HW 4 Solutions pg 157: 1a, c, 2, 3, 8, 9

1a. $(0,1)$ & $(0,1]$

Suppose $f:(0,1] \to (0,1)$ is a homeomorphism.

Then $0 < f(1) < 1$ & hence $(0,1) - f(1)$ is disconnected

Now $f|_{(0,1)} : (0,1) \longrightarrow (0,1) - f(1)$ is bij & cts

as f was bij & cts.

But $(0,1)$ is connected & $f((0,1)) = (0,1) - f(1)$

is not $\longrightarrow\leftarrow$ as cts image of connected set is connected.

Similar argument work for the other two pairs.

$(0,1]$ & $[0,1]$: take out pts $0, 1$ from $[0,1]$

$(0,1)$ & $[0,1]$: take out 1 (or 0) from $[0,1]$

1c. Suppose $f:\mathbb{R}^n \to \mathbb{R}$ is a homeomorphism

$f(\bar{0}) \in \mathbb{R}$ & $\mathbb{R} - f(\bar{0})$ is disconnected.

So $f(\mathbb{R}^n - \{\bar{0}\}) = \mathbb{R} - f(\bar{0})$

for $n \geq 1$ $\mathbb{R}^n - \{\bar{0}\}$ is connected &

$\mathbb{R} - f(\bar{0})$ is disconnected $\longrightarrow\leftarrow$.

[2] Suppose not. Define $g : S^1 \to \mathbb{R}$ as $g(x) = f(x) - f(-x)$. g cts

As $f(x) \neq f(-x)$ $\forall x \in S^1$

$\Rightarrow g(x) \neq 0 \quad \forall x \in S^1$

Suppose $f(x) - f(-x) > 0$ for some x

then $g(-x) = f(-x) - f(x) < 0$

$\Rightarrow$ g takes positive as well as negative values. — *

As $g(x) \neq 0 \ \forall x \in S^1 \Rightarrow g(S^1) \subset \mathbb{R} - 0$

& $\{x / x > 0\} \cap g(S^1) \neq \emptyset$

& $\{x / x < 0\} \cap g(S^1) \neq \emptyset$ } by *

$\Rightarrow g(S^1)$ is disconnected $\longrightarrow \leftarrow$

as S^1 connected & g cts

Suppose not. Let $g(x) = f(x) - x$.

$g : [0,1] \to \mathbb{R}$ cts. , $g(x) \neq 0 \ \forall x \in [0,1]$

$g(0) = f(0) - 0 = f(0) > 0$

$g(1) = f(1) - 1 < 0$

$\Rightarrow$ ~~$g([0,$~~ ~~$g(x)$~~ $g(1) < 0 < g(0)$ & $g(x) \neq 0 \ \forall x \in [0,1]$

$\longrightarrow \leftarrow$ to Intermediate Value Thm

$X = (0,1)$ $f(x) = \frac{1}{2}x^2$ has no fixed pts.

$X = [0,1)$ $f(x) = \frac{1}{2}x + 1$ has no fixed pts.

[8] a. Yes. X, Y be p.c.

Let (x_1, y_1) & $(x_2, y_2) \in X \times Y$

As X is p.c., $\exists\ r_1 : [0,1] \to X$ cts $r_1(0) = x_1$, $r_1(1) = x_2$

As Y is p.c., $\exists\ r_2 : [0,1] \to Y$ cts $r_2(0) = y_1$, $r_2(1) = y_2$

Let $r : [0,1] \to X \times Y$ be $r(t) = (r_1(t), r_2(t))$ cts

& $r(0) = (x_1, y_1)$, $r(1) = (x_2, y_2)$.

$\to X \times Y$ is p.c.

b. No. Topologists sine curve. Example 7, Section 24.

c. Yes. Done in class

d. Yes. Let $a \in \cap A_\alpha$. Let $x, y \in \cup A_\alpha$

$x \in A_{\alpha_1}$, $y \in A_{\alpha_2}$, As both are p.c.

$\exists\ r_1 : [0,1] \to A_{\alpha_1}$ $r_1(0) = x$, $r_1(1) = a$

& $r_2 : [1,2] \to A_{\alpha_2}$ $r_2(1) = a$ & $r_2(2) = y$

$r : [0,2] \to \cup A_\alpha$ s.t. $r(t) = r_1(t)$ $0 \le t \le 1$

$r_2(t)$ $1 \le t \le 2$

$r(0) = x$, $r(2) = y$ cts

So $\cup A_\alpha$ is p.c.



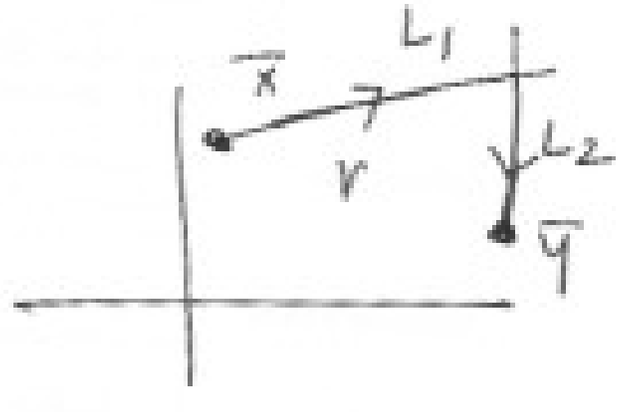
[9] Let $\bar{x}, \bar{y} \in \mathbb{R}^2 - A$

Look at all lines passing through $\bar{x}$.

As $\exists$ uncountably many lines passing through $\bar{x}$ & A is countable, $\exists$ a line say L_1 s.t. $L_1 \cap A = \phi$

i.e. $L_1 \subset \mathbb{R}^2 - A$

Similarly for $\bar{y}$. Now choose a line L_2, s.t. $L_2 \cap A =$ passing through $\bar{y}$ which is not parallel to L_1. Such a line can always be chosen as $\exists$ uncountably many lines through $\bar{y}$ & are disjoint from A.



Now r is the path as shown.

Hence $\mathbb{R}^2 - A$ is p.c.