HW11 : Quotient Spaces
Solutions

(1) $f: \mathbb{R}^2 \to \mathbb{R}$ $\quad f(x,y) = x-y$

f surj & cts.

f is an open map as $f(B_\varepsilon(\bar{0})) = (-\varepsilon\sqrt{2}, \varepsilon\sqrt{2})$ (wh

& $f(B_\varepsilon(x,y)) = ((x-y)-\varepsilon\sqrt{2}, (x-y)+\varepsilon\sqrt{2})$

$\{f^{-1}(c) \mid c \in \mathbb{R}\} = \mathcal{P}$

So by Thm $\mathbb{R}^2/_{\mathcal{N}}$ is homeomorphic to $\mathbb{R}$ ∴

(2) a) $X =$  A , $X/A =$ 

b) $X =$  A , $X/A =$ 

(3) $[0,1] \times [0,1] \xrightarrow{F} \mathbb{R}^3$ $\quad$ a) $[0,1] \times [0,1]$ compact,

$\mathbb{R}^3$ Hausdorff

F cts & surj onto its image

$\Rightarrow$ by Thm F is quotient map

$\downarrow p$

T^2

b) & c).

$F(0,t) = (b + a\cos 2\pi t, 0, a\sin 2\pi t) = F(1,t) \quad 0 \le t \le 1$

& $F(s,0) = (\cancel{b\cos}(b+a)\cos 2\pi s, (b+a)\sin 2\pi s, 0) = F(s,1)$ $\quad 0 \le s \le 1$

& $F(s,t) \neq F(s_1,t_1)$ if (s,t) & $(s_1,t_1) \in (0,1) \times (0,1)$

$\Rightarrow \{ F^{-1}(x,y,z) / (x,y,z) \in \text{Image}(F) \}$ = partition for T^2 (2)

$\Rightarrow$ By Thm, F factors to map from T^2 to $\mathbb{R}^3$ & is a homeomorphism onto its image.

4) Direct computation.

5) a) As $\bar{x} \neq N$, $x_{n+1} \neq 1 \Rightarrow f$ is cts as every component is cts.
g is cts as every component is cts.
It can be checked that they are inverses of each other by direct computation.

b) D^n compact & $S^n \subset \mathbb{R}^{n+1}$ Hausdorff

1) F is cts: g & h are cts $\Rightarrow$ F is cts on $|\bar{x}|<1$
let U be a nbd of N in S^n of the form
$F^{-1}(N) = \{ \bar{x} \in \mathbb{R}^n / B_\varepsilon(N) \cap S^n \} \cap D^n$ which is open.



2) F surj
$\Rightarrow F$ is a quotient map.



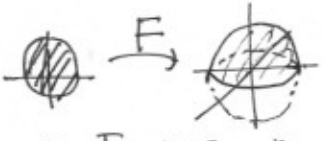
c) As g & h are homeomorphisms
$F(\bar{x}) \neq F(\bar{y})$ for $\bar{x}, \bar{y} \in \text{Interior}(D^n)$ = open ball
$F(\bar{x}) = F(\bar{y})$ $\forall$ $\bar{x}, \bar{y} \in S^{n-1}$ = boundary of D^n
$\Rightarrow \{ F^{-1}(\bar{x}) / \bar{x} \in S^n \} = \{ \{\bar{x}\} / |\bar{x}|<1 \} \cup \{ S^{n-1} \}$

Thm $\Rightarrow$ D^n/S^{n-1} is homeomorphic to S^n. (:

6) a) We will show (1) $S^n/\sim$ homeomorphic to $D^n/\sim$
& (2) $S^n/\sim$ — // — to $\mathbb{R}^{n+1}/\sim$

(1) Let $F: D^n \to S^n$ be defined as

$F(x_1, \ldots x_n) = (x_1, \ldots x_n, \sqrt{1 - \sum_{i=1}^{n} x_i^2})$

F maps D^n as upper hemisphere 

Let $p: S^n \to S^n/\sim$ & $G = p \circ F$ $\quad D^n \xrightarrow{F} S^n \xrightarrow{p} S^n/\sim$

$\{\bar{G}^{-1}(\bar{x}) / \bar{x} \in \text{Image}(G)\} = \{\{\bar{x} / |\bar{x}| < 1\} \cup \{\{\bar{x}, -\bar{x}\} / |\bar{x}| = 1\}$

= partition in second description (verify this) use defn of F above

$D^n \xrightarrow{G} S^n/\sim$ G cts & surj (why?)

D^n compact & $S^n/\sim$ Hausdorff (verify)

$\Rightarrow$ G quotient map.

By Thm $D^n/\sim$ is homeomorphic to $S^n/\sim$.

(2) Let $i: S^n \to \mathbb{R}^{n+1} - \{0\}$ be inclusion map
i.e. $i(\bar{x}) = \bar{x}$ &

Let $p: \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}^{n+1} - \{0\}/\sim$.

Let $F: S^n \to \mathbb{R}^{n+1} - \{0\}/\sim$ be $F = p \circ i$.

F cts as p & i are cts

Also $\bar{x} \sim \frac{\bar{x}}{\|\bar{x}\|}$ & $\frac{\bar{x}}{\|\bar{x}\|} \in S^n$

$\Rightarrow$ F is surjective.

S^n compact & $\mathbb{R}^{n+1}$ Hausdorff

Thm $\Rightarrow$ F is quotient map.

$$\left\{ F^{-1}(C[\bar{x}]) \;\middle|\; [\bar{x}] \in \mathbb{R}^{n+1} - \{\bar{0}\}/\sim \right\} = \left\{ \{\bar{x}, -\bar{x}\} \;\middle|\; \bar{x} \in S^n \right\}$$

as if $\bar{x}, \bar{y} \in F^{-1}[\bar{x}]$ & $\bar{x}, \bar{y} \in S^n$

$\Rightarrow \bar{x} \sim \bar{y} \Rightarrow \bar{x} = \alpha \bar{y}$

$\Rightarrow \|\bar{x}\| = |\alpha| \|\bar{y}\|$

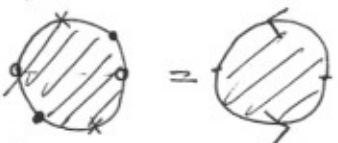
$\Rightarrow |\alpha| = 1$ as $\|\bar{x}\| = \|\bar{y}\| = 1$

$\Rightarrow \alpha = \pm 1$

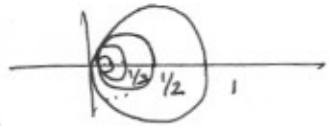
$\Rightarrow \bar{y} = \pm \bar{x}$.

partition is same as in first description,

Thm $\Rightarrow$ $S^n/\sim$ is homeomorphic to $\mathbb{R}^{n+1} - \{\bar{0}\}/\sim$.

b) $\mathbb{RP}^2 = D^2/\sim$ =  = D^2

diametrically opposite points on boundary of D^2 are identified.

7) Hawaiian earring $X =$ 

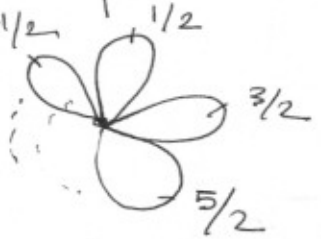
$\subset \mathbb{R}^2$

X is bounded & closed (why?)

$\Rightarrow X$ is compact.

$Y = \mathbb{R}/\mathbb{Z} =$ all integers go to one point & $(n, n+1)$ remain as they are

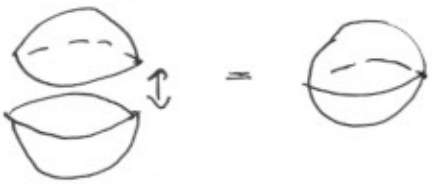
= infinite petaled rose

$\cong$ 

The infinite open cover $\mathcal{A} = \{(n, n+1)\} \cup \{(n-1/2, n)\}_{n \in \mathbb{Z}}$ does not have a finite subcover as $(n, n+1)$ are the petals of the rose & there are infinitely many of them.

$\Rightarrow Y$ not compact.

$\Rightarrow X$ & Y cannot be homeomorphic.

8) a)  $= S^2$

b) 

$= P^2$

c)  = double torus.