HW13 Solutions

pg 335: 1, 3, 4, 5.

1a. 

1b. Use straight line homotopy with base point a_0.

$[r] \in \pi_1(A, a_0)$

$r \simeq e_{a_0}$ as $F(s,t) = (1-t)a_0 + tr(s)$

F is homotopy betn r & e_{a_0}.

$\Rightarrow [r] = [e_{a_0}]$

$\Rightarrow \pi_1(A, a_0) = [e_{a_0}] \Rightarrow$ A is simply connected

3. Let $[f], [g] \in \pi_1(X, x_0)$

Let f & e_{x_0} be two paths at x_0.

Then $\hat{f} = \hat{e}_{x_0} = id_{\pi_1(X, x_0)}$

$$\hat{f}([g]) = [f] * [g] * [f] = [g] = \hat{e}_{x_0}([g])$$

$$\Rightarrow \quad ([f] * \bar{[f]}) * [g] * [f] = [f] * [g]$$

$$\Rightarrow \quad [e_{x_0}] * [g] * [f] = [f] * [g].$$

$$\Rightarrow \quad [g] * [f] = [f] * [g]$$

$\Rightarrow \pi_1(X, x_0)$ abelian.

Conversely, consider $\alpha * \bar{\beta}$ is a loop at x_0.

$\pi_1(X, x_0)$ abelian $\Rightarrow$ for any $[f] \in \pi_1(X, x_0)$

$$[\alpha * \bar{\beta}] * [f] = [f] * [\alpha * \bar{\beta}]$$

$$\Rightarrow [\alpha] * [\bar{\beta}] * [f] = [f] * [\alpha] * [\bar{\beta}]$$

$$\Rightarrow [\bar{\beta}] * [f] = [\bar{\alpha}] * [f] * [\alpha] * [\bar{\beta}]$$

Multiply both sides on left by $[\bar{\alpha}]$

Multiply both sides on right by $[\beta]$

$$\Rightarrow [\bar{\beta}] * [f] * [\beta] = [\bar{\alpha}] * [f] * [\alpha].$$

$$\Rightarrow \hat{\beta}([f]) = \hat{\alpha}([f])$$

$$\Rightarrow \hat{\alpha} = \hat{\beta}$$

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4. let $j: A \to X$ be inclusion

$$j(a) = a.$$

$$A \xrightarrow{j} X \xrightarrow{r} A \qquad r \circ j = id_A$$

$$\Rightarrow r_* \circ j_* = (r \circ j)_* = (id_A)_* = id_{\pi_1(A, a_0)}.$$

Let $[f] \in \pi_1(A, a_0)$.

$$j_*([f]) \in \pi_1(X, a_0)$$

$$r_*(j_*([f])) = id_*([f]) = [f]$$

$\Rightarrow r_*$ is surjective.

5. Let $\tilde{h}:(\mathbb{R}^n, a_0) \to (Y, y_0)$ be extention

$j:(A, a_0) \to (\mathbb{R}^n, a_0)$ be inclusion.

$$A \xrightarrow{j} \mathbb{R} \xrightarrow{\tilde{h}} Y \qquad \tilde{h} \circ j = h.$$

$$(\tilde{h} \cdot j)_* = h_*.$$

$$\tilde{h}_* \cdot j_* = h_*.$$

$$h_*([f]) = \tilde{h}_*(j_*([f]))$$

As $\pi_1(\mathbb{R}^n, a_0) = 0$

$\Rightarrow j_*([f]) = 0.$

$\Rightarrow \tilde{h}_*(0) = 0$

$\Rightarrow h_*([f]) = 0.$
