Solutions to Homework 5

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Please report any typos or mistakes to me.

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1. (pg 100, 2) A is closed in Y implies that $A = A_1 \cap Y$ where A_1 is closed in X. Since Y is closed in X, A is an intersection of two closed sets in X and hence closed.

2. (pg 100, 6b) Since $A \subset A \cup B \subset \overline{A \cup B}$. Since $\overline{A \cup B}$ is closed and contains A, hence $\overline{A} \subset \overline{A \cup B}$. Similarly $\overline{B} \subset \overline{A \cup B}$. Hence $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$. Conversely, as $A \subset \overline{A}$ and $B \subset \overline{B}$, $A \cup B \subset \overline{A} \cup \overline{B}$ which is union of closed sets, hence closed. Hence $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. Hence the proof.

3. (pg 100, 6c) $A_\alpha \subset \cup_\alpha A_\alpha \subset \overline{\cup_\alpha A_\alpha}$ which is a closed set, hence $\overline{A_\alpha} \subset \overline{\cup_\alpha A_\alpha}$. This holds for all α, hence $\cup_\alpha \overline{A_\alpha} \subset \overline{\cup_\alpha A_\alpha}$. The equality may not hold since the right hand side is always closed but the left hand side is an arbitrary union of closed sets which may not be closed. E.g. Let $A_r = r$, $r \in \mathbb{Q}$ then $\overline{A_r} = r$ and $\cup_{r\in\mathbb{Q}} \overline{A_r} = \mathbb{Q}$ whereas $\overline{\cup_{r\in\mathbb{Q}} A_r} = \overline{\mathbb{Q}} = \mathbb{R}$.

4. (pg 100, 7) A neighbourhood U of x may intersect A_α but some other neighbourhood V of x may intersect A_β for $\alpha \neq \beta$. So x may neither be in the closure of A_α nor A_β. Note that this argument even fails for two sets.

5. (pg 100, 11) Let $(x_1, y_1) \neq (x_2, y_2)$ be points of $X \times Y$. Since they are not equal, without loss of generality, let $x_1 \neq x_2$. Then since X is Hausdorff, there exist U_1 neighbourhood of x and U_2 neighbourhood of x_2 and $U_1 \cap U_2 = \phi$. $U_1 \times Y$ and $U_2 \times Y$ are neighbourhoods of (x_1, y_1) and (x_2, y_2) respectively. $(U_1 \times Y) \cap (U_2 \times Y) = (U_1 \cap U_2) \times Y = \phi$. Hence $X \times Y$ is Hausdorff.

6. (pg 100, 13) Let X be Hausdorff. Let $(x, y) \in \Delta$ implies $x \neq y$. Hence there exist disjoint neighbourhoods U and V of x and y respectively.

Now $(U \times V) \cap \Delta = \phi$ as any point in the intersection has same co-ordinates and hence lines in $U \cap V$ which is empty. Hence $(x, y) \in U \times V \subset X - \Delta$ which implies $X - \Delta$ is open and hence Δ is closed. Conversely, let Δ be closed. Let $x \neq y$. So $(x, y) \in X - \Delta$ which is open and hence there exist a basis element such that $(x, y) \in U \times V \subset X - \Delta$. This means $U \cap V = \phi$ as any point in the intersection will give a point in $U \times V$ which has same co-ordinates and hence in Δ. Hence X is Hausdorff.

7. (pg 100, 14) Let $x \in \mathbb{R}$ and let U be any neighbourhood of x in the finite complement topology on $\mathbb{R}$. Since $\mathbb{R} - U$ is finite it can contain only finitely many points of the sequence $\{1/n\}$ i.e. there exists N such that $1/n \in U$ for all $n > N$. This means that $1/n \to x$. So the sequence converges to every point of $\mathbb{R}$.

8. (pg 100, 20) (Find only boundary). $Bd(A) = A$. $Bd(B) = \{(x, y)|x = 0\} \cup \{(x, y)|y = 0, x \geq 0\}$. $Bd(C) = \{(x, y)|x = 0\} \cup \{(x, y)|y = 0, x \leq 0\}$. $Bd(D) = \mathbb{R}$. $Bd(E) = \{(x, y)|x^2 - y^2 = 0\} \cup \{(x, y)|x^2 - y^2 = 1\}$. $Bd(F) = \{\{(x, y)|xy = 0\} \cup \{(x, y)|xy = 1\}$.