Solutions to Homework 3

Topology MA 434/542 Fall 2006
Abhijit Champanerkar

Please report any typos or mistakes to me.

---

1. (pg 51, 3) Same argument as in Problem 6a on pg 44. Let $\mathbf{x} \in \{0,1\}^\omega$ i.e. an infinite sequence $\{x_n\}$ of 0's and 1's. Define $f : \{0,1\}^\omega \to \mathcal{P}(A)$ by $f(\mathbf{x}) = B$ where $i \in B$ if $x_i = 1$ and $i \notin B$ if $x_i = 0$. Define $g : \mathcal{P}(A) \to \{0,1\}^\omega$ by $g(B) = \mathbf{x}$ where $x_i = 1$ if $i \in B$ and $x_i = 0$ if $a_i \notin B$. It is easy to check that f and g are inverses of each other and hence bijections.

2. (pg 51, 5a) Define $F : A \to \mathbb{Z}_+ \times \mathbb{Z}_+$ by $F(f) = (f(0), f(1))$ and define $G : \mathbb{Z}_+ \times \mathbb{Z}_+ \to A$ by $G(m,n) = f$ where $f(0) = m, f(1) = n$. It is easy to check that F and G are inverses of each other and hence bijections. Since $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable (finite product of countable sets), A is countable.

3. (pg 51, 5f) Let $F_n = \{f : \mathbb{Z}_+ \to \{0,1\} \mid f(m) = 0 \ \forall \ m > n \ \}$. Then $F_n \cong \{f : \{1,\ldots,n\} \to \{0,1\}\} \cong \{0,1\}^n$ and hence F_n is finite. $F = \cup_{n=1}^\infty F_n$ which is a countable collection of finite (countable) sets and hence countable.

4. (pg 51, 5h) Let $H_{nm} = \{f : \mathbb{Z}_+ \to \mathbb{Z}_+ \mid f(k) = m \ \forall \ k > n \ \}$. Then $H_{nm} \cong \{f : \{1,\ldots,n\} \to \mathbb{Z}_+\} \cong \mathbb{Z}_+ \times \ldots \times \mathbb{Z}_+ (\ n \text{ times })$ and hence H_{nm} is countable (finite product of countable sets). $H = \cup_{n=1}^\infty \cup_{m=1}^\infty H_{nm}$ which is a countable union (as $\mathbb{Z}_+ \times \mathbb{Z}_+$ is countable) of countable sets and hence countable.

5. (pg 83, 1) By hypothesis, for each $x \in A$, $\exists$ open set U_x such that $x \in U_x \subset A$. Hence $\cup_{x\in A} U_x \subset A$ and since $x \in \cup_{x\in A} U_x$ for every $x \in A$, $A \subset \cup_{x\in A} U_x$. Hence $A = \cup_{x\in A} U_x$ which is a union of open sets, hence open.

6. (pg 83, 3) 1) , X are open. 2) $X - \cup_\alpha U_\alpha = \cap_\alpha (X - U_\alpha) \subset X - U_\alpha$ for every α. Since U_α is open this implies $X - U_\alpha$ is countable which implies

that $X - \cup_\alpha U_\alpha$ is a subset of a countable set and hence countable. So $\cup_\alpha U_\alpha$ is open. 3) $X - \cap_i U_i = \cup_i (X - U_i)$ which is a finite union of countable sets, which is countable. Hence $\cap_i U_i$ is open. This shows $\mathcal{T}_c$ is a topology.
$\mathcal{T}_\infty$ is not a topology as condition 2 may not hold i.e. $U_1 \cup U_2$ may not be open e.g. let $X = \mathbb{Z}_+$ and let $U_1 = \{2n \mid n \in \mathbb{Z}_+ \}$ and $U_2 = \{2n+1 \mid n \in \mathbb{Z}_+ \}$ then both are open but $\mathbb{Z}_+ - (U_1 \cup U_2) = \{1\}$ which is finite and hence $U_1 \cup U_2$ is not open.

7. (pg 83, 7) $\mathcal{T}_1 \subset \mathcal{T}_2$, $\mathcal{T}_1 \subset \mathcal{T}_4$, $\mathcal{T}_3 \subset \mathcal{T}_1$, $\mathcal{T}_3 \subset \mathcal{T}_2$, $\mathcal{T}_3 \subset \mathcal{T}_4$, $\mathcal{T}_5 \subset \mathcal{T}_1$, $\mathcal{T}_5 \subset \mathcal{T}_3$, $\mathcal{T}_5 \subset \mathcal{T}_4$.

8. (pg 83, 8a) It follows from the Archimedean property that given any real numbers $x < y$, $\exists r \in \mathbb{Q}$ such that $x < r < y$. This is called the **denseness** of $\mathbb{Q}$ in $\mathbb{R}$. Hence $(a', b') \subset (a, b)$. Let U be a open set in $\mathbb{R}$ and $x \in U$, then there exists an open interval (a, b) such that $x \in (a, b) \subset U$. So $a < x < b$ and by denseness of $\mathbb{Q}$, $\exists\ a', b' \in \mathbb{Q}$ such that $a < a' < x < b' < b$ i.e. $x \in (a', b') \subset (a, b) \subset U$ and $(a', b') \in \mathcal{B}$. By Lemma 13.2, $\mathcal{B}$ is a basis of $\mathbb{R}$.

9. (pg 83, 8b) The set $[\sqrt{2}, 2)$ is open in $\mathbb{R}_l$. As $\sqrt{2}$ is irrational, there does not exist any $[a, b) \in \mathcal{C}$ such that $\sqrt{2} \in [a, b) \subset [\sqrt{2}, 2)$.