HW 7

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pg126

1a. Compare to square metric.

$$d'(\bar{x},\bar{y}) = \sum_{i=1}^{n} |x_i - y_i| \leq \sum_{i=1}^{n} \max_i \{|x_i - y_i|\} = \sum_{i=1}^{n} \rho(\bar{x},\bar{y})$$

$$= n\,\rho(\bar{x},\bar{y})$$

$$\rho(\bar{x},\bar{y}) = \max_i \{|x_i - y_i|\} \leq \sum |x_i - y_i| = d'(\bar{x},\bar{y})$$

$$\rho(\bar{x},\bar{y}) \leq d'(\bar{x},\bar{y}) \leq n\,\rho(\bar{x},\bar{y}).$$

Proceed as in proof of Thm 20.3 to show that the topologies induced by d' & ρ are same. (& equal to the product topology). ✦

1b. $$d'(\bar{x},\bar{y}) = \left(\sum |x_i - y_i|^p\right)^{1/p}$$

$$d'(\bar{x},\bar{y}) \leq \left(\sum \left(\max_i \{|x_i - y_i|\}\right)^p\right)^{1/p}$$

$$= \left(\sum (\rho(\bar{x},\bar{y})^p\right)^{1/p}$$

$$= \left(n\,\rho(\bar{x},\bar{y})^p\right)^{1/p}.$$

$$= n^{1/p}\,\rho(\bar{x},\bar{y})$$

$$\rho(\bar{x},\bar{y})^p = \left(\max_i \{|x_i - y_i|\}\right)^p \leq \sum |x_i - y_i|^p = (d')^p$$

$$\rho(\bar{x},\bar{y}) \leq d'(\bar{x},\bar{y}) \leq n^{1/p}\,\rho(\bar{x},\bar{y}).$$

3a. 1st Proof.

3a. $d: X \times X \to \mathbb{R}$ cts

Think of $X \times X$ as a metric space with the square metric i.e. $\rho((x_1,y_1),(x_2,y_2)) = \max\{d(x_1,x_2), d(y_1,y_2)\}$

~~$\rho((x_1,y_1),(x_2,y_2)) = \max\{d(x_1,x_2), d(y_1,y_2)\}$~~

~~We want to show.~~ Fix $(x_0,y_0) \in X \times X$.

Given $\varepsilon > 0$, $\exists\ \delta$ s.t.

$$\rho((x,y),(x_0,y_0)) < \delta \Rightarrow |d(x,y) - d(x_0,y_0)| < \varepsilon.$$

~~$d(x,y) \leq d(x_0,y_0)$~~ $d(x,y) \leq d(x,x_0) + d(x_0,y)$

$$\leq d(x,x_0) + d(x_0,y_0) + d(y_0,y)$$

$$d(x,y) - d(x_0,y_0) \leq d(x,x_0) + d(y,y_0)$$

$$\leq 2\rho((x,y),(x_0,y_0))$$

$$< \varepsilon$$

if $\rho((x,y),(x_0,y_0)) < \frac{\varepsilon}{2} = \delta$.

Similar argument works for

$$d(x_0,y_0) - d(x,y) < 2\ \rho((x,y),(x_0,y_0)) < \varepsilon$$

Choose $\delta = \frac{\varepsilon}{2}$

—.

This works as I am implicitly assuming that the product topology on $X \times X$ is the same as the metric topology induced by ρ.

3a. 2nd proof

Think of $X \times X$ as a top sp. with product topology.

Let $(a,b) \in X \times X$ & let $d(a,b) = \alpha$.

It is enough to show that $d^{-1}(\alpha-\varepsilon, \alpha+\varepsilon)$ is open in $X \times X$ (as sets like $(\alpha-\varepsilon, \alpha+\varepsilon)$ are basis of $\mathbb{R}$).

Let $(x,y) \in d^{-1}(\alpha-\varepsilon, \alpha+\varepsilon)$

We want to find a basis elt of $X \times X$ containing (x,y) & which is contained in $d^{-1}(\alpha-\varepsilon, \alpha+\varepsilon)$.

Let $(x',y') \in B_d(x,\delta) \times B_d(y,\delta)$ we will find a δ which works.

We want to show $d(x',y') \in (\alpha-\varepsilon, \alpha+\varepsilon)$

$$d(x',y') \leq d(x,x') + d(x,y) + d(y,y') \quad \text{(triangle ineq. applied twice)}$$
$$< 2\delta + d(x,y)$$

Similarly $d(x,y) < 2\delta + d(x',y')$

So $|d(x,y) - d(x',y')| < 2\delta$

$$|d(x,y) - d(a,b)| \leq |d(x,y) - d(x',y')| + |d(x,y) - d(a,b)|$$
(Triangle ineq. for $\mathbb{R}$)
$$< 2\delta + |d(x,y) - d(a,b)| = \varepsilon$$

Choose $\delta = \dfrac{\varepsilon - |d(x,y) - d(a,b)|}{2} > 0$ as $|d(x,y) - d(a,b)| < \varepsilon$

So $(x',y') \in d^{-1}(\alpha-\varepsilon, \alpha+\varepsilon)$

$\Rightarrow B_d(x,\delta) \times B_d(y,\delta) \subset d^{-1}(\alpha-\varepsilon, \alpha+\varepsilon) \Rightarrow d^{-1}(\alpha-\varepsilon, \alpha+\varepsilon)$ open $\Rightarrow d$ cts.

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3a If $\rho(\bar{x},\bar{y})=0$ iff $\bar{x}=\bar{y}$

$\rho(\bar{x},\bar{y}) = \rho(\bar{y},\bar{x})$.

$$\rho(\bar{x},\bar{y}) = \max\{d_i(x_i,y_i)\}$$

$$\leq$$

$$d_i(x_i,z_i) \leq d_i(x_i,y_i) + d(y_i,z_i)$$

$$\max_i\{d_i(x_i,z_i)\} \leq \max_i\{d_i(x_i,y_i) + d_i(y_i,z_i)\}$$

$$\leq \max_i\{d_i(x_i,y_i)\} + \max_i\{d_i(y_i,z_i)\}.$$

$$\rho(\bar{x},\bar{z}) \leq \rho(\bar{x},\bar{y}) + \rho(\bar{y},\bar{z})$$

ρ is a metric.

Note that it induces same topology as the product topology.

Pf for $n=2$: $X\times Y$

$$B_\rho((x_0,y_0),\varepsilon) = B_{d_1}(x_0,\varepsilon) \times B_{d_2}(y_0,\varepsilon),$$

basis in product topology

& $B_\rho(x_0,\varepsilon_1) \times B_\rho(y_0,\varepsilon_2)$

$\varepsilon = \min\{\varepsilon_1,\varepsilon_2\}$ $\quad B_\rho((x_0,y_0),\varepsilon) = B_{d_1}(x_0,\varepsilon) \times B_{d_2}(y_0,\varepsilon)$

$$\subset B_\rho(x_0,\varepsilon_1) \times B_\rho(y_0,\varepsilon_2).$$

12a. $|(x+y)-(x_0+y_0)| = |(x-x_0)+(y-y_0)|$

$\leq |x-x_0| + |y-y_0|$

$\leq 2\rho((x,y),(x_0,y_0))$

$< \varepsilon$

if

$\rho((x,y),(x_0,y_0)) < \frac{\varepsilon}{2} = \delta$

Choose $\delta = \varepsilon/2$.

12b. $|xy - x_0y_0| = |xy - x_0y + x_0y - x_0y_0|$

$\leq |y||x-x_0| + |x_0||y-y_0|$

$\leq |y-y_0||x-x_0| + |y_0||x-x_0| + |x_0||y-y_0|$

$\leq \rho^2 + |y_0|\rho + |x_0|\rho$

$\leq \rho(\rho + |x_0| + |y_0|) \leq \rho(1 + |x_0| + |y_0|)$

$< \varepsilon$

Choose $\rho\ \delta < 1$, so $\rho < \delta < 1$

$\rho < \frac{\varepsilon}{1+|x_0|+|y_0|} = \delta$

Choose $\delta = \min\left\{1, \frac{\varepsilon}{1+|x_0|+|y_0|}\right\}$