y_j and Y_{j_0} are not separated for any $j \in J$. Then the union $\bigcup_{j\in J} Y_j$ is connected.*

Proof. We have to prove that the subspace $Y = \bigcup_{j\in J} Y_j$ is connected. We use the criterion of Lemma 16.5. Suppose that $Y = A \cup B$ is the union of two separated subsets A and B of X. Each of the sets Y_j is by Corollary 16.7 contained in either A or B. Let's say that Y_{j_0} is contained in A. Then Y_j must be contained in A for all $j \in J$ for if $Y_j \subset B$ for some j then Y_j and Y_{j_0} would be separated. Therefore $Y = A$. □

16.9. **Corollary.** *The union of a collection of connected subspaces with a point in common is connected.*

Proof. Apply 16.8 with any of the subspaces as Y_{j_0}. □

16.10. **Corollary.** *Let C be a connected subspace of X. Then $\overline{C}$ is connected. Indeed, if $C \subset Y \subset \overline{C}$ then Y is connected.*

Proof. Apply Theorem 16.8 to the collection consisting of C and $\{y\}$ for $y \in Y - C$. Then C and $\{y\}$ are not separated since $y \in \overline{C}$. □

16.11. **Corollary.** *Suppose that for any two points in X there is a connected subspace containing both of them. Then X is connected.*

Proof. Let x_0 be some fixed point of X. For each point $x \in X$, let C_x be a connected subspace containing x_0 and x. Then $X = \bigcup C_x$ is connected since $\bigcap C_x \neq \emptyset$ (16.9). □

Theorem 16.12. *Products of connected spaces are connected.*

Proof. We prove first that the product $X \times Y$ of two connected spaces X and Y is connected. In fact, for any two points (x_1, y_1) and (x_2, y_2) the subspace $X \times \{y_1\} \cup \{x_2\} \times Y$ contains the two points and this subspace is connected since (16.9) it is the union of two connected subspaces with a point in common. Thus $X \times Y$ is connected by Corollary 16.11.

Next, induction shows that the product of finitely many connected spaces is connected.

Finally [9, Ex 23.10], consider an arbitrary product $\prod X_j$ of connected spaces X_j, $j \in J$. Choose (3.4) a point x_j in each of the spaces X_j (assuming that all the spaces of the product are non-empty). For every finite subset F of J let $C_F \subset \prod X_j$ be the product of the subspaces X_j if $j \in F$ and $\{x_j\}$ if $j \notin F$. Since C_F is connected for each finite subset F and these subsets have the point $(x_j)_{j\in J}$ in common, the union $\bigcup_{F\in\mathcal{F}} C_F$, where $\mathcal{F}$ is the collection of all finite subsets of J, is connected (16.9). This union is not all of $\prod X_j$ but its closure is, so $\overline{\bigcup_{F\in\mathcal{F}} C_F} = \prod_{j\in J} X_j$ is connected (16.10). □

17. Connected subsets of linearly ordered spaces

The aim of this section is to determine the connected subsets of $\mathbf{R}$, in fact, more generally, of any linear continuum.

Recall that a subset C of a linearly ordered set X is convex in X if $a, b \in C \Longrightarrow [a, b] \subset C$. It is easy to see that connected subspaces are convex. But are all convex subspaces also connected? The convex subset $[0,]$ is not connected in $\mathbf{Z}$ but it is connected in $\mathbf{R}$.

17.2. **Lemma.** *Let X be a linearly ordered space and C a subspace of X. Then*

$$C \text{ is connected} \Longrightarrow C \text{ is convex}$$

The converse holds if X is a linear continuum (1.15).

Proof. Let $C \subset X$ a subspace that is not convex. Then there exist points $a < x < b$ in X such that a and b are in C and x is outside C. Since $C \subset (-\infty, x) \cup (x, +\infty)$ is contained in the union of two separated subsets and meet both of them, C is not connected (16.7).

Let now X be a linear continuum and C a convex subset of X. We claim that C is connected. Pick a fixed point $a \in C$ and note that $C = \bigcup_{b \in C} [a, b]$ is a union of closed intervals with a point in common. Therefore it suffices (16.9) to prove the claim in case $C = [a, b]$ is a closed interval. Suppose that $[a, b] = A \cup B$ is the union of two separated sets A and B where $b \in B$. We must show that $A = \emptyset$ (16.5). The sets A and B are closed and open in $[a, b]$ (since they form a separation of $[a, b]$ (16.6)). As the closed interval $[a, b]$ is closed in X (9.1), A and B are also closed in X (12.7). If we assume that $A \neq \emptyset$ then the nonempty bounded set A has a least upper bound, $c = \sup A$. Then $c \leq b$ and $c \in A$ since A is closed in X. (The least upper bound of any bounded set always belongs to the closure of the set since otherwise it wouldn't be the *least* upper bound.) So $c < b$ (for $b \notin A$) and $A \subset [a, b)$. Now, A is open in $[a, b]$ and in the subspace $[a, b)$ (which has the subspace topology which is the order topology (11.6)). But this contradicts $c \in A$ for no element of an open subset A of $[a, b)$ can be an upper bound for A: For any point $d \in A$ there exists an open interval (x, y) around d such that $[a, b) \cap (x, y) \subset A$ and since $[d, y) \neq \emptyset$, d is not an upper bound for A. (In short, $c \in A$ since A is closed in X and $c \notin A$ since A is open in $[a, b)$.) Therefore A must be empty. □

The converse to 17.2 is of course not true in general as for instance $\mathbf{Z}_+$ is convex but not connected (there are gaps).

We shall now identify the linearly ordered spaces that are connected. (You may read the following theorem and its proof as if we discussed $X = \mathbf{R}$ only.)

Theorem 17.3. [2, Ex 7 p 382, Prop 1 p 336] *Let X be a linearly ordered space. Then*

$$X \text{ is connected in the order topology} \iff X \text{ is a linear continuum}$$

The connected subsets of a linear continuum X are: X, $\emptyset$, intervals and rays.

Proof. $\Longrightarrow$: [9, Ex 24.4] [5, Problem 6.3.2] [2, Ex 7 p 382]. Suppose that X is a linearly ordered set that is not a linear continuum. Then there are nonempty, proper, clopen subsets of X:

- If $(x, y) = \emptyset$ for some points $x < y$ then $(-\infty, x] = (\infty, y)$ is clopen and $\neq \emptyset, X$.
- If $A \subset X$ is a nonempty subset bounded from above which has no least upper bound then the set of upper bounds $B = \bigcap_{a \in A} [a, \infty) = \bigcup_{b \in B} (b, \infty)$ is clopen and $\neq \emptyset, X$.

Therefore X is not connected (16.2).

$\Longleftarrow$: Assume X is a linear continuum. From 17.2 we know that the the connected and the convex subsets of X are the same. In particular, the linear continuum X, certainly convex in itself, is connected in the order topology. Let C be a nonempty convex subset of X. We look at two cases:

- C is neither bounded from above nor below. Let x be any point of X. Since x is neither a lower nor an upper bound for C there exist $a, b \in C$ so that $a < x < b$. Then $x \in C$ by convexity. Thus $C = X$.
- C is bounded from above but not from below. Let $c = \sup C$ be its least upper bound. Then $C \subset (-\infty, c]$. Let $x < c$ be any point. Since x is neither a lower nor an upper bound for C there exist $a, b \in C$ so that $a < x < b$. Then $x \in C$ by convexity. Thus $(\infty, c) \subset C \subset (-\infty, c]$ and C is either $(-\infty, c)$ or $(-\infty, c]$.

The arguments are similar for the other cases. Recall that X also has the greatest lower bound property [9, Ex 3.13]. □

The real line $\mathbf{R}$, the ordered square I_o^2 (9.4.(7)), the (ordinary) (half) line $[1,\omega) = \mathbf{Z}_+ \times [0,1)$ (6.5.(6)), and the long (half) line $[0,\Omega) \times [0,1)$ (6.8) [9, Ex 24.6, 24.12] are examples of linear continua.

Theorem 17.4. *[Intermediate Value Theorem] Let $f\colon X \to Y$ be a continuous map of a connected topological space X to a linearly ordered space Y. Then $f(X)$ is convex. If Y is a linear continuum, $f(X)$ is an interval (bounded or unbounded, closed, half-open, or open)*

Proof. X connected $\overset{(16.3)}{\Longrightarrow}$ $f(X)$ connected $\overset{(17.3)}{\Longrightarrow}$ $f(X)$ convex. For subsets of a linear continuum we know (17.3) that connected = convex = interval. □

Any linearly ordered space containing two consecutive points, two points a and b with $(a,b) = \emptyset$ is not connected as $X = (-\infty, b) \cup (a, +\infty)$ is the union of two disjoint open sets.

Any well-ordered set X containing at least two points is totally disconnected in the order topology. For if $C \subset X$ contains $a < b$ then $a \notin C \cap (a,b] \ni b$ is closed and open in C since $(a,b]$ is closed and open in X.

17.5. **Path connected spaces.** Path connectedness is a stronger property than connectedness.

17.6**. Definition.** *The topological space X is path connected if for any two points x_0 and x_1 in X there is a continuous map (a path) $f\colon [0,1] \to X$ with $f(0) = x_0$ and $f(1) = x_1$.*

The image under a continuous map of a path connected space is path connected, cf 16.3. Any product of path connected spaces is path connected [9, Ex 24.8], cf 16.8.

17.7**. Example.** The punctured euclidian plane $\mathbf{R}^n - \{0\}$ is path connected when $n \geq 2$ since any two points can be joined by a path of broken lines. Thus also the $(n-1)$-sphere S^{n-1}, which is the image of $\mathbf{R}^n - \{0\}$ under the continuous map $x \mapsto x/|x|$, is path connected for $(n-1) \geq 1$.

Since the unit interval I is connected (17.3), all paths $f(I)$ are also connected (16.3) so all path connected spaces are, as unions of paths emanating from one fixed point, connected (16.11). The converse is not true, not all connected spaces are path connected.

17.8**. Example.** (1) (Topologist's sine curve) Let S be the graph of the function $\sin(1/x)$, $0 < x \leq 1$, considered as a subspace of the plane $\mathbf{R}^2$. The (closed) topologist's sine curve is the subspace $\overline{S} = S \cup (\{0\} \times [-1,1])$ of the plane. It follows from 16.3 and 16.10 that $\overline{S}$ is connected. We shall soon see that $\overline{S}$ is not path connected (17.17.(2)).

(2) The ordered square I_o^2 (9.4.(7)) is connected since it is a linear continuum [9, Ex 3.15] but it is not path connected. For suppose that $f\colon [0,1] \to I_o^2$ is a path from the smallest element (the lower left corner) $f(0) = 0 \times 0$ to the largest element (the upper right corner) $f(1) = 1 \times 1$. Then f is surjective for the image contains (17.4) the interval $[0 \times 0, 1 \times 1] = I_o^2$. But this is impossible since the ordered square I_o^2 contains uncountably many open disjoint subsets (eg $x \times (0,1) = (x \times 0, x \times 1)$, $0 \leq x \leq 1$) but $[0,1]$ does not contain uncountably many open disjoint subsets (choose a rational number in each of them).

(3) $\mathbf{R}_K$ (8.9) is connected but not path-connected space [9, Ex 27.3].

(4) The Stone–Čech compactification $\beta\mathbf{R}$ of $\mathbf{R}$ is connected but not path-connected [9, Ex 38.9].

17.9. **Components and path components.** The relations on X defined by

$$x \sim y \Leftrightarrow \text{There is a connected subset } C \subset X \text{ such that } x \in C \text{ and } y \in C$$

$$x \overset{p}{\sim} y \Leftrightarrow \text{There is a path in } X \text{ between } x \text{ and } y$$

are equivalence relations. Check transitivity! (You may need the Glueing lemma (13.5.(6)).)

17.10**. Definition.** *The* components *of X are the equivalence classes of the first equivalence relation. The* path components *of X are the equivalence classes of the second equivalence relation*

The component containing the point $x \in X$ is, by definition, the union of all connected subspaces containing x. The path component of the point $x \in X$ consists, by definition, of all points with a path to x.

Theorem 17.11. *(1) The components are connected closed disjoint subsets with union X. Any connected subset of X is contained in precisely one of the components.*
(2) If $X = \bigcup C_\alpha$, where the C_α are connected subspaces such that for all $\alpha \neq \beta$ there is a separation $A \cup B$ of X with $C_\alpha \subset A$ and $C_\beta \subset B$, then the C_α are the components of X.
(3) The path components of X are path connected disjoint subsets with union X. Any path connected subset is contained in precisely one of the path components.
(4) The path components are connected. The components are unions of path components.

Proof. (1) The components form a partition of X since they are equivalence classes (1.11). The components are connected by the definition of the relation $\sim$ and 16.11. All the points of a connected subset are equivalent so they are contained in the same equivalence class. In particular (16.11), $\overline{C} \subset C$ for any component C so C is closed.
(2) Since each C_α is connected, it is contained in precisely one component (17.11.(1)). Could there be two of the C_αs, say C_α and C_β, inside one component, say C? No: Choose a separation $X = A \cup B$ such that $C_\alpha \subset A$ and $C_\beta \subset B$. Then $C \subset X = A \cup B$ and C meets both A and B which is impossible (16.7).
(3) The path components are path-connected by their very definition. All path-connected spaces are connected, so each path-component is contained in precisely one component. Thus the components are unions of path-components. □

If X has only finitely many components the components are closed and open. If $X = C_1 \cup \cdots \cup C_n$ where the finitely many subspaces C_i are connected and separated then the C_i are the components of X.

The path components can be closed, open, closed and open, or neither closed nor open. However, for locally connected spaces (17.13) components and path components coincide (17.15).

17.12. **Locally connected and locally path connected spaces.**

17.13**. Definition.** *The space X is locally (path) connected at the point x if every neighborhood of x contains a (path) connected neighborhood of x. The space X is locally (path) connected if it is locally (path) connected at each of its points.*

Thus a space is locally (path) connected iff it has a basis of (path) connected subsets.
Consider this table

	$\mathbf{R}$	$\mathbf{R} - \{0\}$	$\overline{S}$	$\mathbf{Q}$
Connected	YES	NO	YES	NO
Locally connected	YES	YES	NO	NO

before you draw any conclusions about the relationship between connected and locally connected spaces. Note in particular that a space can be connected (even path connected (17.17)) and not locally connected.

On the positive side, note that open subsets of locally (path) connected spaces are locally (path) connected and that quotient spaces of locally connected spaces are locally connected (17.16).

17.14**. Proposition.** *The space X is locally (path) connected if and only if open subsets have open (path) components (open in X or, what is the same (11.3), open in the open set). In a locally (path-) connected space the (path-) components are clopen.*

Proof. Assume that X is locally connected and let U be an open subset. Consider a component, C, of U. The claim is that C is open. Let x be a point in C. Choose a connected neighborhood V of x such that $V \subset U$. Since V is connected and intersects C, $V \subset C$ (17.11). This shows that C is open. In particular, the components (of the open set X) are open and, since they form a partition, also closed.

Conversely, assume that open subsets have open components. Let x be a point of X and U a neighborhood of x. Let C be the component of U containing x. Then C is an open connected neighborhood of x contained in U. □

It follows for instance that the open subsets of $\mathbf{R}$ are unions of at most countably many open intervals. For any open set is the union of its components which are open, since $\mathbf{R}$ is locally connected, and connected, so they are open intervals. Since each of the open intervals contains a

rational number, there are at most countably many components. Any closed subset of $\mathbf{R}$ is the complement to a union of at most countably many open intervals.

Theorem 17.15. *In a locally path-connected space the path-components and the components are the same.*

Proof. Suppose that X is locally path-connected. Each path-component P is contained (17.11) in a unique component, C. Since C is connected and P is clopen (17.14), $P = C$ (16.2). □

For instance, the components and the path components of a locally Euclidean space, such as a manifold, are the same. A locally path-connected space is path-connected if and only if it is connected.

17.16**. Proposition.** [9, Ex 25.8] [2, I.§11.6] *Locally (path) connected spaces have locally (path) connected quotient spaces.*

Proof. Let X be a locally connected space and $p\colon X \to Y$ a quotient map. Let V be a subset of Y and C a component of V. Then $p^{-1}(C)$ is a union of components of $p^{-1}(V)$ because continuous maps preserve connectedness (16.3). If V is open in Y, $p^{-1}(V)$ is open in X, and since X is locally connected it follows (17.14) that $p^{-1}(C)$ is open in X. This means (14.5) that C is open in Y. Since thus open subsets of Y have open components, Y is locally connected (17.14). (The same proof applies with 'connected' replaced by 'path connected'.) □

17.17**. Example.** (1) The comb space

$$X = ([0,1] \times \{0\}) \cup ((\{0\} \cup \{1/n \mid n \in \mathbf{Z}_+\}) \times [0,1]) \subset \mathbf{R}^2$$

is clearly path connected but it is not locally connected (17.14): The open set $U = (X - [0,1] \times \{0\}) = ((\{0\} \cup \{1/n \mid n \in \mathbf{Z}_+\}) \times (0,1])$ has a component (use (17.11.(2)) to identify the components) that is not open. (The same is true for any (small) neighborhood of $(0, 1/2)$.) See [9, Ex 25.5] for a similar example.

(2) The closed topologist's sine curve $\overline{S}$ is not locally connected: Let U be a small neighborhood (well, not too big) around $(0, 1/2)$. Then U has a component that is not open. For instance, $U = \overline{S} - ([0,1] \times \{0\}$ has a component (17.11.(2)), $\{0\} \times (0,1]$ that is not open (no neighborhood of $(0, \pm\frac{1}{2})$ is disjoint from S). It follows (17.16, 18.8.(2)) that $\overline{S}$ is not path-connected for it can not be a quotient space of the locally connected compact space $[0,1]$. In fact, $\overline{S}$ has two path-components: $S = \overline{S} \cap (\mathbf{R}_+ \times \mathbf{R})$, which is open and not closed for its closure is $\overline{S}$, and $\{0\} \times [0,1]$, which is closed and not open for S is not closed. (The Warsaw circle, the union of $\overline{S}$ with an arc from one end of S to $[0,1]$) is an example of a path connected but not locally connected space, so is the next example.)

(3) The space $X = \overline{\bigcup C_n}$ (12.13.(2)) is path connected but not locally connected for the component $\mathbf{R}_+ \times \{0\}$ of the open set $X - \{(0,0)\}$ (use (17.11.(2)) to identify the components) is not open (any neighborhood of $(1,0)$ contains points from the circles). Alternatively, X is not locally connected since any small neighborhood of $(1,0)$ has a separation.

(4) Any linear continuum is locally connected since the basis for the topology consists of connected sets (17.3). It need not be locally path connected [9, Ex 25.3].

(5) The (generalized) Jordan curve theorem says that $\mathbf{R}^n - h(S^{n-1})$ has two path-components for any embedding $h\colon S^{n-1} \to \mathbf{R}^n$ of an $(n-1)$-sphere into $\mathbf{R}^n$, $n \geq 1$. This is usually proved using algebraic topology.

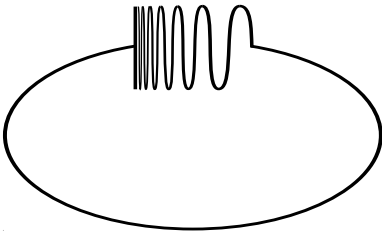


FIGURE 3. The Warsaw circle

18. Compact spaces

18.2. **Definition.** *A topological space X is* compact *if it satisfies one of the following two equivalent conditions:*

(1) *Every collection of open subsets whose union is X contains a finite subcollection whose union is X.*
(2) *Every collection of closed subsets whose intersection is $\emptyset$ contains a finite subcollection whose intersection is $\emptyset$.*

A collection of open subsets whose union is the whole space is called an *open covering*. Thus a space is compact if any open covering contains a finite open covering.

Any finite space is compact as there are only finitely many open subsets. Any compact discrete space is finite. The real line $\mathbf{R}$ is not compact as the open covering $\{(-n, n) \mid n \in \mathbf{Z}_+\}$ has no finite subcovering.

Which subspaces are compact?

18.3. **Proposition.** *Let Y be a subspace of X. The following are equivalent:*

(1) *Y is compact*
(2) *Every collection of open subsets of X whose union contains Y contains a finite subcollection whose union contains Y*
(3) *Every collection of closed subsets of X whose intersection is disjoint from Y contains a finite subcollection whose intersection is disjoint from Y*

Proof. (1) $\Longrightarrow$ (2): Let $\{U_j \mid j \in J\}$ be a collection of open subsets of X such that $Y \subset \bigcup U_j$. Then $\{Y \cap U_j \mid j \in J\}$ is an open covering of Y. Since Y is compact, $Y = \bigcup_{j\in J'} Y \cap U_j$ for some finite index set $J' \subset J$. This means that $Y \subset \bigcup_{j\in J'} U_j$.
(2) $\Longleftarrow$ (1): Let $\{V_j \mid j \in J\}$ be an open covering of Y. Then $V_j = Y \cap U_j$ for some open set $U_j \subset X$ and $Y \subset \bigcap_{j\in J} U_j$. By assumption, $Y \subset \bigcup_{j\in J'} U_j$ for some finite index set $J' \subset J$. Therefore $Y = Y \cap \bigcup_{j\in J'} U_j = \bigcup_{j\in J'} Y \cap U_j = \bigcup_{j\in J'} V_j$. This shows that Y is compact.
(2) $\Longleftrightarrow$ (3): Clear from DeMorgan's laws. □

Theorem 18.4. *Closed subspaces of compact spaces are compact.*

Proof. Suppose that X is a compact space and $Y \subset X$ a closed subset. Let F_j, $j \in J$, be a collection of closed subsets of X such that $Y \cap \bigcap_{j\in J} F_j = \bigcap_{j\in J} Y \cap F_j = \emptyset$. By compactness of X, $\emptyset = \bigcap_{j\in J'} Y \cap F_j = Y \cap \bigcap_{j\in J'} F_j$ for some finite index set $J' \subset J$. Thus Y is compact by 18.3. □

Theorem 18.5. *If $f\colon X \to Y$ is continuous and X is compact, then $f(X)$ is compact.*

Proof. Let U_j, $j \in j$, be a collection of open subsets of Y that cover the image $f(X)$. Then $X = \bigcup_{j\in J} f^{-1}(U_j)$ is an open covering of X. Since X is compact, $X = \bigcup_{j\in J'} f^{-1}(U_j)$ for some finite index set $J' \subset J$. Then $f(X) = f\big(\bigcup_{j\in J'} f^{-1}(U_j)\big) = \bigcup_{j\in J'} ff^{-1}(U_j) \subset \bigcup_{j\in J'} U_j$. This shows that $f(X)$ is compact (18.3). □

Theorem 18.6. *Let X be a Hausdorff space. Every compact subset of X is closed. If K and L are disjoint compact subspaces of X, then there exist disjoint open subsets $U, V \subset X$ such that $K \subset U$ and $L \subset V$.*

Proof. Consider first the special case where $L = \{x_0\}$ is a point. For each point $x \in K$, the Hausdorff property implies that there are disjoint open sets U_x, V_x such that and $x \in U_x$ and $x_0 \in V_x$. Since K is compact (18.3), $K \subset U_{x_1} \cup \ldots \cup U_{x_t}$ for finitely many points $x_1, \ldots, x_t \in K$. Set $U = U_{x_1} \cup \ldots \cup U_{x_t}$ and $V = V_{x_1} \cap \ldots \cap V_{x_t}$. The existence of V alone says that K is closed.

Assume next that L is any compact subspace of X. For each point $y \in L$ there are disjoint open sets $U_y \supset K$ and $V_y \ni y$. By compactness, L is covered by finitely many of the V_y. Then K is contained in the intersection of the corresponding finitely many U_y. □

18.7. **Corollary.** *Let X be a compact Hausdorff space.*

(1) *Let $C \subset X$ a subset. Then C is compact $\Longleftrightarrow$ C is closed.*
(2) *If A and B are disjoint closed sets in X then there exist disjoint open sets U, V such that $A \subset U$ and $B \subset V$.*

Proof. (1) $\Longrightarrow$ is 18.6 and $\Longleftarrow$ is 18.4.
(2) Let A and B be disjoint closed subsets of X. Then A and B are compact as just shown. Now apply 18.6. □

18.8. **Lemma** (Closed Map Lemma). *Suppose that X is a compact space, Y is a Hausdorff space, and $f\colon X \to Y$ is a continuous map.*

(1) *f is closed (14.1).*
(2) *If f is surjective, it is a closed quotient map (14.7).*
(3) *If f is injective, it is an embedding (13.10).*
(4) *If f is bijective, it is a homeomorphism (13.7).*

Proof. Let $f\colon X \to Y$ be a continuous map of a compact space X into a Hausdorff space Y.
(1) We have

$$C \text{ is closed in } X \overset{18.4}{\Rightarrow} C \text{ is compact } \overset{18.5}{\Rightarrow} f(C) \text{ is compact } \overset{18.6}{\Rightarrow} f(C) \text{ is closed in } Y$$

which shows that f is a closed map.
(2) Every surjective closed continuous map is a closed quotient map (14.11).
(3) Every injective closed continuous map is an embedding.
(4) Every bijective closed continuous map is a homeomorphism (14.9).

□

18.9. **Example.** (1) The (unreduced) suspension of the space X is the quotient space

$$SX = [0,1] \times X/(\{0\} \times X, \{1\} \times X)$$

What is the suspension of the n-sphere S^n? Define $f\colon [0,1] \times S^n \to S^{n+1}$ to be the (continuous) map that takes (t,x), $t \in [0,1]$, to the geodesic path from the north pole $(0,1) \in S^{n+1}$, through the equatorial point $(x,0) \in S^n \subset S^{n+1}$, to the south pole $(0,-1) \in S^{n+1}$. (Coordinates in $\mathbf{R}^{n+2} = \mathbf{R}^{n+1} \times \mathbf{R}$.) By the universal property of quotient spaces there is an induced continuous and bijective map $\overline{f}\colon SS^n \to S^{n+1}$. Since SS^n is compact (as a quotient of a product of two compact spaces) and S^{n+1} is Hausdorff (as a subspace of a Hausdorff space), $\overline{f}$ is a homeomorphism (18.8). We often write $SS^n = S^{n+1}$, $n \geq 0$, where the equality sign stands for 'is homeomorphic to'.

(2) Every injective continuous map $S^1 \to \mathbf{R}^2$ is an embedding. Can you find an injective continuous map $\mathbf{R}^1 \to \mathbf{R}^2$ that is *not* an embedding?

(3) Alexander's horned sphere [7, Example 2B.2] is a wild embedding of the 2-sphere S^2 in $\mathbf{R}^3$ such that the unbounded component of the complement $\mathbf{R}^3 - S^2$ (17.17.(5)) contains non-contractible loops. It may be easier instead to consider Alexander's horned disc [3, p 232]. (The horned sphere is obtained by cutting out a disc of the standard sphere and replacing it a horned disc.)

18.10. **Corollary.** *If $f\colon X \to Y$ is continuous and X is compact and locally connected and Y is Hausdorff, then $f(X)$ is compact and locally connected.*

Proof. We know (18.8) that $f\colon X \to f(X)$ is a quotient map. The quotient image of a locally connected space is locally connected (17.16). □

18.11. **Lemma** (The Tube Lemma). *Let X and Y be topological spaces where Y is compact. Let x_0 be a point of X. For any neighborhood $N \subset X \times Y$ of the slice $\{x_0\} \times Y$ there is a neighborhood U of x_0 such that $\{x_0\} \times Y \subset U \times Y \subset N$.*

Proof. For each point $y \in Y$ there is a product neighborhood such that $x_0 \times y \subset U_y \times V_y \subset N$. Since Y is compact, there are finitely many points $y_1, \ldots, y_k \in Y$ such that $Y = V_1 \cup \ldots \cup V_k$ where $V_i = V_{y_i}$. Put $U = U_1 \cap \ldots U_k$. □

Theorem 18.12. *(Cf 18.16) The product of finitely many compact spaces is compact.*

Proof. We show that the product $X \times Y$ of two compact spaces X and Y is compact. The theorem in its general form follows by induction. Suppose that A_j, $j \in J$, is an open covering of $X \times Y$. We show that there is a finite subcovering. For each point $x \in X$, the slice $\{x\} \times Y$, homeomorphic to Y, is compact and therefore $\{x\} \times Y \subset \bigcup_{j \in J_x} A_j$ for some finite index set $J_x \subset J$ (18.3). Thanks

to the tube lemma (18.11) we know that the open set $\bigcup_{j\in J_x} A_j$ actually contains a whole tube $U_x \times Y$ for some neighborhood U_x of x. By compactness, $X = U_{x_1} \cup \ldots \cup U_{x_k}$ can be covered by finitely many of the neighborhoods U_x. Now

$$X \times Y = \big(\bigcup_{1\le i\le k} U_{x_i}\big) \times Y = \bigcup_{1\le i\le k} (U_{x_i} \times Y) = \bigcup_{1\le i\le k} \bigcup_{j\in J_{x_i}} A_j = \bigcup_{j\in \bigcup_{1\le i\le k} J_{x_i}} A_j$$

where $\bigcup_{1\le i\le k} J_{x_i} \subset J$ is finite (4.6). □

Here are two small lemmas that are used quite often.

18.13. **Lemma** (Criterion for noncompactness). *If X contains an infinite closed discrete subspace, then X is not compact.*

Proof. Any closed discrete subspace of a compact space is compact (18.4) and discrete, hence finite. □

18.14. **Lemma** (Intersection of a nested sequence of compact sets). [9, Ex 28.5] [5, 3.10.2] *Let $C_1 \supset C_2 \supset \cdots \supset C_n \supset \cdots$ be a descending sequence of closed nonempty subsets of a compact space. Then $\bigcap C_n \neq \emptyset$.*

Proof. If $\bigcap C_n = \emptyset$ then $C_n = \emptyset$ for some $n \in \mathbf{Z}_+$ by 18.2.(2). □

Theorem 18.15. *[Cf* [9, Ex 27.5]*] A nonempty compact Hausdorff space without isolated points (12.11) is uncountable.*

Proof. Let x_n be a sequence of points in X. It suffices (5.3) to show that $\{x_n \mid n \in \mathbf{Z}_+\} \neq X$.

Since X is Hausdorff and has no isolated points there is a descending sequence

$$V_1 \supset V_2 \supset \cdots \supset V_{n-1} \supset V_n \supset \cdots$$

of nonempty open sets such that $x_n \notin \overline{V}_n$ for all n. These are constructed recursively. Put $V_0 = X$. Suppose that V_{n-1} has been constructed for some $n \in \mathbf{Z}_+$. Since x_n is not isolated, $\{x_n\} \neq V_{n-1}$. Choose a point $y_n \in V_{n-1}$ distinct from x_n and choose disjoint (separated) open sets $U_n \subset X$, $V_n \subset V_{n-1}$ such that $x_n \in U_n$ and $y_n \in V_n$. Then $U_n \cap \overline{V}_n = \emptyset$ so $x_n \notin \overline{V}_n$.

By construction, the intersection $\bigcap \overline{V}_n$ contains none of the points $\{x_n\}$ of the sequence and by compactness (18.14), $\bigcap \overline{V}_n \neq \emptyset$. Thus X contains a point that is not in the sequence. □

The Alexandroff compactification (21.3) $\omega\mathbf{Z}_+ = \overline{K} = K \cup \{0\}$, where K is as in 8.9, is a countable compact Hausdorff space with isolated points.

Is it true that a *connected* Hausdorff space is uncountable?

Theorem 18.16 (Tychonoff theorem). *The product $\prod_{j\in J} X_j$ of any collection $(X_j)_{j\in J}$ of compact spaces is compact.*

Proof. Put $X = \prod_{j\in J} X_j$. Let us say that a collection of subsets of X is an FIP-collection (finite intersection property) if any finite subcollection has nonempty intersection. It is enough (18.2.(2)) to show that

$$\mathcal{A} \text{ is FIP} \Longrightarrow \bigcap_{A\in\mathcal{A}} \overline{A} \neq \emptyset$$

holds for any collection $\mathcal{A}$ of subsets of X.

So let $\mathcal{A}$ be a FIP-collection iof subsets of X. Somehow we must find a point x in $\bigcap_{A\in\mathcal{A}} \overline{A}$.

Step 1. The FIP-collection $\mathcal{A}$ is contained in a maximal FIP-collection.

The set (supercollection?) $\mathbb{A}$ of FIP-collections containing $\mathcal{A}$ is a strictly partially ordered by strict inclusion. Any linearly ordered subset (subsupercollection?) $\mathbb{B} \subset \mathbb{A}$ has an upper bound, namely the FIP-collection $\bigcup \mathbb{B} = \bigcup_{\mathcal{B}\in\mathbb{B}} \mathcal{B}$ containing $\mathcal{A}$. Now Zorn's lemma (7.4) says that $\mathbb{A}$ has maximal elements.

We can therefore assume that $\mathcal{A}$ is a *maximal* FIP-collection.

Step 2. Maximality of $\mathcal{A}$ implies

$$A_1, \ldots, A_k \in \mathcal{A} \Longrightarrow A_1 \cap \cdots \cap A_k \in \mathcal{A} \tag{18.17}$$

and for any subset $A_0 \subset X$ we have

$$\forall A \in \mathcal{A}\colon A_0 \cap A \neq \emptyset \Longrightarrow A_0 \in \mathcal{A} \tag{18.18}$$

To prove (18.17), note that the collection $\mathcal{A}\cup\{A_1\cap\cdots\cap A_k\}$ is FIP, hence equals $\mathcal{A}$ by maximality. To prove (18.18), note that the collection $\mathcal{A}\cup\{A_0\}$ is FIP: Let $A_1,\ldots,A_k\in\mathcal{A}$. By (18.17), $A_1\cap\cdots\cap A_k\in\mathcal{A}$ so $A_1\cap\cdots\cap A_k\cap A\neq\emptyset$ by assumption.
Step 3. $\{\pi_j(A)\mid A\in\ \mathcal{A}\}$ is FIP for all $j\in J$.
$\pi_j(A_1)\cap\cdots\cap\pi_j(A_k)\supset\pi_j(A_1\cap\cdots\cap A_j)\neq\emptyset$ for any finite collection of sets $A_1,\ldots,A_k\in\mathcal{A}$.

Since X_j is compact and $\{\overline{\pi_j(A)}\mid A\in\ \mathcal{A}\}$ is an FIP collection of closed subsets of X_j, the intersection $\bigcap_{A\in\mathcal{A}}\overline{\pi_j(A)}\subset X_j$ is nonempty (18.2.(2)). Choose a point $x_j\in\bigcap\overline{\pi_j(A)}$ for each $j\in J$ and put $x=(x_j)\in X$.
Step 3. $x\in\bigcap_{A\in\mathcal{A}}\overline{A}$.
The point is that $\pi_j^{-1}(U_j)\in\mathcal{A}$ for any neighborhood U_j of x_j. This follows from (18.18) since $\pi_j^{-1}(U_j)\cap A\neq\emptyset$, or equivalently, $U_j\cap\pi_j(A)\neq\emptyset$, for any $A\in\mathcal{A}$ as $x_j\in\overline{\pi_j(A)}$. Since $\mathcal{A}$ has the FIP it follows that $\pi_{j_1}^{-1}(U_{j_1})\cap\cdots\cap\pi_{j_k}^{-1}(U_{j_k})\cap A\neq\emptyset$ whenever U_{j_i} are neighborhoods of the finitely many points $x_{j_i}\in X_{j_i}$ and $A\in\mathcal{A}$. Thus $x\in\overline{A}$ for all A in the collection $\mathcal{A}$; in other words $x\in\bigcap_{A\in\mathcal{A}}\overline{A}$. □

19. Compact subspaces of linearly ordered spaces

We show first that any compact subspace of a linearly ordered space is contained in a closed interval.

19.1. **Lemma.** *Let X be a linearly ordered space and $\emptyset\neq C\subset X$ a nonempty compact subspace. Then $C\subset[m,M]$ for some elements $m,M\in C$.*

Proof. The claim is that C has a smallest and a largest element. The proof is by contradiction. Assume that C has no largest element. Then $C\subset\bigcup_{c\in C}(-\infty,c)$ as there for any $a\in C$ is a $c\in C$ such that $a<c$. By compactness

$$C\subset(-\infty,c_1)\cup\cdots\cup(-\infty,c_k)\subset(-\infty,c)$$

where $c=\max\{c_1,\ldots,c_k\}$ is the largest (6.6) of the finitely many elements $c_1,\ldots,c_k\in C$. But this is a contradiction as $c\in C$ and $c\notin(\infty,c)$. □

We now show that the eg the unit interval $[0,1]$ is compact in $\mathbf{R}$. Note that the unit interval $[0,1]$ is not compact in $\mathbf{R}-\{\frac{1}{2}\}$. (The intersection all the closed subspaces $[-\frac{1}{n}+\frac{1}{2},\frac{1}{n}+\frac{1}{2}]\subset[0,1]$, $n\geq 2$, is empty but the intersection of finitely many of them is not empty.) The reason for this difference is that $\mathbf{R}$, but not $\mathbf{R}-\{\frac{1}{2}\}$ (1.16.(6)), has the least upper bound property (1.15).

Theorem 19.2. *Let X be a linearly ordered space with the least upper bound property. Then every closed interval $[a,b]$ in X is compact.*

Proof. Let $[a,b]\subset X$ be a closed interval and $\mathcal{A}$ and open covering of $[a,b]$ (with the subspace topology which is the order topology (11.6)). We must show that $[a,b]$ is covered by finitely many of the sets from the collection $\mathcal{A}$. The set

$$C=\{x\in[a,b]\mid[a,x]\text{ can be covered by finitely many members of }\mathcal{A}\}$$

is nonempty ($a\in C$) and bounded from above (by b). Let $c=\sup C$ be the least upper bound of C. Then $a\leq c\leq b$. We would like to show that $c=b$.
Step 1 If $x\in C$ and $x<b$ then $C\cap(x,b]\neq\emptyset$.
Proof of Step 1. Suppose first that x has an immediate successor $y>x$. We can not have $x<b<y$ for then y would not be an *immediate* successor. So $x<y\leq b$. Clearly $[a,y]=[a,x]\cup\{y\}$ can be covered by finitely many members of $\mathcal{A}$. Suppose next that x has no immediate successor. Choose an open set $A\in\mathcal{A}$ containing x. Since A is open in $[a,b]$ and contains $x<b$, A contains an interval of the form $[x,d)$ for some $d\leq b$. Since d is not an immediate successor of x there is a point $y\in(x,d)$. Now $[a,y]\subset[a,x]\cup[x,d)\subset[a,x]\cup A$ can be covered by finitely sets from $\mathcal{A}$.
Step 2 $c\in C$.
Proof of Step 2. The claim is that $[a,c]$ can be covered by finitely many members of $\mathcal{A}$. From Step 1 we have that C contains elements $>a$ so the upper bound c is also $>a$. Choose $A\in\mathcal{A}$ such that $c\in A$. Since A is open in $[a,b]$ and $a<c$, A contains an interval of the form $(d,c]$ for some d where $a\leq d$. Since d is not an uppe bound for C, there are points from C in $(d,c]$. Let y be such a point. Now $[a,c]=[a,y]\cup(d,c]\subset[a,y]\cup A$ can be covered by finitely many sets from $\mathcal{A}$.

Step 3 $c = b$.
Proof of Step 3. By Step 2, $c \in C$. But then $c = b$ for Step 1 says that if $c < b$, then c can not be an upper bound. □

For instance, the ordered square $I_o^2 = [0 \times 0, 1 \times 1]$, $\overline{\mathbf{Z}_+} = [1, \infty]$, and $\overline{S}_\Omega = [1, \Omega]$ are compact linearly ordered spaces [9, Ex 10.1]. More importantly, since the linearly ordered space $\mathbf{R}$ has the least upper bound property, we get that all closed intervals in $\mathbf{R}$ are compact.

See [9, Ex 27.1] for the converse.

19.3. **Corollary.** *Let X be a linearly ordered space and $\emptyset \neq C \subset X$ a nonempty subspace. Then*

$$C \text{ is compact and connected} \Longrightarrow C \text{ is a closed interval}$$

The converse holds when X is a linear continuum.

Proof. Since C is compact, $C \subset [m, M]$ for some $m, M \in C$ (19.1) and $[m, M] \subset C$ because C is connected, hence convex (17.2). In a a linear continuum every closed interval is compact (19.2) and connected (17.2). □

In particular, the compact and connected subsets of the linear continuum $\mathbf{R}$ are precisely the closed intervals.

Theorem 19.4 (Heine–Borel)**.** *A subset $C \subset \mathbf{R}^n$ is compact if and only if it is closed and bounded in the standard metric.*

Proof. Let $C \subset \mathbf{R}^n$ be compact. Since C is compact and $\{(-R, R)^n \mid R > 0\}$ an open covering of $\mathbf{R}^n$, there is an $R > 0$ such that $C \subset [-R, R]^n$. Thus C is bounded. Compact subspaces Hausdorff spaces are closed (18.6), so C is closed.

Conversely, if C is closed and bounded then C is a closed subset of $[-R, R]^n$ for some $R > 0$. But $[-R, R]^n$ is compact (19.3, 18.12) and closed subsets of compact spaces are compact (18.4); in particular, C is compact. □

Can you now answer question 13.11.(10)?

Theorem 19.5 (Extreme Value theorem)**.** *Let $f\colon X \to Y$ be a continuous map of a nonempty topological space X into a linearly ordered space Y.*

1. *If X is compact then there exist $m, M \in X$ such that $f(m) \leq f(x) \leq f(M)$ for all $x \in X$.*
2. *If X is compact and connected then there exist $m, M \in X$ such that $f(X) = [f(m), f(M)]$.*

Proof. (1) Since the image $f(X)$ is compact (18.5) we can apply (19.1).
(2) Since the image $f(X)$ is compact (18.5) and connected (16.3) we can apply (19.3). □

19.6. **Compactness in metric spaces.**

19.7. **Lemma** (Lebesgue lemma)**.** *Let X be a compact metric space. For any open covering $\mathcal{A}$ of X there exists an $\varepsilon > 0$ (the Lebesgue number of the open covering $\mathcal{A}$) such that any subset of X with diameter $< \varepsilon$ is contained in a member of $\mathcal{A}$.*

Proof. For each point $x \in X$, choose $\varepsilon_x > 0$ such that the ball $B(x, 2\varepsilon_x)$ is contained in a member of $\mathcal{A}$. The collection $\{B(x, \varepsilon_x)\}_{x \in X}$ covers X so by compactness

$$X = B(x_1, \varepsilon_{x_1}) \cup B(x_2, \varepsilon_{x_2}) \cup \cdots \cup B(x_k, \varepsilon_{x_k})$$

for finitely many points $x_1, x_2, \ldots, x_k \in X$. Let ε be the smallest of the numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k$. The triangle inequality implies that this ε works. (Suppose that $A \subset X$ with $\operatorname{diam} A < \varepsilon$. Choose any point a in A. There is an i such that $d(x_i, a) < \varepsilon$. Let b be any point in A. Since $d(b, x_i) \leq d(b, a) + d(a, x_i) < \varepsilon + \varepsilon_i \leq 2\varepsilon_i$, the point b is contained in $B(x_i, 2\varepsilon_i)$ which is contained in a member of $\mathcal{A}$.) □

19.8. **Definition.** *A map $f\colon X \to Y$ between metric spaces is uniformly continuous if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $d_Y(f(x_1), f(x_2)) < \varepsilon$ whenever $d_X(x_1, x_2) < \delta$:*

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x_1, x_2 \in X\colon d_X(x_1, x_2) < \delta \Rightarrow d_Y(f(x_1), f(x_2)) < \varepsilon$$

Theorem 19.9 (Uniform continuity theorem)**.** *Let $f\colon X \to Y$ be a continuous map between metric spaces. If X is compact, then f is uniformly continuous.*

Proof. Given $\varepsilon > 0$. Let δ be the Lebesgue number of the open covering $\{f^{-1}B(y,\varepsilon) \mid y \in Y\}$. Then we have for all $x_1, x_2 \in X$

$$\begin{aligned} d_X(x_1,x_2) < \delta &\Rightarrow \operatorname{diam}\{x_1,x_2\} < \delta \\ &\Rightarrow \exists y \in Y \colon \{x_1,x_2\} \subset f^{-1}B(y,\varepsilon) \\ &\Rightarrow \exists y \in Y \colon \{f(x_1),f(x_2)\} \subset B(y,\varepsilon) \\ &\Rightarrow d_Y(f(x_1),f(x_2)) < 2\varepsilon \end{aligned}$$

□

20. Limit point compactness and sequential compactness

20.1. **Definition.** *A space X is*

(1) limit point compact *if any infinite subset of X has a limit point*
(2) sequentially compact *if any sequence in X has a convergent subsequence.*

In other words, a space is limit point compact if it contains no infinite closed discrete subspaces (12.12).

We shall not say much about these other forms of compactness (see [5, §3.10] for a more thorough discussion).

Theorem 20.2. *(Cf* [9, Ex 28.4, Ex 35.3]*) For any topological space X we have*

$$X \text{ is compact} \Longrightarrow X \text{ is limit point compact} \overset{X \text{ 1st countable}}{\Longrightarrow} X \text{ is sequentially compact}$$

All three forms of compactness are equivalent for a metrizable space.

Proof. X is compact $\Longrightarrow$ X is limit point compact: A subset with no limit points is closed and discrete (12.12), hence finite (18.13).
X is limit point compact and 1st countable $\Longrightarrow$ X is sequentially compact: Let (x_n) be a sequence in X. Consider the set $A = \{x_n \mid n \in \mathbf{Z}_+\}$. If A is finite there is a constant subsequence (4.6.(3)). If A is infinite, A has a limit point x by hypothesis. If X is 1st countable, then x is (15.12) the limit of a sequence of points from A. By rearranging, if necessary, we get (!) a subsequence converging to x.
X is sequentially compact and metrizable $\Longrightarrow$ X compact: This is more complicated (but should also be well-known from your experience with metric spaces) so we skip the proof. □

20.3. **Example.** (1) The well-ordered space S_Ω of all countable ordinals is limit point compact and sequentially compact but it is not compact. S_Ω is not compact (19.1) for it is a linearly ordered space with no greatest element [9, Ex 10.6]. On the other hand, any countably infinite subset of S_Ω is contained in a compact subset (6.8.(2), 19.2). Therefore (20.2) any countably infinite subset, indeed any infinite subset (12.12), has a limit point and any sequence has a convergent subsequence. (Alternatively, use that S_Ω is first countable (15.10.(4)).) It follows (20.2) that S_Ω is not metrizable.
(2) The Stone–Čech compactification (26.10) $\beta\mathbf{Z}_+$ of the positive integers is compact but it is not limit point compact as the sequence $(n)_{n\in\mathbf{Z}_+}$ has no convergent subsequence. Indeed, no point of the remainder $\beta\mathbf{Z}_+ - \mathbf{Z}_+$ is the limit of a sequence in $\mathbf{Z}_+$ [9, Ex 38.9]. (It follows that $\beta\mathbf{Z}_+$ is not first countable and not metrizable).

21. Locally compact spaces and the Alexandroff compactification

21.1. **Definition.** *A space X is* locally compact at the point $x \in X$ *if x lies in the interior of some compact subset of X. A space is* locally compact *if it is locally compact at each of its points.*

This means that X is locally compact at the point x if $x \in U \subset C$ where U is an open and C a compact subset of X.

Compact spaces are locally compact ($U = X = C$). The real line $\mathbf{R}$ (more generally, $\mathbf{R}^n$) is locally compact but not compact. All linearly ordered spaces with the least upper bound property are locally compact. The space $\mathbf{Q} \subset \mathbf{R}$ of rational numbers is not locally compact [9, Ex 29.1].

21.2. **Definition.** *A* compactification *of the space X is an embedding $c\colon X \to cX$ of X into a compact Hausdorff space cX such that $c(X)$ is dense in cX.*

If X itself is compact then $X = cX$ as X is closed and dense in cX. So compactifications are only interesting for noncompact spaces.

Let X be a locally compact Hausdorff space. Let $\omega X = X \cup \{\omega\}$ denote the union of X with a set consisting of a single point ω. The collection

$$\mathcal{T} = \{U \mid U \subset X \text{ open}\} \cup \{\omega X - C \mid C \subset X \text{ compact}\}$$

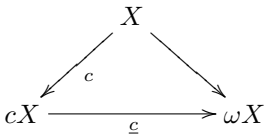
is a topology on ωX: It is easy to see that finite intersections and arbitrary unions of open sets of the first (second) kind are again open of the first (second) kind. It follows that $\mathcal{T}$ is closed under

arbitrary unions. It is also closed under finite intersections since $U \cap (\omega X - C) = U - C$ is open in X (18.6). And $\omega X = \omega X - \emptyset$ and $\emptyset$ are open.

If X is compact, then ωX is X with an added isolated point. The next theorem describes ωX in the more interesting case when X is not compact.

Theorem 21.3 (Alexandroff)**.** *Let X be a locally compact but not compact Hausdorff space.*

(1) *The map $X \to \omega X$ is a compactification missing one point.*
(2) *If $c\colon X \to cX$ is another compactification of X then there is a unique closed quotient map $\underline{c}\colon cX \to \omega X$ such that the diagram*



commutes.

Proof. We must verify the following points:
The subspace topology on X is the topology on X: The subspace topology $X \cap \mathcal{T}$ is clearly the original topology on X.
X is dense in ωX: The intersection of X and some neighborhood of ω has the form $X - C$ where C is compact. Since X is assumed non-compact, $X - C$ is not empty.
ωX is Hausdorff: Let x_1 and x_2 be two distinct points in ωX. If both points are in X then there are disjoint open sets $U_1, U_2 \subset X \subset \omega X$ containing x_1 and x_2, respectively. If $x_1 \in X$ and $x_2 = \omega$, choose an open set U and a compact set C in X such that $x \in U \subset C$. Then $U \ni x$ and $\omega X - C \ni \omega$ are disjoint open sets in ωX.
ωX is compact: Let $\{U_j\}_{j\in J}$ be any open covering of ωX. At least one of these open sets contains ω. If $\omega \in U_k$, $k \in J$, then $U_k = (X - C) \cup \{\omega\}$ for some compact set $C \subset X$. There is a finite set $K \subset J$ such that $\{U_j\}_{j\in K}$ covers C. Then $\{U_j\}_{j\in K\cup\{k\}}$ is a finite open covering of $\omega X = X \cup \{\omega\}$.

Let now $c\colon X \to cX$ be another compactification X. Define $\underline{c}\colon cX \to \omega X$ by $\underline{c}(x) = x$ for all $x \in X$ and $\underline{c}(cX - X) = \omega$. We check that $\underline{c}$ is continuous. For any open set $U \subset X \subset \omega X$, $\underline{c}^{-1}(U)$ is open in X and hence (11.3.(2)) in cX since X is open in cX (21.4). For any compact set $C \subset X$, $\underline{c}^{-1}(\omega X - C) = cX - C$ is open in cX since the compact set C is closed in the Hausdorff space cX (18.7.(1)). This shows that $\underline{c}$ is continuous. By construction, $\underline{c}$ is surjective and hence (18.8) a closed quotient map by the Closed Map Lemma (18.8). □

The theorem says that $\omega X = cX/(cX - X)$ where cX is any compactification of X. In particular, if cX consists of X and one extra point then the map $\underline{c}$ is a bijective quotient map, ie a homeomorphism (14.9). The space ωX is called the *Alexandroff compactification* or the *one-point compactification* of X.

21.4**. Lemma.** *Any locally compact and dense subspace of a Hausdorff space is open.*

Proof. Suppose that Y is Hausdorff and that the subspace $X \subset Y$ is locally compact and dense. Let $x \in X$. Since X is locally compact Hausdorff the point x has a neighborhood $X \cap U$, where U is open in Y, such that its relative closure

$$\mathrm{Cl}_X(X \cap U) \overset{12.7}{=} X \cap \overline{X \cap U} \overset{12.9}{=} X \cap \overline{U}$$

is compact and hence closed in the Hausdorff space Y (18.6). But no part of U can stick outside $X \cap \overline{U}$ for since $U - (X \cap \overline{U})$ is open

$$U - (X \cap \overline{U}) \neq \emptyset \overset{X \text{ dense}}{\Longrightarrow} (X \cap U) - (X \cap \overline{U}) \neq \emptyset$$

which is absurd. Thus we must have $U \subset X \cap \overline{U}$, in particular, $U \subset X$. This shows that X is open. □

21.5**. Example.** (1) If X is compact Hausdorff then ωX is X together with an isolated point ω. ($\{\omega\} = \omega X - X$ is a neighborhood of ω.)
(2) The n-sphere $S^n = \omega\mathbf{R}^n$ is the Alexandroff compactification of the locally compact Hausdorff space $\mathbf{R}^n$ for there is (13.11.(8)) a homeomorphism of $\mathbf{R}^n$ onto the complement of a point in S^n.

(3) Let X be a linearly ordered space with the least upper bound property and let $[a, b)$ be a half-open interval in the locally compact space X. Then $\omega[a, b) = [a, b]$. For instance the Alexandroff compactifications of the half-open intervals $[0, 1) \subset \mathbf{R}$, $\mathbf{Z}_+ = [1, \omega) \subset \mathbf{Z}_+ \times \mathbf{Z}_+$, and $S_\Omega = [1, \Omega) \subset \overline{S}_\Omega$ are $[0, 1]$, $[1, \omega]$, $[1, \Omega] = \overline{S}_\Omega$, respectively.
(4) $\omega\mathbf{Z}_+ \wedge S^1 = \omega(\mathbf{Z}_+ \times \mathbf{R}) = \bigcup_{n\in\mathbf{Z}_+} C_{1/n}$, the Hawaiian Earring (12.13.(2)) (which is compact by the Heine-Borel theorem (19.4)).
(5) The Warsaw circle W (which compact by the Heine-Borel theorem (19.4)) (17.17.(2)) is a compactification of $\mathbf{R}$ and $W/(W - \mathbf{R}) = \omega\mathbf{R} = S^1$.

21.6. **Corollary** (Characterization of locally compact Hausdorff spaces)**.** *Let X be a topological space. The following conditions are equivalent:*

(1) *X is locally compact Hausdorff*
(2) *X is homeomorphic to an open subset of a compact Hausdorff space*
(3) *X is Hausdorff and for any point $x \in X$ and any neighborhood U of x there is an open set V such that $x \in V \subset \overline{V} \subset U$ and $\overline{V}$ is compact.*

Proof. (1) $\Longrightarrow$ (2): The locally compact Hausdorff space X is homeomorphic to the open subspace $\omega X - \{\omega\}$ of the compact Hausdorff space ωX (21.3).
(2) $\Longrightarrow$ (3): Suppose that Y is a compact Hausdorff space and that $X \subset Y$ is an open subset. X is Hausdorff since subspaces of Hausdorff spaces are Hausdorff (12.19). Let x be a point of X and $U \subset X$ a neighborhood of x. Using property 18.7.(2) we find (exactly as in 23.2.(2)) an open set V such that $x \in V \subset \overline{V} \subset U$. Here $\overline{V}$ is compact since it is a closed subset of a compact Hausdorff space (18.7).
(3) $\Longrightarrow$ (1): Let x be a point of X. By property (3) with $U = X$ there is an open set V such that $x \in V \subset \overline{V} \subset X$ and $\overline{V}$ is compact. Thus X is locally compact at x. □

21.7. **Corollary.** *Any closed subset of a locally compact space is locally compact. Any open or closed subset of a locally compact Hausdorff space is locally compact Hausdorff.*

Proof. Let X be a locally compact space and $A \subset X$ a closed subspace. We use the definition (21.1) directly to show that A is locally compact. Let $a \in A$. There are subsets $a \in U \subset C \subset X$ such that U is open and C is compact. Then $A \cap U \subset A \cap C$ where $A \cap U$ is a neighborhood in A and $C \cap A$ is compact as a closed subset of the compact set C (18.4).

If X is locally compact Hausdorff and $A \subset X$ open, it is immediate from 21.6.(2) that A is locally compact Hausdorff. □

An arbitrary subspace of a locally compact Hausdorff space need not be locally compact (21.8.(1)). The product of finitely many locally compact spaces is locally compact but an arbitrary product of locally compact spaces need not be locally compact [9, Ex 29.2]; for instance $\mathbf{Z}_+^\omega$ is not locally compact. The image of a locally compact space under an *open* continuous [9, 29.3] or a perfect map [9, Ex 31.7] [5, 3.7.21] is locally compact but the image under a general continuous map of a locally compact space need not be locally compact; indeed, the quotient of a locally compact space need not be locally compact (21.8.(2)) [5, 3.3.16].

21.8. **Example.** (1) $\bigcup_{n\in\mathbf{Z}_+} C_n \subset \mathbf{R}^2$ (12.13.(2)) is not locally compact at the origin: Any neighborhood of 0×0 contains a countably infinite closed discrete subspace so it can not be contained in any compact subspace (18.13).
(2) The quotient space $\mathbf{R}/\mathbf{Z}_+$ (15.10.(6)) is not locally compact at the point corresponding to $\mathbf{Z}_+$: Any neighborhood of this point contains an infinite closed discrete subspace so it can not be contained in any compact subspace (18.13).
(3) In diagram (15.11), the space $\coprod S^1$ is locally compact, $\prod S^1$ is compact, and $\bigvee S^1$ is not locally compact (at the one point common to all the circles).
(4) (Wedge sums and smash products) A pointed space is a topological space together with one of its points, called the base point. The wedge sum of two pointed disjoint spaces (X, x_0) and (Y, y_0) is the quotient space

$$X \vee Y = (X \amalg Y)/(\{x_0\} \cup \{y_0\})$$

obtained from the disjoint union of X and Y (10.6) by identifying the two base points. Let $f\colon X \cup Y \to X \times Y$ be the continuous map given by $f(x) = (x, y_0)$, $x \in X$, and $f(y) = (x_0, y)$,

$y \in Y$. The image $f(X \cup Y) = (X \times \{y_0\}) \cup (\{x_0\} \times Y) = \pi_2^{-1}(\{y_0\}) \cup \pi_1^{-1}(\{x_0\})$ is closed when X and Y are T_1–spaces. In fact, f is a closed map for if $C \subset X$ is closed then $f(C) = \{(x, y_0) \mid x \in C\} = \pi_1^{-1}(C) \cap \pi_2^{-1}(\{y_0\})$ is also closed. Thus f is a quotient map onto its image and its factorization $\overline{f}$ as in the commutative diagram (14.15.(2))

$$\begin{array}{ccc} X \cup Y & \xrightarrow{\quad f \quad} & X \times Y \\ & \searrow \quad \nearrow_{\overline{f}} & \\ & X \vee Y & \end{array}$$

is an embedding. We can therefore identify $X \vee Y$ and $X \times \{y_0\} \cup \{x_0\} \times Y$. The smash product is defined to be

$$X \wedge Y = (X \times Y)/(X \vee Y)$$

the quotient of the product space $X \times Y$ by the closed subspace $X \vee Y$ ($\times - \vee = \wedge$).

If $A \subset X$ and $B \subset Y$ are closed subspaces, the universal property of quotient maps (14.15.(2)) produces a continuous bijection $\overline{g}$ such that the diagram

$$\begin{array}{ccccc} X \times Y & \xrightarrow{g_X \times g_Y} & X/A \times Y/B & \xrightarrow{g_{X/A \times Y/B}} & X/A \wedge Y/B \\ & \searrow \quad & & \nearrow_{\overline{g}} & \\ & & (X \times Y)/(X \times B \cup A \times Y) & & \end{array}$$

commutes. However, $\overline{g}$ may not be a homeomorphism since the product $g_X \times g_Y$ of the two closed quotient maps $g_X \colon X \to X/A$ and $g_Y \colon Y \to Y/B$ may not be a quotient map. If X/A and Y are locally compact Hausdorff spaces ([Ex 31.7] may be useful here) then $g_X \times g_Y$ is quotient [Ex 29.11] so that $\overline{g}$ is a homeomorphism (14.10, 14.15.(3), 14.9) and

$$(X \times Y)/(X \times B \cup A \times Y) = X/A \wedge Y/B \tag{21.9}$$

in this case.

(5) (The Alexandroff compactification of a product space) If X and Y are locally compact Hausdorff spaces the map $X \times Y \to \omega X \times \omega Y \to \omega X \wedge \omega Y$ is an embedding. This follows from (14.13, 14.9) when we note that the first map embeds $X \times Y$ into an open saturated subset of $\omega X \times \omega Y$ (13.12). The universal property of the Alexandroff compactification (21.3) implies that

$$\omega(X \times Y) = \omega X \wedge \omega Y \tag{21.10}$$

for any two locally compact Hausdorff spaces X and Y. In particular,

$$S^m \wedge S^n = \omega \mathbf{R}^m \wedge \omega \mathbf{R}^n \overset{(21.10)}{=} \omega(\mathbf{R}^m \times \mathbf{R}^n) = \omega \mathbf{R}^{m+n} = S^{m+n}$$

when we view the spheres as Alexandroff compactifications of euclidian spaces.

(6) The 1–sphere S^n is (homeomorphic to) the quotient space $I/\partial I$. Hence

$$S^n = S^1 \wedge \ldots \wedge S^1 = I/\partial I \wedge \ldots \wedge I\partial I = I^n/\partial I^n$$

using the formula [Exam June 2003] for the boundary of the product of two sets.

Some mathematicians prefer to include Hausdorffness in the definition of (local) compactness. What we call a (locally) compact Hausdorff space they simply call a (locally) compact space; what we call a (locally) compact space they call a (locally) quasicompact space.

22. Countability axioms

Let us recall the basic question: Which topological spaces are metrizable? In order to further analyze this question we shall look at a few more properties of topological spaces.

We have already encountered the first countability axiom. There are three more axioms using the term 'countable'. Here are the four countability axioms.

22.1. **Definition.** *A topological space*

- *that has a countable neighborhood basis at each of its points is called a* first countable *space*
- *that has a countable basis is called a* second countable *space*
- *that contains a countable dense subset is said to contain a countable dense subset*[4]
- *in which every open covering has a countable subcovering is called a* Lindelöf *space*

A subset $A \subset X$ is *dense* if $\overline{A} = X$, that is if every nonempty open subset contains a point from A.

Any second countable space is first countable. Any subspace of a (first) second countable space is (first) second countable. A countable product of (first) second countable spaces is (first) second countable (15.9).

The real line $\mathbf{R}$ is second countable for the open intervals (a, b) with rational end-points form a countable basis for the topology. $\mathbf{R}_\ell$ is first but not second countable (23.5).

Metric spaces are first countable but not necessarily second countable [9, Example 2 p 190].

Any compact space is Lindelöf. Any closed subset of a Lindelöf space is Lindelöf [9, Ex 30.9] (with a proof that is similar to that of 18.4). A product of two Lindelöf spaces need not be Lindelöf (23.6).

The next result implies that $\mathbf{R}$, in fact any subset of the 2nd countable space $\mathbf{R}$, is Lindelöf.

Theorem 22.2. *Let X be a topological space. Then*

$$X \text{ has a countable dense subset} \Longleftarrow X \text{ is 2nd countable} \Longrightarrow X \text{ is Lindelöf}$$
$$\Downarrow$$
$$X \text{ is 1st countable}$$

If X is metrizable, the three conditions of the top line equivalent.

Proof. Suppose first that X is second countable and let $\mathcal{B}$ be a countable basis for the topology.
X has a countable dense subset: Pick a point $b_B \in B$ in each basis set. Then $\{b_B \mid B \in \mathcal{B}\}$ is countable (5.5.(2)) and dense.
X is Lindelöf: Let $\mathcal{U}$ be an open covering of X. For each basis set $B \in \mathcal{B}$ for which it is possible pick a member U_B of the open covering $\mathcal{U}$ such that $B \subset U_B$. Then the at most countable collection $\{U_B\}$ of these open sets is an open covering: Let x be any point in X. Since x is contained in a member of $\mathcal{U}$ and every open set is a union of basis sets we have $x \in B \subset U$ for some basis set B and some $U \in \mathcal{U}$. But then also $x \in \mathrm{B} \subset U_B$.
Any metric space with a countable dense subset is 2nd countable: Let X be a metric space with a countable dense subset $A \subset X$. Then the collection $\{B(a, r) \mid a \in A, r \in \mathbf{Q}_+\}$ of balls centered at points in A and with a rational radius is a countable (5.5.(4)) basis for the topology: It suffices to show that for any open ball $B(x, \varepsilon)$ in X and any $y \in B(x, \varepsilon)$ there are $a \in A$ and $r \in \mathbf{Q}_+$ such that $y \in B(a, r) \subset B(x, \varepsilon)$. Let r be a positive rational number such that $2r + d(x, y) < \varepsilon$ and let $a \in A \cap B(y, r)$. Then $y \in B(a, r)$, of course, and $B(a, r) \subset B(x, \varepsilon)$ for if $d(a, z) < r$ then $d(x, z) \leq d(x, y) + d(y, z) \leq d(x, y) + d(y, a) + d(a, z) < d(x, y) + 2r < \varepsilon$.
Any metric Lindelöf space is 2nd countable: Let X be a metric Lindelöf space. For each positive rational number r, let A_r be a countable subset of X such that $X = \bigcup_{a \in A_r} B(a, r)$. Then $A = \bigcup_{r \in \mathbf{Q}_+} A_r$ is a dense countable (5.5.(3)) subset: For any open ball $B(x, \varepsilon)$ and any positive rational $r < \varepsilon$ there is an $a \in A_r$ such that $x \in B(a, r)$. Then $a \in B(x, r) \subset B(x, \varepsilon)$. □

22.3. **Example.** The ordered square I_o^2 is compact (19.2) and therefore Lindelöf but it is not second countable since it contains uncountably many disjoint open sets $(x \times 0, x \times 1)$, $x \in I$. Thus I_o^2 is not metrizable [9, Ex 30.6].

[4] Or to be *separable*

23. Separation Axioms

23.1. **Definition.** *A space X is called a*

- *T_1-space if points $\{x\}$ are closed in X*
- *T_2-space or a Hausdorff space if for any pair of distinct points $x, y \in X$ there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $y \in V$*
- *T_3-space or a* regular *space if points are closed and for any point $x \in X$ and any closed set $B \subset X$ not containing x there exist disjoint open sets $U, V \subset X$ such that $x \in U$ and $B \subset V$*
- *T_4-space or a* normal *space if points are closed and for every par of disjoint closed sets $A, B \subset X$ there exist disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$.*

We have the following sequence of implications

$$X \text{ is normal} \Longrightarrow X \text{ is regular} \Longrightarrow X \text{ is Hausdorff} \Longrightarrow X \text{ is } T_1$$

where none of the arrows reverse (23.6, 23.7, [9, Ex 22.6]).

23.2. **Lemma.** *Let X be a T_1-space. Then*

(1) *X is regular $\iff$ For every point $x \in X$ and every neighborhood U of x there exists an open set V such that $x \in V \subset \overline{V} \subset U$.*
(2) *X is normal $\iff$ For every closed set A and every neighborhood U of A there is an open set V such that $A \subset V \subset \overline{V} \subset U$.*

Proof. Let $B = X - U$. □

Theorem 23.3. *(Cf 12.19) Any subspace of a regular space is regular. Any product of regular spaces is regular.*

Proof. Let X be a regular space and $Y \subset X$ a subset. Then Y is Hausdorff (12.19)). Consider a point $y \in Y$ and a and a closed set $B \subset X$ such that $y \notin Y \cap B$. Then $y \notin B$ and since Y is regular there exist disjoint open sets U and V such that $y \in U$ and $\overline{B} \subset V$. The relatively open sets $U \cap Y$ and $V \cap Y$ are disjoint and they contain y and $B \cap Y$, respectively.

Let $X = \prod X_j$ be the Cartesian product of regular space X_j, $j \in J$. Then X is Hausdorff (12.19)). We use 23.2.(1) to show that X is regular. Let $x = (x_j)$ be a point in X and $U = \prod U_j$ a basis neighborhood of x. Put $V_j = X_j$ whenever $U_j = X_j$. Otherwise, choose V_j such that $x_j \in V_j \subset \overline{V}_j \subset U_j$. Then $V = \prod V_j$ is a neighborhood of x in the product topology and (13.16) $\overline{V} = \prod \overline{V}_j \subset \prod U_j = U$. Thus X is regular. □

Theorem 23.4. [9, Ex 32.1] *A* closed *subspace of a normal space is normal.*

Proof. Quite similar to the proof (23.3) that a subspace of a regular space is regular. □

An arbitraty subspace of a normal space need not be normal (26.5) and the product of two normal spaces need not be normal ((23.6),[9, Example 2 p 203], [5, 2.3.36]).

23.5. **Example** (Sorgenfrey's half-open interval topology)**.** The half-open intervals $[a, b)$ form a basis for the space $\mathbf{R}_\ell$ (8.9, 12.2.(4), 15.10.(2), 16.4.(3)).

$\mathbf{R}_\ell$ is 1st countable: At the point x the collection of open sets of the form $[x, b)$ where $b > x$ is rational, is a countable local basis at x.

$\mathbf{R}_\ell$ is not 2nd countable: Let $\mathcal{B}$ be any basis for the topology. For each point x choose a member B_x of $\mathcal{B}$ such that $x \in B_x \subset [x, x+1)$. The map $\mathbf{R} \to \mathcal{B}\colon x \mapsto B_x$ is injective for if $B_x = B_y$ then $x = \inf B_x = \inf B_y = y$.

$\mathbf{R}_\ell$ has a countable dense subset: $\mathbf{Q}$ is dense since any (basis) open set in $\mathbf{R}_\ell$ contains rational points.

$\mathbf{R}_\ell$ is Lindelöf: It suffices (!) to show that any open covering by basis open sets contains a countable subcovering, ie that if $\mathbf{R} = \bigcup_{j \in J} [a_j, b_j)$ is covered by a collection of right half-open intervals $[a_j, b_j)$ then $\mathbf{R}$ is actually already covered by countably many of these intervals. (Note that this is true had the intervals been open as $\mathbf{R}$ is Lindelöf.) Write

$$\mathbf{R} = \bigcup_{j \in J} (a_j, b_j) \cup \Big(\mathbf{R} - \bigcup_{j \in J} (a_j, b_j) \Big)$$

as the (disjoint) union of the corresponding *open* intervals and the complement of this union. The first set can be covered by countably many of the intervals (a_j, b_j) for any subset of $\mathbf{R}$ (with the standard topology) is 2nd countable and hence Lindelöf (22.2). Also the second set can be covered by countably many of the intervals (a_j, b_j) simply because it is countable: The second set

$$\mathbf{R} - \bigcup_{j\in J}(a_j, b_j) = \bigcup_{j\in J}[a_j, b_j) - \bigcup_{j\in J}(a_j, b_j) = \{a_k \mid \forall j \in J\colon a_k \notin (a_j, b_j)\}$$

consists of some of the left end-points of the intervals. The open intervals (a_k, b_k) are disjoint for a_k in this set. But there is only room for countably many open disjoint intervals in $\mathbf{R}$ (choose a rational point in each of them) so there are at most countaly many left end-dpoints a_k in the second set.

$\mathbf{R}_\ell$ is normal: Let $A, B \subset \mathbf{R}_\ell$ be disjoint closed sets. For each point $a \in A \subset \mathbf{R} - B$ there is an $x_a \in \mathbf{R}$ such that $[a, x_a) \subset \mathbf{R} - B$. Let $U_A = \bigcup_{a\in A}[a, x_a)$. Define $U_B = \bigcup_{b\in B}[b, x_b)$ similarly. Then U_A and U_B are open sets containing A and B, respectively. If $U_A \cap U_B \neq \emptyset$ then $[a, x_a) \cap [b, x_b) \neq \emptyset$ for some $a \in A$, $b \in B$. Say $a < b$; then $b \in [a, x_a) \subset \mathbf{R} - B$ which is a contradiction So U_A and U_B are disjoint. ($\mathbf{R}_\ell$ is even *completely normal* [9, Ex 32.6] by this argument.)

Since $\mathbf{R}_\ell$ has a dense countable subset and is not second countable (23.5) it is not metrizable (22.2).

23.6. **Example** (Sorgenfrey's half-open square topology)**.** The half-open rectangles $[a, b) \times [c, d)$ form a basis (10.3) for the product topology $\mathbf{R}_\ell \times \mathbf{R}_\ell$. The anti-diagonal $L = \{(x, -x) \mid x \in \mathbf{R}\}$ is a closed (clear!) discrete ($L \cap [x, \infty) \times [-x, \infty) = \{(x, -x)\}$) subspace of the same cardinality as $\mathbf{R}$. $\mathbf{Q} \times \mathbf{Q}$ is a countable dense subspace.

$\mathbf{R}_\ell \times \mathbf{R}_\ell$ is not Lindelöf: A Lindelöf space can not contain an uncountable closed discrete subspace (cf 18.13, [9, Ex 30.9]).

$\mathbf{R}_\ell \times \mathbf{R}_\ell$ is not normal: A normal space with a countable dense subset can not contain a closed discrete subspace of the same cardinality as $\mathbf{R}$ [5, 2.1.10]. (Let X be a space with a countable dense subset. Since any continuous map of X into a Hausdorff space is determined by its values on a dense subspace [9, Ex 18.13], the set of continuous maps $X \to \mathbf{R}$ has at most the cardinality of $\mathbf{R}^{\mathbf{Z}} = \prod_{\mathbf{Z}} \mathbf{R}$. Let X be any normal space and L a closed discrete subspace. The Tietze extension theorem (24.5) says that *any* map $L \to \mathbf{R}$ extends to continuous map $X \to \mathbf{R}$. Thus the set of continuous maps $X \to \mathbf{R}$ has at least the cardinality of $\mathbf{R}^{\mathbf{R}} = \prod_{\mathbf{R}} \mathbf{R}$ which is greater (5.6) than the cardinality of $\prod_{\mathbf{Z}} \mathbf{R}$.) See [9, Ex 31.9] for a concrete example of two disjoint closed subspaces that can not be separated by open sets.

$\mathbf{R}_\ell \times \mathbf{R}_\ell$ is (completely) regular: since it is the product (26.3) of two (compeletely) regular (26.1) (even normal (23.5)) spaces.

Example 23.5 shows that the arrows of Theorem 22.2 do not reverse. Example 23.6 shows that the product of two normal spaces need not be normal, that the product of two Lindelöf spaces need not be Lindelöf, and provides an example of a (completely) regular space that is not normal.

23.7. **Example** (A Hausdorff space that is not regular)**.** The open intervals plus the sets $(a, b) - K$ where $K = \{\frac{1}{n} \mid n \in \mathbf{Z}_+\}$ form a basis for the topology $\mathbf{R}_K$ (8.9, 17.8.(3)). This is a Hausdorff topology on $\mathbf{R}$ since it is finer than the standard topology. But $\mathbf{R}_K$ is not regular. The set K is closed and $0 \notin K$. Suppose that $U \ni 0$ and $V \supset K$ are disjoint open sets. We may choose U to be a basis open set. Then U must be of the form $U = (a, b) - K$ for the other basis sets containing 0 intersect K. Let k be a point in $(a, b) \cap K$ (which is nonempty). Since k is in the open set V, there is a basis open set containing k and contained in V; it must be of the form (c, d). But $U \cap V \supset ((a, b) - K) \cap (c, d)$ and $((a, b) - K) \cap (c, d) \neq \emptyset$ for cardinality reasons. This is a contradiction

24. Normal spaces

Normal spaces can be characterized in two ways.

24.1. **Corollary.** *Let X be a T_1-space. Then the following conditions are equivalent:*

(1) *X is normal: For any pair of disjoint closed sets $A, B \subset X$ there exist disjoint open sets $U, V \subset X$ such that $A \subset U$ and $B \subset V$.*
(2) *(Urysohn's characterization of normality) For any pair of disjoint closed sets $A, B \subset X$ there exists a continuous function $f\colon X \to [0,1]$ such that $f(A) = 0$ and $f(B) = 1$.*
(3) *(Tietze's characterization of normality) For any closed subset A of X and any continuous function $f\colon A \to [0,1]$ there exists a continuous function $F\colon X \to [0,1]$ such that $F|A = f$.*

The proof relies on Urysohn's lemma (24.2) and Tietze's extension theorem (24.5).

Theorem 24.2 (Urysohn lemma)**.** *Let X be a normal space and let A and B be disjoint closed subsets of X. Then there exists a continuous function (a Urysohn function) $f\colon X \to [0,1]$ such that $f(a) = 0$ for $a \in A$ and $f(b) = 1$ for $b \in B$.*

Proof. We shall recursively define open sets $U_r \subset X$ for all $r \in \mathbf{Q} \cap [0,1]$ such that (make a drawing!)

$$r < s \Longrightarrow A \subset U_0 \subset U_r \subset \overline{U}_r \subset U_s \subset \overline{U}_s \subset U_1 \subset X - B \tag{24.3}$$

To begin, let U_1 be any open set such that $A \subset U_1 \subset X - B$, for instance $U_1 = X - B$. By normality (23.2.(2)) there is an open set U_0 such that $A \subset U_0 \subset \overline{U}_0 \subset U_1$. To proceed, arrange the elements of $\mathbf{Q} \cap [0,1]$ into a sequence

$$\mathbf{Q} \cap [0,1] = \{r_0 = 0, r_1 = 1, r_2, r_3, r_4, \ldots\}$$

such that the first two elements are $r_0 = 0$ and $r_1 = 1$. Assume that the open sets U_{r_i} satisfying condition (24.3) have been defined for $i \leq n$, where $n \geq 1$. We shall now define $U_{r_{n+1}}$. Suppose that if we put the numbers $r_0, r_1, \ldots, r_n, r_{n+1}$ in order

$$0 = r_0 < \cdots < r_\ell < r_{n+1} < r_m \cdots < r_1 = 1$$

the immediate predecessor of r_{n+1} is r_ℓ and r_m is the immediate successor. Then $\overline{U}_{r_\ell} \subset U_{r_m}$ by (24.3). By normality (23.2.(2)) there is an open set $U_{r_{n+1}}$ such that $\overline{U}_{r_\ell} \subset U_{r_{n+1}} \subset \overline{U}_{r_{n+1}} \subset U_{r_m}$. The sets U_{r_i}, $i \leq n+1$, still satisfy (24.3).

We are now ready to define the function. Consider the function $f\colon X \to [0,1]$ given by

$$f(x) = \begin{cases} \inf\{r \in \mathbf{Q} \cap [0,1] \mid x \in U_r\} & x \in U_1 \\ 1 & x \in X - U_1 \end{cases}$$

Then $f(B) = 1$ by definition and $f(A) = 0$ since $A \subset U_0$. But why is f continuous? It suffices (13.2) to show that the subbasis intervals (9.1) $[0, a)$, $a > 0$, and $(b, 1]$, $b < 1$, have open preimages. Since

$$f(x) < a \iff \exists r < a\colon x \in U_r \iff x \in \bigcup_{r<a} U_r$$

$$f(x) > b \iff \exists r' > b\colon x \notin U_{r'} \iff \exists r > b\colon x \notin \overline{U}_r \iff x \in \bigcup_{r>b} (X - \overline{U}_r)$$

the sets

$$f^{-1}([0,a)) = \bigcup_{r<a} U_r \quad \text{and} \quad f^{-1}((b,1]) = \bigcup_{r>b} (X - \overline{U}_r)$$

are open. □

24.4. **Example.** Let $X = \mathbf{R}$ and let $A = (-\infty, -1]$ and $B = [2, \infty)$. If we let $U_r = (-\infty, r)$ for $r \in \mathbf{Q} \cap [0,1]$ then

$$f(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & 1 \leq x \end{cases}$$

is the Urysohn function with $f(A) = 0$ and $f(B) = 1$.

Theorem 24.5 (Tietze extension theorem)**.** *Every continuous map from a closed subspace A of a normal space X into $(0,1)$, $[0,1)$ or $[0,1]$ can be extended to X.*

24.6**. Lemma.** *Let X be a normal space and $A \subset X$ a closed subspace. For $r > 0$ and any continuous map $f_0\colon A \to [-r,r]$ there exists a continuous map $g\colon X \to \mathbf{R}$ such that*

$$\forall x \in X\colon |g(x)| \le \frac{1}{3}r \quad \textit{and} \quad \forall a \in A\colon |f_0(a) - g(a)| \le \frac{2}{3}r \tag{24.7}$$

Proof. Since the sets $f_0^{-1}([-r,-\frac{1}{3}r]) \subset A$ and $f_0^{-1}([\frac{1}{3}r,r]) \subset A$ are closed and disjoint in A they are also closed and disjoint in X. Choose (24.2) a Urysohn function $g\colon X \to [-\frac{1}{3}r, \frac{1}{3}r]$ such that $g(x) = -\frac{1}{3}r$ on $f_0^{-1}([-r,-\frac{1}{3}r])$ and $g(x) = \frac{1}{3}r$ on $f_0^{-1}([\frac{1}{3}r,r])$. From

$$f_0^{-1}([-r,r]) = \underbrace{f_0^{-1}([-r,-\frac{1}{3}r])}_{g=-\frac{1}{3}r} \cup \underbrace{f_0^{-1}([-\frac{1}{3}r,\frac{1}{3}r])}_{|g|\le\frac{1}{3}r} \cup \underbrace{f_0^{-1}([\frac{1}{3}r,r])}_{g=\frac{1}{3}r}$$

we see that the second inequality of (24.7) is satisfied. □

Proof of 24.5. We shall first prove the theorem for functions from A to $[0,1]$ or rather to the homeomorphic interval $[-1,1]$ which is more convenient for reasons of notation.

Given a continuous function $f\colon A \to [-1,1]$. We have just seen (24.6) that there exists a continuous real function $g_1\colon X \to \mathbf{R}$ such that

$$\forall x \in X\colon |g_1(x)| \le \frac{1}{3} \quad \text{and} \quad \forall a \in A\colon |f(a) - g_1(a)| \le \frac{2}{3}$$

Now Lemma 24.6 applied to the function $f - g_1$ on A says that there exists a continuous real function $g_2\colon X \to \mathbf{R}$ such that

$$\forall x \in X\colon |g_2(x)| \le \frac{1}{3}\frac{2}{3} \quad \text{and} \quad \forall a \in A\colon |f(a) - (g_1(a) + g_2(a))| \le \left(\frac{2}{3}\right)^2$$

Proceeding this way we recursively (2.8) define a sequence $g_1, g_2, \cdots$ of continuous real functions on X such that

$$\forall x \in X\colon |g_n(x)| \le \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} \quad \text{and} \quad \forall a \in A\colon |f(a) - \sum_{i=1}^{n} g_i(a)| \le \left(\frac{2}{3}\right)^n \tag{24.8}$$

By the first inequality in (24.8), the series $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly and the sum function $F = \sum_{n=1}^{\infty} g_n$ is continuous by the uniform limit theorem (15.16). By the first inequality in (24.8),

$$|F(x)| \le \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1} = \frac{1}{3}\cdot 3 = 1,$$

and by the second inequality in (24.8), $F(a) = f(a)$ for all $a \in A$.

Assume now that $f\colon A \to (-1,1)$ maps A into the *open* interval between -1 and 1. We know that we can extend f to a continuous function $F_1\colon X \to [-1,1]$ into the *closed* interval between -1 and 1. We want to modify F_1 so that it does not take the values -1 and 1 and stays the same on A. The closed sets $F_1^{-1}(\{\pm 1\})$ and A are disjoint. There exists (24.2) a Urysohn function $U\colon X \to [0,1]$ such that $U = 0$ on $F_1^{-1}(\{\pm 1\})$ and $U = 1$ on A. Then $F = U \cdot F_1$ is an extension of f that maps X into $(-1,1)$.

A similar procedure applies to functions $f\colon A \to [-1,1)$ into the half-open interval. □

Proof of Corollary 24.1. The Urysohn lemma (24.2) says that (1) $\Longrightarrow$ (2) and the converse is clear. The Tietze extension theorem (24.5) says that (1) $\Longrightarrow$ (3). Only eg (3) $\Longrightarrow$ (2) remains. Assume that X is a T_1-space with property (3). Let A, B be two disjoint closed subsets. The function $f\colon A \cup B \to [0,1]$ given by $f(A) = 0$ and $f(B) = 1$ is continuous (13.5.(6)). Let $f\colon X \to [0,1]$ be a continuous extension of f. Then f is a Urysohn function for A and B. □

Many familiar classes of topological spaces are normal.

Theorem 24.9. *Compact Hausdorff spaces are normal.*

Proof. See 18.7.(2). □

In fact, every regular Lindelöf space is normal [9, Ex 32.4] [5, 3.8.2].

Theorem 24.10. *Metrizable spaces are normal.*

Proof. Let X be a metric space with metric d and let A, B be disjoint closed sets. The continuous function $f\colon X \to [0,1]$ given by

$$f(x) = \frac{d(x,A)}{d(x,A) + d(x,B)}$$

is a Urysohn function with $f(A) = \{0\}$ and $f(B) = \{1\}$. This shows that X is normal (24.1). □

Theorem 24.11. [5, Problem 1.7.4]. *Linearly ordered spaces are normal.*

Proof. We shall only prove the special case that every well-ordered space is normal. The half-open intervals $(a,b]$, $a < b$, are (closed and) open (12.2.(5)). Let A and B be two disjoint closed subsets and let a_0 denote the smallest element of X. Suppose that neither A nor B contain a_0. For any point $a \in A$ there exists a point $x_a < a$ such that $(x_a, a]$ is disjoint from B. Similarly, for any point $b \in B$ there exists a point $x_b < b$ such that $(x_b, b]$ is disjoint from A. The proof now proceeds as the proof (23.5) for normality of $\mathbf{R}_\ell$. Suppose next that $a_0 \in A \cup B$, say $a_0 \in A$. The one-point set $\{a_0\} = [a_0, a_0^+)$ is open and closed (as X is Hausdorff). By the above, we can find disjoint open sets U, V such that $A - \{a_0\} \subset U$ and $B \subset V$. Then $A \subset U \cup \{a_0\}$ and $B \subset V - \{a_0\}$ where the open sets $U \cup \{a_0\}$ and $V - \{a_0\}$ are disjoint. □

In particular, the well-ordered spaces S_Ω and $\overline{S_\Omega}$ are normal (the latter space is even compact Hausdorff).

25. Second countable regular spaces and the Urysohn metrization theorem

We investigate closer the class of regular spaces. We start with an embedding theorem that is used in other contexts as well.

25.1. **An embedding theorem.** We discuss embeddings into product spaces.

25.2. **Definition.** *A set $\{f_j\colon X \to Y_j \mid j \in J\}$ of continuous functions is said to separate points and closed sets if for any point $x \in X$ and any closed subset $C \subset X$ we have*

$$x \notin C \Longrightarrow \exists j \in J\colon f_j(x) \notin \overline{f_j(C)}$$

25.3. **Lemma.** *Let $f\colon X \to Y$ be a map that separates points and closed sets. Assume also that f is injective (eg that X is T_1). Then f is an embedding.*

Proof. If X is T_1, points are closed so that f separates points, ie f is injective. For any point $x \in X$ and any closed set $C \subset X$ we have that

$$f(x) \in \overline{f(C)} \overset{f \text{ separates}}{\Longrightarrow} x \in C \Longrightarrow f(x) \in f(C)$$

so we get that $f(X) \cap \overline{f(C)} = f(C)$. But this equality says that $f(C)$ is closed in $f(X)$. Hence the bijective continuous map $f\colon X \to f(X)$ is closed, so it is a homeomorphism. □

Theorem 25.4 (Diagonal embedding theorem)**.** *Let $\{f_\jmath\colon X \to Y_\jmath \mid \jmath \in J\}$ be a family of continuous functions that separates points and closed sets. Assume that X is T_1 or that at least one of the functions f_j is injective. Then the diagonal map $f = (f_j)\colon X \to \prod_{j\in J} Y_j$ is an embedding.*

Proof. Also the map f separates points and closed sets because

$$f(x) \in \overline{f(C)} \iff f(x) \in \overline{\prod f_j(C)} \overset{13.16}{\iff} f(x) \in \prod \overline{f_j(C)}$$
$$\iff \forall j \in J\colon f_j(x) \in \overline{f_j(C)} \overset{(f_j) \text{ separates}}{\Longrightarrow} x \in C$$

for all points $x \in X$ and all closed subsets $C \subset X$. The theorem now follows from 25.3. □

In particular, if one of the functions f_j is injective and separates points and closed sets then $f = (f_j)$ is an embedding. For instance, the graph $X \to X \times Y\colon x \to (x, g(x))$ is an embedding for any continuous map $g\colon X \to Y$.

25.5. **A universal second countable regular space.** We say that a space X is universal for some property if X has this property and any space that has this property embeds into X.

Theorem 25.6 (Urysohn metrization theorem)**.** *The following conditions are equivalent for a second countable space X:*

(1) *X is regular*
(2) *X is normal*
(3) *X is homeomorphic to a subspace of $[0,1]^\omega$*
(4) *X is metrizable*

The Hilbert cube $[0,1]^\omega$ is a universal second countable metrizable (or normal or regular) space.

Proof. (1) $\Longrightarrow$ (2): Let X be a regular space with a countable basis $\mathcal{B}$. We claim that X is normal. Let A and B be disjoint closed sets in X. By regularity (23.2.(1)), each point $a \in A \subset X - B$ has a basis neighborhood $U_a \in \mathcal{B}$ such that $a \in U_a \subset \overline{U}_a \subset X - B$. Let $U_1, U_2, \dots$ be the elements in the image of the map $A \to \mathcal{B}\colon a \mapsto U_a$. Then

$$A \subset \bigcup_{n=1}^{\infty} U_n \quad \text{and} \quad \overline{U}_n \cap B = \emptyset \quad \text{for } n = 1, 2 \dots$$

Similarly, there is a sequence $V_1, V_2, \dots$ of basis open sets such that

$$B \subset \bigcup_{n=1}^{\infty} V_n \quad \text{and} \quad \overline{V}_n \cap A = \emptyset \quad \text{for } n = 1, 2 \dots$$

The open sets $\bigcup_{n=1}^{\infty} U_n$ and $\bigcup_{n=1}^{\infty} V_n$ may not be disjoint. Consider instead the open sets

$$\begin{aligned} U_1' &= U_1 - \overline{V}_1 & V_1' &= V_1 - \overline{U}_1 \\ U_2' &= U_2 - (\overline{V}_1 \cup \overline{V}_2) & V_2' &= V_2 - (\overline{U}_1 \cup \overline{U}_2) \\ &\vdots & &\vdots \\ U_n' &= U_n - (\overline{V}_1 \cup \dots \cup \overline{V}_n) & V_n' &= V_n - (\overline{U}_1 \cup \dots \cup \overline{U}_n) \end{aligned}$$

Even though we have removed part of U_n we have removed no points from A from U_n, and we have removed no pints from B from V_n. Therefore the open sets U_n' still cover A and the open sets V_n' still cover B:

$$A \subset \bigcup_{n=1}^{\infty} U_n' \quad \text{and} \quad B \subset \bigcup_{n=1}^{\infty} V_n'$$

and in fact these sets are disjoint:

$$\bigcup_{n=1}^{\infty} U_n' \cap \bigcup_{n=1}^{\infty} V_n' = \Big(\bigcup_{m \le n} U_m' \cap V_n' \Big) \cup \Big(\bigcup_{m > n} U_m' \cap V_n' \Big) = \emptyset$$

because $U_m' \subset U_m$ does not intersect $V_n' \subset X - U_m$ if $m \le n$ and $U_m' \subset X - V_n$ does not intersect $V_n' \subset V_n$ if $m > n$. (Make a drawing of U_1, U_2 and V_1, V_2.)
(2) $\Longrightarrow$ (3): Let X be a normal space with a countable basis $\mathcal{B}$. We show that there is a *countable* set $\{f_{UV}\}$ of continuous functions $X \to [0,1]$ that separate points and closed sets. Namely, for each pair U, V of basis open sets $U, V \in \mathcal{B}$ such that $\overline{U} \subset V$, choose a Urysohn function $f_{UV}\colon X \to [0,1]$ (24.2) such that $f_{UV}(\overline{U}) = 0$ and $f_{UV}(X - V) = 1$. Then $\{f_{UV}\}$ separates points and closed sets: For any closed subset C and any point $x \notin C$, or $x \in X - C$, there are (23.2) basis open sets U, V such that $x \in U \subset \overline{U} \subset V \subset X - C$ (choose V first). Then $f_{UV}(x) = 0$ and $f_{UV}(C) = 1$ so that $f_{UV}(x) \notin \overline{f_{UV}(C)}$. According to the Diagonal embedding theorem (25.4) there is an embedding $X \to [0,1]^\omega$ with the f_{UV} as coordinate functions.
(3) $\Longrightarrow$ (4): $[0,1]^\omega$ is metrizable and any subspace of a metrizable space is metrizable. (15.5).
(4) $\Longrightarrow$ (1): Any metrizable space is regular, even normal (24.10). □

The point is that in a second countable regular hence normal space there is a *countable* set of (Urysohn) functions that separate points and closed sets. Therefore such a space embeds in the Hilbert cube.

We have (25.6, 22.2) the following identities

$$\left\{\begin{matrix}\text{2nd countable}\\ \text{regular}\end{matrix}\right\} = \left\{\begin{matrix}\text{2nd countable}\\ \text{normal}\end{matrix}\right\} = \left\{\begin{matrix}\text{2nd countable}\\ \text{metrizable}\end{matrix}\right\}$$
$$= \left\{\begin{matrix}\text{Countable dense subset}\\ \text{metrizable}\end{matrix}\right\} = \left\{\begin{matrix}\text{Lindelöf}\\ \text{metrizable}\end{matrix}\right\} = \left\{\begin{matrix}\text{Subspace of}\\ \text{Hilbert cube}\end{matrix}\right\}$$

between classes of topological spaces.

26. Completely regular spaces and the Stone–Čech compactification

There is no version of the Tietze extension theorem (24.5) for regular spaces, ie it is in general not true that continuous functions separate closed sets and points in regular spaces. Instead we have a new class of spaces where this is true.

26.1. **Definition.** *A space X is* completely regular *if points are closed in X and for any closed subset C of X and any point $x \notin C$ there exists a continuous function $f\colon X \to [0,1]$ such that $f(x) = 0$ and $f(C) = 1$.*

Clearly

$$\text{normal} \implies \text{completely regular} \implies \text{regular} \implies \text{Hausdorff} \implies T_1$$

and none of these arrows reverse (23.6, [9, Ex 33.11]).

In a completely regular space there are enough continuous functions $X \to [0,1]$ to separate points and closed sets.

It is easy to see that any subspace of a completely regular space is completely regular [9, 33.2].

Theorem 26.2. *The following conditions are equivalent for a topological space X:*

(1) *X is completely regular*
(2) *X is homeomorphic to a subspace of $[0,1]^J$ for some set J*
(3) *X is homeomorphic to a subspace of a compact Hausdorff space*

Proof. (1) $\implies$ (2): If X is completely regular then the set $\mathcal{C}(X)$ of continuous maps $X \to [0,1]$ separates points and closed sets. The evaluation map

$$\Delta\colon X \to \textstyle\prod_{j\in\mathcal{C}(X)}[0,1], \quad \pi_j(\Delta(x)) = j(x), \quad j \in \mathcal{C}(X), \quad x \in X,$$

is therefore an embedding (25.4).
(2) $\implies$ (3): $[0,1]^J$ is compact Hausdorff by the Tychonoff theorem (18.16).
(3) $\implies$ (1): A compact Hausdorff space is normal (24.9), hence completely regular and subspaces of completely regular spaces are completely regular. □

Closed subspaces of compact Hausdorff spaces are compact Hausdorff (18.7), open subspaces are locally compact Hausdorff (21.6), and arbitrary subspaces are completely regular (26.2).

26.3. **Corollary.** *(Cf 12.19, 23.3) Any subspace of a completely regular space is completely regular. Any product of completely regular spaces is completely regular.*

Proof. The first part is easily proved [9, 33.2] (and we already used it above). The second part follows from (26.2) because (13.12) the product of embeddings is an embedding. □

26.4. **Corollary.** *Any locally compact Hausdorff space is completely regular.*

Proof. Locally compact Hausdorff spaces are open subspaces of compact spaces (21.6). Completely regular spaces are subspaces of compact spaces (26.2). □

26.5. **Example** (A normal space with a non-normal subspace)**.** Take any completely regular but not normal space (for instance (23.6) $\mathbf{R}_\ell \times \mathbf{R}_\ell$) and embed it into $[0,1]^J$ for some set J (26.2). Then you have an example of a normal (even compact Hausdorff) space with a non-normal subspace.

26.6. **The Stone–Čech construction.** For *any* topological space X let $\mathcal{C}(X)$ denote the set of continuous maps $j\colon X \to I$ of X to the unit interval $I = [0,1]$ and let

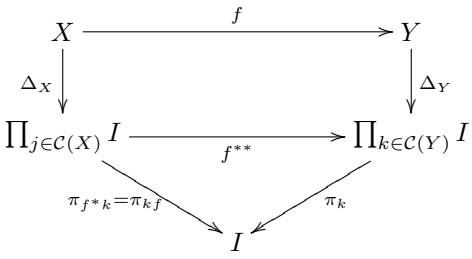
$$\Delta\colon X \to \textstyle\prod_{j\in\mathcal{C}(X)} I = \operatorname{map}(\mathcal{C}(X), I)$$

be the *continuous* evaluation map given by $\Delta(x)(j) = j(x)$ or $\pi_j\Delta = j$ for all $j \in \mathcal{C}(X)$. (The space $\operatorname{map}(\mathcal{C}(X), I)$ is a kind of double-dual of X). This construction is natural: For any continuous map $f\colon X \to Y$ of X into a space Y, there is an induced map $\mathcal{C}(X) \leftarrow \mathcal{C}(Y)\colon f^*$ of sets and yet an induced *continuous* map (13.15)

$$\operatorname{map}(\mathcal{C}(X), I) = \prod_{j\in\mathcal{C}(X)} I \xrightarrow{f^{**}} \prod_{k\in\mathcal{C}(Y)} I = \operatorname{map}(\mathcal{C}(Y), I) \tag{26.7}$$

such that the diagram

(26.8)

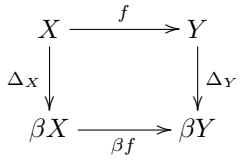


commutes: The lower triangle commutes by the definition of the map f^{**} (13.15) and then the upper square commutes because $\pi_k f^{**}\Delta_X = \pi_{kf}\Delta_X = kf = \pi_k\Delta_Y f$. Put $\beta X = \overline{\Delta(X)}$ and define $\Delta\colon X \to \beta X$ to be the corestriction of $\Delta\colon X \to \prod_{j\in\mathcal{C}(X)} I$ to βX. Then βX is a compact Hausdorff space (it is a closed subspace of the compact (18.16) Hausdorff space $\prod I$) and Δ is, by design, a continuous map with a dense image ΔX in βX. This construction is natural: For any continuous map $f\colon X \to Y$, the induced map f^{**} (26.7) takes βX into βY for

$$f^{**}(\beta X) = f^{**}(\overline{\Delta_X X}) \subset \overline{f^{**}\Delta_X X} \overset{(26.8)}{=} \overline{\Delta_Y f(X)} \subset \overline{\Delta_Y Y} = \beta Y$$

and thus we obtain from (26.8) a new commutative diagram of continuous maps

(26.9)



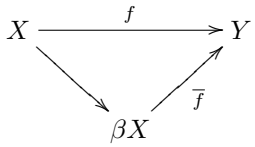
where $\beta f\colon \beta X \to \beta Y$ is the corestriction to βY of the restriction of f^{**} to βX. (This makes β into a functor with a natural transformation from the identity functor from the category of topological spaces to the category of compact Hausdorff spaces.)

The next result says that the map $X \to \beta X$ is the universal map from X to a compact Hausdorff space.

Theorem 26.10. *Let X be a topological space.*

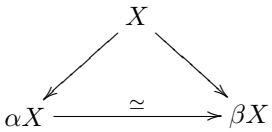
(1) *The map $X \to \beta X$ is a continuous map of X into a compact Hausdorff space.*
(2) *For any continuous map $f\colon X \to Y$ of X into a compact Hausdorff space Y there exists a unique continuous map $\overline{f}\colon \beta X \to Y$ such that the diagram*

(26.11)



commutes. (We say that $f\colon X \to Y$ factors uniquely through βX.)

(3) *Suppose $X \to \alpha X$ is a map of X to a compact Hausdorff space αX such that any map of X to a compact Hausdorff space factors uniquely through αX. Then there exists a homeomorphism $\alpha X \to \beta X$ such that*



commutes.

(4) *If X is completely regular then $X \to \beta X$ is an embedding. If X is compact Hausdorff then $X \to \beta X$ is a homeomorphism.*

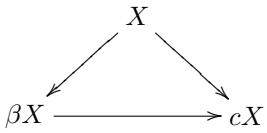
Proof. (4): Since the corestriction of an embedding is an embedding (13.12) and $X \to \prod I$ is an embedding when X is completely regular (26.2), also $X \to \beta X$ is an embedding. If X is compact Hausdorff, X is normal (24.9), hence completely regular, so we have just seen that $\Delta\colon X \to \beta X$ is

an embedding. The image of this embedding is closed (18.8.(1)) and dense. Thus the embedding is bijective so it is a homeomorphism.
(1) and (2): If Y is compact Hausdorff then by item (4) Δ_Y is a homeomorphism in diagram (26.9) and hence $\overline{f} = \Delta_Y^{-1} \circ \beta f$ is a possibility in (26.11). It is the only possibility [9, Ex 18.13], for the image of X is dense in βX.
(3) Let $X \to \alpha X$ be a map to a compact Hausdorff space satisfying the above universal property. Then there exist continuous maps $\alpha X \rightleftarrows \beta X$ that, by uniqueness, are inverse to each other. (All universal constructions are essentially unique.) □

If X is completely regular then $X \to \beta X$ is a compactification (26.10.(4)) and it is called the Stone–Čech compactification of X. Conversely, if X has a compactification then X is homeomorphic to a subspace of a compact space and therefore (26.2) completely regular. We state these observations in

26.12. **Lemma.** *X has a compactification if and only if X is completely regular.*

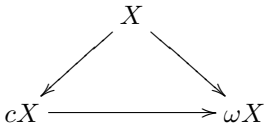
The Stone–Čech compactification βX of a completely regular space X is the *maximal* compactification in the sense that for any other compactification $X \to cX$ there is a closed quotient map $\beta X \to cX$ map such that



commutes. The map $\beta X \to cX$ is closed by 18.8 and surjective because its image is closed and dense.

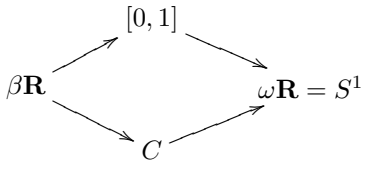
The Alexandroff compactification ωX of a noncompact locally compact Hausdorff space X is the *minimal* compactification in the sense that $\omega X = cX/(cX - X)$ for any other compactification $X \to cX$.

Indeed we saw in 21.3 that there is a closed quotient map $cX \to \omega X$, taking $cX - X$ to ω, such that



commutes. (What is the minimal compactification of a compact Hausdorff space?)

For instance, there are quotient maps



of compactifications of $\mathbf{R}$. Here

$$C = \{0\} \times [-1, 1] \cup \{(x, \sin \frac{1}{x}) \mid 0 < x \leq \pi^{-1}\} \cup L$$

is the Warsaw circle, a compactification of $\mathbf{R}$ with remainder $C - \mathbf{R} = [-1, 1]$, which is obtained by closing up the closed topologist's sine curve $\overline{S}$ by (a piece-wise linear) arc from $(0, 0)$ to $(\pi^{-1}, 0)$. In particular, $C/[-1, 1] = S^1$. More generally, there are quotient maps $\beta\mathbf{R}^n \to [0, 1]^n \to \omega\mathbf{R}^n = S^n$ of compactifications of euclidean n-space $\mathbf{R}^n$.

Investigations of the Stone–Čech compactification of the integers $\beta\mathbf{Z}$ raise several issues of a very fundamental nature [13, 14, 11] whose answers depend on your chosen model of set theory.

27. Manifolds

27.1. **Definition.** *A manifold is a locally euclidean second countable Hausdorff space.*

Manifolds are locally compact and locally path connected since they are locally euclidean. All locally compact Hausdorff spaces are (completely) regular. Thus manifolds are second countable regular spaces and hence they are normal, metrizable, Lindelöfand they have countable dense subsets.

The line with two zeroes is an example of a locally euclidean space that is not Hausdorff. The long line [9, Ex 24.12] is a locally euclidean space that is not 2nd countable.

The n–sphere S^n and the real projective n–space $\mathbf{R}P^n$ (14.14) are examples of manifolds. S^n lies embedded in $\mathbf{R}^{n+1}$ from birth but what about $\mathbf{R}P^n$? Does $\mathbf{R}P^n$ embed in some euclidean space?

Theorem 27.2 (Embeddings of compact manifolds)**.** *Any compact manifold embeds in $\mathbf{R}^N$ for some N.*

Proof. We may assume that M is connected. Let n be the dimension of M (we assume that this is well-defined). Let $B = \{x \in \mathbf{R}^n \mid |x| \leq 1\}$ be the unit ball in $\mathbf{R}^n$. Cover M by finitely many closed sets $B_1, \ldots, B_k$ such that each B_i is homeomorphic to B and such that the interiors of the B_i cover M. Let $f_i \colon M \to S^n$ be the continuous map obtained by collapsing the complement to the interior of B_i to a point. The map $f = (f_1, \ldots, f_k) \colon M \to S^n \times \cdots \times S^n$ is injective: If x lies in the interior of B_i then $f_i(x) \neq f_i(y)$ for all $y \neq x$. Since M is compact and $(S^n)^k$ Hausdorff, f is an embedding. Finally, each sphere S^n embeds into $\mathbf{R}^{n+1}$ so that $(S^n)^k$ embeds into $(\mathbf{R}^{n+1})^k$. □

In particular $\mathbf{R}P^2$ (14.14) embeds into some euclidean space. Which one?

There is up to homeomorphism just one compact manifold of dimension 1, the circle. Compact manifolds of dimension 2 are described by a number, the genus, and orientability. In particular, any simply connected 2-dimensional compact manifold is homeomorphic to S^2. We do not know (December 2003) if there are any simply connected compact 3-dimensional manifolds besides S^3 (Poincaré conjecture). Classification of 4-dimensional compact manifolds is logically impossible.

Any manifold admits a partition of unity. We shall prove the existence in case of compact manifolds.

Theorem 27.3 (Partition of unity)**.** *Let $X = U_1 \cup \cdots \cup U_k$ be a finite open covering of the normal space X. The there exist continuous functions $\phi_i \colon X \to [0,1]$, $1 \leq i \leq k$, such that*

1. $\overline{\{\phi_i > 0\}} \subset U_i$
2. $\sum_{i=1}^k \phi_i(x) = 1$ *for all* $x \in X$.

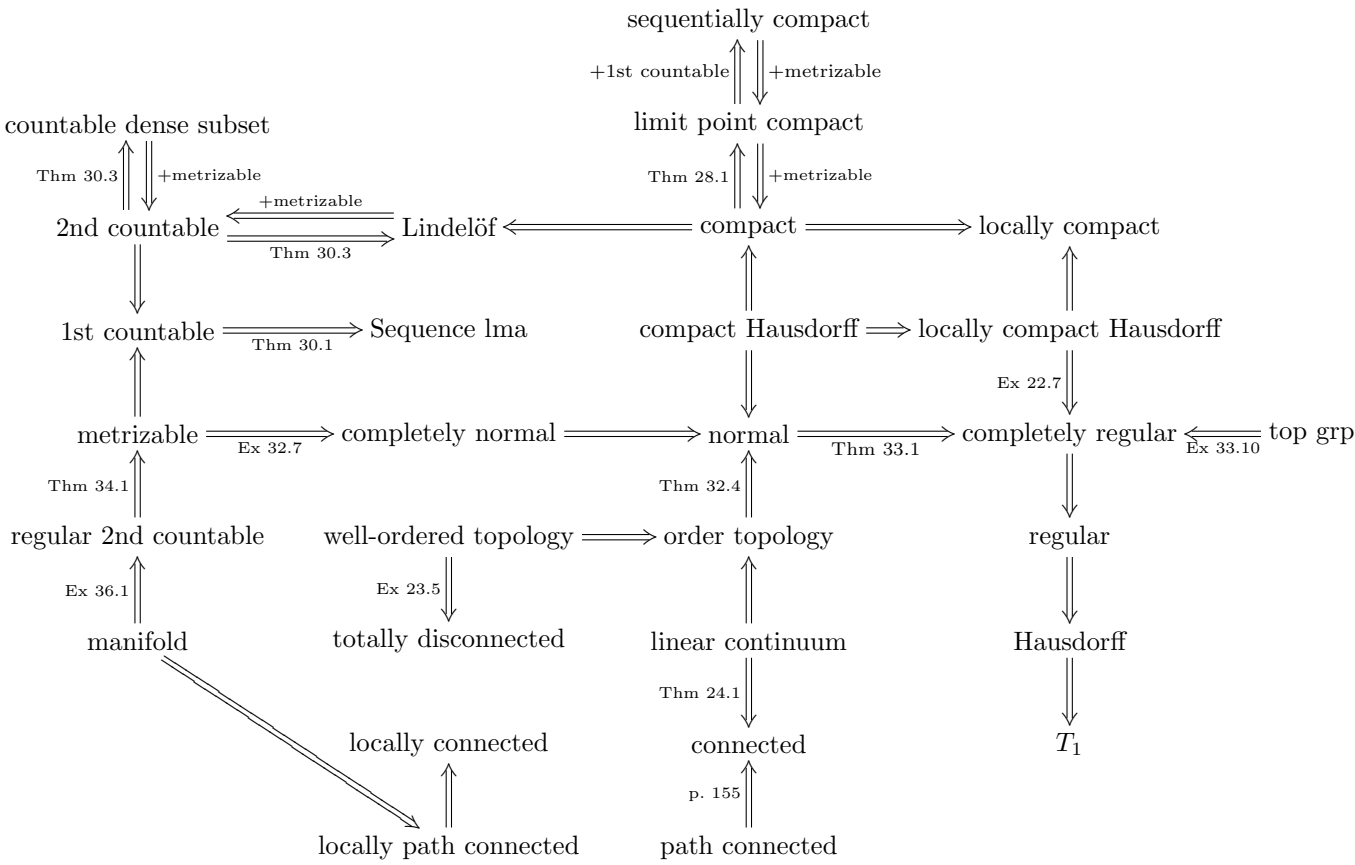
Proof. We show first that there is an open covering $\{V_i\}$ of X such that $\overline{V}_i \subset U_i$ (we shrink the sets of the covering). Since the closed set $X - (U_2 \cup \cdots \cup U_k)$ is contained in the open set U_1 there is an open set V_1 such that

$$X - (U_2 \cup \cdots \cup U_k) \subset V_1 \subset \overline{V}_1 \subset U_1$$

by normality (23.2). Now $V_1 \cup U_2 \cup \cdots \cup U_k$ is an open covering of X. Apply this procedure once again to find an open set V_2 such that $\overline{V}_2 \subset U_2$ and $V_1 \cup V_2 \cup U_3 \cup \cdots \cup U_k$ is still an open covering of X. After finitely many steps we have an open covering $\{V_i\}$ such that $\overline{V}_i \subset U_i$ for all i.

Do this one more time to obtain an open covering $\{W_i\}$ such that $W_i \subset \overline{W}_i \subset V_i \subset \overline{V}_i \subset U_i$ for all i. Now choose a Urysohn function (24.2) $\psi_i \colon X \to [0,1]$ such that $\psi_i(\overline{W}_i) = 1$ and $\psi_i(X - V_i) = 0$. Then $\overline{\{x \mid \psi_i(x) > 0\}} \subset \overline{V}_i \subset U_i$ and $\psi(x) = \sum \psi_i(x) > 1$ for any $x \in X$ since $x \in W_i$ for some i. Hence $\phi_i = \frac{\psi_i}{\psi}$ is a well-defined continuous function on M taking values in the unit interval such that $\sum_{i=1}^k \phi_i(x) = \sum_{i=1}^k \frac{\psi_i(x)}{\psi(x)} = \frac{1}{\psi(x)} \sum_{i=1}^k \psi_i(x) = \frac{\psi(x)}{\psi(x)} = 1$. □

28. Relations between topological spaces



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