Solutions to Homework 12

Topology MA 434/542 Fall 2006
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Please report any typos or mistakes to me.

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1. (pg 330, 2a) Since I is convex we can use straight line homotopy. Let $f : X \to I$ be a continuous map. Let $F : I \times I \to X$ be defined as $F(x,t) = tf(x)$. F is continous as f and multiplication is continuous. $F(x,0) = 0$ and $F(x,1) = f(x)$. This shows that every continuous map is homotopic to the constant map $z(x) \equiv 0$. Hence $[f] = [z]$ for all $[f] \in [X, I]$. Hence $[X, I]$ has a single element.

2. (pg 330, 2b) Let $y_0 \in Y$. We have to show every path in Y is homotopic to e_{y_0}. We will prove this in two steps. First we will show that any path $f : I \to Y$ is homotopic to the constant path $e_{f(0)}$. Let $F : I \times I \to Y$ be defined as $F(s,t) = f(st)$. Since f is coninuous, F is continuous, $F(s,0) = f(0)$ and $F(s,1) = f(s)$. F is a homotopy between f and $e_{f(0)}$. Note that F is not a path homotopy as $F(1,t) = f(t) \neq f(0)$. Secondly we will show that any two constant paths are homotopic. Let e_{y_1} and e_{y_2} be constant paths. Let $g : I \to Y$ be the path from y_1 to y_2. As Y is path connected, there is such a path. Define homotopy $G : I \times I \to Y$ as $G(s,t) = g(t)$. G is continuous, $G(s,0) = g(0) = e_{y_1}(t)$ and $G(s,1) = g(1) = e_{y_2}(t)$. Hence G is a homotopy. Since homotopy is an equivalence relation, all paths are homotopic to e_{y_0} for some fixed $y_0 \in Y$. This shows that $[I, Y]$ has a single element. Since homotopy is a weaker relation than path homotopy, this does not imply that Y has trivial fundamental group.

3. (pg 330, 3a) We use convexity of I and $\mathbb{R}$. Define $F : I \times I \to I$ as $F(s,t) = st$. F is continuous, $F(s,0) = 0$ and $F(s,1) = s = id_{[0,1]}(s)$. F is homotopy between identity map and the constant map, showing I is contractible. The argument is similar for $\mathbb{R}$.

4. (pg 330, 3b) It is enough to show that there is a path from any point in Y to a fixed point y_0. Any two points then can be joined through

y_0. Let $F : Y \times I \to Y$ be the homotopy between the constant map and the id_Y. Let $F(y, 0) = y_0$. Let $y_1 \in Y$. Define $f : I \to Y$ by $f(s) = F(y_1, s)$. Then f is the required path from y_0 to y_1.

5. (pg 330, 3c) Let $F : Y \times I \to Y$ be the homotopy between the constant map and the id_Y and let $F(y, 0) = y_0$. We will show that every continuous map $f : X \to Y$ is homotopic to the constant map $c(x) = y_0$. Let $f : X \to Y$ be continuous map. Define $G : X \times I \to Y$ as $G(x, t) = F(f(x), t)$. G is continuous as f is continuous. $G(x, 0) = F(f(x), 0) = y_0 = c(x)$ and $G(x, 1) = F(f(x), 1) = f(x)$. Hence $[X, Y]$ has a single element.

6. (pg 330, 3c) We proceed as in problem 2a. Let $F : X \times I \to X$ be the homotopy between the constant map and the id_X and let $F(x, 0) = x_0$. First we will show every continuous function$f : X \to Y$ is homotopic to the constant function $c : X \to Y$ defined as $c(x) = f(x_0)$. Define the homotopy $G : X \times I \to Y$ as $G(x, t) = f(F(x, t))$. G is continuous as both f and F are continuous. $G(x, 0) = f(F(x, 0)) = f(x_0) = c(x)$ and $G(x, 1) = f(F(x, 1)) = f(x)$. Secondly we will show that any two constant functions are homotopic. We will use the path connectedness of Y for this. Let c_1 and c_2 be two constant functions from X to Y. Let $c_1(x) = y_1$ and $c_2(x) = y_2$ for all x. Let $h : I \to Y$ be a path from y_1 and y_2. Define homotopy $H : X \times I \to Y$ as $H(x, t) = h(t)$. Then H is continuous, $H(x, 0) = h(0) = y_1 = c_1(x)$ and $H(x, 1) = h(1) = y_2 = c_2(x)$. Since homotopy is an equivalence relation, all functions are homotopic to some constant function. This shows that $[X, Y]$ has a single element.