HW10 Solutions

pg 170: 2a, 3, 4, 5, 7
pg 177: 2abcde.

2a. $A \subseteq \mathbb{R}$ & let $\mathcal{B}$ be an open cover of A.

let $B \in \mathcal{B}$. B open $\Rightarrow$ $\mathbb{R}-B$ is finite.

Let $A \cap (\mathbb{R}-B) = \{x_1, \ldots x_n\}$. As $\mathcal{B}$ is open cover of A

$\exists\ B_1, \ldots B_n \in \mathcal{B}$ s.t. $x_i \in B_i\ \forall\ i = 1, \ldots n$

So $A \subset B \cup B_1 \cup \cdots \cup B_n$

$\Rightarrow \exists$ finite subcover of A.

$\Rightarrow A$ is compact. ___.

3. Let $A_1, \ldots, A_n$ be compact subsp. of X.

Let $A = A_1 \cup \ldots \cup A_n$.

Let $\mathcal{B}$ be an open cover of A.

Then $\mathcal{B}$ is an open cover of $A_i\ \forall\ i = 1, \ldots n$

A_i compt $\Rightarrow \exists$ finite subcover say

$$A_i \subset B_{i1} \cup \ldots \cup B_{in_i} \quad i = 1, \ldots n$$

Then $A = A_1 \cup \ldots \cup A_n \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} B_{ij}$

which is a finite subcover of A

$\Rightarrow A$ is compact.

___.

4. Let (X,d) be a metric sp & let $A \subset X$ be cmpt. Let $x_0 \in X$ (2)

Let $\mathcal{B} = \{ B_d(x_0, n) / n \in \mathbb{Z}_+ \}$

For any $a \in A$, $d(x_0, a) < N$ for some $N \in \mathbb{Z}_+$

$\Rightarrow a \in B_d(x_0, N)$

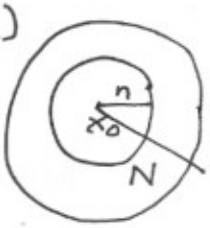
$\Rightarrow A \subset \bigcup_{n \in \mathbb{Z}_+} B_d(x_0, n)$

$\Rightarrow \mathcal{B}$ is an open cover of A

A cmpt $\Rightarrow \exists$ a finite subcover say

$A \subset B_d(x_0, n_1) \cup \dots \cup B_d(x_0, n_k)$

Let $N = \max\{n_1, \dots n_k\}$



$B_d(x_0, n_i) \subset B_d(x_0, N)$

So $A \subset B_d(x_0, N)$

For any $a, b \in A$

$d(a,b) \leq d(a, x_0) + d(x_0, b) \leq N + N = 2N$

$\Rightarrow A$ is bounded.

Let $X = (0,1) \subset \mathbb{R}$

$d(x,y) = |x-y| \leq 1 \Rightarrow (0, \frac{1}{2}]$ is bounded & closed. (in $(0,1)$)

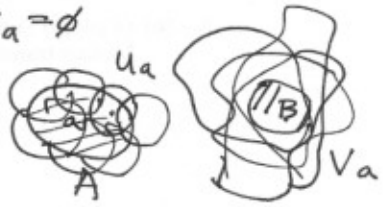
But $(0, \frac{1}{2}]$ is not compact.

5. Use Thm 26.3.

For $a \in A$, Proof of Thm 26.3 $\exists$

an open set V_a & U_a s.t.

$a \in U_a$, $B \subset V_a$ & $U_a \cap V_a = \emptyset$



Let $\mathcal{A} = \{U_a \mid a \in A\}$

$\mathcal{A}$ is an open cover of A

A compact $\Rightarrow$ $\exists$ finite subcover

$A \subset U_{a_1} \cup \cdots \cup U_{a_n}$.

The corresponding V_{a_i} s.t. $U_{a_i} \cap V_{a_i} = \emptyset$

& $B \subset V_{a_i}$ $\forall i = 1, \cdots, n$.

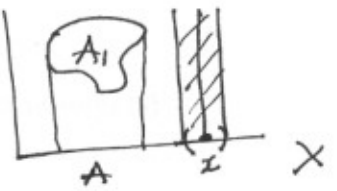
Let $V = V_{a_1} \cap \cdots \cap V_{a_n}$ open

& let $U = U_{a_1} \cup \cdots \cup U_{a_n}$ open

$A \subset U$ & $B \subset V$

& $U \cap V = \emptyset$ as $U_{a_i} \cap V_{a_i} = \emptyset$ $\forall i = 1, \cdots n$.

7. Use Tube Lemma.



Let $A_1 \subset X \times Y$ be closed

Let $\pi_1(A_1) = A$

To show that A is closed in X. Let $x \in X - A$

as $x \notin A = \pi_1(A_1) \rightarrow \{x\} \times Y \cap A_1 = \emptyset$

$\Rightarrow \{x\} \times Y \subset X \times Y - A_1$ which is open as A_1 closed

By Tube Lemma $\exists$ open nbd of x say U s.t.

pg 177
2a). Suppose not. Say $d(x,A) = \varepsilon > 0$
$\Rightarrow B_d(x,\varepsilon) \cap A = \emptyset \Rightarrow x \notin \overline{A}$
(recall $x \in \overline{A}$ iff every open nbd of x intersects A).
Conversely if $x \in \overline{A} \Rightarrow B_d(x,1/n) \cap A \neq \emptyset \ \forall n \in \mathbb{Z}^+$
$\Rightarrow d(x,A) \leq 1/n \ \forall n \in \mathbb{Z}^+ \Rightarrow d(x,A) = 0$.

b) ~~pg 175 ⇒ $f: A \to \mathbb{R}$, $f(x) = d(x,A)$ is cts~~
Fix $x \in X$, pg 175 $\Rightarrow f: A \to \mathbb{R}$, $f(a) = d(x,a)$
is cts fn. A compact $\Rightarrow \exists$ abs min
i.e. $\exists\ a \in A$ s.t. $d(x,a) \leq d(x,b) \ \forall b \in A$
$\Rightarrow d(x,a) \leq \inf d(x,A) \leq d(x,a)$
$\Rightarrow d(x,a) = d(x,A)$.

c) Show that $U(A,\varepsilon) = \bigcup_{a \in A} B_d(a,\varepsilon)$
$x \in U(A,\varepsilon)$, $d(x,A) < \varepsilon$ if $x \notin \bigcup_{a \in A} B_d(a,\varepsilon)$
$\Rightarrow d(x,a) > \varepsilon \ \forall a \in A \Rightarrow d(x,A) > 1/\varepsilon \ \rightarrow\leftarrow$
$\Rightarrow x \in B_d(a,\varepsilon)$ for some $a \Rightarrow U(A,\varepsilon) \subset \bigcup_{a \in A} B_d(a,\varepsilon)$.
Converse is easy.

d) Use Lebesgue Number Lemma on pg 175.
A cmpt & $A \subset U$ open, $a \in A \subset U$, $\exists\ \varepsilon_a$ s.t.
$B_d(a,\varepsilon_a) \subset U$, $\mathcal{B} = \{B_d(a,\varepsilon_a) / a \in A\}$ is
open cover of A. A cmpt, Lebesgue Number
Lemma $\Rightarrow \exists\ \varepsilon$ s.t. $B_d(a,\varepsilon) \subset B_d(a',\varepsilon_{a'})$ for some a'
$\subset U \ \forall a \in$
$\Rightarrow A \subset \bigcup_{a \in A} B_d(a,\varepsilon) = U(A,\varepsilon) \subset U$
(by c))

e) Let $A = \{(x,1/x) / x > 0\}$, $U = \{(x,y) / y > 0\}$.