Solutions to Homework 4

Topology MA 434/542 Fall 2006
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Please report any typos or mistakes to me.

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1. (pg 91, 1) $A \subset Y \subset X$. Let $\mathcal{T}_{A_X}$ and $\mathcal{T}_{A_Y}$ be the subspace topologies on A viewed as a subspace of X and a subspace of Y respectively. Let $U \in \mathcal{T}_{A_X}$, then $U = U_1 \cap A$ where U_1 is open in X. $U = U_1 \cap A = U_1 \cap (Y \cap A) = (U_1 \cap Y) \cap A$. Since $U_1 \cap Y$ is open in Y, this implies $U \in \mathcal{T}_{A_Y}$. Hence $\mathcal{T}_{A_X} \subset \mathcal{T}_{A_Y}$. Conversely, let $U \in \mathcal{T}_{A_Y}$. Then $U = U_1 \cap A$ where U_1 is open in Y. This means $U_1 = U_2 \cap X$ where U_2 is open in X. Hence $U = (U_2 \cap Y) \cap A = U_2 \cap (Y \cap A) = U_2 \cap A$. This implies $U \in \mathcal{T}_{A_X}$. Hence $\mathcal{T}_{A_Y} \subset \mathcal{T}_{A_X}$. Hence the topologies are the same.

2. (pg 91, 2) May or may not be finer. For example let $X = \mathbb{R}$ and $Y = \mathbb{Z}$. Then the subspace topologies induced on $\mathbb{Z}$ by $\mathbb{R}$ with the standard topology and $\mathbb{R}_l$ (lower limit topology) is the discrete topology on $\mathbb{Z}$ because $\{n\} = (n - 1/2, n + 1/2) \cap \mathbb{Z} = [n - 1/2, n + 1/2) \cap \mathbb{Z}$.

3. (pg 91, 3) $A = (-1, -1/2) \cup (1/2, 1)$ open,open. $B = [-1, -1/2) \cup (1/2, 1]$ open,not open. $C = (-1, -1/2] \cup [1/2, 1)$ not open, not open. $D = [-1, -1/2] \cup [1/2, 1]$ not open, not open (closed in both). $E = ((0, 1) \cup (-1, 0)) - \{1/n \mid 0 \notin n \in \mathbb{Z}\}$ open, open.

4. (pg 91, 4) Let $A \subset X \times Y$ be an open set. As the basis for the product topology is $\{U \times V \mid U \text{ open in } X,\ V \text{ open in } Y\ \}$, $A = \cup_\alpha U_\alpha \times V_\alpha$. $\pi_1(A) = \pi_1(\cup_\alpha U_\alpha \times V_\alpha) = \cup_\alpha \pi_1(U_\alpha \times V_\alpha) = \cup_\alpha U_\alpha$ which is a union of open sets and hence open. Hence π_1 is an open map. A similar argument works for π_2.

5. (pg 91, 6) Let $A \subset \mathbb{R}^2$ be open and let $(x, y) \in A \subset \mathbb{R}^2$. Since $\{(a, b) \times (c, d) \mid a, b, c, d \in \mathbb{R}\ \}$ is a basis for the topology on $\mathbb{R}^2$, there exist some $a, b, c, d \in \mathbb{R}$ such that $(x, y) \in (a, b) \times (c, d) \subset A$. The denseness of $\mathbb{Q}$ in $\mathbb{R}$ gives $a', b', c', d' \in \mathbb{Q}$ such that $(x, y) \in (a', b') \times (c', d') \subset (a, b) \times (c, d) \subset A$ (see Problem 8, pg 83). Hence by Lemma 13.2 the given set is a basis.