MAT 5011 (2008-09) Real Analysis

Suggested Answer to Assignment 1

• Page 31, Problem 1. Does there exist an infinite σ-algebra which has only countably many members?

Answer and Proof: No. Suppose that $\mathfrak{M}$ is an infinite σ-algebra on X and $\mathfrak{M}$ has only countably many members. Then by the definition of σ-algebra, for all $x \in X$, $A_x \equiv \bigcap_{x \in A \in \mathfrak{M}} A \in \mathfrak{M}$. Set $\mathfrak{N} \equiv \{A_x : x \in X\}$ and define $f : \mathfrak{M} \to 2^{\mathfrak{N}}$ by

$$f(B) = \{A_x : x \in B\}.$$

Claim: f is bijective.

Proof of Claim: Note that $\mathfrak{N} \subset \mathfrak{M}$, so $\mathfrak{N}$ is at most countable. Hence for any subset $\{A_x : x \in I\}$ of $\mathfrak{N}$, $\bigcup_{x \in I} A_x \in \mathfrak{M}$, we have $f(\bigcup_{x \in I} A_x) = \{A_x : x \in I\}$. Thus f is surjective. For B_1 and B_2 in $\mathfrak{M}$, $x \in B_1 - B_2$ implies $A_x \subset B_1 - B_2$, hence f is injective.

Therefore, if $\mathfrak{N}$ has n elements, then $\mathfrak{M}$ has 2^n elements. But if $\mathfrak{N}$ is countable, then $\mathfrak{M}$ is uncountable since f is bijective. Thus we complete the proof. □

• Page 31, Problem 3. Prove that if f is a real function on a measurable space X such that $\{x : f(x) \geq r\}$ is measurable for every rational r, then f is measurable.

Proof: Similar to the proof of (c) in Theorem 1.12, we only need to prove $E_a = \{x : f(x) > a\}$ is measurable, for all $a \in \mathbb{R}$. Note that

$$\begin{aligned} E_a &= \{x : f(x) > a\} = f^{-1}((a, +\infty)) \\ &= f^{-1}(\bigcup_{n=1}^{\infty} [r_n, +\infty)) = \bigcup_{n=1}^{\infty} f^{-1}([r_n, +\infty)) \end{aligned}$$

where $\{r_n\} \subseteq \mathbb{Q}$, $r_n > a$ and $r_n \to a$ as $n \to \infty$, we have the desired result. □

• Page 32, Problem 5. (a) Suppose $f : X \to [-\infty, \infty]$ and $g : X \to [-\infty, \infty]$ are measurable. Prove that the sets

$$\{x : f(x) < g(x)\}, \{x : f(x) = g(x)\}$$

are measurable.

(b) Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

Proof: (a) Note that

$$\{x : f(x) < g(x)\} = \bigcup_{r\in\mathbb{Q}} \{x : f(x) < r \le g(x)\} = \bigcup_{r\in\mathbb{Q}} (\{x : f(x) < r\} \cap \{x : g(x) \ge r\}),$$

and

$$\{x : f(x) = g(x)\} = \Big(\{x : f(x) < g(x)\} \bigcup \{x : g(x) < f(x)\}\Big)^c.$$

Thus both $\{x : f(x) < g(x)\}$ and $\{x : f(x) = g(x)\}$ are measurable.

(b) Let f_n be a sequence of real measurable functions. Then $g = \limsup_{n\to\infty} f_n$ and $h = \liminf_{n\to\infty} f_n$ are measurable. Recall that a sequence of real numbers (a_n) converges if and only if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$, hence

$$\{x : \ f_n(x) \text{ converges (to a finite limit)}\} = \{x : \ g(x) = h(x) \ (|\, g(x)| < \infty)\}.$$

Therefore, we completes the proof by the result in (a). □

• Page 32, Problem 6. Let X be an uncountable set, let $\mathfrak{M}$ be the collection of all sets $E \subset X$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that $\mathfrak{M}$ is a σ-algebra in X and that μ is a measure on $\mathfrak{M}$. Describe the corresponding measurable functions and their integrals.

Proof: Let $\mathfrak{F} \equiv \{A \subset X : A \text{ is at most countable}\}$,
$\mathfrak{G} \equiv \{A \subset X : A^c \text{ is at most countable}\}$. Since X is uncountable, $\mathfrak{F}$ and $\mathfrak{G}$ are disjoint, hence $\{\mathfrak{F}, \mathfrak{G}\}$ form a partition of $\mathfrak{M}$.

Now we show that $\mathfrak{M}$ is a σ-algebra. Obviously, $\emptyset \in \mathfrak{F}$ and $X \in \mathfrak{G}$.

For $A \in \mathfrak{M}$, then $A \in \mathfrak{F}$ or $A \in \mathfrak{G}$, then $A^c \in \mathfrak{G}$ in the first case and $A^c \in \mathfrak{F}$ in the second. Hence $A^c \in \mathfrak{M}$.

Let $A_n \in \mathfrak{M}, n = 1, 2, ...$ and $A = \bigcup_{n=1}^{\infty} A_n$. If A_n is at most countable for all n, then A is at most countable, hence $A \in \mathfrak{F}$. If $A_{n_0} \in \mathfrak{G}$ for some n_0, then $A^c = \bigcap_{n=1}^{\infty} A_n^c \subset A_{n_0}^c$, with $A_{n_0}^c \in \mathfrak{F}$, hence $A^c \in \mathfrak{F}$ and $A \in \mathfrak{G}$. In other words, $A \in \mathfrak{M}$. Hence $\mathfrak{M}$ is a

σ-algebra.

To show μ is a measure we need only prove that μ is countably additive.

Let $A_n \in \mathfrak{M}, \quad n = 1, 2, ..., A_i \bigcap A_j = \emptyset$ if $i \neq j$ and $A = \bigcup_{n=1}^{\infty} A_n$.

If A_n is at most countable for all n, then A is at most countable. By definition of μ we get for any n, $\mu(A_n) = 0$ and $\mu(A) = 0$, hence $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$.

If $A_{n_0} \in \mathfrak{G}$ for some n_0, from the above discussion we know $A \in \mathfrak{G}$. Since for any $m \neq n_0, A_m \bigcap A_{n_0} = \emptyset$, i.e. $A_m \subset A_{n_0}^c$ and then $A_m \in \mathfrak{F}$. By definition of μ, we get $\mu(A_{n_0}) = 1, \mu(A_m) = 0, m \neq n_0$ and $\mu(A) = 1$. Hence $\mu(A) = \sum_{n=1}^{\infty} \mu(A_n)$.

Thus μ is a measure.

Let $f : \mathfrak{M} \to [-\infty, +\infty]$ be a measurable function.

Claim: There is exact one value α_0 such that $f^{-1}(\{\alpha_0\}) \in \mathfrak{G}$. Then $f = \alpha_0$ a.e. and

$$\int_X f \, d\mu = \alpha_0.$$

Proof of the claim: First we prove the uniqueness. Suppose α_0 satisfies $f^{-1}(\{\alpha_0\}) \in \mathfrak{G}$, then by the definition of $\mathfrak{G}$, $\{x : f(x) \neq \alpha_0\}$ is at most countable. Hence there do not exist another such value. Now we prove the existence. Let $A \equiv \{a \in \mathbb{R} : f^{-1}((a, +\infty]) \in \mathfrak{G}\}$.

If $A = \emptyset$, then we have $f^{-1}((n, +\infty]) \in \mathfrak{G}, \forall \ n \ \in \mathbb{N}$. Hence $f^{-1}([-\infty, n]) \in \mathfrak{F}$, i.e., $f^{-1}([-\infty, n])$ is at most countable. Thus $f^{-1}([-\infty, +\infty))$ is at most countable. Note that X is uncountable, so $f^{-1}(+\infty) \in \mathfrak{G}$.

If $A \neq \emptyset$, then $\alpha_0 \equiv \inf A \in [-\infty, +\infty)$. And for each $n \in \mathbb{N}, f^{-1}((\alpha_0 - \frac{1}{n}, +\infty]) \in \mathfrak{G}$ and $f^{-1}((\alpha_0 + \frac{1}{n}, +\infty]) \in \mathfrak{F}$. Hence both $f^{-1}([-\infty, \alpha_0))$ and $f^{-1}((\alpha_0, +\infty])$ are at most countable. Note that X is uncountable, so $f^{-1}(\alpha_0) \in \mathfrak{G}$. $\square$

• Page 32, Problem 7. Suppose $f_n : X \to [0, \infty]$ is measurable for $n = 1, 2, 3, \ldots, f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0, f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that then

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

and show that this conclusion does not follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

Proof: Let $g_n = f_1 - f_n$, then g_n satisfy

(a) $0 \le g_1 \le g_2 \le g_3 \le \cdots$ on X,

(b) $g_n(x) \to f_1(x) - f(x)$ as $n \to \infty$, for every $x \in X$.

Using Lebegue's Monotone Convergence Theorem we get that

$$\lim_{n\to\infty} \int_X (f_1 - f_n) d\mu = \int_X (f_1 - f) d\mu.$$

Since $f \in L^1(\mu)$, we have

$$\int_X (f_1 - f_n) d\mu = \int_X f_1 d\mu - \int_X f_n d\mu, \quad \int_X (f_1 - f) d\mu = \int_X f_1 d\mu - \int_X f d\mu.$$

Then we get the desired result.

Next, we give an example to show that this conclusion does not follow if the condition "$f_1 \in L^1(\mu)$" is omitted. Let μ be the counting measure on the set $X \equiv \{1, 2, 3, \cdots\}$, Let $A_n \equiv \{n, n+1, n+2, \cdots\}$ and $f_n \equiv \chi_{A_n}$. Obviously, $f_n \to f \equiv 0$ as $n \to \infty$. But $\int_X f_n d\mu = \infty \neq \int_X f d\mu = 0$. $\square$

• Page 32, Problem 9. Suppose μ is a positive measure on X, $f : X \to [0, \infty]$ is measurable, $\int_X f \, d\mu = c$, where $0 < c < \infty$, and α is a constant. Prove that

$$\lim_{n\to\infty} \int_X n \log[1 + (\frac{f}{n})^\alpha] d\mu = \int_X \lim_{n\to\infty} g_n d\mu = \begin{cases} \infty, & 0 < \alpha < 1, \\ c, & \alpha = 1, \\ 0, & 1 < \alpha < \infty. \end{cases}$$

Proof: Let $g_n \equiv n \log[1 + (f/n)^\alpha]$. Then $g_n \ge 0$ and satisfy

$$\lim_{n\to\infty} g_n(x) = \begin{cases} \infty, & \text{if} \quad f(x) = \infty, \\ 0, & \text{if} \quad f(x) = 0, \\ \infty, & \text{if} \quad 0 < f(x) < \infty, \ 0 < \alpha < 1, \\ f(x), & \text{if} \quad 0 < f(x) < \infty, \ \alpha = 1, \\ 0, & \text{if} \quad 0 < f(x) < \infty, \ 1 < \alpha < \infty. \end{cases}$$

For $\alpha \ge 1$, by the inequality $1 + x^\alpha \le (1 + x)^\alpha, x \ge 0$ and $\log(1 + x) \le x, x \ge 0$, we have

$$g_n(x) \le n \log(1 + \frac{f}{n})^\alpha \le n\alpha \cdot \frac{f}{n} \le \alpha f$$

Since $\int_X f\ d\mu = c < \infty$, we have

$$\int\limits_{\{x:\ f(x)=\infty\}} f\ d\mu = 0.$$

Obviously, $\alpha f \in L^1(\mu)$. Using Lebesgue Dominated Convergence Theorem we have

$$\lim_{n\to\infty}\int_X n\log[1+(\frac{f}{n})^\alpha]d\mu = \int_X \lim_{n\to\infty} g_n d\mu = \begin{cases} 0, & \alpha > 1, \\ c, & \alpha = 1. \end{cases}$$

For $\alpha < 1$, Since $\int_X f\ d\mu = c < \infty$, we have

$$\mu\{x : f(x) > 0\} > 0.$$

By Fatou's Lemma we have

$$\liminf_{n\to\infty}\int_X g_n d\mu \geq \liminf_{n\to\infty}\int\limits_{\{x:\ f(x)>0\}} g_n d\mu \geq \int\limits_{\{x:\ f(x)>0\}} \liminf_{n\to\infty} g_n d\mu = \infty.$$

This completes the proof. □

• Extra Problem 1. Let $G \subset \mathbb{R}^n$ be an open set and $f : G \to \mathbb{R}$ be a real function. Show that the set $\{x \in G : f \text{ is continuous at } x\}$ is a G_δ set.

Proof: Let

$$\varphi(x,\delta) \equiv \sup\{|f(s) - f(t)| : s, t \in B_\delta(x)\},$$

and

$$\varphi(x) \equiv \inf\{\varphi(x,\delta) : \delta > 0\}.$$

We claim that $\varphi(x)$ is upper semicontinuous, i.e., for every real a, $E_a \equiv \{x : \varphi(x) < a\}$ is open. Let $x_0 \in E_a$, i.e., $\varphi(x_0) = \inf\{\varphi(x_0,\delta) : \delta > 0\} < a$, then there exists $\delta_0 > 0$ such that

$$\sup\{|f(s) - f(t)| : s, t \in B_\delta(x)\} < a.$$

Thus for any $x \in B_{\delta/2}(x_0)$,

$$\begin{aligned} \varphi(x, \frac{\delta_0}{2}) &= \sup\{|f(s) - f(t)| : s, t \in B_{\delta/2}(x)\} \\ &\leq \sup\{|f(s) - f(t)| : s, t \in B_\delta(x_0)\} \\ &< a. \end{aligned}$$

Hence $\varphi(x) < a$. Then E_a is open and $\varphi(x)$ is upper semicontinuous.

By definition of continuity, we know that f is continuous at x if and only if $\varphi(x) = 0$. Hence

$$\{x : \varphi(x) = 0\} = \{x : f \text{ is continuous at } \ x\}.$$

On the other hand

$$\{x : \varphi(x) = 0\} = \bigcap_{k=1}^{\infty} \{x : \varphi(x) < \frac{1}{k}\}.$$

Therefore, $\{x : f \text{ is continuous at } x\}$ is a G_δ set. $\square$

- Extra Problem 2. Let $f_n \in C([a,b])(n = 1, 2, \dots)$ and $\lim_n f_n(x) = f(x), x \in [a,b]$. Then for any $t \in \mathbb{R}$, the set $\{x \in [a,b] : f(x) < t\}$ is a F_σ set.

Proof: By the definition of limit, we have

$$\{x \in [0,1] : f(x) < t\} = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{x : f_k(x) \leq t - \frac{1}{n}\}.$$

Note that $\{x : f_k(x) \leq t - \frac{1}{n}\}$ is closed since $f_k \in C([a,b])$, we obtain the desired result. $\square$