MEASURE THEORY

Volume 4

D.H.Fremlin



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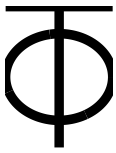
MEASURE THEORY

Volume 4

Topological Measure Spaces

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Contents

General Introduction 10

Introduction to Volume 4 11

Chapter 41: Topologies and measures I
Introduction 13
411 Definitions 13
Topological, inner regular, τ-additive, outer regular, locally finite, effectively locally finite, quasi-Radon, Radon, completion regular, Baire, Borel and strictly positive measures; measurable and almost continuous functions; self-supporting sets and supports of measures; Stone spaces; Dieudonné's measure.
412 Inner regularity 19
Exhaustion; Baire measures; Borel measures on metrizable spaces; completions and c.l.d. versions; complete locally determined spaces; inverse-measure-preserving functions; subspaces; indefinite-integral measures; products; outer regularity.
413 Inner measure constructions 31
Inner measures; constructing a measure from an inner measure; the inner measure defined by a measure; complete locally determined spaces; extension of functionals to measures; countably compact classes; constructing measures dominating given functionals.
414 τ-additivity 50
Semi-continuous functions; supports; strict localizability; subspace measures; regular topologies; density topologies; lifting topologies.
415 Quasi-Radon measure spaces 58
Strict localizability; subspaces; regular topologies; hereditarily Lindelöf spaces; products of separable metrizable spaces; comparison and specification of quasi-Radon measures; construction of quasi-Radon measures extending given functionals; indefinite-integral measures; L^p spaces; Stone spaces.
416 Radon measure spaces 73
Radon and quasi-Radon measures; specification of Radon measures; c.l.d. versions of Borel measures; locally compact topologies; constructions of Radon measures extending or dominating given functionals; additive functionals on Boolean algebras and Radon measures on Stone spaces; subspaces; products; Stone spaces of measure algebras; compact and perfect measures; representation of homomorphisms of measure algebras; the split interval.
417 τ-additive product measures 88
The product of two effectively locally finite τ-additive measures; the product of many τ-additive probability measures; Fubini's theorem; generalized associative law; measures on subproducts as image measures; products of strictly positive measures; quasi-Radon and Radon product measures; when 'ordinary' product measures are τ-additive; continuous functions and Baire σ-algebras in product spaces.
418 Measurable functions and almost continuous functions 110
Measurable functions; into (separable) metrizable spaces; and image measures; almost continuous functions; continuity, measurability, image measures; expressing Radon measures as images of Radon measures; Prokhorov's theorem on projective limits of Radon measures; representing measurable functions into L^0 spaces.
419 Examples 126
A nearly quasi-Radon measure; a Radon measure space in which the Borel sets are inadequate; a nearly Radon measure; the Stone space of the Lebesgue measure algebra; measures with domain $\mathcal{P}\omega_1$; notes on Lebesgue measure.

Chapter 42: Descriptive set theory
Introduction 138
421 Souslin's operation 138
Souslin's operation; is idempotent; as a projection operator; Souslin-F sets; *constituents.
422 K-analytic spaces 148
Usco-compact relations; K-analytic sets; and Souslin-F sets; *First Separation Theorem.
423 Analytic spaces 155
Analytic spaces; are K-analytic spaces with countable networks; Souslin-F sets; Borel measurable functions; injective images of Polish spaces; non-Borel analytic sets; von Neumann-Jankow selection theorem; *constituents of coanalytic sets.
424 Standard Borel spaces 165
Elementary properties; isomorphism types; subspaces; Borel measurable actions of Polish groups.

Chapter 43: Topologies and measures II

Introduction 172

431 Souslin's operation 172
The domain of a complete locally determined measure is closed under Souslin's operation; the kernel of a Souslin scheme is approximable from within.

432 K-analytic spaces 176
Topological measures on K-analytic spaces; extensions to Radon measures; expressing Radon measures as images of Radon measures.

433 Analytic spaces 180
Measures on spaces with countable networks; inner regularity of Borel measures; expressing Radon measures as images of Radon measures; measurable and almost continuous functions; the von Neumann-Jankow selection theorem; products; extension of measures on σ-subalgebras; standard Borel spaces.

434 Borel measures 184
Classification of Borel measures; Radon spaces; universally measurable sets and functions; Borel-measure-compact, Borel-measure-complete and pre-Radon spaces; countable compactness and countable tightness; quasi-dyadic spaces and completion regular measures; first-countable spaces and Borel product measures.

435 Baire measures 203
Classification of Baire measures; extension of Baire measures to Borel measures (Mařik's theorem); measure-compact spaces; sequential spaces and Baire product measures.

436 Representation of linear functionals 209
Smooth and sequentially smooth linear functionals; measures and sequentially smooth functionals; Baire measures; products of Baire measures; quasi-Radon measures and smooth functionals; locally compact spaces and Radon measures.

437 Spaces of measures 219
Smooth and sequentially smooth duals; signed measures; embedding spaces of measurable functions in the bidual of $C_b(X)$; vague and narrow topologies; product measures; extreme points; uniform tightness; Prokhorov spaces.

438 Measure-free cardinals 240
Measure-free cardinals; point-finite families of sets with measurable unions; measurable functions into metrizable spaces; Radon and measure-compact metric spaces; metacompact spaces; hereditarily weakly θ-refinable spaces; when $\mathfrak{c}$ is measure-free.

439 Examples 254
Measures with no extensions to Borel measures; universally negligible sets; Hausdorff measures are rarely semi-finite; a smooth linear functional not expressible as an integral; a first-countable non-Radon space; Baire measures not extending to Borel measures; $\mathbb{N}^{\mathfrak{c}}$ is not Borel-measure-compact; the Sorgenfrey line; $\mathbb{Q}$ is not a Prokhorov space.

Chapter 44: Topological groups

Introduction 271

441 Invariant measures on locally compact spaces 271
Measures invariant under homeomorphisms; Haar measures; measures invariant under isometries.

442 Uniqueness of Haar measure 280
Two (left) Haar measures are multiples of each other; left and right Haar measures; Haar measurable and Haar negligible sets; the modular function of a group; formulae for $\int f(x^{-1})dx$, $\int f(xy)dx$.

443 Further properties of Haar measure 287
The Haar measure algebra of a group carrying Haar measures; actions of the group on the Haar measure algebra; locally compact groups; actions of the group on L^0 and L^p; the bilateral uniformity; Borel sets are adequate; completing the group; expressing an arbitrary Haar measure in terms of a Haar measure on a locally compact group; completion regularity of Haar measure; invariant measures on the set of left cosets of a closed subgroup of a locally compact group; modular functions of subgroups and quotient groups; transitive actions of compact groups on compact spaces.

444 Convolutions 311
Convolutions of quasi-Radon measures; the Banach algebra of signed τ-additive measures; continuous actions and corresponding actions on $L^0(\nu)$ for an arbitrary quasi-Radon measure ν; convolutions of measures and functions; indefinite-integral measures over a Haar measure μ; convolutions of functions; $L^p(\mu)$; approximate identities.

445 The duality theorem 334
Dual groups; Fourier-Stieltjes transforms; Fourier transforms; identifying the dual group with the maximal ideal space of L^1; the topology of the dual group; positive definite functions; Bochner's theorem; the Inversion Theorem; the Plancherel Theorem; the Duality Theorem.
446 The structure of locally compact groups 357
Finite-dimensional representations separate the points of a compact group; groups with no small subgroups have B-sequences; chains of subgroups.
447 Translation-invariant liftings 372
Translation-invariant liftings and lower densities; Vitali's theorem and a density theorem for groups with B-sequences; Haar measures have translation-invariant liftings.
448 Invariant measures on Polish spaces 383
Countably full local semigroups of $\operatorname{Aut}\mathfrak{A}$; σ-equidecomposability; countably non-paradoxical groups; G-invariant additive functions from $\mathfrak{A}$ to $L^\infty(\mathfrak{C})$; measures invariant under Polish group actions (the Nadkarni-Becker-Kechris theorem).
449 Amenable groups 393
Amenable groups; permanence properties; locally compact amenable groups; Tarski's theorem; discrete amenable groups.

Chapter 45: Perfect measures, disintegrations and processes
Introduction 413
451 Perfect, compact and countably compact measures 414
Basic properties of the three classes; subspaces, completions, c.l.d. versions, products; measurable functions from compact measure spaces to metrizable spaces; *weakly α-favourable spaces.
452 Integration and disintegration of measures 427
Integrating families of probability measures; τ-additive and Radon measures; disintegrations and regular conditional probabilities; disintegrating countably compact measures; disintegrating Radon measures; *images of countably compact measures.
453 Strong liftings 441
Strong and almost strong liftings; existence; on product spaces; disintegrations of Radon measures over spaces with almost strong liftings; Stone spaces; Losert's example.
454 Measures on product spaces 454
Perfect, compact and countably compact measures on product spaces; extension of finitely additive functions with perfect countably additive marginals; Kolmogorov's extension theorem; measures defined from conditional distributions; distributions of random processes; measures on $C(T)$ for Polish T.
455 Markov process and Brownian motion 464
Definition of Markov process from conditional distributions; existence of a measure representing Brownian motion; continuous sample paths.
456 Gaussian distributions 472
Gaussian distributions; supports; universal Gaussian distributions; cluster sets of n-dimensional processes; τ-additivity; Gaussian processes.
457 Simultaneous extension of measures 488
Extending families of finitely additive functionals; Strassen's theorem; extending families of measures; examples.
458 Relative independence and relative products 497
Relatively independent families of σ-algebras and random variables; relative distributions; relatively independent families of closed subalgebras of a probability algebra; relative free products of probability algebras; relative products of probability spaces; existence of relative products.
459 Symmetric measures and exchangeable random variables 510
Exchangeable families of inverse-measure-preserving functions; de Finetti's theorem; countably compact symmetric measures on product spaces disintegrate into product measures; symmetric quasi-Radon measures.

Chapter 46: Pointwise compact sets of measurable functions
Introduction 522
461 Barycenters and Choquet's theorem 522
Barycenters; elementary properties; sufficient conditions for existence; closed convex hulls of compact sets; Kreĭn's theorem; measures on sets of extreme points.
462 Pointwise compact sets of continuous functions 532
Angelic spaces; the topology of pointwise convergence on $C(X)$; weak convergence and weakly compact sets in $C_0(X)$; Radon measures on $C(X)$; separately continuous functions; convex hulls.

463 $\mathfrak{T}_p$ and $\mathfrak{T}_m$ 538
Pointwise convergence and convergence in measure on spaces of measurable functions; compact and sequentially compact sets; perfect measures and Fremlin's Alternative; separately continuous functions.
464 Talagrand's measure 550
The usual measure on $\mathcal{P}I$; the intersection of a sequence of non-measurable filters; Talagrand's measure; the L-space of additive functionals on $\mathcal{P}I$; measurable and purely non-measurable functionals.
465 Stable sets 563
Stable sets of functions; elementary properties; pointwise compactness; pointwise convergence and convergence in measure; a law of large numbers; stable sets and uniform convergence in the strong law of large numbers; stable sets in L^0 and L^1; *R-stable sets.
466 Measures on linear topological spaces 590
Quasi-Radon measures for weak and strong topologies; Kadec norms; constructing weak-Borel measures; characteristic functions of measures on locally convex spaces; universally measurable linear operators.
*467 Locally uniformly rotund norms 598
Locally uniformly rotund norms; separable normed spaces; long sequences of projections; K-countably determined spaces; weakly compactly generated spaces; Banach lattices with order-continuous norms; Eberlein compacta.

Chapter 47: Geometric measure theory
Introduction 610
471 Hausdorff measures 610
Metric outer measures; Increasing Sets Lemma; analytic spaces; inner regularity; Vitali's theorem and a density theorem; Howroyd's theorem.
472 Besicovitch's Density Theorem 626
Besicovitch's Covering Lemma; Besicovitch's Density Theorem; *a maximal theorem.
473 Poincaré's inequality 633
Differentiable and Lipschitz functions; smoothing by convolution; the Gagliardo-Nirenberg-Sobolev inequality; Poincaré's inequality for balls.
474 The distributional perimeter 647
The divergence of a vector field; sets with locally finite perimeter, perimeter measures and outward-normal functions; the reduced boundary; invariance under isometries; isoperimetric inequalities; Federer exterior normals; the Compactness Theorem.
475 The essential boundary 670
Essential interior, closure and boundary; the reduced boundary; perimeter measures; characterizing sets with locally finite perimeter; the Divergence Theorem; calculating perimeters from cross-sectional counts; Cauchy's Perimeter Theorem; the Isoperimetric Theorem for convex sets.
476 Concentration of measure 690
Hausdorff metrics; Vietoris topologies; concentration by partial reflection; concentration of measure in $\mathbb{R}^r$; the Isoperimetric Theorem; concentration of measure on spheres.

Chapter 48: Gauge integrals
Introduction 704
481 Tagged partitions 704
Tagged partitions and Riemann sums; gauge integrals; gauges; residual sets; subdivisions; examples (the Riemann integral, the Henstock integral, the symmetric Riemann-complete integral, the McShane integral, box products, the approximately continuous Henstock integral).
482 General theory 714
Saks-Henstock lemma; when gauge-integrable functions are measurable; when integrable functions are gauge-integrable; $I_\nu(f \times \chi H)$; integrating derivatives; B.Levi's theorem; Fubini's theorem.
483 The Henstock integral 729
The Henstock and Lebesgue integrals; indefinite Henstock integrals; Saks-Henstock lemma; fundamental theorem of calculus; the Perron integral; ACG_* functions.
484 The Pfeffer integral 746
The Tamanini-Giacomelli theorem; a family of tagged-partition structures; the Pfeffer integral; the Saks-Henstock indefinite integral of a Pfeffer integrable function; Pfeffer's Divergence Theorem; differentiating the indefinite integral; invariance under lipeomorphisms.

Chapter 49: Further topics
Introduction 765
491 Equidistributed sequences 765
The asymptotic density ideal $\mathcal{Z}$; equidistributed sequences; when equidistributed sequences exist; $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$; effectively regular measures; equidistributed sequences and induced embeddings of measure algebras in $\mathfrak{Z}$.
492 Combinatorial concentration of measure 782
Concentration of measure in product spaces; concentration of measure in permutation groups.
493 Extremely amenable groups 789
Extremely amenable groups; concentrating additive functionals; measure algebras under $\triangle$; L^0; automorphism groups of measure algebras; isometry groups of spheres in inner product spaces; locally compact groups.
494 Cubes in product spaces 800
Subsets of measure algebras with non-zero infima; product sets included in given sets of positive measure.
495 Poisson point processes 802
Poisson distributions; Poisson point processes; disintegrations; transforming disjointness into stochastic independence; representing Poisson point processes by Radon measures; exponential distributions and Poisson point processes on $[0, \infty[$.

Appendix to Volume 4
Introduction 823
4A1 Set theory 823
Cardinals; closed cofinal sets and stationary sets; Δ-system lemma; free sets; Ramsey's theorem; the Marriage Lemma; filters; normal ultrafilters; Ostaszewski's $\clubsuit$; cardinals of σ-algebras.
4A2 General topology 827
Glossary; general constructions; F_σ, G_δ, zero and cozero sets; countable chain condition; separation axioms; compact and locally compact spaces; Lindelöf spaces; Stone-Čech compactifications; uniform spaces; first-countable, sequential, countably tight, metrizable spaces; countable networks; second-countable spaces; separable metrizable spaces; Polish spaces; order topologies.
4A3 Topological σ-algebras 848
Borel σ-algebras; measurable functions; hereditarily Lindelöf spaces; second-countable spaces; Polish spaces; ω_1; Baire σ-algebras; product spaces; compact spaces; Baire property algebras; cylindrical σ-algebras.
4A4 Locally convex spaces 857
Linear topological spaces; locally convex spaces; Hahn-Banach theorem; normed spaces; inner product spaces; max-flow min-cut theorem.
4A5 Topological groups 863
Group actions; topological groups; uniformities; quotient groups; metrizable groups.
4A6 Banach algebras 869
Stone-Weierstrass theorem (fourth form); multiplicative linear functionals; spectral radius; invertible elements; exponentiation.

Concordance 873

References for Volume 4 874

Index to Volumes 1-4
Principal topics and results 881
General index 893

General introduction In this treatise I aim to give a comprehensive description of modern abstract measure theory, with some indication of its principal applications. The first two volumes are set at an introductory level; they are intended for students with a solid grounding in the concepts of real analysis, but possibly with rather limited detailed knowledge. As the book proceeds, the level of sophistication and expertise demanded will increase; thus for the volume on topological measure spaces, familiarity with general topology will be assumed. The emphasis throughout is on the mathematical ideas involved, which in this subject are mostly to be found in the details of the proofs.

My intention is that the book should be usable both as a first introduction to the subject and as a reference work. For the sake of the first aim, I try to limit the ideas of the early volumes to those which are really essential to the development of the basic theorems. For the sake of the second aim, I try to express these ideas in their full natural generality, and in particular I take care to avoid suggesting any unnecessary restrictions in their applicability. Of course these principles are to to some extent contradictory. Nevertheless, I find that most of the time they are very nearly reconcilable, *provided* that I indulge in a certain degree of repetition. For instance, right at the beginning, the puzzle arises: should one develop Lebesgue measure first on the real line, and then in spaces of higher dimension, or should one go straight to the multidimensional case? I believe that there is no single correct answer to this question. Most students will find the one-dimensional case easier, and it therefore seems more appropriate for a first introduction, since even in that case the technical problems can be daunting. But certainly every student of measure theory must at a fairly early stage come to terms with Lebesgue area and volume as well as length; and with the correct formulations, the multidimensional case differs from the one-dimensional case only in a definition and a (substantial) lemma. So what I have done is to write them both out (§§114-115). In the same spirit, I have been uninhibited, when setting out exercises, by the fact that many of the results I invite students to look for will appear in later chapters; I believe that throughout mathematics one has a better chance of understanding a theorem if one has previously attempted something similar alone.

As I write this Introduction (September 2003), the plan of the work is as follows:

Volume 1: The Irreducible Minimum
Volume 2: Broad Foundations
Volume 3: Measure Algebras
Volume 4: Topological Measure Spaces
Volume 5: Set-theoretic Measure Theory.

Volume 1 is intended for those with no prior knowledge of measure theory, but competent in the elementary techniques of real analysis. I hope that it will be found useful by undergraduates meeting Lebesgue measure for the first time. Volume 2 aims to lay out some of the fundamental results of pure measure theory (the Radon-Nikodým theorem, Fubini's theorem), but also gives short introductions to some of the most important applications of measure theory (probability theory, Fourier analysis). While I should like to believe that most of it is written at a level accessible to anyone who has mastered the contents of Volume 1, I should not myself have the courage to try to cover it in an undergraduate course, though I would certainly attempt to include some parts of it. Volumes 3 and 4 are set at a rather higher level, suitable to postgraduate courses; while Volume 5 will assume a wide-ranging competence over large parts of analysis and set theory.

There is a disclaimer which I ought to make in a place where you might see it in time to avoid paying for this book. I make no attempt to describe the history of the subject. This is not because I think the history uninteresting or unimportant; rather, it is because I have no confidence of saying anything which would not be seriously misleading. Indeed I have very little confidence in anything I have ever read concerning the history of ideas. So while I am happy to honour the names of Lebesgue and Kolmogorov and Maharam in more or less appropriate places, and I try to include in the bibliographies the works which I have myself consulted, I leave any consideration of the details to those bolder and better qualified than myself.

The work as a whole is not yet complete; and when it is finished, it will undoubtedly be too long to be printed as a single volume in any reasonable format. I am therefore publishing it one part at a time. However, drafts of most of the rest are available on the Internet; see `http://www.essex.ac.uk/maths/staff/fremlin/mt.htm` for detailed instructions. For the time being, at least, printing will be in short runs. I hope that readers will be energetic in commenting on errors and omissions, since it should be possible to correct these relatively promptly. An inevitable consequence of this is that paragraph references may go out of date rather quickly. I shall be most flattered if anyone chooses to rely on this book as a source

for basic material; and I am willing to attempt to maintain a concordance to such references, indicating where migratory results have come to rest for the moment, if authors will supply me with copies of papers which use them.

I mention some minor points concerning the layout of the material. Most sections conclude with lists of 'basic exercises' and 'further exercises', which I hope will be generally instructive and occasionally entertaining. How many of these you should attempt must be for you and your teacher, if any, to decide, as no two students will have quite the same needs. I mark with a $>$ those which seem to me to be particularly important. But while you may not need to write out solutions to all the 'basic exercises', if you are in any doubt as to your capacity to do so you should take this as a warning to slow down a bit. The 'further exercises' are unbounded in difficulty, and are unified only by a presumption that each has at least one solution based on ideas already introduced. Occasionally I add a final 'problem', a question to which I do not know the answer and which seems to arise naturally in the course of the work.

The impulse to write this book is in large part a desire to present a unified account of the subject. Cross-references are correspondingly abundant and wide-ranging. In order to be able to refer freely across the whole text, I have chosen a reference system which gives the same code name to a paragraph wherever it is being called from. Thus 132E is the fifth paragraph in the second section of the third chapter of Volume 1, and is referred to by that name throughout. Let me emphasize that cross-references are supposed to help the reader, not distract her. Do not take the interpolation '(121A)' as an instruction, or even a recommendation, to lift Volume 1 off the shelf and hunt for §121. If you are happy with an argument as it stands, independently of the reference, then carry on. If, however, I seem to have made rather a large jump, or the notation has suddenly become opaque, local cross-references may help you to fill in the gaps.

Each volume will have an appendix of 'useful facts', in which I set out material which is called on somewhere in that volume, and which I do not feel I can take for granted. Typically the arrangement of material in these appendices is directed very narrowly at the particular applications I have in mind, and is unlikely to be a satisfactory substitute for conventional treatments of the topics touched on. Moreover, the ideas may well be needed only on rare and isolated occasions. So as a rule I recommend you to ignore the appendices until you have some direct reason to suppose that a fragment may be useful to you.

During the extended gestation of this project I have been helped by many people, and I hope that my friends and colleagues will be pleased when they recognise their ideas scattered through the pages below. But I am especially grateful to those who have taken the trouble to read through earlier drafts and comment on obscurities and errors.

Introduction to Volume 4

I return in this volume to the study of measure *spaces* rather than measure *algebras*. For fifty years now measure theory has been intimately connected with general topology. Not only do a very large proportion of the measure spaces arising in applications carry topologies related in interesting ways to their measures, but many questions in abstract measure theory can be effectively studied by introducing suitable topologies. Consequently any course in measure theory at this level must be frankly dependent on a substantial knowledge of topology. With this proviso, I hope that the present volume will be accessible to graduate students, and will lead them to the most important ideas of modern abstract measure theory.

The first and third chapters of the volume seek to provide a thorough introduction into the ways in which topologies and measures can interact. They are divided by a short chapter on descriptive set theory, on the borderline between set theory, logic, real analysis and general topology, which I single out for detailed exposition because I believe that it forms an indispensable part of the background of any measure theorist. Chapter 41 is dominated by the concepts of inner regularity and τ-additivity, coming together in Radon measures (§416). Chapter 43 concentrates rather on questions concerning properties of a topological space which force particular relationships with measures on that space. But plenty of side-issues are treated in both, such as Lusin measurability (§418), the definition of measures from linear functionals (§436) and measure-free cardinals (§438). Chapters 45 and 46 continue some of the same themes, with particular investigations into 'disintegrations' or regular conditional probabilities (§§452-453), the abstract theory of stochastic processes (§§454-455), Talagrand's theory of Glivenko-Cantelli classes (§465) and the theory of measures on normed spaces (§§466-467).

In contrast with the relatively amorphous structure of Chapters 41, 43, 45 and 46, four chapters of this volume have definite topics. I have already said that Chapter 42 is an introduction to descriptive set theory;

like Chapters 31 and 35 in the last volume, it is a kind of appendix brought into the main stream of the argument. Chapter 44 deals with topological groups. Most of it is of course devoted to Haar measure, giving the Pontryagin-van Kampen duality theorem (§445) and the Ionescu Tulcea theorem on the existence of translation-invariant liftings (§447). But there are also sections on Polish groups (§448) and amenable groups (§449), and some of the general theory of measures on measurable groups (§444). Chapter 47 is a second excursion, after Chapter 26, into geometric measure theory. It starts with Hausdorff measures (§471), gives a proof of the Di Giorgio-Federer Divergence Theorem (§475), and then examines a number of examples of 'concentration of measure' (§476). In Chapter 48, I set out the elementary theory of gauge integrals, with sections on the Henstock and Pfeffer integrals (§§483-484). Finally, in Chapter 49, I give notes on five special topics: equidistributed sequences (§491), combinatorial forms of concentration of measure (§492), extremely amenable groups (§493), subproducts in product spaces (§494) and Poisson point processes (§495).

I had better mention prerequisites, as usual. To embark on this material you will certainly need a solid foundation in measure theory. Since I do of course use my own exposition as my principal source of references to the elementary ideas, I advise readers to ensure that they have easy access to all three previous volumes before starting serious work on this one. But you may not need to read very much of them. It might be prudent to glance through the detailed contents of Volume 1 and the first five chapters of Volume 2 to check that most of the material there is more or less familiar. But Volume 3, and the last three chapters of Volume 2, can probably be left on one side for the moment. Of course you will need the Lifting Theorem (Chapter 34) for §§447, 452 and 453, and Chapter 26 is essential background for Chapter 47, while Chapter 28 (on Fourier analysis) may help to make sense of Chapter 44, and parts of Chapter 27 (on probability theory) are necessary for §§455-456. And measure algebras are mentioned in every chapter except (I think) Chapter 48; but I hope that the cross-references are precise enough to lead you to just what you need to know at any particular point. Even Maharam's theorem is hardly used in this volume.

What you will need, apart from any knowledge of measure theory, is a sound background in general topology. This volume calls on a great many miscellaneous facts from general topology, and the list in §4A2 is not a good place to start if continuity and compactness and the separation axioms are unfamiliar. My primary reference for topology is ENGELKING 89. I do not insist that you should have read this book (though of course I hope you will do so sometime); but I do think you should make sure that you can use it.

In the general introduction to this treatise, I wrote 'I make no attempt to describe the history of the subject', and I have generally been casual – some would say negligent – in my attributions of results to their discoverers. Through much of the first three volumes I did at least have the excuse that the history exists in print in far more detail than I am qualified to describe. In the present volume I find my position more uncomfortable, in that I have been watching the evolution of the subject relatively closely over the last thirty years, and ought to be able to say something about it. Nevertheless I remain reluctant to make definite statements crediting one person rather than another with originating an idea. My more intimate knowledge of the topic makes me even more conscious than elsewhere of the danger of error and of the breadth of reading that would be necessary to produce a balanced account. In some cases I do attach a result to a specific published paper, but these attributions should never be regarded as an assertion that any particular author has priority; at most, they declare that a historian should examine the source cited before coming to any decision. I assure my friends and colleagues that my omissions are not intended to slight either them or those we all honour. What I have tried to do is to include in the bibliography to this volume all the published work which (as far as I am consciously aware) has influenced me while writing it, so that those who wish to go into the matter will have somewhere to start their investigations.

Chapter 41

Topologies and Measures I

I begin this volume with an introduction to some of the most important ways in which topologies and measures can interact, and with a description of the forms which such constructions as subspaces and product spaces take in such contexts. By far the most important concept is that of Radon measure (411H, §416). In Radon measure spaces we find both the richest combinations of ideas and the most important applications. But, as usual, we are led both by analysis of these ideas and by other interesting examples to consider wider classes of topological measure space, and the greater part of the chapter, by volume, is taken up by a description of the many properties of Radon measures individually and in partial combinations.

I begin the chapter with a short section of definitions (§411), including a handful of more or less elementary examples. The two central properties of a Radon measure are 'inner regularity' (411B) and 'τ-additivity' (411C). The former is an idea of great versatility which I look at in an abstract setting in §412. I take a section (§413) to describe some methods of constructing measure spaces, extending the rather limited range of constructions offered in earlier volumes. There are two sections on τ-additive measures, §§414 and 417; the former covers the elementary ideas, and the latter looks at product measures, where it turns out that we need a new technique to supplement the purely measure-theoretic constructions of Chapter 25. On the way to Radon measures in §416, I pause over 'quasi-Radon' measures (411H, §415), where inner regularity and τ-additivity first come effectively together.

The possible interactions of a topology and a measure on the same space are so varied that even a brief account makes a long chapter; and this is with hardly any mention of results associated with particular types of topological space, most of which must wait for later chapters. But I include one section on the two most important classes of functions acting between topological measure spaces (§418), and another describing some examples to demonstrate special phenomena (§419).

411 Definitions

In something of the spirit of §211, but this time without apologising, I start this volume with a list of definitions. The rest of Chapter 41 will be devoted to discussing these definitions and relationships between them, and integrating the new ideas into the concepts and constructions of earlier volumes; I hope that by presenting the terminology now I can give you a sense of the directions the following sections will take. I ought to remark immediately that there are many cases in which the exact phrasing of the definitions is important in ways which may not be immediately apparent.

411A I begin with a phrase which will be a useful shorthand for the context in which most, but not all, of the theory here will be developed.

Definition A **topological measure space** is a quadruple $(X, \mathfrak{T}, \Sigma, \mu)$ where (X, Σ, μ) is a measure space and $\mathfrak{T}$ is a topology on X such that $\mathfrak{T} \subseteq \Sigma$, that is, every open set (and therefore every Borel set) is measurable.

411B Now I come to what are in my view the two most important concepts to master; jointly they will dominate the chapter.

Definition Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of sets. I say that μ is **inner regular with respect to** $\mathcal{K}$ if

$$\mu E = \sup\{\mu K : K \in \Sigma \cap \mathcal{K},\, K \subseteq E\}$$

for every $E \in \Sigma$. (Cf. 256Ac, 342Aa.)

Remark Note that in this definition I do not assume that $\mathcal{K} \subseteq \Sigma$, nor even that $\mathcal{K} \subseteq \mathcal{P}X$. But of course μ will be inner regular with respect to $\mathcal{K}$ iff it is inner regular with respect to $\mathcal{K} \cap \Sigma$.

It is convenient in this context to interpret $\sup \emptyset$ as 0, so that we have to check the definition only when $\mu E > 0$, and need not insist that $\emptyset \in \mathcal{K}$.

411C Definition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X. I say that μ is **τ-additive** (the phrase **τ-regular** has also been used) if whenever $\mathcal{G}$ is a non-empty upwards-directed family of open sets such that $\mathcal{G} \subseteq \Sigma$ and $\bigcup \mathcal{G} \in \Sigma$ then $\mu(\bigcup \mathcal{G}) = \sup_{G \in \mathcal{G}} \mu G$.

Remark Note that in this definition I do not assume that every open set is measurable. Consequently we cannot take it for granted that an extension of a τ-additive measure will be τ-additive; on the other hand, the restriction of a τ-additive measure to any σ-subalgebra will be τ-additive.

411D Complementary to 411B we have the following.

Definition Let (X, Σ, μ) be a measure space and $\mathcal{H}$ a family of subsets of X. Then μ is **outer regular with respect to $\mathcal{H}$** if

$$\mu E = \inf\{\mu H : H \in \Sigma \cap \mathcal{H},\ H \supseteq E\}$$

for every $E \in \Sigma$.

411E I delay discussion of most of the relationships between the concepts here to later in the chapter. But it will be useful to have a basic fact set out immediately.

Proposition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X. If μ is inner regular with respect to the compact sets, it is τ-additive.

proof Let $\mathcal{G}$ be a non-empty upwards-directed family of measurable open sets such that $H = \bigcup \mathcal{G} \in \Sigma$. If $\gamma < \mu H$, there is a compact set $K \subseteq H$ such that $\mu K \geq \gamma$; now there must be a $G \in \mathcal{G}$ which includes K, so that $\mu G \geq \gamma$. As γ is arbitrary, $\sup_{G \in \mathcal{G}} \mu G = \mu H$.

411F In order to deal efficiently with measures which are not totally finite, I think we need the following ideas.

Definitions Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) I say that μ is **locally finite** if every point of X has a neighbourhood of finite measure, that is, if the open sets of finite outer measure cover X.

(b) I say that μ is **effectively locally finite** if for every non-negligible measurable set $E \subseteq X$ there is a measurable open set $G \subseteq X$ such that $\mu G < \infty$ and $E \cap G$ is not negligible.

Note that an effectively locally finite measure must measure many open sets, while a locally finite measure need not.

(c) This seems a convenient moment at which to introduce the following term. A real-valued function f defined on a subset of X is **locally integrable** if for every $x \in X$ there is an open set G containing x such that $\int_G f$ is defined (in the sense of 214D) and finite.

411G Elementary facts (a) If μ is a locally finite measure on a topological space X, then $\mu^* K < \infty$ for every compact set $K \subseteq X$. **P** The family $\mathcal{G}$ of open sets of finite outer measure is upwards-directed and covers X, so there must be some $G \in \mathcal{G}$ including K, in which case $\mu^* K \leq \mu^* G$ is finite. **Q**

(b) A measure μ on $\mathbb{R}^r$ is locally finite iff every bounded set has finite outer measure (cf. 256Ab). **P** (i) If every bounded set has finite outer measure then, in particular, every open ball has finite outer measure, so that μ is locally finite. (ii) If μ is locally finite and $A \subseteq \mathbb{R}^r$ is bounded, then its closure $\overline{A}$ is compact (2A2F), so that $\mu^* A \leq \mu^* \overline{A}$ is finite, by (a) above. **Q**

(c) I should perhaps remark immediately that a locally finite topological measure need not be effectively locally finite (419A), and an effectively locally finite measure need not be locally finite (411P).

(d) An effectively locally finite measure must be semi-finite.

(e) A locally finite measure on a Lindelöf space X (definition: 4A2A) is σ-finite. **P** Let $\mathcal{G}$ be the family of open sets of finite outer measure. Because μ is locally finite, $\mathcal{G}$ is a cover of X. Because X is Lindelöf, there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$ covering X. For each $n \in \mathbb{N}$, there is a measurable set $E_n \supseteq G_n$ of finite measure, and now $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence of sets of finite measure covering X. **Q**

(f) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space such that μ is locally finite and inner regular with respect to the compact sets. Then μ is effectively locally finite. **P** Suppose that $\mu E > 0$. Then there is a measurable compact set $K \subseteq E$ such that $\mu K > 0$. As in the argument for (a) above, there is an open set G of finite measure including K, so that $\mu(E \cap G) > 0$. **Q**

(g) Corresponding to (a) above, we have the following fact. If μ is a measure on a topological space and $f \in \mathcal{L}^0(\mu)$ is locally integrable, then $\int_K f d\mu$ is finite for every compact $K \subseteq X$, because K can be covered by a finite family of open sets G such that $\int_G |f| d\mu < \infty$.

(h) If μ is a locally finite measure on a topological space X, and $f \in \mathcal{L}^p(\mu)$ for some $p \in [1, \infty]$, then f is locally integrable; this is because $\int_G |f| \le \int_E f \le \|f\|_p \|\chi E\|_q$ is finite whenever $G \subseteq E$ and $\mu E < \infty$, where $\frac{1}{p} + \frac{1}{q} = 1$, by Hölder's inequality (244Eb).

(i) If $(X, \mathfrak{T})$ is a completely regular space and μ is a locally finite topological measure on X, then the set of open sets with negligible boundaries is a base for $\mathfrak{T}$. **P** If $x \in G \in \mathfrak{T}$, let $H \subseteq G$ be an open set of finite measure containing x, and $f : X \to [0, 1]$ a continuous function such that $f(x) = 1$ and $f(y) = 0$ for $y \in X \setminus H$. Then $\{f^{-1}[\{\alpha\}] : 0 < \alpha < 1\}$ is an uncountable disjoint family of measurable subsets of H, so there must be some $\alpha \in]0, 1[$ such that $f^{-1}[\{\alpha\}]$ is negligible. Set $U = \{y : f(y) > \alpha\}$; then U is an open neighbourhood of x included in G and $\partial U \subseteq f^{-1}[\{\alpha\}]$ is negligible. **Q**

411H Two particularly important combinations of the properties above are the following.

Definitions (a) A **quasi-Radon measure space** is a topological measure space $(X, \mathfrak{T}, \Sigma, \mu)$ such that (i) (X, Σ, μ) is complete and locally determined (ii) μ is τ-additive, inner regular with respect to the closed sets and effectively locally finite.

(b) A **Radon measure space** is a topological measure space $(X, \mathfrak{T}, \Sigma, \mu)$ such that (i) (X, Σ, μ) is complete and locally determined (ii) $\mathfrak{T}$ is Hausdorff (iii) μ is locally finite and inner regular with respect to the compact sets.

411I Remarks(a) You may like to seek your own proof that a Radon measure space is always quasi-Radon, before looking it up in §416 below.

(b) Note that a measure on Euclidean space $\mathbb{R}^r$ is a Radon measure on the definition above iff it is a Radon measure as described in 256Ad. **P** In 256Ad, I said that a measure μ on $\mathbb{R}^r$ is 'Radon' if it is a locally finite complete topological measure, inner regular with respect to the compact sets. (The definition of 'locally finite' in 256A was not the same as the one above, but I have already covered this point in 411Gb.) So the only thing to add is that μ is necessarily locally determined, because it is σ-finite (256Ba). **Q**

411J The following special types of inner regularity are of sufficient importance to have earned separate names.

Definitions (a) If $(X, \mathfrak{T})$ is a topological space, I will say that a measure μ on X is **tight** if it is inner regular with respect to the closed compact sets.

(b) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a topological measure space, I will say that μ is **completion regular** if it is inner regular with respect to the zero sets (definition: 3A3Pa).

411K Borel and Baire measures If $(X, \mathfrak{T})$ is a topological space, I will call a measure with domain (exactly) the Borel σ-algebra of X (4A3A) a **Borel measure** on X, and a measure with domain (exactly) the Baire σ-algebra of X (4A3K) a **Baire measure** on X.

Of course a Borel measure is a topological measure in the sense of 411A. On a metric space, the Borel and Baire measures coincide (4A3Kb). The most important measures in this chapter will be c.l.d. versions of Borel measures.

411L When we come to look at functions defined on a topological measure space, we shall have to relate ideas of continuity and measurability. Two basic concepts are the following.

Definition Let X be a set, Σ a σ-algebra of subsets of X and $(Y, \mathfrak{S})$ a topological space. I will say that a function $f : X \to Y$ is **measurable** if $f^{-1}[G] \in \Sigma$ for every open set $G \subseteq Y$.

Remarks (a) Note that a function $f : X \to \mathbb{R}$ is measurable on this definition (when $\mathbb{R}$ is given its usual topology) iff it is measurable according to the familiar definition in 121C, which asks only that sets of the form $\{x : f(x) < \alpha\}$ should be measurable (121Ef).

(b) For any topological space $(Y, \mathfrak{S})$, a function $f : X \to Y$ is measurable iff f is $(\Sigma, \mathcal{B}(Y))$-measurable, where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y (4A3Cb).

411M Definition Let (X, Σ, μ) be a measure space, $\mathfrak{T}$ a topology on X, and $(Y, \mathfrak{S})$ another topological space. I will say that a function $f : X \to Y$ is **almost continuous** or **Lusin measurable** if μ is inner regular with respect to the family of subsets A of X such that $f \restriction A$ is continuous.

411N Finally, I introduce some terminology to describe ways in which (sometimes) measures can be located in one part of a topological space rather than another.

Definitions Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) I will call a set $A \subseteq X$ **self-supporting** if $\mu^*(A \cap G) > 0$ for every open set G such that $A \cap G$ is non-empty. (Such sets are sometimes called **of positive measure everywhere**.)

(b) A **support** of μ is a closed self-supporting set F such that $X \setminus F$ is negligible.

(c) Note that μ can have at most one support. **P** If F_1, F_2 are supports then $\mu^*(F_1 \setminus F_2) \leq \mu^*(X \setminus F_2) = 0$ so $F_1 \setminus F_2$ must be empty. Similarly, $F_2 \setminus F_1 = \emptyset$, so $F_1 = F_2$. **Q**

(d) If μ is a τ-additive topological measure it has a support. **P** Let $\mathcal{G}$ be the family of negligible open sets, and F the closed set $X \setminus \bigcup \mathcal{G}$. Then $\mathcal{G}$ is an upwards-directed family in $\mathfrak{T} \cap \Sigma$ and $\bigcup \mathcal{G} \in \mathfrak{T} \cap \Sigma$, so

$$\mu(X \setminus F) = \mu(\bigcup \mathcal{G}) = \sup_{G \in \mathcal{G}} \mu G = 0.$$

If G is open and $G \cap F \neq \emptyset$ then $G \notin \mathcal{G}$ so $\mu^*(G \cap F) = \mu(G \cap F) = \mu G > 0$; thus F is self-supporting and is the support of μ. **Q**

(e) Let X and Y be topological spaces with topological measures μ, ν respectively and a continuous inverse-measure-preserving function $f : X \to Y$. Suppose that μ has a support E. Then $\overline{f[E]}$ is the support of ν. **P** We have only to observe that for an open set $H \subseteq Y$

$$\begin{aligned} \nu H > 0 &\iff \mu f^{-1}[H] > 0 \iff f^{-1}[H] \cap E \neq \emptyset \\ &\iff H \cap f[E] \neq \emptyset \iff H \cap \overline{f[E]} \neq \emptyset. \end{aligned}$$ **Q**

(f) μ is **strictly positive** (with respect to $\mathfrak{T}$) if $\mu^* G > 0$ for every non-empty open set $G \subseteq X$, that is, X itself is the support of μ.

***(g)** If $(X, \mathfrak{T})$ is a topological space, and μ is a strictly positive σ-finite measure on X such that the domain Σ of μ includes a π-base $\mathcal{U}$ for $\mathfrak{T}$, then X is ccc. **P** Let $\langle E_n \rangle_{n \in \mathbb{N}}$ be a sequence of sets of finite measure covering X. Let $\mathcal{G}$ be a disjoint family of non-empty open sets. For each $G \in G$, take $U_G \in \mathcal{U} \setminus \{\emptyset\}$ such that $U_G \subseteq G$; then $\mu U_G > 0$, so there is an $n(G)$ such that $\mu(E_{n(G)} \cap U_G) > 0$. Now $\sum_{G \in G, n(G) = k} \mu(E_k \cap U_G) \leq \mu E_k$ is finite for every k, so $\{G : n(G) = k\}$ must be countable and $\mathcal{G}$ is countable. **Q**

411O Example Lebesgue measure on $\mathbb{R}^r$ is a Radon measure (256Ha); in particular, it is locally finite and tight. It is therefore τ-additive and effectively locally finite (411E, 411Gf). It is completion regular (because every compact set is a zero set, see 4A2Lc), outer regular with respect to the open sets (134F) and strictly positive.

411P Example: Stone spaces (a) Let $(Z, \mathfrak{T}, \Sigma, \mu)$ be the Stone space of a semi-finite measure algebra $(\mathfrak{A}, \bar{\mu})$, so that $(Z, \mathfrak{T})$ is a zero-dimensional compact Hausdorff space, (Z, Σ, μ) is complete and semi-finite, the open-and-closed sets are measurable, the negligible sets are the nowhere dense sets, and every measurable set differs by a nowhere dense set from an open-and-closed set (311I, 321K, 322Bd, 322Qa).

(b) μ is inner regular with respect to the open-and-closed sets (322Qa); in particular, it is completion regular and tight. Consequently it is τ-additive (411E).

(c) μ is strictly positive, because the open-and-closed sets form a base for $\mathfrak{T}$ (311I) and a non-empty open-and-closed set has non-zero measure. μ is effectively locally finite. **P** Suppose that $E \in \Sigma$ is not negligible. There is a measurable set $F \subseteq E$ such that $0 < \mu F < \infty$; now there is a non-empty open-and-closed set G included in F, in which case $\mu G < \infty$ and $\mu(E \cap G) > 0$. **Q**

(d) The following are equiveridical, that is, if one is true so are the others:
(i) $(\mathfrak{A}, \bar{\mu})$ is localizable;
(ii) μ is strictly localizable;
(iii) μ is locally determined;
(iv) μ is a quasi-Radon measure.
P The equivalence of (i)-(iii) is Theorem 322N. (iv)⇒(iii) is trivial. If one, therefore all, of (i)-(iii) are true, then μ is a topological measure, because if $G \subseteq Z$ is open, then $\overline{G}$ is open-and-closed, by 314S, therefore measurable, and $\overline{G} \setminus G$ is nowhere dense, therefore also measurable. We know already that μ is complete, effectively locally finite and τ-additive, so that if it is also locally determined it is a quasi-Radon measure. **Q**

(e) The following are equiveridical:
(i) μ is a Radon measure;
(ii) μ is totally finite;
(iii) μ is locally finite;
(iv) μ is outer regular with respect to the open sets.
P (ii)⇒(iv) If μ is totally finite and $E \in \Sigma$, then for any $\epsilon > 0$ there is a closed set $F \subseteq Z \setminus E$ such that $\mu F \geq \mu(Z \setminus E) - \epsilon$, and now $G = Z \setminus F$ is an open set including E with $\mu G \leq \mu E + \epsilon$. (iv)⇒(iii) Suppose that μ is outer regular with respect to the open sets, and $z \in Z$. Because Z is Hausdorff, $\{z\}$ is closed. If it is open it is measurable, and because μ is semi-finite it must have finite measure. Otherwise it is nowhere dense, therefore negligible, and must be included in open sets of arbitrarily small measure. Thus in both cases z belongs to an open set of finite measure; as z is arbitrary, μ is locally finite. (iii)⇒(ii) Becasue Z is compact, this is a consequence of 411Ga. (i)⇒(iii) is part of the definition of 'Radon measure'. Finally, (ii)+(iii)⇒(i), again directly from the definition and the facts set out in (a)-(b) above. **Q**

411Q Example: Dieudonné's measure Recall that a set $E \subseteq \omega_1$ is a Borel set iff either E or its complement includes a cofinal closed set (4A3J). So we may define a Borel measure μ on ω_1 by saying that $\mu E = 1$ if E includes a cofinal closed set and $\mu E = 0$ if E is disjoint from a cofinal closed set. If E is disjoint from some cofinal closed set, so is any subset of E, so μ is complete. Since μ takes only the values 0 and 1, it is a purely atomic probability measure.

μ is a topological measure; being totally finite, it is surely locally finite and effectively locally finite. It is inner regular with respect to the closed sets (because if $\mu E > 0$, there is a cofinal closed set $F \subseteq E$, and now F is a closed set with $\mu F = \mu E$), therefore outer regular with respect to the open sets. It is not τ-additive (because $\xi = [0, \xi[$ is an open set of zero measure for every $\xi < \omega_1$, and the union of these sets is a measurable open set of measure 1).

μ is not completion regular, because the set of countable limit ordinals is a closed set (4A1Bb) which does not include any uncountable zero set (see 411Ra below).

The only self-supporting subset of ω_1 is the empty set (because there is a cover of ω_1 by negligible open sets). In particular, μ does not have a support.

Remark There is a measure of this type on any ordinal of uncountable cofinality; see 411Xj.

411R Example: The Baire σ-algebra of ω_1 The Baire σ-algebra $\mathcal{Ba}(\omega_1)$ of ω_1 is the countable-cocountable algebra (4A3P). The countable-cocountable measure μ on ω_1 is therefore a Baire measure on the definition of 411K. Since all sets of the form $]\xi,\omega_1[$ are zero sets, μ is inner regular with respect to the zero sets and outer regular with respect to the cozero sets. Since sets of the form $[0,\xi[$ $(= \xi)$ form a cover of ω_1 by measurable open sets of zero measure, μ is not τ-additive.

411X Basic exercises (a) Let (X,Σ,μ) be a totally finite measure space and $\mathfrak{T}$ a topology on X. Show that μ is inner regular with respect to the closed sets iff it is outer regular with respect to the open sets, and is inner regular with respect to the zero sets iff it is outer regular with respect to the cozero sets.

(b) Let μ be a Radon measure on $\mathbb{R}^r$, where $r \geq 1$, and $f \in \mathcal{L}^0(\mu)$. Show that f is locally integrable in the sense of 411Fc iff it is locally integrable in the sense of 256E, that is, $\int_E f d\nu < \infty$ for every bounded set $E \subseteq \mathbb{R}^r$.

(c) Let μ be a measure on a topological space, $\hat{\mu}$ its completion and $\tilde{\mu}$ its c.l.d. version. Show that μ is locally finite iff $\hat{\mu}$ is locally finite, and in this case $\tilde{\mu}$ is locally finite.

>(d) Let μ be an effectively locally finite measure on a topological space X. (i) Show that the completion and c.l.d. version of μ are effectively locally finite. (ii) Show that if μ is complete and locally determined, then the union of the measurable open sets of finite measure is conegligible. (iii) Show that if X is hereditarily Lindelöf then μ must be σ-finite.

(e) Let X be a topological space and μ a measure on X. Let $U \subseteq L^0(\mu)$ be the set of equivalence classes of locally integrable functions in $\mathcal{L}^0(\mu)$. Show that U is a solid linear subspace of $L^0(\mu)$. Show that if μ is locally finite then U includes $L^p(\mu)$ for every $p \in [0,\infty]$.

(f) Let X be a topological space. (i) Let μ, ν be two totally finite Borel measures which agree on the closed sets. Show that they are equal. (*Hint*: 136C.) (ii) Let μ, ν be two totally finite Baire measures which agree on the zero sets. Show that they are equal.

(g) Let $(X,\mathfrak{T})$ be a topological space, μ a measure on X, and Y a subset of X; let $\mathfrak{T}_Y$, μ_Y be the subspace topology and measure. Show that if μ is a topological measure, or locally finite, or a Borel measure, so is μ_Y.

(h) Let $\langle (X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of measure spaces, with direct sum (X,Σ,μ); suppose that we are given a topology $\mathfrak{T}_i$ on each X_i, and let $\mathfrak{T}$ be the disjoint union topology on X. Show that μ is a topological measure, or locally finite, or effectively locally finite, or a Borel measure, or a Baire measure, or strictly positive, iff every μ_i is.

(i) Let (X,Σ,μ) and (Y,T,ν) be two measure spaces, with c.l.d. product measure λ on $X\times Y$. Suppose we are given topologies $\mathfrak{T}$, $\mathfrak{S}$ on X, Y respectively, and give $X\times Y$ the product topology. Show that λ is locally finite, or effectively locally finite, if μ and ν are.

(j) Let κ be any cardinal of uncountable cofinality (definition: 3A1Fb). Show that there is a complete topological probability measure μ on κ defined by saying that $\mu E = 1$ if E includes a cofinal closed set in κ, 0 if E is disjoint from some cofinal closed set. Show that μ is inner regular with respect to the closed sets but is not completion regular.

411Y Further exercises (a) Show that a function $f:\mathbb{R}^r\to\mathbb{R}^s$ is measurable iff it is almost continuous (where $\mathbb{R}^r$ is endowed with Lebesgue measure and its usual topology, of course). (*Hint*: 256F.)

(b) Let (X,ρ) be a metric space, $r \geq 0$, and write μ_{Hr} for r-dimensional Hausdorff measure on X (264K, §471). (i) Show that μ_{Hr} is a topological measure, outer regular with respect to the Borel sets. (ii) Show that if X is complete then the c.l.d. version of μ_{Hr} is tight, therefore completion regular.

(c) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space. Set $\mathcal{E} = \{E : E \subseteq X, \mu(\partial E) = 0\}$, where ∂E is the boundary of A. (i) Show that $\mathcal{E}$ is a subalgebra of $\mathcal{P}X$, and that every member of $\mathcal{E}$ is measured by the completion of μ. ($\mathcal{E}$ is sometimes called the **Jordan algebra** of $(X, \mathfrak{T}, \Sigma, \mu)$. Do not confuse with the 'Jordan algebras' of abstract algebra.) (ii) Suppose that μ is complete and totally finite and inner regular with respect to the closed sets, and that $\mathfrak{T}$ is normal. Show that $\{E^\bullet : E \in \mathcal{E}\}$ is dense in the measure algebra of μ endowed with its usual topology. (*Hint*: if $f : X \to \mathbb{R}$ is continuous, then $\{x : f(x) \leq \alpha\} \in \mathcal{E}$ for all but countably many α.) (iii) Suppose that μ is a quasi-Radon measure and $\mathfrak{T}$ is completely regular. Show that $\{E^\bullet : E \in \mathcal{E}\}$ is dense in the measure algebra of μ. (*Hint*: 414Aa.)

411 Notes and comments Of course the list above can give only a rough idea of the ways in which topologies and measures can interact. In particular I have rather arbitrarily given a sort of priority to three particular relationships between the domain Σ of a measure and the topology: 'topological measure space' (in which Σ includes the Borel σ-algebra), 'Borel measure' (in which Σ is precisely the Borel σ-algebra) and 'Baire measure' (in which Σ is the Baire σ-algebra).

Abstract topological measure theory is a relatively new subject, and there are many technical questions on which different authors take different views. For instance, the phrase 'Radon measure' is commonly used to mean what I would call a 'tight locally finite Borel measure' (cf. 416F); and some writers enlarge the definition of 'topological measure' to include Baire measures as defined above.

I give very few examples at this stage, two drawn from the constructions of Volumes 1-3 (Lebesgue measure and Stone spaces, 411O-411P) and one new one ('Dieudonné's measure', 411Q), with a glance at the countable-cocountable measure of ω_1 (411R). The most glaring omission is that of the product measures on $\{0,1\}^I$ and $[0,1]^I$. I pass these by at the moment because a proper study of them requires rather more preparation than can be slipped into a parenthesis. (I return to them in 416U.) I have also omitted any discussion of 'measurable' and 'almost continuous' functions, except for a reference to a theorem in Volume 2 (411Ya), which will have to be repeated and amplified later on (§418). There is an obvious complementarity between the notions of 'inner' and 'outer' regularity (411B, 411D), but it works well only for totally finite spaces (411Xa); in other cases it may not be obvious what will happen (411O, 411Pe, 412W).

412 Inner regularity

As will become apparent as the chapter progresses, the concepts introduced in §411 are synergic; their most interesting manifestations are in combinations of various kinds. Any linear account of their properties will be more than usually like a space-filling curve. But I have to start somewhere, and enough results can be expressed in terms of inner regularity, more or less by itself, to be a useful beginning.

After a handful of elementary basic facts (412A) and a list of standard applications (412B), I give some useful sufficient conditions for inner regularity of topological and Baire measures (412D, 412E, 412G), based on an important general construction (412C). The rest of the section amounts to a review of ideas from Volume 2 and Chapter 32 in the light of the new concept here. I touch on completions (412H), c.l.d. versions and complete locally determined spaces (412H, 412J, 412L), strictly localizable spaces (412I), inverse-measure-preserving functions (412K, 412M), measure algebras (412N), subspaces (412O, 412P), indefinite-integral measures (412Q) and product measures (412R-412V), with a brief mention of outer regularity (412W); most of the hard work has already been done in Chapters 21 and 25.

412A I begin by repeating a lemma from Chapter 34, with some further straightforward facts.

Lemma (a) Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of sets such that

whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K > 0$.

Then whenever $E \in \Sigma$ there is a countable disjoint family $\langle K_i \rangle_{i \in I}$ in $\mathcal{K} \cap \Sigma$ such that $K_i \subseteq E$ for every i and $\sum_{i \in I} \mu K_i = \mu E$. If moreover

(†) $K \cup K' \in \mathcal{K}$ whenever K, K' are disjoint members of $\mathcal{K}$,

then μ is inner regular with respect to $\mathcal{K}$. If $\bigcup_{i \in I} K_i \in \mathcal{K}$ for every countable disjoint family $\langle K_i \rangle_{i \in I}$ in $\mathcal{K}$, then for every $E \in \Sigma$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K = \mu E$.

(b) Let (X, Σ, μ) be a measure space, T a σ-subalgebra of Σ, and $\mathcal{K}$ a family of sets. If μ is inner regular with respect to T and $\mu\restriction \mathrm{T}$ is inner regular with respect to $\mathcal{K}$, then μ is inner regular with respect to $\mathcal{K}$.

(c) Let (X, Σ, μ) be a semi-finite measure space and $\langle \mathcal{K}_n \rangle_{n\in\mathbb{N}}$ a sequence of families of sets such that μ is inner regular with respect to $\mathcal{K}_n$ and

(‡) if $\langle K_i \rangle_{i\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}_n$, then $\bigcap_{i\in\mathbb{N}} K_i \in \mathcal{K}_n$

for every $n \in \mathbb{N}$. Then μ is inner regular with respect to $\bigcap_{n\in\mathbb{N}} \mathcal{K}_n$.

proof (a) This is 342B-342C.

(b) If $E \in \Sigma$ and $\gamma < \mu E$, there are an $F \in \mathrm{T}$ such that $F \subseteq E$ and $\mu F > \gamma$, and a $K \in \mathcal{K} \cap \mathrm{T}$ such that $K \subseteq F$ and $\mu K \geq \gamma$.

(c) Suppose that $E \in \Sigma$ and that $0 \leq \gamma < \mu E$. Because μ is semi-finite, there is an $F \in \Sigma$ such that $F \subseteq E$ and $\gamma < \mu F < \infty$ (213A). Choose $\langle K_i \rangle_{i\in\mathbb{N}}$ inductively, as follows. Start with $K_0 = F$. Given that $K_i \in \Sigma$ and $\gamma < \mu K_i$, then let $n_i \in \mathbb{N}$ be such that $2^{-n_i}(i+1)$ is an odd integer, and choose $K_{i+1} \in \mathcal{K}_{n_i}$ such that $K_{i+1} \subseteq K_i$ and $\mu K_{i+1} > \gamma$; this will be possible because μ is inner regular with respect to $\mathcal{K}_{n_i}$. Consider $K = \bigcap_{i\in\mathbb{N}} K_i$. Then $K \subseteq E$ and $\mu K = \lim_{i\to\infty} \mu K_i \geq \gamma$. But also

$$K = \bigcap_{j\in\mathbb{N}} K_{2^n(2j+1)} \in \mathcal{K}_n$$

because $\langle K_{2^n(2j+1)} \rangle_{j\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}_n$, for each n. So $K \in \bigcap_{n\in\mathbb{N}} \mathcal{K}_n$. As E and γ are arbitrary, μ is inner regular with respect to $\bigcap_{n\in\mathbb{N}} \mathcal{K}_n$.

412B Corollary Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X. Suppose that $\mathcal{K}$ is

either the family of Borel subsets of X

or the family of closed subsets of X

or the family of compact subsets of X

or the family of zero sets in X,

and suppose that whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K > 0$. Then μ is inner regular with respect to $\mathcal{K}$.

proof In every case, $\mathcal{K}$ satisfies the condition (†) of 412Aa.

412C The next lemma provides a particularly useful method of proving that measures are inner regular with respect to 'well-behaved' families of sets.

Lemma Let (X, Σ, μ) be a semi-finite measure space, and suppose that $\mathcal{A} \subseteq \Sigma$ and $\mathcal{K}$ are such that

$\emptyset \in \mathcal{A} \subseteq \Sigma$,

(†) $K \cup K' \in \mathcal{K}$ whenever $K, K' \in \mathcal{K}$,

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ for every sequence $\langle K_n \rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$,

$X \setminus A \in \mathcal{A}$ for every $A \in \mathcal{A}$,

whenever $A \in \mathcal{A}$, $F \in \Sigma$ and $\mu(A \cap F) > 0$, there is a $K \in \mathcal{K} \cap \mathcal{A}$ such that $K \subseteq A$ and $\mu(K \cap F) > 0$.

Let T be the σ-subalgebra of Σ generated by $\mathcal{A}$. Then $\mu\restriction \mathrm{T}$ is inner regular with respect to $\mathcal{K}$.

proof (a) Write $\mathfrak{A}$ for the measure algebra of (X, Σ, μ), and $\mathcal{L} = \mathcal{K} \cap \mathrm{T}$, so that $\mathcal{L}$ is also closed under finite unions and countable intersections. Set

$$\mathcal{H} = \{E : E \in \Sigma, \sup_{L\in\mathcal{L}, L\subseteq E} L^\bullet = E^\bullet\} \text{ in } \mathfrak{A},$$

$$\mathrm{T}' = \{E : E \in \mathcal{H}, X \setminus E \in \mathcal{H}\},$$

so that the last two conditions tell us that $\mathcal{A} \subseteq \mathrm{T}'$.

(b) The intersection of any sequence in $\mathcal{H}$ belongs to $\mathcal{H}$. **P** Let $\langle H_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ with intersection H. Write A_n for $\{L^\bullet : L \in \mathcal{L}, L \subseteq H_n\} \subseteq \mathfrak{A}$ for each $n \in \mathbb{N}$. Since $\mathfrak{A}$ is weakly (σ, ∞)-distributive (322F), A_n is upwards-directed, and $\sup A_n = H_n^\bullet$ for each $n \in \mathbb{N}$,

$$H^\bullet = \inf_{n\in\mathbb{N}} H_n^\bullet$$

(because $F \mapsto F^\bullet : \Sigma \to \mathfrak{A}$ is sequentially order-continuous, by 321H)

$$= \inf_{n\in\mathbb{N}} \sup A_n = \sup\{\inf_{n\in\mathbb{N}} a_n : a_n \in A_n \text{ for every } n \in \mathbb{N}\}$$

(316J)

$$= \sup\{(\bigcap_{n\in\mathbb{N}} L_n)^\bullet : L_n \in \mathcal{L},\ L_n \subseteq H_n \text{ for every } n \in \mathbb{N}\}$$

$$\subseteq \{L^\bullet : L \in \mathcal{L},\ L \subseteq H\}$$

(by (‡))

$$\subseteq H^\bullet,$$

and $H \in \mathcal{H}$. **Q**

(c) The union of any sequence in $\mathcal{H}$ belongs to $\mathcal{H}$. **P** If $\langle H_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{H}$ with union H then

$$\sup_{L\in\mathcal{L},L\subseteq H} L^\bullet \supseteq \sup_{n\in\mathbb{N}} \sup_{L\in\mathcal{L},L\subseteq E_n} L^\bullet = \sup_{n\in\mathbb{N}} H_n^\bullet = H^\bullet,$$

so $H \in \mathcal{H}$. **Q**

(d) T' is a σ-subalgebra of Σ. **P** (i) $\emptyset$ and X belong to $\mathcal{A} \subseteq \mathcal{H}$, so $\emptyset \in \mathrm{T}'$. (ii) Obviously $X \setminus E \in \mathrm{T}'$ whenever $E \in \mathrm{T}'$. (iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in T' with union E then $E \in \mathcal{H}$, by (c); but also $X \setminus E = \bigcap_{n\in\mathbb{N}}(X \setminus E_n)$ belongs to $\mathcal{H}$, by (b). So $E \in \mathrm{T}'$. **Q**

(e) Accordingly $\mathrm{T} \subseteq \mathrm{T}'$, and $E^\bullet = \sup_{L\in\mathcal{L},L\subseteq E} L^\bullet$ for every $E \in \mathrm{T}$. It follows at once that if $E \in \mathrm{T}$ and $\mu E > 0$, there must be an $L \in \mathcal{L}$ such that $L \subseteq E$ and $\mu L > 0$; since (†) is true, and $\mathcal{L} \subseteq \mathrm{T}$, we can apply 412Aa to see that $\mu\restriction\mathrm{T}$ is inner regular with respect to $\mathcal{L}$, therefore with respect to $\mathcal{K}$.

412D As corollaries of the last lemma I give two-and-a-half basic theorems.

Theorem Let $(X, \mathfrak{T})$ be a topological space and μ a semi-finite Baire measure on X. Then μ is inner regular with respect to the zero sets.

proof Write Σ for the Baire σ-algebra of X, the domain of μ, $\mathcal{K}$ for the family of zero sets, and $\mathcal{A}$ for $\mathcal{K} \cup \{X \setminus K : K \in \mathcal{K}\}$. Since the union of two zero sets is a zero set (4A2C(b-ii)), the intersection of a sequence of zero sets is a zero set (4A2C(b-iii)), and the complement of a zero set is the union of a sequence of zero sets (4A2C(b-vi)), the conditions of 412C are satisfied; and as the σ-algebra generated by $\mathcal{A}$ is just Σ, μ is inner regular with respect to $\mathcal{K}$.

412E Theorem Let $(X, \mathfrak{T})$ be a perfectly normal topological space (e.g., any metrizable space). Then any semi-finite Borel measure on X is inner regular with respect to the closed sets.

proof Because the Baire and Borel σ-algebras are the same (4A3Kb), this is a special case of 412D.

412F Lemma Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X such that μ is effectively locally finite with respect to $\mathfrak{T}$. Then

$$\mu E = \sup\{\mu(E \cap G) : G \text{ is a measurable open set of finite measure}\}$$

for every $E \in \Sigma$.

proof Apply 412Aa with $\mathcal{K}$ the family of subsets of measurable open sets of finite measure.

412G Theorem Let (X, Σ, μ) be a measure space with a topology $\mathfrak{T}$ such that μ is effectively locally finite with respect to $\mathfrak{T}$ and Σ is the σ-algebra generated by $\mathfrak{T} \cap \Sigma$. If

$$\mu G = \sup\{\mu F : F \in \Sigma \text{ is closed}, F \subseteq G\}$$

for every measurable open set G of finite measure, then μ is inner regular with respect to the closed sets.

proof In 412C, take $\mathcal{K}$ to be the family of measurable closed subsets of X, and $\mathcal{A}$ to be the family of measurable sets which are *either* open *or* closed. If $G \in \Sigma \cap \mathfrak{T}$, $F \in \Sigma$ and $\mu(G \cap F) > 0$, then there is an open set H of finite measure such that $\mu(H \cap G \cap F) > 0$, because μ is effectively locally finite; now there is a $K \in \mathcal{K}$ such that $K \subseteq H \cap G$ and $\mu K > \mu(H \cap G) - \mu(H \cap G \cap F)$, so that $\mu(K \cap F) > 0$. This is the only non-trivial item in the list of hypotheses in 412C, so we can conclude that $\mu \restriction \mathrm{T}$ is inner regular with respect to $\mathcal{K}$, where T is the σ-algebra generated by $\mathcal{A}$; but of course this is just Σ.

Remark There is a similar result in 416F(iii) below.

412H Proposition Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of sets.

(a) If μ is inner regular with respect to $\mathcal{K}$, so are its completion $\hat{\mu}$ (212C) and c.l.d. version $\tilde{\mu}$ (213E).

(b) Now suppose that

(‡) $\bigcap_{n \in \mathbb{N}} K_n \in \mathcal{K}$ whenever $\langle K_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$.

If *either* $\hat{\mu}$ is inner regular with respect to $\mathcal{K}$ *or* μ is semi-finite and $\tilde{\mu}$ is inner regular with respect to $\mathcal{K}$, then μ is inner regular with respect to $\mathcal{K}$.

proof (a) If F belongs to the domain of $\hat{\mu}$, then there is an $E \in \Sigma$ such that $E \subseteq F$ and $\hat{\mu}(F \setminus E) = 0$. So if $0 \leq \gamma < \hat{\mu}F = \mu E$, there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E \subseteq F$ and $\hat{\mu}K = \mu K \geq \gamma$.

If H belongs to the domain of $\tilde{\mu}$ and $0 \leq \gamma < \tilde{\mu}H$, there is an $E \in \Sigma$ such that $\mu E < \infty$ and $\hat{\mu}(E \cap H) > \gamma$ (213D). Now there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E \cap H$ and $\mu K \geq \gamma$. As $\mu K < \infty$, $\tilde{\mu}K = \mu K \geq \gamma$.

(b) Write $\check{\mu}$ for whichever of $\hat{\mu}$, $\tilde{\mu}$ is supposed to be inner regular with respect to $\mathcal{K}$. Then $\check{\mu}$ is inner regular with respect to Σ (212Ca, 213Fc), so is inner regular with respect to $\mathcal{K} \cap \Sigma$ (412Ac). Also $\check{\mu}$ extends μ (212D, 213Hc). Take $E \in \Sigma$ and $\gamma < \mu E = \check{\mu}E$. Then there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\gamma < \check{\mu}K = \mu K$. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$.

412I Lemma Let (X, Σ, μ) be a strictly localizable measure space and $\mathcal{K}$ a family of sets such that whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K > 0$.

(a) There is a decomposition $\langle X_i \rangle_{i \in I}$ of X such that at most one X_i does not belong to $\mathcal{K}$, and that exceptional one, if any, is negligible.

(b) There is a disjoint family $\mathcal{L} \subseteq \mathcal{K} \cap \Sigma$ such that $\mu^* A = \sum_{L \in \mathcal{L}} \mu^*(A \cap L)$ for every $A \subseteq X$.

(c) If μ is σ-finite then the family $\langle X_i \rangle_{i \in I}$ of (a) and the set $\mathcal{L}$ of (b) can be taken to be countable.

proof (a) Let $\langle E_j \rangle_{j \in J}$ be any decomposition of X. For each $j \in J$, let $\mathcal{K}_j$ be a maximal disjoint subset of

$$\{K : K \in \mathcal{K} \cap \Sigma, K \subseteq E_j, \mu K > 0\}.$$

Because $\mu E_j < \infty$, $\mathcal{K}_j$ must be countable. Set $E_j' = E_j \setminus \bigcup \mathcal{K}_j$. By the maximality of $\mathcal{K}_j$, E_j' cannot include any non-negligible set in $\mathcal{K} \cap \Sigma$; but this means that $\mu E_j' = 0$. Set $X' = \bigcup_{j \in J} E_j'$. Then

$$\mu X' = \sum_{j \in J} \mu(X' \cap E_j) = \sum_{j \in J} \mu E_j' = 0.$$

Note that if $j, j' \in J$ are distinct, and $K \in \mathcal{K}_j$, $K' \in \mathcal{K}_{j'}$, then $K \cap K' = \emptyset$; thus $\mathcal{L} = \bigcup_{j \in J} \mathcal{K}_j$ is disjoint. Let $\langle X_i \rangle_{i \in I}$ be any indexing of $\{X'\} \cup \mathcal{L}$. This is a partition (that is, disjoint cover) of X into sets of finite measure. If $E \subseteq X$ and $E \cap X_i \in \Sigma$ for every $i \in I$, then for every $j \in J$

$$E \cap E_j = (E \cap X' \cap E_j) \cup \bigcup_{K \in \mathcal{K}_j} E \cap K$$

belongs to Σ, so that $E \in \Sigma$ and

$$\mu E = \sum_{j \in J} \sum_{K \in \mathcal{K}_j} \mu(E \cap K) = \sum_{i \in I} \mu(E \cap X_i).$$

Thus $\langle X_i \rangle_{i \in I}$ is a decomposition of X, and it is of the right type because every X_i but one belongs to $\mathcal{L} \subseteq \mathcal{K}$.

(b) If now $A \subseteq X$ is any set,

$$\mu^* A = \mu_A A = \sum_{i \in I} \mu_A(A \cap X_i) = \sum_{i \in I} \mu^*(A \cap X_i)$$

by 214Ia, writing μ_A for the subspace measure on A. So we have

$$\mu^* A = \mu^*(A \cap X') + \sum_{L \in \mathcal{L}} \mu^*(A \cap L) = \sum_{L \in \mathcal{L}} \mu^*(A \cap L),$$

while $\mathcal{L} \subseteq \mathcal{K}$ is disjoint.

(c) If μ is σ-finite we can take J to be countable, so that I and $\mathcal{L}$ will also be countable.

412J Proposition Let (X, Σ, μ) be a complete locally determined measure space, and $\mathcal{K}$ a family of sets such that μ is inner regular with respect to $\mathcal{K}$.

(a) If $E \subseteq X$ is such that $E \cap K \in \Sigma$ for every $K \in \mathcal{K} \cap \Sigma$, then $E \in \Sigma$.

(b) If $E \subseteq X$ is such that $E \cap K$ is negligible for every $K \in \mathcal{K} \cap \Sigma$, then E is negligible.

(c) For any $A \subseteq X$, $\mu^* A = \sup_{K \in \mathcal{K} \cap \Sigma} \mu^*(A \cap K)$.

(d) Let f be a non-negative $[0, \infty]$-valued function defined on a subset of X. If $\int_K f$ is defined in $[0, \infty]$ for every $K \in \mathcal{K}$, then $\int f$ is defined and equal to $\sup_{K \in \mathcal{K}} \int_K f$.

(e) If f is a μ-integrable function and $\epsilon > 0$, there is a $K \in \mathcal{K}$ such that $\int_{X \setminus K} |f| \leq \epsilon$.

Remark In (c), we must interpret $\sup \emptyset$ as 0 if $\mathcal{K} \cap \Sigma = \emptyset$.

proof (a) If $F \in \Sigma$ and $\mu F < \infty$, then $E \cap F \in \Sigma$. **P** If $\mu F = 0$, this is trivial, because μ is complete and $E \cap F$ is negligible. Otherwise, there is a sequence $\langle K_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{K} \cap \Sigma$ such that $K_n \subseteq F$ for each n and $\sup_{n \in \mathbb{N}} \mu K_n = \mu F$. Now $E \cap F \setminus \bigcup_{n \in \mathbb{N}} K_n$ is negligible, therefore measurable, while $E \cap K_n$ is measurable for every $n \in \mathbb{N}$, by hypothesis; so $E \cap F$ is measurable. **Q** As μ is locally determined, $E \in \Sigma$, as claimed.

(b) By (a), $E \in \Sigma$; and because μ is inner regular with respect to $\mathcal{K}$, μE must be 0.

(c) Let μ_A be the subspace measure on A. Because μ is complete and locally determined, μ_A is semi-finite (214Ic). So if $0 \leq \gamma < \mu^* A = \mu_A A$, there is an $H \subseteq A$ such that $\mu_A H$ is defined, finite and greater than γ. Let $E \in \Sigma$ be a measurable envelope of H (132Ee), so that $\mu E = \mu^* H > \gamma$. Then there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K \geq \gamma$. In this case

$$\mu^*(A \cap K) \geq \mu^*(H \cap K) = \mu(E \cap K) = \mu K \geq \gamma.$$

As γ is arbitrary,

$$\mu^* A \leq \sup_{K \in \mathcal{K} \cap \Sigma} \mu^*(A \cap K);$$

but the reverse inequality is trivial, so we have the result.

(d) Applying (b) with $E = X \setminus \operatorname{dom} f$, we see that f is defined almost everywhere in X. Applying (a) with $E = \{x : x \in \operatorname{dom} f,\ f(x) \geq \alpha\}$ for each $\alpha \in \mathbb{R}$, we see that f is measurable. So $\int f$ is defined in $[0, \infty]$, and of course $\int f \geq \sup_{K \in \mathcal{K}} \int_K f$. If $\gamma < \int f$, there is a non-negative simple function g such that $g \leq_{\text{a.e.}} f$ and $\int g > \gamma$; taking $E = \{x : g(x) > 0\}$, there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu(E \setminus K) \|g\|_\infty \leq \gamma - \int g$, so that $\int_K f \geq \int_K g \geq \gamma$. As γ is arbitrary, $\int f = \sup_{K \in \mathcal{K}} \int_K f$.

(e) By (d), there is a $K \in \mathcal{K}$ such that $\int_K |f| \geq \int |f| - \epsilon$.

Remark See also 413F below.

412K Proposition Let (X, Σ, μ) be a complete locally determined measure space, (Y, T, ν) a measure space and $f : X \to Y$ a function. Suppose that $\mathcal{K} \subseteq \mathrm{T}$ is such that

(i) ν is inner regular with respect to $\mathcal{K}$;

(ii) $f^{-1}[K] \in \Sigma$ and $\mu f^{-1}[K] = \nu K$ for every $K \in \mathcal{K}$;

(iii) whenever $E \in \Sigma$ and $\mu E > 0$ there is a $K \in \mathcal{K}$ such that $\nu K < \infty$ and $\mu(E \cap f^{-1}[K]) > 0$.

Then f is inverse-measure-preserving for μ and ν.

proof (a) If $F \in \mathrm{T}$, $E \in \Sigma$ and $\mu E < \infty$, then $E \cap f^{-1}[F] \in \Sigma$. **P** Let H_1, $H_2 \in \Sigma$ be measurable envelopes for $E \cap f^{-1}[F]$ and $E \setminus f^{-1}[F]$ respectively. **?** If $\mu(H_1 \cap H_2) > 0$, there is a $K \in \mathcal{K}$ such that νK is finite

and $\mu(H_1 \cap H_2 \cap f^{-1}[K]) > 0$. Because ν is inner regular with respect to $\mathcal{K}$, there are K_1, $K_2 \in \mathcal{K}$ such that $K_1 \subseteq K \cap F$, $K_2 \subseteq K \setminus F$ and

$$\nu K_1 + \nu K_2 > \nu(K \cap F) + \nu(K \setminus F) - \mu(H_1 \cap H_2 \cap f^{-1}[K]) = \nu K - \mu(H_1 \cap H_2 \cap f^{-1}[K]).$$

Now

$$\mu(H_1 \cap f^{-1}[K_2]) = \mu^*(E \cap f^{-1}[F] \cap f^{-1}[K_2]) = 0,$$

$$\mu(H_2 \cap f^{-1}[K_1]) = \mu^*(E \cap f^{-1}[K_1] \setminus f^{-1}[F]) = 0,$$

so $\mu(H_1 \cap H_2 \cap f^{-1}[K_1 \cup K_2]) = 0$ and

$$\begin{aligned}\mu(H_1 \cap H_2 \cap f^{-1}[K]) &\leq \mu(f^{-1}[K] \setminus f^{-1}[K_1 \cup K_2]) \\ &= \mu f^{-1}[K] - \mu f^{-1}[K_1] - \mu f^{-1}[K_2] \\ &= \nu K - \nu K_1 - \nu K_2 < \mu(H_1 \cap H_2 \cap f^{-1}[K]),\end{aligned}$$

which is absurd. **X**

Now $(E \cap H_1) \setminus (E \cap f^{-1}[F]) \subseteq H_1 \cap H_2$ is negligible, therefore measurable (because μ is complete), and $E \cap f^{-1}[F] \in \Sigma$, as claimed. **Q**

(b) It follows (because μ is locally determined) that $f^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$.

(c) If $F \in \mathrm{T}$ and $\nu F = 0$ then $\mu f^{-1}[F] = 0$. **P?** Otherwise, there is a $K \in \mathcal{K}$ such that $\nu K < \infty$ and

$$0 < \mu(f^{-1}[F] \cap f^{-1}[K]) = \mu f^{-1}[F \cap K].$$

Let $K' \in \mathcal{K}$ be such that $K' \subseteq K \setminus F$ and $\nu K' > \nu K - \mu f^{-1}[F \cap K]$. Then $f^{-1}[K'] \cap f^{-1}[F \cap K] = \emptyset$, so

$$\nu K = \mu f^{-1}[K] \geq \mu f^{-1}[K'] + \mu f^{-1}[F \cap K] > \nu K' + \nu K - \nu K' = \nu K,$$

which is absurd. **XQ**

(d) Finally, $\mu f^{-1}[F] = \nu F$ for every $F \in \mathrm{T}$. **P** Let $\langle K_i \rangle_{i \in I}$ be a countable disjoint family in $\mathcal{K}$ such that $K_i \subseteq F$ for every i and $\sum_{i \in I} \nu K_i = \nu F$ (412Aa). Set $F' = F \setminus \bigcup_{i \in I} K_i$. Then

$$\mu f^{-1}[F] = \mu f^{-1}[F'] + \textstyle\sum_{i \in I} \mu f^{-1}[K_i] = \mu f^{-1}[F'] + \sum_{i \in I} \nu K_i = \mu f^{-1}[F'] + \nu F.$$

If $\nu F = \infty$ then surely $\mu f^{-1}[F] = \infty = \nu F$. Otherwise, $\nu F' = 0$ so $\mu f^{-1}[F'] = 0$ (by (c)) and again $\mu f^{-1}[F] = \nu F$. **Q**

Thus f is inverse-measure-preserving.

412L Corollary Let X be a set and $\mathcal{K}$ a family of subsets of X. Suppose that μ, ν are two complete locally determined measures on X, with domains including $\mathcal{K}$, agreeing on $\mathcal{K}$, and both inner regular with respect to $\mathcal{K}$. Then they are identical (and, in particular, have the same domain).

proof Apply 412K with $X = Y$ and f the identity function to see that μ extends ν; similarly, ν extends μ and the two measures are the same.

412M Corollary Let (X, Σ, μ) be a complete probability space, (Y, T, ν) a probability space and $f : X \to Y$ a function. Suppose that whenever $F \in \mathrm{T}$ and $\nu F > 0$ there is a $K \in \mathrm{T}$ such that $K \subseteq F$, $\nu K > 0$, $f^{-1}[K] \in \Sigma$ and $\mu f^{-1}[K] \geq \nu K$. Then f is inverse-measure-preserving.

proof Set $\mathcal{K}^* = \{K : K \in \mathrm{T}, f^{-1}[K] \in \Sigma, \mu f^{-1}[K] \geq \nu K\}$. Then $\mathcal{K}^*$ is closed under countable disjoint unions and includes $\mathcal{K}$, so for every $F \in \mathrm{T}$ there is a $K \in \mathcal{K}^*$ such that $K \subseteq F$ and $\nu K = \nu F$, by 412Aa. But this means that $\mu f^{-1}[K] = \nu K$ for every $K \in \mathcal{K}^*$. **P** There is a $K' \in \mathcal{K}^*$ such that $K' \subseteq Y \setminus K$ and $\nu K' = 1 - \nu K$; but in this case

$$\mu f^{-1}[K'] + \mu f^{-1}[K] \leq 1 = \nu K' + \nu K,$$

so $\mu f^{-1}[K]$ must be equal to νK. **Q** Moreover, there is a $K^* \in \mathcal{K}^*$ such that $\nu K^* = \nu Y = 1$, so $\mu f^{-1}[K^*] = \mu X = 1$ and $\mu(E \cap f^{-1}[K^*]) > 0$ whenever $\mu E > 0$. Applying 412K to $\mathcal{K}^*$ we have the result.

412N Lemma Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of subsets of X such that μ is inner regular with respect to $\mathcal{K}$. Then

$$E^\bullet = \sup\{K^\bullet : K \in \mathcal{K} \cap \Sigma,\ K \subseteq E\}$$

in the measure algebra $\mathfrak{A}$ of μ, for every $E \in \Sigma$. In particular, $\{K^\bullet : K \in \mathcal{K} \cap \Sigma\}$ is order-dense in $\mathfrak{A}$; and if $\mathcal{K}$ is closed under finite unions, then $\{K^\bullet : K \in \mathcal{K} \cap \Sigma\}$ is topologically dense in $\mathfrak{A}$ for the measure-algebra topology.

proof ? If $E^\bullet \neq \sup\{K^\bullet : K \in \mathcal{K} \cap \Sigma,\ K \subseteq E\}$, there is a non-zero $a \in \mathfrak{A}$ such that $a \subseteq E^\bullet \setminus K^\bullet$ whenever $K \in \mathcal{K} \cap \Sigma$ and $K \subseteq E$. Express a as $F^\bullet$ where $F \subseteq E$. Then $\mu F > 0$, so there is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq F$ and $\mu K > 0$. But in this case $0 \neq K^\bullet \subseteq a$, while $K \subseteq E$. **X**

It follows at once that $D = \{K^\bullet : K \in \mathcal{K} \cap \Sigma\}$ is order-dense. If $\mathcal{K}$ is closed under finite unions, and $a \in \mathfrak{A}$, then $D_a = \{d : d \in D,\ d \subseteq a\}$ is upwards-directed and has supremum a, so $a \in \overline{D}_a \subseteq \overline{D}$ (323D(a-ii)).

412O Lemma Let (X, Σ, μ) be a measure space and $\mathcal{K}$ a family of subsets of X such that μ is inner regular with respect to $\mathcal{K}$.

(a) If $E \in \Sigma$, then the subspace measure μ_E (131B) is inner regular with respect to $\mathcal{K}$.

(b) Let $Y \subseteq X$ be any set such that the subspace measure μ_Y (214A-214B) is semi-finite. Then μ_Y is inner regular with respect to $\mathcal{K}_Y = \{K \cap Y : K \in \mathcal{K}\}$.

proof (a) This is elementary.

(b) Suppose that F belongs to the domain Σ_Y of μ_Y and $0 \leq \gamma < \mu_Y F$. Because μ_Y is semi-finite there is an $F' \in \Sigma_Y$ such that $F' \subseteq F$ and $\gamma < \mu_Y F' < \infty$. Let E be a measurable envelope for F' with respect to μ, so that

$$\mu E = \mu^* F' = \mu_Y F' > \gamma.$$

There is a $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq E$ and $\mu K \geq \gamma$, in which case $K \cap Y \in \mathcal{K}_Y \cap \Sigma_Y$ and

$$\mu_Y(K \cap Y) = \mu^*(K \cap Y) = \mu^*(K \cap F') = \mu(K \cap E) = \mu K \geq \gamma.$$

As F and γ are arbitrary, μ_Y is inner regular with respect to $\mathcal{K}_Y$.

Remark Recall from 214I that if (X, Σ, μ) has locally determined negligible sets (in particular, is either strictly localizable or complete and locally determined), then all its subspaces are semi-finite.

412P Proposition Let (X, Σ, μ) be a measure space, $\mathfrak{T}$ a topology on X and Y a subset of X; write $\mathfrak{T}_Y$ for the subspace topology of Y and μ_Y for the subspace measure on Y. Suppose that *either* $Y \in \Sigma$ *or* μ_Y is semi-finite.

(a) If μ is a topological measure, so is μ_Y.

(b) If μ is inner regular with respect to the Borel sets, so is μ_Y.

(c) If μ is inner regular with respect to the closed sets, so is μ_Y.

(d) If μ is inner regular with respect to the zero sets, so is μ_Y.

(e) If μ is effectively locally finite, so is μ_Y.

proof (a) is an immediate consequence of the definitions of 'subspace measure', 'subspace topology' and 'topological measure'. The other parts follow directly from 412O if we recall that

(i) a subset of Y is Borel for $\mathfrak{T}_Y$ whenever it is expressible as $Y \cap E$ for some Borel set $E \subseteq X$ (4A3Ca);

(ii) a subset of Y is closed in Y whenever it is expressible as $Y \cap F$ for some closed set $F \subseteq X$;

(iii) a subset of Y is a zero set in Y whenever it is expressible as $Y \cap F$ for some zero set $F \subseteq X$ (4A2C(b-v));

(iv) μ is effectively locally finite iff it is inner regular with respect to subsets of open sets of finite measure.

412Q Proposition Let (X, Σ, μ) be a measure space, and ν an indefinite-integral measure over μ (definition: 234B). If μ is inner regular with respect to a family $\mathcal{K}$ of sets, so is ν.

proof Because μ and its completion $\hat{\mu}$ give the same integrals, ν is an indefinite-integral measure over $\hat{\mu}$; and as $\hat{\mu}$ is still inner regular with respect to $\mathcal{K}$ (412H), we may suppose that μ itself is complete. Let f

be a Radon-Nikodým derivative of ν with respect to μ; by 234Ca, we may suppose that $f : X \to [0, \infty[$ is Σ-measurable.

Suppose that $F \in \mathrm{dom}(\nu)$ and that $\gamma < \nu F$. Set $G = \{x : f(x) > 0\}$, so that $F \cap G \in \Sigma$ (234D). For $n \in \mathbb{N}$, set $H_n = \{x : x \in F, 2^{-n} \leq f(x) \leq 2^n\}$, so that $H_n \in \Sigma$ and

$$\nu F = \int f \times \chi F d\mu = \lim_{n\to\infty} \int f \times \chi H_n d\mu.$$

Let $n \in \mathbb{N}$ be such that $\int f \times \chi H_n d\mu > \gamma$.

If $\mu H_n = \infty$, there is a $K \in \mathcal{K}$ such that $K \subseteq H_n$ and $\mu K \geq 2^n\gamma$, so that $\nu K \geq \gamma$. If μH_n is finite, there is a $K \in \mathcal{K}$ such that $2^n(\mu H_n - \mu K) \leq \int f \times \chi H_n d\mu - \gamma$, so that $\int f \times \chi(H_n \setminus K)d\mu + \gamma \leq \int f \times \chi H_n$ and $\nu K = \int f \times \chi K \, d\mu \geq \gamma$. Thus in either case we have a $K \in \mathcal{K}$ such that $K \subseteq F$ and $\nu K \geq \gamma$; as F and γ are arbitrary, ν is inner regular with respect to $\mathcal{K}$.

412R Lemma Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with c.l.d. product space $(X \times Y, \Lambda, \lambda)$ (251F). Suppose that $\mathcal{K} \subseteq \mathcal{P}X$, $\mathcal{L} \subseteq \mathcal{P}Y$, $\mathcal{M} \subseteq \mathcal{P}(X \times Y)$ are such that

(i) μ is inner regular with respect to $\mathcal{K}$;

(ii) ν is inner regular with respect to $\mathcal{L}$;

(iii) $K \times L \in \mathcal{M}$ for all $K \in \mathcal{K}$, $L \in \mathcal{L}$;

(iv) $M \cup M' \in \mathcal{M}$ whenever M, $M' \in \mathcal{M}$;

(v) $\bigcap_{n\in\mathbb{N}} M_n \in \mathcal{M}$ for every sequence $\langle M_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{M}$.

Then λ is inner regular with respect to $\mathcal{M}$.

proof Write $\mathcal{A} = \{E \times Y : E \in \Sigma\} \cup \{X \times F : F \in \mathrm{T}\}$. Then if $V \in \mathcal{A}$, $W \in \Lambda$ and $\lambda(W \cap V) > 0$, there is an $M \in \mathcal{M} \cap \mathcal{A}$ such that $M \subseteq W$ and $\lambda(M \cap V) > 0$. **P** Suppose that $V = E \times Y$ where $E \in \Sigma$. There must be $E_0 \in \Sigma$ and $F_0 \in \mathrm{T}$, both of finite measure, such that $\lambda(W \cap V \cap (E_0 \times F_0)) > 0$ (251F). Now there is a $K \in \mathcal{K}$ such that $K \subseteq E \cap E_0$ and $\mu((E \cap E_0) \setminus K) \cdot \nu F_0 < \lambda(W \cap V \cap (E_0 \times F_0))$; but this means that $M = K \times Y$ is included in V and $\mu(W \cap M) > 0$. Reversing the roles of the coordinates, the same argument deals with the case in which $V = X \times F$ for some $F \in \mathrm{T}$. **Q**

By 412C, $\lambda_0 = \lambda{\restriction}\Sigma\widehat{\otimes}\mathrm{T}$ is inner regular with respect to $\mathcal{K}$. But λ is inner regular with respect to $\Sigma\widehat{\otimes}\mathrm{T}$ (251Ib) so is also inner regular with respect to $\mathcal{M}$ (412Ab).

412S Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with c.l.d. product space $(X \times Y, \Lambda, \lambda)$. Let $\mathfrak{T}$, $\mathfrak{S}$ be topologies on X and Y respectively, and give $X \times Y$ the product topology.

(a) If μ and ν are inner regular with respect to the closed sets, so is λ.

(b) If μ and ν are tight (that is, inner regular with respect to the closed compact sets), so is λ.

(c) If μ and ν are inner regular with respect to the zero sets, so is λ.

(d) If μ and ν are inner regular with respect to the Borel sets, so is λ.

(e) If μ and ν are effectively locally finite, so is λ.

proof We have only to read the conditions (i)-(v) of 412R carefully and check that they apply in each case. (In part (e), recall that 'effectively locally finite' is the same thing as 'inner regular with respect to the subsets of open sets of finite measure'.)

412T Lemma Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product probability space (X, Λ, λ) (§254). Suppose that $\mathcal{K}_i \subseteq \mathcal{P}X_i$, $\mathcal{M} \subseteq \mathcal{P}X$ are such that

(i) μ_i is inner regular with respect to $\mathcal{K}_i$ for each $i \in I$;

(ii) $\pi_i^{-1}[K] \in \mathcal{M}$ for every $i \in I$, $K \in \mathcal{K}_i$, writing $\pi_i(x) = x(i)$ for $x \in X$;

(iii) $M \cup M' \in \mathcal{M}$ whenever M, $M' \in \mathcal{M}$;

(iv) $\bigcap_{n\in\mathbb{N}} M_n \in \mathcal{M}$ for every sequence $\langle M_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{M}$.

Then λ is inner regular with respect to $\mathcal{M}$.

proof The argument is nearly identical to that of 412R. Write $\mathcal{A} = \{\pi_i^{-1}[E] : i \in I, E \in \Sigma_i\}$. Then if $V \in \mathcal{A}$, $W \in \Lambda$ and $\lambda(W \cap V) > 0$, express V as $\pi_i^{-1}[E]$, where $i \in I$ and $E \in \Sigma_i$, and take $K \in \mathcal{K}_i$ such that $K \subseteq E$ and $\mu_i(E \setminus K) < \lambda(W \cap V)$; then $M = \pi_i^{-1}[K]$ belongs to $\mathcal{M} \cap \mathcal{A}$, is included in W, and meets V in a non-negligible set. So, just as in 412R, the conditions of 412C are met.

It follows that $\lambda_0 = \lambda{\restriction}\widehat{\bigotimes}_{i\in I}\Sigma_i$ is inner regular with respect to $\mathcal{K}$. But λ is the completion of λ_0 (254Fd, 254Ff), so is also inner regular with respect to $\mathcal{M}$ (412Ha).

412U Proposition Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product probability space (X, Λ, λ). Suppose that we are given a topology $\mathfrak{T}_i$ on each X_i, and let $\mathfrak{T}$ be the product topology on X.

(a) If every μ_i is inner regular with respect to the closed sets, so is λ.

(b) If every μ_i is inner regular with respect to the zero sets, so is λ.

(c) If every μ_i is inner regular with respect to the Borel sets, so is λ.

proof This follows from 412T just as 412S follows from 412R.

412V Corollary Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, with product probability space (X, Λ, λ). Suppose that we are given a Hausdorff topology $\mathfrak{T}_i$ on each X_i, and let $\mathfrak{T}$ be the product topology on X. Suppose that every μ_i is tight, and that X_i is compact for all but countably many $i \in I$. Then λ is tight.

proof By 412Ua, λ is inner regular with respect to the closed sets. If $W \in \Lambda$ and $\gamma < \lambda W$, let $V \subseteq W$ be a measurable closed set such that $\lambda V > \gamma$. Let J be the set of those $i \in I$ such that X_i is not compact; we are supposing that J is countable. Let $\langle \epsilon_i\rangle_{i\in J}$ be a family of strictly positive real numbers such that $\sum_{i\in J} \epsilon_j \leq \lambda V - \gamma$ (4A1P). For each $i \in J$, let $K_i \subseteq X_i$ be a compact measurable set such that $\mu_i(X_i \setminus K_i) \leq \epsilon_i$; and for $i \in I \setminus J$, set $K_i = X_i$. Then $K = \prod_{i\in I} K_i$ is a compact measurable subset of X, and

$$\lambda(X \setminus K) \leq \textstyle\sum_{i\in J} \mu(X_i \setminus K_i) \leq \lambda V - \gamma,$$

so $\lambda(K \cap V) \geq \gamma$; while $K \cap V$ is a compact measurable subset of W. As W and γ are arbitrary, λ is tight.

***412W Outer regularity** I have already mentioned the complementary notion of 'outer regularity' (411D). In this book it will not be given much prominence. It is however a useful tool when dealing with Lebesgue measure (see, for instance, the proof of 225K), for reasons which the next proposition will make clear.

Proposition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) Suppose that μ is outer regular with respect to the open sets. Then for any integrable function $f : X \to [0, \infty]$ and $\epsilon > 0$, there is a lower semi-continuous measurable function $g : X \to [0, \infty]$ such that $f \leq g$ and $\int g \leq \epsilon + \int f$.

(b) Now suppose that there is a sequence of measurable open sets of finite measure covering X. Then the following are equiveridical:

(i) μ is inner regular with respect to the closed sets;

(ii) μ is outer regular with respect to the open sets;

(iii) for any measurable set $E \subseteq X$ and $\epsilon > 0$, there are a measurable closed set $F \subseteq E$ and a measurable open set $H \supseteq E$ such that $\mu(H \setminus F) \leq \epsilon$;

(iv) for every measurable function $f : X \to [0, \infty[$ and $\epsilon > 0$, there is a lower semi-continuous measurable function $g : X \to [0, \infty]$ such that $f \leq g$ and $\int g - f \leq \epsilon$;

(v) for every measurable function $f : X \to \mathbb{R}$ and $\epsilon > 0$, there is a lower semi-continuous measurable function $g : X \to]-\infty, \infty]$ such that $f \leq g$ and $\mu\{x : g(x) \geq f(x) + \epsilon\} \leq \epsilon$.

proof (a) Let $\eta \in]0, 1]$ be such that $\eta(7 + \int f d\mu) \leq \epsilon$. For $n \in \mathbb{Z}$, set $E_n = \{x : (1+\eta)^n \leq f(x) < (1+\eta)^{n+1}\}$, and let $E'_n \in \Sigma$ be a measurable cover of E_n; let $G_n \supseteq E'_n$ be a measurable open set such that $\mu G_n \leq 3^{-|n|}\eta + \mu E'_n$. Set $g = \sum_{n=-\infty}^{\infty} (1+\eta)^{n+1} \chi G_n$. Then g is lower semi-continuous (4A2B(d-iii), 4A2B(d-v)), $f \leq g$ and

$$\begin{aligned}\int g d\mu &= \sum_{n=-\infty}^{\infty} (1+\eta)^{n+1} \mu G_n \\ &\leq (1+\eta) \sum_{n=-\infty}^{\infty} (1+\eta)^n \mu E'_n + \sum_{n=-\infty}^{\infty} (1+\eta)^{n+1} 3^{-|n|}\eta \\ &\leq (1+\eta) \int f d\mu + 7\eta \leq \int f d\mu + \epsilon,\end{aligned}$$

as required.

(b) Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a sequence of open sets of finite measure covering X; replacing it by $\langle\bigcup_{i<n} G_i\rangle_{n\in\mathbb{N}}$ if necessary, we may suppose that $\langle G_n\rangle_{n\in\mathbb{N}}$ is non-decreasing and that $G_0=\emptyset$.

(i)⇒(iii) Suppose that μ is inner regular with respect to the closed sets, and that $E\in\Sigma$, $\epsilon>0$. For each $n\in\mathbb{N}$ let $F_n\subseteq G_n\setminus E$ be a measurable closed set such that $\mu F_n\geq\mu(G_n\setminus E)-2^{-n-2}\epsilon$. Then $H=\bigcup_{n\in\mathbb{N}}(G_n\setminus F_n)$ is a measurable open set including E and $\mu(H\setminus E)\leq\frac{1}{2}\epsilon$. Applying the same argument to $X\setminus E$, we get a closed set $F\subseteq E$ such that $\mu(E\setminus F)\leq\frac{1}{2}\epsilon$, so that $\mu(H\setminus F)\leq\epsilon$.

(ii)⇒(iii) The same idea works. Suppose that μ is outer regular with respect to the open sets, and that $E\in\Sigma$, $\epsilon>0$. For each $n\in\mathbb{N}$, let $H_n\supseteq G_n\cap E_n$ be an open set such that $\mu(H_n\setminus E_n)\leq 2^{-n-2}\epsilon$; then $H=\bigcup_{n\in\mathbb{N}}H_n$ is a measurable open set including E, and $\mu(H\setminus E)\leq\frac{1}{2}\epsilon$. Now repeat the argument on $X\setminus E$ to find a measurable closed set $F\subseteq E$ such that $\mu(E\setminus F)\leq\frac{1}{2}\epsilon$.

(iii)⇒(iv) Assume (iii), and let $f:X\to[0,\infty[$ be a measurable function, $\epsilon>0$. Set $\eta_n=2^{-n}\epsilon/(16+4\mu G_n)$ for each $n\in\mathbb{N}$. For $k\in\mathbb{N}$ set $E_k=\bigcup_{n\in\mathbb{N}}\{x:x\in G_n,\ k\eta_n\leq f(x)<(k+1)\eta_n\}$, and choose an open set $H_k\supseteq E_k$ such that $\mu(H_k\setminus E_k)\leq 2^{-k}$. Set

$$g=\sup_{k,n\in\mathbb{N}}(k+1)\eta_n\chi(G_n\cap H_k).$$

Then $g:X\to[0,\infty]$ is lower semi-continuous (4A2B(d-v) again). Since

$$\sup_{k,n\in\mathbb{N}}k\eta_n\chi(G_n\cap E_k)\leq f\leq\sup_{k,n\in\mathbb{N}}(k+1)\eta_n\chi(G_n\cap E_k),$$

$f\leq g$ and

$$g-f\leq\sup_{k,n\in\mathbb{N}}(k+1)\eta_n\chi(G_n\cap H_k\setminus E_k)+\sup_{k,n\in\mathbb{N}}\eta_n\chi(G_n\cap E_k)$$

has integral at most

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(k+1)\eta_n2^{-k}+\sum_{n=0}^{\infty}\eta_n\mu G_n\leq\epsilon.$$

(i)⇒(v) Assume (i), and suppose that $f:X\to\mathbb{R}$ is measurable and $\epsilon>0$. For each $n\in\mathbb{N}$, let $\alpha_n\geq 0$ be such that $\mu E_n<2^{-n-1}\epsilon$, where $E_n=\{x:x\in G_{n+1}\setminus G_n,\ f(x)\leq-\alpha_n\}$. Let $F_n\subseteq(G_{n+1}\setminus G_n)\setminus E_n$ be a measurable closed set such that $\mu((G_{n+1}\setminus G_n)\setminus F_n)\leq 2^{-n-2}\epsilon$. Because $\langle F_n\rangle_{n\in\mathbb{N}}$ is disjoint, $h=\sum_{n=0}^{\infty}\alpha_n\chi F_n$ is defined as a function from X to $\mathbb{R}$. $\{F_n:n\in\mathbb{N}\}$ is locally finite, so $\{x:h(x)\geq\alpha\}=\bigcup_{n\in\mathbb{N},\alpha_n\geq\alpha}F_n$ is closed for every $\alpha>0$ (4A2B(h-ii)), and h is upper semi-continuous. Now $f_1=f+h$ is a measurable function. Since (i)⇒(iii)⇒(iv), there is a measurable lower semi-continuous function $g_1:X\to[0,\infty]$ such that $f_1^+\leq g_1$ and $\int g_1-f_1^+\leq\frac{1}{2}\epsilon^2$, where $f_1^+=\max(0,f_1)$. But if we now set $g=g_1-h$, g is lower semi-continuous, $f\leq g$ and

$$\begin{aligned}\{x:f(x)+\epsilon\leq g(x)\}&\subseteq\{x:f_1^+(x)+\epsilon\leq g_1(x)\}\cup\{x:f_1(x)<0\}\\&\subseteq\{x:f_1^+(x)+\epsilon\leq g_1(x)\}\cup\bigcup_{n\in\mathbb{N}}(G_{n+1}\setminus G_n)\setminus F_n\end{aligned}$$

has measure at most ϵ, as required.

(iv)⇒(ii) and **(v)⇒(ii)** Suppose that either (iv) or (v) is true, and that $E\in\Sigma$, $\epsilon>0$. Then there is a measurable lower semi-continuous function $g:X\to]0,\infty]$ such that $\chi E\leq g$ and $\mu\{x:\chi E(x)+\frac{1}{2}\leq g(x)\}\leq\epsilon$, since this is certainly true if $\int g-\chi E\leq\frac{1}{2}\epsilon$. Set $G=\{x:g(x)>\frac{1}{2}\}$; then $E\subseteq G$ and $\mu(G\setminus E)\leq\epsilon$.

(iii)⇒(i) is trivial. Assembling these fragments, the proof is complete.

412X Basic exercises (a) Let (X,Σ,μ) be a measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets *and* with respect to the compact sets. Show that it is tight.

(b) Explain how 213A is a special case of 412Aa.

>(c) Let (X,Σ,μ) be a measure space, and Σ_0 a σ-subalgebra of Σ such that μ is inner regular with respect to Σ_0. Show that if $1\leq p<\infty$ then every member of $L^p(\mu)$ is of the form $f^\bullet$ for some Σ_0-measurable $f:X\to\mathbb{R}$.

>(d) Let (X, Σ, μ) be a semi-finite measure space and $\mathcal{A} \subseteq \Sigma$ an algebra of sets such that the σ-algebra generated by $\mathcal{A}$ is Σ. Write $\mathcal{K}$ for $\{\bigcap_{n\in\mathbb{N}} E_n : E_n \in \mathcal{A} \text{ for every } n \in \mathbb{N}\}$. Show that μ is inner regular with respect to $\mathcal{K}$.

(e) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite Hausdorff topological measure space such that μ is inner regular with respect to the Borel sets. Suppose that $\mu G = \sup\{\mu K : K \subseteq G \text{ is compact}\}$ for every open set $G \subseteq X$. Show that μ is tight.

(f) Let $(X, \mathfrak{T})$ be a topological space such that every open set is an F_σ set. Show that any effectively locally finite Borel measure on X is inner regular with respect to the closed sets.

(g) Let $(X, \mathfrak{T})$ be a normal topological space and μ a topological measure on X which is inner regular with respect to the closed sets. Show that $\mu G = \sup\{\mu H : H \subseteq G \text{ is a cozero set}\}$ for every open set $G \subseteq X$. Show that if μ is totally finite, then $\mu F = \inf\{\mu H : H \supseteq F \text{ is a zero set}\}$ for every closed set $F \subseteq X$.

(h) Let (X, Σ, μ) be a complete locally determined measure space, and suppose that μ is inner regular with respect to a family $\mathcal{K}$ of sets. Let Σ_0 be the σ-algebra of subsets of X generated by $\mathcal{K} \cap \Sigma$. (i) Show that μ is the c.l.d. version of $\mu\restriction\Sigma_0$. (*Hint*: 412J-412L.) (ii) Show that if μ is σ-finite, it is the completion of $\mu\restriction\Sigma_0$.

>(i) (i) Let (X, Σ, μ) be a σ-finite measure space and T a σ-subalgebra of Σ. Show that if μ is inner regular with respect to T then the completion of $\mu\restriction\mathrm{T}$ extends μ, so that μ and $\mu\restriction\mathrm{T}$ have the same negligible sets. (ii) Show that if μ is a σ-finite topological measure which is inner regular with respect to the Borel sets, then every μ-negligible set is included in a μ-negligible Borel set.

(j) Devise a direct proof of 412L, not using 412K, by (i) showing that $\mu^*(A \cap K) = \nu^*(A \cap K)$ whenever $A \subseteq X$, $K \in \mathcal{K}$ (ii) showing that $\mu^* = \nu^*$ (iii) quoting 213C.

(k) Let (X, Σ, μ) be a complete locally determined measure space, Y a set and $f : X \to Y$ a function. Show that the following are equiveridical: (i) μ is inner regular with respect to $\{f^{-1}[B] : B \subseteq Y\}$ (ii) $f^{-1}[f[E]] \setminus E$ is negligible for every $E \in \Sigma$.

(l) Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of measure spaces, with direct sum (X, Σ, μ). Suppose that for each $i \in I$ we are given a topology $\mathfrak{T}_i$ on X_i, and let $\mathfrak{T}$ be the corresponding disjoint union topology on X. Show that (i) μ is inner regular with respect to the closed sets iff every μ_i is (ii) μ is inner regular with respect to the compact sets iff every μ_i is (iii) μ is inner regular with respect to the zero sets iff every μ_i is (iv) μ is inner regular with respect to the Borel sets iff every μ_i is.

(m) Use 412L and 412Q to shorten the proof of 253I.

(n) Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and suppose that we are given, for each $i \in I$, a σ-algebra Σ_i of subsets of X_i and a topology $\mathfrak{T}_i$ on X_i. Let $\mathfrak{T}$ be the product topology on $X = \prod_{i\in I} X_i$, and $\Sigma = \widehat{\bigotimes}_{i\in I}\Sigma_i$. Let μ be a totally finite measure with domain Σ, and set $\mu_i = \mu\pi_i^{-1}$ for each $i \in I$, where $\pi_i(x) = x(i)$ for $i \in I$, $x \in X$. (i) Show that μ is inner regular with respect to the family $\mathcal{K}$ of sets expressible as $X \setminus \bigcup_{n\in\mathbb{N}} \prod_{i\in I} E_{ni}$ where $E_{ni} \in \Sigma_i$ for every n, i and $\{i : E_{ni} \neq X_i\}$ is finite for each n. (ii) Show that if every μ_i is inner regular with respect to the closed sets, so is μ. (iii) Show that if every μ_i is inner regular with respect to the zero sets, so is μ. (iv) Show that if every μ_i is inner regular with respect to the Borel sets, so is μ. (v) Show that if every μ_i is tight, and all but countably many of the X_i are compact, then μ is tight.

(o) Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a Lindelöf topology on X such that μ is locally finite. (i) Show that μ is σ-finite. (ii) Show that μ is inner regular with respect to the closed sets iff it is outer regular with respect to the open sets.

(p) Let X be a topological space and μ a measure on X which is outer regular with respect to the open sets. Show that for any $Y \subseteq X$ the subspace measure on Y is outer regular with respect to the open sets.

(q) Let X be a topological space and μ a measure on X which is outer regular with respect to the open sets. Show that if $f : X \to \mathbb{R}$ is integrable and $\epsilon > 0$ then there is a lower semi-continuous $g : X \to]-\infty, \infty]$ such that $f \leq g$ and $\int g - f \leq \epsilon$.

(r) Let (X, Σ, μ) be a semi-finite measure space and $\langle f_n \rangle_{n \in \mathbb{N}}$ a sequence in $\mathcal{L}^0(\mu)$ which converges almost everywhere to $f \in \mathcal{L}^0(\mu)$. Show that μ is inner regular with respect to $\{E : \langle f_n \restriction E \rangle_{n \in \mathbb{N}}$ is uniformly convergent$\}$. (Cf. 215Yb.)

(s) In 216E, give $\{0,1\}^I$ its usual compact Hausdorff topology. Show that the measure μ described there is inner regular with respect to the zero sets.

412Y Further exercises (a) Let $\mathcal{K}$ be the family of subsets of $\mathbb{R}$ which are homeomorphic to the Cantor set. Show that Lebesgue measure is inner regular with respect to $\mathcal{K}$. (*Hint*: show that if $F \subseteq \mathbb{R} \setminus \mathbb{Q}$ is an uncountable compact set, then $\{x : [x - \delta, x + \delta] \cap F$ is uncountable for every $\delta > 0\}$ belongs to $\mathcal{K}$.)

(b) (i) Show that if X is a perfectly normal space then any semi-finite topological measure on X which is inner regular with respect to the Borel sets is inner regular with respect to the closed sets. (ii) Show that any subspace of a perfectly normal space is perfectly normal. (iii) Show that the split interval $I^{\|}$ (343J, 343Yc, 419L) is perfectly normal. (iv) Show that ω_1, with its order topology, is completely regular, normal and Hausdorff, but not perfectly normal. (v) Show that $I^{\|} \times I^{\|}$ is not perfectly normal. (vi) Show that $[0,1]^I$ is perfectly normal iff I is countable.

(c) Let (X, Σ, μ) be a measure space, and suppose that μ is inner regular with respect to a family $\mathcal{K} \subseteq \Sigma$ such that $K \cup K' \in \mathcal{K}$ for all K, $K' \in \mathcal{K}$. Write Σ^f for $\{E : E \in \Sigma, \mu E < \infty\}$. Show that $\{E^\bullet : E \in \mathcal{K} \cap \Sigma^f\}$ is dense in $\{E^\bullet : E \in \Sigma^f\}$ for the strong measure-algebra topology.

(d) Let X be a normal topological space and Y a closed subset of X. Show that every Baire subset of Y is the intersection of Y with a Baire subset of X. (*Hint*: use Tietze's theorem.)

(e) Let (X, Σ, μ) be $[0,1]$ with Lebesgue measure, and $Y = [0,1]$ with counting measure ν; give X its usual topology and Y the discrete topology, and let λ be the c.l.d. product measure on $X \times Y$. (i) Show that μ, ν and λ are all tight (for the appropriate topologies) and therefore completion regular. (ii) Let λ_0 be the primitive product measure on $X \times Y$ (definition: 251C). Show that λ_0 is not tight. (*Hint*: 252Yf.) *Remark*: it is undecidable in ZFC whether λ_0 is inner regular with respect to the closed sets.

(f) Give an example of a Hausdorff topological measure space $(X, \mathfrak{T}, \Sigma, \mu)$ such that μ is complete, strictly localizable and outer regular with respect to the open sets, but not inner regular with respect to the closed sets.

412 Notes and comments In this volume we are returning to considerations which have been left on one side for almost the whole of Volume 3 – the exceptions being in Chapter 34, where I looked at realization of homomorphisms of measure algebras by functions between measure spaces, and was necessarily dragged into an investigation of measure spaces which had enough points to be adequate codomains (343B). The idea of 'inner regularity' is to distinguish families $\mathcal{K}$ of sets which will be large enough to describe the measure entirely, but whose members will be of recognisable types. For an example of this principle see 412Ya. Of course we cannot always find a single type of set adequate to fill a suitable family $\mathcal{K}$, though this happens oftener than one might expect, but it is surely easier to think about an arbitrary zero set (for instance) than an arbitrary measurable set, and whenever a measure is inner regular with respect to a recognisable class it is worth knowing about it.

I have tried to use the symbols † and ‡ (412A, 412C) consistently enough for them to act as a guide to some of the ideas which will be used repeatedly in this chapter. Note the emphasis on disjoint unions and countable intersections; I mentioned similar conditions in 136Xi-136Xj. You will recognise 412Aa as an exhaustion principle; note that it is enough to use disjoint unions, as in 313K. In the examples of this section this disjointness is not important. Of course inner regularity has implications for the measure algebra (412N), but it is important to recognise that 'μ is inner regular with respect to $\mathcal{K}$' is saying much more than

'$\{K^\bullet : K \in \mathcal{K}\}$ is order-dense in the measure algebra'; the latter formulation tells us only that whenever $\mu E > 0$ there is a $K \in \mathcal{K}$ such that $K \setminus E$ is negligible and $\mu K > 0$, while the former tells us that we can take K to be actually a subset of E.

412D, 412E and 412G are all of great importance. 412D looks striking, but of course the reason it works is just that the Baire σ-algebra is very small. In 412E the Baire and Borel σ-algebras coincide, so it is nothing but a special case of 412D; but as metric spaces are particularly important it is worth having it spelt out explicitly. In 412D and 412E the hypothesis 'semi-finite' is sufficient, while in 412G we need 'effectively locally finite'; this is because in both 412D and 412E the open sets we are looking at are countable unions of measurable closed sets. There are interesting non-metrizable spaces in which the same thing happens (412Yb). As you know, I am strongly prejudiced in favour of complete and locally determined measures, and the Baire and Borel measures dealt with in these three results are rarely complete; but they can still be applied to completions and c.l.d. versions of these measures, using 412Ab or 412H.

412O-412V are essentially routine. For subspace measures, the only problem we need to come to terms with is the fact that subspaces of semi-finite measure spaces need not be semi-finite (216Xa). For product measures the point is that the c.l.d. product of two measure spaces, and the product of any family of probability spaces, as I defined them in Chapter 25, are inner regular with respect to the σ-algebra of sets generated by the cylinder sets. This is not in general true of the 'primitive' product measure (412Ye), which is one of my reasons for being prejudiced against it. I should perhaps warn you of a trap in the language I use here. I say that if the factor measures are inner regular with respect to the closed sets, so is the c.l.d. product measure. But I do not say that all closed sets in the product are measurable for the product measure, even if closed sets in the factors are measurable for the factor measures. So the path is open for a different product measure to exist, still inner regular with respect to the closed sets; and indeed I shall be going down that path in §417. The uniqueness result in 412L specifically refers to complete locally determined measures defined on all sets of the family $\mathcal{K}$.

There is one special difficulty in 412V: in order to ensure that there are enough compact measurable sets in $X = \prod_{i\in I} X_i$, we need to know that all but countably many of the X_i are actually compact. When we come to look more closely at products of Radon probability spaces we shall need to consider this point again (417Q, 417Xq).

In fact some of the ideas of 412U-412V are not restricted to the product measures considered there. Other measures on the product space will have inner regularity properties if their images on the factors, their 'marginals' in the language of probability theory, are inner regular; see 412Xn. I will return to this in §454.

This section is almost exclusively concerned with *inner* regularity. The complementary notion of *outer* regularity is not much use except in σ-finite spaces (415Xh), and not always then (416Yd). In totally finite spaces, of course, and some others, any version of inner regularity corresponds to a version of outer regularity, as in 412Wb(i)-(ii); and when we have something as strong as 412Wb(iii) available it is worth knowing about it.

413 Inner measure constructions

I now turn in a different direction, giving some basic results on the construction of inner regular measures. The first step is to describe 'inner measures' (413A) and a construction corresponding to the Carathéodory construction of measures from outer measures (413C). Just as every measure gives rise to an outer measure, it gives rise to an inner measure (413D). Inner measures form an effective tool for studying complete locally determined measures (413F).

The most substantial results of the section concern the construction of measures as extensions of functionals defined on various classes $\mathcal{K}$ of sets. Typically, $\mathcal{K}$ is closed under finite unions and countable intersections, though it is sometimes possible to relax the hypotheses to some extent. The methods here make it possible to distinguish arguments which produce finitely additive functionals (413H, 413N, 413P, 413Q) from the succeeding steps to countably additive measures (413I, 413O, 413S). 413H-413M investigate conditions on a functional $\phi : \mathcal{K} \to [0, \infty[$ sufficient to produce a measure extending ϕ, necessarily unique, which is inner regular with respect to $\mathcal{K}$ or $\mathcal{K}_\delta$, the set of intersections of sequences in $\mathcal{K}$. 413N-413O look instead at functionals defined on sublattices of the class $\mathcal{K}$ of interest, and at sufficient conditions to ensure the existence of a measure, not normally unique, defined on the whole of $\mathcal{K}$, inner regular with respect to $\mathcal{K}$ and extending

the given functional. Finally, 413P-413S are concerned with majorizations rather than extensions; we seek a measure μ such that $\mu K \geq \lambda K$ for $K \in \mathcal{K}$, while μX is as small as possible.

413A I begin with some material from the exercises of earlier volumes.

Definition Let X be a set. An **inner measure** on X is a function $\phi : \mathcal{P}X \to [0,\infty]$ such that

$\phi\emptyset = 0$;

(α) $\phi(A \cup B) \geq \phi A + \phi B$ for all disjoint A, $B \subseteq X$;

(β) if $\langle A_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of subsets of X and $\phi A_0 < \infty$ then $\phi(\bigcap_{n\in\mathbb{N}} A_n) = \inf_{n\in\mathbb{N}} \phi A_n$;

($*$) $\phi A = \sup\{\phi B : B \subseteq A, \phi B < \infty\}$ for every $A \subseteq X$.

413B The following fact will be recognised as an element of Carathéodory's method. There will be an application later in which it will be useful to know that it is not confined to proving countable additivity.

Lemma Let X be a set and $\phi : X \to [0,\infty]$ any function such that $\phi\emptyset = 0$. Then

$$\Sigma = \{E : E \subseteq X, \phi A = \phi(A \cap E) + \phi(A \setminus E) \text{ for every } A \subseteq X\}$$

is an algebra of subsets of X, and $\phi(E \cup F) = \phi E + \phi F$ for all disjoint E, $F \in \Sigma$.

proof The symmetry of the definition of Σ ensures that $X \setminus E \in \Sigma$ whenever $E \in \Sigma$. If E, $F \in \Sigma$ and $A \subseteq X$, then

$$\begin{aligned}\phi(A \cap (E\cup F)) &+ \phi(A \setminus (E \cup F)) \\ &= \phi(A \cap (E \cup F) \cap E) + \phi(A \cap (E \cup F) \setminus E) + \phi(A \setminus (E \cup F)) \\ &= \phi(A \cap E) + \phi((A \setminus E) \cap F) + \phi((A \setminus E) \setminus F) \\ &= \phi(A \cap E) + \phi(A \setminus E) = \phi A.\end{aligned}$$

As A is arbitrary, $E \cup F \in \Sigma$. Finally, if $A \subseteq X$,

$$\phi(A \cap \emptyset) + \phi(A \setminus \emptyset) = \phi\emptyset + \phi A = \phi A$$

because $\phi\emptyset = 0$; so $\emptyset \in \Sigma$.

Thus Σ is an algebra of sets. If E, $F \in \Sigma$ and $E \cap F = \emptyset$, then

$$\phi(E \cup F) = \phi((E \cup F) \cap E) + \phi((E \cup F) \setminus E) = \phi E + \phi F.$$

413C Measures from inner measures I come now to a construction corresponding to Carathéodory's method of defining measures from outer measures.

Theorem Let X be a set and $\phi : X \to [0,\infty]$ an inner measure. Set

$$\Sigma = \{E : E \subseteq X, \phi(A \cap E) + \phi(A \setminus E) = \phi A \text{ for every } A \subseteq X\}.$$

Then $(X, \Sigma, \phi{\restriction}\Sigma)$ is a complete measure space.

proof (Compare 113C.)

(a) The first step is to note that if $A \subseteq B \subseteq X$ then

$$\phi B \geq \phi A + \phi(B \setminus A) \geq \phi A.$$

Next, a subset E of X belongs to Σ iff $\phi A \leq \phi(A \cap E) + \phi(A \setminus E)$ whenever $A \subseteq X$ and $\mu A < \infty$. **P** Of course any element of Σ satisfies the condition. If E satisfies the condition and $A \subseteq X$, then

$$\begin{aligned}\phi A &= \sup\{\phi B : B \subseteq A, \phi B < \infty\} \\ &\leq \sup\{\phi(B \cap E) + \phi(B \setminus E) : B \subseteq A\} \\ &= \phi(A \cap E) + \phi(A \setminus E) \leq \phi A,\end{aligned}$$

so $E \in \Sigma$. **Q**

(b) By 413B, Σ is an algebra of subsets of X. Now suppose that $\langle E_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in Σ, with union E. If $A \subseteq X$ and $\phi A < \infty$, then

$$\phi(A \setminus E) = \inf_{n \in \mathbb{N}} \phi(A \setminus E_n) = \lim_{n \to \infty} \phi(A \setminus E_n)$$

because $\langle A \setminus E_n \rangle_{n \in \mathbb{N}}$ is non-increasing and $\phi(A \setminus E_0)$ is finite; so

$$\phi(A \cap E) + \phi(A \setminus E) \geq \lim_{n \to \infty} \phi(A \cap E_n) + \phi(A \setminus E_n) = \phi A.$$

By (a), $E \in \Sigma$. So Σ is a σ-algebra.

(c) If E, $F \in \Sigma$ and $E \cap F = \emptyset$ then $\phi(E \cup F) = \phi E + \phi F$, by 413B. If $\langle E_n \rangle_{n \in \mathbb{N}}$ is a disjoint sequence in Σ with union E, then

$$\mu E \geq \mu(\textstyle\bigcup_{i \leq n} E_i) = \sum_{i=0}^{n} \mu E_i$$

for every n, so $\mu E \geq \sum_{i=0}^{\infty} \mu E_i$. **?** If $\mu E > \sum_{i=0}^{\infty} \mu E_i$, there is an $A \subseteq E$ such that $\sum_{i=0}^{\infty} \mu E_i < \phi A < \infty$. But now, setting $F_n = \bigcup_{i \leq n} E_i$ for each n, we have $\lim_{n \to \infty} \phi(A \setminus F_n) = 0$, so that

$$\phi A = \lim_{n \to \infty} \phi(A \cap F_n) + \phi(A \setminus F_n) = \sum_{i=0}^{\infty} \phi(A \cap E_i) < \phi A,$$

which is absurd. **X** Thus $\mu E = \sum_{i=0}^{\infty} \mu E_i$. As $\langle E_n \rangle_{n \in \mathbb{N}}$ is arbitrary, μ is a measure.

(d) Finally, suppose that $B \subseteq E \in \Sigma$ and $\mu E = 0$. Then for any $A \subseteq X$ we must have

$$\phi(A \cap B) + \phi(A \setminus B) \geq \phi(A \setminus E) = \phi(A \cap E) + \phi(A \setminus E) = \phi A,$$

so $B \in \Sigma$. Thus μ is complete.

Remark For a simple example see 213Yc.

413D The inner measure defined by a measure Let (X, Σ, μ) be any measure space. Just as μ has an associated outer measure μ^* defined by the formula

$$\mu^* A = \inf\{\mu E : A \subseteq E \in \Sigma\}$$

(132A-132B), it gives rise to an inner measure μ_* defined by the formula

$$\mu_* A = \sup\{\mu E : E \in \Sigma^f, \, E \subseteq A\},$$

where I write Σ^f for $\{E : E \in \Sigma, \, \mu E < \infty\}$. **P** $\mu_* \emptyset = \mu \emptyset = 0$. (†) If $A \cap B = \emptyset$, and $E \subseteq A$, $F \subseteq B$ belong to Σ^f, then $E \cup F \subseteq A \cup B$ also has finite measure, so

$$\mu_*(A \cup B) \geq \mu(E \cup F) = \mu E + \mu F;$$

taking the supremum over E and F, $\mu_*(A \cup B) \geq \mu_* A + \mu_* B$. (‡) If $\langle A_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence of sets with intersection A and $\mu_* A_0 < \infty$, then for each $n \in \mathbb{N}$ we can find an $E_n \subseteq A_n$ such that $\mu E_n \geq \mu_* A_n - 2^{-n}$. In this case,

$$\mu(\textstyle\bigcup_{m \in \mathbb{N}} E_m) = \sup_{n \in \mathbb{N}} \mu(\bigcup_{m \leq n} E_m) \leq \mu_* A_0 < \infty.$$

Set

$$E = \textstyle\bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} E_m \subseteq A.$$

Then $E \in \Sigma^f$, so

$$\mu_* A \geq \mu E \geq \limsup_{n \to \infty} \mu E_n = \lim_{n \to \infty} \mu_* A_n \geq \mu_* A.$$

(*) If $A \subseteq X$ and $\mu_* A = \infty$ then

$$\sup\{\mu_* B : B \subseteq A, \, \mu_* B < \infty\} \geq \sup\{\mu E : E \in \Sigma^f, \, E \subseteq A\} = \infty. \; \textbf{Q}$$

Warning Many authors use the formula

$$\mu_* A = \sup\{\mu E : A \supseteq E \in \Sigma\}.$$

In 'ordinary' cases, when (X, Σ, μ) is semi-finite, this agrees with my usage (413Ed); but for non-semi-finite spaces there is a difference.

For elementary properties of the construction here see 213Xe.

413E I note the following elementary facts concerning inner measures defined from measures.

Proposition Let (X, Σ, μ) be a measure space. Write Σ^f for $\{E : E \in \Sigma, \mu E < \infty\}$.

(a) For every $A \subseteq X$ there is an $E \in \Sigma$ such that $E \subseteq A$ and $\mu E = \mu_* A$.

(b) $\mu_* A \leq \mu^* A$ for every $A \subseteq X$.

(c) If $E \in \Sigma$ and $A \subseteq X$, then $\mu_*(E \cap A) + \mu^*(E \setminus A) \leq \mu E$, with equality if either (i) $\mu E < \infty$ or (ii) μ is semi-finite.

(d) In particular, $\mu_* E \leq \mu E$ for every $E \in \Sigma$, with equality if either $\mu E < \infty$ or μ is semi-finite.

(e) If μ is inner regular with respect to $\mathcal{K}$, then $\mu_* A = \sup\{\mu K : K \in \mathcal{K} \cap \Sigma^f, K \subseteq A\}$ for every $A \subseteq X$.

(f) If $A \subseteq X$ is such that $\mu_* A = \mu^* A < \infty$, then A is measured by the completion of μ.

(g) If $\hat{\mu}$, $\tilde{\mu}$ are the completion and c.l.d. version of μ, then $\hat{\mu}_* = \tilde{\mu}_* = \mu_*$.

(h) If (Y, T, ν) is another measure space, and $f : X \to Y$ is an inverse-measure-preserving function, then

$$\mu^*(f^{-1}[B]) \leq \nu^* B, \quad \mu_*(f^{-1}[B]) \geq \nu_* B$$

for every $B \subseteq Y$, and

$$\nu^*(f[A]) \geq \mu^* A$$

for every $A \subseteq X$.

(i) Suppose that μ is semi-finite. If $A \subseteq E \in \Sigma$, then E is a measurable envelope of A iff $\mu_*(E \setminus A) = 0$.

proof (a) There is a sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in Σ^f such that $E_n \subseteq A$ for each n and $\lim_{n \to \infty} \mu E_n = \mu_* A$; now set $E = \bigcup_{n \in \mathbb{N}} E_n$.

(b) If $E \subseteq A \subseteq F$ we must have $\mu E \leq \mu F$.

(c) If $F \subseteq E \cap A$ and $F \in \Sigma^f$, then

$$\mu F + \mu^*(E \setminus A) \leq \mu F + \mu(E \setminus F) = \mu E;$$

taking the supremum over F, $\mu_*(E \cap A) + \mu^*(E \setminus A) \leq \mu E$. If $\mu E < \infty$, then

$$\begin{aligned} \mu_*(E \cap A) &= \sup\{\mu F : F \in \Sigma, F \subseteq E \cap A\} \\ &= \mu E - \inf\{\mu(E \setminus F) : F \in \Sigma, F \subseteq E \cap A\} \\ &= \mu E - \inf\{\mu F : F \in \Sigma, E \setminus A \subseteq F \subseteq E\} = \mu E - \mu^*(E \setminus A). \end{aligned}$$

If μ is semi-finite, then

$$\begin{aligned} \mu_*(E \cap A) + \mu^*(E \setminus A) &\geq \sup\{\mu_*(F \cap A) + \mu^*(F \setminus A) : F \in \Sigma^f, F \subseteq E\} \\ &= \sup\{\mu F : F \in \Sigma^f, F \subseteq E\} = \mu E. \end{aligned}$$

(d) Take $A = E$ in (c).

(e)

$$\begin{aligned} \mu_* A &= \sup\{\mu E : E \in \Sigma^f, E \subseteq A\} \\ &= \sup\{\mu K : K \in \mathcal{K} \cap \Sigma, \exists\, E \in \Sigma^f, K \subseteq E \subseteq A\} \\ &= \sup\{\mu K : K \in \mathcal{K} \cap \Sigma^f, K \subseteq A\}. \end{aligned}$$

(f) By (a) above and 132Aa, there are E, $F \in \Sigma$ such that $E \subseteq A \subseteq F$ and

$$\mu E = \mu_* A = \mu^* A = \mu F < \infty;$$

now $\mu(F \setminus E) = 0$, so $F \setminus A$ and A are measured by the completion of μ.

(g) Write $\check{\mu}$ for either $\hat{\mu}$ or $\tilde{\mu}$, and $\check{\Sigma}$ for its domain, and let $A \subseteq X$. (i) If $\gamma < \mu_* A$, there is an $E \in \Sigma$ such that $E \subseteq A$ and $\gamma \leq \mu E < \infty$; now $\check{\mu} E = \mu E$ (212D, 213Fa), so $\check{\mu}_* A \geq \gamma$. As γ is arbitrary, $\mu_* A \leq \check{\mu}_* A$.

(ii) If $\gamma < \check{\mu}_* A$, there is an $E \in \check{\Sigma}$ such that $E \subseteq A$ and $\gamma \leq \check{\mu} E < \infty$. Now there is an $F \in \Sigma$ such that $F \subseteq E$ and $\mu F = \check{\mu} E$ (212C, 213Fc), so that $\mu_* A \geq \gamma$. As γ is arbitrary, $\mu_* A \geq \check{\mu}_* A$.

(h) This is elementary; all we have to note is that if F, $F' \in \mathrm{T}$ and $F \subseteq B \subseteq F'$, then $f^{-1}[F] \subseteq f^{-1}[B] \subseteq f^{-1}[F']$, so that

$$\nu F = \mu f^{-1}[F] \leq \mu_* f^{-1}[B] \leq \mu^* f^{-1}[B] \leq \mu f^{-1}[F'] = \nu F'.$$

Now, for $A \subseteq X$,

$$\mu^* A \leq \mu^*(f^{-1}[f[A]]) \leq \nu^*(f[A]).$$

(i)(i) If E is a measurable envelope of A and $F \in \Sigma$ is included in $E \setminus A$, then

$$\mu F = \mu(F \cap E) = \mu^*(F \cap A) = 0;$$

as F is arbitrary, $\mu_*(E \setminus A) = 0$. (ii) If E is not a measurable envelope of A, there is an $F \in \Sigma$ such that $\mu^*(F \cap A) < \mu(F \cap E)$. Let $G \in \Sigma$ be such that $F \cap A \subseteq G$ and $\mu G = \mu^*(F \cap A)$. Then $\mu(F \cap E \setminus G) > 0$; because μ is semi-finite, $\mu_*(E \setminus A) \geq \mu_*(F \cap E \setminus G) > 0$.

413F The language of 413D makes it easy to express some useful facts about complete locally determined measure spaces, complementing 412J.

Lemma Let (X, Σ, μ) be a complete locally determined measure space and $\mathcal{K}$ a family of subsets of X such that μ is inner regular with respect to $\mathcal{K}$. Then for $E \subseteq X$ the following are equiveridical:
(i) $E \in \Sigma$;
(ii) $E \cap K \in \Sigma$ whenever $K \in \Sigma \cap \mathcal{K}$;
(iii) $\mu^*(K \cap E) + \mu^*(K \setminus E) = \mu^* K$ for every $K \in \mathcal{K}$;
(iv) $\mu_*(K \cap E) + \mu_*(K \setminus E) = \mu_* K$ for every $K \in \mathcal{K}$;
(v) $\mu^*(E \cap K) = \mu_*(E \cap K)$ for every $K \in \mathcal{K} \cap \Sigma$;
(vi) $\min(\mu^*(K \cap E), \mu^*(K \setminus E)) < \mu K$ whenever $K \in \mathcal{K} \cap \Sigma$ and $0 < \mu K < \infty$;
(vii) $\max(\mu_*(K \cap E), \mu_*(K \setminus E)) > 0$ whenever $K \in \mathcal{K} \cap \Sigma$ and $\mu K > 0$.

proof (a) Assume (i). Then of course $E \cap K \in \Sigma$ for every $K \in \Sigma \cap \mathcal{K}$, and (ii) is true. For any $K \in \mathcal{K}$ there is an $F \in \Sigma$ such that $F \supseteq K$ and $\mu F = \mu^* K$ (132Aa); now

$$\mu^* K \leq \mu^*(K \cap E) + \mu^*(K \setminus E) \leq \mu(F \cap E) + \mu(F \setminus E) = \mu F = \mu^* K,$$

so (iii) is true. Next, for any $K \in \mathcal{K}$,

$$\begin{aligned}\mu_*(K \cap E) + \mu_*(K \setminus E) &\leq \mu_* K = \sup\{\mu F : F \in \Sigma^f, F \subseteq K\}\\ \text{(writing } \Sigma^f \text{ for } \{F : F \in \Sigma, \mu F < \infty\}) \\ &= \sup\{\mu(F \cap E) + \mu(F \setminus E) : F \in \Sigma^f, F \subseteq K\}\\ &\leq \mu_*(K \cap E) + \mu_*(K \setminus E).\end{aligned}$$

So (iv) is true. If $K \in \mathcal{K} \cap \Sigma$, then

$$\mu_*(E \cap K) = \sup\{\mu F : F \in \Sigma^f, F \subseteq E \cap K\} = \mu(E \cap K) = \mu^*(E \cap K)$$

because μ is semi-finite. So (v) is true. Since (iii)⇒(vi) and (iv)⇒(vii), we see that all the conditions are satisfied.

(b) Now suppose that $E \notin \Sigma$; I have to show that (ii)-(vii) are all false. Because μ is locally determined, there is an $F \in \Sigma^f$ such that $E \cap F \notin \Sigma$. Take measurable envelopes H, H' of $F \cap E$, $F \setminus E$ respectively (132Ee). Then $F \setminus H' \subseteq F \cap E \subseteq F \cap H$, so

$$G = (F \cap H) \setminus (F \setminus H') = F \cap H \cap H'$$

cannot be negligible. Take $K \in \mathcal{K} \cap \Sigma$ such that $K \subseteq G$ and $\mu K > 0$. As $G \subseteq F$, $\mu K < \infty$. Now

$$\mu^*(K \cap E) = \mu^*(K \cap F \cap E) = \mu(K \cap H) = \mu K,$$

$$\mu^*(K \setminus E) = \mu^*(K \cap F \setminus E) = \mu(K \cap H') = \mu K.$$

But this means that

$$\mu_*(K \cap E) = \mu K - \mu^*(K \setminus E) = 0, \quad \mu_*(K \setminus E) = \mu K - \mu^*(K \cap E) = 0,$$

by 413Eb. Now we see that this K witnesses that (ii)-(vii) are all false.

413G The ideas of 413F can be used to give criteria for measurability of real-valued functions. I spell out one which is particularly useful.

Lemma Let (X, Σ, μ) be a complete locally determined measure space and suppose that μ is inner regular with respect to $\mathcal{K} \subseteq \Sigma$. Suppose that $f : X \to \mathbb{R}$ is a function, and for $\alpha \in \mathbb{R}$ set $E_\alpha = \{x : f(x) \leq \alpha\}$, $F_\alpha = \{x : f(x) \geq \beta\}$. Then f is Σ-measurable iff

$$\min(\mu^*(E_\alpha \cap K), \mu^*(F_\beta \cap K)) < \mu K$$

whenever $K \in \mathcal{K}$, $0 < \mu K < \infty$ and $\alpha < \beta$.

proof (a) If f is measurable, then

$$\mu^*(E_\alpha \cap K) + \mu^*(F_\beta \cap K) = \mu(E_\alpha \cap K) + \mu(F_\beta \cap K) \leq \mu K$$

whenever $K \in \Sigma$ and $\alpha < \beta$, so if $0 < \mu K < \infty$ then we must have $\min(\mu^*(E_\alpha \cap K), \mu^*(F_\beta \cap K)) < \mu K$.

(b) If f is not measurable, then there is some $\alpha \in \mathbb{R}$ such that E_α is not measurable. 413F(iii) tells us that there is a $K \in \mathcal{K}$ such that $0 < \mu K < \infty$ and $\mu^*(E_\alpha \cap K) = \mu^*(K \setminus E_\alpha) = \mu K$. Note that K is a measurable envelope of $K \cap E_\alpha$ (132Eb). Now $\langle K \cap F_{\alpha + 2^{-n}} \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence with union $K \setminus E_\alpha$, so there is some $\beta > \alpha$ such that $K \cap F_\beta$ is not negligible. Let $H \subseteq K$ be a measurable envelope of $K \cap F_\beta$, and $K' \in \mathcal{K}$ such that $K' \subseteq H$ and $\mu K' > 0$; then

$$\mu^*(K' \cap E_\alpha) = \mu^*(K' \cap K \cap E_\alpha) = \mu(K' \cap K) = \mu K',$$

$$\mu^*(K' \cap F_\beta) = \mu^*(K' \cap H \cap F_\beta) = \mu(K' \cap H) = \mu K',$$

so K', α and β witness that the condition is not satisfied.

413H Inner measure constructions based on 413C are important because they offer an efficient way of setting up measures which are inner regular with respect to given families of sets. Two of the fundamental results are 413I and 413J. I proceed by means of a lemma on finitely additive functionals.

Lemma Let X be a set and $\mathcal{K}$ a family of subsets of X such that

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $K \cap K' \in \mathcal{K}$ for all K, $K' \in \mathcal{K}$.

Let $\phi_0 : \mathcal{K} \to [0, \infty[$ be a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K \in \mathcal{K}, K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$.

Set

$$\phi A = \sup\{\phi_0 K : K \in \mathcal{K}, K \subseteq A\} \text{ for } A \subseteq X,$$

$$\Sigma = \{E : E \subseteq X, \phi A = \phi(A \cap E) + \phi(A \setminus E) \text{ for every } A \subseteq X\}.$$

Then Σ is an algebra of subsets of X, including $\mathcal{K}$, and $\phi \restriction \Sigma : \Sigma \to [0, \infty]$ is an additive functional extending ϕ_0.

proof (a) To see that Σ is an algebra of subsets and $\phi \restriction \Sigma$ is additive, all we need to know is that $\phi\emptyset = 0$ (413B); and this is because, applying hypothesis (α) with $K = L = \emptyset$, $\phi_0\emptyset = \phi_0\emptyset + \phi_0\emptyset$, so $\phi_0\emptyset = 0$. (α) also assures us that $\phi_0 L \leq \phi_0 K$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$, so $\phi K = \phi_0 K$ for every $K \in \mathcal{K}$.

(b) To check that $\mathcal{K} \subseteq \Sigma$, we have a little more work to do. First, observe that (†) and (α) together tell us that $\phi_0(K \cup K') = \phi_0 K + \phi_0 K'$ for all disjoint K, $K' \in \mathcal{K}$. So if A, $B \subseteq X$ and $A \cap B = \emptyset$ then

$$\begin{aligned}\phi A + \phi B &= \sup_{K\in\mathcal{K},K\subseteq A}\phi_0 K + \sup_{L\in\mathcal{K},L\subseteq A}\phi_0 L\\ &= \sup_{K,L\in\mathcal{K},K\subseteq A,L\subseteq B}\phi_0(K\cup L) \leq \phi(A\cup B).\end{aligned}$$

(c) $\mathcal{K}\subseteq\Sigma$. **P** Take $K\in\mathcal{K}$, $A\subseteq X$. If $L\in\mathcal{K}$ and $L\subseteq A$, then

$$\phi_0 L = \phi_0(K\cap L) + \sup\{\phi_0 L' : L'\in\mathcal{K},\ L'\subseteq L\setminus K\} \leq \phi(A\cap K)+\phi(A\setminus K).$$

(Note the use of the hypothesis (‡).) As L is arbitrary, $\phi A\leq\phi(A\cap K)+\phi(A\setminus K)$. We already know that $\phi(A\cap K)+\phi(A\setminus K)\leq\phi A$; as A is arbitrary, $K\in\Sigma$. **Q**

This completes the proof.

413I Theorem Let X be a set and $\mathcal{K}$ a family of subsets of X such that

$\emptyset\in\mathcal{K}$,

(†) $K\cup K'\in\mathcal{K}$ whenever K, $K'\in\mathcal{K}$ are disjoint,

(‡) $\bigcap_{n\in\mathbb{N}}K_n\in\mathcal{K}$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$.

Let $\phi_0:\mathcal{K}\to[0,\infty[$ be a functional such that

(α) $\phi_0 K=\phi_0 L+\sup\{\phi_0 K' : K\in\mathcal{K},\ K'\subseteq K\setminus L\}$ whenever K, $L\in\mathcal{K}$ and $L\subseteq K$,

(β) $\inf_{n\in\mathbb{N}}\phi_0 K_n=0$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection.

Then there is a unique complete locally determined measure μ on X extending ϕ_0 and inner regular with respect to $\mathcal{K}$.

proof (a) Set

$$\phi A=\sup\{\phi_0 K : K\in\mathcal{K},\ K\subseteq A\}\text{ for }A\subseteq X,$$

$$\Sigma=\{E : E\subseteq X,\ \phi A=\phi(A\cap E)+\phi(A\setminus E)\text{ for every }A\subseteq X\}.$$

Then 413H tells us that Σ is an algebra of subsets of X, including $\mathcal{K}$, and $\mu=\phi\restriction\Sigma$ is an additive functional extending ϕ_0.

(b) Now $\mu(\bigcap_{n\in\mathbb{N}}K_n)=\inf_{n\in\mathbb{N}}\mu K_n$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$. **P** Set $L=\bigcap_{n\in\mathbb{N}}K_n$. Of course $\mu L\leq\inf_{n\in\mathbb{N}}\mu K_n$. For the reverse inequality, take $\epsilon>0$. Then (α) tells us that there is a $K'\in\mathcal{K}$ such that $K'\subseteq K_0\setminus L$ and $\mu K_0\leq\mu L+\mu K'+\epsilon$. Since $\langle K_n\cap K'\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, (β) tells us that there is an $n\in\mathbb{N}$ such that $\mu(K_n\cap K')\leq\epsilon$. Now

$$\begin{aligned}\mu K_0-\mu L&=\mu(K_0\setminus L)=\mu(K_0\setminus(K'\cup L))+\mu K'\\ &\leq\epsilon+\mu(K_n\cap K')+\mu(K'\setminus K_n)\leq 2\epsilon+\mu(K_0\setminus K_n)=2\epsilon+\mu K_0-\mu K_n.\end{aligned}$$

(These calculations depend, of course, on the additivity of μ and the finiteness of μK_0.) So $\mu L\geq\mu K_n-2\epsilon$. As ϵ is arbitrary, $\mu L=\inf_{n\in\mathbb{N}}\mu K_n$. **Q**

(c) If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of subsets of X, with intersection A, and $\phi A_0<\infty$, then $\phi A=\inf_{n\in\mathbb{N}}\phi A_n$. **P** Of course $\phi A\leq\phi A_n$ for every n. Given $\epsilon>0$, then for each $n\in\mathbb{N}$ choose $K_n\in\mathcal{K}$ such that $K_n\subseteq A_n$ and $\phi_0 K_n\geq\phi A_n-2^{-n}\epsilon$ (this is where I use the hypothesis that ϕA_0 is finite); set $L_n=\bigcap_{i\leq n}K_i$ for each n, $L=\bigcap_{n\in\mathbb{N}}L_n$. Then we have

$$\begin{aligned}\phi A_{n+1}-\mu L_{n+1}&=\phi A_{n+1}-\mu(K_{n+1}\cap L_n)\\ &=\phi A_{n+1}-\mu K_{n+1}-\mu L_n+\mu(K_{n+1}\cup L_n)\\ &\leq 2^{-n-1}\epsilon-\mu L_n+\phi A_n\end{aligned}$$

because $K_{n+1}\subseteq A_{n+1}\subseteq A_n$ and $L_n\subseteq K_n\subseteq A_n$. Inducing on n, we see that $\mu L_n\geq\phi A_n-2\epsilon+2^{-n}\epsilon$ for every n. So

$$\phi A\geq\mu L=\inf_{n\in\mathbb{N}}\mu L_n\geq\inf_{n\in\mathbb{N}}\phi A_n-2\epsilon,$$

using (b) above for the middle equality. As ϵ is arbitrary, $\phi A = \inf_{n\in\mathbb{N}} \phi A_n$. **Q**

(d) It follows that ϕ is an inner measure. **P** The arguments of parts (a) and (b) of the proof of 413H tell us that $\phi\emptyset = 0$ and $\phi(A \cup B) \leq \phi A + \phi B$ whenever A, $B \subseteq X$ are disjoint. We have just seen that $\phi(\bigcap_{n\in\mathbb{N}} A_n) = \inf_{n\in\mathbb{N}} \phi A_n$ whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of sets and $\phi A_0 < \infty$. Finally, $\phi K = \phi_0 K$ is finite for every $K \in \mathcal{K}$, so $\phi A = \sup\{\phi B : B \subseteq A, \phi B < \infty\}$ for every $A \subseteq X$. Putting these together, ϕ is an inner measure. **Q**

(e) So 413C tells us that μ is a complete measure, and of course it is inner regular with respect to $\mathcal{K}$, by the definition of ϕ. It is semi-finite because $\mu K = \phi_0 K$ is finite for every $K \in \mathcal{K}$. Now suppose that $E \subseteq X$ and that $E \cap F \in \Sigma$ whenever $\mu F < \infty$. Take any $A \subseteq X$. If $L \in \mathcal{K}$ and $L \subseteq A$, we have $L \in \Sigma$ and $\mu L < \infty$, so

$$\phi_0 L = \mu L = \mu(L \cap E) + \mu(L \setminus E) = \phi(L \cap E) + \phi(L \setminus E) \leq \phi(A \cap E) + \phi(A \setminus E);$$

taking the supremum over L, $\phi A \leq \phi(A \cap E) + \phi(A \setminus E)$. As A is arbitrary, $E \in \Sigma$; as E is arbitrary, μ is locally determined.

(f) Finally, μ is unique by 412L.

413J Theorem Let X be a set and $\mathcal{K}$ a family of subsets of X such that

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $K \cap K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$.

Let $\phi_0 : \mathcal{K} \to [0, \infty[$ be a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K \in \mathcal{K}, K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$,

(β) $\inf_{n\in\mathbb{N}} \phi_0 K_n = 0$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection.

Then there is a unique complete locally determined measure μ on X extending ϕ_0 and inner regular with respect to $\mathcal{K}_\delta$, the family of sets expressible as intersections of sequences in $\mathcal{K}$.

proof (a) Set

$$\psi A = \sup\{\phi_0 K : K \in \mathcal{K}, K \subseteq A\} \text{ for } A \subseteq X,$$

$$\mathrm{T} = \{E : E \subseteq X, \psi A = \psi(A \cap E) + \psi(A \setminus E) \text{ for every } A \subseteq X\}.$$

Then 413H tells us that T is an algebra of subsets of X, including $\mathcal{K}$, and $\nu = \psi\restriction \mathrm{T}$ is an additive functional extending ϕ_0.

(b) Write T^f for $\{E : E \in \mathrm{T}, \nu E < \infty\}$. If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T^f with empty intersection, $\lim_{n\to\infty} \nu E_n = 0$. **P** Given $\epsilon > 0$, we can choose a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $K_n \subseteq E_n$ and

$$\nu K_n = \phi_0 K_n \geq \nu E_n - 2^{-n}\epsilon$$

for each n. Set $L_n = \bigcap_{i\leq n} K_i$ for each n; then

$$\lim_{n\to\infty} \nu L_n = \lim_{n\to\infty} \phi_0 L_n = 0$$

by hypothesis (β). But also, for each n,

$$\nu E_n \leq \nu L_n + \textstyle\sum_{i=0}^{n} \nu(E_i \setminus K_i) \leq \nu L_n + 2\epsilon,$$

because ν is additive and non-negative and $E_n \subseteq L_n \cup \bigcup_{i\leq n}(E_i \setminus K_i)$. So $\limsup_{n\to\infty} \nu E_n \leq 2\epsilon$; as ϵ is arbitrary, $\lim_{n\to\infty} \nu E_n = 0$. **Q**

(c) Write T^f_δ for the family of sets expressible as intersections of sequences in T^f, and for $H \in \mathrm{T}^f_\delta$ set $\phi_1 H = \inf\{\nu E : H \subseteq E \in \mathrm{T}\}$. Note that because $E \cap F \in \mathrm{T}^f$ for every E, $F \in \mathrm{T}^f$, every member of T^f_δ can be expressed as the intersection of a non-increasing sequence in T^f.

(i) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T^f with intersection $H \in \mathrm{T}^f_\delta$, $\phi_1 H = \lim_{n\to\infty} \nu E_n$. **P** Of course

$$\phi_1 H \leq \inf_{n\in\mathbb{N}} \nu E_n = \lim_{n\to\infty} \nu E_n.$$

On the other hand, if $H \subseteq E \in \mathrm{T}$, then $\langle E_n \setminus E\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T^f with empty intersection, and

$$\nu E \geq \lim_{n\to\infty} \nu(E_n \cap E) = \lim_{n\to\infty} \nu E_n - \lim_{n\to\infty} \nu(E_n \setminus E) = \lim_{n\to\infty} \nu E_n$$

by (b) above. As E is arbitrary, $\phi_1(\bigcap_{n\in\mathbb{N}} E_n) = \lim_{n\to\infty} \nu E_n$. **Q**

(ii) Because $\mathcal{K} \subseteq \mathrm{T}^f$, $\mathcal{K}_\delta \subseteq \mathrm{T}^f_\delta$. Now for any $H \in \mathrm{T}^f_\delta$, $\phi_1 H = \sup\{\phi_1 L : L \in \mathcal{K}_\delta,\ L \subseteq H\}$. **P** Express H as $\bigcap_{n\in\mathbb{N}} E_n$ where $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T^f. Given $\epsilon > 0$, we can choose a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$ such that $K_n \subseteq E_n$ and $\nu K_n \geq \nu E_n - 2^{-n}\epsilon$ for each n. Setting $L_n = \bigcap_{i\leq n} K_i$ for each n and $L = \bigcap_{n\in\mathbb{N}} L_n$, we have $L \in \mathcal{K}_\delta$, $L \subseteq H$ and

$$\phi_1 H = \lim_{n\to\infty} \nu E_n \leq \lim_{n\to\infty} (\nu L_n + \textstyle\sum_{i=0}^n \nu(E_i \setminus K_i)) \leq \phi_1 L + 2\epsilon.$$

As ϵ is arbitrary, this gives the result. **Q**

(d) We find that T^f_δ, ϕ_1 satisfy the conditions of 413I. **P** Of course $\emptyset \in \mathrm{T}^f_\delta$. If G, $H \in \mathrm{T}^f_\delta$ and $G \cap H = \emptyset$, express them as $\bigcap_{n\in\mathbb{N}} E_n$, $\bigcap_{n\in\mathbb{N}} F_n$ where $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle F_n\rangle_{n\in\mathbb{N}}$ are non-increasing sequences in T^f. Then

$$G \cup H = \textstyle\bigcap_{n\in\mathbb{N}} E_n \cup F_n$$

belongs to T^f_δ, and

$$\begin{aligned}\phi_1(G\cup H) &= \lim_{n\to\infty} \nu(E_n \cup F_n) = \lim_{n\to\infty} \nu E_n + \nu F_n - \nu(E_n \cap F_n)\\ &= \lim_{n\to\infty} \nu E_n + \nu F_n\end{aligned}$$

(by (b))

$$= \phi_1 G + \phi_1 H.$$

The definition of T^f_δ as the set of intersections of sequences in T^f ensures that the intersection of any sequence in T^f_δ will belong to T^f_δ.

Now suppose that G, $H \in \mathrm{T}^f_\delta$ and that $G \subseteq H$. Express them as intersections $\bigcap_{n\in\mathbb{N}} E_n$, $\bigcap_{n\in\mathbb{N}} F_n$ of non-increasing sequences in T^f, so that $\phi_1 G = \lim_{n\to\infty} \nu E_n$, $\phi_1 H = \lim_{n\to\infty} \nu F_n$. For each n, set $H_n = \bigcap_{m\in\mathbb{N}} F_m \setminus E_n$, so that $H_n \in \mathrm{T}^f_\delta$, $H_n \subseteq H \setminus G$, and

$$\begin{aligned}\phi_1 H_n &= \lim_{m\to\infty} \nu(F_m \setminus E_n) = \lim_{m\to\infty} \nu F_m - \nu(F_m \cap E_n)\\ &\geq \lim_{m\to\infty} \nu F_m - \nu E_n = \phi_1 H - \nu E_n.\end{aligned}$$

Accordingly

$$\sup\{\phi_1 G' : G' \in \mathrm{T}^f_\delta,\ G' \subseteq H \setminus G\} \geq \sup_{n\in\mathbb{N}} \phi_1 H - \nu E_n = \phi_1 H - \phi_1 G.$$

On the other hand, if $G' \in \mathrm{T}^f_\delta$ and $G' \subseteq H \setminus G$, then

$$\phi_1 G + \phi_1 G' = \phi_1(G \cup G') \leq \phi_1 H$$

because of course ϕ_1 is non-decreasing, as well as being additive on disjoint sets. So

$$\sup\{\phi_1 G' : G' \in \mathrm{T}^f_\delta,\ G' \subseteq H \setminus G\} = \phi_1 H - \phi_1 G$$

as required by condition (α) of 413I. Finally, suppose that $\langle H_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in T^f_δ with empty intersection. For each $n \in \mathbb{N}$, let $\langle E_{ni}\rangle_{i\in\mathbb{N}}$ be a non-increasing sequence in T^f with intersection H_n, and set $F_m = \bigcap_{n\leq m} E_{nn}$ for each m. Then $\langle F_m\rangle_{m\in\mathbb{N}}$ is a non-increasing sequence in T^f with empty intersection, while $H_m \subseteq F_m$ for each m, so

$$\lim_{m\to\infty} \phi_1 H_m \leq \lim_{m\to\infty} \nu F_m = 0.$$

Thus condition 413I(β) is satisfied, and we have the full list. **Q**

(e) By 413I, we have a complete locally determined measure μ, extending ϕ_1, and inner regular with respect to T^f_δ. Since $\phi_1 K = \nu K = \phi_0 K$ for $K \in \mathcal{K}$, μ extends ϕ_0. If G belongs to the domain of μ, and $\gamma < \mu G$, there is an $H \in \mathrm{T}^f_\delta$ such that $H \subseteq G$ and $\gamma < \mu H = \phi_1 H$; by (c-ii), there is an $L \in \mathcal{K}_\delta$ such that $L \subseteq H$ and $\gamma \leq \phi_1 L = \mu L$. Thus μ is inner regular with respect to $\mathcal{K}_\delta$. To see that μ is unique, observe that if μ' is any other measure with these properties, and $L \in \mathcal{K}_\delta$, then L is expressible as $\bigcap_{n\in\mathbb{N}} K_n$ where $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$. Now

$$\mu L = \lim_{n\to\infty} \mu(\textstyle\bigcap_{i\leq n} K_i) = \lim_{n\to\infty} \phi_0(\bigcap_{i\leq n} K_i) = \mu' L.$$

So μ and μ' must agree on $\mathcal{K}_\delta$, and by 412L they are identical.

413K Corollary (a) Let X be a set, Σ a subring of $\mathcal{P}X$, and $\nu : \Sigma \to [0,\infty[$ a non-negative finitely additive functional such that $\lim_{n\to\infty} \nu E_n = 0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection. Then ν has a unique extension to a complete locally determined measure on X which is inner regular with respect to the family Σ_δ of intersections of sequences in Σ.

(b) Let X be a set, Σ a subalgebra of $\mathcal{P}X$, and $\nu : \Sigma \to [0,\infty[$ a non-negative finitely additive functional such that $\lim_{n\to\infty} \nu E_n = 0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection. Then ν has a unique extension to a measure defined on the σ-algebra of subsets of X generated by Σ.

proof (a) Take Σ, ν in place of $\mathcal{K}$, ϕ_0 in 413J.

(b) Let ν_1 be the complete extension as in (a), and let ν_1' be the restriction of ν_1 to the σ-algebra Σ' generated by Σ; this is the extension required here. To see that ν_1' is unique, use the Monotone Class Theorem (136C).

Remark You will sometimes see (b) above stated as 'an additive functional on an algebra of sets extends to a measure iff it is countably additive'. But this formulation depends on a different interpretation of the phrase 'countably additive' from the one used in this book; see the note after the definition in 326E.

413L It will be useful to have a definition extending an idea in §342.

Definition A **countably compact class** (or **semicompact paving**) is a family $\mathcal{K}$ of sets such that $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $\bigcap_{i\leq n} K_i \neq \emptyset$ for every $n \in \mathbb{N}$.

413M Corollary Let X be a set and $\mathcal{K}$ a countably compact class of subsets of X such that

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $K \cap K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$.

Let $\phi_0 : \mathcal{K} \to [0,\infty[$ be a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K \in \mathcal{K}, K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$.

Then there is a unique complete locally determined measure μ on X extending ϕ_0 and inner regular with respect to $\mathcal{K}_\delta$, the family of sets expressible as intersections of sequences in $\mathcal{K}$.

proof The point is that the hypothesis (β) of 413J is necessarily satisfied: if $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, then, because $\mathcal{K}$ is countably compact, there must be some n such that $K_n = \emptyset$. Since hypothesis (α) here is already enough to ensure that $\phi_0\emptyset = 0$, $\phi_0 K \geq 0$ for every $K \in \mathcal{K}$, we must have $\inf_{n\in\mathbb{N}} \phi_0 K_n = 0$. So we apply 413J to get the result.

413N I now turn to constructions of a different kind, being extension theorems in which the extension is not uniquely defined. Again I start with a theorem on finitely additive functionals.

Theorem Let X be a set, T_0 a subring of $\mathcal{P}X$, and $\nu_0 : \mathrm{T}_0 \to [0,\infty[$ a finitely additive functional. Suppose that $\mathcal{K} \subseteq \mathcal{P}X$ is a family of sets such that

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $K \cap K' \in \mathcal{K}$ for all K, $K' \in \mathcal{K}$,

every member of $\mathcal{K}$ is included in some member of T_0,

and ν_0 is inner regular with respect to $\mathcal{K}$ in the sense that

(α) $\nu_0 E = \sup\{\nu_0 K : K \in \mathcal{K} \cap \mathrm{T}_0, K \subseteq E\}$ for every $E \in \mathrm{T}_0$.

Then ν_0 has an extension to a non-negative finitely additive functional ν_1, defined on a subring T_1 of $\mathcal{P}X$ including $\mathrm{T}_0 \cup \mathcal{K}$, inner regular with respect to $\mathcal{K}$, and such that whenever $E \in \mathrm{T}_1$, $\epsilon > 0$ there is an $E_0 \in \mathrm{T}_0$ such that $\nu_1(E \triangle E_0) \leq \epsilon$.

proof (a) Let P be the set of all non-negative additive real-valued functionals ν, defined on subrings of $\mathcal{P}X$, inner regular with respect to $\mathcal{K}$, and such that

($*$) whenever $E \in \operatorname{dom}\nu$, $\epsilon > 0$ there is an $E_0 \in \mathrm{T}_0$ such that $\nu(E \triangle E_0) \leq \epsilon$.

Order P by extension of functions, so that P is a partially ordered set.

(b) It will be convenient to borrow some notation from the theory of countably additive functionals. If T is a subring of $\mathcal{P}X$ and $\nu : \mathrm{T} \to [0, \infty[$ is a non-negative additive functional, set

$$\nu^* A = \inf\{\nu E : A \subseteq E \in \mathrm{T}\}, \quad \nu_* A = \sup\{\nu E : A \supseteq E \in \mathrm{T}\}$$

for every $A \subseteq X$ (interpreting $\inf \emptyset$ as ∞ if necessary). Now if $A \subseteq X$ and E, $F \in \mathrm{T}$ are disjoint,

$$\nu^*(A \cap (E \cup F)) = \nu^*(A \cap E) + \nu^*(A \cap F),$$

$$\nu_*(A \cap (E \cup F)) = \nu_*(A \cap E) + \nu_*(A \cap F).$$

P
$$\begin{aligned}
\nu^*(A \cap (E \cup F)) &= \inf\{\nu G : G \in \mathrm{T}, A \cap (E \cup F) \subseteq G\} \\
&= \inf\{\nu G : G \in \mathrm{T}, A \cap (E \cup F) \subseteq G \subseteq E \cup F\} \\
&= \inf\{\nu(G \cap E) + \nu(G \cap F) : G \in \mathrm{T}, A \cap (E \cup F) \subseteq G \subseteq E \cup F\} \\
&= \inf\{\nu G_1 + \nu G_2 : G_1, G_2 \in \mathrm{T}, A \cap E \subseteq G_1 \subseteq E, A \cap F \subseteq G_2 \subseteq F\} \\
&= \inf\{\nu G_1 : G_1 \in \mathrm{T}, A \cap E \subseteq G_1 \subseteq E\} \\
&\qquad + \inf\{\nu G_2 : G_2 \in \mathrm{T}, A \cap F \subseteq G_2 \subseteq F\} \\
&= \nu^*(E \cap A) + \nu^*(F \cap A),
\end{aligned}$$

$$\begin{aligned}
\nu_*(A \cap (E \cup F)) &= \sup\{\nu G : G \in \mathrm{T}, A \cap (E \cup F) \supseteq G\} \\
&= \sup\{\nu(G \cap E) + \nu(G \cap F) : G \in \mathrm{T}, A \cap (E \cup F) \supseteq G\} \\
&= \sup\{\nu G_1 + \nu G_2 : G_1, G_2 \in \mathrm{T}, A \cap E \supseteq G_1, A \cap F \supseteq G_2\} \\
&= \sup\{\nu G_1 : G_1 \in \mathrm{T}, A \cap E \supseteq G_1\} \\
&\qquad + \sup\{\nu G_2 : G_2 \in \mathrm{T}, A \cap F \supseteq G_2\} \\
&= \nu_*(E \cap A) + \nu_*(F \cap A). \ \textbf{Q}
\end{aligned}$$

(c) The key to the proof is the following fact: if $\nu \in P$ and $M \in \mathcal{K}$, there is a $\nu' \in P$ such that ν' extends ν and $M \in \operatorname{dom}\nu'$. **P** Set $\mathrm{T} = \operatorname{dom}\nu$, $\mathrm{T}' = \{(E \cap M) \cup (F \setminus M) : E, F \in \mathrm{T}\}$. For $H \in \mathrm{T}'$, set

$$\nu' H = \nu^*(H \cap M) + \nu_*(H \setminus M).$$

Now we have to check the following.

(i) T' is a subring of $\mathcal{P}X$, because if E, F, E', $F' \in \mathrm{T}$ then

$$((E \cap M) \cup (F \setminus M)) * ((E' \cap M) \cup (F' \setminus M)) = ((E * E') \cap M) \cup ((F * F') \setminus M)$$

for both the Boolean operations $* = \triangle$ and $* = \cap$. $\mathrm{T}' \supseteq \mathrm{T}$ because $E = (E \cap M) \cup (E \setminus M)$ for every $E \in \mathrm{T}$. (Cf. 312M.) $M \in \mathrm{T}'$ because there is some $E \in \mathrm{T}_0$ such that $M \subseteq E$, so that $M = (E \cap M) \cup (\emptyset \setminus M) \in \mathrm{T}'$.

(ii) ν' is finite-valued because if $H = (E \cap M) \cup (F \setminus M)$, where E, $F \in \mathrm{T}$, then $\nu' H \leq \nu E + \nu F$. If H, $H' \in \mathrm{T}$ are disjoint, they can be expressed as $(E \cap M) \cup (F \setminus M)$, $(E' \cap M) \cup (F' \setminus M)$ where E, F, E', F' belong to T; replacing E', F' by $E' \setminus E$ and $F' \setminus F$ if necessary, we may suppose that $E \cap E' = F \cap F' = \emptyset$. Now

$$\begin{aligned}\nu'(H \cup H') &= \nu^*((E \cup E') \cap M) + \nu_*((F \cup F') \cap (X \setminus M)) \\ &= \nu^*(E \cap M) + \nu^*(E' \cap M) + \nu_*(F \cap (X \setminus M)) + \nu_*(F' \cap (X \setminus M)) \\ &\text{(by (b) above)} \\ &= \nu' H + \nu' H'.\end{aligned}$$

Thus ν' is additive.

(iii) If $E \in \mathrm{T}$, then

$$\begin{aligned}\nu_*(E \setminus M) &= \sup\{\nu F : F \in \mathrm{T},\ F \subseteq E \setminus M\} \\ &= \sup\{\nu E - \nu(E \setminus F) : F \in \mathrm{T},\ F \subseteq E \setminus M\} \\ &= \sup\{\nu E - \nu F : F \in \mathrm{T},\ E \cap M \subseteq F \subseteq E\} \\ &= \nu E - \inf\{\nu F : F \in \mathrm{T},\ E \cap M \subseteq F \subseteq E\} = \nu E - \nu^*(E \cap M).\end{aligned}$$

So

$$\nu' E = \nu^*(E \cap M) + \nu_*(E \setminus M) = \nu E.$$

Thus ν' extends ν.

(iv) If $H \in \mathrm{T}'$ and $\epsilon > 0$, express H as $(E \cap M) \cup (F \setminus M)$, where E, $F \in \mathrm{T}$. Then we can find (α) a $K \in \mathcal{K} \cap \mathrm{T}$ such that $K \subseteq E$ and $\nu(E \setminus K) \leq \epsilon$ (β) an $F' \in \mathrm{T}$ such that $F' \subseteq F \setminus M$ and $\nu F' \geq \nu_*(F \setminus M) - \epsilon$ (γ) a $K' \in \mathcal{K} \cap \mathrm{T}$ such that $K' \subseteq F'$ and $\nu K' \geq \nu F' - \epsilon$. Set $L = (K \cap M) \cup K' \in \mathrm{T}'$; by the hypotheses (†) and (‡), $L \in \mathcal{K}$. Now $L \subseteq H$ and

$$\begin{aligned}\nu' L &= \nu'(K \cap M) + \nu' K' = \nu'(E \cap M) - \nu'((E \setminus K) \cap M) + \nu K' \\ &= \nu^*(H \cap M) - \nu^*((E \setminus K) \cap M) + \nu K' \geq \nu^*(H \cap M) - \nu(E \setminus K) + \nu F' - \epsilon \\ &\geq \nu^*(H \cap M) + \nu_*(F \setminus M) - 3\epsilon = \nu' H - 3\epsilon.\end{aligned}$$

As H and ϵ are arbitrary, ν is inner regular with respect to $\mathcal{K}$.

(v) Finally, given $H \in \mathrm{T}'$ and $\epsilon > 0$, take E, $F \in \mathrm{T}$ such that $H \cap M \subseteq E$, $F \subseteq H \setminus M$, $\nu E \leq \nu^*(H \cap M) + \epsilon$ and $\nu F \geq \nu_*(H \setminus M) - \epsilon$. In this case,

$$\nu'(E \setminus (H \cap M)) = \nu' E - \nu'(H \cap M) = \nu E - \nu^*(H \cap M) \leq \epsilon,$$

$$\nu'((H \setminus M) \setminus F) = \nu'(H \setminus M) - \nu' F = \nu_*(H \setminus M) - \nu F \leq \epsilon.$$

But as

$$H \triangle (E \cup F) \subseteq (E \setminus (H \cap M)) \cup ((H \setminus M) \setminus F),$$

$\nu'(H \triangle (E \cup F)) \leq 2\epsilon$. Now ν satisfies the condition $(*)$, so there is an $E_0 \in \mathrm{T}_0$ such that $\nu((E \cup F) \triangle E_0) \leq \epsilon$, and $\nu'(H \triangle E_0) \leq 3\epsilon$. As H and ϵ are arbitrary, ν' satisfies $(*)$.

This completes the proof that ν' is a member of P extending ν. **Q**

(d) It is easy to check that if $Q \subseteq P$ is a non-empty totally ordered subset, the smallest common extension ν' of the functions in Q belongs to P. (To see that ν' is inner regular with respect to $\mathcal{K}$, observe that if $E \in \operatorname{dom} \nu'$ and $\gamma < \nu' E$, there is some $\nu \in Q$ such that $E \in \operatorname{dom} \nu$; now there is a $K \in \mathcal{K} \cap \operatorname{dom} \nu$ such that $K \subseteq E$ and $\nu K \geq \gamma$, so that $K \in \mathcal{K} \cap \operatorname{dom} \nu'$ and $\nu' K \geq \gamma$.) And of course P is not empty, because $\nu_0 \in P$. So by Zorn's Lemma P has a maximal element ν_1 say; write T_1 for the domain of ν_1. If $M \in \mathcal{K}$ there is an element of P, with a domain containing M, extending ν_1; as ν_1 is maximal, this must be ν_1 itself, so $M \in \mathrm{T}_1$. Thus $\mathcal{K} \subseteq \mathrm{T}_1$, and ν_1 has all the required properties.

413O Corollary Let (X, Σ_0, μ_0) be a measure space and $\mathcal{K}$ a countably compact class of subsets of X such that

($\dagger$) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

($\ddagger$) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ for every sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$,

$\mu_0^* K < \infty$ for every $K \in \mathcal{K}$,

μ_0 is inner regular with respect to $\mathcal{K}$.

Then μ_0 has an extension to a complete locally determined measure μ, defined on every member of $\mathcal{K}$, inner regular with respect to $\mathcal{K}$, and such that whenever $E \in \operatorname{dom}\mu$ and $\mu E < \infty$ there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$.

proof (a) Set $\mathrm{T}_0 = \{E : E \in \Sigma_0, \mu_0 E < \infty\}$, $\nu_0 = \mu_0 \restriction \mathrm{T}_0$. Then ν_0, T_0 satisfy the conditions of 413N; take ν_1, T_1 as in 413N. If K, $L \in \mathcal{K}$ and $L \subseteq K$, then

$$\nu_1 L + \sup\{\nu_1 K' : K' \in \mathcal{K},\, K' \subseteq K \setminus L\} = \nu_1 L + \nu_1(K \setminus L) = \nu_1 K.$$

So $\nu_1 \restriction \mathcal{K}$ satisfies the conditions of 413M and there is a complete locally determined measure μ, extending $\nu_1 \restriction \mathcal{K}$, and inner regular with respect to $\mathcal{K}$.

(b) Write Σ for the domain of μ. Then $\mathrm{T}_1 \subseteq \Sigma$. **P** If $E \in \mathrm{T}_1$ and $K \in \mathcal{K}$,

$$\begin{aligned}\mu_*(K \cap E) + \mu_*(K \setminus E) &\geq \sup\{\mu K' : K' \in \mathcal{K},\, K' \subseteq K \cap E\} + \sup\{\mu K' : K' \in \mathcal{K},\, K' \subseteq K \setminus E\}\\ &= \sup\{\nu_1 K' : K' \in \mathcal{K},\, K' \subseteq K \cap E\} + \sup\{\nu_1 K' : K' \in \mathcal{K},\, K' \subseteq K \setminus E\}\\ &= \nu_1(K \cap E) + \nu_1(K \setminus E) = \nu_1 K = \mu K.\end{aligned}$$

By 413F(iv), $E \in \Sigma$. **Q** It follows at once that μ extends ν_1, since if $E \in \mathrm{T}_1$

$$\nu_1 E = \sup\{\nu_1 K : K \in \mathcal{K},\, K \subseteq E\} = \sup\{\mu K : K \in \mathcal{K},\, K \subseteq E\} = \mu E.$$

(c) In particular, μ agrees with μ_0 on T_0. Now in fact μ extends μ_0. **P** Take $E \in \Sigma_0$. If $K \in \mathcal{K}$, there is an $E_0 \in \Sigma_0$ such that $K \subseteq E_0$ and $\mu_0 E_0 < \infty$. Since $E \cap E_0 \in \mathrm{T}_0 \subseteq \Sigma$, $E \cap K = E \cap E_0 \cap K \in \Sigma$. As K is arbitrary, $E \in \Sigma$, by 413F(ii). Next, because every member of $\mathcal{K}$ is included in a member of T_0,

$$\begin{aligned}\mu_0 E &= \sup\{\mu_0 K : K \in \mathcal{K} \cap \Sigma_0,\, K \subseteq E\} = \sup\{\mu_0(E \cap E_0) : E_0 \in \mathrm{T}_0\}\\ &= \sup\{\mu(E \cap E_0) : E_0 \in \mathrm{T}_0\} = \sup\{\mu K : K \in \mathcal{K},\, K \subseteq E\} = \mu E.\end{aligned}$$ **Q**

(d) Finally, suppose that $E \in \Sigma$ and $\mu E < \infty$. For each $n \in \mathbb{N}$ we can find $K_n \in \mathcal{K}$, $E_n \in \Sigma_0$ such that $K_n \subseteq E$, $\mu(E \setminus K_n) \leq 2^{-n}$ and $\nu_1(K_n \triangle E_n) \leq 2^{-n}$. In this case $\sum_{n=0}^{\infty} \mu(E_n \triangle E) < \infty$, so $\mu(E \triangle E') = 0$, where $E' = \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} E_m \in \Sigma_0$.

Thus μ has all the required properties.

413P I now describe an alternative route to some of the applications of 413N. As before, I do as much as possible in the context of finitely additive functionals.

Lemma Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$. Let $\lambda : \mathcal{K} \to [0, \infty[$ be a bounded functional such that

$$\lambda\emptyset = 0, \quad \lambda K \leq \lambda K' \text{ whenever } K,\, K' \in \mathcal{K} \text{ and } K \subseteq K',$$

$$\lambda(K \cup K') + \lambda(K \cap K') \geq \lambda K + \lambda K' \text{ for all } K,\, K' \in \mathcal{K}.$$

Then there is a finitely additive functional $\nu : \mathcal{P}X \to [0, \infty[$ such that

$$\nu X = \sup_{K\in\mathcal{K}} \lambda K, \quad \nu K \geq \lambda K \text{ for every } K \in \mathcal{K}.$$

proof (a) The key to the proof is the following fact: if $K_0, \ldots, K_n \in \mathcal{K}$, then there are $K_0', \ldots, K_n' \in \mathcal{K}$ such that $K_0' \subseteq K_1' \subseteq \ldots \subseteq K_n'$, $\sum_{i=0}^n \chi K_i' = \sum_{i=0}^n \chi K_i$ and $\sum_{i=0}^n \lambda K_i' \geq \sum_{i=0}^n \lambda K_i$. **P** Induce on n. If

$n = 0$ the result is trivial. For the inductive step to $n+1$, given $K_0, \ldots, K_{n+1} \in \mathcal{K}$, set $L_{n+1} = K_{n+1}$ and use the inductive hypothesis to find $L_0, \ldots, L_n \in \mathcal{K}$ such that

$$L_0 \subseteq \ldots \subseteq L_n,\ \textstyle\sum_{i=0}^n \chi L_i = \sum_{i=0}^n \chi K_i \text{ and } \sum_{i=0}^n \lambda L_i \geq \sum_{i=0}^n \lambda K_i.$$

Now set $L'_n = L_{n+1} \cap L_n$, $L'_i = L_i$ for $i < n$, and use the inductive hypothesis again to find $K'_0, \ldots, K'_n \in \mathcal{K}$ such that

$$K'_0 \subseteq \ldots \subseteq K'_n,\ \textstyle\sum_{i=0}^n \chi K'_i = \sum_{i=0}^n \chi L'_i \text{ and } \sum_{i=0}^n \lambda K'_i \geq \sum_{i=0}^n \lambda L'_i.$$

Set

$$K'_{n+1} = L'_{n+1} = L_n \cup L_{n+1}.$$

Then all the L_i, L'_i and K'_i belong to $\mathcal{K}$. We know that $\chi K'_n \leq \sum_{i=0}^n \chi L'_i$, so

$$K'_n \subseteq \textstyle\bigcup_{i \leq n} L'_i \subseteq L_n \subseteq K'_{n+1};$$

accordingly we have $K'_0 \subseteq K'_1 \subseteq \ldots \subseteq K'_n \subseteq K'_{n+1}$. Next,

$$\begin{aligned}\sum_{i=0}^{n+1} \chi K'_i &= \sum_{i=0}^{n} \chi L'_i + \chi L'_{n+1} = \sum_{i=0}^{n-1} \chi L_i + \chi(L_n \cap L_{n+1}) + \chi(L_n \cup L_{n+1}) \\ &= \sum_{i=0}^{n-1} \chi L_i + \chi L_n + \chi L_{n+1} = \sum_{i=0}^{n} \chi L_i + \chi K_{n+1} = \sum_{i=0}^{n+1} \chi K_i,\end{aligned}$$

$$\begin{aligned}\sum_{i=0}^{n+1} \lambda K'_i &\geq \sum_{i=0}^{n} \lambda L'_i + \lambda L'_{n+1} = \sum_{i=0}^{n-1} \lambda L_i + \lambda(L_n \cap L_{n+1}) + \lambda(L_n \cup L_{n+1}) \\ &\geq \sum_{i=0}^{n-1} \lambda L_i + \lambda L_n + \lambda L_{n+1}\end{aligned}$$

(using the hypothesis on λ)

$$= \sum_{i=0}^{n} \lambda L_i + \lambda K_{n+1} \geq \sum_{i=0}^{n+1} \lambda K_i,$$

so we have an appropriate family $K'_0, \ldots, K'_{n+1}$, and the induction continues. **Q**

(b) For the moment, suppose that $\sup_{K \in \mathcal{K}} \lambda K = 1$. In this case, if $\langle K_i \rangle_{i \in I}$ is a finite indexed family in $\mathcal{K}$, there is a set $J \subseteq I$ such that $\#(J) \geq \sum_{i \in I} \lambda K_i$ and $\bigcap_{i \in J} K_i \neq \emptyset$. **P** If $I = \emptyset$ this is trivial, so we may suppose that $I = \{0, \ldots, n\}$ for some n. Take $K'_0, \ldots, K'_n$ as in (a). If $K'_n = \emptyset$ then every K_i is empty so (because $\lambda\emptyset = 0$) $\sum_{i=0}^n \lambda K_i = 0$ and we may take $J = \emptyset$. Otherwise, let m be the first number such that $K'_m \neq \emptyset$, and take $x \in K'_m$, $J = \{i : i \leq n, x \in K_i\}$. Then $\bigcap_{i \in J} K_i \neq \emptyset$ and

$$\begin{aligned}\#(J) &= \sum_{i=0}^{n} \chi K_i(x) = \sum_{i=0}^{n} \chi K'_i(x) = n - m + 1 \\ &\geq \sum_{i=m}^{n} \lambda K'_i = \sum_{i=0}^{n} \lambda K'_i \geq \sum_{i=0}^{n} \lambda K_i,\end{aligned}$$

as claimed. **Q**

By 391F, there is an additive functional $\nu : \mathcal{P}X \to [0,1]$ such that $\nu X = 1$ and $\nu K \geq \lambda K$ for every $K \in \mathcal{K}$, as required.

(c) For the general case, set $\gamma = \sup_{K \in \mathcal{K}} \lambda K$. If $\gamma = 0$, take ν to be the zero functional; if $\gamma > 0$, apply (a)-(b) to the functional $\gamma^{-1}\lambda$.

Remark If P is a lattice, a function $f : P \to \mathbb{R}$ such that $f(p \vee q) + f(p \wedge q) \geq f(p) + f(q)$ for all $p, q \in P$ is called **supermodular**.

413Q Theorem Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$. Let Σ be the algebra of subsets of X generated by $\mathcal{K}$, and $\nu_0 : \Sigma \to [0,\infty[$ a finitely additive functional. Then there is a finitely additive functional $\nu : \Sigma \to [0,\infty[$ such that (i) $\nu X = \sup_{K\in\mathcal{K}} \nu_0 K$ (ii) $\nu K \geq \nu_0 K$ for every $K \in \mathcal{K}$ (iii) ν is inner regular with respect to $\mathcal{K}$ in the sense that $\nu E = \sup\{\nu K : K \in \mathcal{K}, K \subseteq E\}$ for every $E \in \Sigma$.

proof (a) Set $\gamma = \sup_{K\in\mathcal{K}} \nu_0 K$. Let P be the set of all functionals $\lambda : \mathcal{K} \to [0,\gamma]$ such that

$$\lambda K + \lambda K' \leq \lambda(K \cup K') + \lambda(K \cap K')$$

for every K, $K' \in \mathcal{K}$. Give P the natural partial ordering inherited from $\mathbb{R}^{\mathcal{K}}$. Note that $\nu_0 \restriction \mathcal{K}$ belongs to P. If $Q \subseteq P$ is non-empty and upwards-directed, then $\sup Q$, taken in $\mathbb{R}^{\mathcal{K}}$, belongs to P; so there is a maximal $\lambda \in P$ such that $\nu_0 \restriction \mathcal{K} \leq \lambda$. By 413P, there is a non-negative additive functional ν on $\mathcal{P}X$ such that $\nu K \geq \lambda K$ for every $K \in \mathcal{K}$ and $\nu X = \gamma$. Since $\nu \restriction \mathcal{K}$ also belongs to P, we must have $\nu K = \lambda K$ for every $K \in \mathcal{K}$.

(b) Now for any $K_0 \in \mathcal{K}$,

$$\nu K_0 + \sup\{\nu L : L \in \mathcal{K}, L \subseteq X \setminus K_0\} = \gamma.$$

P (i) Set $\mathcal{L} = \{L : L \in \mathcal{K}, L \subseteq X \setminus K_0\}$. For $A \subseteq X$, set $\theta_0 A = \sup_{L\in\mathcal{L}} \nu(A \cap L)$. Because $\mathcal{L}$ is upwards-directed, $\theta_0 : \mathcal{P}X \to \mathbb{R}$ is additive, and of course $0 \leq \theta_0 \leq \nu$. Set $\theta_1 = \nu - \theta_0$, so that θ_1 is another additive functional, and write

$$\lambda' K = \theta_0 K + \sup\{\theta_1 M : M \in \mathcal{K}, M \cap K_0 \subseteq K\}$$

for $K \in \mathcal{K}$.

(ii) If K, $K' \in \mathcal{K}$ and $\epsilon > 0$, there are M, $M' \in \mathcal{K}$ such that $M \cap K_0 \subseteq K$, $M' \cap K_0 \subseteq K'$ and

$$\theta_0 K + \theta_1 M \geq \lambda' K - \epsilon, \quad \theta_0 K' + \theta_1 M' \geq \lambda' K' - \epsilon.$$

Now

$$M \cup M',\ M \cap M' \in \mathcal{K},$$

$$(M \cup M') \cap K_0 \subseteq K \cup K', \quad (M \cap M') \cap K_0 \subseteq K \cap K',$$

so

$$\begin{aligned}\lambda'(K \cup K') + \lambda'(K \cap K') &\geq \theta_0(K \cup K') + \theta_1(M \cup M') + \theta_0(K \cap K') + \theta_1(M \cap M') \\ &= \theta_0 K + \theta_1 M + \theta_0 K' + \theta_1 M' \geq \lambda' K + \lambda' K' - 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\lambda'(K \cup K') + \lambda'(K \cap K') \geq \lambda K + \lambda K'$.

(iii) Suppose that K, $M \in \mathcal{K}$ are such that $M \cap K_0 \subseteq K$. If $L \in \mathcal{L}$, then

$$\begin{aligned}\nu(K \cap L) + \theta_1 M &= \nu(K \cap L) + \nu M - \theta_0 M \\ &= \nu(M \cap K \cap L) + \nu(M \cup (K \cap L)) - \theta_0 M \leq \gamma\end{aligned}$$

because $K \cap L \in \mathcal{L}$; taking the supremum over L and M, $\lambda' K \leq \gamma$. As K is arbitrary, $\lambda' \in P$.

(iv) If $K \in \mathcal{K}$, then of course $K \cap K_0 \subseteq K$, so

$$\lambda' K \geq \theta_0 K + \theta_1 K = \nu K = \lambda K.$$

Thus $\lambda' \geq \lambda$. Because λ is maximal, $\lambda' = \lambda$. But this means that

$$\lambda K_0 = \lambda' K_0 = \theta_0 K_0 + \sup\{\theta_1 M : M \in \mathcal{K}, M \cap K_0 \subseteq K_0\} = \sup_{M\in\mathcal{K}} \theta_1 M.$$

Now given $\epsilon > 0$ there is an $M \in \mathcal{K}$ such that

$$\gamma - \epsilon \leq \nu_0 M \leq \lambda M = \nu M,$$

so that

$$\nu K_0 = \lambda K_0 \geq \theta_1 M = \nu M - \theta_0 M \geq \gamma - \epsilon - \theta_0 M \geq \gamma - \epsilon - \sup_{L\in\mathcal{L}} \nu L,$$

and $\nu K_0 + \sup_{L\in\mathcal{L}} \nu L \geq \gamma - \epsilon$. As ϵ is arbitrary, $\nu K_0 + \sup_{L\in\mathcal{L}} \nu L \geq \gamma$. But of course $\nu K_0 + \nu L \leq \nu X = \gamma$ for every $L \in \mathcal{L}$, so $\nu K_0 + \sup_{L\in\mathcal{L}} \nu L = \gamma$, as claimed. **Q**

(c) It follows that if K, $L \in \mathcal{K}$ and $L \subseteq K$,

$$\nu K = \nu L + \sup\{\nu K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}.$$

P Because ν is additive and non-negative, we surely have

$$\nu K \geq \nu L + \sup\{\nu K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}.$$

On the other hand, given $\epsilon > 0$, there is an $M \in \mathcal{K}$ such that $M \subseteq X \setminus L$ and $\nu L + \nu M \geq \gamma - \epsilon$, so that $M \cap K \in \mathcal{K}$, $M \cap K \subseteq K \setminus L$ and

$$\nu L + \nu(M \cap K) = \nu L + \nu K + \nu M - \nu(M \cup K) \geq \nu K + \gamma - \epsilon - \gamma = \nu K - \epsilon.$$

As ϵ is arbitrary,

$$\nu K \leq \nu L + \sup\{\nu K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$$

and we have equality. **Q**

(d) By 413H, we have an additive functional $\nu' : \Sigma \to [0, \infty[$ such that $\nu' E = \sup\{\nu K : K \in \mathcal{K},\ K \subseteq E\}$ for every $E \in \Sigma$. It is easy to show that ν' and ν must agree on Σ, but even without doing so we can see that ν' has the properties (i)-(iii) required in the theorem.

413R The following lemma on countably compact classes, corresponding to 342Db, will be useful.

Lemma (Marczewski 53) Let $\mathcal{K}$ be a countably compact class of sets. Then there is a countably compact class $\mathcal{K}^* \supseteq \mathcal{K}$ such that $K \cup L \in \mathcal{K}^*$ and $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}^*$ whenever K, $L \in \mathcal{K}^*$ and $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}^*$.

proof (a) Write $\mathcal{K}_s$ for $\{K_0 \cup \ldots \cup K_n : K_0, \ldots, K_n \in \mathcal{K}\}$. Then $\mathcal{K}_s$ is countably compact. **P** Let $\langle L_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}_s$ such that $\bigcap_{i\leq n} L_i \neq \emptyset$ for each $n \in \mathbb{N}$. Then there is an ultrafilter $\mathcal{F}$ on $X = \bigcup\mathcal{K}$ containing every L_n. For each n, L_n is a finite union of members of $\mathcal{K}$, so there must be a $K_n \in \mathcal{K}$ such that $K_n \subseteq L_n$ and $K_n \in \mathcal{F}$. Now $\bigcap_{i\leq n} K_i \neq \emptyset$ for every n, so $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$ and $\bigcap_{n\in\mathbb{N}} L_n \neq \emptyset$. As $\langle L_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{K}_s$ is countably compact. **Q**

Note that $L \cup L' \in \mathcal{K}_s$ for all L, $L' \in \mathcal{K}_s$.

(b) Write $\mathcal{K}^*$ for

$$\{\bigcap\mathcal{L}_0 : \mathcal{L}_0 \subseteq \mathcal{K}_s \text{ is non-empty and countable}\}.$$

Then $\mathcal{K}^*$ is countably compact. **P** If $\langle M_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{K}^*$ such that $\bigcap_{i\leq n} M_i \neq \emptyset$ for every $n \in \mathbb{N}$, then for each $n \in \mathbb{N}$ let $\mathcal{L}_n \subseteq \mathcal{K}_s$ be a countable non-empty set such that $M_n = \bigcap\mathcal{L}_n$. Let $\langle L_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\bigcup_{n\in\mathbb{N}} \mathcal{L}_n$; then $\bigcap_{i\leq n} L_i \neq \emptyset$ for every n, so $\bigcap_{n\in\mathbb{N}} L_n = \bigcap_{n\in\mathbb{N}} M_n$ is non-empty. As $\langle M_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{K}^*$ is countably compact. **Q**

(c) Of course $\mathcal{K} \subseteq \mathcal{K}_s \subseteq \mathcal{K}^*$. It is immediate from the definition of $\mathcal{K}^*$ that it is closed under countable intersections. Finally, if M_1, $M_2 \in \mathcal{K}^*$, let $\mathcal{L}_1$, $\mathcal{L}_2 \subseteq \mathcal{K}_s$ be countable sets such that $M_1 = \bigcap\mathcal{L}_1$ and $M_2 = \bigcap\mathcal{L}_2$; then $\mathcal{L} = \{L_1 \cup L_2 : L_1 \in \mathcal{L}_1,\ L_2 \in \mathcal{L}_2\}$ is a countable subset of $\mathcal{K}_s$, so $M_1 \cup M_2 = \bigcap\mathcal{L}$ belongs to $\mathcal{K}^*$.

413S Corollary Let X be a set and $\mathcal{K}$ a countably compact class of subsets of X. Let T be a subalgebra of $\mathcal{P}X$ and $\nu : \mathrm{T} \to \mathbb{R}$ a non-negative finitely additive functional.

(a) There is a complete measure μ on X such that $\mu X \leq \nu X$, $\mathcal{K} \subseteq \mathrm{dom}\,\mu$ and $\mu K \geq \nu K$ for every $K \in \mathcal{K} \cap \mathrm{T}$.

(b) If

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$,

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ for every sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{K}$,

we may arrange that μ is inner regular with respect to $\mathcal{K}$.

proof By 413R, there is always a countably compact class $\mathcal{K}^* \supseteq \mathcal{K}$ satisfying (†) and (‡); for case (b), take $\mathcal{K}^* = \mathcal{K}$. By 391G, there is an extension of ν to a finitely additive functional $\nu' : \mathcal{P}X \to \mathbb{R}$. Let T_1 be the subalgebra of $\mathcal{P}X$ generated by $\mathcal{K}^*$. By 413Q, there is a non-negative additive functional $\nu_1 : \mathrm{T}_1 \to \mathbb{R}$ such that $\nu_1 X \leq \nu' X = \nu X$, $\nu_1 K \geq \nu' K = \nu K$ for every $K \in \mathcal{K}^* \cap \mathrm{T}$ and $\nu_1 E = \sup\{\nu_1 K : K \in \mathcal{K}^*, K \subseteq E\}$ for every $E \in \mathrm{T}_1$. In particular, if K, $L \in \mathcal{K}^*$,

$$\nu_1 L + \sup\{\nu_1 K' : K' \in \mathcal{K}^*, K' \subseteq K \setminus L\} = \nu_1 L + \nu_1(K \setminus L) = \nu_1 K.$$

So $\mathcal{K}^*$ and $\nu_1 \restriction \mathcal{K}^*$ satisfy the hypotheses of 413M. Accordingly we have a complete measure μ extending $\nu_1 \restriction \mathcal{K}^*$ and inner regular with respect to $\mathcal{K}^* = \mathcal{K}^*_\delta$; in which case

$$\mu K = \nu_1 K \geq \nu K$$

for every $K \in \mathcal{K} \cap \mathrm{T}$,

$$\mu X = \sup_{K \in \mathcal{K}^*} \mu K = \sup_{K \in \mathcal{K}^*} \nu_1 K \leq \nu_1 X \leq \nu X,$$

as required.

413X Basic exercises (a) Define $\phi : \mathcal{P}\mathbb{N} \to [0, \infty[$ by setting $\phi A = 0$ if A is finite, ∞ otherwise. Check that ϕ satisfies conditions (α) and (β) of 413A, but that if we attempt to reproduce the construction of 413C then we obtain $\Sigma = \mathcal{P}\mathbb{N}$, $\mu = \phi$, so that μ is not countably additive.

(b) Let ϕ_1, ϕ_2 be two inner measures on a set X, inducing measures μ_1, μ_2 by the method of 413C. (i) Show that $\phi = \phi_1 + \phi_2$ is an inner measure. (ii) Show that the measure μ induced by ϕ extends the measure $\mu_1 + \mu_2$ defined on $\operatorname{dom}\mu_1 \cap \operatorname{dom}\mu_2$.

>(c) Let X be a set, ϕ an inner measure on X, and μ the measure constructed from it by the method of 413C. (i) Let Y be a subset of X. Show that $\phi \restriction \mathcal{P}Y$ is an inner measure on Y, and that the measure on Y defined from it extends the subspace measure μ_Y induced on Y by μ. (ii) Let Y be a set and $f : X \to Y$ a function. Show that $B \mapsto \phi f^{-1}[B]$ is an inner measure on Y, and that it defines a measure on Y which extends the image measure μf^{-1}.

(d) Let (X, Σ, μ) be any measure space. Set $\theta A = \frac{1}{2}(\mu^* A + \mu_* A)$ for every $A \subseteq X$. Show that θ is an outer measure on X, and that the measure defined from θ by Carathéodory's method extends μ.

>(e) Show that there is a partition $\langle A_n \rangle_{n \in \mathbb{N}}$ of $[0,1]$ such that $\mu_*(\bigcup_{i \leq n} A_i) = 0$ for every n, where μ_* is Lebesgue inner measure. (*Hint*: set $A_n = (A + q_n) \cap [0,1]$ where $\langle q_n \rangle_{n \in \mathbb{N}}$ is an enumeration of $\mathbb{Q}$ and A is a suitable set; cf. 134B.)

(f) Let (X, Σ, μ) be a measure space. (i) Show that $\mu_* \restriction \Sigma$ is the semi-finite version μ_{sf} of μ as constructed in 213Xc. (ii) Show that if A is any subset of X, and Σ_A the subspace σ-algebra, then $\mu_* \restriction \Sigma_A$ is a semi-finite measure on A.

>(g) Let (X, Σ, μ) and (Y, T, ν) be two measure spaces, and λ the c.l.d. product measure on $X \times Y$. Show that $\lambda_*(A \times B) = \mu_* A \cdot \nu_* B$ for all $A \subseteq X$ and $B \subseteq Y$. (*Hint*: use Fubini's theorem to show that $\lambda_*(A \times B) \leq \mu_* A \cdot \nu_* B$.)

(h) Let X be a set and μ, ν two complete locally determined measures on X with domains Σ, T respectively, both inner regular with respect to $\mathcal{K} \subseteq \Sigma \cap \mathrm{T}$. Suppose that, for $K \in \mathcal{K}$, $\mu K = 0$ iff $\nu K = 0$. Show that $\Sigma = \mathrm{T}$ and that μ and ν give rise to the same negligible sets.

>(i) Let (X, T, ν) be a measure space. (i) Show that the measure constructed by the method of 413C from the inner measure ν_* is the c.l.d. version of ν. (ii) Set $\mathcal{K} = \{E : E \in \mathrm{T}, \nu E < \infty\}$, $\phi_0 = \nu \restriction \mathcal{K}$. Show that $\mathcal{K}$, ϕ_0 satisfy the conditions of 413I, and that the measure constructed by the method there is again the c.l.d. version of ν.

(j) Let (X, Σ, μ) be a complete locally determined measure space and $\mathcal{L}$ a family of subsets of X such that μ is inner regular with respect to $\mathcal{L}$. Set $\mathcal{K} = \{K : K \in \mathcal{L} \cap \Sigma, \mu K < \infty\}$ and $\phi_0 = \mu \restriction \mathcal{K}$. Show that $\mathcal{K}$, ϕ_0 satisfy the conditions of 413I and that the measure constructed from $\mathcal{K}$, ϕ_0 by the method there is just μ.

>(k) Let $\mathcal{K}$ be the family of subsets of $\mathbb{R}$ expressible as finite unions of bounded closed intervals. (i) Show from first principles that there is a unique functional $\phi_0 : \mathcal{K} \to [0,\infty[$ such that $\phi_0[\alpha,\beta] = \beta - \alpha$ whenever $\alpha \leq \beta$ and ϕ_0 satisfies the conditions of 413J. (ii) Show that the measure on $\mathbb{R}$ constructed from ϕ_0 by the method of 413J is Lebesgue measure.

(l) Let X be a set, Σ a subring of $\mathcal{P}X$, and $\nu : \Sigma \to [0,\infty[$ a non-negative additive functional such that $\lim_{n\to\infty} \nu E_n = 0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection, as in 413K. Define $\theta : \mathcal{P}X \to [0,\infty]$ by setting

$$\theta A = \inf\{\textstyle\sum_{n=0}^{\infty} \nu E_n : \langle E_n\rangle_{n\in\mathbb{N}} \text{ is a sequence in } \Sigma \text{ covering } A\}$$

for $A \subseteq X$, interpreting $\inf \emptyset$ as ∞ if necessary. Show that θ is an outer measure. Let μ_θ be the measure defined from θ by Carathéodory's method. Show that the measure defined from ν by the process of 413K is the c.l.d. version of μ_θ. (*Hint*: the c.l.d. version of μ_θ is inner regular with respect to Σ_δ.)

>(m) Let X be a set, Σ a subring of $\mathcal{P}X$, and $\nu : \Sigma \to [0,\infty[$ a non-negative additive functional. Show that the following are equiveridical: (i) ν has an extension to a measure on X; (ii) $\lim_{n\to\infty} \nu E_n = 0$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in Σ with empty intersection; (iii) $\nu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \nu E_n$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ such that $\bigcup_{n\in\mathbb{N}} E_n \in \Sigma$.

(n) Let $\langle(X_n,\Sigma_n,\mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of probability spaces, and $\mathcal{F}$ a non-principal ultrafilter on $\mathbb{N}$. For x, $y \in \prod_{n\in\mathbb{N}} X_n$, write $x \sim y$ if $\{n : x(n) = y(n)\} \in \mathcal{F}$. (i) Show that $\sim$ is an equivalence relation; write X for the set of equivalence classes, and $x^\bullet \in X$ for the equivalence class of $x \in \prod_{n\in\mathbb{N}} X_n$. (Compare 351M.) (ii) Let Σ be the set of subsets of X expressible in the form $Q(\langle E_n\rangle_{n\in\mathbb{N}}) = \{x^\bullet : x \in \prod_{n\in\mathbb{N}} E_n\}$, where $E_n \in \Sigma_n$ for each $n \in \mathbb{N}$. Show that Σ is an algebra of subsets of X, and that there is a well-defined additive functional $\nu : \Sigma \to [0,1]$ defined by setting $\nu(Q(\langle E_n\rangle_{n\in\mathbb{N}})) = \lim_{n\to\mathcal{F}} \mu_n E_n$. (iii) Show that for any non-decreasing sequence $\langle H_i\rangle_{i\in\mathbb{N}}$ in Σ there is an $H \in \Sigma$ such that $H \subseteq \bigcap_{i\in\mathbb{N}} H_i$ and $\nu H = \lim_{n\to\infty} \nu H_n$. (*Hint*: express each H_i as $Q(\langle E_{in}\rangle_{n\in\mathbb{N}})$. Do this in such a way that $E_{i+1,n} \subseteq E_{in}$ for all i, n. Take a decreasing sequence $\langle J_i\rangle_{i\in\mathbb{N}}$ in $\mathcal{F}$, with empty intersection, such that $\nu H_i \leq \mu E_{in} + 2^{-i}$ for $n \in J_i$. Set $E_n = E_{in}$ for $n \in J_i \setminus J_{i+1}$.) (iv) Show that there is a unique extension of ν to a complete probability measure μ on X which is inner regular with respect to Σ. (This is a kind of **Loeb measure**.)

(o) Let $\mathfrak{A}$ be a Boolean algebra and $K \subseteq \mathfrak{A}$ a sublattice containing 0. Suppose that $\lambda : K \to [0,\infty[$ is a bounded functional such that $\lambda 0 = 0$, $\lambda a \leq \lambda a'$ whenever a, $a' \in K$ and $a \subseteq a'$, and $\lambda(a \cup a') + \lambda(a \cap a') \geq \lambda a + \lambda a'$ for all a, $a' \in K$. Show that there is a non-negative additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ such that $\nu a \geq \lambda a$ for every $a \in K$ and $\nu 1 = \sup_{a\in K} \lambda a$.

(p) Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing $\emptyset$. Let $\lambda : \mathcal{K} \to \mathbb{R}$ be a bounded order-preserving function such that $\lambda\emptyset = 0$ and $\lambda(K \cup K') + \lambda(K \cap K') = \lambda K + \lambda K'$ for all K, $K' \in \mathcal{K}$. Show that there is a non-negative additive functional $\nu : \mathcal{P}X \to \mathbb{R}$ extending λ. (*Hint*: for finite $\mathcal{K}$, induce on $\#(\mathcal{K})$. For the general case, recall that if $\gamma = \sup_{K\in\mathcal{K}} \lambda K$, the additive functionals form a compact subset of $[0,\gamma]^{\mathcal{P}X}$.) (If P is a lattice, a functional $f : P \to \mathbb{R}$ such that $f(p \vee q) + f(p \wedge q) = f(p) + f(q)$ for all p, $q \in P$ is called **modular**.)

(q) Let X be a set and $\mathcal{K}$ a sublattice of $\mathcal{P}X$ containing K. Let $\lambda : \mathcal{K} \to [0,1]$ be a functional such that

$$\lambda K \leq \lambda K' \text{ whenever } K,\ K' \in \mathcal{K} \text{ and } K \subseteq K', \quad \inf_{K\in\mathcal{K}} \lambda K = 0,$$

$$\lambda(K \cup K') + \lambda(K \cap K') \leq \lambda K + \lambda K' \text{ for all } K,\ K' \in \mathcal{K}$$

(λ is **submodular**). Show that there is a finitely additive functional $\nu : \mathcal{P}X \to [0,1]$ such that

$$\nu X = \sup_{K\in\mathcal{K}} \lambda K, \quad \nu K \leq \lambda K \text{ for every } K \in \mathcal{K}.$$

413Y Further exercises (a) Give an example of two inner measures ϕ_1, ϕ_2 on a set X such that the measure defined by $\phi_1 + \phi_2$ strictly extends the sum of the measures defined by ϕ_1, ϕ_2.

(b) Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be any family of probability spaces, and λ the product measure on $X = \prod_{i\in I} X_i$. Show that $\lambda_*(\prod_{i\in I} A_i) \leq \prod_{i\in I}(\mu_i)_* A_i$ whenever $A_i \subseteq X_i$ for every i, with equality if I is countable.

(c) Let (X, Σ, μ) be a totally finite measure space, and Z the Stone space of the Boolean algebra Σ. For $E \in \Sigma$ write $\widehat{E}$ for the corresponding open-and-closed subset of Z. Show that there is a unique function $f : X \to Z$ such that $f^{-1}[\widehat{E}] = E$ for every $E \in \Sigma$. Show that there is a measure ν on Z, inner regular with respect to the open-and-closed sets, such that f is inverse-measure-preserving with respect to μ and ν, and that f represents an isomorphism between the measure algebras of μ and ν. Use this construction to prove (vi)⇒(i) in Theorem 343B without appealing to the Lifting Theorem.

(d) Let X be a set, T a subalgebra of $\mathcal{P}X$, and $\nu : \mathrm{T} \to [0, \infty[$ a finitely additive functional. Suppose that there is a set $\mathcal{K} \subseteq \mathrm{T}$, containing $\emptyset$, such that (i) $\mu F = \sup\{\mu K : K \in \mathcal{K}, K \subseteq F\}$ for every $F \in \mathrm{T}$ (ii) $\mathcal{K}$ is **monocompact**, that is, $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$ for every non-decreasing sequence in $\mathcal{K}$. Show that ν extends to a measure on X.

(e) (i) Let X be a topological space. Show that the family of closed countably compact subsets of X is a countably compact class. (ii) Let X be a Hausdorff space. Show that the family of sequentially compact subsets of X is a countably compact class.

413 Notes and comments I gave rather few methods of constructing measures in the first three volumes of this treatise; in the present volume I shall have to make up for lost time. In particular I used Carathéodory's construction for Lebesgue measure (Chapter 11), product measures (Chapter 25) and Hausdorff measures (Chapter 26). The first two, at least, can be tackled in quite different ways if we choose. The first alternative approach I offer is the 'inner measure' method of 413C. Note the exact definition in 413A; I do not think it is an obvious one. In particular, while (α) seems to have something to do with subadditivity, and (β) is a kind of sequential order-continuity, there is no straightforward way in which to associate an outer measure with an inner measure, unless they both happen to be derived from measures (132B, 413D), even when they are finite-valued; and for an inner measure which is allowed to take the value ∞ we have to add the semi-finiteness condition ($*$) of 413A (see 413Xa).

Once we have got these points right, however, we have a method which rivals Carathéodory's in scope, and in particular is especially well adapted to the construction of inner regular measures. As an almost trivial example, we have a route to the c.l.d. version of a measure μ (413Xi(i)), which can be derived from the inner measure μ_* defined from μ (413D). Henceforth μ_* will be a companion to the familiar outer measure μ^*, and many calculations will be a little easier with both available, as in 413D-413F.

The intention behind 413I-413J is to find a minimal set of properties of a functional ϕ_0 which will ensure that it has an extension to a measure. Indeed it is easy to see that, in the context of 413I, given a family $\mathcal{K}$ with the properties (†) and (‡) there, a functional ϕ_0 on $\mathcal{K}$ can have an extension to an inner regular measure iff it satisfies the conditions (α) and (β), so in this sense 413I is the best possible result. Note that while Carathéodory's construction is liable to produce wildly infinite measures (like Hausdorff measures, or primitive product measures), the construction here always gives us locally determined measures, provided only that ϕ_0 is finite-valued.

We have to work rather hard for the step from 413I to 413J. Of course 413I is a special case of 413J, and I could have saved a little space by giving a direct proof of the latter result. But I do not think that this would have made it easier; 413J really does require an extra step, because somehow we have to extend the functional ϕ_0 from $\mathcal{K}$ to $\mathcal{K}_\delta$. The method I have chosen uses 413B and 413H to cast as much of the argument as possible into the context of algebras of sets with additive functionals, where I hope the required manipulations will seem natural. (But perhaps I should insist that you must not take them too much for granted, as some of the time we have a finitely additive functional taking infinite values, and must take care not to subtract illegally, as well as not to take limits in the wrong places.) Note that the progression $\phi_0 \to \phi_1 \to \mu$ in the proof of 413J involves first an approximation from outside (if $K \in \mathcal{K}_\delta$, then $\phi_1 K$ will be $\inf\{\phi_0 K' : K \subseteq K' \in \mathcal{K}\}$) and then an approximation from inside (if $E \in \Sigma$, then $\mu E = \sup\{\phi_1 K : K \in \mathcal{K}_\delta, K \subseteq E\}$). The essential difficulty in the proof is just that we have to take successive non-exchangeable limits. I have slipped 413K in as a corollary of 413J; but it can be regarded as one of the fundamental results of measure theory. A non-negative finitely additive functional ν on an algebra Σ of sets can be extended to a countably additive

measure iff it is 'relatively countably additive' in the sense that $\nu(\bigcup_{n\in\mathbb{N}} E_n) = \sum_{n=0}^{\infty} \mu E_n$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ such that $\bigcup_{n\in\mathbb{N}} E_n \in \Sigma$ (413Xm). Of course the same result can easily be got from an outer measure construction (413Xl). Note that the outer measure construction also has repeated limits, albeit simpler ones: in the formula

$$\theta A = \inf\{\textstyle\sum_{n=0}^{\infty} \nu E_n : \langle E_n\rangle_{n\in\mathbb{N}} \text{ is a sequence in } \Sigma \text{ covering } A\}$$

the sum $\sum_{n=0}^{\infty} \nu E_n = \sup_{n\in\mathbb{N}} \sum_{i=0}^{n} \nu E_i$ can be regarded as a crude approximation from inside, while the infimum is an approximation from outside. To get the result as stated in 413K, of course, the outer measure construction needs a third limiting process, to obtain the c.l.d. version automatically provided by the inner measure method, and the inner regularity with respect to Σ_δ, while easily checked, also demands a few words of argument.

Many applications of the method of 413I-413J pass through 413M; if the family $\mathcal{K}$ is a countably compact class then the sequential order-continuity hypothesis (β) of 413I or 413J becomes a consequence of the other hypotheses. The essence of the method is the inner regularity hypothesis (α). I have tried to use the labels †, ‡, α and β consistently enough to suggest the currents which I think are flowing in this material.

In 413N we strike out in a new direction. The object here is to build an extension which is not going to be unique, and for which choices will have to be made. As with any such argument, the trick is to specify the allowable intermediate stages, that is, the partially ordered set P to which we shall apply Zorn's Lemma. But here the form of the theorem makes it easy to guess what P should be: it is the set of functionals satisfying the hypotheses of the theorem which have not wandered outside the boundary set by the conclusion, that is, which satisfy the condition $(*)$ of part (a) of the proof of 413N. The finitistic nature of the hypotheses makes it easy to check that totally ordered subsets of P have upper bounds (that is to say, if we did this by transfinite induction there would be no problem at limit stages), and all we have to prove is that maximal elements of P are defined on adequately large domains; which amounts to showing that a member of P not defined on every element of $\mathcal{K}$ has a proper extension, that is, setting up a construction for the step to a successor ordinal in the parallel transfinite induction (part (c) of the proof).

Of course the principal applications of 413N in this book will be in the context of countably additive functionals, as in 413O.

It is clear that 413N and 413Q overlap to some extent. I include both because they have different virtues. 413N can be applied to infinite measures in a way that 413Q, as given, cannot; but its chief advantage, from the point of view of the work to come, is the approximation of members of T_1, in measure, by members of T_0. This will eventually enable us to retain control of the Maharam types of measures constructed by the method of 413O. In 413S we have a different kind of control; we can specify a lower bound for the measure of each member of our basic class $\mathcal{K}$, provided only that our specifications are consistent with some finitely additive functional.

414 τ-additivity

The second topic I wish to treat is that of 'τ-additivity'. Here I collect results which do not depend on any strong kind of inner regularity. I begin with what I think of as the most characteristic feature of τ-additivity, its effect on the properties of semi-continuous functions (414A), with a variety of corollaries, up to the behaviour of subspace measures (414K). A very important property of τ-additive topological measures is that they are often strictly localizable (414J).

The theory of inner regular τ-additive measures belongs to the next section, but here I give two introductory results: conditions under which a τ-additive measure will be inner regular with respect to closed sets (414M) and conditions under which a measure which is inner regular with respect to closed sets will be τ-additive (414N). I end the section with notes on 'density' and 'lifting' topologies (414P-414R).

414A Theorem Let $(X, \mathfrak{T})$ be a topological space and μ an effectively locally finite τ-additive measure on X with domain Σ.

(a) Suppose that $\mathcal{G}$ is a non-empty family in $\Sigma \cap \mathfrak{T}$ such that $H = \bigcup \mathcal{G}$ also belongs to Σ. Then $\sup_{G\in\mathcal{G}} G^\bullet = H^\bullet$ in the measure algebra $\mathfrak{A}$ of μ.

(b) Write $\mathcal{L}$ for the family of Σ-measurable lower semi-continuous functions from X to $\mathbb{R}$. Suppose that $\emptyset \neq A \subseteq \mathcal{L}$ and set $g(x) = \sup_{f\in A} f(x)$ for every $x \in X$. If g is Σ-measurable and finite almost everywhere, then $\tilde{g}^\bullet = \sup_{f\in A} f^\bullet$ in $L^0(\mu)$, where $\tilde{g}(x) = g(x)$ whenever $g(x)$ is finite.

(c) Suppose that $\mathcal{F}$ is a non-empty family of measurable closed sets such that $\bigcap \mathcal{F} \in \Sigma$. Then $\inf_{F\in\mathcal{F}} F^\bullet = (\bigcap \mathcal{F})^\bullet$ in $\mathfrak{A}$.

(d) Write $\mathcal{U}$ for the family of Σ-measurable upper semi-continuous functions from X to $\mathbb{R}$. Suppose that $A \subseteq \mathcal{U}$ is non-empty and set $g(x) = \inf_{f\in A} f(x)$ for every $x \in X$. If g is Σ-measurable and finite almost everywhere, then $\tilde{g}^\bullet = \inf_{f\in A} f^\bullet$ in $L^0(\mu)$, where $\tilde{g}(x) = g(x)$ whenever $g(x)$ is finite.

proof (a) ? If $H^\bullet \neq \sup_{G\in\mathcal{G}} G^\bullet$, there is a non-zero $a \in \mathfrak{A}$ such that $a \subseteq H^\bullet$ but $a \cap G^\bullet = 0$ for every $G \in \mathcal{G}$. Express a as $E^\bullet$ where $E \in \Sigma$ and $E \subseteq H$. Because μ is effectively locally finite, there is a measurable open set H_0 of finite measure such that $\mu(H_0 \cap E) > 0$. Now $\{H_0 \cap G : G \in \mathcal{G}\}$ is an upwards-directed family of measurable open sets with union $H_0 \cap H \supseteq H_0 \cap E$; as μ is τ-additive, there is a $G \in \mathcal{G}$ such that $\mu(H_0 \cap G) > \mu H_0 - \mu(H_0 \cap E)$. But in this case $\mu(G \cap E) > 0$, which is impossible, because $G^\bullet \cap E^\bullet = 0$. **X**

(b) For any $\alpha \in \mathbb{R}$,

$$\{x : g(x) > \alpha\} = \bigcup_{f\in A} \{x : f(x) > \alpha\},$$

and these are all measurable open sets. Identifying $\{x : g(x) > \alpha\}^\bullet \in \mathfrak{A}$ with $[\![\tilde{g}^\bullet > \alpha]\!]$ (364Jb), we see from (a) that $[\![\tilde{g}^\bullet > \alpha]\!] = \sup_{f\in A} [\![f^\bullet > \alpha]\!]$ for every α. But this means that $\tilde{g}^\bullet = \sup_{f\in A} f^\bullet$, by 364Mb.

(c) Apply (a) to $\mathcal{G} = \{X \setminus F : F \in \mathcal{F}\}$.

(d) Apply (b) to $\{-f : f \in A\}$.

414B Corollary Let X be a topological space and μ an effectively locally finite τ-additive topological measure on X.

(a) Suppose that A is a non-empty upwards-directed family of lower semi-continuous functions from X to $[0, \infty]$. Set $g(x) = \sup_{f\in A} f(x)$ in $[0, \infty]$ for every $x \in X$. Then $\int g = \sup_{f\in A} \int f$ in $[0, \infty]$.

(b) Suppose that A is a non-empty downwards-directed family of non-negative continuous real-valued functions on X, and that $g(x) = \inf_{x\in A} f(x)$ for every $x \in X$. If any member of A is integrable, then $\int g = \inf_{f\in A} \int f$.

proof (a) Of course all the $f \in A$, and also g, are measurable functions. Set $g_n = g \wedge n\chi X$ for every $n \in \mathbb{N}$. Then

$$g_n(x) = \sup_{f\in A} (f \wedge n\chi X)(x)$$

for every $x \in X$, so $g_n^\bullet = \sup_{f\in A} (f \wedge n\chi X)^\bullet$, by 414Ab, and

$$\int g_n = \int g_n^\bullet = \sup_{f\in A} \int (f \wedge n\chi X)^\bullet = \sup_{f\in A} \int f \wedge n\chi X$$

by 365Dh. But now, of course,

$$\int g = \sup_{n\in\mathbb{N}} \int g_n = \sup_{n\in\mathbb{N}, f\in A} \int f \wedge n\chi X = \sup_{f\in A} \int f,$$

as claimed.

(b) Take an integrable $f_0 \in A$, and apply (a) to $\{(f_0 - f)^+ : f \in A\}$.

414C Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ-additive topological measure space and $\mathcal{F}$ a non-empty downwards-directed family of closed sets. If $\inf_{F\in\mathcal{F}} \mu F$ is finite, this is the measure of $\bigcap \mathcal{F}$.

proof Setting $F_0 = \bigcap \mathcal{F}$, then $F_0^\bullet = \inf_{F\in\mathcal{F}} F^\bullet$, by 414Ac; now

$$\mu F_0 = \bar{\mu} F_0^\bullet = \inf_{F\in\mathcal{F}} \bar{\mu} F^\bullet = \inf_{F\in\mathcal{F}} \mu F$$

by 321F.

414D Corollary Let μ be an effectively locally finite τ-additive measure on a topological space X. If ν is a totally finite measure with the same domain as μ, truly continuous with respect to μ, then ν is τ-additive. In particular, if μ is σ-finite and ν is absolutely continuous with respect to μ, then ν is τ-additive.

proof We have a functional $\bar\nu : \mathfrak{A} \to [0,\infty[$, where $\mathfrak{A}$ is the measure algebra of μ, such that $\bar\nu E^\bullet = \nu E$ for every E in the common domain Σ of μ and ν. Now $\bar\nu$ is continuous for the measure-algebra topology of $\mathfrak{A}$ (327Cd), therefore completely additive (327Ba), therefore order-continuous (326Kc). So if $\mathcal{G}$ is an upwards-directed family of open sets belonging to Σ with union $G_0 \in \Sigma$,

$$\sup_{G\in\mathcal{G}} \nu G = \sup_{G\in\mathcal{G}} \bar\nu G^\bullet = \bar\nu G_0^\bullet = \nu G_0$$

because $G_0^\bullet = \sup_{G\in\mathcal{G}} G^\bullet$.

The last sentence follows at once, because on a σ-finite space an absolutely continuous countably additive functional is truly continuous (232Bc).

414E Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ-additive topological measure space. Suppose that $\mathcal{G} \subseteq \mathfrak{T}$ is non-empty and upwards-directed, and $H = \bigcup \mathcal{G}$. Then

(a) $\mu(E \cap H) = \sup_{G\in\mathcal{G}} \mu(E \cap G)$ for every $E \in \Sigma$;

(b) if f is a non-negative virtually measurable real-valued function defined almost everywhere on X, then $\int_H f = \sup_{G\in\mathcal{G}} \int f$ in $[0,\infty]$.

proof (a) In the measure algebra $(\mathfrak{A}, \bar\mu)$ of μ,

$$\begin{aligned}(E \cap H)^\bullet &= E^\bullet \cap H^\bullet = E^\bullet \cap \sup_{G\in\mathcal{G}} G^\bullet \\ &= \sup_{G\in\mathcal{G}} E^\bullet \cap G^\bullet = \sup_{G\in\mathcal{G}} (E \cap G)^\bullet,\end{aligned}$$

using 414Aa and the distributive law 313Ba. So

$$\mu(E \cap H) = \bar\mu(E \cap H)^\bullet = \sup_{G\in\mathcal{G}} \bar\mu(E \cap G)^\bullet = \sup_{G\in\mathcal{G}} \mu(E \cap G)$$

by 321D, because $\mathcal{G}$ and $\{(E \cap G)^\bullet : G \in \mathcal{G}\}$ are upwards-directed.

(b) For each $G \in \mathcal{G}$,

$$\int_G f = \int f \times \chi G = \int (f \times \chi G)^\bullet = \int f^\bullet \times \chi G^\bullet,$$

where $\chi G^\bullet$ can be interpreted either as $(\chi G)^\bullet$ (in $L^0(\mu)$) or as $\chi(G^\bullet)$ (in $L^0(\mathfrak{A})$, where $\mathfrak{A}$ is the measure algebra of μ); see 364K. Now $H^\bullet = \sup_{G\in\mathcal{G}} G^\bullet$ (414Aa); since χ and $\times$ are order-continuous (364Kc, 364P), $f^\bullet \times \chi H^\bullet = \sup_{G\in\mathcal{G}} f^\bullet \times \chi G^\bullet$; so

$$\int_H f = \int f^\bullet \times \chi H^\bullet = \sup_{G\in\mathcal{G}} \int f^\bullet \times \chi G^\bullet = \sup_{G\in\mathcal{G}} \int_G f$$

by 365Dh.

414F Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ-additive topological measure space. Then for every $E \in \Sigma$ there is a unique relatively closed self-supporting set $F \subseteq E$ such that $\mu(E \setminus F) = 0$.

proof Let $\mathcal{G}$ be the set $\{G : G \in \mathfrak{T}, \mu(G \cap E) = 0\}$. Then $\mathcal{G}$ is upwards-directed, so $\mu(E \cap G^*) = \sup_{G\in\mathcal{G}} \mu(E \cap G) = 0$, where $G^* = \bigcup \mathcal{G}$. Set $F = E \setminus G^*$. Then $F \subseteq E$ is relatively closed, and $\mu(E \setminus F) = 0$. If $H \in \mathfrak{T}$ and $H \cap F \neq \emptyset$, then $H \notin \mathcal{G}$ so $\mu(F \cap H) = \mu(E \cap H) > 0$; thus F is self-supporting. If $F' \subseteq E$ is another self-supporting relatively closed set such that $\mu(E \setminus F') = 0$, then $\mu(F \setminus F') = \mu(F' \setminus F) = 0$; but as $F \setminus F'$ is relatively open in F, and $F' \setminus F$ is relatively open in F', these must both be empty, and $F = F'$.

414G Corollary If $(X, \mathfrak{T}, \Sigma, \mu)$ is a Hausdorff effectively locally finite τ-additive topological measure space and $E \in \Sigma$ is an atom for μ (definition: 211I), then there is an $x \in E$ such that $E \setminus \{x\}$ is negligible.

proof Let $F \subseteq E$ be a self-supporting set such that $\mu(E \setminus F) = 0$. Since $\mu F = \mu E > 0$, F is not empty; take $x \in F$. **?** If $F \neq \{x\}$, let $y \in F \setminus \{x\}$. Because $\mathfrak{T}$ is Hausdorff, there are disjoint open sets G, H containing x, y respectively; and in this case $\mu(E \cap G) = \mu(F \cap G)$ and $\mu(E \cap H) = \mu(F \cap H)$ are both non-zero, which is impossible, since E is an atom. **X**

So $F = \{x\}$ and $E \setminus \{x\}$ is negligible.

414H Corollary If $(X, \mathfrak{T}, \Sigma, \mu)$ is an effectively locally finite τ-additive topological measure space and ν is an indefinite-integral measure over μ (definition: 234B), then ν is a τ-additive topological measure.

proof Because ν measures every set in Σ (234Da), it is a topological measure. To see that it is τ-additive, apply 414Eb to a Radon-Nikodým derivative of ν.

414I Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined effectively locally finite τ-additive topological measure space. If $E \subseteq X$, $\mathcal{G} \subseteq \mathfrak{T}$ are such that $E \subseteq \bigcup \mathcal{G}$ and $E \cap G \in \Sigma$ for every $G \in \mathcal{G}$, then $E \in \Sigma$.

proof Set $\mathcal{K} = \{K : K \in \Sigma, E \cap K \in \Sigma\}$. Then whenever $F \in \Sigma$ and $\mu F > 0$ there is a $K \in \mathcal{K}$ included in F with $\mu K > 0$. **P** Set $K_1 = F \setminus \bigcup \mathcal{G}$. Then K_1 is a member of $\mathcal{K}$ included in F. If $\mu K_1 > 0$ then we can stop. Otherwise, $\mathcal{G}^* = \{G_0 \cup \ldots \cup G_n : G_0, \ldots, G_n \in \mathcal{G}\}$ is an upwards-directed family of open sets, and

$$\sup_{G \in \mathcal{G}^*} \mu(F \cap G) = \mu(F \cap \bigcup \mathcal{G}^*) = \mu F > 0,$$

by 414E. So there is a $G \in \mathcal{G}^*$ such that $\mu(F \cap G) > 0$; but now $E \cap G \in \Sigma$ so $F \cap G \in \mathcal{K}$. **Q**

By 412Aa, μ is inner regular with respect to $\mathcal{K}$; by 412Ja, $E \in \Sigma$.

414J Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined effectively locally finite τ-additive topological measure space. Then μ is strictly localizable.

proof Let $\mathcal{F}$ be a maximal disjoint family of self-supporting measurable sets of finite measure. Then whenever $E \in \Sigma$ and $\mu E > 0$, there is an $F \in \mathcal{F}$ such that $\mu(E \cap F) > 0$. **P?** Otherwise, let G be an open set of finite measure such that $\mu(G \cap E) > 0$, and set $\mathcal{F}_0 = \{F : F \in \mathcal{F}, F \cap G \neq \emptyset\}$. Then $\mu(F \cap G) > 0$ for every $F \in \mathcal{F}_0$, while $\mu G < \infty$ and $\mathcal{F}_0$ is disjoint, so $\mathcal{F}_0$ is countable and $\bigcup \mathcal{F}_0 \in \Sigma$. Set $E' = E \setminus \bigcup \mathcal{F}_0$; then $E \setminus E' = E \cap \bigcup \mathcal{F}_0$ is negligible, so $\mu(G \cap E') > 0$. By 414F, there is a self-supporting set $F' \subseteq G \cap E'$ such that $\mu F' > 0$. But in this case $F' \cap F = \emptyset$ for every $F \in \mathcal{F}$, so we ought to have added F' to $\mathcal{F}$. **XQ**

This means that $\mathcal{F}$ satisfies the criterion of 213O. Because (X, Σ, μ) is complete and locally determined, it is strictly localizable.

414K Proposition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X, and $Y \subseteq X$ a subset such that the subspace measure μ_Y is semi-finite (see the remark following 412O). If μ is an effectively locally finite τ-additive topological measure, so is the subspace measure μ_Y.

proof By 412P, μ_Y is an effectively locally finite topological measure. Now suppose that $\mathcal{H}$ is a non-empty upwards-directed family in $\mathfrak{T}_Y$ with union H^*. Set

$$\mathcal{G} = \{G : G \in \mathfrak{T}, G \cap Y \in \mathcal{H}\}, \quad G^* = \bigcup \mathcal{G},$$

so that $\mathcal{G}$ is upwards-directed and $H^* = Y \cap G^*$. Let $\mathcal{K}$ be the family of sets $K \subseteq X$ such that $K \cap G^* \setminus G = \emptyset$ for some $G \in \mathcal{G}$. If $E \in \Sigma$,

$$\begin{aligned} \mu E &= \mu(E \setminus G^*) + \mu(E \cap G^*) = \mu(E \setminus G^*) + \sup_{G \in \mathcal{G}} \mu(E \cap G) \\ &= \sup_{G \in \mathcal{G}} \mu(E \setminus (G^* \setminus G)), \end{aligned} \tag{414Ea}$$

so μ is inner regular with respect to $\mathcal{K}$. By 412Ob, μ_Y is inner regular with respect to $\{K \cap Y : K \in \mathcal{K}\}$. So if $\gamma < \mu_Y H^*$, there is a $K \in \mathcal{K}$ such that $K \cap Y \subseteq H^*$ and $\mu_Y(K \cap Y) \geq \gamma$. But now there is a $G \in \mathcal{G}$ such that $K \cap G^* \setminus G = \emptyset$, so that $K \cap Y \subseteq G \cap Y \in \mathcal{H}$ and $\sup_{H \in \mathcal{H}} \mu H \geq \gamma$. As $\mathcal{H}$ and γ are arbitrary, μ is τ-additive.

Remarks Recall from 214I that if (X, Σ, μ) has locally determined negligible sets (in particular, is either strictly localizable or complete and locally determined), then all its subspaces are semi-finite. In 419C below I describe a tight locally finite Borel measure with a subset on which the subspace measure is not semi-finite, therefore not effectively locally finite or τ-additive. In 419A I describe a σ-finite locally finite τ-additive topological measure, inner regular with respect to the closed sets, with a closed subset on which the subspace measure is totally finite but not τ-additive.

414L Lemma Let $(X, \mathfrak{T})$ be a topological space, and μ, ν two effectively locally finite Borel measures on X which agree on the open sets. Then they are equal.

proof Write $\mathfrak{T}^f$ for the family of open sets of finite measure. (I do not need to specify which measure I am using here.) For $G \in \mathfrak{T}^f$, set $\mu_G E = \mu(G \cap E)$, $\nu_G E = \nu(G \cap E)$ for every Borel set E. Then μ_G and ν_G are totally finite Borel measures which agree on $\mathfrak{T}$. By the Monotone Class Theorem (136C), μ_G and ν_G agree on the σ-algebra generated by $\mathfrak{T}$, that is, the Borel σ-algebra $\mathcal{B}$. Now, for any $E \in \mathcal{B}$,

$$\mu E = \sup_{G \in \mathfrak{T}^f} \mu_G E = \sup_{G \in \mathfrak{T}^f} \nu_G E = \nu E,$$

by 412F. So $\mu = \nu$.

414M Proposition Let (X, Σ, μ) be a measure space with a regular topology $\mathfrak{T}$ such that μ is effectively locally finite and τ-additive and Σ includes a base for $\mathfrak{T}$.

(a) $\mu G = \sup\{\mu F : F \in \Sigma \text{ is closed}, F \subseteq G\}$ for every open set $G \in \Sigma$.

(b) If μ is inner regular with respect to the σ-algebra generated by $\mathfrak{T} \cap \Sigma$, it is inner regular with respect to the closed sets.

proof (a) For $U \in \Sigma \cap \mathfrak{T}$, the set

$$\mathcal{H}_U = \{H : H \in \Sigma \cap \mathfrak{T}, \overline{H} \subseteq U\}$$

is an upwards-directed family of open sets, and $\bigcup \mathcal{H}_U = U$ because $\mathfrak{T}$ is regular and Σ includes a base for $\mathfrak{T}$. Because μ is τ-additive, $\mu U = \sup\{\mu H : H \in \mathcal{H}_U\}$. Now, given $\gamma < \mu G$, we can choose $\langle U_n \rangle_{n \in \mathbb{N}}$ in $\Sigma \cap \mathfrak{T}$ inductively, as follows. Start by taking $U_0 \subseteq G$ such that $\gamma < \mu U_0 < \infty$ (using the hypothesis that μ is effectively locally finite). Given $U_n \in \Sigma \cap \mathfrak{T}$ and $\mu U_n > \gamma$, take $U_{n+1} \in \Sigma \cap \mathfrak{T}$ such that $\overline{U}_{n+1} \subseteq U_n$ and $\mu U_{n+1} > \gamma$. On completing the induction, set

$$F = \bigcap_{n \in \mathbb{N}} U_n = \bigcap_{n \in \mathbb{N}} \overline{U}_n;$$

then F is a closed set belonging to Σ, $F \subseteq G$ and $\mu F \geq \gamma$. As γ is arbitrary, we have the result.

(b) Let Σ_0 be the σ-algebra generated by $\Sigma \cap \mathfrak{T}$ and set $\mu_0 = \mu \restriction \Sigma_0$. Then $\Sigma_0 \cap \mathfrak{T} = \Sigma \cap \mathfrak{T}$ is still a base for $\mathfrak{T}$ and μ_0 is still τ-additive and effectively locally finite, so by (a) and 412G it is inner regular with respect to the closed sets. Now we are supposing that μ is inner regular with respect to Σ_0, so μ is inner regular with respect to the closed sets, by 412Ab.

414N Proposition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X. Suppose that (i) μ is semi-finite and inner regular with respect to the closed sets (ii) whenever $\mathcal{F}$ is a non-empty downwards-directed family of measurable closed sets with empty intersection and $\inf_{F \in \mathcal{F}} \mu F < \infty$, then $\inf_{F \in \mathcal{F}} \mu F = 0$. Then μ is τ-additive.

proof Let $\mathcal{G}$ be a non-empty upwards-directed family of measurable open sets with measurable union H. Take any $\gamma < \mu H$. Because μ is semi-finite, there is a measurable set $E \subseteq H$ such that $\gamma < \mu E < \infty$. Now there is a measurable closed set $F \subseteq E$ such that $\mu F \geq \gamma$. Consider $\mathcal{F} = \{F \setminus G : G \in \mathcal{G}\}$. This is a downwards-directed family of closed sets of finite measure with empty intersection. So $\inf_{G \in \mathcal{G}} \mu(F \setminus G) = 0$, that is,

$$\gamma \leq \mu F = \sup_{G \in \mathcal{G}} \mu(F \cap G) \leq \sup_{G \in \mathcal{G}} \mu G.$$

As γ is arbitrary, $\mu H = \sup_{G \in \mathcal{G}} \mu G$; as $\mathcal{G}$ is arbitrary, μ is τ-additive.

414O The following elementary result is worth noting.

Proposition If X is a hereditarily Lindelöf space (e.g., if it is separable and metrizable) then every measure on X is τ-additive.

proof If μ is a measure on X, with domain Σ, and $\mathcal{G} \subseteq \Sigma$ is a non-empty upwards-directed family of measurable open sets, then there is a sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{G}$ such that $\bigcup \mathcal{G} = \bigcup_{n \in \mathbb{N}} G_n$. Now

$$\mu(\bigcup \mathcal{G}) = \lim_{n \to \infty} \mu(\bigcup_{i \leq n} G_i) \leq \sup_{G \in \mathcal{G}} \mu G.$$

As $\mathcal{G}$ is arbitrary, μ is τ-additive.

414P Density topologies Recall that a lower density of a measure space (X,Σ,μ) is a function $\underline{\phi}:\Sigma\to\Sigma$ such that $\underline{\phi}E=\underline{\phi}F$ whenever E, $F\in\Sigma$ and $\mu(E\triangle F)=0$, $\mu(E\triangle\underline{\phi}E)=0$ for every $E\in\Sigma$, $\underline{\phi}\emptyset=\emptyset$ and $\underline{\phi}(E\cap F)=\underline{\phi}E\cap\underline{\phi}F$ for all E, $F\in\Sigma$ (341C).

Proposition Let (X,Σ,μ) be a complete locally determined measure space and $\underline{\phi}:\Sigma\to\Sigma$ a lower density such that $\underline{\phi}X=X$. Set

$$\mathfrak{T}=\{E:E\in\Sigma,\ E\subseteq\underline{\phi}E\}.$$

Then $\mathfrak{T}$ is a topology on X, the **density topology** associated with $\underline{\phi}$, and $(X,\mathfrak{T},\Sigma,\mu)$ is an effectively locally finite τ-additive topological measure space; μ is strictly positive and inner regular with respect to the open sets.

proof (a)(i) For any $E\in\Sigma$, $\underline{\phi}(E\cap\underline{\phi}E)=\underline{\phi}E$ because $E\setminus\underline{\phi}E$ is negligible; consequently $E\cap\underline{\phi}E\in\mathfrak{T}$. In particular, $\emptyset=\emptyset\cap\underline{\phi}\emptyset$ and $X=X\cap\underline{\phi}X$ belong to $\mathfrak{T}$. If E, $F\in\mathfrak{T}$ then

$$\underline{\phi}(E\cap F)=\underline{\phi}E\cap\underline{\phi}F\supseteq E\cap F,$$

so $E\cap F\in\mathfrak{T}$.

(ii) Suppose that $\mathcal{G}\subseteq\mathfrak{T}$ and $H=\bigcup\mathcal{G}$. By 341M, μ is (strictly) localizable, so $\mathcal{G}$ has an essential supremum $F\in\Sigma$ such that $F^\bullet=\sup_{G\in\mathcal{G}}G^\bullet$ in the measure algebra $\mathfrak{A}$ of μ; that is, for $E\in\Sigma$, $\mu(G\setminus E)=0$ for every $G\in\mathcal{G}$ iff $\mu(F\setminus E)=0$. Now $F\setminus H$ is negligible, by 213K. On the other hand,

$$G\subseteq\underline{\phi}G=\underline{\phi}(G\cap F)\subseteq\underline{\phi}F$$

for every $G\in\mathcal{G}$, so $H\subseteq\underline{\phi}F$, and $H\setminus F\subseteq\underline{\phi}F\setminus F$ is negligible. But as μ is complete, this means that $H\in\Sigma$. Also $\underline{\phi}H=\underline{\phi}F\supseteq H$, so $H\in\mathfrak{T}$. Thus $\mathfrak{T}$ is closed under arbitrary unions and is a topology.

(b) By its definition, $\mathfrak{T}$ is included in Σ, so μ is a topological measure. If $E\in\Sigma$ then $E\cap\underline{\phi}E$ belongs to $\mathfrak{T}$, is included in E and has the same measure as E; so μ is inner regular with respect to the open sets. If $E\in\mathfrak{T}$ is non-empty, then $\underline{\phi}E\supseteq E$ is non-empty, so $\mu E>0$; thus μ is strictly positive. Finally, if $\mathcal{G}$ is a non-empty upwards-directed family in $\mathfrak{T}$, then the argument of (a-ii) shows that $(\bigcup\mathcal{G})^\bullet=\sup_{G\in\mathcal{G}}G^\bullet$ in $\mathfrak{A}$, so that $\mu(\bigcup\mathcal{G})=\sup_{G\in\mathcal{G}}\mu G$. Thus μ is τ-additive. If $E\in\Sigma$ and $\mu E>0$ then there is an $F\subseteq E$ such that $0<\mu F<\infty$, and now $E\cap\underline{\phi}F$ is an open set of non-zero finite measure included in E; so μ is effectively locally finite.

414Q Lifting topologies Let (X,Σ,μ) be a measure space and $\phi:\Sigma\to\Sigma$ a lifting, that is, a Boolean homomorphism such that $\phi E=\emptyset$ whenever $\mu E=0$ and $\mu(E\triangle\phi E)=0$ for every $E\in\Sigma$ (341A). The **lifting topology** associated with ϕ is the topology generated by $\{\phi E:E\in\Sigma\}$. Note that $\{\phi E:E\in\Sigma\}$ is a topology base, so is a base for the lifting topology.

414R Proposition Let (X,Σ,μ) be a complete locally determined measure space and $\phi:\Sigma\to\Sigma$ a lifting with lifting topology $\mathfrak{S}$ and density topology $\mathfrak{T}$. Then $\mathfrak{S}\subseteq\mathfrak{T}\subseteq\Sigma$, and μ is τ-additive, effectively locally finite and strictly positive with respect to $\mathfrak{S}$. Moreover, $\mathfrak{S}$ is zero-dimensional.

proof Of course ϕ is a lower density, so we can talk of its density topology, and since $\phi^2E=\phi E$, $\phi E\in\mathfrak{T}$ for every $E\in\Sigma$, so $\mathfrak{S}\subseteq\mathfrak{T}$. Because μ is τ-additive and strictly positive with respect to $\mathfrak{T}$, it must also be τ-additive and strictly positive with respect to $\mathfrak{S}$. If $E\in\Sigma$ and $\mu E>0$ there is an $F\subseteq E$ such that $0<\mu F<\infty$, and now ϕF is an $\mathfrak{S}$-open set of finite measure meeting E in a non-negligible set; so μ is effectively locally finite with respect to $\mathfrak{S}$. Of course $\mathfrak{S}$ is zero-dimensional because $\phi[\Sigma]$ is a base for $\mathfrak{S}$ consisting of open-and-closed sets.

414X Basic exercises (a) Let (X,Σ,μ) and (Y,T,ν) be measure spaces with topologies $\mathfrak{T}$ and $\mathfrak{S}$, and $f:X\to Y$ a continuous inverse-measure-preserving function. Show that if μ is τ-additive with respect to $\mathfrak{T}$ then ν is τ-additive with respect to $\mathfrak{S}$. Show that if ν is locally finite, so is μ.

(b) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of measure spaces, with direct sum (X,Σ,μ); suppose that we are given a topology $\mathfrak{T}_i$ on X_i for each i, and let $\mathfrak{T}$ be the disjoint union topology on X. Show that μ is τ-additive iff every μ_i is.

>(c) Let $(X, \mathfrak{T})$ be a topological space and μ a totally finite measure on X which is inner regular with respect to the closed sets. Suppose that $\mu X = \sup_{G \in \mathcal{G}} \mu G$ whenever $\mathcal{G}$ is an upwards-directed family of measurable open sets covering X. Show that μ is τ-additive.

(d) Let μ be an effectively locally finite τ-additive σ-finite measure on a topological space X, and $\nu : \Sigma \to [0, \infty[$ a countably additive functional which is absolutely continuous with respect to μ. Show from first principles that ν is τ-additive.

(e) Give an example of an indefinite-integral measure over Lebesgue measure on $\mathbb{R}$ which is not effectively locally finite. (*Hint*: arrange for every non-trivial interval to have infinite measure.)

(f) Let $(X, \mathfrak{T})$ be a topological space and μ a complete locally determined effectively locally finite τ-additive topological measure on X. Show that if f is a real-valued function, defined on a subset of X, which is locally integrable in the sense of 411Fc, then f is measurable.

(g) Let $(X, \mathfrak{T})$ be a topological space and μ an effectively locally finite τ-additive measure on X. Let $\mathcal{G}$ be a cover of X consisting of measurable open sets, and $\mathcal{K}$ the ideal of subsets of X generated by $\mathcal{G}$. Show that μ is inner regular with respect to $\mathcal{K}$.

(h) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined effectively locally finite τ-additive topological measure space, and A a subset of X. Suppose that for every $x \in X$ there is an open set G containing x such that $A \cap G$ is negligible. Show that A is negligible.

(i) Give an alternative proof of 414K based on the fact that the canonical map from the measure algebra of μ to the measure algebra of μ_Y is order-continuous (322Yd).

>(j) (i) If μ is an effectively locally finite τ-additive Borel measure on a regular topological space, show that the c.l.d. version of μ is a quasi-Radon measure. (ii) If μ is a locally finite, effectively locally finite τ-additive Borel measure on a locally compact Hausdorff space, show that μ is tight, so that the c.l.d. version of μ is a Radon measure.

>(k) Let (X, Σ, μ) be a complete locally determined measure space and $\underline{\phi}$ a lower density on X such that $\underline{\phi}X = X$; let $\mathfrak{T}$ be the corresponding density topology. (i) Show that a dense open subset of X must be conegligible. (ii) Show that a subset of X is nowhere dense for $\mathfrak{T}$ iff it is negligible iff it is meager for $\mathfrak{T}$. (iii) Show that a function $f : X \to \mathbb{R}$ is Σ-measurable iff it is $\mathfrak{T}$-continuous at almost every point of X. (*Hint*: if f is measurable, set $E_q = \{x : f(x) > q\}$, $F_q = \{x : f(x) < q\}$; show that f is continuous at every point of $X \setminus \bigcup_{q \in \mathbb{Q}}((E_q \setminus \underline{\phi}E_q) \cup (F_q \setminus \underline{\phi}F_q))$.)

(l) Let (X, Σ, μ) be a complete locally determined measure space and $\underline{\phi} : \Sigma \to \Sigma$ a lower density such that $\underline{\phi}X = X$, with density topology $\mathfrak{T}$. Show that if $A \subseteq X$ and E is a measurable envelope of A then the $\mathfrak{T}$-closure of A is just $A \cup (X \setminus \underline{\phi}(X \setminus E))$.

(m) Let μ be Lebesgue measure on $\mathbb{R}^r$, Σ its domain, $\underline{\phi} : \Sigma \to \Sigma$ lower Lebesgue density (341E) and $\mathfrak{T}$ the corresponding density topology. Show that for any $A \subseteq \mathbb{R}$, the closure of A for $\mathfrak{T}$ is just $A \cup \{x : \limsup_{\delta \downarrow 0} \frac{\mu^*(A \cap B(x,\delta))}{\mu B(x,\delta)} > 0\}$, and the interior is $A \cap \{x : \lim_{\delta \downarrow 0} \frac{\mu_*(A \cap B(x,\delta))}{\mu B(x,\delta)} = 1\}$.

(n) Let (X, Σ, μ) be a complete locally determined measure space and $\underline{\phi} : \Sigma \to \Sigma$ a lower density such that $\underline{\phi}X = X$; let $\mathfrak{T}$ be the associated density topology. Let A be a subset of X and E a measurable envelope of A; let Σ_A be the subspace σ-algebra and μ_A the subspace measure on A. (i) Show that we have a lower density $\underline{\phi}_A : \Sigma_A \to \Sigma_A$ defined by setting $\underline{\phi}_A(F \cap A) = A \cap \underline{\phi}(E \cap F)$ for every $F \in \Sigma$. (ii) Show that $\underline{\phi}_A A = A$ iff $A \subseteq \underline{\phi}E$, and that in this case the density topology on A derived from $\underline{\phi}_A$ is just the subspace topology.

(o) Let (X, Σ, μ) be a complete locally determined measure space and $\phi : \Sigma \to \Sigma$ a lifting, with density topology $\mathfrak{T}$ and lifting topology $\mathfrak{S}$. (i) Show that

$$\mathfrak{T} = \{H \cap G : G \in \mathfrak{S},\ H \text{ is conegligible}\} = \{H \cap \phi E : E \in \Sigma,\ H \text{ is conegligible}\}.$$

(ii) Show that if $A \subseteq X$ and E is a measurable envelope of A then the $\mathfrak{T}$-closure of A is $A \cup \phi E$.

(p) Let (X, Σ, μ) be a complete locally determined measure space and $\phi : \Sigma \to \Sigma$ a lifting; let $\mathfrak{S}$ be its lifting topology. Let A be a subset of X such that $A \subseteq \phi E$ for some (therefore any) measurable envelope E of A. Let Σ_A be the subspace σ-algebra and μ_A the subspace measure on A. (i) Show that we have a lifting $\phi_A : \Sigma_A \to \Sigma_A$ defined by setting $\phi_A(F \cap A) = A \cap \phi F$ for every $F \in \Sigma$. (ii) Show that the lifting topology on A derived from ϕ_A is just the subspace topology.

(q) Let (X, Σ, μ) and (Y, T, ν) be complete locally determined measure spaces and $f : X \to Y$ an inverse-measure-preserving function. (i) Suppose that we have lower densities $\underline{\phi} : \Sigma \to \Sigma$ and $\underline{\psi} : \mathrm{T} \to \mathrm{T}$ such that $\underline{\phi}X = X$, $\underline{\psi}Y = Y$ and $\underline{\phi} f^{-1}[F] = f^{-1}[\underline{\psi}F]$ for every $F \in \mathrm{T}$. Show that f is continuous for the density topologies of $\underline{\phi}$ and $\underline{\psi}$. (ii) Show that if $\underline{\phi}$ and $\underline{\psi}$ are liftings then f is continuous for the lifting topologies.

(r) Let (X, Σ, μ) be a complete locally determined measure space and $\phi : \Sigma \to \Sigma$ a lifting, with associated lifting topology $\mathfrak{S}$. Show that a function $f : X \to \mathbb{R}$ is Σ-measurable iff there is a conegligible set H such that $f \restriction H$ is $\mathfrak{S}$-continuous. (Compare 414Xk.)

(s) Let (X, Σ, μ) be a complete locally determined measure space and $\phi : \Sigma \to \Sigma$ a lifting. Let (Z, T, ν) be the Stone space of the measure algebra of μ, and $f : X \to Z$ the inverse-measure-preserving function associated with ϕ (341P). Show that the lifting topology on X is just $\{f^{-1}[G] : G \subseteq Z \text{ is open}\}$.

(t) Let (X, Σ, μ) be a strictly localizable measure space and $\phi : \Sigma \to \Sigma$ a lifting. Write $\mathcal{L}^\infty$ for the Banach lattice of bounded Σ-measurable real-valued functions on X, identified with $\mathcal{L}^\infty(\Sigma)$ (363H); let $T : \mathcal{L}^\infty \to \mathcal{L}^\infty$ be the Riesz homomorphism associated with ϕ (363F). (i) Show that $T^2 = T$. (ii) Show that if X is given the lifting topology $\mathfrak{S}$ defined by ϕ, then $T[\mathcal{L}^\infty]$ is precisely the space of bounded continuous real-valued functions on X. (iii) Show that if $f \in \mathcal{L}^\infty$, $x \in X$ and $\epsilon > 0$ there is an $\mathfrak{S}$-open set U containing x such that $|(Tf)(x) - \frac{1}{\mu V} \int_V f d\mu| \leq \epsilon$ for every non-negligible measurable set V included in U.

414Y Further exercises (a) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a totally finite topological measure space. For $E \in \Sigma$ set

$$\mu_\tau E = \inf\{\mu(E \setminus \bigcup \mathcal{G}) + \sup_{G \in \mathcal{G}} \mu(E \cap G) : \mathcal{G} \subseteq \mathfrak{T} \text{ is an upwards-directed set}\}.$$

Suppose *either* that μ is inner regular with respect to the closed sets *or* that $\mathfrak{T}$ is regular. Show that μ_τ is a τ-additive measure, the largest τ-additive measure with domain Σ which is dominated by μ.

(b) Let X be a set, Σ an algebra of subsets of X, and $\mathfrak{T}$ a topology on X. Let M be the L-space of bounded finitely additive real-valued functionals on Σ (362B). Let $N \subseteq M$ be the set of those functionals ν such that $\inf_{G \in \mathcal{G}} |\nu|(H \setminus G) = 0$ whenever $\mathcal{G} \subseteq \mathfrak{T} \cap \Sigma$ is a non-empty upwards-directed family with union $H \in \Sigma$. Show that N is a band in M. (Cf. 362Xi.)

(c) Find a probability space (X, Σ, μ) and a topology $\mathfrak{T}$ on X such that Σ includes a base for $\mathfrak{T}$ and μ is τ-additive, but there is a set $E \in \Sigma$ such that the subspace measure μ_E is not τ-additive.

(d) Let (X, Σ, μ) be a complete locally determined measure space and $\underline{\phi}$ a lower density on X such that $\underline{\phi}X = X$; let $\mathfrak{T}$ be the density topology. Show that Σ is precisely the Baire property algebra for $\mathfrak{T}$, so that $(X, \mathfrak{T})$ is a Baire space.

(e) Let $\underline{\phi}$ be lower Lebesgue density on $\mathbb{R}^r$, and $\mathfrak{T}$ the associated density topology. Show that every $\mathfrak{T}$-Borel set is an F_σ set for $\mathfrak{T}$.

(f) Let (X, ρ) be a metric space and μ a strictly positive locally finite quasi-Radon measure on X; write $\mathfrak{T}$ for the topology of X and Σ for the domain of μ. For $E \in \Sigma$ set $\underline{\phi}(E) = \{x : x \in X, \lim_{\delta \downarrow 0} \frac{\mu(E \cap B(x,\delta))}{\mu B(x,\delta)} = 1\}$. Suppose that $E \setminus \phi(E)$ is negligible for every $E \in \Sigma$ (cf. 261D, 472D). (i) Show that $\underline{\phi}$ is a lower density for μ, with $\underline{\phi}(X) = X$. Let $\mathfrak{T}_d$ be the associated density topology. (ii) Suppose that $H \in \mathfrak{T}_d$ and that $K \subseteq H$ is $\mathfrak{T}$-closed and ρ-totally bounded. Show that there is a $\mathfrak{T}$-closed, ρ-totally bounded $K' \subseteq H$ such that K is included in the $\mathfrak{T}_d$-interior of K'. (iii) Show that $\mathfrak{T}_d$ is completely regular. (*Hint*: Lukeš Malý & Zajíček 86.)

(g) Show that the density topology on $\mathbb{R}$ associated with lower Lebesgue density is not normal.

(h) Let μ be Lebesgue measure on $\mathbb{R}^r$, Σ its domain, $\underline{\phi} : \Sigma \to \Sigma$ lower Lebesgue density and $\mathfrak{T}$ the corresponding density topology. (i) Show that if $f : \mathbb{R}^r \to \mathbb{R}^r$ is a bijection such that f and f^{-1} are both differentiable everywhere, with continuous derivatives, then f is a homeomorphism for $\mathfrak{T}$. (*Hint*: 263D.) (ii) Show that if $\phi : \Sigma \to \Sigma$ is a lifting and $\mathfrak{S}$ the corresponding lifting topology, then $x \mapsto -x$ is not a homeomorphism for $\mathfrak{S}$. (*Hint*: 345Xc.)

414 Notes and comments I have remarked before that it is one of the abiding frustrations of measure theory, at least for anyone ambitious to apply the power of modern general topology to measure-theoretic problems, that the basic convergence theorems are irredeemably confined to sequences. In Volume 3 I showed that if we move to measure algebras and function spaces, we can hope that the countable chain condition or the countable sup property will enable us to replace arbitrary directed sets with monotonic sequences, thereby giving theorems which apply to apparently more general types of convergence. In 414A and its corollaries we come to a quite different context in which a measure, or integral, behaves like an order-continuous functional. Of course the theorems here depend directly on the hypothesis of τ-additivity, which rather begs the question; but we shall see in the rest of the chapter that this property does indeed often appear. For the moment, I remark only that as Lebesgue measure is τ-additive we certainly have a non-trivial example to work with.

The hypotheses of the results above move a touch awkwardly between those with the magic phrase 'topological measure' and those without. The point is that (as in 412G, for instance) it is sometimes useful to be able to apply these ideas to Baire measures on completely regular spaces, which are defined on a base for the topology but may not be defined on every open set.

I hope that no confusion will arise between the two topologies associated with a lifting on a complete locally determined space. I have called them the 'density topology' and the 'lifting topology' because the former can be defined directly from a lower density; but it would be equally reasonable to call them the 'fine' and 'coarse' lifting topologies. The density topology has the apparent advantage of giving us a measure which is inner regular with respect to the Borel sets, but at the cost of being wildly irregular regarded as a topological space. It has the important advantage that there are densities (like the Lebesgue lower density) which have some claim to be called canonical, and others with useful special properties, as in §346, while liftings are always arbitrary and invariance properties for them sometimes unachievable. So, for instance, the Lebesgue density topology on $\mathbb{R}^r$ is invariant under diffeomorphisms, which no lifting topology can be (414Yh). The lifting topology is well-behaved as a topology, but only in special circumstances (as in 453Xd) is the measure inner regular with respect to its Borel sets, and even the closure of a set can be difficult to determine.

As with inner regularity, τ-additivity can be associated with the band structure of the space of bounded additive functionals on an algebra (414Yb); there will therefore be corresponding decompositions of measures into τ-additive and 'purely non-τ-additive' parts (cf. 414Ya).

415 Quasi-Radon measure spaces

We are now I think ready to draw together the properties of inner regularity and τ-additivity. Indeed this section will unite several of the themes which have been running through the treatise so far: (strict) localizability, subspaces and products as well as the new concepts of this chapter. In these terms, the principal results are that a quasi-Radon space is strictly localizable (415A), any subspace of a quasi-Radon space is quasi-Radon (415B), and the product of a family of strictly positive quasi-Radon probability measures on separable metrizable spaces is quasi-Radon (415E). I describe a basic method of constructing quasi-Radon measures (415K), with details of one of the standard ways of applying it (415L, 415N) and some notes on how to specify a quasi-Radon measure uniquely (415H-415I). I spell out useful results on indefinite-integral measures (415O) and L^p spaces (415P), and end the section with a discussion of the Stone space Z of a localizable measure algebra $\mathfrak{A}$ and an important relation in $Z \times X$ when $\mathfrak{A}$ is the measure algebra of a quasi-Radon measure space X (415Q-415R).

It would be fair to say that the study of quasi-Radon spaces for their own sake is a minority interest. If you are not already well acquainted with Radon measure spaces, it would make good sense to read this

section in parallel with the next. In particular, the constructions of 415K and 415L derive much of their importance from the corresponding constructions in §416.

415A Theorem A quasi-Radon measure space is strictly localizable.

proof This is a special case of 414J.

415B Theorem Any subspace of a quasi-Radon measure space is quasi-Radon.

proof Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and $(Y, \mathfrak{T}_Y, \Sigma_Y, \mu_Y)$ a subspace with the induced topology and measure. Because μ is complete, locally determined and localizable (by 415A), so is μ_Y (214Id). Because μ_Y is semi-finite and μ is an effectively locally finite τ-additive topological measure, so is μ_Y (414K). Because μ is inner regular with respect to the closed sets and μ_Y is semi-finite, μ_Y is inner regular with respect to the relatively closed subsets of Y (412Pc). So μ_Y is a quasi-Radon measure.

415C In regular topological spaces, the condition 'inner regular with respect to the closed sets' in the definition of 'quasi-Radon measure' can be weakened or omitted.

Proposition Let $(X, \mathfrak{T})$ be a regular topological space.

(a) If μ is a complete locally determined effectively locally finite τ-additive topological measure on X, inner regular with respect to the Borel sets, then it is a quasi-Radon measure.

(b) If μ is an effectively locally finite τ-additive Borel measure on X, its c.l.d. version is a quasi-Radon measure.

proof (a) By 414Mb, μ is inner regular with respect to the closed sets, which is the only feature missing from the given hypotheses.

(b) The c.l.d. version of μ satisfies the hypotheses of (a).

415D In separable metric spaces, among others, we can even omit τ-additivity.

Proposition Let $(X, \mathfrak{T})$ be a hereditarily Lindelöf topological space; e.g., a separable metrizable space (4A2P(a-iii)).

(i) If μ is a complete effectively locally finite measure on X, inner regular with respect to the Borel sets, and its domain includes a base for $\mathfrak{T}$, then it is a quasi-Radon measure.

(ii) If μ is an effectively locally finite Borel measure on X, then its completion is a quasi-Radon measure.

(iii) Any quasi-Radon measure on X is σ-finite.

(iv) If X is regular, any quasi-Radon measure on X is completion regular.

proof (a) The basic fact we need is that if $\mathcal{G}$ is any family of open sets in X, then there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\bigcup \mathcal{G}_0 = \bigcup \mathcal{G}$ (4A2H(c-i)). Consequently any effectively locally finite measure μ on X is σ-finite. **P** Let $\mathcal{G}$ be the family of measurable open sets of finite measure. Let $\mathcal{G}_0 \subseteq \mathcal{G}$ be a countable set with the same union as $\mathcal{G}$. Then $E = X \setminus \bigcup \mathcal{G}_0$ is measurable, and $E \cap G = \emptyset$ for every $G \in \mathcal{G}$, so $\mu E = 0$; accordingly $\mathcal{G}_0 \cup \{E\}$ is a countable cover of X by sets of finite measure, and μ is σ-finite. **Q**

Moreover, any measure on X is τ-additive. **P** If $\mathcal{G}$ is a non-empty upwards-directed family of open measurable sets, there is a sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{G}$ with union $\bigcup \mathcal{G}$. If $n \in \mathbb{N}$ there is a $G \in \mathcal{G}$ such that $\bigcup_{i \le n} G_i \subseteq G$, so

$$\mu(\bigcup \mathcal{G}) = \mu(\bigcup_{n \in \mathbb{N}} G_n) = \sup_{n \in \mathbb{N}} \mu(\bigcup_{i \le n} G_i) \le \sup_{G \in \mathcal{G}} \mu G.$$

As $\mathcal{G}$ is arbitrary, μ is τ-additive. **Q**

(b)(i) Now let μ be a complete effectively locally finite measure on X, inner regular with respect to the Borel sets, and with domain Σ including a base for the topology of X. If $H \in \mathfrak{T}$, then $\mathcal{G} = \{G : G \in \Sigma \cap \mathfrak{T}, G \subseteq H\}$ has union H, because $\Sigma \cap \mathfrak{T}$ is a base for $\mathfrak{T}$; but in this case there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $H = \bigcup \mathcal{G}_0$, so that $H \in \Sigma$. Thus μ is a topological measure. We know also from (a) that it is τ-additive and σ-finite, therefore locally determined. By 415Ca, it is a quasi-Radon measure.

(ii) If μ is an effectively locally finite Borel measure on X, then its completion $\hat{\mu}$ satisfies the conditions of (i), so is a quasi-Radon measure.

(iii) If μ is a quasi-Radon measure on X, it is surely effectively locally finite, therefore σ-finite.

(iv) If X is regular, then every closed set is a zero set (4A2H(c-iii)), so any measure which is inner regular with respect to the closed sets is completion regular.

415E I am delaying most of the theory of products of (quasi-)Radon measures to §417. However, there is one result which is so important that I should like to present it here, even though some of the ideas will have to be repeated later.

Theorem Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of separable metrizable quasi-Radon probability spaces such that every μ_i is strictly positive, and λ the product measure on $X = \prod_{i\in I} X_i$. Then

(i) λ is a completion regular quasi-Radon measure;

(ii) if $F \subseteq X$ is a closed self-supporting set, there is a countable set $J \subseteq I$ such that F is determined by coordinates in J, so F is a zero set.

proof (a) Write Λ for the domain of λ, and $\mathcal{U}$ for the family of subsets of X of the form $\prod_{i\in I} G_i$ where $G_i \in \mathfrak{T}_i$ for every $i \in I$ and $\{i : G_i \neq X_i\}$ is finite. Then $\mathcal{U}$ is a base for the topology of X, included in Λ. For $J \subseteq I$ let λ_J be the product measure on $X_J = \prod_{i\in J} X_i$ and Λ_J its domain. Write $\mathcal{U}^{(J)}$ for the family of subsets of X_J of the form $\prod_{i\in J} G_i$ where $G_i \in \mathfrak{T}_i$ for each $i \in J$ and $\{i : G_i \neq X_i\}$ is finite.

(b) Consider first the case in which I is countable. In this case X also is separable and metrizable (4A2P(a-v)), while Λ includes a base for its topology. Also λ is a complete probability measure and inner regular with respect to the closed sets (412Ua), so must be a quasi-Radon measure, by 415D(i).

(c) Now consider uncountable I. The key to the proof is the following fact: if $\mathcal{V} \subseteq \mathcal{U}$ has union W, then $W \in \Lambda$ and $\lambda W = \sup_{V\in\mathcal{V}^*} \lambda V$, where $\mathcal{V}^*$ is the set of unions of finite subsets of $\mathcal{V}$.

P (i) By 215B(iv), there is a countable set $\mathcal{V}_1 \subseteq \mathcal{V}$ such that $\lambda(U \setminus W_1) = 0$ for every $U \in \mathcal{V}$, where $W_1 = \bigcup \mathcal{V}_1$. Every member of $\mathcal{U}$ is determined by coordinates in some finite set (see 254M for this concept), so there is a countable set $J \subseteq I$ such that every member of $\mathcal{V}_1$ is determined by coordinates in J, and W_1 also is determined by coordinates in J. Let $\pi_J : X \to X_J$ be the canonical map. Because it is an open map (4A2B(f-i)), $\pi_J[W]$ and $\pi_J[W_1]$ are open in X_J, and belong to Λ_J, by (b).

(ii) ? Suppose, if possible, that $\lambda_J\pi_J[W] > \lambda_J\pi_J[W_1]$. Since $\pi_J[W] = \bigcup\{\pi_J[U] : U \in \mathcal{V}\}$, while λ_J is quasi-Radon and all the sets $\pi_J[U]$ are open, there must be some $U \in \mathcal{V}$ such that $\lambda_J(\pi_J[U] \setminus \pi_J[W_1]) > 0$ (414Ea). Now π_J is inverse-measure-preserving (254Oa), so

$$0 < \lambda\pi_J^{-1}[\pi_J[U] \setminus \pi_J[W_1]] = \lambda(\pi_J^{-1}[\pi_J[U]] \setminus \pi_J^{-1}[\pi_J[W_1]]) = \lambda(\pi_J^{-1}[\pi_J[U]] \setminus W_1),$$

because W_1 is determined by coordinates in J.

At this point note that U is of the form $\prod_{i\in I} G_i$, where $G_i \in \mathfrak{T}_i$ for each I, so we can express U as $U' \cap U''$, where $U' = \pi_J^{-1}[\pi_J[U]]$ and $U'' = \pi_{I\setminus J}^{-1}[\pi_{I\setminus J}[U]]$. U' depends on coordinates in J and U'' depends on coordinates in $I \setminus J$. In this case

$$\lambda(U \setminus W_1) = \lambda(U'' \cap U' \setminus W_1) = \lambda U'' \cdot \lambda(U' \setminus W_1),$$

because U'' depends on coordinates in $I \setminus J$ and $U' \setminus W_1$ depends on coordinates in J, and we can identify λ with the product $\lambda_{I\setminus J} \times \lambda_J$ (254N). But now recall that every μ_i is strictly positive. Since U is surely not empty, no G_i can be empty and no $\mu_i G_i$ can be 0. Consequently $\prod_{i\in I} \mu_i G_i > 0$ (because only finitely many terms in the product are less than 1) and $\lambda U > 0$; more to the point, $\lambda U'' > 0$. Since we already know that $\lambda(U' \setminus W_1) > 0$, we have $\lambda(U \setminus W_1) > 0$. But this contradicts the first sentence of (i) just above. **X**

(iii) Thus $\lambda_J\pi_J[W] = \lambda_J\pi_J[W_1]$. But this means that $\lambda\pi_J^{-1}[\pi_J[W]] = \lambda W_1$. Since λ is complete and $W_1 \subseteq W \subseteq \pi_J^{-1}[\pi_J[W]]$, λW is defined and equal to λW_1.

Taking $\langle V_n\rangle_{n\in\mathbb{N}}$ to be a sequence running over $\mathcal{V}_1 \cup \{\emptyset\}$, we have

$$\lambda W = \lambda W_1 = \lambda(\textstyle\bigcup_{n\in\mathbb{N}} V_n) = \sup_{n\in\mathbb{N}} \lambda(\bigcup_{i\le n} V_i) \le \sup_{V\in\mathcal{V}^*} \lambda V \le \lambda W,$$

so $\lambda W = \sup_{V\in\mathcal{V}^*} \lambda V$, as required. **Q**

(d) Thus we see that λ is a topological measure. But it is also τ-additive. **P** If $\mathcal{W}$ is an upwards-directed family of open sets in X with union W^*, set

$$\mathcal{V} = \{U : U \in \mathcal{U}, \exists W \in \mathcal{W}, U \subseteq W\}.$$

Then $W^* = \bigcup \mathcal{V}$, so $\lambda W^* = \sup_{V \in \mathcal{V}^*} \lambda V$, where $\mathcal{V}^*$ is the set of finite unions of members of $\mathcal{V}$. But because $\mathcal{W}$ is upwards-directed, every member of $\mathcal{V}^*$ is included in some member of $\mathcal{W}$, so

$$\lambda W^* = \sup_{V \in \mathcal{V}^*} \lambda V \leq \sup_{W \in \mathcal{W}} \lambda W \leq \lambda W^*.$$

As $\mathcal{W}$ is arbitrary, λ is τ-additive. **Q**

(e) As in (b) above, we know that λ is a complete probability measure and is inner regular with respect to the closed sets, so it is a quasi-Radon measure. Because λ is inner regular with respect to the zero sets (412Ub), it is completion regular.

(f) Now suppose that $F \subseteq X$ is a closed self-supporting set. By 254Oc, there is a set $W \subseteq X$, determined by coordinates in some countable set $J \subseteq I$, such that $W \triangle F$ is negligible. **?** Suppose, if possible, that $x \in F$ and $y \in X \setminus F$ are such that $x{\restriction}J = y{\restriction}J$. Then there is a $U \in \mathcal{U}$ such that $y \in U \subseteq X \setminus F$. As in (b-ii) above, we can express U as $U' \cap U''$ where $U', U'' \in \mathcal{U}$ are determined by coordinates in J and $I \setminus J$ respectively. In this case,

$$\begin{aligned}\lambda(F \cap U) &= \lambda(W \cap U) = \lambda(W \cap U') \cdot \lambda U'' \\ &= \lambda(F \cap U') \cdot \lambda U'' > 0,\end{aligned}$$

because $x \in F \cap U'$ and F is self-supporting, while $U'' \neq \emptyset$ and λ is strictly positive. But $F \cap U = \emptyset$, so this is impossible. **X**

Thus F is determined by coordinates in the countable set J. Consequently it is of the form $\pi_J^{-1}[\pi_J[F]]$. But $\pi_J[X \setminus F]$ is open (4A2B(f-i)), so its complement $\pi_J[F]$ is closed. Now X_J is metrizable (4A2P(a-v)), so $\pi_J[F]$ is a zero set (4A2Lc) and F is a zero set (4A2C(b-iv)).

415F Corollary(a) If Y is either $[0,1[$ or $]0,1[$, endowed with Lebesgue measure, and I is any set, then Y^I, with the product topology and measure, is a quasi-Radon measure space.

(b) If $\langle \nu_i \rangle_{i \in I}$ is a family of probability distributions on $\mathbb{R}$, in the sense of §271 (that is, Radon probability measures), and every ν_i is strictly positive, then the product measure on $\mathbb{R}^I$ is a quasi-Radon measure.

Remark See also 416U below, and 453I, where there is an alternative proof of the main step in 415E, applicable to some further cases. Yet another approach, most immediately applicable to $[0,1[^I$, is in 443Xq. For further facts about these product measures, see §417, particularly 417G and 417M.

415G Comparing quasi-Radon measures: Proposition Let X be a topological space, and μ, ν two quasi-Radon measures on X. Then the following are equiveridical:

(i) $\mu F \leq \nu F$ for every closed set $F \subseteq X$;

(ii) $\operatorname{dom}\nu \subseteq \operatorname{dom}\mu$ and $\mu E \leq \nu E$ for every $E \in \operatorname{dom}\nu$.

If ν is locally finite, we can add

(iii) $\mu G \leq \nu G$ for every open set $G \subseteq X$;

(iv) there is a base $\mathcal{U}$ for the topology of X such that $G \cup H \in \mathcal{U}$ for all G, $H \in \mathcal{U}$ and $\mu G \leq \nu G$ for $G \in \mathcal{U}$.

proof (a) Of course (ii)⇒(i). Suppose that (i) is true. Observe that if $E \in \operatorname{dom}\mu \cap \operatorname{dom}\nu$ (for instance, if $E \subseteq X$ is Borel), then

$$\mu E = \sup\{\mu F : F \subseteq E \text{ is closed}\} \leq \sup\{\nu F : F \subseteq E \text{ is closed}\} = \nu E.$$

Set $\mathcal{H} = \{H : H \subseteq X \text{ is open}, \nu H < \infty\}$, and $W = \bigcup \mathcal{H}$. Then $\nu(X \setminus W) = 0$, because ν is effectively locally finite, so $\mu(X \setminus W) = 0$. Set $\mathcal{F} = \{F : F \subseteq X \text{ is closed}, \mu F < \infty\}$.

Take any $E \in \operatorname{dom}\nu$. If $F \in \mathcal{F}$ and $\epsilon > 0$, then $\mu(F \cap W) = \mu F$, so there is an $H \in \mathcal{H}$ such that $\mu(F \cap H) \geq \mu F - \epsilon$. Now there are closed sets $F_1 \subseteq F \cap H \cap E$, $F_2 \subseteq F \cap H \setminus E$ such that $\nu F_1 + \nu F_2 \geq \nu(F \cap H) - \epsilon$, that is, $\nu((F \cap H) \setminus (F_1 \cup F_2)) \leq \epsilon$, so that $\mu((F \cap H) \setminus (F_1 \cup F_2)) \leq \epsilon$ and $\mu F_1 + \mu F_2 \geq \mu(F \cap H) - \epsilon$. This means that

$$\mu_*(F \cap E) + \mu_*(F \setminus E) \geq \mu(F \cap H) - \epsilon \geq \mu F - \epsilon.$$

As ϵ is arbitrary, $\mu_*(F \cap E) + \mu_*(F \setminus E) \geq \mu F$; as μ is inner regular with respect to $\mathcal{F}$, μ measures E, by 413F(vii).

Thus $\operatorname{dom}\nu \subseteq \operatorname{dom}\mu$; and we have already observed that $\mu E \leq \nu E$ whenever E is measured by both.

(b) The first sentence in the proof of (a) shows that (i)⇒(iii), and (iii)⇒(iv) is trivial. If (iv) is true and $G \subseteq X$ is open, then $\mathcal{V} = \{V : V \in \mathcal{U}, V \subseteq G\}$ is upwards-directed and has union G, so

$$\mu G = \sup_{V \in \mathcal{V}} \mu V \leq \sup_{V \in \mathcal{V}} \nu V = \nu G.$$

Thus (iv)⇒(iii).

Now assume that ν is locally finite and that (iii) is true. **?** Suppose, if possible, that $F \subseteq X$ is a closed set such that $\nu F < \mu F$. Then $\mathcal{H}$, as defined in part (a) of the proof, is upwards-directed and has union X, so there is an $H \in \mathcal{H}$ such that $\nu F < \mu(F \cap H)$. Now there is a closed set $F' \subseteq H \setminus F$ such that

$$\nu F' > \nu(H \setminus F) - \mu(F \cap H) + \nu F \geq \nu H - \mu(F \cap H).$$

Set $G = H \setminus F'$, so that $F \cap H \subseteq G$ and

$$\nu G = \nu H - \nu F' < \mu(F \cap H) \leq \mu G,$$

which is impossible. **X**

This shows that (provided that ν is locally finite) (iii)⇒(i).

415H Uniqueness of quasi-Radon measures: Proposition Let $(X, \mathfrak{T})$ be a topological space and μ, ν two quasi-Radon measures on X. Then the following are equiveridical:

(i) $\mu = \nu$;

(ii) $\mu F = \nu F$ for every closed set $F \subseteq X$;

(iii) $\mu G = \nu G$ for every open set $G \subseteq X$;

(iv) there is a base $\mathcal{U}$ for the topology of X such that $G \cup H \in \mathcal{U}$ for every G, $H \in \mathcal{U}$ and $\mu\restriction\mathcal{U} = \nu\restriction\mathcal{U}$;

(v) there is a base $\mathcal{U}$ for the topology of X such that $G \cap H \in \mathcal{U}$ for every G, $H \in \mathcal{U}$ and $\mu\restriction\mathcal{U} = \nu\restriction\mathcal{U}$.

proof Of course (i) implies all the others. (ii)⇒(i) is immediate from 415G (see also 412L). If (iii) is true, then, for any closed set $F \subseteq X$,

$$\begin{aligned}\mu F &= \sup\{\mu(G \cap F) : G \in \mathfrak{T}, \mu G < \infty\}\\ &= \sup\{\mu G - \mu(G \setminus F) : G \in \mathfrak{T}, \mu G < \infty\}\\ &= \sup\{\nu G - \nu(G \setminus F) : G \in \mathfrak{T}, \nu G < \infty\} = \nu F;\end{aligned}$$

so (iii)⇒(ii). (iv)⇒(iii) by the argument of (iv)⇒(iii) in the proof of 415G.

Finally, suppose that (v) is true. Then $\mu(G_0 \cup \ldots \cup G_n) = \nu(G_0 \cup \ldots \cup G_n)$ for all $G_0, \ldots, G_n \in \mathcal{U}$. **P** Induce on n. For the inductive step to $n \geq 1$, if any G_i has infinite measure (for either measure) the result is trivial. Otherwise,

$$\begin{aligned}\mu(G_0 \cup \ldots \cup G_n) &= \mu(\bigcup_{i<n} G_i) + \mu G_n - \mu(\bigcup_{i<n}(G_n \cap G_i))\\ &= \nu(\bigcup_{i<n} G_i) + \nu G_n - \nu(\bigcup_{i<n}(G_n \cap G_i)) = \nu(G_0 \cup \ldots \cup G_n).\end{aligned}$$ **Q**

So μ and ν agree on the base $\{G_0 \cup \ldots \cup G_n : G_0, \ldots, G_n \in \mathcal{U}\}$, and (iv) is true.

415I Proposition Let X be a completely regular topological space and μ, ν two quasi-Radon measures on X such that $\int f d\mu = \int f d\nu$ whenever $f : X \to \mathbb{R}$ is a bounded continuous function integrable with respect to both measures. Then $\mu = \nu$.

proof **?** Otherwise, there is an open set $G \subseteq X$ such that $\mu G \neq \nu G$; suppose $\mu G < \nu G$. Because ν is effectively locally finite, there is an open set $G' \subseteq G$ such that $\mu G < \nu G' < \infty$. Now the cozero sets form a base for the topology of X, so $\mathcal{H} = \{H : H \subseteq G' \text{ is a cozero set}\}$ has union G'; as ν is τ-additive, there is an $H \in \mathcal{H}$ such that $\nu H > \mu G$. Express H as $\{x : g(x) > 0\}$ where $g : X \to [0, \infty[$ is continuous. For each $n \in \mathbb{N}$, set $f_n = ng \wedge \chi X$; then $\langle f_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence with limit χH, so there is an $n \in \mathbb{N}$ such that $\int f_n d\nu > \mu G \geq \int f_n d\mu$. But f_n is both μ-integrable and ν-integrable because μG and νH are both finite. **X**

415J Proposition Let X be a regular topological space, Y a subspace of X, and ν a quasi-Radon measure on Y. Then there is a quasi-Radon measure μ on X such that $\mu E = \nu(E \cap Y)$ whenever μ measures E, that is, Y has full outer measure in X and ν is the subspace measure on Y.

proof Write $\mathcal{B}$ for the Borel σ-algebra of X, and set $\mu_0 E = \nu(E \cap Y)$ for every $E \in \mathcal{B}$. Then it is easy to see that μ_0 is a τ-additive Borel measure on X. Moreover, μ_0 is effectively locally finite. **P** If $E \in \mathcal{B}$ and $\mu_0 E > 0$, there is a relatively open set $H \subseteq Y$ such that $\nu H < \infty$ and $\nu(H \cap E \cap Y) > 0$. Now H is of the form $G \cap Y$ where $G \subseteq X$ is open, and we have $\mu_0 G = \nu H < \infty$, $\mu_0(E \cap G) = \nu(H \cap E \cap Y) > 0$. **Q**

By 415Cb, the c.l.d. version μ of μ_0 is a quasi-Radon measure on X. If $E \in \operatorname{dom}\mu$, then $E \cap Y \in \operatorname{dom}\nu$. **P** Let $\mathcal{F}_Y$ be the set of relatively closed subsets of Y of finite measure for ν. If $F \in \mathcal{F}_Y$, it is expressible as $F' \cap Y$ where F' is a closed subset of X, and $\mu F' = \mu_0 F' = \nu F$ is finite. So there are E_1, $E_2 \in \mathcal{B}$ such that $E_1 \subseteq E \cap F' \subseteq E_2$ and $\mu E_1 = \mu(E \cap F') = \mu E_2$. Accordingly

$$E_1 \cap Y \subseteq E \cap Y \cap F \subseteq E_2 \cap Y$$

and

$$\nu(E_1 \cap Y) = \nu(E_2 \cap Y) = \mu(E \cap F')$$

is finite. This means that $E \cap Y \cap F \in \operatorname{dom}\nu$; because ν is complete and locally determined and inner regular with respect to $\mathcal{F}_Y$, $E \cap Y \in \operatorname{dom}\nu$, by 412Ja. **Q**

If $E \in \operatorname{dom}\mu$, then

$$\begin{aligned}\mu E &= \sup\{\mu F : F \subseteq E \text{ is closed}\}\\ &= \sup\{\nu(F \cap Y) : F \subseteq E \text{ is closed}\} \leq \nu(E \cap Y).\end{aligned}$$

On the other hand, if $\gamma < \nu(E \cap Y)$, there is a relatively open set $H \subseteq Y$ such that $\nu H < \infty$ and $\nu(E \cap Y \cap H) \geq \gamma$ (412F). Let $G \subseteq X$ be an open set such that $G \cap Y = H$. Then

$$\mu E \geq \mu G - \mu(G \setminus E) = \nu H - \nu(H \setminus E) = \nu(E \cap Y \cap H) \geq \gamma.$$

As γ is arbitrary, $\mu E = \nu(E \cap Y)$.

Thus $\mu E = \nu(E \cap Y)$ whenever μ measures E. So if E, $F \in \operatorname{dom}\mu$ and $E \cap Y \subseteq F$,

$$\mu E = \nu(E \cap Y) \leq \nu(F \cap Y) = \mu F;$$

as F is arbitrary, $\mu^*(E \cap Y) = \mu E$; as E is arbitrary, Y has full outer measure in X. Moreover, if μ_Y is the subspace measure on Y, $\mu_Y H = \mu^* H = \nu H$ whenever $H \in \operatorname{dom}\mu_Y$, that is, $H = E \cap Y$ for some $E \in \operatorname{dom}\mu$. Now μ_Y, like ν, is a quasi-Radon measure on Y (415B), and they agree on the (relatively) closed subsets of Y, so are equal, by 415H.

415K I come now to a couple of basic results on the construction of quasi-Radon measures. The first follows 413J.

Theorem Let X be a topological space and $\mathcal{K}$ a family of closed subsets of X such that

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $F \in \mathcal{K}$ whenever $K \in \mathcal{K}$ and $F \subseteq K$ is closed.

Let $\phi_0 : \mathcal{K} \to [0, \infty[$ be a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K}, K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$,

(β) $\inf_{K \in \mathcal{K}'} \phi_0 K = 0$ whenever $\mathcal{K}'$ is a non-empty downwards-directed subset of $\mathcal{K}$ with empty intersection,

(γ) whenever $K \in \mathcal{K}$ and $\phi_0 K > 0$, there is an open set G such that the supremum $\sup_{K' \in \mathcal{K}, K' \subseteq G} \phi_0 K'$ is finite, while $\phi_0 K' > 0$ for some $K' \in \mathcal{K}$ such that $K' \subseteq K \cap G$.

Then there is a unique quasi-Radon measure on X extending ϕ_0 and inner regular with respect to $\mathcal{K}$.

proof By 413J, there is a complete locally determined measure μ on X, inner regular with respect to $\mathcal{K}$, and extending ϕ_0; write Σ for the domain of μ. If $F \subseteq X$ is closed, then $K \cap F \in \mathcal{K} \subseteq \Sigma$ for every $K \in \mathcal{K}$, so $F \in \Sigma$, by 413F(ii); accordingly every open set is measurable. Because μ is inner regular with respect to

$\mathcal{K}$ it is surely inner regular with respect to the closed sets. If $E \in \Sigma$ and $\mu E > 0$, there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu K > 0$; now (γ) tells us that there is an open set G such that $\mu G < \infty$ and $\mu(G \cap K) > 0$, so that $\mu(G \cap E) > 0$. As E is arbitrary, μ is effectively locally finite. Now suppose that $\mathcal{G}$ is a non-empty upwards-directed family of open sets with union H, and that $\gamma < \mu H$. Then there is a $K \in \mathcal{K}$ such that $K \subseteq H$ and $\mu K > \gamma$. Applying the hypothesis (β) to $\mathcal{K}' = \{K \setminus G : G \in \mathcal{G}\}$, we see that $\inf_{G\in\mathcal{G}} \mu(K \setminus G) = 0$, so that

$$\sup_{G\in\mathcal{G}} \mu G \geq \sup_{G\in\mathcal{G}} \mu(K \cap G) = \mu K \geq \gamma.$$

As $\mathcal{G}$ and γ are arbitrary, μ is τ-additive. So μ is a quasi-Radon measure.

415L Proposition Let (X, Σ_0, μ_0) be a measure space and $\mathfrak{T}$ a topology on X such that μ_0 is τ-additive, effectively locally finite and inner regular with respect to the closed sets, and Σ_0 includes a base for $\mathfrak{T}$. Then μ_0 has a unique extension to a quasi-Radon measure μ on X such that

(i) $\mu F = \mu_0^* F$ whenever $F \subseteq X$ is closed and $\mu_0^* F < \infty$,

(ii) $\mu G = (\mu_0)_* G$ whenever $G \subseteq X$ is open,

(iii) the embedding $\Sigma_0 \subseteq \Sigma$ identifies the measure algebra $(\mathfrak{A}_0, \bar{\mu}_0)$ of μ_0 with an order-dense subalgebra of the measure algebra $(\mathfrak{A}, \bar{\mu})$ of μ, so that the subrings $\mathfrak{A}_0^f$, $\mathfrak{A}^f$ of elements of finite measure coincide, and $L^p(\mu_0)$ may be identified with $L^p(\mu)$ for $1 \leq p < \infty$,

(iv) whenever $E \in \Sigma$ and $\mu E < \infty$, there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$,

(v) for every μ-integrable real-valued function f there is a μ_0-integrable function g such that $f = g$ μ-a.e.

If μ_0 is complete and locally determined, then we have

(i)′ $\mu F = \mu_0^* F$ for every closed $F \subseteq X$.

If μ_0 is localizable, then we have

(iii)′ $\mathfrak{A}_0 = \mathfrak{A}$, so that $L^0(\mu) \cong L^0(\mu_0)$, $L^\infty(\mu) \cong L^\infty(\mu_0)$,

(iv)′ for every $E \in \Sigma$ there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$,

(v)′ for every Σ-measurable real-valued function f there is a Σ_0-measurable real-valued function g such that $f = g$ μ-a.e.

proof (a) Let $\mathcal{K}$ be the set of closed subsets of X of finite outer measure for μ_0. Note that μ_0 is inner regular with respect to $\mathcal{K}$, because it is inner regular with respect to the closed sets and also with respect to the sets of finite measure.

It is obvious from its definition that $\mathcal{K}$ satisfies (†) and (‡) of 415K. For $K \in \mathcal{K}$, set $\phi_0 K = \mu_0^* K$. Then ϕ_0 satisfies (α)-(γ) of 415K.

P (α) If K, $L \in \mathcal{K}$ and $L \subseteq K$, take measurable envelopes E_0, $E_1 \in \Sigma_0$ of K, L respectively. (i) Let $\epsilon > 0$. Because μ_0 is inner regular with respect to the closed sets, there is a closed set $F \in \Sigma_0$ such that $F \subseteq E_0 \setminus E_1$ and $\mu F \geq \mu_0(E_0 \setminus E_1) - \epsilon$. Set $K' = F \cap K$. Then $K' \in \mathcal{K}$ and

$$\phi_0 K' = \mu_0^*(F \cap K) = \mu_0(F \cap E_0) = \mu_0 F \geq \mu_0 E_0 - \mu_0 E_1 - \epsilon = \phi_0 K - \phi_0 L - \epsilon.$$

As ϵ is arbitrary, we have

$$\phi_0 K \leq \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}.$$

(ii) On the other hand, **?** suppose, if possible, that there is a closed set $K' \subseteq K \setminus L$ such that $\mu_0^* L + \mu_0^* K' > \mu^* K$. Let E_2 be a measurable envelope of K', so that $\mu_0 E_1 + \mu_0 E_2 > \mu_0 E_0$; since

$$\mu_0(E_1 \setminus E_0) = \mu_0^*(L \setminus E_0) = \mu_0^* \emptyset = 0, \quad \mu_0(E_2 \setminus E_0) = \mu_0^*(K' \setminus E_0) = 0,$$

$\mu_0(E_1 \cap E_2) > 0$. Because μ_0 is effectively locally finite, there is a measurable open set G_0, of finite measure, such that $\mu_0(G_0 \cap E_1 \cap E_2) > 0$. Set

$$\mathcal{G} = \{G \cup G' : G,\ G' \in \Sigma_0 \cap \mathfrak{T},\ G \subseteq G_0 \setminus L,\ G' \subseteq G_0 \setminus K'\}.$$

Then $\mathcal{G}$ is an upwards-directed family of measurable open sets, and because Σ_0 includes a base for the topology of X, its union is $(G_0 \setminus L) \cup (G_0 \setminus K') = G_0$. So there is an $H \in \mathcal{G}$ such that $\mu_0 H > \mu_0 G_0 - \mu_0(E_1 \cap E_2)$,

that is, there are open sets G, $G' \in \Sigma_0$ such that $G \subseteq G_0 \setminus L$, $G' \subseteq G_0 \setminus K'$ and $\mu_0((G \cup G') \cap E_1 \cap E_2)) > 0$. But we must have

$$\mu_0(G \cap E_1) = \mu_0^*(G \cap L) = 0, \quad \mu_0(G' \cap E_2) = \mu_0^*(G' \cap K') = 0,$$

so this is impossible. **X**

Accordingly

$$\phi_0 K \geq \phi_0 L + \sup\{\phi_0 K' : K' \in \mathcal{K},\, K' \subseteq K \setminus L\},$$

so that ϕ_0 satisfies condition (α) of 415K.

(β) Let $\mathcal{K}' \subseteq \mathcal{K}$ be a non-empty downwards-directed family with empty intersection. Fix $K_0 \in \mathcal{K}'$ and $\epsilon > 0$. Let E_0 be a measurable envelope of K_0 and G_0 a measurable open set of finite measure such that $\mu_0(G_0 \cap E_0) \geq \mu_0 E_0 - \epsilon$. Then

$$\mathcal{G} = \{G : G \in \Sigma_0 \cap \mathfrak{T},\, G \subseteq G_0 \setminus K \text{ for some } K \in \mathcal{K}' \text{ such that } K \subseteq K_0\}$$

is an upwards-directed family of measurable open sets, and its union is $G_0 \setminus \bigcap \mathcal{K}' = G_0$, again because Σ_0 includes a base for the topology $\mathfrak{T}$. So there is a $G \in \mathcal{G}$ such that $\mu_0 G \geq \mu_0 G_0 - \epsilon$. Let $K \in \mathcal{K}'$ be such that $K \subseteq K_0$ and $G \cap K = \emptyset$; then

$$\phi_0 K = \mu_0^* K \leq \mu_0(E_0 \setminus G) \leq \mu_0(E_0 \setminus G_0) + \mu_0(G_0 \setminus G) \leq 2\epsilon.$$

As ϵ is arbitrary, $\inf_{K \in \mathcal{K}'} \phi_0 K = 0$.

(γ) If $K \in \mathcal{K}$ and $\phi_0 K > 0$, let E_0 be a measurable envelope of K. Then there is a measurable open set G of finite measure such that $\mu_0(G \cap E_0) > 0$. Of course $\sup_{K' \in \mathcal{K}, K' \subseteq G} \phi_0 K' \leq \mu_0 G < \infty$; but also there is a measurable closed set $K' \subseteq G \cap E_0$ such that $\mu_0 K' > 0$, in which case $\phi_0(K \cap K') = \mu_0(E_0 \cap K') > 0$. So ϕ_0 satisfies condition (γ). **Q**

(b) By 415K, ϕ_0 has an extension to a quasi-Radon measure μ on X which is inner regular with respect to $\mathcal{K}$. Write Σ for the domain of μ. Note that, for $K \in \mathcal{K}$,

$$\mu K = \phi_0 K = \mu_0^* K,$$

so we can already be sure that the conclusion (i) of the proposition is satisfied. Now μ extends μ_0.

P(i) Take any $K \in \mathcal{K}$. Let $E_0 \in \Sigma_0$ be a measurable envelope of K for the measure μ_0. If $E \in \Sigma_0$, then surely

$$\begin{aligned}\mu_*(K \cap E) &= \sup\{\mu K' : K' \in \mathcal{K},\, K' \subseteq K \cap E\} \\ &= \sup\{\mu_0^* K' : K' \in \mathcal{K},\, K' \subseteq K \cap E\} \leq \mu_0^*(K \cap E).\end{aligned}$$

On the other hand, given $\gamma < \mu_0^*(K \cap E) = \mu_0(E_0 \cap E)$, there is a closed set $F \in \Sigma_0$ such that $F \subseteq E_0 \cap E$ and $\mu_0 F \geq \gamma$, so that

$$\mu_*(K \cap E) \geq \mu(K \cap F) = \mu_0^*(K \cap F) = \mu_0(E_0 \cap F) \geq \gamma.$$

Thus $\mu_*(K \cap E) = \mu_0^*(K \cap E)$ for every $K \in \mathcal{K}$, $E \in \Sigma_0$.

(ii) If $K \in \mathcal{K}$, $E \in \Sigma_0$ then

$$\mu_*(K \cap E) + \mu_*(K \setminus E) = \mu_0^*(K \cap E) + \mu_0^*(K \setminus E) = \mu_0^* K = \mu K.$$

Because μ is complete and locally determined and inner regular with respect to $\mathcal{K}$, $E \in \Sigma$ (413F(iv)). Thus $\Sigma_0 \subseteq \Sigma$.

(iii) For any $E \in \Sigma_0$, we now have

$$\begin{aligned}\mu E &= \sup\{\mu K : K \in \mathcal{K},\, K \subseteq E\} = \sup\{\mu_0^* K : K \in \mathcal{K},\, K \subseteq E\} \\ &\leq \mu_0 E = \sup\{\mu_0 K : K \in \mathcal{K} \cap \Sigma_0,\, K \subseteq E\} \leq \mu E.\end{aligned}$$

As E is arbitrary, μ extends μ_0. **Q**

(c) Because $\Sigma_0 \cap \mathfrak{T}$ is a base for $\mathfrak{T}$, closed under finite unions, μ is unique, by 415Hc.

(d) Now for the conditions (i)-(v). I have already noted that (i) is guaranteed by the construction. Concerning (ii), if $G \subseteq X$ is open, we surely have $(\mu_0)_*G \leq \mu_*G = \mu G$ because μ extends μ_0. On the other hand, writing $\mathcal{G} = \{G' : G' \in \Sigma_0 \cap \mathfrak{T}, G' \subseteq G\}$, $\mathcal{G}$ is upwards-directed and has union G, so

$$\mu G = \sup_{G' \in \mathcal{G}} \mu G' = \sup_{G' \in \mathcal{G}} \mu_0 G' \leq (\mu_0)_* G.$$

So (ii) is true.

Because μ extends μ_0, the embedding $\Sigma_0 \subseteq \Sigma$ corresponds to a measure-preserving embedding of $\mathfrak{A}_0$ as a σ-subalgebra of $\mathfrak{A}$. To see that $\mathfrak{A}_0$ is order-dense in $\mathfrak{A}$, take any non-zero $a \in \mathfrak{A}$. This is expressible as $E^\bullet$ for some $E \in \Sigma$ with $\mu E > 0$. Now there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu K > 0$. There is an $E_0 \in \Sigma_0$ which is a measurable envelope for K with respect to μ_0, so that

$$\mu E_0 = \mu_0 E_0 = \mu_0^* K = \mu K.$$

But this means that

$$0 \neq E_0^\bullet = K^\bullet \subseteq E^\bullet = a$$

in $\mathfrak{A}$, while $E_0^\bullet \in \mathfrak{A}_0$. As a is arbitrary, $\mathfrak{A}_0$ is order-dense in $\mathfrak{A}$.

If $a \in \mathfrak{A}^f$, then $B = \{b : b \in \mathfrak{A}_0, b \subseteq a\}$ is upwards-directed and $\sup_{b \in B} \bar{\mu}_0 b \leq \bar{\mu} a$ is finite; accordingly B has a supremum in $\mathfrak{A}_0$ (321C), which must also be its supremum in $\mathfrak{A}$, which is a (313O, 313K). So $a \in \mathfrak{A}_0$. Thus $\mathfrak{A}^f$ can be identified with $\mathfrak{A}_0^f$. But this means that, for any $p \in [1, \infty[$, $L^p(\mu) \cong L^p(\mathfrak{A}, \bar{\mu})$ is identified with $L^p(\mathfrak{A}_0, \bar{\mu}_0) \cong L^p(\mu)$ (366H). This proves (iii).

Of course (iv) and (v) are just translations of this. If $E \in \Sigma$ and $\mu E < \infty$, then $E^\bullet \in \mathfrak{A}^f \subseteq \mathfrak{A}_0$, that is, there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$. If f is μ-integrable, then $f^\bullet \in L^1(\mu) = L^1(\mu_0)$, that is, there is a μ_0-integrable function f_0 such that $f = f_0$ μ-a.e.

(e) If μ_0 is complete and locally determined and $F \subseteq X$ is an arbitrary closed set, then

$$\mu_0^* F = \sup_{K \in \mathcal{K}} \mu_0^*(F \cap K) = \sup_{K \in \mathcal{K}} \mu(F \cap K) = \sup_{K \in \mathcal{K}, K \subseteq F} \mu K = \mu F$$

by 412Jc, because μ and μ_0 are both inner regular with respect to $\mathcal{K}$.

(f) If μ_0 is localizable, $\mathfrak{A}_0$ is Dedekind complete; as it is order-dense in $\mathfrak{A}$, the two must coincide (314Ia). Consequently

$$L^0(\mu) \cong L^0(\mathfrak{A}) = L^0(\mathfrak{A}_0) \cong L^0(\mu_0), \quad L^\infty(\mu) \cong L^\infty(\mathfrak{A}) = L^\infty(\mathfrak{A}_0) \cong L^\infty(\mu_0).$$

So (iii)$'$ is true; now (iv)$'$ and (v)$'$ follow at once.

415M Corollary Let $(X, \mathfrak{T})$ be a regular topological space and μ_0 an effectively locally finite τ-additive measure on X, defined on the σ-algebra Σ_0 generated by a base for $\mathfrak{T}$. Then μ_0 has a unique extension to a quasi-Radon measure on X.

proof By 414Mb, μ_0 is inner regular with respect to the closed sets. So 415L gives the result.

415N Corollary Let $(X, \mathfrak{T})$ be a completely regular topological space, and μ_0 a τ-additive effectively locally finite Baire measure on X. Then μ_0 has a unique extension to a quasi-Radon measure on X.

proof This is a special case of 415M, because the domain Σ_0 of μ_0, the Baire σ-algebra, is generated by the family of cozero sets, which form a base for $\mathfrak{T}$ (4A2Fc).

415O Proposition (a) Let $(X, \mathfrak{T})$ be a topological space, and μ, ν two quasi-Radon measures on X. Then ν is an indefinite-integral measure over μ iff $\nu F = 0$ whenever $F \subseteq X$ is closed and $\mu F = 0$.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space, and ν an indefinite-integral measure over μ (definition: 234B). If ν is effectively locally finite it is a quasi-Radon measure.

proof (a) If ν is an indefinite-integral measure over μ, then of course it is zero on all μ-negligible closed sets. So let us suppose that the condition is satisfied. Write $\Sigma = \operatorname{dom} \mu$ and $\mathrm{T} = \operatorname{dom} \nu$.

(i) If $E \subseteq X$ is a μ-negligible Borel set it is ν-negligible, because every closed subset of E must be μ-negligible, therefore ν-negligible, and ν is inner regular with respect to the closed sets. In particular,

taking U^* to be the union of the family $\mathcal{U} = \{U : U \in \mathfrak{T}, \mu U < \infty\}$, $\nu(X \setminus U^*) = \mu(X \setminus U^*) = 0$ because μ is effectively locally finite. Also, of course, taking V^* to be the union of the family $\mathcal{V} = \{V : V \in \mathfrak{T}, \nu V < \infty\}$, $\nu(X \setminus V^*) = 0$ because ν is effectively locally finite. Setting $\mathcal{G} = \mathcal{U} \cap \mathcal{V}$ and $G^* = \bigcup \mathcal{G}$, we have $G^* = U^* \cap V^*$, so G^* is ν-conegligible.

(ii) In fact, every μ-negligible set E is ν-negligible. **P?** Otherwise, $\nu^*(E \cap G^*) > 0$. Because the subspace measure ν_E is quasi-Radon (415B), there is a $G \in \mathcal{G}$ such that $\nu^*(E \cap G) > 0$. But there is an F_σ set $H \subseteq G \setminus E$ such that $\mu H = \mu(G \setminus E)$, and now $E \cap G$ is included in the μ-negligible Borel set $G \setminus H$, so that $\nu(E \cap G) = \nu(G \setminus H) = 0$. **XQ**

(iii) Let $\mathcal{K}$ be the family of closed subsets F of X such that either F is included in some member of $\mathcal{G}$ or $F \cap G^* = \emptyset$. If $E \in \operatorname{dom}\mu$ and $\mu E > 0$, then there is an $F \in \mathcal{K}$ such that $F \subseteq E$ and $\mu F > 0$. **P** If $\mu(E \setminus G^*) > 0$ take any closed set $F \subseteq E \setminus G^*$ with $\mu F > 0$. Otherwise, $\mu(E \cap G^*) > 0$. Because the subspace measure μ_E is quasi-Radon, there is a $G \in \mathcal{G}$ such that $\mu(E \cap G) > 0$; and now we can find a closed set $F \subseteq E \cap G$ with $\mu F > 0$, and $F \in \mathcal{K}$. **Q**

(iv) By 412I, there is a decomposition $\langle X_i \rangle_{i \in I}$ for μ such that every X_i except perhaps one belongs to $\mathcal{K}$ and that exceptional one, if any, is μ-negligible. Now $\langle X_i \rangle_{i \in I}$ is a decomposition for ν. **P** Every X_i is measured by ν because it is either closed or μ-negligible, and of finite measure for ν because it is included in either a member of $\mathcal{G}$ or the ν-negligible set $X \setminus G^*$. If $E \subseteq X$ and $\nu E > 0$, then $\nu(E \cap G^*) > 0$, so there must be some $G \in \mathcal{G}$ such that $\nu(E \cap G) > 0$. Now $J = \{i : i \in I, \mu(X_i \cap G) > 0\}$ is countable, and $\nu(G \setminus \bigcup_{i \in J} X_i) = \mu(G \setminus \bigcup_{i \in J} X_i) = 0$, so there is an $i \in J$ such that $\nu(X_i \cap E) > 0$. By 213O, $\langle X_i \rangle_{i \in I}$ is a decomposition for ν. **Q**

(v) It follows that $\Sigma \subseteq \mathrm{T}$. **P** If $E \in \Sigma$, then for every $i \in I$ there is an F_σ set $H \subseteq E \cap X_i$ such that $E \cap X_i \setminus H$ is μ-negligible, therefore ν-negligible, and $E \cap X_i \in \mathrm{T}$. As i is arbitrary, $E \in \mathrm{T}$. **Q** In fact, ν is the completion of $\nu \restriction \Sigma$. **P** If $F \in \mathrm{T}$, then for every $i \in I$ there is an F_σ set $H_i \subseteq F \cap X_i$ such that $F \cap X_i \setminus H_i$ is ν-negligible. Set $H = \bigcup_{i \in I} H_i$; because $H \cap X_i = H_i$ belongs to Σ for every i, $H \in \Sigma$; and $\nu(F \setminus H) = \sum_{i \in I} \nu(F \cap X_i \setminus H) = 0$. Similarly, there is an $H' \in \Sigma$ such that $H' \subseteq X \setminus F$ and $\nu((X \setminus F) \setminus H') = 0$, so that $H \subseteq F \subseteq X \setminus H'$ and $\nu((X \setminus H') \setminus H) = 0$. So F is measured by the completion of $\nu \restriction \Sigma$. Since ν itself is complete, it must be the completion of $\nu \restriction \Sigma$. **Q**

(vi) By (iv), ν is inner regular with respect to $\{E : E \in \Sigma, \mu E < \infty\}$. By 234G, ν is an indefinite-integral measure over μ.

(b) Let $f \in \mathcal{L}^0(\mu)$ be a non-negative function such that $\nu F = \int f \times \chi F \, d\mu$ whenever this is defined. Because μ is complete and locally determined, so is ν (234Fb). Because μ is an effectively locally finite τ-additive topological measure, ν is a τ-additive topological measure (414H). Because μ is inner regular with respect to the closed sets, so is ν (412Q). Since we are assuming in the hypotheses that ν is effectively locally finite, it is a quasi-Radon measure.

415P Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space.

(a) Suppose that $(X, \mathfrak{T})$ is completely regular. If $1 \leq p < \infty$ and $f \in \mathcal{L}^p(\mu)$, then for any $\epsilon > 0$ there is a bounded continuous function $g : X \to \mathbb{R}$ such that $\mu\{x : g(x) \neq 0\} < \infty$ and $\|f - g\|_p \leq \epsilon$.

(b) Suppose that $(X, \mathfrak{T})$ is regular and Lindelöf. Let $f \in \mathcal{L}^0(\mu)$ be locally integrable. Then for any $\epsilon > 0$ there is a continuous function $g : X \to \mathbb{R}$ such that $\|f - g\|_1 \leq \epsilon$.

proof (a) Write $\mathcal{C}$ for the set of bounded continuous functions $g : X \to \mathbb{R}$ such that $\{x : g(x) \neq 0\}$ has finite measure. Then $\mathcal{C}$ is a linear subspace of $\mathbb{R}^X$ included in $\mathcal{L}^p = \mathcal{L}^p(\mu)$. Let $\mathcal{U}$ be the closure of $\mathcal{C}$ in $\mathcal{L}^p$, that is, the set of $h \in \mathcal{L}^p$ such that for every $\epsilon > 0$ there is a $g \in \mathcal{C}$ such that $\|h - g\|_p \leq \epsilon$. Then $\mathcal{U}$ is closed under addition and scalar multiplication. Also $\chi E \in \mathcal{U}$ whenever $\mu E < \infty$. **P** Let $\epsilon > 0$. Set $\delta = \frac{1}{4}\epsilon^{1/p}$. Write $\mathcal{G}$ for the family of open sets of finite measure. Because μ is effectively locally finite, there is a $G \in \mathcal{G}$ such that $\mu(E \setminus G) \leq \delta$. Let $F \subseteq G \setminus E$ be a closed set such that $\mu F \geq \mu(G \setminus E) - \delta$; then $\mu(E \triangle (G \setminus F)) \leq 2\delta$. Write $\mathcal{H}$ for the family of cozero sets. Because $\mathfrak{T}$ is completely regular, $\mathcal{H}$ is a base for $\mathfrak{T}$; because $\mathcal{H}$ is closed under finite unions (4A2C(b-iii)) and μ is τ-additive, there is an $H \in \mathcal{H}$ such that $H \subseteq G \setminus F$ and $\mu H \geq \mu(G \setminus F) - \delta$, so that $\mu(E \triangle H) \leq 3\delta$. Express H as $\{x : g(x) > 0\}$ where $g : X \to \mathbb{R}$ is a continuous function. For each $n \in \mathbb{N}$, set $g_n = ng \wedge \chi X \in \mathcal{C}$; then

$$|\chi E - g_n|^p \leq \chi(E \triangle H) + (\chi H - g_n)^p$$

for every n, so

$$\int |\chi E - g_n|^p \leq \mu(E \triangle H) + \int (\chi H - g_n)^p \to \mu(E \triangle H)$$

as $n \to \infty$, because $g_n \to \chi H$. So there is an $n \in \mathbb{N}$ such that $\int |\chi E - g_n|^p \leq 4\delta$, that is, $\|\chi E - g_n\|_p \leq \epsilon$. As ϵ is arbitrary, $\chi E \in \mathcal{U}$. **Q**

Accordingly every simple function belongs to $\mathcal{U}$. But if $f \in \mathcal{L}^p$ and $\epsilon > 0$, there is a simple function h such that $\|f - h\|_p \leq \frac{1}{2}\epsilon$ (244Ha); now there is a $g \in \mathcal{C}$ such that $\|h - g\|_p \leq \frac{1}{2}\epsilon$ and $\|f - g\|_p \leq \epsilon$, as claimed.

(b) This time, write $\mathcal{G}$ for the family of open subsets of X such that $\int_G f$ is finite, so that $\mathcal{G}$ is an open cover of X. As X is paracompact (4A2H(b-i)), there is a locally finite family $\mathcal{G}_0 \subseteq \mathcal{G}$ covering X, which must be countable (4A2H(b-ii)).

Let $\langle \epsilon_G \rangle_{G \in \mathcal{G}_0}$ be a family of strictly positive real numbers such that $\sum_{G \in \mathcal{G}_0} \epsilon_G \leq \epsilon$ (4A1P). Since X is completely regular (4A2H(b-i)), we can apply (a) to see that, for each $G \in \mathcal{G}_0$, there is a continuous function $g_G : X \to \mathbb{R}$ such that $\int |g_G - f \times \chi G| \leq \epsilon_G$. Next, because X is normal (4A2H(b-i)), there is a family $\langle h_G \rangle_{G \in \mathcal{G}_0}$ of continuous functions from X to $[0,1]$ such that $h_G \leq \chi G$ for every $G \in \mathcal{G}_0$ and $\sum_{G \in \mathcal{G}_0} h_G(x) = 1$ for every $x \in X$ (4A2F(d-ix)).

Set $g(x) = \sum_{G \in \mathcal{G}_0} g_G(x) h_G(x)$ for every $x \in X$. Because $\mathcal{G}_0$ is locally finite, $g : X \to \mathbb{R}$ is continuous (4A2Bh). Now

$$\begin{aligned}\int |f - g| &= \int |\sum_{G \in \mathcal{G}_0} (f - g_G) \times h_G| \leq \sum_{G \in \mathcal{G}_0} \int |(f - g_G) \times h_G| \\ &\leq \sum_{G \in \mathcal{G}_0} \int_G |f - g_G| \leq \sum_{G \in \mathcal{G}_0} \epsilon_G \leq \epsilon,\end{aligned}$$

as required.

415Q Recall (411P) that if $(\mathfrak{A}, \bar{\mu})$ is a localizable measure algebra, with Stone space $(Z, \mathfrak{S}, \mathrm{T}, \nu)$, then ν is a strictly positive completion regular quasi-Radon measure, inner regular with respect to the open-and-closed sets (which are all compact). The following construction is primarily important for Radon measure spaces (see 416V), but is also of interest for general quasi-Radon spaces.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and $(\mathfrak{A}, \bar{\mu})$ its measure algebra. Let $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ be the Stone space of $(\mathfrak{A}, \bar{\mu})$. For $E \in \Sigma$ let $E^* \subseteq Z$ be the open-and-closed set corresponding to the image $E^\bullet$ of E in $\mathfrak{A}$. Define $R \subseteq Z \times X$ by saying that $(z, x) \in R$ iff $x \in F$ whenever $F \subseteq X$ is closed and $z \in F^*$. Set $Q = R^{-1}[X]$.

(a) R is a closed subset of $Z \times X$.

(b) For any $E \in \Sigma$, $R[E^*]$ is the smallest closed set such that $\mu(E \setminus R[E^*]) = 0$. In particular, if $F \subseteq X$ is closed then $R[F^*]$ is the self-supporting closed set included in F such that $\mu(F \setminus R[F^*]) = 0$; and $R[Z]$ is the support of μ.

(c) Q is of full outer measure in Z.

(d) For any $E \in \Sigma$, $R^{-1}[E] \triangle (Q \cap E^*)$ is negligible; consequently $\nu^* R^{-1}[E] = \mu E$ and $R^{-1}[E] \cap R^{-1}[X \setminus E]$ is negligible.

(e) For any $A \subseteq X$, $\nu^* R^{-1}[A] = \mu^* A$.

(f) If $(X, \mathfrak{T})$ is regular, then $R^{-1}[G]$ is relatively open in Q for every open set $G \subseteq X$, $R^{-1}[F]$ is relatively closed in Q for every closed set $F \subseteq X$ and $R^{-1}[X \setminus E] = Q \setminus R^{-1}[E]$ for every Borel set $E \subseteq X$.

proof (a)

$$R = \bigcap_{F \subseteq X \text{ is closed}} ((Z \setminus F^*) \times X) \cup (Z \times F)$$

is an intersection of closed sets, therefore closed.

(b) Let $\mathcal{G}$ be the family of open sets $G \subseteq X$ such that $\mu(E \cap G) = 0$, and $G_0 = \bigcup \mathcal{G}$; then $G_0 \in \mathcal{G}$ (414Ea). Set $F_0 = X \setminus G_0$, so that F_0 is the smallest closed set such that $E \setminus F_0$ is negligible, and $F_0^* \supseteq E^*$.

If $(z,x) \in R$ and $z \in E^*$ we must have $x \in F_0$. Thus $R[E^*] \subseteq F_0$. On the other hand, if $x \in F_0$, and G is an open set containing x, then $G \notin \mathcal{G}$ so $\mu(G \cap E) > 0$ and $(E \cap G)^* \neq \emptyset$. Accordingly $\{(G \cap E)^* : x \in G \in \mathfrak{T}\}$ is a downwards-directed family of non-empty open-and-closed sets in the compact space Z and has non-empty intersection, containing a point z say. If $H \subseteq X$ is closed and $z \in H^*$, then $X \setminus H$ is open and $z \notin (X \setminus H)^*$, so x cannot belong to $X \setminus H$, that is, $x \in H$; as H is arbitrary, $(z,x) \in R$ and $x \in R[E^*]$; as x is arbitrary, $R[E^*] = F_0$, as claimed.

Of course, when E is actually closed, $R[E^*] = F_0 \subseteq E$. Taking $E = X$ we see that $R[Z] = R[X^*]$ is the support of μ.

(c) If $W \in \mathrm{T}$ and $\nu W > 0$, there is a non-empty open-and-closed set $U \subseteq W$, by 322Qa, which must be of the form E^* for some $E \in \Sigma$. By (b), $R[E^*]$ cannot be empty; but $E^* \subseteq W$, so $R[W] \neq \emptyset$, that is, $W \cap Q \neq \emptyset$. As W is arbitrary, $\nu_*(Z \setminus Q) = 0$, that is, Z is a measurable envelope of Q (413Ei).

(d)(i) Let $\mathcal{F}$ be the set of closed subsets of X included in E. Then $\sup_{F \in \mathcal{F}} F^\bullet = E^\bullet$ in $\mathfrak{A}$ (412N), so $E^* \setminus \bigcup_{F \in \mathcal{F}} F^*$ is nowhere dense and negligible. Now for each $F \in \mathcal{F}$, $R[F^*] \subseteq F$, so $Q \cap F^* \subseteq R^{-1}[F] \subseteq R^{-1}[E]$. Accordingly

$$Q \cap E^* \setminus R^{-1}[E] \subseteq E^* \setminus \bigcup_{F \in \mathcal{F}} F^*$$

is nowhere dense and negligible. (ii) **?** Suppose, if possible, that $\nu^*(R^{-1}[E] \setminus E^*) > 0$. Then there is an open-and-closed set U of finite measure such that $\nu^*(R^{-1}[E] \cap U \setminus E^*) > 0$ (use 412Jc). Express U as H^*, where $\mu H < \infty$, and let $F \subseteq H \setminus E$ be a closed set such that $\mu((H \setminus E) \setminus F) < \nu^*(R^{-1}[E] \cap H^* \setminus E^*)$. Then we must have $\nu^*(R^{-1}[E] \cap F^*) > 0$. But $R[F^*] \subseteq F \subseteq X \setminus E$ so $F^* \cap R^{-1}[E] = \emptyset$, which is impossible. **X** (iii) Putting these together, $(Q \cap E^*) \triangle R^{-1}[E]$ is negligible. (iv) It follows at once that (because Z is a measurable envelope for Q)

$$\nu^* R^{-1}[E] = \nu^*(Q \cap E^*) = \nu E^* = \mu E.$$

Moreover, applying the result to $X \setminus E$,

$$R^{-1}[X \setminus E] \cap R^{-1}[E] \subseteq (R^{-1}[X \setminus E] \triangle (Q \cap (X \setminus E)^*)) \cup (R^{-1}[E] \triangle (Q \cap E^*))$$

is negligible.

(e)(i) Take $E \in \Sigma$ such that $A \subseteq E$ and $\mu E = \mu^* A$; then $R^{-1}[A] \subseteq R^{-1}[E]$, so

$$\nu^* R^{-1}[A] \leq \nu^* R^{-1}[E] \leq \mu E = \mu^* A.$$

(ii) **?** Suppose, if possible, that $\nu^* R^{-1}[A] < \mu^* A$. Let $W \in \mathrm{T}$ be such that $R^{-1}[A] \subseteq W$ and $\nu W = \nu^* R^{-1}[A]$. Then there is an $F \in \Sigma$ such that $\nu(W \triangle F^*) = 0$. Since

$$\mu F = \nu F^* = \nu W < \mu^* A,$$

$\mu^*(A \setminus F) > 0$; let G be a measurable envelope of $A \setminus F$ disjoint from F. Then $G^* \cap F^* = \emptyset$ so

$$\nu(G^* \setminus W) = \nu G^* = \mu G > 0$$

and there is a non-empty open-and-closed $V \subseteq G^* \setminus W$; let $H \in \Sigma$ be such that $H \subseteq G$ and $V = H^*$. In this case, $R[V]$ is closed and $\mu(H \setminus R[V]) = 0$, by (a), so that $H \cap R[V]$ is measurable, not negligible, and included in G. But $H \cap R[V] \cap A$ is empty, because $V \cap R^{-1}[A]$ is empty, so $\mu^*(H \cap R[V] \cap A) < \mu(H \cap R[V])$, and G cannot be a measurable envelope of $A \setminus F$. **X**

Thus $\nu^* R^{-1}[A] = \mu^* A$, as claimed.

(f) Suppose now that $(X, \mathfrak{T})$ is regular.

(i) If $G \subseteq X$ is open, $R^{-1}[G] \cap R^{-1}[X \setminus G] = \emptyset$. **P** If $z \in R^{-1}[G]$, then there is an $x \in G$ such that $(z,x) \in R$. Let H be an open set containing x such that $\overline{H} \subseteq G$. Then $x \notin X \setminus H$ so $z \notin (X \setminus H)^*$, that is, $z \in H^*$. But

$$R[H^*] \subseteq R[\overline{H}^*] \subseteq \overline{H} \subseteq G,$$

so $\overline{H}^* \cap R^{-1}[X \setminus G] = \emptyset$ and $z \notin R^{-1}[X \setminus G]$. **Q**

(ii) It is easy to check that

$$\mathcal{E} = \{E : E \subseteq X,\ R^{-1}[E] \cap R^{-1}[X \setminus E] = \emptyset\} = \{E : E \subseteq X,\ R^{-1}[X \setminus E] = Q \setminus R^{-1}[E]\}$$

is a σ-algebra of subsets of X (indeed, an algebra closed under arbitrary unions), just because $R \subseteq Z \times X$ and $R^{-1}[X] = Q$. Because it contains all open sets, $\mathcal{E}$ must contain all Borel sets.

(iii) Now suppose once again that $G \subseteq X$ is open and that $z \in R^{-1}[G]$. As in (i) above, there is an open set $H \subseteq G$ such that $z \in H^* \subseteq Z \setminus R^{-1}[X \setminus G]$, so that $z \in H^* \cap Q \subseteq R^{-1}[G]$. As z is arbitrary, $R^{-1}[G]$ is relatively open in Q.

(iv) Finally, if $F \subseteq X$ is closed, $R^{-1}[F] = Q \setminus R^{-1}[X \setminus F]$ is relatively closed in Q.

415R Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Hausdorff quasi-Radon measure space and $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ the Stone space of its measure algebra. Let $R \subseteq Z \times X$ be the relation described in 415Q. Then

(a) R is (the graph of) a function f;

(b) f is inverse-measure-preserving for the subspace measure ν_Q on $Q = \operatorname{dom} f$, and in fact μ is the image measure $\nu_Q f^{-1}$;

(c) if $(X, \mathfrak{T})$ is regular, then f is continuous.

proof (a) If $z \in Z$ and x, $y \in X$ are distinct, let G, H be disjoint open sets containing x, y respectively. Then

$$(X \setminus G)^* \cup (X \setminus H)^* = ((X \setminus G) \cup (X \setminus H))^* = Z,$$

defining * as in 415Q, so z must belong to at least one of $(X \setminus G)^*$, $(X \setminus H)^*$. In the former case $(z, x) \notin R$ and in the latter case $(z, y) \notin R$. This shows that R is a function; to remind us of its new status I will henceforth call it f. The domain of f is just $Q = R^{-1}[X]$.

(b) By 415Qd, f is inverse-measure-preserving for ν_Q and μ. Suppose that $A \subseteq X$ and $f^{-1}[A]$ is in the domain T_Q of ν_Q, that is, is of the form $Q \cap U$ for some $U \in \mathrm{T}$. Take any $E \in \Sigma$ such that $\mu E > 0$; then either $\nu(E^* \cap U) > 0$ or $\nu(E^* \setminus U) > 0$. (α) Suppose that $\nu(E^* \cap U) > 0$. Because ν is inner regular with respect to the open-and-closed sets, there is an $H \in \Sigma$ such that $H^* \subseteq E^* \cap U$ and $\mu H = \nu H^* > 0$. Now there is a closed set $F \subseteq E \cap H$ with $\mu F > 0$. In this case, $f[F^*] \subseteq F \subseteq E$, by 415Qb, while $F^* \cap U \subseteq f^{-1}[A]$, so $f[F^*] \subseteq E \cap A$. But this means that

$$\mu_*(E \cap A) \geq \mu f[F^*] = \mu F > 0.$$

(β) If $\nu(E^* \setminus U) > 0$, then the same arguments show that $\mu_*(E \setminus A) > 0$. (γ) Thus $\mu_*(E \cap A) + \mu_*(E \setminus A) > 0$ whenever $\mu E > 0$. Because μ is complete and locally determined, $A \in \Sigma$ (413F(vii)).

Thus we see that $\{A : A \subseteq X,\ f^{-1}[A] \in \mathrm{T}_Q\}$ is included in Σ, and μ is the image measure $\nu_Q f^{-1}$.

(c) If $\mathfrak{T}$ is regular, then 415Qf tells us that f is continuous.

415X Basic exercises >(a) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and $E \in \Sigma$ an atom for the measure. Show that there is a closed set $F \subseteq E$ such that $\mu F > 0$ and F is an atom of Σ, in the sense that the only measurable subsets of F are $\emptyset$ and F. (*Hint*: 414G.) Show that μ is atomless iff all countable subsets of X are negligible.

(b) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be any family of quasi-Radon measure spaces. Show that the direct sum measure μ on $X = \{(x, i) : i \in I,\ x \in X_i\}$ is a quasi-Radon measure when X is given its disjoint union topology.

(c) Let $\mathfrak{S}$ be the **right-facing Sorgenfrey topology** or **lower limit topology** on $\mathbb{R}$, that is, the topology generated by the half-open intervals of the form $[\alpha, \beta[$. Show that Lebesgue measure is completion regular and quasi-Radon for $\mathfrak{S}$. (*Hint*: 114Yj or 221Yb, or 419L.)

(d) Let X be a topological space and μ a complete measure on X, and suppose that there is a conegligible closed measurable set $Y \subseteq X$ such that the subspace measure on Y is quasi-Radon. Show that μ is quasi-Radon.

(e) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space. Show that μ is inner regular with respect to the family of self-supporting closed sets included in open sets of finite measure.

(f) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space. Show that for any $E \in \Sigma$, $\epsilon > 0$ there is an open set G such that $\mu G \leq \mu E + \epsilon$ and $E \setminus G$ is negligible.

(g) Find a compact Hausdorff quasi-Radon measure space which is not σ-finite.

(h) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an atomless quasi-Radon measure space which is outer regular with respect to the open sets. Show that it is σ-finite. (*Hint*: if not, take a decomposition $\langle X_i \rangle_{i \in I}$ in which every X_i except one is self-supporting, and a set A meeting every X_i in just one point.)

(i) Let (X, Σ, μ) be a σ-finite measure space in which Σ is countably generated as a σ-algebra. Show that, for a suitable topology on X, the completion of μ is a quasi-Radon measure. (*Hint*: take the topology generated by a countable subalgebra of Σ, and use the arguments of 415D.)

(j) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of quasi-Radon probability spaces such that every μ_i is strictly positive, and λ the product measure on $X = \prod_{i \in I} X_i$. Show that if every $\mathfrak{T}_i$ has a countable network, λ is a quasi-Radon measure.

(k) Let $\langle X_i \rangle_{i \in I}$ be a family of separable metrizable spaces, and μ a quasi-Radon measure on $X = \prod_{i \in I} X_i$. Show that μ is completion regular iff every self-supporting closed set in X is determined by coordinates in a countable set. (*Hint*: 4A2Eb.)

(l) Find two quasi-Radon measures μ, ν on the unit interval such that $\mu G \leq \nu G$ for every open set G but there is a closed set F such that $\nu F < \mu F$.

(m) Let X be a topological space and μ, ν two quasi-Radon measures on X. (i) Suppose that $\mu F = \nu F$ whenever $F \subseteq X$ is closed and both μF and νF are finite. Show that $\mu = \nu$. (ii) Suppose that $\mu G = \nu G$ whenever $G \subseteq X$ is open and both μG and νG are finite. Show that $\mu = \nu$.

(n) In 415L, write $\tilde{\mu}_0$ for the c.l.d. version of μ_0 (213E). Show that μ extends $\tilde{\mu}_0$. Show that $\tilde{\mu}_0$ is τ-additive and inner regular with respect to the closed sets.

(o) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite paracompact Hausdorff quasi-Radon measure space, and $f \in \mathcal{L}^0(\mu)$ a locally integrable function. Show that for any $\epsilon > 0$ there is a continuous function $g : X \to \mathbb{R}$ such that $\int |f - g| \leq \epsilon$.

>(p) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite completely regular quasi-Radon measure space. (i) Show that for every $E \in \Sigma$ there is an F in the Baire σ-algebra $\mathcal{Ba}(X)$ of X such that $\mu(E \triangle F) = 0$. (*Hint*: start with an open set E of finite measure.) (ii) Show that for every Σ-measurable function $f : X \to \mathbb{R}$ there is a $\mathcal{Ba}(X)$-measurable function equal almost everywhere to f.

(q) Let (X, Σ, μ) be a measure space and f a μ-integrable real-valued function. Show that there is a unique quasi-Radon measure λ on $\mathbb{R}$ such that $\lambda\{0\} = 0$ and $\lambda[\alpha, \infty[= \mu^*\{x : x \in \operatorname{dom} f,\ f(x) \geq \alpha\}$, $\lambda]-\infty, -\alpha] = \mu^*\{x : x \in \operatorname{dom} f,\ f(x) \leq -\alpha\}$ whenever $\alpha > 0$; and that $\int h \, d\lambda = \int hf d\mu$ whenever $h \in \mathcal{L}^0(\lambda)$ and $h(0) = 0$ and either integral is defined in $[-\infty, \infty]$. (*Hint*: set $\lambda E = \mu^* f^{-1}[E \setminus \{0\}]$ for Borel sets $E \subseteq \mathbb{R}$, and use 414Mb, 414O and 235Ib.)

415Y Further exercises (a) Give an example of two quasi-Radon measures μ, ν on $\mathbb{R}$ such that their sum, as defined in 112Xe, is not effectively locally finite, therefore not a quasi-Radon measure.

(b) Show that any quasi-Radon measure space is isomorphic, as topological measure space, to a subspace of a compact quasi-Radon measure space. (*Hint*: if X is a T_1 quasi-Radon measure space, let $\hat{X}$ be its Wallman compactification (Engelking 89, 3.6.21).)

(c) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space. Show that the following are equiveridical: (i) μ is outer regular with respect to the open sets; (ii) every negligible subset of X is included in an open set of finite measure; (iii) $\{x : \mu\{x\} = 0\}$ can be covered by a sequence of open sets of finite measure.

(d) Show that $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous for the right-facing Sorgenfrey topology.

(e) Let $r \geq 1$. On $\mathbb{R}^r$ let $\mathfrak{S}$ be the topology generated by the half-open intervals $[a, b[$ where $a, b \in \mathbb{R}^r$ (as defined in §115). (i) Show that $\mathfrak{S}$ is the product topology if each factor is given the right-facing Sorgenfrey topology (415Xc). (ii) Show that Lebesgue measure is quasi-Radon for $\mathfrak{S}$. (*Hint*: induce on r. See also 417Yi.)

(f) Let $Y \subseteq [0, 1]$ be a set of full outer measure and zero inner measure for Lebesgue measure μ. Give $[0, 1]$ the topology $\mathfrak{T}$ generated by $\mathfrak{S} \cup \{Y\}$ where $\mathfrak{S}$ is the usual topology. Show that the subspace measure $\nu = \mu_Y$ is quasi-Radon for the subspace topology $\mathfrak{T}_Y$, but that there is no measure λ on X which is quasi-Radon for $\mathfrak{T}$ and such that the subspace measure λ_Y is equal to ν.

(g) Find a base $\mathcal{U}$ for the topology of $X = \{0, 1\}^{\mathbb{N}}$ and two totally finite (quasi-)Radon measures μ, ν on X such that $G \cap H \in \mathcal{U}$ for all G, $H \in \mathcal{U}$, $\mu G \leq \nu G$ for every $G \in \mathcal{U}$, but $\nu X < \mu X$.

(h) Let X be a topological space and $\mathcal{G}$ an open cover of X. Suppose that for each $G \in \mathcal{G}$ we are given a quasi-Radon measure μ_G on G such that $\mu_G(U) = \mu_H(U)$ whenever G, $H \in \mathcal{G}$ and $U \subseteq G \cap H$ is open. Show that there is a unique quasi-Radon measure on X such that each μ_G is the subspace measure on G. (*Hint*: if $\langle \mu_G \rangle_{G \in \mathcal{G}}$ is a maximal family with the given properties, then $\mathcal{G}$ is upwards-directed.)

(i) Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X, and suppose that there is a family $\mathcal{U} \subseteq \Sigma \cap \mathfrak{T}$ such that (i) $\mu U < \infty$ for every $U \in \mathcal{U}$ (ii) for every $U \in \mathcal{U}$, $\mathfrak{T} \cap \Sigma \cap \mathcal{P}U$ is a base for the subspace topology of U (iii) if $\mathcal{G}$ is an upwards-directed family in $\mathfrak{T} \cap \Sigma$ and $\bigcup \mathcal{G} \in \mathcal{U}$, then $\mu(\bigcup \mathcal{G}) = \sup_{G \in \mathcal{G}} \mu G$ (iv) μ is inner regular with respect to the closed sets (v) if $E \in \Sigma$ and $\mu E > 0$ then there is a $U \in \mathcal{U}$ such that $\mu(E \cap U) > 0$. Show that μ has an extension to a quasi-Radon measure on X.

(j) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space such that $\mathfrak{T}$ is normal (but not necessarily Hausdorff or regular). Show that if $1 \leq p < \infty$, $f \in \mathcal{L}^p(\mu)$ and $\epsilon > 0$, there is a bounded continuous function $g : X \to \mathbb{R}$ such that $\|f - g\|_p \leq \epsilon$ and $\{x : g(x) \neq 0\}$ has finite measure.

(k) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a completely regular quasi-Radon measure space and suppose that we are given a uniformity defining the topology $\mathfrak{T}$. Show that if $1 \leq p < \infty$, $f \in \mathcal{L}^p(\mu)$ and $\epsilon > 0$, there is a bounded uniformly continuous function $g : X \to \mathbb{R}$ such that $\|f - g\|_p \leq \epsilon$ and $\{x : g(x) \neq 0\}$ has finite measure.

(l) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a completely regular quasi-Radon measure space and τ an extended Fatou norm on $L^0(\mu)$ such that (i) $\tau \restriction L^\tau$ is an order-continuous norm (ii) whenever $E \in \Sigma$ and $\mu E > 0$ there is an open set G such that $\mu(E \cap G) > 0$ and $\tau(\chi G^\bullet) < \infty$. Show that $L^\tau \cap \{f^\bullet : f : X \to \mathbb{R} \text{ is continuous}\}$ is norm-dense in L^τ.

(m) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space. Show that μ is a compact measure in the sense of §342 iff there is a locally compact topology $\mathfrak{S}$ on X such that $(X, \mathfrak{S}, \Sigma, \mu)$ is quasi-Radon.

415 Notes and comments 415B is particularly important because a very high proportion of the quasi-Radon measure spaces we study are actually subspaces of Radon measure spaces. I would in fact go so far as to say that when you have occasion to wonder whether all quasi-Radon measure spaces have a property, you should as a matter of habit look first at subspaces of Radon measure spaces; if the answer is affirmative for them, you will have most of what you want, even if the generalization to arbitrary quasi-Radon spaces gives difficulties. Of course the reverse phenomenon can also occur. Stone spaces (411P) can be thought of as quasi-Radon compactifications of Radon measure spaces (416V). But this is relatively rare. Indeed the reason why I give so few examples of quasi-Radon spaces at this point is just that the natural ones arise from Radon measure spaces. Note however that the quasi-Radon product of an uncountable family of Radon

probability spaces need not be Radon (see 417Xq), so that 415E here and 417O below are sources of non-Radon quasi-Radon measure spaces. Density and lifting topologies can also provide us with quasi-Radon measure spaces (453Xd, 453Xg).

415K is the second in a series of inner-regular-extension theorems; there will be a third in 416J.

I have been saying since Volume 1 that the business of measure theory, since Lebesgue's time, has been to measure as many sets and integrate as many functions as possible. I therefore take seriously any theorem offering a canonical extension of a measure. 415L and its corollaries can all be regarded as improvement theorems, showing that a good measure can be made even better. We have already had such improvement theorems in Chapter 21: the completion and c.l.d. version of a measure (212C, 213E). In all such theorems we need to know exactly what effect our improvement is having on the other constructions we are interested in; primarily, the measure algebra and the function spaces. The machinery of Chapter 36 shows that if we understand the measure algebra(s) involved then the function spaces will give us no further surprises. Completion of a measure does not affect the measure algebra at all (322Da). Taking the c.l.d. version does not change $\mathfrak{A}^f = \{a : \bar{\mu}a < \infty\}$ or L^1 (213Fc, 213G, 322Db, 366H), but can affect the rest of the measure algebra and therefore L^0 and L^∞. In this respect, what we might call the 'quasi-Radon version' behaves like the c.l.d. version (as could be expected, since the quasi-Radon version must itself be complete and locally determined; cf. 415Xn). The archetypal application of 415L is 415N. We shall see later how Baire measures arise naturally when studying Banach spaces of continuous functions (436E). 415N will be one of the keys to applying the general theory of topological measure spaces in such contexts. A virtue of Baire measures is that inner regularity with respect to closed sets comes almost free (412D); but there can be unsurmountable difficulties if we wish to extend them to Borel measures (439M), and it is important to know that τ-additivity, even in the relatively weak form allowed by the definition I use here (411C), is often enough to give a canonical extension to a well-behaved measure defined on every Borel set. In 415C we have inner regularity for a different reason, and the measure is already known to be defined on every Borel set, so in fact the quasi-Radon version of the measure is just the c.l.d. version (415Xd).

For a volume and a half I have neglected indefinite-integral measures, though they are mentioned in the exercises; but we shall need them later, and in 415O I spell out a result which it will be useful to be able to quote. The exact hypotheses are not perhaps instantly predictable; see 414Xe.

One interpretation of the Lifting Theorem is that for a complete strictly localizable measure space (X, Σ, μ) there is a function $g : X \to Z$, where Z is the Stone space of the measure algebra of μ, such that $E \triangle g^{-1}[E^*]$ is negligible for every $E \in \Sigma$, where $E^* \subseteq Z$ is the open-and-closed set corresponding to the image of E in the measure algebra (341Q). For a Hausdorff quasi-Radon measure space we have a function $f : Q \to X$, where Q is a dense subset of Z, such that $(Q \cap E^*) \triangle f^{-1}[E]$ is negligible for every $E \in \Sigma$ (415Qd, 415R); moreover, there is a canonical construction for this function. For completeness' sake, I have given the result for general, not necessarily Hausdorff, spaces X (415Q); but evidently it will be of greatest interest for regular Hausdorff spaces (415Rc), especially if they happen to be 'compact' in the sense of §342. Perhaps I should remark that in the most important applications, Q is the whole of Z (416Xw). Of course the question arises, whether fg can be the identity. (Z typically has larger cardinal than X, so asking for gf to be the identity is a bit optimistic.) This is in fact an important question; I will return to it in 453M.

416 Radon measure spaces

We come now to the results for which the chapter so far has been preparing. The centre of topological measure theory is the theory of 'Radon' measures (411Hb), measures inner regular with respect to compact sets. Most of the section is devoted to pulling the earlier work together, and in particular to re-stating theorems on quasi-Radon measures in the new context. Of course this has to begin with a check that Radon measures are quasi-Radon (416A). It follows immediately that Radon measures are (strictly) localizable (416B). After presenting a miscellany of elementary facts, I turn to the constructions of §413, which take on simpler and more dramatic forms in this context (416J-416P). I proceed to investigate subspace measures (416R-416T) and some special product measures (416U). I end the section with further notes on the forms which earlier theorems on Stone spaces (416V) and compact measure spaces (416W) take when applied to Radon measure spaces.

416A Proposition A Radon measure space is quasi-Radon.

proof Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space. Because $\mathfrak{T}$ is Hausdorff, every compact set is closed, so μ is inner regular with respect to the closed sets. By 411E, μ is τ-additive; by 411Gf, it is effectively locally finite. Thus all parts of condition (ii) of 411Ha are satisfied, and μ is a quasi-Radon measure.

416B Corollary A Radon measure space is strictly localizable.

proof Put 416A and 415A together.

416C In order to use the results of §415 effectively, it will be helpful to spell out elementary conditions ensuring that a quasi-Radon measure is Radon.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a locally finite Hausdorff quasi-Radon measure space. Then the following are equiveridical:

(i) μ is a Radon measure;

(ii) whenever $E \in \Sigma$ and $\mu E > 0$ there is a compact set K such that $\mu(E \cap K) > 0$;

(iii) $\sup\{K^\bullet : K \subseteq X \text{ is compact}\} = 1$ in the measure algebra of μ.

If μ is totally finite we can add

(iv) $\sup\{\mu K : K \subseteq X \text{ is compact}\} = \mu X$.

proof (i)⇒(ii) and (ii) ⟺ (iii) are trivial. For (ii)⇒(i), observe that if $E \in \Sigma$ and $\mu E > 0$ there is a compact set $K \subseteq E$ such that $\mu K > 0$. **P** There is a compact set K' such that $\mu(E \cap K') > 0$, by hypothesis; now there is a closed set $K \subseteq E \cap K'$ such that $\mu K > 0$, because μ is inner regular with respect to the closed sets, and K is compact. **Q** By 412B, μ is tight. Being a complete, locally determined, locally finite topological measure, it is a Radon measure.

When $\mu X < \infty$, of course, we also have (ii) ⟺ (iv).

416D Some further elementary facts are worth writing out plainly.

Lemma (a) In a Radon measure space, every compact set has finite measure.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $E \subseteq X$ a set such that $E \cap K \in \Sigma$ for every compact $K \subseteq X$. Then $E \in \Sigma$.

(c) A Radon measure is inner regular with respect to the self-supporting compact sets.

(d) Let X be a Hausdorff space and μ a locally finite complete locally determined measure on X which is tight (that is, inner regular with respect to the compact sets). If every compact set belongs to the domain of μ, μ is a Radon measure.

proof (a) 411Ga.

(b) We have only to remember that μ is complete, locally determined and tight, and apply 413F(ii).

(c) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a Radon measure space, $E \in \Sigma$ and $\gamma < \mu E$, there is a compact set $K \subseteq E$ such that $\mu K \geq \gamma$. By 414F, there is a self-supporting relatively closed set $L \subseteq K$ such that $\mu L = \mu K$; but now of course L is compact, while $L \subseteq E$ and $\mu L \geq \gamma$.

(d) Let $\mathcal{K}$ be the family of compact subsets of X; write Σ for the domain of μ. If $F \subseteq X$ is closed, then $F \cap K \in \mathcal{K} \subseteq \Sigma$ for every $K \in \mathcal{K}$; accordingly $F \in \Sigma$. But this means that every closed set, therefore every open set, belongs to Σ, and μ is a Radon measure.

416E Specification of Radon measures In 415H I described some conditions which enable us to be sure that two quasi-Radon measures on a given topological space are the same. In the case of Radon measures we have a similar list. This time I include a note on the natural ordering of Radon measures.

Proposition Let X be a Hausdorff space and μ, ν two Radon measures on X.

(a) The following are equiveridical:

(i) $\operatorname{dom}\nu \subseteq \operatorname{dom}\mu$ and $\mu E \leq \nu E$ for every $E \in \operatorname{dom}\nu$;

(ii) $\mu K \leq \nu K$ for every compact set $K \subseteq X$.

(iii) $\mu G \leq \nu G$ for every open set $G \subseteq X$;
(iv) $\mu F \leq \nu F$ for every closed set $F \subseteq X$.

If X is locally compact, we can add

(v) $\int f d\mu \leq \int f d\nu$ for every non-negative continuous function $f : X \to \mathbb{R}$ with compact support.

(b) The following are equiveridical:
(i) $\mu = \nu$;
(ii) $\mu K = \nu K$ for every compact set $K \subseteq X$.
(iii) $\mu G = \nu G$ for every open set $G \subseteq X$;
(iv) $\mu F = \nu F$ for every closed set $F \subseteq X$.

If X is locally compact, we can add

(v) $\int f d\mu = \int f d\nu$ for every continuous function $f : X \to \mathbb{R}$ with compact support.

proof (a)(i)⇒(iv)⇒(ii) and **(i)⇒(iii)** are trivial.

(ii)⇒(i) If (ii) is true, then

$$\mu E = \sup_{K \subseteq E \text{ is compact}} \mu K \leq \sup_{K \subseteq E \text{ is compact}} \nu K = \nu E$$

for every set E measured by both μ and ν. Also $\operatorname{dom}\nu \subseteq \operatorname{dom}\mu$. **P** Suppose that $E \in \operatorname{dom}\nu$ and that $K \subseteq X$ is a compact set such that $\mu K > 0$. Then there are compact sets $K_1 \subseteq K \cap E$, $K_2 \subseteq K \setminus E$ such that

$$\nu K_1 + \nu K_2 \geq \nu(K \cap E) + \nu(K \setminus E) - \mu K = \nu K - \mu K.$$

So

$$\mu(K \setminus (K_1 \cup K_2)) \leq \nu(K \setminus (K_1 \cup K_2)) < \mu K$$

and $\mu K_1 + \mu K_2 > 0$. This shows that $\mu_*(K \cap E) + \mu_*(K \setminus E) > 0$. As K is arbitrary, $E \in \operatorname{dom}\mu$ (413F(vii)). **Q**

So (i) is true.

(iii)⇒(ii) The point is that if $K \subseteq X$ is compact, then $\mu K = \inf\{\mu G : G \subseteq X \text{ is open}, K \subseteq G\}$. **P** Because $X = \bigcup\{\mu G : G \subseteq X \text{ is open}, \mu G < \infty\}$, there is an open set G_0 of finite measure including K. Now, for any $\gamma > \mu K$, there is a compact set $L \subseteq G_0 \setminus K$ such that $\mu L \geq \mu G_0 - \gamma$, so that $\mu G \leq \gamma$, where $G = G_0 \setminus L$ is an open set including K. **Q**

The same is true for ν. So, if (iii) is true,

$$\mu K = \inf_{G \supseteq K \text{ is open}} \mu G \leq \inf_{G \supseteq K \text{ is open}} \nu G = \nu K$$

for every compact $K \subseteq X$, and (ii) is true.

(iii)⇒(v) If (iii) is true and $f : X \to [0, \infty[$ is a non-negative continuous function, then

$$\int f d\mu = \int_0^\infty \mu\{x : f(x) > t\} dt$$

(252O)

$$\leq \int_0^\infty \nu\{x : f(x) > t\} dt = \int f d\nu.$$

(v)⇒(iii) If X is locally compact and (v) is true, take any open set $G \subseteq X$, and consider

$$A = \{f : f \text{ is a continuous function with compact support from } X \text{ to } [0,1] \text{ and } f \leq \chi G\}.$$

Then A is upwards-directed and $\sup_{f \in A} f(x) = \chi G(x)$ for every $x \in X$, by 4A2G(e-i). So

$$\mu G = \sup_{f \in A} \int f d\mu \leq \sup_{f \in A} \int f d\nu = \nu G$$

by 414Ba. As G is arbitrary, (iii) is true.

(b) now follows at once, or from 415H.

416F Proposition Let X be a Hausdorff space and μ a Borel measure on X. Then the following are equiveridical:

(i) μ has an extension to a Radon measure on X;

(ii) μ is locally finite and tight;

(iii) μ is locally finite and effectively locally finite, and $\mu G = \sup\{\mu K : K \subseteq G \text{ is compact}\}$ for every open set $G \subseteq X$;

(iv) μ is locally finite, effectively locally finite and τ-additive, and $\mu G = \sup\{\mu(G \cap K) : K \subseteq X \text{ is compact}\}$ for every open set $G \subseteq X$.

In this case the extension is unique; it is the c.l.d. version of μ.

proof (a)(i)⇒(iv) If $\mu = \tilde{\mu}\restriction\mathcal{B}$ where $\tilde{\mu}$ is a Radon measure and $\mathcal{B}$ is the Borel σ-algebra of X, then of course μ is locally finite and effectively locally finite and τ-additive because $\tilde{\mu}$ is (see 416A) and every open set belongs to $\mathcal{B}$. Also

$$\mu G = \sup\{\mu K : K \subseteq G \text{ is compact}\} \leq \sup\{\mu(G \cap K) : K \subseteq X \text{ is compact}\} \leq \mu G$$

for every open set $G \subseteq X$, because $\tilde{\mu}$ is tight and compact sets belong to $\mathcal{B}$.

(b)(iv)⇒(iii) Suppose that (iv) is true. Of course μ is locally finite and effectively locally finite. Suppose that $G \subseteq X$ is open and that $\gamma < \mu G$. Then there is a compact $K \subseteq X$ such that $\mu(G \cap K) > \gamma$. By 414K, the subspace measure μ_K is τ-additive. Now K is a compact Hausdorff space, therefore regular. By 414Ma there is a closed set $F \subseteq G \cap K$ such that $\mu_K F \geq \gamma$. Now F is compact, $F \subseteq G$ and $\mu F \geq \gamma$. As G and γ are arbitrary, (iii) is true.

(c)(iii)⇒(ii) I have to show that if μ satisfies the conditions of (iii) it is tight. Let $\mathcal{K}$ be the family of compact subsets of X and $\mathcal{A}$ the family of subsets of X which are either open or closed. Then whenever $A \in \mathcal{A}$, $F \in \Sigma$ and $\mu(A \cap F) > 0$, there is a $K \in \mathcal{K}$ such that $K \subseteq A$ and $\mu(K \cap F) > 0$. **P** Because μ is effectively locally finite, there is an open set G of finite measure such that $\mu(G \cap A \cap F) > 0$. (α) If A is open, then there will be a compact set $K \subseteq G \cap A$ such that $\mu K > \mu(G \cap A) - \mu(G \cap A \cap F)$, so that $\mu(K \cap F > 0$. (β) If A is closed, then let $L \subseteq G$ be a compact set such that $\mu L > \mu G - \mu(G \cap A \cap F)$; then $K = L \cap A$ is compact and $\mu(K \cap F) > 0$. **Q**

By 412C, μ is inner regular with respect to $\mathcal{K}$, as required.

(d)(ii)⇒(i) If μ is locally finite and tight, let $\tilde{\mu}$ be the c.l.d. version of μ. Then $\tilde{\mu}$ is complete, locally determined, locally finite (because μ is), a topological measure (because μ is) and tight (because μ is, using 412Ha); so is a Radon measure. Every compact set has finite measure for μ, so μ is semi-finite and $\tilde{\mu}$ extends μ (213Hc).

(e) By 416Eb there can be at most one Radon measure extending μ, and we have observed in (c) above that in the present case it is the c.l.d. version of μ.

416G One of the themes of §434 will be the question: on which Hausdorff spaces is every locally finite quasi-Radon measure a Radon measure? I do not think we are ready for a general investigation of this, but I can give one easy special result.

Proposition Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space and μ a locally finite quasi-Radon measure on X. Then μ is a Radon measure.

proof μ satisfies condition (ii) of 416C. **P** Take $E \in \operatorname{dom}\mu$ such that $\mu E > 0$. Let $\mathcal{G}$ be the family of relatively compact open subsets of X; then $\mathcal{G}$ is upwards-directed and has union X. By 414Ea, there is a $G \in \mathcal{G}$ such that $\mu(E \cap G) > 0$. But now $\overline{G}$ is compact and $\mu(E \cap \overline{G}) > 0$. **Q** By 416C, μ is a Radon measure.

416H Corollary Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space, and μ a locally finite, effectively locally finite, τ-additive Borel measure on X. Then μ is tight and its c.l.d. version is a Radon measure, the unique Radon measure on X extending μ.

proof By 415Cb, the c.l.d. version $\tilde{\mu}$ of μ is a quasi-Radon measure extending μ. Because μ is locally finite, so is $\tilde{\mu}$; by 416G, $\tilde{\mu}$ is a Radon measure. By 416Eb, the extension is unique. Now

$$\mu E = \tilde{\mu} E = \sup_{K \subseteq E \text{ is compact}} \tilde{\mu} K = \sup_{K \subseteq E \text{ is compact}} \mu K$$

for every Borel set $E \subseteq X$, so μ itself is tight.

416I While on the subject of locally compact spaces, I mention an important generalization of a result from Chapter 24.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a locally compact Radon measure space. Write C_k for the space of continuous real-valued functions on X with compact supports. If $1 \leq p < \infty$, $f \in \mathcal{L}^p(\mu)$ and $\epsilon > 0$, there is a $g \in C_k$ such that $\|f - g\|_p \leq \epsilon$.

proof By 415Pa, there is a bounded continuous function $h_1 : X \to \mathbb{R}$ such that $G = \{x : h_1(x) \neq 0\}$ has finite measure and $\|f - h_1\|_p \leq \frac{1}{2}\epsilon$. Let $K \subseteq G$ be a compact set such that $\|h_1\|_\infty (\mu(G \setminus K))^{1/p} \leq \frac{1}{2}\epsilon$, and let $h_2 \in C_k$ be such that $\chi K \leq h_2 \leq \chi G$ (4A2G(e-i)). Set $g = h_1 \times h_2$. Then $g \in C_k$ and

$$\int |h_1 - g|^p \leq \int_{G \setminus K} |h_1|^p \leq \mu(G \setminus K) \|h_1\|_\infty^p,$$

so $\|h_1 - g\|_p \leq \frac{1}{2}\epsilon$ and $\|f - g\|_p \leq \epsilon$, as required.

416J I turn now to constructions of Radon measures based on ideas in §413.

Theorem Let X be a Hausdorff space. Let $\mathcal{K}$ be the family of compact subsets of X and $\phi_0 : \mathcal{K} \to [0, \infty[$ a functional such that

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K \in \mathcal{K}, K' \subseteq K \setminus L\}$ whenever $K, L \in \mathcal{K}$ and $L \subseteq K$,

(γ) for every $x \in X$ there is an open set G containing x such that $\sup\{\phi_0 K : K \in \mathcal{K}, K \subseteq G\}$ is finite.

Then there is a unique Radon measure on X extending ϕ_0.

proof By 413M, there is a unique complete locally determined measure μ on X, extending ϕ_0, which is inner regular with respect to $\mathcal{K}$. By (γ), μ is locally finite; by 416Dd, it is a Radon measure.

416K Proposition Let X be a regular Hausdorff space. Let $\mathcal{K}$ be the family of compact subsets of X, and $\phi_0 : \mathcal{K} \to [0, \infty[$ a functional such that

(α_1) $\phi_0 K \leq \phi_0(K \cup L) \leq \phi_0 K + \phi_0 L$ for all $K, L \in \mathcal{K}$,

(α_2) $\phi_0(K \cup L) = \phi_0 K + \phi_0 L$ whenever $K, L \in \mathcal{K}$ and $K \cap L = \emptyset$,

(γ) for every $x \in X$ there is an open set G containing X such that $\sup\{\phi_0 K : K \in \mathcal{K}, K \subseteq G\} < \infty$.

Then there is a unique Radon measure μ on X such that

$$\mu K = \inf_{G \subseteq X \text{ is open}, K \subseteq G} \sup_{L \subseteq G \text{ is compact}} \phi_0 L$$

for every $K \in \mathcal{K}$.

proof (a) For open sets $G \subseteq X$ set

$$\psi G = \sup_{L \in \mathcal{K}, L \subseteq G} \phi_0 L,$$

and for compact sets $K \subseteq X$ set

$$\phi_1 K = \inf\{\psi G : G \subseteq X \text{ is open}, K \subseteq G\}.$$

Evidently $\psi G \leq \psi H$ whenever $G \subseteq H$. We need to know that $\psi(G \cup H) \leq \psi G + \psi H$ for all open sets G, $H \subseteq X$. **P** If $L \subseteq G \cup H$ is compact, then the disjoint compact sets $L \setminus G$, $L \setminus H$ can be separated by disjoint open sets H', G' (4A2F(h-i)); now $L \setminus G' \subseteq H$, $L \setminus H' \subseteq G$ are compact and cover L, so

$$\phi_0 L \leq \phi_0(L \setminus G') + \phi_0(L \setminus H') \leq \psi H + \psi G.$$

As L is arbitrary, $\psi(G \cup H) \leq \psi G + \psi H$. **Q**

Moreover, $\psi(G \cup H) = \psi G + \psi H$ if $G \cap H = \emptyset$. **P** If $K \subseteq G$, $L \subseteq H$ are compact, then

$$\phi_0 K + \phi_0 L = \phi_0(K \cup L) \leq \psi(G \cup H).$$

As K and L are arbitrary, $\psi G + \psi H \leq \psi(G \cup H)$. **Q**

(b) It follows that $\phi_1 K$ is finite for every compact $K \subseteq X$. **P** Set $\mathcal{G} = \{G : G \subseteq X \text{ is open}, \psi G < \infty\}$. Then (a) tells us that $\mathcal{G}$ is upwards-directed. But also we are supposing that $\mathcal{G}$ covers X, by (γ). So if $K \subseteq X$ is compact there is a member of $\mathcal{G}$ including K and $\phi_1 K < \infty$. **Q**

(c) Now ϕ_1 satisfies the conditions of 416J.

P(α) Suppose that K, $L \in \mathcal{K}$ and $L \subseteq K$. Set $\gamma = \sup\{\phi_1 M : M \in \mathcal{K}, M \subseteq K \setminus L\}$. Take any $\epsilon > 0$.

Let G be an open set such that $K \subseteq G$ and $\psi G \leq \phi_1 K + \epsilon$. If $M \in \mathcal{K}$ and $M \subseteq K \setminus L$, there are disjoint open sets U, V such that $L \subseteq U$ and $M \subseteq V$ (4A2F(h-i) again); we may suppose that $U \cup V \subseteq G$. In this case,

$$\phi_1 L + \phi_1 M \leq \psi U + \psi V = \psi(U \cup V)$$

(by the second part of (a) above)

$$\leq \psi G \leq \phi_1 K + \epsilon.$$

As M is arbitrary, $\gamma \leq \phi_1 K - \phi_1 L + \epsilon$.

On the other hand, there is an open set H such that $L \subseteq H$ and $\psi H \leq \phi_1 L + \epsilon$. Set $F = K \setminus H$, so that F is a compact subset of $K \setminus L$. Then there is an open set V such that $F \subseteq V$ and $\psi V \leq \phi_1 F + \epsilon$. In this case $K \subseteq H \cup V$, so

$$\begin{aligned}\phi_1 K &\leq \psi(H \cup V) \leq \psi H + \psi V \\ &\leq \phi_1 L + \phi_1 F + 2\epsilon \leq \phi_1 L + \gamma + 2\epsilon,\end{aligned}$$

so $\gamma \geq \phi_1 K - \phi_1 L - 2\epsilon$.

As ϵ is arbitrary, $\gamma = \phi_1 K - \phi_1 L$; as K and L are arbitrary, ϕ_1 satisfies condition (α) of 416J.

(γ) Any $x \in X$ is contained in an open set G such that $\psi G < \infty$; but now $\sup\{\phi_1 K : K \in \mathcal{K}, K \subseteq G\} \leq \psi G$ is finite. So ϕ_1 satisfies condition (γ) of 416J. **Q**

(d) By 416J, there is a unique Radon measure on X extending ϕ_1, as claimed.

416L Corollary Let X be a locally compact Hausdorff space. Let $\mathcal{K}$ be the family of compact subsets of X, and $\phi_0 : \mathcal{K} \to [0, \infty[$ a functional such that

$\phi_0 K \leq \phi_0(K \cup L) \leq \phi_0 K + \phi_0 L$ for all K, $L \in \mathcal{K}$,

$\phi_0(K \cup L) = \phi_0 K + \phi_0 L$ whenever K, $L \in \mathcal{K}$ and $K \cap L = \emptyset$.

Then there is a unique Radon measure μ on X such that

$$\mu K = \inf\{\phi_0 K' : K' \in \mathcal{K}, K \subseteq \operatorname{int} K'\}$$

for every $K \in \mathcal{K}$.

proof Observe that ϕ_0 satisfies the conditions of 416K; 416K(γ) is true because X is locally compact. Define ψ, ϕ_1 as in the proof of 416K, and set

$$\phi_1' K = \inf\{\phi_0 K' : K' \in \mathcal{K}, K \subseteq \operatorname{int} K'\}$$

for every $K \in \mathcal{K}$. Then $\phi_1' = \phi_1$. **P** Let $K \in \mathcal{K}$, $\epsilon > 0$. (i) There is an open set $G \subseteq X$ such that $K \subseteq G$ and $\psi G \leq \phi_1 K + \epsilon$. Now the relatively compact open subsets with closures included in G form an upwards-directed cover of K, so there is a $K' \in \mathcal{K}$ such that $K \subseteq \operatorname{int} K'$ and $K' \subseteq G$. Accordingly

$$\phi_1' K \leq \phi_0 K' \leq \psi G \leq \phi K + \epsilon.$$

(ii) There is an $L \in \mathcal{K}$ such that $K \subseteq \operatorname{int} L$ and $\phi_0 L \leq \phi_1' K + \epsilon$, so that

$$\phi_1 K \leq \psi(\operatorname{int} L) \leq \phi_0 L \leq \phi_1' K + \epsilon.$$

(iii) As K, ϵ are arbitrary, $\phi_1' = \phi_1$. **Q**

Now 416K tells us that there is a unique Radon measure extending ϕ_1, and this is the measure we seek.

416M The extension theorems in the second half of §413 also have important applications to Radon measures.

Henry's Theorem (HENRY 69) Let X be a Hausdorff space and μ_0 a measure on X which is locally finite and tight. Then μ_0 has an extension to a Radon measure μ on X; and the extension may be made in such a way that whenever $\mu E < \infty$ there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$.

proof All the work has been done in §413; we need to check here only that the family $\mathcal{K}$ of compact subsets of X and the measure μ_0 satisfy the hypotheses of 413O. But (†) and (‡) there are elementary, and $\mu_0^* K < \infty$ for every $K \in \mathcal{K}$ by 411Ga.

Now take the measure μ from 413O. It is complete, locally determined and inner regular with respect to $\mathcal{K}$; also $\mathcal{K} \subseteq \operatorname{dom}\mu$. Because μ_0 is locally finite and μ extends μ_0, μ is locally finite. By 416Dd, μ is a Radon measure. And the construction of 413O ensures that every set of finite measure for μ differs from a member of Σ_0 by a μ-negligible set.

416N Proposition Let X be a Hausdorff space and T a subalgebra of $\mathcal{P}X$. Let $\nu : \mathrm{T} \to \mathbb{R}$ be a non-negative finitely additive functional. Then there is a Radon measure μ on X such that $\mu X \leq \nu X$ and $\mu K \geq \nu K$ for every compact set $K \in \mathrm{T}$.

proof Use 413S, with $\mathcal{K}$ the family of compact subsets of X, and 416Dd.

416O Theorem Let X be a Hausdorff space and T a subalgebra of $\mathcal{P}X$. Let $\nu : \mathrm{T} \to \mathbb{R}$ be a non-negative finitely additive functional such that

$\nu E = \sup\{\nu F : F \in \mathrm{T}, F \subseteq E, F \text{ is closed}\}$ for every $E \in \mathrm{T}$,

$\nu X = \sup_{K \subseteq X \text{ is compact}} \inf_{F \in \mathrm{T}, F \supseteq K} \nu F$.

Then there is a Radon measure μ on X extending ν.

proof (a) For $A \subseteq X$, write

$$\nu^* A = \inf_{F \in \mathrm{T}, F \supseteq A} \nu F.$$

Let $\langle K_n \rangle_{n \in \mathbb{N}}$ be a sequence of compact subsets of X such that $\lim_{n \to \infty} \nu^* K_n = \nu X$; replacing K_n by $\bigcup_{i<n} K_i$ if necessary, we may suppose that $\langle K_n \rangle_{n \in \mathbb{N}}$ is non-decreasing and that $K_0 = \emptyset$. For each $n \in \mathbb{N}$, set

$$\nu_n' E = \nu^*(E \cap K_n)$$

for every $E \in \mathrm{T}$. Then ν_n' is additive. **P** (I copy from the proof of 413N.) If E, $F \in \mathrm{T}$ and $E \cap F = \emptyset$,

$$\begin{aligned}
\nu_n'(E \cup F) &= \inf\{\nu G : G \in \mathrm{T}, K_n \cap (E \cup F) \subseteq G\} \\
&= \inf\{\nu G : G \in \mathrm{T}, K_n \cap (E \cup F) \subseteq G \subseteq E \cup F\} \\
&= \inf\{\nu(G \cap E) + \nu(G \cap F) : G \in \mathrm{T}, K_n \cap (E \cup F) \subseteq G \subseteq E \cup F\} \\
&= \inf\{\nu G_1 + \nu G_2 : G_1, G_2 \in \mathrm{T}, K_n \cap E \subseteq G_1 \subseteq E, K_n \cap F \subseteq G_2 \subseteq F\} \\
&= \inf\{\nu G_1 : G_1 \in \mathrm{T}, K_n \cap E \subseteq G_1 \subseteq E\} \\
&\qquad + \inf\{\nu G_2 : G_2 \in \mathrm{T}, K_n \cap F \subseteq G_2 \subseteq F\} \\
&= \nu_n' E + \nu_n' F.
\end{aligned}$$

As E and F are arbitrary, ν_n' is additive. **Q**

(b) For each $n \in \mathbb{N}$, set $\nu_n E = \nu_{n+1}' E - \nu_n' E$ for every $E \in \mathrm{T}$; then ν_n is additive. Because $K_{n+1} \supseteq K_n$, ν_n is non-negative.

If $E \in \mathrm{T}$ and $E \cap K_{n+1} = \emptyset$, then $\nu_n E = \nu_{n+1}' E = 0$. So if we set $\mathrm{T}_n = \{E \cap K_{n+1} : E \in \mathrm{T}\}$, we have an additive functional $\tilde{\nu}_n : \mathrm{T}_n \to [0, \infty[$ defined by setting $\tilde{\nu}_n(E \cap K_{n+1}) = \nu_n E$ for every $E \in \mathrm{T}$. Also $\tilde{\nu}_n H = \sup\{\tilde{\nu}_n K : K \in \mathrm{T}_n, K \subseteq H, K \text{ is compact}\}$ for every $H \in \mathrm{T}_n$. **P** Express H as $E \cap K_{n+1}$ where $E \in \mathrm{T}$. Given $\epsilon > 0$, there is a closed set $F \in \mathrm{T}$ such that $F \subseteq E$ and $\nu F \geq \nu E - \epsilon$; but now $K = F \cap K_{n+1} \in \mathrm{T}_n$ is a compact subset of H, and

$$\tilde{\nu}_n(H \setminus K) = \nu_n(E \setminus F) \leq \nu_{n+1}'(E \setminus F) \leq \nu(E \setminus F) \leq \epsilon,$$

so $\tilde\nu_n K \geq \tilde\nu_n H - \epsilon$. **Q**

(c) For each $n \in \mathbb{N}$, we have a Radon measure μ_n on K_{n+1}, with domain Σ_n say, such that $\mu_n K_{n+1} \leq \tilde\nu_n K_{n+1}$ and $\mu_n K \geq \tilde\nu_n K$ for every compact set $K \subseteq K_{n+1}$ (416N). Since K_{n+1} is itself compact, we must have $\mu_n K_{n+1} = \tilde\nu_n K_{n+1}$. But this means that μ_n extends $\tilde\nu_n$. **P** If $H \in \mathrm{T}_n$, $\epsilon > 0$ there is a compact set $K \in \mathrm{T}_n$ such that $\tilde\nu_n K \geq \tilde\nu_n H - \epsilon$, so that $(\mu_n)_* H \geq \mu_n K \geq \tilde\nu_n H - \epsilon$; as ϵ is arbitrary, $(\mu_n)_* H \geq \tilde\nu_n H$. So there is an $F_1 \in \Sigma_n$ such that $F_1 \subseteq H$ and $\mu_n F_1 \geq \tilde\nu_n H$. Similarly, there is an $F_2 \in \Sigma_n$ such that $F_2 \subseteq K_{n+1} \setminus H$ and $\mu_n F_2 \geq \tilde\nu_n(K_{n+1} \setminus H)$. But in this case $H \setminus F_1 \subseteq K_{n+1} \setminus (F_1 \cup F_2)$ is μ_n-negligible, because

$$\mu_n F_1 + \mu_n F_2 \geq \tilde\nu_n H + \tilde\nu_n(K_{n+1} \setminus H) = \tilde\nu_n K_{n+1} = \mu_n K_{n+1}.$$

So $H \setminus F_1$ and H belong to Σ_n and $\mu_n H = \mu_n F_1 = \tilde\nu_n H$. **Q**

(d) Set

$$\Sigma = \{E : E \subseteq X,\ E \cap K_{n+1} \in \Sigma_n \text{ for every } n \in \mathbb{N}\},$$

$$\mu E = \textstyle\sum_{n=0}^{\infty} \mu_n(E \cap K_{n+1}) \text{ for every } E \in \Sigma.$$

Then μ is a Radon measure on X extending ν.

P **(i)** It is easy to check that Σ is a σ-algebra of subsets of X including T, just because each Σ_n is a σ-algebra of subsets of K_{n+1} including T_n; and that μ is a complete measure because every μ_n is.

(ii) If $E \in \mathrm{T}$, then

$$\begin{aligned}\mu E &= \sum_{n=0}^{\infty} \mu_n(E \cap K_{n+1}) = \sum_{n=0}^{\infty} \tilde\nu_n(E \cap K_{n+1}) = \sum_{n=0}^{\infty} \nu_n E = \lim_{n\to\infty} \sum_{i=0}^{n} \nu_i E \\ &= \lim_{n\to\infty} \sum_{i=0}^{n} \nu^*(E \cap K_{i+1}) - \nu^*(E \cap K_i) = \lim_{n\to\infty} \nu^*(E \cap K_n) \leq \nu E.\end{aligned}$$

On the other hand,

$$\mu X = \lim_{n\to\infty} \nu^* K_{n+1} = \nu X,$$

so in fact $\mu E = \nu E$ for every $E \in \mathrm{T}$, that is, μ extends ν. In particular, μ is totally finite, therefore locally determined and locally finite.

(iii) If $G \subseteq X$ is open, then $G \cap K_{n+1} \in \Sigma_n$ for every n, so $G \in \Sigma$; thus μ is a topological measure. If $\mu E > 0$, there is some $n \in \mathbb{N}$ such that $\mu_n(E \cap K_{n+1}) > 0$; now there is a compact set $K \subseteq E \cap K_{n+1}$ such that $\mu_n K > 0$, so that $\mu K > 0$. This shows that μ is tight, so is a Radon measure, as required. **Q**

Remark Observe that in this construction

$$\begin{aligned}\mu K_{n+1} &= \sum_{i=0}^{\infty} \mu_i(K_{n+1} \cap K_{i+1}) = \sum_{i=0}^{\infty} \tilde\nu_i(K_{n+1} \cap K_{i+1}) = \sum_{i=0}^{\infty} \nu_i(K_{n+1} \cap K_{i+1}) \\ &= \sum_{i=0}^{\infty} \nu'_{i+1}(K_{n+1} \cap K_{i+1}) - \nu'_i(K_{n+1} \cap K_{i+1}) \\ &= \sum_{i=0}^{\infty} \nu^*(K_{n+1} \cap K_{i+1}) - \nu^*(K_{n+1} \cap K_i) \\ &= \sum_{i=0}^{n} \nu^*(K_{n+1} \cap K_{i+1}) - \nu^*(K_{n+1} \cap K_i) = \nu^* K_{n+1}\end{aligned}$$

for every $n \in \mathbb{N}$. What this means is that if instead of the hypothesis

$$\nu X = \sup_{K \subseteq X \text{ is compact}} \inf_{F \in \mathrm{T}, F \supseteq K} \nu F$$

we are presented with a specified non-decreasing sequence $\langle L_n \rangle_{n\in\mathbb{N}}$ of compact subsets of X such that $\nu X = \sup_{n\in\mathbb{N}} \nu^* L_n$, then we can take $K_{n+1} = L_n$ in the argument above and we shall have $\mu L_n = \nu^* L_n$ for every n.

416P Theorem Let X be a Hausdorff space and μ a locally finite measure on X which is inner regular with respect to the closed sets. Then the following are equiveridical:

(i) μ has an extension to a Radon measure on X;

(ii) for every non-negligible measurable set $E \subseteq X$ there is a compact set $K \subseteq E$ such that $\mu^* K > 0$.

If μ is totally finite, we can add

(iii) $\sup\{\mu^* K : K \subseteq X \text{ is compact}\} = \mu X$.

proof Write Σ for the domain of μ.

(a)(i)⇒(ii) If λ is a Radon measure extending μ, and $\mu E > 0$, then $\lambda E > 0$, so there is a compact set $K \subseteq E$ such that $\lambda K > 0$; but now, because λ is an extension of μ,

$$\mu^* K \geq \lambda^* K = \lambda K > 0.$$

(b)(ii)⇒(i) & (iii)(α) Let $\mathcal{E}$ be the family of measurable envelopes of compact sets. Then $\mu E < \infty$ for every $E \in \mathcal{E}$. **P** If $E \in \mathcal{E}$, there is a compact set K such that E is a measurable envelope of K. Now $\mu E = \mu^* K$ is finite by 411G. **Q**

Next, $\mathcal{E}$ is closed under finite unions, by 132Ed. The hypothesis (ii) tells us that if $\mu E > 0$ then there is some $F \in \mathcal{E}$ such that $F \subseteq E$ and $\mu F > 0$; for there is a compact set $K \subseteq E$ such that $\mu^* K > 0$, K has a measurable envelope F_0, and $F = E \cap F_0$ is still a measurable envelope of K. So in fact μ is inner regular with respect to $\mathcal{E}$ (412Aa). In particular, μ is semi-finite.

If $\gamma < \mu X$ there is an $F \in \mathcal{E}$ such that $\mu F \geq \epsilon$, and now there is a compact set K such that F is a measurable envelope of K, so that $\mu^* K = \mu F \geq \gamma$. As γ is arbitrary, (iii) is true.

(β) Because μ is inner regular with respect to $\mathcal{E}$, $D = \{E^\bullet : E \in \mathcal{E}\}$ is order-dense in the measure algebra $(\mathfrak{A}, \bar{\mu})$ of μ (412N), so there is a family $\langle d_i \rangle_{i \in I}$ in D which is a partition of unity in $\mathfrak{A}$ (313K). For each $i \in I$, take $E_i \in \mathcal{E}$ such that $E_i^\bullet = d_i$. Then

$$\begin{aligned} \sum_{i \in I} \mu(E \cap E_i) &= \sum_{i \in I} \bar{\mu}(E^\bullet \cap d_i) = \bar{\mu} E^\bullet \\ &= \mu E \end{aligned} \tag{321E}$$

for every $E \in \Sigma$.

(γ) For each $i \in I$, let μ_i be the subspace measure on E_i. Then there is a Radon measure λ_i on E_i extending μ_i. **P** Because μ is inner regular with respect to the closed sets, μ_i is inner regular with respect to the relatively closed subsets of E_i (412O). Also there is a compact subset $K \subseteq E_i$ such that

$$\mu_i E_i = \mu E_i = \mu^* K = \mu_i^* K,$$

so μ_i satisfies the conditions of 416O and has an extension to a Radon measure. **Q**

(δ) Define

$$\lambda E = \textstyle\sum_{i \in I} \lambda_i(E \cap E_i)$$

whenever $E \subseteq X$ is such that λ_i measures $E \cap E_i$ for every $i \in I$. Then λ is a Radon measure on X extending μ. **P** It is easy to check that it is a measure, just because every λ_i is a measure, and it extends μ by (β) above. If $G \subseteq X$ is open, then $G \cap E_i$ is relatively open for every $i \in I$, so λ measures G; thus λ is a topological measure. If $\lambda E = 0$ and $A \subseteq E$, then $\lambda_i(A \cap E_i) \leq \lambda(E \cap E_i) = 0$ for every i, so $\lambda A = 0$; thus λ is complete. For all distinct i, $j \in I$,

$$\lambda_i(E_i \cap E_j) = \mu_i(E_i \cap E_j) = \mu(E_i \cap E_j) = \bar{\mu}(d_i \cap d_j) = 0,$$

so $\lambda E_i = \lambda_i E_i = \mu_i E_i$ is finite. This means that if $E \subseteq X$ is such that λ measures $E \cap F$ whenever $\lambda F < \infty$, then λ must measure $E \cap E_i$ for every i, and λ measures E; thus λ is locally determined. If $\lambda E > 0$ there are an $i \in I$ such that $\lambda_i(E \cap E_i) > 0$ and a compact $K \subseteq E \cap E_i$ such that $0 < \lambda_i K = \lambda K$; consequently λ

is tight. Finally, if $x \in X$, there is an $E \in \Sigma$ such that $x \in \operatorname{int} E$ and $\lambda E = \mu E < \infty$, so λ is locally finite. Thus λ is a Radon measure. **Q**

So (i) is true.

(c) Finally, suppose that μ is totally finite and (iii) is true. Then we can appeal directly to 416O to see that (i) is true.

416Q Proposition (a) Let X be a compact Hausdorff space and $\mathcal{E}$ the algebra of open-and-closed subsets of X. Then any non-negative finitely additive functional from $\mathcal{E}$ to $\mathbb{R}$ has an extension to a Radon measure on X. If X is zero-dimensional then the extension is unique.

(b) Let $\mathfrak{A}$ be a Boolean algebra, and Z its Stone space. Then there is a one-to-one correspondence between non-negative additive functionals ν on $\mathfrak{A}$ and Radon measures μ on Z given by the formula

$$\nu a = \mu \widehat{a} \text{ for every } a \in \mathfrak{A},$$

where for $a \in \mathfrak{A}$ I write $\widehat{a}$ for the corresponding open-and-closed subset of Z.

proof (a) Let $\nu : \mathcal{E} \to [0, \infty[$ be a non-negative additive functional. Then ν satisfies the conditions of 416O (because every member of $\mathcal{E}$ is closed, while X is compact), so has an extension to a Radon measure μ. If X is zero-dimensional, $\mathcal{E}$ is a base for the topology of X closed under finite unions and intersections, so μ is unique, by 415H(iv) or 415H(v).

(b) The map $a \mapsto \widehat{a}$ is a Boolean isomorphism between $\mathfrak{A}$ and the algebra $\mathcal{E}$ of open-and-closed subsets of Z, so we have a one-to-one correspondence between non-negative additive functionals ν on $\mathfrak{A}$ and non-negative additive functionals ν' on $\mathcal{E}$ defined by the formula $\nu'\widehat{a} = \nu a$. Now Z is compact, Hausdorff and zero-dimensional, so ν' has a unique extension to a Radon measure on Z, by part (a). And of course every Radon measure μ on Z gives us a non-negative additive functional $\mu\restriction\mathcal{E}$ on $\mathcal{E}$, corresponding to a non-negative additive functional on $\mathfrak{A}$.

416R Theorem (a) Any subspace of a Radon measure space is a quasi-Radon measure space.

(b) A measurable subspace of a Radon measure space is a Radon measure space.

(c) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a Hausdorff complete locally determined topological measure space, and $Y \subseteq X$ is such that the subspace measure μ_Y on Y is a Radon measure, then $Y \in \Sigma$.

proof (a) Put 416A and 415B together.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $(E, \mathfrak{T}_E, \Sigma_E, \mu_E)$ a member of Σ with the induced topology and measure. Because μ is complete and locally determined, so is μ_E (214Ja). Because $\mathfrak{T}$ is Hausdorff, so is $\mathfrak{T}_E$ (4A2F(a-i)). Because μ is locally finite, so is μ_E. Because μ is tight (and a subset of E is compact for $\mathfrak{T}_E$ whenever it is compact for $\mathfrak{T}$), μ_E is tight (412Oa).

(c) **?** If $Y \notin \Sigma$, then there is a set $F \in \Sigma$ such that $\mu_*(Y \cap F) < \mu^*(Y \cap F)$ (413F(v)). But now $\mu^*(Y \cap F) = \mu_Y(Y \cap F)$, so there is a compact set $K \subseteq Y \cap F$ such that $\mu_Y K > \mu_*(Y \cap F)$. When regarded as a subset of X, K is still compact; because $\mathfrak{T}$ is Hausdorff, K is closed, so belongs to Σ, and

$$\mu_*(Y \cap F) \geq \mu K = \mu_Y K > \mu_*(Y \cap F),$$

which is absurd. **X**

416S Corresponding to 415O, we have the following.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and ν an indefinite-integral measure over μ (definition: 234B). If ν is locally finite, it is a Radon measure.

proof Because μ is complete and locally determined, so is ν (234Fb). Because μ is tight (412Q). So if ν is also locally finite, it is a Radon measure.

416T I said in the notes to §415 that the most important quasi-Radon measure spaces are subspaces of Radon measure spaces. I do not know of a useful necessary and sufficient condition, but the following deals with completely regular spaces.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a locally finite completely regular Hausdorff quasi-Radon measure space. Then it is isomorphic, as topological measure space, to a subspace of a locally compact Radon measure space.

proof (a) Write βX for the Stone-Čech compactification of X (4A2I); I will take it that X is actually a subspace of βX. Let $\mathcal{U}$ be the set of those open subsets U of βX such that $\mu(U \cap X) < \infty$; then $\mathcal{U}$ is upwards-directed and covers X, because μ is locally finite. Set $W = \bigcup \mathcal{U} \supseteq X$. Then W is an open subset of βX, so is locally compact.

(b) Let $\mathcal{B}$ be the Borel σ-algebra of W. Then $V \cap X$ is a Borel subset of X for every $V \in \mathcal{B}$ (4A3Ca), so we have a measure $\nu : \mathcal{B} \to [0, \infty]$ defined by setting $\nu V = \mu(X \cap V)$ for every $V \in \mathcal{B}$. Now ν satisfies the conditions of 415Cb. **P** (α) If $\nu V > 0$, then, because μ is effectively locally finite, there is an open set $G \subseteq X$ such that $\mu(G \cap V) > 0$ and $\mu G < \infty$. There is an open set $U \subseteq \beta X$ such that $U \cap X = G$, in which case $U \subseteq W$, $\nu U < \infty$ and $\nu(U \cap V) > 0$. Thus ν is effectively locally finite. (β) If $\mathcal{U}$ is an upwards-directed family of open subsets of W, then $\{U \cap X : U \in \mathcal{U}\}$ is an upwards-directed family of open subsets of X, so

$$\begin{aligned}\nu(\bigcup \mathcal{U}) &= \mu(X \cap \bigcup \mathcal{U}) = \mu(\bigcup \{U \cap X : U \in \mathcal{U}\}) \\ &= \sup_{U \in \mathcal{U}} \mu(X \cap U) = \sup_{U \in \mathcal{U}} \nu U. \ \textbf{Q}\end{aligned}$$

So the c.l.d. version $\tilde{\nu}$ of ν is a quasi-Radon measure on W (415Cb).

(c) The construction of W ensures that ν and $\tilde{\nu}$ are locally finite. By 416G, $\tilde{\nu}$ is a Radon measure. So the subspace measure $\tilde{\nu}_X$ is a quasi-Radon measure on X (416Ra). But $\tilde{\nu}_X G = \mu G$ for every open set $G \subseteq X$. **P** Note first that as ν effectively locally finite, therefore semi-finite, $\tilde{\nu}$ extends ν (213Hc). If $K \subseteq W$ is a compact set not meeting X, then

$$\tilde{\nu} K = \nu K = \mu(K \cap X) = 0;$$

accordingly $\tilde{\nu}_*(W \setminus X) = 0$, by 413Ee. Now there is an open set $U \subseteq W$ such that $G = X \cap U$, and

$$\begin{aligned}\tilde{\nu}_X G = \tilde{\nu}^* G &\leq \tilde{\nu} U = \nu U = \mu(U \cap X) = \mu G \\ &= \tilde{\nu}^*(U \cap X) + \tilde{\nu}_*(U \setminus X)\end{aligned}$$

(by 413E(c-ii), because $\tilde{\nu}$ is semi-finite)

$$\leq \tilde{\nu}^*(U \cap X) + \tilde{\nu}_*(W \setminus X) = \tilde{\nu}^* G. \ \textbf{Q}$$

So 415H(iii) tells us that $\mu = \tilde{\nu}_X$ is the subspace measure induced by ν.

416U Theorem If $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ is a family of compact metrizable Radon probability spaces such that every μ_i is strictly positive, the product measure on $X = \prod_{i \in I} X_i$ is a completion regular Radon measure. In particular, the usual measures on $\{0,1\}^I$ and $[0,1]^I$ and $\mathcal{P}I$ are completion regular Radon measures, for any set I.

proof By 415E, it is a completion regular quasi-Radon probability measure; but X is a compact Hausdorff space, so it is a Radon measure, by 416G or otherwise.

Remark I suppose it is obvious that by the 'usual measure on $[0,1]^I$' I mean the product measure when each copy of $[0,1]$ is given Lebesgue measure. Recall also that the 'usual measure on $\mathcal{P}I$' is just the copy of the usual measure on $\{0,1\}^I$ induced by the standard bijection $A \leftrightarrow \chi A$ (254Jb), which is a homeomorphism (4A2Ud).

416V Stone spaces The results of 415Q-415R become simpler and more striking in the present context.

Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ the Stone space of its measure algebra $(\mathfrak{A}, \bar{\mu})$. For $E \in \Sigma$ let E^* be the open-and-closed set in Z corresponding to the image $E^\bullet$ of E in $\mathfrak{A}$. Define $R \subseteq Z \times X$ by saying that $(z, x) \in R$ iff $x \in F$ whenever $F \subseteq X$ is closed and $z \in F^*$.

(a) R is the graph of a function $f : Q \to X$, where $Q = R^{-1}[X]$. If we set $W = \bigcup\{K^* : K \subseteq X$ is compact$\}$, then $W \subseteq Q$ is a ν-conegligible open set, and the subspace measure ν_W on W is a Radon measure.

(b) Setting $g = f\restriction W$, g is continuous and μ is the image measure $\nu_W g^{-1}$.

(c) If X is compact, $W = Q = Z$ and $\mu = \nu g^{-1}$.

proof (a) By 415Ra, R is the graph of a function. If $K \subseteq X$ is compact and $z \in K^*$, then $\mathcal{F} = \{F : F \subseteq X$ is closed, $z \in F^*\}$ is a family of non-empty closed subsets of X, closed under finite intersections, and containing the compact set K; so it has non-empty intersection, and there is an $x \in K$ such that $(z, x) \in R$, that is, $z \in Q$ and $f(z) \in K$. Thus $W \subseteq Q$. Of course W is an open set, being the union of a family of open-and-closed sets; but it is also conegligible, because $\sup\{K^\bullet : K \subseteq X$ is compact$\} = 1$ in $\mathfrak{A}$ (412N), so $Z \setminus W$ must be nowhere dense, therefore negligible. Now the subspace measure ν_W is quasi-Radon because ν is (411P(d-iv), 415B); but W is a union of compact open sets of finite measure, so ν_W is locally finite and W is locally compact; by 416G, ν_W is a Radon measure.

(b) g is continuous. **P** Let $G \subseteq X$ be an open set and $z \in g^{-1}[G]$. Let $K \subseteq X$ be a compact set such that $z \in K^*$. As remarked above, $g(z) = f(z)$ belongs to K. K, being a compact Hausdorff space, is regular (3A3Bb), so there is an open set H containing $g(z)$ such that $L = \overline{H \cap K} \subseteq G$. Note that L is compact, so $L^* \subseteq W$. Now $g(z)$ does not belong to the closed set $X \setminus H$, so $z \notin (X \setminus H)^*$ and $z \in H^*$; accordingly $z \in (H \cap K)^* \subseteq L^*$. If $w \in L^*$, $g(w) \in L \subseteq G$; so $L^* \subseteq g^{-1}[G]$, and $z \in \operatorname{int} g^{-1}[G]$. As z is arbitrary, $g^{-1}[G]$ is open; as G is arbitrary, g is continuous. **Q**

By 415Rb, we know that $\mu = \nu_Q f^{-1}$, where ν_Q is the subspace measure on Q. But as ν is complete and both Q and W are conegligible, we have

$$\nu_Q f^{-1}[A] = \nu f^{-1}[A] = \nu g^{-1}[A] = \nu_W g^{-1}[A]$$

whenever $A \subseteq X$ and any of the four terms is defined, so that $\mu = \nu_Q f^{-1} = \nu_W g^{-1}$.

(c) If X is compact, then $Z = X^* \subseteq W$, so $W = Q = Z$ and $\nu g^{-1} = \nu_W g^{-1} = \mu$.

416W Compact measure spaces Recall that a semi-finite measure space (X, Σ, μ) is 'compact' (as a measure space) if there is a family $\mathcal{K} \subseteq \Sigma$ such that μ is inner regular with respect to $\mathcal{K}$ and $\bigcap \mathcal{K}' \neq \emptyset$ whenever $\mathcal{K}' \subseteq \mathcal{K}$ has the finite intersection property (342A); while (X, Σ, μ) is 'perfect' if whenever $f : X \to \mathbb{R}$ is measurable and $\mu E > 0$, there is a compact set $K \subseteq f[E]$ such that $\mu f^{-1}[K] > 0$ (342K). In §342 I introduced these concepts in order to study the realization of homomorphisms between measure algebras. The following result is now very easy.

Proposition (a) Any Radon measure space is a compact measure space, therefore perfect.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, with measure algebra $(\mathfrak{A}, \bar{\mu})$, and (Y, T, ν) a complete strictly localizable measure space, with measure algebra $(\mathfrak{B}, \bar{\nu})$. If $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, there is a function $f : Y \to X$ such that $f^{-1}[E] \in \mathrm{T}$ and $f^{-1}[E]^\bullet = \pi E^\bullet$ for every $E \in \Sigma$. If π is measure-preserving, f is inverse-measure-preserving.

proof (a) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a Radon measure space, μ is inner regular with respect to the compact class consisting of the compact subsets of X, so (X, Σ, μ) is a compact measure space. By 342L, it is perfect.

(b) Use (i)⇒(v) of Theorem 343B. (Of course f is inverse-measure-preserving iff π is measure-preserving.)

416X Basic exercises >(a) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, and $E \in \Sigma$ an atom for the measure. Show that there is a point $x \in E$ such that $\mu\{x\} = \mu E$.

(b) Let X be a topological space and μ a point-supported measure on X, as described in 112Bd. (i) Show that μ is tight, so is a Radon measure iff it is locally finite. In particular, show that if X has the discrete topology then counting measure on X is a Radon measure. (ii) Show that every purely atomic Radon measure is of this type.

(c) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i \in I}$ be a family of Radon measure spaces, with direct sum (X, Σ, μ) (214K). Give X its disjoint union topology. Show that μ is a Radon measure.

(d) Let $(X, \mathfrak{T})$ be a Hausdorff space, and μ_1, μ_2 two Radon measures on X, with domains Σ_1, Σ_2. (i) Show that $\mu_1 + \mu_2$, defined on $\Sigma_1 \cap \Sigma_2$ (112Xe), is a Radon measure. (ii) Show that $\alpha\mu_1$, defined on Σ_1, is a Radon measure for any $\alpha > 0$.

(e) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space. (i) Show that μ has a decomposition $\langle X_i\rangle_{i\in I}$ in which every X_i except at most one is a self-supporting compact set, and the exceptional one, if any, is negligible. (ii) Show that μ has a decomposition $\langle X_i\rangle_{i\in I}$ in which every X_i is expressible as the intersection of a closed set with an open set. (*Hint*: enumerate the open sets of finite measure as $\langle G_\xi\rangle_{\xi<\kappa}$, and set $X_\xi = G_\xi \setminus \bigcup_{\eta<\xi} G_\eta$.)

(f) Let X be a Hausdorff space and μ, ν two Radon measures on X such that $\nu G = \mu G$ whenever $G \subseteq X$ is open and $\min(\mu G, \nu G) < \infty$. Show that $\mu = \nu$.

(g) Explain how to prove 416H from 416F, without appealing to §415.

(h) Give a direct proof of 416I not relying on 415O.

(i) Let $(X, \mathfrak{T})$ be a completely regular topological space and μ a locally finite topological measure on X which is inner regular with respect to the closed sets. Show that

$$\begin{aligned}\mu K &= \inf\{\mu G : G \supseteq K \text{ is a cozero set}\}\\ &= \inf\{\mu F : F \supseteq K \text{ is a zero set}\} = \inf\{\int f d\mu : \chi K \le f \in C(X)\}\end{aligned}$$

for every compact set $K \subseteq X$.

(j) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a locally compact Radon measure space, and C_k the space of continuous real-valued functions on X with compact supports. Show that $\{f^\bullet : f \in C_k\}$ is dense in $L^0(\mu)$ for the topology of convergence in measure.

(k) Let X be a completely regular Hausdorff space and ν a locally finite Baire measure on X. (i) Show that $\nu^* K = \inf\{\nu G : G \subseteq X \text{ is a cozero set}, K \subseteq G\}$ for every compact set $K \subseteq X$. (ii) Show that there is a Radon measure μ on X such that $\mu K = \nu^* K$ for every compact set $K \subseteq X$. (*Hint*: in the language of the proof of 416K, $\phi_1 = \nu^* \restriction \mathcal{K}$.)

(l) Let X be a Hausdorff space and ν a non-negative finitely additive functional defined on some algebra of subsets of X. Show that there is a Radon measure μ on X such that $\mu X \le \nu X$ and $\mu K \ge \nu E$ whenever $E \in \mathrm{T}$, $K \subseteq X$ is compact and $E \subseteq K$. (*Hint*: start by extending ν to $\mathcal{P}X$.)

(m) Let $(X, \mathfrak{T})$ be a Hausdorff space, $\Sigma \supseteq \mathfrak{T}$ a σ-algebra of subsets of X, and $\nu : \Sigma \to [0, \infty[$ a finitely additive functional such that $\nu E = \sup\{\nu K : K \subseteq E \text{ is compact}\}$ for every $E \in \Sigma$. Show that ν is countably additive and that its completion is a Radon measure on X.

(n) Explain how to prove 416Rb from 416C and 415B.

(o) Let X be a Hausdorff space, μ a complete locally finite measure on X, and Y a conegligible subset of X. Show that μ is a Radon measure iff the subspace measure on Y is a Radon measure.

(p) Let X be a Hausdorff space, Y a subset of X, and ν a Radon measure on Y. Define a measure μ on X by setting $\mu E = \nu(E \cap Y)$ whenever ν measures $E \cap Y$. Show that if *either* Y is closed *or* ν is totally finite, μ is a Radon measure on X. (Cf. 418I.)

(q) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space and $\mathcal{E} \subseteq \Sigma$ a non-empty upwards-directed family. Set $\nu F = \sup_{E\in\mathcal{E}} \mu(E \cap F)$ whenever $F \subseteq X$ is such that μ measures $E \cap F$ for every $E \in \mathcal{E}$. Show that ν is a Radon measure on X.

(r) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a sequence of Hausdorff spaces with product X; write $\mathcal{B}(X_n)$ for the Borel σ-algebra of X_n. Let T be the σ-algebra $\widehat{\bigotimes}_{n\in\mathbb{N}} \mathcal{B}(X_n)$ (definition: 254E). Let $\nu : \mathrm{T} \to [0, \infty[$ be a finitely additive functional such that $E \mapsto \nu\pi_n^{-1}[E] : \mathcal{B}(X_n) \to [0, \infty[$ is countably additive and tight for each $n \in \mathbb{N}$, writing $\pi_n(x) = x(n)$ for $x \in X$, $n \in \mathbb{N}$. Show that there is a unique Radon measure on X extending ν.

(s) Set $S_2^* = \bigcup_{n\in\mathbb{N}}\{0,1\}^n$, and let $\phi : S_2^* \to [0,\infty[$ be a functional such that $\phi(\sigma) = \phi(\sigma^\frown 0) + \phi(\sigma^\frown 1)$ for every $\sigma \in S_2^*$, writing $\sigma^\frown 0$ and $\sigma^\frown 1$ for the two members of $\{0,1\}^{n+1}$ extending any $\sigma \in \{0,1\}^n$. Show that there is a unique Radon measure μ on $\{0,1\}^{\mathbb{N}}$ such that $\mu\{x : x\restriction\{0,\dots,n-1\} = \sigma\} = \phi(\sigma)$ whenever $n \in \mathbb{N}$, $\sigma \in \{0,1\}^{\mathbb{N}}$. (*Hint*: use 416Xr or 416Q.)

(t) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space. Show that a measure ν on X is an indefinite-integral measure over μ iff (α) ν is a complete, locally determined topological measure (β) ν is tight (γ) $\nu K = 0$ whenever $K \subseteq X$ is compact and $\mu K = 0$.

(u) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a compact Hausdorff quasi-Radon measure space. Let $W \subseteq X$ be the union of the open subsets of X of finite measure. Show that the subspace measure on W is a Radon measure.

(v) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a completely regular Radon measure space. Show that it is isomorphic, as topological measure space, to a measurable subspace of a locally compact Radon measure space.

(w) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a compact Radon measure space and $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ the Stone space of its measure algebra. For $E \in \Sigma$ let E^* be the corresponding open-and-closed subset of Z, as in 416V. Show that the function described in 416V is the unique continuous function $h : Z \to X$ such that $\nu(E^* \triangle h^{-1}[E]) = 0$ for every $E \in \Sigma$.

(x) Show that the right-facing Sorgenfrey line (415Xc), with Lebesgue measure, is a quasi-Radon measure space which, regarded as a measure space, is compact, but, regarded as a topological measure space, is not a Radon measure space.

416Y Further exercises (a) Let X be a Hausdorff space and ν a countably additive real-valued functional defined on a σ-algebra Σ of subsets of X. Show that the following are equiveridical: (i) $|\nu| : \Sigma \to [0,\infty[$, defined as in 362B, is a Radon measure on X; (ii) ν is expressible as $\mu_1 - \mu_2$, where μ_1, μ_2 are Radon measures on X and $\Sigma = \operatorname{dom}\mu_1 \cap \operatorname{dom}\mu_2$.

(b) Let X be a topological space and μ_0 a semi-finite measure on X which is inner regular with respect to the family $\mathcal{K}_{\mathrm{ccc}}$ of closed countably compact sets. Show that μ_0 has an extension to a complete locally determined topological measure μ on X, still inner regular with respect to $\mathcal{K}_{\mathrm{ccc}}$; and that the extension may be done in such a way that whenever $\mu E < \infty$ there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$. (*Hint*: use the argument of 416M, but with $\mathcal{K} = \{K : K \in \mathcal{K}_{\mathrm{ccc}}, \mu_0^* K < \infty\}$.)

(c) Let X be a topological space and μ_0 a semi-finite measure on X which is inner regular with respect to the family $\mathcal{K}_{\mathrm{sc}}$ of sequentially compact sets. Show that μ_0 has an extension to a complete locally determined topological measure μ on X, still inner regular with respect to $\mathcal{K}_{\mathrm{sc}}$; and that the extension may be done in such a way that whenever $\mu E < \infty$ there is an $E_0 \in \Sigma_0$ such that $\mu(E \triangle E_0) = 0$.

(d) Let $X \subseteq \beta\mathbb{N}$ be the union of all those open sets $G \subseteq \beta\mathbb{N}$ such that $\sum_{n\in G\cap\mathbb{N}} \frac{1}{n+1}$ is finite. For $E \subseteq X$ set $\mu E = \sum_{n\in E\cap\mathbb{N}} \frac{1}{n+1}$. Show that μ is a σ-finite Radon measure on the locally compact Hausdorff space X. Show that μ is not outer regular with respect to the open sets.

(e) Set $S^* = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$, and let $\phi : S^* \to [0,\infty[$ be a functional such that $\phi(\sigma) = \sum_{i=0}^\infty \phi(\sigma^\frown i)$ for every $\sigma \in S^*$, writing $\sigma^\frown i$ for the members of $\mathbb{N}^{n+1}$ extending any $\sigma \in \mathbb{N}^n$. Show that there is a unique Radon measure μ on $\mathbb{N}^{\mathbb{N}}$ such that $\mu I_\sigma = \phi(\sigma)$ for every $\sigma \in S^*$, where $I_\sigma = \{x : x\restriction\{0,\dots,n-1\} = \sigma\}$ for any $n \in \mathbb{N}$, $\sigma \in \mathbb{N}^{\mathbb{N}}$. (*Hint*: set $\theta A = \inf\{\sum_{\sigma\in R} \phi(\sigma) : R \subseteq S^*, A \subseteq \bigcup_{\sigma\in R} I_\sigma\}$ for every $A \subseteq \mathbb{N}^{\mathbb{N}}$, and use Carathéodory's method.)

(f) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space. Show that a measure ν on X is an indefinite-integral measure over μ iff (i) there is a topology $\mathfrak{S}$ on X, including $\mathfrak{T}$, such that ν is a Radon measure with respect to $\mathfrak{S}$ (ii) $\nu K = 0$ whenever K is a $\mathfrak{T}$-compact set and $\mu K = 0$.

(g) Let $\langle x_n\rangle_{n\in\mathbb{N}}$ enumerate a dense subset of $X = \{0,1\}^{\mathfrak{c}}$ (4A2B(e-ii)). Let λ be the usual measure on X, and set $\mu E = \frac{1}{2}\lambda E + \sum_{n=0}^\infty 2^{-n-2}\chi E(x_n)$ for $E \in \operatorname{dom}\lambda$. (i) Show that μ is a strictly positive Radon

probability measure on X of Maharam type $\mathfrak{c}$. (ii) Let $I \in [\mathfrak{c}]^{\leq\omega}$ be such that $x_m{\restriction}I \neq x_n{\restriction}I$ whenever $m \neq n$. Set $Z = \{0,1\}^I$ and let $\pi : X \to Z$ be the canonical map. Show that if $f \in C(X)$ is such that $\int f \times g\pi \, d\mu = 0$ for every $g \in C(Z)$, then $f = 0$. (*Hint*: otherwise, take $n \in \mathbb{N}$ such that $|f(x_n)| \geq \frac{1}{2}\|f\|_\infty$, and let $g \geq 0$ be such that $g(\pi x_n) = 1$ and $\int g \, d(\mu\pi^{-1}) < 3 \cdot 2^{-n-3}$; show that $\int f \times g\pi \, d\mu > 0$.) (iii) Show that there is no orthonormal basis for $L^2(\mu)$ in $\{f^\bullet : f \in C(X)\}$. (See Hart & Kunen 99.)

416 Notes and comments The original measures studied by Radon (Radon 13) were, in effect, what I call differences of Radon measures on $\mathbb{R}^r$, as introduced in §256. Successive generalizations moved first to Radon measures on general compact Hausdorff spaces, then to locally compact Hausdorff spaces, and finally to arbitrary Hausdorff spaces, as presented in this section. I ought perhaps to remark that, following Bourbaki 65, many authors use the term 'Radon measure' to describe a linear functional on a space of continuous functions; I will discuss the relationship between such functionals and the measures of this chapter in §436. For the moment, observe that by 415I a Radon measure on a completely regular space can be determined from the integrals it assigns to continuous functions. It is also common for the phrase 'Radon measure' to be used for what I would call a tight Borel measure; you have to check each author to see whether local finiteness is also assumed. In my usage, a Radon measure is necessarily the c.l.d. version of a Borel measure. The Borel measures which correspond to Radon measures are described in 416F.

In §256, I discussed Radon measures on $\mathbb{R}^r$ as a preparation for a discussion of convolutions of measures. It should now be becoming clear why I felt that it was impossible, in that context, to give you a proper idea of what a Radon measure, in the modern form, 'really' is. In Euclidean space, too many concepts coincide. As a trivial example, the simplest definition of 'local finiteness' (256Ab, 411Fa) is not the right formulation in other spaces. Next, because every closed set is a countable union of compact sets, there is no distinction between 'inner regular with respect to closed sets' and 'inner regular with respect to compact sets', so one cannot get any intuition for which is important in which arguments. (When we come to subspace measures on non-measurable subsets, of course, this changes; quasi-Radon measures on subsets of Euclidean space are important and interesting.) Third, the fact that the c.l.d. product of two Radon measures on Euclidean space is already a Radon measure (256K) leaves us with no idea of what to do with a general product of Radon measures. (There are real difficulties at this point, which I will attack in the next section. For the moment I offer just 416U.) And finally, we simply cannot represent a product of uncountably many Radon probability measures on Euclidean spaces as a measure on Euclidean space.

As you would expect, a very large proportion of the results of this chapter, and many theorems from earlier volumes, were originally proved for compact Radon measure spaces. The theory of general totally finite Radon measures is, in effect, the theory of measurable subspaces of compact Radon measure spaces, while the theory of quasi-Radon measures is pretty much the theory of non-measurable subspaces of Radon measure spaces. Thus the theorem that every quasi-Radon measure space is strictly localizable is almost a consequence of the facts that every Radon measure space is strictly localizable and any subspace of a strictly localizable space is strictly localizable.

The cluster of results between 416J and 416P form only a sample, I hope a reasonably representative sample, of the many theorems on construction of Radon measures from functionals on algebras or lattices of sets. (See also 416Ye.) The essential simplification, compared with the theorems in §413 and §415, is that we do not need to mention any σ- or τ-additivity condition of the type 413I(β) or 415K(β), because we are dealing with a 'compact class', the family of compact subsets of a Hausdorff space. We can use this even at some distance, as in 416O (where the hypotheses do not require any non-empty compact set to belong to the domain of the original functional). The particular feature of 416O which makes it difficult to prove from such results as 413J and 413O above is that we have to retain control of the outer measures of a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of non-measurable sets. In general this is hard to do, and is possible here principally because the sequence is non-decreasing, so that we can make sense of the functionals $\nu_n E = \nu^*(E \cap K_{n+1}) - \nu^*(E \cap K_n)$.

417 τ-additive product measures

The 'ordinary' product measures introduced in Chapter 25 have served us well for a volume and a half. But we come now to a fundamental obstacle. If we start with two Radon measure spaces, their product measure, as defined in §251, need not be a Radon measure (419E). Furthermore, the counterexample is one of the basic compact measure spaces of the theory; and while it is dramatically non-metrizable, there is no other reason to set it aside. Consequently, if we wish (as we surely do) to create Radon measure spaces as products of Radon measure spaces, we need a new construction. This is the object of the present section. It turns out that the construction can be adapted to work well beyond the special context of Radon measure spaces; the methods here apply to general effectively locally finite τ-additive topological measures (for the product of finitely many factors) and to τ-additive topological probability measures (for the product of infinitely many factors). Elsewhere in this volume I will show how these constructions are related to some others which have been described for special spaces.

The fundamental theorems are 417C and 417E, listing the essential properties of what I call 'τ-additive product measures', which are extensions of the c.l.d. product measures and product probability measures of Chapter 25. They depend on a straightforward lemma on the extension of a measure to make every element of a given class of sets negligible (417A). It is relatively easy to prove that the extensions are more or less canonical (417D, 417F). We still have Fubini's theorem for the new product measures (417H), and the basic operations from §254 still apply (417J, 417K, 417M).

It is easy to check that if we start with quasi-Radon measures, then the τ-additive product measure is again quasi-Radon (417N, 417O). The τ-additive product of two Radon measures is Radon (417P), and the τ-additive product of Radon probability measures with compact supports is Radon (417Q).

In the last part of the section I look at continuous real-valued functions and Baire σ-algebras; it turns out that for these the ordinary product measures are adequate (417U, 417V).

417A Lemma Let (X, Σ, μ) be a semi-finite measure space, and $\mathcal{A} \subseteq \mathcal{P}X$ a family of sets such that $\mu_*(\bigcup_{n\in\mathbb{N}} A_n) = 0$ for every sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$. Then there is a measure μ' on X, extending μ, such that

(i) $\mu' A$ is defined and zero for every $A \in \mathcal{A}$,

(ii) μ' is complete if μ is,

(iii) for every F in the domain Σ' of μ' there is an $E \in \Sigma$ such that $\mu'(F \triangle E) = 0$.

In particular, μ and μ' have isomorphic measure algebras, so that μ' is localizable if μ is.

proof Let $\mathcal{A}^*$ be the collection of subsets of X which can be covered by a countable subfamily of $\mathcal{A}$. Then $\mathcal{A}^*$ is a σ-ideal of subsets of X and $\mu_* A = 0$ for every $A \in \mathcal{A}^*$. Set

$$\Sigma' = \{E \triangle A : E \in \Sigma,\ A \in \mathcal{A}^*\}.$$

Then Σ' is a σ-algebra of subsets of X. **P** (i) $\emptyset = \emptyset \triangle \emptyset \in \Sigma'$. (ii) If $E \in \Sigma$, $A \in \mathcal{A}^*$ then $X \setminus (E \triangle A) = (X \setminus E) \triangle A \in \Sigma'$. (iii) If $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle A_n\rangle_{n\in\mathbb{N}}$ are sequences in Σ, $\mathcal{A}^*$ respectively, then

$$E = \bigcup_{n\in\mathbb{N}} E_n \in \Sigma, \quad A = E \triangle \bigcup_{n\in\mathbb{N}} (E_n \triangle A_n) \subseteq \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}^*,$$

so $\bigcup_{n\in\mathbb{N}} (E_n \triangle A_n) = E \triangle A \in \Sigma'$. **Q**

If $E, E' \in \Sigma$, $A, A' \in \mathcal{A}^*$ and $E \triangle A = E' \triangle A'$, then $E \triangle E' = A \triangle A' \in \mathcal{A}^*$ and $\mu_*(E \triangle E') = 0$; because μ is semi-finite, $\mu(E \triangle E') = 0$ and $\mu E = \mu E'$. Accordingly we can define $\mu' : \Sigma' \to [0, \infty]$ by setting

$$\mu'(E \triangle A) = \mu E \text{ whenever } E \in \Sigma,\ A \in \mathcal{A}^*.$$

Evidently μ' extends μ and $\mu' A = 0$ for every $A \in \mathcal{A}$. Also μ' is a measure. **P** (i) $\mu' \emptyset = \mu \emptyset = 0$. (ii) If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in Σ', with union F, express each F_n as $E_n \triangle A_n$ where $E_n \in \Sigma$, $A_n \in \mathcal{A}^*$; set $E = \bigcup_{n\in\mathbb{N}} E_n$, so that $F \triangle E \in \mathcal{A}^*$ and $\mu' F = \mu E$. If $m \neq n$, then $E_m \cap E_n \subseteq A_m \cup A_n$, so $\mu(E_m \cap E_n) = 0$; accordingly

$$\mu' F = \mu E = \textstyle\sum_{n=0}^{\infty} \mu E_n = \sum_{n=0}^{\infty} \mu' F_n. \quad \textbf{Q}$$

A subset of X is μ'-negligible iff it can be included in a set of the form $E \triangle A$ where $\mu E = 0$ and $A \in \mathcal{A}^*$, so μ' is complete if μ is. The embedding $\Sigma \subseteq \Sigma'$ induces a measure-preserving homomorphism from the measure algebra of μ to the measure algebra of μ' which is an isomorphism just because every member of Σ' is the symmetric difference of a member of Σ and a μ'-negligible set.

417B Lemma Let X and Y be topological spaces, and ν a τ-additive topological measure on Y.

(a) If $W \subseteq X \times Y$ is open, then $x \mapsto \nu W[\{x\}] : X \to [0,\infty]$ is lower semi-continuous.

(b) If ν is effectively locally finite and σ-finite and $W \subseteq X \times Y$ is a Borel set, then $x \mapsto \nu W[\{x\}]$ is Borel measurable.

(c) If $f : X \times Y \to [0,\infty]$ is a lower semi-continuous function, then $x \mapsto \int f(x,y)\nu(dy) : X \to [0,\infty]$ is lower semi-continuous.

(d) If ν is totally finite and $f : X \times Y \to \mathbb{R}$ is a bounded continuous function, then $x \mapsto \int f(x,y)\nu(dy)$ is continuous.

(e) If ν is totally finite and $W \subseteq X \times Y$ is a Baire set, then $x \mapsto \nu W[\{x\}]$ is Baire measurable.

proof (a) If $x \in X$ and $\nu W[\{x\}] > \alpha$, then

$$\mathcal{H} = \{H : H \subseteq Y \text{ is open, there is an open set } G \text{ containing } x \text{ such that } G \times H \subseteq W\}$$

is an upwards-directed family of open sets with union $W[\{x\}]$, so there is an $H \in \mathcal{H}$ such that $\nu H \geq \alpha$. Now there is an open set G containing x such that $G \times H \subseteq W$, so that $\nu W[\{x'\}] \geq \alpha$ for every $x' \in G$.

(b)(i) Suppose to begin with that ν is totally finite. In this case, the set

$$\{W : W \subseteq X \times Y,\ x \mapsto \nu W[\{x\}] \text{ is a Borel measurable function defined everywhere on } X\}$$

is a Dynkin class containing every open set, so contains every Borel set, by the Monotone Class Theorem (136B).

(ii) For the general case, let $\langle Y_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence of sets of finite measure covering Y, and for $n \in \mathbb{N}$ let ν_n be the subspace measure on Y_n. Then ν_n is effectively locally finite and τ-additive (414K). If $W \subseteq X \times Y$ is a Borel set, then $W_n = W \cap (X \times Y_n)$ is a relatively Borel set for each n, so that $x \mapsto \nu_n W_n[\{x\}]$ is Borel measurable, by (i). Since $\nu W[\{x\}] = \sum_{n=0}^{\infty} \nu_n W_n[\{x\}]$ for every x, $x \mapsto \nu W[\{x\}]$ is Borel measurable.

(c) For $i,\ n \in \mathbb{N}$ set $W_{ni} = \{(x,y) : f(x,y) > 2^{-n}i\}$, so that $W_{ni} \subseteq X \times Y$ is open. Set $f_n = 2^{-n}\sum_{i=1}^{4^n} \chi W_{ni}$; then $\langle f_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with supremum f. For $n \in \mathbb{N}$ and $x \in X$, $\int f_n(x,y)\nu(dy) = 2^{-n}\sum_{i=1}^{4^n} \nu W_{ni}[\{x\}]$, so $x \mapsto \int f_n(x,y)\nu(dy)$ is lower semi-continuous, by (a) and 4A2B(d-iii). By 414Ba, $\int f(x,y)\nu(dy) = \sup_{n\in\mathbb{N}} \int f_n(x,y)\nu(dy)$ for every x, so $x \mapsto \int f(x,y)\nu(dy)$ is lower semi-continuous (4A2B(d-v)).

(d) Applying (c) to $f + \|f\|_\infty \chi(X\times Y)$, we see that $x \mapsto \int f(x,y)\nu(dy)$ is lower semi-continuous. Similarly, $x \mapsto -\int f(x,y)\nu(dy)$ is lower semi-continuous, so $x \mapsto \int f(x,y)\nu(dy)$ is continuous (4A2B(d-vi)).

(e) Suppose first that W is a cozero set; let $f : X \times Y \to [0,1]$ be a continuous function such that $W = \{(x,y) : f(x,y) > 0\}$. For $n \in \mathbb{N}$ set $f_n = nf \wedge \chi(X \times Y)$. Then $\langle f_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of continuous functions with supremum χW. By (d), all the functions $x \mapsto \int f_n(x,y)\nu(dy)$ are continuous, so their limit $x \mapsto \nu W[\{x\}]$ is Baire measurable.

Now

$$\{W : W \subseteq X \times Y,\ x \mapsto \nu W[\{x\}] \text{ is a Baire measurable function defined everywhere on } X\}$$

is a Dynkin class containing every cozero set, so contains every Baire set, by the Monotone Class Theorem again.

417C Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces, with c.l.d. product $(X \times Y, \Lambda, \lambda)$. Then λ has an extension to a τ-additive topological measure $\tilde{\lambda}$ on $X \times Y$. Moreover, we can arrange that:

(i) $\tilde{\lambda}$ is complete, locally determined and effectively locally finite, therefore strictly localizable;

(ii) if Q belongs to the domain $\tilde{\Lambda}$ of $\tilde{\lambda}$, there is a $Q_1 \in \Lambda$ such that $\tilde{\lambda}(Q \triangle Q_1) = 0$; that is to say, the embedding $\Lambda \subseteq \tilde{\Lambda}$ induces an isomorphism between the measure algebras of λ and $\tilde{\lambda}$;

(iii) if $Q \in \tilde{\Lambda}$, then

$$\tilde{\lambda}Q = \sup\{\tilde{\lambda}(Q \cap (G \times H)) : G \in \mathfrak{T},\ \mu G < \infty,\ H \in \mathfrak{S},\ \nu H < \infty\};$$

(iv) if $W \subseteq X \times Y$ is open, then there is an open set $W_0 \in \Lambda$ such that $W_0 \subseteq W$ and $\lambda W_0 = \tilde{\lambda}W$, and

$$\tilde{\lambda}W = \lambda_* W = \int \nu W[\{x\}]\mu(dx) = \int \mu W^{-1}[\{y\}]\nu(dy);$$

in particular, $\tilde{\lambda}(G \times H) = \mu G \cdot \nu H$ for every $G \in \mathfrak{T}$, $H \in \mathfrak{S}$;

(v) the support of λ is the product of the supports of μ and ν;

(vi) if μ and ν are both inner regular with respect to the Borel sets, so is $\tilde{\lambda}$;

(vii) if μ and ν are both inner regular with respect to the closed sets, so is $\tilde{\lambda}$;

(viii) if μ and ν are both tight (that is, inner regular with respect to the closed compact sets), so is $\tilde{\lambda}$.

proof Write

$$\Sigma^f = \{E : E \in \Sigma,\ \mu E < \infty\}, \quad \mathrm{T}^f = \{F : F \in \mathrm{T},\ \nu F < \infty\},$$

$$\mathfrak{T}^f = \mathfrak{T} \cap \Sigma^f, \quad \mathfrak{S}^f = \mathfrak{S} \cap \mathrm{T}^f.$$

(a) Let $\mathcal{U}$ be $\{G \times H : G \in \mathfrak{T}^f,\ H \in \mathfrak{S}^f\}$. Because $\mathfrak{T} \subseteq \Sigma$ and $\mathfrak{S} \subseteq \mathrm{T}$, $\mathcal{U} \subseteq \Lambda$. $\mathcal{U}$ need not be a base for the topology of $X \times Y$, unless μ and ν are locally finite, but if an open subset of $X \times Y$ is included in a member of $\mathcal{U}$ it is the union of the members of $\mathcal{U}$ it includes. Moreover, if $Q \in \Lambda$, then $\lambda Q = \sup_{U \in \mathcal{U}} \lambda(Q \cap U)$. **P** By 412R, λ is inner regular with respect to $\bigcup_{U \in \mathcal{U}} \mathcal{P}U$. **Q**

Write $\mathcal{U}_s$ for the set of finite unions of members of $\mathcal{U}$, and $\mathfrak{V}$ for the set of non-empty upwards-directed families $\mathcal{V} \subseteq \mathcal{U}_s$ such that $\sup_{V \in \mathcal{V}} \lambda V < \infty$. For each $\mathcal{V} \in \mathfrak{V}$, fix on a countable $\mathcal{V}' \subseteq \mathcal{V}$ such that $\sup_{V \in \mathcal{V}'} \lambda V = \sup_{V \in \mathcal{V}} \lambda V$; because $\mathcal{V}$ is upwards-directed, we may suppose that $\mathcal{V}' = \{V_n : n \in \mathbb{N}\}$ for some non-decreasing sequence $\langle V_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{V}$. Set $A(\mathcal{V}) = \bigcup \mathcal{V} \setminus \bigcup \mathcal{V}'$.

(b)(i) For $V \in \mathcal{U}_s$, set $f_V(x) = \nu V[\{x\}]$ for every $x \in X$. This is always defined because V is open; moreover, f_V is lower semi-continuous, by 417Ba. Because V is a finite union of products of sets of finite measure, $\int f_V d\mu = \lambda V$.

(ii) The key to the proof is the following fact: for any $\mathcal{V} \in \mathfrak{V}$, almost every vertical section of $A(\mathcal{V})$ is negligible. **P** $\langle f_V \rangle_{V \in \mathcal{V}}$ is a non-empty upwards-directed set of lower semi-continuous functions. Set

$$g(x) = \nu(\textstyle\bigcup_{V \in \mathcal{V}} V[\{x\}]), \quad h(x) = \nu(\textstyle\bigcup_{V \in \mathcal{V}'} V[\{x\}])$$

for every $x \in X$. Because $\mathcal{V}$ is upwards-directed and ν is τ-additive,

$$g(x) = \sup_{V \in \mathcal{V}} \nu V[\{x\}] = \sup_{V \in \mathcal{V}} f_V(x)$$

in $[0, \infty]$ for each x, so, by 414Ba,

$$\int g\, d\mu = \sup_{V \in \mathcal{V}} \int f_V d\mu = \sup_{V \in \mathcal{V}} \lambda V = \sup_{V \in \mathcal{V}'} \lambda V = \int h\, d\mu.$$

Since $h \leq g$ and $\sup_{V \in \mathcal{V}} \lambda V$ is finite, $g(x) = h(x) < \infty$ for μ-almost every x. But for any such x, we must have

$$\nu(\textstyle\bigcup \mathcal{V})[\{x\}] = \nu(\bigcup \mathcal{V}')[\{x\}] < \infty,$$

so that

$$A(\mathcal{V})[\{x\}] = (\textstyle\bigcup \mathcal{V})[\{x\}] \setminus (\bigcup \mathcal{V}')[\{x\}]$$

is negligible. **Q**

(c) **?** Suppose, if possible, that there is a sequence $\langle \mathcal{V}_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{V}$ such that $\lambda_*(\bigcup_{n \in \mathbb{N}} A(\mathcal{V}_n)) > 0$. Take $W \in \Lambda$ such that $W \subseteq \bigcup_{n \in \mathbb{N}} A(\mathcal{V}_n)$ and $\lambda W > 0$. Because almost every vertical section of every $A(\mathcal{V}_n)$ is negligible, almost every vertical section of W is negligible. But this contradicts Fubini's theorem (252F). **X**

(d) We may therefore apply the construction of 417A to form a measure λ' on the σ-algebra $\Lambda' = \{W \triangle A : W \in \Lambda,\ A \in \mathcal{A}^*\}$, where $\mathcal{A}^*$ is the σ-ideal generated by $\{A(\mathcal{V}) : \mathcal{V} \in \mathfrak{V}\}$. Let $\tilde{\lambda}$ be the c.l.d. version of λ' (213E), and $\tilde{\Lambda}$ its domain.

(i) If $W \in \Lambda$, then $W \in \Lambda' \subseteq \tilde{\Lambda}$. Also, because λ is semi-finite,

$$\begin{aligned}\lambda' W = \lambda W &= \sup\{\lambda W' : W' \subseteq W, W \in \Lambda, \lambda W' < \infty\} \\ &\leq \sup\{\lambda' W' : W' \subseteq W, W \in \Lambda', \lambda' W' < \infty\} = \tilde{\lambda} W \leq \lambda' W.\end{aligned}$$

Thus $\lambda W = \tilde{\lambda} W$; as W is arbitrary, $\tilde{\lambda}$ extends λ.

(ii) It will be useful if we go directly to one of the targets: if $Q \in \tilde{\Lambda}$ and $\gamma < \tilde{\lambda} Q$, there is a $U \in \mathcal{U}$ such that $\tilde{\lambda}(Q \cap U) \geq \gamma$. **P** There is a $Q_1 \in \Lambda'$ such that $Q_1 \subseteq Q$ and $\gamma < \lambda' Q_1 < \infty$. There is a $Q_2 \in \Lambda$ such that $\lambda'(Q_1 \triangle Q_2) = 0$, so that

$$\lambda Q_2 = \lambda' Q_2 = \lambda' Q_1 > \gamma.$$

There is a $U \in \mathcal{U}$ such that $\lambda(Q_2 \cap U) \geq \gamma$, by (a). Now

$$\tilde{\lambda}(Q \cap U) \geq \lambda'(Q_1 \cap U) = \lambda'(Q_2 \cap U) = \lambda(Q_2 \cap U) \geq \gamma. \ \textbf{Q}$$

(iii) $\tilde{\lambda}$ is a topological measure. **P** Let $W \subseteq X \times Y$ be an open set. Suppose that $Q \in \tilde{\Lambda}$ and $\tilde{\lambda} Q > 0$. By (ii), there is a $U \in \mathcal{U}$ such that $\tilde{\lambda}(Q \cap U) > 0$. Let $\mathcal{V}$ be $\{V : V \in \mathcal{U}_s, V \subseteq W \cap U\}$. Then $\mathcal{V} \in \mathfrak{V}$, so $\tilde{\lambda} A(\mathcal{V}) = \lambda' A(\mathcal{V}) = 0$; since $\bigcup \mathcal{V}' \in \Lambda$,

$$W \cap U = \bigcup \mathcal{V} \in \Lambda' \subseteq \tilde{\Lambda}.$$

But this means that

$$\tilde{\lambda}_*(Q \cap W) + \tilde{\lambda}_*(Q \setminus W) \geq \tilde{\lambda}_*(Q \cap U \cap W) + \tilde{\lambda}_*(Q \cap U \setminus W) = \tilde{\lambda}(Q \cap U) > 0.$$

Because $\tilde{\lambda}$ is complete and locally determined, and Q is arbitrary, this is enough to ensure that $W \in \tilde{\Lambda}$ (413F(vii)). **Q**

(iv) $\tilde{\lambda}$ is τ-additive. **P?** Suppose, if possible, otherwise; that there is a non-empty upwards-directed family $\mathcal{W}$ of open sets in $X \times Y$ such that $\tilde{\lambda} W^* > \gamma = \sup_{W \in \mathcal{W}} \tilde{\lambda} W$, where $W^* = \bigcup \mathcal{W}$. In this case, we can find a $Q_0 \in \Lambda'$ such that $Q_0 \subseteq W^*$ and $\lambda' Q_0 > \gamma$, a $Q_1 \in \Lambda$ such that $\lambda'(Q_0 \triangle Q_1) = 0$, and a $U \in \mathcal{U}$ such that $\lambda(Q_1 \cap U) > \gamma$ (using (a) again). Let $\mathcal{V} \in \mathfrak{V}$ be the set of those $V \in \mathcal{U}_s$ such that $V \subseteq W \cap U$ for some $W \in \mathcal{W}$. Then $\bigcup \mathcal{V} = W^* \cap U$, so

$$\begin{aligned}\gamma < \lambda(Q_1 \cap U) &= \lambda'(Q_1 \cap U) = \lambda'(Q_0 \cap U) \\ &\leq \tilde{\lambda}(W^* \cap U) = \tilde{\lambda}(\bigcup \mathcal{V}) = \tilde{\lambda}(\bigcup \mathcal{V}')\end{aligned}$$

(because $\tilde{\lambda} A(\mathcal{V}) = 0$)

$$= \sup_{V \in \mathcal{V}'} \tilde{\lambda} V$$

(because $\mathcal{V}'$ is countable and upwards-directed)

$$\leq \sup_{W \in \mathcal{W}} \tilde{\lambda} W \leq \gamma,$$

which is absurd. **XQ**

(e) Now for the supplementary properties (i)-(viii), in order.

(i) $\tilde{\lambda}$ was constructed to be complete and locally determined. If $\tilde{\lambda} Q > 0$, then by (d-ii) there is a $U \in \mathcal{U}$ such that $\tilde{\lambda}(Q \cap U) > 0$; since U is open and $\tilde{\lambda} U = \lambda U$ is finite, this shows that $\tilde{\lambda}$ is effectively locally finite. By 414J, it is strictly localizable.

(ii) The point is that λ is also (strictly) localizable. **P** Let $\tilde{\mu}$, $\tilde{\nu}$ be the c.l.d. versions of μ and ν. These are τ-additive topological measures (because μ and ν are), complete and locally determined (by construction), and are still effectively locally finite (cf. 412Ha), so are strictly localizable (414J again). Now λ is the c.l.d. product of $\tilde{\mu}$ and $\tilde{\nu}$ (251S), therefore strictly localizable (251N). **Q**

As remarked in 417A, it follows that λ' is localizable, so that every member Q of $\tilde{\Lambda}$ differs by a $\tilde{\lambda}$-negligible set from a member Q_0 of Λ' (213Hb). Now there is a $Q_1 \in \Lambda$ such that $\lambda'(Q_1 \triangle Q_0) = 0$, in which case $\tilde{\lambda}(Q_1 \triangle Q) = 0$.

(iii) This is just (d-ii) above.

(iv) Let $W \subseteq X \times Y$ be an open set. Set $\mathcal{V} = \{V : V \in \mathcal{U}_s, V \subseteq W\}$. Then $\mathcal{V}$ is upwards-directed and has union $W \cap (X^* \times Y^*)$, where X^* and Y^* are the unions of the open sets of finite measure in X and Y respectively. Because μ and ν are effectively locally finite, X^* and Y^* are both conegligible. Now, as before,

$$\tilde{\lambda}W = \tilde{\lambda}(W \cap (X^* \times Y^*)) = \sup_{V \in \mathcal{V}} \tilde{\lambda}V = \sup_{V \in \mathcal{V}} \lambda V \leq \lambda_* W \leq \tilde{\lambda}W.$$

If we take a countable set $\mathcal{V}_0 \subseteq \mathcal{V}$ such that $\sup_{V \in \mathcal{V}_0} \lambda V = \sup_{V \in \mathcal{V}} \lambda V$, and set $W_0 = \bigcup \mathcal{V}_0$, then W_0 is an open set, belonging to λ and included in W, and $\lambda W_0 = \tilde{\lambda}W$.

Next, defining the functions f_V as in part (a) of this proof, we have $\int f_V d\mu = \lambda V$ for every $V \in \mathcal{V}$; and setting $g(x) = \nu W[\{x\}]$, we have $g(x) = \sup_{V \in \mathcal{V}} f_V(x)$ for every $x \in X^*$. So 414Ba tells us that

$$\int \nu W[\{x\}]\mu(dx) = \int g \, d\mu = \sup_{V \in \mathcal{V}} \int f_V d\mu = \sup_{V \in \mathcal{V}} \lambda V = \tilde{\lambda}W.$$

$$\int \mu W^{-1}[\{y\}]\nu(dy) = \tilde{\lambda}W.$$

(The point here is that while the arguments of part (b) of this proof give different roles to μ and ν, the actual construction performed in part (d) is symmetric between them.)

Now, given $G \in \mathfrak{T}$ and $H \in \mathfrak{S}$, set $W = G \times H$; then

$$\tilde{\lambda}(G \times H) = \int \mu W^{-1}[\{y\}]\nu(dy) = \int_H \mu G \nu(dy) = \mu G \cdot \nu H.$$

(v) Let E and F be the supports of μ, ν respectively. (By 411Nd these are defined.) Then $E \times F$ is a closed subset of $X \times Y$. Because $X \setminus E$ and $Y \setminus F$ are negligible, $E \times F$ is conegligible. If $W \subseteq X \times Y$ is an open set and $(x, y) \in W \cap (E \times F)$, there are open sets $G \subseteq X$, $H \subseteq Y$ such that $(x, y) \in G \times H \subseteq W$. Now

$$\lambda(W \cap (E \times F)) \geq \lambda((G \times H) \cap (E \times F)) = \mu(G \cap E) \cdot \nu(H \cap F) > 0.$$

This shows that $E \times F$ is self-supporting, so is the support of λ.

(vi), (vii), (viii) The same method works for all of these. Take $\mathcal{K}$ to be either the family of Borel subsets of $X \times Y$ (for (vi)) or the family of closed subsets of $X \times Y$ (for (vii)) or the family of closed compact subsets of $X \times Y$ (for (viii)); the essential features of $\mathcal{K}$, valid for all three cases, are that

(α) $K \setminus W \in \mathcal{K}$ whenever $K \in \mathcal{K}$ and $W \subseteq X \times Y$ is open,

(β) $K \cup K' \in \mathcal{K}$ for all K, $K' \in \mathcal{K}$,

(γ) $\bigcap_{n \in \mathbb{N}} K_n \in \mathcal{K}$ for every sequence $\langle K_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{K}$.

Now the hypotheses of each part are just what we need in order to be sure that λ is inner regular with respect to $\mathcal{K}$ (412Sd, 412Sa, 412Sb), and I am trying to show that $\tilde{\lambda}$ also is, without assuming that $\tilde{\lambda}$ is inner regular with respect to Λ, because this need not be true.

Take $Q \in \tilde{\Lambda}$ and $\gamma < \tilde{\lambda}Q$; then there is a $Q_0 \in \Lambda'$ such that $Q_0 \subseteq Q$ and $\gamma < \lambda' Q_0 < \infty$. We have a $Q_1 \in \Lambda$ and a sequence $\langle \mathcal{V}_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{V}$ such that $Q_0 \triangle Q_1 \subseteq \bigcup_{n \in \mathbb{N}} A(\mathcal{V}_n)$. Of course $\lambda Q_1 = \lambda' Q_0 > \gamma$; set $\epsilon = \frac{1}{4}(\lambda Q_1 - \gamma) > 0$. For each $n \in \mathbb{N}$, take K_n, $L_n \in \mathcal{K} \cap \Lambda$ such that

$$K_n \subseteq Q_1 \cap \bigcup \mathcal{V}_n', \quad \lambda((Q_1 \cap \bigcup \mathcal{V}_n') \setminus K_n) \leq 2^{-n}\epsilon,$$

$$L_n \subseteq Q_1, \quad \lambda(Q_1 \setminus L_n) \leq 2^{-n}\epsilon.$$

Set $K_n' = K_n \cup (L_n \setminus \bigcup \mathcal{V}_n)$, so that $K_n' \in \mathcal{K}$ and

$$\begin{aligned}\tilde{\lambda}(Q_1 \setminus K_n') &\leq \tilde{\lambda}(Q_1 \setminus L_n) + \tilde{\lambda}A(\mathcal{V}_n) + \tilde{\lambda}(Q_1 \cap \bigcup \mathcal{V}_n' \setminus K_n) \\ &= \lambda(Q_1 \setminus L_n) + \lambda(Q_1 \cap \bigcup \mathcal{V}_n' \setminus K_n) \leq 2^{-n+1}\epsilon.\end{aligned}$$

Now $K = \bigcap_{n \in \mathbb{N}} K_n'$ belongs to $\mathcal{K}$ and

$$K \subseteq Q_1 \setminus \bigcup_{n \in \mathbb{N}} A(\mathcal{V}_n) \subseteq Q_0 \subseteq Q$$

because $K'_n \subseteq Q_1 \setminus A(\mathcal{V}_n)$ for every n. And

$$\lambda(Q_1 \setminus K) \leq \sum_{n=0}^{\infty} \tilde{\lambda}(Q_1 \setminus K'_n) \leq 4\epsilon,$$

so $\tilde{\lambda}K = \lambda K \geq \gamma$. As Q and γ are arbitrary, $\tilde{\lambda}$ is inner regular with respect to $\mathcal{K}$.

417D Multiple products Just as with the c.l.d. product measure (see 251W), we can apply the construction of 417C repeatedly to obtain measures on the products of finite families of τ-additive measure spaces.

Proposition (a) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a finite family of effectively locally finite τ-additive topological measure spaces. Then there is a unique complete locally determined effectively locally finite τ-additive topological measure $\tilde{\lambda}$ on $X = \prod_{i\in I} X_i$, inner regular with respect to the Borel sets, such that $\tilde{\lambda}(\prod_{i\in I} G_i) = \prod_{i\in I} \mu_i G_i$ whenever $G_i \in \mathfrak{T}_i$ for every $i \in I$.

(b) If now $\langle I_k\rangle_{k\in K}$ is a partition of I, and $\tilde{\lambda}_k$ is the product measure defined by the construction of (a) on $Z_k = \prod_{i\in I_k} X_i$ for each $k \in K$, then the natural bijection between X and $\prod_{k\in K} Z_k$ identifies $\tilde{\lambda}$ with the product of the $\tilde{\lambda}_k$ defined by the construction of (a).

proof (a)(i) Suppose first that every μ_i is inner regular with respect to the Borel sets. Then a direct induction on $\#(I)$, using 417C for the inductive step, tells us that there is a measure $\tilde{\lambda}$ with the required properties. Note that, in 417C, (vi) ensures that (in the present context) all our product measures will be inner regular with respect to the Borel sets.

(ii) For the general case, apply (a) to $\mu_i\restriction\mathcal{B}(X_i)$, where $\mathcal{B}(X_i)$ is the Borel σ-algebra of X_i for each i.

(iii) To see that $\tilde{\lambda}$ is unique, suppose that λ' is another measure with the same properties. Let $\mathcal{U}$ be the set $\{\prod_{i\in I} G_i : G_i \in \mathfrak{T}_i \text{ for every } i \in I\}$, and $\mathcal{U}_s$ the set of finite unions of members of $\mathcal{U}$. Then $\tilde{\lambda}$ and λ' agree on $\mathcal{U}_s$. **P** Suppose that $U = \bigcup_{j\leq n} \prod_{i\in I} G_{ji}$, where $G_{ji} \in \mathfrak{T}_i$ for $j \leq n$, $i \in I$. If there is any j such that $\prod_{i\in I} \mu_i G_{ji} = \infty$, then $\tilde{\lambda}U = \lambda' U = \infty$. Otherwise, set $L = \{j : j \leq n, \prod_{i\in I} \mu_i G_{ji} > 0\}$ and $G^*_i = \bigcup_{j\in L} G_{ji}$ for $i \in I$. Then

$$\tilde{\lambda}(U \setminus (\textstyle\prod_{i\in I} G^*_i)) = 0 = \lambda'(U \setminus (\prod_{i\in I} G^*_i)).$$

On the other hand, $\mu_i G_{ji}$ must be finite whenever $j \in L$ and $i \in I$, so $\mu_i G^*_i$ is finite for every i. Consider $\mathcal{I} = \{\prod_{i\in I} G_i : G_i \in \mathfrak{T}_i, G_i \subseteq G^*_i \text{ for every } i \in I\}$. Then $V \cap V' \in \mathcal{I}$ for all V, $V' \in \mathcal{I}$, and $\tilde{\lambda}$, λ' agree on $\mathcal{I}$. It follows from the Monotone Class Theorem (136C), or otherwise, that $\tilde{\lambda}$ and λ' agree on the algebra of subsets of $\prod_{i\in I} G^*_i$ generated by $\mathcal{I}$. In particular, $\tilde{\lambda}(U \cap (G^* \times H^*)) = \lambda'(U \cap (G^* \times H^*))$, so that $\tilde{\lambda}U = \lambda' U$. **Q**

But now, because $\tilde{\lambda}$ and λ' are τ-additive,

$$\tilde{\lambda}W = \sup\{\tilde{\lambda}U : U \in \mathcal{U}_s, U \subseteq W\} = \sup\{\lambda' U : U \in \mathcal{U}_s, U \subseteq W\} = \lambda' W$$

for every open set $W \subseteq X$. Writing $\mathcal{B}(X)$ for the Borel σ-algebra of X, $\tilde{\lambda}\restriction\mathcal{B}(X)$ and $\lambda'\restriction\mathcal{B}(X)$ are effectively locally finite Borel measures which agree on the open sets, so must be equal, by 414L. Since both $\tilde{\lambda}$ and λ' are complete locally determined measures defined on $\mathcal{B}$ and inner regular with respect to $\mathcal{B}$, they also are equal, by 412L.

(b) Let λ' be the measure on X corresponding to the τ-additive product of the $\tilde{\lambda}_k$ on $\prod_{k\in K} Z_k$. Then λ' is an effectively locally finite complete locally determined τ-additive topological measure inner regular with respect to the zero sets, and if $G_i \in \mathfrak{T}_i$ for every $i \in I$ then

$$\lambda'(\textstyle\prod_{i\in I} G_i) = \prod_{k\in K} \tilde{\lambda}_k(\prod_{i\in I_k} G_i) = \prod_{k\in K} \prod_{i\in I_k} \mu_i G_i = \prod_{i\in I} \mu_i G_i,$$

so $\lambda' = \tilde{\lambda}$.

417E Theorem Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces, with product probability space (X, Λ, λ). Then λ is τ-additive, and has an extension to a τ-additive topological measure $\tilde{\lambda}$ on X. Moreover, we can arrange that:

(i) $\tilde{\lambda}$ is complete;

(ii) if Q is measured by $\tilde{\lambda}$, there is a $Q_1 \in \Lambda$ such that $\tilde{\lambda}(Q \triangle Q_1) = 0$; that is to say, the embedding $\Lambda \subseteq \tilde{\Lambda}$ induces an isomorphism between the measure algebras of λ and $\tilde{\lambda}$;

(iii) $\tilde{\lambda}W = \lambda_* W$ for every open set $W \subseteq X$, and $\tilde{\lambda}F = \lambda^* F$ for every closed set $F \subseteq X$;

(iv) the support of λ is the product of the supports of the μ_i;

(v) if λ is inner regular with respect to the Borel sets, so is $\tilde{\lambda}$;

(vi) if λ is inner regular with respect to the closed sets, so is $\tilde{\lambda}$;

(vii) if λ is tight, so is $\tilde{\lambda}$.

proof The strategy of the proof is the same as in 417C, subject to some obviously necessary modifications. The key step, showing that every union $\bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)$ has zero inner measure, is harder, but we do save a little work because we no longer have to worry about sets of infinite measure.

(a) I begin by setting up some machinery. Let $\mathcal{C}$ be the family of subsets of X expressible in the form $\prod_{i\in I} E_i$, where $E_i \in \Sigma_i$ for every i and $\{i : E_i \neq X_i\}$ is finite. Let $\mathcal{U} \subseteq \mathcal{C}$ be the standard basis for the topology $\mathfrak{T}$ of X, consisting of sets expressible as $\prod_{i\in I} G_i$ where $G_i \in \mathfrak{T}_i$ for every $i \in I$ and $\{i : G_i \neq X_i\}$ is finite. Write $\mathcal{U}_s$ for the set of finite unions of members of $\mathcal{U}$, and $\mathfrak{V}$ for the set of non-empty upwards-directed families in $\mathcal{U}_s$. Note that every member of $\mathcal{U}_s$ is determined by coordinates in some finite subset of I (definition: 254M).

If $J \subseteq I$, write λ_J for the product measure on $\prod_{i\in J} X_i$; we shall need $\lambda_\emptyset$, which is the unique probability measure on the single-point set $\{\emptyset\} = \prod_{i\in\emptyset} X_i$. For $J \subseteq I$, $v \in \prod_{i\in J} X_i$, $W \subseteq X$ set

$$f_W(v) = \lambda_{I\setminus J}\{w : (v, w) \in W\}$$

if this is defined, identifying $\prod_{i\in J} X_i \times \prod_{i\in I\setminus J} X_i$ with X.

(b) We need two easy facts.

(i) $f_W(v) = \int f_W(v^\frown t)\mu_j(dt)$ whenever $W \in \widehat{\bigotimes}_{i\in I}\Sigma_i$, $J \subseteq I$, $v \in \prod_{i\in J} X_i$ and $j \in I \setminus J$, writing $v^\frown t$ for the member of $\prod_{i\in J\cup\{j\}} X_i$ extending v and taking the value t at the coordinate j. **P** Let $\mathcal{A}$ be the family of sets W satisfying the property. Then $\mathcal{A}$ is a Dynkin class including $\mathcal{C}$, so includes the σ-algebra generated by $\mathcal{C}$, which is $\widehat{\bigotimes}_{i\in I}\Sigma_i$. **Q**

(ii) If $J \subseteq I$, $v \in \prod_{i\in J} X_i$, $j \in I \setminus J$ and $V \in \mathcal{U}_s$, and we set $g(t) = f_V(v^\frown t)$ for $t \in X_j$, then g is lower semi-continuous. **P** We can express V as $\bigcup_{n\le m}\prod_{i\in I} G_{ni}$, where $G_{ni} \subseteq X_i$ is open for every $n \le m$, $i \in I$. Now if $t \in X_j$, we shall have

$$\{w : (v^\frown t, w) \in V\} \subseteq \{w : (v^\frown t', w) \in V\}$$

whenever

$$t' \in H = X_j \cap \bigcap\{G_{nj} : n \le m,\ t \in G_{nj}\}.$$

So $g(t') \ge g(t)$ for every $t' \in H$, which is an open neighbourhood of t. As t is arbitrary, g is lower semi-continuous. **Q**

(c) For each $\mathcal{V} \in \mathfrak{V}$, fix, for the remainder of this proof, a countable $\mathcal{V}' \subseteq \mathcal{V}$ such that $\sup_{V\in\mathcal{V}'} \lambda V = \sup_{V\in\mathcal{V}} \lambda V$; because $\mathcal{V}$ is upwards-directed, we may suppose that $\mathcal{V}' = \{V_n : n \in \mathbb{N}\}$ for some non-decreasing sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{V}$. Set $A(\mathcal{V}) = \bigcup\mathcal{V} \setminus \bigcup\mathcal{V}'$.

? Suppose, if possible, that there is a sequence $\langle \mathcal{V}_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{V}$ such that $\lambda_*(\bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)) > 0$.

(i) We have $\lambda^*(X \setminus \bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)) < 1$; let $\langle C_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$ such that

$$X \setminus \bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n) \subseteq \bigcup_{n\in\mathbb{N}} C_n, \quad \sum_{n=0}^{\infty} \lambda C_n = \gamma_0 < 1$$

(see 254A-254C). For each $n \in \mathbb{N}$, express $\mathcal{V}'_n$ as $\{V_{nr} : r \in \mathbb{N}\}$ where $\langle V_{nr}\rangle_{r\in\mathbb{N}}$ is non-decreasing, and set $W_n = \bigcup\mathcal{V}'_n = \bigcup_{r\in\mathbb{N}} V_{nr}$. Let $J \subseteq I$ be a countable set such that every C_n and every V_{nr} is dependent on coordinates in J. Express J as $\bigcup_{k\in\mathbb{N}} J_k$ where $J_0 = \emptyset$ and, for each k, J_{k+1} is equal either to J_k or to J_k

with one point added. (As in the proof of 254Fa, I am using a formulation which will apply equally to finite and infinite I, though of course the case of finite I is elementary once we have 417C.)

(ii) For each $n \in \mathbb{N}$, set

$$W_n' = \bigcup_{k\in\mathbb{N}} \{x : x \in X,\ f_{W_n}(x\restriction J_k) = 1\}.$$

Then $\lambda(W_n' \setminus W_n) = 0$. **P** For any $k \in \mathbb{N}$, if we think of λ as the product of λ_{J_k} and $\lambda_{I\setminus J_k}$ and of f_{W_n} as a measurable function on $\prod_{i\in J_k} X_i$, we see that $\{x : f_{W_n}(x\restriction J_k) = 1\}$ is of the form $F_k \times \prod_{i\in I\setminus J_k} X_i$, where $F_k \subseteq \prod_{i\in J_k} X_i$ is measurable; and

$$\lambda((F_k \times \prod_{i\in I\setminus J_k} X_i) \setminus W_n) = \int_{F_k} (1 - f_{W_n}(v))\lambda_{J_k}(dv) = 0.$$

Summing over k, we see that $W_n' \setminus W_n$ is negligible. **Q**

Observe that every W_n', like W_n, is determined by coordinates in J. So $\bigcup_{n\in\mathbb{N}} W_n' \setminus W_n$ is of the form $E \times \prod_{i\in I\setminus J} X_i$ where $\lambda_J E = 0$ (254Ob). There is therefore a sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of measurable cylinders in $\prod_{i\in J} X_i$ such that $E \subseteq \bigcup_{n\in\mathbb{N}} D_n$ and $\sum_{n=0}^\infty \lambda_J D_n < 1 - \gamma_0$. Set $C_n' = \{x : x \in X,\ x\restriction J \in D_n\} \in \mathcal{C}$ for each n. Then $\bigcup_{n\in\mathbb{N}} W_n' \setminus W_n \subseteq \bigcup_{n\in\mathbb{N}} C_n'$, so

$$(X \setminus \bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)) \cup \bigcup_{n\in\mathbb{N}} W_n' \setminus W_n \subseteq \bigcup_{n\in\mathbb{N}} C_n \cup \bigcup_{n\in\mathbb{N}} C_n',$$

$$\gamma = \sum_{n=0}^\infty \lambda C_n + \sum_{n=0}^\infty \lambda C_n' < 1,$$

while each C_n and each C_n' is dependent on coordinates in a finite subset of J.

(iii) For $k \in \mathbb{N}$, let P_k be the set of those $v \in \prod_{i\in J_k} X_i$ such that

$$\sum_{n=0}^\infty f_{C_n}(v) + f_{C_n'}(v) \le \gamma, \quad f_V(v) \le f_{W_n}(v) \text{ for every } n \in \mathbb{N},\ V \in \mathcal{V}_n.$$

Our hypothesis is that

$$\sum_{n=0}^\infty f_{C_n}(\emptyset) + f_{C_n'}(\emptyset) = \sum_{n=0}^\infty \lambda C_n + \lambda C_n' \le \gamma,$$

and the $\mathcal{V}_n'$ were chosen such that

$$f_V(\emptyset) = \lambda V \le \lambda W_n = f_{W_n}(\emptyset)$$

for every $n \in \mathbb{N}$, $V \in \mathcal{V}_n$; that is, $\emptyset \in P_0$.

(iv) Now if $k \in \mathbb{N}$, $v \in P_k$ there is a $v' \in P_{k+1}$ extending v. **P** If $J_{k+1} = J_k$ we can take $v' = v$. Otherwise, $J_{k+1} = J_k \cup \{j\}$ for some $j \in I \setminus J_k$. Now

$$\gamma \ge \sum_{n=0}^\infty f_{C_n}(v) + f_{C_n'}(v) = \sum_{n=0}^\infty \int f_{C_n}(v^\frown t) + f_{C_n'}(v^\frown t)\,\mu_j(dt)$$

((b-i) above)

$$= \int \sum_{n=0}^\infty f_{C_n}(v^\frown t) + f_{C_n'}(v^\frown t)\,\mu_j(dt),$$

so

$$H = \{t : t \in X_j,\ \textstyle\sum_{n=0}^\infty f_{C_n}(v^\frown t) + f_{C_n'}(v^\frown t)\mu_j(dt) \le \gamma\}$$

has positive measure.

Next, for $V \in \mathcal{U}_s$, set $g_V(t) = f_V(v^\frown t)$ for each $t \in X_j$. Then g_V is lower semi-continuous, by (b-ii) above. For each $n \in \mathbb{N}$, $\{g_V : V \in \mathcal{V}_n\}$ is an upwards-directed family of lower semi-continuous functions, so its supremum g_n^* is also lower semi-continuous, and because μ_j is τ-additive,

$$\int g_n^* d\mu_j = \sup_{V\in\mathcal{V}_n} \int g_V d\mu_j = \sup_{V\in\mathcal{V}_n} f_V(v) \le f_{W_n}(v) = \int f_{W_n}(v^\frown t)\mu_j(dt)$$

(using 414B and (b-i) again). But also, because $\langle V_{nr}\rangle_{r\in\mathbb{N}}$ is non-decreasing and has union W_n,

$$f_{W_n}(v^\frown t) = \sup_{r\in\mathbb{N}} f_{V_{nr}}(v^\frown t) \le g_n^*(t)$$

for every $t \in X_j$. So we must have

$$f_{W_n}(v^\frown t) = g_n^*(t) \text{ a.e.}(t).$$

And this is true for every $n \in \mathbb{N}$.

There is therefore a $t \in H$ such that

$$f_{W_n}(v^\frown t) = g_n^*(v^\frown t) \text{ for every } n \in \mathbb{N}.$$

Fix on such a t and set $v' = v^\frown t \in \prod_{i \in J_{k+1}} X_i$; then $v' \in P_{k+1}$, as required. **Q**

(v) We can therefore choose a sequence $\langle v_k \rangle_{k \in \mathbb{N}}$ such that $v_k \in P_k$ and v_{k+1} extends v_k for each k. Choose $x \in X$ such that $x(i) = v_k(i)$ whenever $k \in \mathbb{N}$, $i \in J_k$ and $x(i)$ belongs to the support of μ_i whenever $i \in I \setminus J$. (Recall from 411Nd that μ_i does have a support.)

We need to know that if k, $n \in \mathbb{N}$ and $V \in \mathcal{V}_n$ then $f_{V \setminus W_n}(v_k) = 0$. **P** For any $r \in \mathbb{N}$ there is a $V' \in \mathcal{V}_n$ such that $V \cup V_{nr} \subseteq V'$, so

$$f_{V \cup V_{nr}}(v_k) \leq f_{V'}(v_k) \leq f_{W_n}(v_k),$$

and

$$f_{V \setminus W_n}(v_k) \leq f_{V \setminus V_{nr}}(v_k) = f_{V \cup V_{nr}}(v_k) - f_{V_{nr}}(v_k) \leq f_{W_n}(v_k) - f_{V_{nr}}(v_k) \to 0$$

as $r \to \infty$. **Q**

(vi) If $n \in \mathbb{N}$, then $x \notin C_n \cup C_n'$. **P** C_n and C_n' are determined by coordinates in a finite subset of J, so must be determined by coordinates in J_k for some $k \in \mathbb{N}$. Now $f_{C_n}(v_k) + f_{C_n'}(v_k) \leq \gamma < 1$, so $\{y : y \restriction J_k = v_k\}$ cannot be included in $C_n \cup C_n'$, and must be disjoint from it; accordingly $x \notin C_n \cup C_n'$. **Q**

(vii) Because

$$(X \setminus \bigcup_{n \in \mathbb{N}} A(\mathcal{V}_n)) \cup \bigcup_{n \in \mathbb{N}} W_n' \setminus W_n \subseteq \bigcup_{n \in \mathbb{N}} C_n \cup \bigcup_{n \in \mathbb{N}} C_n',$$

there is some $n \in \mathbb{N}$ such that

$$x \in A(\mathcal{V}_n) \setminus (W_n' \setminus W_n) \subseteq (\bigcup \mathcal{V}_n) \setminus W_n',$$

that is, there is some $V \in \mathcal{V}_n$ such that $x \in V \setminus W_n'$. Let $U \in \mathcal{U}$ be such that $x \in U \subseteq V$. Express U as $U' \cap U''$ where $U' \in \mathcal{U}$ is determined by coordinates in a finite subset of J, and $U'' \in \mathcal{U}$ is determined by coordinates in a finite subset of $I \setminus J$. Let $k \in \mathbb{N}$ be such that U' is determined by coordinates in J_k. Then

$$f_{U \setminus W_n}(v_k) \leq f_{V \setminus W_n}(v_k) = 0$$

by (v) above. Now

$$\{w : w \in \textstyle\prod_{i \in I \setminus J_k} X_i, (v_k, w) \in U \setminus W_n\} = \{w : (v_k, w) \in U'' \setminus W_n\}$$

(because $(v_k, w) = (x \restriction J_k, w) \in U'$ for every w), while

$$\{w : (v_k, w) \in U''\}, \quad \{w : (v_k, w) \in W_n\}$$

are stochastically independent because the former depends on coordinates in $I \setminus J$, while the latter depends on coordinates in $J \setminus J_k$. So we must have

$$\begin{aligned} 0 = f_{U \setminus W_n}(v_k) &= \lambda_{I \setminus J_k}\{w : (v_k, w) \in U \setminus W_n\} \\ &= \lambda_{I \setminus J_k}\{w : (v_k, w) \in U'' \setminus W_n\} \\ &= \lambda_{I \setminus J_k}\{w : (v_k, w) \in U''\}(1 - \lambda_{I \setminus J_k}\{w : (v_k, w) \in W_n\}). \end{aligned}$$

At this point, recall that $x(i)$ belongs to the support of μ_i for every $i \in I \setminus J$, while $x \in U''$. So if $U'' = \{y : y(i) \in H_i \text{ for } i \in K\}$, where $K \subseteq I \setminus J$ is finite and $H_i \subseteq X_i$ is open for every i, we must have $\mu_i H_i > 0$ for every i, and

$$\lambda_{I \setminus J_k}\{w : (v_k, w) \in U''\} = \textstyle\prod_{i \in K} \mu_i H_i > 0.$$

On the other hand, we are also supposing that $x \notin W_n'$, so

$$\lambda_{I \setminus J_k}\{w : (v_k, w) \in W_n\} = f_{W_n}(v_k) < 1.$$

But this means that we have expressed 0 as the product of two non-zero numbers, which is absurd. **X**

(d) Thus $\lambda_*(\bigcup_{n\in\mathbb{N}} A(\mathcal{V}_n)) = 0$ for every sequence $\langle\mathcal{V}_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{V}$. Accordingly there is an extension of λ to a measure $\tilde{\lambda}$ on X such that $\tilde{\lambda}A(\mathcal{V}) = 0$ for every $\mathcal{V} \in \mathfrak{V}$, the domain of $\tilde{\lambda}$ being

$$\tilde{\Lambda} = \{W\triangle A : W \in \Lambda,\ A \in \mathcal{A}^*\},$$

where $\mathcal{A}^*$ is the σ-ideal generated by $\{A(\mathcal{V}) : V \in \mathfrak{V}\}$ (417A).

Now $\tilde{\lambda}$ is a topological measure. **P** If $W \subseteq X$ is open, then $\mathcal{V} = \{V : V \in \mathcal{U}_s,\ V \subseteq W\}$ belongs to $\mathfrak{V}$, and $\bigcup\mathcal{V} = W$. Now $\bigcup\mathcal{V}' \in \Lambda$ (because $\mathcal{V}'$ is countable), so

$$W = \bigcup\mathcal{V}' \cup A(\mathcal{V})$$

is measured by $\tilde{\lambda}$. **Q**

Also, $\tilde{\lambda}$ is τ-additive. **P** Let $\mathcal{W}$ be a non-empty upwards-directed family of open subsets of X with union W^*. Set

$$\mathcal{V} = \{V : V \in \mathcal{U}_s,\ \exists\, W \in \mathcal{W},\ V \subseteq W\}.$$

Then $\mathcal{V} \in \mathfrak{V}$ and $\bigcup\mathcal{V} = W^*$, so $\tilde{\lambda}A(\mathcal{V}) = 0$ and

$$\tilde{\lambda}W^* = \tilde{\lambda}(\bigcup\mathcal{V}') = \sup_{V\in\mathcal{V}'}\tilde{\lambda}V \le \sup_{W\in\mathcal{W}}\tilde{\lambda}W \le \tilde{\lambda}W^*$$

(using the fact that $\mathcal{V}'$ is upwards-directed). As $\mathcal{W}$ is arbitrary, $\tilde{\lambda}$ is τ-additive. **Q**

Of course it follows at once that λ is also τ-additive.

(e) Now for the supplementary properties (i)-(vi) listed in the theorem.

(i) If $Q \in \tilde{\Lambda}$, $\tilde{\lambda}Q = 0$ and $Q' \subseteq Q$, then Q is expressible as $W\triangle A$ where $\lambda W = 0$ and $A \in \mathcal{A}^*$. But in this case (because λ is complete) $\lambda(Q' \cap W) = 0$ while $Q' \setminus W \subseteq A \in \mathcal{A}^*$, so $Q' \in \tilde{\Lambda}$. Thus $\tilde{\lambda}$ is complete.

(ii) As always, the construction ensures that every member of $\tilde{\Lambda}$ differs by a $\tilde{\lambda}$-negligible set from some member of Λ.

(iii) Let $W \subseteq X$ be an open set. Set $\mathcal{V} = \{V : V \in \mathcal{U}_s,\ V \subseteq W\}$. Then

$$\tilde{\lambda}W = \sup_{V\in\mathcal{V}}\tilde{\lambda}V = \sup_{V\in\mathcal{V}}\lambda V \le \lambda_*W \le \tilde{\lambda}W$$

just because $\tilde{\lambda}$ is a τ-additive extension of λ. Now if $F \subseteq X$ is closed,

$$\tilde{\lambda}F = 1 - \tilde{\lambda}(X \setminus F) = 1 - \lambda_*(X \setminus F) = \lambda^*F.$$

(iv) For each $i \in I$ write F_i for the support of μ_i, and set $F = \prod_{i\in I} F_i$. This is closed because every F_i is. Its complement is covered by the negligible open sets $\{x : x \in X,\ x(i) \in X_i \setminus F_i\}$ as i runs over I; as λ is τ-additive, the union of the negligible open sets is negligible, and F is conegligible. If $W \subseteq X$ is open and $x \in F \cap W$, let $U \in \mathcal{U}$ be such that $x \in U \subseteq W$. Express U as $\prod_{i\in I} G_i$ where $G_i \in \mathfrak{T}_i$ for every $i \in I$ and $J = \{i : G_i \ne X_i\}$ is finite. Then $x(i) \in G_i \cap F_i$, so $\mu_iG_i > 0$, for every i. Accordingly

$$\lambda(W \cap F) = \lambda W = \lambda U = \prod_{i\in J}\mu_iU_i > 0.$$

Thus F is self-supporting and is the support of λ.

(v), (vi), (vi) Use the same arguments as in the corresponding parts of 417C, this time using 412U to confirm that λ is inner regular with respect to the given family of sets.

417F Corollary Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that μ_i is inner regular with respect to the Borel sets for each i. Then there is a unique complete τ-additive topological measure $\tilde{\lambda}$ on $X = \prod_{i\in I} X_i$ which extends the ordinary product measure and is inner regular with respect to the Borel sets.

proof By 417E(v) we have a measure $\tilde{\lambda}$ with the right properties. If λ' is any other complete τ-additive topological measure, extending λ and inner regular with respect to the family $\mathcal{B}$ of Borel sets, then $\lambda'W = \tilde{\lambda}W$ for every open set $W \subseteq X$. **P** By the argument of (e-iii) of the proof of 417E, $\lambda'W = \lambda_*W = \tilde{\lambda}W$. **Q** By 414L, applied to the Borel measures $\lambda'\restriction\mathcal{B}$ and $\tilde{\lambda}\restriction\mathcal{B}$, $\lambda'W = \tilde{\lambda}W$ for every Borel set W. Now λ' and $\tilde{\lambda}$ are supposed to be complete topological probability measures inner regular with respect to $\mathcal{B}$, so they must be identical, by 412L or otherwise.

417G Notation In the context of 417D or 417F, I will call $\tilde\lambda$ the **τ-additive product measure** on $\prod_{i\in I} X_i$.

Note that the uniqueness assertions in 417D and 417F mean that for the products of finitely many probability spaces we do not need to distinguish between the two constructions. The latter also shows that we can relate 415E to the new method: if every $\mathfrak{T}_i$ is separable and metrizable and every μ_i is strictly positive, then the 'ordinary' product measure λ is a complete topological measure. Since it is also inner regular with respect to the Borel sets (412Uc), it must be exactly the τ-additive product measure as described here.

417H Fubini's theorem for τ-additive product measures Let $(X,\mathfrak{T},\Sigma,\mu)$ and $(Y,\mathfrak{S},\mathrm{T},\nu)$ be two complete locally determined effectively locally finite τ-additive topological measure spaces such that both μ and ν are inner regular with respect to the Borel sets. Let $\tilde\lambda$ be the τ-additive product measure on $X\times Y$, and $\tilde\Lambda$ its domain.

(a) Let f be a $[-\infty,\infty]$-valued function such that $\int f\,d\tilde\lambda$ is defined in $[-\infty,\infty]$ and $(X\times Y)\setminus\{(x,y) : (x,y)\in\operatorname{dom} f,\ f(x,y)=0\}$ can be covered by a set of the form $X\times\bigcup_{n\in\mathbb{N}} Y_n$ where $\nu Y_n<\infty$ for every $n\in\mathbb{N}$. Then the repeated integral $\iint f(x,y)\nu(dy)\mu(dx)$ is defined and equal to $\int f d\tilde\lambda$.

(b) Let $f: X\times Y\to[0,\infty]$ be lower semi-continuous. Then

$$\iint f(x,y)\nu(dy)\mu(dx)=\iint f(x,y)\mu(dx)\nu(dy)=\int f d\tilde\lambda$$

in $[0,\infty]$.

(c) Let f be a $\tilde\Lambda$-measurable real-valued function defined $\tilde\lambda$-a.e. on $X\times Y$. If either $\iint|f(x,y)|\nu(dy)\mu(dx)$ or $\iint|f(x,y)|\mu(dx)\nu(dy)$ is defined and finite, then f is $\tilde\lambda$-integrable.

proof (a) I use 252B.

(i) Write $\mathcal{W}$ for the set of those $W\in\tilde\Lambda$ such that $\int\nu W[\{x\}]\mu(dx)$ is defined in $[0,\infty]$ and equal to $\tilde\lambda W$. Then open sets belong to $\mathcal{W}$, by 417C(iv). Next, any Borel subset of an open set of finite measure belongs to $\mathcal{W}$. **P** If W_0 is an open set of finite measure, then $\{W : W\subseteq X\times Y,\ W\cap W_0\in\mathcal{W}\}$ is a Dynkin class containing every open set, so contains all Borel subsets of $X\times Y$. **Q**

Now suppose that $W\subseteq X\times Y$ is $\tilde\lambda$-negligible and included in $X\times\bigcup_{n\in\mathbb{N}}Y_n$, where $\nu Y_n<\infty$ for every n. Then $W\in\mathcal{W}$. **P** Set $A=\{x : x\in X,\ \nu^*W[\{x\}]>0\}$. For each n, let $H_n\subseteq Y$ be an open set of finite measure such that $\nu(Y_n\setminus H_n)\le 2^{-n}$; we may arrange that $H_{n+1}\supseteq H_n$ for each n. Set $H=\bigcup_{n\in\mathbb{N}}H_n$, so that $W[\{x\}]\setminus H\subseteq\bigcup_{n\in\mathbb{N}}Y_n\setminus H$ is negligible for every $x\in X$.

Fix an open set $G\subseteq X$ of finite measure and $n\in\mathbb{N}$ for the moment. Because $\tilde\lambda$ is inner regular with respect to the Borel sets, there is a Borel set $V\subseteq(G\times H_n)\setminus W$ such that $\tilde\lambda V=\tilde\lambda((G\times H_n)\setminus W)$, that is, $\tilde\lambda V'=0$, where $V'=(G\times H_n)\setminus V\supseteq(G\times H_n)\cap W$. We know that $V'\in\mathcal{W}$, so

$$\int\nu V'[\{x\}]dx=\tilde\lambda V'=0,$$

and $\nu V'[\{x\}]=0$ for almost every $x\in X$; but this means that $H_n\cap W[\{x\}]$ is negligible for almost every $x\in G$.

At this point, recall that n was arbitrary, so $H\cap W[\{x\}]$ and $W[\{x\}]$ are negligible for almost every $x\in G$, that is, $A\cap G$ is negligible. This is true for every open set $G\subseteq X$ of finite measure. Because μ is inner regular with respect to subsets of open sets of finite measure, and is complete and locally determined, A is negligible (412Jb). But this means that $\int\nu W[\{x\}]\mu(dx)$ is defined and equal to zero, so that $W\in\mathcal{W}$. **Q**

(ii) Now suppose that $\int f d\tilde\lambda$ is defined in $[-\infty,\infty]$ and that there is a sequence $\langle Y_n\rangle_{n\in\mathbb{N}}$ of sets of finite measure in Y such that $f(x,y)$ is defined and zero whenever $x\in X$ and $y\in Y\setminus\bigcup_{n\in\mathbb{N}}Y_n$. Set $Z=\bigcup_{n\in\mathbb{N}}Y_n$. Write λ for the c.l.d. product measure on $X\times Y$ and Λ for its domain. Then there is a Λ-measurable function $g: X\times Y\to[-\infty,\infty]$ which is equal $\tilde\lambda$-almost everywhere to f. **P** For $q\in\mathbb{Q}$ set $W_q=\{(x,y) : (x,y)\in\operatorname{dom} f,\ f(x,y)\ge q\}\in\tilde\Lambda$, and choose $V_q\in\Lambda$ such that $\tilde\lambda(W_q\triangle V_q)=0$ (417C(ii)); set $g(x,y)=\sup\{q : q\in\mathbb{Q},\ (x,y)\in V_q\}$ for $x\in X$, $y\in Y$, interpreting $\sup\emptyset$ as $-\infty$. **Q** Adjusting g if necessary, we may suppose that it is zero on $X\times(Y\setminus Z)$. Set

$$A=(X\times Y)\setminus\{(x,y) : f(x,y)=g(x,y)\},$$

so that A is $\tilde{\lambda}$-negligible and included in $X \times Z$. By (i), $\nu A[\{x\}] = 0$, that is, $y \mapsto f(x,y)$ and $y \mapsto g(x,y)$ are equal ν-a.e., for μ-almost every x. Write $\lambda_{X\times Z}$ for the subspace measure induced by λ on $X \times Z$; note that this is the c.l.d. product of μ with the subspace measure ν_Z on Z, by 251P(ii-α).

Now we have

$$\int f d\tilde{\lambda} = \int g\, d\tilde{\lambda} = \int g\, d\lambda$$

(by 235Ib, because the identity map from $(X \times Y, \tilde{\lambda})$ to $(X \times Y, \lambda)$ is inverse-measure-preserving)

$$= \int_{X\times Z} g\, d\lambda = \int_{X\times Z} g\, d\lambda_{X\times Z} = \iint_Z g(x,y)\nu_Z(dy)\mu(dx)$$

(by 252B, because ν_Z is σ-finite)

$$= \iint g(x,y)\nu(dy)\mu(dx)$$

(because $g(x,y) = 0$ if $y \in Y \setminus Z$)

$$= \iint f(x,y)\nu(dy)\mu(dx).$$

(b) If f is non-negative and lower semi-continuous, set

$$W_{ni} = \{(x,y) : f(x,y) > 2^{-n}i\}$$

for n, $i \in \mathbb{N}$, and

$$f_n = 2^{-n} \textstyle\sum_{i=1}^{4^n} \chi W_{ni}$$

for $n \in \mathbb{N}$. Applying 417C(iv) we see that

$$\int f_n d\tilde{\lambda} = \iint f_n(x,y)dydx = \iint f_n(x,y)dxdy$$

in $[0,\infty]$ for every n; taking the limit as $n \to \infty$,

$$\int f d\tilde{\lambda} = \iint f(x,y)dydx = \iint f(x,y)dxdy,$$

because $\langle f_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence with limit f.

(c) ❓ Suppose, if possible, that $\gamma = \iint |f(x,y)|dydx$ is finite, but that f is not integrable. Because $\tilde{\lambda}$ is semi-finite, there must be a non-negative $\tilde{\lambda}$-simple function g such that $g \leq_{\text{a.e.}} |f|$ and $\int g\, d\tilde{\lambda} > \gamma$ (213B). For each $n \in \mathbb{N}$, there are open sets $G_n \subseteq X$, $H_n \subseteq Y$ of finite measure such that $\tilde{\lambda}(\{(x,y) : g(x,y) \geq 2^{-n}\} \setminus (G_n \times H_n)) \leq 2^{-n}$, by 417C(iii); now $g \times \chi(G_n \times H_n) \to g$ a.e., so there is some n such that $\int_{G_n\times H_n} g\, d\tilde{\lambda} > \gamma$. In this case, setting $g'(x,y) = \min(g(x,y), |f(x,y)|)$ for $(x,y) \in (G_n \times H_n) \cap \operatorname{dom} f$, 0 otherwise, we have $g = g'$ a.e. on $G_n \times H_n$, so that $\int g' d\tilde{\lambda} > \gamma$. But we can apply (a) to g' to see that

$$\gamma < \int g' d\tilde{\lambda} = \iint g'(x,y)dydx \leq \iint |f(x,y)|dydx \leq \gamma,$$

which is absurd. ❌

So if $\iint |f(x,y)|dydx$ is finite, f must be $\tilde{\lambda}$-integrable. Of course the same arguments, reversing the roles of X and Y, show that f is $\tilde{\lambda}$-integrable if $\iint |f(x,y)|dxdy$ is defined and finite.

417I The constructions here have most of the properties one would hope for. I give several in the exercises (417Xd-417Xf, 417Xj). One fact which is particularly useful, and also has a trap in it, is the following.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces in which the measures are inner regular with respect to the Borel sets, and $\tilde{\lambda}$ the τ-additive product measure on $X \times Y$. Suppose that $A \subseteq X$ and $B \subseteq Y$, and write μ_A, ν_B for the corresponding subspace measures; assume that both μ_A and ν_B are semi-finite. Then these are also effectively locally finite, τ-additive and inner regular with respect to the Borel sets, and the subspace measure $\tilde{\lambda}_{A\times B}$ induced by $\tilde{\lambda}$ on $A \times B$ is just the τ-additive product measure of μ_A and ν_B.

proof (a) To check that μ_A and ν_B are effectively locally finite, τ-additive and inner regular with respect to the Borel sets, see 414K and 412P. Of course $\tilde{\lambda}_{A\times B}$ inherits the same properties from $\tilde{\lambda}$, and is also complete and locally determined, by 214Id.

(b) Now if $C \in \operatorname{dom}\mu_A$ and $D \in \operatorname{dom}\nu_B$, then $\tilde{\lambda}^*(C\times D) = \mu_A C \cdot \nu_B D$. **P** ($\alpha$) There are $E \in \Sigma$, $F \in \mathrm{T}$ such that $C \subseteq E$, $D \subseteq F$, $\mu E = \mu^* C$ and $\nu F = \nu^* D$; in which case

$$\tilde{\lambda}^*(C\times D) \leq \tilde{\lambda}(E\times F) = \lambda(E\times F) = \mu E\cdot\nu F$$

(251J)

$$= \mu^* C\cdot\nu^* D = \mu_A C\cdot\nu_B D.$$

(β) If $\gamma < \mu_A C\cdot\nu_B D$ then, because μ_A and ν_B are semi-finite, there are $C' \subseteq C$, $D' \subseteq D$ such that both have finite outer measure and $\mu^* C'\cdot\nu^* D' \geq \gamma$. In this case, take $E' \in \Sigma$, $F' \in \mathrm{T}$ such that $C' \subseteq E'$, $D' \subseteq F'$ and both E' and F' have finite measure. Now if $W \in \operatorname{dom}\tilde{\lambda}$ and $C\times D \subseteq W$, we have $C'\times D' \subseteq W \cap (E\times F)$, so that $\nu(W\cap(E\times F))[\{x\}] \geq \nu^* D'$ for every $x \in C'$, and

$$\tilde{\lambda}W \geq \int_E \nu(W\cap(E\times F))[\{x\}]\mu(dx) \geq \mu^* C'\cdot\nu^* D' \geq \gamma,$$

by 417Ha. As W is arbitrary, $\tilde{\lambda}^*(C\times D) \geq \gamma$; as γ is arbitrary, $\tilde{\lambda}^*(C\times D) \geq \mu_A C\cdot\nu_B D$. **Q**

(c) In particular, if $U \subseteq A$ and $V \subseteq B$ are relatively open,

$$\tilde{\lambda}_{A\times B}(U\times V) = \tilde{\lambda}^*(U\times V) = \mu_A U\cdot\nu_B V.$$

But now 417D tells us that $\tilde{\lambda}_{A\times B}$ must be exactly the τ-additive product measure of μ_A and ν_B.

417J In order to use 417H effectively in the theory of infinite products, we need an 'associative law' corresponding to 254N.

Theorem Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and $\langle K_j\rangle_{j\in J}$ a partition of I. For each $j \in J$ let $\tilde{\lambda}_j$ be the τ-additive product measure on $Z_j = \prod_{i\in K_j} X_i$, and write $\tilde{\lambda}$ for the τ-additive product measure on $X = \prod_{i\in I} X_i$. Then the natural bijection

$$x \mapsto \phi(x) = \langle x{\restriction}K_j\rangle_{j\in J} : X \to \prod_{j\in J} Z_j$$

identifies $\tilde{\lambda}$ with the τ-additive product of the family $\langle\tilde{\lambda}_j\rangle_{j\in J}$.

In particular, if $K \subseteq I$ is any set, then $\tilde{\lambda}$ can be identified with the τ-additive product of the τ-additive product measures on $\prod_{i\in K} X_i$ and $\prod_{i\in I\setminus K} X_i$.

proof We have a lot of measures to keep track of; I hope that the following notation will not be too opaque. Write λ for the ordinary product measure on X, and for $j \in J$ write λ_j, $\tilde{\lambda}_j$ for the ordinary and τ-additive product measures on Z_j. Write θ for the *ordinary* product measure on $Z = \prod_{j\in J} Z_j$ of the *τ-additive* product measures $\tilde{\lambda}_j$, and $\tilde{\theta}$ for the τ-additive product of the $\tilde{\lambda}_j$. Write $\tilde{\lambda}^\#$ for the measure on X corresponding to $\tilde{\theta}$ on Z. (If you like, $\tilde{\lambda}^\#$ is the image measure $\tilde{\theta}(\phi^{-1})^{-1}$ defined from $\tilde{\theta}$ and the function $\phi^{-1} : Z \to X$.) Then $\tilde{\lambda}^\#$, like $\tilde{\theta}$, is a complete τ-additive topological measure, inner regular with respect to the Borel sets, because $\phi : X \to Z$ is a homeomorphism. If $C \subseteq X$ is a measurable cylinder, it is of the form $\prod_{i\in I} E_i$ where $E_i \in \Sigma_i$ for each i and $\{i : i \in I, E_i \neq \Sigma_i\}$ is finite. So $\phi[C]$ is of the form $\prod_{j\in J} C_j$, where $C_j = \prod_{i\in K_j} E_i$, and

$$\tilde{\lambda}^\# C = \tilde{\theta}(\prod_{j\in J} C_j) = \theta(\prod_{j\in J} C_j) = \prod_{j\in J}\tilde{\lambda}_j C_j$$

$$= \prod_{j\in J}\lambda_j C_j = \prod_{j\in J}\prod_{i\in K_j} E_i = \prod_{i\in I} E_i = \lambda C.$$

But this means, applying 254G to the identity map from $(X, \tilde{\lambda}^\#)$ to (X, λ), that $\tilde{\lambda}^\#$ extends λ. So it is a complete τ-additive topological measure, inner regular with respect to the Borel sets, extending the ordinary product measure, and by the uniqueness declared in 417F, must be identical to the τ-additive product measure on X, as claimed.

417K Proposition Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and let $(X, \tilde\Lambda, \tilde\lambda)$ be their τ-additive product. For $J \subseteq I$ let $\tilde\lambda_J$ be the τ-additive product measure on $X_J = \prod_{i\in J} X_i$, and $\tilde\Lambda_J$ its domain; let $\pi_J : X \to X_J$ be the canonical map. Then $\tilde\lambda_J$ is the image measure $\tilde\lambda\pi_J^{-1}$. In particular, if $W \in \tilde\Lambda$ is determined by coordinates in $J \subseteq I$, then $\pi_J[W] \in \tilde\Lambda_J$ and $\tilde\lambda_J\pi_J[W] = \tilde\lambda W$.

proof Because $\tilde\lambda$ is an extension of the ordinary product measure λ on X, $\tilde\lambda\pi_J^{-1}$ is an extension of $\lambda\pi_J^{-1}$, which is the ordinary product measure on X_J (254Oa). Because $\tilde\lambda$ is a τ-additive topological measure and π_J is continuous, $\tilde\lambda\pi_J^{-1}$ is a τ-additive topological measure; because $\tilde\lambda$ is a complete probability measure, so is $\tilde\lambda\pi_J^{-1}$. Finally, $\tilde\lambda\pi_J^{-1}$ is inner regular with respect to the Borel sets. **P** Recall that we may identify $\tilde\lambda$ with the τ-additive product of $\tilde\lambda_J$ and $\tilde\lambda_{I\setminus J}$ (417J). If $V \in \operatorname{dom}\tilde\lambda\pi_J^{-1}$, that is, $\pi_J^{-1}[V] \in \tilde\Lambda$, we can think of $\pi_J^{-1}[V] \subseteq X$ as $V \times X_{I\setminus J} \subseteq X_I \times X_J$. In this case, we must have

$$\tilde\lambda\pi_J^{-1}[V] = \int \tilde\lambda_J V \, d\lambda_{I\setminus J},$$

by Fubini's theorem for τ-additive products (417Ha); that is, $\tilde\lambda_J V$ must be defined and equal to $\tilde\lambda\pi_J^{-1}[V]$. Now if $\gamma < \tilde\lambda\pi_J^{-1}[V]$, there must be a Borel set $V' \subseteq V$ such that $\tilde\lambda_J V' \geq \gamma$. In this case, because π_J is continuous, $\pi_J^{-1}[V']$ is also Borel, and $\tilde\lambda\pi_J^{-1}[V']$ is defined. As with V, this measure must be $\tilde\lambda_J V' \geq \gamma$. Since V and γ are arbitrary, $\tilde\lambda\pi_J^{-1}$ is inner regular with respect to the Borel sets, as claimed. **Q**

By the uniqueness assertion in 417F, $\tilde\lambda\pi_J^{-1}$ must be $\tilde\lambda_J$ exactly.

If now $W \in \tilde\Lambda$ is determined by coordinates in J, then

$$\tilde\lambda_J\pi_J[W] = \tilde\lambda\pi_J^{-1}[\pi_J[W]] = \tilde\lambda W.$$

417L Corollary Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and $(X, \tilde\Lambda, \tilde\lambda)$ their τ-additive product. Let $\langle K_j\rangle_{j\in J}$ be a disjoint family of subsets of I, and for $j \in J$ write $\tilde\Lambda_j$ for the σ-algebra of members of Λ determined by coordinates in K_j. Then $\langle\tilde\Lambda_j\rangle_{j\in J}$ is a stochastically independent family of σ-algebras (definition: 272Ab).

proof It is enough to consider the case in which J is finite (272Bb), no K_j is empty (since if $K_j = \emptyset$ then $\tilde\Lambda_j = \{\emptyset, X\}$) and $\bigcup_{j\in J} K_j = I$ (adding an extra term if necessary). In this case, if $W_j \in \tilde\Lambda_j$ for each j, then the identification between X and $\prod_{j\in J}\prod_{i\in K_j} X_i$, as described in 417J, matches $\bigcap_{j\in J} W_j$ with $\prod_{j\in J}\pi_{K_j}[W_j]$, writing $\pi_{K_j}(x)$ for $x{\restriction}K_j$. Now if $\tilde\lambda_j$ is the τ-additive product measure on $Z_j = \prod_{i\in K_j} X_i$, we have $\tilde\lambda_j\pi_{K_j}[W_j] = \tilde\lambda W_j$, by 417K. Since $\tilde\lambda$ can be identified with the τ-additive product of $\langle\tilde\lambda_j\rangle_{j\in J}$ (417J),

$$\tilde\lambda(\textstyle\bigcap_{j\in J} W_j) = \prod_{j\in J}\tilde\lambda_j\pi_{K_j}[W_j] = \prod_{j\in J}\tilde\lambda W_j.$$

As $\langle W_j\rangle_{j\in J}$ is arbitrary, $\langle\tilde\Lambda_j\rangle_{j\in J}$ is independent.

417M Proposition Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets and strictly positive. For $J \subseteq I$ let π_J be the canonical map from X onto $X_J = \prod_{i\in J} X_i$; write λ_J, $\tilde\lambda_J$ for the ordinary and τ-additive product measures on X_J, and Λ_J, $\tilde\Lambda_J$ for their domains. Set $\tilde\lambda = \tilde\lambda_I$, $\tilde\Lambda = \tilde\Lambda_I$, $\lambda = \lambda_I$, $\Lambda = \Lambda_I$.

(a) Let $F \subseteq X$ be a closed self-supporting set, and J the smallest subset of I such that F is determined by coordinates in J (4A2B(g-ii)). Then

(i) if $W \in \tilde\Lambda$ is such that $W\triangle F$ is $\tilde\lambda$-negligible and determined by coordinates in $K \subseteq I$, then $K \supseteq J$;

(ii) J is countable;

(iii) there is a $W \in \Lambda$, determined by coordinates in J, such that $W\triangle F$ is $\tilde\lambda$-negligible.

(b) $\tilde\lambda$ is inner regular with respect to the family of sets of the form $\bigcap_{n\in\mathbb{N}} V_n$ where each $V_n \in \tilde\Lambda$ is determined by finitely many coordinates.

(c) If $W \in \tilde\Lambda$, there are a countable $J \subseteq I$ and sets $W', W'' \in \tilde\Lambda$, determined by coordinates in J, such that $W' \subseteq W \subseteq W''$ and $\tilde\lambda(W'' \setminus W') = 0$. Consequently $\tilde\lambda\pi_J^{-1}[\pi_J[W]] = \tilde\lambda W$.

proof (a)(i) ? Suppose, if possible, otherwise. Then F is not determined by coordinates in K, so there are $x \in F$, $y \in X \setminus F$ such that $x\restriction K = y\restriction K$. Let U be an open set containing y, disjoint from F, and of the form $\prod_{i\in I} G_i$, where $G_i \in \mathfrak{T}_i$ for every i and $L = \{i : G_i \neq X_i\}$ is finite. Set

$$U' = \{z : z \in X, \, z(i) \in G_i \text{ for every } i \in L \cap K\},$$

$$U'' = \{z : z(i) \in G_i \text{ for every } i \in L \setminus K\}.$$

Then $U' \cap W$ is determined by coordinates in K, while U'' is determined by coordinates in $I \setminus K$, so

$$0 = \tilde{\lambda}(F \cap U) = \tilde{\lambda}(W \cap U) = \tilde{\lambda}(W \cap U' \cap U'') = \tilde{\lambda}(W \cap U') \cdot \tilde{\lambda}U''$$

(by 417L)

$$= \tilde{\lambda}(F \cap U') \cdot \tilde{\lambda}U'' = \tilde{\lambda}(F \cap U') \cdot \prod_{i\in L\setminus K} \mu_i G_i.$$

But $y \in U'$, and $x\restriction K = y\restriction K$, so $x \in F \cap U'$; as F is self-supporting, $\tilde{\lambda}(F \cap U') > 0$. Because every μ_i is strictly positive, and no G_i is empty, $\prod_{i\in L\setminus K} \mu_i G_i > 0$; and this is impossible. **X**

(ii) By 417E(ii), there is a $W_0 \in \Lambda$ such that $\tilde{\lambda}(F \triangle W_0) = 0$. By 254Oc there is a $W_1 \in \Lambda$, determined by coordinates in a countable subset K of I, such that $\lambda(W_0 \triangle W_1) = 0$. Now $\tilde{\lambda}(F \triangle W_1) = 0$, so (i) tells us that $J \subseteq K$ is countable.

(iii) By 417K, $\pi_J[F] \in \tilde{\Lambda}_J$. By 417E(ii), there is a $V \in \Lambda_J$ such that $V \triangle \pi_J[F]$ is $\tilde{\lambda}_J$-negligible. Set $W = \pi_J^{-1}[V]$. Then $W \in \Lambda$, W is determined by coordinates in J, and $W \triangle F = \pi_J^{-1}[V \triangle \pi_J[F]]$ is $\tilde{\lambda}$-negligible.

(b)(i) Write $\mathcal{V}$ for the set of those members of $\tilde{\Lambda}$ which are determined by finitely many coordinates, and $\mathcal{V}_\delta$ for the set of intersections of sequences in $\mathcal{V}$. Because $\mathcal{V}$ is closed under finite unions, so is $\mathcal{V}_\delta$; $\mathcal{V}_\delta$ is surely closed under countable intersections, and $\emptyset$, X belong to $\mathcal{V}_\delta$.

(ii) We need to know that every self-supporting closed set $F \subseteq X$ belongs to $\mathcal{V}_\delta$. **P** By (a), F is determined by a countable set J of coordinates. Express J as the union of a non-decreasing sequence $\langle J_n\rangle_{n\in\mathbb{N}}$ of finite sets. Then $F_n = \pi_{J_n}^{-1}[\overline{\pi_{J_n}[F]}] \in \mathcal{V}$ for each n, and $F = \bigcap_{n\in\mathbb{N}} F_n \in \mathcal{V}_\delta$. **Q**

(iii) Let $\mathcal{A}$ be the family of subsets of X which are either open or closed. Then if $A \in \mathcal{A}$, $V \in \tilde{\Lambda}$ and $\tilde{\lambda}(A \cap V) > 0$, there is a $K \in \mathcal{V}_\delta \cap \mathcal{A}$ such that $K \subseteq A$ and $\tilde{\lambda}(K \cap V) > 0$. **P** ($\alpha$) If A is open, set

$$\mathcal{U} = \{U : U \in \mathcal{V} \text{ is open}, \, U \subseteq A\}.$$

Because $\mathcal{V}$ includes a base for the topology of X, $\bigcup\mathcal{U} = A$; because $\tilde{\lambda}$ is τ-additive and $\mathcal{V}$ is closed under finite unions, there is a $U \in \mathcal{U}$ such that $U \subseteq A$ and $\tilde{\lambda}U > \tilde{\lambda}A - \tilde{\lambda}(A \cap V)$, so that $\tilde{\lambda}(U \cap V) > 0$. ($\beta$) If A is closed, then it includes a self-supporting closed set V of the same measure (414F), which belongs to $\mathcal{V}_\delta$, by (ii) just above. **Q**

(iv) By 412C, the restriction $\tilde{\lambda}\restriction\mathcal{B}$ of $\tilde{\lambda}$ to the Borel σ-algebra of X is inner regular with respect to $\mathcal{V}_\delta$. But $\tilde{\lambda}$ is just the completion of $\tilde{\lambda}\restriction\mathcal{B}$, so is also inner regular with respect to $\mathcal{V}_\delta$ (412Ha).

(c) By (b), we have sequences $\langle V_n\rangle_{n\in\mathbb{N}}$, $\langle V_n'\rangle_{n\in\mathbb{N}}$ in $\mathcal{V}_\delta$ such that $V_n \subseteq W$, $V_n' \subseteq X \setminus W$, $\tilde{\lambda}V_n \geq \tilde{\lambda}W - 2^{-n}$ and $\tilde{\lambda}V_n' \geq \tilde{\lambda}(X \setminus W) - 2^{-n}$ for every $n \in \mathbb{N}$. Each V_n, V_n' is determined by a countable set of coordinates, so there is a single countable set $J \subseteq I$ such that every V_n and every V_n' is determined by coordinates in J. Set $W' = \bigcup_{n\in\mathbb{N}} V_n$, $W'' = X \setminus \bigcup_{n\in\mathbb{N}} V_n'$; then W', W'' are both determined by coordinates in J, $W' \subseteq W \subseteq W''$ and $\tilde{\lambda}(W'' \setminus W') = 0$, as required.

417N Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be two quasi-Radon measure spaces. Then the τ-additive product measure $\tilde{\lambda}$ on $X \times Y$ is a quasi-Radon measure, the unique quasi-Radon measure on $X \times Y$ such that $\tilde{\lambda}(E \times F) = \mu E \cdot \nu F$ for every $E \in \Sigma$, $F \in \mathrm{T}$.

proof $\tilde{\lambda}$ is a complete, locally determined, effectively locally finite, τ-additive topological measure, inner regular with respect to the closed sets (417C(vii)). But this says just that it is a quasi-Radon measure. By 417D, it is the unique quasi-Radon measure with the right values on measurable rectangles.

417O Theorem Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of quasi-Radon probability spaces. Then the τ-additive product measure $\tilde{\lambda}$ on $X = \prod_{i\in I} X_i$ is a quasi-Radon measure, the unique quasi-Radon measure on X extending the ordinary product measure.

proof By 417E(vi), $\tilde{\lambda}$ is inner regular with respect to the closed sets, so is a quasi-Radon measure, which is unique by 417F.

417P Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be Radon measure spaces. Then the τ-additive product measure $\tilde{\lambda}$ on $X\times Y$ is a Radon measure, the unique Radon measure on $X\times Y$ such that $\tilde{\lambda}(E\times F) = \mu E\cdot\nu F$ whenever $E\in\Sigma$, $F\in\mathrm{T}$.

proof Of course $X\times Y$ is Hausdorff, and $\tilde{\lambda}$ is locally finite (because $\tilde{\lambda}(G\times H) = \mu G\cdot\nu H$ is finite whenever μG and νH are finite). By 417C(viii), $\tilde{\lambda}$ is tight, so is a Radon measure. As in 417N, it is uniquely defined by its values on measurable rectangles.

417Q Theorem Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of Radon probability spaces, and $\tilde{\lambda}$ the τ-additive product measure on $X = \prod_{i\in I} X_i$. For each $i\in I$, let $Z_i\subseteq X_i$ be the support of μ_i. Suppose that $J = \{i : i\in I, Z_i \text{ is not compact}\}$ is countable. Then $\tilde{\lambda}$ is a Radon measure, the unique Radon measure on X extending the ordinary product measure.

proof Of course X, being a product of Hausdorff spaces, is Hausdorff, and $\tilde{\lambda}$, being totally finite, is locally finite. Now, given $\epsilon\in\left]0,1\right]$, let $\langle\epsilon_j\rangle_{j\in J}$ be a family of strictly positive numbers such that $\sum_{j\in J}\epsilon_j\leq\epsilon$, and for $j\in J$ choose a compact set $K_j\subseteq X_j$ such that $\mu_j K_j\geq 1-\epsilon_j$; for $i\in I\setminus J$, set $K_i = Z_i$, so that K_i is compact and $\mu_i K_i = 1$. Consider $K = \prod_{i\in I} K_i$. Then, using 417E(iii) and 254Lb for the two equalities,

$$\tilde{\lambda}K = \lambda^* K = \prod_{i\in I}\mu_i K_i \geq \prod_{j\in J} 1-\epsilon_j \geq 1-\epsilon,$$

where λ is the ordinary product measure on X. As ϵ is arbitrary, $\tilde{\lambda}$ satisfies the condition (iv) of 416C, and is a Radon measure. As in 417F, it is the unique Radon measure on X extending λ.

417R Notation I will use the phrase **quasi-Radon product measure** for a τ-additive product measure which is in fact a quasi-Radon measure; similarly, a **Radon product measure** is a τ-additive product measure which is a Radon measure.

417S Later I will give an example in which a τ-additive product measure is different from the corresponding c.l.d. product measure (419E). In 415E-415F, 415Ye and 416U I have described cases in which c.l.d. measures are τ-additive product measures. It remains very unclear when to expect this to happen. I can however give a couple of results which show that sometimes, at least, we can be sure that the two measures coincide.

Proposition (a) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces such that both μ and ν are inner regular with respect to the Borel sets, and λ the c.l.d. product measure on $X\times Y$. If every open subset of $X\times Y$ is measured by λ, then λ is the τ-additive product measure on $X\times Y$.

(b) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and λ the ordinary product measure on $X = \prod_{i\in I} X_i$. If every open subset of X is measured by λ, then λ is the τ-additive product measure on X.

(c) In (b), let λ_J be the ordinary product measure on $X_J = \prod_{i\in J} X_i$ for each $J\subseteq I$, and $\tilde{\lambda}_J$ the τ-additive product measure. If $\lambda_J = \tilde{\lambda}_J$ for every finite $J\subseteq I$, and every μ_i is strictly positive, then $\lambda = \tilde{\lambda}_I$ is the τ-additive product measure on X.

proof (a), (b) In both cases, λ is a complete locally determined effectively locally finite τ-additive measure which is inner regular with respect to the Borel sets (assembling facts from 251I, 254F, 412S, 412U, 417C and 417E). The extra hypothesis added here is that λ is a topological measure, so itself satisfies the conditions of 417D or 417F, and is the τ-additive product measure.

(c)(i) The first step is to note that $\lambda_J = \tilde{\lambda}_J$ for every countable $J \subseteq I$. **P** Express J as $\bigcup_{n\in\mathbb{N}} J_n$ where $\langle J_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of finite sets. If $F \subseteq X_J$ is closed, then it is $\bigcap_{n\in\mathbb{N}} \pi_n^{-1}[\overline{\pi_n[F]}]$, where $\pi_n : X_J \to X_{J_n}$ is the canonical map for each n. But every $\overline{\pi_n[F]}$ is a closed subset of X_{J_n}, therefore measured by λ_{J_n}; because π_n is inverse-measure-preserving (417K), $\pi_n^{-1}[\overline{\pi_n[F]}] \in \operatorname{dom}\lambda_J$ for each n, and $F \in \operatorname{dom}\lambda_J$. Thus every closed set, therefore every open set is measured by λ_J, and λ_J is a topological measure; by (b), $\lambda_J = \tilde{\lambda}_J$. **Q**

(ii) Suppose that $W \subseteq X$ is open. By 417M, there are W', W'' measured by $\tilde{\lambda}$ such that $W' \subseteq W \subseteq W''$, both W'' and W' are determined by coordinates in a countable set, and $\tilde{\lambda}_I(W'' \setminus W') = 0$. Let $J \subseteq I$ be a countable set such that W' and W'' depend on coordinates in J. Then $\lambda_J = \tilde{\lambda}_J$ measures $\pi_J[W']$, by 417K, so λ measures $W' = \pi_J^{-1}[\pi_J[W]]$, by 254Oa. Similarly, λ measures W''. Now $\lambda(W''\setminus W') = \tilde{\lambda}_I(W''\setminus W') = 0$, so λ measures W. As W is arbitrary, λ is a topological measure and must be the τ-additive product measure, by (a).

417T Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces such that both μ and ν are inner regular with respect to the Borel sets, and λ the c.l.d. product measure on $X \times Y$. If X has a conegligible subset with a countable network (e.g., if X is separable and metrizable), then λ is the τ-additive product measure on $X \times Y$.

proof (a) Suppose to begin with that μ and ν are totally finite, and that X has a countable network; let $\langle A_n\rangle_{n\in\mathbb{N}}$ run over a network for X. Let $\hat{\mu}$ be the completion of μ and $\hat{\Sigma}$ its domain. Let $\tilde{\lambda}$ be the τ-additive product measure on $X \times Y$. (We are going to need Fubini's theorem both for λ and for $\tilde{\lambda}$. I will use a sprinkling of references to §§251-252 to indicate which parts of the argument below depend on the properties of λ.)

Let $W \subseteq X \times Y$ be an open set. For each $n \in \mathbb{N}$, set

$$H_n = \bigcup\{H : H \in \mathfrak{S},\ A_n \times H \subseteq W\},$$

so that H_n is open. Then $W = \bigcup_{n\in\mathbb{N}} A_n \times H_n$. **P** Of course $A_n \times H_n \subseteq W$ for every $n \in \mathbb{N}$. If $(x, y) \in W$, there are open sets $G \subseteq X$, $H \subseteq Y$ such that $(x, y) \in G \times H \subseteq W$; now there is an $n \in \mathbb{N}$ such that $x \in A_n \subseteq G$, so that $H \subseteq H_n$ and $(x, y) \in A_n \times H_n$. **Q**

By 417C(iv), there is an open set W_0 in the domain Λ of λ such that $W_0 \subseteq W$ and $\tilde{\lambda}(W \setminus W_0) = 0$. By 417Ha, applied to $\chi(W \setminus W_0)$, $A = \{x : \nu(W[\{x\}] \setminus W_0[\{x\}]) > 0\}$ is μ-negligible. For each $n \in \mathbb{N}$, $x \in X$ set $f_n(x) = \nu(H_n \cap W_0[\{x\}])$; then 252B tells us that $\int f_n d\mu$ is defined and equal to $\lambda(W_0 \cap (X \times H_n))$. In particular, f_n is $\hat{\Sigma}$-measurable. Set $E_n = \{x : f_n(x) = \nu H_n\} \in \hat{\Sigma}$. If $x \in A_n$, then $H_n \subseteq W[\{x\}]$, so $A_n \setminus E_n \subseteq A$.

Now, by 252B again,

$$\begin{aligned}\lambda((E_n \times H_n) \setminus W_0) &= \int_{E_n} \nu(H_n \setminus W_0[\{x\}])\mu(dx)\\ &= \int_{E_n} \nu H_n - \nu(H_n \cap W_0[\{x\}])\mu(dx) = 0.\end{aligned}$$

So if we set $W_1 = \bigcup_{n\in\mathbb{N}} E_n \times H_n$, $W_1 \setminus W \subseteq W_1 \setminus W_0$ is λ-negligible. On the other hand,

$$W \setminus W_1 \subseteq \bigcup_{n\in\mathbb{N}} (A_n \setminus E_n) \times H_n \subseteq A \times Y$$

is also λ-negligible. Because λ is complete, $W \in \Lambda$. As W is arbitrary, λ is a topological measure and is equal to $\tilde{\lambda}$, by 417Sa.

(b) Now consider the general case. Let Z be a conegligible subset of X with a countable network; since any subset of a space with a countable network again has a countable network (4A2Na), we may suppose that $Z \in \Sigma$. Again let W be an open set in $X \times Y$. This time, take arbitrary $E \in \Sigma$, $F \in \mathrm{T}$ of finite measure, and consider the subspace measures $\mu_{E\cap Z}$ and ν_F. These are still effectively locally finite and τ-additive (414K), and are now totally finite. Also $E \cap Z$ has a countable network. So (a) tells us that the relatively open set $W \cap ((E \cap Z) \times F)$ is measured by the c.l.d. product of $\mu_{E\cap Z}$ and ν_F, which is the subspace measure on $(E \cap Z) \times F$ induced by λ (251P). Since λ surely measures $E \times F$, it measures $W \cap (Z \times Y) \cap (E \times F)$. As

E and F are arbitrary, λ measures $W \cap (Z \times Y)$ (251H). But $\lambda((X \setminus Z) \times Y) = \mu(X \times Z) \cdot \nu Y = 0$ (251Ia), so λ also measures W. As W is arbitrary, λ is the τ-additive product measure.

417U Proposition Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces. Let λ be the ordinary product probability measure on $X = \prod_{i\in I} X_i$ and Λ its domain. Then every continuous function $f : X \to \mathbb{R}$ is Λ-measurable, so Λ includes the Baire σ-algebra of X.

proof (a) Let $\tilde\lambda$ be a τ-additive topological measure extending λ (417E), and $\tilde\Lambda$ its domain; then f is $\tilde\Lambda$-measurable, just because $\tilde\lambda$ is a topological measure. For $\alpha \in \mathbb{R}$, set

$$G_\alpha = \{x : x \in X,\ f(x) < \alpha\}, \quad H_\alpha = \{x : x \in X,\ f(x) > \alpha\},$$

$$F_\alpha = \{x : x \in X,\ f(x) = \alpha\}.$$

Then $\langle F_\alpha\rangle_{\alpha\in\mathbb{R}}$ is disjoint, so $A = \{\alpha : \alpha \in \mathbb{R},\ \tilde\lambda F_\alpha > 0\}$ is countable, and $A' = \mathbb{R} \setminus A$ is dense in $\mathbb{R}$; let $Q \subseteq A'$ be a countable dense set.

For each $q \in Q$, let $V_q \subseteq G_q$, $W_q \subseteq H_q$ be such that

$$\lambda V_q = \lambda_* G_q = \tilde\lambda G_q, \quad \lambda W_q = \lambda_* H_q = \tilde\lambda H_q$$

(413Ea, 417E(iii)). Then

$$\lambda^*(G_q \setminus V_q) \le \lambda(X \setminus (V_q \cup W_q)) = 1 - \lambda V_q - \lambda W_q = \tilde\lambda(X \setminus (G_q \cup H_q)) = 0.$$

Because λ is complete, $G_q \setminus V_q$ and G_q belong to Λ. But now, if $\alpha \in \mathbb{R}$,

$$\{x : f(x) < \alpha\} = \textstyle\bigcup_{q\in Q, q<\alpha} G_q \in \Lambda,$$

so f is Λ-measurable.

(b) It follows that every zero set belongs to Λ, so that Λ must include the Baire σ-algebra of X.

417V Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces, and $(X \times Y, \Lambda, \lambda)$ their c.l.d. product. Then every continuous function $f : X \times Y \to \mathbb{R}$ is Λ-measurable, and the Baire σ-algebra of $X \times Y$ is included in Λ.

proof Let $Z \subseteq X \times Y$ be a zero set. If $E \in \Sigma$, $F \in \mathrm{T}$ are sets of finite measure, then $Z \cap (E \times F)$ is a zero set for the relative topology of $E \times F$. Now the subspace measures μ_E and ν_F are τ-additive topological measures (414K), so $Z \cap (E \times F)$ is measured by the c.l.d. product $\mu_E \times \nu_F$ of μ_E and ν_F. **P** If either μE or νF is zero, this is trivial. Otherwise, they have scalar multiples μ'_E, ν'_F which are probability measures, and of course are still τ-additive topological measures. By 417U, $Z \cap (E \times F)$ is measured by $\mu'_E \times \nu'_F$. Since $\mu_E \times \nu_F$ is just a scalar multiple of $\mu'_E \times \nu'_F$, $Z \cap (E \times F)$ is measured by $\mu_E \times \nu_F$. **Q** But $\mu_E \times \nu_F$ is the subspace measure $\lambda_{E\times F}$ (251P), so $Z \cap (E \times F) \in \Lambda$. As E and F are arbitrary, $Z \in \Lambda$ (251H).

Thus every zero set belongs to Λ; accordingly Λ must include the Baire σ-algebra, and every continuous function must be Λ-measurable.

417X Basic exercises (a) Let (X, Σ, μ) be a semi-finite measure space and $\mathcal{A}$ a family of subsets of X. Show that the following are equiveridical: (i) there is a measure μ' on X, extending μ, such that $\mu' A = 0$ for every $A \in \mathcal{A}$; (ii) $\mu_*(\bigcup_{n\in\mathbb{N}} A_n) = 0$ for every sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{A}$.

(b) Let (X, Σ, μ) and (Y, T, ν) be measure spaces with topologies with respect to which μ and ν are locally finite. Show that the c.l.d. product measure on $X \times Y$ is locally finite.

>(c) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be topological measure spaces such that μ and ν are both effectively locally finite τ-additive Borel measures. Show that there is a unique effectively locally finite τ-additive Borel measure λ' on $X \times Y$ such that $\lambda'(G \times H) = \mu G \cdot \nu H$ for all open sets $G \subseteq X$, $H \subseteq Y$.

>(d) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of topological probability spaces in which every μ_i is a τ-additive Borel measure. Show that there is a unique τ-additive Borel measure λ' on $X = \prod_{i\in I} X_i$ such that $\lambda'(\prod_{i\in I} F_i) = \prod_{i\in I} \mu_i F_i$ whenever $F_i \subseteq X_i$ is closed for every $i \in I$.

(e) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces in which the measures are inner regular with respect to the Borel sets, and $\tilde{\lambda}$ the τ-additive product measure on $X \times Y$. Let $\langle X_i \rangle_{i \in I}$, $\langle Y_j \rangle_{j \in J}$ be decompositions for μ, ν respectively (definition: 211E). Show that $\langle X_i \times Y_j \rangle_{i \in I, j \in J}$ is a decomposition for $\tilde{\lambda}$. (Cf. 251N.)

>(f) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of τ-additive topological probability spaces such that μ_i is inner regular with respect to the Borel sets for every i, and $\tilde{\lambda}$ the τ-additive product measure on $X = \prod_{i \in I} X_i$. Take $A_i \subseteq X_i$ for each $i \in I$. (i) Show that if $\mu_i^* A_i = 1$ for every i, then the subspace measure induced by $\tilde{\lambda}$ on $A = \prod_{i \in I} A_i$ is just the τ-additive product $\tilde{\lambda}^{\#}$ of the subspace measures on the A_i. (*Hint*: show that if we set $\lambda' W = \tilde{\lambda}^{\#}(W \cap A)$ for Borel sets $W \subseteq X$, then λ' satisfies the conditions of 417Xd.) (ii) Show that in any case $\tilde{\lambda}^* A = \prod_{i \in I} \mu_i^* A_i$. (Cf. 254L.)

(g) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ and $\langle (Y_i, \mathfrak{S}_i, \mathrm{T}_i, \nu_i) \rangle_{i \in I}$ be two families of τ-additive topological probability spaces in which every μ_i and every ν_i is inner regular with respect to the Borel sets. Let $\tilde{\lambda}$, $\tilde{\lambda}'$ be the τ-additive product measures on $X = \prod_{i \in I} X_i$ and $Y = \prod_{i \in I} Y_i$ respectively. Suppose that for each $i \in I$ we are given a continuous inverse-measure-preserving function $\phi_i : X_i \to Y_i$. Show that the function $\phi : X \to Y$ defined by setting $\phi(x)(i) = \phi_i(x(i))$ for $x \in X$, $i \in I$ is inverse-measure-preserving.

(h) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be two complete locally determined effectively locally finite τ-additive topological measure spaces such that both μ and ν are inner regular with respect to the Borel sets. Let $\tilde{\lambda}$ be the τ-additive product measure on $X \times Y$, and $\tilde{\Lambda}$ its domain. Suppose that ν is σ-finite. Show that for any $W \in \tilde{\Lambda}$, $W[\{x\}] \in \mathrm{T}$ for almost every $x \in X$, and $x \mapsto \nu W[\{x\}]$ is measurable.

>(i) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be $[0,1]$ with its usual topology and Lebesgue measure, and let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be $[0,1]$ with the discrete topology and counting measure. (i) Show that both are Radon measure spaces. (ii) Show that the c.l.d. product measure on $X \times Y$ is a Radon measure. (*Hint*: 252Kc, or use 417T and 417P.) (iii) Show that 417Ha can fail if we omit the hypothesis on $\{(x,y) : f(x,y) \neq 0\}$.

(j) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be two effectively locally finite τ-additive topological measure spaces. Let λ be the c.l.d. product measure and $\tilde{\lambda}$ the τ-additive product measure on $X \times Y$. Show that $\lambda^*(A \times B) = \tilde{\lambda}^*(A \times B)$ for all sets $A \subseteq X$, $B \subseteq Y$. (*Hint*: start with A, B of finite outer measure, so that 417I applies.)

(k) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of τ-additive topological probability spaces with strictly positive measures all inner regular with respect to the Borel sets, and $(X, \mathfrak{T}, \tilde{\Lambda}, \tilde{\lambda})$ their τ-additive product. For $J \subseteq I$ let $\tilde{\lambda}_J$ be the τ-additive product measure on $X_J = \prod_{i \in J} X_i$, and $\tilde{\Lambda}_J$ its domain. (i) Show that if f is a real-valued $\tilde{\Lambda}$-measurable function defined $\tilde{\lambda}$-almost everywhere on X, we can find a countable set $J \subseteq I$ and a $\tilde{\Lambda}_J$-measurable function g, defined $\tilde{\lambda}_J$-almost everywhere on X_J, such that f extends $g\pi_J$. (ii) In (i), show that $\int f d\tilde{\lambda} = \int g \, d\tilde{\lambda}_J$ if either is defined in $[-\infty, \infty]$.

(l) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and $(X, \mathfrak{T}, \tilde{\Lambda}, \tilde{\lambda})$ their τ-additive product. Show that for any $W \in \tilde{\Lambda}$ there is a smallest set $J \subseteq I$ for which there is a $W' \in \tilde{\Lambda}$, determined by coordinates in J, with $\tilde{\lambda}(W \triangle W') = 0$. (*Hint*: 254R.)

(m) What needs to be added to 417M and 415Xk to complete a proof of 415E?

(n) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an atomless τ-additive topological probability space such that μ is inner regular with respect to the Borel sets, and I a set of cardinal at most that of the support of μ. Show that the set of injective functions from I to X has full outer measure for the τ-additive product measure on X^I.

>(o) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be Radon measure spaces. Show that the Radon product measure on $X \times Y$ is the unique Radon measure $\tilde{\lambda}$ such that $\tilde{\lambda}(K \times L) = \mu K \cdot \nu L$ for all compact sets $K \subseteq X$, $L \subseteq Y$.

>(p) Let I be an uncountable set, and for each $i \in I$ let X_i be $\{0,1\}$, $\mathfrak{T}_i = \mathcal{P}\{0,1\}$ the usual topology, and μ_i the measure on $\Sigma_i = \mathcal{P}\{0,1\}$ defined by saying that $\mu_i A = 1$ if $0 \in A$, 0 otherwise. Check that

$(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)$ is a Radon probability space. Let λ, $\tilde{\lambda}$ be the ordinary and τ-additive product measures on $X = \prod_{i\in I} X_i = \{0,1\}^I$. Show that they are different. Show that the support of $\tilde{\lambda}$ is not determined by any countable set of coordinates. Find a $\tilde{\lambda}$-negligible open set $W \subseteq X$ such that its projection onto $\{0,1\}^J$ is conegligible for every countable $J \subseteq I$.

(q) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of Radon probability spaces, and $\tilde{\lambda}$ the quasi-Radon product measure on $X = \prod_{i\in I} X_i$. For each $i \in I$, let $Z_i \subseteq X_i$ be the support of μ_i. Show that $\tilde{\lambda}$ is a Radon measure iff $\{i : i \in I,\ Z_i \text{ is not compact}\}$ is countable. In particular, show that the ordinary product measure on $[0,1[^I$, where I is uncountable and each copy of $[0,1[$ is given Lebesgue measure, is a quasi-Radon measure, but not a Radon measure.

(r) Let $\langle (X_n, \mathfrak{T}_n, \Sigma_n, \mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of Radon probability spaces. Show that the Radon product measure on $X = \prod_{n\in\mathbb{N}} X_n$ is the unique Radon measure $\tilde{\lambda}$ on X such that $\tilde{\lambda}(\prod_{n\in\mathbb{N}} K_n) = \prod_{n=0}^{\infty} \mu_n K_n$ whenever $K_n \subseteq X_n$ is compact for every n.

(s) Let $(X, \mathfrak{T}, \Sigma, \nu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be two topological measure spaces, of which $(X, \mathfrak{T})$ has a countable network. (i) Show that the c.l.d. product measure λ on $X \times Y$ is a topological measure. (ii) Show directly, without relying on ideas from 417D, that λ is τ-additive if ν is.

(t) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and λ, $\tilde{\lambda}$ the ordinary and τ-additive product measures on $X = \prod_{i\in I} X_i$. Show that if $A \subseteq X$ has $\tilde{\lambda}$-negligible boundary, then A is measured by λ.

(u) Let us say that a topological space X is **chargeable** if there is an additive functional $\nu : \mathcal{P}X \to [0,\infty[$ such that $\nu G > 0$ for every non-empty open set $G \subseteq X$. (i) Show that if there is a σ-finite measure μ on X such that $\mu_* G > 0$ for every non-empty open set G, then X is chargeable. (*Hint*: 215B(vii), 391G.) (ii) Show that any separable space is chargeable. (iii) Show that X is chargeable iff its regular open algebra is chargeable in the sense of 391X. (*Hint*: see the proof of 314P.) (iv) Show that any open subspace of a chargeable space is chargeable. (v) Show that if $Y \subseteq X$ is dense, then X is chargeable iff Y is chargeable. (vi) Show that if X is expressible as the union of countably many chargeable subspaces, then it is chargeable. (vii) Show that any product of chargeable spaces is chargeable. (Cf. 391Xb(iii).) (viii) Show that if $\langle X_i\rangle_{i\in I}$ is a family of chargeable spaces with product X, then every regular open subset of X and every Baire subset of X is determined by coordinates in a countable set. (*Hint*: 4A2Eb, 4A3Mb.)

(v) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be quasi-Radon measure spaces such that $\mu X \cdot \nu Y > 0$. Show that the quasi-Radon product measure on $X \times Y$ is completion regular iff it is equal to the c.l.d. product measure and μ and ν are both completion regular. (*Hint*: 412Sc; if μE, νF are finite and $Z \subseteq E \times F$ is a zero set of positive measure, use Fubini's theorem to show that Z has sections of positive measure.)

(w) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of quasi-Radon probability spaces. Show that the quasi-Radon product measure on $\prod_{i\in I} X_i$ is completion regular iff it is equal to the ordinary product measure and every μ_i is completion regular.

(x) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and λ the τ-additive product measure on $X = \prod_{i\in I} X_i$; write Λ for its domain. (i) Show that if $W \in \Lambda$, $\lambda W > 0$ and $\epsilon > 0$ then there are a finite $J \subseteq I$ and a $W' \in \Lambda$ such that $\lambda W' \geq 1 - \epsilon$ and for every $x \in W'$ there is a $y \in W$ such that $x{\restriction}I \setminus J = y{\restriction}I \setminus J$. (Cf. 254Sb.) (ii) Show that if $A \subseteq X$ is determined by coordinates in $I \setminus \{i\}$ for every $i \in I$ then $\lambda^* A \in \{0,1\}$. (Cf. 254Sa.)

417Y Further exercises (a) Give an example to show that, in 417A, λ can be strictly localizable while λ' is not.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be effectively locally finite τ-additive topological measure spaces such that μ and ν are both inner regular with respect to the Borel sets.

(i) Fix open sets $G \subseteq X$, $H \subseteq Y$ of finite measure. Let $\mathcal{W}_{GH}$ be the set of those $W \subseteq X \times Y$ such that $\theta_{GH}(W) = \int_G \hat{\nu}(W[\{x\}] \cap H)dx$ is defined, where $\hat{\nu}$ is the completion of ν. (α) Show that every

open set belongs to $\mathcal{W}_{GH}$. (β) Show that θ_{GH} is countably additive in the sense that $\theta_{GH}(\bigcup_{n\in\mathbb{N}} W_n) = \sum_{n=0}^{\infty} \theta_{GH}(W_n)$ for every disjoint sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{W}_{GH}$, and τ-additive in the sense that $\theta_{GH}(\bigcup \mathcal{V}) = \sup_{V\in\mathcal{V}} \theta_{GH}(V)$ for every non-empty upwards-directed family $\mathcal{V}$ of open sets in $X\times Y$. (γ) Show that every Borel set belongs to $\mathcal{W}_{GH}$. (*Hint*: Monotone Class Theorem.) (δ) Writing $\mathcal{B}$ for the Borel σ-algebra of $X\times Y$, show that $\theta_{GH}\restriction\mathcal{B}$ is a τ-additive Borel measure; let λ_{GH} be its completion. (ϵ) Show that $\lambda_{GH} = \theta_{GH}\restriction\Lambda_{GH}$, where $\Lambda_{GH} = \operatorname{dom}\lambda_{GH}$. ($\zeta$) Show that $\lambda_{GH}(E\times F)$ is defined and equal to $\mu E\cdot\nu F$ whenever $E\in\Sigma$, $F\in\mathrm{T}$, $E\subseteq G$ and $F\subseteq H$. (*Hint*: start with open E and F, move to Borel E and F with the Monotone Class Theorem.) (η) Writing λ for the c.l.d. product measure on $X\times Y$, show that $\lambda_{GH}(W)$ is defined and equal to $\lambda(W\cap(G\times H))$ whenever $W\in\operatorname{dom}\lambda$.

(ii) Now take $\tilde{\Lambda}$ to be $\bigcap\{\Lambda_{GH} : G\in\mathfrak{T}, H\in\mathfrak{S}, \mu G<\infty, \nu H<\infty\}$ and $\tilde{\lambda}W = \sup_{G,H}\lambda_{GH}(W)$ for $W\in\tilde{\Lambda}$. Show that $\tilde{\lambda}$ is an extension of λ to a complete locally determined effectively locally finite τ-additive topological measure on $X\times Y$ which is inner regular with respect to the Borel sets, so is the τ-additive product measure as defined in 417G.

(c) Let (X,Σ,μ) and (Y,T,ν) be complete measure spaces with topologies $\mathfrak{T}$, $\mathfrak{S}$. Suppose that μ and ν are effectively locally finite and τ-additive and moreover that their domains include bases for the two topologies. Show that the c.l.d. product measure on $X\times Y$ has the same properties. (*Hint*: start by assuming that μX and νY are both finite. If $\mathcal{V}$ is an upwards-directed family of measurable open sets with measurable open union W, look at $g_V(x) = \nu V[\{x\}]$ for $V\in\mathcal{V}$.)

(d) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces such that every μ_i is inner regular with respect to the Borel sets, and $(X,\mathfrak{T},\tilde{\Lambda},\tilde{\lambda})$ their τ-additive product. (i) Show that the following are equiveridical: (α) μ_i is strictly positive for all but countably many $i\in I$; (β) whenever $W\in\tilde{\Lambda}$ there are a countable $J\subseteq I$ and $W_1, W_2\in\tilde{\Lambda}$, determined by coordinates in J, such that $W_1\subseteq W\subseteq W_2$ and $\tilde{\lambda}(W_2\setminus W_1) = 0$. (ii) Show that when these are false, $\tilde{\lambda}$ cannot be equal to the ordinary product measure on X.

(e) Let (X,Σ,μ) and (Y,T,ν) be measure spaces with topologies $\mathfrak{T}$, $\mathfrak{S}$ such that both μ and ν are inner regular with respect to the families of sequentially compact sets in each space. Show that the c.l.d. product measure λ on $X\times Y$ is also inner regular with respect to the sequentially compact sets, so has an extension to a topological measure which is inner regular with respect to the sequentially compact sets. (*Hint*: 412R, 416Yc.)

(f) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of probability spaces with topologies $\mathfrak{T}_i$ such that every μ_i is inner regular with respect to the family of closed countably compact sets in X_i and every X_i is compact. Show that the ordinary product measure λ on $X = \prod_{i\in I} X_i$ is also inner regular with respect to the closed countably compact sets, so has an extension to a topological measure $\tilde{\lambda}$ which is inner regular with respect to the closed countably compact sets in X. Show that this can be done in such a way that for every $W\in\operatorname{dom}\tilde{\lambda}$ there is a $V\in\operatorname{dom}\lambda$ such that $\tilde{\lambda}(W\triangle V) = 0$. (*Hint*: 412T, 416Yb.)

(g) Let $\langle(X_n,\Sigma_n,\mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of probability spaces with topologies $\mathfrak{T}_n$ such that every μ_n is inner regular with respect to the family of sequentially compact sets in X_n. Show that the ordinary product measure λ on $X = \prod_{n\in\mathbb{N}} X_n$ is also inner regular with respect to the sequentially compact sets, so has an extension to a topological measure $\tilde{\lambda}$ which is inner regular with respect to the sequentially compact sets in X. Show that this can be done in such a way that for every $W\in\operatorname{dom}\tilde{\lambda}$ there is a $V\in\operatorname{dom}\lambda$ such that $\tilde{\lambda}(W\triangle V) = 0$.

(h) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of quasi-Radon probability spaces, and λ, $\tilde{\lambda}$ the ordinary and quasi-Radon product measures on $X = \prod_{i\in I} X_i$. Suppose that all but *one* of the $\mathfrak{T}_i$ have countable networks and all but *countably* many of the μ_i are strictly positive. Show that $\lambda = \tilde{\lambda}$.

(i) Let us say that a quasi-Radon measure space $(X,\mathfrak{T},\Sigma,\mu)$ has the **simple product property** if the c.l.d. product measure on $X\times Y$ is equal to the quasi-Radon product measure for every quasi-Radon measure space $(Y,\mathfrak{S},\mathrm{T},\nu)$. (i) Show that if $(X,\mathfrak{T})$ has a countable network then $(X,\mathfrak{T},\Sigma,\mu)$ has the simple

product property. (ii) Show that if a quasi-Radon measure space has the simple product property so do all its subspaces. (iii) Show that the quasi-Radon product of two quasi-Radon measure spaces with the simple product property has the simple product property. (iv) Show that the quasi-Radon product of any family of quasi-Radon probability spaces with the simple product property has the simple product property. (v) Show that the real line with the right-facing Sorgenfrey topology (415Xc) and Lebesgue measure has the simple product property.

417 Notes and comments The general problem of determining just when a measure can be extended to a measure with given properties is one which will recur throughout this volume. I have more than once mentioned the Banach-Ulam problem; if you like, this is the question of whether there can ever be an extension of the countable-cocountable measure on a set X to a measure defined on the whole algebra $\mathcal{P}X$. This particular question appears to be undecidable from the ordinary axioms of set theory; but for many sets (for instance, if $X = \omega_1$) it is known that the answer is 'no'. (See 419G, 438C.) This being so, we have to take each manifestation of the general question on its own merits. In 417C and 417E the challenge is to take a product measure λ defined in terms of the factor measures alone, disregarding their topological properties, and extend it to a topological measure, preferably τ-additive. Of course there are important cases in which λ is itself already a topological measure; for instance, we know that the c.l.d. product of Lebesgue measure on $\mathbb{R}$ with itself is Lebesgue measure on $\mathbb{R}^2$ (251M), and other examples are in 415E, 415Ye, 416U, 417S-417T and 417Yk. But in general not every open set in the product belongs to the domain of λ, even when we have the product of two Radon measures on compact Hausdorff spaces (419E).

Once we have resolved to grasp the nettle, however, there is a natural strategy for the proof. It is easy to see that if λ, in 417C or 417E, is to have an extension to a τ-additive topological measure $\tilde{\lambda}$, then we must have $\tilde{\lambda}A(\mathcal{V}) = 0$ for every $\mathcal{V}$ belonging to the class $\mathfrak{V}$. Now 417A descibes a sufficient (and obviously necessary) condition for there to be an extension of λ with this property. So all we have to do is check. The check is not perfectly straightforward; in 417E it uses all the resources of the original proof that there is a product measure on an arbitrary product of probability spaces (which I suppose is to be expected), with 414B (of course) to apply the hypothesis that the factor measures are τ-additive, and a couple of extra wrinkles (the W'_n and C'_n of part (c-ii) of the proof of 417E, and the use of supports in part (c-v)).

It is worth noting that (both for finite and for infinite products) the measure algebras of λ and $\tilde{\lambda}$ are identical (417C(ii), 417E(ii)), so there is no new work to do in identifying the measure algebra of $\tilde{\lambda}$ and the associated function spaces.

An obstacle we face in 417C-417E is the fact that *not* every τ-additive measure μ has an extension to a τ-additive topological measure, even when μ is totally finite and its domain includes a base for the topology. (I give an example in 419H.) Consequently it is not enough, in 417C or 417E, to show that the ordinary product measure λ is τ-additive. But perhaps I should remark that if λ is inner regular with respect to the closed sets, this obstacle evaporates (415L). Accordingly, for the principal applications (to quasi-Radon and Radon product measures, and in particular whenever the topological spaces involved are regular) we have rather easier proofs available, based on the constructions of §415. For completely regular spaces, there is yet another approach, because the product measures can be described in terms of the integrals of continuous functions (415I), which by 417U and 417V can be calculated from the ordinary product measures. Of course the proof that λ itself is τ-additive is by no means trivial, especially in the case of infinite products, corresponding to 417E; but for finite products there are relatively direct arguments, applying indeed to slightly more general situations (417Yc). If we have measures which are inner regular with respect to countably compact classes of sets, then there may be other ways of approaching the extension, using theorems from §413 (see 417Ye-417Yg), and for compact Radon measure spaces, λ becomes tight (412Sb, 412V), so its τ-additivity is elementary.

As always, it is important to recognise which constructions are in some sense canonical. The arguments of 417C and 417E allow for the possibility that the factor measures are defined on σ-algebras going well beyond the Borel sets. For all the principal applications, however, the measures will be c.l.d. versions of Borel measures, and in particular will be inner regular with respect to the Borel sets. In such a context it is natural to ask for product measures with the same property, and in this case we can identify a canonical τ-additive topological product measure, as in 417D and 417F. (If you prefer to restrict your measures to Borel σ-algebras, you again get canonical product Borel measures (417Xc-417Xd).) Having done so, we

can reasonably expect 'commutative' and 'associative' and 'distributive' laws, as in 417D, 417J and 417Xe. Subspaces mostly behave themselves (417I, 417Xf).

Of course extending the product measure means that we get new integrable functions on the product, so that Fubini's theorem has to be renegotiated. Happily, it remains valid, at least in the contexts in which it was effective before (417Ha); we still need, in effect, one of the measures to be σ-finite. The theorem still fails for arbitrary integrable functions on products of Radon measure spaces, and the same example works as before (417Xi). In fact this means that we have an alternative route to the construction of the τ-additive product of two measures (417Yb). But note that on this route 'commutativity', the identification of the product measure on $X \times Y$ with that on $Y \times X$, becomes something which can no longer be taken for granted, because if we *define* $\tilde{\lambda}W$ to be $\int \nu W[\{x\}]dx$ we have to worry about when, and why, this will be equal to $\int \mu W^{-1}[\{y\}]dy$.

A version of Tonelli's theorem follows from Fubini's theorem, as before (417Hc). We also have results corresponding to most of the theorems of §254. But note that there are two traps. In the theorem that a measurable set can be described in terms of a projection onto a countable subproduct (254O, 417M) we need to suppose that the factor measures are strictly positive, and in the theorem that a product of Radon measures is a Radon measure (417Q) we need to suppose that the factor measures have compact supports. The basic examples to note in this context are 417Xp and 417Xq.

It is not well understood when we can expect c.l.d. product measures to be topological measures, even in the case of compact Radon probability spaces. Example 419E remains a rather special case, but of course much more effort has gone into seeking positive results. Note that the ordinary product measures of this section are always effectively locally finite and τ-additive (417C, 417E), so that they will be equal to the τ-additive products iff they measure every open set (417S). Regarding infinite products, the τ-additive product measure can fail to be the ordinary product measure in just two ways: if one of the *finite* product measures is not a topological measure, or if uncountably many of the factor measures are not strictly positive (417Sc, 417Xp, 417Yd). So it is finite products which need to be studied.

Whenever we have a subset F of an infinite product $X = \prod_{i \in I} X_i$, it is important to know when F is determined by coordinates in a proper subset of I; in measure theory, we are particularly interested in sets determined by coordinates in countable subsets of I (254Mb). It may happen that there is a *smallest* set J such that F is determined by coordinates in J; for instance, when we have a topological product and F is closed (4A2Bg). When we have a product of probability spaces, we sometimes wish to identify sets J such that F is 'essentially' determined by coordinates in J, in the sense that there is an F', determined by coordinates in J, such that $F \triangle F'$ is negligible. In this context, again, there is a smallest such set (254Rd), which can be identified in terms of the probability algebra free product of the measure algebras (325Mb). In 417Ma the two ideas come together: under the conditions there, we get the same smallest J by either route.

In 417Ma, we have a product of strictly positive τ-additive topological probability measures. If we keep the 'strictly positive' but abandon everything else, we still have very striking results just because the product topology is ccc, so that we can apply 4A2Eb. An abstract expression of this idea is in 417Xu.

418 Measurable functions and almost continuous functions

In this section I work through the basic properties of measurable and almost continuous functions, as defined in 411L and 411M. I give the results in the full generality allowed by the terminology so far introduced, but most of the ideas are already required even if you are interested only in Radon measure spaces as the domains of the functions involved. Concerning the codomains, however, there is a great difference between metrizable spaces and others, and among metrizable spaces separability is of essential importance.

I start with the elementary properties of measurable functions (418A-418C) and almost continuous functions (418D). Under mild conditions on the domain space, almost continuous functions are measurable (418E); for a separable metrizable codomain, we can expect that measurable functions should be almost continuous (418J). Before coming to this, I spend a couple of paragraphs on image measures: a locally finite image measure under a measurable function is Radon if the measure on the domain is Radon and the function is almost continuous (418I).

418L-418Q are important results on expressing given Radon measures as image measures associated with continuous functions, first dealing with ordinary functions $f : X \to Y$ (418L) and then coming to Prokhorov's theorem on projective limits of probability spaces (418M).

The machinery of the first part of the section can also be used to investigate representations of vector-valued functions in terms of product spaces (418R-418T).

418A Proposition Let X be a set, Σ a σ-algebra of subsets of X, Y a topological space and $f : X \to Y$ a measurable function.

(a) $f^{-1}[F] \in \Sigma$ for every Borel set $F \subseteq Y$.

(b) If $A \subseteq X$ is any set, endowed with the subspace σ-algebra, then $f \restriction A : A \to Y$ is measurable.

(c) Let $(Z, \mathfrak{T})$ be another topological space. Then $gf : X \to Z$ is measurable for every Borel measurable function $g : Y \to Z$; in particular, for every continuous function $g : Y \to Z$.

proof (a) The set $\{F : F \subseteq Y, f^{-1}[F] \in \Sigma\}$ is a σ-algebra of subsets of Y containing every open set, so contains every Borel subset of Y.

(b) is obvious from the definition of 'subspace σ-algebra' (121A).

(c) If $H \subseteq Z$ is open, then $g^{-1}[H]$ is a Borel subset of Y so $(gf)^{-1}[H] = f^{-1}[g^{-1}[H]]$ belongs to Σ.

418B Proposition Let X be a set and Σ a σ-algebra of subsets of X.

(a) If Y is a metrizable space and $\langle f_n \rangle_{n \in \mathbb{N}}$ is a sequence of measurable functions from X to Y such that $f(x) = \lim_{n \to \infty} f_n(x)$ is defined in Y for every $x \in X$, then $f : X \to Y$ is measurable.

(b) If Y is a topological space, Z is a separable metrizable space and $f : X \to Y$, $g : X \to Z$ are functions, then $x \mapsto (f(x), g(x)) : X \to Y \times Z$ is measurable iff f and g are measurable.

(c) If Y is a hereditarily Lindelöf space, $\mathcal{U}$ a family of open sets generating its topology, and $f : X \to Y$ a function such that $f^{-1}[U] \in \Sigma$ for every $U \in \mathcal{U}$, then f is measurable.

(d) If $\langle Y_i \rangle_{i \in I}$ is a countable family of separable metrizable spaces, with product Y, then a function $f : X \to Y$ is measurable iff $\pi_i f : X \to Y_i$ is measurable for every i, writing $\pi_i(y) = y(i)$ for $y \in Y$, $i \in \mathbb{N}$.

proof (a) Let ρ be a metric defining the topology of Y. Let $G \subseteq Y$ be any open set, and for each $n \in \mathbb{N}$ set

$$F_n = \{y : y \in Y, \rho(y, z) \geq 2^{-n} \text{ for every } z \in Y \setminus G\}.$$

Then F_n is closed, so $f_i^{-1}[F_n] \in \Sigma$ for every n, $i \in \mathbb{N}$. But this means that

$$f^{-1}[G] = \bigcup_{n \in \mathbb{N}} \bigcap_{i \geq n} f_i^{-1}[F_i] \in \Sigma.$$

As G is arbitrary, f is measurable.

(b)(i) The functions $(y, z) \mapsto y$, $(y, z) \mapsto z$ are continuous, so if $x \mapsto (f(x), g(x))$ is measurable, so are f and g, by 418Ac.

(ii) Now suppose that f and g are measurable, and that $W \subseteq Y \times Z$ is open. By 4A2P(a-i), the topology of Z has a countable base $\mathcal{H}$; let $\langle H_n \rangle_{n \in \mathbb{N}}$ be a sequence running over $\mathcal{H} \cup \{\emptyset\}$. For each n, set

$$G_n = \bigcup \{G : G \subseteq Y \text{ is open}, G \times H_n \subseteq W\};$$

then G_n is open and $G_n \times H_n \subseteq W$. Accordingly $W \supseteq \bigcup_{n \in \mathbb{N}} G_n \times H_n$. But in fact $W = \bigcup_{n \in \mathbb{N}} G_n \times H_n$. **P** If $(y, z) \in W$, there are open sets $G \subseteq Y$, $H \subseteq Z$ such that $(y, z) \in G \times H \subseteq W$. Now there is an $n \in \mathbb{N}$ such that $z \in H_n \subseteq H$, in which case $G \times H_n \subseteq W$ and $G \subseteq G_n$ and $(y, z) \in G_n \times H_n$. **Q**

Accordingly

$$\{x : (f(x), g(x)) \in W\} = \bigcup_{n \in \mathbb{N}} f^{-1}[G_n] \cap g^{-1}[H_n] \in \Sigma.$$

As W is arbitrary, $x \mapsto (f(x), g(x))$ is measurable.

(c) This is just 4A3Db.

(d) If f is measurable, so is every $\pi_i f$, by 418Ac. If every $\pi_i f$ is measurable, set

$$\mathcal{U} = \{\pi_i^{-1}[H] : i \in I, H \subseteq Y_i \text{ is open}\}.$$

Then $\mathcal{U}$ generates the topology of Y, and if $U = \pi_i^{-1}[H]$ then $f^{-1}[U] = (\pi_i f)^{-1}[H]$, so $f^{-1}[U] \in \Sigma$ for every U. Also Y is hereditarily Lindelöf (4A2P(a-iii)), so f is measurable, by (c).

418C Proposition Let (X, Σ, μ) be a measure space and Y a Polish space. Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence of measurable functions from X to Y. Then

$$\{x : x \in X, \lim_{n\to\infty} f_n(x) \text{ is defined in } Y\}$$

belongs to Σ.

proof (Compare 121H.) Let ρ be a complete metric on Y defining the topology of Y.

(a) For m, $n \in \mathbb{N}$ and $\delta > 0$, the set $\{x : \rho(f_m(x), f_n(x)) \leq \delta\}$ belongs to Σ. **P** The function $x \mapsto (f_m(x), f_n(x)) : X \to Y^2$ is measurable, by 418Bb, and the function $\rho : Y^2 \to \mathbb{R}$ is continuous, so $x \mapsto \rho(f_m(x), f_n(x))$ is measurable and $\{x : \rho(f_m(x), f_n(x)) \leq \delta\} \in \Sigma$. **Q**

(b) Now $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ is convergent iff it is Cauchy, because Y is complete. But

$$\{x : x \in X, \langle f_n(x)\rangle_{n\in\mathbb{N}} \text{ is Cauchy}\} = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{i\geq m} \{x : \rho(f_i(x), f_m(x)) \leq 2^{-n}\}$$

belongs to Σ.

418D Proposition Let (X, Σ, μ) be a measure space and $\mathfrak{T}$ a topology on X.

(a) Suppose that Y is a topological space. Then any continuous function from X to Y is almost continuous.

(b) Suppose that Y and Z are topological spaces, $f : X \to Y$ is almost continuous and $g : Y \to Z$ is continuous. Then $gf : X \to Z$ is almost continuous.

(c) Suppose that $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ is a σ-finite topological measure space, Z is a topological space, $g : Y \to Z$ is almost continuous and $f : X \to Y$ is inverse-measure-preserving and almost continuous. Then $gf : X \to Z$ is almost continuous.

(d) Suppose that μ is semi-finite, and that $\langle Y_i\rangle_{i\in I}$ is a countable family of topological spaces with product Y. Then a function $f : X \to Y$ is almost continuous iff $f_i = \pi_i f$ is almost continuous for every $i \in I$, writing $\pi_i(y) = y(i)$ for $i \in I$, $y \in Y$.

proof (a) is trivial.

(b) The set $\{A : A \subseteq X, gf\restriction A \text{ is continuous}\}$ includes $\{A : A \subseteq X, f\restriction A \text{ is continuous}\}$; so if μ is inner regular with respect to the latter, it is inner regular with respect to the former.

(c) Take $E \in \Sigma$ and $\gamma < \mu E$; take $\epsilon > 0$. We have a cover of Y by a non-decreasing sequence $\langle Y_n\rangle_{n\in\mathbb{N}}$ of measurable sets of finite measure; now $\langle f^{-1}[Y_n]\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence covering E, so there is an $n \in \mathbb{N}$ such that $\mu(E \cap f^{-1}[Y_n]) \geq \gamma$. Because f is inverse-measure-preserving, $E \cap f^{-1}[Y_n]$ has finite measure. Now we can find measurable sets $F \subseteq Y_n$, $E_1 \subseteq E \cap f^{-1}[Y_n]$ such that $f\restriction E_1$, $g\restriction F$ are continuous and $\nu F \geq \nu Y_n - \epsilon$, $\mu E_1 \geq \mu(E \cap f^{-1}[Y_n] \setminus E_1) - \epsilon$. In this case $E_0 = E_1 \cap f^{-1}[F]$ has measure at least $\gamma - 2\epsilon$ and $gf\restriction E_0$ is continuous. As E, γ and ϵ are arbitrary, gf is almost continuous.

(d)(i) If f is almost continuous, every f_i must be almost continuous, by (b).

(ii) Now suppose that every f_i is almost continuous. Take $E \in \Sigma$ and $\gamma < \mu E$. There is an $E_0 \subseteq E$ such that $E_0 \in \Sigma$ and $\gamma < \mu E_0 < \infty$. Let $\langle \epsilon_i\rangle_{i\in I}$ be a family of strictly positive real numbers such that $\sum_{i\in I} \epsilon_i \leq \mu E_0 - \gamma$. For each $i \in I$ choose a measurable set $F_i \subseteq E_0$ such that $\mu F_i \geq \mu E_0 - \epsilon_i$ and $f_i\restriction F_i$ is continuous. Then $F = E_0 \cap \bigcap_{i\in I} F_i$ is a subset of E with measure at least γ, and $f\restriction F$ is continuous because $f_i\restriction F$ is continuous for every i (3A3Ib).

418E Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined topological measure space, Y a topological space, and $f : X \to Y$ an almost continuous function. Then f is measurable.

proof Set $\mathcal{K} = \{K : K \in \Sigma, f\restriction K \text{ is continuous}\}$; then μ is inner regular with respect to $\mathcal{K}$. If $H \subseteq Y$ is open and $K \in \mathcal{K}$, then $K \cap f^{-1}[H]$ is relatively open in K, that is, there is an open set $G \subseteq X$ such that $K \cap f^{-1}[H] = K \cap G$. Because μ is a topological measure, $G \in \Sigma$ so $K \cap f^{-1}[H] \in \Sigma$. As K is arbitrary, and μ is complete and locally determined, $f^{-1}[H] \in \Sigma$ (412Ja). As H is arbitrary, f is measurable.

418F Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a semi-finite topological measure space, Y a metrizable space, and $f : X \to Y$ a function. Suppose there is a sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ of almost continuous functions from X to Y such that $f(x) = \lim_{n\to\infty} f_n(x)$ for almost every $x \in X$. Then f is almost continuous.

proof Suppose that $E \in \Sigma$ and that $\gamma < \mu E$, $\epsilon > 0$. Then there is a measurable set $F \subseteq E$ such that $\gamma \leq \mu F < \infty$; discarding a negligible set if necessary, we may arrange that $f(x) = \lim_{n\to\infty} f_n(x)$ for every $x \in F$. Let ρ be a metric on Y defining its topology. For each $n \in \mathbb{N}$, let $F_n \subseteq F$ be a measurable set such that $f_n \restriction F_n$ is continuous and $\mu(F_n \setminus F) \leq 2^{-n}\epsilon$; set $G = \bigcap_{n\in\mathbb{N}} F_n$, so that $\mu G \geq \gamma - 2\epsilon$ and $f_n \restriction G$ is continuous for every $n \in \mathbb{N}$.

For m, $n \in \mathbb{N}$, the functions $x \mapsto (f_m(x), f_n(x)) : G \to Y^2$ and $x \mapsto \rho(f_m(x), f_n(x)) : G \to \mathbb{R}$ are continuous, therefore measurable, because μ is a topological measure. Also $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence for every $x \in G$. So if we set $G_{kn} = \{x : x \in G, \rho(f_i(x), f_j(x)) \leq 2^{-k} \text{ for all } i, j \geq n\}$, $\langle G_{kn}\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of measurable sets with union G for each $k \in \mathbb{N}$, and we can find a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ such that $\mu(G \setminus G_{kn_k}) \leq 2^{-k}\epsilon$ for every k. Setting $H = \bigcap_{k\in\mathbb{N}} G_{kn_k}$, $\mu H \geq \mu G - 2\epsilon \geq \gamma - 4\epsilon$ and $\rho(f_i(x), f_{n_k}(x)) \leq 2^{-k}$ whenever $x \in H$ and $i \geq n_k$; consequently $\rho(f(x), f_{n_k}(x)) \leq 2^{-k}$ whenever $x \in H$ and $k \in \mathbb{N}$. But this means that $\langle f_{n_k}\rangle_{k\in\mathbb{N}}$ converges to f uniformly on H, while every f_{n_k} is continuous on H, so $f \restriction H$ is continuous (3A3Nb). And of course $H \subseteq E$.

As E, γ and ϵ are arbitrary, f is almost continuous.

418G Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite quasi-Radon measure space, Y a metrizable space and $f : X \to Y$ an almost continuous function. Then there is a conegligible set $X_0 \subseteq X$ such that $f[X_0]$ is separable.

proof (a) Let $\mathcal{K}$ be the family of self-supporting measurable sets K of finite measure such that $f \restriction K$ is continuous. Then μ is inner regular with respect to $\mathcal{K}$. **P** If $E \in \Sigma$ and $\gamma < \mu E$, there is an $F \in \Sigma$ such that $F \subseteq E$ and $\gamma < \mu F < \infty$; there is an $H \in \Sigma$ such that $H \subseteq F$, $\gamma \leq \mu H$ and $f \restriction H$ is continuous; and there is a measurable self-supporting $K \subseteq H$ with the same measure as H (414F), in which case $K \in \mathcal{K}$ and $K \subseteq E$ and $\mu K \geq \gamma$. **Q**

(b) Now $f[K]$ is ccc for every $K \in \mathcal{K}$. **P** If $\mathcal{G}$ is a disjoint family of non-empty relatively open subsets of $f[K]$, then $\langle K \cap f^{-1}[G]\rangle_{G\in\mathcal{G}}$ is a disjoint family of non-empty relatively open subsets of K, because $f \restriction K$ is continuous, and $\sum_{G\in\mathcal{G}} \mu(K \cap f^{-1}[G]) \leq \mu K$. Because K is self-supporting, $\mu(K \cap f^{-1}[G]) > 0$ for every $G \in \mathcal{G}$; because μK is finite, $\mathcal{G}$ is countable. As $\mathcal{G}$ is arbitrary, $f[K]$ is ccc. **Q**

Because Y is metrizable, $f[K]$ must be separable (4A2Pd).

(c) Because μ is σ-finite, there is a countable family $\mathcal{L} \subseteq \mathcal{K}$ such that $X_0 = \bigcup \mathcal{L}$ is conegligible (412Ic). Now $f[X_0] = \bigcup_{L\in\mathcal{L}} f[L]$ is a countable union of separable spaces, so is separable (4A2B(e-i)).

418H Proposition (a) Let X and Y be topological spaces, μ an effectively locally finite τ-additive measure on X, and $f : X \to Y$ an almost continuous function. Then the image measure μf^{-1} is τ-additive.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a totally finite quasi-Radon measure space, $(Y, \mathfrak{S})$ a regular topological space, and $f : X \to Y$ an almost continuous function. Then there is a unique quasi-Radon measure ν on Y such that f is inverse-measure-preserving for μ and ν.

proof (a) Let $\mathcal{H}$ be an upwards-directed family of open subsets of Y, all measured by μf^{-1}, and suppose that $H^* = \bigcup \mathcal{H}$ is also measurable. Take any $\gamma < (\mu f^{-1})(H^*) = \mu f^{-1}[H^*]$. Then there is a measurable set $E \subseteq f^{-1}[H^*]$ such that $\mu E \geq \gamma$ and $f \restriction E$ is continuous. Consider $\{E \cap f^{-1}[H] : H \in \mathcal{H}\}$. This is an upwards-directed family of relatively open measurable subsets of E with measurable union E. By 414K, the subspace measure on E is τ-additive, so

$$\gamma \leq \mu E \leq \sup_{H\in\mathcal{H}} \mu(E \cap f^{-1}[H]) \leq \sup_{H\in\mathcal{H}} \mu f^{-1}[H].$$

As γ is arbitrary, $\mu f^{-1}[H^*] \leq \sup_{H\in\mathcal{H}} \mu f^{-1}[H]$; as $\mathcal{H}$ is arbitrary, μf^{-1} is τ-additive.

(b) By 418E, f is measurable. Let ν_0 be the restriction of μf^{-1} to the Borel σ-algebra of Y; by (a), ν_0 is τ-additive, and f is inverse-measure-preserving with respect to μ and ν_0. Because Y is regular, the completion ν of ν_0 is a quasi-Radon measure (415Cb). Because μ is complete, f is still inverse-measure-preserving with respect to μ and ν (235Hc).

To see that ν is unique, observe that its values on Borel sets are determined by the requirement that f be inverse-measure-preserving, so that 415H gives the result.

418I The next theorem is one of the central properties of Radon measures. I have already presented what amounts to a special case in 256G.

Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, Y a Hausdorff space, and $f : X \to Y$ an almost continuous function. If the image measure $\nu = \mu f^{-1}$ is locally finite, it is a Radon measure.

proof (a) By 418E, f is measurable, that is, $f^{-1}[H] \in \Sigma$ for every open set $H \subseteq Y$; but this means that the domain T of ν contains every open set, and ν is a topological measure.

(b) ν is tight (that is, inner regular with respect to the compact sets). **P** If $F \in \mathrm{T}$ and $\nu F > 0$, then $\mu f^{-1}[F] > 0$, so there is an $E \subseteq f^{-1}[F]$ such that $\mu E > 0$ and $f \restriction E$ is continuous. Next, there is a compact set $K \subseteq E$ such that $\mu K > 0$. In this case, $L = f[K]$ is a compact subset of F, and

$$\nu L = \mu f^{-1}[L] \geq \mu K > 0.$$

By 412B, this is enough to prove that ν is tight. **Q** Note that because ν is locally finite, $\nu L < \infty$ for every compact $L \subseteq Y$ (411Ga).

(c) Because μ is complete, so is ν (212Bd). Next, ν is locally determined. **P** Suppose that $H \subseteq Y$ is such that $H \cap F \in \mathrm{T}$ whenever $\nu F < \infty$. Then, in particular, $H \cap f[K] \in \mathrm{T}$ whenever $K \subseteq X$ is compact and $f \restriction K$ is continuous. But setting

$$\mathcal{K} = \{K : K \subseteq X \text{ is compact, } f \restriction K \text{ is continuous}\},$$

μ is inner regular with respect to $\mathcal{K}$ (412Ac). And if $K \in \mathcal{K}$,

$$K \cap f^{-1}[H] = K \cap f^{-1}[H \cap f[K]] \in \Sigma.$$

Because μ is complete and locally determined, this is enough to show that $f^{-1}[H] \in \Sigma$ (412Ja), that is, $H \in \mathrm{T}$. As H is arbitrary, ν is locally determined. **Q**

(d) Thus ν is a complete locally determined tight locally finite topological measure; that is, it is a Radon measure.

418J Theorem Let (X, Σ, μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y is a separable metrizable space and $f : X \to Y$ is measurable. Then f is almost continuous.

proof Let $\mathcal{H}$ be a countable base for the topology of Y, and $\langle H_n \rangle_{n \in \mathbb{N}}$ a sequence running over $\mathcal{H} \cup \{\emptyset\}$. Take $E \in \Sigma$ and $\gamma < \mu E$. Choose $\langle E_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. There is an $E_0 \in \Sigma$ such that $E_0 \subseteq E$ and $\gamma < \mu E_0 < \infty$. Given $E_n \in \Sigma$ with $\gamma < \mu E_n < \infty$, $E_n \setminus f^{-1}[H_n] \in \Sigma$, so there is a closed set $F_n \in \Sigma$ such that

$$F_n \subseteq E_n \setminus f^{-1}[H_n], \quad \mu((E_n \setminus f^{-1}[H_n]) \setminus F_n) < \mu E_n - \gamma;$$

set $E_{n+1} = (E_n \cap f^{-1}[H_n]) \cup F_n$, so that

$$E_{n+1} \in \Sigma, \quad E_{n+1} \subseteq E_n, \quad \mu E_{n+1} > \gamma, \quad E_{n+1} \setminus f^{-1}[H_n] = F_n.$$

Continue.

At the end of the induction, set $F = \bigcap_{n \in \mathbb{N}} E_n$. Then $F \subseteq E$, $\mu F \geq \gamma$, and for every $n \in \mathbb{N}$

$$F \cap f^{-1}[H_n] = F \cap E_{n+1} \cap f^{-1}[H_n] = F \setminus F_n$$

is relatively open in F. It follows that $f \restriction F$ is continuous (4A2B(a-ii)). As E, γ are arbitrary, f is almost continuous.

Remark For variations on this idea, see 418Yg, 433E and 434Yb; also 418Yh.

418K Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and Y a separable metrizable space. Then a function $f : X \to Y$ is measurable iff it is almost continuous.

proof Put 418E and 418J together.

Remark This generalizes 256F.

418L In all the results above, the measure starts on the left of the diagram $f : X \to Y$; in 418I-418K, it is transferred to an image measure on Y. If X has enough compact sets, a measure can move in the reverse direction, as follows.

Theorem Let $(X, \mathfrak{T})$ be a Hausdorff space, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ a Radon measure space and $f : X \to Y$ a continuous function such that whenever $F \in \mathrm{T}$ and $\nu F > 0$ there is a compact set $K \subseteq X$ such that $\nu(F \cap f[K]) > 0$. Then there is a Radon measure μ on X such that ν is the image measure μf^{-1} and the inverse-measure-preserving function f induces an isomorphism between the measure algebras of ν and μ.

proof (a) Note first that ν is inner regular with respect to $\mathcal{L} = \{f[K] : K \in \mathcal{K}\}$, where $\mathcal{K}$ is the family of compact subsets of X. **P** If $\nu F > 0$, there is a $K \in \mathcal{K}$ such that $\nu(F \cap f[K]) > 0$; now there is a closed set $F' \subseteq F \cap f[K]$ such that $\nu F' > 0$, and $K' = K \cap f^{-1}[F']$ is compact, while $f[K'] \subseteq F$ has non-zero measure. As $\mathcal{L}$ is closed under finite unions, this is enough to show that ν is inner regular with respect to $\mathcal{L}$ (412Aa). **Q**

(b) Consequently there is a disjoint set $\mathcal{L}_0 \subseteq \mathcal{L}$ such that every non-negligible $F \in \mathrm{T}$ meets some member of $\mathcal{L}_0$ in a non-negligible set (412Ib). We can express $\mathcal{L}_0$ as $\{f[K] : K \in \mathcal{K}_0\}$ where $\mathcal{K}_0 \subseteq \mathcal{K}$ is disjoint. Set $X_0 = \bigcup \mathcal{K}_0$.

(c) Set

$$\Sigma_0 = \{X_0 \cap f^{-1}[F] : F \in \mathrm{T}\}.$$

Then Σ_0 is a σ-algebra of subsets of X_0. If F, $F' \in \mathrm{T}$ and $\nu F \neq \nu F'$, then there must be some $K \in \mathcal{K}_0$ such that $f[K] \cap (F \triangle F') \neq \emptyset$, so that $X_0 \cap f^{-1}[F] \neq X_0 \cap f^{-1}[F']$; we therefore have a functional $\mu_0 : \Sigma_0 \to [0, \infty]$ defined by setting $\mu_0(X_0 \cap f^{-1}[F]) = \nu F$ whenever $F \in \mathrm{T}$. It is easy to check that μ_0 is a measure on X_0. Now μ_0 is inner regular with respect to $\mathcal{K}$. **P** If $E \in \Sigma_0$ and $\mu E > 0$, there is an $F \in \mathrm{T}$ such that $E = X_0 \cap f^{-1}[F]$ and $\nu F > 0$. There are a $K \in \mathcal{K}_0$ such that $\nu(F \cap f[K]) > 0$, and a closed set $F' \subseteq F \cap f[K]$ such that $\nu F' > 0$; now $K \cap f^{-1}[F'] = X_0 \cap f^{-1}[F']$ belongs to $\Sigma_0 \cap \mathcal{K}$, is included in E and has measure greater than 0. Because $\mathcal{K}$ is closed under finite unions, this is enough to show that μ_0 is inner regular with respect to $\mathcal{K}$. **Q**

(c) Set

$$\Sigma_1 = \{E : E \subseteq X,\ E \cap X_0 \in \Sigma_0\}, \quad \mu_1 E = \mu_0(E \cap X_0) \text{ for every } E \in \Sigma_1.$$

Then μ_1 is a measure on X (being the image measure $\mu_0 \iota^{-1}$, where $\iota : X_0 \to X$ is the identity map), and is inner regular with respect to $\mathcal{K}$. If $F \in \mathrm{T}$, then

$$\mu_1 f^{-1}[F] = \mu_0(X_0 \cap f^{-1}[F]) = \nu F,$$

so f is inverse-measure-preserving for μ_1 and ν. Consequently μ_1 is locally finite. **P** If $x \in X$, there is an open set $H \subseteq Y$ such that $f(x) \in H$ and $\nu H < \infty$; now $f^{-1}[H]$ is an open subset of X of finite measure containing x. **Q** In particular, $\mu_1^* K < \infty$ for every compact $K \subseteq X$ (411Ga).

(d) By 413O, there is an extension of μ_1 to a complete locally determined measure μ on X which is inner regular with respect to $\mathcal{K}$, defined on every member of $\mathcal{K}$, and such that whenever E belongs to the domain Σ of μ and $\mu E < \infty$, there is an $E_1 \in \Sigma_1$ such that $\mu(E \triangle E_1) = 0$. Now μ is locally finite because μ_1 is, so μ is a Radon measure; and f is inverse-measure-preserving for μ and ν because it is inverse-measure-preserving for μ_1 and ν.

(e) The image measure μf^{-1} extends ν, so is locally finite, and is therefore a Radon measure (418I); since it agrees with ν on the compact subsets of Y, it must be identical with ν.

(f) I have still to check that the corresponding measure-preserving homomorphism π from the measure algebra $\mathfrak{B}$ of ν to the measure algebra $\mathfrak{A}$ of μ is actually an isomorphism, that is, is surjective. If $a \in \mathfrak{A}$ and $\bar{\mu} a < \infty$, we can find $E \in \Sigma$ such that $E^\bullet = a$ and $E_1 \in \Sigma_1$ such that $\mu(E \triangle E_1) = 0$. Now $E_1 \cap X_0 = f^{-1}[F] \cap X_0$ for some $F \in \mathrm{T}$; but in this case

$$\mu(E_1 \triangle f^{-1}[F]) = \mu_1(E_1 \triangle f^{-1}[F]) = 0, \quad a = E_1^\bullet = (f^{-1}[F])^\bullet = \pi F^\bullet.$$

Accordingly $\pi[\mathfrak{B}]$ includes $\{a : \bar{\mu} a < \infty\}$, and is order-dense in $\mathfrak{A}$. But as π is injective and $\mathfrak{B}$ is Dedekind complete (being the measure algebra of a Radon measure, which is strictly localizable), it follows that $\pi[\mathfrak{B}] = \mathfrak{A}$ (314Ib). Thus π is an isomorphism, as required.

Remarks Of course this result is most commonly applied when X and Y are both compact and f is a surjection, in which case the condition

(*) whenever $F \in \mathrm{T}$ and $\nu F > 0$ there is a compact set $K \subseteq X$ such that $\nu(F \cap f[K]) > 0$

is trivially satisfied.

Evidently (*) is necessary if there is to be any Radon measure on X for which f is inverse-measure-preserving, so in this sense the result is best possible. In 433D, however, there is a version of the theorem in which f is not required to be continuous.

418M Prokhorov's theorem Suppose that $(I, \leq)$, $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$, $\langle f_{ij}\rangle_{i\leq j\in I}$, $(X, \mathfrak{T})$ and $\langle g_i\rangle_{i\in I}$ are such that

$(I, \leq)$ is a non-empty upwards-directed partially ordered set,

every $(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)$ is a Radon probability space,

$f_{ij} : X_j \to X_i$ is an inverse-measure-preserving function whenever $i \leq j$ in I,

$(X, \mathfrak{T})$ is a Hausdorff space,

$g_i : X \to X_i$ is a continuous function for every $i \in I$,

$g_i = f_{ij}g_j$ whenever $i \leq j$ in I.

Suppose moreover that

for every $\epsilon > 0$ there is a compact set $K \subseteq X$ such that $\mu_i g_i[K] \geq 1 - \epsilon$ for every $i \in I$.

Then there is a Radon probability measure μ on X such that every g_i is inverse-measure-preserving for μ.

proof (a) Set

$$\mathrm{T} = \{g_i^{-1}[E] : i \in I,\ E \in \Sigma_i\} \subseteq \mathcal{P}X.$$

Then T is a subalgebra of $\mathcal{P}X$. **P** (i) There is an $i \in I$, so $\emptyset = g_i^{-1}[\emptyset]$ belongs to T. (ii) If $H \in \mathrm{T}$ there are $i \in I$, $E \in \Sigma_i$ such that $H = g_i^{-1}[E]$; now $X \setminus H = g_i^{-1}[X_i \setminus E]$ belongs to T. (iii) If G, $H \in \mathrm{T}$, there are i, $j \in I$ and $E \in \Sigma_i$, $F \in \Sigma_j$ such that $G = g_i^{-1}[E]$ and $H = g_j^{-1}[F]$. Now I is upwards-directed, so there is a $k \in I$ such that $i \leq k$ and $j \leq k$. Because f_{ik} and f_{jk} are inverse-measure-preserving, $f_{ik}^{-1}[E]$ and $f_{jk}^{-1}[F]$ belong to Σ_k, so that

$$\begin{aligned} G \cap H &= g_i^{-1}[E] \cap g_j^{-1}[F] = (f_{ik}g_k)^{-1}[E] \cap (f_{jk}g_k)^{-1}[F] \\ &= g_k^{-1}[f_{ik}^{-1}[E] \cap f_{jk}^{-1}[F]] \in \mathrm{T}. \ \textbf{Q} \end{aligned}$$

(b) There is an additive functional $\nu : \mathrm{T} \to [0, 1]$ defined by writing $\nu g_i^{-1}[E] = \mu_i E$ whenever $i \in I$ and $E \in \Sigma_i$.

P (i) Suppose that i, $j \in I$ and $E \in \Sigma_i$, $F \in \Sigma_j$ are such that $g_i^{-1}[E] = g_j^{-1}[F]$. Let $k \in I$ be such that $i \leq k$ and $j \leq k$. Then

$$g_k^{-1}[f_{ik}^{-1}[E] \triangle f_{jk}^{-1}[F]] = g_i^{-1}[E] \triangle g_j^{-1}[F] = \emptyset,$$

so $g_k[X] \cap (f_{ik}^{-1}[E] \triangle f_{jk}^{-1}[F]) = \emptyset$. But now remember that for every $\epsilon > 0$ there is a set $K \subseteq X$ such that $\mu_k g_k[K] \geq 1 - \epsilon$. This means that $\mu_k g_k[X]$ must be 1, so that $f_{ik}^{-1}[E] \triangle f_{jk}^{-1}[F]$ must be negligible, and

$$\mu_i E = \mu_k f_{ik}^{-1}[E] = \mu_k f_{jk}^{-1}[F] = \mu_j F.$$

Thus the proposed formula for ν defines a function on T.

(ii) Now suppose that G, $H \in \mathrm{T}$ are disjoint. Again, take i, $j \in I$ and $E \in \Sigma_i$, $F \in \Sigma_j$ such that $G = g_i^{-1}[E]$ and $H = g_j^{-1}[F]$, and $k \in I$ such that $i \leq k$ and $j \leq k$. Then

$$\begin{aligned} \nu G + \nu H &= \mu_i E + \mu_j F = \mu_k f_{ik}^{-1}[E] + \mu_k f_{jk}^{-1}[F] \\ &= \mu_k(f_{ik}^{-1}[E] \cup f_{jk}^{-1}[F]) + \mu_k(f_{ik}^{-1}[E] \cap f_{jk}^{-1}[F]) \\ &= \nu g_k^{-1}[f_{ik}^{-1}[E] \cup f_{jk}^{-1}[F]] + \nu g_k^{-1}[f_{ik}^{-1}[E] \cap f_{jk}^{-1}[F]] \\ &= \nu(G \cup H) + \nu(G \cap H). \end{aligned}$$

But as $\nu\emptyset$ is certainly 0, we get $\nu(G \cup H) = \nu G + \nu H$. As G, H are arbitrary, ν is additive. **Q**

Note that $\nu X = 1$.

(c) $\nu G = \sup\{\nu H : H \in \mathrm{T},\ H \subseteq G,\ H \text{ is closed}\}$ for every $G \in \mathrm{T}$. **P** If $\gamma < \nu G$, there are an $i \in I$ and an $E \in \Sigma_i$ such that $G = g_i^{-1}[E]$. In this case $\mu_i E = \nu G > \gamma$; let $L \subseteq E$ be a compact set such that $\mu_i L \geq \gamma$; then $H = g_i^{-1}[L]$ is a closed subset of G and $\nu H = \mu_i L \geq \gamma$. **Q**

$\nu X = \sup_{K \subseteq X \text{ is compact}} \inf_{G \in \mathrm{T}, G \supseteq K} \nu G$. **P** If $\epsilon > 0$, there is a compact $K \subseteq X$ such that $\mu_i g_i[K] \geq 1 - \epsilon$ for every $i \in I$, by the final hypothesis of the theorem. If $G \in \mathrm{T}$ and $G \supseteq K$, there are an $i \in I$ and an $E \in \Sigma_i$ such that $G = g_i^{-1}[E]$, in which case $g_i[K] \subseteq E$, so that

$$\nu G = \mu_i E \geq \mu_i g_i[K] \geq 1 - \epsilon.$$

Thus $\inf_{G \in \mathrm{T}, G \supseteq K} \nu G \geq 1 - \epsilon$; as ϵ is arbitrary, we have the result. **Q**

This means that the conditions of 416O are satisfied, and there is a Radon measure μ extending ν. Of course this means that every g_i is inverse-measure-preserving.

418N Remarks (a) Taking I to be a singleton, we get a version of 418L in which Y is a probability space, and omitting the check that the function g induces an isomorphism of the measure algebras. Taking I to be the family of finite subsets of a set T, and every X_i to be a product $\prod_{t \in i} Z_t$ of Radon probability spaces with its product Radon measure, we obtain a method of constructing products of arbitrary families of compact probability spaces from finite products.

(b) In the hypotheses of 418M, I asked only that the f_{ij} should be measurable, and omitted any check on the compositions $f_{ij} f_{jk}$ when $i \leq j \leq k$. But it is easy to see that the f_{ij} must in fact be almost continuous, and that $f_{ij} f_{jk}$ must be equal almost everywhere to f_{ik} (418Xu), just as in 418P below.

(c) In the theorem as written out above, the space X and the functions $g_i : X \to X_i$ are part of the data. Of course in many applications we start with a structure

$$(\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i \in I}, \langle f_{ij} \rangle_{i \leq j \in I}),$$

and the first step is to find a suitable X and g_i, as in 418O and 418P.

(d) There are important questions concerning possible relaxations of the hypotheses in 418M, especially in the special case already mentioned, in which $X_i = \prod_{t \in i} Z_t$, $f_{ij}(x) = x \restriction i$ when $i \subseteq j \in [T]^{<\omega}$, $X = \prod_{t \in T} Z_t$, and $g_i(x) = x \restriction i$ for $x \in X$ and $i \in I$, but there is no suggestion that the μ_i are product measures. For a case in which we can dispense with auxiliary topologies on the X_i, see 451Yb.

(e) A typical class of applications of Prokhorov's theorem is in the theory of stochastic processes, in which we have large families $\langle X_t \rangle_{t \in T}$ of random variables; for definiteness, imagine that $T = [0, \infty[$, so that we are looking at a system evolving over time. Not infrequently our intuition leads us to a clear description of the joint distributions ν_J of finite subfamilies $\langle X_t \rangle_{t \in J}$ without providing any suggestion of a measure space on which the whole family $\langle X_t \rangle_{t \in T}$ might be defined. (As I tried to explain in the introduction to Chapter 27, probability spaces themselves are often very shadowy things in true probability theory.) Each ν_J can be thought of as a Radon measure on $\mathbb{R}^J$, and for $I \subseteq J \in [T]^{<\omega}$ we have a natural map $f_{IJ} : \mathbb{R}^J \to \mathbb{R}^I$, setting $f_{IJ}(y) = y \restriction I$ for $y \in \mathbb{R}^J$. If our distributions ν_J mean anything at all, every f_{IJ} will surely be inverse-measure-preserving; this is simply saying that ν_I is the joint distribution of a subfamily of $\langle X_t \rangle_{t \in J}$. If we can find a Hausdorff space Ω and a continuous function $g : \Omega \to \mathbb{R}^T$ such that, for every finite $J \subseteq T$ and $\epsilon > 0$, there is a compact set $K \subseteq \Omega$ such that $\nu_J g_J[K] \geq 1 - \epsilon$ (where $g_J(x) = g(x) \restriction J$), then Prokhorov's theorem will give us a measure μ on Ω which will then provide us with a suitable realization of $\langle X_t \rangle_{t \in T}$ as a family of random variables on a genuine probability space, writing $X_t(\omega) = g(\omega)(t)$. That they become continuous functions on a Radon measure space is a valuable shield against irrelevant complications.

Clearly, if this can be done at all it can be done with $\Omega = \mathbb{R}^T$; but some of the central results of probability theory are specifically concerned with the possibility of using other sets Ω (e.g., $\Omega = C(T)$, as in 455D).

(f) In (e) above, we do always have the option of regarding each ν_J as a measure on the compact space $[-\infty, \infty]^J$. In this case, by 418O or otherwise, we can be sure of finding a measure on $[-\infty, \infty]^T$ to support functions X_t, at the cost of either allowing the values $\pm\infty$ or (as I should myself ordinarily do) accepting

that each X_t would be undefined on a negligible set. The advantage of this is just that it gives us confidence in applying the Kolmogorov-Lebesgue theory to the whole family $\langle X_t\rangle_{t\in T}$ at once, rather than to finite or countable subfamilies. For an example of what can happen if we try to do similar things with non-compact measures, see 419K. For an example of the problems which can arise with uncountable families, see 418Xv.

418O I mention two cases in which we can be sure that the projective limit $(X, \langle g_i\rangle_{i\in I})$ required in Prokhorov's theorem will exist.

Proposition Suppose that $(I, \leq)$, $\langle (X_i, \mathfrak{T}_i)\rangle_{i\in I}$ and $\langle f_{ij}\rangle_{i\leq j\in I}$ are such that

$(I, \leq)$ is a non-empty upwards-directed partially ordered set,

every $(X_i, \mathfrak{T}_i)$ is a compact Hausdorff space,

$f_{ij} : X_j \to X_i$ is a continuous surjection whenever $i \leq j$ in I,

$f_{ij}f_{jk} = f_{ik}$ whenever $i \leq j \leq k$ in I.

Then there are a compact Hausdorff space $(X, \mathfrak{T})$ and a family $\langle g_i\rangle_{i\in I}$ such that $g_i : X \to X_i$ is a continuous surjection and $f_{ij}g_j = f_i$ whenever $i \leq j \in I$; so that if we endow the X_i with Radon probability measures for which the f_{ij} are inverse-measure-preserving, $(I, \langle X_i\rangle_{i\in I}, \langle f_{ij}\rangle_{i\leq j\in I}, X, \langle g_i\rangle_{i\in I})$ will satisfy all the hypotheses of 418M.

proof Set

$$X = \{x : x \in \textstyle\prod_{i\in I} X_i,\ f_{ij}x(j) = x(i) \text{ whenever } i \leq j \in I\},$$

$$g_i(x) = x(i) \text{ for } x \in X,\ i \in I.$$

Of course $g_i = f_{ij}g_j$ whenever $i \leq j$. To see that every g_i is surjective, observe that if $y \in X_i$ and $J \subseteq I$ is finite, then

$$F_J = \{x : x \in \textstyle\prod_{i\in I} X_i,\ x(i) = y,\ f_{jk}x(k) = x(j) \text{ whenever } j \leq k \in J\}$$

is a closed set. F_J is always non-empty, because if k is an upper bound of $J \cup \{i\}$ there is a $z \in X_k$ such that $f_{ik}(z) = y$, in which case $x \in F_J$ whenever $x(j) = f_{jk}(z)$ for every $j \in J \cup \{i\}$. Now $\{F_J : J \in [I]^{<\omega}\}$ is a downwards-directed family of non-empty closed sets in the compact space $\prod_{j\in I} X_j$, so has non-empty intersection, and if x is any point of the intersection then $x \in X$ and $g_i(x) = y$.

418P Proposition Let $(I, \leq)$, $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ and $\langle f_{ij}\rangle_{i\leq j\in I}$ be such that

$(I, \leq)$ is a countable non-empty upwards-directed partially ordered set,

every $(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)$ is a Radon probability space,

$f_{ij} : X_j \to X_i$ is an inverse-measure-preserving almost continuous function whenever $i \leq j$ in I,

$f_{ij}f_{jk} = f_{ik}$ μ_k-a.e. whenever $i \leq j \leq k$ in I.

Then there are a Radon probability space $(X, \mathfrak{T}, \Sigma, \mu)$ and continuous inverse-measure-preserving functions $g_i : X \to X_i$ such that $g_i = f_{ij}g_j$ whenever $i \leq j$ in I.

proof (a) We can use the same formula as in 418O:

$$X = \{x : x \in \textstyle\prod_{i\in I} X_i,\ f_{ij}x(j) = x(i) \text{ whenever } i \leq j \in I\},$$

$$g_i(x) = x(i) \text{ for } x \in X,\ i \in I.$$

As before, the consistency relation $g_i = f_{ij}g_j$ is a trivial consequence of the definition of X. For the rest, we have to check that the final condition of 418M is satisfied. Fix $\epsilon \in]0, 1[$. Start by taking a family $\langle \epsilon_{ij}\rangle_{i\leq j\in I}$ of strictly positive numbers such that $\sum_{i\leq j\in I} \epsilon_{ij} \leq \frac{1}{2}\epsilon$. (This is where we need to know that I is countable.) Set $\epsilon_j = \sum_{i\leq j} \epsilon_{ij}$ for each j, so that $\sum_{j\in I} \epsilon_j \leq \frac{1}{2}\epsilon$.

For $i \leq j \leq k$ in I, set

$$E_{ijk} = \{x : x \in X_k,\ f_{ik}(x) = f_{ij}f_{jk}(x)\},$$

so that E_{ijk} is μ_k-conegligible; set $E_k = \bigcap_{i\le j\le k} E_{ijk}$, so that E_k is μ_k-conegligible. For $i \le j \in I$, choose compact sets $K_{ij} \subseteq E_j$ such that $\mu_j K_{ij} \ge 1 - \epsilon_{ij}$ and $f_{ij}\restriction K_{ij}$ is continuous. Now we seem to need a three-stage construction, as follows:

for $j \in I$, set $K_j = \bigcap_{i\le j} K_{ij}$;

for $j \in I$, set $K_j^* = K_j \cap \bigcap_{i\le j} f_{ij}^{-1}[K_i]$;

finally, set $K = X \cap \prod_{i\in I} K_i^*$.

Let us trace the properties of these sets stage by stage.

(b) For each $j \in I$, $K_j \subseteq K_{jj} \subseteq E_j$ is compact and

$$\mu_j(X_j \setminus K_j) \le \sum_{i\le j} \mu_j(X_j \setminus K_{ij}) \le \sum_{i\le j} \epsilon_{ij} = \epsilon_j,$$

so that $\mu_j K_j \ge 1 - \epsilon_j$. Note that f_{ik} agrees with $f_{ij}f_{jk}$ on K_k whenever $i \le j \le k$, and that $f_{ij}\restriction K_j$ is continuous whenever $i \le j$.

(c) Every K_j^* is compact, and if $i \le j \le k$ then f_{ik} agrees with $f_{ij}f_{jk}$ on K_k^*, while $f_{ij}\restriction K_j^*$ is always continuous. Also

$$\begin{aligned}\mu_j(X_j \setminus K_j^*) &\le \mu_j(X_j \setminus K_j) + \sum_{i\le j} \mu_j(X_j \setminus f_{ij}^{-1}[K_i]) \\ &\le \epsilon_j + \sum_{i\le j} \epsilon_i \le \epsilon,\end{aligned}$$

so $\mu_j K_j^* \ge 1 - \epsilon$, for every $j \in I$.

The point of moving from K_j to K_j^* is that $f_{jk}[K_k^*] \subseteq K_j^*$ whenever $j \le k$ in I. **P** $K_k^* \subseteq f_{jk}^{-1}[K_j]$, so $f_{jk}[K_k^*] \subseteq K_j$. If $i \le j$, then

$$K_k^* = K_k^* \cap f_{ik}^{-1}[K_i] = K_k^* \cap f_{jk}^{-1}[f_{ij}^{-1}[K_i]]$$

because $f_{ij}f_{jk}$ agrees with f_{ik} on K_k^*. So $f_{jk}[K_k^*] \subseteq f_{ij}^{-1}[K_i]$. As i is arbitrary, $f_{jk}[K_k^*] \subseteq K_j^*$. **Q**

Again because f_{ik} agrees with $f_{ij}f_{jk}$ on K_k^*, we have $f_{ik}[K_k^*] = f_{ij}[f_{jk}[K_k^*]] \subseteq f_{ij}[K_j^*]$ whenever $i \le j \le k$. And because $f_{ij}\restriction K_j^*$ is always continuous, all the sets $f_{ij}[K_j^*]$ are compact.

(d)(i) K is compact. **P**

$$K = \{x : x \in \textstyle\prod_{i\in I} K_i^*,\ f_{ij}x(j) = x(i) \text{ whenever } i \le j \in I\}$$

is closed in $\prod_{i\in I} K_i^*$ because $f_{ij}\restriction K_j^*$ is always continuous (and every X_i is Hausdorff). Since $\prod_{i\in I} K_i^*$ is compact, so is K. **Q**

(ii) $\mu_i g_i[K] \ge 1-\epsilon$ for every $i \in I$. **P** By (c), $f_{ik}[K_k^*] \subseteq f_{ij}[K_j^*]$ whenever $i \le j \le k$. So $\{f_{ij}[K_j^*] : j \ge i\}$ is a downwards-directed family of compact sets; write L for their intersection. Since

$$\mu_i f_{ij}[K_j^*] = \mu_j f_{ij}^{-1}[f_{ij}[K_j^*]] \ge \mu_j K_j^* \ge 1 - \epsilon$$

for every $j \ge i$, $\mu_i L \ge 1 - \epsilon$ (414C). If $z \in L$, then for every $k \ge i$ the set

$$F_k = \{x : x \in \textstyle\prod_{j\in I} K_j^*,\ x(k) = z,\ f_{jk}x(k) = x(j) \text{ whenever } j \le k\}$$

is a non-empty closed set in $\prod_{j\in I} K_j^*$, while $F_k \subseteq F_j$ when $j \le k$; so that $\{F_k : k \ge i\}$ is a downwards-directed family of non-empty closed sets in a compact space, and has non-empty intersection. But if $x \in \bigcap_{k\ge i} F_k$, then $x \in K$ and $x(k) = z$, so $z \in g_i[K]$. Thus $g_i[K] \supseteq L$ and $\mu_i g_i[K] \ge 1 - \epsilon$. **Q**

(e) As ϵ is arbitrary, the final condition of 418M is satisfied. But now 418M tells us that (β) is true.

418Q Corollary Let $\langle (X_n, \mathfrak{T}_n, \Sigma_n, \mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of Radon probability spaces, and suppose we are given an inverse-measure-preserving almost continuous function $f_n : X_{n+1} \to X_n$ for each n. Set

$$X = \{x : x \in \textstyle\prod_{n\in\mathbb{N}} X_n,\ f_n(x(n+1)) = x(n) \text{ for every } n \in \mathbb{N}\}.$$

Then there is a unique Radon probability measure μ on X such that all the coordinate maps $x \mapsto x(n) : X \to X_n$ are inverse-measure-preserving.

proof (a) For $i \leq j \in \mathbb{N}$, define $f_{ij} : X_j \to X_i$ by writing

$$f_{ii}(x) = x \text{ for every } x \in X_i,$$
$$f_{i,j+1} = f_{ij} f_j \text{ for every } j \geq i.$$

It is easy to check that $f_{ij} f_{jk} = f_{ik}$ whenever $i \leq j \leq k$, and that every f_{ij} is inverse-measure-preserving and almost continuous (using 418Dc). So we are exactly in the situation of 418P, and we know that there is a Radon probability measure on X for which every g_i is inverse-measure-preserving.

(b) To see that μ is unique, observe that if $K \subseteq X$ is compact and $x \in X \setminus K$ then there is some finite set $I \subseteq \mathbb{N}$ such that $\{y : y{\restriction}I = x{\restriction}I\}$ is disjoint from K. But as $g_n = f_n g_{n+1}$ for every n, this means that $\{y : g_n(y) = g_n(x)\}$ is disjoint from K for all n large enough. As x is arbitrary, we see that $\langle g_n^{-1}[g_n[K]]\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence with intersection K, so that

$$\mu K = \lim_{n\to\infty} \mu g_n^{-1}[g_n[K]] = \lim_{n\to\infty} \mu_n g_n[K].$$

Thus the values of μ on compact sets are determined by the construction. By 416E(b-ii), μ is uniquely defined.

418R I turn now to a special kind of measurable function, corresponding to a new view of product spaces.

Theorem Let X be a set, Σ a σ-algebra of subsets of X, and (Y, T, ν) a σ-finite measure space. Give $L^0(\nu)$ the topology of convergence in measure (§245). Write $\mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})$ for the space of $\Sigma\widehat{\otimes}\mathrm{T}$-measurable functions $h : X \times Y \to \mathbb{R}$, where $\Sigma\widehat{\otimes}\mathrm{T}$ is the σ-algebra of subsets of $X \times Y$ generated by $\{E \times F : E \in \Sigma, F \in \mathrm{T}\}$. Then for a function $f : X \to L^0(\nu)$ the following are equiveridical:

(i) $f[X]$ is separable and f is measurable;

(ii) there is an $h \in \mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})$ such that $f(x) = h_x^\bullet$ for every $x \in X$, where $h_x(y) = h(x, y)$ for $x \in X$, $y \in Y$.

proof Let $\langle Y_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of subsets of Y of finite measure covering Y.

(a)(i)⇒(ii) For each $n \in \mathbb{N}$, let ρ_n be the continuous pseudometric on $L^0(\nu)$ defined by saying that $\rho_n(g_1^\bullet, g_2^\bullet) = \int_{Y_n} \min(1, |g_1 - g_2|)d\nu$ for g_1, $g_2 \in \mathcal{L}^0(\mathrm{T})$, writing $\mathcal{L}^0(\mathrm{T})$ for the space of T-measurable real-valued functions on Y (245A). Then $\{\rho_n : n \in \mathbb{N}\}$ defines the topology of $L^0(\nu)$ (see the proof of 245Eb). Because $f[X]$ is separable, there is a sequence $\langle v_k\rangle_{k\in\mathbb{N}}$ in $L^0(\nu)$ such that $f[X] \subseteq \overline{\{v_k : k \in \mathbb{N}\}}$. For each k, choose $g_k \in \mathcal{L}^0(\mathrm{T})$ such that $g_k^\bullet = v_k$. For n, $k \in \mathbb{N}$ set

$$E_{nk} = \{x : x \in X, \rho_n(f(x), v_k) < 2^{-n}\},$$

$$H_{nk} = E_{nk} \setminus \textstyle\bigcup_{i<k} E_{ni}.$$

Then every E_{nk} belongs to Σ (because f is measurable) and $\bigcup_{k\in\mathbb{N}} E_{nk} = X$ (because $\{v_k : k \in \mathbb{N}\}$ is dense); so $\langle H_{nk}\rangle_{k\in\mathbb{N}}$ is a disjoint sequence of measurable sets covering X. Set $h^{(n)}(x, y) = g_k(y)$ for $x \in H_{nk}$; then $h^{(n)} \in \mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})$.

Fix $x \in X$ for the moment. Then for each $n \in \mathbb{N}$ there is a unique k_n such that $x \in H_{nk_n}$, and $\rho_n(f(x), v_{k_n}) \leq 2^{-n}$. So if $n \leq m$,

$$\begin{aligned}\int_{Y_n} \min(1, |h_x^{(m+1)} - h_x^{(m)}|) &= \int_{Y_n} \min(1, |g_{k_{m+1}} - g_{k_m}|) = \rho_n(g_{k_{m+1}}^\bullet, g_{k_m}^\bullet)\\ &\leq \rho_n(g_{k_{m+1}}^\bullet, f(x)) + \rho_n(f(x), g_{k_m}^\bullet)\\ &\leq \rho_{m+1}(g_{k_{m+1}}^\bullet, f(x)) + \rho_m(f(x), g_{k_m}^\bullet) \leq 3 \cdot 2^{-m-1}.\end{aligned}$$

But this means that $\sum_{m=0}^\infty \int_{Y_n} \min(1, |h_x^{(m+1)} - h_x^{(m)}|)$ is finite, so that $\langle h_x^{(m)}\rangle_{m\in\mathbb{N}}$ must be convergent almost everywhere on Y_n. As this is true for every n, $\langle h_x^{(m)}\rangle_{m\in\mathbb{N}}$ is convergent a.e. on Y. Moreover,

$$\lim_{m\to\infty} (h_x^{(m)})^\bullet = \lim_{m\to\infty} g_{k_m}^\bullet = f(x)$$

in $L^0(\nu)$.

Since this is true for every x,

$$W = \{(x,y) : \langle h^{(m)}(x,y)\rangle_{m\in\mathbb{N}} \text{ converges in } \mathbb{R}\}$$

has conegligible vertical sections, while of course $W \in \Sigma\widehat{\otimes}\mathrm{T}$ because every $h^{(m)}$ is $\Sigma\widehat{\otimes}\mathrm{T}$-measurable (418C). If we set $h(x,y) = \lim_{m\to\infty} h^{(m)}(x,y)$ for $(x,y) \in W$, 0 for other $(x,y) \in X\times Y$, then $h \in \mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})$, while (by 245Ca)

$$h_x^\bullet = \lim_{m\to\infty}(h_x^{(m)})^\bullet = f(x)$$

in $L^0(\nu)$ for every $x \in X$. So we have a suitable h.

(ii)⇒(i) Let Φ be the set of those $h \in \mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})$ such that (i) is satisfied; that is, $x \mapsto h_x^\bullet$ is measurable, and $\{h_x^\bullet : x \in X\}$ is separable. Then

(α) Φ is closed under addition. **P** If h, $\tilde h$ belong to Φ, set $A = \{h_x^\bullet : x \in X\}$, $\tilde A = \{\tilde h_x^\bullet : x \in X\}$. Then both A and $\tilde A$ are separable metrizable spaces, so $A\times\tilde A$ is separable and metrizable and $x \mapsto (h_x^\bullet, \tilde h_x^\bullet) : X \to A\times\tilde A$ is measurable (418Bb). But addition on $L^0(\nu)$ is continuous (245D), so

$$x \mapsto h_x^\bullet + \tilde h_x^\bullet = (h+\tilde h)_x^\bullet$$

is measurable (418Ac), and

$$\{(h+\tilde h)_x^\bullet : x \in X\} \subseteq \{u+\tilde u : u \in A,\ \tilde u \in \tilde A\}$$

is separable (4A2Be). Thus $h + \tilde h \in \Phi$. **Q**

(β) Φ is closed under scalar multiplication, just because $u \mapsto \alpha u : L^0(\nu) \to L^0(\nu)$ is always continuous.

(γ) If $\langle h^{(n)}\rangle_{n\in\mathbb{N}}$ is a sequence in Φ and $h(x,y) = \lim_{n\to\infty} h^{(n)}(x,y)$ for all $x \in X$, $y \in Y$, then $h \in \Phi$. **P** Setting $A_n = \{(h_x^{(n)})^\bullet : x \in X\}$ for each n, then $A = \{h_x^\bullet : x \in X\}$ is included in $\overline{\bigcup_{n\in\mathbb{N}} A_n}$, which is separable (4A2B(e-i)), so A is separable (4A2P(a-iv)); moreover, $h_x^\bullet = \lim_{n\to\infty}(h_x^{(n)})^\bullet$ for every $x \in X$, so $x \mapsto h_x^\bullet$ is measurable, by 418Ba. **Q**

(δ) What this means is that if we set $\mathcal{W} = \{W : W \in \Sigma\widehat{\otimes}\mathrm{T},\ \chi W \in \Phi\}$, then $W \setminus W' \in \mathcal{W}$ whenever W, $W' \in \mathcal{W}$ and $W' \subseteq W$, and that $\bigcup_{n\in\mathbb{N}} W_n \in \mathcal{W}$ whenever $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{W}$. Also, it is easy to see that $E\times F \in \mathcal{W}$ whenever $E \in \Sigma$ and $F \in \mathrm{T}$. By the Monotone Class Theorem (136B), $\mathcal{W}$ includes the σ-algebra generated by $\{E\times F : E \in \Sigma,\ F \in \mathrm{T}\}$, that is, is equal to $\Sigma\widehat{\otimes}\mathrm{T}$. It follows at once, from (α) and (β), that $\sum_{i=0}^n \alpha_i\chi W_i \in \Phi$ whenever $W_0,\dots,W_n \in \Sigma\widehat{\otimes}\mathrm{T}$ and $\alpha_0,\dots,\alpha_n \in \mathbb{R}$, and hence (using ($\gamma$)) that $\mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T}) \subseteq \Phi$, which is what we had to prove.

418S Corollary Let (X,Σ,μ) and (Y,T,ν) be σ-finite measure spaces with c.l.d. product $(X\times Y,\Lambda,\lambda)$. Give $L^0(\nu)$ the topology of convergence in measure. Write $\mathcal{L}^0(\lambda)$ for the space of Λ-measurable real-valued functions defined λ-a.e. on $X\times Y$, as in §241.

(a) If $h \in \mathcal{L}^0(\lambda)$, set $h_x(y) = h(x,y)$ whenever this is defined. Then

$$\{x : f(x) = h_x^\bullet \text{ is defined in } L^0(\nu)\}$$

is μ-conegligible, and includes a conegligible set X_0 such that $f : X_0 \to L^0(\nu)$ is measurable and $f[X_0]$ is separable.

(b) If $f : X \to L^0(\nu)$ is measurable and there is a conegligible set $X_0 \subseteq X$ such that $f[X_0]$ is separable, then there is an $h \in \mathcal{L}^0(\lambda)$ such that $f(x) = h_x^\bullet$ for almost every $x \in X$.

proof (a) The point is that λ is just the completion of its restriction to $\Sigma\widehat{\otimes}\mathrm{T}$ (251K). So there is a conegligible set $W \in \Sigma\widehat{\otimes}\mathrm{T}$ such that $h\restriction W$ is $\Sigma\widehat{\otimes}\mathrm{T}$-measurable (212Fa). Setting $\tilde h(x,y) = h(x,y)$ for $(x,y) \in W$, 0 otherwise, and setting $\tilde f(x) = \tilde h_x^\bullet$ for every $x \in X$, we see from 418R that $\tilde f$ is measurable and that $\tilde f[X]$ is separable. But 252D tells us that

$$X_0 = \{x : ((X\times Y)\setminus W)[\{x\}] \text{ is negligible}\}$$

is conegligible; and if $x \in X_0$ then $h_x = \tilde h_x$ ν-a.e., so that $f(x)$ is defined and equal to $\tilde f(x)$. This proves the result.

(b) $f\restriction X_0$ satisfies 418R(i). So, setting $f_1(x) = f(x)$ for $x \in X_0$, 0 otherwise, there is some $h \in \mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})$ such that $f_1(x) = h^\bullet_x$ for every x, so that $f(x) = h^\bullet_x$ for almost every x, and (ii) is true.

418T Corollary (Mauldin & Stone 81) Let (Y, T, ν) be a σ-finite measure space, and $(\mathfrak{B}, \bar\nu)$ its measure algebra, with its measure-algebra topology (§323).

(a) Let X be a set, Σ a σ-algebra of subsets of X, and $f : X \to \mathfrak{B}$ a function. Then the following are equiveridical:

(i) $f[X]$ is separable and f is measurable;

(ii) there is a $W \in \Sigma\widehat{\otimes}\mathrm{T}$ such that $f(x) = W[\{x\}]^\bullet$ for every $x \in X$.

(b) Let (X, Σ, μ) be a σ-finite measure space and Λ the domain of the c.l.d. product measure λ on $X \times Y$.

(i) Suppose that ν is complete. If $W \in \Lambda$, then

$$\{x : f(x) = W[\{x\}]^\bullet \text{ is defined in } \mathfrak{B}\}$$

is μ-conegligible, and includes a conegligible set X_0 such that $f : X_0 \to \mathfrak{B}$ is measurable and $f[X_0]$ is separable.

(ii) If $f : X \to \mathfrak{B}$ is measurable and there is a conegligible set $X_0 \subseteq X$ such that $f[X_0]$ is separable, then there is a $W \in \Sigma\widehat{\otimes}\mathrm{T}$ such that $f(x) = W[\{x\}]^\bullet$ for almost every $x \in X$.

proof Everything follows directly from 418R and 418S if we observe that $\mathfrak{B}$ is homeomorphically embedded in $L^0(\nu)$ by the function $F^\bullet \mapsto (\chi F)^\bullet$ for $F \in \mathrm{T}$ (323Xf, 367R). We do need to check, for (i)⇒(ii) of part (a), that if $h \in \mathcal{L}^0(\Sigma\widehat{\otimes}\mathrm{T})$ and $h^\bullet_x$ is always of the form $(\chi F)^\bullet$, then there is some $W \in \Sigma\widehat{\otimes}\mathrm{T}$ such that $h^\bullet_x = (\chi W[\{x\}])^\bullet$ for every x; but of course this is true if we just take $W = \{(x, y) : h(x, y) = 1\}$. Now (b-ii) follows from (a) just as 418Sb followed from 418R.

418X Basic exercises >(a) Let (X, Σ, μ) be a measure space, Y a set and $h : X \to Y$ a function; give Y the image measure μh^{-1}. Show that for any function g from Y to a topological space Z, g is measurable iff $gh : X \to Z$ is measurable.

>(b) Let X be a set, Σ a σ-algebra of subsets of X, $\langle Y_n\rangle_{n\in\mathbb{N}}$ a sequence of topological spaces with product Y, and $f : X \to Y$ a function. Show that f is measurable iff $\psi_n f : X \to \prod_{i\le n} Y_i$ is measurable for every $n \in \mathbb{N}$, where $\psi_n(y) = (y(0), \dots, y(n))$ for $y \in Y$, $n \in \mathbb{N}$.

(c) Let (X, Σ, μ) be a semi-finite measure space, $(Y, \mathfrak{S})$ a metrizable space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of measurable functions from X to Y such that $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ is convergent for almost every $x \in X$. Show that μ is inner regular with respect to $\{E : \langle f_n\restriction E\rangle_{n\in\mathbb{N}} \text{ is uniformly convergent}\}$. (Cf. 412Xr.)

>(d) Set $Y = [0, 1]^{[0,1]}$, with the product topology. For $x \in [0, 1]$, $n \in \mathbb{N}$ define $f_n(x) \in Y$ by saying that $f_n(x)(t) = \max(0, 1 - 2^n|x - t|)$ for $t \in [0, 1]$. Check that (i) each f_n is continuous, therefore measurable; (ii) $f(x) = \lim_{n\to\infty} f_n(x)$ is defined in Y for every $x \in [0, 1]$; (iii) for each $t \in [0, 1]$, the coordinate functional $x \mapsto f(x)(t)$ is continuous except at t, and in particular is almost continuous and measurable; (iv) $f\restriction F$ is not continuous for any infinite closed set $F \subseteq [0, 1]$, and in particular f is not almost continuous; (v) every subset of $[0, 1]$ is of the form $f^{-1}[H]$ for some open set $H \subseteq Y$; (vi) f is not measurable; (vii) the image measure μf^{-1}, where μ is Lebesgue measure on $[0, 1]$, is neither a topological measure nor tight.

(e) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space, Y a topological space, and $f : X \to Y$ a function. Suppose that for every $x \in X$ there is an open set G containing x such that $f\restriction G$ is almost continuous with respect to the subspace measure on G. Show that f is almost continuous.

(f) For $i = 1, 2$ let $(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)$ and $(Y_i, \mathfrak{S}_i, \mathrm{T}_i, \nu_i)$ be quasi-Radon measure spaces, and $f_i : X_i \to Y_i$ an almost continuous inverse-measure-preserving function. Show that $(x_1, x_2) \mapsto (f_1(x_1), f_2(x_2))$ is inverse-measure-preserving for the quasi-Radon product measures.

(g) Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ and $\langle(Y_i, \mathfrak{T}_i, \Sigma_i, \nu_i)\rangle_{i\in I}$ be two families of topological spaces with τ-additive Borel probability measures, and let μ, ν be the τ-additive product measures on $X = \prod_{i\in I} X_i$, $Y = \prod_{i\in I} Y_i$. Suppose that every ν_i is strictly positive. Show that if $f_i : X_i \to Y_i$ is almost continuous and inverse-measure-preserving for each i, then $x \mapsto \langle f_i(x(i))\rangle_{i\in I} : X \to Y$ is inverse-measure-preserving, but need not be almost continuous.

(h) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be quasi-Radon measure spaces, $(Z, \mathfrak{U})$ a topological space and $f : X \times Y \to Z$ a function which is almost continuous with respect to the quasi-Radon product measure on $X \times Y$. Suppose that ν is σ-finite. Show that $y \mapsto f(x, y)$ is almost continuous for almost every $x \in X$.

(i) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an effectively locally finite τ-additive topological measure space, Y a topological space and $f : X \to Y$ an almost continuous function. (i) Show that the image measure μf^{-1} is τ-additive. (ii) Show that if μ is a totally finite quasi-Radon measure and the topology on Y is regular, then μf^{-1} is quasi-Radon.

(j) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space and U a linear topological space. Show that if $f : X \to U$ and $g : X \to U$ are almost continuous, then $f + g : X \to U$ is almost continuous.

(k) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be topological measure spaces, and $(Z, \mathfrak{U})$ a topological space; let $f : X \to Y$ be almost continuous and inverse-measure-preserving, and $g : Y \to Z$ almost continuous. Show that if *either* μ is a Radon measure and ν is locally finite *or* μ is τ-additive and effectively locally finite and ν is effectively locally finite, then $gf : X \to Z$ is almost continuous. (*Hint*: show that if $\mu E > 0$ there is a set F such that $\nu F < \infty$ and $\mu(E \cap f^{-1}[F]) > 0$.)

(l) Let (X, Σ, μ) be a complete strictly localizable measure space, $\underline{\phi} : \Sigma \to \Sigma$ a lower density such that $\underline{\phi}X = X$, and $\mathfrak{T}$ the associated density topology on X (414P). Let $f : X \to \overline{\mathbb{R}}$ be a function. Show that the following are equiveridical: (i) f is measurable; (ii) f is almost continuous; (iii) f is continuous at almost every point; (iv) there is a conegligible set $H \subseteq X$ such that $f \restriction H$ is continuous. (Cf. 414Xk.)

(m) Let (X, Σ, μ) be a complete strictly localizable measure space, $\phi : \Sigma \to \Sigma$ a lifting, and $\mathfrak{S}$ the lifting topology on X (414Q). Let $f : X \to \mathbb{R}$ be a function. Show that the following are equiveridical: (i) f is measurable; (ii) f is almost continuous; (iii) there is a conegligible set $H \subseteq X$ such that $f \restriction H$ is continuous. (Cf. 414Xr.)

(n) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space, $(Y, \mathfrak{S})$ a regular topological space and $f : X \to Y$ an almost continuous function. Show that there is a quasi-Radon measure ν on Y such that f is inverse-measure-preserving for μ and ν iff $\bigcup\{f^{-1}[H] : H \subseteq Y \text{ is open}, \mu f^{-1}[H] < \infty\}$ is conegligible in X.

(o) Let X and Y be Hausdorff spaces and $f : X \to Y$ a continuous injective function. Show that if μ_1 and μ_2 are distinct totally finite Radon measures on X then $\mu_1 f^{-1} \neq \mu_2 f^{-1}$.

(p) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, $(Y, \mathfrak{S})$ and $(Z, \mathfrak{U})$ Hausdorff spaces, $f : X \to Y$ an almost continuous function such that $\nu = \mu f^{-1}$ is locally finite, and $g : Y \to Z$ a function. Show that g is almost continuous with respect to ν iff gf is almost continuous with respect to μ.

(q) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be topological probability spaces, and $f : X \to Y$ a measurable function such that $\mu f^{-1}[H] \geq \nu H$ for every $H \in \mathfrak{S}$. Show that (i) $\int gf\,d\mu = \int g\,d\nu$ for every $g \in C_b(Y)$ (ii) $\mu f^{-1}[F] = \nu F$ for every Baire set $F \subseteq Y$ (iii) if μ is a Radon measure and f is almost continuous, then $\mu f^{-1}[F] = \nu F$ for every Borel set $F \subseteq Y$, so that if in addition ν is complete and inner regular with respect to the Borel sets then it is a Radon measure.

(r) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a totally finite topological measure space in which the topology $\mathfrak{T}$ is normal and μ is inner regular with respect to the closed sets. Show that if $f : X \to \mathbb{R}$ is a measurable function and $\epsilon > 0$ there is a continuous $g : X \to \mathbb{R}$ such that $\mu\{x : g(x) \neq f(x)\} \leq \epsilon$.

(s) Let X and Y be Hausdorff spaces, ν a totally finite Radon measure on Y, and $f : X \to Y$ an injective continuous function. Show that the following are equiveridical: (i) there is a Radon measure μ on X such that f is inverse-measure-preserving; (ii) $f[X]$ is conegligible and $f^{-1} : f[X] \to X$ is almost continuous.

(t) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be Radon measure spaces and $f : X \to Y$ an almost continuous inverse-measure-preserving function. Show that (i) $\mu_* A \leq \nu_* f[A]$ for every $A \subseteq X$ (ii) ν is precisely the image measure μf^{-1}.

(u) In 418M, show that all the f_{ij} must be almost continuous. Show that if $i \leq j \leq k$ then $f_{ij}f_{jk} = f_{ik}$ almost everywhere on X_k.

>(v) Let $\mathcal{I}$ be the family of finite subsets of $[0,1]$, and let $(X_I, \mathfrak{T}_I, \Sigma_I, \mu_I)$ be $[0,1] \setminus I$ with its subspace topology and measure for each $I \in \mathcal{I}$. For $I \subseteq J \in \mathcal{I}$, $y \in X_J$ set $f_{IJ}(y) = y$. Show that these X_I, f_{IJ} satisfy nearly all the hypotheses of 418O, but that there are no X, g_I which satisfy the hypotheses of 418M.

(w) Let T be any set, and X the set of total orders on T. (i) Regarding each member of X as a subset of $T \times T$, show that X is a closed subset of $\mathcal{P}(T \times T)$. (ii) Show that there is a unique Radon measure μ on X such that $\Pr(t_1 \leq t_2 \leq \ldots \leq t_n) = \frac{1}{n!}$ for all distinct $t_1, \ldots, t_n \in T$. (*Hint*: for $I \in [T]^{<\omega}$, let X_I be the set of total orderings on I with the uniform probability measure giving the same measure to each singleton; show that the natural map from X_I to X_J is inverse-measure-preserving whenever $J \subseteq I$.)

(x) In 418S, suppose that $f_1 : X \to L^0(\nu)$ and $f_2 : X \to L^0(\nu)$ correspond to h_1, $h_2 \in \mathcal{L}^0(\lambda)$. Show that $f_1(x) \leq f_2(x)$ μ-a.e.(x) iff $h_1 \leq h_2$ λ-a.e. Hence show that (if we assign appropriate algebraic operations to the space of functions from X to $L^0(\nu)$) we have an f-algebra isomorphism between $L^0(\lambda)$ and the space of equivalence classes of measurable functions from X to $L^0(\nu)$ with separable ranges.

418Y Further exercises (a) Let X be a set, Σ a σ-algebra of subsets of X, Y a topological space and $f : X \to Y$ a function. Set $\mathrm{T} = \{F : F \subseteq Y, f^{-1}[F] \in \Sigma\}$. Suppose that Y is hereditarily Lindelöf and its topology is generated by some subset of T. Show that f is measurable.

(b) Let (X, Σ, μ) be a measure space, Y and Z topological spaces and $f : X \to Y$, $g : X \to Z$ measurable functions. Show that if Z has a countable network consisting of Borel sets (e.g., Z is second-countable, or Z is regular and has a countable network), then $x \mapsto (f(x), g(x)) : X \to Y \times Z$ is measurable.

(c) Let X be a set, Σ a σ-algebra of subsets of X, and $\langle Y_i \rangle_{i \in I}$ a countable family of topological spaces with product Y. Suppose that every Y_i has a countable network, and that $f : X \to Y$ is a function such that $\pi_i f$ is measurable for every $i \in I$, writing $\pi_i(y) = y(i)$. Show that f is measurable.

(d) Find strictly localizable Hausdorff topological measure spaces $(X, \mathfrak{T}, \Sigma, \mu)$, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ and $(Z, \mathfrak{U}, \Lambda, \lambda)$ and almost continuous inverse-measure-preserving functions $f : X \to Y$, $g : Y \to Z$ such that gf is not almost continuous.

(e) Let (X, Σ, μ) be a σ-finite measure space and $\mathfrak{T}$ a topology on X such that μ is effectively locally finite and τ-additive. Let Y be a topological space and $f : X \to Y$ an almost continuous function. Show that there is a conegligible subset X_0 of X such that $f[X_0]$ is ccc.

(f) Show that if μ is Lebesgue measure on $\mathbb{R}$, $\mathfrak{T}$ is the usual topology on $\mathbb{R}$ and $\mathfrak{S}$ is the right-facing Sorgenfrey topology, then the identity map from $(\mathbb{R}, \mathfrak{T}, \mu)$ to $(\mathbb{R}, \mathfrak{S})$ is measurable, but not almost continuous, and the image measure is not a Radon measure.

(g) Let (X, Σ, μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y is a topological space with a countable network consisting of Borel sets, and that $f : X \to Y$ is measurable. Show that f is almost continuous.

(h) Find a topological probability space $(X, \mathfrak{T}, \Sigma, \mu)$ in which μ is inner regular with respect to the closed sets, a topological space Y with a countable network and a measurable function $f : X \to Y$ which is not almost continuous.

(i) Let (X, Σ, μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Let $A \subseteq L^\infty(\mu)$ be a norm-compact set. Show that there is a set B of bounded real-valued measurable functions on X such that (i) $A = \{f^\bullet : f \in B\}$ (ii) B is norm-compact in $\ell^\infty(X)$ (iii) μ is inner regular with respect to $\{E : f \restriction E \text{ is continuous for every } f \in B\}$.

(j) Let μ be Lebesgue measure on $[0,1]$. For $t \in [0,1]$ set $u_t = \chi[0,t]^\bullet \in L^0(\mu)$. Show that $A = \{u_t : t \in [0,1]\}$ is norm-compact in $L^p(\mu)$ for every $p \in [1,\infty[$ and also compact for the topology of convergence in measure on $L^0(\mu)$. Show that if B is a set of measurable functions such that $A = \{f^\bullet : f \in B\}$ then μ is not inner regular with respect to $\{E : f\restriction E \text{ is continuous for every } f \in B\}$.

(k) Suppose that $(I,\leq)$, $\langle (X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ and $\langle f_{ij}\rangle_{i\leq j\in I}$ are such that (α) $(I,\leq)$ is a non-empty upwards-directed partially ordered set (β) every $(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)$ is a completely regular Hausdorff quasi-Radon probability space (γ) $f_{ij} : X_j \to X_i$ is a continuous inverse-measure-preserving function whenever $i \leq j$ in I (δ) $f_{ij}f_{jk} = f_{ik}$ whenever $i \leq j \leq k$ in I. Let X_i' be the support of μ_i for each i; show that $f_{ij}[X_j']$ is a dense subset of X_i' whenever $i \leq j$. Let Z_i be the Stone-Čech compactification of X_i' and let $\tilde{f}_{ij} : Z_j \to Z_i$ be the continuous extension of $f_{ij}\restriction X_j'$ for $i \leq j$; let $\tilde{\mu}_i$ be the Radon probability measure on Z_i corresponding to $\mu_i\restriction \mathcal{P}X_i'$ (416V). Show that Z_i, $\tilde{f}_{ij}$ satisfy the conditions of 418O, so that we have a projective limit Z, $\langle g_i\rangle_{i\in I}$, μ as in 418M.

(l) Suppose that $(I,\leq)$, $\langle (X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$, X, $\langle f_i\rangle_{i\in I}$ and $\langle f_{ij}\rangle_{i\leq j\in I}$ are such that (α) $(I,\leq)$ is a non-empty upwards-directed partially ordered set (β) every (X_i,Σ_i,μ_i) is a probability space (γ) $f_{ij} : X_j \to X_i$ is inverse-measure-preserving whenever $i \leq j$ in I (δ) $f_{ij}f_{jk} = f_{ik}$ whenever $i \leq j \leq k$ (ϵ) $f_i : X \to X_i$ is a function for every $i \in I$ (ζ) $f_i = f_{ij}f_j$ whenever $i \leq j$ (η) whenever $\langle i_n\rangle_{n\in\mathbb{N}}$, $\langle x_n\rangle_{n\in\mathbb{N}}$ are such that $\langle i_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in I, $x_n \in X_{i_n}$ for every $n \in \mathbb{N}$ and $f_{i_n i_{n+1}}(x_{n+1}) = x_n$ for every $n \in \mathbb{N}$, then there is an $x \in X$ such that $f_{i_n}(x) = x_n$ for every n. Show that there is a probability measure on X such that every f_i is inverse-measure-preserving.

418 Notes and comments The message of this section is that measurable functions are dangerous, but that almost continuous functions behave themselves. There are two fundamental problems with measurable functions: a function $x \mapsto (f(x),g(x))$ may not be measurable when the components f and g are measurable (419Xg), and an image measure under a measurable function can lose tightness, even when both domain and codomain are Radon measure spaces and the function is inverse-measure-preserving (419Xh). (This is the 'image measure catastrophe' mentioned in 235J and the notes to §343.) Consequently, as long as we are dealing with measurable functions, we often have to impose strong conditions on the range spaces – commonly, we have to restrict ourselves to separable metrizable spaces (418B, 418C), or something similar, which indeed often means that a measurable function is actually almost continuous (418J, 433E). Indeed, for functions taking values in metrizable spaces, 'almost continuity' is very close to 'measurable with essentially separable range' (418G, 418J). The condition 'separable and metrizable' is a little stronger than is strictly necessary (418Yb, 418Yc, 418Yg), but covers the principal applications other than 433E. If we keep the 'metrizable' we can very substantially relax the 'separable' (438E, 438F), and it is in fact the case that a measurable function from a Radon measure space to any metrizable space is almost continuous (451S). These extensions apply equally to the results in 418R-418T (438Xg-438Xh, 451Xn). But both take us deeper into set theory than seems appropriate at the moment.

For almost continuous functions, the two problems mentioned above do not arise (418Dd, 418I, 418Xt). Indeed we rather expect almost continuous functions to behave as if they were continuous. But we still have to be careful. The limit of a sequence of almost continuous functions need not be almost continuous (418Xd), unless the codomain is metrizable (418F); and if we have a function f from a topological measure space to an uncountable product of topological spaces, it can happen that every coordinate of f is an almost continuous function while f is not (418Xd again). But for many purposes, intuitions gained from the study of measurable functions between Euclidean spaces can be transferred to general almost continuous functions.

Theorems 418L and 418M are of a quite different kind, but seem to belong here as well as anywhere. Even in the simplest application of 418L (when $Y = [0,1]$ with Lebesgue measure, and $X \subseteq [0,1]^2$ is a closed set meeting every vertical line) it is not immediately obvious that there will be a measure with the right projection onto the horizontal axis, though there are at least two proofs which are easier than the general case treated in 413N-413O-418L.

As I explain in 418N, the really interesting question concerning 418M is when, given the projective system $\langle (X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$, $\langle f_{ij}\rangle_{i\leq j\in I}$, we can expect to find X and $\langle g_i\rangle_{i\in I}$ satisfying the rest of the hypotheses, and once past the elementary results 418O-418Q this can be hard to determine. I describe a method in

418Yk which can sometimes be used, but (like the trick in 418Nf) it is too easy and too abstract to be often illuminating. See 454G below for something rather deeper.

The results of 418R-418T stand somewhat aside from anything else considered in this chapter, but they form part of an important technique. A special case has already been mentioned in 253Yg. I do not discuss vector-valued measurable functions in this book, except incidentally, but 418R is one of the fundamental results on their representation; it means, for instance, that if V is any of the Banach function spaces of Chapter 36 we can expect to represent Bochner integrable V-valued functions (253Yf) in terms of functions on product spaces, because V will be continuously embedded in an L^0 space (367P). The measure-algebra version in 418T will be very useful in Volume 5 when establishing relationships between properties of measure spaces and corresponding properties of measure algebras.

419 Examples

In §216, I went much of the way to describing examples of spaces with all the possible combinations of the properties considered in Chapter 21. When we come to topological measure spaces, the number of properties involved makes it unreasonable to seek any such comprehensive list. I therefore content myself with seven examples to indicate some of the boundaries of the theory developed here.

The first example (419A) is supposed to show that the hypothesis 'effectively locally finite' which appears in so many of the theorems of this chapter cannot as a rule be replaced by 'locally finite'. The next two (419C-419D) address technical questions concerning the definition of 'Radon measure', and show how small variations in the definition can lead to very different kinds of measure space. The fourth example (419E) shows that the τ-additive product measures of §417 are indeed new constructions. 419H is there to show that extension theorems of the types proved in §415 and §417 cannot be taken for granted. The classical example 419K exhibits one of the obstacles to generalizations of Prokhorov's theorem (418M, 418Q). Finally, I return to the split interval (419L) to describe its standard topology and its relation to the measure introduced in 343J.

419A Example There is a locally compact Hausdorff space X with a complete, σ-finite, locally finite, τ-additive topological measure μ, inner regular with respect to the closed sets, which has a closed subset Y, of measure 1, such that the subspace measure μ_Y on Y is not τ-additive. In particular, μ is not effectively locally finite.

proof (a) Let Q be a countably infinite set, not containing any ordinal. Fix an enumeration $\langle q_n\rangle_{n\in\mathbb{N}}$ of Q, and for $A \subseteq Q$ set $\nu A = \sum\{\frac{1}{n+1} : q_n \in A\}$. Let $\mathcal{I}$ be the ideal $\{A : A \subseteq Q, \nu A < \infty\}$. For any sets I, J, say that $I \subseteq^* J$ if $I \setminus J$ is finite; then $\subseteq^*$ is a reflexive transitive relation.

Let κ be the smallest cardinal of any family $\mathcal{K} \subseteq \mathcal{I}$ for which there is no $I \in \mathcal{I}$ such that $K \subseteq^* I$ for every $K \in \mathcal{K}$. Then κ is uncountable. **P** (i) Of course κ is infinite. (ii) If $\langle I_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{I}$, then for each $n \in \mathbb{N}$ we can find a finite $I_n' \subseteq I_n$ such that $\nu(I_n \setminus I_n') \leq 2^{-n}$; setting $I = \bigcup_{n\in\mathbb{N}} I_n \setminus I_n'$, we have $\nu I \leq 2 < \infty$, while $I_n \subseteq^* I$ for every n. Thus $\kappa > \omega$. **Q**

(b) There is a family $\langle I_\xi\rangle_{\xi<\kappa}$ in $\mathcal{I}$ such that (i) $I_\eta \subseteq^* I_\xi$ whenever $\eta \leq \xi < \kappa$ (ii) there is no $I \in \mathcal{I}$ such that $I_\xi \subseteq^* I$ for every $\xi < \kappa$. **P** Take a family $\langle K_\xi\rangle_{\xi<\kappa}$ in $\mathcal{I}$ such that there is no $I \in \mathcal{I}$ such that $K_\xi \subseteq^* I$ for every $\xi < \kappa$. Choose $\langle I_\xi\rangle_{\xi<\kappa}$ in $\mathcal{I}$ inductively in such a way that

$$K_\xi \subseteq^* I_\xi, \quad I_\eta \subseteq^* I_\xi \text{ for every } \eta < \xi.$$

(This can be done because $\{K_\xi\} \cup \{I_\eta : \eta < \xi\}$ will always be a subset of $\mathcal{I}$ of cardinal less than κ.) If now $I_\xi \subseteq^* I$ for every $\xi < \kappa$, then $K_\xi \subseteq^* I$ for every $\xi < \kappa$, so $I \notin \mathcal{I}$. **Q**

Hence, or otherwise, we see that κ is regular. **P** If $A \subseteq \kappa$ and $\#(A) < \kappa$, then there is an $I \in \mathcal{I}$ such that $I_\zeta \subseteq^* I$ for every $\zeta \in A$; now there must be a $\xi < \kappa$ such that $I_\xi \not\subseteq^* I$, in which case $\zeta < \xi$ for every $\zeta \in A$, and A is not cofinal with κ. **Q**

(c) Set $X = Q \cup \kappa$. (This is where it is helpful to have arranged at the start that no ordinal belongs to Q, so that $Q \cap \kappa = \emptyset$.) Let $\mathfrak{T}$ be the family of sets $G \subseteq X$ such that

$G \cap \kappa$ is open for the order topology of κ,

for every $\xi \in G \cap \kappa \setminus \{0\}$ there is an $\eta < \kappa$ such that $I_\xi \setminus I_\eta \subseteq^* G$,

if $0 \in G$ then $I_0 \subseteq^* G$.

(i) This is a Hausdorff topology on X. **P** (α) It is easy to check that $X \in \mathfrak{T}$, $\emptyset \in \mathfrak{T}$ and $\bigcup \mathcal{G} \in \mathfrak{T}$ for every $\mathcal{G} \subseteq \mathfrak{T}$. ($\beta$) Suppose that G, $H \in \mathfrak{T}$. Then $(G \cap H) \cap \kappa = (G \cap \kappa) \cap (H \cap \kappa)$ is open for the order topology of κ. If $\xi \in G \cap H \cap \kappa \setminus \{0\}$ there are η, $\zeta < \xi$ such that $I_\xi \setminus I_\eta \subseteq^* G$ and $I_\xi \setminus I_\zeta \subseteq^* H$, and now

$$\alpha = \max(\eta, \zeta) < \xi, \quad I_\eta \cup I_\zeta \subseteq^* I_\alpha,$$

so

$$I_\xi \setminus I_\alpha \subseteq^* (I_\xi \setminus I_\eta) \cap (I_\xi \setminus I_\zeta) \subseteq^* G \cap H.$$

Finally, if $0 \in G \cap H$ then $I_0 \subseteq^* G \cap H$. So $G \cap H \in \mathfrak{T}$. Thus $\mathfrak{T}$ is a topology on X. (γ) For any $\xi < \kappa$, the set $E_\xi = (\xi + 1) \cup I_\xi$ is open-and-closed for $\mathfrak{T}$; for any $q \in Q$, $\{q\}$ is open-and-closed. Since these sets separate the points of X, $\mathfrak{T}$ is Hausdorff. **Q**

(ii) The sets E_ξ of the last paragraph are all compact for $\mathfrak{T}$. **P** Let $\mathcal{F}$ be an ultrafilter on X containing E_ξ. (α) If a finite set K belongs to $\mathcal{F}$, then $\mathcal{F}$ must contain $\{x\}$ for some $x \in K$, and converges to x. So suppose henceforth that $\mathcal{F}$ contains no finite set. (β) If $E_0 \in \mathcal{F}$, then for any open set G containing 0, $E_0 \setminus G$ is finite, so does not belong to $\mathcal{F}$, and $G \in \mathcal{F}$; as G is arbitrary, $\mathcal{F} \to 0$. (γ) If $E_0 \notin \mathcal{F}$, let $\eta \leq \xi$ be the least ordinal such that $E_\eta \in \mathcal{F}$. If G is an open set containing η, there are ζ', $\zeta'' < \eta$ such that $I_\eta \setminus I_{\zeta'} \subseteq^* G$, $]\zeta'', \eta] \subseteq G$; so that $E_\eta \setminus E_\zeta \subseteq^* G$, where $\zeta = \max(\zeta', \zeta'') < \eta$. Now $E_\eta \in \mathcal{F}$, $E_\zeta \notin \mathcal{F}$ and $(E_\eta \setminus E_\zeta) \setminus G \notin \mathcal{F}$, so that $G \in \mathcal{F}$. As G is arbitrary, $\mathcal{F} \to \eta$. (δ) As $\mathcal{F}$ is arbitrary, E_ξ is compact. **Q**

(iii) It follows that $\mathfrak{T}$ is locally compact. **P** For $q \in Q$, $\{q\}$ is a compact open set containing q; for $\xi < \kappa$, E_ξ is a compact open set containing ξ. **Q**

(iv) The definition of $\mathfrak{T}$ makes it clear that $Q \in \mathfrak{T}$, that is, that κ is a closed subset of X. We need also to check that the subspace topology $\mathfrak{T}_\kappa$ on κ induced by $\mathfrak{T}$ is just the order topology of κ. **P** (α) By the definition of $\mathfrak{T}$, $G \cap \kappa$ is open for the order topology of κ for every $G \in \mathfrak{T}$. (β) For any $\xi < \kappa$, E_ξ is open-and-closed for $\mathfrak{T}$ so $\xi + 1 = E_\xi \cap \kappa$ is open-and-closed for $\mathfrak{T}_\kappa$. But this means that all sets of the forms $[0, \xi[= \bigcup_{\eta < \xi} \eta + 1$ and $]\xi, \kappa[= \kappa \setminus (\xi + 1)$ belong to $\mathfrak{T}_\kappa$; as these generate the order topology, every open set for the order topology belongs to $\mathfrak{T}_\kappa$, and the two topologies are equal. **Q**

(d) Now let $\mathcal{F}$ be the filter on X generated by the cofinal closed sets in κ. Because the intersection of any sequence of closed cofinal sets in κ is another (4A1Bc), the intersection of any sequence in $\mathcal{F}$ belongs to $\mathcal{F}$. So

$$\Sigma = \mathcal{F} \cup \{X \setminus F : F \in \mathcal{F}\}$$

is a σ-algebra of subsets of X, and we have a measure $\mu_1 : \Sigma \to \{0, 1\}$ defined by saying that $\mu_1 F = 1$, $\mu_1(X \setminus F) = 0$ if $F \in \mathcal{F}$.

(e) Set $\mu E = \nu(E \cap Q) + \mu_1 E$ for $E \in \Sigma$. Then μ is a measure. Let us work through the properties called for.

(i) If $\mu E = 0$ and $A \subseteq E$, then $X \setminus A \supseteq X \setminus E \in \mathcal{F}$, so $A \in \Sigma$. Thus μ is complete.

(ii) $\mu(\kappa) = 1$ and $\mu\{q\}$ is finite for every $q \in Q$, so μ is σ-finite.

(iii) If $G \subseteq X$ is open, then $\kappa \setminus G$ is closed, in the order topology of κ; if it is cofinal with κ, it belongs to $\mathcal{F}$; otherwise, $\kappa \cap G \in \mathcal{F}$. Thus in either case $G \in \Sigma$, and μ is a topological measure.

(iv) The next thing to note is that $\mu G = \nu(G \cap Q)$ for every open set $G \subseteq X$. **P** If $G \notin \mathcal{F}$ this is trivial. If $G \in \mathcal{F}$, then $\kappa \setminus G$ cannot be cofinal with κ, so there is a $\xi < \kappa$ such that $\kappa \setminus \xi \subseteq G$. **?** If $G \cap Q \in \mathcal{I}$, then $(G \cap Q) \cup I_\xi \in \mathcal{I}$. There must be a least $\eta < \kappa$ such that $I_\eta \not\subseteq^* (G \cap Q) \cup I_\xi$; of course $\eta > \xi$, so $\eta \in G$. There is some $\zeta < \eta$ such that $I_\eta \setminus I_\zeta \subseteq^* G$; but as $I_\zeta \subseteq^* G \cup I_\xi$, by the choice of η, we must also have $I_\eta \subseteq^* G \cup I_\xi$, which is impossible. **X** Thus $G \cap Q \notin \mathcal{I}$ and $\mu G = \nu(G \cap Q) = \infty$. **Q**

(v) It follows that μ is τ-additive. **P** Suppose that $\mathcal{G} \subseteq \mathfrak{T}$ is a non-empty upwards-directed set with union H. Then

$$\mu H = \nu(H \cap Q) = \sup_{G \in \mathcal{G}} \nu(G \cap Q) = \sup_{G \in \mathcal{G}} \mu G$$

because ν is τ-additive (indeed, is a Radon measure) with respect to the discrete topology on Q. **Q**

(vi) μ is inner regular with respect to the closed sets. **P** Take $E \in \Sigma$, $\gamma < \mu E$. Then $\gamma - \mu_1(E \cap \kappa) < \nu(E \cap Q)$, so there is a finite $I \subseteq E \cap Q$ such that $\nu I > \gamma - \mu(E \cap \kappa)$. If $\mu_1(E \cap \kappa) = 0$, then $I \subseteq E$ is already a closed set with $\mu I > \gamma$. Otherwise, $E \cap \kappa \in \mathcal{F}$, so there is a cofinal closed set $F \subseteq \kappa$ such that $F \subseteq E$; now F is closed in X (because κ is closed in X and the subspace topology on κ is the order topology), so $I \cup F$ is closed, and $\mu(I \cup F) > \gamma$. As E and γ are arbitrary, μ is inner regular with respect to the closed sets. **Q**

(vii) μ is locally finite. **P** For any $\xi < \kappa$, E_ξ is an open set containing ξ, and $\mu E_\xi = \nu I_\xi$ is finite. For any $q \in Q$, $\{q\}$ is an open set containing q, and $\mu\{q\} = \nu\{q\}$ is finite. **Q**

(viii) Now consider $Y = \kappa$. This is surely a closed set, and $\mu\kappa = 1$. I noted in (b-iv) above that the subspace topology $\mathfrak{T}_\kappa$ is just the order topology of κ. But this means that $\{\xi : \xi < \kappa\}$ is an upwards-directed family of negligible relatively open sets with union κ, so that the subspace measure $\mu_\kappa = \mu_1$ is not τ-additive.

(ix) It follows from 414K that μ cannot be effectively locally finite; but it is also obvious from the work above that κ is a measurable set, of non-zero measure, such that $\mu(\kappa \cap G) = 0$ whenever G is an open set of finite measure.

419B Lemma For any non-empty set I, there is a dense G_δ set in $[0,1]^I$ which is negligible for the usual measure on $[0,1]^I$.

proof Fix on some $i_0 \in I$, and set $\pi(x) = x(i_0)$ for each $x \in [0,1]^I$, so that π is continuous and inverse-measure-preserving for the usual topologies and measures on $[0,1]^I$ and $[0,1]$. For each $n \in \mathbb{N}$ let $G_n \supseteq [0,1] \cap \mathbb{Q}$ be an open subset of $[0,1]$ with measure at most 2^{-n}, so that $\pi^{-1}[G_n]$ is an open set of measure at most 2^{-n}, and $E = \bigcap_{n\in\mathbb{N}} \pi^{-1}[G_n]$ is a G_δ set of measure 0. If $H \subseteq [0,1]^I$ is any non-empty open set, its image $\pi[H]$ is open in $[0,1]$, so contains some rational number, and meets $\bigcap_{n\in\mathbb{N}} G_n$; but this means that $H \cap E \neq \emptyset$, so E is dense.

419C Example (Fremlin 75B) There is a completion regular Radon measure space $(X, \mathfrak{T}, \Sigma, \mu)$ such that

(i) there is an $E \in \Sigma$ such that $\mu(F \triangle E) > 0$ for every Borel set $F \subseteq X$, that is, not every element of the measure algebra of μ can be represented by a Borel set;

(ii) μ is not outer regular with respect to the Borel sets;

(iii) writing ν for the restriction of μ to the Borel σ-algebra of X, ν is a locally finite, effectively locally finite, tight (that is, inner regular with respect to the compact sets) τ-additive topological measure, and there is a set $Y \subseteq X$ such that the subspace measure ν_Y is not semi-finite.

proof (a) For each $\xi < \omega_1$ set $X_\xi = [0,1]^{\omega_1 \setminus \xi}$, and take μ_ξ to be the usual measure on X_ξ; write Σ_ξ for its domain. Note that μ_ξ is a completion regular Radon measure for the usual topology $\mathfrak{T}_\xi$ of X_ξ (416U). Set $X = \bigcup_{\xi<\omega_1} X_\xi$, and let μ be the direct sum measure on X (214K), that is, write

$$\Sigma = \{E : E \subseteq X,\ E \cap X_\xi \in \Sigma_\xi \text{ for every } \xi < \omega_1\},$$

$$\mu E = \textstyle\sum_{\xi<\omega_1} \mu_\xi(E \cap X_\xi) \text{ for every } E \in \Sigma.$$

Then μ is a complete locally determined (in fact, strictly localizable) measure on X. Write Σ for its domain.

(b) For each $\eta < \omega_1$ let $\langle \beta_{\xi\eta} \rangle_{\xi\le\eta}$ be a summable family of strictly positive real numbers with $\beta_{\eta\eta} = 1$ (4A1P). Define $g_\eta : X \to \mathbb{R}$ by setting

$$g_\eta(x) = \frac{1}{\beta_{\xi\eta}} x(\eta) \text{ if } x \in X_\xi \text{ where } \xi \le \eta,$$
$$= 0 \text{ if } x \in X_\xi \text{ where } \xi > \eta.$$

Now define $f : X \to \omega_1 \times \mathbb{R}^{\omega_1}$ by setting

$$f(x) = (\xi, \langle g_\eta(x) \rangle_{\eta<\omega_1})$$

if $x \in X_\xi$. Note that f is injective. Let $\mathfrak{T}$ be the topology on X defined by f, that is, the family $\{f^{-1}[W] : W \subseteq \omega_1 \times \mathbb{R}^{\omega_1} \text{ is open}\}$, where ω_1 and $\mathbb{R}^{\omega_1}$ are given their usual topologies (4A2S, 3A3K), and their product

is given its product topology. Because f is injective, $\mathfrak{T}$ can be identified with the subspace topology on $f[X]$; it is Hausdorff and completely regular.

(c) For $\xi, \eta < \omega_1$, $g_\eta \restriction X_\xi$ is continuous for the compact topology $\mathfrak{T}_\xi$. Consequently $f \restriction X_\xi$ is continuous, and the subspace topology on X_ξ induced by $\mathfrak{T}$ must be $\mathfrak{T}_\xi$ exactly. It follows that μ is a Radon measure for $\mathfrak{T}$. **P** (i) We know already that μ is complete and locally determined. (ii) If $G \in \mathfrak{T}$ then $G \cap X_\xi \in \mathfrak{T}_\xi \subseteq \Sigma_\xi$ for every $\xi < \omega_1$, so $G \in \Sigma$; thus μ is a topological measure. (iii) If $E \in \Sigma$ and $\mu E > 0$, there is a $\xi < \omega_1$ such that $\mu_\xi(E \cap X_\xi) > 0$. Because μ_ξ is a Radon measure, there is a $\mathfrak{T}_\xi$-compact set $F \subseteq E \cap X_\xi$ such that $\mu_\xi F > 0$. Now F is $\mathfrak{T}$-compact and $\mu F > 0$. As E is arbitrary, μ is tight (using 412B). (iv) If $x \in X$, take that $\xi < \omega_1$ such that $x \in X_\xi$, and consider

$$G = f^{-1}[(\xi + 1) \times \{w : w \in \mathbb{R}^{\omega_1}, w(\xi) < 2\}].$$

Because $\xi + 1$ is open in ω_1, $G \in \mathfrak{T}$. Because $g_\xi(x) = x(\xi) \leq 1$, $x \in G$. Now for $\zeta \leq \xi$,

$$\begin{aligned}\mu_\zeta(G \cap X_\zeta) &= \mu_\zeta\{x : x \in X_\zeta, g_\xi(x) < 2\}\\ &= \mu_\zeta\{x : x \in X_\zeta, \beta_{\zeta\xi}^{-1}x(\xi) < 2\}\\ &= \mu_\zeta\{x : x \in X_\zeta, x(\xi) < 2\beta_{\zeta\xi}\} \leq 2\beta_{\zeta\xi},\end{aligned}$$

so

$$\mu G = \textstyle\sum_{\zeta \leq \xi} \mu_\zeta(G \cap X_\zeta) \leq 2 \sum_{\zeta \leq \xi} \beta_{\zeta\xi} < \infty.$$

As x is arbitrary, μ is locally finite, therefore a Radon measure. **Q**

We also find that μ is completion regular. **P** If $E \subseteq X$ and $\mu E > 0$, then there is a $\xi < \omega_1$ such that $\mu(E \cap X_\xi) > 0$. Because μ_ξ is completion regular, there is a set $F \subseteq E \cap X_\xi$, a zero set for $\mathfrak{T}_\xi$, such that $\mu F > 0$. Now X_ξ is a G_δ set in X (being the intersection of the open sets $\bigcup_{\eta < \zeta < \xi+1} X_\zeta$ for $\eta < \xi$, unless $\xi = 0$, in which case X_ξ itself is open), so F is a G_δ set in X (4A2C(a-iv)); being a compact G_δ set in a completely regular space, it is a zero set (4A2F(h-v)).

Thus every set of positive measure includes a zero set of positive measure. So μ is inner regular with respect to the zero sets (412B). **Q**

(d) The key to the example is the following fact: if $G \subseteq X$ is open, then *either* there is a cofinal closed set $V \subseteq \omega_1$ such that $G \cap X_\xi = \emptyset$ for every $\xi \in V$ *or* $\{\xi : \mu(G \cap X_\xi) \neq 1\}$ is countable. **P** Suppose that $A = \{\xi : G \cap X_\xi \neq \emptyset\}$ meets every cofinal closed set, that is, is stationary (4A1C). Then $B = A \cap \Omega$ is stationary, where Ω is the set of non-zero countable limit ordinals (4A1Bb, 4A1Cb). Let $H \subseteq \omega_1 \times \mathbb{R}^{\omega_1}$ be an open set such that $G = f^{-1}[H]$.

For each $\xi \in B$ choose $x_\xi \in G \cap X_\xi$. Then $f(x_\xi) \in H$, so there must be a $\zeta_\xi < \xi$, a finite set $I_\xi \subseteq \omega_1$, and a $\delta_\xi > 0$ such that $z \in H$ whenever $z = (\gamma, \langle t_\eta \rangle_{\eta \in \omega_1}) \in \omega_1 \times \mathbb{R}^{\omega_1}$, $\zeta_\xi < \gamma \leq \xi$ and $|t_\eta - g_\eta(x)| < \delta_\xi$ for every $\eta \in I_\xi$. Because ξ is a non-zero limit ordinal, $\zeta'_\xi = \sup(\{\zeta_\xi\} \cup (I_\xi \cap \xi)) < \xi$.

By the Pressing-Down Lemma (4A1Cc), there is a $\zeta < \omega_1$ such that $C = \{\xi : \xi \in B, \zeta'_\xi = \zeta\}$ is uncountable. **?** Suppose, if possible, that $\zeta < \eta < \omega_1$ and $\mu(G \cap X_\eta) < 1$. Then there is a measurable subset F of $X_\eta \setminus G$, determined by coordinates in a countable set $J \subseteq \omega_1 \setminus \eta$, such that $\mu F = \mu_\eta F > 0$ (254Ob). Let $\xi \in C$ be such that $\eta < \xi$ and $J \subseteq \xi$, and take any $y \in F$. If we define $y' \in X_\eta$ by setting

$$\begin{aligned}y'(\gamma) &= y(\gamma) \text{ for } \gamma \in \xi \setminus \eta,\\ &= x_\xi(\gamma) \text{ for } \gamma \in \omega_1 \setminus \xi,\end{aligned}$$

then $y' \in F$. But also $\zeta_\xi < \eta < \xi$ and $\xi \setminus \eta \subseteq \xi \setminus \zeta'_\xi$ is disjoint from I_ξ, so $g_\gamma(y') = g_\gamma(x_\xi)$ for every $\gamma \in I_\xi$, since both are zero if $\gamma \leq \eta$ and otherwise $y'(\gamma) = x_\xi(\gamma)$. By the choice of ζ_ξ, I_ξ we must have $f(y') \in H$ and $y' \in F \cap G$; which is impossible. **X**

Thus $\mu(G \cap X_\eta) = 1$ for every $\eta > \zeta$, as required by the second alternative. **Q**

(e) For each $\xi < \omega_1$, let $\mathcal{I}_\xi$ be the family of negligible meager subsets of X_ξ. Then $\mathcal{I}_\xi$ is a σ-ideal; note that it contains every closed negligible set, because μ_ξ is strictly positive. Set

$$\mathrm{T}_\xi = \mathcal{I}_\xi \cup \{X_\xi \setminus F : F \in \mathcal{I}_\xi\},$$

so that T_ξ is a σ-algebra of subsets of X_ξ, containing every conegligible open set, and $\mu_\xi F \in \{0, 1\}$ for every $F \in \mathrm{T}_\xi$. Set

$$\mathrm{T} = \{E : E \in \Sigma, \{\xi : E \cap X_\xi \notin \mathrm{T}_\xi\} \text{ is non-stationary}\}.$$

Then T is a σ-subalgebra of Σ (because the non-stationary sets form a σ-ideal of subsets of ω_1, 4A1Cb), and contains every open set, by (d); so includes the Borel σ-algebra $\mathcal{B}$ of X.

If we set

$$E_\xi = \{x : x \in X_\xi, x(\xi) \leq \tfrac{1}{2}\} \text{ for each } \xi < \omega_1, \quad E = \bigcup_{\xi<\omega_1} E_\xi,$$

then $E \in \Sigma$. But if $F \subseteq X$ is a Borel set, $F \in \mathrm{T}$ so $\mu(E \triangle F) = \infty$. This proves the property (i) claimed for the example.

(f) Next, for each $\xi < \omega_1$, take a negligible dense G_δ set $E'_\xi \subseteq X_\xi$ (419B). Set $Y = \bigcup_{\xi<\omega_1} E'_\xi$, so that $\mu Y = 0$. If $F \supseteq Y$ is a Borel set, then $F \cap X_\xi \supseteq E_\xi \notin \mathcal{I}_\xi$ for every $\xi < \omega_1$, while $F \in \mathrm{T}$, so $\{\xi : \mu_\xi(F \cap X_\xi) = 0\}$ is non-stationary and $\mu F = \infty$. Thus μ is not outer regular with respect to the Borel sets. Taking $\nu = \mu\restriction\mathcal{B}$, the subspace measure ν_Y is not semi-finite. **P** We have just seen that $\nu_Y Y = \nu^* Y$ is infinite. If $F \in \mathcal{B}$ and $\nu F < \infty$, then $A = \{\xi : \mu_\xi(F \cap X_\xi) > 0\}$ is countable, so $F_0 = \bigcup_{\xi\in A} E'_\xi$ and $F_1 = F \setminus \bigcup_{\xi\in A} X_\xi$ are negligible Borel sets; since $F \cap Y \subseteq F_0 \cup F_1$, $\nu_Y(F \cap Y) = 0$. But this means that ν_Y takes no values in $]0, \infty[$ and is not semi-finite. **Q**

Remark X here is not locally compact. But as it is Hausdorff and completely regular, it can be embedded as a subspace of a locally compact Radon measure space $(X', \mathfrak{T}', \Sigma', \mu')$ (416T). Now μ' still has the properties (i)-(iii).

419D Example (Fremlin 75B) There is a complete locally determined τ-additive completion regular topological measure space $(X, \mathfrak{T}, \Sigma, \mu)$ in which μ is tight and compact sets have finite measure, but μ is not localizable.

proof (a) Let I be a set of cardinal greater than $\mathfrak{c}$. Set $X = [0,1]^I$. For $i \in I$, $t \in [0,1]$ set $X_{it} = \{x : x \in X, x(i) = t\}$. Give X_{it} its natural topology $\mathfrak{T}_{it}$ and measure μ_{it}, with domain Σ_{it}, defined from the expression of X_{it} as $[0,1]^{I\setminus\{i\}} \times \{t\}$, each factor $[0,1]$ being given its usual topology and Lebesgue measure, and the singleton factor $\{t\}$ being given its unique (discrete) topology and (atomic) probability measure. By 416U, μ_{it} is a completion regular Radon measure. Set

$$\mathfrak{T} = \{G : G \subseteq X, G \cap X_{it} \in \mathfrak{T}_{it} \text{ for all } i \in I, t \in [0,1]\},$$

$$\Sigma = \{E : E \subseteq X, E \cap X_{it} \in \Sigma_{it} \text{ for all } i \in I, t \in [0,1]\},$$

$$\mu E = \textstyle\sum_{i\in I, t\in[0,1]} \mu_{it}(E \cap X_{it}) \text{ for every } E \in \Sigma.$$

(Compare 216D.) Then it is easy to check that $\mathfrak{T}$ is a topology. $\mathfrak{T}$ is Hausdorff because it is finer(= larger) than the usual topology $\mathfrak{S}$ on X; because each $\mathfrak{T}_{it}$ is the subspace topology induced by $\mathfrak{S}$, it is also the subspace topology induced by $\mathfrak{T}$. Next, the definition of μ makes it a locally determined measure; it is a tight complete topological measure because every μ_{it} is.

(b) If $K \subseteq X$ is compact, $\mu K < \infty$. **P?** Otherwise, $M = \{(i,t) : i \in I, t \in [0,1], \mu_{it}(K \cap X_{it}) > 0\}$ must be infinite. Take any sequence $\langle (i_n, t_n)\rangle_{n\in\mathbb{N}}$ of distinct elements of M. Choose a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in K inductively, as follows. Given $\langle x_m\rangle_{m<n}$, then set $C_{ni} = \{x_m(i) : m < n\}$ for each $i \in I$, and

$$A_n = \{x : x \in X_{i_n t_n}, x(i) \notin C_{ni} \text{ for } i \in I \setminus \{i_n\}\};$$

then $\mu^*_{i_n t_n} A_n = 1$ (254Lb). Since $\mu_{i_n t_n}(X_{it} \cap X_{ju}) = 0$ whenever $(i,t) \neq (j,u)$, there must be some $x_n \in K \cap A_n \setminus \bigcup_{m\neq n} X_{i_m, t_m}$. Continue.

This construction ensures that that if $i \in I$ and $m < n$, either $i \neq i_n$ so $x_n(i) \notin C_{ni}$ and $x_n(i) \neq x_m(i)$, or $i = i_n \neq i_m$ and $x_m \notin X_{i_n t_n}$ so $x_n(i) = t_n \neq x_m(i)$, or $i = i_m = i_n$ and $x_n(i) = t_n \neq t_m = x_m(i)$. But this means that $\{x_n : n \in \mathbb{N}\}$ is an infinite set meeting each X_{it} in at most one point, and is closed for $\mathfrak{T}$; so $\langle x_n\rangle_{n\in\mathbb{N}}$ has no cluster point for $\mathfrak{T}$, which is impossible. **XQ**

(c) μ is not localizable. **P** Fix on any $k \in I$ and consider $\mathcal{E} = \{X_{kt} : t \in [0,1]\}$. **?** If $E \in \Sigma$ is an essential supremum for $\mathcal{E}$, then $E \cap X_{kt}$ must be μ_{kt}-conegligible for every $t \in [0,1]$. We can therefore find a countable set $J_t \subseteq I$ and a μ_{kt}-conegligible set $F_t \subseteq E \cap X_{kt}$, determined by coordinates in J_t. At this

point recall that $\#(I) > \mathfrak{c}$, so there is some $j \in I \setminus (\{k\} \cup \bigcup_{t\in[0,1]} J_t)$. Since $X_{j0} \cap X_{kt}$ is negligible for every $t \in [0,1]$, $X_{j0} \cap E$ must be negligible, and $\int_0^1 \nu H_t dt = 0$, where

$$H_t = \{y : y \in [0,1]^{I\setminus\{j,k\}}, (y,0,t) \in E\}$$

and ν is the usual measure on $[0,1]^{I\setminus\{j,k\}}$, identifying X with $[0,1]^{I\setminus\{j,k\}} \times [0,1] \times [0,1]$. But because F_t is determined by coordinates in $I \setminus \{j\}$, we can identify it with $F_t' \times [0,1] \times \{t\}$ where F_t' is a ν-conegligible subset of $[0,1]^{I\setminus\{j,k\}}$, and $F_t' \subseteq H_t$, so $\nu H_t = 1$ for every t, which is absurd. **X**

Thus $\mathcal{E}$ has no essential supremum in Σ, and μ cannot be localizable. **Q**

(d) I have still to check that μ is completion regular. **P** If $E \in \Sigma$ and $\mu E > 0$, there are $i \in I$, $t \in [0,1]$ such that $\mu_{it}(E \cap X_{it}) > 0$, and an $F \subseteq E \cap X_{it}$, a zero set for the subspace topology of X_{it}, such that $\mu_{it}F > 0$. But now observe that X_{it} is a zero set in X for the usual topology $\mathfrak{S}$, so that F is a zero set for $\mathfrak{S}$ (4A2Gc) and therefore for the finer topology $\mathfrak{T}$. By 412B, this is enough to show that μ is inner regular with respect to the zero sets. **Q**

Remark It may be worth noting that the topology $\mathfrak{T}$ here is not regular. See Fremlin 75b, p. 106.

419E Example (Fremlin 76) Let $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$, so that ν is a strictly positive completion regular Radon probability measure (411P). Then the c.l.d. product measure λ on $Z \times Z$ is not a topological measure, so is not equal to the τ-additive product measure $\tilde{\lambda}$, and $\tilde{\lambda}$ is not completion regular.

proof Consider the sets W, $\tilde{W}$ described in 346K. We have $W \in \Lambda = \operatorname{dom}\lambda$ and $\tilde{W} = \bigcup \mathcal{V}$, where

$$\mathcal{V} = \{G \times H : G, H \subseteq Z \text{ are open-and-closed}, (G \times H) \setminus W \text{ is negligible}\}.$$

$\tilde{W}$ is a union of open sets, therefore must be open in Z^2. And $\lambda_*\tilde{W} \leq \lambda W$. **P?** Otherwise, there is a $V \in \Lambda$ such that $V \subseteq \tilde{W}$ and $\lambda V > \lambda W$. Now λ is tight, by 412Sb, so there is a compact set $K \subseteq V$ such that $K \in \Lambda$ and $\lambda K > \lambda W$. There must be $U_0, \ldots, U_n \in \mathcal{V}$ such that $K \subseteq \bigcup_{i\leq n} U_i$. But $\lambda(U_i \setminus W) = 0$ for every i, so $\lambda(K \setminus W) = 0$ and $\lambda K \leq \lambda W$. **XQ**

However, the construction of 346K arranged that $\lambda^*\tilde{W}$ should be 1 and λW strictly less than 1. So $\lambda_*\tilde{W} < \lambda^*\tilde{W}$ and $\tilde{W} \notin \Lambda$. Accordingly λ is not a topological measure and cannot be equal to the Radon measure $\tilde{\lambda}$ of 417P.

We know that λ is inner regular with respect to the zero sets (412Sc) and is defined on every zero set (417V), while $\tilde{\lambda}$ properly extends λ. But this means that $\tilde{\lambda}$ cannot be inner regular with respect to the zero sets, by 412L, that is, cannot be completion regular.

419F Theorem (Rao 69) $\mathcal{P}(\omega_1 \times \omega_1) = \mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1$, the σ-algebra of subsets of ω_1 generated by $\{E \times F : E, F \subseteq \omega_1\}$.

proof (a) Because $\omega_1 \leq \mathfrak{c}$, there is an injection $h : \omega_1 \to \{0,1\}^{\mathbb{N}}$; set $E_i = \{\xi : h(\xi)(i) = 1\}$ for each $i \in \mathbb{N}$.

(b) Suppose that $A \subseteq \omega_1$ has countable vertical sections. Then $A \in \mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1$. **P** Set $B = A^{-1}[\omega_1]$ and for $\xi \in B$ let $f_\xi : \mathbb{N} \to A[\{\xi\}]$ be a surjection. Set $g_n(\xi) = f_\xi(n)$ for $\xi \in B$ and $n \in \mathbb{N}$, and $A_n = \{(\xi, f_\xi(n)) : \xi \in B\}$ for $n \in \mathbb{N}$. Then

$$\begin{aligned}
A_n &= \{(\xi,\eta) : \xi \in B, \eta = g_n(\xi)\} \\
&= \{(\xi,\eta) : \xi \in B, \eta < \omega_1, h(g_n(\xi)) = h(\eta)\} \\
&= \bigcap_{i\in\mathbb{N}} \big(\{(\xi,\eta) : \xi \in g_n^{-1}[E_i], \eta \in E_i\} \cup \{(\xi,\eta) : \xi \in B \setminus g_n^{-1}[E_i], \eta \in \omega_1 \setminus E_i\}\big) \\
&\in \mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1.
\end{aligned}$$

So

$$A = \textstyle\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1. \ \textbf{Q}$$

(c) Similarly, if a subset of $\omega_1 \times \omega_1$ has countable horizontal sections, it belongs to $\mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1$. But for any $A \subseteq \omega_1 \times \omega_1$, $A = A' \cup A''$ where

$$A' = \{(\xi,\eta) : (\xi,\eta) \in A,\ \eta \leq \xi\} \text{ has countable vertical sections,}$$
$$A'' = \{(\xi,\eta) : (\xi,\eta) \in A,\ \xi \leq \eta\} \text{ has countable horizontal sections,}$$

so both A' and A'' belong to $\mathcal{P}\omega_1 \widehat{\otimes} \mathcal{P}\omega_1$ and A also does.

419G Corollary (ULAM 30) Let Y be a set of cardinal at most ω_1 and μ a σ-finite measure with domain $\mathcal{P}Y$. Then μ is point-supported: in particular, there is a countable conegligible set $A \subseteq Y$.

proof (a) Let μ_0 be the point-supported part of μ, that is, $\mu_0 A = \sum_{y \in A} \mu\{y\}$ for every $A \subseteq Y$; then μ_0 is a measure (112Bd) and so is $\nu = \mu - \mu_0$. If $A \subseteq Y$ is countable, then of course $\mu_0 A = \mu A$, so $\nu A = 0$. Because μ is σ-finite, so is ν.

(b) If Y is countable, we can stop. For the case in which $\#(Y) = \omega_1$, it is enough to consider the case in which Y is actually equal to ω_1. Let $\lambda = \nu \times \nu$ be the product measure on $\omega_1 \times \omega_1$. By 419F, the domain of λ is the whole of $\mathcal{P}(\omega_1 \times \omega_1)$; in particular, it contains the set $V = \{(\xi,\eta) : \xi \leq \eta < \omega_1\}$. Now by Fubini's theorem

$$\lambda V = \int \nu V[\{\xi\}]\nu(d\xi) = \int \nu(\omega_1 \setminus \xi)\nu(d\xi) = (\nu\omega_1)^2,$$

and also

$$\lambda V = \int \nu V^{-1}[\{\eta\}]\nu(d\eta) = \int \nu(\eta+1)\nu(d\eta) = 0.$$

So $\nu\omega_1 = 0$ and $\mu = \mu_0$. Now $\{y : \mu\{y\} > 0\}$ is a countable set (because μ is σ-finite) and is conegligible.

Remark I ought to remark that this result, though not 419F, is valid for many other cardinals besides ω_1; see, in particular, 438C below. There will be more on this topic in Volume 5.

419H Example There is a complete probability space (X, Σ, μ) with a topology $\mathfrak{T}$ such that μ is τ-additive and inner regular with respect to the Borel sets, $\mathfrak{T}$ is generated by $\mathfrak{T} \cap \Sigma$, but μ has no extension to a topological measure.

proof (a) Set $Y = \omega_1 + 1 = \omega_1 \cup \{\omega_1\}$. Let T be the σ-algebra of subsets of Y generated by the σ-ideal of countable subsets of ω_1. Let ν be the probability measure with domain T defined by the formula

$$\nu F = \tfrac{1}{2}\#(F \cap \{0, \omega_1\}) \text{ for every } F \in \mathrm{T}.$$

Set

$$\mathfrak{S} = \{\emptyset, Y\} \cup \{H : 0 \in H \subseteq \omega_1\}.$$

This is a topology on Y, and every subset of Y is a Borel set for $\mathfrak{S}$; so ν is surely inner regular with respect to the Borel sets.

Note that

$$\{\{0,\alpha\} : \alpha < \omega_1\} \cup \{Y\}$$

is a base for $\mathfrak{S}$ included in T.

(b) Let λ be the product probability measure on $Y^{\mathbb{N}}$, and Λ its domain; let $\Lambda_0 \subseteq \Lambda$ be the σ-algebra of subsets of $Y^{\mathbb{N}}$ generated by sets of the form $\{y : y(i) \in F\}$, where $i \in \mathbb{N}$ and $F \in \mathrm{T}$. Let $\mathfrak{S}^*$ be the product topology on $Y^{\mathbb{N}}$, so that $\mathfrak{S}^* \cap \Lambda_0$ is a base for $\mathfrak{S}^*$. Then λ is inner regular with respect to the Borel sets (412Uc, or otherwise). Define $\phi : \{0,1\}^{\mathbb{N}} \to Y$ by setting

$$\phi(u)(n) = 0 \text{ if } u(n) = 0,$$
$$= \omega_1 \text{ if } u(n) = 1.$$

Let ν_ω be the usual measure on $\{0,1\}^{\mathbb{N}}$; then ϕ is inverse-measure-preserving for ν_ω and λ, by 254H.

If $V \in \Lambda_0$ there is an $\alpha < \omega_1$ such that

whenever $x \in V$, $y \in Y^{\mathbb{N}}$ and $y(i) = x(i)$ whenever $\min(x(i), y(i)) < \alpha$, then $y \in V$.

P Let $\mathcal{W}$ be the family of sets $V \in \Lambda$ with this property. Then $\mathcal{W}$ is a σ-algebra of subsets of $Y^{\mathbb{N}}$ including the family $\mathcal{C}$ of measurable cylinders, so includes Λ_0. **Q**

(c) Let X be $Y^{\mathbb{N}} \setminus \{0,\omega_1\}^{\mathbb{N}}$. Then $\lambda^* X = 1$. **P** Take $V \in \Lambda$ such that $V \supseteq X$ and $\lambda V = \lambda^* X$. Then there is a $V' \in \Lambda_0$ such that $V \subseteq V'$ and $\lambda V' = \lambda V$ (254Ff). As remarked in (b), there is an $\alpha \in]0,\omega_1[$ such that $y \in V'$ whenever $x \in V'$, $y \in Y^{\mathbb{N}}$ and $y(i) = x(i)$ whenever $\min(x(i), y(i)) < \alpha$. But as $\{0,\alpha\}^{\mathbb{N}} \setminus \{\mathbf{0}\} \subseteq X \subseteq V'$, $\{0,\omega_1\}^{\mathbb{N}} \setminus \{\mathbf{0}\} \subseteq V'$. Accordingly

$$\lambda^* X = \lambda V = \lambda V' = \nu_\omega \phi^{-1}[V'] = \nu_\omega(\{0,1\}^{\mathbb{N}} \setminus \{\mathbf{0}\}) = 1. \ \textbf{Q}$$

Give X the subspace measure μ induced by λ, with domain $\Sigma = \{X \cap W : W \in \Lambda\}$, and the subspace topology $\mathfrak{T}$ induced by $\mathfrak{S}^*$. Then $\mu X = \lambda^* X = 1$, and $\mu(X \cap V) = \lambda V$ for every $V \in \Lambda$. $\Sigma \cap \mathfrak{T}$ is a base for $\mathfrak{T}$, just because $\Lambda \cap \mathfrak{S}^*$ is a base for $\mathfrak{S}^*$; and μ is inner regular with respect to the Borel sets, by 412Pb.

(d) ? Suppose, if possible, that μ is not τ-additive. Let $\mathcal{G} \subseteq \Sigma \cap \mathfrak{T}$ be a non-empty upwards-directed set with union $W \in \Sigma \cap \mathfrak{T}$ such that $\mu W > \sup_{G \in \mathcal{G}} \mu G$. Let $\mathcal{G}_0$ be a countable subset of $\mathcal{G}$ such that $\mu(G \setminus \bigcup \mathcal{G}_0) = 0$ for every $G \in \mathcal{G}_0$, and set $W_1 = W \setminus \bigcup \mathcal{G}_0$; then $\mu(\bigcup \mathcal{G}_0) \leq \sup_{G \in \mathcal{G}} \mu G$, so $\mu W_1 > 0$, while $\mu(G \cap W_1) = 0$ for every $G \in \mathcal{G}$.

Express W_1 as $V_1 \cap X$ where $V_1 \in \Lambda$. Then there is a $V_2 \in \Lambda_0$ such that $V_2 \subseteq V_1$ and $\lambda(V_1 \setminus V_2) = 0$, so that

$$\lambda V_2 = \lambda V_1 = \mu W_1 > 0.$$

Let $\alpha < \omega_1$ be such that $y \in V_2$ whenever $x \in V_2$, $y \in Y^{\mathbb{N}}$ and $y(i) = x(i)$ whenever $\min(x(i), y(i)) < \alpha$.

Let $F \subseteq \phi^{-1}[V_2]$ be a non-negligible measurable self-supporting set for the Radon measure ν_ω (416U, 414F). By Lemma 345E, there are u, $u' \in F$ which differ at exactly one coordinate; let $i \in \mathbb{N}$ be that coordinate, and suppose that $u(i) = 1$, $u'(i) = 0$.

We know that $\phi(u) \in V_2$, and $\phi(u)(i) = \omega_1$. Define $x \in X$ by saying that $x(j) = \phi(u)(j)$ for $j \neq i$, $x(i) = \alpha$. Then $x \in V_2$, by the choice of α. Now $V_2 \cap X \subseteq W_1 \subseteq \bigcup \mathcal{G}$, so there is a $G \in \mathcal{G}$ containing x. Let $V \subseteq Y^{\mathbb{N}}$ be a basic open set such that $x \in X \cap V \subseteq G$; express V as $\prod_{j \in \mathbb{N}} H_j$ where $H_j \in \mathfrak{S}$ for every j and $J = \{j : H_j \neq Y\}$ is finite. Observe that $0 \in H_j$ and $x(j) < \omega_1$ for every $j \in J$, just because 0 belongs to every non-empty open subset of Y and the only open set containing ω_1 is Y itself. But this means that $u(j) = 0$ for every $j \in J \setminus \{i\}$, so $u'(j) = 0$ for every $j \in J$, and $\phi(u') \in V$; thus the open set $U = \phi^{-1}[V]$ meets F. Because F is self-supporting,

$$0 < \nu_\omega(F \cap U) \leq \lambda(V_2 \cap V) = \mu(V_2 \cap V \cap X) \leq \mu(W_1 \cap G),$$

which is impossible. **X**

Thus μ is τ-additive.

(e) ? But suppose, if possible, that there were a topological measure $\tilde{\mu}$ on X agreeing with μ on every open set in the domain of μ. For each $i \in \mathbb{N}$, set $\pi_i(x) = x(i)$ for $x \in X$. Every subset of Y is a Borel set for $\mathfrak{S}$; because π_i is continuous, the image measure $\tilde{\mu}\pi_i^{-1}$ is defined on $\mathcal{P}Y$. Now $\#(Y) = \omega_1$, so there must be a countable conegligible set (419G), and there must be some $\alpha_i < \omega_1$ such that $\tilde{\mu}\pi_i^{-1}(\omega_1 \setminus \alpha_i) = 0$. On the other hand,

$$\tilde{\mu}\pi_i^{-1}(\alpha_i \setminus \{0\}) = \mu\pi_i^{-1}(\alpha_i \setminus \{0\}) = \lambda\{y : 0 < y(i) < \alpha_i\} = \nu(\alpha_i \setminus \{0\}) = 0,$$

so $\tilde{\mu}\pi_i^{-1}(\omega_1 \setminus \{0\}) = 0$.

But the definition of X was exactly devised so that

$$X = \textstyle\bigcup_{i \in \mathbb{N}} \pi_i^{-1}(\omega_1 \setminus \{0\}),$$

so this is impossible. **X**

So we have the required example.

Remark I note that the topology of X is not regular. Of course the phenomenon here cannot arise with regular spaces, by 415M.

419I For the next example it will be helpful to know some basic facts about Lebesgue measure which seemed a little advanced for Volume 1 and for which I have not found a suitable place since.

Lemma (a) If $(X, \mathfrak{T}, \Sigma, \mu)$ is an atomless Radon measure space and $E \in \Sigma$ has non-zero measure, then $\#(E) \geq \mathfrak{c}$.

(b) The number of closed subsets of $\mathbb{R}$ is $\mathfrak{c}$.

proof (a) There must be a compact set $K \subseteq E$ such that $\mu K > 0$, and a self-supporting closed $K' \subseteq K$ such that $\mu K' = \mu K$. Because $\mu\{x\} = 0$ for every $x \in X$, K' can have no isolated points. So $\#(K') \geq \mathfrak{c}$ (4A2G(i-ii)) and $\#(E) \geq \mathfrak{c}$.

(b) Write $\mathcal{E}$ for the family of closed subsets of $\mathbb{R}$. Let $\langle U_n \rangle_{n \in \mathbb{N}}$ enumerate a base for the topology of $\mathbb{R}$ (4A2Ua). For each $I \subseteq \mathbb{N}$, set

$$F_I = \mathbb{R} \setminus \bigcup_{n \in I} U_n.$$

Because every open set is expressible as a union of some U_i, the map $I \mapsto F_I : \mathcal{P}\mathbb{N} \to \mathcal{E}$ is surjective. So $\#(\mathcal{E}) \leq \#(\mathcal{P}\mathbb{N}) = \mathfrak{c}$. On the other hand, the map $x \mapsto [0, x] : [0, 1] \to \mathcal{E}$ is injective, so $\#(\mathcal{E}) \geq \#([0, 1]) = \mathfrak{c}$.

Remark In fact, of course, the number of Borel subsets of $\mathbb{R}$ is $\mathfrak{c}$; see 4A1O.

419J The next result is a strengthening of 134D.

Lemma Let μ be Lebesgue measure on $\mathbb{R}$, and H any measurable subset of $\mathbb{R}$. Then there is a disjoint family $\langle A_\alpha \rangle_{\alpha < \mathfrak{c}}$ of subsets of H such that H is a measurable envelope of every A_α; in particular, $\mu_* A_\alpha = 0$ and $\mu^* A_\alpha = \mu H$ for every $\alpha < \mathfrak{c}$.

proof If $\mu H = 0$, we can take every A_α to be empty; so suppose that $\mu H > 0$. Let $\mathcal{E}$ be the family of closed subsets of H of non-zero measure. By 419Ib, $\#(\mathcal{E}) \leq \mathfrak{c}$; enumerate $\mathcal{E} \times \mathfrak{c}$ as $\langle (F_\xi, \alpha_\xi) \rangle_{\xi < \mathfrak{c}}$ (3A1Ca). Choose $\langle x_\xi \rangle_{\xi < \mathfrak{c}}$ inductively, as follows. Given $\langle x_\eta \rangle_{\eta < \xi}$, where $\xi < \mathfrak{c}$, F_ξ has cardinal (at least) $\mathfrak{c}$, by 419Ia, so cannot be included in $\{x_\eta : \eta < \xi\}$; take any $x_\xi \in F_\xi \setminus \{x_\eta : \eta < \xi\}$, and continue.

At the end of the induction, set

$$A_\alpha = \{x_\xi : \xi < \mathfrak{c}, \alpha_\xi = \alpha\}.$$

Then the A_α are disjoint just because the x_ξ are distinct.

? Suppose, if possible, that H is not a measurable envelope of A_α for some α. Then $\mu_*(H \setminus A_\alpha) > 0$ (413Ei), so there is a non-negligible measurable set $E \subseteq H \setminus A_\alpha$. Now there is an $F \in \mathcal{E}$ such that $F \subseteq E$. Let $\xi < \mathfrak{c}$ be such that $F = F_\xi$ and $\alpha = \alpha_\xi$; then $x_\xi \in A_\alpha \cap F$, which is impossible. **X**

Thus H is always a measurable envelope of A_α. It follows from the definition of 'measurable envelope' that $\mu^* A_\alpha = \mu H$. But also, if $\alpha < \mathfrak{c}$, $\mu_* A_\alpha \leq \mu_*(H \setminus A_{\alpha+1})$, which is 0, as we have just seen. So we have a suitable family.

419K Example (Blackwell 56) There are sequences $\langle X_n \rangle_{n \in \mathbb{N}}$, $\langle \mathfrak{T}_n \rangle_{n \in \mathbb{N}}$ and $\langle \nu_n \rangle_{n \in \mathbb{N}}$ such that (i) for each n, $(X_n, \mathfrak{T}_n)$ is a separable metrizable space and ν_n is a quasi-Radon probability measure on $Z_n = \prod_{i \leq n} X_i$ (ii) for $m \leq n$ the canonical map $\pi_{mn} : Z_n \to Z_m$ is inverse-measure-preserving (iii) there is no probability measure on $Z = \prod_{i \in \mathbb{N}} X_i$ such that all the canonical maps from Z to Z_n are inverse-measure-preserving.

proof Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a disjoint sequence of subsets of $[0, 1]$ such that $\mu_*([0, 1] \setminus A_n) = 0$, that is, $\mu^* A_n = 1$ for every n, where μ is Lebesgue measure (using 419J). Set $X_n = \bigcup_{i \geq n} A_i$, so that $\langle X_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence of sets of outer measure 1 with empty intersection. For each $n \geq 1$, we have a map $f_n : X_n \to Z_n$ defined by setting $f_n(x)(i) = x$ for every $i \leq n$, $x \in X_n$. Let ν_n be the image measure $\mu_{X_n} f_n^{-1}$, where μ_{X_n} is the subspace measure on X_n induced by μ. Note that f_n is a homeomorphism between X_n and the diagonal $\Delta_n = \{z : z \in Z_n, z(i) = z(j) \text{ for all } i, j \leq n\}$, which is a closed subset of Z_n; so that ν_n, like μ_{X_n}, is a quasi-Radon probability measure.

If $m \leq n$, then π_{mn} is inverse-measure-preserving, where $\pi_{mn}(z)(i) = z(i)$ for $z \in Z_n$, $i \leq m$. **P** If $W \subseteq Z_m$ is measured by ν_m, then $f_m^{-1}[W]$ is measured by μ_{X_m}, so is of the form $X_m \cap E$ where E is Lebesgue measurable. But in this case $f_m^{-1}[\pi_{mn}^{-1}[W]] = X_n \cap E$, so that

$$\nu_n(\pi_{mn}^{-1}[W]) = \mu_{X_n}(f_n^{-1}[\pi_{mn}^{-1}[W]]) = \mu^*(X_n \cap E) = \mu E = \mu^*(X_m \cap E) = \nu_m W. \ \textbf{Q}$$

? But suppose, if possible, that there is a probability measure ν on $Z = \prod_{i \in \mathbb{N}} X_i$ such that $\pi_n : Z \to Z_n$ is inverse-measure-preserving for every n, where $\pi_n(z)(i) = z(i)$ for $z \in Z$, $i \leq n$. Then

$$\nu\pi_n^{-1}[\Delta_n] = \nu_n\Delta_n = \mu_{X_n} f_n^{-1}[\Delta_n] = 1$$

for each n, so

$$1 = \nu(\textstyle\bigcap_{n\in\mathbb{N}} \pi_n^{-1}[\Delta_n]) = \nu\{z : z \in Z,\, z(i) = z(j) \text{ for all } i,\, j \in \mathbb{N}\} = \nu\emptyset,$$

because $\bigcap_{n\in\mathbb{N}} X_n = \emptyset$; which is impossible. **X**

419L The split interval again (a) For the sake of an example in §343, I have already introduced the 'split interval' or 'double arrow space'. As this construction gives us a topological measure space of great interest, I repeat it here. Let $I^{\Vdash}$ be the set $\{a^+ : a \in [0,1]\} \cup \{a^- : a \in [0,1]\}$. Order it by saying that

$$a^+ \le b^+ \iff a^- \le b^+ \iff a^- \le b^- \iff a \le b, \quad a^+ \le b^- \iff a < b.$$

Then it is easy to check that $I^{\Vdash}$ is a totally ordered space, and that it is Dedekind complete. (If $A \subseteq [0,1]$ is a non-empty set, then $\sup_{a\in A} a^- = (\sup A)^-$, while $\sup_{a\in A} a^+$ is either $(\sup A)^+$ or $(\sup A)^-$, depending on whether $\sup A$ belongs to A or not.) Its greatest element is 1^+ and its least element is 0^-. Consequently the order topology on $I^{\Vdash}$ is a compact Hausdorff topology (4A2Rc, 4A2Ri). Note that $Q = \{q^+ : q \in [0,1] \cap \mathbb{Q}\} \cup \{q^- : q \in [0,1] \cap \mathbb{Q}\}$ is dense, because it meets every non-trivial interval in $I^{\Vdash}$. By 4A2E(a-ii) and 4A2Rn, $I^{\Vdash}$ is ccc and hereditarily Lindelöf.

(b) If we define $h : I^{\Vdash} \to [0,1]$ by writing $h(a^+) = h(a^-) = a$ for every $a \in [0,1]$, then h is continuous, because $\{x : h(x) < a\} = \{x : x < a^-\}$, $\{x : h(x) > a\} = \{x : x > a^+\}$ for every $a \in [0,1]$. Now we can describe the Borel sets of $I^{\Vdash}$, as follows: a set $E \subseteq I^{\Vdash}$ is Borel iff there is a Borel set $F \subseteq [0,1]$ such that $E \triangle h^{-1}[F]$ is countable. **P** Write Σ_0 for the family of subsets E of $I^{\Vdash}$ such that $E\triangle h^{-1}[F]$ is countable for some Borel set $F \subseteq [0,1]$. It is easy to check that Σ_0 is a σ-algebra of subsets of $I^{\Vdash}$. (If $E\triangle h^{-1}[F]$ is countable, so is $(I^{\Vdash} \setminus E)\triangle h^{-1}[[0,1] \setminus F]$; if $E_n\triangle h^{-1}[F_n]$ is countable for every n, so is $(\bigcup_{n\in\mathbb{N}} E_n)\triangle h^{-1}[\bigcup_{n\in\mathbb{N}} F_n]$.) Because the topology of $I^{\Vdash}$ is Hausdorff, every singleton set is closed, so every countable set is Borel. Also $h^{-1}[F]$ is Borel for every Borel set $F \subseteq [0,1]$, because h is continuous (4A3Cd). So if $E\triangle h^{-1}[F]$ is countable for some Borel set $F \subseteq [0,1]$, $E = h^{-1}[F]\triangle(E\triangle h^{-1}[F])$ is a Borel set in $I^{\Vdash}$. Thus Σ_0 is included in the Borel σ-algebra $\mathcal{B}$ of $I^{\Vdash}$. On the other hand, if $J \subseteq I^{\Vdash}$ is an interval, $h[J]$ is also an interval, therefore a Borel set, and $h^{-1}[h[J]] \setminus J$ can contain at most two points, so $J \in \Sigma_0$. If $G \subseteq I^{\Vdash}$ is open, it is expressible as $\bigcup_{i\in I} J_i$, where $\langle J_i\rangle_{i\in I}$ is a disjoint family of non-empty open intervals (4A2Rj). As X is ccc, I must be countable. Thus G is expressed as a countable union of members of Σ_0 and belongs to Σ_0. But this means that the Borel σ-algebra $\mathcal{B}$ must be included in Σ_0, by the definition of 'Borel algebra'. So $\mathcal{B} = \Sigma_0$, as claimed. **Q**

(c) In 343J I described the standard measure μ on $I^{\Vdash}$; its domain is the set $\Sigma = \{h^{-1}[F]\triangle M : F \in \Sigma_L,\, M \subseteq I^{\Vdash},\, \mu_L h[M] = 0\}$, where Σ_L is the set of Lebesgue measurable subsets of $[0,1]$ and μ_L is Lebesgue measure, and $\mu E = \mu_L h[E]$ for $E \in \Sigma$. h is inverse-measure-preserving for μ and μ_L.

The new fact I wish to mention is: μ is a completion regular Radon measure. **P** I noted in 343Ja that it is a complete probability measure; *a fortiori*, it is locally determined and locally finite. If $G \subseteq I^{\Vdash}$ is open, then we can express it as $h^{-1}[F]\triangle C$ for some Borel set $F \subseteq [0,1]$, countable set $C \subseteq I^{\Vdash}$ ((b) above), so it belongs to Σ; thus μ is a topological measure. If $E \in \Sigma$ and $\mu E > \gamma$, then $F = [0,1] \setminus h[I^{\Vdash} \setminus E]$ is Lebesgue measurable, and $\mu E = \mu_L F$. So there is a compact set $L \subseteq F$ such that $\mu_L L \ge \gamma$. But now $K = h^{-1}[L] \subseteq E$ is closed, therefore compact, and $\mu K \ge \gamma$. Moreover, L is a zero set, being a closed set in a metrizable space (4A2Lc), so K is a zero set (4A2C(b-iv)). As E and γ are arbitrary, μ is inner regular with respect to the compact zero sets, and is a completion regular Radon measure. **Q**

419X Basic exercises (a) Show that the topological space X of 419A is zero-dimensional.

(b) Give an example of a compact Radon probability space in which every dense G_δ set is conegligible. (*Hint*: 411P.)

(c) In 419E, show that we can start from any atomless probability measure in place of Lebesgue measure on $[0,1]$.

>(d) (i) Show that if $E \subseteq \mathbb{R}^2$ is Lebesgue measurable, with non-zero measure, then it cannot be covered by fewer than $\mathfrak{c}$ lines. (*Hint*: if $H = \{t : \mu_1 E[\{t\}] > 0\}$, where μ_1 is Lebesgue measure on $\mathbb{R}$, then $\mu_1 H > 0$, so $\#(H) = \mathfrak{c}$. So if we have a family $\mathcal{L}$ of lines, with $\#(\mathcal{L}) < \mathfrak{c}$, there must be a $t \in H$ such that $L_t = \{t\} \times \mathbb{R}$ does not belong to $\mathcal{L}$. Now $\#(L_t \cap E) = \mathfrak{c}$ and each member of $\mathcal{L}$ meets $L_t \cap E$ in at most one point.) (ii) Show that there is a subset A of $\mathbb{R}^2$, of full outer measure, which meets every vertical line and every horizontal line in exactly one point. (*Hint*: enumerate $\mathbb{R}$ as $\langle t_\xi \rangle_{\xi<\mathfrak{c}}$ and the closed sets of non-zero measure as $\langle F_\xi \rangle_{\xi<\mathfrak{c}}$. Choose $\langle I_\xi \rangle_{\xi<\mathfrak{c}}$ such that every I_ξ is finite, no two points of $I_\xi \cup \bigcup_{\eta<\xi} I_\eta$ lie on any horizontal or vertical line, the lines $\{t_\xi\} \times \mathbb{R}$ and $\mathbb{R} \times \{t_\xi\}$ both meet $I_\xi \cup \bigcup_{\eta<\xi} I_\eta$, and I_ξ meets F_ξ.) (iii) Show that there is a subset B of $\mathbb{R}^2$, of full outer measure, such that every straight line meets A in exactly two points. (*Hint*: enumerate the straight lines in $\mathbb{R}^2$ as $\langle L_\xi \rangle_{\xi<\mathfrak{c}}$. Choose $\langle J_\xi \rangle_{\xi<\mathfrak{c}}$ such that every J_ξ is finite, no three points of $J_\xi \cup \bigcup_{\eta<\xi} J_\eta$ lie on any line, $L_\xi \cap (J_\xi \cup \bigcup_{\eta<\xi} J_\eta)$ has just two points and $J_\xi \cap F_\xi \neq \emptyset$.)

(e) Let μ be Lebesgue measure on $[0,1]$, and $A \subseteq [0,1]$ a set with inner measure 0 and outer measure 1; let $\mathfrak{T}$ be the usual topology on $[0,1]$. Let $\mathcal{I}$ be the family of sets $I \subseteq A$ such that every point of A has a neighbourhood containing at most one point of I. Show that $\mathfrak{S} = \{G \setminus I : G \in \mathfrak{T}, I \in \mathcal{I}\}$ is a topology on $[0,1]$ with a countable network. Show that the identity map from $[0,1]$ to itself, regarded as a map from $([0,1], \mathfrak{T}, \mu)$ to $([0,1], \mathfrak{S})$, is measurable but not almost continuous.

(f) Let $\mathbb{R}^{\|}$ be the **split line**, that is, the set $\{a^+ : a \in \mathbb{R}\} \cup \{a^- : a \in \mathbb{R}\}$, ordered by the rules in 419L. Show that $\mathbb{R}^{\|}$ is a Dedekind complete totally ordered set, so that its order topology $\mathfrak{T}$ is locally compact. Write μ_L for Lebesgue measure on $\mathbb{R}$, Σ_L for its domain. Set $h(a^+) = h(a^-) = a$ for $a \in \mathbb{R}$, $\Sigma = \{E : E \subseteq X, h[E] \in \Sigma_L, \mu_L(h[E] \cap h[X \setminus E]) = 0\}$, $\mu E = \mu_L h[E]$ for $E \in \Sigma$. Show that μ is a completion regular Radon measure on $\mathbb{R}^{\|}$ and that h is continuous and inverse-measure-preserving for μ and μ_L. Show that the set $\{a^+ : a \in \mathbb{R}\}$, with the induced topology and measure, is isomorphic, as quasi-Radon measure space, to the right-facing Sorgenfrey line (415Xc) with Lebesgue measure. Show that $\mathbb{R}^{\|}$ and the Sorgenfrey line are hereditarily Lindelöf.

(g) Let μ be Lebesgue measure on $[0,1]$ and Σ its domain. Let $I^{\|}$ be the split interval. (i) Show that the functions $x \mapsto x^+ : [0,1] \to I^{\|}$ and $x \mapsto x^- : [0,1] \to I^{\|}$ are measurable. (*Hint*: 419Lb.) (ii) Show that the function $x \mapsto (x^+, x^-) : [0,1] \to (I^{\|})^2$ is not measurable. (*Hint*: the subspace topology on $\{(x^+, x^-) : x \in [0,1]\}$ is discrete.)

>(h) (i) Again writing $I^{\|}$ for the split interval, show that the function which exchanges x^+ and x^- for every $x \in [0,1]$ is a Borel automorphism and an automorphism for the usual Radon measure ν on $I^{\|}$, but is not almost continuous. (ii) Show that if we set $f(x) = x^+$ for $x \in [0,1]$, then f is inverse-measure-preserving for Lebesgue measure μ_L on $[0,1]$, but the image measure $\mu_L f^{-1}$ is not ν (nor, indeed, a Radon measure).

419Y Further exercises (a) In the example of 419E, show that there is a Borel set $V \subseteq Z^2$ such that $\tilde{\lambda} V = 0$ and $\lambda^* V = 1$.

(b) Show that if $\mathcal{A} \subseteq \mathcal{P}\omega_1$ is any family with $\#(\mathcal{A}) \leq \omega_1$, there is a countably generated σ-algebra Σ of subsets of ω_1 such that $\mathcal{A} \subseteq \Sigma$.

(c) Show that the split interval with its usual topology and measure has the simple product property (417Yi).

419 Notes and comments The construction of the locally compact space X in 419A from the family $\langle I_\xi \rangle_{\xi<\kappa}$ is a standard device which has been used many times. The relation $\subseteq^*$ also appears in many contexts. In effect, part of the argument is taking place in the quotient algebra $\mathfrak{A} = \mathcal{P}Q/[Q]^{<\omega}$, since $I \subseteq^* J$ iff $I^\bullet \subseteq J^\bullet$ in $\mathfrak{A}$; setting $\mathcal{I}^\# = \{I^\bullet : I \in \mathcal{I}\}$, the cardinal κ is $\min\{\#(A) : A \subseteq \mathcal{I}^\#$ has no upper bound in $\mathcal{I}^\#\}$, the 'additivity' of the partially ordered set $\mathcal{I}^\#$. Additivities of partially ordered sets will be one of the important concerns of Volume 5. I remark that we do not need to know whether (for instance) $\kappa = \omega_1$ or $\kappa = \mathfrak{c}$. This is an early taste of the kind of manoeuvre which has become a staple of set-theoretic analysis. It happens that the cardinal κ here is one of the most important cardinals of set-theoretic measure

theory; it is 'the additivity of Lebesgue measure' (529Xa), and under that name will appear repeatedly in Chapter 52.

Observe that the measure μ of 419A only just fails to be a quasi-Radon measure; it is locally finite instead of being effectively locally finite. And it would be a Radon measure if it were inner regular with respect to the compact sets, rather than just with respect to the closed sets.

419C and 419D are relevant to the question: have I given the 'right' definition of Radon measure space? 419C is perhaps more important. Here we have a Radon measure space (on my definition) for which the associated Borel measure is not localizable. (If $\mathfrak{A}$ is the measure algebra of the measure μ, and $\mathfrak{B}$ the measure algebra of $\mu\restriction\mathcal{B}$ where $\mathcal{B}$ is the Borel σ-algebra of X, then the embedding $\mathcal{B} \subseteq \Sigma$ induces an embedding of $\mathfrak{B}$ in $\mathfrak{A}$ which represents $\mathfrak{B}$ as an order-dense subalgebra of $\mathfrak{A}$, just because μ is inner regular with respect to $\mathcal{B}$. Property (i) of 419C shows that $\mathfrak{B} \neq \mathfrak{A}$, so $\mathfrak{B}$ cannot be Dedekind complete in itself, by 314Ia.) Since (I believe) localizable versions of measure spaces should almost always be preferred, I take this as strong support for my prejudice in favour of insisting that 'Radon' measure spaces should be locally determined as well as complete. Property (ii) of 419C is not I think of real significance, but is further evidence, to be added to 415Xh, that outer regularity is like an exoskeleton: it may inhibit growth above a certain size.

In 419D I explore the consequences of omitting the condition 'locally finite' from the definition of Radon measure. Even if we insist instead that compact sets should have finite measure, we are in danger of getting a non-localizable measure. Of course this particular space is pathological in terms of most of the criteria of this chapter – for instance, every non-empty open set has infinite measure, and the topology is not regular.

Perhaps the most important example in the section is 419E. The analysis of τ-additive product measures in §417 was long and difficult, and if these were actually equal to the familiar product measures in all important cases the structure of the theory would be very different. But we find that for one of the standard compact Radon probability spaces of the theory, the c.l.d. product measure on its square is not a Radon measure, and something has to be done about it.

I present 419H here to indicate one of the obstacles to any simplification of the arguments in 417C and 417E. It is not significant in itself, but it offers a welcome excuse to describe some fundamental facts about ω_1 (419F-419G). Similarly, 419K asks for some elementary facts about Lebesgue measure (419I-419J) which seem to have got left out. This example really is important in itself, as it touches on the general problem of representing stochastic processes, to which I will return in Chapter 45.

Chapter 42

Descriptive set theory

At this point, I interpolate an auxiliary chapter, in the same spirit as Chapters 31 and 35 in the last volume. As with Boolean algebras and Riesz spaces, it is not just that descriptive set theory provides essential tools for modern measure theory; it also offers deep intuitions, and for this reason demands study well beyond an occasional glance at an appendix. Several excellent accounts have been published; the closest to what we need here is probably ROGERS 80; at a deeper level we have MOSCHOVAKIS 80, and an admirable recent treatment is KECHRIS 95. Once again, however, I indulge myself by extracting those parts of the theory which I shall use directly, giving proofs and exercises adapted to the ideas I am trying to emphasize in this volume and the next.

The first section describes Souslin's operation and its basic set-theoretic properties up to the theory of 'constituents' (421N-421Q), mostly steering away from topological ideas, but with some remarks on σ-algebras and Souslin-F sets. §422 deals with usco-compact relations and K-analytic spaces, working through the topological properties which will be useful later, and giving a version of the First Separation Theorem (422I-422J). §423 looks at 'analytic' or 'Souslin' spaces, treating them primarily as a special kind of K-analytic space, with the von Neumann-Jankow selection theorem (423N). §424 is devoted to 'standard Borel spaces'; it is largely a series of easy applications of results in §423, but there is one substantial theorem on Borel measurable actions of Polish groups (424H).

421 Souslin's operation

I introduce Souslin's operation $\mathcal{S}$ (421B) and show that it is idempotent (421D). I describe alternative characterizations of members of $\mathcal{S}(\mathcal{E})$, where $\mathcal{E} \subseteq \mathcal{P}X$, as projections of sets in $\mathbb{N}^{\mathbb{N}} \times X$ (421G-421J). I briefly mention Souslin-F sets (421J-421L) and a special property of 'inner Souslin kernels' (421M). At the end of the section I set up an abstract theory of 'constituents' for kernels of Souslin schemes and their complements (421N-421Q).

421A Notation Throughout this chapter, and frequently in the next, I shall regard a member of $\mathbb{N}$ as the set of its predecessors, so that $\mathbb{N}^k$ can be identified with the set of functions from k to $\mathbb{N}$, and if $\phi \in \mathbb{N}^{\mathbb{N}}$ and $k \in \mathbb{N}$, we can speak of the restriction $\phi{\restriction}k \in \mathbb{N}^k$. In the same spirit, identifying functions with their graphs, I can write '$\sigma \subseteq \phi$' when $\sigma \in \mathbb{N}^k$, $\phi \in \mathbb{N}^{\mathbb{N}}$ and ϕ extends σ. On occasion I may write $\#(\sigma)$ for the 'length' of a finite function σ – again identifying σ with its graph – so that $\#(\sigma) = k$ if $\sigma \in X^k$. And if $k = 0$, identified with $\emptyset$, then the only function from k to X is the empty function, so X^0 becomes $\{\emptyset\}$.

I shall sometimes refer to the 'usual topology of $\mathbb{N}^{\mathbb{N}}$'; this is the product topology if each copy of $\mathbb{N}$ is given its discrete topology. Writing $I_\sigma = \{\phi : \phi \in \mathbb{N}^{\mathbb{N}}, \phi \supseteq \sigma\}$ for $\sigma \in S^* = \bigcup_{k\in\mathbb{N}} \mathbb{N}^k$, $\{I_\sigma : \sigma \in S^*\}$ is a base for the topology of $\mathbb{N}^{\mathbb{N}}$ consisting of open-and-closed sets. If $\sigma \in \mathbb{N}^k$, $i \in \mathbb{N}$ I write $\sigma^\frown i$ for the member τ of $\mathbb{N}^{k+1}$ such that $\tau(k) = i$ and $\tau(j) = \sigma(j)$ for $j < k$.

421B Definition Let S be the set $\bigcup_{k\geq 1} \mathbb{N}^k$. If $\mathcal{E}$ is a family of sets, I write $\mathcal{S}(\mathcal{E})$ for the family of sets expressible in the form

$$\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\geq 1} E_{\phi{\restriction}k}$$

for some family $\langle E_\sigma\rangle_{\sigma\in S}$ in $\mathcal{E}$.

A family $\langle E_\sigma\rangle_{\sigma\in S}$ is called a **Souslin scheme**; the corresponding set $\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\geq 1} E_{\phi{\restriction}k}$ is its **kernel**; the operation

$$\langle E_\sigma\rangle_{\sigma\in S} \mapsto \bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\geq 1} E_{\phi{\restriction}k}$$

is **Souslin's operation** or **operation** $\mathcal{A}$. Thus $\mathcal{S}(\mathcal{E})$ is the family of sets obtainable from sets in $\mathcal{E}$ by Souslin's operation. If $\mathcal{E} = \mathcal{S}(\mathcal{E})$, we say that $\mathcal{E}$ is **closed under Souslin's operation**.

Remark I should perhaps warn you that some authors use $S^* = \bigcup_{k\in\mathbb{N}} \mathbb{N}^k$ here in place of S; so that their Souslin kernels are of the form $\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\geq 0} E_{\phi\restriction k} \subseteq E_\emptyset$. Consequently, for such authors, any member of $\mathcal{S}(\mathcal{E})$ is included in some member of $\mathcal{E}$. If $\mathcal{E}$ has a greatest member (or, fractionally more generally, if any sequence in $\mathcal{E}$ is bounded above in $\mathcal{E}$) this makes no difference; but if, for instance, $\mathcal{E}$ is the family of compact subsets of a topological space, the two definitions of $\mathcal{S}$ may not quite coincide. I believe that on this point, for once, I am following the majority.

421C Elementary facts (a) It is worth noting straight away that if $\mathcal{E}$ is any family of sets, then $\bigcup_{n\in\mathbb{N}} E_n$ and $\bigcap_{n\in\mathbb{N}} E_n$ belong to $\mathcal{S}(\mathcal{E})$ for any sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$. **P** Set

$$F_\sigma = E_{\sigma(0)} \text{ for every } \sigma \in S,$$

$$G_\sigma = E_k \text{ whenever } k \in \mathbb{N},\ \sigma \in \mathbb{N}^{k+1};$$

then

$$\textstyle\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\geq 1} F_{\phi\restriction k} \in \mathcal{S}(\mathcal{E}),$$

$$\textstyle\bigcap_{n\in\mathbb{N}} E_n = \bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{k\geq 1} G_{\phi\restriction k} \in \mathcal{S}(\mathcal{E}).$$ **Q**

In particular, $\mathcal{E} \subseteq \mathcal{S}(\mathcal{E})$. But note that there is no reason why $E \setminus F$ should belong to $\mathcal{S}(\mathcal{E})$ for $E, F \in \mathcal{E}$.

(b) Let X and Y be sets, and $f : X \to Y$ a function. Let $\langle F_\sigma\rangle_{\sigma\in S}$ be a Souslin scheme in $\mathcal{P}Y$, with kernel B. Then $f^{-1}[B]$ is the kernel of the Souslin scheme $\langle f^{-1}[F_\sigma]\rangle_{\sigma\in S}$. **P**

$$f^{-1}[B] = f^{-1}[\textstyle\bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n\geq 1} F_{\phi\restriction n}] = \bigcup_{\phi\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n\geq 1} f^{-1}[F_{\phi\restriction n}].$$ **Q**

(c) Let X and Y be sets, and $f : X \to Y$ a function. Let $\mathcal{F}$ be a family of subsets of Y. Then

$$\{f^{-1}[B] : B \in \mathcal{S}(\mathcal{F})\} = \mathcal{S}(\{f^{-1}[F] : F \in \mathcal{F}\}).$$

P For a set $A \subseteq X$, $A \in \mathcal{S}(\{f^{-1}[F] : F \in \mathcal{F}\})$ iff there is some Souslin scheme $\langle E_\sigma\rangle_{\sigma\in S}$ in $\{f^{-1}[F] : F \in \mathcal{F}\}$ such that A is the kernel of $\langle E_\sigma\rangle_{\sigma\in S}$, that is, iff there is some Souslin scheme $\langle F_\sigma\rangle_{\sigma\in S}$ in $\mathcal{F}$ such that A is the kernel of $\langle f^{-1}[F_\sigma]\rangle_{\sigma\in S}$, that is, iff $A = f^{-1}[B]$ where B is the kernel of some Souslin scheme in $\mathcal{F}$. **Q**

(d) Let X and Y be sets, and $f : X \to Y$ a surjective function. Let $\mathcal{F}$ be a family of subsets of Y. Then

$$\mathcal{S}(\mathcal{F}) = \{B : B \subseteq Y,\ f^{-1}[B] \in \mathcal{S}(\{f^{-1}[F] : F \in \mathcal{F}\})\}.$$

P If $B \in \mathcal{S}(\mathcal{F})$, then $f^{-1}[B] \in \mathcal{S}(\{f^{-1}[F] : F \in \mathcal{F}\})$, by (c) above. If $B \subseteq Y$ and $f^{-1}[B] \in \mathcal{S}(\{f^{-1}[F] : F \in \mathcal{F}\})$, then there is a Souslin scheme $\langle F_\sigma\rangle_{\sigma\in S}$ in $\mathcal{F}$ such that $f^{-1}[B]$ is the kernel of $\langle f^{-1}[F_\sigma]\rangle_{\sigma\in S}$, that is, $f^{-1}[B] = f^{-1}[C]$ where C is the kernel of $\langle F_\sigma\rangle_{\sigma\in S}$. Because f is surjective, $B = C \in \mathcal{S}(\mathcal{F})$. **Q**

(e) Souslin's operation can be thought of as a projection operator, as follows. Let $\langle E_\sigma\rangle_{\sigma\in S}$ be a Souslin scheme with kernel A. Set

$$R = \textstyle\bigcap_{n\geq 1} \bigcup_{\sigma\in\mathbb{N}^n} I_\sigma \times E_\sigma,$$

writing $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$ as usual. Then $R[\mathbb{N}^{\mathbb{N}}] = A$. **P** For any x, and any $\phi \in \mathbb{N}^{\mathbb{N}}$,

$$\begin{aligned}(\phi, x) \in R &\iff \text{for every } n \geq 1 \text{ there is a } \sigma \in \mathbb{N}^n \text{ such that } x \in E_\sigma,\ \phi \in I_\sigma\\ &\iff x \in E_{\phi\restriction n} \text{ for every } n \geq 1.\end{aligned}$$

But this means that

$$\begin{aligned}x \in R[\mathbb{N}^{\mathbb{N}}] &\iff \text{there is a } \phi \in \mathbb{N}^{\mathbb{N}} \text{ such that } (\phi, x) \in R\\ &\iff \text{there is a } \phi \in \mathbb{N}^{\mathbb{N}} \text{ such that } x \in \bigcap_{n\geq 1} E_{\phi\restriction n} \iff x \in A.\end{aligned}$$ **Q**

421D The first fundamental theorem is that the operation $\mathcal{S}$ is idempotent.

Theorem (SOUSLIN 17) For any family $\mathcal{E}$ of sets, $\mathcal{S}(\mathcal{E})$ is closed under Souslin's operation.

proof (a) Let $\langle A_\sigma\rangle_{\sigma\in S}$ be a family in $\mathcal{S}(\mathcal{E})$, and set $A = \bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{k\geq 1} A_{\phi\restriction k}$; I have to show that $A \in \mathcal{S}(\mathcal{E})$. For each $\sigma \in S$, let $\langle E_{\sigma\tau}\rangle_{\tau\in S}$ be a family in $\mathcal{E}$ such that $A_\sigma = \bigcup_{\psi\in\mathbb{N}^\mathbb{N}} \bigcap_{m\geq 1} E_{\sigma,\psi\restriction m}$. Then

$$A = \bigcup_{\phi\in\mathbb{N}^\mathbb{N}} \bigcap_{k\geq 1} \bigcup_{\psi\in\mathbb{N}^\mathbb{N}} \bigcap_{m\geq 1} E_{\phi\restriction k,\psi\restriction m} = \bigcup_{\substack{\phi\in\mathbb{N}^\mathbb{N}\\ \Psi\in(\mathbb{N}^\mathbb{N})^{\mathbb{N}\setminus\{0\}}}} \bigcap_{k,m\geq 1} E_{\phi\restriction k,\psi_k\restriction m},$$

writing $\Psi = \langle\psi_k\rangle_{k\geq 1}$ for $\Psi \in (\mathbb{N}^\mathbb{N})^{\mathbb{N}\setminus\{0\}}$. The idea of the proof is simply that $\mathbb{N}^\mathbb{N} \times (\mathbb{N}^\mathbb{N})^{\mathbb{N}\setminus\{0\}}$ is essentially identical to $\mathbb{N}^\mathbb{N}$, so that all we have to do is to organize new names for the $E_{\sigma\tau}$. But as it is by no means a trivial matter to devise a coding scheme which really works, I give the details at length.

(b) The first step is to note that S and S^2 are countable, so there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ running over $\{E_{\sigma\tau} : \sigma, \tau \in S\}$. Next, choose any injective function $q : \mathbb{N}\times\mathbb{N} \to \mathbb{N}\setminus\{0\}$ such that $q(0,0) = 1$ and $q(0,1) = 2$. For k, $m \geq 1$ set $J_{km} = \{(i,0) : i < k\} \cup \{(i,k) : i < m\}$, so that $J_{11} = \{(0,0),(0,1)\}$, and choose a family $\langle(k_n,m_n)\rangle_{n\geq 3}$ running over $(\mathbb{N}\setminus\{0\})^2$ such that $q[J_{k_n,m_n}] \subseteq n$ for every $n \geq 3$. (The pairs (k_n,m_n) need not all be distinct, so this is easy to achieve.)

Now, for $\upsilon \in \mathbb{N}^n$, where $n \geq 3$, set $F_\upsilon = E_{\sigma\tau}$ where

$$\sigma \in \mathbb{N}^{k_n},\ \sigma(i) = \upsilon(q(i,0)) \text{ for } i < k_n,$$

$$\tau \in \mathbb{N}^{m_n},\ \tau(i) = \upsilon(q(i,k_n)) \text{ for } i < m_n;$$

these are well-defined because $q[J_{k_n,m_n}] \subseteq n$. For $\upsilon \in \mathbb{N}^1 \cup \mathbb{N}^2$, set $F_\upsilon = H_{\upsilon(0)}$.

(c) This defines a Souslin scheme $\langle F_\upsilon\rangle_{\upsilon\in S}$ in $\mathcal{E}$. Let A' be its kernel, so that $A' \in \mathcal{S}(\mathcal{E})$. The point is that $A' = A$.

P (i) If $x \in A$, there must be $\phi \in \mathbb{N}^\mathbb{N}$, $\Psi \in (\mathbb{N}^\mathbb{N})^{\mathbb{N}\setminus\{0\}}$ such that $x \in \bigcap_{k,m\geq 1} E_{\phi\restriction k,\psi_k\restriction m}$. Choose $\theta \in \mathbb{N}^\mathbb{N}$ such that

$$H_{\theta(0)} = E_{\phi\restriction 1,\psi_1\restriction 1},$$
$$\theta(q(i,0)) = \phi(i) \text{ for every } i \in \mathbb{N},$$
$$\theta(q(i,k)) = \psi_k(i) \text{ for every } k \geq 1,\ i \in \mathbb{N}.$$

(This is possible because $q : \mathbb{N}^2 \to \mathbb{N}\setminus\{0\}$ is injective.) Now

$$F_{\theta\restriction 1} = F_{\theta\restriction 2} = H_{\theta(0)} = E_{\phi\restriction 1,\psi_1\restriction 1}$$

certainly contains x. And for $n \geq 3$, $F_{\theta\restriction n} = E_{\sigma\tau}$ where $\sigma(i) = \theta(q(i,0))$ for $i < k_n$, $\tau(i) = \theta(q(i,k_n))$ for $i < m_n$, that is, $\sigma = \phi\restriction k_n$ and $\tau = \psi_{k_n}\restriction m_n$, so again $x \in F_{\theta\restriction n}$. Thus

$$x \in \textstyle\bigcap_{n\geq 1} F_{\theta\restriction n} \subseteq A'.$$

As x is arbitrary, $A \subseteq A'$.

(ii) Now take any $x \in A'$. Let $\theta \in \mathbb{N}^\mathbb{N}$ be such that $x \in \bigcap_{n\geq 1} F_{\theta\restriction n}$. Define $\phi \in \mathbb{N}^\mathbb{N}$, $\Psi \in (\mathbb{N}^\mathbb{N})^{\mathbb{N}\setminus\{0\}}$ by setting

$$\phi(i) = \theta(q(i,0)) \text{ for } i \in \mathbb{N},$$
$$\psi_k(i) = \theta(q(i,k)) \text{ for } k \geq 1,\ i \in \mathbb{N}.$$

If k, $m \geq 1$, let $n \geq 3$ be such that $k = k_n$, $m = m_n$. Then $x \in F_{\theta\restriction n} = E_{\sigma\tau}$, where

$$\sigma(i) = \theta(q(i,0)) \text{ for } i < k_n, \quad \tau(i) = \theta(q(i,k_n)) \text{ for } i < m_n,$$

that is, $\sigma = \phi\restriction k_n = \phi\restriction k$ and $\tau = \psi_{k_n}\restriction m_n = \psi_k\restriction m$. As m and n are arbitrary,

$$x \in \textstyle\bigcap_{m,n\geq 1} E_{\phi\restriction k,\psi_k\restriction m} \subseteq A.$$

As x is arbitrary, $A' \subseteq A$. **Q**

Accordingly we must have $A \in \mathcal{S}(\mathcal{E})$, and the proof is complete.

421E Corollary For any family $\mathcal{E}$ of sets, $\mathcal{S}(\mathcal{E})$ is closed under countable unions and intersections.

proof For 421Ca tells us that the union and intersection of any sequence in $\mathcal{S}(\mathcal{E})$ will belong to $\mathcal{S}\mathcal{S}(\mathcal{E}) = \mathcal{S}(\mathcal{E})$.

421F Corollary Let X be a set and $\mathcal{E}$ a family of subsets of X. Suppose that X and $\emptyset$ belong to $\mathcal{S}(\mathcal{E})$ and that $X \setminus E \in \mathcal{S}(\mathcal{E})$ for every $E \in \mathcal{E}$. Then $\mathcal{S}(\mathcal{E})$ includes the σ-algebra of subsets of X generated by $\mathcal{E}$.

proof The set

$$\Sigma = \{F : F \in \mathcal{S}(\mathcal{E}),\ X \setminus F \in \mathcal{S}(\mathcal{E})\}$$

is closed under complements (necessarily), contains $\emptyset$ (because $\emptyset$ and X belong to $\mathcal{S}(\mathcal{E})$), and is also closed under countable unions, by 421E. So it is a σ-algebra; but the hypotheses also ensure that $\mathcal{E} \subseteq \Sigma$, so that the σ-algebra generated by $\mathcal{E}$ is included in Σ and in $\mathcal{S}(\mathcal{E})$.

421G Proposition Let $\mathcal{E}$ be a family of sets such that $\emptyset \in \mathcal{E}$. For $\sigma \in S = \bigcup_{k\geq 1} \mathbb{N}^k$ set $I_\sigma = \{\phi : \phi \in \mathbb{N}^{\mathbb{N}},\ \phi \supseteq \sigma\}$. Then

$$\begin{aligned}\mathcal{S}(\mathcal{E}) &= \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\{I_\sigma \times E : \sigma \in S,\ E \in \mathcal{E}\})\}\\ &= \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\{I_\sigma \times E : \sigma \in S,\ E \in \mathcal{E}\}),\ R^{-1}[\{x\}] \text{ is closed for every } x\}.\end{aligned}$$

proof Set $\mathcal{F} = \{I_\sigma \times E : \sigma \in S,\ E \in \mathcal{E}\}$.

(a) Suppose first that $A \in \mathcal{S}(\mathcal{E})$. Let $\langle E_\sigma\rangle_{\sigma\in S}$ be a Souslin scheme in $\mathcal{E}$ with kernel A. Set

$$R = \textstyle\bigcap_{k\geq 1} \bigcup_{\sigma\in\mathbb{N}^k} I_\sigma \times E_\sigma.$$

Then $R \in \mathcal{S}(\mathcal{F})$, by 421E, and $R[\mathbb{N}^{\mathbb{N}}] = A$, by 421Ce. Also

$$R^{-1}[\{x\}] = \textstyle\bigcap_{k\geq 1} \bigcup\{I_\sigma : \sigma \in \mathbb{N}^k,\ x \in E_\sigma\}$$

is closed, for every x.

(b) Now suppose that $A = R[\mathbb{N}^{\mathbb{N}}]$ for some $R \in \mathcal{S}(\mathcal{F})$. Let $\langle I_{\tau(\sigma)} \times E_\sigma\rangle_{\sigma\in S}$ be a Souslin scheme in $\mathcal{F}$ with kernel R. For $k \geq 1$, $\sigma \in \mathbb{N}^k$ set

$$\begin{aligned}F_\sigma = E_\sigma \text{ if } &\bigcap_{1\leq n\leq k} I_{\tau(\sigma\restriction n)} \neq \emptyset,\\ &= \emptyset \text{ otherwise.}\end{aligned}$$

Then $\langle F_\sigma\rangle_{\sigma\in S}$ is a Souslin scheme in $\mathcal{E}$, so its kernel A' belongs to $\mathcal{S}(\mathcal{E})$.

The point is that $A' = A$. **P** (i) If $x \in A$, there are a $\phi \in \mathbb{N}^{\mathbb{N}}$ such that $(\phi, x) \in R$ and a $\psi \in \mathbb{N}^{\mathbb{N}}$ such that $(\phi, x) \in \bigcap_{n\geq 1} I_{\tau(\psi\restriction n)} \times E_{\psi\restriction n}$. Now, for any $k \geq 1$, we have

$$\phi \in \textstyle\bigcap_{1\leq n\leq k} I_{\tau(\psi\restriction n)} = \bigcap_{1\leq n\leq k} I_{\tau((\psi\restriction k)\restriction n)},$$

so that $F_{\psi\restriction k} = E_{\psi\restriction k}$ contains x; thus $x \in \bigcap_{k\geq 1} F_{\psi\restriction k} \subseteq A'$. As x is arbitrary, $A \subseteq A'$. (ii) If $x \in A'$, take $\psi \in \mathbb{N}^{\mathbb{N}}$ such that $x \in \bigcap_{n\geq 1} F_{\psi\restriction n}$. In this case we must have $F_{\psi\restriction k} \neq \emptyset$, so $\bigcap_{1\leq n\leq k} I_{\tau(\psi\restriction n)} \neq \emptyset$, for every $k \geq 1$. But what this means is that, setting $\tau_n = \tau(\psi\restriction n)$ for each $n \geq 1$, $\tau_n(i) = \tau_m(i)$ whenever $i \in \mathbb{N}$ is such that both are defined. So $\{\tau_n : n \geq 1\}$ must have a common extension $\phi \in \mathbb{N}^{\mathbb{N}}$, and $\phi \in \bigcap_{n\geq 1} I_{\tau(\psi\restriction n)}$. Now

$$(\phi, x) \in \textstyle\bigcap_{n\geq 1} I_{\tau(\psi\restriction n)} \times E_{\psi\restriction n} \subseteq R,$$

so $x \in A$. Thus $A' \subseteq A$ and the two are equal. **Q**

This shows that

$$\{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\mathcal{F})\} \subseteq \mathcal{S}(\mathcal{E}),$$

and the proof is complete.

421H When the class $\mathcal{E}$ is a σ-algebra, the last proposition can be extended.

Proposition Let X be a set, and Σ a σ-algebra of subsets of X. Let $\mathcal{B}$ be the algebra of Borel subsets of $\mathbb{N}^{\mathbb{N}}$. For $\sigma \in S = \bigcup_{k\geq 1} \mathbb{N}^k$, set $I_\sigma = \{\phi : \phi \in \mathbb{N}^{\mathbb{N}}, \phi \supseteq \sigma\}$. Then

$$\begin{aligned}\mathcal{S}(\Sigma) &= \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{B}\widehat{\otimes}\Sigma\}\\ &= \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\{I_\sigma \times E : \sigma \in S, E \in \Sigma\})\}\\ &= \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\mathcal{B}\widehat{\otimes}\Sigma)\}.\end{aligned}$$

Notation Recall that $\mathcal{B}\widehat{\otimes}\Sigma$ is the σ-algebra of subsets of $\mathbb{N}^{\mathbb{N}} \times X$ generated by $\{H \times E : H \in \mathcal{B}, E \in \Sigma\}$.

proof (a) Suppose first that $A \in \mathcal{S}(\Sigma)$. As in 421G, let $\langle E_\sigma\rangle_{\sigma\in S}$ be a Souslin scheme in Σ with kernel A, and set

$$R = \bigcap_{k\geq 1} \bigcup_{\sigma\in\mathbb{N}^k} I_\sigma \times E_\sigma,$$

so that $A = R[\mathbb{N}^{\mathbb{N}}]$. Because every I_σ is an open-and-closed set in $\mathbb{N}^{\mathbb{N}}$, $R \in \mathcal{B}\widehat{\otimes}\Sigma$. Thus

$$\mathcal{S}(\Sigma) \subseteq \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{B}\widehat{\otimes}\Sigma\}.$$

(b) Set $\mathcal{F} = \{I_\sigma \times E : \sigma \in S, E \in \Sigma\}$. Then $\mathcal{S}(\mathcal{B}\widehat{\otimes}\Sigma) = \mathcal{S}(\mathcal{F})$. **P** If $E \in \Sigma$ and $\sigma \in \mathbb{N}^k$ then

$$(\mathbb{N}^{\mathbb{N}} \times X) \setminus (I_\sigma \times E) = (I_\sigma \times (X \setminus E)) \cup \bigcup_{\tau\in\mathbb{N}^k,\tau\neq\sigma} I_\tau \times X \in \mathcal{S}(\mathcal{F}).$$

Also

$$\mathbb{N}^{\mathbb{N}} \times X = \bigcup_{\sigma\in\mathbb{N}^1} I_\sigma \times X, \quad \emptyset = I_\tau \times \emptyset$$

(where τ is any member of S) belong to $\mathcal{S}(\mathcal{F})$. By 421F, $\mathcal{S}(\mathcal{F})$ includes the σ-algebra Λ of sets generated by $\mathcal{F}$. Now if $E \in \Sigma$ and $H \subseteq \mathbb{N}^{\mathbb{N}}$ is open, $H = \bigcup_{\sigma\in T} I_\sigma$ for some $T \subseteq S$; as T is necessarily countable,

$$H \times E = \bigcup_{\sigma\in T} I_\sigma \times E \in \Lambda.$$

Since $\{F : F \subseteq \mathbb{N}^{\mathbb{N}}, F \times E \in \Lambda\}$ is a σ-algebra of subsets of $\mathbb{N}^{\mathbb{N}}$, and we have just seen that it contains all the open sets, it must include $\mathcal{B}$; thus $F \times E \in \Lambda$ for every $F \in \mathcal{B}$, $E \in \Sigma$. So $\mathcal{B}\widehat{\otimes}\Sigma \subseteq \Lambda \subseteq \mathcal{S}(\mathcal{F})$, and

$$\mathcal{S}(\mathcal{F}) \subseteq \mathcal{S}(\mathcal{B}\widehat{\otimes}\Sigma) \subseteq \mathcal{S}\mathcal{S}(\mathcal{F}) = \mathcal{S}(\mathcal{F})$$

(421D). **Q**

(c) Now we have

$$\begin{aligned}\mathcal{S}(\Sigma) &\subseteq \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{B}\widehat{\otimes}\Sigma\}\\ &\text{(by (a))}\\ &\subseteq \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\mathcal{B}\widehat{\otimes}\Sigma)\} = \{R[\mathbb{N}^{\mathbb{N}}] : R \in \mathcal{S}(\mathcal{F})\}\\ &\text{(by (b))}\\ &= \mathcal{S}(\Sigma)\end{aligned}$$

by 421G.

421I There is a particularly simple description of sets obtainable by Souslin's operation from closed sets in a topological space.

Lemma Let X be a topological space and $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ a closed set. Then

$$R[A] = \bigcup_{\phi\in A} \bigcap_{n\geq 1} \overline{R[I_{\phi\restriction n}]}.$$

for any $A \subseteq \mathbb{N}^{\mathbb{N}}$, writing $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$ as usual. In particular, $R[\mathbb{N}^{\mathbb{N}}]$ is the kernel of the Souslin scheme $\langle\overline{R[I_\sigma]}\rangle_{\sigma\in S}$.

proof Set

$$B = \bigcup_{\phi\in A}\bigcap_{n\geq 1}\overline{R[I_{\phi\restriction n}]}.$$

(i) If $x \in R[A]$, there is a $\phi \in A$ such that $(\phi, x) \in R$. In this case, $\phi \in I_{\phi\restriction n}$ so

$$x \in R[I_{\phi\restriction n}] \subseteq \overline{R[I_{\phi\restriction n}]}$$

for every n, and $x \in B$. Thus $R[A] \subseteq B$. (ii) If $x \in B$, let $\phi \in A$ be such that $x \in \overline{R[I_{\phi\restriction n}]}$ for every $n \in \mathbb{N}$. **?** If $(\phi, x) \notin R$, then (because R is closed) there are a $\sigma \in S$ and an open $G \subseteq X$ such that $\phi \in I_\sigma$, $x \in G$ and $(I_\sigma \times G) \cap R = \emptyset$. But this means that $G \cap R[I_\sigma] = \emptyset$ so $G \cap \overline{R[I_\sigma]} = \emptyset$ and $x \notin \overline{R[I_\sigma]}$; which is absurd, because $\sigma = \phi\restriction n$ for some $n \geq 1$. **X** Thus $(\phi, x) \in R$ and $x \in R[A]$. As x is arbitrary, $B \subseteq R[A]$ and $B = R[A]$, as required.

421J Proposition Let X be a topological space, and $\mathcal{F}$ the family of closed subsets of X. Then a set $A \subseteq X$ belongs to $\mathcal{S}(\mathcal{F})$ iff there is a closed set $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that A is the projection of R on X.

proof (a) Suppose that $A \in \mathcal{S}(\mathcal{F})$. Let $\langle F_\sigma\rangle_{\sigma\in S}$ be a Souslin scheme in $\mathcal{F}$ with kernel A. Set

$$R = \bigcap_{n\geq 1}\bigcup_{\sigma\in\mathbb{N}^n} I_\sigma \times F_\sigma.$$

For each $n \geq 1$,

$$\bigcup_{\sigma\in\mathbb{N}^n} I_\sigma \times F_\sigma = (\mathbb{N}^{\mathbb{N}} \times X) \setminus \bigcup_{\sigma\in\mathbb{N}^n} I_\sigma \times (X \setminus F_\sigma)$$

is closed in $\mathbb{N}^{\mathbb{N}} \times X$, so R is closed; and the projection $R[\mathbb{N}^{\mathbb{N}}]$ is A, by 421Ce.

(b) Suppose that $R \subseteq \mathbb{N}^{\mathbb{N}}$ is a closed set with projection A. Then A is the kernel of the Souslin scheme $\langle\overline{R[I_\sigma]}\rangle_{\sigma\in S}$, by 421I, so belongs to $\mathcal{S}(\mathcal{F})$.

421K Definition Let X be a topological space. A subset of X is a **Souslin-F** set in X if it is obtainable from closed subsets of X by Souslin's operation; that is, is the projection of a closed subset of $\mathbb{N}^{\mathbb{N}} \times X$.

For a subset of $\mathbb{R}^r$, or, more generally, of any Polish space, it is common to say 'Souslin set' for 'Souslin-F set'; see 421Xm.

421L Proposition Let X be any topological space. Then every Baire subset of X is Souslin-F.

proof Let $\mathcal{Z}$ be the family of zero sets in X. If $F \in \mathcal{Z}$ then $X \setminus F$ is a countable union of zero sets (4A2C(b-vi)), so belongs to $\mathcal{S}(\mathcal{Z})$. By 421F, the σ-algebra generated by $\mathcal{Z}$ is included in $\mathcal{S}(\mathcal{Z}) \subseteq \mathcal{S}(\mathcal{F})$, where $\mathcal{F}$ is the family of closed subsets of X; that is, every Baire set is Souslin-F.

421M Proposition Let $\mathcal{E}$ be any family of sets such that $\emptyset \in \mathcal{E}$ and $E \cup E'$, $\bigcap_{n\in\mathbb{N}} E_n$ belong to $\mathcal{E}$ for every E, $E' \in \mathcal{E}$ and all sequences $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$. (For instance, $\mathcal{E}$ could be the family of closed subsets of a topological space, or a σ-algebra of sets.) Let $\langle E_\sigma\rangle_{\sigma\in S}$ be a Souslin scheme in $\mathcal{E}$, and $K \subseteq \mathbb{N}^{\mathbb{N}}$ a set which is compact for the usual topology on $\mathbb{N}^{\mathbb{N}}$. Then $\bigcup_{\phi\in K}\bigcap_{n\geq 1} E_{\phi\restriction n} \in \mathcal{E}$.

proof Set $A = \bigcup_{\phi\in K}\bigcap_{n\geq 1} E_{\phi\restriction n}$. For $k \in \mathbb{N}$, set $K_k = \{\phi\restriction k : \phi \in K\}$; note that $K_k \subseteq \mathbb{N}^k$ is compact, because $\phi \mapsto \phi\restriction k$ is continuous, therefore finite, because the topology of $\mathbb{N}^k$ is discrete. Set

$$H = \bigcap_{k\geq 1}\bigcup_{\sigma\in K_k}\bigcap_{1\leq n\leq k} E_{\sigma\restriction n}.$$

Because $\mathcal{E}$ is closed under finite unions and countable intersections, $H \in \mathcal{E}$. Now $A = H$. **P** (i) If $x \in A$, take $\phi \in K$ such that $x \in E_{\phi\restriction n}$ for every $n \geq 1$; then $\phi\restriction k \in K_k$ and $x \in \bigcap_{1\leq n\leq k} E_{(\phi\restriction k)\restriction n}$ for every $k \geq 1$, so $x \in H$. Thus $A \subseteq H$. (ii) If $x \in H$, then for each $k \in \mathbb{N}$ we have a $\sigma_k \in K_k$ such that $x \in \bigcap_{1\leq n\leq k} E_{\sigma_k\restriction n}$. Choose $\phi_k \in K$ such that $\phi_k\restriction k = \sigma_k$ for each k. Now K is supposed to be compact, so the sequence $\langle\phi_k\rangle_{k\in\mathbb{N}}$ has a cluster point ϕ in K.

If $n \geq 1$, then $I_{\phi\restriction n}$ is a neighbourhood of ϕ in $\mathbb{N}^{\mathbb{N}}$, so must contain ϕ_k for infinitely many k; let $k \geq n$ be such that $\phi_k\restriction n = \phi\restriction n$. In this case

$$x \in E_{\sigma_k\restriction n} = E_{\phi_k\restriction n} = E_{\phi\restriction n}.$$

As n is arbitrary,

$$x \in \bigcap_{n\geq 1} E_{\phi\restriction n} \subseteq A.$$

As x is arbitrary, $H \subseteq A$ and $H = A$, as claimed. **Q**

So $A \in \mathcal{E}$.

***421N** I now embark on preparations for the theory of 'constituents' of analytic and coanalytic sets. It turns out that much of the work can be done in the abstract context of this section.

Trees and derived trees (a) Let $\mathcal{T}$ be the family of subsets T of $S = \bigcup_{n\geq 1} \mathbb{N}^n$ such that $\sigma\restriction k \in T$ whenever $\sigma \in T$ and $1 \leq k \leq \#(\sigma)$. Note that the intersection and union of any non-empty family of members of $\mathcal{T}$ again belong to $\mathcal{T}$. Members of $\mathcal{T}$ are often called **trees**.

(b) For $T \in \mathcal{T}$, set

$$\partial T = \{\sigma : \sigma \in S, \exists\, i \in \mathbb{N}, \sigma^\frown i \in T\},$$

so that $\partial T \in \mathcal{T}$ and $\partial T \subseteq T$. Of course $\partial T_0 \subseteq \partial T_1$ whenever $T_0, T_1 \in \mathcal{T}$ and $T_0 \subseteq T_1$.

(c) For $T \in \mathcal{T}$, define $\langle \partial^\xi T\rangle_{\xi<\omega_1}$ inductively by setting $\partial^0 T = T$ and, for $\xi > 0$, $\partial^\xi T = \partial(\bigcap_{\eta<\xi} \partial^\eta T)$. An easy induction shows that $\partial^\xi T \in \mathcal{T}$, $\partial^\xi T \subseteq \partial^\eta T$ and $\partial^{\xi+1} T = \partial(\partial^\xi T)$ whenever $\eta \leq \xi < \omega_1$.

(d) For any $T \in \mathcal{T}$, there is a $\xi < \omega_1$ such that $\partial^\xi T = \partial^\eta T$ whenever $\xi \leq \eta < \omega_1$. **P** Set $T_1 = \bigcap_{\xi<\omega_1} \partial^\xi T$. For each $\sigma \in S \setminus T_1$, there is a $\xi_\sigma < \omega_1$ such that $\sigma \notin \partial^{\xi_\sigma} T$. Set $\xi = \sup\{\xi_\sigma : \sigma \in S \setminus T_1\}$; because S is countable, $\xi < \omega_1$, and we must now have $\partial^\xi T = T_1$, so that $\partial^\xi T = \partial^\eta T$ whenever $\xi \leq \eta < \omega_1$. **Q**

(e) For $T \in \mathcal{T}$, its **rank** is the first ordinal $r(T) < \omega_1$ such that $\partial^{r(T)} T = \partial^{r(T)+1} T$; of course $\partial^{r(T)} T = \partial^\eta T$ whenever $r(T) \leq \eta < \omega_1$, and $\partial(\partial^{r(T)} T) = \partial^{r(T)} T$.

(f) For $T \in \mathcal{T}$, the following are equiveridical: (α) $\partial^{r(T)} T \neq \emptyset$; ($\beta$) there is a $\phi \in \mathbb{N}^\mathbb{N}$ such that $\phi\restriction n \in T$ for every $n \geq 1$. **P** (i) If $\sigma \in \partial^{r(T)} T$ then $\sigma \in \partial(\partial^{r(T)} T)$ so there is an $i \in \mathbb{N}$ such that $\sigma^\frown i \in \partial^{r(T)} T$. We can therefore choose $\langle \sigma_n\rangle_{n\in\mathbb{N}}$ inductively so that $\sigma_n \in \partial^{r(T)} T$ and σ_{n+1} properly extends σ_n for every n. At the end of the induction, $\phi = \bigcup_{n\in\mathbb{N}} \sigma_n$ belongs to $\mathbb{N}^\mathbb{N}$ and

$$\phi\restriction n = \sigma_n\restriction n \in \partial^{r(T)} T \subseteq T$$

for every $n \geq 1$. (ii) If $\phi \in \mathbb{N}^\mathbb{N}$ is such that $\phi\restriction n \in T$ for every $n \geq 1$, then an easy induction shows that $\phi\restriction n \in \partial^\xi T$ for every $\xi < \omega_1$ and every $n \geq 1$, so that $\partial^{r(T)} T$ is non-empty. **Q**

(g) Now suppose that $\langle A_\sigma\rangle_{\sigma\in S}$ is a Souslin scheme. For any x we have a tree $T_x \in \mathcal{T}$ defined by saying that

$$T_x = \{\sigma : \sigma \in S, , x \in \textstyle\bigcap_{1\leq i\leq \#(\sigma)} A_{\sigma\restriction i}\}.$$

Now the kernel of $\langle A_\sigma\rangle_{\sigma\in S}$ is just

$$\begin{aligned} A &= \{x : \exists\, \phi \in \mathbb{N}^\mathbb{N}, x \in \bigcap_{n\geq 1} A_{\phi\restriction n}\} \\ &= \{x : \exists\, \phi \in \mathbb{N}^\mathbb{N}, \phi\restriction n \in T_x \;\forall\, n \geq 1\} = \{x : \partial^{r(T)} T \neq \emptyset\} \end{aligned}$$

by (f).

The sets

$$\{x : x \in X \setminus A, r(T_x) = \xi\} = \{x : x \in X, r(T_x) = \xi, \partial^\xi T_x = \emptyset\},$$

for $\xi < \omega_1$, are called **constituents** of $X \setminus A$. (Of course they should properly be called 'the constituents of the Souslin scheme $\langle A_\sigma\rangle_{\sigma\in S}$'.)

***421O Theorem** Let X be a set and Σ a σ-algebra of subsets of X. Let $\langle A_\sigma\rangle_{\sigma\in S}$ be a Souslin scheme in Σ with kernel A, and for $x\in X$ set

$$T_x=\{\sigma:\sigma\in S,\,x\in\textstyle\bigcap_{1\le i\le\#(\sigma)}A_{\sigma\restriction i}\}\in\mathcal{T}$$

as in 421Ng.

(a) For every $\xi<\omega_1$ and $\sigma\in S$, $\{x:x\in X,\,\sigma\in\partial^\xi T_x\}\in\Sigma$.

(b) For every $\xi<\omega_1$, $\{x:x\in A,\,r(T_x)\le\xi\}$ and $\{x:x\in X\setminus A,\,r(T_x)\le\xi\}$ belong to Σ. In particular, all the constituents of $X\setminus A$ belong to Σ.

proof (a) Induce on ξ. For $\xi=0$, we have

$$\{x:x\in X,\,\sigma\in\partial^0T_x\}=\{x:x\in X,\,\sigma\in T_x\}=\textstyle\bigcap_{1\le i\le\#(\sigma)}A_{\sigma\restriction i}\in\Sigma.$$

For the inductive step to $\xi>0$, we have

$$\begin{aligned}\{x:\sigma\in\partial^\xi T_x\}&=\{x:\sigma\in\partial(\bigcap_{\eta<\xi}\partial^\eta T_x)\}=\bigcup_{i\in\mathbb{N}}\{x:\sigma^\frown i\in\bigcap_{\eta<\xi}\partial^\eta T_x\}\\&=\bigcup_{i\in\mathbb{N}}\bigcap_{\eta<\xi}\{x:\sigma^\frown i\in\partial^\eta T_x\}\in\Sigma\end{aligned}$$

because ξ is countable and all the sets $\{x:\sigma^\frown i\in\partial^\eta T_x\}$ belong to Σ by the inductive hypothesis.

(b) Now, given $\xi<\omega_1$, we see that $r(T_x)\le\xi$ iff $\partial^{\xi+1}T_x\subseteq\partial^\xi T_x$, so that if we set $E_\xi=\{x:x\in X,\,r(T_x)\le\xi\}$ then

$$E_\xi=\textstyle\bigcap_{\sigma\in S}\{x:x\in X,\,\sigma\in\partial^{\xi+1}T_x\text{ or }\sigma\notin\partial^\xi T_x\}$$

belongs to Σ. Now, given that $x\in E_\xi$, so that $\partial^{r(T_x)}T_x=\partial^\xi T_x$, 421Ng tells us that $x\in A$ iff $\emptyset\in\partial^\xi T_x$; so that $E_\xi\cap A$ and $E_\xi\setminus A$ both belong to Σ.

Now the constituents of $X\setminus A$ are the sets $(E_\xi\setminus A)\setminus\bigcup_{\eta<\xi}E_\eta$ for $\xi<\omega_1$, so all belong to Σ.

***421P Corollary** Let X be a set and Σ a σ-algebra of subsets of X. If $A\in\mathcal{S}(\Sigma)$ then both A and $X\setminus A$ can be expressed as the union of at most ω_1 members of Σ.

proof In the language of 421O, we have

$$A=\textstyle\bigcup_{\xi<\omega_1}E_\xi\cap A,\quad X\setminus A=\bigcup_{\xi<\omega_1}E_\xi\setminus A.$$

***421Q Lemma** Let X be a set and $\langle A_\sigma\rangle_{\sigma\in S}$ and $\langle B_\sigma\rangle_{\sigma\in S}$ two Souslin schemes of subsets of X. Suppose that whenever $\phi,\psi\in\mathbb{N}^\mathbb{N}$ there is an $n\ge1$ such that $\bigcap_{1\le i\le n}A_{\phi\restriction i}\cap B_{\psi\restriction i}=\emptyset$. For $x\in X$ set

$$T_x=\textstyle\bigcup_{n\ge1}\{\sigma:\sigma\in\mathbb{N}^n,\,x\in\bigcap_{1\le i\le n}A_{\sigma\restriction i}\}$$

as in 421Ng, and let B be the kernel of $\langle B_\sigma\rangle_{\sigma\in S}$. Then $\sup_{x\in B}r(T_x)<\omega_1$.

proof For $\sigma\in S$ set $A'_\sigma=\bigcap_{1\le i\le\#(\sigma)}A_{\sigma\restriction i}$, $B'_\sigma=\bigcap_{1\le i\le\#(\sigma)}B_{\sigma\restriction i}$. Then $T_x=\{\sigma:\sigma\in S,\,x\in A'_\sigma\}$ for each $x\in X$, B is the kernel of $\langle B'_\sigma\rangle_{\sigma\in S}$, and for every $\phi,\psi\in\mathbb{N}^\mathbb{N}$ there is an $n\in\mathbb{N}$ such that $A'_{\phi\restriction n}\cap B'_{\psi\restriction n}=\emptyset$.

Define $\langle Q_\xi\rangle_{\xi<\omega_1}$ inductively by setting

$$Q_0=\{(\sigma,\tau):\sigma,\,\tau\in S,\,A'_\sigma\cap B'_\tau\ne\emptyset\},$$

and, for $0<\xi<\omega_1$,

$$Q_\xi=\{(\sigma,\tau):\sigma,\,\tau\in S,\,\exists\,i,\,j\in\mathbb{N},\,(\sigma^\frown i,\tau^\frown j)\in\textstyle\bigcap_{\eta<\xi}Q_\eta\}.$$

Then the same arguments as in 421Na-421Nd show that there is a $\zeta<\omega_1$ such that $Q_{\zeta+1}=Q_\zeta$. **?** If $Q_\zeta\ne\emptyset$, then, just as in 421Nf, there must be $\phi,\psi\in\mathbb{N}^\mathbb{N}$ such that $(\phi\restriction m,\psi\restriction n)\in Q_\zeta\subseteq Q_0$ for every m, $n\ge1$; but this means that $A'_{\phi\restriction n}\cap B'_{\psi\restriction n}\ne\emptyset$ for every $n\ge1$, which is supposed to be impossible. **X**

Now suppose that $x\in B$. Then there is a $\psi\in\mathbb{N}^\mathbb{N}$ such that $x\in B'_{\psi\restriction n}$ for every $n\ge1$. But this means that $(\sigma,\psi\restriction n)\in Q_0$ for every $\sigma\in T_x$ and every $n\ge1$. An easy induction shows that $(\sigma,\psi\restriction n)\in Q_\xi$ whenever $\xi<\omega_1$, $\sigma\in\partial^\xi T_x$ and $n\ge1$. But as $Q_\zeta=\emptyset$ we must have $\partial^\zeta T_x=\emptyset$ and $r(T_x)\le\zeta$. Thus $\sup_{x\in B}r(T_x)\le\zeta<\omega_1$, and the proof is complete.

421X Basic exercises (a) Let X be a set and $\mathcal{E}$ a family of subsets of X. (i) Show that $\emptyset \in \mathcal{S}(\mathcal{E})$ iff there is a sequence in $\mathcal{E}$ with empty intersection. (ii) Show that $X \in \mathcal{S}(\mathcal{E})$ iff there is a sequence in $\mathcal{E}$ with union X.

(b) Let $\mathcal{E}$ be a family of sets and F any set. Show that

$$\mathcal{S}(\{E \cap F : E \in \mathcal{E}\}) = \{A \cap F : A \in \mathcal{S}(\mathcal{E})\},$$

$$\mathcal{S}(\{E \cup F : E \in \mathcal{E}\}) = \{A \cup F : A \in \mathcal{S}(\mathcal{E})\}.$$

(c) Let $\mathcal{E}$ be a family of sets and F any set. Show that

$$\mathcal{S}(\mathcal{E} \cup \{F\}) = \{F\} \cup \{A \cap F : A \in \mathcal{S}(\mathcal{E})\} \cup \{B \cup F : B \in \mathcal{S}(\mathcal{E})\} \cup \{(A \cap F) \cup B : A, B \in \mathcal{S}(\mathcal{E})\}.$$

(d) Suppose that $\mathcal{E}$ is a family of sets with $\#(\mathcal{E}) \leq \mathfrak{c}$. Show that $\#(\mathcal{S}(\mathcal{E})) \leq \mathfrak{c}$. (*Hint*: $\#(\mathcal{E}^S) \leq \#((\mathcal{P}\mathbb{N})^S) = \#(\mathcal{P}(\mathbb{N} \times S))$.)

(e) Let $\mathcal{E}$ be the family of half-open intervals $[2^{-n}k, 2^{-n}(k+1)[$, where $n \in \mathbb{N}$, $k \in \mathbb{Z}$; let $\mathcal{G}$ be the set of open subsets of $\mathbb{R}$; let $\mathcal{F}$ be the set of closed subsets of $\mathbb{R}$; let $\mathcal{K}$ be the set of compact subsets of $\mathbb{R}$; let $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{R}$. Show that $\mathcal{S}(\mathcal{E}) = \mathcal{S}(\mathcal{F}) = \mathcal{S}(\mathcal{G}) = \mathcal{S}(\mathcal{K}) = \mathcal{S}(\mathcal{B})$. (*Hint*: 421F.)

(f) Let $\mathcal{I}$ be the family $\{I_\sigma : \sigma \in \bigcup_{k \in \mathbb{N}} \mathbb{N}^k\}$ (421A); let $\mathcal{G}$ be the set of open subsets of $\mathbb{N}^{\mathbb{N}}$; let $\mathcal{F}$ be the set of closed subsets of $\mathbb{N}^{\mathbb{N}}$; let $\mathcal{K}$ be the set of compact subsets of $\mathbb{N}^{\mathbb{N}}$; let $\mathcal{B}$ be the Borel σ-algebra of $\mathbb{N}^{\mathbb{N}}$. Show that $\mathcal{S}(\mathcal{I}) = \mathcal{S}(\mathcal{F}) = \mathcal{S}(\mathcal{G}) = \mathcal{S}(\mathcal{B})$, but that $\mathcal{S}(\mathcal{K})$ is strictly smaller than these. (*Hint*: if $\langle K_n \rangle_{n \in \mathbb{N}}$ is any sequence in $\mathcal{K}$, set $\phi(i) = 1 + \sup_{\psi \in K_i} \psi(i)$ for each $i \in \mathbb{N}$, so that $\phi \notin \bigcup_{n \in \mathbb{N}} K_n$; hence show that $\mathbb{N}^{\mathbb{N}} \notin \mathcal{S}(\mathcal{K})$.)

(g) Let X be a separable metrizable space with at least two points; let $\mathcal{U}$ be any base for its topology, and $\mathcal{B}$ its Borel σ-algebra. Show that $\mathcal{S}(\mathcal{U}) = \mathcal{S}(\mathcal{B})$. What can happen if $\#(X) \leq 1$?

(h) Let X be a topological space; let $\mathcal{Z}$ be the set of zero sets in X, $\mathcal{G}$ the set of cozero sets, and $\mathcal{B}\mathfrak{a}$ the Baire σ-algebra. Show that $\mathcal{S}(\mathcal{Z}) = \mathcal{S}(\mathcal{G}) = \mathcal{S}(\mathcal{B}\mathfrak{a})$.

(i) Let X be a set, $\mathcal{E}$ a family of subsets of X, and Σ the σ-algebra of subsets of X generated by $\mathcal{E}$. Show that if $\#(\mathcal{E}) \leq \mathfrak{c}$ then $\#(\Sigma) \leq \mathfrak{c}$. (*Hint*: $\#(\mathfrak{c}^S) = \#(\mathcal{P}(\mathbb{N} \times \mathbb{N})) = \mathfrak{c}$ and $\Sigma \subseteq \mathcal{S}(\mathcal{E} \cup \{X \setminus E : E \in \mathcal{E}\})$.)

(j) Let X be a topological space such that every open set is Souslin-F. Show that every Borel set is Souslin-F.

(k) Let X be a topological space and $\mathcal{B}(X)$ its Borel σ-algebra. Show that $\mathcal{S}(\mathcal{B}(X))$ is just the set of projections on X of Borel subsets of $\mathbb{N}^{\mathbb{N}} \times X$. (*Hint*: 4A3G.)

(l) Let X and Y be topological spaces, $f : X \to Y$ a continuous function and $F \subseteq Y$ a Souslin-F set. Show that $f^{-1}[F]$ is a Souslin-F set in X.

(m) Let X be any metrizable space; let $\mathcal{G}$ be the set of open subsets of X, $\mathcal{F}$ the set of closed subsets, and $\mathcal{B}$ the Borel σ-algebra. Show that $\mathcal{S}(\mathcal{G}) = \mathcal{S}(\mathcal{F}) = \mathcal{S}(\mathcal{B})$.

(n) Let us say that a Souslin scheme $\langle E_\sigma \rangle_{\sigma \in S}$ is **regular** if $E_\sigma \subseteq E_\tau$ whenever $\sigma, \tau \in S$, $\#(\tau) \leq \#(\sigma)$ and $\sigma(i) \leq \tau(i)$ for every $i < \#(\sigma)$. Let $\mathcal{E}$ be a family of sets such that $E \cup F$, $E \cap F \in \mathcal{E}$ for all $E, F \in \mathcal{E}$. Show that every member of $\mathcal{S}(\mathcal{E})$ can be expressed as the kernel of a regular Souslin scheme in $\mathcal{E}$. (*Hint*: if $\langle E_\sigma \rangle_{\sigma \in S}$ is any Souslin scheme in $\mathcal{E}$ with kernel A, set $F_\sigma = \bigcap_{\tau \subseteq \sigma} E_\tau$, $G_\sigma = \bigcup_{\tau \leq \sigma} F_\tau$, where $\tau \leq \sigma$ if $\tau(i) \leq \sigma(i)$ for $i < \#(\tau) = \#(\sigma)$; show that A is the kernel of $\langle F_\sigma \rangle_{\sigma \in S}$ and of $\langle G_\sigma \rangle_{\sigma \in S}$, using an idea from 421M for the latter.)

(o) Let X be a Hausdorff topological space and $\langle K_\sigma\rangle_{\sigma\in S}$ a Souslin scheme in which every K_σ is a compact subset in X. Show that $\bigcup_{\phi\in K}\bigcap_{n\geq 1}K_{\phi\restriction n}$ is compact for any compact $K\subseteq\mathbb{N}^{\mathbb{N}}$.

421Y Further exercises (a) Let X be a topological space, Y a Hausdorff space and $f:X\to Y$ a continuous function. Let $\mathcal{K}$ be the family of closed countably compact subsets of X. Show that for any $\mathcal{E}\subseteq\mathcal{K}$,

$$\{f[A]:A\in\mathcal{S}(\mathcal{E})\}=\mathcal{S}(\{f[E]:E\in\mathcal{E}\}).$$

(b) Let X be a topological space, and $\mathcal{B}\mathfrak{a}$ its Baire σ-algebra. Show that $\mathcal{S}(\mathcal{B}\mathfrak{a})$ is just the family of sets expressible as $f^{-1}[B]$ where f is a continuous function from X to some metrizable space Y and $B\subseteq Y$ is Souslin-F.

(c) Let X be a set, $\mathcal{E}$ a family of subsets of X, and Σ the smallest σ-algebra of subsets of X including $\mathcal{E}$ and closed under Souslin's operation. Show that if $\#(\mathcal{E})\leq\mathfrak{c}$ then $\#(\Sigma)\leq\mathfrak{c}$. (*Hint*: define $\langle\mathcal{E}_\xi\rangle_{\xi<\omega_1}$ by setting $\mathcal{E}_\xi=\mathcal{S}(\{X\setminus E:E\in\mathcal{E}\cup\bigcup_{\eta<\xi}\mathcal{E}_\xi\})$ for each ξ. Show that $\#(\mathcal{E}_\xi)\leq\mathfrak{c}$ for every ξ and that $\Sigma=\bigcup_{\xi<\omega_1}\mathcal{E}_\xi$.)

(d) Let X be a compact space and A a Souslin-F set in X. Show that there is a family $\langle F_\xi\rangle_{\xi<\omega_1}$ of Borel sets such that $X\setminus A=\bigcup_{\xi<\omega}F_\xi$ and whenever $B\subseteq X\setminus A$ is a Souslin-F set there is a $\xi<\omega_1$ such that $B\subseteq F_\xi$. (*Hint*: take $F_\xi=\{x:r(T_x)\leq\xi\}\setminus A$ as in 421Ob, and apply 421Q.)

421 Notes and comments In 111G, I defined the Borel sets of $\mathbb{R}$ to be the members of the smallest σ-algebra containing every open set. In 114E, I defined a set to be Lebesgue measurable if it behaves in the right way with respect to Lebesgue outer measure. The latter formulation, at least, provides some sort of testing principle to determine whether a set is Lebesgue measurable. But the definition of 'Borel set' does not. The only tool so far available for proving that a set $E\subseteq\mathbb{R}$ is *not* Borel is to find a σ-algebra containing all open sets and not containing E; conversely, the only method we have for proving properties of Borel sets is to show that a property is possessed by every member of some σ-algebra containing every open set. The revolutionary insight of Souslin 17 was a construction which could build every Borel set from rational intervals. (See 421Xe.) For fundamental reasons, no construction of this kind can provide all Borel sets without also producing other sets, and to actually characterize the Borel σ-algebra a further idea is needed (423Fa); but the class of analytic sets, being those constructible by Souslin's operation from rational intervals (or open sets, or closed sets, or Borel sets – the operation is robust under such variations), turns out to have remarkable properties which make it as important in modern real analysis as the Borel algebra itself.

The guiding principle of 'descriptive set theory' is that the properties of a set may be analysed in the light of a construction for that set. Thus we can think of a closed set $F\subseteq\mathbb{R}$ as

$$\mathbb{R}\setminus\bigcup_{(q,q')\in I}]q,q'[$$

where $I\subseteq\mathbb{Q}\times\mathbb{Q}$. The principle can be effective because we often have such descriptions in terms of objects fundamentally simpler than the set being described. In the formula above, for instance, $\mathbb{Q}\times\mathbb{Q}$ is simpler than the set F, being a countable set with a straightforward description from $\mathbb{N}$. The set $\mathcal{P}(\mathbb{Q}\times\mathbb{Q})$ is relatively complex; but a single subset I of $\mathbb{Q}\times\mathbb{Q}$ can easily be coded as a single subset of $\mathbb{N}$ (taking some more or less natural enumeration of $\mathbb{Q}^2$ as a sequence $\langle(q_n,q_n')\rangle_{n\in\mathbb{N}}$, and matching I with $\{n:(q_n,q_n')\in I\}$). So, subject to an appropriate coding, we have a description of closed subsets of $\mathbb{R}$ in terms of subsets of $\mathbb{N}$. At the most elementary level, this shows that there are at most $\mathfrak{c}$ closed subsets of $\mathbb{R}$. But we can also set out to analyse such operations as intersection, union, closure in terms of these descriptions. The details are complex, and I shall go no farther along this path here; but investigations of this kind are at the heart of some of the most exciting developments of twentieth-century real analysis.

The particular descriptive method which concerns us in the present section is Souslin's operation. Starting from a relatively simple class $\mathcal{E}$, we proceed to the larger class $\mathcal{S}(\mathcal{E})$. The most fundamental property of $\mathcal{S}$ is 421D: $\mathcal{S}\mathcal{S}(\mathcal{E})=\mathcal{S}(\mathcal{E})$. This means, for instance, that if $\mathcal{E}\subseteq\mathcal{S}(\mathcal{F})$ and $\mathcal{F}\subseteq\mathcal{S}(\mathcal{E})$, then $\mathcal{S}(\mathcal{E})$ will be equal to $\mathcal{S}(\mathcal{F})$; consequently, different classes of sets will often have the same Souslin closures, as in 421Xe-421Xh.

After a little practice you will find that it is often easy to see when two classes $\mathcal{E}$ and $\mathcal{F}$ are at the same level in this sense; but watch out for traps like the class of compact subsets of $\mathbb{N}^{\mathbb{N}}$ (421Xf) and odd technical questions (421Xg).

Souslin's operation, and variations on it, will be the basis of much of the next chapter; it has dramatic applications in general topology and functional analysis as well as in real analysis and measure theory. An important way of looking at the kernel of a Souslin scheme $\langle E_\sigma\rangle_{\sigma\in S}$ is to regard it as the projection on the second coordinate of the corresponding set $R = \bigcap_{k\geq 1}\bigcup_{\sigma\in\mathbb{N}^k} I_\sigma \times E_\sigma$ (421Ce). We find that many other sets $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ will also have projections in $\mathcal{S}(\mathcal{E})$ (421G, 421H). Let me remark that it is essential here that the first coordinate should be of the right type. In one sense, indeed, $\mathbb{N}^{\mathbb{N}}$ is the only thing that will do; but its virtue transfers to analytic spaces, as we shall see in 423M-423O below. We shall often want to deal with members of $\mathcal{S}(\mathcal{E})$ which are most naturally defined in terms of some such auxiliary space.

I have moved into slightly higher gear for 421N-421Q because these are not essential for most of the work of the next chapter. From the point of view of this section 421P is very striking but the significance of 421Q is unlikely to be apparent. It becomes important in contexts in which the condition

$$\forall\, \phi, \psi \in \mathbb{N}^{\mathbb{N}}\ \exists\, n \geq 1,\ \textstyle\bigcap_{1\leq i\leq n} A_{\phi\restriction i} \cap B_{\psi\restriction i} = \emptyset$$

is satisfied for natural reasons. I will expand on these in the next two sections. In the meantime, I offer 421Yd as an example of what 421O and 421Q together can tell us.

422 K-analytic spaces

I introduce K-analytic spaces, defined in terms of usco-compact relations. The first step is to define the latter (422A) and give their fundamental properties (422B-422E). I reach K-analytic spaces themselves in 422F, with an outline of the most important facts about them in 422G-422K.

422A Definition Let X and Y be Hausdorff spaces. A relation $R \subseteq X \times Y$ is **usco-compact** if

(α) $R[\{x\}]$ is a compact subset of Y for every $x \in X$,

(β) $R^{-1}[F]$ is a closed subset of X for every closed set $F \subseteq Y$.

(Relations satisfying condition (β) are sometimes called 'upper semi-continuous'.)

422B The following elementary remark will be useful.

Lemma Let X and Y be Hausdorff spaces and $R \subseteq X \times Y$ an usco-compact relation. If H is an open subset of Y including $R[\{x\}]$ where $x \in X$, there is an open set $G \subseteq X$, containing x, such that $R[G] \subseteq H$.

proof Set $G = X \setminus R^{-1}[Y \setminus H]$. Because $Y \setminus H$ is closed, so is $R^{-1}[Y \setminus H]$, and G is open. Of course $R[G] \subseteq H$, and $x \in G$ because $R[\{x\}] \subseteq H$.

422C Proposition Let X and Y be Hausdorff spaces. Then a subset R of $X \times Y$ is an usco-compact relation iff whenever $\mathcal{F}$ is an ultrafilter on $X \times Y$, containing R, such that the first-coordinate image $\pi_1[[\mathcal{F}]]$ of $\mathcal{F}$ has a limit in X, then $\mathcal{F}$ has a limit in R.

proof Recall that, writing $\pi_1(x,y) = x$ and $\pi_2(x,y) = y$ for $(x,y) \in X \times Y$,

$$\pi_1[[\mathcal{F}]] = \{A : A \subseteq X,\ \pi_1^{-1}[A] \in \mathcal{F}\} = \{A : A \subseteq X,\ A \times Y \in \mathcal{F}\}$$

(2A1Ib), and that $\mathcal{F} \to (x,y)$ iff $\pi_1[[\mathcal{F}]] \to x$ and $\pi_2[[\mathcal{F}]] \to y$ (3A3Ic).

(a) Suppose that R is usco-compact and that $\mathcal{F}$ is an ultrafilter on $X \times Y$, containing R, such that $\pi_1[[\mathcal{F}]]$ has a limit $x \in X$. **?** If $\mathcal{F}$ has no limit in R, then, in particular, it does not converge to (x,y) for any $y \in R[\{x\}]$; that is, $\pi_2[[\mathcal{F}]]$ does not converge to any point of $R[\{x\}]$, that is, every point of $R[\{x\}]$ belongs to an open set not belonging to $\pi_2[[\mathcal{F}]]$. Because $R[\{x\}]$ is compact, it is covered by a finite union of open sets not belonging to $\pi_2[[\mathcal{F}]]$; but as $\pi_2[[\mathcal{F}]]$ is an ultrafilter (2A1N), there is an open set $H \supseteq R[\{x\}]$ such that $Y \setminus H \in \pi_2[[\mathcal{F}]]$.

Now 422B tells us that there is an open set G containing x such that $R[G] \subseteq H$. In this case, $G \in \pi_1[[\mathcal{F}]]$ so $G \times Y \in \mathcal{F}$; at the same time, $X \times (Y \setminus H) \in \mathcal{F}$. So

$$R \cap (G \times Y) \cap (X \times (Y \setminus H)) \in \mathcal{F}.$$

But this is empty, by the choice of G; which is intolerable. **X**

Thus $\mathcal{F}$ has a limit in R, as required.

(b) Now suppose that R has the property described.

(i) Let $x \in X$, and suppose that $\mathcal{G}$ is an ultrafilter on Y containing $R[\{x\}]$. Set $h(y) = (x, y)$ for $y \in Y$; then $\mathcal{F} = h[[\mathcal{G}]]$ is an ultrafilter on $X \times Y$ containing R. The image $\pi_1[[\mathcal{F}]]$ is just the principal filter generated by $\{x\}$, so certainly converges to x; accordingly $\mathcal{F}$ must converge to some point $(x, y) \in R$, and $\mathcal{G} = \pi_2[[\mathcal{F}]]$ converges to $y \in R[\{x\}]$. As $\mathcal{G}$ is arbitrary, $R[\{x\}]$ is compact (2A3R).

(ii) Let $F \subseteq Y$ be closed, and $x \in \overline{R^{-1}[F]} \subseteq X$. Consider

$$\mathcal{E} = \{R, X \times F\} \cup \{G \times Y : G \subseteq X \text{ is open}, x \in G\}.$$

Then $\mathcal{E}$ has the finite intersection property. **P** If $G_0, \ldots, G_n$ are open sets containing x, then $R^{-1}[F]$ meets $G_0 \cap \ldots \cap G_n$ in z say, and now $(z, y) \in R \cap (X \times F) \cap \bigcap_{i \le n}(G_i \times Y)$ for some $y \in F$. **Q** Let $\mathcal{F}$ be an ultrafilter on $X \times Y$ including $\mathcal{E}$ (4A1Ia). Because $G \times Y \in \mathcal{E} \subseteq \mathcal{F}$ for every open set G containing x, $\pi_1[[\mathcal{F}]] \to x$, so $\mathcal{F}$ converges to some point (x, y) of R. Because $X \times F$ is a closed set belonging to $\mathcal{E} \subseteq \mathcal{F}$, $y \in F$ and $x \in R^{-1}[F]$. As x is arbitrary, $R^{-1}[F]$ is closed; as F is arbitrary, R satisfies condition (β) of 422A, and is usco-compact.

422D Lemma (a) Let X and Y be Hausdorff spaces. If $R \subseteq X \times Y$ is an usco-compact relation, then R is closed in $X \times Y$.

(b) Let X and Y be Hausdorff spaces. If $R \subseteq X \times Y$ is an usco-compact relation and $R' \subseteq R$ is a closed set, then R' is usco-compact.

(c) Let X and Y be Hausdorff spaces. If $f : X \to Y$ is a continuous function, then its graph is an usco-compact relation.

(d) Let $\langle X_i \rangle_{i \in I}$ and $\langle Y_i \rangle_{i \in I}$ be families of Hausdorff spaces, and $R_i \subseteq X_i \times Y_i$ an usco-compact relation for each i. Set $X = \prod_{i \in I} X_i$, $Y = \prod_{i \in I} Y_i$ and

$$R = \{(x, y) : x \in X, y \in Y, (x(i), y(i)) \in R_i \text{ for every } i \in I\}.$$

Then R is usco-compact in $X \times Y$.

(e) Let X and Y be Hausdorff spaces, and $R \subseteq X \times Y$ an usco-compact relation. Then (i) $R[K]$ is a compact subset of Y for any compact subset K of X (ii) $R[L]$ is a Lindelöf subset of Y for any Lindelöf subset L of X.

(f) Let X, Y and Z be Hausdorff spaces, and $R \subseteq X \times Y$, $S \subseteq Y \times Z$ usco-compact relations. Then the composition

$$S \circ R = \{(x, z) : \text{there is some } y \in Y \text{ such that } (x, y) \in R \text{ and } (y, z) \in S\}$$

is usco-compact in $X \times Z$.

(g) Let X and Y be Hausdorff spaces and Y_0 any subset of Y. Then a relation $R \subseteq X \times Y_0$ is usco-compact when regarded as a relation between X and Y_0 iff it is usco-compact when regarded as a relation between X and Y.

proof (a) If $(x, y) \in \overline{R}$, there is an ultrafilter $\mathcal{F}$ containing R and converging to (x, y) (4A2Bc). By 422C, $\mathcal{F}$ must have a limit in R; but as $X \times Y$ is Hausdorff, this limit must be (x, y), and $(x, y) \in R$. As (x, y) is arbitrary, R is closed.

(b) It is obvious that R' will satisfy the condition of 422C if R does.

(c) $f[\{x\}] = \{f(x)\}$ is surely compact for every $x \in X$, and $f^{-1}[F]$ is closed for every closed set $F \subseteq Y$ because f is continuous.

(d) For $i \in I$, $x \in X$, $y \in Y$ set $\phi_i(x, y) = (x(i), y(i))$. If $\mathcal{F}$ is an ultrafilter on $X \times Y$ containing R such that $\pi_1[[\mathcal{F}]]$ has a limit in X, then

$$\pi_1 \phi_i[[\mathcal{F}]] = \psi_i \pi_1[[\mathcal{F}]]$$

has a limit in X_i for every $i \in I$, writing $\psi_i(x) = x(i)$ for $i \in I$, $x \in X$. But $\phi_i[[\mathcal{F}]]$ contains R_i, so has a limit $(x_0(i), y_0(i))$ in $X_i \times Y_i$, for each i. Accordingly (x_0, y_0) is a limit of $\mathcal{F}$ in $X \times Y$ (3A3Ic). As $\mathcal{F}$ is arbitrary, R is usco-compact.

(e) For the moment, let L be any subset of X. Let $\mathcal{H}$ be a family of open sets in Y covering $R[L]$. Let $\mathcal{G}$ be the family of those open sets $G \subseteq X$ such that $R[G]$ can be covered by finitely many members of $\mathcal{H}$. Then $\mathcal{G}$ covers L. **P** If $x \in L$, then $R[\{x\}]$ is a compact subset of $R[L] \subseteq \bigcup \mathcal{H}$, so there is a finite set $\mathcal{H}' \subseteq \mathcal{H}$ covering $R[\{x\}]$. Now there is an open set G containing x such that $R[G] \subseteq \bigcup \mathcal{H}'$, by 422B. **Q**

(i) If L is compact, then there must be a finite subfamily $\mathcal{G}'$ of $\mathcal{G}$ covering L; now $R[L] \subseteq R[\bigcup \mathcal{G}']$ is covered by finitely many members of $\mathcal{H}$. As $\mathcal{H}$ is arbitrary, $R[L]$ is compact.

(ii) If L is Lindelöf, then there must be a countable subfamily $\mathcal{G}'$ of $\mathcal{G}$ covering L; now $R[L] \subseteq R[\bigcup \mathcal{G}']$ is covered by countably many members of $\mathcal{H}$. As $\mathcal{H}$ is arbitrary, $R[L]$ is Lindelöf.

(f) If $x \in X$ then $R[\{x\}] \subseteq Y$ is compact, so $(SR)[\{x\}] = S[R[\{x\}]$ is compact, by (e-i). If $F \subseteq Z$ is closed then $S^{-1}[F] \subseteq Y$ is closed so $(SR)^{-1}[F] = R^{-1}[S^{-1}[F]]$ is closed.

(g)(i) Suppose that R is usco-compact when regarded as a subset of $X \times Y_0$. Set $S = \{(y, y) : y \in Y_0\}$; by (c), S is usco-compact when regarded as a subset of $Y_0 \times Y$, so by (f) $R = SR$ is usco-compact when regarded as a subset of $X \times Y$.

(ii) If R is usco-compact when regarded as a subset of $X \times Y$, and $x \in X$, then $R[\{x\}]$ is a subset of Y_0 which is compact for the topology of Y, therefore for the subspace topology of Y_0. If $F \subseteq Y_0$ is closed for the subspace topology, it is of the form $F' \cap Y_0$ for some closed $F' \subseteq Y$, so $R^{-1}[F] = R^{-1}[F']$ is closed in X. As x and F are arbitrary, R is usco-compact in $X \times Y_0$.

422E The following lemma is actually very important in the structure theory of K-analytic spaces (see 422Yb). It will be useful to us in 423C below.

Lemma Let X and Y be Hausdorff spaces, and $R \subseteq X \times Y$ an usco-compact relation. If X is regular, so is R (in its subspace topology).

proof ? Suppose, if possible, otherwise; that there are a closed set $F \subseteq R$ and an $(x, y) \in R \setminus F$ which cannot be separated from F by open sets (in R). If G, H are open sets containing x, y respectively, then $R \cap (G \times H)$, $R \setminus (\overline{G} \times \overline{H})$ are disjoint relatively open sets in R, so the latter cannot include F; that is, $F \cap (\overline{G} \times \overline{H}) \neq \emptyset$ whenever G, H are open and $x \in G$, $y \in H$. Accordingly there is an ultrafilter $\mathcal{F}$ on $X \times Y$ such that $F \cap (\overline{G} \times \overline{H}) \in \mathcal{F}$ whenever $G \subseteq X$, $H \subseteq Y$ are open sets containing x, y respectively. In this case $R \in \mathcal{F}$, and $\overline{G} \in \pi_1[[\mathcal{F}]]$ for every open set G containing x. Because the topology of X is regular, every open set containing x includes $\overline{G}$ for some smaller open set G containing x, and belongs to $\pi_1[[\mathcal{F}]]$; thus $\pi_1[[\mathcal{F}]] \to x$ in X. Because R is usco-compact, $\mathcal{F}$ has a limit in R, which must be of the form (x, y'). Because $F \in \mathcal{F}$ is closed (in R), $(x, y') \in F$. But also $y' \in \overline{H}$ for every open set H containing y, since $X \times \overline{H}$ is a closed set belonging to $\mathcal{F}$; because the topology of Y is Hausdorff, y' must be equal to y, and $(x, y) \in F$, which is absurd. **X**

422F Definition (Frolík 61) Let X be a Hausdorff space. Then X is **K-analytic** if there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = X$.

If X is a Hausdorff space, we call a subset of X **K-analytic** if it is a K-analytic space in its subspace topology.

422G Theorem (a) Let X be a Hausdorff space. Then a subset A of X is K-analytic iff there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = A$.

(b) $\mathbb{N}^{\mathbb{N}}$ is K-analytic.

(c) Compact Hausdorff spaces are K-analytic.

(d) If X and Y are Hausdorff spaces and $R \subseteq X \times Y$ is an usco-compact relation, then $R[A]$ is K-analytic whenever $A \subseteq X$ is K-analytic. In particular, a Hausdorff continuous image of a K-analytic Hausdorff space is K-analytic.

(e) A product of countably many K-analytic Hausdorff spaces is K-analytic.

(f) A closed subset of a K-analytic Hausdorff space is K-analytic.

(g) A K-analytic Hausdorff space is Lindelöf, so a regular K-analytic Hausdorff space is completely regular.

proof (a) A is K-analytic iff there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times A$ with projection A. But a subset of $\mathbb{N}^{\mathbb{N}} \times X$ with projection A is usco-compact in $\mathbb{N}^{\mathbb{N}} \times A$ iff it is usco-compact in $\mathbb{N}^{\mathbb{N}} \times X$, by 422Dg.

(b) The identity function from $\mathbb{N}^{\mathbb{N}}$ to itself is an usco-compact relation, by 422Dc.

(c) If X is compact, then $R = \mathbb{N}^{\mathbb{N}} \times X$ is an usco-compact relation (because $R[\{\phi\}] = X$ is compact for every $\phi \in \mathbb{N}^{\mathbb{N}}$, while $R^{-1}[F]$ is either $\mathbb{N}^{\mathbb{N}}$ or $\emptyset$ for every closed $F \subseteq X$), so $X = R[\mathbb{N}^{\mathbb{N}}]$ is K-analytic.

(d) By (a), there is an usco-compact relation $S \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $S[\mathbb{N}^{\mathbb{N}}] = A$. Now $R \circ S \subseteq \mathbb{N}^{\mathbb{N}} \times Y$ is usco-compact, by 422Df, and $RS[\mathbb{N}^{\mathbb{N}}] = R[A]$.

In particular, if X itself is K-analytic and $f : X \to Y$ is a continuous surjection, f is an usco-compact relation (422Dc), so $Y = f[X]$ is K-analytic.

(e) Let $\langle X_i \rangle_{i \in I}$ be a countable family of K-analytic Hausdorff spaces with product X. If $I = \emptyset$ then $X = \{\emptyset\}$ is compact, therefore K-analytic. Otherwise, choose for each $i \in I$ an usco-compact relation $R_i \subseteq \mathbb{N}^{\mathbb{N}} \times X_i$ such that $R_i[\mathbb{N}^{\mathbb{N}}] = X_i$. Set

$$R = \{(\phi, x) : \phi \in (\mathbb{N}^{\mathbb{N}})^I,\ x \in X,\ (\phi(i), x(i)) \in R_i \text{ for every } i \in I\}.$$

By 422Dd, R is an usco-compact relation in $(\mathbb{N}^{\mathbb{N}})^I \times X$, and it is easy to see that $R[(\mathbb{N}^{\mathbb{N}})^I] = X$. But $(\mathbb{N}^{\mathbb{N}})^I \cong \mathbb{N}^{\mathbb{N} \times I}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$, because I is countable, so we can identify R with a relation in $\mathbb{N}^{\mathbb{N}} \times X$ which is still usco-compact, and X is K-analytic.

(f) Let X be a K-analytic Hausdorff space and F a closed subset. Let $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}] = X$. Set $R' = R \cap (\mathbb{N}^{\mathbb{N}} \times F)$. Then R' is a closed subset of R, so is usco-compact (422Da). By (a), $F = R'[\mathbb{N}^{\mathbb{N}}]$ is K-analytic.

(g) Let X be a K-analytic Hausdorff space. $\mathbb{N}^{\mathbb{N}}$ is Lindelöf (4A2Ub), and there is an usco-compact relation R such that $R[\mathbb{N}^{\mathbb{N}}] = X$, so that X is Lindelöf, by 422D(e-ii). 4A2H(b-i) now tells us that if X is regular it is completely regular.

422H Theorem (a) If X is a Hausdorff space, then any K-analytic subset of X is Souslin-F in X.

(b) If X is a K-analytic Hausdorff space, then a subset of X is K-analytic iff it is Souslin-F in X.

(c) For any Hausdorff space X, the family of K-analytic subsets of X is closed under Souslin's operation.

proof (a) If $A \subseteq X$ is K-analytic, there is an usco-compact relation $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = A$, by 422Ga. By 422Da, R is a closed set; so A is Souslin-F by 421J.

(b) Now suppose that X itself is K-analytic, and that $A \subseteq X$ is Souslin-F in X. Then there is a closed set $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = A$ (421J, in the other direction). $\mathbb{N}^{\mathbb{N}} \times X$ is K-analytic (422Gb, 422Ge), and R is closed, therefore itself K-analytic (422Gf); so its continuous image A is K-analytic, by 422Gd.

(c)(i) The first step is to show that the union of a sequence of K-analytic subsets of X is K-analytic. **P** Let $\langle A_n \rangle_{n \in \mathbb{N}}$ be a sequence of K-analytic sets, with union A. For each $n \in \mathbb{N}$, let $R_n \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be an usco-compact relation such that $R_n[\mathbb{N}^{\mathbb{N}}] = A_n$. In $(\mathbb{N} \times \mathbb{N}^{\mathbb{N}}) \times X$ let R be the set

$$\{((n, \phi), x) : n \in \mathbb{N},\ (\phi, x) \in R_n\}.$$

If $(n, \phi) \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$, then $R[\{(n, \phi)\}] = R_n[\{\phi\}]$ is compact; if $F \subseteq X$ is closed, then

$$R^{-1}[F] = \{(n, \phi) : n \in \mathbb{N},\ \phi \in R_n^{-1}[F]\}$$

is closed in $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$. So R is usco-compact, and of course

$$R[\mathbb{N} \times \mathbb{N}^{\mathbb{N}}] = \bigcup_{n \in \mathbb{N}} R_n[\mathbb{N}^{\mathbb{N}}] = A.$$

As $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ is K-analytic (in fact, homeomorphic to $\mathbb{N}^{\mathbb{N}}$), A is K-analytic. **Q**

(ii) Now suppose that $\langle A_\sigma \rangle_{\sigma \in S}$ is a Souslin scheme consisting of K-analytic sets with kernel A. Then $X' = \bigcup_{\sigma \in S} A_\sigma$ is K-analytic, by (i). By (a), every A_σ is Souslin-F when regarded as a subset of X'. But since the family of Souslin-F subsets of X' is closed under Souslin's operation, by 421D, A is also Souslin-F in X'. By (b) of this theorem, A is K-analytic. As $\langle A_\sigma \rangle_{\sigma \in S}$ is arbitrary, we have the result.

422I It seems that for the measure-theoretic results of §432, at least, the following result (the 'First Separation Theorem') is not essential. However I do not think it possible to get a firm grasp on K-analytic and analytic spaces without knowing some version of it, so I present it here. It is most often used through the forms in 422J and 422Xd below.

Lemma Let X be a Hausdorff space. Let $\mathcal{E}$ be a family of subsets of X such that (i) $\bigcup_{n\in\mathbb{N}} E_n, \bigcap_{n\in\mathbb{N}} E_n \in \mathcal{E}$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\in \mathcal{E}$ (ii) whenever x, y are distinct points of X, there are disjoint E, $F \in \mathcal{E}$ such that $x \in \operatorname{int} E$ and $y \in \operatorname{int} F$. Then whenever A, B are disjoint non-empty K-analytic subsets of X, there are disjoint E, $F \in \mathcal{E}$ such that $A \subseteq E$ and $B \subseteq F$.

proof (a) We need to know that if K, L are disjoint non-empty compact subsets of X, there are disjoint E, $F \in \mathcal{E}$ such that $K \subseteq \operatorname{int} E$ and $L \subseteq \operatorname{int} F$. **P** For any point $(x, y) \in K \times L$, we can find disjoint E_{xy}, $F_{xy} \in \mathcal{E}$ such that $x \in \operatorname{int} E$ and $y \in \operatorname{int} F$. Because L is compact and non-empty, there is for each $x \in K$ a non-empty finite set $I_x \subseteq L$ such that $L \subseteq \bigcup_{y\in I_x} \operatorname{int} F_{xy}$. Set $E_x = \bigcap_{y\in I_x} E_{xy}$, $F_x = \bigcup_{y\in I_x} F_{xy}$; then E_x, F_x are disjoint members of $\mathcal{E}$, $x \in \operatorname{int} E_x$ and $L \subseteq \operatorname{int} F_x$. Because K is compact and not empty, there is a non-empty finite set $J \subseteq K$ such that $K \subseteq \bigcup_{x\in J} \operatorname{int} E_x$. Set $E = \bigcup_{x\in J} E_x$, $F = \bigcap_{x\in J} F_x$; then E, $F \in \mathcal{E}$, $E \cap F = \emptyset$, $K \subseteq \operatorname{int} E$ and $L \subseteq \operatorname{int} F$, as required. **Q**

(b) Let Q, $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be usco-compact relations such that $Q[\mathbb{N}^{\mathbb{N}}] = A$ and $R[\mathbb{N}^{\mathbb{N}}] = B$. For each $\sigma \in S^* = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$, set

$$I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}, \quad A_\sigma = Q[I_\sigma], \quad B_\sigma = R[I_\sigma],$$

so that $A = A_\emptyset$ and $A_\sigma = \bigcup_{i\in\mathbb{N}} A_{\sigma^\frown i}$ for every σ.

(c) Write T for the set of pairs

$$\{(\sigma, \tau) : \sigma, \tau \in S^* \text{ and there are disjoint } E, F \in \mathcal{E} \text{ such that } A_\sigma \subseteq E, B_\tau \subseteq F\}.$$

If σ, $\tau \in S^*$ are such that $(\sigma^\frown i, \tau^\frown j) \in T$ for every i, $j \in \mathbb{N}$, then $(\sigma, \tau) \in T$. **P** For each i, $j \in \mathbb{N}$ take disjoint E_{ij}, $F_{ij} \in \mathcal{E}$ such that

$$A_{\sigma^\frown i} \subseteq E_{ij}, \quad B_{\tau^\frown j} \subseteq F_{ij}.$$

Then $E = \bigcup_{i\in\mathbb{N}} \bigcap_{j\in\mathbb{N}} E_{ij}$, $F = \bigcap_{i\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} F_{ij}$ are disjoint and belong to $\mathcal{E}$, and $A_\sigma \subseteq E$, $B_\tau \subseteq F$. So $(\sigma, \tau) \in T$. **Q**

(d) ? Now suppose, if possible, that there are no disjoint E, $F \in \mathcal{E}$ such that $A \subseteq E$ and $B \subseteq F$; that is, that $(\emptyset, \emptyset) \notin T$. By (c), used repeatedly, we can find sequences $\langle\phi(i)\rangle_{i\in\mathbb{N}}$, $\langle\psi(i)\rangle_{i\in\mathbb{N}}$ such that $(\phi{\restriction}n, \psi{\restriction}n) \notin T$ for every $n \in \mathbb{N}$. Set $K = Q[\{\phi\}]$, $L = R[\{\psi\}]$. These are compact (because R is usco-compact) and disjoint (because $K \subseteq A$ and $L \subseteq B$). By (a), there are disjoint E, $F \in \mathcal{E}$ such that $K \subseteq \operatorname{int} E$ and $L \subseteq \operatorname{int} F$.

By 422B, there are open sets U, $V \subseteq \mathbb{N}^{\mathbb{N}}$ such that

$$\phi \in U, \quad Q[U] \subseteq \operatorname{int} E, \quad \psi \in V, \quad R[V] \subseteq \operatorname{int} F.$$

But now there is some $n \in \mathbb{N}$ such that $I_{\phi{\restriction}n} \subseteq U$ and $I_{\psi{\restriction}n} \subseteq V$, in which case

$$A_{\phi{\restriction}n} \subseteq E, \quad B_{\psi{\restriction}n} \subseteq F,$$

and $(\phi{\restriction}n, \psi{\restriction}n) \in T$, contrary to the choice of ϕ and ψ. **X**

This contradiction shows that the lemma is true.

422J Corollary Let X be a Hausdorff space and A, B disjoint K-analytic subsets of X. Then there is a Borel set which includes A and is disjoint from B.

proof Apply 422I with $\mathcal{E}$ the Borel σ-algebra of X.

***422K** I give the next step in the theory of 'constituents' begun in 412O-412R.

Theorem Let X be a Hausdorff space.

(i) Suppose that X is regular. Let $A \subseteq X$ be a K-analytic set. Then there is a non-decreasing family $\langle E_\xi\rangle_{\xi<\omega_1}$ of Borel sets in X, with union $X \setminus A$, such that every Souslin-F subset of X disjoint from A is included in some E_ξ.

(ii) Suppose that X is regular. Let $A \subseteq X$ be a Souslin-F set. Then there is a non-decreasing family $\langle E_\xi \rangle_{\xi<\omega_1}$ of Borel sets in X, with union $X \setminus A$, such that every K-analytic subset of $X \setminus A$ is included in some E_ξ.

(iii) Let $A \subseteq X$ be a K-analytic set. Then there is a non-decreasing family $\langle E_\xi \rangle_{\xi<\omega_1}$ of Borel sets in X, with union $X \setminus A$, such that every K-analytic subset of $X \setminus A$ is included in some E_ξ.

proof (a) The first two parts depend on the following fact: if X is regular, $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ is usco-compact, $\langle F_\sigma \rangle_{\sigma \in S}$ is a Souslin scheme consisting of closed sets with kernel B, and $R[\mathbb{N}^{\mathbb{N}}] \cap B = \emptyset$, then for any ϕ, $\psi \in \mathbb{N}^{\mathbb{N}}$ there is an $n \geq 1$ such that $\overline{R[I_{\phi\restriction n}]} \cap \bigcap_{1 \leq i \leq n} F_{\psi\restriction i}$ is empty, where I write $I_\sigma = \{\theta : \sigma \subseteq \theta \in \mathbb{N}^{\mathbb{N}}\}$ for $\sigma \in S = \bigcup_{n \geq 1} \mathbb{N}^n$. **P** We know that $K = R[\{\phi\}]$ is a compact set disjoint from the closed set $\bigcap_{n \geq 1} F_{\psi\restriction n}$. So there is some $m \geq 1$ such that $K \cap F = \emptyset$ where $F = \bigcap_{1 \leq i \leq m} F_{\psi\restriction i}$. Because X is regular, there are disjoint open sets G, $H \subseteq X$ such that $K \subseteq G$ and $F \subseteq H$ (4A2F(h-ii)). Now $R^{-1}[X \setminus G]$ is a closed set not containing ϕ, so there is some n such that $R[I_{\phi\restriction n}] \subseteq G$. Of course we can take $n \geq m$, and in this case

$$\overline{R[I_{\phi\restriction n}]} \cap \bigcap_{1 \leq i \leq n} F_{\psi\restriction i} \subseteq \overline{G} \cap F = \emptyset,$$

as required. **Q**

(b)(i) Suppose that $A \subseteq X$ is K-analytic. Then there is an usco-compact set $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = A$, and R is closed (422Da), so that A is the kernel of the Souslin scheme $\langle \overline{R[I_\sigma]} \rangle_{\sigma \in S}$ (421I). For $x \in X$ set $T_x = \{\sigma : \sigma \in S, x \in \overline{R[I_\sigma]}\}$, as in 421Ng, and let $r(T_x) < \omega_1$ be the rank of the tree T_x. Then $E_\xi = \{x : x \in X \setminus A, r(T_x) \leq \xi\}$ is a Borel set for every $\xi < \omega_1$, by 421Ob. Now suppose that $B \subseteq X \setminus A$ is a Souslin-F set. Then it is the kernel of a Souslin scheme $\langle F_\sigma \rangle_{\sigma \in S}$ consisting of closed sets. If ϕ, $\psi \in \mathbb{N}^{\mathbb{N}}$ then by (a) above there is an $n \geq 1$ such that $\overline{R[I_{\phi\restriction n}]} \cap \bigcap_{1 \leq i \leq n} F_{\psi\restriction i}$ is empty. By 421Q, there must be some $\xi < \omega_1$ such that $B \subseteq E_\xi$. So $\langle E_\xi \rangle_{\xi<\omega_1}$ is a suitable family.

(ii) The other part is almost the same. Suppose that $A \subseteq X$ is Souslin-F. Then it is the kernel of a Souslin scheme $\langle F_\sigma \rangle_{\sigma \in S}$ consisting of closed sets. For $x \in X$ set

$$T_x = \bigcup_{n \geq 1} \{\sigma : \sigma \in \mathbb{N}^n, x \in \bigcap_{1 \leq i \leq n} F_{\sigma\restriction i}\},$$

and let $r(T_x) < \omega_1$ be the rank of the tree T_x. Then $E_\xi = \{x : x \in X \setminus A, r(T_x) \leq \xi\}$ is a Borel set for every $\xi < \omega_1$. Now let $B \subseteq X \setminus A$ be a K-analytic set. There is an usco-compact set $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ such that $R[\mathbb{N}^{\mathbb{N}}] = B$, and B is the kernel of the Souslin scheme $\langle \overline{R[I_\sigma]} \rangle_{\sigma \in S}$. If ϕ, $\psi \in \mathbb{N}^{\mathbb{N}}$ then by (a) above there is an $n \geq 1$ such that $\bigcap_{1 \leq i \leq n} F_{\psi\restriction i} \cap \overline{R[I_{\phi\restriction n}]}$ is empty. So there must be some $\xi < \omega_1$ such that $B \subseteq E_\xi$. Thus here again $\langle E_\xi \rangle_{\xi<\omega_1}$ is a suitable family.

(c) If X is not regular, we still have a version of the result in (a), as follows: if R, $S \subseteq \mathbb{N}^{\mathbb{N}} \times X$ and $R[\mathbb{N}^{\mathbb{N}}] \cap S[\mathbb{N}^{\mathbb{N}}] = \emptyset$, then for any ϕ, $\psi \in \mathbb{N}^{\mathbb{N}}$ there is an $n \geq 1$ such that $\overline{R[I\phi\restriction n]} \cap \overline{S[I_{\psi\restriction n}]}$ is empty. **P** This time, $R[\{\phi\}]$ and $S[\{\psi\}]$ are disjoint compact sets, so there are disjoint open sets G, H with $R[\{\phi\}] \subseteq G$ and $S[\{\psi\}] \subseteq H$ (4A2F(h-i)). Now $R[I_{\phi\restriction n}] \subseteq G$ and $S[I_{\psi\restriction n}] \subseteq H$ for all n large enough. **Q**

Now the argument of (b-i), with $F_\sigma = S[I_\sigma]$, gives part (iii).

422X Basic exercises (a) Let X and Y be Hausdorff spaces and X_0 a closed subset of X. Show that a relation $R \subseteq X_0 \times Y$ is usco-compact when regarded as a relation between X_0 and Y iff it is usco-compact when regarded as a relation between X and Y.

(b) Show that a locally compact Hausdorff space is K-analytic iff it is Lindelöf iff it is σ-compact.

>(c) Prove 422Hc from first principles, without using 421D. (*Hint*: if $\langle R_\sigma \rangle_{\sigma \in S}$ is a Souslin scheme of usco-compact relations in $\mathbb{N}^{\mathbb{N}} \times X$,

$$\{((\phi, \langle \psi_\sigma \rangle_{\sigma \in S}), x) : (\psi_{\phi\restriction n}, x) \in R_{\phi\restriction n} \text{ for every } n\}$$

is usco-compact in $(\mathbb{N}^{\mathbb{N}} \times (\mathbb{N}^{\mathbb{N}})^S) \times X$.)

(d) Let X be a completely regular Hausdorff space and A, B disjoint K-analytic subsets of X. Show that there is a Baire set including A and disjoint from B.

(e) Let X be a Hausdorff K-analytic space. (i) Show that every Baire subset of X is K-analytic. (*Hint*: 136G.) (ii) Show that if X is regular, it is perfectly normal iff it is hereditarily Lindelöf iff every open subset of X is K-analytic.

(f) Let X be a Hausdorff space in which every open set is K-analytic. Show that every Borel set is K-analytic. (*Hint*: apply 136Xi to the family of K-analytic subsets of X.)

422Y Further exercises (a) Let X be a completely regular Hausdorff space, and βX its Stone-Čech compactification. Show that X is K-analytic iff it is a Souslin-F set in βX.

(b) Show that a Hausdorff space is K-analytic iff it is a continuous image of a $\mathrm{K}_{\sigma\delta}$ set in a compact Hausdorff space, that is, a set expressible as $\bigcap_{m\in\mathbb{N}}\bigcup_{n\in\mathbb{N}} K_{mn}$ where every K_{mn} is compact. (*Hint*: Write $\mathcal{K}^*$ for the class of Hausdorff continuous images of $\mathrm{K}_{\sigma\delta}$ subsets of compact Hausdorff spaces. (i) Show that $\mathbb{N}^{\mathbb{N}}$ is a $\mathrm{K}_{\sigma\delta}$ set in $Y^{\mathbb{N}}$, where Y is the one-point compactification of $\mathbb{N}$. (ii) Show that if X is a compact Hausdorff space and $R\subseteq\mathbb{N}^{\mathbb{N}}\times X$ is closed, then $R\in\mathcal{K}^*$. (iii) Show that if X is a compact Hausdorff space, then every Souslin-F subset of X belongs to $\mathcal{K}^*$. (iv) Show that if X is a regular K-analytic Hausdorff space, then $X\in\mathcal{K}^*$. (v) Show that if X is any Hausdorff space and $R\subseteq\mathbb{N}^{\mathbb{N}}\times X$ is an usco-compact relation, then $R\in\mathcal{K}^*$. See Jayne 76.)

(c) Let X be a normal space and $\mathcal{C}$ the family of countably compact closed subsets of X. Let A, B be disjoint sets obtainable from $\mathcal{C}$ by Souslin's operation. (For instance, if X itself is countably compact, A and B could be disjoint Souslin-F sets.) Show that there is a Borel set including A and disjoint from B. (*Hint*: adapt the proof of 422I.)

(d) Let X be a Hausdorff space and $\langle A_n\rangle_{n\in\mathbb{N}}$ a sequence of K-analytic subsets of X such that $\bigcap_{n\in\mathbb{N}} A_n=\emptyset$. Show that there is a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ of Borel sets such that $A_n\subseteq E_n$ for every $n\in\mathbb{N}$ and $\bigcap_{n\in\mathbb{N}} E_n=\emptyset$. (*Hint*: for each $n\in\mathbb{N}$ choose an usco-compact $R_n\subseteq\mathbb{N}^{\mathbb{N}}\times X$ with projection A_n. Consider the set $T=\{\langle\sigma_n\rangle_{n\in\mathbb{N}}:\exists\text{ Borel }E_n,\ R_n[I_{\sigma_n}]\subseteq E_n\ \forall\, n,\ \bigcap_{n\in\mathbb{N}}E_n=\emptyset\}$.)

(e) Explain how to prove 422J from 421Q, without using 422I.

(f) Let X be a set and $\mathfrak{S}$, $\mathfrak{T}$ two Hausdorff topologies on X such that $\mathfrak{S}\subseteq\mathfrak{T}$ and $(X,\mathfrak{T})$ is K-analytic. Show that $\mathfrak{S}$, $\mathfrak{T}$ yield the same K-analytic subspaces of X.

422 Notes and comments In a sense, this section starts at the deep end of its topic. 'Descriptive set theory' originally developed in the context of the real line and associated spaces, and this remains the centre of the subject. But it turns out that some of the same arguments can be used in much more general contexts, and in particular greatly illuminate the theory of Radon measures on Hausdorff spaces. I find that a helpful way to look at K-analytic spaces is to regard them as a common generalization of compact Hausdorff spaces and Souslin-F subsets of $\mathbb{R}$; if you like, any theorem which is true of both these classes has a fair chance of being true of all K-analytic spaces. In the next section we shall come to the special properties of the more restricted class of 'analytic' spaces, which are much closer to the separable metric spaces of the original theory.

The phrase 'usco-compact' is neither elegant nor transparent, but is adequately established and (in view of the frequency with which it is needed) seems preferable to less concise alternatives. If we think of a relation $R\subseteq X\times Y$ as a function $x\mapsto R[\{x\}]$ from x to $\mathcal{P}Y$, then an usco-compact relation is one which takes compact values and is 'upper semi-continuous' in the sense that $\{x:R[\{x\}]\subseteq H\}$ is open for every open set $H\subseteq Y$; just as a real-valued function is upper semi-continuous if $\{x:f(x)<\alpha\}$ is open for every α.

This is not supposed to be a book on general topology, and in my account of the topological properties of K-analytic spaces I have concentrated on facts which are useful when proving that spaces are K-analytic, on the assumption that these will be valuable when we seek to apply the results of §432 below. Other properties are mentioned only when they are relevant to the measure-theoretic results which are my real concern, and readers already acquainted with this area may be startled by some of my omissions. For a proper treatment of the subject, I refer you to Rogers 80. As usual, however, I take technical details

seriously in the material I do cover. I hope you will not find that such results as 422Dg and 422Ga try your patience too far. I think a moment's thought will persuade you that it is of the highest importance that K-analyticity (like compactness) is an intrinsic property. In contrast, the property of being 'Souslin-F', like the property of being closed, depends on the surrounding space. A completely regular Hausdorff space is compact iff it must be a closed set in any surrounding Hausdorff space iff it is closed in its Stone-Čech compactification; and it is K-analytic iff it must be a Souslin-F set in any surrounding Hausdorff space iff it is a Souslin-F set in its Stone-Čech compactification (422Ya).

For regular spaces, 422K gives us another version of the First Separation Theorem. But this one is simultaneously more restricted in its scope (it does not seem to have applications to Baire σ-algebras, for instance) and very much more powerful in its application. When all Borel sets are Souslin-F, as in the next section, it tells us something very important about the cofinal structure of the Souslin-F subsets of the complement of a K-analytic set.

423 Analytic spaces

We come now to the original class of K-analytic spaces, the 'analytic' spaces. I define these as continuous images of $\mathbb{N}^{\mathbb{N}}$ (423A), but move as quickly as possible to their characterization as K-analytic spaces with countable networks (423C), so that many other fundamental facts (423E-423G) can be regarded as simple corollaries of results in §422. I give two versions of Lusin's theorem on injective images of Borel sets (423I), and a form of the von Neumann-Jankow measurable selection theorem (423N). I end with notes on constituents of coanalytic sets (423P-423Q).

423A Definition A Hausdorff space is **analytic** or **Souslin** if it is either empty or a continuous image of $\mathbb{N}^{\mathbb{N}}$.

423B Proposition (a) A Polish space (definition: 4A2A) is analytic.
(b) A Hausdorff continuous image of an analytic Hausdorff space is analytic.
(c) A product of countably many analytic Hausdorff spaces is analytic.
(d) A closed subset of an analytic Hausdorff space is analytic.
(e) An analytic Hausdorff space has a countable network consisting of analytic sets.

proof (a) Let X be a Polish space. If $X = \emptyset$, we can stop. Otherwise, let ρ be a metric on X, inducing its topology, under which X is complete. For $\sigma \in S^* = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ choose $X_\sigma \subseteq X$ as follows. $X_\emptyset = X$. Given that X_σ is a closed non-empty subset of X, where $\sigma \in S^*$, then X_σ is separable, because X is separable and metrizable (4A2P(a-iv)), and we can choose a sequence $\langle x_{\sigma i}\rangle_{i\in\mathbb{N}}$ in X_σ such that $\{x_{\sigma i} : i \in \mathbb{N}\}$ is dense in X_σ. Set $X_{\sigma^\frown i} = X_\sigma \cap B(x_{\sigma i}, 2^{-n})$ for each $i \in \mathbb{N}$, where $B(x,\delta) = \{y : \rho(y,x) \leq \delta\}$, and continue. Note that because $\{x_{\sigma i} : i \in \mathbb{N}\}$ is dense in X_σ, $X_\sigma = \bigcup_{i\in\mathbb{N}} X_{\sigma^\frown i}$, for every $\sigma \in S^*$.

For each $\phi \in \mathbb{N}^{\mathbb{N}}$, $\langle X_{\phi\restriction n}\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of non-empty closed sets, and $\operatorname{diam}(X_{\phi\restriction n+1}) \leq 2^{-n+1}$ for every n. Because X is complete under ρ, $\bigcap_{n\in\mathbb{N}} X_{\phi\restriction n}$ is a singleton $\{f(\phi)\}$ say. ($f(\phi)$ is the limit of the Cauchy sequence $\langle x_{\phi\restriction n,\phi(n)}\rangle_{n\in\mathbb{N}}$.) Thus we have a function $f : \mathbb{N}^{\mathbb{N}} \to X$. f is continuous because $\rho(f(\psi), f(\phi)) \leq 2^{-n+1}$ whenever $\phi\restriction n+1 = \psi\restriction n+1$ (since in this case both $f(\psi)$ and $f(\phi)$ belong to $X_{\phi\restriction n+1}$). f is surjective because, given $x \in X$, we can choose $\langle\phi(i)\rangle_{i\in\mathbb{N}}$ inductively so that $x \in X_{\phi\restriction n}$ for every n; at the inductive step, we have $x \in X_{\phi\restriction n} = \bigcup_{i\in\mathbb{N}} X_{(\phi\restriction n)^\frown i}$, so we can take $\phi(n)$ such that $x \in X_{(\phi\restriction n)^\frown\phi(n)} = X_{\phi\restriction n+1}$.

Thus X is a continuous image of $\mathbb{N}^{\mathbb{N}}$, as claimed.

(b) If X is an analytic Hausdorff space and Y is a Hausdorff continuous image of X, then either X is a continuous image of $\mathbb{N}^{\mathbb{N}}$ and Y is a continuous image of $\mathbb{N}^{\mathbb{N}}$, or $X = \emptyset$ and $Y = \emptyset$.

(c) Let $\langle X_i\rangle_{i\in I}$ be a countable family of analytic Hausdorff spaces, with product X. Then X is Hausdorff (3A3Id). If $I = \emptyset$ then $X = \{\emptyset\}$ is a continuous image of $\mathbb{N}^{\mathbb{N}}$, therefore analytic. If there is some $i \in I$ such that $X_i = \emptyset$, then $X = \emptyset$ is analytic. Otherwise, we have for each $i \in I$ a continuous surjection $f_i : \mathbb{N}^{\mathbb{N}} \to X_i$. Setting $f(\phi) = \langle f_i(\phi(i))\rangle_{i\in I}$ for $\phi \in (\mathbb{N}^{\mathbb{N}})^I$, $f : (\mathbb{N}^{\mathbb{N}})^I \to X$ is a continuous surjection. But $(\mathbb{N}^{\mathbb{N}})^I \cong \mathbb{N}^{\mathbb{N}\times I}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$, so X is analytic.

(d) Let X be an analytic Hausdorff space and F a closed subset of X. Then F is Hausdorff in its subspace topology (4A2F(a-i)). If $X = \emptyset$ then $F = \emptyset$ is analytic. Otherwise, there is a continuous surjection $f : \mathbb{N}^{\mathbb{N}} \to X$. Now $H = f^{-1}[F]$ is a closed subset of the Polish space $\mathbb{N}^{\mathbb{N}}$, therefore Polish in its induced topology (4A2Qd). By (a), H is analytic, so its continuous image $F = f[H]$ is also analytic, by (b).

(e) Let X be an analytic Hausdorff space. If it is empty then of course it has a countable network consisting of analytic sets. Otherwise, there is a continuous surjection $f : \mathbb{N}^{\mathbb{N}} \to X$. For $\sigma \in S^*$ set $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$; then $\{I_\sigma : \sigma \in S^*\}$ is a base for the topology of $\mathbb{N}^{\mathbb{N}}$, so $\{f[I_\sigma] : \sigma \in S^*\}$ is a network for the topology of X (see the proof of 4A2Nd). But I_σ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$, so $f[I_\sigma]$ is analytic, for every $\sigma \in S^*$, and $\{f[I_\sigma] : \sigma \in S^*\}$ is a countable network consisting of analytic sets.

423C Theorem A Hausdorff space is analytic iff it is K-analytic and has a countable network.

proof (a) Let X be an analytic Hausdorff space. By 423Be, it has a countable network. If $X = \emptyset$ then surely it is K-analytic. Otherwise, X is a continuous image of $\mathbb{N}^{\mathbb{N}}$. But $\mathbb{N}^{\mathbb{N}}$ is K-analytic (422Gb), so X also is K-analytic, by 422Gd.

(b) Now suppose that X is a K-analytic Hausdorff space and has a countable network.

(i) If $X \subseteq \mathbb{N}^{\mathbb{N}}$ then X is analytic. **P** Let $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}] = X$. Then R is still usco-compact when regarded as a subset of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ (422Dg), so is closed in $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ (422Da). But $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \cong \mathbb{N}^{\mathbb{N}}$ is analytic, so R is in itself an analytic space (423Bd), and its continuous image X is analytic, by 423Bb. **Q**

(ii) Now suppose that X is regular. By 4A2Ng, X has a countable network $\mathcal{E}$ consisting of closed sets. Adding $\emptyset$ to $\mathcal{E}$ if need be, we may suppose that $\mathcal{E} \neq \emptyset$. Let $\langle E_n \rangle_{n \in \mathbb{N}}$ be a sequence running over $\mathcal{E}$. For each $n \in \mathbb{N}$, let $\langle F_{ni} \rangle_{i \in \mathbb{N}}$ be a sequence running over $\{E_n\} \cup \{E : E \in \mathcal{E}, E \cap E_n = \emptyset\}$. Now consider the relation

$$R = \{(\phi, x) : \phi \in \mathbb{N}^{\mathbb{N}}, x \in \textstyle\bigcap_{n \in \mathbb{N}} F_{n,\phi(n)}\} \subseteq \mathbb{N}^{\mathbb{N}} \times X.$$

(α) R is closed in $\mathbb{N}^{\mathbb{N}} \times X$. **P** Because every F_{ni} is closed,

$$(\mathbb{N}^{\mathbb{N}} \times X) \setminus R = \textstyle\bigcup_{i,n \in \mathbb{N}} \{(\phi, x) : \phi(n) = i \text{ and } x \notin F_{ni}\}$$

is open. **Q**

(β) $R[\mathbb{N}^{\mathbb{N}}] = X$. **P** For every $n \in \mathbb{N}$,

$$X \setminus E_n = \textstyle\bigcup \{E : E \in \mathcal{E}, E \subseteq X \setminus E_n\}$$

because $\mathcal{E}$ is a network and E_n is closed, so $\bigcup_{i \in \mathbb{N}} F_{ni} = X$. So, given $x \in X$, we can find for each n a $\phi(n)$ such that $x \in F_{n,\phi(n)}$, and $(\phi, x) \in R$. **Q**

(γ) R is the graph of a function. **P?** Suppose that we have (ϕ, x) and (ϕ, y) in R where $x \neq y$. Because the topology of X is Hausdorff, there is an $n \in \mathbb{N}$ such that $x \in E_n$ and $y \notin E_n$. But in this case $x \in E_n \cap F_{n,\phi(n)}$, so $F_{n,\phi(n)} = E_n$, while $y \in F_{n,\phi(n)} \setminus E_n$, so $F_{n,\phi(n)} \neq E_n$; which is absurd. **XQ**

(δ) Set $A = R^{-1}[X]$, so that R is the graph of a function from A to X; in recognition of its new status, give it a new name f. Then f is continuous. **P** Suppose that $\phi \in A$ and that $x = f(\phi) \in G$, where $G \subseteq X$ is open. Then there is an $n \in \mathbb{N}$ such that $x \in E_n \subseteq G$. In this case $x \in F_{n,\phi(n)}$, because $(\phi, x) \in R$, so $F_{n,\phi(n)} = E_n$. Now if $\psi \in A$ and $\psi(n) = \phi(n)$, we must have

$$f(\psi) \in F_{n,\psi(n)} = E_n \subseteq G.$$

Thus $f^{-1}[G]$ includes a neighbourhood of ϕ in A. As ϕ and G are arbitrary, f is continuous. **Q**

(ϵ) At this point recall that X is K-analytic. It follows that $\mathbb{N}^{\mathbb{N}} \times X$ is K-analytic (422Ge), so that its closed subset R is K-analytic (422Gf) and A, which is a continuous image of R, is K-analytic (422Gd). But now A is a K-analytic subset of $\mathbb{N}^{\mathbb{N}}$, so is analytic, by (i) just above. And, finally, X is a continuous image of A, so is analytic.

(iii) Thus any regular K-analytic space with a countable network is analytic. Now suppose that X is an arbitrary K-analytic Hausdorff space with a countable network. Let $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be an usco-compact

relation such that $R[\mathbb{N}^{\mathbb{N}}] = X$. Then R is a closed subset of $\mathbb{N}^{\mathbb{N}} \times X$, so is itself a K-analytic space with a countable network. But it is also regular, by 422E. So R is analytic and its continuous image X is analytic.

This completes the proof.

423D Corollary (a) Any analytic Hausdorff space is hereditarily Lindelöf.

(b) In a regular analytic Hausdorff space, closed subsets are zero sets and the Baire and Borel σ-algebras coincide.

(c) A compact subset of an analytic Hausdorff space is metrizable.

(d) A metrizable space is analytic iff it is K-analytic.

proof (a)-(c) These are true just because there is a countable network (4A2Nb, 4A2Na, 4A2Nh, 4A3Kb).

(d) Let X be a metrizable space. If X is analytic, of course it is K-analytic. If X is K-analytic, it is Lindelöf (422Gg) therefore separable (4A2Pd) and has a countable network (4A2P(a-iii)), so is analytic.

423E Theorem (a) For any Hausdorff space X, the family of subsets of X which are analytic in their subspace topologies is closed under Souslin's operation.

(b) Let $(X, \mathfrak{T})$ be an analytic Hausdorff space. For a subset A of X, the following are equiveridical:

(i) A is analytic;

(ii) A is K-analytic;

(iii) A is Souslin-F;

(iv) A can be obtained by Souslin's operation from the family of Borel subsets of X.

In particular, all Borel sets in X are analytic.

proof (a) Let X be a Hausdorff space and $\mathcal{A}$ the family of analytic subsets of X. Let $\langle A_\sigma \rangle_{\sigma \in S}$ be a Souslin scheme in $\mathcal{A}$ with kernel A. Then every A_σ is K-analytic, so A is K-analytic, by 422Hc. Also every A_σ has a countable network, so $A' = \bigcup_{\sigma \in S} A_\sigma$ has a countable network (4A2Nc); as $A \subseteq A'$, A also has a countable network (4A2Na) and is analytic.

(b) Because X has a countable network, so does A. So 423C tells us at once that (i) $\iff$ (ii). In particular, X is K-analytic, so 422Hb tells us that (ii) $\iff$ (iii). Of course (iii)$\Rightarrow$(iv).

Now suppose that $G \subseteq X$ is open. Then $G \in \mathcal{A}$. **P** If $X = \emptyset$ then $G = \emptyset$ is open. Otherwise, there is a continuous surjection $f : \mathbb{N}^{\mathbb{N}} \to X$. Set $H = f^{-1}[G]$, so that $H \subseteq \mathbb{N}^{\mathbb{N}}$ is open and $G = f[H]$. Being an open set in a metric space, H is F_σ (4A2Lc), so, in particular, is Souslin-F; but $\mathbb{N}^{\mathbb{N}}$ is analytic, so H is analytic and its continuous image G is analytic. **Q**

We have already seen that closed subsets of X belong to $\mathcal{A}$ (423Bd). Because $\mathcal{A}$ is closed under Souslin's operation, it contains every Borel set, by 421F. It therefore contains every set obtainable by Souslin's operation from Borel sets, and (i)$\Rightarrow$(i).

Remark See also 423Yb below.

423F Proposition Let $(X, \mathfrak{T})$ be an analytic Hausdorff space.

(a) A set $E \subseteq X$ is Borel iff both E and $X \setminus E$ are analytic.

(b) If $\mathfrak{S}$ is a coarser(= smaller) Hausdorff topology on X, then $\mathfrak{S}$ and $\mathfrak{T}$ have the same Borel sets.

proof (a) If E is Borel, then E and $X \setminus E$ are analytic, by 423Eb. If E and $X \setminus E$ are analytic, they are K-analytic (423Eb) and disjoint, so there is a Borel set $F \supseteq E$ which is disjoint from $X \setminus E$ (422J); but now of course $F = E$, so E must be Borel.

(b) Because the identity map from $(X, \mathfrak{T})$ to $(X, \mathfrak{S})$ is continuous, $\mathfrak{S}$ is an analytic topology (423Bb) and every $\mathfrak{S}$-Borel set is $\mathfrak{T}$-Borel. If $E \subseteq X$ is $\mathfrak{T}$-Borel, then it and its complement are $\mathfrak{T}$-analytic, therefore $\mathfrak{S}$-analytic (423Bb), and E is $\mathfrak{S}$-Borel by (a).

423G Lemma Let X and Y be analytic Hausdorff spaces and $f : X \to Y$ a Borel measurable function.

(a) (The graph of) f is an analytic set.

(b) $f[A]$ is an analytic set in Y for any analytic set (in particular, any Borel set) $A \subseteq X$.

(c) $f^{-1}[B]$ is an analytic set in X for any analytic set (in particular, any Borel set) $B \subseteq Y$.

proof (a) Let $\mathcal{E}$ be a countable network for the topology of Y. Set

$$R = \bigcap_{E\in\mathcal{E}} (X \times \overline{E}) \cup ((X \setminus f^{-1}[\overline{E}]) \times Y).$$

Then R is a Borel set in $X \times Y$. But also R is the graph of f. **P** If $f(x) = y$, then surely $y \in \overline{E}$ whenever $x \in f^{-1}[\overline{E}]$, so $(x, y) \in R$. On the other hand, if $x \in X$, $y \in Y$ and $f(x) \neq y$, there are disjoint open sets G, $H \subseteq Y$ such that $f(x) \in G$ and $y \in H$; now there is an $E \in \mathcal{E}$ such that $f(x) \in E \subseteq G$, so that $f(x) \in \overline{E}$ but $y \notin \overline{E}$, and $(x, y) \notin R$. **Q**

Because $X \times Y$ is analytic (423Bc), R is analytic (423Eb).

(b) If $A \subseteq X$ is analytic, then $A \times Y$ and $R \cap (A \times Y)$ are analytic (423Ea), so $f[A] = R[A]$, which is a continuous image of $R \cap (A \times Y)$, is analytic.

(c) Similarly, if $B \subseteq Y$ is analytic, then $f^{-1}[B]$ is a continuous image of $R \cap (X \times B)$, so is analytic.

423H Lemma Let $(X, \mathfrak{T})$ be an analytic Hausdorff space, and $\langle E_n\rangle_{n\in\mathbb{N}}$ any sequence of Borel sets in X. Then the topology $\mathfrak{T}'$ generated by $\mathfrak{T} \cup \{E_n : n \in \mathbb{N}\}$ is analytic.

proof If $X = \emptyset$ this is trivial. Otherwise, there is a continuous surjection $f : \mathbb{N}^{\mathbb{N}} \to X$. Set $F_n = f^{-1}[E_n]$ for each n; then F_n is a Borel subset of $\mathbb{N}^{\mathbb{N}}$, so there is a Polish topology $\mathfrak{S}'$ on $\mathbb{N}^{\mathbb{N}}$, finer than the usual topology, for which every F_n is open, by 4A3I. But now f is continuous for $\mathfrak{S}'$ and $\mathfrak{T}'$, so $\mathfrak{T}'$ is analytic, by 423Ba and 423Bb. (Of course $\mathfrak{T}'$ is Hausdorff, because it is finer than $\mathfrak{T}$.)

423I Theorem Let X be a Polish space, $E \subseteq X$ a Borel set, Y a Hausdorff space and $f : E \to Y$ an injective function.

(a) If f is continuous, then $f[E]$ is Borel.

(b) If Y has a countable network (e.g., is an analytic space or a separable metrizable space), and f is Borel measurable, then $f[E]$ is Borel.

proof (a)(i) Since there is a finer Polish topology on X for which E is closed (4A3I), therefore Polish in the subspace topology (4A2Qd), and f will still be continuous for this topology, we may suppose that $E = X$.

(ii) Let $\langle U_n\rangle_{n\in\mathbb{N}}$ run over a base for the topology of X (4A2P(a-i)). For each pair m, $n \in \mathbb{N}$ such that $U_m \cap U_n$ is empty, $f[U_m]$ and $f[U_n]$ are analytic sets in Y (423Eb, 423Bb) and are disjoint (because f is injective), so there is a Borel set H_{mn} including $f[U_m]$ and disjoint from $f[U_n]$ (422J). Set

$$E_n = \overline{f[U_n]} \cap \bigcap\{H_{nm} \setminus H_{mn} : m \in \mathbb{N}, U_m \cap U_n = \emptyset\}$$

for each $n \in \mathbb{N}$; then E_n is a Borel set in Y including $f[U_n]$. Note that if $U_m \cap U_n$ is empty, then $E_m \cap E_n \subseteq (H_{mn} \setminus H_{nm}) \cap (H_{nm} \setminus H_{mn})$ is also empty.

Fix a metric ρ on X, inducing its topology, for which X is complete, and for $k \in \mathbb{N}$ set

$$F_k = \bigcup\{E_n : n \in \mathbb{N}, \operatorname{diam} U_n \leq 2^{-k}\},$$

so that F_k is Borel. Let $F = \bigcap_{k\in\mathbb{N}} F_k$; then F is also a Borel subset of Y.

The point is that $F = f[X]$. **P** (i) If $x \in X$, then for every $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that

$$x \in U_n \subseteq \{y : \rho(y, x) \leq 2^{-k-1}\};$$

now $\operatorname{diam} U_n \leq 2^{-k}$, so

$$f(x) \in f[U_n] \subseteq E_n \subseteq F_k.$$

As k is arbitrary, $f(x) \in F_k$; as x is arbitrary, $f[X] \subseteq F$. (ii) If $y \in F$, then for each $k \in \mathbb{N}$ we can find an $n(k)$ such that $y \in E_{n(k)}$ and $\operatorname{diam} U_{n(k)} \leq 2^{-k}$. Since $\overline{f[U_{n(k)}]} \supseteq E_{n(k)}$ is not empty, nor is $U_{n(k)}$, and we can choose $x_k \in U_{n(k)}$. Indeed, for any j, $k \in \mathbb{N}$, $E_{n(j)} \cap E_{n(k)}$ contains y, so is not empty, and $U_{n(j)} \cap U_{n(k)}$ cannot be empty; but this means that there is some x in the intersection, and

$$\rho(x_j, x_k) \leq \rho(x_j, x) + \rho(x, x_k) \leq \operatorname{diam} U_{n(j)} + \operatorname{diam} U_{n(k)} \leq 2^{-j} + 2^{-k}.$$

This means that $\langle x_k\rangle_{k\in\mathbb{N}}$ is a Cauchy sequence. But X is supposed to be complete, so $\langle x_k\rangle_{k\in\mathbb{N}}$ has a limit x say.

? If $f(x) \neq y$, then (because Y is Hausdorff) there is an open set H containing $f(x)$ such that $y \notin \overline{H}$. Now f is continuous, so there is a $\delta > 0$ such that $f(x') \in H$ whenever $\rho(x', x) \leq \delta$. There is a $k \in \mathbb{N}$ such that $2^{-k} + \rho(x_k, x) \leq \delta$. If $x' \in U_{n(k)}$, then

$$\rho(x', x) \leq \rho(x', x_k) + \rho(x_k, x) \leq \delta;$$

thus $f[U_{n(k)}] \subseteq H$, and

$$E_{n(k)} \subseteq \overline{f[U_{n(k)}]} \subseteq \overline{H}.$$

But $y \in E_{n(k)} \setminus \overline{H}$. **X**

Thus $y = f(x)$ belongs to $f[X]$; as y is arbitrary, $F \subseteq f[X]$. **Q** Accordingly $f[X] = F$ is a Borel subset of Y, as claimed.

(b) By 4A2Nf, there is a countable family $\mathcal{V}$ of open sets in Y such that whenever y, y' are distinct points of Y there are disjoint V, $V' \in \mathcal{V}$ such that $y \in V$ and $y' \in V'$. Let $\mathfrak{S}'$ be the topology generated by $\mathcal{V}$; then $\mathfrak{S}'$ is Hausdorff. For each $V \in \mathcal{V}$, $f^{-1}[V]$ is a Borel set in X, so there is a Polish topology $\mathfrak{T}'$ on X, finer than the original topology, for which every $f^{-1}[V]$ is open (4A3I). Now f is continuous for $\mathfrak{T}'$ and $\mathfrak{S}'$ (4A2B(a-ii)), and E is $\mathfrak{T}'$-Borel, so $f[E]$ is a $\mathfrak{S}'$-Borel set in Y, by (a). Since $\mathfrak{S}'$ is coarser than the original topology $\mathfrak{S}$ on Y, $f[X]$ is also $\mathfrak{S}$-Borel.

423J Lemma If X is an uncountable analytic Hausdorff space, it has subsets homeomorphic to $\{0,1\}^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$.

proof (a) Let $f : \mathbb{N}^{\mathbb{N}} \to X$ be a continuous surjection. Write $S^* = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$, $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$ for $\sigma \in S^*$,

$$T = \{\sigma : \sigma \in S^*, f[I_\sigma] \text{ is uncountable}\}.$$

Then if $\sigma \in T$ there are τ, $\tau' \in T$, both extending σ, such that $f[I_\tau] \cap f[I_{\tau'}] = \emptyset$. **P** Set

$$A = \bigcup \{f[I_\tau] : \tau \in S^* \setminus T\}.$$

Then A is a countable union of countable sets, so is countable. There must therefore be distinct points x, y of $f[I_\sigma] \setminus A$; express x as $f(\phi)$ and y as $f(\psi)$ where ϕ and ψ belong to I_σ. Because X is Hausdorff, there are disjoint open sets G, H such that $x \in G$ and $y \in H$. Because f is continuous, there are m, $n \in \mathbb{N}$ such that $I_{\phi\restriction m} \subseteq f^{-1}[G]$ and $I_{\psi\restriction n} \subseteq f^{-1}[H]$. Of course both $\tau = \phi\restriction m$ and $\tau' = \psi\restriction n$ must extend σ, and they belong to T because $x \in f[I_\tau] \setminus A$, $y \in f[I_{\tau'}] \setminus A$. **Q**

(b) We can therefore choose inductively a family $\langle \tau(\upsilon) \rangle_{\upsilon \in S_2^*}$ in T, where $S_2^* = \bigcup_{n \in \mathbb{N}} \{0,1\}^n$, such that

$$\tau(\emptyset) = \emptyset,$$

$$\tau(\upsilon^\frown i) \supseteq \tau(\upsilon) \text{ whenever } \upsilon \in S_2^*, i \in \{0,1\},$$

$$f[I_{\tau(\upsilon^\frown 0)}] \cap f[I_{\tau(\upsilon^\frown 1]} = \emptyset \text{ for every } \upsilon \in S_2^*.$$

Note that $\#(\tau(\upsilon)) \geq \#(\upsilon)$ for every $\upsilon \in S_2$. For each $z \in \{0,1\}^{\mathbb{N}}$, $\langle \tau(z\restriction n) \rangle_{n \in \mathbb{N}}$ is a sequence in S^* in which each term strictly extends its predecessor, so there is a unique $g(z) \in \mathbb{N}^{\mathbb{N}}$ such that $\tau(z\restriction n) \subseteq g(z)$ for every n. Now $g(z')\restriction n = g(z)\restriction n$ whenever $z\restriction n = z'\restriction n$, so g and $fg : \{0,1\}^{\mathbb{N}} \to X$ are continuous. If w, z are distinct points of $\{0,1\}^{\mathbb{N}}$, there is a first n such that $w(n) \neq z(n)$, in which case $fg(w) \in f[I_{\tau(w\restriction n)^\frown w(n)}]$ and $fg(z) \in f[I_{\tau(w\restriction n)^\frown z(n)}]$ are distinct. So $fg : \{0,1\}^{\mathbb{N}} \to X$ is a continuous injection, therefore a homeomorphism between $\{0,1\}^{\mathbb{N}}$ and its image, because $\{0,1\}^{\mathbb{N}}$ is compact (3A3Dd).

(c) Thus X has a subspace homeomorphic to $\{0,1\}^{\mathbb{N}}$. Now $\{0,1\}^{\mathbb{N}}$ has a subspace homeomorphic to $\mathbb{N}$. **P** For instance, setting $d_n(n) = 1$, $d_n(i) = 0$ for $i \neq n$, $D = \{d_n : n \in \mathbb{N}\}$ is homeomorphic to $\mathbb{N}$. **Q** Now $D^{\mathbb{N}}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ and is a subspace of $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}} \cong \{0,1\}^{\mathbb{N} \times \mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$, so $\{0,1\}^{\mathbb{N}}$ has a subspace homeomorphic to $\mathbb{N}^{\mathbb{N}}$. Accordingly X also has a subspace homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

423K Corollary Any uncountable Borel set in any analytic space has cardinal $\mathfrak{c}$.

proof If X is an analytic space and $E \subseteq X$ is an uncountable Borel set, then E is analytic (423E), so includes a copy of $\{0,1\}^{\mathbb{N}}$ and must have cardinal at least $\#(\{0,1\}^{\mathbb{N}}) = \mathfrak{c}$. On the other hand, E is also a continuous image of $\mathbb{N}^{\mathbb{N}}$, so has cardinal at most $\#(\mathbb{N}^{\mathbb{N}}) = \mathfrak{c}$.

423L Proposition Let X be an uncountable analytic Hausdorff space. Then it has a non-Borel analytic subset.

proof (a) I show first that there is an analytic set $A \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that every analytic subset of $\mathbb{N}^{\mathbb{N}}$ is a vertical section of A. **P** Let $\mathcal{U}$ be a countable base for the topology of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, containing $\emptyset$, and $\langle U_n \rangle_{n\in\mathbb{N}}$ an enumeration of $\mathcal{U}$. Write

$$M = (\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}) \setminus \bigcup_{m,n\in\mathbb{N}} (\{x : x(m) = n\} \times U_n).$$

Then M is a closed subset of $(\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}) \times \mathbb{N}^{\mathbb{N}}$, therefore analytic (423Ba, 423Bd), so its continuous image

$$A = \{(x,z) : \text{there is some } y \text{ such that } (x,y,z) \in M\}$$

is analytic (423Bb).

Now let E be any analytic subset of $\mathbb{N}^{\mathbb{N}}$. By 423E, E is Souslin-F; by 421J, there is a closed set $F \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $E = \{z : \exists\, y,\ (y,z) \in F\}$. Let $\langle x(m) \rangle_{m\in\mathbb{N}}$ be a sequence running over $\{n : n \in \mathbb{N},\ U_n \cap F = \emptyset\}$, so that

$$F = (\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}) \setminus \bigcup_{m\in\mathbb{N}} U_{x(m)} = \{(y,z) : (x,y,z) \in M\}.$$

Now

$$\begin{aligned}\{z : (x,z) \in A\} &= \{z : \text{ there is some } y \text{ such that } (x,y,z) \in M\} \\ &= \{z : \text{ there is some } y \text{ such that } (y,z) \in F\} = E,\end{aligned}$$

and E is a vertical section of A, as required. **Q**

(b) It follows that there is a non-Borel analytic set $B \subseteq \mathbb{N}^{\mathbb{N}}$. **P** Take A from (a) above, and try

$$B = \{x : (x,x) \in A\}.$$

Because B is the inverse image of A under the continuous map $x \mapsto (x,x)$, it is analytic (423Gc). **?** If B were a Borel set, then $B' = \mathbb{N}^{\mathbb{N}} \setminus B$ would also be Borel, therefore analytic (423E), and there would be an $x \in \mathbb{N}^{\mathbb{N}}$ such that $B' = \{y : (x,y) \in A\}$. But in this case

$$x \in B \iff (x,x) \in A \iff x \in B',$$

which is a difficulty you may have met before. **XQ**

(c) Now return to our arbitrary uncountable analytic Hausdorff space X. By 423J, X has a subset Z homeomorphic to $\mathbb{N}^{\mathbb{N}}$. By (b), Z has an analytic subset A which is not Borel in Z, therefore cannot be a Borel subset of X.

423M I devote a few paragraphs to an important method of constructing selectors.

Theorem Let X be an analytic Hausdorff space, Y a set, and $\mathcal{C} \subseteq \mathcal{P}Y$. Write T for the σ-algebra of subsets of Y generated by $\mathcal{S}(\mathcal{C})$, where $\mathcal{S}$ is Souslin's operation, and $\mathcal{V}$ for $\mathcal{S}(\{F \times C : F \subseteq X \text{ is closed}, C \in \mathcal{C}\})$. If $W \in \mathcal{V}$, then $W[X] \in \mathrm{T}$ and there is a T-measurable function $f : W[X] \to X$ such that $(f(y), y) \in W$ for every $y \in W[X]$.

proof Write $\mathcal{F}$ for $\{F \times C : F \subseteq X \text{ is closed}, C \in \mathcal{C}\}$.

(a) Consider first the case in which $X = \mathbb{N}^{\mathbb{N}}$ and all the horizontal sections $W^{-1}[\{y\}]$ of W are closed. Let $\mathcal{E}$ be the family of closed subsets of Y. For $\sigma \in S^* = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ set $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$. Then $W \cap (I_\sigma \times Y) \in \mathcal{V}$. **P** Because Souslin's operation is idempotent (421D), $\mathcal{S}(\mathcal{V}) = \mathcal{V}$. The set $\{V : V \cap (I_\sigma \times Y) \in \mathcal{V}\}$ is therefore closed under Souslin's operation (apply 421Cc to the identity map from $I_\sigma \times Y$ to $X \times Y$, or otherwise); since it includes $\mathcal{F}$, it is the whole of $\mathcal{V}$, and contains W. **Q**

By 421G, $W[I_\sigma] = (W \cap (I_\sigma \times Y))[\mathbb{N}^{\mathbb{N}}]$ belongs to $\mathcal{S}(\mathcal{C}) \subseteq \mathrm{T}$ for every σ. In particular, $W[\mathbb{N}^{\mathbb{N}}] = W[I_\emptyset] \in \mathrm{T}$. Choose $\langle Y_\sigma \rangle_{\sigma\in S^*}$ in T inductively, as follows. $Y_\emptyset = W[\mathbb{N}^{\mathbb{N}}]$. Given that $Y_\sigma \in \mathrm{T}$ and that $Y_\sigma \subseteq W[I_\sigma]$, set

$$Y_{\sigma^\frown j} = Y_\sigma \cap W[I_{\sigma^\frown j}] \setminus \bigcup_{i<j} W[I_{\sigma^\frown i}]$$

for every $j \in \mathbb{N}$. Continue.

At the end of the induction, we have

$$\bigcup_{j\in\mathbb{N}} Y_{\sigma^\frown j} = Y_\sigma \cap \bigcup_{j\in\mathbb{N}} W[I_{\sigma^\frown j}] = Y_\sigma \cap W[I_\sigma] = Y_\sigma$$

for every $\sigma \in S^*$, while $\langle Y_{\sigma^\frown j}\rangle_{j\in\mathbb{N}}$ is always disjoint. So for each $y \in Y_\emptyset = W[\mathbb{N}^\mathbb{N}]$ we have a unique $f(y) \in \mathbb{N}^\mathbb{N}$ such that $y \in Y_{f(y)\restriction n}$ for every n. Since $f^{-1}[I_\sigma] = Y_\sigma \in \mathrm{T}$ for every $\sigma \in S^*$, f is T-measurable (4A2H(c-ii)). Also $(f(y), y) \in W$ for every $y \in W[\mathbb{N}^\mathbb{N}]$. **P** For each $n \in \mathbb{N}$, $y \in Y_{f(y)\restriction n} = W[I_{f(y)\restriction n}]$, so there is an $x_n \in \mathbb{N}^\mathbb{N}$ such that $x_n\restriction n = f(y)\restriction n$ and $(x_n, y) \in W$. But this means that $f(y) = \lim_{n\to\infty} x_n$; since we are supposing that the horizontal sections of W are closed, $(f(y), y) \in W$. **Q**

Thus the theorem is true if $X = \mathbb{N}^\mathbb{N}$ and W has closed horizontal sections.

(b) Now suppose that $X = \mathbb{N}^\mathbb{N}$ and that $W \subseteq \mathbb{N}^\mathbb{N} \times Y$ is any set in $\mathcal{V}$. Then there is a Souslin scheme $\langle F_\sigma \times C_\sigma\rangle_{\sigma\in S}$ in $\mathcal{F}$ with kernel W; of course I mean you to suppose that $F_\sigma \subseteq \mathbb{N}^\mathbb{N}$ is closed and $C_\sigma \in \mathcal{C}$ for every σ. Set

$$\tilde{W} = \bigcap_{k\geq 1} \bigcup_{\sigma\in\mathbb{N}^k} I_\sigma \times F_\sigma \times C_\sigma \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times Y.$$

Then W is the projection of $\tilde{W}$ onto the last two coordinates, by 421Ce. If $y \in Y$, then

$$\{(\phi,\psi) : (\phi,\psi,y) \in \tilde{W}\} = \bigcap_{k\geq 1} \bigcup\{I_\sigma \times F_\sigma : \sigma \in \mathbb{N}^k,\ y \in C_\sigma\}$$

is closed in $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$. (If J is any subset of $\mathbb{N}^k$, then

$$(\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}) \setminus \bigcup_{\sigma\in J} I_\sigma \times F_\sigma = \bigcup_{\sigma\in J} I_\sigma \times (\mathbb{N}^\mathbb{N} \setminus F_\sigma) \cup \bigcup_{\sigma\in\mathbb{N}^k\setminus J} I_\sigma \times \mathbb{N}^\mathbb{N}$$

is open.) Also $I_\sigma \times F_\sigma$ is a closed subset of $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ for every σ, and $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ is homeomorphic to $\mathbb{N}^\mathbb{N}$. We can therefore apply (a) to $\tilde{W}$, regarded as a subset of $(\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}) \times Y$, to see that $W[\mathbb{N}^\mathbb{N}] = \tilde{W}[\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}] \in \mathrm{T}$ and that there is a T-measurable function $h = (g, f) : W[\mathbb{N}^\mathbb{N}] \to \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ such that $(g(y), f(y), y) \in \tilde{W}$ for every $y \in W[\mathbb{N}^\mathbb{N}]$. Now, of course, $f : W[\mathbb{N}^\mathbb{N}] \to \mathbb{N}^\mathbb{N}$ is T-measurable and $(f(y), y) \in W$ for every $y \in W[\mathbb{N}^\mathbb{N}]$.

(c) Finally, suppose only that X is an analytic Hausdorff space and that $W \in \mathcal{V}$. If X is empty, so is Y, and the result is trivial. Otherwise, there is a continuous surjection $h : \mathbb{N}^\mathbb{N} \to X$. Set $\tilde{h}(\phi, y) = (h(\phi), y)$ for $\phi \in \mathbb{N}^\mathbb{N}$ and $y \in Y$; then $\tilde{h} : \mathbb{N}^\mathbb{N} \times Y \to X \times Y$ is a continuous surjection, and $\tilde{W} = \tilde{h}^{-1}[W]$ is the kernel of a Souslin scheme in

$$\{\tilde{h}^{-1}[F \times C] : F \subseteq \mathbb{N}^\mathbb{N} \text{ is closed}, C \in \mathcal{C}\} = \{h^{-1}[F] \times C : F \subseteq \mathbb{N}^\mathbb{N} \text{ is closed}, C \in \mathcal{C}\}$$

by 421Cb. So we can apply (b) to see that $W[X] = \tilde{W}[\mathbb{N}^\mathbb{N}] \in \mathrm{T}$ and there is a T-measurable $g : W[X] \to \mathbb{N}^\mathbb{N}$ such that $(g(y), y) \in \tilde{W}$ for every $y \in Y$. Finally $f = hg : W[X] \to X$ is T-measurable and $(f(y), y) \in W$ for every $y \in Y$. This completes the proof.

423N The expression

$$\mathcal{V} = \mathcal{S}(\{F \times C : F \subseteq X \text{ is closed}, C \in \mathcal{C}\})$$

in 423M is a new formulation, and I had better describe one of the basic cases in which we can use the result.

Corollary Let X be an analytic Hausdorff space and Y any topological space. Let T be the σ-algebra of subsets of Y generated by $\mathcal{S}(\mathcal{B}(Y))$, where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y. If $W \in \mathcal{S}(\mathcal{B}(X \times Y))$, then $W[X] \in \mathrm{T}$ and there is a T-measurable function $f : W[X] \to X$ such that $(f(y), y) \in W$ for every $y \in W[X]$.

proof (a) Suppose to begin with that $X = \mathbb{N}^\mathbb{N}$. In 423M, set $\mathcal{C} = \mathcal{B}(Y)$. Then every open subset and every closed subset of $X \times Y$ belongs to $\mathcal{V}$ as defined in 423M. **P** For $\sigma \in S^* = \bigcup_{k\in\mathbb{N}} \mathbb{N}^k$, set $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^\mathbb{N}\}$. If $V \subseteq X \times Y$ is open, set

$$H_\sigma = \bigcup\{H : H \subseteq Y \text{ is open}, I_\sigma \times H_\sigma \subseteq V\}$$

for each $\sigma \in S^*$. Because $\{I_\sigma : \sigma \in S^*\}$ is a base for the topology of $\mathbb{N}^\mathbb{N}$, $V = \bigcup_{\sigma\in S^*} I_\sigma \times H_\sigma \in \mathcal{V}$.

As for the complement of V, we have

$$\begin{aligned}(\mathbb{N}^{\mathbb{N}} \times Y) \setminus V &= \bigcap_{\substack{k\in\mathbb{N}\\ \sigma\in\mathbb{N}^k}} (\mathbb{N}^{\mathbb{N}} \times Y) \setminus (I_\sigma \times H_\sigma)\\ &= \bigcap_{\substack{k\in\mathbb{N}\\ \sigma\in\mathbb{N}^k}} ((\mathbb{N}^{\mathbb{N}} \setminus I_\sigma) \times Y) \cup (\mathbb{N}^{\mathbb{N}} \times (Y \setminus H_\sigma))\\ &= \bigcap_{\substack{k\in\mathbb{N}\\ \sigma\in\mathbb{N}^k}} \bigcup_{\substack{\tau\in\mathbb{N}^k\\ \tau\neq\sigma}} (I_\tau \times Y) \cup (\mathbb{N}^{\mathbb{N}} \times (Y \setminus H_\sigma))\end{aligned}$$

again belongs to $\mathcal{V}$, because $\mathcal{V}$ is closed under countable unions and intersections and contains $I_\tau \times Y$ and $\mathbb{N}^{\mathbb{N}} \times (Y \setminus H_\sigma)$ for all σ, $\tau \in S^*$. **Q**

By 421F, $\mathcal{V}$ contains every Borel subset of $\mathbb{N}^{\mathbb{N}} \times Y$, so includes $\mathcal{S}(\mathcal{B}(\mathbb{N}^{\mathbb{N}} \times Y))$. So in this case we can apply 423M directly to get the result.

(b) Now suppose that X is any analytic space. If X is empty, the result is trivial. Otherwise, let $h : \mathbb{N}^{\mathbb{N}} \to X$ be a continuous surjection. Set $\tilde{h}(\phi, y) = (h(\phi), y)$ for $\phi \in \mathbb{N}^{\mathbb{N}}$, $y \in Y$, so that $\tilde{h} : \mathbb{N}^{\mathbb{N}} \times Y \to X \times Y$ is continuous. Set $\tilde{W} = \tilde{h}^{-1}[W]$. If $V \in \mathcal{B}(X \times Y)$ then $\tilde{h}^{-1}[V] \in \mathcal{B}(\mathbb{N}^{\mathbb{N}} \times Y)$ (4A3Cd), so $\tilde{W} \in \mathcal{S}(\mathcal{B}(\mathbb{N}^{\mathbb{N}} \times Y))$ (421Cc). By (a), $\tilde{W}[\mathbb{N}^{\mathbb{N}}] \in \mathrm{T}$ and there is a T-measurable function $g : \tilde{W}[\mathbb{N}^{\mathbb{N}}] \to \mathbb{N}^{\mathbb{N}}$ such that $(g(y), y) \in \tilde{W}$ for every $y \in \tilde{W}[\mathbb{N}^{\mathbb{N}}]$. It is now easy to check that $W[X] = \tilde{W}[\mathbb{N}^{\mathbb{N}}] \in \mathrm{T}$ (this is where we need to know that h is surjective), that $f = hg : W[X] \to X$ is T-measurable, and that $(f(y), y) \in W$ for every $y \in W[X]$, as required.

Remark This is a version of the **von Neumann-Jankow selection theorem**.

423O Corollary Let X and Y be analytic Hausdorff spaces, and $f : X \to Y$ a Borel measurable surjection. Let T be the σ-algebra of subsets of Y generated by the Souslin-F sets in Y. Then there is a T-measurable function $g : Y \to X$ such that fg is the identity on Y.

proof The point is that the graph $\Gamma = \{(x, f(x)) : x \in X\}$ of f is analytic (423Ga), therefore a Souslin-F set in $X \times Y$ (423Eb). Also $\Gamma[X] = f[X] = Y$. By 423N, there is a T-measurable function $g : Y \to X$ such that $(g(y), y) \in \Gamma$, that is, $f(g(y)) = y$, for every $y \in Y$.

***423P Constituents of coanalytic sets: Theorem** Let X be a Hausdorff space, and $A \subseteq X$ an analytic subset of X. Then there is a non-decreasing family $\langle E_\xi \rangle_{\xi<\omega_1}$ of Borel subsets of X, with union $X \setminus A$, such that every analytic subset of $X \setminus A$ is included in some E_ξ.

proof Put 422K(iii) and 423C together.

***423Q Remarks (a)** Let A be an analytic set in an analytic space X and $\langle E_\xi \rangle_{\xi<\omega_1}$ a family of Borel sets as in 423P. There is nothing unique about the E_ξ. But if $\langle E'_\xi \rangle_{\xi<\omega_1}$ is another such family, then every E'_ξ is an analytic subset of $X \setminus A$, by 423E, so is included in some E_η; and, similarly, every E_ξ is included in some E'_η. We therefore have a function $f : \omega_1 \to \omega_1$ such that $E'_\xi \subseteq E_{f(\xi)}$ and $E_\xi \subseteq E'_{f(\xi)}$ for every $\xi < \omega$. If we set $C = \{\xi : \xi < \omega_1, f(\eta) < \xi \text{ for every } \eta < \xi\}$, then C is a closed cofinal set in ω_1 (4A1Bd), and $\bigcup_{\eta<\xi} E_\eta = \bigcup_{\eta<\xi} E'_\eta$ for every $\xi \in C$. If $X \setminus A$ is itself analytic, that is, if A is a Borel set, then we shall have to have $X \setminus A = E_\xi = E'_\xi$ for some $\xi < \omega_1$.

Another way of expressing the result in 423P is to say that if we write $\mathcal{I} = \{B : B \subseteq X \setminus A \text{ is analytic}\}$, then $\{E : E \in \mathcal{I}, E \text{ is Borel}\}$ is cofinal with $\mathcal{I}$ (this is the First Separation Theorem) and $\mathrm{cf}\mathcal{I} \leq \omega_1$.

(b) It is a remarkable fact that, in some models of set theory, we can have non-Borel coanalytic sets in Polish spaces such that all their constituents are countable (Jech 78, p. 529, Cor. 2). (Note that, by (a), this is the same thing as saying that $X \setminus A$ is uncountable but all its Borel subsets are countable.) But in 'ordinary' cases we shall have, for every Borel subset E of $X \setminus A$, an uncountable Borel subset of $(X \setminus A) \setminus E$; so that for any family $\langle G_\xi \rangle_{\xi<\omega_1}$ of Borel constituents of $X \setminus A$, there must be uncountably many uncountable G_ξ. To see that this happens at least sometimes, take any non-Borel analytic subset A_0 of $\mathbb{N}^{\mathbb{N}}$ (423L), and consider $A = A_0 \times \mathbb{N}^{\mathbb{N}} \subseteq (\mathbb{N}^{\mathbb{N}})^2$. Then A is analytic (423B). If $E \subseteq (\mathbb{N}^{\mathbb{N}})^2 \setminus A$ is Borel, then

$\pi_1[E] = \{x : (x,y) \in E\}$ is an analytic subset of $\mathbb{N}^{\mathbb{N}} \setminus A$, so is not the whole of $\mathbb{N}^{\mathbb{N}} \setminus A_0$ (by 423Fa). Taking any $x \in (\mathbb{N}^{\mathbb{N}} \setminus A_0) \setminus \pi_1[E]$, $\{x\} \times \mathbb{N}^{\mathbb{N}}$ is an uncountable Borel subset of $((\mathbb{N}^{\mathbb{N}})^2 \setminus A) \setminus E$. For an alternative construction, see 423Ye.

423X Basic exercises >(a) For a Hausdorff space X, show that the following are equiveridical: (i) X is analytic; (ii) X is a continuous image of a Polish space; (iii) X is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$.

(b) Write out a direct proof of 423Ea, not quoting 423C or 421D.

>(c) Let X be a set and $\mathfrak{S}$ a Hausdorff topology on X, $\mathfrak{T}$ an analytic topology on X such that $\mathfrak{S} \subseteq \mathfrak{T}$. Show that $\mathfrak{S}$ and $\mathfrak{T}$ have the same analytic sets. (*Hint*: 423F.)

>(d) Let X be an analytic Hausdorff space. (i) Show that its Borel σ-algebra $\mathcal{B}(X)$ is countably generated as σ-algebra. (*Hint*: use 4A2Nf and 423Fb.) (ii) Show that there is an analytic subset Y of $\mathbb{R}$ such that $(X, \mathcal{A}_X)$ is isomorphic to $(Y, \mathcal{A}_Y)$, where $\mathcal{A}_X$, $\mathcal{A}_Y$ are the families of Souslin-F subsets of X, Y respectively. (*Hint*: show that there is an injective Borel measurable function from X to $\mathbb{R}$ (cf. 343E), and use 423G.) (iii) Show that $(X, \mathcal{B}(X))$ is isomorphic to $(Y, \mathcal{B}(Y))$, where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y. (*Hint*: 423Fa.) (iv) Let T_X, T_Y be the σ-algebras generated by $\mathcal{A}_X$, $\mathcal{A}_Y$ respectively. Show that (X, T_X) and (Y, T_Y) are isomorphic.

(e) Let $\mathfrak{S}$ be the right-facing Sorgenfrey topology on $\mathbb{R}$ (415Xc). Show that $\mathfrak{S}$ has the same Borel sets as the usual topology $\mathfrak{T}$ on $\mathbb{R}$. (*Hint*: $\mathfrak{S}$ is hereditarily Lindelöf (419Xf) and has a base consisting of $\mathfrak{T}$-Borel sets.) Show that $\mathfrak{S}$ is not analytic.

(f) Let X be an analytic Hausdorff space and Y any topological space. Let T be the σ-algebra of subsets of Y generated by the Souslin-F sets. Show that if $W \subseteq X \times Y$ is Souslin-F, then $W[X] \in \mathrm{T}$ and there is a T-measurable function $f : W[X] \to Y$ such that $(f(y), y) \in W$ for every $y \in W[X]$. (*Hint*: start with $X = \mathbb{N}^{\mathbb{N}}$, as in 423N.)

(g) Let X and Y be Hausdorff spaces, and $R \subseteq X \times Y$ an analytic set such that $R^{-1}[Y] = X$. Show that there is a function $g : X \to Y$, measurable with respect to the σ-algebra generated by the Souslin-F subsets of X, such that $(x, g(x)) \in R$ for every $x \in X$.

>(h) (i) Show that the family of analytic subsets of $[0,1]$ has cardinal $\mathfrak{c}$. (*Hint*: 421Xd.) (ii) Show that the σ-algebra T of subsets of $[0,1]$ generated by the analytic sets has cardinal $\mathfrak{c}$. (*Hint*: 421Xi.) (iii) Show that there is a set $A \subseteq [0,1]$ which does not belong to T.

(i) Let $X = Y = [0,1]$. Give Y the usual topology, and give X the topology corresponding to the one-point compactification of the discrete topology on $[0,1[$, that is, a set $G \subseteq X$ is open if either $1 \notin G$ or G is cofinite. Show that the identity map $f : X \to Y$ is a Borel measurable bijection, but that f^{-1} is not measurable for the σ-algebra of subsets of Y generated by the Souslin-F sets.

423Y Further exercises (a) Show that a space with a countable network is **hereditarily separable** (that is, every subset is separable), therefore countably tight.

(b) Show that if X is a Hausdorff space with a countable network, then every analytic subset of X is obtainable by Souslin's operation from the open subsets of X.

(c) Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be analytic Hausdorff spaces and $f : X \to Y$ a Borel measurable function. (i) Show that there is a zero-dimensional separable metrizable topology $\mathfrak{S}'$ on Y with the same Borel sets and the same analytic sets as $\mathfrak{S}$. (*Hint*: 423Xd.) (ii) Show that there is a zero-dimensional separable metrizable topology $\mathfrak{T}'$ on X, with the same Borel sets and the same analytic sets as $\mathfrak{T}$, such that f is continuous for the topologies $\mathfrak{T}'$, $\mathfrak{S}'$. (iii) Explain how to elaborate these ideas to deal with any countable family of analytic spaces and Borel measurable functions between them.

(d) Let X be an analytic Hausdorff space, Y a Hausdorff space with a countable network, and $f : X \to Y$ a Borel measurable surjection. Let T be the σ-algebra of subsets of Y generated by the Souslin-F sets in Y. Show that there is a T-measurable function $g : Y \to X$ such that fg is the identity on Y.

(e) Set $S = \bigcup_{n\geq 1} \mathbb{N}^n$ and consider $\mathcal{P}S$ with its usual topology. Let $\mathcal{T} \subseteq \mathcal{P}S$ be the set of trees (412O); show that $\mathcal{T}$ is closed, therefore a compact metrizable space. Set $F_\sigma = \{T : \sigma \in T \in \mathcal{T}\}$ for $\sigma \in S$, and let A be the kernel of the Souslin scheme $\langle F_\sigma\rangle_{\sigma\in s}$. Show that the constituents of $\mathcal{T} \setminus A$ for this scheme are just the sets $G_\xi = \{T : r(T) = \xi\}$, where r is the rank function of 421Ne. Show by induction on ξ that all the G_ξ is non-empty, so that A is not a Borel set. Show that $\#(G_\xi) = \mathfrak{c}$ for $1 \leq \xi < \omega_1$. Show that if X is any topological space and $B \subseteq X$ is a Souslin-F set, there is a Borel measurable function $f : X \to \mathcal{T}$ such that $B = f^{-1}[A]$.

423 Notes and comments We have been dealing, in this section and the last, with three classes of topological space: the class of analytic spaces, the class of K-analytic spaces and the class of spaces with countable networks. The first is more important than the other two put together, and I am sure many people would find it more comfortable, if more time-consuming, to learn the theory of analytic spaces thoroughly first, before proceeding to the others. This was indeed my own route into the subject. But I think that the theory of K-analytic spaces has now matured to the point that it can stand on its own, without constant reference to its origin as an extension of descriptive set theory on the real line; and that our understanding of analytic spaces is usefully advanced by seeing how easily their properties can be deduced from the fact that they are K-analytic spaces with countable networks.

'Selection theorems' appear everywhere in mathematics. The axiom of choice is a selection theorem: if $\langle A_i\rangle_{i\in I}$ is a family of non-empty sets, there is a selector for the relation $\{(i, x) : i \in I,\ x \in A_i\}$. The Lifting Theorem (§341) is a special kind of selection theorem, where we seek a selector which is a Boolean homomorphism. In general topology we look for continuous selectors; in measure theory, for measurable selectors. Any selection theorem will have expressions either as a theorem on right inverses of functions, as in 423O, or as a theorem on selectors for relations, as in 423M-423N. In the language of this section, however, the strongest results are most easily set out in terms of relations, because the essence of the method is that we can find selectors taking values in analytic spaces, and the relations of 423M-423N can be very far from being analytic spaces in themselves, even when we have a natural topology on the space Y. The value of these results in measure theory will become clearer in §431, where we shall see that there are important σ-algebras which are closed under Souslin's operation, so that they can include the algebras T of 423M-423O. Typical applications are in

As in §422, I have made no attempt to cover the general theory of analytic spaces, nor even to give a balanced introduction. I have tried instead to give a condensed account of the principal methods for showing that spaces are analytic, with some of the ideas which can be applied to make them more accessible to the imagination (423J, 423Xc-423Xd, 423Yb-423Yc). Lusin's theorem 423I does not mention 'analytic' sets in its statement, but it depends essentially on the separation theorem 422J, so cannot really be put with the other results on Polish spaces in 4A2Q. You must of course know that not all analytic sets are Borel (423L) and that not all sets are analytic (423Xh). For further information about this fascinating subject, see Rogers 80, Kechris 95 and Moschovakis 80.

'Selection theorems' appear everywhere in mathematics. The axiom of choice is a selection theorem; it says that whenever $R \subseteq X \times Y$ is a relation and $R[X] = Y$, there is a function $f : Y \to X$ such that $(f(y), y) \in R$ for every $y \in Y$. The Lifting Theorem (§341) asks for a selector which is a Borel homomorphism. In general topology we look for continuous selectors. In measure theory, naturally, we are interested in measurable selectors, as in 423M-423O. Any selection theorem will have expressions either as a theorem on right inverses of functions, as in 423O, or as a theorem on selectors for relations, as in 423M-423N. In the language here, however, we get better theorems by examining relations, because the essence of the method is that we can find measurable functions into analytic spaces, and the relations of 423M can be very far from being analytic, even when there is a natural topology on the space Y. The value of these results will become clearer in §431, when we shall see that the σ-algebras T of 423M-423O are often included in familiar σ-algebras. Typical applications are in 433F-433G below.

424 Standard Borel spaces

This volume is concerned with topological measure spaces, and it will come as no surprise that the topological properties of Polish spaces are central to the theory. But even from the point of view of unadorned measure theory, not looking for topological structures on the underlying spaces, it turns out that the Borel algebras of Polish spaces have a very special position. It will be useful later on to be able to refer to some fundamental facts concerning them.

424A Definition Let X be a set and Σ a σ-algebra of subsets of X. We say that (X, Σ) is a **standard Borel space** if there is a Polish topology on X for which Σ is the algebra of Borel sets.

Warning! Many authors reserve the phrase 'standard Borel space' for the case in which X is uncountable. I have seen the phrase 'Borel space' used for what I call a 'standard Borel space'.

424B Proposition (a) If (X, Σ) is a standard Borel space, then Σ is countably generated as σ-algebra of sets.

(b) If $\langle (X_i, \Sigma_i) \rangle_{i \in I}$ is a countable family of standard Borel spaces, then $(\prod_{i \in I} X_i, \widehat{\bigotimes}_{i \in I} \Sigma_i)$ (definition: 254E) is a standard Borel space.

(c) Let (X, Σ) and (Y, T) be standard Borel spaces and $f : X \to Y$ a (Σ, T)-measurable surjection. Then
(i) if $E \in \Sigma$ is such that $f[E] \cap f[X \setminus E] = \emptyset$, then $f[E] \in \mathrm{T}$;
(ii) $\mathrm{T} = \{F : F \subseteq Y, f^{-1}[F] \in \Sigma\}$;
(iii) if f is a bijection it is an isomorphism.

(d) Let (X, Σ) and (Y, T) be standard Borel spaces and $f : X \to Y$ a (Σ, T)-measurable injection. Then $Z = f[X] \in \mathrm{T}$ and f is an isomorphism between (X, Σ) and (Z, T_Z), where T_Z is the subspace σ-algebra.

proof (a) Let $\mathfrak{T}$ be a Polish topology on X such that Σ is the algebra of Borel sets. Then $\mathfrak{T}$ has a countable base $\mathcal{U}$, which generates Σ (4A3Da/4A3E).

(b) For each $i \in I$ let $\mathfrak{T}_i$ be a Polish topology on X_i such that Σ_i is the algebra of $\mathfrak{T}_i$-Borel sets. Then $X = \prod_{i \in I} \mathfrak{T}_i$, with the product topology $\mathfrak{T}$, is Polish (4A2Qc). By 4A3Dc/4A3E, $\Sigma = \widehat{\bigotimes}_{i \in I} \Sigma_i$ is just the Borel σ-algebra of X, so (X, Σ) is a standard Borel space.

(c) Let $\mathfrak{T}$, $\mathfrak{S}$ be Polish topologies on X, Y respectively for which Σ and T are the Borel σ-algebras. Then f is Borel measurable.

(i) By 423Gb and 423Eb, $f[E]$ and $f[X \setminus E]$ are analytic subsets of Y. But they are complementary, so they are Borel sets, by 423Fa.

(ii) $f^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$, just because f is measurable. On the other hand, if $F \subseteq Y$ and $E = f^{-1}[F] \in \Sigma$, then $F = f[E] \in \mathrm{T}$ by (i).

(iii) follows at once.

(d) Give X and Y Polish topologies for which Σ, T are the Borel σ-algebras. By 423Ib, $f[E] \in \mathrm{T}$ for every $E \in \Sigma$; in particular, $Z = f[X]$ belongs to T. Also $f^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}_Z$, so f is an isomorphism between (X, Σ) and (Z, T_Z).

424C Theorem Let (X, Σ) be a standard Borel space.

(a) If X is countable then $\Sigma = \mathcal{P}X$.

(b) If X is uncountable then (X, Σ) is isomorphic to $(\mathbb{N}^{\mathbb{N}}, \mathcal{B}(\mathbb{N}^{\mathbb{N}}))$, where $\mathcal{B}(\mathbb{N}^{\mathbb{N}})$ is the algebra of Borel subsets of $\mathbb{N}^{\mathbb{N}}$.

proof Let $\mathfrak{T}$ be a Polish topology on X such that Σ is its Borel σ-algebra.

(a) Every singleton subset of X is closed, so must belong to Σ. If X is countable, every subset of X is a countable union of singletons, so belongs to Σ.

(b) (Rao & Srivastava 94) The strategy of the proof is to find Borel sets $Z \subseteq X$, $W \subseteq \mathbb{N}^{\mathbb{N}}$ such that $(Z, \Sigma_Z) \cong (\mathbb{N}^{\mathbb{N}}, \mathcal{B}(\mathbb{N}^{\mathbb{N}}))$ and $(W, \mathcal{B}(W)) \cong (X, \Sigma)$ (writing Σ_Z, $\mathcal{B}(W)$ for the subspace σ-algebras), and use a form of the Schröder-Bernstein theorem.

(i) By 423J, X has a subset Z homeomorphic to $\mathbb{N}^{\mathbb{N}}$; let $h : \mathbb{N}^{\mathbb{N}} \to Z$ be a homeomorphism. By 424Bd, h is an isomorphism between $(\mathbb{N}^{\mathbb{N}}, \mathcal{B}(\mathbb{N}^{\mathbb{N}}))$ and (Z, Σ_Z).

(ii) Let $\langle U_n \rangle_{n\in\mathbb{N}}$ run over a base for the topology of X. Define $g : X \to \{0,1\}^{\mathbb{N}} \subseteq \mathbb{N}^{\mathbb{N}}$ by setting $g(x) = \langle \chi U_n(x) \rangle_{n\in\mathbb{N}}$ for every $x \in \mathbb{N}$. Then g is injective, because X is Hausdorff. Also g is Borel measurable, by 4A3D(c-ii). By 424Bd, g is an isomorphism between (X, Σ) and $(W, \mathcal{B}(W))$, where $W = g[X]$ belongs to $\mathcal{B}(\mathbb{N}^{\mathbb{N}})$.

(iii) We have $Z \in \Sigma$, $W \in \mathcal{B}(\mathbb{N}^{\mathbb{N}})$ such that $(Z, \Sigma_Z) \cong (\mathbb{N}^{\mathbb{N}}, \mathcal{B}(\mathbb{N}^{\mathbb{N}}))$ and $(W, \mathcal{B}(W)) \cong (X, \Sigma)$. By 344D, $(X, \Sigma) \cong (\mathbb{N}^{\mathbb{N}}, \mathcal{B}(\mathbb{N}^{\mathbb{N}}))$, as claimed.

424D Corollary (a) If (X, Σ) and (Y, T) are standard Borel spaces and $\#(X) = \#(Y)$, then (X, Σ) and (Y, T) are isomorphic.

(b) If (X, Σ) is an uncountable standard Borel space then $\#(X) = \#(\Sigma) = \mathfrak{c}$.

proof These follow immediately from 424C, if we recall that if $\mathcal{B}(\mathbb{N}^{\mathbb{N}})$ is the Borel σ-algebra of $\mathbb{N}^{\mathbb{N}}$ then $\#(\mathcal{B}(\mathbb{N}^{\mathbb{N}})) = \mathfrak{c}$ (4A3Fb).

424E Proposition Let X be a set and Σ a σ-algebra of subsets of X; suppose that (X, Σ) is countably separated in the sense that there is a countable set $\mathcal{E} \subseteq \Sigma$ separating the points of X. If $A \subseteq X$ is such that (A, Σ_A) is a standard Borel space, where Σ_A is the subspace σ-algebra, then $A \in \Sigma$.

proof Give A a Polish topology $\mathfrak{T}$ such that Σ_A is the Borel σ-algebra of A, and let $\mathfrak{S}$ be the topology on X generated by $\mathcal{E} \cup \{X \setminus E : E \in \mathcal{E}\}$. Then $\mathfrak{S}$ is second-countable (4A2Oa), so has a countable network (4A2Oc), and is Hausdorff because $\mathcal{E}$ separates the points of X. The identity map from A to X is Borel measurable for $\mathfrak{T}$ and $\mathfrak{S}$, so 423Ib tells us that A is $\mathfrak{S}$-Borel; but of course the $\mathfrak{S}$-Borel σ-algebra is just the σ-algebra generated by $\mathcal{E}$ (4A3Da), so is included in Σ.

424F Corollary Let X be a Polish space and $A \subseteq X$ any set which is not Borel. Let $\mathcal{B}(A)$ be the Borel σ-algebra of A. Then $(A, \mathcal{B}(A))$ is not a standard Borel space.

424G Proposition Let (X, Σ) be a standard Borel space. Then (E, Σ_E) is a standard Borel space for every $E \in \Sigma$, writing Σ_E for the subspace σ-algebra.

proof Let $\mathfrak{T}$ be a Polish topology on X for which Σ is the Borel σ-algebra. Then there is a Polish topology $\mathfrak{T}' \supseteq \mathfrak{T}$ for which E is closed (4A3I), therefore itself Polish in the subspace topology $\mathfrak{T}'_E$ (4A2Qd). But $\mathfrak{T}'$ and $\mathfrak{T}$ have the same Borel sets (423Fb), so Σ_E is just the Borel σ-algebra of E for $\mathfrak{T}'_E$, and (E, Σ_E) is a standard Borel space.

***424H** For the full strength of a theorem in §448 we need a remarkable result concerning group actions on Polish spaces.

Theorem (Becker & Kechris 96) Let G be a Polish group, $(X, \mathfrak{T})$ a Polish space and $\bullet$ a Borel measurable action of G on X. Then there is a Polish topology $\mathfrak{T}'$ on X, yielding the same Borel sets as $\mathfrak{T}$, such that the action is continuous for $\mathfrak{T}'$ and the given topology of G.

proof (a) Fix on a right-translation-invariant metric ρ on G defining the topology of G (4A5Q), and let D be a countable dense subset of G; write e for the identity of G. Let Z be the set of 1-Lipschitz functions from G to $[0,1]$, that is, functions $f : G \to [0,1]$ such that $|f(g) - f(h)| \leq \rho(g,h)$ for all g, $h \in G$. Then Z, with the topology of pointwise convergence inherited from the product topology of $[0,1]^G$, is a compact metrizable space. **P** It is a closed subset of $[0,1]^G$, so is a compact Hausdorff space. Writing $q(f) = f \restriction D$ for $f \in Z$, $q : Z \to [0,1]^D$ is injective, because D is dense and every member of Z is continuous; but this means that Z is homeomorphic to $q[Z]$, which is metrizable, by 4A2Pc. **Q**

We see also that the Borel σ-algebra of Z is the σ-algebra generated by sets of the form $W_{g\alpha} = \{f : f(g) < \alpha\}$ where $g \in G$ and $\alpha \in [0,1]$. **P** This σ-algebra contains every set of the form $\{f : f \in Z, \alpha < f(g) < \beta\}$, where $g \in G$ and α, $\beta \in \mathbb{R}$; since these sets generate the topology of Z, the σ-algebra they generate is the Borel σ-algebra of Z, by 4A3Da. **Q**

(b) There is a continuous action of G on Z defined by setting

$$(g\bullet f)(h) = f(hg)$$

for $f \in Z$ and $g, h \in G$. **P** (i) If $f \in Z$ and $g, h_1, h_2 \in G$, then

$$|(g\bullet f)(h_1) - (g\bullet f)(h_2)| = |f(h_1g) - f(h_2g)| \leq \rho(h_1g, h_2g) = \rho(h_1, h_2)$$

because ρ is right-translation-invariant. So $g\bullet f \in Z$ for every $f \in Z$, $g \in G$. (ii)

$$(e\bullet f)(h) = f(he) = f(h)$$

for every $h \in G$, so $e\bullet f = f$ for every $f \in Z$. (iii) If $f \in Z$ and $g_1, g_2, h \in G$, then

$$(g_1\bullet(g_2\bullet f))(h) = (g_2\bullet f)(hg_1) = f(hg_1g_2) = ((g_1g_2)\bullet f)(h);$$

as h is arbitrary, $(g_1g_2)\bullet f = g_1\bullet(g_2\bullet f)$. Thus $\bullet$ is an action of G on Z. (iv) Suppose that $g_0 \in G$, $f_0 \in Z$, $h \in G$ and $\epsilon > 0$. Set

$$V = \{g : g \in G, \rho(hg, hg_0) < \tfrac{1}{2}\epsilon\},$$

$$W = \{f : f \in Z, |f(hg_0) - f_0(hg_0)| < \tfrac{1}{2}\epsilon\}.$$

Then V is an open set in G containing g_0 (because $g \mapsto \rho(hg, hg_0)$ is continuous) and W is an open set in Z containing f_0. If $g \in V$ and $f \in W$,

$$\begin{aligned}|(g\bullet f)(h) - (g_0\bullet f_0)(h)| &= |f(hg) - f_0(hg_0)| \\ &\leq |f(hg) - f(hg_0)| + |f(hg_0) - f_0(hg_0)| \\ &\leq \rho(hg, hg_0) + \tfrac{1}{2}\epsilon \leq \epsilon.\end{aligned}$$

As f_0, g_0, ϵ are arbitrary, the map $(g, f) \mapsto (g\bullet f)(h)$ is continuous; as h is arbitrary, the map $(g, f) \mapsto g\bullet f$ is continuous. **Q**

(c) Let $\mathcal{B}(X)$ be the Borel σ-algebra of X. For $x \in X$ and $B \in \mathcal{B}(X)$, set

$$P_B(x) = \{g : g \in G, g\bullet x \in B\},$$

$$Q_B(x) = \bigcup\{V : V \subseteq G \text{ is open, } V \setminus P_B(x) \text{ is meager}\},$$

$$f_B(x)(g) = \inf(\{1\} \cup \{\rho(g, h) : h \in G \setminus Q_B(x)\})$$

for $g \in G$. It is easy to check that

$$f_B(x)(g') \leq \rho(g, g') + f_B(x)(g')$$

for all $g, g' \in G$, so that every $f_B(x)$ belongs to Z.

Every $P_B(x)$ is a Borel set, because $\bullet$ is Borel measurable. It follows that $Q_B(x) \triangle P_B(x)$ is meager. **P** By 4A3Ra, there is an open set $V_0 \subseteq G$ such that $V_0 \triangle (X \setminus P_B(x))$ is meager and $V_0 \cap V$ is empty whenever $V \subseteq G$ is open and $V \setminus P_B(x)$ is meager. Now $Q_B(x) \cap V_0$ is empty, so $Q_B(x) \setminus P_B(x) \subseteq (X \setminus P_B(x)) \setminus V_0$ is meager. Also

$$(X \setminus \overline{V}_0) \setminus P_B(x) \subseteq (X \setminus P_B(x)) \setminus V_0$$

is meager, so $X \setminus \overline{V}_0 \subseteq Q_B(x)$ and

$$P_B(x) \setminus Q_B(x) \subseteq (\overline{V}_0 \setminus V_0) \cup (V_0 \setminus (X \setminus P_B(x))$$

is meager. **Q** Consequently $Q_B(x) \setminus \overline{P_B(x)}$ is empty, because the only meager open set in G is empty, by Baire's theorem for complete metric spaces (4A2Ma).

Let $\mathcal{V}$ be a countable base for the topology of G containing G.

(d) For each $B \in \mathcal{B}(X)$, the map $f_B : X \to Z$ is Borel measurable. **P** Because the Borel σ-algebra of Z is generated by the sets $W_{g\alpha}$ of (a) above, it is enough to show that

$$\{x : f_B(x) \in W_{g\alpha}\} = \{x : f_B(x)(g) < \alpha\}$$

always belongs to $\mathcal{B}(X)$, because $\{W : f_B^{-1}[W] \in \mathcal{B}\}$ is surely a σ-algebra of subsets of Z. But if $\alpha > 1$ this set is X, while if $\alpha \leq 1$ it is

$$\begin{aligned}\{x :\ & \text{there is some } h \in X \setminus Q_B(x) \text{ such that } \rho(g,h) < \alpha\}\\ &= \{x :\ \text{there is some } V \in \mathcal{V} \text{ such that } V \setminus Q_B(x) \neq \emptyset\\ &\qquad\qquad \text{and } \rho(g,h) < \alpha \text{ for every } h \in V\}\\ &= \bigcup_{V \in \mathcal{V}'} \{x : V \not\subseteq Q_B(x)\}\end{aligned}$$

(where $\mathcal{V}' = \{V : V \in \mathcal{V},\ \rho(g,h) < \alpha \text{ for every } h \in V\}$)

$$= \bigcup_{V \in \mathcal{V}'} \{x : V \setminus P_B(x) \text{ is not meager}\}.$$

But for any fixed $V \in \mathcal{V}$,

$$\{x : V \setminus P_B(x) \text{ is not meager}\} = X \setminus \{x : W[\{x\}] \text{ is meager}\}$$

where

$$W = \{(y,g) : y \in X,\ g \in V,\ g{\bullet}y \in X \setminus B\}$$

is a Borel subset of $X \times G$, because $\bullet$ is supposed to be Borel measurable; and therefore $W \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(G)$, writing $\mathcal{B}(G)$ for the Borel σ-algebra of G (4A3G). Now $\mathcal{B}(G) \subseteq \widehat{\mathcal{B}}(G)$, the Baire property algebra of G (4A3Rb), so $W \in \mathcal{B}(X)\widehat{\otimes}\widehat{\mathcal{B}}(G)$. By 4A3Rc, the quotient algebra $\widehat{\mathcal{B}}/\mathcal{M}$ has a countable order-dense set, where $\mathcal{M}$ is the σ-ideal of meager sets, so 4A3Sa tells us that $\{x : W[\{x\}] \text{ is meager}\} \in \mathcal{B}(X)$. Accordingly $\{x : V \setminus P_B(x) \text{ is not meager}\}$ is Borel for every V, and $\{x : f_B(x)(g) < \alpha\}$ is Borel. **Q**

(e) If $g \in G$, $B \in \mathcal{B}(X)$ and $x \in X$, then $g{\bullet}f_B(x) = f_B(g{\bullet}x)$. **P**

$$P_B(g{\bullet}x) = \{h : h{\bullet}(g{\bullet}x) \in B\} = \{h : hg \in P_B(x)\} = P_B(x)g^{-1}.$$

Because the map $h \mapsto hg^{-1} : G \to G$ is a homeomorphism, $Q_B(g{\bullet}x) = Q_B(x)g^{-1}$; because it is an isometry,

$$\begin{aligned}f_B(g{\bullet}x)(h) &= \min(1, \rho(h, X \setminus Q_B(g{\bullet}x))) = \min(1, \rho(h, X \setminus Q_B(x)g^{-1}))\\ &= \min(1, \rho(hg, X \setminus Q_B(x))) = f_B(x)(hg) = (g{\bullet}f_B(x))(h)\end{aligned}$$

for every $h \in G$. **Q**

(f) Let $\langle B_{0m}\rangle_{m\in\mathbb{N}}$ run over a base for $\mathfrak{T}$ containing X. We can now find a countable set $\mathcal{E} \subseteq \mathcal{B}(X)$ such that (i) the topology $\mathfrak{T}^*$ generated by $\mathcal{E}$ is a Polish topology finer than $\mathfrak{T}$ (ii) f_B is $\mathfrak{T}^*$-continuous for every $B \in \mathcal{E}$ (iii) $X \in \mathcal{E}$. **P** Let $\mathcal{W}$ be a countable base for the topology of Z. Enumerate $\mathbb{N} \times \mathbb{N} \times \mathcal{W}$ as $\langle (k_n, m_n, W_n)\rangle_{n\in\mathbb{N}}$ in such a way that $k_n \leq n$ for every n. Having chosen Borel sets $B_{ij} \subseteq X$ for $i \leq n$, $j \in \mathbb{N}$ in such a way that the topology $\mathfrak{S}_n$ generated by $\{B_{ij} : i \leq n,\ j \in \mathbb{N}\}$ is a Polish topology finer than $\mathfrak{T}$, consider the set

$$C_n = \{x : f_{B_{k_n,m_n}}(x) \in W_n\}.$$

This is $\mathfrak{T}$-Borel, therefore $\mathfrak{S}_n$-Borel, so by 4A3H there is a Polish topology $\mathfrak{S}_{n+1} \supseteq \mathfrak{S}_n$ such that $C_n \in \mathfrak{S}_{n+1}$. Let $\langle B_{n+1,m}\rangle_{m\in\mathbb{N}}$ run over a base for $\mathfrak{S}_{n+1}$; by 423Fb, every $B_{n+1,m}$ belongs to $\mathcal{B}(X)$. Continue.

Let $\mathfrak{T}^*$ be the topology generated by $\mathcal{E} = \{B_{ij} : i, j \in \mathbb{N}\}$. By 4A2Qf, $\mathfrak{T}^*$ is Polish, because it is the topology generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{S}_n$. If $W \in \mathcal{W}$, $B \in \mathcal{E}$ there are $i, j \in \mathbb{N}$ such that $B = B_{ij}$ and an $n \in \mathbb{N}$ such that $(i,j,W) = (k_n, m_n, W_n)$; now

$$f_B^{-1}[W] = C_n \in \mathfrak{S}_{n+1} \subseteq \mathfrak{T}^*.$$

As W is arbitrary, f_B is $\mathfrak{T}^*$-continuous. Also

$$X \in \{B_{0m} : m \in \mathbb{N}\} \subseteq \mathcal{E},$$

as required. **Q**

(g) Define $\theta : X \to Z^{\mathcal{E}}$ by setting $\theta(x)(E) = f_E(x)$ for $x \in X$, $E \in \mathcal{E}$. Then θ is injective. **P** Suppose that x, $y \in X$ and that $x \neq y$. For every $g \in G$,

$$g^{-1}\bullet(g\bullet x) = x \neq y = g^{-1}\bullet(g\bullet y),$$

so $g\bullet x \neq g\bullet y$ and there is some $m \in \mathbb{N}$ such that $g\bullet x \in B_{0m}$ while $g\bullet y \notin B_{0m}$, that is, $g \in P_{B_{0m}}(x) \setminus P_{B_{0m}}(y)$. Thus $G = \bigcup_{m\in\mathbb{N}} P_{B_{0m}}(x) \setminus P_{B_{0m}}(y)$; by Baire's theorem, there is some $m \in \mathbb{N}$ such that $P_{B_{0m}}(x) \setminus P_{B_{0m}}(y)$ is non-meager. Because $Q_{B_{0m}}(x) \triangle P_{B_{0m}}(x)$ and $Q_{B_{0m}}(y) \triangle P_{B_{0m}}(y)$ are both meager,

$$\{g : f_{B_{0m}}(x)(g) > 0\} = Q_{B_{0m}}(x) \neq Q_{B_{0m}}(y) = \{g : f_{B_{0m}}(y)(g) > 0\},$$

and

$$\theta(x)(B_{0m}) = f_{B_{0m}}(x) \neq f_{B_{0m}}(y) = \theta(y)(B_{0m}).$$

So $\theta(x) \neq \theta(y)$. **Q** Because f_E is $\mathfrak{T}^*$-continuous for every $E \in \mathcal{E}$, θ is $\mathfrak{T}^*$-continuous.

(h) Let $\mathfrak{T}'$ be the topology on X induced by θ; that is, the topology which renders θ a homeomorphism between X and $\theta[X]$. Because $\theta[X] \subseteq Z^{\mathcal{E}}$ is separable and metrizable, $\mathfrak{T}'$ is separable and metrizable. Because θ is $\mathfrak{T}^*$-continuous, $\mathfrak{T}' \subseteq \mathfrak{T}^*$.

(i) The action of G on X is continuous for the given topology $\mathfrak{S}$ on G and $\mathfrak{T}'$ on X. **P** For any $E \in \mathcal{E}$,

$$(g, x) \mapsto \theta(g\bullet x)(E) = f_E(g\bullet x) = g\bullet f_E(x)$$

((e) above) is $\mathfrak{S} \times \mathfrak{T}'$-continuous because the action of G on Z is continuous ((b) above) and $f_E : X \to Z$ is $\mathfrak{T}'$-continuous (by the definition of $\mathfrak{T}'$). But this means that $(g, x) \mapsto \theta(g\bullet x)$ is $\mathfrak{S} \times \mathfrak{T}'$-continuous, so that $(g, x) \mapsto g\bullet x$ is $(\mathfrak{S} \times \mathfrak{T}', \mathfrak{T}')$-continuous. **Q**

(j) Let σ be a complete metric on G defining the topology $\mathfrak{S}$, and τ a complete metric on X defining the topology $\mathfrak{T}^*$. (We do not need to relate σ to ρ in any way beyond the fact that they both give rise to the same topology $\mathfrak{S}$.) For $E \in \mathcal{E}$, $V \in \mathcal{V}$ and $n \in \mathbb{N}$ let S_{EVn} be the set of those $\phi \in Z^{\mathcal{E}}$ such that *either* $\phi(E)(g) = 0$ for every $g \in V$ *or* there is an $F \in \mathcal{E}$ such that $F \subseteq E$, $\operatorname{diam}_\tau(F) \leq 2^{-n}$ and $\phi(F)(g) > 0$ for some $g \in V$. Then S_{EVn} is the union of a closed set and an open set, so is a G_δ set in $Z^{\mathcal{E}}$ (4A2C(a-i)). Consequently

$$Y = \{\phi : \phi \in Z^{\mathcal{E}},\ \phi(X) = \chi G\} \cap \overline{\theta[X]} \cap \textstyle\bigcap_{E\in\mathcal{E},V\in\mathcal{V},n\in\mathbb{N}} S_{EVn}$$

is a G_δ subset of $Z^{\mathcal{E}}$, being the intersection of countably many G_δ sets.

(k) $\theta[X] \subseteq Y$. **P** Let $x \in X$. (i) $P_X(x) = G$ so $Q_X(x) = G$ and $\theta(x)(X) = f_X(x) = \chi G$. (ii) Of course $\theta(x) \in \overline{\theta[X]}$. (iii) Suppose that $E \in \mathcal{E}$, $V \in \mathcal{V}$ and $n \in \mathbb{N}$. If $Q_E(x) \cap V = \emptyset$ then $\theta(x)(E)(g) = f_E(x)(g) = 0$ for every $g \in V$, and $\theta(x)(E) \in S_{EVn}$. Otherwise, $V \cap P_E(x)$ is non-meager. But

$$E = \textstyle\bigcup\{F : F \in \mathcal{E},\ F \subseteq E,\ \operatorname{diam}_\tau(F) \leq 2^{-n}\},$$

so

$$P_E(x) = \textstyle\bigcup\{P_F(x) : F \in \mathcal{E},\ F \subseteq E,\ \operatorname{diam}_\tau(F) \leq 2^{-n}\};$$

because $\mathcal{E}$ is countable, this is a countable union and there is an $F \in \mathcal{E}$ such that $F \subseteq E$, $\operatorname{diam}_\tau(F) \leq 2^{-n}$ and $P_F(x) \cap V$ is non-meager. In this case $Q_F(x) \cap V$ is non-empty and $\theta(x)(F) = f_F(x)$ is non-zero at some point of V; thus again $\theta(x) \in S_{EVn}$. As E, V and n are arbitrary, we have the result. **Q**

(l) (The magic bit.) $Y \subseteq \theta[X]$. **P** Take any $\phi \in Y$. Choose $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{V}$ and $\langle \tilde{g}_n\rangle_{n\in\mathbb{N}}$ in G as follows. $E_0 = X$ and $V_0 = G$. Given that $\phi(E_n)$ is non-zero at some point of V_n, then, because $\phi \in S_{E_nV_nn}$, there is an $E_{n+1} \in \mathcal{E}$ such that $E_{n+1} \subseteq E_n$, $\operatorname{diam}_\tau(E_{n+1}) \leq 2^{-n}$ and $\phi(E_{n+1})$ is non-zero at some point of V_n; say $\tilde{g}_n \in V_n$ is such that $\phi(E_{n+1})(\tilde{g}_n) > 0$. Now we can find a $V_{n+1} \in \mathcal{V}$ such that $\tilde{g}_n \in V_{n+1} \subseteq V_n$ and $\operatorname{diam}_\sigma(V_{n+1}) \leq 2^{-n}$. Continue.

We are supposing also that $\phi \in \overline{\theta[X]}$, so we have a sequence $\langle x_i\rangle_{i\in\mathbb{N}}$ in X such that $\langle\theta(x_i)\rangle_{i\in\mathbb{N}} \to \phi$, that is, $\langle f_E(x_i)\rangle_{i\in\mathbb{N}} \to \phi(E)$ for every $E \in \mathcal{E}$. In particular,

$$\lim_{i\to\infty} f_{E_{n+1}}(x_i)(\tilde{g}_n) = \phi(E_{n+1})(\tilde{g}_n) > 0$$

for every n. Let $\langle i_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that $f_{E_{n+1}}(x_{i_n})(\tilde{g}_n) > 0$ for every n. Then

$$\tilde{g}_n \in V_{n+1} \cap Q_{E_{n+1}}(x_{i_n}) \subseteq V_{n+1} \cap \overline{P_{E_{n+1}}(x_{i_n})};$$

there is therefore some $g_n \in V_{n+1} \cap P_{E_{n+1}}(x_{i_n})$, so that $g_n \bullet x_{i_n} \in E_{n+1}$.

$\langle V_n \rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of sets with σ-diameters converging to 0, so $\langle g_n \rangle_{n\in\mathbb{N}}$ is a Cauchy sequence for the complete metric σ. Similarly, $\langle g_n \bullet x_{i_n} \rangle_{n\in\mathbb{N}}$ is a Cauchy sequence for the complete metric τ, because $\operatorname{diam}_\tau(E_{n+1}) \leq 2^{-n}$. We therefore have $g \in G$, $y \in X$ such that $\langle g_n \rangle_{n\in\mathbb{N}} \to g$ for $\mathfrak{S}$ and $\langle g_n \bullet x_{i_n} \rangle_{n\in\mathbb{N}} \to y$ for $\mathfrak{T}^*$. In this case, $\langle g_n \bullet x_{i_n} \rangle_{n\in\mathbb{N}} \to y$ for the coarser topology $\mathfrak{T}'$, while $\langle g_n^{-1} \rangle_{n\in\mathbb{N}} \to g^{-1}$ for $\mathfrak{S}$. Because the action is $(\mathfrak{S} \times \mathfrak{T}', \mathfrak{T}')$-continuous,

$$\langle x_{i_n} \rangle_{n\in\mathbb{N}} = \langle g_n^{-1} \bullet (g_n \bullet x_{i_n}) \rangle_{n\in\mathbb{N}} \to g^{-1} \bullet y$$

for $\mathfrak{T}'$. But θ is continuous for $\mathfrak{T}'$, so

$$\theta(g^{-1} \bullet y) = \lim_{n\to\infty} \theta(x_{i_n}) = \phi,$$

and $\phi \in \theta[X]$. As ϕ is arbitrary, $Y \subseteq \theta[X]$. **Q**

(m) Thus $\theta[X] = Y$ is a G_δ set in the compact metric space $Z^{\mathcal{E}}$, and is a Polish space in its induced topology (4A2Qd). But this means that $(X, \mathfrak{T}')$, which is homeomorphic to $\theta[X]$, is also Polish.

(n) I have still to check that $\mathfrak{T}'$ has the same Borel sets as $\mathfrak{T}$. But $\mathfrak{T}$, $\mathfrak{T}^*$ and $\mathfrak{T}'$ are all Polish topologies and $\mathfrak{T}^*$ is finer than both the other two. By 423Fb, $\mathfrak{T}^*$ has the same Borel sets as either of the others.

This completes the proof.

424X Basic exercises >(a) Let $\langle (X_i, \Sigma_i) \rangle_{i\in I}$ be a countable family of standard Borel spaces, and (X, Σ) their direct sum, that is, $X = \{(x, i) : i \in I, x \in X_i\}$, $\Sigma = \{E : E \subseteq X, \{x : (x, i) \in E\} \in \Sigma_i \text{ for every } i\}$. Show that (X, Σ) is a standard Borel space.

>(b) Let (X, Σ) be a standard Borel space and T a countably generated σ-subalgebra of Σ. Show that there is an analytic Hausdorff space Z such that T is isomorphic to the Borel σ-algebra of Z. (*Hint*: by 4A3I, we can suppose that X is a Polish space and T is generated by a sequence of open-and-closed sets, corresponding to a continuous function from X to $\{0, 1\}^{\mathbb{N}}$.)

>(c) Let (X, Σ) be a standard Borel space and T_1, T_2 two countably generated σ-subalgebras of Σ which separate the same points, in the sense that if x, $y \in X$ then there is an $E \in \mathrm{T}_1$ such that $x \in E$, $y \in X \setminus E$ iff there is an $E' \in \mathrm{T}_2$ such that $x \in E'$, $y \in X \setminus E'$. Show that $\mathrm{T}_1 = \mathrm{T}_2$. (*Hint*: 424Xb, 423Fb.) In particular, if T_1 separates the points of X then $\mathrm{T}_1 = \Sigma$.

(d) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra. Show that $\mathfrak{A}$ is isomorphic to the Borel σ-algebra of an analytic Hausdorff space iff it is isomorphic to a countably generated σ-subalgebra of the Borel σ-algebra of $[0, 1]$.

>(e) Let U be a separable Banach space. Show that its Borel σ-algebra is generated, as σ-algebra, by the sets of the form $\{u : h(u) \leq \alpha\}$ as h runs over the dual U^* and α runs over $\mathbb{R}$.

>(f) Let X be a compact metrizable space. Show that the Borel σ-algebra of $C(X)$ (the Banach space of continuous real-valued functions on X) is generated, as σ-algebra, by the sets $\{u : u \in C(X), u(x) \geq \alpha\}$ as x runs over X and α runs over $\mathbb{R}$.

(g) Let (X, Σ) be a standard Borel space, Y any set, and T a σ-algebra of subsets of Y. Write T^* for the σ-algebra of subsets of Y generated by $\mathcal{S}(\mathrm{T})$, the family of sets obtainable from sets in T by Souslin's operation. Let $W \in \mathcal{S}(\Sigma \widehat{\otimes} \mathrm{T})$. Show that $W[X] \in \mathrm{T}^*$ and that there is a (T^*, Σ)-measurable function $f : W[X] \to X$ such that $(f(y), y) \in W$ for every $y \in W[X]$. (*Hint*: 423M.)

(h) Let (X, Σ) be a standard Borel space, Y any set, and T a countably generated σ-algebra of subsets of Y. Let $f : X \to Y$ be a (Σ, T)-measurable function, and write T^* for the σ-algebra of subsets of Y generated by $\mathcal{S}(\mathrm{T})$, the family of sets obtainable from sets in T by Souslin's operation. Show that (i) $f[X] \in \mathrm{T}^*$ (ii) there is a (T^*, Σ)-measurable function $g : f[X] \to X$ such that gf is the identity on X. (*Hint*: start with the case in which T separates the points of Y, so that the graph of f belongs to $\Sigma \widehat{\otimes} \mathrm{T}$.)

>(i) Show that 424Xc and 424Xh are both false if we omit the hypothesis 'countably generated'. (*Hint*: consider (i) the countable-cocountable algebra of $\mathbb{R}$ (ii) the split interval (iii) 423Xi.)

(j) Let (X, Σ) be a standard Borel space. Show that if X is uncountable, Σ has a countably generated σ-subalgebra not isomorphic either to Σ or to $\mathcal{P}I$ for any set I. (*Hint*: 423L.)

(k) Let (X, Σ) be a standard Borel space. (i) Show that the Boolean algebra $\mathfrak{A} = \Sigma/[X]^{\leq\omega}$ is homogeneous. (ii) Show that for every sequentially order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{A}$ there is a (Σ, Σ)-measurable $f : X \to X$ representing π in the sense that $\pi(E^\bullet) = (f^{-1}[E])^\bullet$ for every $E \in \Sigma$. (*Hint*: reduce to the case $X = \{0,1\}^{\mathbb{N}}$.) (iii) Show that for every $\pi \in \operatorname{Aut}\mathfrak{A}$ there is an automorphism f of (X, Σ) representing π. (*Hint*: 344Ya.)

424Y Further exercises (a) Let $(X, \mathfrak{T})$ be a Polish space and $\mathcal{F}$ the family of closed subsets of X. Let Σ be the σ-algebra of subsets of $\mathcal{F}$ generated by the sets $\mathcal{E}_H = \{F : F \in \mathcal{F}, F \cap H \neq \emptyset\}$ as H runs over the open subsets of X. (i) Show that $(\mathcal{F}, \Sigma)$ is a standard Borel space. (*Hint*: take a complete metric ρ defining the topology of X. Set $S^* = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ and choose a family $\langle U_\sigma\rangle_{\sigma\in S^*}$ of open sets in X such that $U_\emptyset = X$, $\operatorname{diam} U_\sigma \leq 2^{-n}$ whenever $\#(\sigma) = n+1$, $U_\sigma = \bigcup_{i\in\mathbb{N}} U_{\sigma^\frown i}$ for every σ, $\overline{U}_{\sigma^\frown i} \subseteq U_\sigma$ for every σ, i. Define $f : \mathcal{F} \to \{0,1\}^{S^*}$ by setting $f(F)(\sigma) = 1$ if $F \cap U_\sigma \neq \emptyset$, 0 otherwise. Show that $Z = f[\mathcal{F}]$ is a Borel set and that f is an isomorphism between Σ and the Borel σ-algebra of Z.) (This is the **Effros Borel structure** on $\mathcal{F}$.) (ii) Show that $[X]^n \in \Sigma$ for every $n \in \mathbb{N}$.

(b) Let (X, Σ) be a standard Borel space. Let $\mathbb{T}$ be the family of Polish topologies on X for which Σ is the Borel σ-algebra. Show that any sequence in $\mathbb{T}$ has an upper bound in $\mathbb{T}$, and that any sequence with a lower bound has a least upper bound.

(c) Let (X, Σ) be a standard Borel space. Say that $C \subseteq X$ is **coanalytic** if its complement belongs to $\mathcal{S}(\Sigma)$. Show that for any such C the partially ordered set $\Sigma \cap \mathcal{P}C$ has cofinality 1 if $C \in \Sigma$ and cofinality ω_1 otherwise. (*Hint*: 423P.)

(d) Let $I^{\|}$ be the split interval. Show that there is a σ-algebra Σ of subsets of $I^{\|}$ such that $(I^{\|}, \Sigma)$ is a standard Borel space and $\{(x,y) : x, y \in I^{\|}, x \leq y\} \in \Sigma\widehat{\otimes}\Sigma$.

424 Notes and comments In this treatise I have generally indulged my prejudice in favour of 'complete' measures. Consequently Borel σ-algebras, as such, have taken subordinate roles. But important parts of the theory of Lebesgue measure, and Radon measures on Polish spaces in general, are associated with the fact that these are completions of measures defined on standard Borel spaces. Moreover, such spaces provide a suitable framework for a large part of probability theory. Of course they become deficient in contexts where we need to look at uncountable independent families of random variables, and there are also difficulties with σ-subalgebras, even countably generated ones, since these can correspond to the Borel algebras of general analytic spaces, which will not always be standard Borel structures (424F, 423L). 424Xf and 424Ya suggest the ubiquity of standard Borel structures; the former shows that they are not always presented as countably generated algebras, while the latter is an example in which we have to make a special construction in order to associate a topology with the algebra. The theory is of course dominated by the results of §423, especially 423Fb and 423I.

I include 424H in this section because there is no other convenient place for it, but I have an excuse: the idea of 'Borel measurable action' can, in this context, be described entirely in terms of σ-algebras, since the Borel algebra of $G \times X$ is just the σ-algebra product of the Borel algebras of the factors (as in 424Bb). Of course for the theorem as expressed here we do need to know that G has a Polish group structure; but X could be presented just as a standard Borel space. The result is a dramatic expression of the fact that, given a standard Borel space (X, Σ), we have a great deal of freedom in defining a corresponding Polish topology on X.

Chapter 43

Topologies and measures II

The first chapter of this volume was 'general' theory of topological measure spaces; I attempted to distinguish the most important properties a topological measure can have – inner regularity, τ-additivity – and describe their interactions at an abstract level. I now turn to rather more specialized investigations, looking for features which offer explanations of the behaviour of the most important spaces, radiating outwards from Lebesgue measure.

In effect, this chapter consists of three distinguishable parts and two appendices. The first three sections are based on ideas from descriptive set theory, in particular Souslin's operation (§431); the properties of this operation are the foundation for the theory of two classes of topological space of particular importance in measure theory, the K-analytic spaces (§432) and the analytic spaces (§433). The second part of the chapter, §§434-435, collects miscellaneous results on Borel and Baire measures, looking at the ways in which topological properties of a space determine properties of the measures it carries. In §436 I present the most important theorems on the representation of linear functionals by integrals; if you like, this is the inverse operation to the construction of integrals from measures in §122. The ideas continue into §437, where I discuss spaces of signed measures representing the duals of spaces of continuous functions. The first appendix, §438, looks at a special topic: the way in which the patterns in §§434-435 are affected if we assume that our spaces are not unreasonably complex in a rather special sense defined in terms of measures on discrete spaces. Finally, I end the chapter with a further collection of examples, mostly to exhibit boundaries to the theorems of the chapter, but also to show some of the variety of the structures we are dealing with.

431 Souslin's operation

I begin the chapter with a short section on Souslin's operation (§421). The basic facts we need to know are that (in a complete locally determined measure space) the family of measurable sets is closed under Souslin's operation (431A), and that the kernel of a Souslin scheme can be approximated from within in measure (431D).

431A Theorem Let (X, Σ, μ) be a complete locally determined measure space. Then Σ is closed under Souslin's operation.

proof Let $\langle E_\sigma \rangle_{\sigma \in S}$ be a Souslin scheme in Σ with kernel A. Write

$$S^* = \bigcup_{k \in \mathbb{N}} \mathbb{N}^k = S \cup \{\emptyset\}.$$

If $F \in \Sigma$ and $\mu F < \infty$, then $A \cap F \in \Sigma$. **P** For each $\sigma \in S^*$, set

$$A_\sigma = \bigcup_{\phi \in \mathbb{N}^{\mathbb{N}}, \phi \supseteq \sigma} \bigcap_{n \geq 1} E_{\phi \restriction n},$$

and let G_σ be a measurable envelope of $A_\sigma \cap F$. Because $A_\sigma \subseteq E_\sigma$ (writing $E_\emptyset = X$), we may suppose that $G_\sigma \subseteq E_\sigma \cap F$. Now, for any $\sigma \in S^*$,

$$A_\sigma \cap F = \bigcup_{i \in \mathbb{N}} A_{\sigma^\frown i} \cap F \subseteq \bigcup_{i \in \mathbb{N}} G_{\sigma^\frown i},$$

so

$$H_\sigma = G_\sigma \setminus \bigcup_{i \in \mathbb{N}} G_{\sigma^\frown i}$$

is negligible.

Set $H = \bigcup_{\sigma \in S^*} H_\sigma$, so that H is negligible. Take any $x \in G_\emptyset \setminus H$. Choose $\langle \phi(i) \rangle_{i \in \mathbb{N}}$ inductively, as follows. Given that $\sigma = \langle \phi(i) \rangle_{i<k}$ has been chosen and $x \in G_\sigma$, then $x \notin H_\sigma$, so there must be some $j \in \mathbb{N}$ such that $x \in G_{\sigma^\frown j}$; set $\phi(k) = j$, and continue. Now

$$x \in \bigcap_{k \geq 1} G_{\phi \restriction k} \subseteq \bigcap_{k \geq 1} E_{\phi \restriction k} \subseteq A.$$

Thus we see that $G_\emptyset \setminus H \subseteq A$; as $G_\emptyset \subseteq F$, $G_\emptyset \setminus H \subseteq A \cap F$. On the other hand, $A \cap F \subseteq G_\emptyset$. Because H is negligible and μ is complete, $A \cap F \in \Sigma$. **Q**

Because μ is locally determined, it follows that $A \in \Sigma$. As $\langle E_\sigma \rangle_{\sigma \in S}$ is arbitrary, Σ is closed under Souslin's operation.

431B Corollary If $(X, \mathfrak{T}, \Sigma, \mu)$ is a complete locally determined topological measure space, every Souslin-F set in X (definition: 421K) is measurable.

431C Corollary Let X be a set and θ an outer measure on X. Let μ be the measure defined by Carathéodory's method, and Σ its domain. Then Σ is closed under Souslin's operation.

proof Let $\langle E_\sigma \rangle_{\sigma \in S}$ be a Souslin scheme in Σ with kernel A. Take any $C \subseteq X$ such that $\theta C < \infty$. Then $\theta_C = \theta \restriction \mathcal{P}C$ is an outer measure on C; let μ_C be the measure on C defined from θ_C by Carathéodory's method, and Σ_C its domain. If $\sigma \in S$, $D \subseteq C$ then

$$\begin{aligned}\theta_C(D \cap C \cap E_\sigma) + \theta_C(D \setminus (C \cap E_\sigma)) &= \theta(D \cap E_\sigma) + \theta(D \setminus E_\sigma) \\ &= \theta D = \theta_C D;\end{aligned}$$

as D is arbitrary, $C \cap E_\sigma \in \Sigma_C$. μ_C is a complete totally finite measure, so 431A tells us that the kernel of the Souslin scheme $\langle C \cap E_\sigma \rangle_{\sigma \in S}$ belongs to Σ_C. But this is just $C \cap A$ (applying 421Cb to the identity map from C to X.) So

$$\theta(C \cap A) + \theta(C \setminus A) = \theta_C(C \cap A) + \theta_C(C \setminus A) = \theta_C C = \theta C.$$

As C is arbitrary, $A \in \Sigma$ (113D). As $\langle E_\sigma \rangle_{\sigma \in S}$ is arbitrary, we have the result.

431D Theorem Let (X, Σ, μ) be a complete locally determined measure space, and $\langle E_\sigma \rangle_{\sigma \in S}$ a Souslin scheme in Σ with kernel A. Then

$$\begin{aligned}\mu A &= \sup\{\mu(\bigcup_{\phi \in K} \bigcap_{n \geq 1} E_{\phi \restriction n}) : K \subseteq \mathbb{N}^{\mathbb{N}} \text{ is compact}\} \\ &= \sup\{\mu(\bigcup_{\phi \leq \psi} \bigcap_{n \geq 1} E_{\phi \restriction n}) : \psi \in \mathbb{N}^{\mathbb{N}}\},\end{aligned}$$

writing $\phi \leq \psi$ if $\phi(i) \leq \psi(i)$ for every $i \in \mathbb{N}$.

proof (a) By 431A, A is measurable. For $K \subseteq \mathbb{N}^{\mathbb{N}}$, set $H_K = \bigcup_{\phi \in K} \bigcap_{n \geq 1} E_{\phi \restriction n}$. Of course $H_K \subseteq A$, and we know from 421M (or otherwise) that $H_K \in \Sigma$ if K is compact. So surely $\mu A \geq \mu H_K$ for every compact $K \subseteq \mathbb{N}^{\mathbb{N}}$. If $\psi \in \mathbb{N}^{\mathbb{N}}$, then $\{\phi : \phi \leq \psi\} = \prod_{i \in \mathbb{N}} (\psi(i) + 1)$ is compact. We therefore have

$$\begin{aligned}\mu A &\geq \sup\{\mu(\bigcup_{\phi \in K} \bigcap_{n \geq 1} E_{\phi \restriction n}) : K \subseteq \mathbb{N}^{\mathbb{N}} \text{ is compact}\} \\ &\geq \sup\{\mu(\bigcup_{\phi \leq \psi} \bigcap_{n \geq 1} E_{\phi \restriction n}) : \psi \in \mathbb{N}^{\mathbb{N}}\}.\end{aligned}$$

So what I need to prove is that

$$\mu A \leq \sup\{\mu(\textstyle\bigcup_{\phi \leq \psi} \bigcap_{n \geq 1} E_{\phi \restriction n}) : \psi \in \mathbb{N}^{\mathbb{N}}\}.$$

(b) Fix on a set $F \in \Sigma$ of finite measure. For $\sigma \in S^* = \bigcup_{k \in \mathbb{N}} \mathbb{N}^k$ set

$$A_\sigma = \textstyle\bigcup_{\phi \in \mathbb{N}^{\mathbb{N}}, \phi \supseteq \sigma} \bigcap_{n \geq 1} E_{\phi \restriction n}.$$

We need to know that A_σ belongs to Σ; this follows from 431A, because writing $E'_\tau = E_\tau$ if $\tau \subseteq \sigma$ or $\sigma \subseteq \tau$, $\emptyset$ otherwise,

$$A_\sigma = \textstyle\bigcup_{\phi \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \geq 1} E'_{\phi \restriction n} \in \mathcal{S}(\Sigma) = \Sigma,$$

writing $\mathcal{S}$ for Souslin's operation, as in §421.

Let $\epsilon > 0$, and take a family $\langle \epsilon_\sigma \rangle_{\sigma \in S^*}$ of strictly positive real numbers such that $\sum_{\sigma \in S^*} \epsilon_\sigma \leq \epsilon$. For each $\sigma \in S^*$ we have $A_\sigma = \bigcup_{i \in \mathbb{N}} A_{\sigma^\frown i}$, so there is an $m_\sigma \in \mathbb{N}$ such that

$$\mu(F \cap A_\sigma \setminus \textstyle\bigcup_{i \leq m_\sigma} A_{\sigma^\frown i}) \leq \epsilon_\sigma.$$

Define $\psi \in \mathbb{N}^{\mathbb{N}}$ by saying that

$$\psi(k) = \max\{m_\sigma : \sigma \in \mathbb{N}^k, \sigma(i) \leq \psi(i) \text{ for every } i < k\}$$

for each $k \in \mathbb{N}$. Set

$$H = \bigcup_{\phi \in \mathbb{N}^{\mathbb{N}}, \phi \leq \psi} \bigcap_{n \geq 1} E_{\phi\restriction n}.$$

(c) Set

$$G = \bigcup_{\sigma \in S^*} F \cap A_\sigma \setminus \bigcup_{i \leq m_\sigma} A_{\sigma^\frown i},$$

so that $\mu G \leq \epsilon$, by the choice of the ϵ_σ and the m_σ. Then $F \cap A \setminus G \subseteq H$. **P** If $x \in F \cap A \setminus G$, choose $\langle \phi(i) \rangle_{i \in \mathbb{N}}$ inductively, as follows. Given that $\phi(i) \leq \psi(i)$ for $i < k$ and $x \in A_\sigma$, where $\sigma = \langle \phi(i) \rangle_{i<k}$, then $x \notin A_\sigma \setminus \bigcup_{j \leq m_\sigma} A_{\sigma^\frown j}$, so there must be some $j \leq m_\sigma$ such that $x \in A_{\sigma^\frown j}$; set $\phi(k) = j$; because $\sigma \in \prod_{i<k} (\psi(i) + 1)$, $j \leq m_\sigma \leq \psi(k)$, and the induction continues. At the end of the induction, $\phi \leq \psi$ and

$$x \in \bigcap_{n \geq 1} A_{\phi\restriction n} \subseteq \bigcap_{n \geq 1} E_{\phi\restriction n} \subseteq H. \text{ } \mathbf{Q}$$

(d) It follows that

$$\mu(A \cap F) \leq \mu G + \mu H \leq \epsilon + \mu H.$$

As F and ϵ are arbitrary, and μ is semi-finite,

$$\mu A \leq \sup\{\mu(\bigcup_{\phi \leq \psi} \bigcap_{n \geq 1} E_{\phi\restriction n}) : \psi \in \mathbb{N}^{\mathbb{N}}\},$$

and the proof is complete.

431E Corollary If $(X, \mathfrak{T}, \Sigma, \mu)$ is a topological measure space and $E \subseteq X$ is a Souslin-F set with finite outer measure, then $\mu^* E = \sup\{\mu F : F \subseteq E \text{ is closed}\}$.

proof Let $\tilde{\mu}$ be the c.l.d. version of μ (213E). Let $\langle E_\sigma \rangle_{\sigma \in S}$ be a Souslin scheme of closed sets with kernel E. Then 213Fb and 431D tell us that

$$\mu^* E = \tilde{\mu} E = \sup_{K \subseteq \mathbb{N}^{\mathbb{N}} \text{ is compact}} \mu F_K,$$

where $F_K = \bigcup_{\phi \in K} \bigcap_{n \geq 1} E_{\phi\restriction n}$ for $K \subseteq \mathbb{N}^{\mathbb{N}}$. But every F_K is closed, by 421M. So $\mu^* E \leq \sup_{F \subseteq E \text{ is closed}} \mu F$; as the reverse inequality is trivial, we have the result.

***431F** There is a topological version of 431A, as follows.

Theorem Let X be any topological space, and $\widehat{\mathcal{B}}$ its Baire property algebra.

(a) For any $A \subseteq X$, there is a Baire property envelope of A, that is, a set $E \in \widehat{\mathcal{B}}$ such that $A \subseteq E$ and $E \setminus F$ is meager whenever $A \subseteq F \in \widehat{\mathcal{B}}$.

(b) $\widehat{\mathcal{B}}$ is closed under Souslin's operation.

proof (a) By 4A3Ra, there is an open set $H \subseteq X$ such that $A \setminus H$ is meager and $H \cap G$ is empty whenever $G \subseteq X$ is open and $A \cap G$ is meager. Set $E = A \cup H$; then $E \supseteq A$ and $E \triangle H = A \setminus H$ is meager, so $E \in \widehat{\mathcal{B}}$. If $A \subseteq F \in \widehat{\mathcal{B}}$, let G be an open set such that $G \triangle (X \setminus F)$ is meager. Then $G \cap A \subseteq G \cap F$ is meager, so $G \cap H$ is empty and $E \setminus F \subseteq (E \triangle H) \cup (G \triangle (X \setminus F))$ is meager. Thus E is a Baire property envelope of A.

(b) Let $\langle E_\sigma \rangle_{\sigma \in S}$ be a Souslin scheme in $\widehat{\mathcal{B}}$ with kernel A. Write

$$S^* = \bigcup_{k \in \mathbb{N}} \mathbb{N}^k = S \cup \{\emptyset\}.$$

For each $\sigma \in S^*$, set

$$A_\sigma = \bigcup_{\phi \in \mathbb{N}^{\mathbb{N}}, \phi \supseteq \sigma} \bigcap_{n \geq 1} E_{\phi\restriction n},$$

and let G_σ be a Baire property envelope of A_σ as described in (a). Because $A_\sigma \subseteq E_\sigma$ (writing $E_\emptyset = X$), we may suppose that $G_\sigma \subseteq E_\sigma$. Now, for any $\sigma \in S^*$,

$$A_\sigma = \bigcup_{i \in \mathbb{N}} A_{\sigma^\frown i} \subseteq \bigcup_{i \in \mathbb{N}} G_{\sigma^\frown i},$$

so

$$H_\sigma = G_\sigma \setminus \bigcup_{i\in\mathbb{N}} G_{\sigma^\frown i}$$

is meager.

Set $H = \bigcup_{\sigma\in S^*} H_\sigma$, so that H is negligible. Take any $x \in G_\emptyset \setminus H$. Choose $\langle\phi(i)\rangle_{i\in\mathbb{N}}$ inductively, as follows. Given that $\sigma = \langle\phi(i)\rangle_{i<k}$ has been chosen and $x \in G_\sigma$, then $x \notin H_\sigma$, so there must be some $j \in \mathbb{N}$ such that $x \in G_{\sigma^\frown j}$; set $\phi(k) = j$, and continue. Now

$$x \in \bigcap_{k\geq 1} G_{\phi\restriction k} \subseteq \bigcap_{k\geq 1} E_{\phi\restriction k} \subseteq A.$$

Thus we see that $G_\emptyset \setminus H \subseteq A$. On the other hand, $A \subseteq G_\emptyset$, so $G_\emptyset \triangle A$ is meager and $A \in \widehat{\mathcal{B}}$. As $\langle E_\sigma\rangle_{\sigma\in S}$ is arbitrary, $\widehat{\mathcal{B}}$ is closed under Souslin's operation.

431X Basic exercises (a) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined topological measure space, $(Y, \mathfrak{S})$ a topological space, and $f : X \to Y$ a measurable function. Let $\mathcal{B}(Y)$ be the Borel algebra of Y. Show that $f^{-1}[B] \in \Sigma$ for every $B \in \mathcal{S}(\mathcal{B}(Y))$.

(b) Let X be a topological space and μ a semi-finite topological measure on X which is inner regular with respect to the Souslin-F sets. Show that μ is inner regular with respect to the closed sets.

(c) Let (X, Σ, μ) be a complete locally determined measure space, and $\langle E_\sigma\rangle_{\sigma\in S}$ a regular Souslin scheme in Σ (definition: 421Xn) with kernel A. Show that

$$\mu A = \sup_{\phi\in\mathbb{N}^\mathbb{N}} \mu(\textstyle\bigcap_{n\geq 1} E_{\phi\restriction n}).$$

>(d) Let (X, Σ, μ) be a measure space with locally determined negligible sets (definition: 213I), and $\langle E_\sigma\rangle_{\sigma\in S}$ a Souslin scheme in Σ with kernel A. Show that

$$\mu^* A = \sup_{\psi\in\mathbb{N}^\mathbb{N}} \mu(\textstyle\bigcup_{\phi\leq\psi} \bigcap_{n\geq 1} E_{\phi\restriction n}).$$

>(e) Let (X, Σ, μ) be a semi-finite measure space, and $\langle E_\sigma\rangle_{\sigma\in S}$ a Souslin scheme in Σ with kernel A. Show that

$$\mu_* A = \sup_{\psi\in\mathbb{N}^\mathbb{N}} \mu(\textstyle\bigcup_{\phi\leq\psi} \bigcap_{n\geq 1} E_{\phi\restriction n}).$$

431Y Further exercises (a) Let (X, Σ, μ) be a complete measure space with the measurable envelope property (213Xl). Show that Σ is closed under Souslin's operation.

(b) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ a σ-ideal of subsets of X such that $\mathcal{I} \subseteq \Sigma$. Suppose that for every $A \subseteq X$ there is an $F \in \Sigma$ such that $A \subseteq F$ and $F \setminus E \in \mathcal{I}$ whenever $A \subseteq E \in \Sigma$. Show that Σ is closed under Souslin's operation.

(c) Let X be a set, Σ a σ-algebra of subsets of X, and $\mathcal{I}$ an ω_1-saturated σ-ideal of Σ; suppose that $A \in \Sigma$ whenever $A \subseteq B \in \mathcal{I}$. Show that Σ is closed under Souslin's operation.

431 Notes and comments From the point of view of measure theory, the most important property of Souslin's operation, after its idempotence, is the fact that (for many measure spaces) the family of measurable sets is closed under the operation (431A). The proof I give here is based on the concept of measurable envelope, which can be used in other cases of great interest (431Yb, 431Yc). But for some applications it is also very important to know that if A is the kernel of a Souslin scheme $\langle E_\sigma\rangle_{\sigma\in S}$, then A can be approximated from inside by sets of the form $H = \bigcup_{\phi\leq\psi} \bigcap_{n\geq 1} E_{\phi\restriction n}$ (431D, 431Xd), which belong to the σ-algebra generated by the E_σ (421M). A typical application of this idea is when every E_σ is a Borel subset of $\mathbb{R}$; then we find not only that A is Lebesgue measurable (indeed, measurable for every Radon measure on $\mathbb{R}$) but that (for any given Radon measure μ) the Souslin scheme itself provides a Borel subset H of A of measure approximating the measure of A.

Let me repeat that the essence of descriptive set theory is that we are not satisfied merely to know that a set of a certain type exists. We want also to know how to build it, because we expect that an explicit

construction will be valuable later on. For instance, the construction given in 431D shows that if the Souslin scheme consists of closed compact sets, the set H will be compact (421Xo).

I mention 431B as a typical application of 431A, even though it is both obvious and obviously less than what can be said. The algebras Σ of this section are algebras closed under Souslin's operation. In a complete locally determined topological measure space, the algebra Σ of measurable sets includes the open sets (by definition), therefore the Borel algebra $\mathcal{B}$, therefore $\mathcal{S}(\mathcal{B})$; but now we can take the algebra $\mathcal{A}_1$ generated by $\mathcal{S}(\mathcal{B})$, and $\mathcal{A}_1$ and $\mathcal{S}(\mathcal{A}_1)$ will also be included in Σ, so that Σ will included the algebra $\mathcal{A}_2$ generated by $\mathcal{S}(\mathcal{A}_1)$, and so on. (Note that $\mathcal{S}(\mathcal{A}_1)$ includes the σ-algebra generated by $\mathcal{A}_1$, by 421F, so I do not need to mention that separately.) We have to run through all the countable ordinals before we can be sure of getting to the smallest algebra $\mathcal{A}_{\omega_1} = \bigcup_{\xi<\omega_1} \mathcal{A}_\xi$ which contains every open set and is closed under Souslin's operation, and we shall then have $\mathcal{A}_{\omega_1} \subseteq \Sigma$.

The result in 431D is one of the special features of measures. (A similar result, based on rather different hypotheses, is in 432J.) But the argument of 431A can be applied in many other cases; see 431Yb-431Yc. A striking one is 431F, which will be useful in Volume 5.

432 K-analytic spaces

I describe the basic measure-theoretic properties of K-analytic spaces (§422). I start with 'elementary' results (432A-432C), assembling ideas from §§421, 422 and 431. The main theorem of the section is 432D, one of the leading cases of the general extension theorem 416P. An important corollary (432G) gives a sufficient condition for the existence of pull-back measures. I briefly mention 'capacities' (432I-432J).

432A Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined Hausdorff topological measure space. Then every K-analytic subset of X is measurable.

proof If $A \subseteq X$ is K-analytic, it is Souslin-F (422Ha), therefore measurable (431B).

432B Theorem Let X be a K-analytic Hausdorff space, and μ a semi-finite topological measure on X. Then

$$\mu X = \sup\{\mu K : K \subseteq X \text{ is compact}\}.$$

proof If $\gamma < \mu X$, there is an $E \in \operatorname{dom}\mu$ such that $\gamma < \mu E < \infty$; set $\nu F = \mu(E \cap F)$ for every Borel set $F \subseteq X$, so that ν is a totally finite Borel measure on X, and $\nu X > \gamma$. Let $\hat{\nu}$ be the completion of ν. Let $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}] = X$. Set $F_\sigma = \overline{R[I_\sigma]}$ for $\sigma \in S = \bigcup_{k\geq 1} \mathbb{N}^k$, where $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$. Because R is closed in $\mathbb{N}^{\mathbb{N}} \times X$ (422Da), X is the kernel of the Souslin scheme $\langle F_\sigma\rangle_{\sigma\in S}$ (421I). By 431D, there is a compact $L \subseteq \mathbb{N}^{\mathbb{N}}$ such that $\hat{\nu}(\bigcup_{\phi\in L}\bigcap_{n\in\mathbb{N}} F_{\phi\restriction n}) \geq \gamma$. But, by 421I, this is just $\hat{\nu}(R[L])$; and $R[L]$ is compact, by 422D(e-i). So $\mu R[L]$ is defined, with $\mu R[L] \geq \nu R[L] = \hat{\nu} R[L]$, and we have a compact subset of X of measure at least γ. As γ is arbitrary, the proposition is proved.

432C Proposition Let X be a Hausdorff space such that all its open sets are K-analytic, and μ a Borel measure on X.

(a) If μ is semi-finite, it is tight (that is, inner regular with respect to the compact sets).

(b) If μ is locally finite, its completion is a Radon measure on X.

proof (a) By 422Hb, every open subset of X is Souslin-F. Applying 421F to the family $\mathcal{E}$ of closed subsets of X, we see that every Borel subset of X is Souslin-F, therefore K-analytic (by 422Hb in the opposite direction). Now suppose that $E \subseteq X$ is a Borel set. Then the subspace measure μ_E is a semi-finite Borel measure on the K-analytic space E, so by 432B $\mu E = \sup_{K\subseteq E \text{ is compact}} \mu K$. As E is arbitrary, μ is tight.

(b) Because X is Lindelöf (422Gg), μ is σ-finite (411Ge), therefore semi-finite. So (a) tells us that μ is tight. By 416F, its c.l.d. version is a Radon measure. But (because μ is σ-finite) this is just its completion (213Ha).

432D Theorem (ALDAZ & RENDER 00) Let X be a K-analytic Hausdorff space and μ a locally finite measure on X which is inner regular with respect to the closed sets. Then μ has an extension to a Radon measure on X. In particular, μ is τ-additive.

proof The point is that if $\mu E > 0$ then there is a compact $K \subseteq E$ such that $\mu^* K > 0$. **P** Write Σ for the domain of μ. Take $\gamma < \mu E$. Because X is Lindelöf (422Gg), μ is σ-finite (411Ge), therefore semi-finite. Let $E' \subseteq E$ be such that $\gamma < \mu E' < \infty$. Because μ is inner regular with respect to the closed sets, there is a closed set $F \subseteq E$ such that $\mu F > \gamma$. F is K-analytic (422Gf); let $R \subseteq \mathbb{N}^{\mathbb{N}} \times F$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}] = F$. For $\sigma \in S^* = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ set

$$F_\sigma = \{x : (\phi, x) \in R \text{ for some } \phi \in \mathbb{N}^{\mathbb{N}} \text{ such that } \phi(i) \leq \sigma(i) \text{ for every } i < \#(\sigma)\}.$$

Then $\langle F_{\sigma^\frown i}\rangle_{i\in\mathbb{N}}$ is a non-decreasing sequence with union F_σ, so

$$\mu^* F_\sigma = \sup_{i\in\mathbb{N}} \mu^* F_{\sigma^\frown i}$$

for every $\sigma \in S^*$. We can therefore find a sequence $\psi \in \mathbb{N}^{\mathbb{N}}$ such that

$$\mu^* F_{\psi\restriction n} > \gamma$$

for every $n \in \mathbb{N}$. Set

$$K = \{\phi : \phi \in \mathbb{N}^{\mathbb{N}}, \phi(i) \leq \psi(i) \text{ for every } i \in \mathbb{N}\};$$

then $K = \prod_{n\in\mathbb{N}}(\psi(n)+1)$ is compact, so $R[K]$ is compact (422D(e-i)).

? Suppose, if possible, that $\mu^* R[K] < \gamma$. Then there is an $H \in \Sigma$ such that $R[K] \subseteq H \subseteq F$ and $\mu(F \setminus H) > \mu F - \gamma$. Because μ is inner regular with respect to the closed sets, there is a closed set $F' \in \Sigma$ such that $F' \subseteq F \setminus H$ and $\mu F' > \mu F - \gamma$. Since $R[K] \cap F' = \emptyset$, $K \cap R^{-1}[F'] = \emptyset$. $R^{-1}[F']$ is closed, because R is usco-compact, so there is some n such that

$$L = \{\phi : \phi \in \mathbb{N}^{\mathbb{N}}, \phi\restriction n = \phi'\restriction n \text{ for some } \phi' \in K\}$$

does not meet $R^{-1}[F']$ (4A2F(h-vi)), and $R[L] \cap F' = \emptyset$. But L is just $\{\phi : \phi(i) \leq \psi(i) \text{ for every } i < n\}$, so $R[L] = F_{\psi\restriction n}$, and

$$\gamma < \mu^* F_{\psi\restriction n} \leq \mu(F \setminus F') < \gamma,$$

which is absurd. **X**

Thus $\mu^* R[K] \geq \gamma$. As $\gamma > 0$, we have the result. **Q**

Now the theorem follows at once from 416P(ii)⇒(i).

432E Corollary Let X be a K-analytic Hausdorff space, and μ a locally finite quasi-Radon measure on X. Then μ is a Radon measure.

proof By 432D, μ has an extension to a Radon measure μ'. But of course μ and μ' must coincide, by 415H or otherwise.

432F Corollary Let X be a K-analytic Hausdorff space, and ν a locally finite Baire measure on X. Then ν has an extension to a Radon measure on X; in particular, it is τ-additive. If the topology of X is regular, the extension is unique.

proof Because X is Lindelöf (422Gg), ν is σ-finite, therefore semi-finite; by 412D, it is inner regular with respect to the closed sets. So 432D tells us that it has an extension to a Radon measure on X. Since the extension is τ-additive, so is ν.

If X is regular, then it must be completely regular (4A2H(b-i)), and the family $\mathcal{G}$ of cozero sets is a base for the topology closed under finite unions. If μ, μ' are Radon measures extending ν, they agree on $\mathcal{G}$, and must be equal, by 415H(iv).

432G Corollary Let X be a K-analytic Hausdorff space, Y a Hausdorff space and ν a locally finite measure on Y which is inner regular with respect to the closed sets. Let $f : X \to Y$ be a continuous function such that $f[X]$ has full outer measure in Y. Then there is a Radon measure μ on X such that f is inverse-measure-preserving for μ and ν. If ν is Radon, it is precisely the image measure μf^{-1}.

proof (a) Write T for the domain of ν, and set $\Sigma_0 = \{f^{-1}[F] : f \in \mathrm{T}\}$, so that Σ_0 is a σ-algebra of subsets of X, and we have a measure μ_0 on X defined by setting $\mu_0 f^{-1}[F] = \nu F$ whenever $F \in \mathrm{T}$ (132Yd).

(b) If $E \in \Sigma_0$ and $\gamma < \mu_0 E$, there is an $F \in \mathrm{T}$ such that $E = f^{-1}[F]$. Now there is a closed set $F' \subseteq F$ such that $\nu F' \geq \gamma$. Because f is continuous, $f^{-1}[F']$ is closed, and we have $f^{-1}[F'] \subseteq E$ and $\mu_0 f^{-1}[F'] \geq \gamma$. As E and γ are arbitrary, μ_0 is inner regular with respect to the closed sets.

If $x \in X$, then (because ν is locally finite) there are an open set $H \subseteq Y$ and an $F \in \mathrm{T}$ such that

$$f(x) \in H \subseteq F, \quad \nu F < \infty.$$

In this case $f^{-1}[H]$ is an open set containing x and

$$\mu_0^* f^{-1}[H] \leq \mu_0 f^{-1}[F] = \nu F < \infty.$$

As x is arbitrary, μ_0 is locally finite.

(c) By 432D, there is a Radon measure μ on X extending μ_0. Because f is inverse-measure-preserving for μ_0 and ν, it is surely inverse-measure-preserving for μ and ν.

The image measure μf^{-1} extends ν, so must be locally finite; it is therefore a Radon measure (418I). So if ν itself is a Radon measure, it must be identical with μf^{-1}, by 416Eb.

432H Corollary Suppose that X is a set and that $\mathfrak{S}$, $\mathfrak{T}$ are Hausdorff topologies on X such that $(X, \mathfrak{T})$ is K-analytic and $\mathfrak{S} \subseteq \mathfrak{T}$. Then the totally finite Radon measures on X are the same for $\mathfrak{S}$ and $\mathfrak{T}$.

proof Write f for the identity function on X regarded as a continuous function from $(X, \mathfrak{T})$ to $(X, \mathfrak{S})$. If μ is a totally finite $\mathfrak{T}$-Radon measure on X, then $\mu = \mu f^{-1}$ is $\mathfrak{S}$-Radon, by 418I. If ν is a totally finite $\mathfrak{S}$-Radon measure on X, then 432G tells us that it is of the form $\mu = \mu f^{-1}$ for some $\mathfrak{T}$-Radon measure μ, that is, is itself $\mathfrak{T}$-Radon.

432I Capacitability The next result is not exactly measure theory as studied in most of this treatise; but it is clearly very close to the other ideas of this section, and it has important applications to measure theory in the narrow sense.

Definition Let $(X, \mathfrak{T})$ be a topological space. A (**Choquet**) **capacity** on X is a function $c : \mathcal{P}X \to [0, \infty]$ such that

(i) $c(A) \leq c(B)$ whenever $A \subseteq B \subseteq X$;

(ii) $\lim_{n\to\infty} c(A_n) = c(A)$ whenever $\langle A_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of X with union A;

(iii) $c(K) = \inf\{c(G) : G \supseteq K \text{ is open}\}$ for every compact set $K \subseteq X$.

432J Theorem (Choquet 55) Let X be a Hausdorff space and c a capacity on X. If $A \subseteq X$ is K-analytic, then $c(A) = \sup\{c(K) : K \subseteq A \text{ is compact}\}$.

proof Take $\gamma < c(A)$. Let $R \subseteq \mathbb{N}^{\mathbb{N}} \times X$ be an usco-compact relation such that $R[\mathbb{N}^{\mathbb{N}}] = A$; for $\sigma \in S^* = \bigcup_{n\in\mathbb{N}} \mathbb{N}^n$ set

$$A_\sigma = \{x : (\phi, x) \in R \text{ for some } \phi \in \mathbb{N}^{\mathbb{N}} \text{ such that } \phi(i) \leq \sigma(i) \text{ for every } i < \#(\sigma)\}.$$

Then $\langle A_{\sigma^\frown i} \rangle_{i\in\mathbb{N}}$ is a non-decreasing sequence with union A_σ, so

$$c(A_\sigma) = \sup_{i\in\mathbb{N}} c(A_{\sigma^\frown i})$$

for every $\sigma \in S^*$. We can therefore find a sequence $\psi \in \mathbb{N}^{\mathbb{N}}$ such that $c(A_{\psi\restriction n}) > \gamma$ for every $n \in \mathbb{N}$. Set

$$K = \{\phi : \phi \in \mathbb{N}^{\mathbb{N}}, \ \phi(i) \leq \psi(i) \text{ for every } i \in \mathbb{N}\};$$

then $K = \prod_{n\in\mathbb{N}} (\psi(n) + 1)$ is compact, so $R[K]$ is compact (422D(e-i)).

? Suppose, if possible, that $c(R[K]) < \gamma$. Then, by (iii) of 432I, there is an open set $G \supseteq R[K]$ such that $c(G) < \gamma$. Set $F = X \setminus G$, so that F is closed and $K \cap R^{-1}[F] = \emptyset$. $R^{-1}[F]$ is closed, because R is usco-compact, so there is some n such that

$$L = \{\phi : \phi \in \mathbb{N}^{\mathbb{N}}, \ \phi\restriction n = \phi'\restriction n \text{ for some } \phi' \in K\}$$

does not meet $R^{-1}[F]$ (4A2F(h-vi)), and $R[L] \cap F = \emptyset$, that is, $R[L] \subseteq G$. But L is just $\{\phi : \phi(i) \leq \psi(i) \text{ for every } i < n\}$, so $R[L] = A_{\psi\restriction n}$, and

$$\gamma < c(A_{\psi\restriction n}) \leq c(G) < \gamma,$$

which is absurd. **X**

Thus $c(R[K]) \geq \gamma$. As γ is arbitrary and $R[K]$ is compact, we have the result.

432X Basic exercises (a) Put 422Xf, 431Xb and 432D together to prove 432C.

(b) Let X be a K-analytic Hausdorff space, and μ a measure on X which is outer regular with respect to the open sets. Show that $\mu X = \sup_{K\subseteq X \text{ is compact}} \mu^* K$. (*Hint*: use the argument of part (a) of the proof of 432D.)

>(c) Let X be a K-analytic Hausdorff space, and μ a semi-finite topological measure on X. Show that if *either* μ is inner regular with respect to the closed sets *or* X is regular and μ is a τ-additive Borel measure, then μ is tight.

(d) Use 422Gf, 432B and 416C to prove 432E.

>(e) Suppose that X is a set and that $\mathfrak{S}$, $\mathfrak{T}$ are Hausdorff topologies on X such that $(X, \mathfrak{T})$ is K-analytic and $\mathfrak{S} \subseteq \mathfrak{T}$. Let $(Z, \mathfrak{U}, \mathrm{T}, \nu)$ be a Radon measure space and $f : Z \to X$ a function which is almost continuous for $\mathfrak{U}$ and $\mathfrak{S}$. Show that f is almost continuous for $\mathfrak{U}$ and $\mathfrak{T}$. (*Hint*: it is enough to consider totally finite ν; show that νf^{-1} is $\mathfrak{T}$-Radon, so is inner regular for $\{K : \mathfrak{T}_K = \mathfrak{S}_K\}$, writing $\mathfrak{T}_K$ for the subspace topology induced by $\mathfrak{T}$ on K.)

(f) Let X be a topological space and μ a locally finite measure on X which is inner regular with respect to the closed sets. Show that μ^* is a capacity.

(g) Let X be a topological space and F a closed subset of X. Define $c : \mathcal{P}X \to \{0, 1\}$ by setting $c(A) = 1$ if A meets F, 0 otherwise. Show that c is a capacity on X.

(h) Let X and Y be Hausdorff spaces, and $R \subseteq X \times Y$ an usco-compact relation. Show that if c is a capacity on Y, then $A \mapsto c(R[A])$ is a capacity on X.

(i) Use 432J and 432Xf to shorten the proof of 432D.

432Y Further exercises (a) Show that there are a K-analytic Hausdorff space X and a probability measure μ on X such that (i) μ is inner regular with respect to the Borel sets (ii) the domain of μ includes a base for the topology of X (iii) every compact subset of X is negligible. Show that there is no extension of μ to a topological measure on X.

432 Notes and comments The measure-theoretic properties of K-analytic spaces can largely be summarised in the slogan 'K-analytic spaces have lots of compact sets'. I said above that it is sometimes helpful to think of K-analytic spaces as an amalgam of compact Hausdorff spaces and Souslin-F subsets of $\mathbb{R}$. For the former, it is obvious that they have many compact subsets; for the latter, it is not obvious, but is of course one of their fundamental properties, deducible from 422De. 432B and part (a) of the proof of 432D are typical manifestations of the phenomenon. The real point of these theorems is that we can extend a Borel or Baire measure to a Radon measure with no assumption of τ-additivity (432F). A Radon measure must be τ-additive just because it is tight. A (locally finite) Borel or Baire measure must be τ-additive whenever the measurable open sets are K-analytic.

The condition 'every open set is K-analytic' in 432C is of course a very strong one in the context of compact Hausdorff spaces (422Xe). But for analytic spaces it is automatically satisfied (423E), and that is the side on which the principal applications of 432C appear.

The results which I call corollaries of 432D can mostly be proved by more direct methods (see 432Xd), but the line I choose here seems to be the most powerful technique. Indeed it can be used to deal with 432C as well (432Xa).

In §434 I will discuss 'universally measurable' sets in topological spaces. In fact K-analytic sets are universally measurable in a particularly strong sense (432A). The point here is that K-analyticity is intrinsic;

a K-analytic space is measurable whenever embedded as a subspace of a (complete locally determined) topological measure space.

The theorems here touch on two phenomena of particular importance. First, in 432G we have an example of 'pulling back' a measure, that is, we have a measure ν on a set Y and a function $f : X \to Y$ and seek a Radon measure μ on X such that f is inverse-measure-preserving, or, even better, such that $\nu = \mu f^{-1}$. There was a similar result in 418L. In both cases we have to suppose that f is continuous and (in effect) that ν is a Radon measure. (This is not part of the hypotheses of 432G, but of course it is an easy consequence of them, using 432B.) In 418L, we need a special hypothesis to ensure that there are enough compact subsets of X to carry an appropriate Radon measure; in 432G, this is an automatic result of assuming that X is K-analytic. Both 418L and 432G can be regarded as consequences of Henry's theorem (416M). The difficulty arises from the requirement that μ should be a Radon measure; if we do not insist on this there is a much simpler solution, since we need suppose only that $f[X]$ is of full outer measure (132Yd).

The next theme I wish to mention is a related one, the investigation of comparable topologies. If $\mathfrak{S}$ and $\mathfrak{T}$ are (Hausdorff) topologies on a set X, and $\mathfrak{S}$ is coarser than $\mathfrak{T}$ (so that $(X, \mathfrak{S})$ is a continuous image of $(X, \mathfrak{T})$), then 418I tells us that any totally finite $\mathfrak{T}$-Radon measure is $\mathfrak{S}$-Radon. We very much want to know when the reverse is true, so that the (totally finite) Radon measures for the two topologies are the same. 432H provides one of the important cases in which this occurs. The hypothesis '$(X, \mathfrak{T})$ is K-analytic' generalizes the alternative '$(X, \mathfrak{T})$ is compact'; in the latter case, $\mathfrak{S} = \mathfrak{T}$, so that the result is, from our point of view here, trivial. (But from the point of view of elementary general topology, of course, it is one of the pivots of the theory of compact Hausdorff spaces.) In a similar vein we have a variety of important topological consequences of the same hypotheses (422Ye, 423Fb).

The paragraphs 432I-432J may appear to be no more that a minor extension of ideas already set out. I ought therefore to say plainly that the topological and measure theory of K-analytic spaces have co-evolved with the notion of capacity, and that 432J ('K-analytic spaces are capacitable') is one of the cornerstones of a theory of which I am giving only a minuscule part. For a idea of the vitality and scope of this theory, see Dellacherie 80.

433 Analytic spaces

We come now to the special properties of measures on 'analytic' spaces, that is, continuous images of $\mathbb{N}^{\mathbb{N}}$, as described in §423. I start with a couple of facts about spaces with countable networks.

433A Proposition Let $(X, \mathfrak{T})$ be a topological space with a countable network, and μ a localizable topological measure on X which is inner regular with respect to the Borel sets. Then μ has countable Maharam type.

proof Let $\tilde{\mu}$ be the c.l.d. version of μ (213E). Then the measure algebra $\mathfrak{A}$ of $\tilde{\mu}$ can be identified with the measure algebra of μ (322D(b-iii)). Also $\tilde{\mu}$ is complete, locally determined and localizable, so every subset of X has a measurable envelope with respect to $\tilde{\mu}$ (213L). Let $\tilde{\Sigma}$ be the domain of $\tilde{\mu}$, and $\mathcal{E}$ a countable network for $\mathfrak{T}$. For each $E \in \mathcal{E}$, let $F_E \in \tilde{\Sigma}$ be a measurable envelope of E.

Let $\mathfrak{B}$ be the order-closed subalgebra of $\mathfrak{A}$ generated by $\{F_E^\bullet : E \in \mathcal{E}\}$, and set $\mathrm{T} = \{F : F \in \tilde{\Sigma}, F^\bullet \in \mathfrak{B}\}$. Because $\mathfrak{B}$ is an order-closed subalgebra of $\mathfrak{A}$, T is a σ-subalgebra of $\tilde{\Sigma}$. Now $\mathfrak{T} \subseteq \mathrm{T}$. **P** If $G \in \mathfrak{T}$, set $\mathcal{E}_0 = \{E : E \in \mathcal{E}, E \subseteq G\}$. Set $F = \bigcup_{E \in \mathcal{E}_0} F_E$, so that $F \in \mathrm{T}$ and $G \subseteq F$. For each $E \in \mathcal{E}_0$, $F_E \setminus G$ is negligible, so $F \setminus G$ is negligible, and $G^\bullet = F^\bullet \in \mathfrak{B}$, so $G \in \mathrm{T}$. **Q**

It follows that T includes the Borel σ-algebra of X. Because μ is inner regular with respect to the Borel sets, $\mathfrak{B}$ is order-dense in $\mathfrak{A}$, and $\mathfrak{B} = \mathfrak{A}$. Thus the countable set $\{F_E^\bullet : E \in \mathcal{E}\}$ τ-generates $\mathfrak{A}$, and the Maharam type of $\mathfrak{A}$, which is the Maharam type of μ, is countable.

433B Lemma If $(X, \mathfrak{T})$ is a Hausdorff space with a countable network, then any topological measure on X is countably separated in the sense of 343D.

proof By 4A2Nf, there is a countable family of open sets separating the points of X.

433C Theorem Let X be an analytic Hausdorff space, and μ a Borel measure on X.
(a) If μ is semi-finite, it is tight.
(b) If μ is locally finite, its completion is a Radon measure on X.

proof X is K-analytic (423C); moreover, every open subset of X is again analytic (423Eb). So 432C gives the result at once.

Remark Compare 256C.

433D Theorem Let X and Y be analytic Hausdorff spaces, ν a totally finite Radon measure on Y and $f : X \to Y$ a Borel measurable function such that $f[X]$ is of full outer measure in Y. Then there is a Radon measure μ on X such that $\nu = \mu f^{-1}$.

proof By 423Ga, the graph R of f is an analytic set in $X \times Y$, therefore K-analytic. Set $\pi_1(x,y) = x$, $\pi_2(x,y) = y$ for $(x,y) \in R$, so that π_1 and π_2 are continuous. Now $\pi_2[R] = f[X]$ is of full outer measure, so by 432G there is a Radon measure λ on R such that $\nu = \lambda\pi_2^{-1}$. Next, because π_1 is continuous, the image $\mu = \lambda\pi_1^{-1}$ is a Radon measure on X, by 418I. But $\pi_2 = f\pi_1$, so

$$\mu f^{-1} = (\lambda\pi_1^{-1})f^{-1} = \lambda(f\pi_1)^{-1} = \lambda\pi_2^{-1} = \nu,$$

as required.

433E Proposition Let (X, Σ, μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Let $(Y, \mathfrak{S})$ be an analytic space and $f : X \to Y$ a measurable function. Then f is almost continuous.

proof Take $E \in \Sigma$ and $\gamma < \mu E$. Then there is an $F \in \Sigma$ such that $F \subseteq E$ and $\gamma < \mu F < \infty$. For Borel sets $H \subseteq Y$, set $\nu H = \mu(F \cap f^{-1}[H])$. Then ν is a totally finite Borel measure on Y, so is tight (that is, inner regular with respect to the compact sets) (433C); let $K \subseteq Y$ be a compact set such that $\nu K > \gamma$, so that $\mu(F \cap f^{-1}[K]) > \gamma$. The subspace measure on $L = F \cap f^{-1}[K]$ is still inner regular with respect to the (relatively) closed sets (412Pc), and $f{\restriction}L$ is still measurable; but $f{\restriction}L$ is a function from L to K, and K is metrizable, by 423Dc. So $f{\restriction}L$ is almost continuous, by 418J, and there is a set $F' \subseteq L$, of measure at least γ, such that $f{\restriction}F'$ is continuous.

As E and γ are arbitrary, f is almost continuous.

Remark Compare 418Yg.

433F I give two simple corollaries of the von Neumann-Jankow selection theorem (423N-423O).

Proposition Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be analytic Hausdorff spaces, and $f : X \to Y$ a Borel measurable surjection. Let ν be a complete locally determined topological measure on Y, and T its domain. Then there is a T-measurable function $g : Y \to X$ such that gf is the identity on X.

proof By 423O we know that there is a function $g : Y \to X$ such that gf is the identity and g is T_1-measurable, where T_1 is the σ-algebra generated by the Souslin-F subsets of Y. But T contains every Souslin-F subset of Y, by 431B, therefore includes T_1, and g is actually T-measurable.

433G Proposition Let $(X, \mathfrak{T})$ be an analytic Hausdorff space, (Y, T, ν) a complete locally determined measure space, and $f : X \to Y$ a surjection. Suppose that there is some countable family $\mathcal{F} \subseteq \mathrm{T}$ such that $\mathcal{F}$ separates the points of Y (that is, whenever y, y' are distinct points of Y there is a member of $\mathcal{F}$ containing one and not the other) and $f^{-1}[F]$ is a Borel subset of X for every $F \in \mathcal{F}$. Then there is a T-measurable function $g : Y \to X$ such that fg is the identity on Y.

proof Set $\mathcal{A} = \mathcal{F} \cup \{Y \setminus F : F \in \mathcal{F}\}$. The topology $\mathfrak{T}_1$ on X generated by $\mathfrak{T} \cup \{f^{-1}[A] : A \in \mathcal{A}\}$ is still analytic (423H). If we take $\mathfrak{S}$ to be the topology on Y generated by $\mathcal{A}$, then $\mathfrak{S}$ is Hausdorff and f is $(\mathfrak{T}_1, \mathfrak{S})$-continuous, so $\mathfrak{S}$ is analytic (423Bb).

Because $\mathfrak{S}$ is generated by a countable subset $\mathcal{A}$ of T, it is second-countable, and $\mathfrak{S} \subseteq \mathrm{T}$ (4A3Da/4A3E). So ν is a topological measure with respect to $\mathfrak{S}$. By 433F, there is a function $g : Y \to X$, measurable for T and the topology $\mathfrak{T}_1$, such that gf is the identity on X; and of course g is still measurable for T and the coarser original topology $\mathfrak{T}$ on X.

433H Because analytic spaces have countable networks (423C), and their compact subsets are therefore metrizable (4A2Nh), their measure theory is very close to that of $\mathbb{R}$ or $[0,1]$ or $\{0,1\}^{\mathbb{N}}$. I give some simple manifestations of this principle.

Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of analytic spaces, and for each $i \in I$ let μ_i be a Radon probability measure on X_i. Let λ be the ordinary product measure on $X = \prod_{i\in I} X_i$, as defined in §254.

(a) If I is countable then λ is a Radon measure.

(b) If every μ_i is strictly positive, then λ is a quasi-Radon measure.

proof (a) In this case, X is analytic (423Bc), therefore hereditarily Lindelöf (423Da). Let Λ be the domain of λ and $\mathfrak{T}$ the topology of X. Then $\Lambda \cap \mathfrak{T}$ is a base for $\mathfrak{T}$; by 4A3Da, $\mathfrak{T} \subseteq \Lambda$. By 417Sb, λ is the τ-additive product measure on X; by 417Q, this is a Radon measure.

(b) By (a), the ordinary product measure on $\prod_{i\in J} X_i$ is a Radon measure for every finite set $J \subseteq I$. So 417Sc tells us that λ is the τ-additive product measure on X; by 417O, this is a quasi-Radon measure.

433I Proposition Let X be an analytic Hausdorff space, and T a countably generated σ-subalgebra of the Borel σ-algebra $\mathcal{B}(X)$ of X. Then any locally finite measure with domain T has an extension to a Radon measure on X.

proof Let μ_0 be a locally finite measure with domain T.

(a) Consider first the case in which μ_0 is totally finite. Let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a sequence in T generating T as σ-algebra. Define $f : X \to \{0,1\}^{\mathbb{N}}$ by setting

$$f(x)(n) = \chi F_n(x) \text{ for } n \in \mathbb{N},\ x \in X.$$

Then f is T-measurable (use 418Bd), so we have a Borel measure ν_0 on $\{0,1\}^{\mathbb{N}}$ defined by setting $\nu_0 E = \mu_0 f^{-1}[E]$ for every Borel set $E \subseteq \{0,1\}^{\mathbb{N}}$. Now the completion ν of ν_0 is a Radon measure (433C). Also $f[X]$ must be analytic, by 423Gb, because f is $\mathcal{B}(X)$-measurable. So ν measures $f[X]$ (432A), and

$$\nu f[X] = \nu_0^* f[X] = \nu_0\{0,1\}^{\mathbb{N}},$$

that is, $f[X]$ is ν-conegligible. By 433D, there is a Radon measure μ on X such that $\nu = \mu f^{-1}$.

Because every F_n is expressible as $f^{-1}[E]$ for some $E \in \mathcal{B}(\{0,1\}^{\mathbb{N}})$, so is every member of T. If $F \in$ T, take $H \in \mathcal{B}(\{0,1\}^{\mathbb{N}})$ such that $F = f^{-1}[H]$; then

$$\mu F = \nu H = \nu_0 H = \mu_0 F.$$

Thus μ extends μ_0 and $\mu\restriction\Sigma$ will serve.

(b) In general, because X is Lindelöf and μ_0 is locally finite, μ_0 is σ-finite. Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a partition of X into members of T such that $\mu_0 X_n$ is finite for every n, and set $\mu_0^{(n)} F = \mu_0(F \cap X_n)$ for every n and every $F \in$ T; then every $\mu_0^{(n)}$ has an extension to a Radon measure $\mu^{(n)}$, and we can set $\mu F = \sum_{n=0}^{\infty} \mu^{(n)} F$ for every $F \in \Sigma$. Because μ_0 is locally finite, so is μ, and it is now easy to check that μ is complete, locally determined and tight, just because every $\mu^{(n)}$ is.

433J I turn now to a brief mention of 'standard Borel spaces'. From the point of view of this chapter, it is natural to regard the following results as simple corollaries of theorems about Polish spaces. But, as remarked in §424, there are cases in which a standard Borel space is presented without any specific topology being attached; and in any case it is interesting to look at the ways in which we can express these ideas as theorems about σ-algebras rather than about topological spaces.

Proposition Let (X, Σ) be a standard Borel space and T a countably generated σ-subalgebra of Σ. Then any σ-finite measure with domain T has an extension to Σ.

proof Let μ_0 be a σ-finite measure with domain T.

(a) If μ_0 is totally finite, give X a Polish topology for which Σ is the Borel σ-algebra of X, and use 433I.

(b) In general, let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a partition of X into members of T such that $\mu_0 X_n < \infty$ for every n, and set $\mu_0^{(n)} F = \mu_0(F \cap X_n)$ for every n and every $F \in$ T; then every $\mu_0^{(n)}$ has an extension to a measure $\mu^{(n)}$ with domain Σ, and we can set $\mu F = \sum_{n=0}^{\infty} \mu^{(n)} F$ for every $F \in \Sigma$.

433K Proposition Let $\langle (X_n, \Sigma_n, \mu_n)\rangle_{n\in\mathbb{N}}$ be a sequence of probability spaces such that (X_n, Σ_n) is a standard Borel space for every n. Suppose that for each $n \in \mathbb{N}$ we are given an inverse-measure-preserving function $f_n : X_{n+1} \to X_n$. Then we can find a standard Borel space (X, Σ), a probability measure μ with domain Σ, and inverse-measure-preserving functions $g_n : X \to X_n$ such that $f_n g_{n+1} = g_n$ for every n.

proof For each n, choose a Polish topology $\mathfrak{T}_n$ on X_n such that Σ_n is the algebra of $\mathfrak{T}_n$-Borel sets. Let $\hat{\mu}_n$ be the completion of μ_n; then $\hat{\mu}_n$ is a Radon measure (433C). Every f_n is inverse-measure-preserving for $\hat{\mu}_{n+1}$ and $\hat{\mu}_n$, by 235Hc, and almost continuous, by 418J.

By 418Q, we have a Radon measure $\hat{\mu}$ on

$$X = \{x : x \in \textstyle\prod_{n\in\mathbb{N}} X_n,\ f_n(x(n+1)) = x(n) \text{ for every } n \in \mathbb{N}\}$$

such that the continuous maps $x \mapsto x(n) = g_n(x) : X \to X_n$ are inverse-measure-preserving for every n. Now X is a Borel subset of $Z = \prod_{n\in\mathbb{N}} X_n$. **P** For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be a countable base for $\mathfrak{T}_n$. Then

$$Z \setminus X = \textstyle\bigcup_{n\in\mathbb{N}} \bigcup_{U,V\in\mathcal{U}_n, U\cap V=\emptyset} \{z : z(n) \in U,\ f_n(z(n+1)) \in V\}$$

is a countable union of Borel sets because $\{z : z(n) \in U\}$ is open and $\{z : z(n+1) \in f_n^{-1}[V]\}$ is Borel whenever $n \in \mathbb{N}$ and $U, V \in \mathcal{U}_n$. So $Z \setminus X$ and X are Borel sets. **Q**

Accordingly (X, Σ) is a standard Borel space, where Σ is the Borel σ-algebra of X (424G). So if we take $\mu = \hat{\mu}\restriction\Sigma$, we shall have a suitable measure on X.

433X Basic exercises (a) Let $(X, \mathfrak{T})$ be a topological space with a countable network, and μ a topological measure on X which is inner regular with respect to the Borel sets and has the measurable envelope property (213Xl). Show that μ has countable Maharam type.

(b) Show that an effectively locally finite measure on a hereditarily Lindelöf space (in particular, on any analytic space) is σ-finite.

(c) Let $X \subseteq [0, 1]$ be a set of Lebesgue outer measure 1 and inner measure 0. Show that the subspace measure on X is a totally finite Borel measure which is not tight.

(d) Let X be a Hausdorff space and μ a locally finite measure on X, inner regular with respect to the Borel sets, such that $\operatorname{dom}\mu$ includes a base for the topology of X. Suppose that $Y \subseteq X$ is an analytic set of full outer measure. Show that μ has a unique extension to a Radon measure $\tilde{\mu}$ on X, and that Y is $\tilde{\mu}$-conegligible.

(e) Let (X, Σ) be a standard Borel space. (i) Show that any semi-finite measure with domain Σ is a compact measure (definition: 342Ac), therefore perfect. (*Hint*: if X is given a suitable topology, the measure is tight.) (ii) Show that any measure with domain Σ is countably separated.

(f) Let (X, Σ, μ) and (Y, T, ν) be atomless probability spaces such that (X, Σ) and (Y, T) are standard Borel spaces. Show that (X, Σ, μ) and (Y, T, ν) are isomorphic. (*Hint*: by 344I, their completions are isomorphic; by 344H, they have negligible sets of cardinal $\mathfrak{c}$; show that any isomorphism between the completions is (Σ, T)-measurable on a conegligible set; use 424Da to match residual negligible sets.)

(g) Let X be $[0, 1] \times \{0, 1\}$, with its usual topology, and $I^{\|}$ the split interval (419L); define $f : X \to I^{\|}$ by setting $f(t, 0) = t^-$, $f(t, 1) = t^+$ for $t \in [0, 1]$. (i) Give $I^{\|}$ its usual Radon measure ν (343J, 419Lc). Show that there is no Radon measure λ on X such that $\nu = \lambda f^{-1}$. (ii) Let μ be the product Radon probability measure on X, starting from Lebesgue measure on $[0, 1]$ and the usual fair-coin measure on $\{0, 1\}$. Show that f is inverse-measure-preserving for μ and ν. Show that f is not almost continuous.

433Y Further exercises (a) Find a Hausdorff topological space X with a countable network and a semi-finite Borel measure on X which does not have countable Maharam type.

(b) Let X be an analytic Hausdorff space and μ an atomless Radon measure on X. Show that (X, μ) is isomorphic to Lebesgue measure on some measurable subset of $\mathbb{R}$. (*Hint*: 344I.)

(c) Let $(X, \mathfrak{T})$ be a Polish space without isolated points, and μ a σ-finite topological measure on X. Show that there is a conegligible meager set. (*Hint*: Show that every non-empty open set is uncountable. Find a countable dense negligible set and a negligible G_δ set including it.)

(d) Let $\langle X_n \rangle_{n \in \mathbb{N}}$ be a sequence of analytic spaces and for each $n \in \mathbb{N}$ let μ_n be a Borel probability measure on X_n. Suppose that for each $n \in \mathbb{N}$ we are given an inverse-measure-preserving function $f_n : X_{n+1} \to X_n$. Show that we can find a standard Borel space (X, Σ), a probability measure μ with domain Σ, and inverse-measure-preserving functions $g_n : X \to X_n$ such that $f_n g_{n+1} = g_n$ for every n.

433 Notes and comments The measure-theoretic results of 433C-433E are of much the same type as those in §432. A characteristic difference is that Borel measurable functions between analytic spaces behave in many ways like continuous functions. (Compare 433D and 432G.) You may feel that 423Yc offers some explanation of this. For any question which refers to the Borel algebra of an analytic space X, or to the class of its analytic subsets, we can expect to be able to suppose that X is separable and metrizable (see 423Xd), and that any single Borel measurable function appearing is continuous. (424H is a particularly remarkable instance of this principle.)

433H here amounts to spelling out a special case of ideas already treated in 417S. As this territory is relatively unfamiliar, I give detailed examples (423Xi, 433Xc, 433Xg, 439A, 439K) to show that the theorems of this section are not generally valid for compact Hausdorff spaces (the archetype of K-analytic spaces which need not be analytic), nor for separable metric spaces (the archetypical spaces with countable network). They really do depend on the particular combination of properties possessed by analytic spaces.

For large parts of probability theory, standard Borel spaces provide an adequate framework, and have a number of advantages; some of the technical problems concerning measurability which loom rather large in this treatise disappear in such contexts. Many authors accordingly give them great prominence. I myself believe that the simplifications are an entrapment rather than a liberation, that sooner or later everyone has to leave the comfortable environment of Borel algebras on Polish spaces, and that it is better to be properly equipped with a suitable general theory when one does. But it is surely important to know what the simplifications are, and the results in 433J-433K will I hope show at least that there are wonderful ideas here, even if my own presentation tends to leave them on one side.

434 Borel measures

What one might call the fundamental question of topological measure theory is the following.

What kinds of measures can arise on what kinds of topological space?

Of course this question has inexhaustible ramifications, corresponding to all imaginable properties of measures and topologies and connexions between them. The challenge I face here is that of identifying particular ideas as being more important than others, and the chief difficulty lies in the bewildering variety of topological properties which have been studied, any of which may have implications for the measure theory of the spaces involved. In this section and the next I give a sample of what is known, necessarily biased and incomplete. I try however to include the results which are most often applied and enough others for the proofs to contain, between them, most of the non-trivial arguments which have been found effective in this area.

In 434A I set out a crude classification of Borel measures on topological spaces. For compact Hausdorff spaces, at least, the first question is whether they carry Borel measures which are not, in effect, Radon measures; this leads us to the definition of 'Radon' space (434C) which is also of interest in the context of general Hausdorff spaces. I give a brief account of the properties of Radon spaces (434F, 434N). I look also at two special topics: 'quasi-dyadic' spaces (434O-434Q) and a construction of Borel product measures by integration of sections (434R).

In the study of Radon spaces we find ourselves looking at 'universally measurable' subsets of topological spaces (434D). These are interesting in themselves, and also interact with constructions from earlier parts of this treatise (434S-434T). Three further classes of topological space, defined in terms of the types of topological measure which they carry, are the 'Borel-measure-compact', 'Borel-measure-complete' and 'pre-Radon' spaces; I discuss them briefly in 434G-434J. They provide useful methods for deciding whether Hausdorff spaces are Radon (434K).

434A Types of Borel measures In §411 I introduced the following properties which a Borel measure may or may not have:

(i) inner regularity with respect to closed sets;

(ii) inner regularity with respect to zero sets;

(iii) tightness (that is, inner regularity with respect to closed compact sets);

(iv) τ-additivity.

These are of course interrelated. (ii)⇒(i) just because zero sets are closed, and (iii)⇒(iv) by 411E; in a Hausdorff space, (iii)⇒(i); and for an effectively locally finite measure on a regular topological space, (iv)⇒(i) (414Mb).

On a regular Hausdorff space, therefore, we can divide totally finite Borel measures into four classes:

(A) measures which are not inner regular with respect to the closed sets;

(B) measures which are inner regular with respect to the closed sets, but not τ-additive nor tight;

(C) measures which are τ-additive and inner regular with respect to the closed sets, but not inner regular with respect to the compact sets;

(D) tight measures;

and each of the classes (B)-(D) can be further subdivided into those which are completion regular (B_1, C_1, D_1) and those which are not (B_0, C_0, D_0). Examples may be found in 434Xf (type A), 411Q and 439K (type B_0), 439J (type B_1), 415Xc and 434Xa (type C_1) and 434Xb (type D_0), while Lebesgue measure itself is of type D_1, and any direct sum of spaces of types D_0 and C_1 will have type C_0. (The space in 439J depends for its construction on supposing that there is a cardinal which is not measure-free. It seems that no convincing example of a space of class B_1, that is, a completion regular, non-τ-additive Borel probability measure on a completely regular Hausdorff space, is known which does not depend on some special axiom beyond ordinary ZFC. For one of the obstacles to finding such a space, see 434Q below.)

Note that a totally finite Borel measure μ on a regular Hausdorff space can be extended to a quasi-Radon measure iff μ is of class C or D (415M), and that in this case the quasi-Radon measure must be just the completion $\hat{\mu}$ of μ. $\hat{\mu}$ will be of the same type, on the classification here, as μ; in particular, $\hat{\mu}$ will be a Radon measure iff μ is of class D (416F).

434B Compact, analytic and K-analytic spaces For any class of topological spaces, we can enquire which of the seven types of measure described above can be realized by measures on spaces of that class. The enquiry is limited only by our enterprise and diligence in seeking out new classes of topological space. For the spaces studied in §§432-433, however, we have something worth repeating here. On a K-analytic Hausdorff space, a semi-finite Borel measure which is inner regular with respect to the closed sets is tight (432B, 432D); consequently classes B and C of 434A cannot appear, and we are left with only the types A, D_0 and D_1, all of which appear on compact Hausdorff spaces (434Xb, 434Xf). On an analytic Hausdorff space we have further simplifications: every semi-finite Borel measure is tight (433C), and (if X is regular) every closed set is a zero set (423Db). Thus on an analytic regular Hausdorff space only type D_1, of the seven types in 434A, can appear. (If the topology is not regular, we may also get measures of type D_0; see 434Ya.)

434C Radon spaces: Definition For K-analytic Hausdorff spaces, therefore, we have a large gap between the 'bad' measures of class A and the 'good' measures of class D; furthermore, we have an important class of spaces in which type A cannot appear. It is natural to enquire further into the spaces in which every (totally finite) Borel measure is of class D, and (given that no exact description can be found) we are led, as usual, to a definition. A Hausdorff space X is **Radon** if every totally finite Borel measure on X is tight.

434D Universally measurable sets Before going farther with the study of Radon spaces it will be useful to spend a couple of paragraphs on the following concept. Let X be a topological space.

(a) I will say that a subset E of X is **universally measurable** (in X) if it is measured by the completion of every totally finite Borel measure on X; that is, for every totally finite Borel measure μ on X there is a Borel set $F \subseteq X$ such that $E \triangle F$ is μ-negligible.

(b) A subset of X is universally measurable iff it is measured by every complete locally determined topological measure on X. **P** (i) Suppose that $A \subseteq X$ is universally measurable and that μ is a complete locally determined topological measure on X. Let $F \subseteq X$ be such that μF is defined and finite. Then we have a totally finite Borel measure ν on X defined by setting $\nu E = \mu(F \cap E)$ for every Borel set $E \subseteq X$. Now there are Borel sets $E, B \subseteq X$ such that $A \triangle E \subseteq B$ and $\nu B = 0$. In this case, $(A \cap F) \triangle (E \cap F) \subseteq B \cap F$ and $\mu(B \cap F) = 0$, so that (because μ is complete) $A \cap F$ is measured by μ. Because F is arbitrary and μ is locally determined, A is measured by μ. (ii) Suppose that $A \subseteq X$ is measured by every complete locally determined topological measure on X. Then, in particular, it is measured by the completion of any totally finite Borel measure, so is universally measurable. **Q**

(c) If X is a topological space then the family $\Sigma_{\rm um}$ of universally measurable subsets of X is a σ-algebra closed under Souslin's operation and including the Borel σ-algebra. (For it is the intersection of the domains of a family of complete totally finite measures, and all these are σ-algebras including the Borel σ-algebra and closed under Souslin's operation, by 431A.) In particular, Souslin-F sets are universally measurable, so (if X is Hausdorff) K-analytic and analytic sets are (422Ha, 423C).

(d) Note that a function $f : X \to \mathbb{R}$ is $\Sigma_{\rm um}$-measurable iff it is μ-virtually measurable for every totally finite Borel measure μ on X (122Q, 212F). Generally, if Y is another topological space, I will say that $f : X \to Y$ is **universally measurable** if $f^{-1}[H] \in \Sigma_{\rm um}$ for every open set $H \subseteq Y$; that is, if f is $(\Sigma_{\rm um}, \mathcal{B}(Y))$-measurable, where $\mathcal{B}(Y)$ is the Borel σ-algebra of Y.

(e) In fact, if $f : X \to Y$ is universally measurable, then it is $(\Sigma_{\rm um}, \Sigma_{\rm um}^{(Y)})$-measurable, where $\Sigma_{\rm um}^{(Y)}$ is the algebra of universally measurable subsets of Y. **P** Take $F \in \Sigma_{\rm um}^{(Y)}$ and a totally finite Borel measure μ on X. If $\hat{\mu}$ is the completion of μ, then the image measure $\nu = \hat{\mu} f^{-1}$ is a complete totally finite topological measure on Y, so measures F, and $f^{-1}[F] \in \operatorname{dom} \hat{\mu}$. As μ is arbitrary, $f^{-1}[F] \in \Sigma_{\rm um}$; as F is arbitrary, f is $(\Sigma_{\rm um}, \Sigma_{\rm um}^{(Y)})$-measurable. **Q**

(f) It follows that if Z is a third topological space and $f : X \to Y$, $g : Y \to Z$ are universally measurable, then $gf : X \to Z$ is universally measurable.

434E Universally Radon-measurable sets A companion idea is the following.

(a) Let X be a Hausdorff space. I will say that a subset E of X is **universally Radon-measurable** if it is measured by every Radon measure on X.

(b) If X is a Hausdorff space, the family $\Sigma_{\rm uRm}$ of universally Radon-measurable subsets of X is a σ-algebra closed under Souslin's operation and including the algebra of universally measurable subsets of X (and, *a fortiori*, including the Borel σ-algebra). (Use 434Db and the idea of 434Dc.)

(c) If X is a Hausdorff space, I will say that a function f from X to a topological space Y is **universally Radon-measurable** if $f^{-1}[H] \in \Sigma_{uRm}$ for every open set $H \subseteq Y$. A function $f : X \to \mathbb{R}$ is $\Sigma_{\rm uRm}$-measurable iff it is μ-virtually measurable for every totally finite tight Borel measure μ on X. (Compare 434Dd.)

434F Elementary properties of Radon spaces: Proposition Let X be a Hausdorff space.

(a) The following are equiveridical:

(i) X is a Radon space;

(ii) every semi-finite Borel measure on X is tight;

(iii) if μ is a locally finite Borel measure on X, its c.l.d. version $\tilde{\mu}$ is a Radon measure;

(iv) whenever μ is a totally finite Borel measure on X, and $G \subseteq X$ is an open set with $\mu G > 0$, then there is a compact set $K \subseteq G$ such that $\mu K > 0$;

(v) whenever μ is a non-zero totally finite Borel measure on X, there is a Radon subspace Y of X such that $\mu^* Y > 0$.

(b) If $Y \subseteq X$ is a subspace which is a Radon space in its induced topology, then Y is universally measurable in X.

(c) If X is a Radon space and $Y \subseteq X$, then Y is Radon iff it is universally measurable in X iff it is universally Radon-measurable in X. In particular, all Borel subsets and all Souslin-F subsets of X are Radon spaces.

(d) The family of Radon subspaces of X is closed under Souslin's operation and set difference.

proof (a)(i)⇒(ii) Let μ be a semi-finite Borel measure on X, $E \subseteq X$ a Borel set and $\gamma < \mu E$. Because μ is semi-finite, there is a Borel set H of finite measure such that $\mu(E \cap H) > \gamma$. Set $\nu F = \mu(F \cap H)$ for every Borel set $F \subseteq X$; then ν is a totally finite Borel measure on X, and $\nu E > \gamma$. Because X is a Radon space, there is a compact set $K \subseteq E$ such that $\nu K \geq \gamma$, and now $\mu K \geq \gamma$. As μ, E and γ are arbitrary, (ii) is true.

(ii)⇒(i) and **(i)⇒(v)** are trivial.

(v)⇒(iv) Assume (v), and let μ be a totally finite Radon measure on X and G a non-negligible open set. Set $\nu E = \mu(E \cap G)$ for every Borel set $E \subseteq X$. Then ν is a non-zero totally finite Borel measure on X, so there is a Radon subspace Y of X such that $\nu^* Y > 0$. The subspace measure ν_Y on Y is a Borel measure on Y, so is tight. Since $\nu_Y(Y \setminus G) = \nu(X \setminus G) = 0$, $\nu_Y(Y \cap G) > 0$ and there is a compact set $K \subseteq Y \cap G$ such that $\nu_Y K > 0$. Now $\mu K > 0$. As μ and G are arbitrary, (iv) is true.

not-(i)⇒not-(iv) If X is not Radon, there is a totally finite Borel measure μ on X which is not tight. By 416F(iii), there is an open set $G \subseteq X$ such that

$$\mu G > \sup_{K \subseteq G \text{ is compact}} \mu K = \gamma$$

say. Let $\mathcal{K}$ be the family of compact subsets of G. By 215B(v), there is a non-decreasing sequence $\langle K_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{K}$ such that $\mu(K \setminus F) = 0$ for every $K \in \mathcal{K}$, where $F = \bigcup_{n \in \mathbb{N}} K_n$. Observe that

$$\mu F = \lim_{n \to \infty} \mu K_n \leq \gamma < \mu G.$$

Now set $\nu E = \mu(E \cap G \setminus F)$ for every Borel set $E \subseteq X$. Then ν is a Borel measure on X, and $\nu G > 0$. If $K \subseteq G$ is compact, then $\nu K = \mu(K \setminus F) = 0$. So ν and G witness that (iii) is false.

(i)⇒(iii) The point is that $\tilde{\mu}$ is tight. **P** If $\tilde{\mu} E > \gamma$, then, because $\tilde{\mu}$ is semi-finite, there is a set $E' \subseteq E$ such that $\gamma < \tilde{\mu} E' < \infty$; now there is a Borel set $H \subseteq E'$ such that $\mu H = \tilde{\mu} E'$ (213Fc). Setting $\nu F = \mu(H \cap F)$ for every Borel set F, ν is a totally finite Borel measure on X and $\nu H > \gamma$, so there is a compact set $K \subseteq H$ such that $\nu K \geq \gamma$. Now $\gamma \leq \mu K < \infty$, so $\tilde{\mu} K = \mu K \geq \gamma$ (213Fa), while $K \subseteq E$. As E and γ are arbitrary, $\tilde{\mu}$ is tight. **Q**

On the other hand, every point of X belongs to an open set of finite measure for μ, which is still of finite measure for $\tilde{\mu}$ (213Fa again). So $\tilde{\mu}$ is locally finite; since it is surely complete and locally determined, it is a Radon measure.

(iii)⇒(i) Assume (iii), and let μ be a totally finite Borel measure on X. Then its c.l.d. version $\tilde{\mu}$ is tight. But $\tilde{\mu}$ extends μ (213Hc), so μ also is tight. As μ is arbitrary, X is a Radon space.

(b) Let μ be a totally finite Borel measure on X, and $\hat{\mu}$ its completion; let $\epsilon > 0$. Let μ_Y be the subspace measure on Y, so that μ_Y is a totally finite Borel measure on Y, and is tight. There is a compact set $K \subseteq Y$ such that $\nu K \geq \mu_Y Y - \epsilon$. But this means that

$$\mu^* Y = \mu_Y Y \leq \mu_Y K + \epsilon = \mu^*(K \cap Y) + \epsilon = \mu K + \epsilon \leq \mu_* Y + \epsilon.$$

As ϵ is arbitrary, $\mu_* Y = \mu^* Y$, and Y is measured by $\hat{\mu}$ (413Ef); as μ is arbitrary, Y is universally measurable.

(c) (i) If Y is Radon, it is universally measurable, by (b). (ii) If Y is universally measurable, it is universally Radon-measurable, by 434Eb. (iii) Suppose that Y is universally Radon-measurable, and that ν is a totally finite Borel measure on Y. For Borel sets $E \subseteq X$, set $\mu E = \nu(E \cap Y)$. Then μ is a totally finite Borel measure on X, so its c.l.d. version $\tilde{\mu}$ is a Radon measure on X, by (a-iii). We are supposing that Y is universally Radon-measurable, so, in particular, it must be measured by $\tilde{\mu}$. We have

$$\tilde{\mu}(X \setminus Y) = \sup_{K \subseteq X \setminus Y \text{ is compact}} \tilde{\mu} K = \sup_{K \subseteq X \setminus Y \text{ is compact}} \mu K$$

(213Ha, because μ is totally finite)

$$= \sup_{K \subseteq X \setminus Y \text{ is compact}} \nu(K \cap Y) = 0,$$

and Y is $\tilde{\mu}$-conegligible.

Now suppose that $E \subseteq Y$ is a (relatively) Borel subset of Y. Then E is of the form $F \cap Y$ where F is a Borel subset of X, so that

$$\begin{aligned}\nu E = \mu F = \tilde{\mu} F = \tilde{\mu}(Y \cap F) = \tilde{\mu} E \\ = \sup_{K \subseteq E \text{ is compact}} \mu K = \sup_{K \subseteq E \text{ is compact}} \nu K.\end{aligned}$$

As E is arbitrary, ν is tight; as ν is arbitrary, Y is a Radon space.

By 434Dc, it follows that all Borel subsets and all Souslin-F subsets of X are Radon spaces.

(d) The first step is to note that if $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence of Radon subspaces of X with union E, then E is Radon; this is immediate from (a-v) above.

Now let $\langle E_\sigma \rangle_{\sigma \in S}$ be a Souslin scheme, consisting of Radon subsets of X, with kernel A. We know that $E = \bigcup_{\sigma \in S} E_\sigma$ is a Radon space. Every E_σ is universally measurable in E, by (b), so A also is (434Dc), and must be Radon, by (c). Thus the family of Radon subspaces of X is closed under Souslin's operation.

If E and F are Radon subsets of X, then $E \cup F$ is Radon, and, just as above, F is universally measurable in $E \cup F$. But this means that $E \setminus F = (E \cup F) \setminus F$ is universally measurable in $E \cup F$, so that $E \setminus F$ is Radon.

434G Just as we can address the question 'when can we be sure that every Borel measure is of class D' in terms of the definition of 'Radon' space (434C), we can form other classes of topological space by declaring that the Borel measures they support must be of certain kinds. Three definitions which lead to interesting patterns of ideas are the following.

Definitions (a) A topological space X is **Borel-measure-compact** (Gardner & Pfeffer 84) if every totally finite Borel measure on X which is inner regular with respect to the closed sets is τ-additive, that is, X carries no measure of class B in the classification of 434A.

(b) A topological space X is **Borel-measure-complete** (Gardner & Pfeffer 84) if every totally finite Borel measure on X is τ-additive. (If X is regular and Hausdorff, this amounts to saying that X carries no measures of classes A or B in the classification of 434A.)

(c) A Hausdorff space X is **pre-Radon** (also called '**hypo-radonian**', '**semi-radonian**') if every τ-additive totally finite Borel measure on X is tight. (If X is regular, this amounts to saying that X carries no measure of class C in the classification of 434A.)

434H Proposition Let X be a topological space and $\mathcal{B}$ its Borel σ-algebra.

(a) The following are equiveridical:

(i) X is Borel-measure-compact;

(ii) every semi-finite Borel measure on X which is inner regular with respect to the closed sets is τ-additive;

(iii) every effectively locally finite Borel measure on X which is inner regular with respect to the closed sets has an extension to a quasi-Radon measure;

(iv) every totally finite Borel measure on X which is inner regular with respect to the closed sets has a support;

(v) if μ is a non-zero totally finite Borel measure on X, inner regular with respect to the closed sets, and $\mathcal{G}$ is an open cover of X, then there is some $G \in \mathcal{G}$ such that $\mu G > 0$.

(b) If X is Lindelöf (in particular, if X is a K-analytic Hausdorff space), it is Borel-measure-compact.

(c) If X is Borel-measure-compact and $A \subseteq X$ is a Souslin-F set, then A is Borel-measure-compact in its subspace topology. In particular, any Baire subset of X is Borel-measure-compact.

proof (a)(i)⇒(ii) Assume (i), and let μ be a semi-finite Borel measure on X which is inner regular with respect to the closed sets. Let $\mathcal{G}$ be an upwards-directed family of open sets with union G_0, and $\gamma < \mu G_0$.

Because μ is semi-finite, there is an $H \in \mathcal{B}$ such that $\mu H < \infty$ and $\mu(H \cap G_0) \geq \gamma$. Set $\nu E = \mu(E \cap H)$ for every $E \in \mathcal{B}$; then ν is a totally finite Borel measure on X. For any $E \in \mathcal{B}$,

$$\nu E = \mu(E \cap H) = \sup\{\mu F : F \subseteq E \cap H \text{ is closed}\} \leq \sup\{\nu F : F \subseteq E \text{ is closed}\},$$

so ν is inner regular with respect to the closed sets, and must be τ-additive. Now

$$\gamma \leq \nu G_0 = \sup_{G \in \mathcal{G}} \nu G \leq \sup_{G \in \mathcal{G}} \mu G.$$

As γ and $\mathcal{G}$ is arbitrary, μ is τ-additive.

(ii)⇒(iii) Assume (ii), and let μ be an effectively locally finite Borel measure on X which is inner regular with respect to the closed sets. Then it is semi-finite (411Gd), therefore τ-additive. By 415L, it has an extension to a quasi-Radon measure on X.

(iii)⇒(i) If (i) is true and μ is a totally finite Borel measure on X which is inner regular with respect to the closed sets, then μ has an extension to a quasi-Radon measure, which is τ-additive, so μ is also τ-additive (411C).

(i)⇒(iv) Use 411Nd.

(iv)⇒(v) Suppose that (iv) is true, that μ is a non-zero totally finite Borel measure on X which is inner regular with respect to the closed sets, and that $\mathcal{G}$ is an open cover of X. If F is the support of μ, then $\mu F > 0$ so $F \neq \emptyset$; there must be some $G \in \mathcal{G}$ meeting F, and now $\mu G > 0$.

not-(i)⇒not-(v) Suppose that there is a totally finite Borel measure μ on X, inner regular with respect to the closed sets, which is not τ-additive. Let $\mathcal{G}$ be an upwards-directed family of open sets such that $\mu G^* > \gamma$, where $G^* = \bigcup \mathcal{G}$ and $\gamma = \sup_{G \in \mathcal{G}} \mu G$. Let $\langle G_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence in $\mathcal{G}$ such that $\mu(G \setminus G_0^*)$ for every $G \in \mathcal{G}$, where $G_0^* = \bigcup_{n \in \mathbb{N}} G_n$ (215B(v)). Then $\mu G_0^* \leq \gamma$, so there is a closed set $F \subseteq G^* \setminus G_0^*$ such that $\mu F > 0$.

Let ν be the Borel measure on X defined by setting $\mu E = \mu(E \cap F)$ for every $E \in \mathcal{B}$. As in the argument for (i)⇒(ii), ν is inner regular with respect to the closed sets. Consider $\mathcal{H} = \mathcal{G} \cup \{X \setminus F\}$; this is an open cover of X. If $G \in \mathcal{G}$ then $\nu G \leq \mu(G \setminus G_0^*) = 0$, so $\nu H = 0$ for every $H \in \mathcal{H}$; thus ν and $\mathcal{H}$ witness that (iv) is false.

(b) Use (a-v) and 422Gg.

(c) Let μ be a Borel measure on A which is inner regular with respect to the closed sets, that is to say, the relatively closed sets in A. Let ν be the corresponding Borel measure on X, defined by setting $\nu E = \mu(A \cap E)$ for every $E \in \mathcal{B}$. Let $\hat{\nu}$ be the completion of ν. Putting 431D and 421M together, we see that $\hat{\nu} A = \sup\{\hat{\nu} F : F \subseteq A \text{ is closed in } X\}$, that is, $\nu X = \sup\{\mu F : F \subseteq A \text{ is closed in } X\}$. But this means that if $E \in \mathcal{B}$ and $\gamma < \nu E$, there is a closed set F in X such that $F \subseteq A$ such that $\mu(E \cap F) > \gamma$; now there is a relatively closed set $F' \subseteq A$ such that $F' \subseteq E \cap F$ and $\mu F' \geq \gamma$, and as F' must be relatively closed in F it is closed in X, while $\nu F' \geq \gamma$. Since E and γ are arbitrary, ν is inner regular with respect to the closed sets, and will be τ-additive.

Now suppose that $\mathcal{G}$ is an upwards-directed family of relatively open subsets of A. Set $\mathcal{H} = \{H : H \subseteq X \text{ is open}, H \cap A \in \mathcal{G}\}$. Then $\mathcal{H}$ is upwards-directed, so

$$\mu(\bigcup \mathcal{G}) = \nu(\bigcup \mathcal{H}) = \sup_{H \in \mathcal{H}} \nu H = \sup_{G \in \mathcal{G}} \mu G.$$

As μ and $\mathcal{G}$ are arbitrary, A is Borel-measure-compact.

By 421L, it follows that any Baire subset of X is Borel-measure-compact.

434I Proposition Let X be a topological space.

(a) The following are equiveridical:

(i) X is Borel-measure-complete;

(ii) every semi-finite Borel measure on X is τ-additive;

(iii) every totally finite Borel measure on X has a support;

(iv) whenever μ is a totally finite Borel measure on X there is a base $\mathcal{U}$ for the topology of X such that $\mu(\bigcup\{U : U \in \mathcal{U}, \mu U = 0\}) = 0$.

(b) If X is regular, it is Borel-measure-complete iff every effectively locally finite Borel measure on X has an extension to a quasi-Radon measure.

(c) If X is Borel-measure-complete, it is Borel-measure-compact.

(d) If X is Borel-measure-complete, so is every subspace of X.

(e) If X is hereditarily Lindelöf (for instance, if X is separable and metrizable, see 4A2P(a-iii)), it is Borel-measure-complete, therefore Borel-measure-compact.

proof (a)(i)⇒(ii) Use the argument of (i)⇒(ii) of 434Ha; this case is simpler, because we do not need to check that the auxiliary measure ν is inner regular.

(ii)⇒(i) is trivial.

(i)⇒(iv) If X is Borel-measure-complete and μ is a totally finite Borel measure on X, take $\mathcal{U}$ to be the family of all open subsets of X. This is surely a base for the topology, and setting $\mathcal{U}_0 = \{U : U \in \mathcal{U}, \mu U = 0\}$, $\mathcal{U}_0$ is upwards-directed so $\mu(\bigcup \mathcal{U}_0) = \sup_{U\in\mathcal{U}_0} \mu U = 0$, as required.

(iv)⇒(iii) Assume (iv), and let μ be a totally finite Borel measure on X. Take a base $\mathcal{U}$ as in (iv), so that $\mu(\bigcup \mathcal{U}_0) = 0$, where $\mathcal{U}_0$ is the family of negligible members of $\mathcal{U}$. Set $F = X \setminus \bigcup \mathcal{U}_0$, so that F is a conegligible closed set. If $G \subseteq X$ is an open set meeting F, there is a member U of $\mathcal{U}$ such that $U \subseteq G$ and $U \cap F \neq \emptyset$; now $U \notin \mathcal{U}$ so

$$\mu(G \cap F) = \mu G \geq \mu U > 0.$$

As G is arbitrary, F is self-supporting and is the support of μ.

(iii)⇒(i) Assume (iii), and let μ be a totally finite Borel measure on X. Let $\mathcal{G}$ be an upwards-directed family of open sets with union G^*. Set $\gamma = \sup_{G\in\mathcal{G}} \mu G$. Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathcal{G}$ such that $\mu(G \setminus G_0^*)$ for every $G \in \mathcal{G}$, where $G_0^* = \bigcup_{n\in\mathbb{N}} G_n$ (215B(v)). Then $\mu G_0^* \leq \gamma$. Let ν be the Borel measure on X defined by setting $\mu E = \mu(E \cap G^* \setminus G_0^*)$ for every $E \in \mathcal{B}$. Then ν has a support F say. Now $\nu G = 0$ for every $G \in \mathcal{G}$, so $F \cap G = \emptyset$ for every $G \in \mathcal{G}$, and $F \cap G^* = \emptyset$; but this means that

$$\mu(G^* \setminus G_0^*) = \nu X = \nu F = 0.$$

Accordingly $\mu G^* = \gamma$. As μ and $\mathcal{G}$ are arbitrary, X is Borel-measure-complete.

(b) If X is Borel-measure-complete and μ is an effectively locally finite Borel measure on X, then μ is τ-additive, by (a-ii), so extends to a quasi-Radon measure on X, by 415Cb. If effectively locally finite Borel measures on X extend to quasi-Radon measures, then any totally finite Borel measure is τ-additive, by 411C, and X is Borel-measure-complete.

(c) Immediate from the definitions.

(d) If $Y \subseteq X$ and μ is a totally finite Borel measure on Y, let ν be the Borel measure on X defined by setting $\nu E = \mu(E \cap Y)$ for every Borel set $E \subseteq X$. It is easy to check that ν is τ-additive. So if $\mathcal{G}$ is an upwards-directed family of relatively open subsets of Y, and we set $\mathcal{H} = \{H : H \subseteq X \text{ is open}, H \cap Y \in \mathcal{G}\}$, we shall get

$$\mu(\bigcup \mathcal{G}) = \nu(\bigcup \mathcal{H}) = \sup_{H\in\mathcal{H}} \nu H = \sup_{G\in\mathcal{G}} \mu G.$$

As μ and $\mathcal{G}$ are arbitrary, Y is Borel-measure-complete.

(e) If μ is a totally finite Borel measure on X and $\mathcal{G}$ is a non-empty upwards-directed family of open subsets of X with union G^*, then there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$ with union G^*, by 4A2H(c-i). Because $\mathcal{G}$ is upwards-directed, there is a non-decreasing sequence $\langle G'_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{G}$ such that $G'_n \supseteq G_n$ for every $n \in \mathbb{N}$, so that

$$\mu G^* = \lim_{n\to\infty} \mu G'_n \leq \sup_{G\in\mathcal{G}} \mu G.$$

As μ and $\mathcal{G}$ are arbitrary, X is Borel-measure-complete.

434J Proposition Let X be a Hausdorff space.

(a) The following are equiveridical:

(i) X is pre-Radon;

(ii) every effectively locally finite τ-additive Borel measure on X is tight;

(iii) whenever μ is a non-zero totally finite τ-additive Borel measure on X, there is a compact set $K \subseteq X$ such that $\mu K > 0$;

(iv) whenever μ is a totally finite τ-additive Borel measure on X, $\mu X = \sup_{K \subseteq X \text{ is compact}} \mu K$;

(v) whenever μ is a locally finite effectively locally finite τ-additive Borel measure on X, the c.l.d. version of μ is a Radon measure on X.

(b) If X is pre-Radon, then every locally finite quasi-Radon measure on X is a Radon measure.

(c) If X is regular and every totally finite quasi-Radon measure on X is a Radon measure, then X is pre-Radon.

(d) If X is pre-Radon, then any universally Radon-measurable subspace (in particular, any Borel subset or Souslin-F subset) A of X is pre-Radon.

(e) If $A \subseteq X$ is pre-Radon in its subspace topology, it is universally Radon-measurable in X.

(f) If X is K-analytic (for instance, if it is compact), it is pre-Radon.

(g) If X is completely regular and Čech-complete (for instance, if it is locally compact (4A2Gj), or metrizable and complete under a metric inducing its topology (4A2Md)), it is pre-Radon.

(h) If $X = \prod_{i \in I} X_i$ where $\langle X_i \rangle_{i \in I}$ is a countable family of pre-Radon Hausdorff spaces, then X is pre-Radon.

(i) If every point of X belongs to a pre-Radon open subset of X, then X is pre-Radon.

proof (a)(i)⇒(ii) Suppose that X is pre-Radon, that μ is an effectively locally finite τ-additive Borel measure on X, that $E \subseteq X$ is Borel, and that $\gamma < \mu E$. Because μ is semi-finite, there is a Borel set $H \subseteq X$ of finite measure such that $\mu(H \cap E) > \gamma$. Set $\nu F = \mu(F \cap H)$ for every Borel set $F \subseteq X$; then ν is a totally finite Borel measure on X, and is τ-additive, by 414Ea. Now $\nu E > \gamma$, so there is a compact set $K \subseteq E$ such that $\gamma \leq \nu K \leq \mu K$. As E is arbitrary, μ is tight.

(ii)⇒(iii) is trivial.

(iii)⇒(iv) Assume (iii), and let μ be a totally finite τ-additive Borel measure on X. Let $\mathcal{K}$ be the family of compact subsets of X and set $\alpha = \sup_{K \in \mathcal{K}} \mu K$. **?** Suppose, if possible, that $\mu X > \alpha$. Let $\langle K_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathcal{K}$ such that $\sup_{n \in \mathbb{N}} \mu K_n = \alpha$, and set $L = \bigcup_{n \in \mathbb{N}} K_n$; then

$$\mu L = \lim_{n \to \infty} \mu(\textstyle\bigcup_{i \leq n} K_i) = \alpha.$$

Set $\nu E = \mu(E \setminus L)$ for every Borel set $E \subseteq X$. Then ν is a non-zero totally finite Borel measure on X, and is τ-additive, by 414E again. So there is a $K \in \mathcal{K}$ such that $\nu K > 0$. But now there is an $n \in \mathbb{N}$ such that $\nu K + \mu K_n > \alpha$, and in this case $K \cup K_n \in \mathcal{K}$ and

$$\mu(K \cup K_n) = \mu K + \mu K_n - \mu(K \cap K_n) \geq \nu K + \mu K_n > \alpha,$$

which is impossible. **X** So $\mu X = \alpha$, as required.

(iv)⇒(i) Assume (iv), and let μ be a totally finite τ-additive Borel measure on X. Suppose that $E \subseteq X$ is Borel and that $\gamma < \mu E$. By (iii), there is a compact set $K \subseteq X$ such that $\mu K > \mu X - \mu E + \gamma$, so that $\mu(E \cap K) > \gamma$. Consider the subspace measure μ_K on K. By 414K, this is τ-additive, so inner regular with respect to the closed subsets of K (414Mb). There is therefore a relatively closed subset F of K such that $F \subseteq K \cap E$ and $\mu_K F \geq \gamma$; but now F is a compact subset of E and $\mu F \geq \gamma$. As E and γ are arbitrary, μ is tight. As μ is arbitrary, X is pre-Radon.

(ii)⇒(v) Assume (ii), and let μ be a locally finite effectively locally finite τ-additive Borel measure on X. Then μ is tight, so by 416F(ii) its c.l.d. version is a Radon measure.

(v)⇒(iv) Assume (v), and let μ be a totally finite τ-additive Borel measure on X. Then the c.l.d. version $\tilde{\mu}$ of μ is a Radon measure; but $\tilde{\mu}$ extends μ (213Hc), so

$$\sup_{K \subseteq X \text{ is compact}} \mu K \sup_{K \subseteq X \text{ is compact}} \tilde{\mu} K = \tilde{\mu} X = \mu X.$$

(b) Let μ be a locally finite quasi-Radon measure on X. By (a-ii), μ is tight; by 416C, μ is a Radon measure.

(c) Let μ be a totally finite τ-additive Borel measure on X. Because X is regular, its c.l.d. version $\tilde{\mu}$ is a quasi-Radon measure (415C), therefore a Radon measure; but $\tilde{\mu}$ extends μ (213Hc again), so μ, like $\tilde{\mu}$, must be tight. As μ is arbitrary, X is pre-Radon.

(d) A be a universally Radon-measurable subset of X, and μ a totally finite τ-additive Borel measure on A. Set $\nu E = \mu(E \cap A)$ for every Borel set $E \subseteq X$; then ν is a totally finite τ-additive Borel measure on X. So its c.l.d. version (that is, its completion $\hat{\nu}$, by 213Ha) is a Radon measure on X, by (a-v). Now $\hat{\nu}$ measures A, so

$$\mu A = \nu^* A = \hat{\nu} A = \sup\{\hat{\nu} K : K \subseteq A \text{ is compact}\} = \sup\{\mu K : K \subseteq A \text{ is compact}\}.$$

By (b-iv), A is pre-Radon.

(e) Let μ be a totally finite Radon measure on X. Then the subspace measure μ_A is τ-additive (414K), so its restriction ν to the Borel σ-algebra of A is still τ-additive. Because A is pre-Radon,

$$\begin{aligned}\mu^* A = \mu_A A = \nu A &= \sup\{\nu K : K \subseteq A \text{ is compact}\} \\ &= \sup\{\mu K : K \subseteq A \text{ is compact}\} = \mu_* A,\end{aligned}$$

and μ measures A (413Ef). As μ is arbitrary, A is universally Radon-measurable.

(f) Put 432B and (a-iv) together.

(g) If we identify X with a G_δ set in a compact Hausdorff space Z, then Z is pre-Radon, by (f), so X is pre-Radon, by (d).

(h) Let μ be a totally finite τ-additive Borel measure on X, and $\epsilon > 0$. Let $\langle \epsilon_i \rangle_{i \in I}$ be a family of strictly positive real numbers such that $\sum_{i \in I} \epsilon_i \leq \epsilon$ (4A1P). For each $i \in I$, Borel set $F \subseteq X_i$ set $\mu_i F = \mu \pi_i^{-1}[F]$, where $\pi_i(x) = x(i)$ for $x \in X$; because $\pi_i : X \to X_i$ is continuous, μ_i is a totally finite τ-additive Borel measure on X_i. Because X_i is pre-Radon, we can find a compact set $K_i \subseteq X_i$ such that $\mu_i(X_i \setminus K_i) \leq \epsilon_i$, by (a-iv). Now $K = \prod_{i \in I} K_i$ is compact (3A3J), and $X \setminus K \subseteq \bigcup_{i \in I} \pi_i^{-1}[X_i \setminus K_i]$, so

$$\mu(X \setminus K) \leq \textstyle\sum_{i \in I} \mu_i(X_i \setminus K_i) \leq \sum_{i \in I} \epsilon_i \leq \epsilon.$$

As ϵ and μ are arbitrary, X satisfies the condition of (b-iv), and is pre-Radon.

(i) Let $\mathcal{G}$ be a cover of X by pre-Radon open sets. Let μ be a non-zero totally finite τ-additive Borel measure on X. Then $\mu X = \sup\{\mu(\bigcup \mathcal{G}_0) : \mathcal{G}_0 \subseteq \mathcal{G} \text{ is finite}\}$, so there is some $G \in \mathcal{G}$ such that $\mu G > 0$. Now the subspace measure μ_G is a non-zero totally finite τ-additive Borel measure on G, so there is a compact set $K \subseteq G$ such that $\mu_G K > 0$, in which case $\mu K > 0$. As μ is arbitrary, X is pre-Radon, by (a-iii).

434K I return to criteria for deciding whether Hausdorff spaces are Radon.

Proposition (a) A Hausdorff space is Radon iff it is Borel-measure-complete and pre-Radon.

(b) An analytic Hausdorff space is Radon. In particular, any compact metrizable space is Radon and any Polish space is Radon.

(c) ω_1 and $\omega_1 + 1$, with their order topologies, are not Radon.

(d) For a set I, $[0,1]^I$ is Radon iff $\{0,1\}^I$ is Radon iff I is countable.

(e) A hereditarily Lindelöf K-analytic Hausdorff space is Radon; in particular, the split interval (343J, 419L) is Radon.

proof (a) Put the definitions 434C, 434Gb and 434Gc together, recalling that a tight measure is necessarily τ-additive (411E).

(b) 433Cb.

(c) Dieudonné's measure (411Q) is a Borel measure on ω_1 which is not tight, so ω_1 is certainly not a Radon space; as it is an open set in $\omega_1 + 1$, and the subspace topology inherited from $\omega_1 + 1$ is the order topology of ω_1 (4A2S(a-iii)), $\omega_1 + 1$ cannot be Radon (434Fc).

(d) If I is countable, then $\{0,1\}^I$ and $[0,1]^I$ are compact metrizable spaces, so are Radon. If I is uncountable, then $\omega_1 + 1$, with its order topology, is homeomorphic to a closed subset of $\{0,1\}^I$. **P** Set

$\kappa = \#(I)$. For $\xi \leq \omega_1$, $\eta < \kappa$ set $x_\xi(\eta) = 1$ if $\eta < \xi$, 0 if $\xi \leq \eta$. The map $\xi \mapsto x_\xi : \omega_1 + 1 \to \{0,1\}^\kappa$ is injective because $\kappa \geq \omega_1$, and is continuous because all the sets $\{\xi : x_\xi(\eta) = 0\} = (\omega_1 + 1) \cap (\eta + 1)$ are open-and-closed in $\omega_1 + 1$. Since $\omega_1 + 1$ is compact in its order topology (4A2S(a-i)), it is homeomorphic to its image in $\{0,1\}^\kappa \cong \{0,1\}^I$. **Q**

By 434Fc, $\{0,1\}^I$ cannot be a Radon space. Since $\{0,1\}^I$ is a closed subset of $[0,1]^I$, $[0,1]^I$ is also not a Radon space.

(e) Suppose that X is a hereditarily Lindelöf K-analytic Hausdorff space. Then it is Borel-measure-complete by 434Ie and pre-Radon by 434Jf, so by (a) here it is Radon.

Since the split interval is compact and Hausdorff and hereditarily Lindelöf (419La), it is a Radon space.

434L It is worth noting an elementary special property of metric spaces.

Proposition If (X, ρ) is a metric space, then any quasi-Radon measure on X is inner regular with respect to the totally bounded subsets of X.

proof Let μ be a quasi-Radon measure on X and Σ its domain. Suppose that $E \in \Sigma$ and $\gamma < \mu E$. Then there is an open set G of finite measure such that $\mu(E \cap G) > \gamma$; set $\delta = \mu(E \cap G) - \gamma$. For $n \in \mathbb{N}$, $I \subseteq X$ set $H(n, I) = \bigcup_{x \in I} \{y : \rho(y, x) < 2^{-n}\}$. Then $\{H(n, I) : I \in [X]^{<\omega}\}$ is an upwards-directed family of open sets covering X. Because μ is τ-additive, there is a finite set $I_n \subseteq X$ such that $\mu(G \setminus H(n, I_n)) \leq 2^{-n-1}\delta$. Consider $F = \bigcap_{n \in \mathbb{N}} H(n, I_n)$. This is totally bounded and $\mu(G \setminus F) \leq \delta$, so $E \cap F$ is totally bounded and $\mu(E \cap F) \geq \gamma$. As E and γ are arbitrary, μ is inner regular with respect to the totally bounded sets.

434M I turn next to a couple of ideas depending on countable compactness.

Lemma Let X be a countably compact topological space and $\mathcal{E}$ a non-empty family of closed subsets of X with the finite intersection property. Then there is a Borel probability measure μ on X, inner regular with respect to the closed sets, such that $\mu F = 1$ for every $F \in \mathcal{E}$.

proof (a) By Zorn's lemma, $\mathcal{E}$ is included in a maximal family $\mathcal{E}^*$ of closed subsets of X with the finite intersection property.

(i) If $F \subseteq X$ is closed and $F \cap F_0 \cap \ldots \cap F_n \neq \emptyset$ for every $F_0, \ldots, F_n \in \mathcal{E}^*$, then $\mathcal{E}^* \cup \{F\}$ has the finite intersection property, so $F \in \mathcal{E}^*$.

(ii) If $F, F' \in \mathcal{E}^*$, then $F \cap F' \cap F_0 \cap \ldots \cap F_n \neq \emptyset$ for all $F_0, \ldots, F_n \in \mathcal{E}^*$, so $F \cap F' \in \mathcal{E}^*$.

(iii) If $F \subseteq X$ is closed and $F \cap F' \in \mathcal{E}^*$ for every $F' \in \mathcal{E}^*$, then $F \cap F_0 \cap \ldots \cap F_n \in \mathcal{E}^*$ for every $F_0, \ldots, F_n \in \mathcal{E}^*$ (because $F_0 \cap \ldots \cap F_n \in \mathcal{E}^*$, by (ii)), so $F \in \mathcal{E}^*$.

(iv) If $\langle F_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathcal{E}^*$, with intersection F, and $F' \in \mathcal{E}^*$, then $F' \cap \bigcap_{i \leq n} F_i \neq \emptyset$ for every $n \in \mathbb{N}$. Because X is countably compact, $F' \cap F \neq \emptyset$ (4A2G(f-ii)). As F' is arbitrary, $F \in \mathcal{E}^*$, by (iii). Thus $\mathcal{E}^*$ is closed under countable intersections.

(b) Set

$$\Sigma = \{E : E \subseteq X, \text{ there is an } F \in \mathcal{E}^* \text{ such that either } F \subseteq E \text{ or } F \cap E = \emptyset\},$$

and define $\hat{\mu} : \Sigma \to \{0,1\}$ by saying that $\hat{\mu}E = 1$ if there is some $F \in \mathcal{E}^*$ such that $F \subseteq E$, 0 otherwise. Then $\hat{\mu}$ is a probability measure on X. **P**

(i) $\emptyset \in \Sigma$ because $\mathcal{E}^* \supseteq \mathcal{E}$ is not empty.

(ii) $X \setminus E \in \Sigma$ whenever $E \in \Sigma$ because the definition of Σ is symmetric between E and $X \setminus E$.

(iii) If $\langle E_n \rangle_{n \in \mathbb{N}}$ is any sequence in Σ with union E, then either there are $n \in \mathbb{N}$, $F \in \mathcal{E}^*$ such that $F \subseteq E_n \subseteq E$ and $E \in \Sigma$, or for every $n \in \mathbb{N}$ there is an $F_n \in \mathcal{E}^*$ such that $F_n \cap E_n = \emptyset$. In this case $F = \bigcap_{n \in \mathbb{N}} F_n \in \mathcal{E}^*$, by (a-iv), and $E \cap F = \emptyset$, so again $E \in \Sigma$. Thus Σ is a σ-algebra of subsets of X.

(iv) $\hat{\mu}\emptyset = 0$ because $\emptyset$ cannot belong to $\mathcal{E}^*$.

(v) If $\langle E_n \rangle_{n \in \mathbb{N}}$ is any disjoint sequence in Σ with union E, then either there is some n such that $\hat{\mu}E_n = 1$, in which case $\hat{\mu}E_i = 0$ for every $i \neq n$ (because any two members of $\mathcal{E}^*$ must meet) and $\hat{\mu}E = 1 = \sum_{i=0}^\infty \hat{\mu}E_i$, or $\hat{\mu}E_i = 0$ for every i, in which case, just as in (iii), $\hat{\mu}E = 0 = \sum_{i=0}^\infty \hat{\mu}E_i$. Thus $\hat{\mu}$ is a measure.

(vi) Because $\mathcal{E}^* \neq \emptyset$, $\hat{\mu}X = 1$. Thus $\hat{\mu}$ is a probability measure. **Q**

(c) If $F \subseteq X$ is a closed set, then either F itself belongs to $\mathcal{E}^*$, so $F \in \Sigma$, or there is some $F' \in \mathcal{E}^*$ such that $F \cap F' = \emptyset$, in which case again $F \in \Sigma$. So Σ contains every closed set, therefore every Borel set, and $\hat{\mu}$

is a topological measure. By construction, $\hat{\mu}$ is inner regular with respect to $\mathcal{E}^*$ and therefore with respect to the closed sets. Finally, if $F \in \mathcal{E}$ then $F \in \mathcal{E}^*$, so $\hat{\mu}F = 1$. We may therefore take μ to be the restriction of $\hat{\mu}$ to the Borel σ-algebra of X, and μ will be a Borel measure on X, inner regular with respect to the closed sets, such that $\mu E = 1$ for every $E \in \mathcal{E}$.

434N Proposition (a) Let X be a Borel-measure-compact topological space. Then closed countably compact subsets of X are compact.

(b) Let X be a Borel-measure-complete topological space. Then closed countably compact subsets of X are compact.

(c) Let X be a Hausdorff Borel-measure-complete topological space. Then compact subsets of X are countably tight.

(d) In particular, any Radon compact Hausdorff space is countably tight.

proof (a) Let C be a closed countably compact subset of X. Let $\mathcal{F}$ be an ultrafilter on C. Let $\mathcal{E}$ be the family of closed subsets of C belonging to $\mathcal{F}$. Then $\mathcal{E}$ has the finite intersection property, so by 434M there is a Borel probability measure μ on C, inner regular with respect to the closed sets, such that $\mu E = 1$ for every $E \in \mathcal{E}$. Let ν be the Borel measure on X defined by setting $\nu H = \mu(C \cap H)$ for every Borel set $H \subseteq X$. Then ν is also inner regular with respect to the closed sets (of either C or X); because X is Borel-measure-compact, ν has a support F (434H(a-iv)). Since $\nu F = \nu X = 1$, $F \cap C \neq \emptyset$; take $x \in F \cap C$. If G is any open set (in X) containing x, then $\nu(X \setminus G) < 1$, so $C \setminus G \notin \mathcal{F}$ and $C \cap G \in \mathcal{F}$. As G is arbitrary, $\mathcal{F} \to x$; as $\mathcal{F}$ is arbitrary, C is compact.

(b) Repeat the argument of (a). Let C be a countably compact subset of X and $\mathcal{F}$ be an ultrafilter on C. Let $\mathcal{E}$ be the family of relatively closed subsets of C belonging to $\mathcal{F}$. Then there is a Borel probability measure μ on C such that $\mu E = 1$ for every $E \in \mathcal{E}$. Let ν be the Borel measure on X defined by setting $\nu H = \mu(C \cap H)$ for every Borel set $H \subseteq X$. Because X is Borel-measure-complete, ν has a support F (434I(a-iii)). Since $\nu F = \nu X = 1$, $F \cap C \neq \emptyset$; take $x \in F \cap C$. If G is any open set containing x, then $\nu(X \setminus G) < 1$, so $C \setminus G \notin \mathcal{F}$ and $C \cap G \in \mathcal{F}$. As G is arbitrary, $\mathcal{F} \to x$; as $\mathcal{F}$ is arbitrary, C is compact.

(c) Again let C be a (countably) compact subset of X. Take $A \subseteq C$, and set $C_0 = \bigcup\{\overline{B} : B \in [A]^{\leq\omega}\}$. Then C_0 is countably compact. **P** If $\langle y_n\rangle_{n\in\mathbb{N}}$ is any sequence in C_0, it has a cluster point $y \in C$. For each $n \in \mathbb{N}$ there is a countable set $B_n \subseteq A$ such that $y_n \in \overline{B_n}$. Now $B = \bigcup_{n\in\mathbb{N}} B_n$ is a countable subset of A, and $y \in \overline{B} \subseteq C_0$, so y is a cluster point of $\langle y_n\rangle_{n\in\mathbb{N}}$ in C_0. As $\langle y_n\rangle_{n\in\mathbb{N}}$ is arbitrary, C_0 is countably compact. **Q**

By (b), C_0 is compact, therefore closed, and must include $\overline{A}$. Thus every point of $\overline{A}$ is in the closure of some countable subset of A. As A is arbitrary, C is countably tight.

(d) Finally, a compact Radon Hausdorff space is Borel-measure-complete (434Ka) and countably compact, therefore countably tight.

434O Quasi-dyadic spaces I wish now to present a result in an entirely different direction. Measures of type B_1 in the classification of 434A (completion regular, but not τ-additive) seem to be hard to come by. The next theorem shows that on a substantial class of spaces they cannot appear. First, we need a definition.

Definition A topological space X is **quasi-dyadic** if it is expressible as a continuous image of a product of separable metrizable spaces.

I give some elementary results to indicate what kind of spaces we have here.

434P Proposition (a) A continuous image of a quasi-dyadic space is quasi-dyadic.

(b) Any product of quasi-dyadic spaces is quasi-dyadic.

(c) A space with a countable network is quasi-dyadic.

(d) A Baire subset of a quasi-dyadic space is quasi-dyadic.

(e) If X is any topological space, a countable union of quasi-dyadic subspaces of X is quasi-dyadic.

proof (a) Immediate from the definition.

(b) Again immediate; if X_i is a continuous image of $\prod_{j\in J_i} Y_{ij}$, where Y_{ij} is a separable metric space for every $i \in I$, $j \in J_i$, then $\prod_{i\in I} X_i$ is a continuous image of $\prod_{i\in I, j\in J_i} Y_{ij}$.

(c) Let $\mathcal{E}$ be a countable network for the topology of X. On X let $\sim$ be the equivalence relation in which $x \sim y$ if they belong to just the same members of $\mathcal{E}$; let Y be the space $X/\sim$ of equivalence classes, and $\phi : X \to Y$ the canonical map. Y has a separable metrizable topology with base $\{\phi[E] : E \in \mathcal{E}\} \cup \{\phi[X \setminus E] : E \in \mathcal{E}\}$. Let I be any set such that $\#(\{0,1\}^I) \geq \#(X)$, and for each $y \in Y$ let $f_y : \{0,1\}^I \to y$ be a surjection. Then we have a continuous surjection $f : Y \times \{0,1\}^I \to X$ given by saying that $f(y,z) = f_y(z)$ for $y \in Y$, $z \in \{0,1\}^I$.

(d) Let $\langle Y_i\rangle_{i\in I}$ be a family of separable metric spaces with product Y and $f : Y \to X$ a continuous surjection. If $W \subseteq Y$ is a Baire set, it is determined by coordinates in a countable subset of I (4A3Nb), so can be regarded as $W' \times \prod_{i\in I\setminus J} Y_i$, where $J \subseteq I$ is countable and $W' \subseteq \prod_{i\in J} Y_i$; as $\prod_{i\in J} Y_i$ and W' are separable metrizable spaces (4A2Pa), W can be thought of as a product of separable metrizable spaces. Now the set $\{E : E \subseteq X, f^{-1}[E] \text{ is a Baire set in } Y\}$ is a σ-algebra containing every zero set in X, so contains every Baire set. Thus every Baire subset of X is a continuous image of a Baire subset of Y, and is therefore quasi-dyadic.

(e) If $E_n \subseteq X$ is quasi-dyadic for each $n \in \mathbb{N}$, then $Z = \mathbb{N} \times \prod_{n\in\mathbb{N}} E_n$ is quasi-dyadic, and $f : Z \to \bigcup_{n\in\mathbb{N}} E_n$ is a continuous surjection, where $f(n, \langle x_i\rangle_{i\in\mathbb{N}}) = x_n$. So $\bigcup_{n\in\mathbb{N}} E_n$ is quasi-dyadic.

434Q Theorem(Fremlin & Grekas 95) A semi-finite completion regular topological measure on a quasi-dyadic space is τ-additive.

proof ? Suppose, if possible, otherwise.

(a) The first step is the standard reduction to the case in which $\mu X = 1$ and X is covered by open sets of zero measure. In detail: suppose that X is a quasi-dyadic space and μ_0 is a semi-finite completion regular topological measure on X which is not τ-additive. Let $\mathcal{G}$ be an upwards-directed family of open sets in X such that $\mu_0(\bigcup\mathcal{G})$ is strictly greater than $\sup_{G\in\mathcal{G}} \mu_0 G = \gamma$ say. Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathcal{G}$ such that $\lim_{n\to\infty} \mu_0 G_n = \gamma$, and set $H_0 = \bigcup_{n\in\mathbb{N}} G_n$, so that $\mu_0 H_0 = \gamma$; take a closed set $Z \subseteq \bigcup\mathcal{G}$ such that $\gamma < \mu_0 Z < \infty$. Set $\mu_1 E = \mu_0(E \cap Z \setminus H_0)$ for every Borel set $E \subseteq X$. Then μ_1 is a non-zero totally finite Borel measure on X, and is completion regular. **P** If $E \subseteq X$ is a Borel set and $\epsilon > 0$, there is a zero set $F \subseteq E \cap Z \setminus H_0$ such that $\mu_0 F \geq \mu_0(E \cap X \setminus H_0) - \epsilon$, and now $\mu_1 F \geq \mu_1 E - \epsilon$. **Q** Note that $\mu_1(X \setminus Z) = \mu_1 G = 0$ for every $G \in \mathcal{G}$.

For Borel sets $E \subseteq X$, set $\mu E = \mu_1 E/\mu_1 X$; then μ is a completion regular Borel probability measure on X, and $\mathcal{G} \cup \{X \setminus Z\}$ is a cover of X by open negligible sets.

(b) Now let $\langle Y_i\rangle_{i\in I}$ be a family of separable metric spaces such that there is a continuous surjection $f : Y \to X$, where $Y = \prod_{i\in I} Y_i$. For each $i \in I$ let $\mathcal{B}_i$ be a countable base for the topology of Y_i. For $J \subseteq I$ let $\mathcal{C}(J)$ be the family of all open cylinders in Y expressible in the form

$$\{s : s(i) \in B_i \ \forall\, i \in K\},$$

where K is a finite subset of J and $B_i \in \mathcal{B}_i$ for each $i \in K$; thus $\mathcal{C}(I)$ is a base for the topology of Y. Set $\mathcal{C}_0(J) = \{U : U \in \mathcal{C}(J), \mu^* f[U] = 0\}$ for each $J \subseteq I$. Note that (because every $\mathcal{B}_i$ is countable) $\mathcal{C}(J)$ and $\mathcal{C}_0(J)$ are countable for every countable subset J of I. It is easy to see that $\mathcal{C}(J) \cap \mathcal{C}(K) = \mathcal{C}(J \cap K)$ for all J, $K \subseteq I$, because if $U \in \mathcal{C}(I)$ is not empty it belongs to $\mathcal{C}(J)$ iff its projection onto X_i is the whole of X_i for every $i \notin J$.

For each negligible set $E \subseteq X$, let $\langle F_n(E)\rangle_{n\in\mathbb{N}}$ be a family of zero sets, subsets of $X \setminus E$, such that $\sup_{n\in\mathbb{N}} \mu F_n = 1$. Then each $f^{-1}[F_n(E)]$ is a zero set in Y, so there is a countable set $M(E) \subseteq I$ such that all the sets $f^{-1}[F_n(E)]$ are determined by coordinates in $M(E)$ (4A3Nc). Let $\mathcal{J}$ be the family of countable subsets J of I such that $M(f[U]) \subseteq J$ for every $U \in \mathcal{C}_0(J)$; then $\mathcal{J}$ is cofinal with $[I]^{\leq\omega}$, that is, every countable subset of I is included in some member of $\mathcal{J}$. **P** If we start from any countable subset J_0 of I and set

$$J_{n+1} = J_n \cup \bigcup\{M(f[U]) : U \in \mathcal{C}_0(J_n)\}$$

for each $n \in \mathbb{N}$, then every J_n is countable, and $\bigcup_{n\in\mathbb{N}} J_n \in \mathcal{J}$, because $\langle J_n\rangle_{n\in\mathbb{N}}$ is non-decreasing, so $\mathcal{C}_0(\bigcup_{n\in\mathbb{N}} J_n) = \bigcup_{n\in\mathbb{N}} \mathcal{C}_0(J_n)$. **Q**

(c) For each $J \in \mathcal{J}$, set

$$Q_J = \bigcap\{\bigcup_{n\in\mathbb{N}} F_n(f[U]) : U \in \mathcal{C}_0(J)\}.$$

Then $\mu Q_J = 1$ and $f^{-1}[Q_J]$ is determined by coordinates in J, while $f^{-1}[Q_J] \cap U = \emptyset$ whenever $U \in \mathcal{C}_0(J)$.

If $G \subseteq X$ is an open set, then $G \cap Q_J = \emptyset$ whenever $J \in \mathcal{J}$ and there is a negligible Baire set $Q \supseteq G$ such that $f^{-1}[Q]$ is determined by coordinates in J. **P** Set $H = \pi_J^{-1}[\pi_J[f^{-1}[G]]]$, where $\pi_J : Y \to \prod_{i\in J} Y_i$ is the canonical map; then H is a union of members of $\mathcal{C}(J)$, because $f^{-1}[G]$ is open in Y and $\pi_J[f^{-1}[G]]$ is open in $\prod_{i\in J} Y_i$. Also, because $f^{-1}[Q]$ is determined by coordinates in J, $H \subseteq f^{-1}[Q]$, so $f[H] \subseteq Q$ and $\mu^* f[H] = 0$; thus all the members of $\mathcal{C}(J)$ included in H actually belong to $\mathcal{C}_0(J)$, and $H \cap f^{-1}[Q_J] = \emptyset$. But this means that $f^{-1}[G] \cap f^{-1}[Q_J] = \emptyset$ and (because f is a surjection) $G \cap Q_J = \emptyset$, as claimed. **Q** In particular, if G is a negligible open set in X, then $G \cap Q_J = \emptyset$ whenever $J \in \mathcal{J}$ and $J \supseteq M(G)$.

(d) If $J \in \mathcal{J}$, there are s, $s' \in f^{-1}[Q_J]$ such that $s{\restriction}J = s'{\restriction}J$ and $f(s)$, $f(s')$ can be separated by open sets in X. **P** Start from any $x \in Q_J$ and take a negligible open set G including x (recall that our hypothesis is that X is covered by negligible open sets). For each $n \in \mathbb{N}$ let $h_n : X \to \mathbb{R}$ be a continuous function such that $F_n(G) = h_n^{-1}[\{0\}]$. We know that $G \cap Q_J \neq \emptyset$, while $G \subseteq X \setminus (\bigcup_{n\in\mathbb{N}} F_n(G) \cap Q_J)$, which is a negligible Baire set; by (c), $f^{-1}[X \setminus (\bigcup_{n\in\mathbb{N}} F_n(G) \cap Q_J)]$ is not determined by coordinates in J, and there must be some n such that $f^{-1}[F_n(G) \cap Q_J]$ is not determined by coordinates in J. Accordingly there must be s, $s' \in Y$ such that $s{\restriction}J = s'{\restriction}J$, $s \in f^{-1}[F_n(G) \cap Q_J]$ and $s' \notin f^{-1}[F_n(G) \cap Q_J]$. Now $s \in f^{-1}[Q_J]$, which is determined by coordinates in J; since $s{\restriction}J = s'{\restriction}J$, $s' \in f^{-1}[Q_J]$ and $s' \notin f^{-1}[F_n(G)]$. Accordingly $h_n(f(s)) = 0 \neq h_n(f(s'))$ and $f(s)$, $f(s')$ can be separated by open sets. **Q**

(e) We are now ready to embark on the central construction of the argument. We may choose inductively, for ordinals $\xi < \omega_1$, sets $J_\xi \in \mathcal{J}$, negligible open sets G_ξ, $G'_\xi \subseteq X$, points s_ξ, $s'_\xi \in Y$ and sets U_ξ, V_ξ, $V'_\xi \in \mathcal{C}(I)$ such that

$J_\eta \subseteq J_\xi$, U_η, V_η, V'_η all belong to $\mathcal{C}(J_\xi)$ and $G_\eta \cap Q_{J_\xi} = \emptyset$ whenever $\eta < \xi < \omega_1$ (using the results of (b) and (c) to choose J_ξ);

$s_\xi{\restriction}J_\xi = s'_\xi{\restriction}J_\xi$, $s_\xi \in f^{-1}[Q_{J_\xi}]$ and $f(s_\xi)$ and $f(s'_\xi)$ can be separated by open sets in X (using (d) to choose s_ξ, s'_ξ);

G_ξ, G'_ξ are disjoint negligible open sets containing $f(s_\xi)$, $f(s'_\xi)$ respectively (choosing G_ξ, G'_ξ);

$U_\xi \in \mathcal{C}(J_\xi)$, V_ξ, $V'_\xi \in \mathcal{C}(I \setminus J_\xi)$, $s_\xi \in U_\xi \cap V_\xi \subseteq f^{-1}[G_\xi]$, $s'_\xi \in U_\xi \cap V'_\xi \subseteq f^{-1}[G'_\xi]$ (choosing U_ξ, V_ξ, V'_ξ, using the fact that $s_\xi{\restriction}J_\xi = s'_\xi{\restriction}J_\xi$).

On completing this construction, take for each $\xi < \omega_1$ a finite set $K_\xi \subseteq J_{\xi+1}$ such that U_ξ, V_ξ and V'_ξ all belong to $\mathcal{C}(K_\xi)$. By the Δ-system Lemma (4A1Db), there is an uncountable $A \subseteq \omega_1$ such that $\langle K_\xi\rangle_{\xi\in A}$ is a Δ-system with root K say. For $\xi \in A$, express U_ξ as $\tilde{U}_\xi \cap U'_\xi$ where $\tilde{U}_\xi \in \mathcal{C}(K)$, $U'_\xi \in \mathcal{C}(K_\xi \setminus K)$. Then there are only countably many possibilities for $\tilde{U}_\xi$, so there is an uncountable $B \subseteq A$ such that $\tilde{U}_\xi$ is constant for $\xi \in B$; write $\tilde{U}$ for the constant value. Let $C \subseteq B$ be an uncountable set, not containing $\min A$, such that $K_\xi \setminus K$ does not meet J_η whenever ξ, $\eta \in C$ and $\eta < \xi$ (4A1Eb). Let $D \subseteq C$ be such that D and $C \setminus D$ are both uncountable.

Note that $K \subseteq K_\eta \subseteq J_\xi$ whenever η, $\xi \in A$ and $\eta < \xi$, so that $K \subseteq J_\xi$ for every $\xi \in C$. Consequently U'_ξ, V_ξ and V'_ξ all belong to $\mathcal{C}(K_\xi \setminus K)$ for every $\xi \in C$.

(f) Consider the open set

$$G = \bigcup_{\xi\in D} G_\xi \subseteq X.$$

At this point the argument divides.

case 1 Suppose $\mu^*(G \cap f[\tilde{U}]) > 0$. Then there is a Baire set $Q \subseteq G$ such that $\mu^*(Q \cap f[\tilde{U}]) > 0$. Let $J \subseteq I$ be a countable set such that $f^{-1}[Q]$ is determined by coordinates in J. Let $\gamma \in C \setminus D$ be so large that $K_\xi \setminus K$ does not meet J for any $\xi \in A$ with $\xi \geq \gamma$. Then $Q \cap Q_{J_\gamma} \cap f[\tilde{U}]$ is not empty; take $s \in \tilde{U} \cap f^{-1}[Q \cap Q_{J_\gamma}]$. Because the $K_\xi \setminus K$ are disjoint from each other and from $J \cup J_\gamma$ for $\xi \in D$, $\xi > \gamma$, we may modify s to form s' such that $s'{\restriction}J \cup J_\gamma = s{\restriction}J \cup J_\gamma$ and $s' \in U'_\xi \cap V'_\xi$ for every $\xi \in D$, $\xi > \gamma$; now $s' \in \tilde{U}$ (because $K \subseteq J_\gamma$), so $s' \in \tilde{U} \cap U'_\xi \cap V'_\xi \subseteq f^{-1}[G'_\xi]$ and $f(s') \notin G_\xi$ whenever $\xi \in D$, $\xi > \gamma$. On the

other hand, if $\xi \in D$ and $\xi < \gamma$, $G_\xi \cap Q_{J_\gamma} = \emptyset$, while $s' \in f^{-1}[Q_{J_\gamma}]$ (because $f^{-1}[Q_{J_\gamma}]$ is determined by coordinates in J_γ), so again $f(s') \notin G_\xi$.

Thus $f(s') \notin G$. But $s' \restriction J = s \restriction J$ so $f(s) \in Q \subseteq G$; which is impossible.

This contradiction disposes of the possibility that $\mu^*(G \cap f[\tilde{U}]) > 0$.

case 2 Suppose that $\mu^*(G \cap f[\tilde{U}]) = 0$. In this case there is a negligible Baire set $Q \supseteq G \cap f[\tilde{U}]$. Let $J \subseteq I$ be a countable set such that $f^{-1}[Q]$ is determined by coordinates in J. Let $\xi \in D$ be such that $K_\xi \setminus K$ does not meet J. Then

$$\tilde{U} \cap U'_\xi \cap V_\xi \subseteq f^{-1}[G_\xi] \cap \tilde{U} \subseteq f^{-1}[G \cap f[\tilde{U}]] \subseteq f^{-1}[Q],$$

so $\tilde{U} \subseteq f^{-1}[Q]$, because $U'_\xi \cap V_\xi$ is a non-empty member of $\mathcal{C}(I \setminus J)$. But this means that $\mu^* f[\tilde{U}] = 0$ and $\mu^* f[U_\xi] = 0$. On the other hand, we have $s_\xi \in U_\xi \cap f^{-1}[Q_{J_\xi}]$, so $U_\xi \notin \mathcal{C}_0(J_\xi)$ and $\mu^* f[U_\xi] > 0$. **X**

Thus this route also is blocked and we must abandon the original hypothesis that there is a quasi-dyadic space with a semi-finite completion regular topological measure which is not τ-additive.

434R There is a useful construction of Borel product measures which can be fitted in here.

Proposition Let X and Y be topological spaces with Borel measures μ and ν; write $\mathcal{B}(X)$, $\mathcal{B}(Y)$ for the Borel σ-algebras of X and Y respectively. If *either* X is first-countable *or* ν is τ-additive and effectively locally finite, there is a Borel measure λ_B on $X \times Y$ defined by the formula

$$\lambda_B W = \sup_{F \in \mathcal{B}(Y), \nu F < \infty} \int \nu(W[\{x\}] \cap F) \mu(dx)$$

for every Borel set $W \subseteq X \times Y$. Moreover

(i) if μ is semi-finite, then λ_B agrees with the c.l.d. product measure λ on $\mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y)$, and the c.l.d. version $\tilde{\lambda}_B$ of λ_B extends λ;

(ii) if ν is σ-finite, then $\lambda_B W = \int \nu W[\{x\}] \mu(dx)$ for every Borel set $W \subseteq X \times Y$;

(iii) if both μ and ν are τ-additive and effectively locally finite, then λ_B is just the restriction of the τ-additive product measure $\tilde{\lambda}$ (417D, 417G) to the Borel σ-algebra of $X \times Y$; in particular, λ_B is τ-additive.

proof (a) The point is that $x \mapsto \nu(W[\{x\}] \cap F)$ is lower semi-continuous whenever $W \subseteq X \times Y$ is open and $\nu F < \infty$. **P** Of course $W[\{x\}]$ is always open, so ν always measures $W[\{x\}] \cap F$. Take any $\alpha \in \mathbb{R}$ and set $G = \{x : x \in X, \nu(W[\{x\}] \cap F) > \alpha\}$; let $x_0 \in G$.

(α) Suppose that X is first-countable. Let $\langle U_n \rangle_{n \in \mathbb{N}}$ be a non-increasing sequence running over a base of open neighbourhoods of x_0. For each $n \in \mathbb{N}$, set

$$V_n = \bigcup \{V : V \subseteq Y \text{ is open}, U_n \times V \subseteq W\}.$$

Then $\langle V_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence with union $W[\{x\}]$, so there is an $n \in \mathbb{N}$ such that $\nu(V_n \cap F) > \alpha$. Now $V_n \subseteq W[\{x\}]$ for every $x \in U_n$, so $U_n \subseteq G$.

(β) Suppose that ν is τ-additive and effectively locally finite. Set

$$\mathcal{V} = \{V : V \subseteq Y \text{ is open}, U \times V \subseteq W \text{ for some open set } U \text{ containing } x_0\}.$$

Then $\mathcal{V}$ is an upwards-directed family of open sets with union $W[\{x_0\}]$, so there is a $V \in \mathcal{V}$ such that $\nu(V \cap F) > \alpha$ (414Ea). Let U be an open set containing x_0 such that $U \times V \subseteq W$; then $V \subseteq W[\{x\}]$ for every $x \in U$, so $U \subseteq G$.

(γ) Thus in either case we have an open set containing x_0 and included in G. As x_0 is arbitrary, G is open; as α is arbitrary, $x \mapsto \nu(W[\{x\}] \cap F)$ is lower semi-continuous. **Q**

(b) It follows that $x \mapsto \nu(W[\{x\}] \cap F)$ is Borel measurable whenever $W \subseteq X \times Y$ is a Borel set and $\nu F < \infty$. **P** Let $\mathcal{W}$ be the family of sets $W \subseteq X \times Y$ such that $W[\{x\}]$ is a Borel set for every $x \in X$ and $x \mapsto \nu(W[\{x\}] \cap F)$ is Borel measurable. Then every open subset of $X \times Y$ belongs to $\mathcal{W}$ (by (a) above), $W \setminus W' \in \mathcal{W}$ whenever $W, W' \in \mathcal{W}$ and $W' \subseteq W$, and $\bigcup_{n \in \mathbb{N}} W_n \in \mathcal{W}$ whenever $\langle W_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in $\mathcal{W}$. By the Monotone Class Theorem (136B), $\mathcal{W}$ includes the σ-algebra generated by the open sets, that is, the Borel σ-algebra of $X \times Y$. **Q**

(c) It is now easy to check that $W \mapsto \int \nu(W[\{x\}] \cap F) \mu(dx)$ is a Borel measure on $X \times Y$ whenever $\nu F < \infty$, and therefore that λ_B, as defined here, is a Borel measure.

(d) Now suppose that μ is semi-finite, and that $W \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$. Then

$$\lambda W = \sup_{\mu E<\infty, \nu F<\infty} \lambda(W \cap (E \times F))$$

(by the definition of 'c.l.d. product measure', 251F)

$$= \sup_{\mu E<\infty, \nu F<\infty} \int_E \nu(W[\{x\}] \cap F)\mu(dx)$$

(by Fubini's theorem, 252C, applied to the product of the subspace measures μ_E and ν_F)

$$= \sup_{\nu F<\infty} \int \nu(W[\{x\}] \cap F)\mu(dx)$$

(by 213B, because μ is semi-finite)

$$= \lambda_B W.$$

(e) If, on the other hand, ν is σ-finite, let $\langle F_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of sets of finite measure covering Y; then

$$\lambda_B W = \sup_{\nu F<\infty} \int \nu(W[\{x\}] \cap F)\mu(dx) \geq \sup_{n\in\mathbb{N}} \int \nu(W[\{x\}] \cap F_n)\mu(dx)$$
$$= \int \sup_{n\in\mathbb{N}} \nu(W[\{x\}] \cap F_n)\mu(dx) = \int \nu W[\{x\}]\mu(dx) \geq \lambda_B W$$

for any Borel set $W \subseteq X$.

(f) If both μ and ν are τ-additive and effectively locally finite, so that we have a τ-additive product measure $\tilde{\lambda}$, then Fubini's theorem for such measures (417H) tells us that $\lambda_B W = \tilde{\lambda} W$ at least when $W \subseteq X \times Y$ is a Borel set and $\tilde{\lambda} W$ is finite. If W is any Borel subset of $X \times Y$, then, as in (d),

$$\lambda_B W = \sup_{\mu E<\infty, \nu F<\infty} \int_E \nu(W[\{x\}] \cap F)\mu(dx)$$
$$= \sup_{\mu E<\infty, \nu F<\infty} \tilde{\lambda}(W \cap (E \times F)) = \tilde{\lambda} W$$

by 417C(iii).

Remark The case in which X is first-countable is due to Johnson 82.

***434S** The concept of 'universally measurable' set enables us to extend a number of ideas from earlier sections. First, recall a problem from the very beginning of measure theory on the real line: the composition of Lebesgue measurable functions need not be Lebesgue measurable (134Ib), while the composition of a Borel measurable function with a Lebesgue measurable function is measurable (121Eg). In fact we can replace 'Borel measurable' by 'universally measurable', as follows.

Proposition Let (X, Σ, μ) be a complete locally determined measure space, Y and Z topological spaces, $f : X \to Y$ a measurable function and $g : Y \to Z$ a universally measurable function. Then $gf : X \to Z$ is measurable. In particular, $f^{-1}[F] \in \Sigma$ for every universally measurable set $F \subseteq Y$.

proof Let $H \subseteq Z$ be an open set and $E \in \Sigma$ a set of finite measure. Let μ_E be the subspace measure on E. Then the image measure $\nu = \mu_E(f\restriction E)^{-1}$ is a complete totally finite topological measure on Y, so its domain contains $g^{-1}[H]$, and

$$E \cap (gf)^{-1}[H] = (f\restriction E)^{-1}[g^{-1}[H]] \in \operatorname{dom}\mu_E \subseteq \Sigma.$$

As E is arbitrary and μ is locally determined, $(gf)^{-1}[H] \in \Sigma$; as H is arbitrary, gf is measurable.

Applying this to $g = \chi F$, we see that $f^{-1}[F] \in \Sigma$ for every universally measurable $F \subseteq Y$.

***434T** The next remark concerns the concept $[\![u \in E]\!]$ of §364.

Proposition Let $(\mathfrak{A}, \bar{\mu})$ be a localizable measure algebra. Write $\Sigma_{\rm um}$ for the algebra of universally measurable subsets of $\mathbb{R}$.

(a) For any $u \in L^0 = L^0(\mathfrak{A})$, we have a sequentially order-continuous Boolean homomorphism $E \mapsto [\![u \in E]\!] : \Sigma_{\rm um} \to \mathfrak{A}$ defined by saying that

$$[\![u \in E]\!] = \sup\{[\![u \in F]\!] : F \subseteq E \text{ is Borel}\} = \inf\{[\![u \in F]\!] : F \supseteq E \text{ is Borel}\}$$

for every $E \in \Sigma_{\rm um}$.

(b) For any $u \in L^0$ and universally measurable function $h : \mathbb{R} \to \mathbb{R}$ we have a corresponding element $\bar{h}(u)$ of L^0 defined by the formula

$$[\![\bar{h}(u) \in E]\!] = [\![u \in h^{-1}[E]]\!] \text{ for every } E \in \Sigma_{\rm um},\ u \in L^0.$$

proof We can regard $(\mathfrak{A}, \bar{\mu})$ as the measure algebra of a complete strictly localizable measure space (X, Σ, μ) (322N), in which case L^0 can be identified with $L^0(\mu)$ (364J). Write $\mathcal{B}$ for the Borel σ-algebra of $\mathbb{R}$.

(a) If $u \in L^0$, let $f : X \to \mathbb{R}$ be a Σ-measurable function representing u. Then $f^{-1}[E] \in \Sigma$ for every $E \in \Sigma_{\rm um}$, by 434S. Setting $\phi E = (f^{-1}[E])^\bullet$, $\phi : \Sigma_{\rm um} \to \mathfrak{A}$ is a sequentially order-continuous Boolean homomorphism.

Now

$$\phi E = \sup\{[\![u \in F]\!] : F \in \mathcal{B},\ F \subseteq E\} = \inf\{[\![u \in F]\!] : F \in \mathcal{B},\ F \supseteq E\}$$

for every $E \in \Sigma_{\rm um}$. **P** If $H \in \Sigma$ and $\mu H < \infty$, then (writing μ_H for the subspace measure on H) the image measure $\mu_H(f{\restriction}H)^{-1}$ is a complete topological measure, and its restriction ν to the Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}$ is a totally finite Borel measure. Now E is measured by the completion of ν, so there are F_1, $F_2 \in \mathcal{B}$ such that $F_1 \subseteq E \subseteq F_2$ and $\nu(F_2 \setminus F_1) = 0$. In this case,

$$[\![u \in F_1]\!] = (f^{-1}[F_1])^\bullet \subseteq \phi E \subseteq (f^{-1}[F_2])^\bullet = [\![u \in F_2]\!],$$

using the formula of 364Jb, while $H^\bullet \cap [\![u \in F_1]\!] = H^\bullet \cap [\![u \in F_2]\!]$. As $(\mathfrak{A}, \bar{\mu})$ is semi-finite, $\sup\{[\![u \in F]\!] : F \in \mathcal{B},\ F \subseteq E\}$ and $\inf\{[\![u \in F]\!] : F \in \mathcal{B},\ F \supseteq E\}$ must both be equal to ϕE. **Q** We can therefore identify the sequentially order-continuous Boolean homomorphism ϕ with $E \mapsto [\![u \in E]\!]$, as described.

(b) If $u \in L^0$ and $h : \mathbb{R} \to \mathbb{R}$ is universally measurable, then, once again identifying u with $f^\bullet$ where f is Σ-measurable, we see that hf is Σ-measurable (by 434S), so we have a corresponding element $(hf)^\bullet$ of L^0. If $E \in \Sigma_{\rm um}$, then

$$[\![(hf)^\bullet \in E]\!] = ((hf)^{-1}[E])^\bullet = (f^{-1}[h^{-1}[E]])^\bullet = [\![u \in h^{-1}[E]]\!],$$

using 434De to check that $h^{-1}[E] \in \Sigma_{\rm um}$, so that we can identify $(hf)^\bullet$ with $\bar{h}(u)$, as described.

434X Basic exercises >(a) Let $A \subseteq [0,1]$ be any non-measurable set. Show that the subspace measure on A is completion regular and τ-additive but not tight.

>(b) Let X be any Hausdorff space with a point x such that $\{x\}$ is not a G_δ set; for instance, $X = \omega_1 + 1$ and $x = \omega_1$, or $X = \{0,1\}^I$ for any uncountable set I and x any point of X. Show that setting $\mu E = \chi E(x)$ we get a tight Borel measure on X which is not completion regular.

>(c) Let X be a topological space. (i) Show that if $A \subseteq X$ is universally measurable in X, then $A \cap Y$ is universally measurable in Y for any set $Y \subseteq X$. (ii) Show that if $Y \subseteq X$ is universally measurable in X, and $A \subseteq Y$ is universally measurable in Y, then A is universally measurable in X. (iii) Suppose that X is the product of a countable family $\langle X_i \rangle_{i \in I}$ of topological spaces, and $E_i \subseteq X_i$ is a universally measurable set for each $i \in I$. Show that $\prod_{i \in I} E_i$ is universally measurable in X.

(d) Let X be a topological space, Y an analytic Hausdorff space, and $W \subseteq X \times Y$ a Borel set. Show that $W^{-1}[Y]$ is a universally measurable subset of X. (*Hint*: 423N.)

(e) Let X be a Hausdorff space. (i) Show that, for $A \subseteq X$, the following are equiveridical: (α) A is universally Radon-measurable in X; (β) A is measured by every atomless Radon probability measure on X;

(γ) $A \cap K$ is universally Radon-measurable in K for every compact $K \subseteq X$. (ii) Show that if $A \subseteq X$ is universally Radon-measurable in X, then $A \cap Y$ is universally Radon-measurable in Y for any set $Y \subseteq X$. (iii) Show that if $Y \subseteq X$ is universally Radon-measurable in X, and $A \subseteq Y$ is universally Radon-measurable in Y, then A is universally Radon-measurable in X. (iv) Show that if $\mathcal{G}$ is an open cover of X, and $A \subseteq X$ is such that $A \cap G$ is universally Radon-measurable (in G or in X) for every $G \in \mathcal{G}$, then A is universally Radon-measurable in X. (v) Show that if Y is another Hausdorff space, and $\Sigma^{(X)}_{\mathrm{uRm}}$, $\Sigma^{(Y)}_{\mathrm{uRm}}$ are the algebras of universally Radon-measurable subsets of X, Y respectively, then every continuous function from X to Y is $(\Sigma^{(X)}_{\mathrm{uRm}}, \Sigma^{(Y)}_{\mathrm{uRm}})$-measurable. (vi) Suppose that X is the product of a countable family $\langle X_i \rangle_{i \in I}$ of topological spaces, and $E_i \subseteq X_i$ is a universally Radon-measurable set for each $i \in I$. Show that $\prod_{i \in I} E_i$ is universally Radon-measurable in X.

>(f) (i) Let μ_0 be Dieudonné's measure on ω_1. Give $\omega_1 + 1 = \omega_1 \cup \{\omega_1\}$ its compact Hausdorff order topology, and define a Borel measure μ on $\omega_1 + 1$ by setting $\mu E = \mu_0(E \cap \omega_1)$ for every Borel set $E \subseteq \omega_1 + 1$. Show that μ is a complete probability measure and is neither τ-additive nor inner regular with respect to the closed sets. (ii) Show that the universally measurable subsets of $\omega_1 + 1$ are just its Borel sets. (*Hint*: 4A3J, 411Q.) (iii) Show that every totally finite τ-additive topological measure on $\omega_1 + 1$ has a countable support. (iv) Show that every subset of $\omega_1 + 1$ is universally Radon-measurable.

(g) (i) Show that there is a set $X \subseteq [0,1]$ such that $K \cap X$ and $K \setminus X$ are both of cardinal $\mathfrak{c}$ for every uncountable compact set $K \subseteq [0,1]$. (*Hint*: 419Ib, 423K.) (ii) Show that if we give X its subspace topology, then every subset of X is universally Radon-measurable, but not every subset is universally measurable. (*Hint*: every compact subset of X is countable, so every Radon measure on X is purely atomic, but X has full outer Lebesgue measure in $[0,1]$.)

(h) Show that a Hausdorff space X is Radon iff (α) every compact subset of X is Radon (β) for every non-zero totally finite Borel measure μ on X there is a compact subset K of X such that $\mu K > 0$. (*Hint*: 434F(a-v).)

>(i) (i) Let X and Y be K-analytic Hausdorff spaces and $f : X \to Y$ a continuous surjection. Suppose that $F \subseteq Y$ and that $f^{-1}[F]$ is universally Radon-measurable in X. Show that F is universally Radon-measurable in Y. (*Hint*: 432G.) (ii) Let X and Y be analytic Hausdorff spaces and $f : X \to Y$ a Borel measurable surjection. Suppose that $F \subseteq Y$ and that $f^{-1}[F]$ is universally Radon-measurable in X. Show that F is universally Radon-measurable in Y. (*Hint*: 433D.)

(j) Show that if X is a perfectly normal space then it is Borel-measure-compact iff it is Borel-measure-complete.

(k) Show that if we give $\omega_1 + 1$ its order topology, it is Borel-measure-compact but not Borel-measure-complete or pre-Radon, and that its open subset ω_1 is not Borel-measure-compact.

(l) Set $X = \omega_1 + 1$, with its order topology, and let Σ be the σ-algebra of subsets of X generated by the countable sets and the set Ω of limit ordinals in X. Show that there is a unique probability measure μ on X with domain Σ such that $\mu\xi = \mu\Omega = 0$ for every $\xi < \omega_1$. Show that μ is inner regular with respect to the Borel sets, is defined on a base for the topology of the compact Hausdorff space X, but has no extension to a topological measure on X.

(m) Show that $\mathbb{R}$, with the right-facing Sorgenfrey topology, is Borel-measure-complete and Borel-measure-compact, but not Radon or pre-Radon.

(n) Let X be a topological space. (i) Show that the family of Borel-measure-compact subsets of X is closed under Souslin's operation. (ii) Show that the union of a sequence of Borel-measure-complete subsets of X is Borel-measure-complete. (iii) Show that if X is Hausdorff then the family of pre-Radon subsets of X is closed under Souslin's operation. (*Hint*: in (i) and (iii), start by showing that the family under consideration is closed under countable unions.)

(o) Show that $]0,1[^{\omega_1}$ is not pre-Radon.

>(p) Show that a K-analytic Hausdorff space is Radon iff all its compact subsets are Radon. (*Hint*: 432B, 434Xh.)

(q) Suppose that X is a K-analytic Hausdorff space such that every Radon measure on X is completion regular. Show that X is a Radon space.

(r) Let X and Y be topological spaces, and suppose that Y has a countable network. (i) Show that if X is Borel-measure-complete, then $X \times Y$ is Borel-measure-complete. (ii) Show that if X and Y are Radon Hausdorff spaces, then $X \times Y$ is Radon.

(s) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a sequence of topological spaces; write $X = \prod_{n\in\mathbb{N}} X_n$ and $Z_n = \prod_{i\le n} X_i$ for each n. (i) Show that if every Z_n is Borel-measure-complete, so is X. (ii) Show that if every Z_n is Hausdorff and pre-Radon, so is X. (iii) Show that if every Z_n is Hausdorff and Radon, so is X.

(t) (i) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a sequence of Radon Hausdorff spaces such that $\prod_{i\le n} K_i$ is Radon whenever $n \in \mathbb{N}$ and $K_i \subseteq X_i$ is compact for every $i \le n$. Show that $X = \prod_{n\in\mathbb{N}} X_n$ is Radon. (ii) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a sequence of Radon Hausdorff spaces with countable networks. Show that $\prod_{n\in\mathbb{N}} X_n$ is Radon.

(u) (i) Let X be a Radon Hausdorff space and Y a Borel-measure-compact space. Show that $X \times Y$ is Borel-measure-compact. (ii) Let X be a Radon Hausdorff space and Y a Borel-measure-complete space. Show that $X \times Y$ is Borel-measure-complete.

(v) Show that if, in 434R, ν is σ-finite, then $\int g\,d\lambda_B = \iint g(x,y)\nu(dy)\mu(dx)$ for every λ_B-integrable function $g : X \times Y \to \mathbb{R}$.

(w) Show that the product measure construction of 434R is 'associative' and 'distributive' in the sense that (under appropriate hypotheses) the product measures on $(X \times Y) \times Z$ and $X \times (Y \times Z)$ agree, and those on $\bigcup_{i,j\in\mathbb{N}}(X_i \times Y_j)$ and $(\bigcup_{i\in\mathbb{N}} X_i) \times (\bigcup_{j\in\mathbb{N}} Y_j)$ agree.

>(x) Show that the product measure construction of 434R is not 'commutative'; indeed, taking $\mu = \nu$ to be Dieudonné's measure on ω_1, show that the Borel measures λ_1, λ_2 on ω_1^2 defined by setting

$$\lambda_1 W = \int \nu W[\{\xi\}]\mu(d\xi), \quad \lambda_2 W = \int \mu W^{-1}[\{\eta\}]\nu(d\eta)$$

are different.

(y) Read through §271, looking for ways to apply the concept '$\Pr(\boldsymbol{X} \in E)$' for random variables $\boldsymbol{X}$ and universally measurable sets E.

434Y Further exercises (a) Set $X = \mathbb{N} \setminus \{0,1\}$. For m, $p \in X$ set $U_{mp} = m + p\mathbb{N}$; show that $\{U_{mp} : m, p \in X \text{ are coprime}\}$ is a base for a connected Hausdorff topology on X. (*Hint*: $pq \in \overline{U}_{mp}$ for every $q \ge 1$. See Steen & Seebach 78, ex. 60.) Show that X is a second-countable analytic Hausdorff space and carries a Radon measure which is not completion regular.

(b) Let (X, Σ, μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y is a topological space with a countable network consisting of universally measurable sets, and that $f : X \to Y$ is measurable. Show that f is almost continuous.

(c) Let X be a completely regular Hausdorff space. Show that the following are equiveridical: (α) X is Radon; (β) X is a universally measurable subset of its Stone-Čech compactification; (γ) whenever Y is a Hausdorff space and X' is a subspace of Y which is homeomorphic to X, then X' is universally measurable in Y.

(d) Let X be a completely regular Hausdorff space. Show that the following are equiveridical: (α) X is pre-Radon; (β) X is a universally Radon-measurable subset of its Stone-Čech compactification; (γ) whenever Y is a Hausdorff space and X' is a subspace of Y which is homeomorphic to X, then X' is universally Radon-measurable in Y.

(e) Let X be a metrizable space without isolated points, and μ a σ-finite Borel measure on X. Show that there is a conegligible meager set. (*Hint*: there is a dense set $D \subseteq X$ such that $\{\{d\} : d \in D\}$ is σ-metrically-discrete.)

(f) (Aldaz 97) A topological space X is **countably metacompact** if whenever $\mathcal{G}$ is a countable open cover of X then there is a point-finite open cover $\mathcal{H}$ of X refining G. (i) Show that X is countably metacompact iff whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of closed sets with empty intersection in X then there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of open sets, with empty intersection, such that $F_n \subseteq G_n$ for every n. (ii) Let X be any topological space and $\nu : \mathcal{P}X \to [0,1]$ a finitely additive functional such that $\nu X = 1$. Show that there is a finitely additive $\nu' : \mathcal{P}X \to [0,1]$ such that $\nu F \leq \nu' F = \inf\{\nu' G : G \supseteq F \text{ is open}\}$ for every closed $F \subseteq X$. (*Hint*: 413Q.) (iii) Show that if X is countably metacompact and μ is any Borel probability measure on X, there is a Borel probability measure μ' on X, inner regular with respect to the closed sets, such that $\mu F \leq \mu' F$ for every closed set $F \subseteq X$; so that μ and μ' agree on the Baire σ-algebra of X.

(g) Show that $\beta\mathbb{N}$ is not countably tight, therefore not Borel-measure-compact.

(h) Let X be a quasi-dyadic space. Suppose that $\langle (G_\xi, H_\xi)\rangle_{\xi<\omega_1}$ is a family of pairs of disjoint non-empty open sets. Show that there is an uncountable $A \subseteq \omega_1$ such that $\bigcap_{\xi\in B} G_\xi \cap \bigcap_{\xi\in A\setminus B} H_\xi$ is non-empty for every $B \subseteq A$. (*Hint*: start by supposing that X is itself a product of separable metric spaces, and that every G_ξ, H_ξ is an open cylinder set; use the Δ-system lemma.)

(i) (i) Show that the split interval is not quasi-dyadic. (ii) Show that $\mathbb{R}$, with the Sorgenfrey right-facing topology, is not quasi-dyadic. (iii) Show that ω_1 and ω_1+1, with their order topologies, are not quasi-dyadic. (*Hint*: 434Yh.)

(j) Show that a perfectly normal quasi-dyadic space is Borel-measure-compact.

(k) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a sequence of first-countable spaces, and μ_n a Borel measure on X_n for each n. For $n \in \mathbb{N}$ set $Z_n = \prod_{i\leq n} X_i$, and let λ_n be the product Borel measure on Z_n constructed by repeatedly using the method of 434R (cf. 434Xw). Show that there is a unique Borel measure λ on $Z = \prod_{n\in\mathbb{N}} X_n$ such that all the canonical maps from Z to Z_n are inverse-measure-preserving. Show that, for any n, λ can be identified with the product of λ_n and a suitable product measure on $\prod_{i>n} X_i$.

(l) Let X be a totally ordered set with its order topology. Show that any τ-additive Borel probability measure on X has countable Maharam type. (*Hint*: $\{]-\infty, x]^\bullet : x \in X\}$ generates the measure algebra.)

434Z Problems (a) Must every Radon compact Hausdorff space be sequentially compact?

(b) Must a Hausdorff continuous image of a Radon compact Hausdorff space be Radon?

434 Notes and comments I said that the fundamental question of topological measure theory is 'which measures can appear on which topological spaces'? In this section I have concentrated on Borel measures, classified according to the scheme laid out in §411. (Of course there are other kinds of classification. One of the most interesting is the Maharam classification of Chapter 33: we can ask what measure algebras can appear from topological measures on a given topological space. But I pass this by for the moment, with only 434Yl to give a taste.) We can ask this question from either of two directions. The obvious approach is to ask, for a given class of topological spaces, which types of measure can appear. But having discovered that (for instance) there are several types of topological space on which all (totally finite) Borel measures are tight, we can use this as a definition of a class of topological spaces, and ask the ordinary questions about this class. Thus we have 'Radon', 'Borel-measure-complete', 'Borel-measure-compact' and 'pre-Radon' spaces (434C, 434H, 434I, 434J). I have given precedence to the first partly to honour the influence of Schwartz 73 and partly because a compact Hausdorff space is always Borel-measure-compact and pre-Radon (434Hb, 434Jf) and is Borel-measure-complete iff it is Radon (434Ka). In effect, 'Borel-measure-complete' means 'Borel measures are quasi-Radon' (434Ib), 'pre-Radon' means 'quasi-Radon measures are Radon' (434Jb), and 'Radon' means 'Borel measures are Radon' (434F(a-iii)). These slogans have to be interpreted with

care; but it is true that a Hausdorff space is Radon iff it is both Borel-measure-complete and pre-Radon (434Ka).

The concept of 'Radon' space is in fact one of the important contributions of measure theory to general topology, offering a variety of challenging questions. One which has attracted some attention is the problem of determining when products of Radon spaces are Radon. Uncountable products hardly ever are (434Ke); for countable products it is enough to understand products of finitely many compact spaces (434Xt); but the product of two compact spaces already seems to lead us into undecidable questions (438Xn, WAGE 80). Two more very natural questions are in 434Z. One of the obstacles to the investigation is the rather small number of Radon compact Hausdorff spaces which are known. I should remark that if the continuum hypothesis (for instance) is true, then every compact Hausdorff space in which countably compact sets are closed is sequentially compact (ISMAIL & NYIKOS 80, or FREMLIN 84, 24Nc), so that in this case we have a quick answer to 434Za.

You will recognise the construction of 434M as a universal version of Dieudonné's measure (411Q). 'Tightness' is of great interest for other reasons (ENGELKING 89), and here is very helpful in giving quick proofs that spaces are not Radon (434Yg).

A large proportion of the definitions in general topology can be regarded as different abstractions from the concept of metrizability. Countable tightness is an obvious example; so is 'first-countability' (434R). In quite a different direction we have 'metacompactness' (438J, 434Yf). The construction of the product measure in 434R is an obvious idea, as soon as you have seen Fubini's theorem, but it is not obvious just when it will work.

'Quasi-dyadic' spaces are a relatively recent invention; I introduce them here only as a vehicle for the argument of 434Q. Of course a dyadic space (4A2A) is quasi-dyadic; for basic facts on dyadic spaces, see 4A2Dd, 4A5T and ENGELKING 89, §3.12 and 4.5.9-4.5.11.

435 Baire measures

Imitating the programme of §434, I apply a similar analysis to Baire measures, starting with a simple-minded classification (435A). This time the central section (435D-435H) is devoted to 'measure-compact' spaces, those on which all (totally finite) Baire measures are τ-additive.

435A Types of Baire measures In 434A I looked at a list of four properties which a Borel measure may or may not possess: inner regularity with respect to closed sets, inner regularity with respect to zero sets, tightness (that is, inner regularity with respect to closed compact sets), and τ-additivity. Since every (semi-finite) Baire measure is inner regular with respect to the zero sets (412D), only two of the four are important considerations for Baire measures: tightness and τ-additivity. On the other hand, there is a new question we can ask. Given a Baire measure on a topological space, when can it be extended to a Borel measure? And in the case of a positive answer, we can ask whether the extension is unique, and whether we can find extensions to Borel measures satisfying the properties considered in 434A.

We already have some information on this. If X is a completely regular space, and μ is a τ-additive effectively locally finite Baire measure on X, then μ has a (unique) extension to a τ-additive Borel measure (415N). While if μ is tight, the extension will also be tight (cf. 416C). Perhaps I should remark immediately that while there can be only one τ-additive Borel measure extending μ, there might be another Borel measure, not τ-additive, also extending μ; see 435Xa. Of course if there is any completion regular Borel measure extending μ, there is only one; moreover, if μ is σ-finite, and there is a completion regular Borel measure extending μ, this is the only Borel measure extending μ. (For every Borel set will belong to the domain of the completion of μ.)

A possible division of Baire measures is therefore into classes

(E) measures which are not τ-additive,

(F) measures which are τ-additive, but not tight,

(G) tight measures,

and within these classes we can distinguish measures with no extension to a Borel measure (type E_0), measures with more than one extension to a Borel measure (types E_1, F_1 and G_1), measures with exactly one extension to a Borel measure which is not completion regular (types E_2, F_2 and G_2) and measures with

an extension to a completion regular Borel measure (types E_3, F_3 and G_3). For examples, see 439M and 439O (E_0), 439N (E_2), 439J (E_3), 435Xc (F_1), 435Xd (F_2), 415Xc and 434Xa (F_3), 435Xa (G_1), 435Xb (G_2) and the restriction of Lebesgue measure to the Baire subsets of $\mathbb{R}$ (G_3); other examples may be constructed as direct sums of these.

A separate question we can ask of a Baire measure is whether it can be extended to a Radon measure. For this there is a straightforward criterion (435B), which shows that (at least for totally finite measures on completely regular spaces) only the types F_1 and F_2 are divided by this question. (If a Baire measure μ can be extended to a Radon measure, it is surely τ-additive. If μ is tight, it satisfies the criteria of 435B, so has an extension to a Radon measure. If μ has an extension to a completion regular Borel measure μ_1 *and* has an extension to a Radon measure μ_2, then the completion $\hat{\mu}$ of μ extends μ_1, while μ_2 extends $\hat{\mu}$; so μ_1 is the restriction of μ_2 to the Borel sets and $\mu_2 = \hat{\mu}_1 = \hat{\mu}$ and μ, like μ_2, is tight, by 412Hb or otherwise. Thus no measure of type F_3 can be extended to a Radon measure.)

As with the classification of Borel measures that I offered in §434, any restriction on the topology of the underlying space may eliminate some of these possibilities. For instance, because a semi-finite Baire measure is inner regular with respect to the closed sets, we can have no (semi-finite) measure of classes E or F on a compact Hausdorff space. On a locally compact Hausdorff space we can have no effectively locally finite Baire measure of class F (435Xe), while on a K-analytic Hausdorff space we can have no locally finite Baire measure of class E (432F). In a metric space, or a regular space with a countable network (e.g., a regular analytic Hausdorff space), the Baire and Borel σ-algebras coincide (4A3Kb), so we can have no measures of type E_0, E_1, F_1 or G_1.

435B Theorem Let X be a Hausdorff space and μ a locally finite Baire measure on X. Then the following are equiveridical:

(i) μ has an extension to a Radon measure on X;

(ii) for every non-negligible Baire set $E \subseteq X$ there is a compact set $K \subseteq E$ such that $\mu^* K > 0$.

If μ is totally finite, we can add

(iii) $\sup\{\mu^* K : K \subseteq X \text{ is compact}\} = \mu X$.

proof Because μ is inner regular with respect to the closed sets (412D), this is just a special case of 416P.

435C Theorem (Maříк 57) Let X be a normal countably paracompact space. Then any semi-finite Baire measure on X has an extension to a semi-finite Borel measure which is inner regular with respect to the closed sets.

proof (a) Let ν be a semi-finite Baire measure on X. Let $\mathcal{K}$ be the family of those closed subsets of X which are included in zero sets of finite measure, and set $\phi_0 K = \nu^* K$ for $K \in \mathcal{K}$. Then $\mathcal{K}$ and ϕ_0 satisfy the conditions of 413I, that is,

$\emptyset \in \mathcal{K}$,

(†) $K \cup K' \in \mathcal{K}$ whenever K, $K' \in \mathcal{K}$ are disjoint,

(‡) $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$,

(α) $\phi_0 K = \phi_0 L + \sup\{\phi_0 K' : K \in \mathcal{K},\ K' \subseteq K \setminus L\}$ whenever K, $L \in \mathcal{K}$ and $L \subseteq K$,

(β) $\inf_{n\in\mathbb{N}} \phi_0 K_n = 0$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection.

P The first three are trivial.

(α) Take K, $L \in \mathcal{K}$ with $L \subseteq K$, and set $\gamma = \sup\{\phi_0 K' : K \in \mathcal{K},\ K' \subseteq K \setminus L\}$. (i) If $K' \subseteq K \setminus L$ is closed, then (because X is normal) there is a zero set F including K' and disjoint from L (4A2F(d-iv)), so

$$\phi_0 K' + \phi_0 L = \nu^*((K' \cup L) \cap F) + \nu^*((K' \cup L) \setminus F) = \nu^*(K' \cup L) \leq \nu^* K.$$

As K' is arbitrary, $\gamma + \phi_0 L \leq \phi_0 K$. (ii) Let $\epsilon > 0$. Let F_0 be a zero set of finite measure including K. Because ν is inner regular with respect to the zero sets (412D), there is a zero set $F \subseteq F_0 \setminus L$ such that $\nu F \geq \nu_*(F_0 \setminus L) - \epsilon$ (413Ee), so that $\nu(F_0 \setminus F) \leq \nu^* L + \epsilon$ (413Ec). Set $K' = K \cap F$. Then

$$\nu^* K = \nu^*(K \setminus F) + \nu^*(K \cap F) \leq \nu(F_0 \setminus F) + \nu^* K' \leq \nu^* L + \epsilon + \gamma.$$

As ϵ is arbitrary, $\nu^* K \leq \nu^* L + \gamma$.

(β) If $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, then (because X is countably paracompact) there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of open sets such that $K_n \subseteq G_n$ for every n and $\bigcap_{n\in\mathbb{N}} G_n = \emptyset$ (4A2Ff). Because X is normal, there are zero sets F_n such that $K_n \subseteq F_n \subseteq G_n$ for each n (4A2F(d-iv)), so that $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$. We may suppose that F_0 has finite measure. In this case,

$$\lim_{n\to\infty} \nu^* K_n \leq \lim_{n\to\infty} \nu(\textstyle\bigcap_{i\leq n} F_i) = 0.$$

Thus $\mathcal{K}$ and ϕ_0 satisfy the conditions (α) and (β) as well. **Q**

(b) By 413I, there is a complete locally determined measure μ on X, extending ϕ_0 and inner regular with respect to $\mathcal{K}$. If $F \subseteq X$ is closed, then $F \cap K \in \mathcal{K}$ for every $K \in \mathcal{K}$, so $F \in \operatorname{dom}\mu$ (413F(ii)); accordingly μ is a topological measure, and because ν also is inner regular with respect to $\mathcal{K}$, μ must extend ν. So the restriction of μ to the Borel sets is a Borel extension of ν which is inner regular with respect to the closed sets.

Remark If X is normal, but not countably paracompact, the result may fail; see 439O. I have stated the result in terms of 'countable paracompactness', but the formally distinct 'countable metacompactness' is also sufficient (435Ya). If we are told that the Baire measure is τ-additive and effectively locally finite, we have a much stronger result (415M).

435D Just as with the 'Radon' spaces of §434, we can look at classes of topological spaces defined by the behaviour of the Baire measures they carry. The class which has aroused most interest is the following.

Definition A completely regular topological space X is **measure-compact** (sometimes called **almost Lindelöf**) if every totally finite Baire measure on X is τ-additive, that is, has an extension to a quasi-Radon measure on X (415N).

435E The following lemma will make our path easier.

Lemma Let X be a completely regular topological space and ν a totally finite Baire measure on X. Suppose that $\sup_{G\in\mathcal{G}} \nu G = \nu X$ whenever $\mathcal{G}$ is an upwards-directed family of cozero sets with union X. Then ν is τ-additive.

proof Let $\mathcal{G}$ be an upwards-directed family of open Baire sets such that $G^* = \bigcup\mathcal{G}$ is also a Baire set, and $\epsilon > 0$. Because ν is inner regular with respect to the zero sets, there is a zero set $F \subseteq G^*$ such that $\nu F \geq \nu G^* - \epsilon$. Let $\mathcal{G}'$ be the family of cozero sets included in members of $\mathcal{G}$; because X is completely regular, so that the cozero sets are a base for its topology, $\bigcup\mathcal{G}' = G^*$, and of course $\mathcal{G}'$ is upwards-directed. Now

$$\mathcal{H} = \{G \cup (X \setminus F) : G \in \mathcal{G}'\}$$

is an upwards-directed family of cozero sets with union X, so there is a $G_0 \in \mathcal{G}'$ such that $\nu(G_0 \cup (X \setminus F)) \geq \nu X - \epsilon$. In this case

$$\sup_{G\in\mathcal{G}} \nu G \geq \nu G_0 \geq \nu X - \epsilon - \nu(X \setminus F) \geq \nu G^* - 2\epsilon.$$

As $\mathcal{G}$ and ϵ are arbitrary, ν is τ-additive.

435F Elementary facts (a) If X is a completely regular space which is not measure-compact, there are a Baire probability measure μ on X and a cover of X by μ-negligible cozero sets. **P** There is a totally finite Baire measure ν on X which is not τ-additive. By 435E, there is an upwards-directed family $\mathcal{G}$ of cozero sets, covering X, such that $\sup_{G\in\mathcal{G}} \nu G < \nu X$. Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathcal{G}$ such that $\sup_{n\in\mathbb{N}} \nu G_n = \sup_{G\in\mathcal{G}} \nu G$. Then $\gamma = \nu(X \setminus \bigcup_{n\in\mathbb{N}} G_n) > 0$. Set

$$\mu H = \frac{1}{\gamma}\nu(H \setminus \textstyle\bigcup_{n\in\mathbb{N}} G_n)$$

for Baire sets $H \subseteq X$; then μ is a Baire probability measure and $\mathcal{G}$ is a cover of X by μ-negligible cozero sets. **Q**

(b) Regular Lindelöf spaces are measure-compact. (For if a Lindelöf space can be covered by negligible open sets, it can be covered by countably many negligible open sets, so is itself negligible.) In particular, compact Hausdorff spaces, indeed all regular K-analytic Hausdorff spaces (422Gg), are measure-compact.

Note that regular Lindelöf spaces are normal and paracompact (4A2H(b-i)), so their measure-compactness is also a consequence of 435C and 434Hb.

(c) An open subset of a measure-compact space need not be measure-compact (435Xi(i)). A continuous image of a measure-compact space need not be measure-compact (435Xi(ii)). $\mathbb{N}^{\mathfrak{c}}$ is not measure-compact (439P). The product of two measure-compact spaces need not be measure-compact (439Q).

(d) If X is a measure-compact completely regular space it is Borel-measure-compact. **P** Let μ be a non-zero totally finite Borel measure on X and $\mathcal{G}$ an open cover of X. Let ν be the restriction of μ to the Baire σ-algebra of X, so that ν is τ-additive. Let $\mathcal{U}$ be the set of cozero sets $U \subseteq X$ included in members of $\mathcal{G}$; because the family of cozero sets is a base for the topology of X, $\bigcup \mathcal{U} = X$, and there is some $U \in \mathcal{U}$ such that $\nu U > 0$. This means that there is some $G \in \mathcal{G}$ such that $\mu G > 0$. By 434H(a-v), X is Borel-measure-compact. **Q**

435G Proposition A Souslin-F subset of a measure-compact completely regular space is measure-compact.

proof (a) Let X be a measure-compact completely regular space, $\langle F_\sigma \rangle_{\sigma \in S}$ a Souslin scheme consisting of closed subsets of X with kernel A, ν a totally finite Baire measure on A, and $\mathcal{G}$ an upwards-directed family of (relatively) cozero subsets of A covering A. Let ν_1 be the Baire measure on X defined by setting $\nu_1 H = \nu(A \cap H)$ for every Baire subset H of X. Because X is measure-compact, ν_1 has an extension to a quasi-Radon measure μ on X. Let μ_A be the subspace measure on A.

(b) By 431A, A is measured by μ. In fact $\mu A = \nu A$. **P** The construction of μ given in 415K-415N ensures that $\mu F = \nu_1^* F$ for every closed set F, and this is in any case a consequence of the facts that μ is τ-additive and $\operatorname{dom} \nu_1$ includes a base for the topology. For each $\sigma \in S$, in particular, $\mu F_\sigma = \nu_1^* F_\sigma$; let $F'_\sigma \supseteq F_\sigma$ be a Baire set such that $\nu_1 F'_\sigma = \nu_1^* F_\sigma$. Then

$$\mu F'_\sigma = \nu_1 F'_\sigma = \nu_1^* F_\sigma = \mu F_\sigma$$

and $\mu(F'_\sigma \setminus F_\sigma) = 0$ for every $\sigma \in S$. Let A' be the kernel of the Souslin scheme $\langle F'_\sigma \rangle_{\sigma \in S}$. Then $A \subseteq A'$ and

$$\mu(A' \setminus A) \leq \textstyle\sum_{\sigma \in S} \mu(F'_\sigma \setminus F_\sigma) = 0,$$

so $\mu A = \mu A'$. On the other hand, writing $\hat{\nu}_1$ for the completion of ν_1, A' is measured by $\hat{\nu}_1$, by 431A again, so that (because μ extends ν_1)

$$\mu A = \mu A' = \mu_* A' \leq (\nu_1)_* A' = \nu_1^* A' \leq \mu^* A' = \mu A.$$

Thus $\mu A = \nu_1^* A'$. But of course

$$\nu A = \nu_1 X = \nu_1^* A = \nu_1^* A',$$

so that $\mu A = \nu A$. **Q**

Since we surely have

$$\mu X = \nu_1 X = \nu A,$$

we see that $\mu(X \setminus A) = 0$.

(c) It follows that $\mu F = \nu F$ for every (relatively) zero set $F \subseteq A$. **P** There is a closed set $F' \subseteq X$ such that $F = A \cap F'$. Now if $H \subseteq X$ is a Baire set including F', $H \cap A$ is a (relatively) Baire set including F, so $\nu F \leq \nu(H \cap A) = \nu_1 H$; as H is arbitrary, $\nu F \leq \nu_1^* F'$. But $\nu_1^* F' = \mu F'$, as remarked above, and $\mu(X \setminus A) = 0$, so

$$\mu F = \mu F' = \nu_1^* F' \geq \nu F.$$

On the other hand, $A \setminus F$ is (relatively) cozero, so there is a non-decreasing sequence $\langle F_n \rangle_{n \in \mathbb{N}}$ of (relatively) zero subsets of A with union $A \setminus F$, and

$$\mu(A \setminus F) = \lim_{n \to \infty} \mu F_n \geq \lim_{n \to \infty} \nu F_n = \nu(A \setminus F).$$

Since we already know that $\mu A = \nu A$, it follows that

$$\mu F = \mu A - \mu(A \setminus F) \leq \nu A - \nu(A \setminus F) = \nu F,$$

and $\mu F = \nu F$. **Q**

(d) The set

$$\{E : E \subseteq A \text{ is a (relative) Baire set, } \mu E = \nu E\}$$

therefore contains every (relatively) zero set, and by the Monotone Class Theorem (136C) contains every (relatively) Baire set. What this means is that μ actually extends ν; so the subspace measure $\mu_A = \mu \restriction \mathcal{P}A$ also extends ν. But μ_A is a quasi-Radon measure (415B), therefore τ-additive, and ν must also be τ-additive.

435H Corollary A Baire subset of a measure-compact completely regular space is measure-compact.

proof Put 435G and 421L together.

435X Basic exercises >(a) Give $\omega_1 + 1$ its order topology. (i) Show that its Baire σ-algebra Σ is just the family of sets $E \subseteq \omega_1 + 1$ such that either E or its complement is a countable subset of ω_1. (ii) Show that there is a unique Baire probability measure ν on $\omega_1 + 1$ such that $\nu\{\xi\} = 0$ for every $\xi < \omega_1$. (iii) Show that ν is τ-additive. (iv) Show that there is exactly one Radon measure on $\omega_1 + 1$ extending ν, but that the measure μ of 434Xf is another Borel measure also extending ν.

>(b) Let I be a set of cardinal ω_1, endowed with the discrete topology, and $X = I \cup \{\infty\}$ its one-point compactification (3A3O). Let μ be the Radon probability measure on X such that $\mu\{\infty\} = 1$. (i) Show that every subset of X is a Borel set. (ii) Show that $\{\infty\}$ is not a zero set. (iii) Let ν be the restriction of μ to the Baire σ-algebra of X. Show that ν is tight. Show that μ is the unique Borel measure extending ν (*hint*: you will need 419G), but is not completion regular. (iv) Show that the subspace measure ν_I on I is the countable-cocountable measure on I, and is not a Baire measure, nor has any extension to a Baire measure on I. (v) Show that X is measure-compact.

(c) On $\mathbb{R}^{\omega_1}$ let μ be the Baire measure defined by saying that $\mu E = 1$ if $\chi\omega_1 \in E$, 0 otherwise. (i) Show that μ is τ-additive, but not tight. (*Hint*: 4A3P.) (ii) Show that the map $\xi \mapsto \chi\xi : \omega_1 + 1 \to \mathbb{R}^{\omega_1}$ is continuous, so that μ has more than one extension to a Borel measure. (iii) Show that μ has an extension to a Radon measure.

(d) Set $X = \omega_1 + 1$ with the topology $\mathcal{P}\omega_1 \cup \{X \setminus A : A \subseteq \omega_1 \text{ is countable}\}$. Let μ be the Baire measure on X defined by saying that, for Baire sets $E \subseteq X$, $\mu E = 1$ if $\omega_1 \in E$, 0 otherwise. (i) Show that a function $f : X \to \mathbb{R}$ is continuous iff $\{\xi : \xi \in X, f(\xi) \neq f(\omega_1)\}$ is countable; show that X is completely regular and Hausdorff. (ii) Show that μ is τ-additive. (iii) Show that every subset of X is Borel. (iv) Show that the only Borel measure extending μ is that which gives measure 1 to $\{\omega_1\}$, and that this is a Radon measure. (v) Show that all compact subsets of X are finite, so that μ is not tight. *(vi) Show that X is Lindelöf.

(e) Let X be a locally compact Hausdorff space and μ an effectively locally finite τ-additive Baire measure on X. Show that μ is tight. (*Hint*: the relatively compact cozero sets cover X; use 414Ea and 412D.)

>(f) Let X be a completely regular space and μ a totally finite τ-additive Borel measure on X. Let μ_0 be the restriction of μ to the Baire σ-algebra of X. Show that $\mu F = \mu_0^* F$ for every closed set $F \subseteq X$.

(g) Show that if a semi-finite Baire measure ν on a normal countably paracompact space is extended to a Borel measure μ by the construction in 435C, then the measure algebra of ν becomes embedded as an order-dense subalgebra of the measure algebra of μ, so that $L^1(\mu)$ can be identified with $L^1(\nu)$.

(h) Show that a Borel-measure-compact normal countably paracompact space is measure-compact.

(i) (i) Show that $\omega_1 + 1$ is measure-compact, in its order topology, but that its open subset ω_1 is not (cf. 434Xk). (ii) Show that a discrete space of cardinal ω_1 is measure-compact, but that it has a continuous image which is not measure-compact.

(j) Let X be a metacompact completely regular space and ν a totally finite strictly positive Baire measure on X. Show that X is Lindelöf, so that ν has an extension to a quasi-Radon measure on X. (*Hint*: if $\mathcal{H}$ is a point-finite open cover of X, not containing $\emptyset$, then for each $H \in \mathcal{H}$ choose a non-empty cozero set $G_H \subseteq H$; show that $\{H : \nu G_H \geq \delta\}$ is finite for every $\delta > 0$.)

(k) A completely regular space X is **strongly measure-compact** (MORAN 69) if $\mu X = \sup\{\mu^* K : K \subseteq X \text{ is compact}\}$ for every totally finite Baire measure μ on X. (i) Show that a completely regular Hausdorff space X is strongly measure-compact iff every totally finite Baire measure on X has an extension to a Radon measure. (ii) Show that a strongly measure-compact completely regular space is measure-compact. (iii) Show that a Souslin-F subset of a strongly measure-compact completely regular space is strongly measure-compact. (iv) Show that a discrete space of cardinal ω_1 is strongly measure-compact. (v) Show that a countable product of strongly measure-compact completely regular spaces is strongly measure-compact. (vi) Show that $\mathbb{N}^{\omega_1}$ is not strongly measure-compact. (*Hint*: take a non-trivial probability measure on $\mathbb{N}$ and consider its power on $\mathbb{N}^{\omega_1}$.) (vii) Show that if X and Y are completely regular spaces, X is measure-compact and Y is strongly measure-compact then $X \times Y$ is measure-compact. (viii) Show that if X is a strongly measure-compact completely regular Hausdorff space, then it is Borel-measure-compact and pre-Radon.

435Y Further exercises (a) Show that a normal countably metacompact space (434Yf) is countably paracompact.

(b) Let X be a completely regular Hausdorff space and βX its Stone-Čech compactification. Show that X is measure-compact iff whenever ν is a Radon measure on βX such that $\nu X = 0$, there is a ν-negligible Baire subset of βX including X.

435 Notes and comments The principal reason for studying Baire measures is actually outside the main line of this chapter. For a completely regular Hausdorff space X, write $C_b(X)$ for the M-space of bounded continuous real-valued functions on X. Then $C_b(X)^* = C_b(X)^\sim$ is an L-space (356N), and inside $C_b(X)^*$ we have the bands generated by the tight, smooth and sequentially smooth functionals (see 437Xq and 437A below), all identifiable, if we choose, with spaces of 'signed Baire measures'. WHEELER 83 argues convincingly that for the questions a functional analyst naturally asks, these Baire measures are often an effective aid.

From the point of view of the arguments in this section, the most fundamental difference between 'Baire' and 'Borel' measures lies in their action on subspaces. If X is a topological space and A is a subset of X, then any Borel or Baire measure μ on A provides us with a measure μ_1 of the same type on X, setting $\mu_1 E = \mu(A \cap E)$ for the appropriate sets E. In the other direction, if μ is a Borel measure on X, then the subspace measure μ_A is a Borel measure on A, because the Borel σ-algebra of A is just the subspace σ-algebra derived from the Borel algebra of X (4A3Ca). But if μ is a Baire measure on X, it does not follow that μ_A is a Baire measure on A; this is because (in general) not every continuous function $f : A \to [0,1]$ has a continuous extension to X, so that not every zero set in A is the intersection of A with a zero set in X (see 435Xb). The analysis of those pairs (X, A) for which the Baire σ-algebra of A is just the subspace algebra derived from the Baire sets in X is a challenging problem in general topology which I pass by here. For the moment I note only that avoiding it is the principal technical problem in the proof of 435G.

I do not know if I ought to apologise for 'countably tight' spaces (434N), 'first-countable' spaces (434R), 'metacompact' spaces (438J), 'normal countably paracompact' spaces (435C), 'quasi-dyadic' spaces (434O) and 'sequential' spaces (436F). General topology is notorious for invoking arcane terminology to stretch arguments to their utmost limit of generality, and even specialists may find their patience tried by definitions which seem to have only one theorem each. In 438J, for instance, it is obvious that the original result concerned metrizable spaces (438H), and you may well feel at first that the extension is a baroque over-elaboration. On the other hand, there are (if you look for them) some very interesting metacompact spaces (ENGELKING 89, §5.3), and metacompactness has taken its place in the standard lists. In this book I try to follow a rule of introducing a class of topological spaces only when it is both genuinely interesting, from the point of view of general topology, and also supports an idea which is interesting from the point of view of measure theory.

436 Representation of linear functionals

I began this treatise with the three steps which make measure theory, as we know it, possible: a construction of Lebesgue measure, a definition of an integral from a measure, and a proof of the convergence theorems. I used what I am sure is the best route: Lebesgue measure from Lebesgue outer measure, and integrable functions from simple functions. But of course there are many other paths to the same ends, and some of them show us slightly different aspects of the subject. In this section I come – rather later than many authors would – to an account of a procedure for constructing measures from integrals.

I start with three fundamental theorems, the first and third being the most important. A positive linear functional on a truncated Riesz space of functions is an integral iff it is sequentially smooth (436D); a smooth linear functional corresponds to a quasi-Radon measure (436H); and if X is a compact Hausdorff space, any positive linear functional on $C(X)$ corresponds to a Radon measure (436J-436K).

436A Definition Let X be a set, U a Riesz subspace of $\mathbb{R}^X$, and $f : U \to \mathbb{R}$ a positive linear functional. I say that f is **sequentially smooth** if whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$, then $\lim_{n\to\infty} f(u_n) = 0$.

If (X, Σ, μ) is a measure space and U is a Riesz subspace of the space of real-valued μ-integrable functions defined everywhere on X, then $\int d\mu : U \to \mathbb{R}$ is sequentially smooth, by Fatou's Lemma or Lebesgue's Dominated Convergence Theorem.

Remark It is essential to distinguish between 'sequentially smooth', as defined here, and 'sequentially order-continuous', as in 313Hb or 355G. In the context here, a positive linear operator $f : U \to \mathbb{R}$ is sequentially order-continuous if $\lim_{n\to\infty} f(u_n) = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that 0 is the greatest lower bound for $\{u_n : n \in \mathbb{N}\}$ in U; while f is sequentially smooth if $\lim_{n\to\infty} f(u_n) = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that 0 is the greatest lower bound for $\{u_n : n \in \mathbb{N}\}$ in $\mathbb{R}^X$. So there can be sequentially smooth functionals which are not sequentially order-continuous, as in 436Xi. A sequentially order-continuous positive linear functional is of course sequentially smooth.

436B Definition Let X be a set. I will say that a Riesz subspace U of $\mathbb{R}^X$ is **truncated** (or satisfies **Stone's condition**) if $u \wedge \chi X \in U$ for every $u \in U$.

In this case, $u \wedge \gamma\chi X \in U$ for every $\gamma \geq 0$ and $u \in U$ (being $-u^-$ if $\gamma = 0$, $\gamma(\gamma^{-1}u \wedge \chi X)$ otherwise).

436C Lemma Let X be a set and U a truncated Riesz subspace of $\mathbb{R}^X$. Write $\mathcal{K}$ for the family of sets of the form $\{x : x \in X, u(x) \geq 1\}$ as u runs over U. Let $f : U \to \mathbb{R}$ be a sequentially smooth positive linear functional, and μ a measure on X such that μK is defined and equal to $\inf\{f(u) : \chi K \leq u \in U\}$ for every $K \in \mathcal{K}$. Then $\int u\,d\mu$ exists and is equal to $f(u)$ for every $u \in U$.

proof It is enough to deal with the case $u \geq 0$, since $U = U^+ - U^+$ and both f and $\int$ are linear. Note that if $v \in U$, $K \in \mathcal{K}$ and $v \leq \chi K$, then $v \leq w$ whenever $\chi K \leq w \in U$, so $f(v) \leq \mu K$. For k, $n \in \mathbb{N}$ set

$$K_{nk} = \{x : u(x) \geq 2^{-n}k\}, \quad u_{nk} = u \wedge 2^{-n}k\chi X.$$

Then, for $k \geq 1$,

$$K_{nk} = \{x : 2^n k^{-1} u \geq 1\} \in \mathcal{K},$$

$$2^n(u_{n,k+1} - u_{nk}) \leq \chi K_{nk} \leq 2^n(u_{nk} - u_{n,k-1}).$$

So

$$2^n f(u_{n,k+1} - u_{nk}) \leq \mu K_{nk} \leq 2^n f(u_{nk} - u_{n,k-1}),$$

and

$$f(u_{n,4^n+1} - u_{n1}) \leq \textstyle\sum_{k=1}^{4^n} 2^{-n}\mu K_{nk} \leq f(u).$$

But setting $w_n = u_{n,4^n+1} - u_{n1}$, $\langle w_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of functions in U and $\sup_{n\in\mathbb{N}} w_n(x) = u(x)$ for every x, so $\lim_{n\to\infty} f(u - w_n) = 0$ and $\lim_{n\to\infty} f(w_n) = f(u)$. Also, setting $v_n = \sum_{k=1}^{4^n} 2^{-n}\chi K_{nk}$, we have $w_n \leq v_n \leq u$ and $f(w_n) \leq \int v_n \leq f(u)$ for each n, so

$$\textstyle\int u = \lim_{n\to\infty} \int v_n = f(u)$$

by B.Levi's theorem.

436D Theorem Let X be a set and U a truncated Riesz subspace of $\mathbb{R}^X$. Let $f : U \to \mathbb{R}$ be a positive linear functional. Then the following are equiveridical:

(i) f is sequentially smooth;

(ii) there is a measure μ on X such that $\int u\,d\mu$ is defined and equal to $f(u)$ for every $u \in U$.

proof I remarked in 436A that (ii)⇒(i) is a consequence of Fatou's Lemma. So the argument here is devoted to proving that (i)⇒(ii).

(a) Let $\mathcal{K}$ be the family of sets $K \subseteq X$ such that $\chi K = \inf_{n\in\mathbb{N}} u_n$ for some sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U, taking the infimum in $\mathbb{R}^X$, so that $(\inf_{n\in\mathbb{N}} u_n)(x) = \inf_{n\in\mathbb{N}} u_n(x)$ for every $x \in X$. Then $\mathcal{K}$ is closed under finite unions and countable intersections. **P** (i) If K, $K' \in \mathcal{K}$ take sequences $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle u'_n\rangle_{n\in\mathbb{N}}$ in U such that $\chi K = \inf_{n\in\mathbb{N}} u_n$ and $\chi K' = \inf_{n\in\mathbb{N}} u'_n$; then $\chi(K \cup K') = \inf_{m,n\in\mathbb{N}} u_m \vee u'_n$, so $K \cup K' \in \mathcal{K}$. (ii) If $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$, then for each $n \in \mathbb{N}$ we can choose a sequence $\langle u_{ni}\rangle_{i\in\mathbb{N}}$ in U such that $\chi K_n = \inf_{i\in\mathbb{N}} u_{ni}$; now $\chi(\bigcap_{n\in\mathbb{N}} K_n) = \inf_{n,i\in\mathbb{N}} u_{ni}$, so $\bigcap_{n\in\mathbb{N}} K_n \in \mathcal{K}$. **Q**

Note that $\emptyset \in \mathcal{K}$ because $0 \in U$.

(b) We need to know that if $u \in U$ then $K = \{x : u(x) \geq 1\}$ belongs to $\mathcal{K}$. **P** Set

$$u_n = 2^n((u \wedge \chi X) - (u \wedge (1 - 2^{-n})\chi X)).$$

Because U is truncated, every u_n belongs to U, and it is easy to check that $\inf_{n\in\mathbb{N}} u_n = \chi K$. **Q** It follows that

$$\{x : u(x) \geq \alpha\} = \{x : \frac{1}{\alpha}u(x) \geq 1\} \in \mathcal{K}$$

whenever $u \in U$ and $\alpha > 0$.

(c) For $K \in \mathcal{K}$, set $\phi_0 K = \inf\{f(u) : u \in U,\ u \geq \chi K\}$. Then ϕ_0 satisfies the conditions of 413I. **P** I have already checked (†) and (‡) of 413I.

(α) Fix K, $L \in \mathcal{K}$ with $L \subseteq K$. Set $\gamma = \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$.

(i) Suppose that $K' \in \mathcal{K}$ is included in $K \setminus L$, and $\epsilon > 0$. Let $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle u'_n\rangle_{n\in\mathbb{N}}$ be sequences in U such that $\chi L = \inf_{n\in\mathbb{N}} u_n$ and $\chi K' = \inf_{n\in\mathbb{N}} u'_n$, and let $u \in U$ be such that $u \geq \chi K$ and $f(u) \leq \phi_0 K + \epsilon$. Set $v_n = u \wedge \inf_{i\leq n} u_i$, $v'_n = u \wedge \inf_{i\leq n} u'_i$ for each n. Then $\langle v_n \wedge v'_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U with infimum $\chi L \wedge \chi K' = 0$, so there is an n such that $f(v_n \wedge v'_n) \leq \epsilon$. In this case

$$\begin{aligned}\phi_0 L + \phi_0 K' &\leq f(v_n) + f(v'_n) = f(v_n + v'_n)\\ &= f(v_n \vee v'_n) + f(v_n \wedge v'_n) \leq f(u) + \epsilon \leq \phi_0 K + 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\phi_0 L + \phi_0 K' \leq \phi_0 K$. As K' is arbitrary, $\phi_0 L + \gamma \leq \phi_0 K$.

(ii) Next, given $\epsilon \in]0, 1[$, there are u, $v \in U$ such that $u \geq \chi K$, $v \geq \chi L$ and $f(v) \leq \phi_0 L + \epsilon$. Consider

$$K' = \{x : x \in K,\ \min(1, u(x)) - v(x) \geq \epsilon\} \subseteq K \setminus L.$$

By (a) and (b), $K' \in \mathcal{K}$. If $w \in U$ and $w \geq \chi K'$, then $v(x) + w(x) \geq 1 - \epsilon$ for every $x \in K$, so

$$\phi_0 K \leq \tfrac{1}{1-\epsilon} f(v + w) \leq \tfrac{1}{1-\epsilon}(\phi_0 L + \epsilon + f(w)).$$

As w is arbitrary,

$$(1 - \epsilon)\phi_0 K \leq \phi_0 L + \epsilon + \phi_0 K' \leq \phi_0 L + \epsilon + \gamma.$$

As ϵ is arbitrary, $\phi_0 K \leq \phi_0 L + \gamma$ and we have equality, as required by (α) in 413I.

(β) Now suppose that $\langle K_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection. For each $n \in \mathbb{N}$ let $\langle u_{ni}\rangle_{i\in\mathbb{N}}$ be a sequence in U with infimum χK_n in $\mathbb{R}^X$. Set $v_n = \inf_{i,j\leq n} u_{ji}$ for each n; then $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U with infimum $\inf_{n\in\mathbb{N}} \chi K_n = 0$, so $\inf_{n\in\mathbb{N}} f(v_n) = 0$. But

$$v_n \geq \inf_{j\leq n} \chi K_j = \chi K_n, \quad \phi_0 K_n \leq f(v_n)$$

for every n, so $\inf_{n\in\mathbb{N}} \phi_0 K_n = 0$, as required by (β) of 413I. **Q**

(d) By 413I, there is a complete locally determined measure μ on X, inner regular with respect to $\mathcal{K}$, extending ϕ_0. By 436C, $f(u) = \int u\,d\mu$ for every $u \in U$, as required.

436E Proposition Let X be any topological space, and $C_b(X)$ the space of bounded continuous real-valued functions on X. Then there is a one-to-one correspondence between totally finite Baire measures μ on X and sequentially smooth positive linear functionals $f : C_b(X) \to \mathbb{R}$, given by the formulae

$$f(u) = \int u\,d\mu \text{ for every } u \in C_b(X),$$

$$\mu Z = \inf\{f(u) : \chi Z \leq u \in C_b(X)\} \text{ for every zero set } Z \subseteq X.$$

proof (a) If μ is a totally finite Baire measure on X, then every continuous bounded real-valued function is integrable, and $f = \int d\mu$ is a sequentially smooth positive linear operator on $C_b(X)$, by Fatou's Lemma, as usual.

(b) If $f : C_b(X) \to \mathbb{R}$ is a sequentially smooth positive linear operator, then 436D tells us that there is a measure μ_0 on X such that $\int u\,d\mu_0$ is defined and equal to $f(u)$ for every $u \in C_b(X)$. By the construction in 436D, or otherwise, we may suppose that μ_0 is complete, so that every $u \in C_b(X)$ is Σ-measurable, where Σ is the domain of μ_0. It follows by the definition of the Baire σ-algebra $\mathcal{B}\mathfrak{a}$ of X (4A3K) that $\mathcal{B}\mathfrak{a} \subseteq \Sigma$, so that $\mu = \mu_0 \restriction \Sigma$ is a Baire measure; of course we still have $f(u) = \int u\,d\mu$ for every $u \in C_b(X)$. Also, if $Z \subseteq X$ is a zero set, $\mu Z = \inf\{f(u) : \chi Z \leq u \in C_b(X)\}$. **P** Express Z as $\{x : v(x) = 0\}$ where $v : X \to [0,1]$ is continuous. Set

$$u_n = (\chi X - 2^n v)^+$$

for $n \in \mathbb{N}$; then $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $C_b(X)$ and $\langle u_n(x)\rangle_{n\in\mathbb{N}} \to (\chi Z)(x)$ for every $x \in X$, so

$$\begin{aligned}\mu Z &\leq \inf\{\int u\,d\mu : \chi Z \leq u \in C_b(X)\} = \inf\{f(u) : \chi Z \leq u \in C_b(X)\}\\ &\leq \inf_{n\in\mathbb{N}} f(u_n) = \lim_{n\to\infty} \int u_n d\mu = \mu Z.\end{aligned}$$ **Q**

(c) The argument of (b) shows that if two totally finite Baire measures give the same integrals to every member of $C_b(X)$, then they must agree on all zero sets. By the Monotone Class Theorem (136C) they agree on the σ-algebra generated by the zero sets, that is, $\mathcal{B}\mathfrak{a}$, and are therefore equal. Thus the operator $\mu \mapsto \int d\mu$ from the set of totally finite Baire measures on X to the set of sequentially smooth positive linear operators on $C_b(X)$ is a bijection, and if $f = \int d\mu$ then $\mu Z = \inf\{f(u) : \chi Z \leq u \in C_b(X)\}$ for every zero set Z, as required.

436F Corresponding to 434R, we have the following construction for product Baire measures, applicable to a slightly larger class of spaces.

Proposition Let X be a sequential space, Y a topological space, and μ, ν totally finite Baire measures on X, Y respectively. Then there is a Baire measure λ on $X \times Y$ such that

$$\lambda W = \int \nu W[\{x\}]\mu(dx), \quad \int f d\lambda = \iint f(x,y)\nu(dy)\mu(dx)$$

for every Baire set $W \subseteq X \times Y$ and every bounded continuous function $f : X \times Y \to \mathbb{R}$.

proof (a) $\phi(f) = \iint f(x,y)dydx$ is defined in $\mathbb{R}$ for every bounded continuous function $f : X \times Y \to \mathbb{R}$. **P** For each $x \in X$, $g(x) = \int f(x,y)dy$ is defined because $y \mapsto f(x,y)$ is continuous. If $\langle x_n\rangle_{n\in\mathbb{N}}$ is any sequence in X converging to $x \in X$, then

$$g(x) = \int f(x,y)dy = \int \lim_{n\to\infty} f(x_n,y)dy = \lim_{n\to\infty} \int f(x_n,y)dy = \lim_{n\to\infty} g(x_n)$$

by Lebesgue's Dominated Convergence Theorem. So g is sequentially continuous; because X is sequential, g is continuous (4A2Kd). So $\iint f(x,y)dydx = \int g(x)dx$ is defined in $\mathbb{R}$. **Q**

(b) Of course ϕ is a positive linear functional on $C_b(X \times Y)$, and B.Levi's theorem shows that it is sequentially smooth. By 436E, there is a Baire measure λ on $X \times Y$ such that $\int f d\lambda = \phi(f)$ for every $f \in C_b(X \times Y)$.

(c) If $W \subseteq X \times Y$ is a zero set, there is a non-increasing sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ in $C_b(X \times Y)$ such that $\chi W = \inf_{n \in \mathbb{N}} f_n$. So B.Levi's theorem tells us that

$$\int \nu W[\{x\}] dx = \lim_{n \to \infty} \int f_n(x,y) dy dx = \lim_{n \to \infty} \int f_n d\lambda = \lambda W.$$

Now the Monotone Class Theorem (136B) tells us that

$$\{W : W \subseteq X \times Y \text{ is Baire}, \int \nu W[\{x\}] dx \text{ exists} = \lambda W\}$$

includes the σ-algebra generated by the zero sets, that is, contains every Baire set in $X \times Y$. So λ has the required properties.

436G Definition Let X be a set, U a Riesz subspace of $\mathbb{R}^X$, and $f : U \to \mathbb{R}$ a positive linear functional. I say that f is **smooth** if whenever A is a non-empty downwards-directed family in U such that $\inf_{u \in A} u(x) = 0$ for every $x \in X$, then $\inf_{u \in A} f(u) = 0$.

Of course a smooth functional is sequentially smooth. If $(X, \mathfrak{T}, \Sigma, \mu)$ is an effectively locally finite τ-additive topological measure space and U is a Riesz subspace of $\mathbb{R}^X$ consisting of integrable continuous functions, then $\int d\mu : U \to \mathbb{R}$ is smooth, by 414Bb. Corresponding to the remark in 436A, note that an order-continuous positive linear functional must be smooth, but that a smooth positive linear functional need not be order-continuous.

436H Theorem Let X be a set and U a truncated Riesz subspace of $\mathbb{R}^X$. Let $f : U \to \mathbb{R}$ be a positive linear functional. Then the following are equiveridical:

(i) f is smooth;

(ii) there are a topology $\mathfrak{T}$ and a measure μ on X such that μ is a quasi-Radon measure with respect to $\mathfrak{T}$, $U \subseteq C(X)$ and $\int u \, d\mu$ is defined and equal to $f(u)$ for every $u \in U$;

(iii) writing $\mathfrak{S}$ for the coarsest topology on X for which every member of U is continuous, there is a measure μ on X such that μ is a quasi-Radon measure with respect to $\mathfrak{S}$, and $\int u \, d\mu$ is defined and equal to $f(u)$ for every $u \in U$.

proof As remarked in 436G, in a fractionally more general context, (ii)⇒(i) is a consequence of 414B. Of course (iii)⇒(ii). So the argument here is devoted to proving that (i)⇒(iii). Except for part (b) it is a simple adaptation of the method of 436D.

(a) Let $\mathcal{K}$ be the family of sets $K \subseteq X$ such that $\chi K = \inf A$ in $\mathbb{R}^X$ for some non-empty set $A \subseteq U$. Then $\mathcal{K}$ is closed under finite unions and arbitrary intersections. **P** (i) If K, $K' \in \mathcal{K}$ take A, $A' \subseteq U$ such that $\chi K = \inf A$, $\chi K' = A'$; then $\chi(K \cup K') = \inf\{u \vee u' : u \in A, u' \in A'\}$, so $K \cup K' \in \mathcal{K}$. (ii) If $\mathcal{L}$ is a non-empty subset of $\mathcal{K}$ with intersection K, set $A = \bigcup_{L \in \mathcal{L}} \{u : \chi L \leq u \in U\}$; then $\chi K = \inf A$, so $K \in \mathcal{K}$. **Q**

Note that $\emptyset \in \mathcal{K}$ because $0 \in U$.

As in part (b) of the proof of 436D, $\{x : u(x) \geq \alpha\} \in \mathcal{K}$ whenever $\alpha > 0$ and $u \in U$.

(b) Every member of $\mathcal{K}$ is closed for $\mathfrak{S}$, being of the form $\{x : u(x) \geq 1 \text{ for every } u \in A\}$ for some $A \subseteq U$. We need to know that if $K \in \mathcal{K}$ and $G \in \mathfrak{S}$, then $K \setminus G \in \mathcal{K}$. **P** Take a non-empty set $B \subseteq U$ such that $\chi K = \inf B$. Because $\mathcal{K}$ is closed under finite unions and arbitrary intersections, $\mathfrak{S}_K = \{G : G \subseteq X, K \setminus G \in \mathcal{K}\}$ is a topology on X. (i) If $u \in U$ and $G = \{x : u(x) > 0\}$, then $\chi(K \setminus G) = \inf\{(v - ku)^+ : v \in B, k \in \mathbb{N}\}$ so $K \setminus G \in \mathcal{K}$ and $G \in \mathfrak{S}_K$. (ii) If $u \in U$ and $\alpha > 0$, then $\{x : u(x) > \alpha\} = \{x : (u - u \wedge \alpha \chi X)(x) > 0\}$ belongs to $\mathfrak{S}_K$, by (i). (iii) If $u \in U$ and $\alpha > 0$, set $G = \{x : u(x) < \alpha\}$. Then

$$K \setminus G = K \cap \{x : u(x) \geq \alpha\} \in \mathcal{K},$$

so $G \in \mathfrak{S}_K$. (iv) Thus every member of U^+ is $\mathfrak{S}_K$-continuous (2A3Bc), so every member of U is $\mathfrak{S}_K$-continuous (2A3Be), and $\mathfrak{S} \subseteq \mathfrak{S}_K$, that is, $K \setminus G \in \mathcal{K}$ for every $G \in \mathfrak{S}$. **Q**

(c) For $K \in \mathcal{K}$, set $\phi_0 K = \inf\{f(u) : u \in U, u \geq \chi K\}$. Then ϕ_0 satisfies the conditions of 415K. **P** I have already checked (†) and (‡) of 415K.

(α) Fix K, $L \in \mathcal{K}$ with $L \subseteq K$. Set $\gamma = \sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq K \setminus L\}$.

(i) Suppose that $K' \in \mathcal{K}$ is included in $K \setminus L$ and $\epsilon > 0$. Set $A = \{u : \chi L \leq u \in U\}$, $A' = \{u : \chi K' \leq u \in U\}$, so that $\chi L = \inf A$ and $\chi K' = \inf A'$, and let $v \in U$ be such that $v \geq \chi K$ and $f(v) \leq \phi_0 K + \epsilon$. Then $\{u \wedge u' : u \in A,\ u' \in A'\}$ is a downwards-directed family with infimum 0 in $\mathbb{R}^X$, so (because f is smooth) there are $u \in A$, $u' \in A'$ such that $f(u \wedge u') \leq \epsilon$. In this case

$$\phi_0 L + \phi_0 K' \leq f(v \wedge u) + f(v \wedge u') = f(v \wedge (u \vee u')) + f(v \wedge u \wedge u') \leq \phi_0 K + 2\epsilon.$$

As ϵ is arbitrary, $\phi_0 L + \phi_0 K' \leq \phi_0 K$. As K' is arbitrary, $\phi_0 L + \gamma \leq \phi_0 K$.

(ii) Next, given $\epsilon \in]0, 1[$, there are u, $v \in U$ such that $u \geq \chi K$, $v \geq \chi L$ and $f(v) \leq \phi_0 L + \epsilon$. Consider

$$K' = \{x : x \in K,\ \min(1, u(x)) - v(x) \geq \epsilon\}.$$

By the last remark in (a), $K' \in \mathcal{K}$. If $w \in U$ and $w \geq \chi K'$, then $v(x) + w(x) \geq 1 - \epsilon$ for every $x \in K$, so

$$\phi_0 K \leq \frac{1}{1-\epsilon} f(v + w) \leq \frac{1}{1-\epsilon}(\phi_0 L + \epsilon + f(w)).$$

As w is arbitrary,

$$(1 - \epsilon)\phi_0 K \leq \phi_0 L + \epsilon + \phi_0 K' \leq \phi_0 L + \epsilon + \gamma.$$

As ϵ is arbitrary, $\phi_0 K \leq \phi_0 L + \gamma$ and we have equality, as required by (α) in 415K.

(β) Now suppose that $\mathcal{K}'$ is a non-empty downwards-directed subset of $\mathcal{K}$ with empty intersection. Set

$$A = \textstyle\bigcup_{K \in \mathcal{K}'} \{u : \chi K \leq u \in U\}.$$

Then A is a downwards-directed subset of U and $\inf A = 0$ in $\mathbb{R}^X$. Because f is smooth,

$$0 = \inf_{u \in A} f(u) = \inf_{K \in \mathcal{K}'} \phi_0 K.$$

Thus (β) of 415K is satisfied.

(γ) If $K \in \mathcal{K}$ and $\phi_0 K > 0$, take $u \in U$ such that $u \geq \chi K$, and consider $G = \{x : u(x) > \frac{1}{2}\}$. Then $K \subseteq G$, while $G \subseteq \{x : 2u(x) \geq 1\}$, so

$$\sup\{\phi_0 K' : K' \in \mathcal{K},\ K' \subseteq G\} \leq 2f(u) < \infty.$$

Thus ϕ_0 satisfies (γ) of 415K. **Q**

(d) By 415K, there is a quasi-Radon measure μ on X extending ϕ_0. By 436C, $f(u) = \int u \, d\mu$ for every $u \in U$.

Remark It is worth noting explicitly that μ, as constructed here, is inner regular with respect to the family $\mathcal{K}$ of sets $K \subseteq X$ such that $\chi K = \inf A$ for some set $A \subseteq U$.

436I Lemma Let X be a topological space. Let $C_0(X)$ be the space of continuous functions $u : X \to \mathbb{R}$ which 'vanish at infinity' in the sense that $\{x : |u(x)| \geq \epsilon\}$ is compact for every $\epsilon > 0$.

(a) $C_0(X)$ is a norm-closed solid linear subspace of $C_b(X)$, so is a Banach lattice in its own right.

(b) $C_0(X)^* = C_0(X)^\sim$ is an L-space (definition: 354M).

(c) If $A \subseteq C_0(X)$ is a non-empty downwards-directed set such that $\inf_{u \in A} u(x) = 0$ for every $x \in X$, then $\inf_{u \in A} \|u\|_\infty = 0$.

proof (a)(i) If $u \in C_0(X)$, then $K = \{x : |u(x)| \geq 1\}$ is compact, so $\|u\|_\infty \leq \sup(\{1\} \cup \{|u(x)| : x \in K\})$ is finite, and $u \in C_b(X)$.

(ii) If u, $v \in C_0(X)$ and $\alpha \in \mathbb{R}$ and $w \in C_b(X)$ and $|w| \leq |u|$, then for any $\epsilon > 0$

$$\{x : |u(x) + v(x)| \geq \epsilon\} \subseteq \{x : |u(x)| \geq \tfrac{1}{2}\epsilon\} \cup \{x : |v(x)| \geq \tfrac{1}{2}\epsilon\},$$

$$\{x : |\alpha u(x)| \geq \epsilon\} \subseteq \{x : |u(x)| \geq \tfrac{\epsilon}{1+|\alpha|}\},$$

$$\{x : |w(x)| \geq \epsilon\} \subseteq \{x : |u(x)| \geq \epsilon\}$$

are closed relatively compact sets, so are compact, and $u+v$, αu, w belong to $C_0(X)$. Thus $C_0(X)$ is a solid linear subspace of $C_b(X)$.

(iii) If $\langle u_n\rangle_{n\in\mathbb{N}}$ is a sequence in $C_0(X)$ which $\| \|_\infty$-converges to $u \in C_b(X)$, then for any $\epsilon > 0$ there is an $n \in \mathbb{N}$ such that $\|u - u_n\|_\infty \leq \frac{1}{2}\epsilon$, so that

$$\{x : |u(x)| \geq \epsilon\} \subseteq \{x : |u_n(x)| \geq \tfrac{1}{2}\epsilon\}$$

is compact, and $u \in C_0(X)$. Thus $C_0(X)$ is norm-closed in $C_b(X)$.

(iv) Being a norm-closed Riesz subspace of a Banach lattice, $C_0(X)$ is itself a Banach lattice.

(b) By 356Dc, $C_0(X)^* = C_0(X)^\sim$ is a Banach lattice. Now $\|f + g\| = \|f\| + \|g\|$ for all non-negative f, $g \in C_0(X)^*$. **P** Of course $\|f + g\| \leq \|f\| + \|g\|$. On the other hand, for any $\epsilon > 0$ there are u, $v \in C_0(X)$ such that $\|u\|_\infty \leq 1$, $\|v\|_\infty \leq 1$ and $|f(u)| \geq \|f\| - \epsilon$, $|g(v)| \geq \|g\| - \epsilon$. Set $w = |u| \vee |v|$; then $w \in C_0(X)$ and

$$\|w\|_\infty = \max(\|u\|_\infty, \|v\|_\infty) \leq 1.$$

So

$$\|f + g\| \geq (f + g)(w) \geq f(|u|) + g(|v|) \geq |f(u)| + |g(v)| \geq \|f\| + \|g\| - 2\epsilon.$$

As ϵ is arbitrary, $\|f + g\| \geq \|f\| + \|g\|$. **Q**

So $C_0(X)^*$ is an L-space.

(c) Let $\epsilon > 0$. For $u \in A$ set $K_u = \{x : u(x) \geq \epsilon\}$. Then $\{K_u : u \in A\}$ is a downwards-directed family of closed compact sets with empty intersection, so there must be some $u \in A$ such that $K_u = \emptyset$, and $\|u\|_\infty \leq \epsilon$. As ϵ is arbitrary, we have the result.

Remark (c) is a version of **Dini's theorem**.

436J Riesz Representation Theorem (first form) Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space, and $C_k(X)$ the space of continuous real-valued functions on X with compact support. If $f : C_k(X) \to \mathbb{R}$ is any positive linear functional, there is a unique Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k(X)$.

proof (a) The point is that f is smooth. **P** Suppose that $A \subseteq C_k(X)$ is non-empty and downwards-directed and that $\inf A = 0$ in $\mathbb{R}^X$. Fix $u_0 \in A$ and set $K = \overline{\{x : u_0(x) > 0\}}$, so that K is compact. Because X is locally compact, there is an open relatively compact set $G \supseteq K$. Now there is a continuous function $u_1 : X \to [0, 1]$ such that $u_1(x) = 1$ for $x \in K$ and $u_1(x) = 0$ for $x \in X \setminus G$ (4A2F(h-iii)). Because G is relatively compact, $u_1 \in C_k(X)$.

Take any $\epsilon > 0$. By 436Ic, there is a $v \in A$ such that $\|v\|_\infty \leq \epsilon$. Now there is a $v' \in A$ such that $v' \leq v \wedge u_0$, so that $v'(x) \leq \epsilon$ for every $x \in K$ and $v'(x) = 0$ for $x \notin K$. In this case $v' \leq \epsilon u_1$, and

$$\inf_{u\in A} f(u) \leq f(v') \leq \epsilon f(u_1).$$

As ϵ is arbitrary, $\inf_{u\in A} f(u) = 0$; as A is arbitrary, f is smooth. **Q**

(b) Note that because $\mathfrak{T}$ is locally compact, it is the coarsest topology on X for which every function in $C_k(X)$ is continuous (4A2G(e-ii)). Also $C_k(X)$ is a truncated Riesz subspace of $\mathbb{R}^X$. So 436H tells us that there is a quasi-Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k(X)$. And μ is locally finite. **P** If $x_0 \in X$, then (as in (a) above) there is a $u_1 \in C_k(X)^+$ such that $u_1(x_0) = 1$; now $G = \{x : u_1(x) > \frac{1}{2}\}$ is an open set containing x_0, and $\mu G \leq 2f(u_1)$ is finite. **Q**

By 416G, or otherwise, μ is a Radon measure.

(c) By 416E(b-v), μ is unique.

436K Riesz Representation Theorem (second form) Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space. If $f : C_0(X) \to \mathbb{R}$ is any positive linear functional, there is a unique totally finite Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_0(X)$.

proof (a) As noted in 436Ib, $C_0(X)^* = C_0(X)^\sim$, so f is $\|\ \|_\infty$-continuous. $C_k(X)$ is a linear subspace of $C_0(X)$, and $f\restriction C_k(X)$ is a positive linear functional; so by 436J there is a unique Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_k(X)$. Now μ is totally finite. **P** By 414Ab,

$$\mu X = \sup\{f(u) : u \in C_k(X),\ 0 \le u \le \chi X\} \le \|f\| < \infty. \ \textbf{Q}$$

(b) Accordingly $\int u\,d\mu$ is defined for every $u \in C_b(X)$, and in particular for every $u \in C_0(X)$. Next, $C_k(X)$ is norm-dense in $C_0(X)$. **P** If $u \in C_0(X)^+$, then $u_n = (u - 2^{-n}\chi X)^+$ belongs to $C_k(X)$ and $\|u - u_n\|_\infty \le 2^{-n}$ for every $n \in \mathbb{N}$, so $u \in \overline{C}_k$; accordingly $C_0(X) = C_0(X)^+ - C_0(X)^+$ is included in $\overline{C}_k$. **Q** Since $\int d\mu$, regarded as a linear functional on $C_0(X)$, is positive, therefore continuous, and agrees with f on $C_k(X)$, it must be identical to f. Thus $f(u) = \int u\,d\mu$ for every $u \in C_0(X)$.

(c) Because there is only one Radon measure giving the right integrals to members of $C_k(X)$ (436J), μ is unique.

***436L** The results here, by opening a path between measure theory and the study of linear functionals on spaces of continuous functions, provide an enormously powerful tool for the analysis of dual spaces $C(X)^*$ and their relatives. I will explore some of these ideas in the next section. Here I will give only a sample theorem to show how measure theory can tell us something about Banach lattices which seems difficult to reach by other methods.

Proposition Let X be a topological space and U a norm-closed linear subspace of $C_b(X)^*$ such that the functional $u \mapsto f(u \times v) : C_b(X) \to \mathbb{R}$ belongs to U whenever $f \in U$ and $v \in C_b(X)$. Then U is a band in the L-space $C_b(X)^*$.

proof (a) Let $e = \chi X$ be the standard order unit of $C_b(X)$, and if $f \in C_b(X)^*$ and u, $v \in C_b(X)$ write $f_v(u)$ for $f(u \times v)$. By 356Na, $C_b(X)^* = C_b(X)^\sim$ is an L-space.

(b) I show first that U is a Riesz subspace of $C_b(X)^\sim$. **P** If $f \in U$ and $\epsilon > 0$, there is a $v \in C_b(X)$ such that $|v| \le e$ and $f(v) \ge |f|(e) - \epsilon$ (356B). Now $f_v \le |f|$ and

$$\||f| - f_v\| = (|f| - f_v)(e) \le \epsilon$$

(356Nb), while $f_v \in U$. As ϵ is arbitrary, $|f| \in \overline{U} = U$; as f is arbitrary, U is a Riesz subspace of $C_b(X)^*$ (352Ic). **Q**

(c) Now suppose that X is a compact Hausdorff space. Then U is a solid linear subspace of $C_b(X)^\sim = C(X)^\sim$. **P** Suppose that $f \in U$ and that $0 \le g \le f$. Let $\epsilon > 0$. By either 436J or 436K, there are Radon measures μ, ν on X such that $f(u) = \int u\,d\mu$ and $g(u) = \int u\,d\nu$ for every $u \in C(X)$. By 416Ea, $\operatorname{dom}\mu \subseteq \operatorname{dom}\nu$ and $\nu E \le \mu E$ for every $E \in \operatorname{dom}\mu$, so ν is an indefinite-integral measure over μ (415Oa, or otherwise); let $w : X \to [0,1]$ be such that $\int_E w\,d\mu = \nu E$ for every $E \in \operatorname{dom}\nu$. There is a continuous function $v : X \to \mathbb{R}$ such that $\int |w - v| d\mu \le \epsilon$ (416I), and now

$$|g(u) - f_v(u)| = |\int u\,d\nu - \int u \times v\,d\mu| = |\int u \times (w - v) d\mu|$$

(235M)

$$\le \|u\|_\infty \int |w - v| d\mu \le \epsilon \|u\|_\infty$$

for every $u \in C(X)$, so $\|g - f_v\| \le \epsilon$, while $f_v \in U$. As ϵ is arbitrary, $g \in U$; as f and g are arbitrary (and U is a Riesz subspace of $C(X)^*$), U is a solid linear subspace of $C(X)^*$. **Q**

Since every norm-closed solid linear subspace of an L-space is a band (use 354Ea), it follows that (provided X is a compact Hausdorff space) U is actually a band.

(c) For the general case, let Z be the set of all Riesz homomorphisms $z : C_b(X) \to \mathbb{R}$ such that $z(e) = 1$. Then we have a Banach lattice isomorphism $T : C_b(X) \to C(Z)$ given by the formula $(Tu)(z) = z(u)$ for $u \in C_b(X)$, $z \in Z$ (see the proofs of 353M and 354K). But note also that T is multiplicative (353Pd), and

$T' : C(Z)^* \to C_b(X)^*$ is a Banach lattice isomorphism. Let $V = (T')^{-1}[U] \subseteq C(Z)^*$; then V is a closed linear subspace of $C(Z)^*$. If $g \in V$ and v, $w \in C(Z)$, then

$$g(v \times w) = (T'g)(T^{-1}v \times T^{-1}w),$$

so g_w, defined in $C(Z)^*$ by the convention of (a) above, is just $(T')^{-1}((T'g)_{T^{-1}w})$, and belongs to V. By (b), V is a band in $C(Z)^*$ so U is a band in $C_b(X)^*$, as required.

436X Basic exercises >(a) Let (X, Σ, μ_0) be a measure space, and U the set of μ_0-integrable Σ-measurable real-valued functions defined everywhere on X. For $u \in U$ set $f(u) = \int u \, d\mu_0$. Show that U and f satisfy the conditions of 436D, and that the measure μ constructed from f by the procedure there is just the c.l.d. version of μ_0.

(b) Let μ and ν be two complete locally determined measures on a set X, and suppose that $\int f \, d\mu = \int f \, d\nu$ for every function $f : X \to \mathbb{R}$ for which either integral is defined in $\mathbb{R}$. Show that $\mu = \nu$.

>(c) Let X be a set, U a truncated Riesz subspace of $\mathbb{R}^X$, and $f : U \to \mathbb{R}$ a sequentially smooth positive linear functional. For $A \subseteq X$ set

$$\theta A = \inf\{\sup_{n \in \mathbb{N}} f(u_n) : \langle u_n \rangle_{n \in \mathbb{N}} \text{ is a non-decreasing sequence in } U^+, \lim_{n \to \infty} u_n(x) = 1 \text{ for every } x \in A\},$$

taking $\inf \emptyset = \infty$ if need be. Show that θ is an outer measure on X. Let μ_0 be the measure defined from θ by Carathéodory's method. Show that $f(u) = \int u \, d\mu_0$ for every $u \in U$. Show that the measure μ constructed in 436D is the c.l.d. version of μ_0.

(d) Let X be a set and U a truncated Riesz subspace of $\mathbb{R}^X$. Let $\tau : U \to [0, \infty[$ be a seminorm such that (i) $\tau(u) \leq \tau(v)$ whenever $|u| \leq |v|$ (ii) $\lim_{n \to \infty} \tau(u_n) = 0$ whenever $\langle u_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in U and $\lim_{n \to \infty} u_n(x) = 0$ for every $x \in X$. Show that for any $u_0 \in U^+$ there is a measure μ on x such that $\int u \, d\mu$ is defined, and less than or equal to $\tau(u)$, for every $u \in U$, and $\int u_0 d\mu = \tau(u_0)$. (*Hint*: put the Hahn-Banach theorem together with 436D.)

(e) Let X be any topological space. Show that every positive linear functional on $C(X)$ is sequentially smooth (compare 375A), so corresponds to a totally finite Baire measure on X.

(f) Let X be a completely regular space. Show that it is measure-compact iff every sequentially smooth positive linear functional on $C_b(X)$ is smooth. (*Hint*: 436Xj.)

(g) A completely regular space is called **realcompact** if every Riesz homomorphism from $C(X)$ to $\mathbb{R}$ is of the form $u \mapsto \alpha u(x)$ for some $x \in X$ and $\alpha \geq 0$. (i) Show that, for any topological space X, any Riesz homomorphism from $C(X)$ to $\mathbb{R}$ is representable by a Baire measure on X which takes at most two values. (ii) Show that a completely regular space X is realcompact iff every $\{0,1\}$-valued Baire measure on X is of the form $E \mapsto \chi E(x)$. (iii) Show that a completely regular space X is realcompact iff every purely atomic totally finite Baire measure on X is τ-additive. (iii) Show that a measure-compact completely regular space is realcompact. (iv) Show that the discrete topology on $[0,1]$ is realcompact. (*Hint*: if ν is a Baire measure taking only the values 0 and 1, set $x_0 = \sup\{x : \nu[x,1] = 1\}$.) (v) Show that any product of realcompact completely regular spaces is realcompact. (vi) Show that a Souslin-F subset of a realcompact completely regular space is realcompact. (For realcompact spaces which are not measure-compact, see 439Xm.)

(h) In 436F, suppose that μ and ν are τ-additive. Let $\tilde{\mu}$ and $\tilde{\nu}$ be the corresponding quasi-Radon measures (415N), and $\tilde{\lambda}$ the quasi-Radon product of $\tilde{\mu}$ and $\tilde{\nu}$ (417N). Show that λ is the restriction of $\tilde{\lambda}$ to the Baire σ-algebra of $X \times Y$.

>(i) For $u \in C([0,1])$ let $f(u)$ be the Lebesgue integral of u. Show that f is smooth (therefore sequentially smooth) but not sequentially order-continuous (therefore not order-continuous). (*Hint*: enumerate $\mathbb{Q} \cap [0,1]$ as $\langle q_n \rangle_{n \in \mathbb{N}}$, and set $u_n(x) = \min_{i \leq n} 2^{i+2}|x - q_i|$ for $n \in \mathbb{N}$, $x \in [0,1]$; show that $\inf_{n \in \mathbb{N}} u_n = 0$ in $C([0,1])$ but $\lim_{n \to \infty} f(u_n) > 0$.)

(j) In 436E, show that μ is τ-additive iff f is smooth.

(k) Suppose that X is a set, U is a truncated Riesz subspace of $\mathbb{R}^X$ and $f : U \to \mathbb{R}$ is a smooth positive linear functional. For $A \subseteq X$ set

$$\theta A = \inf\{\sup_{u\in B} f(u) : B \text{ is an upwards-directed family in } U^+ \text{ such that } \sup_{u\in B} u(x) = 1 \text{ for every } x \in A\},$$

taking $\inf \emptyset = \infty$ if need be. Show that θ is an outer measure on X. Let μ_0 be the measure defined from θ by Carathéodory's method. Show that $f(u) = \int u\,d\mu_0$ for every $u \in U$. Show that the measure μ constructed in 436H is the c.l.d. version of μ_0.

(l) Let X be a completely regular topological space and f a smooth positive linear functional on $C_b(X)$. Show that there is a unique totally finite quasi-Radon measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_b(X)$.

>(m) For $u \in C([0,1])$ let $f(u)$ be the Riemann integral of u (134K). Show that the Radon measure on $[0,1]$ constructed by the method of 436J is just Lebesgue measure on $[0,1]$. Explain how to construct Lebesgue measure on $\mathbb{R}$ from an appropriate version of the Riemann integral on $\mathbb{R}$.

(n) Let X be a topological space. Let $f : C_b(X) \to \mathbb{R}$ be a positive linear functional. Show that the following are equiveridical: (i) f is **tight**, that is, for every $\epsilon > 0$ there is a compact $K \subseteq X$ such that $f(u) \leq \epsilon$ whenever $0 \leq u \leq \chi(X \setminus K)$ (ii) there is a tight Borel measure μ on X such that $f(u) = \int u\,d\mu$ for every $u \in C_b(X)$. (*Hint*: show that a tight functional is smooth.)

>(o) Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be locally compact Radon measure spaces. (i) Show that the function $x \mapsto \int w(x,y)\nu(dy)$ belongs to $C_k(X)$ for every $w \in C_k(X \times Y)$, so we have a positive linear functional $h : C_k(X \times Y) \to \mathbb{R}$ defined by setting

$$h(w) = \iint w(x,y)\nu(dy)\mu(dx)$$

for $w \in C_k(X \times Y)$. (ii) Show that the corresponding Radon measure on $X \times Y$ is just the product Radon measure as defined in 417P/417R.

>(p) Let $\mathfrak{A}$ be a Boolean algebra and Z its Stone space; identify $L^\infty(\mathfrak{A})$ with $C(Z)$, as in 363A. Let ν be a non-negative finitely additive functional on $\mathfrak{A}$, f the corresponding positive linear functional on $L^\infty(\mathfrak{A})$ (363K), and μ the corresponding Radon measure on Z (416Qb). Show that $f(u) = \int u\,d\mu$ for every $u \in L^\infty(\mathfrak{A})$.

(q) Let X be a locally compact Hausdorff space. Show that a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $C_0(X)$ converges to 0 for the weak topology on $C_0(X)$ iff $\sup_{n\in\mathbb{N}} \|f_n\|_\infty$ is finite and $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in X$.

(r) Let $\mathfrak{A}$ be a Boolean algebra, and $M(\mathfrak{A})$ the L-space of bounded finitely additive functionals on $\mathfrak{A}$. Let $U \subseteq M(\mathfrak{A})$ be a norm-closed linear subspace such that $a \mapsto \nu(a \cap b)$ belongs to U whenever $\nu \in U$ and $b \in \mathfrak{A}$. Show that U is a band in $M(\mathfrak{A})$. (*Hint*: identify $L^\infty(\mathfrak{A})$ with $C_b(X)$, where X is the Stone space of $\mathfrak{A}$, and $M(\mathfrak{A})$ with $C_b(X)^*$.)

(s) Let X be a non-empty compact Hausdorff space, and $\phi : X \to X$ a continuous function. Show that there is a Radon probability measure μ on X such that ϕ is inverse-measure-preserving for μ. (*Hint*: let $\mathcal{F}$ be a non-principal ultrafilter on X, x_0 any point of X, and define μ by the formula $\int f d\mu = \lim_{n\to\mathcal{F}} \frac{1}{n+1}\sum_{k=0}^n f(\phi^k(x_0))$ for every $f \in C(X)$. Use 416E(b-v) to show that $\mu\phi^{-1} = \mu$.)

436Y Further exercises (a) Let X be a set, U a Riesz subspace of $\mathbb{R}^X$, and $f : U \to \mathbb{R}$ a sequentially smooth positive linear functional. (i) Write U_σ for the set of functions from X to $[0,\infty]$ expressible as the supremum of a non-decreasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U^+ such that $\sup_{n\in\mathbb{N}} f(u_n) < \infty$. Show that there is a

functional $f_\sigma : U_\sigma \to [0,\infty[$ such that $f_\sigma(u) = \sup_{n\in\mathbb{N}} f(u_n)$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in U^+ with supremum $u \in U_\sigma$. (Compare 122I.) (ii) Show that $u+v \in U_\sigma$ and $f_\sigma(u+v) = f_\sigma(u) + f_\sigma(v)$ for all u, $v \in U_\sigma$. (iii) Suppose that u, $v \in U_\sigma$, $u \le v$ and $u(x) = v(x)$ whenever $v(x)$ is finite. Show that $f_\sigma(u) = f_\sigma(v)$. (*Hint*: take non-decreasing sequences $\langle u_n\rangle_{n\in\mathbb{N}}$, $\langle v_n\rangle_{n\in\mathbb{N}}$ with suprema u, v. Consider $\langle f(v_k - u_n - \delta v_n)^+\rangle_{n\in\mathbb{N}}$ where $k \in \mathbb{N}$, $\delta > 0$.) (iv) Let V be the set of functions $v : X \to \mathbb{R}$ such that there are u_1, $u_2 \in U_\sigma$ such that $v(x) = u_1(x) - u_2(x)$ whenever $u_1(x)$, $u_2(x)$ are both finite. Show that V is a linear subspace of $\mathbb{R}^X$ and that there is a linear functional $g : V \to \mathbb{R}$ defined by setting $g(v) = f_\sigma(u_1) - f_\sigma(u_2)$ whenever $v = u_1 - u_2$ in the sense of the last sentence. (v) Show that V is a Riesz subspace of $\mathbb{R}^X$. (vi) Show that if $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in V and $\gamma = \sup_{n\in\mathbb{N}} g(v_n)$ is finite, then there is a $v \in V$ such that $g(v) = \sup_{n\in\mathbb{N}} g(v \wedge v_n) = \gamma$. (This is a version of the **Daniell integral**.)

(b) Develop further the theory of 436Ya, finding a version of Lebesgue's Dominated Convergence Theorem, a concept of 'negligible' subset of X, and an L-space of equivalence classes of 'integrable' functions.

(c) Let X be a countably compact topological space. Show that every positive linear functional on $C_b(X)$ is sequentially smooth, so corresponds to a totally finite Baire measure on X.

(d) Let X be a sequential space, Y a topological space, μ a semi-finite Baire measure on X and ν a σ-finite Baire measure on Y. Let $\tilde\mu$ be the c.l.d. version of μ. Show that there is a semi-finite Baire measure λ on $X \times Y$ such that

$$\lambda W = \int \nu W[\{x\}]\tilde\mu(dx), \quad \int f d\lambda = \iint f(x,y)\nu(dy)\tilde\mu(dx)$$

for every Baire set $W \subseteq X \times Y$ and every non-negative continuous function $f : X \times Y \to \mathbb{R}$. Show that the c.l.d. version of λ extends the c.l.d. product measure of μ and ν.

(e) Let $X_0, \dots, X_n$ be sequential spaces and μ_i a totally finite Baire measure on X_i for each i. (i) Show that if $f : X_0 \times \dots \times X_n \to \mathbb{R}$ is a bounded separately continuous function, then

$$\phi(f) = \int \dots \int f(x_0, \dots, x_n)\mu_n(dx_n)\dots\mu_0(dx_0)$$

is defined, so that we have a corresponding Baire product measure on $X_0 \times \dots \times X_n$. (ii) Show that this product is associative.

(f) Let X and Y be compact Hausdorff spaces. (i) Show that there is a unique bilinear map $\phi : C(X)^* \times C(Y)^* \to C(X \times Y)^*$ which is separately continuous for the weak* topologies and such that $\phi(\delta_x, \delta_y) = \delta_{(x,y)}$ for all $x \in X$ and $y \in Y$, setting $\delta_x(f) = f(x)$ for $f \in C(X)$ and $x \in X$. (ii) Show that if μ and ν are Radon measures on X, Y respectively with Radon measure product λ, then $\phi(\int d\mu, \int d\nu) = \int d\lambda$. (iii) Show that $\|\phi\| \le 1$ (definition: 253Ab).

436 Notes and comments From the beginning, integration has been at the centre of measure theory. My own view – implicit in the arrangement of this treatise, from Chapter 11 onward – is that 'measure' and 'integration' are not quite the same thing. I freely acknowledge that my treatment of 'integration' is distorted by my presentation of it as part of measure theory; on the other side of the argument, I hold that regarding 'measure' as a concept subsidiary to 'integral', as many authors do, seriously interferes with the development of truly penetrating intuitions for the former. But it is undoubtedly true that every complete locally determined measure can be derived from its associated integral (436Xb). Moreover, it is clearly of the highest importance that we should be able to recognise integrals when we see them; I mean, given a linear functional on a linear space of functions, then if it can be expressed as an integral with respect to a measure this is something we need to know at once. And thirdly, investigation of linear functionals frequently leads us to measures of great importance and interest.

Concerning the conditions in 436D, an integral must surely be 'positive' (because measures in this treatise are always non-negative) and 'sequentially smooth' (because measures are supposed to be countably additive). But it is not clear that we are forced to restrict our attention to Riesz subspaces of $\mathbb{R}^X$, and even less clear that they have to be 'truncated'. In 439I below I give an example to show that this last condition is essential for 436D and 436H as stated. However it is not necessary for large parts of the theory. In many cases, if $U \subseteq \mathbb{R}^X$ is a Riesz subspace which is not truncated, we can take an element $e \in U^+$ and look at

$Y = \{x : e(x) > 0\}$, $V_e = \{u : u(x) = 0 \text{ for every } x \in X \setminus Y\} \cong W_e$, where $W_e = \{u/e : u \in V_e\}$ is a truncated Riesz subspace of $\mathbb{R}^Y$. But there are applications in which this approach is unsatisfactory and a more radical revision of the basic theory of integration, as in 436Ya, is useful.

I have based the arguments of this section on the inner measure constructions of §413. Of course it is also possible to approach them by means of outer measures (436Xc, 436Xk).

I emphasize the exercise 436Xo because it is prominent in 'Bourbakist' versions of the theory of Radon measures, in which (following BOURBAKI 65 rather than BOURBAKI 69) Radon measures are regarded as linear functionals on spaces of continuous functions. By 436J, this is a reasonably effective approach as long as we are interested only in locally compact spaces, and there are parts of the theory of topological groups (notably the duality theory of §445 below) in which it even has advantages. The construction of 436Xo shows that we can find a direct approach to the tensor product of linear functionals which does not require any attempt to measure sets rather than integrate functions. I trust that the prejudices I am expressing will not be taken as too sweeping a disparagement of such methods. Practically all correct arguments in mathematics (and not a few incorrect ones) are valuable in some way, suggesting new possibilities for investigation. In particular, one of the challenges of measure theory (not faced in this treatise) is that of devising effective theories of vector-valued measures. Typically this is much easier with Riemann-type integrals, and any techniques for working directly with these should be noted.

436L revisits ideas from Chapter 35, and the result would be easier to find if it were in §356. I include it here as an example of the way in which familiar material from measure theory (in particular, the Radon-Nikodým theorem) can be drafted to serve functional analysis. I should perhaps remark that there are alternative routes which do not use measure theory explicitly, and while longer are (in my view) more illuminating.

437 Spaces of measures

Once we have started to take the correspondence between measures and integrals as something which operates in both directions, we can go a very long way. While 'measures', as dealt with in this treatise, are essentially positive, an 'integral' can be thought of as a member of a linear space, dual in some sense to a space of functions. Since the principal spaces of functions are Riesz spaces, we find ourselves looking at dual Riesz spaces as discussed in §356; while the corresponding spaces of measures are close to those of §362. Here I try to draw these ideas together with an examination of the spaces $U_\sigma^\sim$ and $U_\tau^\sim$ of sequentially smooth and smooth functionals, and the matching spaces M_σ and M_τ of countably additive and τ-additive measures (437A-437I). Because a (sequentially) smooth functional corresponds to a countably additive measure, which can be expected to integrate many more functions than those in the original Riesz space (typically, a space of continuous functions), we find that relatively large spaces of bounded measurable functions can be canonically embedded into the biduals $(U_\sigma^\sim)^*$ and $(U_\tau^\sim)^*$ (437C, 437H, 437I).

The guiding principles of functional analysis encourage us not only to form linear spaces, but also to examine linear space topologies, starting with norm and weak topologies. The theory of Banach lattices described in §354, particularly the theory of M- and L-spaces, is an important part of the structure here. In addition, our spaces $U_\sigma^\sim$ have natural weak* topologies which can be regarded as topologies on spaces of measures; these are the 'vague' topologies of 437J, which have already been considered, in a special case, in §285.

It turns out that on the positive cone of M_τ, at least, the vague topology may have an alternative description directly in terms of the behaviour of the measures on open sets (437L). This leads us to a parallel idea, the 'narrow' topology on non-negative additive functionals (437Jd). The second half of the section is devoted to the elementary properties of narrow topologies (437K-437R), with especial reference to compact sets in these topologies (437O-437U). Seeking to identify narrowly compact sets, we come to the concept of 'uniform tightness' (437T). Bounded uniformly tight sets are narrowly relatively compact (437U); in 'Prokhorov spaces' (437V) the converse is true; I end the section with a list of the best-known Prokhorov spaces (437W).

437A Smooth and sequentially smooth duals Let X be a set, and U a Riesz subspace of $\mathbb{R}^X$. Recall that $U^\sim$ is the Dedekind complete Riesz space of order-bounded linear functionals on U, that $U_c^\sim$ is the band of differences of sequentially order-continuous positive linear functionals, and that $U^\times$ is the band of

differences of order-continuous positive linear functionals (356A). A functional $f \in (U^\sim)^+$ is 'sequentially smooth' if $\inf_{n\in\mathbb{N}} f(u_n) = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U and $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$, and 'smooth' if $\inf_{u\in A} f(u) = 0$ whenever $A \subseteq U$ is a non-empty downwards-directed set and $\inf_{u\in A} u(x) = 0$ for every $x \in X$ (436A, 436G).

(a) Set $U_\sigma^\sim = \{f : f \in U^\sim, |f| \text{ is sequentially smooth}\}$, the **sequentially smooth dual** of U. Then $U_\sigma^\sim$ is a band in $U^\sim$. **P** (i) If $f \in U_\sigma^\sim$, $g \in U^\sim$ and $|g| \le |f|$, then

$$|g|(u_n) \le |f|(u_n) \to 0$$

whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U and $\lim_{n\to\infty} u_n(x) = 0$ for every x, so $|g|$ is sequentially smooth and $g \in U_\sigma^\sim$. Thus $U_\sigma^\sim$ is a solid subset of $U^\sim$. (ii) If $f, g \in U_\sigma^\sim$ and $\alpha \in \mathbb{R}$, then

$$|f+g|(u_n) \le |f|(u_n) + |g|(u_n) \to 0, \quad |\alpha f|(u_n) = |\alpha||f|(u_n) \to 0$$

whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U and $\lim_{n\to\infty} u_n(x) = 0$ for every x, so $|f+g|$ and $|\alpha f|$ are sequentially smooth. Thus $U_\sigma^\sim$ is a Riesz subspace of $U^\sim$. (iii) Now suppose that $B \subseteq (U_\sigma^\sim)^+$ is an upwards-directed set with supremum $g \in U^\sim$, and that $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$. Then, given $\epsilon > 0$, there is an $f \in B$ such that $(g-f)(u_0) \le \epsilon$ (355Ed), so that $g(u_n) \le f(u_n) + \epsilon$ for every n, and

$$\limsup_{n\to\infty} g(u_n) \le \epsilon + \lim_{n\to\infty} f(u_n) \le \epsilon.$$

As ϵ and $\langle u_n\rangle_{n\in\mathbb{N}}$ are arbitrary, $g \in U_\sigma^\sim$; as B is arbitrary, $U_\sigma^\sim$ is a band (352Ob). **Q**

As remarked in 436A, sequentially order-continuous positive linear functionals are sequentially smooth, so $U_c^\sim \subseteq U_\sigma^\sim$.

(b) Set $U_\tau^\sim = \{f : f \in U^\sim, |f| \text{ is smooth}\}$, the **smooth dual** of U. Then $U_\tau^\sim$ is a band in $U^\sim$. **P** (i) Suppose that $f, g \in U_\tau^\sim$, $\alpha \in \mathbb{R}$, $h \in U^\sim$ and $|h| \le |f|$. If $A \subseteq U$ is a non-empty downwards-directed set and $\inf_{u\in A} u(x) = 0$ for every $x \in X$, and $\epsilon > 0$, then there are $u_0, u_1 \in A$ such that $|f|(u_0) \le \epsilon$ and $|g|(u_1) \le \epsilon$, and a $u \in A$ such that $u \le u_0 \wedge u_1$. In this case

$$|h|(u) \le |f|(u) \le \epsilon,$$

$$|f+g|(u) \le |f|(u) + |g|(u) \le 2\epsilon,$$

$$|\alpha f|(u) = |\alpha||f|(u) \le |\alpha|\epsilon.$$

As A and ϵ are arbitrary, h, $f+g$ and αf all belong to $U_\tau^\sim$; so that $U_\tau^\sim$ is a solid Riesz subspace of $U^\sim$. (ii) Now suppose that $B \subseteq (U_\tau^\sim)^+$ is an upwards-directed set with supremum $g \in U^\sim$, and that $A \subseteq U$ is a non-empty downwards-directed set such that $\inf_{u\in A} u(x) = 0$ for every $x \in X$. Fix any $u_0 \in A$. Then, given $\epsilon > 0$, there is an $f \in B$ such that $(g-f)(u_0) \le \epsilon$, so that $g(u) \le f(u) + \epsilon$ whenever $u \in A$ and $u \le u_0$. But $A_0 = \{u : u \in A, u \le u_0\}$ is also a downwards-directed set, and $\inf_{u\in A_0} u(x) = 0$ for every $x \in X$, so

$$\inf_{u\in A} g(u) \le \epsilon + \inf_{u\in A_0} f(u) \le \epsilon.$$

As ϵ and A are arbitrary, $g \in U_\tau^\sim$; as B is arbitrary, $U_\tau^\sim$ is a band. **Q**

Just as $U_c^\sim \subseteq U_\sigma^\sim$, $U^\times \subseteq U_\tau^\sim$.

437B Signed measures Collecting these ideas together with those of §§362-363, we are ready to approach 'signed measures'. Recall that if X is a set and Σ is a σ-algebra of subsets of X, we can identify $L^\infty = L^\infty(\Sigma)$, as defined in §363, with the space $\mathcal{L}^\infty$ of bounded Σ-measurable real-valued functions (363Ha). Now, because $\mathcal{L}^\infty$ is sequentially order-closed in $\mathbb{R}^X$, sequentially smooth functionals on $\mathcal{L}^\infty$ are actually sequentially order-continuous, so $(\mathcal{L}^\infty)_\sigma^\sim = (\mathcal{L}^\infty)_c^\sim$. Next, we can identify $(L^\infty)_c^\sim$ with the space M_σ of countably additive functionals, or 'signed measures', on Σ (363K); if $\nu \in M_\sigma$, the corresponding member of $(L^\infty)_c^\sim$ is the unique order-bounded (or norm-continuous) linear functional f on L^∞ such that $f(\chi E) = \nu E$ for every $E \in \Sigma$. If $\nu \ge 0$, so that ν is a totally finite measure with domain Σ, then of course f, when interpreted as a functional on $\mathcal{L}^\infty$, must be just integration with respect to ν.

The identification between $(L^\infty)_c^\sim$ and M_σ described in 363K is an L-space isomorphism. So it tells us, for instance, that if we are willing to use the symbol $\int$ for the duality between L^∞ and the space of bounded finitely additive functionals on Σ, as in 363L, then we can write

$$\int u\,d(\mu+\nu) = \int u\,d\mu + \int u\,d\nu$$

for every $u \in \mathcal{L}^\infty$ and all μ, $\nu \in M_\sigma$.

437C Theorem Let X be a set and U a Riesz subspace of $\ell^\infty(X)$, the M-space of bounded real-valued functions on X, containing the constant functions.

(a) Let Σ be the smallest σ-algebra of subsets of X with respect to which every member of U is measurable. Let $M_\sigma = M_\sigma(\Sigma)$ be the L-space of countably additive functionals on Σ (326E, 362B). Then we have a Banach lattice isomorphism $T : M_\sigma \to U_\sigma^\sim$ defined by saying that $(T\mu)(u) = \int u\,d\mu$ whenever $\mu \in M_\sigma^+$ and $u \in U$.

(b) We now have a sequentially order-continuous norm-preserving Riesz homomorphism S, embedding the M-space $\mathcal{L}^\infty = \mathcal{L}^\infty(\Sigma)$ of bounded real-valued Σ-measurable functions on X (363Ha) into the M-space $(U_\sigma^\sim)^\sim = (U_\sigma^\sim)^* = (U_\sigma^\sim)^\times$, defined by saying that $(Sv)(T\mu) = \int v\,d\mu$ whenever $v \in \mathcal{L}^\infty$ and $\mu \in M_\sigma^+$. If $u \in U$, then $(Su)(f) = f(u)$ for every $f \in U_\sigma^\sim$.

proof (a)(i) The norm $\|\ \|_\infty$ is an order-unit norm on U (354Ga), so $U^* = U^\sim$ is an L-space (356N), and the band $U_\sigma^\sim$ (437Aa) is an L-space in its own right (354O).

(ii) As noted in 437B, we have a Banach lattice isomorphism $T_0 : M_\sigma \to (\mathcal{L}^\infty)_c^\sim$ defined by saying that $(T_0\mu)(u) = \int u\,d\mu$ whenever $u \in \mathcal{L}^\infty$ and $\mu \in M_\sigma^+$. If we set $T\mu = T_0\mu \restriction U$, then T is a positive linear operator from M_σ to $U^\sim$, just because U is a linear subspace of $\mathcal{L}^\infty$; and since $T\mu \in U_\sigma^\sim$ for every $\mu \in M_\sigma^+$, T is an operator from M_σ to $U_\sigma^\sim$. Now every $f \in (U_\sigma^\sim)^+$ is of the form $T\mu$ for some $\mu \in M_\sigma^+$. **P** By 436D, there is some measure λ such that $\int u\,d\lambda = f(u)$ for every $u \in U$. Completing λ if necessary, we see that we may suppose that every member of U is $(\operatorname{dom}\lambda)$-measurable, that is, that $\Sigma \subseteq \operatorname{dom}\lambda$; take $\mu = \lambda\restriction\Sigma$. **Q** So T is surjective.

(iii) Write $\mathcal{K}$ for the family of sets $K \subseteq X$ such that $\chi K = \inf_{n\in\mathbb{N}} u_n$ for some sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U. (See the proof of 436D.) We need to know the following. (α) $\mathcal{K} \subseteq \Sigma$. ($\beta$) If $K \in \mathcal{K}$, then there is a non-increasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U such that $\chi K = \inf_{n\in\mathbb{N}} u_n$. (For if $\langle u'_n\rangle_{n\in\mathbb{N}}$ is any sequence in U such that $\chi K = \inf_{n\in\mathbb{N}} u'_n$, we can set $u_n = \inf_{i\le n} u'_i$ for each i.) (γ) The σ-algebra of subsets of X generated by $\mathcal{K}$ is Σ. **P** Let T be the σ-algebra of subsets of X generated by $\mathcal{K}$. $\mathrm{T} \subseteq \Sigma$ because $\mathcal{K} \subseteq \Sigma$. If $u \in U$ and $\alpha > 0$ then $\{x : u(x) \ge \alpha\} \in \mathcal{K}$ (see part (b) of the proof of 436D). So every member of U^+, therefore every member of U, is T-measurable, and $\Sigma \subseteq \mathrm{T}$. **Q**

(iv) T is injective. **P** If μ_1, $\mu_2 \in M_\sigma$ and $T\mu_1 = T\mu_2$, set $\nu_i = \mu_i + \mu_1^- + \mu_2^-$ for each i, so that ν_i is non-negative and $T\nu_1 = T\nu_2$. If $K \in \mathcal{K}$ then there is a non-increasing sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in U such that $\chi K = \inf_{n\in\mathbb{N}} u_n$ in $\mathbb{R}^X$, so

$$\nu_1 K = \inf_{n\in\mathbb{N}} \int u_n d\nu_1 = \inf_{n\in\mathbb{N}} \int u_n d\nu_2 = \nu_2 K.$$

Now $\mathcal{K}$ contains X and is closed under finite intersections and ν_1 and ν_2 agree on $\mathcal{K}$. By the Monotone Class Theorem (136C), ν_1 and ν_2 agree on the σ-algebra generated by $\mathcal{K}$, which is Σ; so $\nu_1 = \nu_2$ and $\mu_1 = \mu_2$. **Q**

Thus T is a linear space isomorphism between M_σ and $U_\sigma^\sim$.

(v) As noted in (ii), $T[M_\sigma^+] = (U_\sigma^\sim)^+$; so T is a Riesz space isomorphism.

(vi) Now if $\mu \in M_\sigma$,

$$\|T\mu\| = |T\mu|(\chi X)$$

(356Nb)

$$= (T|\mu|)(\chi X)$$

(because T is a Riesz homomorphism)

$$= |\mu|(X) = \|\mu\|$$

(362Ba). So T is norm-preserving and is an L-space isomorphism, as claimed.

(b)(i) By 356Pb, $(U_\sigma^\sim)^* = (U_\sigma^\sim)^\sim = (U_\sigma^\sim)^\times$ is an M-space.

(ii) We have a canonical map $S_0 : \mathcal{L}^\infty \to ((\mathcal{L}^\infty)_c^\sim)^\times$ defined by saying that $(S_0v)(h) = h(v)$ for every $v \in \mathcal{L}^\infty$, $h \in (\mathcal{L}^\infty)_c^\sim$; and by 356F, S_0 is a Riesz homomorphism. If $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{L}^\infty$ with infimum 0, then $\inf_{n\in\mathbb{N}}(S_0v_n)(h) = \inf_{n\in\mathbb{N}} h(v_n) = 0$ for every $h \in ((\mathcal{L}^\infty)_c^\sim)^+$, so $\inf_{n\in\mathbb{N}} S_0v_n = 0$ (355Ee); as $\langle v_n\rangle_{n\in\mathbb{N}}$ is arbitrary, S_0 is sequentially order-continuous (351Gb).

Also S_0 is norm-preserving. **P** (α) If $h \in (\mathcal{L}^\infty)_c^\sim$ and $v \in \mathcal{L}^\infty$, then

$$|(S_0v)(h)| = |h(v)| \le \|h\|\|v\|_\infty,$$

so $\|S_0v\| \le \|v\|_\infty$. ($\beta$) If $v \in \mathcal{L}^\infty$ and $0 \le \gamma < \|v\|_\infty$, take $x \in X$ such that $|v(x)| > \gamma$, and define $h_x \in (\mathcal{L}^\infty)_c^\sim$ by setting $h_x(w) = w(x)$ for every $w \in \mathcal{L}^\infty$; then $\|h_x\| = 1$ and

$$|(S_0v)(h_x)| = |h_x(v)| = |v(x)| \ge \gamma,$$

so $\|S_0v\| \ge \gamma$. As γ is arbitrary, $\|S_0v\| \ge \|v\|_\infty$ and $\|S_0v\| = \|v\|_\infty$. **Q**

(iii) Now $T_0 : M_\sigma \to (\mathcal{L}^\infty)_c^\sim$ and $T : M_\sigma \to U_\sigma^\sim$ are both norm-preserving Riesz space isomorphisms, so $T_0T^{-1} : U_\sigma^\sim \to (\mathcal{L}^\infty)_c^\sim$ is another, and its adjoint $S_1 : ((\mathcal{L}^\infty)_c^\sim)^* \to (U_\sigma^\sim)^*$ must also be a norm-preserving Riesz space isomorphism. So if we set $S = S_1S_0$, S will be a norm-preserving sequentially order-continuous Riesz homomorphism from $\mathcal{L}^\infty$ to $(U_\sigma^\sim)^\times = (U_\sigma^\sim)^*$.

(iv) Setting the construction out in this way tells us a lot about the properties of the operator S, but undeniably leaves it somewhat obscure. So let us start again from $v \in \mathcal{L}^\infty$ and $\mu \in M_\sigma^+$, and seek to calculate $(Sv)(T\mu)$. We have

$$(Sv)(T\mu) = (S_1S_0v)(T\mu) = (S_0v)(T_0T^{-1}T\mu)$$

(because S_1 is the adjoint of T_0T^{-1})

$$= (S_0v)(T_0\mu) = (T_0\mu)(v) = \int v\,d\mu,$$

as claimed.

If $u \in U$, then $(T\mu)(u) = (T_0\mu)(u)$ for every $\mu \in M_\sigma$, so if $f \in U_\sigma^\sim$ then

$$(Su)(f) = (S_1S_0u)(f) = (S_0u)(T_0T^{-1}f) = (T_0T^{-1}f)(u) = (TT^{-1}f)(u) = f(u).$$

This completes the proof.

437D Remarks What is happening here is that the canonical Riesz homomorphism $u \mapsto \hat{u}$ from U to $(U_\sigma^\sim)^*$ (356F) has a natural extension to $\mathcal{L}^\infty(\Sigma)$. The original homomorphism $u \mapsto \hat{u}$ is not, as a rule, sequentially order-continuous, just because $U_\sigma^\sim$ is generally larger than $U_c^\sim$; but the extension to $\mathcal{L}^\infty$ is sequentially order-continuous. If you like, it is sequential smoothness which is carried over to the extension, and because the embedding of $\mathcal{L}^\infty$ in $\mathbb{R}^X$ is sequentially order-continuous, a sequentially smooth operator on $\mathcal{L}^\infty$ is sequentially order-continuous.

In the statement of 437C, I have used the formulae $(T\mu)(u) = \int u\,d\mu$ and $(Sv)(T\mu) = \int v\,d\mu$ on the assumption that $\mu \in M_\sigma^+$, so that μ is actually a measure on the definition used in this treatise, and $\int d\mu$ is the ordinary integral as constructed in §122. Since the functions u and v are bounded, measurable and defined everywhere, we can choose to extend the notion of integration to signed measures, as in 363L, in which case the formulae $(T\mu)(u) = \int u\,d\mu$ and $(Sv)(T\mu) = \int v\,d\mu$ become meaningful, and true, for all $\mu \in M_\sigma$, $u \in U$ and $v \in \mathcal{L}^\infty$.

In fact the ideas here can be pushed farther, as in 437Ib, 437Xe and 437Yd.

437E Corollary Let X be a completely regular Hausdorff space, and $\mathcal{Ba} = \mathcal{Ba}(X)$ its Baire σ-algebra. Then we can identify $C_b(X)_\sigma^\sim$ with the L-space $M_\sigma(\mathcal{Ba})$ of countably additive functionals on $\mathcal{Ba}$, and we have a norm-preserving sequentially order-continuous Riesz homomorphism from $\mathcal{L}^\infty(\mathcal{Ba})$ to $(C_b(X)_\sigma^\sim)^*$ defined by setting $(Sv)(f) = \int v\,d\mu_f$ for every $v \in \mathcal{L}^\infty$ and $f \in (C_b(X)_\sigma^\sim)^+$, where μ_f is the Baire measure associated with f.

proof Put 437C and 436E together.

437F Proposition Let X be a topological space and $\mathcal{B} = \mathcal{B}(X)$ its Borel σ-algebra. Let M_σ be the L-space of countably additive functionals on $\mathcal{B}$.

(a) Write $M_\tau \subseteq M_\sigma$ for the set of differences of τ-additive totally finite Borel measures. Then M_τ is a band in M_σ, so is an L-space in its own right.

(b) Write $M_t \subseteq M_\tau$ for the set of differences of totally finite Borel measures which are tight (that is, inner regular with respect to the closed compact sets). Then M_t is a band in M_σ, so is an L-space in its own right.

proof (a)(i) Let μ_1, μ_2 be totally finite τ-additive Borel measures on X, $\alpha \geq 0$, and $\mu \in M_\sigma$ such that $0 \leq \mu \leq \mu_1$. Then $\mu_1 + \mu_2$, $\alpha\mu_1$ and μ are totally finite τ-additive Borel measures. **P** They all belong to M_σ, that is, are Borel measures. Now let $\mathcal{G}$ be a non-empty upwards-directed family of open sets in X with union H, and $\epsilon > 0$. Then there are G_1, $G_2 \in \mathcal{G}$ such that $\mu_1 G_1 \geq \mu_1 H - \epsilon$ and $\mu_2 G_2 \geq \mu_2 H - \epsilon$, and a $G \in \mathcal{G}$ such that $G \supseteq G_1 \cup G_2$. In this case,

$$(\mu_1 + \mu_2)(G) \geq (\mu_1 + \mu_2)(H) - 2\epsilon,$$

$$(\alpha\mu_1)(G) \geq (\alpha\mu_1)(H) - \alpha\epsilon$$

and

$$\mu G = \mu H - \mu(H \setminus G) \geq \mu H - \mu_1(H \setminus G) \geq \mu H - \epsilon.$$

As $\mathcal{G}$ and ϵ are arbitrary, $\mu_1 + \mu_2$, $\alpha\mu_1$ and μ are all τ-additive. **Q**

It follows that M_τ is a solid linear subspace of M_σ.

(ii) Now suppose that $B \subseteq M_\tau^+$ is non-empty and upwards-directed and has a supremum ν in M_σ. Then $\nu \in M_\tau$. **P** If $\mathcal{G}$ is a non-empty upwards-directed family of open sets with union H, then

$$\begin{aligned} \nu H &= \sup_{\mu \in B} \mu H \\ &= \sup_{\mu \in B, G \in \mathcal{G}} \mu G = \sup_{G \in \mathcal{G}} \nu G; \end{aligned} \tag{362Be}$$

as $\mathcal{G}$ is arbitrary, ν is τ-additive and belongs to M_τ. **Q**

As B is arbitrary, M_τ is a band in M_σ. By 354O, it is itself an L-space.

(b) We can use the same arguments. Suppose that μ_1, $\mu_2 \in M_\sigma$ are tight, $\alpha \geq 0$, and $\mu \in M_\sigma$ is such that $0 \leq \mu \leq \mu_1$. If $E \in \mathcal{B}$ and $\epsilon > 0$, there are closed compact sets K_1, $K_2 \subseteq E$ such that $\mu_1(E \setminus K_1) \leq \epsilon$ and $\mu_2(E \setminus K_2) \leq \epsilon$; now $K = K_1 \cup K_2$ is a closed compact subset of E, and

$$\mu K = \mu E - \mu(E \setminus K) \geq \mu E - \mu_1(E \setminus K) - \mu_2(E \setminus K) \geq \mu E - 2\epsilon,$$

$$(\alpha\mu_1)(K) \geq (\alpha\mu_1)(E) - \alpha\epsilon.$$

As E and ϵ are arbitrary, μ and $\alpha\mu_1$ are tight; as μ_1, μ_2 and α are arbitrary, M_t is a solid linear subspace of M_σ.

Now suppose that $B \subseteq M_t^+$ is non-empty and upwards-directed and has a supremum ν in M_σ. Take any $E \in \mathcal{B}$ and $\epsilon > 0$. Then there is a $\mu \in B$ such that $\mu E \geq \nu E - \epsilon$; there is a closed compact set $K \subseteq E$ such that $\mu K \geq \mu E - \epsilon$; and now $\nu K \geq \nu E - 2\epsilon$. As E and ϵ are arbitrary, ν is tight; as B is arbitrary, M_t is a band in M_σ, and is in itself an L-space.

437G Definitions Let X be a topological space. A **signed Baire measure** on X will be a countably additive functional on the Baire σ-algebra $\mathcal{Ba}(X)$, which by the Jordan decomposition theorem (231F) is expressible as the difference of two totally finite Baire measures; a **signed Borel measure** will be a countably additive functional on the Borel σ-algebra $\mathcal{B}(X)$, that is, the difference of two totally finite Borel measures; a **signed τ-additive Borel measure** will be a member of the L-space M_τ as described in 437F, that is, the difference of two τ-additive totally finite Borel measures; and if X is Hausdorff, a **signed tight Borel measure** will be a member of the L-space M_t as described in 437F, that is, the difference of two tight totally finite Borel measures.

437H Theorem Let X be a set and U a Riesz subspace of $\ell^\infty(X)$ containing the constant functions. Let $\mathfrak{T}$ be the coarsest topology on X rendering every member of U continuous, and $\mathcal{B} = \mathcal{B}(X)$ the corresponding Borel σ-algebra.

(a) Let M_τ be the L-space of signed τ-additive Borel measures on X. Then we have a Banach lattice isomorphism $T : M_\tau \to U_\tau^\sim$ defined by saying that $(T\mu)(u) = \int u\,d\mu$ whenever $\mu \in M_\tau^+$ and $u \in U$.

(b) We now have a sequentially order-continuous norm-preserving Riesz homomorphism S, embedding the M-space $\mathcal{L}^\infty = \mathcal{L}^\infty(\mathcal{B})$ of bounded Borel measurable functions on X into $(U_\tau^\sim)^\sim = (U_\tau^\sim)^* = (U_\tau^\sim)^\times$, defined by saying that $(Sv)(T\mu) = \int v\,d\mu$ whenever $v \in \mathcal{L}^\infty$ and $\mu \in M_\tau^+$. If $u \in U$, then $(Su)(f) = f(u)$ for every $f \in U_\tau^\sim$.

proof The proof follows the same lines as that of 437C.

(a)(i) As before, the norm $\|\ \|_\infty$ is an order-unit norm on U, so $U^* = U^\sim$ is an L-space, and the band $U_\tau^\sim$ (437Ab) is an L-space in its own right, like M_τ (437F).

(ii) Let M_σ be the L-space of all countably additive functionals on $\mathcal{B}$, so that M_τ is a band in M_σ. Let $T_0 : M_\sigma \to (\mathcal{L}^\infty)_c^\sim$ be the canonical Banach lattice isomorphism defined by saying that $(T_0\mu)(u) = \int u\,d\mu$ whenever $u \in \mathcal{L}^\infty$ and $\mu \in M_\sigma^+$. If we set $T\mu = T_0\mu\restriction U$ for $\mu \in M_\tau$, then T is a positive linear operator from M_τ to $U^\sim$, just because U is a Riesz subspace of $\mathcal{L}^\infty$; and since $T\mu \in U_\tau^\sim$ for every $\mu \in M_\tau^+$ (436H), T is an operator from M_τ to $U_\tau^\sim$. Now every $f \in (U_\tau^\sim)^+$ is of the form $T\mu$ for some $\mu \in M_\tau^+$. **P** By 436H, there is a quasi-Radon measure λ such that $\int u\,d\lambda = f(u)$ for every $u \in U$; set $\mu = \lambda\restriction\mathcal{B}$. **Q** So T is surjective.

(iii) Let $\mathcal{K}$ be the family of subsets K of X such that $\chi K = \inf A$ in $\mathbb{R}^X$ for some non-empty subset A of U. Then $\mathcal{K}$ is just the family of closed sets for $\mathfrak{T}$. **P** As noted in part (b) of the proof of 436H, every member of $\mathcal{K}$ is closed, and $K \setminus G \in \mathcal{K}$ whenever $K \in \mathcal{K}$ and $G \in \mathfrak{T}$; but as, in the present case, $X \in \mathcal{K}$, every closed set belongs to $\mathcal{K}$. **Q**

(iv) T is injective. **P** If μ_1, $\mu_2 \in M_\tau$ and $T\mu_1 = T\mu_2$, set $\nu_i = \mu_i + \mu_1^- + \mu_2^-$ for each i, so that ν_i is non-negative and $T\nu_1 = T\nu_2$. If $K \in \mathcal{K}$, set $A = \{u : u \in \mathcal{U},\ u \geq \chi K\}$, so that $\chi K = \inf A$ in $\mathbb{R}^X$, and A is downwards-directed. By 414Bb,

$$\nu_1 K = \inf_{u\in A} \int u\,d\nu_1 = \inf_{u\in A} \int u\,d\nu_2 = \nu_2 K.$$

Now $\mathcal{K}$ contains X and is closed under finite intersections and ν_1 and ν_2 agree on $\mathcal{K}$. By the Monotone Class Theorem, ν_1 and ν_2 agree on the σ-algebra generated by $\mathcal{K}$, which is $\mathcal{B}$; so $\nu_1 = \nu_2$ and $\mu_1 = \mu_2$. **Q**

Thus T is a linear space isomorphism between M_τ and $U_\tau^\sim$.

(v) As noted in (ii), $T[M_\tau^+] = (U_\tau^\sim)^+$; so T is a Riesz space isomorphism.

(vi) Now if $\mu \in M_\tau$,

$$\|T\mu\| = |T\mu|(\chi X) = (T|\mu|)(\chi X) = |\mu|(X) = \|\mu\|.$$

So T is norm-preserving and is an L-space isomorphism, as claimed.

(b)(i) By 356Pb, $(U_\tau^\sim)^* = (U_\tau^\sim)^\sim = (U_\tau^\sim)^\times$ is an M-space.

(ii) Because $T_0 : M_\sigma \to (\mathcal{L}^\infty)_c^\sim$ is a Banach lattice isomorphism, and M_τ is a band in M_σ, $W = T_0[M_\tau]$ is a band in $(\mathcal{L}^\infty)_c^\sim$. We therefore have a Riesz homomorphism $S_0 : \mathcal{L}^\infty \to W^\times$ defined by writing $(S_0 v)(h) = h(v)$ for $v \in \mathcal{L}^\infty$, $h \in W$ (356F). Just as in (b-ii) of the proof of 437C, S_0 is sequentially order-continuous and norm-preserving. (We need to observe that h_x in the second half of the argument there always belongs to W; this is because $h_x = T_0(\delta_x)$, where $\delta_x \in M_\tau$ is defined by setting $\delta_x(E) = \chi E(x)$ for every Borel set E.)

(iii) Now $T_0 : M_\sigma \to (\mathcal{L}^\infty)_c^\sim$ and $T : M_\tau \to U_\tau^\sim$ are both norm-preserving Riesz space isomorphisms, so $T_0 T^{-1} : U_\tau^\sim \to W$ is another, and its adjoint $S_1 : W^* \to (U_\tau^\sim)^*$ must also be a norm-preserving Riesz space isomorphism. So if we set $S = S_1 S_0$, S will be a norm-preserving sequentially order-continuous Riesz homomorphism from $\mathcal{L}^\infty$ to $(U_\tau^\sim)^\times = (U_\tau^\sim)^*$.

(iv) If $v \in \mathcal{L}^\infty$ and $\mu \in M_\tau^+$,

$$(Sv)(T\mu) = (S_1S_0v)(T\mu) = (S_0v)(T_0T^{-1}T\mu)$$
$$= (S_0v)(T_0\mu) = (T_0\mu)(v) = \int v\,d\mu;$$

if $u \in U$ and $f \in U_\tau^\sim$, then $(T\mu)(u) = (T_0\mu)(u)$ for every $\mu \in M_\tau$, so

$$(Su)(f) = (S_1S_0u)(f) = (S_0u)(T_0T^{-1}f) = (T_0T^{-1}f)(u) = (TT^{-1}f)(u) = f(u).$$

437I Proposition Let X be a locally compact Hausdorff space, $\mathcal{B}(X)$ its Borel σ-algebra, and $\mathcal{L}^\infty$ the M-space of bounded Borel measurable real-valued functions on X.

(a) Let M_t be the L-space of signed tight Borel measures on X. Then we have a Banach lattice isomorphism $T : M_t \to C_0(X)^*$ defined by saying that $(T\mu)(u) = \int u\,d\mu$ whenever $\mu \in M_t^+$ and $u \in C_0(X)$.

(b) Let $\Sigma_{\rm uRm}$ be the algebra of universally Radon-measurable subsets of X (definition: 434E), and $\mathcal{L}^\infty(\Sigma_{\rm uRm})$ the M-space of bounded $\Sigma_{\rm uRm}$-measurable real-valued functions on X. Then we have a norm-preserving sequentially order-continuous Riesz homomorphism $S : \mathcal{L}^\infty(\Sigma_{\rm uRm}) \to C_0(X)^{**}$ defined by saying that $(Sv)(T\mu) = \int v\,d\mu$ whenever $v \in \mathcal{L}^\infty(\Sigma_{\rm uRm})$ and $\mu \in M_t^+$; and $(Su)(f) = f(u)$ for every $u \in C_0(X)$, $f \in C_0(X)^*$.

proof (a) The point is just that in this context M_t is equal to M_τ, as defined in 437F-437H (416H), while $C_0(X)^* = C_0(X)_\tau^\sim$ (see part (a) of the proof of 436J), and the topology of X is completely regular, so we just have a special case of 437Ha.

(b)(i) Write $\mathcal{B}$ for the Borel σ-algebra of X, and $\mathcal{L}^\infty(\mathcal{B}) \subseteq \mathcal{L}^\infty(\Sigma_{\rm uRm})$ for the space of bounded Borel measurable functions on X. As in 437Hb, we have a sequentially order-continuous Riesz homomorphism $S_0 : \mathcal{L}^\infty(\mathcal{B}) \to C_0(X)^{**}$ defined by saying that $(S_0v)(T\nu) = \int v\,d\nu$ whenever $v \in \mathcal{L}^\infty(\mathcal{B})$ and $\nu \in M_t^+$.

(ii) If $v \in \mathcal{L}^\infty(\Sigma_{\rm uRm})$, then

$$\sup\{S_0w : w \in \mathcal{L}^\infty(\mathcal{B}),\, w \le v\} = \inf\{S_0w : w \in \mathcal{L}^\infty(\mathcal{B}),\, w \ge v\}$$

in $C_0(X)^{**}$. **P** Set

$$A = \{w : w \in \mathcal{L}^\infty(\mathcal{B}),\, w \le v\}, \quad B = \{w : w \in \mathcal{L}^\infty(\mathcal{B}),\, w \ge v\}.$$

Because the constant functions belong to $\mathcal{L}^\infty(\mathcal{B})$, A and B are both non-empty; of course $w \le w'$ and $S_0w \le S_0w'$ for every $w \in A$ and $w' \in B$; because $C_0(X)^{**}$ is Dedekind complete, $\phi = \sup S_0[A]$ and $\psi = \inf S_0[B]$ are both defined in $C_0(X)^{**}$, and $\phi \le \psi$. If $f \ge 0$ in $C_0(X)^*$, then there is a $\nu \in M_t^+$ such that $T\nu = f$. Since v is ν-virtually measurable (see 434Ec), there are (bounded) Borel measurable functions w, w' such that $w \le v \le w'$ and $w = w'$ ν-a.e., that is, $w \in A$, $w' \in B$ and

$$(S_0w)(f) = \int w\,d\nu = \int w'd\nu = (S_0w')(f).$$

But as

$$(S_0w)(f) \le \phi(f) \le \psi(f) \le (S_0w')(f),$$

$\phi(f) = \psi(f)$; as f is arbitrary, $\phi = \psi$. **Q**

(iii) We can therefore define $S : \mathcal{L}^\infty(\Sigma_{\rm uRm}) \to C_0(X)^{**}$ by setting

$$Sv = \sup\{S_0w : w \in \mathcal{L}^\infty(\mathcal{B}),\, w \le v\} = \inf\{S_0w : w \in \mathcal{L}^\infty(\mathcal{B}),\, w \ge v\}$$

for every $v \in \mathcal{L}^\infty$. The argument in (ii) tells us also that $(Sv)(T\nu) = \int v\,d\nu$ for every $\nu \in M_t^+$; that is, that $(Sv)(T\mu) = \int v\,d\mu$ for every Radon measure μ on X.

(iv) Now S is a norm-preserving sequentially order-continuous Riesz homomorphism. **P** (Compare 355F.) (α) The non-trivial part of this is actually the check that S is additive. But the formula

$$Sv = \sup\{S_0w : w \in \mathcal{L}^\infty(\mathcal{B}),\, w \le v\}$$

ensures that $Sv_1 + Sv_2 \le S(v_1 + v_2)$ for all v_1, $v_2 \in \mathcal{L}^\infty$, while the formula

$$Sv = \inf\{S_0 w : w \in \mathfrak{L}^\infty(\mathcal{B}),\, w \geq v\}$$

ensures that $Sv_1 + Sv_2 \geq S(v_1 + v_2)$ for all v_1, v_2. (β) It is easy to check that $S(\alpha v) = \alpha Sv$ whenever $v \in \mathfrak{L}^\infty(\Sigma_{\rm uRm})$ and $\alpha > 0$, so that S is linear. (γ) Because S_0 is a Riesz homomorphism, $S(v_1 \wedge v_2) = 0$ whenever $v_1 \wedge v_2 = 0$, so S is a Riesz homomorphism. (δ) Now suppose that $\langle v_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{L}^\infty(\Sigma_{\rm uRm})$ with infimum 0 in $\mathfrak{L}^\infty(\Sigma_{\rm uRm})$. Then $\inf_{n\in\mathbb{N}} v_n(x) = 0$ for every $x \in X$, so $\inf_{n\in\mathbb{N}} \int v_n d\nu = 0$ for every $\nu \in M_t^+$ and $\inf_{n\in\mathbb{N}}(Sv_n)(f) = 0$ for every $f \in (C_0(X)^*)^+$. So $\inf_{n\in\mathbb{N}} Sv_n = 0$; as $\langle v_n\rangle_{n\in\mathbb{N}}$ is arbitrary, S is sequentially order-continuous. (iv) If $v \in \mathfrak{L}^\infty(\Sigma_{\rm uRm})$, then $|v| \leq \|v\|_\infty \chi X$, so

$$\|Sv\| \leq \|v\|_\infty \|S(\chi X)\| = \|v\|_\infty.$$

On the other hand, for any $x \in X$ we have the corresponding point-supported probability measure δ_x, with the matching functional $h_x \in C_0(X)^*$, and

$$\|Sv\| = \||Sv|\| = \|S|v|\| \geq (S|v|)(h_x) = \int |v| d\delta_x = |v(x)|,$$

so $\|Sv\| \geq \|v\|_\infty$; thus S is norm-preserving. **Q**

Remark Once again, if we interpret integrals in the way set out in 363L, we shall have $(Sv)(T\mu) = \int v\, d\mu$ whenever $v \in \mathfrak{L}^\infty(\mathcal{B})$ and $u \in U$ and $\mu \in M_\tau$ (in 437H) or $v \in \mathfrak{L}^\infty(\Sigma_{\rm uRm})$ and $u \in C_0(X)$ and $\mu \in M_t$ (in 437I).

437J Vague and narrow topologies We are ready for another look at 'vague' topologies on spaces of measures. Let X be a topological space.

(a) Let Σ be an algebra of subsets of X. I will say that Σ **separates zero sets** if whenever F, $F' \subseteq X$ are disjoint zero sets then there is an $E \in \Sigma$ such that $F \subseteq E$ and $E \cap F' = \emptyset$.

(b) If Σ is any algebra of subsets of X, we can identify the Banach algebra and Banach lattice $L^\infty(\Sigma)$, as defined in §363, with the $\|\,\|_\infty$-closed linear subspace of $\ell^\infty(X)$ generated by $\{\chi E : E \in \Sigma\}$ (363F, 363Ha). If we do this, then $C_b(X) \subseteq L^\infty(\Sigma)$ iff Σ separates zero sets. **P** (i) Suppose that $C_b(X) \subseteq L^\infty(\Sigma)$ and that F_1, $F_2 \subseteq X$ are disjoint zero sets. Let u_1, $u_2 : X \to \mathbb{R}$ be continuous functions such that $F_i = u_i^{-1}[\{0\}]$ for both i; then $|u_1(x)| + |u(2)(x)| > 0$ for every x; set $v = \frac{|u_1|}{|u_1|+|u_2|}$, so that $v : X \to [0,1]$ is continuous, $v(x) = 0$ for $x \in F_1$ and $v(x) = 1$ for $x \in F_2$. Now $v \in C_b(X) \subseteq L^\infty(\Sigma)$, so there is a $w \in S(\Sigma)$, the linear subspace of $L^\infty(\Sigma)$ generated by $\{\chi E : E \in \Sigma\}$, such that $\|v - w\|_\infty < \frac{1}{2}$ (363C). Set $E = \{x : w(x) \leq \frac{1}{2}\}$; then $E \in \Sigma$ and $F_1 \subseteq E \subseteq X \setminus F_2$. As F_1 and F_2 are arbitrary, Σ separates zero sets.

(ii) Now suppose that Σ separates zero sets, that $u : X \to [0,1]$ is continuous, and that $n \geq 1$ is an integer. For $i \leq n$, set $F_i = \{x : x \in X,\, u(x) \leq \frac{i}{n}\}$, $F_i' = \{x : x \in X,\, u(x) \geq \frac{i+1}{n}\}$. Then F_i and F_i' are disjoint zero sets so there is an $E_i \in \Sigma$ such that $F_i' \subseteq E_i \subseteq X \setminus F_i$. Set $w = \frac{1}{n}\sum_{i=1}^n \chi E_i \in S(\Sigma)$. If $x \in X$, let $j \leq n$ be such that $\frac{j}{n} \leq u(x) < \frac{j+1}{n}$; then for $i \leq n$

$$i < j \Rightarrow x \in F_i' \Rightarrow x \in E_i \Rightarrow x \notin F_i \Rightarrow i \leq j,$$

and $w(x) = \frac{1}{n}\#(\{i : i \leq n,\, x \in E_i\})$ is either $\frac{j}{n}$ or $\frac{j+1}{n}$. Thus $|w(x) - u(x)| \leq \frac{1}{n}$. As x is arbitrary, $\|u - w\|_\infty \leq \frac{1}{n}$; as n is arbitrary, $u \in L^\infty(\Sigma)$. As $L^\infty(\Sigma)$ is a linear subspace of $\ell^\infty(X)$, this is enough to show that $C_b(X) \subseteq L^\infty(\Sigma)$. **Q**

(c) It follows that if Σ is an algebra of subsets of X separating the zero sets, and $\nu : \Sigma \to \mathbb{R}$ is a bounded additive functional, we have an interpretation of $\int u\, d\nu$ for any $u \in C_b(X)$; $\int d\nu$ is the unique norm-continuous linear functional on $L^\infty(\Sigma)$ such that $\int \chi E\, d\nu = \nu E$ for every $E \in \Sigma$ (363L). The map $\nu \mapsto \int d\nu$ is a Banach lattice isomorphism from the L-space $M(\Sigma)$ of bounded additive functionals on Σ to $L^\infty(\Sigma)^* = L^\infty(\Sigma)^\sim$ (363K). We therefore have a positive linear operator $T : M(\Sigma) \to C_b(X)^*$ defined by setting $(T\nu)(u) = \int u\, d\nu$ for every $\nu \in M(\Sigma)$ and $u \in C_b(X)$. Except in the trivial case $X = \emptyset$, $\|T\| = 1$ (if $x \in X$, we have $\delta_x \in M(\Sigma)$ defined by setting $\delta_x(E) = \chi E(x)$ for $E \in \Sigma$, and $\|T(\delta_x)\| = 1$).

The **vague topology** on $M(\Sigma)$ is now the topology generated by the functionals $\nu \mapsto \int u\, d\nu$ as u runs over $C_b(X)$; that is, the coarsest topology on $M(\Sigma)$ such that the canonical map $T : M(\Sigma) \to C_b(X)^*$ is continuous for the weak* topology of $C_b(X)^*$. Because the functionals $\nu \mapsto |\int u\, d\nu|$ are seminorms on $M(\Sigma)$, the vague topology is a locally convex linear space topology.

(d) There is a variant of the vague topology which can be applied directly to spaces of (non-negative) totally finite measures. Let $\tilde{M}^+$ be the set of all non-negative real-valued additive functionals defined on algebras of subsets of X which contain every open set. The **narrow topology** on $\tilde{M}^+$ is that generated by sets of the form

$$\{\nu : \nu \in \tilde{M}^+, \nu G > \alpha\}, \quad \{\nu : \nu \in \tilde{M}^+, \nu X < \alpha\}$$

for open sets $G \subseteq X$ and real numbers α.

Observe that $\nu \mapsto \nu X : \tilde{M}^+ \to [0, \infty[$ is continuous for the narrow topology, and if $G \subseteq X$ is open then $\nu \mapsto \nu G$ is lower semi-continuous for the narrow topology. Writing $\tilde{M}^+_\sigma$ for the set of totally finite topological measures on X, then $\nu \mapsto \nu E : \tilde{M}^+_\sigma \to [0, \infty[$ is Borel measurable, for the narrow topology on $\tilde{M}^+_\sigma$, for every Borel set $E \subseteq X$ (because the set of E for which $\nu \mapsto \nu E$ is Borel measurable is a Dynkin class containing the open sets). Writing $\tilde{P}$ for the set of topological probability measures, then the narrow topology on $\tilde{P}$ is generated by sets of the form $\{\mu : \mu \in \tilde{P}, \mu G > \alpha\}$ for open sets $G \subseteq X$.

(e) Vague topologies, being linear space topologies, are necessarily completely regular (3A4Ad, 4A2Ja). In the very general context of (c) here, in which we have a space $M(\Sigma)$ of all finitely additive functionals on an algebra Σ, we do not expect the vague topology to be Hausdorff. But if we look at particular subspaces, such as the space $M_\sigma(\mathcal{Ba}(X))$ of signed Baire measures, or the space M_τ of signed τ-additive Borel measures on a completely regular space X, we may well have a Hausdorff vague topology (437Xf).

Similarly, the narrow topology on $\tilde{M}^+$ is rarely Hausdorff. But on important subspaces we can get Hausdorff topologies. In particular, the narrow topology on the space M^+_{qR} of totally finite quasi-Radon measures is Hausdorff. **P** Take distinct μ_0, $\mu_1 \in M^+_{\mathrm{qR}}$. If $\mu_0 X \neq \mu_1 X$ then they can be separated by open sets of the form $\{\mu : \mu X < \alpha\}$, $\{\mu : \mu X > \alpha\}$. Otherwise, set $\gamma = \mu_0 X = \mu_1 X$. There is certainly an open set G such that $\mu_0 G \neq \mu_1 G$ (415H); suppose that $\mu_0 G < \mu_1 G$. Because μ_1 is inner regular with respect to the closed sets, there is a closed set $F \subseteq G$ such that $\mu_0 G < \mu_1 F$. Now

$$\mu_1 G + \mu_0(X \setminus F) = \mu_1 G + \gamma - \mu_0 F \geq \mu_1 F + \gamma - \mu_0 G > \gamma,$$

so there are $\alpha < \mu_0(X \setminus F)$ and $\beta < \mu_1 G$ such that $\alpha + \beta > \gamma$. Now $\{\mu : \mu(X \setminus F) > \alpha, \mu X < \alpha + \beta\}$ and $\{\mu : \mu G > \beta, \mu X < \alpha + \beta\}$ are disjoint open sets containing μ_0, μ_1 respectively. **Q**

(f) For a variant of the narrow topology, adapted to the space of all Radon measures on a Hausdorff space, see 495O below.

437K Proposition Let X be a topological space, and $\tilde{M}^+$ the set of all non-negative real-valued additive functionals defined on algebras of subsets of X containing every open set.

(a) We have a function $T : \tilde{M}^+ \to C_b(X)^*$ defined by the formula $(T\nu)(f) = \int u \, d\nu$ for every $\nu \in \tilde{M}^+$, $u \in C_b(X)$.

(b) T is continuous for the narrow topology on $\tilde{M}^+$ and the weak* topology on $C_b(X)^*$.

(c) Suppose now that X is completely regular, and that $W \subseteq \tilde{M}^+$ is a family of τ-additive totally finite topological measures such that two members of W which agree on the Borel σ-algebra are equal. Then $T \restriction W$ is a homeomorphism between W, with the narrow topology, and $T[W]$, with the weak* topology.

proof (a) We have only to assemble the operators of 437Jc, noting that if an algebra of subsets of X contains every open set then it certainly separates the zero sets (indeed, it actually contains every zero set).

(b) Write $\mathfrak{S}$ for the narrow topology on $\tilde{M}^+$. Suppose that $u \in C_b(X)^+$ and that $\gamma \in \mathbb{R}$. Then $V = \{\nu : \nu \in \tilde{M}^+, \int u \, d\mu > \gamma\}$ belongs to $\mathfrak{S}$. **P** If $\gamma < 0$ this is trivial. Otherwise, for n, $i \in \mathbb{N}$, let G_{ni} be the cozero set $\{x : x \in X, u(x) > 2^{-n} i\}$, so that $G_{ni} \in \operatorname{dom} \nu$. For $n \in \mathbb{N}$ set $u_n = 2^{-n} \sum_{i=1}^{4^n} \chi G_{ni}$. Then

$$V_n = \{\nu : \int u_n d\nu > \gamma\} = \bigcup \{\bigcap_{1 \leq i \leq 4^n} \{\nu : \nu G_{ni} > \gamma_i\} : 2^{-n} \textstyle\sum_{i=1}^{4^n} \gamma_i > \gamma\}$$

belongs to $\mathfrak{S}$. Since $\langle u_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence converging to u for $\| \|_\infty$, $V = \bigcup_{n \in \mathbb{N}} V_n$ and $V \in \mathfrak{S}$. **Q**

Next, $W = \{\nu : \int u \, d\nu < \gamma\}$ belongs to $\mathfrak{S}$. **P** If $u = 0$ this is trivial. Otherwise, set $v = \|u\|_\infty \chi X - u$, so that $v \in C_b(X)^+$. Then

$$W = \bigcup_{\beta>0}\{\nu : \nu X < \frac{\beta}{\|u\|_\infty}, \int v\,d\nu > \beta - \gamma\}$$

belongs to $\mathfrak{S}$. **Q**

(ii) This shows that $\nu \mapsto (T\nu)(u) = \int u\,d\nu$ is $\mathfrak{S}$-continuous for every $u \in C_b(X)^+$ and therefore for every $u \in C_b(X)$. Since the weak* topology on $C_b(X)^*$ is the coarsest topology on $C_b(X)^*$ for which all the functionals $f \mapsto f(u)$ are continuous, T is continuous.

(c)(i) Write $\mathfrak{T}$ for the topology on W induced by T, that is, the family of sets of the form $W \cap T^{-1}[V]$ where $V \subseteq C_b(X)^*$ is weak*-open. If $G \subseteq X$ is open, then $A = \{u : u \in C_b(X), 0 \leq u \leq \chi G\}$ is upwards-directed and has supremum χG, so $\mu G = \sup_{u\in A} \int u\,d\mu$ for every $\mu \in W$ (414Ba). Accordingly $\{\mu : \mu \in W, \mu G > \alpha\} = \bigcup_{u\in A}\{\mu : (T\mu)(u) > \alpha\}$ belongs to $\mathfrak{T}$ for every $\alpha \in \mathbb{R}$. Also, of course, $\{\mu : \mu X < \alpha\} = \{\mu : (T\mu)(\chi X) < \alpha\} \in \mathfrak{T}$ for every α. So if $\mathfrak{S}'$ is the narrow topology on W, $\mathfrak{S}' \subseteq \mathfrak{T}$. Putting this together with (b), we see that $\mathfrak{S}' = \mathfrak{T}$.

(ii) Now the same formulae show that $T\restriction W$ is injective. **P** Suppose that μ_1, $\mu_2 \in W$ and that $T\mu_1 = T\mu_2$. Then $\mu_1 G = \mu_2 G$ for every open set $G \subseteq X$. By the Monotone Class Theorem, μ_1 and μ_2 agree on all Borel sets; but our hypothesis is that this is enough to ensure that $\mu_1 = \mu_2$. **Q**

Since $T : W \to T[W]$ is continuous and open, it is a homeomorphism.

437L Corollary Let X be a completely regular topological space, and M_τ the space of signed τ-additive Borel measures on X. Then the narrow and vague topologies on M_τ^+ coincide.

437M Theorem For a topological space X, write $\tilde{M}_\tau^+(X)$ for the space of totally finite τ-additive topological measures on X, $M_\tau^+(X)$ for the space of totally finite τ-additive Borel measures on X, $M_\tau(X)$ for the L-space of signed τ-additive Borel measures on X, and $\tilde{P}_\tau(X)$ for the space of τ-additive topological probability measures on X.

(a) Let X and Y be topological spaces. If $\mu \in \tilde{M}_\tau^+(X)$ and $\nu \in \tilde{M}_\tau^+(Y)$, write $\mu\times\nu \in \tilde{M}_\tau^+(X\times Y)$ for the τ-additive product measure on $X \times Y$ (417G). Then $(\mu,\nu) \mapsto \mu\times\nu$ is continuous for the narrow topologies on $\tilde{M}_\tau^+(X)$, $\tilde{M}_\tau^+(Y)$ and $\tilde{M}_\tau^+(X\times Y)$.

(b) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces, with product X. If $\langle\mu_i\rangle_{i\in I}$ is a family of probability measures such that $\mu_i \in \tilde{P}_\tau(X_i)$ for each i, write $\prod_{i\in I}\mu_i \in \tilde{P}_\tau(X)$ for their τ-additive product. Then $\langle\mu_i\rangle_{i\in I} \mapsto \prod_{i\in I}\mu_i$ is continuous for the narrow topology on $\tilde{P}_\tau(X)$ and the product of the narrow topologies on $\prod_{i\in I}\tilde{P}_\tau(X_i)$.

(c) Let X and Y be topological spaces.

(i) We have a unique bilinear map $\psi : M_\tau(X)\times M_\tau(Y) \to M_\tau(X\times Y)$ such that $\psi(\mu,\nu)$ is the restriction of $\mu\times\nu$ to the Borel σ-algebra of $X\times Y$ whenever $\mu \in M_\tau^+(X)$ and $\nu \in M_\tau^+(Y)$.

(ii) $\|\psi\| \leq 1$ (definition: 253Ab).

(iii) ψ is separately continuous for the vague topologies on $M_\tau(X)$, $M_\tau(Y)$ and $M_\tau(X\times Y)$.

(d) In (c), suppose that X and Y are compact. If $B \subseteq M_\tau(X)$ and $B' \subseteq M_\tau(Y)$ are norm-bounded, then $\psi\restriction B\times B'$ is continuous for the vague topologies.

proof (a)(i) If $W \subseteq X\times Y$ is open and $\alpha \in \mathbb{R}$, then $Q = \{(\mu,\nu) : (\mu\times\nu)(W) > \alpha\}$ is open in $\tilde{M}_\tau^+(X) \times \tilde{M}_\tau^+(Y)$. **P** Suppose that $(\mu_0,\nu_0) \in Q$. Because $\mu_0\times\nu_0$ is τ-additive, there is a subset $W' \subseteq W$, expressible in the form $\bigcup_{i\leq n} G_i \times H_i$ where $G_i \subseteq X$ and $H_i \subseteq Y$ are open for every i, such that

$$\alpha < (\mu_0\times\nu_0)(W') = \int \nu_0 W'[\{x\}]\mu_0(dx)$$

(417C(iv)). Set $u(x) = \nu_0 W'[\{x\}]$ for $x \in X$, so that u is lower semi-continuous (417Ba). Let $\eta > 0$ be such that $\int u\,d\mu_0 > \alpha + (1+2\mu_0 X)\eta$, and set $E_i = \{x : u(x) > \eta i\}$ for $i \in \mathbb{N}$, so that $\eta\sum_{i=1}^\infty \mu_0 E_i > \int u\,d\mu_0 - \eta\mu_0 X$. Because every E_i is open, there is a neighbourhood U of μ_0 in $\tilde{M}_\tau^+(X)$ such that

$$\int u\,d\mu_0 - \eta\mu_0 X \leq \eta\textstyle\sum_{i=1}^\infty \mu E_i \leq \int u\,d\mu = (\mu\times\nu_0)(W')$$

for every $\mu \in U$; shrinking U if necessary, we can arrange at the same time that $\mu X < \mu_0 X + 1$ for every $\mu \in U$. Next, observe that $\mathcal{H} = \{W'[\{x\}] : x \in X\} \subseteq \{\bigcup_{i\in I} H_i : I \subseteq \{0,\dots,n\}\}$ is finite, so there is a

neighbourhood V of ν_0 in $\tilde{M}^+_\tau(Y)$ such that $\nu H \geq \nu_0 H - \eta$ for every $H \in \mathcal{H}$ and $\nu \in V$. If $\mu \in U$ and $\nu \in V$, we have

$$\begin{aligned}(\mu \times \nu)(W) \geq (\mu \times \nu)(W') &= \int \nu W'[\{x\}]\mu(dx) \geq \int u(x) - \eta\,\mu(dx)\\ &= \int u\,d\mu - \eta\mu X \geq \int u\,d\mu_0 - \eta\mu_0 X - \eta(1+\mu_0 X) > \alpha.\end{aligned}$$

As μ_0 and ν_0 are arbitrary, Q is open. **Q**

(ii) Since $(\mu \times \nu)(X \times Y) = \mu X \cdot \nu Y$, the sets $\{(\mu,\nu) : (\mu \times \nu)(X \times Y) < \alpha\}$ are also open for every $\alpha \in \mathbb{R}$. So $(\mu,\nu) \mapsto \mu \times \nu$ is continuous (4A2B(a-ii)).

(b) For finite sets I, this is a simple induction on $\#(I)$, using 417Db. For infinite I, let $W \subseteq X$ be an open set and $\alpha \in \mathbb{R}$, and consider

$$Q = \{\langle \mu_i\rangle_{i\in I} : \mu_i \in \tilde{P}_\tau(X_i) \text{ for each } i, (\textstyle\prod_{i\in I}\mu_i)(W) > \alpha\}.$$

If $\langle\mu_i\rangle_{i\in I} \in Q$, then there is an open set $W' \subseteq W$, determined by coordinates in a finite set $J \subseteq I$, such that $(\prod_{i\in I}\mu_i)(W') > \alpha$. Setting $V = \{x{\restriction}J : x \in W'\}$, we have $(\prod_{i\in J}\mu_i)(V) > \alpha$. Now we can find open sets U_i in $\tilde{P}_\tau(X_i)$, for $i \in J$, such that $(\prod_{i\in J}\nu_i)(V) > \alpha$ whenever $\nu_i \in U_i$ for $i \in J$. If now $\langle\nu_i\rangle_{i\in I} \in \prod_{i\in I}\tilde{P}_\tau(X_i)$ is such that $\nu_i \in U_i$ for every $i \in J$,

$$(\textstyle\prod_{i\in I}\nu_i)(W) \geq (\prod_{i\in I}\nu_i)(W') = (\prod_{i\in J}\nu_i)(V) > \alpha,$$

so $\prod_{i\in I}\nu_i \in Q$. As $\langle\mu_i\rangle_{i\in I}$ is arbitrary, Q is open.

As W and α are arbitrary, $\langle\mu_i\rangle_{i\in I} \mapsto \prod_{i\in I}\mu_i$ is continuous.

(c)(i) Start by writing $\psi(\mu,\nu) = (\mu\times\nu){\restriction}\mathcal{B}(X\times Y)$ for $\mu \in M^+_\tau(X)$ and $\nu \in M^+_\tau(Y)$, where $\mathcal{B}(X\times Y)$ is the Borel σ-algebra of $X \times Y$. If μ, μ_1, $\mu_2 \in M^+_\tau(X)$ and ν, ν_1, $\nu_2 \in M^+_\tau(Y)$ and $\alpha \geq 0$, then

$$\psi(\mu_1+\mu_2,\nu) = \psi(\mu_1,\nu) + \psi(\mu_2,\nu).$$

P On each side of the equation we have a τ-additive Borel measure, and the two measures agree on the standard base $\mathcal{W}$ for the topology of $X \times Y$ consisting of products of open sets; since $\mathcal{W}$ is closed under finite intersections, they agree on the algebra generated by $\mathcal{W}$ and therefore on all open sets and therefore (using the Monotone Class Theorem yet again) on all Borel sets. **Q** Similarly,

$$\psi(\mu,\nu_1+\nu_2) = \psi(\mu,\nu_1) + \psi(\mu,\nu_2), \quad \psi(\alpha\mu,\nu) = \psi(\mu,\alpha\nu) = \alpha\psi(\mu,\nu).$$

Now if μ_1', $\mu_2' \in M^+_\tau(X)$ and ν_1', $\nu_2' \in M^+_\tau(Y)$ are such that $\mu_1 - \mu_2 = \mu_1' - \mu_2'$ and $\nu_1 - \nu_2 = \nu_1' - \nu_2'$, we shall have

$$\begin{aligned}\psi(\mu_1,\nu_1) - \psi(\mu_1,&\nu_2) - \psi(\mu_2,\nu_1) + \psi(\mu_2,\nu_2)\\ &= \psi(\mu_1,\nu_1+\nu_2') - \psi(\mu_1+\mu_2',\nu_2') + \psi(\mu_2',\nu_2')\\ &\quad - \psi(\mu_1,\nu_1'+\nu_2) + \psi(\mu_1+\mu_2',\nu_1') - \psi(\mu_2',\nu_1')\\ &\quad - \psi(\mu_2,\nu_1+\nu_2') + \psi(\mu_2+\mu_1',\nu_2') - \psi(\mu_1',\nu_2')\\ &\quad + \psi(\mu_2,\nu_1'+\nu_2) - \psi(\mu_2+\mu_1',\nu_1') + \psi(\mu_1',\nu_1')\\ &= \psi(\mu_2',\nu_2') - \psi(\mu_2',\nu_1') - \psi(\mu_1',\nu_2') + \psi(\mu_1',\nu_1').\end{aligned}$$

We can therefore extend ψ to an operator on $M_\tau(X) \times M_\tau(Y)$ by setting

$$\psi(\mu_1-\mu_2,\nu_1-\nu_2) = \psi(\mu_1,\nu_1) - \psi(\mu_1,\nu_2) - \psi(\mu_2,\nu_1) + \psi(\mu_2,\nu_2)$$

whenever μ_1, $\mu_2 \in M^+_\tau(X)$ and ν_1, $\nu_2 \in M^+_\tau(Y)$, and it is straightforward to check that ψ is bilinear.

(ii) If $\mu \in M_\tau(X)$, then $\|\mu\| = \mu^+(X) + \mu^-(X)$, where μ^+ and μ^- are evaluated in the Riesz space $M_\tau(X)$. Now if $\nu \in M_\tau(Y)$,

$$\begin{aligned}|\psi(\mu,\nu)| &= |\psi(\mu^+,\nu^+) - \psi(\mu^+,\nu^-) - \psi(\mu^-,\nu^+) + \psi(\mu^-,\nu^-)|\\ &\leq \psi(\mu^+,\nu^+) + \psi(\mu^+,\nu^-) + \psi(\mu^-,\nu^+) + \psi(\mu^-,\nu^-),\end{aligned}$$

so

$$\begin{aligned}\|\psi(\mu,\nu)\| &= |\psi(\mu,\nu)|(X\times Y)\\ &\leq \psi(\mu^+,\nu^+)(X\times Y)+\psi(\mu^+,\nu^-)(X\times Y)\\ &\qquad +\psi(\mu^-,\nu^+)(X\times Y)+\psi(\mu^-,\nu^-)(X\times Y)\\ &= \mu^+(X)\cdot\nu^+(Y)+\mu^+(X)\cdot\nu^-(Y)+\mu^-(X)\cdot\nu^+(Y)+\mu^-(X)\cdot\nu^-(Y)\\ &= \|\mu\|\|\nu\|.\end{aligned}$$

As μ and ν are arbitrary, $\|\psi\|\leq 1$.

(iii) Fix $\nu\in M_\tau^+(Y)$ and $w\in C_b(X\times Y)^+$, and consider the map $\mu\mapsto\int w\,d\psi(\mu,\nu):M_\tau(X)\to\mathbb{R}$. Note first that if $\mu\in M_\tau^+(X)$,

$$\int w\,d\psi(\mu,\nu)=\int w\,d(\mu\times\nu)=\iint w(x,y)\nu(dy)\mu(dx)$$

(417H). Since both sides of this equation are linear in μ, we have

$$\int w\,d\psi(\mu,\nu)=\int w(x,y)\nu(dy)\mu(dx)$$

for every $\mu\in M_\tau(X)$. Now $x\mapsto\int w(x,y)\nu(dy)$ is continuous. **P** By 417Bc, it is lower semi-continuous; but if $\alpha\geq\|w\|_\infty$ and $w'=\alpha\chi(X\times Y)-w$, then $x\mapsto\int w'(x,y)\nu(dy)$ is lower semi-continuous, so

$$x\mapsto\alpha\nu Y-\int w'(x,y)\nu(dy)=\int w(x,y)\nu(dy)$$

is also upper semi-continuous, therefore continuous. **Q** It follows at once that $\mu\mapsto\int w(x,y)\nu(dy)\mu(dx)$ is continuous for the vague topology on $M_\tau(X)$. The argument has supposed that w and ν are positive; but taking positive and negative parts as usual, we see that $\mu\mapsto\int w\,d\psi(\mu,\nu)$ is vaguely continuous for every $w\in C_b(X\times Y)$ and $\nu\in M_\tau(Y)$. As w is arbitrary, $\mu\mapsto\psi(\mu,\nu)$ is vaguely continuous, for every ν. Similarly, $\nu\mapsto\psi(\mu,\nu)$ is vaguely continuous for every μ, and ψ is separately continuous.

(d) Now suppose that X and Y are compact. Let W be the linear subspace of $C(X\times Y)$ generated by $\{u\otimes v:u\in C(X),\ v\in C(Y)\}$, writing $(u\otimes v)(x,y)=u(x)v(y)$ as in 253B. Then W is a subalgebra of $C(X\times Y)$ separating the points of $X\times Y$ and containing the constant functions, so is $\|\ \|_\infty$-dense in $C(X\times Y)$ (281E). Now

$$(\mu,\nu)\mapsto\int u\otimes v\,d\psi(\mu,\nu)=\int u\,d\mu\cdot\int v\,d\nu$$

is continuous whenever $u\in C(X)$ and $v\in C(Y)$, so

$$(\mu,\nu)\mapsto\int w\,d\psi(\mu,\nu)$$

is continuous whenever $w\in W$.

Now suppose that $B\subseteq M_\tau(X)$ and $B'\subseteq M_\tau(Y)$ are bounded. Let $\gamma\geq 0$ be such that $\|\mu\|\leq\gamma$ for every $\mu\in B$ and $\|\nu\|\leq\gamma$ for every $\nu\in B$. If $w\in C(X\times Y)$ and $\epsilon>0$, there is a $w'\in W$ such that $\|w-w'\|_\infty\leq\epsilon$. In this case

$$\begin{aligned}|\int w\,d\psi(\mu,\nu)-\int w'\,d\psi(\mu,\nu)| &\leq \|w-w'\|_\infty\|\psi(\mu,\nu)\|\\ &\leq\epsilon\|\mu\|\|\nu\|\leq\gamma^2\epsilon\end{aligned}$$

whenever $\mu\in B$ and $\nu\in B'$. As ϵ is arbitrary, the function $(\mu,\nu)\mapsto\int w\,d\psi(\mu,\nu)$ is uniformly approximated on $B\times B'$ by vaguely continuous functions, and is therefore itself vaguely continuous on $B\times B'$.

437N (Quasi)-Radon measures Throughout the above discussion, I have tried to maintain the formal distinctions between 'quasi-Radon measure' and 'τ-additive effectively locally finite Borel measure inner regular with respect to the closed sets', and between 'Radon measure' and 'tight locally finite Borel measure'. There are obvious problems in interpreting the sum and difference of measures with different domains, which are readily soluble (see, for instance, 416Ea and 416Xd) but in the context of this section are unilluminating. If, however, we take M_{qR}^+ to be the set of totally finite quasi-Radon measures on X, and X is completely regular, we have a canonical embedding of M_{qR}^+ into a cone in the L-space $C_b(X)^*$; more generally, even

if our space X is not completely regular, the map $\mu \mapsto \mu\restriction\mathcal{B}(X) : M^+_{\mathrm{qR}} \to M_\sigma(\mathcal{B}(X))$ is still injective, and we can identify M^+_{qR} with a cone in the L-space M_τ of signed τ-additive Borel measures (often the whole positive cone of M_τ, as in 415M). Similarly, when X is Hausdorff, we can identify totally finite Radon measures with tight totally finite Borel measures (416F).

It is even possible to extend these ideas to measures which are not totally finite (437Yj), though there may be new difficulties (415Ya).

437O As usual, the case in which X is compact and Hausdorff is particularly important.

Proposition Let X be a compact Hausdorff space, and $\gamma \in [0, \infty[$. Write $P'_{\mathrm{R}\gamma}$ for the set of Radon measures μ on X such that $\mu X \leq \gamma$, and $P_{\mathrm{R}\gamma}$ for the set of Radon measures μ on X such that $\mu X = \gamma$. Then $P'_{\mathrm{R}\gamma}$ and $P_{\mathrm{R}\gamma}$ are compact in their narrow topologies.

proof By 437Kc the narrow topologies on these sets can be identified with the weak* topologies on their images in $C_b(X)^* = C(X)^*$. Let $T : \tilde{M}^+ \to C(X)^*$ be the map described in 437K. Then

$$T[P'_{\mathrm{R}\gamma}] = \{f : f \in C(X)^*,\ f(u) \geq 0 \text{ for every } u \in C(X)^+,\ f(\chi X) \leq \gamma\}.$$

P If $\mu \in P'_{\mathrm{R}\gamma}$, then of course $(T\mu)(u) = \int u\,d\mu \geq 0$ whenever $u \geq 0$, and $(T\mu)(\chi X) = \mu X \leq \gamma$. If $f \in (C(X)^*)^+$ and $f(\chi X) \leq \gamma$, then the Riesz representation theorem (436J/436K) tells us that $f = T\mu$ for some Radon measure μ, and $\mu X = f(\chi X) \leq \gamma$, so $\mu \in P'_{\mathrm{R}\gamma}$. **Q**

This description of $T[P'_{\mathrm{R}\gamma}]$ makes it plain that it is weak*-closed in $C(X)^*$. But also, of course, $\|T\mu\| = \mu X \leq \gamma$ for every $\mu \in P'_{\mathrm{R}\gamma}$; since the unit ball of $C(X)^*$ is weak*-compact (3A5F), so is $T[P'_{\mathrm{R}\gamma}]$. So $P'_{\mathrm{R}\gamma}$ is compact for the narrow topology.

Since $P_{\mathrm{R}\gamma} = \{\mu : \mu \in P'_{\mathrm{R}\gamma},\ \int \chi X\,d\mu = \gamma\}$, it too is compact.

437P The sets $P'_{\mathrm{R}\gamma}$, $P_{\mathrm{R}\gamma}$ can be identified with convex sets in $C(X)^*$, and can therefore have extreme points. These extreme points are easy to identify; I look at $P_{\mathrm{R}1} = P_{\mathrm{R}}$.

Proposition Let X be a Hausdorff space, and P_{R} the set of Radon probability measures on X. The natural embedding of P_{R} into the linear space $M_\sigma(\mathcal{B}(X))$ of signed Borel measures on X gives P_{R} a convex structure, for which the extreme points are just the point-supported measures δ_x for $x \in X$, where $\delta_x(E) = \chi E(x)$ for every $E \subseteq X$.

proof (a) Suppose that $x \in X$. If μ_1, $\mu_2 \in P_{\mathrm{R}}$ are such that $\delta_x = \frac{1}{2}(\mu_1 + \mu_2)$, then (performing the linear operations in M_σ) we must have $\mu_1 E \leq 2\mu E$ for every Borel set E; in particular, $\mu_1(X \setminus \{x\}) = 0$ and $\mu_1\{x\} = 1$, that is, $\mu_1 = \delta_x$. Similarly, $\mu_2 = \delta_x$; as μ_1 and μ_2 are arbitrary, δ_x is an extreme point of P_{R}.

(b) Suppose that μ is an extreme point of P_{R}. Let K be the support of μ. **?** If K has more than one point, take distinct x, $y \in K$. As X is Hausdorff, there are disjoint open sets G, H such that $x \in G$ and $y \in H$. Set $E = G \cap K$, $\alpha = \mu E$. Because K is the support of μ, $\alpha > 0$. But similarly $\mu(H \cap K) > 0$ and $\alpha < 1$. Let μ_1, μ_2 be the indefinite-integral measures defined over μ by $\frac{1}{\alpha}\chi E$ and $\frac{1}{\beta}\chi(X \setminus E)$ respectively. Then both are Radon probability measures on X (416S), so belong to P_{R}. Now $\mu F = \alpha\mu_1 F + \beta\mu_2 F$ for every Borel set F, so (computing $\alpha\mu_1 + \beta\mu_2$ in M_σ) $\mu = \alpha\mu_1 + \beta\mu_2$; as neither μ_1 nor μ_2 is equal to μ, μ is not extreme in P_{R}. **X**

Thus $K = \{x\}$ for some $x \in X$. But this means that $\mu\{x\} = 1$ and $\mu(X \setminus \{x\}) = 0$, so $\mu = \delta_x$ is of the declared form.

437Q For metrizable spaces we can go farther. It will help to have a straightforward lemma.

Lemma (a) Let X and Y be Hausdorff spaces, and $\phi : X \to Y$ a continuous function. Let $M^+_{\mathrm{R}}(X)$, $M^+_{\mathrm{R}}(Y)$ be the spaces of totally finite Radon measures on X and Y respectively. Writing $\tilde{\phi}(\mu)$ for the image measure $\mu\phi^{-1}$ for $\mu \in M^+_{\mathrm{R}}$, $\tilde{\phi} : M^+_{\mathrm{R}}(X) \to M^+_{\mathrm{R}}(Y)$ is continuous for the narrow topologies on $M^+_{\mathrm{R}}(X)$ and $M^+_{\mathrm{R}}(Y)$.

(b) If Y is a Hausdorff space, X a subset of Y, and $\phi : X \to Y$ the identity map, then $\tilde{\phi}$ is a homeomorphism between $M^+_{\mathrm{R}}(X)$ and $\{\nu : \nu \in M^+_{\mathrm{R}}(Y),\ \nu(Y \setminus X) = 0\}$.

proof (a) All we have to do is to recall from 418I that $\mu\phi^{-1} \in M^+_{\mathrm{R}}(Y)$ for every $\mu \in M^+_{\mathrm{R}}(X)$, and observe that

$$\{\mu : (\mu\phi^{-1})(H) > \alpha\} = \{\mu : \mu\phi^{-1}[H] > \alpha\}, \quad \{\mu : (\mu\phi^{-1})(Y) < \alpha\} = \{\mu : \mu X < \alpha\}$$

are narrowly open in $M^+_{\rm R}(X)$ for every open set $H \subseteq Y$ and $\alpha \in \mathbb{R}$.

(b) First note that if $\mu \in M^+_{\rm R}(X)$, then certainly $\tilde{\phi}(\mu)(Y \setminus X) = 0$; while if $\nu \in M^+_{\rm R}(Y)$ and $\nu(Y \setminus X) = 0$, then $\mu = \nu\restriction\mathcal{P}X$ is a Radon measure on X (416Rb) and $\nu = \tilde{\phi}(\mu)$. Thus $\tilde{\phi}$ is a continuous bijection from $M^+_{\rm R}(X)$ to $\{\nu : \nu \in M^+_{\rm R}(Y), \nu(Y \setminus X) = 0\}$. Now if $G \subseteq X$ is relatively open and $\alpha \in \mathbb{R}$, there is an open set $H \subseteq Y$ such that $G = H \cap X$, so that

$$\{\mu : \mu \in M^+_{\rm R}(X), \mu G > \alpha\} = \{\mu : \tilde{\phi}(\mu)(H) > \alpha\}$$

is the inverse image of a narrowly open set in $M^+_{\rm R}(Y)$; and of course

$$\{\mu : \mu \in M^+_{\rm R}(X), \mu X < \alpha\} = \{\mu : \tilde{\phi}(\mu)(Y) < \alpha\}$$

is also the inverse image of an open set. So $\tilde{\phi}$ is a homeomorphism between $M^+_{\rm R}(X)$ and $\{\nu : \nu \in M^+_{\rm R}(Y), \nu(Y \setminus X) = 0\}$.

437R Theorem For a Hausdorff space X, write $P_{\rm R}(X)$ for the set of Radon probability measures on X, with its narrow topology.

(a) If X is a compact metrizable space, $P_{\rm R}(X)$ is compact and metrizable.

(b) If X is a Polish space, $P_{\rm R}(X)$ is Polish.

proof (a) Let $\mathcal{U}$ be a countable base for the topology of X which contains $\emptyset$ and is closed under finite unions. For $U \in \mathcal{U}$, $q \in \mathbb{Q}$ set $H_{Uq} = \{\mu : \mu \in P_{\rm R}(X), \mu U > q\}$. Then every H_{Uq} is open for the narrow topology $\mathfrak{T}$ on $P_{\rm R}(X)$; let $\mathfrak{S}$ be the topology generated by $\{H_{Uq} : U \in \mathcal{U}, q \in \mathbb{Q}\}$, so that $\mathfrak{S}$ is a second-countable topology coarser than $\mathfrak{T}$. If $G \subseteq X$ is open and $\alpha \in \mathbb{R}$, then $\{U : U \in \mathcal{U}, U \subseteq G\}$ is an upwards-directed set with union G, so

$$\{\mu : \mu \in P_{\rm R}(X), \mu G > \alpha\} = \bigcup_{U \in \mathcal{U}, U \subseteq G} \bigcup_{q \in \mathbb{Q}, q \geq \alpha} H_{Uq} \in \mathfrak{S}.$$

Accordingly $\mathfrak{S}$ must be actually equal to $\mathfrak{T}$, and $\mathfrak{T}$ is second-countable; since we already know that $\mathfrak{T}$ is compact (437O), it must be metrizable (4A2Nh).

(b) By 4A2Qg, we can suppose that X is a G_δ subset of a compact metrizable space Z. Then we have a homeomorphism between $P_{\rm R}(X)$ and $Q = \{\nu : \nu \in P_{\rm R}(Z), \nu(Z \setminus X) = 0\}$ (437Qb). Now Q is a G_δ set in $P_{\rm R}(Z)$. **P** Express X as $\bigcap_{n \in \mathbb{N}} H_n$ where $H_n \subseteq Z$ is open for each n. Then

$$Q = \bigcap_{n \in \mathbb{N}} \{\nu : \nu H_n = 1\} = \bigcap_{m,n \in \mathbb{N}} \{\nu : \nu H_n > 1 - 2^{-m}\}$$

is a G_δ set in $P_{\rm R}(Z)$. **Q**

By (a), $P_{\rm R}(Z)$ is a compact metrizable space; by 4A2Qd, its G_δ subset Q is Polish; so $P_{\rm R}(X)$ also is Polish, as claimed.

437S We now have a language in which to express a fundamental result in the theory of dynamical systems.

Theorem Let X be a non-empty compact Hausdorff space, and $\phi : X \to X$ a continuous function. Write Q_ϕ for the set of Radon probability measures on X for which ϕ is inverse-measure-preserving. Then Q_ϕ is not empty, and is compact for the narrow topology; regarded as a subset of the linear space M_t of signed tight Borel measures on X, Q_ϕ is convex.

proof (a) Write $P_{\rm R}$ for the set of Radon probability measures on X. For $\mu \in P_{\rm R}$, write $T\mu$ for the corresponding linear functional on $C(X)$, so that $(T\mu)(u) = \int u \, d\mu$ for $u \in C(X)$. By the Riesz Representation Theorem (436J/436K), T is a bijection between $P_{\rm R}$ and the set K of positive linear functionals f on $C(X)$ such that $f(\chi X) = 1$. By 437H/437I, the linear structure of M_t matches that of $C(X)^*$, so the convex structure of K, when transferred to $P_{\rm R}$ by T, corresponds to the convex structure of $\{\mu\restriction\mathcal{B}(X) : \mu \in P_{\rm R}\} \subseteq M_t$. By 437Kc, the narrow topology on $P_{\rm R}$ corresponds to the weak* topology $\mathfrak{T}_s(C(X)^*, C(X))$ on K. As noted in 437O, $P_{\rm R}$ and K are compact.

(b) If $\mu \in P_{\rm R}$, then $\mu \in Q_\phi$ iff $(T\mu)(u\phi) = (T\mu)(u)$ for every $u \in C(X)$. **P** For $\mu \in P_{\rm R}$, consider the image measure $\mu\phi^{-1}$. This is a Radon measure (418I). Now

$$\mu \in Q_\phi \iff \mu\phi^{-1}[E] = \mu E \text{ for every } E \in \operatorname{dom}\mu$$
$$\iff \mu\phi^{-1} = \mu$$

(because $\mu\phi^{-1}$ and μ are Radon measures, so if they agree on all Borel sets they must be identical)

$$\iff \int u\, d(\mu\phi^{-1}) = \int u\, d\mu \text{ for every } u \in C(X)$$

(416E(b-v))

$$\iff \int u\phi\, d\mu = \int u\, d\mu \text{ for every } u \in C(X)$$

(235A)

$$\iff (T\mu)(u\phi) = (T\mu)(u) \text{ for every } u \in C(X). \blacksquare$$

(c) Since $\{f : f \in C(X)^*,\ f(u) = f(u\phi) \text{ for every } u \in C(X)\}$ is convex and closed in $C(X)^*$ for the weak* topology, Q_ϕ is convex and closed in P_{R}, and is therefore compact.

(d) To see that Q_ϕ is not empty, take any $x_0 \in X$ and a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$. Define $f : C(X) \to \mathbb{R}$ by setting $f(u) = \lim_{n\to\mathcal{F}} \frac{1}{n+1}\sum_{i=0}^{n} u(\phi^i(x_0))$ for every $u \in C(X) = C_b(X)$. Then f is a positive linear functional and $f(\chi X) = 1$. So there is a $\mu \in P_{\mathrm{R}}$ such that $f = T\mu$.

If $u \in C(X)$, then $f(u) = f(u\phi)$. $\blacktriangleright$

$$|f(u\phi) - f(u)| = |\lim_{n\to\mathcal{F}} \frac{1}{n+1}\sum_{i=0}^{n} u(\phi^{i+1}(x_0)) - u(\phi^i(x_0))|$$
$$= |\lim_{n\to\mathcal{F}} \frac{1}{n+1} u(\phi^{n+1}(x_0)) - u(x_0)|$$
$$\leq \lim_{n\to\mathcal{F}} \frac{1}{n+1}|u(\phi^{n+1}(x_0)) - u(x_0)| \leq \lim_{n\to\mathcal{F}} \frac{2\|u\|_\infty}{n+1} = 0. \blacksquare$$

By the characterization in (b), $\mu \in Q_\phi$.

437T Prokhorov spaces For topological spaces X which are not compact, weak* compactness in $C_b(X)^*$ is still interesting, and we can look for measure-theoretic criteria which will help us to recognise sets of measures which are relatively compact in the vague topology. The most important of these seems to be the following.

Definition Let X be a Hausdorff space.

(a) If A is a norm-bounded subset of $C_b(X)^*$, we say that A is **uniformly tight** if for every $\epsilon > 0$ there is a compact set $K \subseteq X$ such that $|f(u)| \leq \epsilon$ whenever $f \in A$, $u \in C_b(X)$ and $|u| \leq \chi(X \setminus K)$. (Cf. 436Xn.)

(b) Let $\tilde{M}$ be the space of bounded additive functionals defined on subalgebras of $\mathcal{P}X$ containing every open set. For $\nu \in \tilde{M}$ define $|\nu| \in \tilde{M}$ by saying that if Σ is the domain of ν, then $|\nu|E = \sup\{\nu F - \nu(E \setminus F) : F \in \Sigma,\ F \subseteq E\}$ for every $E \in \Sigma$; that is, we calculate $|\nu|$ in the L-space $M(\Sigma)$ of bounded additive functionals on Σ (362Ba). (Warning! if T is a subalgebra of Σ, $|\nu\restriction \mathrm{T}|$ may be different from $|\nu|\restriction \mathrm{T}$.) I say that a functional $\nu \in \tilde{M}$ is **tight** if $|\nu|(E) = \sup\{|\nu|(K) : K \subseteq E \text{ is closed and compact}\}$ for every $E \in \operatorname{dom}\nu$, and that a norm-bounded set A of tight functionals in $\tilde{M}$ is **uniformly tight** if for every $\epsilon > 0$ there is a compact set $K \subseteq X$ such that $|\nu|(X \setminus K) \leq \epsilon$ for every $\nu \in A$.

437U Proposition Let X be a Hausdorff space, and M_{R}^+ the set of totally finite Radon measures on X. For $\mu \in M_{\mathrm{R}}^+$ define $T\mu \in C_b(X)^*$ by setting $(T\mu)(u) = \int u\, d\mu$ for every $u \in C_b(X)$.

(a) Let $A \subseteq C_b(X)^*$ be a norm-bounded set. Then the following are equiveridical:

(i) A is uniformly tight;

(ii) there is a solid convex weak*-closed uniformly tight set $\tilde{A} \subseteq C_b(X)^*$ including A;

(iii) there is a uniformly tight set $B \subseteq M_{\mathrm{R}}^+$ such that $\{|f| : f \in A\} = T[B]$.

(b) If $A \subseteq M_{\mathrm{R}}^+$ is bounded and uniformly tight, then A is relatively compact in M_{R}^+ for the narrow topology on M_{R}^+.

proof (a)(i)⇒(ii) If A is uniformly tight, then for each $n \in \mathbb{N}$ we can choose a compact set $K_n \subseteq X$ such that $|f(u)| \leq 2^{-n}$ whenever $u \in C_b(X)$ and $|u| \leq \chi(X \setminus K_n)$. Set

$$\tilde{A} = \{f : f \in C_b(X)^*, |f(u)| \leq 2^{-n} \text{ whenever } u \in C_b(X), n \in \mathbb{N} \text{ and } |u| \leq \chi(X \setminus K_n)\}.$$

If $f \in \tilde{A}$, $g \in C_b(X)^*$ and $|g| \leq |f|$, then

$$|g(v)| \leq |g|(|v|) \leq |f|(|v|) = \sup\{|f(u)| : |u| \leq |v|\}$$

for every $v \in C_b(X)$; it follows at once that $g \in \tilde{A}$. Thus $\tilde{A}$ is solid (352Ja). It is easy to check that $\tilde{A}$ is convex, weak*-closed and uniformly tight. So (ii) is true.

(ii)⇒(i) is trivial.

(i)⇒(iii) Suppose that A is uniformly tight. For every $n \in \mathbb{N}$ choose a compact set $K_n \subseteq X$ such that $|f(u)| \leq 2^{-n}$ whenever $u \in C_b(X)$ and $|u| \leq \chi(X \setminus K_n)$; replacing K_n by $\bigcup_{i \leq n} K_i$ if necessary, we can arrange that $\langle K_n \rangle_{n \in \mathbb{N}}$ is non-decreasing. Let B_1 be the set of Radon measures μ on X such that $\mu X \leq \sup_{f \in A} \|f\|$ and $\mu(X \setminus K_n) \leq 2^{-n}$ for every $n \in \mathbb{N}$; of course B_1 is uniformly tight.

(α) Set $A' = \{|f| : f \in A\}$. Then every $g \in A'$ is smooth. **P** Let $f \in A$ be such that $g = |f|$. Suppose that $D \subseteq C_b(X)^*$ is non-empty and downwards-directed, and that $\inf_{u \in D} u(x) = 0$ for every $x \in X$. Fix $v_0 \in D$. Let $\epsilon > 0$. Let $n \in \mathbb{N}$ be such that $2^{-n}(\|g\| + \|v_0\|_\infty) \leq \epsilon$. Then there is a $v \in D$ such that $v \leq v_0$ and $v(x) \leq 2^{-n}$ whenever $x \in K_n$ (apply 436Ic to $\{v \restriction K_n : v \in D, v \leq v_0\} \subseteq C(K_n)$). Set $v' = v \wedge 2^{-n}\chi X$, $v'' = v - v'$. Then $g(v') \leq 2^{-n}\|g\|$. Also, if $u \in C_b(X)$ and $|u| \leq v''$, we have $|u| \leq \|v_0\|_\infty \chi(X \setminus K_n)$, so $|f(u)| \leq 2^{-n}\|v_0\|_\infty$; as u is arbitrary, $g(v'') \leq 2^{-n}\|v_0\|_\infty$. Putting these together, $g(v) \leq 2^{-n}(\|g\| + \|v_0\|_\infty \leq \epsilon$. As D and ϵ are arbitrary, g is smooth. **Q**

(β) For every $g \in A'$ there is a $\mu \in B_1$ such that $g = T\mu$. **P** By 436H there is a measure ν on X, quasi-Radon for the topology $\mathfrak{S}$ generated by $C_b(X)$, such that $g(u) = \int u \, d\nu$ for every $u \in C_b(X)$; and of course $\nu X \leq \sup_{f \in A} \|f\|$. As noted in the remark following 436H, we can arrange that ν should be inner regular for the family $\mathcal{L}$ of sets L such that $\chi L = \inf D$ for some non-empty $D \subseteq C_b(X)$. If $L \in \mathcal{L}$ and $n \in \mathbb{N}$ and $K_n \cap L = \emptyset$ and $\epsilon > 0$, set $D = \{u : u \in C_b(X), \chi L \leq u \leq \chi X\}$. Then D is downwards-directed and $\inf_{u \in D} u(x) = 0$ for every $x \in K_n$, so there is a $u \in D$ such that $u(x) \leq \epsilon$ for every $x \in K_n$; in this case, $(1 - \epsilon)\nu L \leq g((u - \epsilon\chi X)^+) \leq 2^{-n}$. As ϵ is arbitrary, $\nu L \leq 2^{-n}$; as L is arbitrary, $\nu_*(X \setminus K_n) \leq 2^{-n}$ and $\nu^* K_n \geq \nu X - 2^{-n}$.

What this means is that $\nu X = \sup_{n \in \mathbb{N}} \nu^* K_n$, so by 416O there is a Radon measure μ on X extending ν such that $\mu K_n = \nu^* K_n$ for every n. In this case $\mu(X \setminus K_n) = \nu_*(X \setminus K_n) \leq 2^{-n}$ for every n, and $\mu \in B_1$. Of course we now have $\int u \, d\mu = \int u \, d\nu = g(u)$ for every $u \in C_b(X)$ (because the identity map from X to itself is inverse-measure-preserving for μ and ν), so $T\mu = g$. **Q**

(γ) So if we set $B = \{\mu : \mu \in B_1, T\mu \in A'\}$, then B witnesses that (iii) is true.

(iii)⇒(i) If $B \subseteq M_{\mathrm{R}}^+$ is uniformly tight, then $\{T\mu : \mu \in B\}$ is uniformly tight. **P** For every $\epsilon > 0$ there is a compact set $K \subseteq X$ such that $\mu(X \setminus K) \leq \epsilon$ for every $\mu \in B$; now $|(T\mu)(u)| \leq \epsilon$ whenever $\mu \in B$, $u \in C_b(X)$ and $|u| \leq \chi(X \setminus K)$. **Q** So (iii) implies that $A' = \{|f| : f \in A\}$ is uniformly tight; because (i)⇒(ii), it follows at once that A is uniformly tight.

(b) Set $\gamma = \sup_{\mu \in A} \mu X$ (taking $\gamma = 0$ in the trivial case $A = \emptyset$), and for each $n \in \mathbb{N}$ let $K_n \subseteq X$ be a compact set such that $\mu(X \setminus K_n) \leq 2^{-n}$ for every $\mu \in A$. Let $\tilde{A} \supseteq A$ be the set of Radon measures μ on X such that $\mu X \leq \gamma$ and $\mu(X \setminus K_n) \leq 2^{-n}$ for every $n \in \mathbb{N}$. Let $\mathcal{F}$ be an ultrafilter on M_{R}^+ containing $\tilde{A}$, and set $\nu E = \lim_{\mu \to \mathcal{F}} \mu E$ for every Borel set $E \subseteq X$. Then ν is a non-negative additive functional on the Borel σ-algebra, and $\nu X \leq \gamma$, while $\nu(X \setminus K_n) \leq 2^{-n}$ for every n.

By 416N, there is a Radon measure λ on X such that $\lambda X \leq \nu X$ and $\lambda K \geq \nu K$ for every compact set K. Now since $\inf_{K \subseteq X \text{ is compact}} \nu(X \setminus K) = 0$, $\sup_{K \subseteq X \text{ is compact}} \nu(E \cap K) = \nu E$ for every $E \in \operatorname{dom} \nu$. In particular,

$$\lambda X \leq \nu X = \sup_{K \subseteq X \text{ is compact}} \nu K \leq \sup_{K \subseteq X \text{ is compact}} \lambda K = \lambda X$$

and $\lambda X = \nu X$. Now

$$\lambda K_n \geq \nu K_n \geq \nu X - 2^{-n} = \lambda X - 2^{-n}$$

for every n, so $\lambda \in \tilde{A}$. Next, if $G \subseteq X$ is open, $\lambda G \leq \nu G$. **P** If $\epsilon > 0$, let $K \subseteq X$ be a compact set such that $\nu K \geq \nu X - \epsilon$. Then

$$\begin{aligned}\lambda G &\leq \lambda X - \lambda(K \setminus G) \leq \nu X - \nu(K \setminus G)\\ &\leq \epsilon + \nu K - \nu(K \setminus G) = \epsilon + \nu(K \cap G) \leq \epsilon + \nu G;\end{aligned}$$

as ϵ is arbitrary, $\lambda G \leq \nu G$. **Q**

Now suppose that U is a subset of M^+_{R} which contains λ and is open for the narrow topology. Then there must be open sets $G_0, \ldots, G_n \subseteq X$ and $\alpha_0, \ldots, \alpha_n, \beta \in \mathbb{R}$ such that

$$V = \{\mu : \mu \in M^+_{\mathrm{R}},\ \mu G_i > \alpha_i \text{ for every } i \leq n,\ \mu X < \beta\}$$

contains λ and is included in U. In this case, however, $\nu X < \beta$ and $\nu G_i > \alpha_i$ for every i, so $V \in \mathcal{F}$ and $U \in \mathcal{F}$. Thus λ is a limit of $\mathcal{F}$; as $\mathcal{F}$ is arbitrary, $\tilde{A}$ is compact and A is relatively compact.

437V In important cases, the narrowly compact subsets of $M^+_{\mathrm{R}}(X)$ are exactly the bounded uniformly tight sets. Once again, it is worth introducing a word to describe when this happens.

Definition Let X be a Hausdorff space and $P_{\mathrm{R}}(X)$ the set of Radon probability measures on X. X is a **Prokhorov space** if every subset of $P_{\mathrm{R}}(X)$ which is compact for the narrow topology is uniformly tight.

437W Theorem (a) Compact Hausdorff spaces are Prokhorov spaces.
(b) A closed subspace of a Prokhorov Hausdorff spaces is a Prokhorov space.
(c) An open subspace of a Prokhorov Hausdorff space is a Prokhorov space.
(d) The product of a countable family of Prokhorov Hausdorff spaces is a Prokhorov space.
(e) Any G_δ subset of a Prokhorov Hausdorff space is a Prokhorov space.
(f) Čech-complete spaces are Prokhorov spaces.
(g) Polish spaces are Prokhorov spaces.

proof (a) This is trivial; on a compact Hausdorff space the set of all Radon probability measures is uniformly tight.

(b) Let X be a Prokhorov Hausdorff space, Y a closed subset of X, and $A \subseteq P_{\mathrm{R}}(Y)$ a narrowly compact set. Taking ϕ to be the identity map from Y to X, and defining $\tilde{\phi} : M^+_{\mathrm{R}}(Y) \to M^+_{\mathrm{R}}(X)$ as in 437Q, $\tilde{\phi}[A]$ is narrowly compact in $P_{\mathrm{R}}(X)$, so is uniformly tight. For any $\epsilon > 0$, there is a compact set $K \subseteq X$ such that $\tilde{\phi}(\mu)(X \setminus K) \leq \epsilon$ for every $\mu \in A$. Now $K \cap Y$ is a compact subset of Y and $\mu(Y \setminus (K \cap Y)) \leq \epsilon$ for every $\mu \in A$. As ϵ is arbitrary, A is uniformly tight in $P_{\mathrm{R}}(Y)$.

(c) Let X be a Prokhorov Hausdorff space, Y an open subset of X, and $A \subseteq P_{\mathrm{R}}(Y)$ a narrowly compact set. Once again, take ϕ to be the identity map from Y to X, so that $\tilde{\phi}[A] \subseteq P_{\mathrm{R}}(X)$ is narrowly compact and uniformly tight in $P_{\mathrm{R}}(X)$.

? Suppose, if possible, that A is not uniformly tight in $P_{\mathrm{R}}(Y)$. Then there is an $\epsilon > 0$ such that $A_K = \{\mu : \mu \in A,\ \mu(Y \setminus K) \geq 5\epsilon\}$ is non-empty for every compact set $K \subseteq Y$. Note that $A_K \subseteq A_{K'}$ whenever $K \supseteq K'$, so $\{A_K : K \subseteq Y \text{ is compact}\}$ has the finite intersection property, and there is an ultrafilter $\mathcal{F}$ on $P_{\mathrm{R}}(Y)$ containing every A_K. Because A is narrowly compact, there is a $\lambda \in P_{\mathrm{R}}(Y)$ such that $\mathcal{F} \to \lambda$. Let $K^* \subseteq Y$ be a compact set such that $\lambda(Y \setminus K^*) \leq \epsilon$.

As $\tilde{\phi}[A]$ is uniformly tight, there is a compact set $L \subseteq X$ such that $\mu(Y \setminus L) = \tilde{\phi}(\mu)(X \setminus L) \leq \epsilon$ for every $\mu \in A$. Now K^* and $L \setminus Y$ are disjoint compact sets in the Hausdorff space X, so there are disjoint open sets G, $H \subseteq X$ such that $K^* \subseteq G$ and $L \setminus Y \subseteq H$ (4A2Fh). Set $K = L \setminus H \supseteq L \cap G$; then K is a compact subset of Y. As $A_K \in \mathcal{F}$, there must be a $\mu \in A_K$ such that $\mu Y \leq \lambda Y + \epsilon$ and $\mu(G \cap Y) \geq \lambda(G \cap Y) - \epsilon$. Accordingly

$$\begin{aligned}\mu(Y \cap L) &\leq \epsilon,\\ \mu(Y \setminus G) &= \mu Y - \mu(G \cap Y) \leq \lambda Y + \epsilon - \lambda(G \cap Y) + \epsilon\\ &= \lambda(Y \setminus G) + 2\epsilon \leq \lambda(Y \setminus K^*) + 2\epsilon \leq 3\epsilon,\\ \mu((Y \cap L) \cup (Y \setminus G)) &= \mu(Y \setminus (L \cap G)) \geq \mu(Y \setminus K) \geq 5\epsilon,\end{aligned}$$

which is impossible. **X**

Thus A is uniformly tight. As A is arbitrary, Y is a Prokhorov space.

(d) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a sequence of Prokhorov Hausdorff spaces with product X. Let $A \subseteq P_{\rm R}(X)$ be a narrowly compact set. Let $\epsilon > 0$. For each $n \in \mathbb{N}$ let $\pi_n : X \to X_n$ be the canonical map and $\tilde{\pi}_n : M^+_{\rm R}(X) \to M^+_{\rm R}(X_n)$ the associated function. Then $\tilde{\pi}_n[A]$ is narrowly compact in $P_{\rm R}(X_n)$, therefore uniformly tight, and there is a compact set $K_n \subseteq X_n$ such that $(\tilde{\pi}_n\mu)(X_n \setminus K_n) \leq 2^{-n-1}\epsilon$. Set $K = \prod_{n\in\mathbb{N}} K_n$, so that K is a compact subset of X and $X \setminus K = \bigcup_{n\in\mathbb{N}} \pi_n^{-1}[X_n \setminus K_n]$. If $\mu \in A$, then

$$\mu(X \setminus K) \leq \sum_{n=0}^{\infty} \mu\pi_n^{-1}[X_n \setminus K_n] \leq \sum_{n=0}^{\infty} 2^{-n-1}\epsilon = \epsilon.$$

As ϵ is arbitrary, A is uniformly tight; as A is arbitrary, X is a Prokhorov space.

(e) Let X be a Prokhorov Hausdorff space and Y a G_δ subset of X. Express Y as $\bigcap_{n\in\mathbb{N}} Y_n$ where every $Y_n \subseteq X$ is open. Set $Z = \{z : z \in \prod_{n\in\mathbb{N}} Y_n,\ z(m) = z(n) \text{ for all } m, n \in \mathbb{N}\}$. Because X is Hausdorff, Z is a closed subspace of $\prod_{n\in\mathbb{N}} Y_n$ homeomorphic to Y. Putting (c), (d) and (b) together, Z and Y are Prokhorov spaces.

(f) Put (a), (e) and the definition of 'Čech-complete' together.

(g) This is a special case of (f) (4A2Md).

437X Basic exercises (a) Let X be a set, U a Riesz subspace of $\mathbb{R}^X$ and $f \in U^\sim$. (i) Show that $f \in U^\sim_\sigma$ iff $\lim_{n\to\infty} f(u_n) = 0$ whenever $\langle u_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in U such that $\lim_{n\to\infty} u_n(x) = 0$ for every $x \in X$. (*Hint*: show that in this case, if $0 \leq v_n \leq u_n$, we can find $k(n)$ such that $f(v_n \wedge u_{k(n)}) \leq f(v_n)+2^{-n}$.) (ii) Show that $f \in U^\sim_\tau$ iff $\inf_{u\in A} |f(u)| = 0$ whenever $A \subseteq U$ is a non-empty downwards-directed set and $\inf_{u\in A} u(x) = 0$ for every $x \in X$. (*Hint*: given $\epsilon > 0$, set $B = \{v : f(v) \geq \inf_{u\in A} f^+(u) - \epsilon,\ \exists\, w \in A,\ v \geq w\}$ and show that B is a downwards-directed set with infimum 0 in $\mathbb{R}^X$.)

(b) Let X be a set, U a Riesz subspace of $\ell^\infty(X)$ containing the constant functions, and Σ the smallest σ-algebra of subsets of X with respect to which every member of U is measurable. Let μ and ν be two totally finite measures on X with domain Σ, and f, g the corresponding linear functionals on U. Show that $f \wedge g = 0$ in $U^\sim$ iff there is an $E \in \Sigma$ such that $\mu E = \nu(X \setminus E) = 0$. (*Hint*: 326I.)

(c) (i) Show that S, in 437C, is the unique sequentially order-continuous positive linear operator from $\mathcal{L}^\infty$ to $(U^\sim_\sigma)^*$ which extends the canonical embedding of U in $(U^\sim_\sigma)^*$. (ii) Show that S, in 437H, is the unique sequentially order-continuous positive linear operator from $\mathcal{L}^\infty$ to $(U^\sim_\tau)^*$ which extends the canonical embedding of U in $(U^\sim_\tau)^*$ and is 'τ-additive' in the sense that whenever $\mathcal{G}$ is a non-empty upwards-directed family of open sets with union H then $S(\chi H) = \sup_{G\in\mathcal{G}} S(\chi G)$ in $(U^\sim_\tau)^*$.

(d) Let X and Y be completely regular topological spaces and $\phi : X \to Y$ a continuous function. Define $T : C_b(Y) \to C_b(X)$ by setting $T(v) = v\phi$ for every $v \in C_b(Y)$, and let $T' : C_b(Y)^* \to C_b(X)^*$ be its adjoint. (i) Show that T' is a norm-preserving Riesz homomorphism. (ii) Show that $T'[C_b(Y)^\sim_\sigma] \subseteq C_b(X)^\sim_\sigma$, and that if $f \in C_b(X)^\sim_\sigma$ corresponds to a Baire measure μ on X, then $T'f$ corresponds to the Baire measure $\mu\phi^{-1}\restriction\mathcal{B}\mathfrak{a}(Y)$. (iii) Show that $T'[C_b(Y)^\sim_\tau] \subseteq C_b(X)^\sim_\tau$, and that if $f \in C_b(X)^\sim_\tau$ corresponds to a Borel measure μ on X, then $T'f$ corresponds to the Borel measure $\mu\phi^{-1}\restriction\mathcal{B}(Y)$. (iv) Write $\mathcal{L}^\infty_X$ and $\mathcal{L}^\infty_Y$ for the M-spaces of bounded real-valued Borel measurable functions on X, Y respectively, and $S_X : \mathcal{L}^\infty_X \to (C_b(X)^\sim_\tau)^*$, $S_Y : \mathcal{L}^\infty_Y \to (C_b(Y)^\sim_\tau)^*$ for the canonical Riesz homomorphisms as constructed in 437Hb. Show that if $T'' : (C_b(Y)^\sim_\tau)^* \to (C_b(X)^\sim_\tau)^*$ is the adjoint of $T'\restriction C_b(X)^\sim_\tau$, then $T''S_Y(v) = S_X(v\phi)$ for every $v \in \mathcal{L}^\infty(Y)$.

(e) Let X be a topological space, $\mathcal{L}^\infty(\Sigma_{\rm um})$ the space of bounded universally measurable real-valued functions on X, and M_σ the space of countably additive functionals on the Borel σ-algebra of X. Show that we have a sequentially order-continuous Riesz homomorphism $S : \mathcal{L}^\infty(\Sigma_{\rm um}) \to M^*_\sigma$ defined by the formula $(Sv)(\mu) = \int v\,d\mu$ whenever $v \in \mathcal{L}^\infty(\Sigma_{\rm um})$ and $\mu \in M^+_\sigma$.

(f) Let X be a completely regular topological space. Show that the vague topology on the space of differences of τ-additive totally finite Borel measures on X is Hausdorff.

(g) Let X be a topological space, and for $x \in X$ set $\delta_x(E) = \chi E(x)$ for every $E \in \mathcal{B} = \mathcal{B}(X)$. (i) Show that the linear span of $\{\delta_x : x \in X\}$ is dense in $M(\mathcal{B})$ for the vague topology on $M(\mathcal{B})$. (ii) Show that the convex hull of $\{\delta_x : x \in X\}$ is dense in $\{\mu : \mu \in M(\mathcal{B})^+, \mu X = 1\}$ for the narrow topology. (*Hint*: for (ii), do not use the Hahn-Banach theorem.)

>(h) Let X and Y be topological spaces, and $\phi : X \to Y$ a continuous function. Write $M_\#(X)$ for any of $M(\mathcal{B}\mathfrak{a}(X))$, $M_\sigma(\mathcal{B}\mathfrak{a}(X))$, $M(\mathcal{B}(X))$, $M_\sigma(\mathcal{B}(X))$, $M_\tau(X)$ or $M_t(X)$, where $M_\tau(X) \subseteq M_\sigma(\mathcal{B}(X))$ is the space of signed τ-additive Borel measures and $M_t(X) \subseteq M_\tau(X)$ is the space of signed tight Borel measures; and $M_\#(Y)$ for the corresponding space based on Y. Show that there is a positive linear operator $\tilde{\phi} : M_\#(X) \to M_\#(Y)$ defined by saying that $\tilde{\phi}(\mu)(E) = \mu\phi^{-1}[E]$ whenever $\mu \in M_\#(X)$ and E belongs to $\mathcal{B}\mathfrak{a}(Y)$ or $\mathcal{B}(Y)$, as appropriate, and that $\tilde{\phi}$ is continuous for the vague topologies on $M_\#(X)$ and $M_\#(Y)$.

(i) Let X be any topological space. Show that the narrow topology on the set of totally finite Borel measures on X is T_0.

(j) Let X be any topological space and $\tilde{M}^+$ the set of non-negative additive functionals defined on subalgebras of $\mathcal{P}X$ containing every open set. For μ, $\nu \in \tilde{M}^+$ define $\mu + \nu \in \tilde{M}^+$ by setting $(\mu + \nu)(E) = \mu E + \nu E$ for $E \in \operatorname{dom}\mu \cap \operatorname{dom}\nu$. (i) Show that addition on $\tilde{M}^+$ is continuous for the narrow topology. (ii) Show that $(\alpha, \mu) \mapsto \alpha\mu : [0, \infty[\times \tilde{M}^+ \to \tilde{M}^+$ is continuous for the narrow topology on $\tilde{M}^+$.

(k) Give $X = \omega_1 + 1$ its order topology. On X let μ be the Borel probability measure corresponding to Dieudonné's measure on ω_1 (see 434Xf) and ν the Borel probability measure such that $\nu\{\omega_1\} = 1$. (i) Show that $\mu G \geq \nu G$ for every open set $G \subseteq X$. (ii) Show that the narrow topology on the set of Borel probability measures on X is not T_1.

>(l) Let X be a zero-dimensional compact Hausdorff space and $\mathcal{E}$ the algebra of open-and-closed subsets of X. (i) Show that $\mathcal{E}$ separates zero sets. (ii) Show that the vague topology on $M(\mathcal{E})$ is just the pointwise topology induced by the usual topology of $\mathbb{R}^{\mathcal{E}}$. (iii) Writing M_t for the space of signed tight Borel measures on X, show that $\mu \mapsto \mu\restriction\mathcal{E} : M_t \to M(\mathcal{E})$ is a Banach lattice isomorphism between the L-spaces M_t and $M(\mathcal{E})$, and is also a homeomorphism when M_t and $M(\mathcal{E})$ are given their vague topologies.

(m) Let X be a topological space, and $\tilde{M}^+$ the space of non-negative real-valued additive functionals defined on algebras of subsets of X which contain every open set, with its narrow topology. Show that if $u : X \to \mathbb{R}$ is a bounded lower semi-continuous function then $\nu \mapsto \int u \, d\nu : \tilde{M}^+ \to \mathbb{R}$ is lower semi-continuous.

(n) In 437Mc, show that $|\psi(\mu, \nu)| = \psi(|\mu|, |\nu|)$ for every $\mu \in M_\tau(X)$ and $\nu \in M_\tau(Y)$.

(o) (i) In 437P, give P_{R} its narrow topology. Show that the map $x \mapsto \delta_x$ is a homeomorphism between X and its image in P_{R}. (ii) Show that if X is completely regular then $x \mapsto \delta_x$ is a homeomorphism between X and its image in P_{R} when P_{R} is given the vague topology corresponding to its embedding in $M_\sigma(\mathcal{B}(X))$.

(p) Let X be a compact Hausdorff space and P'_{R} the set of Radon measures μ on X such that $\mu X \leq 1$. Show that the extreme points of P'_{R} are the point-masses δ_x, as in 437P, together with the zero measure.

(q) Let X be a topological space. Show that the set of tight functionals $f \in C_b(X)^\sim$ (436Xn) is a band in $C_b(X)^\sim$ included in $C_b(X)^\sim_\tau$.

(r) Let X be a Hausdorff space and $\tilde{M}^+$ the set of non-negative additive functionals defined on subalgebras of $\mathcal{P}X$ containing every open set, with the algebraic operations defined in 437Xj. Let A and B be norm-bounded uniformly tight subsets of $\tilde{M}^+$ and $\gamma \geq 0$. Show that $A \cup B$, $A + B = \{\mu + \nu : \mu \in A, \nu \in B\}$ and $\{\alpha\mu : \mu \in A, 0 \leq \alpha \leq \gamma\}$ are uniformly tight.

>(s) Let X be a Prokhorov Hausdorff space, and A a set of Radon measures on X which is compact for the narrow topology. Show that A is uniformly tight. (*Hint*: (i) $\gamma = \sup_{\mu\in A} \mu X$ is finite; (ii) for any $\epsilon > 0$ the set $\{\frac{1}{\mu X}\mu : \mu \in A, \mu X \geq \epsilon\}$ is narrowly compact, therefore uniformly tight; for any $\epsilon > 0$ the set $\{\mu : \mu \in A, \mu X \geq \epsilon\}$ is uniformly tight.)

(t) Give ω_1 its order topology, and let M_t be the L-space of signed tight Borel measures on ω_1. (i) Show that ω_1 is a Prokhorov space. (ii) For $\xi < \omega_1$, define $\mu_\xi \in M_t$ by setting $\mu_\xi(E) = \chi E(\xi) - \chi E(\xi + 1)$ for every Borel set $E \subseteq \omega_1$. Show that $\{\mu_\xi : \xi < \omega_1\}$ is relatively compact in M_t for the vague topology, but is not uniformly tight.

437Y Further exercises (a) Let X be a set and U a Riesz subspace of $\mathbb{R}^X$. Give formulae for the components of a given element of $U^\sim$ in the bands $U_\sigma^\sim$, $(U_\sigma^\sim)^\perp$, $U_\tau^\sim$ and $(U_\tau^\sim)^\perp$. (*Hint*: 356Yb.)

(b) Let X be a compact Hausdorff space. Show that the dual $C(X;\mathbb{C})^*$ of the complex linear space of continuous functions from X to $\mathbb{C}$ can be identified with the space of 'complex tight Borel measures' on X, that is, the space of functionals $\mu : \mathcal{B}(X) \to \mathbb{C}$ expressible as a complex linear combination of tight totally finite Borel measures; explain how this may be identified, as Banach space, with the complexification of the L-space M_t of signed tight Borel measures as described in 354Yk. Show that the complex Banach space $\mathfrak{L}^\infty_{\mathbb{C}}(\mathcal{B}(X))$ is canonically embedded in $C(X;\mathbb{C})^{**}$.

(c) Write μ_c for counting measure on $[0,1]$, and μ_L for Lebesgue measure; write $\mu_c \times \mu_L$ for the product measure on $[0,1]^2$, and μ for the direct sum of μ_c and $\mu_c \times \mu_L$. Show that the L-space $C([0,1])^\sim$ is isomorphic, as L-space, to $L^1(\mu)$. (*Hint*: every Radon measure on $[0,1]$ has countable Maharam type.)

(d) Let X be a set, U a Riesz subspace of $\ell^\infty(X)$ containing the constant functions, and Σ the smallest σ-algebra of subsets of X with respect to which every member of U is measurable. Write $\tilde{\Sigma}$ for the intersection of the domains of the completions of the totally finite measures with domain Σ. Show that there is a unique sequentially order-continuous norm-preserving Riesz homomorphism from $\mathfrak{L}^\infty(\tilde{\Sigma})$ to $(U_\sigma^\sim)^* \cong M_\sigma^*$ such that $(Su)(f) = f(u)$ whenever $u \in U$ and $f \in U_\sigma^\sim$.

(e) Let X be a completely regular Hausdorff space and P_τ the space of τ-additive Borel probability measures on X. Let $B \subseteq P_\tau$ be a non-empty set. Show that the following are equiveridical: (i) B is relatively compact in P_τ for the vague topology; (ii) whenever $A \subseteq C_b(X)$ is non-empty and downwards-directed and $\inf_{u\in A} u(x) = 0$ for every $x \in A$, then $\inf_{u\in A} \sup_{\mu\in B} \int u\, d\mu = 0$; (iii) whenever $\mathcal{G}$ is an upwards-directed family of open sets with union X, then $\sup_{G\in\mathcal{G}} \inf_{\mu\in B} \mu G = 1$.

(f) Explain how to express the proof of 285L(iii)⇒(ii) as (α) a proof that if the characteristic functions of a sequence $\langle\nu_n\rangle_{n\in\mathbb{N}}$ of Radon probability measures on $\mathbb{R}^r$ converge pointwise to a characteristic function, then $\{\nu_n : n \in \mathbb{N}\}$ is uniformly tight (β) the observation that any subalgebra of $C_b(\mathbb{R}^r)$ which separates the points of $\mathbb{R}^r$ and contains the constant functions will define the vague topology on any vaguely compact set of measures.

(g) Let X be a topological space. (i) Let $M_\sigma(\mathcal{B}a)$ be the space of signed Baire measures on X, and $u : X \to \mathbb{R}$ a bounded Baire measurable function. Show that we have a linear functional $\mu \mapsto \int u\, d\mu : M_\sigma(\mathcal{B}\mathfrak{a}) \to \mathbb{R}$ agreeing with ordinary integration with respect to non-negative measures. Show that this functional is Baire measurable with respect to the vague topology on $M_\sigma(\mathcal{B}\mathfrak{a})$. (ii) Let M_τ be the space of signed τ-additive Borel measures on X. Show that we have a linear functional $\mu \mapsto \int u\, d\mu : M_\tau \to \mathbb{R}$ agreeing with ordinary integration with respect to non-negative measures. Show that this functional is Borel measurable with respect to the vague topology on M_τ.

(h) For a topological space X let $M_\tau(X)$ be the L-space of signed τ-additive Borel measures on X, and $\psi : M_\tau(X) \times M_\tau(X) \to M_\tau(X \times X)$ the canonical bilinear map (437M); give $M_\tau(X)$ and $M_\tau(X \times X)$ their vague topologies. (i) Show that if $X = [0,1]$ then ψ is not continuous. (ii) Show that $X = \mathbb{Z}$ and B is the unit ball of $M_\tau(X)$ then $\psi \restriction B \times B$ is not continuous.

(i) Let (X,ρ) be a metric space. For $f \in C_b(X)^\sim_\sigma$, set $\|f\|_H = \sup\{|f(u)| : u \in C_b(X), \|u\|_\infty \leq 1, u \text{ is 1-Lipschitz}\}$ (definition: 471J). (i) Show that $\| \|_H$ is a norm on $C_b(X)^\sim_\sigma$. The metric it induces is sometimes called **Huntingdon's metric**. (ii) Show that the topology induced by Huntingdon's metric on the set M_τ of signed τ-additive Borel measures on X is the vague topology. (*Hint*: 434L.) (iii) Show that if (X,ρ) is complete then the set M_t of signed tight Borel measures is complete under Huntingdon's metric. (*Hint*: show that any Cauchy sequence in M_t must be uniformly tight.) (iv) Show that if (X,ρ) is $\mathbb{R}$ with its usual metric, then Huntingdon's metric, interpreted as a metric on the set of Radon probability measures on X, is not uniformly equivalent to Lévy's metric as described in 274Ya.

(j) Let X be any Hausdorff space, and $M_{\mathrm{R}}^{\infty+}$ the set of Radon measures on X. Define addition and scalar multiplication (by positive scalars) on $M_{\mathrm{R}}^{\infty+}$ as in 416Xd, and $\leq$ by the formulae of 416Ea. (i) Show that there is a Dedekind complete Riesz space V such that the positive cone of V is isomorphic to $M_{\mathrm{R}}^{\infty+}$. (ii) Show that every principal band in V is an L-space. (iii) Show that if X is metrizable then V is perfect.

(k) Let X be a compact Hausdorff space and P_{R} the set of Radon probability measures on X, with its narrow topology. Show that the weight $w(P_{\mathrm{R}})$ of P_{R} is at most $\max(\omega, w(X))$. (*Hint*: the weight of $C(X)$ in its norm topology is at most $\max(\omega, w(X))$.)

(l) Let X be a Čech-complete completely regular Hausdorff space and P_{R} the set of Radon probability measures on X, with its narrow topology. Show that P_{R} is Čech-complete.

(m) Let X and Y be topological spaces, and $\psi : M_\tau(X) \times M_\tau(Y) \to M_\tau(X \times Y)$ the bilinear map of 437Mc. Write $M_t(X)$, etc., for the spaces of signed tight Borel measures. (i) Show that $\psi(\mu,\nu) \in M_t(X \times Y)$ for every $\mu \in M_t(X)$, $\nu \in M_t(Y)$. (ii) Show that if $B \subseteq M_t(X)$, $B' \subseteq M_t(Y)$ are norm-bounded and uniformly tight, then $\psi \restriction B \times B'$ is continuous for the vague topologies.

(n) (i) X be a metrizable space, and A a narrowly compact subset of the set of Radon probability measures on X. Show that there is a separable subset Y of X which is conegligible for every measure in A. (ii) Show that a metrizable space is Prokhorov iff all its closed separable subspaces are Prokhorov.

(o) I say that a completely regular Hausdorff space X is **strongly Prokhorov** if every vaguely compact subset of the space $M_t(X)$ of signed tight Borel measures on X is uniformly tight. (i) Check that a strongly Prokhorov completely regular Hausdorff space is Prokhorov. (ii) Show that a closed subspace of a strongly Prokhorov completely regular Hausdorff space is strongly Prokhorov. (iii) Show that the product of a countable family of strongly Prokhorov completely regular Hausdorff spaces is strongly Prokhorov. (iv) Show that a G_δ subset of a strongly Prokhorov metrizable space is strongly Prokhorov. (v) Show that a Polish space is strongly Prokhorov.

437 Notes and comments The ramifications of the results here are enormous. For completely regular topological spaces X, the theorems of §436 give effective descriptions of the totally finite Baire, quasi-Radon and Radon measures on X as linear functionals on $C_b(X)$ (436E, 436Xl, 436Xn). This makes it possible, and natural, to integrate the topological measure theory of X into functional analysis, through the theory of $C_b(X)^*$. (See Wheeler 83 for an extensive discussion of this approach.) For the rest of this volume we shall never be far away from such considerations. In 437C-437I I give only a sample of the results, heavily slanted towards the abstract theory of Riesz spaces in Chapter 35 and the first part of Chapter 36.

Note that while the constructions of the dual spaces $U^\sim$, $U^\sim_c$ and $U^\times$ are 'intrinsic' to a Riesz space U, in that we can identify these functions as soon as we know the linear and order structure of U, the spaces $U^\sim_\sigma$ and $U^\sim_\tau$ are definable only when U is presented as a Riesz subspace of $\mathbb{R}^X$. In the same way, while the space $M_\sigma(\Sigma)$ of countably additive functionals on a σ-algebra Σ depends only on the Boolean algebra structure, the spaces M_τ here (not to be confused with the space of completely additive functionals considered in 362B) depend on the topology as well as the Borel algebra. (For an example in which radically different topologies give rise to the same Borel algebra, see Juhász Kunen & Rudin 76.)

You may have been puzzled by the shift from 'quasi-Radon' measures in 436H to 'τ-additive' measures in 437H; somewhere the requirement of inner regularity has got lost. The point is that the topologies being

considered here, being defined by declaring certain families of functions continuous, are (completely) regular; so that τ-additive measures are necessarily inner regular with respect to the closed sets (414Mb).

The theory of 'vague' and 'narrow' topologies' in 437J-437W here hardly impinges on the questions considered in §§274 and 285, where vague topologies first appeared. This is because the earlier investigation was dominated by the very special position of the functions $x \mapsto e^{iy \cdot x}$ (what we shall in §445 come to call the 'characters' of the additive group of $\mathbb{R}$ or $\mathbb{R}^r$). One idea which does appear essentially in the proof of 285L, and has a natural interpretation in the general theory, is that of a 'uniformly tight' family of Radon measures; see 437T-437W. In 445Yh I set out a generalization of 285L to abelian locally compact groups.

In §461 I will return to the general theory of extreme points in compact convex sets. Here I remark only that it is never surprising that extreme points should be special in some way, as in 437P; but the precise ways in which they are special are often unexpected. A good deal of work has been done on relationships between the topological properties of a topological space X and the space P of Radon probability measures on X with the vague topology. Here I show only that if X is compact, so is P (437O), that if X is compact and metrizable so is P (437Ra), and that if X is Polish so is P (437Rb), with a couple of hints at generalizations (437Yk-437Yl).

The terms 'vague' and 'narrow' both appear in the literature on this topic, and I take the opportunity to use them both, meaning slightly different things. Vague topologies, in my usage, are linear space topologies on linear spaces of functionals; narrow topologies are topologies on spaces of (finitely additive) measures, which are not linear spaces, though we can make an attempt to define addition and multiplication by non-negative scalars (437Xj). I must warn you that this distinction is not standard. I see that the word 'narrow' appears a good deal oftener than the word 'vague', which is in part a reflection of a simple prejudice against signed measures; but from the point of view of this treatise as a whole, it is more natural to work with a concept well adapted to measures with variable domains, even if we are considering questions (like compactness of sets of measures) which originate in linear analysis. I should mention also that the definition in 437Jc includes a choice. The duality considered there uses the space $C_b(X)$; for locally compact X, we have the rival spaces $C_0(X)$ and $C_k(X)$ (see 436J and 436K), and there are occasions when one of these gives a more suitable topology on a space of measures (as in 495Xl below).

The elementary theory of uniform tightness and Prokhorov spaces (437T-437W) is both pretty and useful. The emphasis I give it here, however, is partly because it provides the background to a remarkable construction by D.Preiss (439S below), showing that $\mathbb{Q}$ is not a Prokhorov space.

438 Measure-free cardinals

At several points in §418, and again in §434, we had theorems about separable metrizable spaces in which the proofs undoubtedly needed some special property of these spaces (e.g., the fact that they are Lindelöf), but left it unclear whether something more general could be said. When we come to investigate further, asking (for instance) whether complete metric spaces in general are Radon (438H), we find ourselves once again approaching the Banach-Ulam problem, already mentioned at several points in previous volumes, and in particular in 363S. It seems to be undecidable, in ordinary set theory with the axiom of choice, whether or not every discrete space is Radon in the sense of 434C. On the other hand it is known that discrete spaces of cardinal at most ω_1 (for instance) are indeed always Radon. While as a rule I am deferring questions of this type to Volume 5, this particular phenomenon is so pervasive that I think it is worth taking a section now to clarify it.

The central definition is that of 'measure-free cardinal' (438A), and the basic results are 438B-438D. In particular, 'small' infinite cardinals are measure-free (438C). From the point of view of measure theory, a metrizable space whose weight is measure-free is almost separable, and most of the results in §418 concerning separable metrizable spaces can be extended (438E-438G). In fact 'measure-free weight' exactly determines whether a metrizable space is measure-compact (438J, 438Xk) and whether a complete metric space is Radon (438H). If $\mathfrak{c}$ is measure-free, some interesting spaces of functions are Radon (438R). I approach these last spaces through the concept of 'hereditary weak θ-refinability' (438K), which enables us to do most of the work without invoking any special axiom.

438A Measure-free cardinals: Definition A cardinal κ is **measure-free** or **of measure zero** if whenever μ is a probability measure with domain $\mathcal{P}\kappa$ then there is a $\xi < \kappa$ such that $\mu\{\xi\} > 0$. In 363S I discussed some statements equiveridical with the assertion 'every cardinal is measure-free'.

438B It is worth getting some basic facts out into the open immediately.

Lemma Let (X, Σ, μ) be a semi-finite measure space and $\langle E_i\rangle_{i\in I}$ a point-finite family of subsets of X such that $\#(I)$ is measure-free and $\bigcup_{i\in J} E_i \in \Sigma$ for every $J \subseteq I$. Set $E = \bigcup_{i\in I} E_i$.

(a) $\mu E = \sup_{J\subseteq I \text{ is finite}} \mu(\bigcup_{i\in J} E_i)$.

(b) If $\langle E_i\rangle_{i\in I}$ is disjoint, then $\mu E = \sum_{i\in I} \mu E_i$. In particular, if $\Sigma = \mathcal{P}X$ and $A \subseteq X$ has measure-free cardinal, then $\mu A = \sum_{x\in A} \mu\{x\}$.

(c) If μ is σ-finite, then $L = \{i : i \in I, \mu E_i > 0\}$ is countable and $\bigcup_{i\in I\setminus L} E_i$ is negligible.

proof (a)(i) The first step is to show, by induction on n, that the result is true if $\mu X < \infty$ and every E_i is negligible and $\#(\{i : i \in I, x \in E_i\}) \leq n$ for every $x \in X$. If $n = 0$ this is trivial, since every E_i must be empty. For the inductive step to $n \geq 1$, define $\nu : \mathcal{P}I \to [0, \infty[$ by setting $\nu J = \mu(\bigcup_{i\in J} E_i)$ for every $J \subseteq I$. Then ν is a measure on I. **P** Write $F_J = \bigcup_{i\in J} E_j$ for $J \subseteq I$. (α) If J, $K \subseteq I$ are disjoint, then for $i \in I$ set $E'_i = E_i \cap F_K$ for $i \in J$, $\emptyset$ for $i \in I \setminus J$. In this case, $\langle E'_i\rangle_{i\in I}$ is a family of negligible subsets of X, $\bigcup_{i\in J'} E'_i = F_{J'\cap J} \cap F_K$ is measurable for every $J' \subseteq I$, and $\#(\{i : x \in E'_i\}) \leq n-1$ for every $x \in X$; so the inductive hypothesis tells us that

$$\mu(\textstyle\bigcup_{i\in I} E'_i) = \sup_{J'\subseteq I \text{ is finite}} \mu(\bigcup_{i\in J'} \mu E'_i) = 0,$$

that is, $F_J \cap F_K$ is negligible. But this means that

$$\nu(J \cup K) = \mu F_{J\cup K} = \mu(F_J \cup F_K) = \mu F_J + \mu F_K = \nu J + \nu K.$$

As J and K are arbitrary, ν is additive. (β) If $\langle J_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{P}I$, then

$$\begin{aligned}\nu(\bigcup_{n\in\mathbb{N}} J_n) &= \mu(\bigcup_{n\in\mathbb{N}} F_{J_n}) = \lim_{n\to\infty} \mu(\bigcup_{m\leq n} F_{J_m})\\ &= \lim_{n\to\infty} \sum_{m=0}^{n} \nu J_m = \sum_{n=0}^{\infty} \nu J_n,\end{aligned}$$

so ν is countably additive and is a measure. **Q**

At the same time, $\nu\{i\} = \mu E_i = 0$ for every i. Because $\#(I)$ is measure-free, $\nu I = 0$. **P?** Otherwise, let $f : I \to \kappa = \#(I)$ be any bijection and set $\lambda A = \frac{1}{\nu I}\nu f^{-1}[A]$ for every $A \subseteq \kappa$; then λ is a probability measure with domain $\mathcal{P}\kappa$ which is zero on singletons, and κ is not measure-free. **XQ** But this means just that $\mu(\bigcup_{i\in I} E_i) = 0$. Thus the induction proceeds.

(ii) ? Now suppose, if possible, that the general result is false. For finite sets $J \subseteq I$ set $F_J = \bigcup_{i\in J} E_i$, as before, and consider $\mathcal{E} = \{F_J : J \in [\kappa]^{<\omega}\}$ (see 3A1J for this notation). Then $\mathcal{E}$ is upwards-directed and $\gamma = \sup_{H\in\mathcal{E}} \mu H$ is finite, because it is less than μE; let $\langle H_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathcal{E}$ such that $\mu(H \setminus H^*) = 0$ for every $H \in \mathcal{E}$, where $H^* = \bigcup_{n\in\mathbb{N}} H_n$ and $\mu H^* = \gamma$ (215Ab).

Because μ is semi-finite, there is an $F \in \Sigma$ such that $F \subseteq E$ and $\gamma < \mu F < \infty$. For each $n \in \mathbb{N}$, set

$$Y_n = \{x : x \in F \setminus H^*, \#(\{i : x \in E_i\}) \leq n\}.$$

Then there is some $n \in \mathbb{N}$ such that $\mu^* Y_n > 0$. Let ν be the subspace measure on Y_n, so that ν is non-zero and totally finite. Now $\langle E_i \cap Y_n\rangle_{i\in I}$ is a family of negligible subsets of Y_n, $\bigcup_{i\in J} E_i \cap Y_n = Y_n \cap \bigcup_{i\in J} E_i$ is measured by ν for every $J \subseteq I$, and $\#(\{i : x \in E_i \cap Y_n\}) \leq n$ for every $x \in Y_n$. But this contradicts (i) above. **X**

This proves (a).

(b) If $\langle E_i\rangle_{i\in I}$ is disjoint, then

$$\sup_{J\in[I]^{<\omega}} \mu(\textstyle\bigcup_{i\in J} E_i) = \sup_{J\in[I]^{<\omega}} \sum_{i\in J} \mu E_i = \sum_{i\in I} \mu E_i.$$

Setting $I = A$, $E_x = \{x\}$ for $x \in A$, we get the special case.

(c) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of measurable sets of finite measure covering X. For each n, set $L_n = \{i : i \in I, \mu(E_i \cap X_n) \geq 2^{-n}\}$. **?** If L_n is infinite, take a sequence $\langle i_k\rangle_{k\in\mathbb{N}}$ of distinct elements in L_n, and consider $G_m = \bigcup_{k\geq m} E_{i_k}$ for $m \in \mathbb{N}$; then every G_m has measure at least 2^{-n}, G_0 has finite measure, $\langle G_m\rangle_{m\in\mathbb{N}}$ is non-increasing, and $\bigcap_{m\in\mathbb{N}} G_m$ is empty, because $\langle E_{i_k}\rangle_{k\in\mathbb{N}}$ is point-finite. But this is impossible. **X**

Thus every L_n is finite and $L = \bigcup_{n\in\mathbb{N}} L_n$ is countable.

Now (a), applied to $\langle E_i'\rangle_{i\in I}$ where $E_i' = E_i$ if $i \in I \setminus L$, $\emptyset$ if $i \in L$, tells us that $\bigcup_{i\in I\setminus L} E_i$ is negligible.

438C I do not think we are ready for the most interesting set-theoretic results concerning measure-free cardinals. But the following facts may help to make sense of the general pattern.

Theorem (ULAM 30) (a) ω is measure-free.

(b) If κ is a measure-free cardinal and $\kappa' \leq \kappa$ is a smaller cardinal, then κ' is measure-free.

(c) If $\langle\kappa_\xi\rangle_{\xi<\lambda}$ is a family of measure-free cardinals, and λ is also measure-free, then $\kappa = \sup_{\xi<\lambda}\kappa_\xi$ is measure-free.

(d) If κ is a measure-free cardinal so is κ^+.

(e) The following are equiveridical:

(i) $\mathfrak{c}$ is not measure-free;

(ii) there is a semi-finite measure space $(X, \mathcal{P}X, \mu)$ which is not purely atomic;

(iii) there is a measure μ on $[0,1]$ extending Lebesgue measure and measuring every subset of $[0,1]$.

(f) If $\kappa \geq \mathfrak{c}$ is a measure-free cardinal then 2^κ is measure-free.

proof (a) This is trivial.

(b) If μ is a probability measure with domain $\mathcal{P}\kappa'$, set $\nu A = \mu(\kappa' \cap A)$ for every $A \subseteq \kappa$. Then ν is a probability measure with domain $\mathcal{P}\kappa$, so there is a $\xi < \kappa$ such that $\nu\{\xi\} > 0$; evidently $\xi < \kappa'$ and $\mu\{\xi\} > 0$.

(c) Let μ be a probability measure on κ with domain $\mathcal{P}\kappa$. Define $f : \kappa \to \lambda$ by setting $f(\alpha) = \min\{\xi : \alpha < \kappa_\xi\}$ for $\alpha < \kappa$. Then the image measure μf^{-1} is a probability measure on λ with domain $\mathcal{P}\lambda$, so there is a $\xi < \lambda$ such that $\mu f^{-1}[\{\xi\}] > 0$. Now $\mu\kappa_\xi > 0$. Applying 438Bb to $A = \kappa_\xi$, we see that there is an $\alpha < \kappa_\xi$ such that $\mu\{\alpha\} > 0$. As μ is arbitrary, κ is measure-free.

(d) By (a) and (b), we need consider only the case $\kappa \geq \omega$. **?** Suppose, if possible, that μ is a probability measure with domain $\mathcal{P}\kappa^+$ such that $\mu\{\alpha\} = 0$ for every $\alpha < \kappa^+$. For each $\alpha < \kappa^+$, let $f_\alpha : \alpha \to \kappa$ be an injection. For $\beta < \kappa^+$, $\xi < \kappa$ set $A(\beta,\xi) = \{\alpha : \beta < \alpha < \kappa^+, f_\alpha(\beta) = \xi\}$. Then $\kappa \setminus \bigcup_{\xi<\kappa} A(\beta,\xi) = \beta + 1$ has cardinal at most κ, which is measure-free, so $\mu(\beta+1) = 0$ and $\mu(\bigcup_{\xi<\kappa} A(\beta,\xi)) > 0$. Also $\langle A(\beta,\xi)\rangle_{\xi<\kappa}$ is disjoint. There is therefore a $\xi_\beta < \kappa$ such that $\mu A(\beta,\xi_\beta) > 0$, by 438Bb. Now $\kappa^+ > \max(\omega,\kappa)$, so there must be an $\eta < \kappa$ such that $B = \{\beta : \xi_\beta = \eta\}$ is uncountable. In this case, however, $\langle A(\beta,\eta)\rangle_{\beta\in B}$ is an uncountable family of sets of measure greater than zero, and cannot be disjoint, because μ is totally finite (215B(iii)); but if $\alpha \in A(\beta,\eta) \cap A(\beta',\eta)$, where $\beta \neq \beta'$, then $f_\alpha(\beta) = f_\alpha(\beta') = \eta$, which is impossible, because f_α is supposed to be injective. **X**

So there is no such measure μ, and κ^+ is measure-free.

(e)(i)⇒(ii) Suppose that $\mathfrak{c}$ is not measure-free; let μ be a probability measure with domain $\mathcal{P}\mathfrak{c}$ such that $\mu\{\xi\} = 0$ for every $\xi < \mathfrak{c}$. Then μ is atomless. **P?** Suppose, if possible, that $A \subseteq \mathfrak{c}$ is an atom for μ. Let $f : \mathfrak{c} \to \mathcal{P}\mathbb{N}$ be a bijection. For each $n \in \mathbb{N}$, set $E_n = \{\xi : n \in f(\xi)\}$. Set $D = \{n : \mu(A \cap E_n) = \mu A\}$. Because A is an atom, $\mu(A \cap E_n) = 0$ for every $n \in \mathbb{N} \setminus D$. This means that $B = \bigcap_{n\in D} E_n \setminus \bigcup_{n\in\mathbb{N}\setminus D} E_n$ has measure $\mu A > 0$; but $f(\xi) = D$ for every $\xi \in B$, so $\#(B) \leq 1$, and $\mu\{\xi\} > 0$ for some ξ, contrary to hypothesis. **XQ**

So (ii) is true.

(ii)⇒(iii) Suppose that there is a semi-finite measure space $(X, \mathcal{P}X, \mu)$ which is not purely atomic. Then there is a non-negligible set $E \subseteq X$ which does not include any atom; let $F \subseteq E$ be a set of non-zero finite measure. If we take ν to be $\frac{1}{\mu F}\mu_F$, where μ_F is the subspace measure on F, then ν is an atomless probability measure with domain $\mathcal{P}F$. Consequently there is a function $g : F \to [0,1]$ which is inverse-measure-preserving for ν and Lebesgue measure (343Cb). But this means that the image measure νg^{-1} is a measure defined on every subset of $[0,1]$ which extends Lebesgue measure.

not-(i)⇒not-(iii) Conversely, if $\mathfrak{c}$ is measure-free, then any probability measure on $[0,1]$ measuring every subset must give positive measure to some singleton, and cannot extend Lebesgue measure.

(f) We are supposing that $\kappa \geq \mathfrak{c}$ is measure-free, so, in particular, $\mathfrak{c}$ is measure-free. Let μ be a probability measure with domain $\mathcal{P}(2^\kappa)$. By (e), it cannot be atomless; let $E \subseteq 2^\kappa$ be an atom. Let $f : 2^\kappa \to \mathcal{P}\kappa$ be a bijection, and for $\xi < \kappa$ set $E_\xi = \{\alpha : \alpha < 2^\kappa, \xi \in f(\alpha)\}$; set $D = \{\xi : \xi < \kappa, \mu(E \cap E_\xi) = \mu E\}$. Note that $\mu(E \cap E_\xi)$ must be zero for every $\xi \in \kappa \setminus D$, so that $E \cap \{\alpha : \xi \in D \triangle f(\alpha)\}$ is always negligible. Consider

$$A_\xi = \{\alpha : \alpha \in E, \xi = \min(D \triangle f(\alpha))\}$$

for $\xi < \kappa$. Then $\langle A_\xi \rangle_{\xi<\kappa}$ is a disjoint family of negligible sets, so its union A is negligible, by 438Bb, because κ is measure-free. But $E \setminus A \subseteq f^{-1}[\{D\}]$ has at most one element, and is not negligible; so $\mu\{\alpha\} > 0$ for some α. As μ is arbitrary, 2^κ is measure-free.

Remark This extends the result of 419G, which used a different approach to show that ω_1 is measure-free.

We see from (d) above that $\omega_2, \omega_3, \ldots$ are all measure-free; so, by (c), ω_ω also is; generally, if κ is any measure-free cardinal, so is ω_κ (438Xa). I ought to point out that there are even more powerful arguments showing that any cardinal which is not measure-free must be enormous (Solovay 71, Fremlin 93). In this context, however, $\mathfrak{c} = 2^\omega$ can be 'large', at least in the absence of an axiom like the continuum hypothesis to locate it in the hierarchy $\langle \omega_\xi \rangle_{\xi \in \mathrm{On}}$; it is generally believed that it is consistent to suppose that $\mathfrak{c}$ is not measure-free.

438D I turn now to the contexts in which measure-free cardinals behave as if they were 'small'.

Proposition Let (X, Σ, μ) be a σ-finite measure space, Y a metrizable space with measure-free weight, and $f : X \to Y$ a measurable function. Then there is a closed separable set $Y_0 \subseteq Y$ such that $f^{-1}[Y_0]$ is conegligible; that is, there is a conegligible measurable set $X_0 \subseteq X$ such that $f[X_0]$ is separable.

proof Let $\mathcal{U}$ be a σ-disjoint base for the topology of Y (4A2L(h-ii)); express it as $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ where each $\mathcal{U}_n$ is a disjoint family of open sets. If $n \in \mathbb{N}$, $\#(\mathcal{U}_n) \leq w(Y)$ (4A2Db) is a measure-free cardinal (438Cb), and $\langle f^{-1}[U] \rangle_{U \in U_n}$ is a disjoint family in Σ such that $\bigcup_{u\in\mathcal{V}} f^{-1}[U] = f^{-1}[\bigcup \mathcal{V}]$ is measurable for every $\mathcal{V} \subseteq \mathcal{U}_n$; so 438Bc tells us that there is a countable set $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that

$$f^{-1}[\bigcup(\mathcal{U}_n \setminus \mathcal{V}_n)] = \bigcup_{U \in \mathcal{U}_n \setminus \mathcal{V}_n} f^{-1}[U]$$

is negligible. Set

$$Y_0 = Y \setminus \bigcup_{n\in\mathbb{N}} \bigcup(\mathcal{U}_n \setminus \mathcal{V}_n).$$

Then $f^{-1}[Y \setminus Y_0] = \bigcup_{n\in\mathbb{N}} f^{-1}[\bigcup(\mathcal{U}_n \setminus \mathcal{V}_n)]$ is negligible. On the other hand,

$$\{U \cap Y_0 : U \in \mathcal{U}\} \subseteq \{\emptyset\} \cup \{V \cap Y_0 : V \in \bigcup_{n\in\mathbb{N}} \mathcal{V}_n\}$$

is countable, and is a base for the subspace topology of Y_0 (4A2B(a-iv)); so Y_0 is second-countable and must be separable (4A2Oc).

Thus we have an appropriate Y_0. Now $X_0 = f^{-1}[Y_0]$ is conegligible and measurable and $f[X_0] \subseteq Y_0$ is separable (4A2P(a-iv)).

438E Proposition (cf. 418B) Let (X, Σ, μ) be a complete locally determined measure space.

(a) If Y is a topological space, Z is a metrizable space, $w(Z)$ is measure-free, and $f : X \to Y$, $g : X \to Z$ are measurable functions, then $x \mapsto (f(x), g(x)) : X \to Y \times Z$ is measurable.

(b) If $\langle Y_n \rangle_{n\in\mathbb{N}}$ is a sequence of metrizable spaces, with product Y, and $w(Y_n)$ is measure-free for every $n \in \mathbb{N}$, and $f_n : X \to Y_n$ is measurable for every $n \in \mathbb{N}$, then $x \mapsto f(x) = \langle f_n(x) \rangle_{n\in\mathbb{N}} : X \to \prod_{n\in\mathbb{N}} Y_n$ is measurable.

proof (a)(i) Consider first the case in which μ is totally finite. Then there is a conegligible set $X_0 \subseteq X$ such that $g[X_0]$ is separable (438D). Applying 418Bb to $f\restriction X_0$ and $g\restriction X_0$, we see that $x \mapsto (f(x), g(x)) : X_0 \to Y \times g[Z_0]$ is measurable. As μ is complete, it follows that $x \mapsto (f(x), g(x)) : X \to Y \times Z$ is measurable.

(ii) In the general case, take any open set $W \subseteq Y \times Z$ and any measurable set $F \subseteq X$ of finite measure. Set $Q = \{x : (f(x), g(x)) \in W\}$. By (i), applied to $f\restriction F$ and $g\restriction F$, $F \cap Q \in \Sigma$; as F is arbitrary and μ is locally determined, $Q \in \Sigma$; as W is arbitrary, $x \mapsto (f(x), g(x))$ is measurable.

(b) As in (a), it is enough to consider the case in which μ is totally finite. In this case, we have for each $n \in \mathbb{N}$ a conegligible set X_n such that $f_n[X_n]$ is separable. Set $X' = \bigcap_{n\in\mathbb{N}} X_n$; then 418Bd tells us that $f\restriction X'$ is measurable, so that f is measurable.

438F Proposition (cf. 418J) Let (X, Σ, μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X such that μ is inner regular with respect to the closed sets. Suppose that Y is a metrizable space, $w(Y)$ is measure-free and $f : X \to Y$ is measurable. Then f is almost continuous.

proof Take $E \in \Sigma$ and $\gamma < \mu E$. Then there is a measurable set $F \subseteq E$ such that $\gamma < \mu F < \infty$. Let $F_0 \subseteq F$ be a measurable set such that $F \setminus F_0$ is negligible and $f[F_0]$ is separable (438D). By 412Pc, the subspace measure on F_0 is still inner regular with respect to the closed sets, so $f\restriction F_0$ is almost continuous (418J), and there is a measurable set $H \subseteq F_0$, of measure at least γ, such that $f\restriction H$ is continuous. As E and γ are arbitrary, f is almost continuous.

438G Corollary (cf. 418K) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and Y a metrizable space such that $w(Y)$ is measure-free. Then a function $f : X \to Y$ is measurable iff it is almost continuous.

438H Now let us turn to questions which arose in §434.

Proposition A complete metric space is Radon iff its weight is measure-free.

proof Let (X, ρ) be a complete metric space, and $\kappa = w(X)$ its weight.

(a) If κ is measure-free, let μ be any totally finite Borel measure on X. Applying 438D to the identity map from X to itself, we see that there is a closed separable conegligible subspace X_0. Now X_0 is complete, so is a Polish space, and by 434Kb is a Radon space. The subspace measure μ_{X_0} is therefore tight (that is, inner regular with respect to the compact sets); as X_0 is conegligible, it follows at once that μ also is. As μ is arbitrary, X is a Radon space.

(b) If κ is not measure-free, take any σ-disjoint base $\mathcal{U}$ for the topology of X. Express $\mathcal{U}$ as $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ where every $\mathcal{U}_n$ is disjoint. Then $\kappa \leq \#(\mathcal{U})$ and there is a probability measure ν on $\mathcal{U}$, with domain $\mathcal{PU}$, such that $\nu\{U\} = 0$ for every $U \in \mathcal{U}$. Let $n \in \mathbb{N}$ be such that $\nu\mathcal{U}_n > 0$. For each $U \in \mathcal{U}_n$ choose $x_U \in U$. For Borel sets $E \subseteq X$ set $\mu E = \nu\{U : U \in \mathcal{U}_n, x_U \in E\}$; then μ is a Borel measure on X and $\mu(\bigcup \mathcal{U}_n) = \nu\mathcal{U}_n > 0$, while $\mu(\bigcup \mathcal{V}) = \nu\mathcal{V} = 0$ for every finite $\mathcal{V} \subseteq \mathcal{U}_n$. Thus μ is not τ-additive and cannot be tight, and X is not a Radon space.

438I Proposition Let X be a metrizable space and $\langle F_\xi\rangle_{\xi<\kappa}$ a non-decreasing family of closed subsets of X, where κ is a measure-free cardinal. Then

$$\mu(\textstyle\bigcup_{\xi<\kappa} F_\xi) = \sum_{\xi<\kappa} \mu(F_\xi \setminus \bigcup_{\eta<\xi} F_\eta)$$

for every semi-finite Borel measure μ on X.

proof (a) I had better begin by remarking that $H_\xi = \bigcup_{\eta<\xi} F_\eta$ is an F_σ set for every ordinal $\xi \leq \kappa$, by 4A2Ld and 4A2Ka. So, setting $E_\xi = F_\xi \setminus H_\xi$, $\sum_{\xi<\kappa} \mu E_\xi$ is defined.

(b) I show by induction on ζ that $\mu H_\zeta = \sum_{\xi<\zeta} \mu E_\xi$ for every $\zeta \leq \kappa$. The induction starts trivially with $\mu H_0 = 0$. The inductive step to a successor ordinal $\zeta + 1$ is also immediate, as $H_{\zeta+1} = H_\zeta \cup E_\zeta$. For the inductive step to a limit ordinal ζ of countable cofinality, let $\langle\zeta_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in ζ with supremum ζ; then

$$\mu H_\zeta = \sup_{n\in\mathbb{N}} \mu H_{\zeta_n} = \sup_{n\in\mathbb{N}} \textstyle\sum_{\xi<\zeta_n} \mu E_\xi = \sum_{\xi<\zeta} \mu E_\xi,$$

as required.

(c) So we are left with the case in which ζ is a limit ordinal of uncountable cofinality. In this case, $\mu(E \cap H_\zeta) \leq \sum_{\xi<\zeta} \mu E_\xi$ whenever μE is finite. **P** Let $\mathcal{U}$ be a σ-disjoint base for the topology of X (4A2L(h-ii)), and express $\mathcal{U}$ as $\bigcup_{n\in\mathbb{N}} \mathcal{U}_n$ where each $\mathcal{U}_n$ is disjoint. For $n \in \mathbb{N}$, $\xi \leq \zeta$ set

$$\mathcal{V}_{n\xi} = \{U : U \in \mathcal{U}_n, U \cap H_\xi \neq \emptyset\}, \quad V_{n\xi} = \bigcup \mathcal{V}_{n\xi}.$$

Define $\phi_n : V_{n\zeta} \to \zeta$ by saying that $\phi_n(x) = \min(\{\xi : \xi < \zeta, x \in V_{n\xi}\})$. Then, for any $D \subseteq \zeta$,

$$\phi_n^{-1}[D] = \bigcup_{\xi \in D} \bigcup (\mathcal{V}_{n\xi} \setminus \bigcup_{\eta<\xi} \mathcal{V}_{n\eta})$$

is a union of members of $\mathcal{U}_n$, so is open. We therefore have a measure ν_n on $\mathcal{P}\zeta$ defined by saying that $\nu_n D = \mu(E \cap \phi_n^{-1}[D])$ for every $D \subseteq \zeta$. At this point, recall that we are supposing that κ is measure-free, so $\#(\zeta)$ is also measure-free (438Cb) and $\nu_n\zeta = \sum_{\xi<\zeta} \nu_n\{\xi\} = \sup_{\xi<\zeta} \nu_n\xi$. Interpreting this in X, we have $\mu(E \cap V_{n\zeta}) = \sup_{\xi<\zeta} \mu(E \cap V_{n\xi})$.

This is true for every $n \in \mathbb{N}$. So there is a countable set $C \subseteq \zeta$ such that $\mu(E \cap V_{n\zeta}) = \sup_{\xi \in C} \mu(E \cap V_{n\xi})$ for every $n \in \mathbb{N}$. Because $\mathrm{cf}(\zeta) > \omega$, there is an $\alpha < \zeta$ such that $C \subseteq \alpha$, and $\mu(E \cap V_{n\zeta}) = \mu(E \cap V_{n\alpha})$, that is, $E \cap V_{n\zeta} \setminus V_{n\alpha}$ is negligible, for every $n \in \mathbb{N}$.

Now note that F_α is closed. So

$$\begin{aligned} H_\zeta \setminus F_\alpha &\subseteq \bigcup \{U : U \in \mathcal{U},\, H_\zeta \cap U \neq \emptyset,\, U \cap F_\alpha = \emptyset\} \\ &= \bigcup_{n \in \mathbb{N}} V_{n\zeta} \setminus V_{n,\alpha+1}, \end{aligned}$$

and $E \cap H_\zeta \setminus F_\alpha$ is negligible. Accordingly, using the inductive hypothesis,

$$\mu(E \cap H_\zeta) \leq \mu F_\alpha = \mu H_{\alpha+1} \leq \sum_{\xi \leq \alpha} \mu E_\xi \leq \sum_{\xi<\zeta} \mu E_\xi,$$

as claimed. **Q**

Because μ is semi-finite, and E is arbitrary, $\mu H_\zeta \leq \sum_{\xi<\zeta} E_\xi$; but the reverse inequality is trivial, so we have equality, and the induction proceeds in this case also.

(d) At the end of the induction we have $\mu H_\kappa = \sum_{\xi<\kappa} \mu E_\xi$, as stated.

438J So far we have been looking at metrizable spaces, the obvious first step. But it turns out that the concept of 'metacompactness' leads to generalizations of some of the results above.

Proposition (Moran 70, Haydon 74) Let X be a metacompact space with measure-free weight.
(a) X is Borel-measure-compact.
(b) If X is normal, it is measure-compact.
(c) If X is perfectly normal (for instance, if it is metrizable), it is Borel-measure-complete.

proof (a) **?** If X is not Borel-measure-compact, there are a non-zero totally finite Borel measure μ on X and a cover $\mathcal{G}$ of X by negligible open sets (434H(a-v)). Let $\mathcal{H}$ be a point-finite open cover of X refining $\mathcal{G}$. By 4A2Dc, $\#(\mathcal{H}) \leq \max(\omega, w(X))$, so is measure-free, by 438C. Because μ is a Borel measure, $\bigcup \mathcal{H}'$ is measurable for every $\mathcal{H}' \subseteq \mathcal{H}$; $\mu H = 0$ for every $H \in \mathcal{H}$; while $\mu(\bigcup \mathcal{H}) = \mu X > 0$. But this contradicts 438Ba. **X**

(b) Now suppose that X is normal, and that μ is a totally finite Baire measure on X. Because a normal metacompact space is countably paracompact (4A2F(g-iii)), μ has an extension to a Borel measure μ_1 which is inner regular with respect to the closed sets, by Mařík's theorem (435C). Now μ_1 is τ-additive, by (a) above, so μ also is (411C). As μ is arbitrary, X is measure-compact.

(c) Since on a perfectly normal space the Baire and Borel measures are the same, X is Borel-measure-complete iff it is measure-compact, and we can use (b).

Remark The arguments here can be adapted in various ways, and in particular the hypotheses can be weakened; see 438Yd-438Yf.

438K Hereditarily weakly θ-refinable spaces A topological space X is **hereditarily weakly θ-refinable** (also called **hereditarily σ-relatively metacompact**, **hereditarily weakly submetacompact**) if for every family $\mathcal{G}$ of open subsets of X there is a σ-isolated family $\mathcal{A}$ of subsets of X, refining $\mathcal{G}$, such that $\bigcup \mathcal{A} = \bigcup \mathcal{G}$.

438L Lemma (a) Any subspace of a hereditarily weakly θ-refinable topological space is hereditarily weakly θ-refinable.

(b) A hereditarily metacompact space (e.g., any metrizable space, see 4A2Lb) is hereditarily weakly θ-refinable.

(c) A hereditarily Lindelöf space is hereditarily weakly θ-refinable.

(d) A topological space with a σ-isolated network is hereditarily weakly θ-refinable.

proof (a) If X is hereditarily weakly θ-refinable, Y is a subspace of X, and $\mathcal{H}$ is a family of open subsets of Y, set $\mathcal{G} = \{G : G \subseteq X \text{ is open}, G \cap Y \in \mathcal{H}\}$. Then there is a σ-isolated family $\mathcal{A}$, refining $\mathcal{G}$, with union $\bigcup \mathcal{G}$; and $\{A \cap Y : A \in \mathcal{A}\}$ is σ-isolated (4A2B(a-viii)), refines $\mathcal{H}$, and has union $\bigcup \mathcal{H}$. As $\mathcal{H}$ is arbitrary, Y is hereditarily weakly θ-refinable.

(b) If X is hereditarily metacompact, and $\mathcal{G}$ is any family of open sets in X with union W, then $\mathcal{G}$ is an open cover of the metacompact space W, so has a point-finite open refinement $\mathcal{H}$ with the same union. For each $x \in W$, set $\mathcal{H}_x = \{H : x \in H \in \mathcal{H}\}$, $V_x = \bigcap \mathcal{H}_x$, so that $\mathcal{H}_x$ is a non-empty finite set and V_x is an open set containing x. For $n \geq 1$, set $A_n = \{x : x \in W, \#(\mathcal{H}_x) = n\}$; then for any distinct $x, y \in A_n$, either $\mathcal{H}_x = \mathcal{H}_y$ and $V_x = V_y$, or $\#(\mathcal{H}_x \cup \mathcal{H}_y) > n$ and $V_x \cap V_y \cap A_n = \emptyset$. This means that $\mathcal{A}_n = \{V_x \cap A_n : x \in A_n\}$ is a partition of A_n into relatively open sets, and is an isolated family. Also, $\mathcal{A}_n$ is a refinement of $\mathcal{H}$ and therefore of $\mathcal{G}$; so $\bigcup_{n \geq 1} \mathcal{A}_n$ is a σ-isolated refinement of $\mathcal{G}$, and its union is $\bigcup_{n \geq 1} A_n = W$. As $\mathcal{G}$ is arbitrary, X is hereditarily weakly θ-refinable.

(c) If X is hereditarily Lindelöf and $\mathcal{G}$ is a family of open subsets of X, there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ with the same union; now $\mathcal{G}_0$, being countable, is σ-isolated. As $\mathcal{G}$ is arbitrary, X is hereditarily weakly θ-refinable.

(d) If X has a σ-isolated network $\mathcal{A}$, and $\mathcal{G}$ is any family of open subsets of X, then

$$\mathcal{E} = \{A : A \in \mathcal{A}, A \text{ is included in some member of } \mathcal{G}\}$$

is a σ-isolated family (4A2B(a-viii)), refining $\mathcal{G}$, with union $\bigcup \mathcal{G}$.

438M Proposition (Gardner 75) If X is a hereditarily weakly θ-refinable topological space with measure-free weight, it is Borel-measure-complete.

proof Let μ be a Borel probability measure on X, and $\mathcal{G}$ the family of μ-negligible open sets. Let $\mathcal{A}$ be a σ-isolated family refining $\mathcal{G}$ with union $\bigcup \mathcal{G}$. Express $\mathcal{A}$ as $\bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ where each $\mathcal{A}_n$ is an isolated family; for each $n \in \mathbb{N}$, set $X_n = \bigcup \mathcal{A}_n$ and let μ_n be the subspace measure on X_n. Then $\mathcal{A}_n$ is a disjoint family of relatively open μ_n-negligible sets; as $\#(\mathcal{A}_n) \leq w(X_n) \leq w(X)$ (4A2D) is measure-free, and μ_n is a totally finite Borel measure on X_n,

$$\mu^* X_n = \mu_n X_n = \mu_n(\bigcup \mathcal{A}_n) = 0,$$

by 438Bb. Now $\mu(\bigcup \mathcal{G}) = \mu^*(\bigcup_{n \in \mathbb{N}} X_n) = 0$. As μ is arbitrary, X is Borel-measure-complete (434I(a-iv)).

438N For the next few paragraphs, I will use the following notation. Let X be a topological space and $\mathcal{G}$ a family of subsets of X. Then $\mathcal{J}(\mathcal{G})$ will be the family of subsets of X expressible as $\bigcup \mathcal{A}$ for some σ-isolated family $\mathcal{A}$ refining $\mathcal{G}$. Observe that X is hereditarily weakly θ-refinable iff $\bigcup \mathcal{G}$ belongs to $\mathcal{J}(\mathcal{G})$ for every family $\mathcal{G}$ of open subsets of X.

(a) $\mathcal{J}(\mathcal{G})$ is always a σ-ideal of subsets of X. **P** If $\mathcal{A}$ is a σ-isolated family of subsets of X, refining $\mathcal{G}$, and B is any set, then $\{B \cap A : A \in \mathcal{A}\}$ is still σ-isolated and still refines $\mathcal{G}$; so any subset of a member of $\mathcal{J}(\mathcal{G})$ belongs to $\mathcal{J}(\mathcal{G})$. If $\langle \mathcal{A}_n \rangle_{n \in \mathbb{N}}$ is a sequence of σ-isolated families refining $\mathcal{G}$, then $\bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ is σ-isolated and refines $\mathcal{G}$; so the union of any sequence in $\mathcal{J}(\mathcal{G})$ belongs to $\mathcal{J}(\mathcal{G})$. **Q**

(b) If $\mathcal{H}$ refines $\mathcal{G}$, then $\mathcal{J}(\mathcal{H}) \subseteq \mathcal{J}(\mathcal{G})$. **P** All we need to remember is that any family refining $\mathcal{H}$ also refines $\mathcal{G}$. **Q**

(c) If X and Y are topological spaces, $A \subseteq X$, $f : A \to Y$ is continuous, and $\mathcal{H}$ is a family of subsets of Y, set $\mathcal{G} = \{f^{-1}[H] : H \in \mathcal{H}\}$. Then $\mathcal{J}(\mathcal{G}) \supseteq \{f^{-1}[B] : B \in \mathcal{J}(\mathcal{H})\}$. **P** If $B \in \mathcal{J}(\mathcal{H})$, there is a

σ-isolated family $\mathcal{D}$ of subsets of Y, refining $\mathcal{H}$, and with union B. Now $\mathcal{A} = \{f^{-1}[D] : D \in \mathcal{D}\}$ refines $\mathcal{G}$ and has union $f^{-1}[B]$. We can express $\mathcal{D}$ as $\bigcup_{n\in\mathbb{N}} \mathcal{D}_n$, where each $\mathcal{D}_n$ is a relatively open family; set $\mathcal{A}_n = \{f^{-1}[D] : D \in \mathcal{D}_n\}$, so that $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$. For each n, $\mathcal{A}_n$ is disjoint, because $\mathcal{D}_n$ is. Moreover, if $D \in \mathcal{D}_n$, then $D = H \cap \bigcup \mathcal{D}_n$ for some open set $H \subseteq Y$, so that $f^{-1}[D] = f^{-1}[H] \cap \bigcup \mathcal{A}_n$ is relatively open in $\bigcup \mathcal{A}_n$; this shows that $\mathcal{A}_n$ is an isolated family. Accordingly $\mathcal{A}$ is σ-isolated and witnesses that $f^{-1}[B] \in \mathcal{J}(\mathcal{G})$. As B is arbitrary, we have the result. **Q**

(d) If X is a topological space, $\mathcal{G}$ is a family of subsets of X, and $\langle D_i\rangle_{i\in I}$ is an isolated family in $\mathcal{J}(\mathcal{G})$, then $\bigcup_{i\in I} D_i \in \mathcal{J}(\mathcal{G})$. **P** For each $i \in I$, let $\langle \mathcal{A}_{ni}\rangle_{n\in\mathbb{N}}$ be a sequence of isolated families, all refining $\mathcal{G}$, such that $D_i = \bigcup_{n\in\mathbb{N}} \bigcup \mathcal{A}_{in}$. Set $\mathcal{A}_n = \bigcup_{i\in I} \mathcal{A}_{in}$ for each n. Then $\mathcal{A}_n$ refines $\mathcal{G}$, and $\bigcup_{n\in\mathbb{N}} \bigcup \mathcal{A}_n = \bigcup_{i\in I} D_i$. It is easy to check that every $\mathcal{A}_n$ is isolated, so that $\bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ witnesses that $\bigcup_{i\in I} D_i$ belongs to $\mathcal{J}(\mathcal{G})$. **Q**

438O Lemma Give $\mathbb{R}$ the topology $\mathfrak{S}$ generated by the closed intervals $]-\infty, t]$ for $t \in \mathbb{R}$, and let $r \geq 1$. Then $\mathbb{R}^r$, with the product topology corresponding to $\mathfrak{S}$, is hereditarily weakly θ-refinable.

proof Induce on r. Write $\leq$ for the usual partial order of $\mathbb{R}^r$, and $]-\infty, x]$ for $\{y : y \leq x\}$; set $V_A = \bigcup_{x\in A}]-\infty, x]$ for $A \subseteq \mathbb{R}^r$. The sets $]-\infty, x]$, as x runs over $\mathbb{R}^r$, form a base for the topology of $\mathbb{R}^r$.

The induction starts easily because $\mathfrak{S}$ itself is hereditarily Lindelöf. **P** If $\mathcal{G} \subseteq \mathfrak{S}$, set

$$A = \{x : x \in \mathbb{R}, \text{ there is some } G \in \mathcal{G} \text{ such that }]-\infty, x] \subseteq G\}.$$

Then A has a countable cofinal set D, so that there is a corresponding countable subset of $\mathcal{G}$ with the same union as $\mathcal{G}$. **Q** By 438Lc, $\mathfrak{S}$ is hereditarily weakly θ-refinable.

For the inductive step to $r+1$, where $r \geq 1$, let $\mathcal{G}$ be a family of open subsets of $\mathbb{R}^{r+1}$, and set

$$A = \{x : x \in \mathbb{R}^r, \text{ there is some } G \in \mathcal{G} \text{ such that }]-\infty, x] \subseteq G\}.$$

For each $k \leq r$, $q \in \mathbb{Q}$ set $K_k = (r+1) \setminus \{k\}$ and let $B_{kq} \subseteq \mathbb{R}^{K_k}$ be the set $\{z : z^\frown q \in V_A\}$, writing $z^\frown q$ for that member x of $\mathbb{R}^{r+1}$ such that $x{\restriction}K_k = z$ and $x(k) = q$. (I am thinking of members of $\mathbb{R}^{r+1}$ as functions from $r+1 = \{0, \dots, r\}$ to $\mathbb{R}$.) Set $A_{kq} = \{x : x \in V_A, x(k) = q\}$, $\mathcal{G}_{kq} = \{]-\infty, x] : x \in A_{kq}\}$. Then, in the notation of 438N, $V_{A_{kq}} \in \mathcal{J}(\mathcal{G})$. **P** Set $f(x) = x{\restriction}K_k$ for each $x \in V_{A_{kq}}$, so that $f : V_{A_{kq}} \to \mathbb{R}^{K_k}$ is continuous. For $x \in A_{kq}$, $]-\infty, x] = f^{-1}[\,]-\infty, f(x)]\,]$. Now $\mathbb{R}^{K_k}$ is hereditarily weakly θ-refinable, by the inductive hypothesis, so if we set $\mathcal{H}_{kq} = \{]-\infty, f(x)] : x \in A_{kq}\}$, $\bigcup \mathcal{H}_{kq} \in \mathcal{J}(\mathcal{H}_{kq})$ and (by 438Nc)

$$V_{A_{kq}} = f^{-1}[\bigcup \mathcal{H}_{kq}] \in \mathcal{J}(\mathcal{G}_{kq}) \subseteq \mathcal{J}(\mathcal{G}). \text{ } \mathbf{Q}$$

Accordingly $W \in \mathcal{J}(\mathcal{G})$, where $W = \bigcup_{k\leq r, q\in\mathbb{Q}} V_{A_{kq}}$, by 438Na.

Now consider $V_A \setminus W$. If $x, x' \in V_A \setminus W$ and $x \leq x'$ then $x = x'$. **P?** Otherwise, there are a $k \leq n$ and a $q \in \mathbb{Q}$ such that $x(k) \leq q \leq x'(k)$. In this case, setting $y{\restriction}K_k = x{\restriction}K_k$ and $y(k) = q$, we have $y \in A_{kq}$ and $x \in V_{A_{kq}}$. **XQ**

But this means that the subspace topology of $V_A \setminus W$ is discrete, so that $\{\{x\} : x \in V_A \setminus W\}$ is an isolated family covering $V_A \setminus W$ and refining $\mathcal{G}$; thus $V_A \setminus W \in \mathcal{J}(\mathcal{G})$ and $\bigcup \mathcal{G} = V_A$ belongs to $\mathcal{J}(\mathcal{G})$. As $\mathcal{G}$ is arbitrary, $\mathbb{R}^{r+1}$ is hereditarily weakly θ-refinable and the induction proceeds.

438P Proposition Let $X \subseteq \mathbb{R}^{\mathbb{R}}$ be the family of functions $x : \mathbb{R} \to \mathbb{R}$ such that $\lim_{s\uparrow t} x(s)$ and $\lim_{s\downarrow t} x(s)$ are defined in $\mathbb{R}$ for every $t \in \mathbb{R}$. Then X, with its topology of pointwise convergence inherited from the product topology of $\mathbb{R}^{\mathbb{R}}$, is a K-analytic hereditarily weakly θ-refinable space.

proof (a) I start by showing that it is K-analytic.

(i) The first step is to find an alternative characterization of X. If $x \in X$ and $U, V \subseteq \mathbb{R}$ are disjoint, then I say that a finite set $D \subseteq \mathbb{R}$ is a **back-and-forth set** for x, U and V if whenever s, t are successive points of D (in the usual ordering of $\mathbb{R}$) then either $x(s) \in U$ and $x(t) \in V$, or $x(s) \in V$ and $x(t) \in U$. Now, if $x \in X$ and $k \in \mathbb{N}$ and $U, V \subseteq \mathbb{R}$ have disjoint closures,

$$\omega_{kUV}(x) = \sup\{\#(D) : D \subseteq [-k, k] \text{ is a back-and-forth set for } x, U \text{ and } V\}$$

is finite. **P** For each $t \in [-k, k]$ there is a $\delta > 0$ such that $\{x(s) : t - \delta < s < t\}$ meets at most one of U and V, and $\{x(s) : t < s < t + \delta\}$ meets at most one of U and V. So if D is a back-and-forth set for x, U and V, D can meet each of the intervals $]t - \delta, t[$ and $]t, t + \delta[$ in at most one point, and $\#(D \cap]t - \delta, t + \delta[) \leq 3$.

By the Heine-Borel theorem, there is a cover of $[-k,k]$ by m intervals of this type for some $m \in \mathbb{N}$, so that $\omega_{kUV}(x) \leq 3m$. **Q**

(ii) If $x \in X$ and $\langle t_n \rangle_{n\in\mathbb{N}}$ is any bounded sequence in $\mathbb{R}$, then it has a monotonic subsequence $\langle t'_n \rangle_{n\in\mathbb{N}}$ and $\langle x(t'_n) \rangle_{n\in\mathbb{N}}$ must be convergent in $\mathbb{R}$. It follows that $\{x(t) : -k \leq t \leq k\}$ is bounded in $\mathbb{R}$ for every $k \in \mathbb{N}$.

(iii) Conversely, let $\mathcal{I}$ be the family of non-empty open intervals in $\mathbb{R}$ with rational endpoints, and suppose that $x \in \mathbb{R}^{\mathbb{R}}$ is such that

whenever U, $V \in \mathcal{I}$ have disjoint closures, and $k \in \mathbb{N}$, then $\{x(t) : |t| \leq k\}$ is bounded and $\omega_{kUV}(x)$ is finite.

Then $x \in X$. **P** Let $t \in \mathbb{R}$. **?** If $\lim_{s\downarrow t} x(s)$ is not defined in $\mathbb{R}$, then (because $\{x(s) : t \leq s \leq t+1\}$ is bounded) there must be decreasing sequences $\langle t_n \rangle_{n\in\mathbb{N}}$, $\langle t'_n \rangle_{n\in\mathbb{N}}$, both convergent to t, such that $z = \lim_{n\to\infty} x(t_n)$, $z' = \lim_{n\to\infty} x(t'_n)$ are both defined and different. Let U, $V \in \mathcal{I}$ be such that $z \in U$, $z' \in V$ and $\overline{U} \cap \overline{V} = \emptyset$; then for any $m \in \mathbb{N}$ there is a back-and-forth set $D \subseteq \{t_n : n \in \mathbb{N}\} \cup \{t'_n : n \in \mathbb{N}\}$ for x, U and V with $\#(D) \geq m$, so $\omega_{kUV}(x)$ is infinite for some k. **X** Thus $\lim_{s\downarrow t} x(s)$ is defined in $\mathbb{R}$. Similarly, $\lim_{s\uparrow t} x(s)$ is defined in $\mathbb{R}$; as t is arbitrary, $x \in X$. **Q**

(iv) Now this means that X is a Souslin-F set in the compact space $[-\infty,\infty]^{\mathbb{R}}$. **P** For k, $m \in \mathbb{N}$ and disjoint U, $V \in \mathcal{I}$, set

$$F_{km} = \{x : x \in [-\infty,\infty]^{\mathbb{R}}, |x(t)| \leq m \text{ for every } t \in [-k,k]\},$$

$$F'_{kUVm} = \{x : x \in [-\infty,\infty]^{\mathbb{R}}, \omega_{kUV}(x) \leq m\}.$$

Of course every F_{km} is closed. But so is every F'_{kUVm}, because if $x \notin F'_{kUVm}$ and $D \subseteq [-k,k]$ is a back-and-forth set for x, U and V of size greater than m, then D will witness that $y \notin F'_{kUVm}$ for any y close enough to x. Now the characterization of X above tells us just that

$$X = \bigcap_{k\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} F_{km} \cap \bigcap_{k\in\mathbb{N},U,V\in\mathcal{I},\overline{U}\cap\overline{V}=\emptyset} \bigcup_{m\in\mathbb{N}} F'_{kUVm},$$

which is a countable intersection of F_σ sets, and is Souslin-F. **Q**

So X is K-analytic (422Hb).

(b) Now I turn to the proof that X is hereditarily weakly θ-refinable. Let $\mathcal{G}$ be a family of open sets in X.

(i) In the notation of 438N, I seek to show that $\bigcup\mathcal{G}$ belongs to $\mathcal{J}(\mathcal{G})$. Of course it will be enough to consider the case in which $\bigcup\mathcal{G}$ is non-empty. The following elementary remarks will be useful.

(α) If $D \subseteq \bigcup\mathcal{G}$, and $\mathcal{E}$ is a partition of D into relatively open sets such that $\mathcal{E}$ refines $\mathcal{G}$, then $D \in \mathcal{J}(\mathcal{G})$.

(β) If $\langle D_i \rangle_{i\in I}$ is any family in $\mathcal{J}(\mathcal{G})$, and $\langle H_i \rangle_{i\in I}$ is a family of open sets, and $D \subseteq \{x : \#(\{i : x \in H_i\}) = 1\}$, then $D \cap \bigcup_{i\in I} H_i \cap D_i$ belongs to $\mathcal{J}(\mathcal{G})$. **P** $\langle D \cap H_i \cap D_i \rangle_{i\in I}$ is an isolated family in $\mathcal{J}(\mathcal{G})$; use 438Nd. **Q**

(ii) Write Q for the family of all finite sequences

$$\boldsymbol{q} = (I_0, U_0, V_0, W_0, I_1, U_1, V_1, W_1, \dots, I_n, U_n, V_n, W_n)$$

where $I_0, I_1, \dots, I_n$ are disjoint members of $\mathcal{I}$, all the U_i, V_i, W_i belong to $\mathcal{I}$, and, for each $i \leq n$, any pair of U_i, V_i, W_i are either equal or disjoint. Fix $\boldsymbol{q} = (I_0, \dots, W_n) \in Q$ for the moment.

(iii) Set $T_{\boldsymbol{q}} = \prod_{i\leq n} I_i$. For $\tau \in T_{\boldsymbol{q}}$, write $F_{\boldsymbol{q}\tau}$ for the set of those $x \in X$ such that, for every $i \leq n$, $x(s) \in U_i$ for $s \in I_i \cap]-\infty, \tau(i)[$, $x(\tau(i)) \in V_i$ and $x(s) \in W_i$ for $s \in I_i \cap]\tau(i), \infty[$. Set $X_{\boldsymbol{q}} = \bigcup\{F_{\boldsymbol{q}\tau} : \tau \in T_{\boldsymbol{q}}\}$, and for $\tau \in T_{\boldsymbol{q}}$ set $H_{\boldsymbol{q}\tau} = \{x : x \in X_{\boldsymbol{q}}, x(\tau(i)) \in V_i \text{ for every } i \leq n\}$. Finally, set

$$S_{\boldsymbol{q}} = \{\tau : \tau \in T_{\boldsymbol{q}} \text{ and } H_{\boldsymbol{q}\tau} \text{ is included in some member of } \mathcal{G}\}.$$

(iv) If $T \subseteq S_{\boldsymbol{q}}$ then $H = \bigcup_{\tau\in T} H_{\boldsymbol{q}\tau}$ belongs to $\mathcal{J}(\mathcal{G})$. **P** Induce on $\#(L(T))$, where

$$L(T) = \{i : i \leq n, \text{ there are } \tau, \tau' \in T \text{ such that } \tau(i) \neq \tau'(i)\}.$$

If $L(T) = \emptyset$, then $\#(T) \leq 1$, so H is either empty or included in some member of $\mathcal{G}$, and the induction starts. For the inductive step to $\#(L(T)) = k \geq 1$, consider three cases.

case 1 Suppose there is a $j \in L(T)$ such that $U_j = V_j = W_j$. Then $x(t) \in V_j$ for every $x \in X_{\boldsymbol{q}}$, $t \in I_j$. Fix any $t^* \in I_j$ and for $\tau \in T_{\boldsymbol{q}}$ define $\tau^* \in T_{\boldsymbol{q}}$ by setting $\tau^*(j) = t^*$, $\tau^*(i) = \tau(i)$ for $i \neq j$; then $H_{\boldsymbol{q}\tau^*} = H_{\boldsymbol{q}\tau}$. Set $T^* = \{\tau^* : \tau \in T\}$; then $L(T^*) = L(T) \setminus \{j\}$, so $\#(L(T^*)) < \#(L(T))$, while $T^* \subseteq S_{\boldsymbol{q}}$. By the inductive hypothesis, $H = \bigcup_{\tau \in T^*} H_{\boldsymbol{q}\tau}$ is negligible.

case 2 Suppose there is a $j \in L(T)$ such that $U_j \neq V_j$ and $V_j \neq W_j$. Then $V_j \cap (U_j \cup W_j) = \emptyset$. For $s \in I_j$ set $T^*_s = \{\tau : \tau \in T, \tau(j) = s\}$. Then $\#(L(T^*_s)) < \#(L(T))$ so, by the inductive hypothesis, $H^*_s \in \mathcal{J}(\mathcal{G})$, where $H^*_s = \bigcup_{\tau \in T^*_s} H_{\boldsymbol{q}\tau}$. But, for $\tau \in T$ and $x \in H_{\boldsymbol{q}\tau}$, $x(s) \in V_j$ iff $s = \tau(j)$; so $H^*_s = \{x : x \in H, x(s) \in V_j\}$ and $\langle H^*_s \rangle_{s \in I_j}$ is a partition of H into relatively open sets. By (i-β), $H \in \mathcal{J}(\mathcal{G})$.

case 3 Otherwise, $L(T) = J \cup J'$ where $J = \{i : i \in L(T), U_i = V_i\}$ and $J' = \{i : i \in L(T), V_i = W_i\}$ are disjoint. For $x \in H$, $i \in J$ we see that there is a largest $t \in I_i$ such that $x(t) \in V_i$; set $\phi_i(x) = -t$. Similarly, for $x \in H$, $i \in J'$ there is a smallest $t \in I_i$ such that $x(t) \in V_i$; in this case, set $\phi_i(x) = t$. Observe that, for $i \in J$, $s \in \mathbb{R}$,

$$\begin{aligned}\{x : x \in H, \phi_i(x) \leq s\} &= \emptyset \text{ if } s < -t \text{ for every } t \in I_i,\\ &= H \text{ if } -t < s \text{ for every } t \in I_i,\\ &= \{x : x \in H, x(-s) \in V_i\} \text{ if } -s \in I_i,\end{aligned}$$

so is always relatively open in H, and ϕ_i is continuous if $\mathbb{R}$ is given the topology $\mathfrak{S}$ of Lemma 438O. Similarly, for $i \in J'$, $s \in \mathbb{R}$,

$$\begin{aligned}\{x : x \in H, \phi_i(x) \leq s\} &= \emptyset \text{ if } s < t \text{ for every } t \in I_i,\\ &= H \text{ if } t < s \text{ for every } t \in I_i,\\ &= \{x : x \in H, x(s) \in V_i\} \text{ if } s \in I_i.\end{aligned}$$

So in this case also ϕ_i is continuous.

Accordingly, giving $\mathbb{R}^{L(T)}$ the product topology corresponding to $\mathfrak{S}$, we have a continuous map $\phi : H \to \mathbb{R}^{L(T)}$ defined by setting $\phi(x) = \langle \phi_i(x) \rangle_{i \in L(T)}$ for $x \in H$. For $\tau \in T$, set $\tilde{\tau}(i) = -\tau(i)$ if $i \in J$, $\tau(i)$ if $i \in J'$, and $\tilde{H}_\tau =]-\infty, \tilde{\tau}] \subseteq \mathbb{R}^{L(T)}$. Then

$$\begin{aligned}H_{\boldsymbol{q}\tau} &= \{x : x \in H, x(\tau(i)) \in V_i \text{ for every } i \leq n\}\\ &= \{x : x \in H, x(\tau(i)) \in V_i \text{ for every } i \in L(T)\}\end{aligned}$$

(because if $x \in H$, $i \leq n$, $i \notin L(T)$ then there is some $\tau' \in T$ such that $x \in H_{\boldsymbol{q}\tau'}$, so that $x(\tau'(i)) \in V_i$ and therefore $x(\tau(i)) \in V_i$)

$$= \{x : x \in H, \phi_i(x) \leq \tilde{\tau}(i) \text{ for every } i \in L(T)\} = \phi^{-1}[\tilde{H}_\tau].$$

Set $\tilde{\mathcal{G}} = \{\tilde{H}_\tau : \tau \in T\}$. Because $\mathbb{R}^{L(T)}$ is hereditarily weakly θ-refinable (438O), and $\tilde{\mathcal{G}}$ is a family of open subsets of $\mathbb{R}^{L(T)}$, $\bigcup \tilde{\mathcal{G}} = \in \mathcal{J}(\tilde{\mathcal{G}})$. By 438Nc, $H = \bigcup_{\tau \in T} H_{\boldsymbol{q}\tau} = \phi^{-1}[\bigcup \tilde{\mathcal{G}}]$ belongs to $\mathcal{J}(\{H_{\boldsymbol{q}\tau} : \tau \in T\}$ and therefore to $\mathcal{J}(\mathcal{G})$, by 438Nb.

Thus in all three cases the induction proceeds. **Q**

(v) This means that, for any $\boldsymbol{q} \in Q$, $Y_{\boldsymbol{q}} = \bigcup\{H_{\boldsymbol{q}\tau} : \tau \in S_{\boldsymbol{q}}\}$ belongs to $\mathcal{J}(\mathcal{G})$. Since Q is countable, $Y \in \mathcal{J}(\mathcal{G})$, where $Y = \bigcup_{\boldsymbol{q} \in Q} Y_{\boldsymbol{q}}$. But $\bigcup \mathcal{G} \subseteq Y$. **P** If $x \in G \in \mathcal{G}$, there are $t_0 < \ldots < t_n$ and $V'_i \in \mathcal{I}$, for $i \leq n$, such that

$$x \in \{y : y \in X, y(t_i) \in V'_i \text{ for every } i \leq n\} \subseteq G.$$

Set $z_i = x(t_i)$, $z_i^- = \lim_{s \uparrow t_i} x(s)$, $z_i^+ = \lim_{s \downarrow t_i} x(s)$ for $i \leq n$; let U_i, V_i, $W_i \in \mathcal{I}$ be such that $z_i^- \in U_i$, $z_i \in V_i \subseteq V'_i$, $z_i^+ \in W_i$ and any pair of U_i, V_i, W_i are either equal or disjoint; and let $I_0, \ldots, I_n \in \mathcal{I}$ be disjoint and such that $t_i \in I_i$, $x(s) \in U_i$ for $s \in I_i \cap]-\infty, t_i[$ and $x(s) \in W_i$ for $s \in I_i \cap]t_i, \infty[$ for each $i \leq n$. Then, setting $\boldsymbol{q} = (I_0, \ldots, W_n)$ and $\tau(i) = t_i$ for $i \leq n$,

$$x \in X_{\boldsymbol{q}\tau} \subseteq H_{\boldsymbol{q}\tau} \subseteq G,$$

so that $\tau \in S_{\boldsymbol{q}}$ and $x \in Y_{\boldsymbol{q}} \subseteq Y$. **Q**

As $\mathcal{G}$ is arbitrary, X is hereditarily weakly θ-refinable.

438Q Corollary (a) Let $I^{\|}$ be the split interval (419L). Then any countable power of $I^{\|}$ is a K-analytic hereditarily weakly θ-refinable space.

(b) Let Y be the 'Helly space', the space of non-decreasing functions from $[0,1]$ to itself with the topology of pointwise convergence inherited from the product topology on $[0,1]^{[0,1]}$ (KELLEY 55, Ex. 5M). Then Y is a K-analytic hereditarily weakly θ-refinable space.

proof These are both (homeomorphic to) subspaces of the space X of Proposition 438P. To see this, argue as follows. For (a), observe that we have a function $f : I^{\|} \to X$ defined by setting $f(t^-)(s) = f(t^+)(s) = 1$ if $s < t$, $f(t^-)(s) = f(t^+)(s) = 0$ if $s > t$, and $f(t^-)(t) = 0$, $f(t^+)(t) = 1$, and that f is a homeomorphism between $I^{\|}$ and its image. Next, for any $L \subseteq \mathbb{N}$, we can define $g : (I^{\|})^L \to X$ by setting

$$\begin{aligned} g(\mathbf{t})(s) &= f(t_n)(s-2n) \text{ if } n \in L \text{ and } 2n \le s \le 2n+1 \\ &= 0 \text{ if } s \in \mathbb{R} \setminus \bigcup_{n \in L} [2n, 2n+1] \end{aligned}$$

for $\mathbf{t} = \langle t_n \rangle_{n \in \mathbb{N}} \in (I^{\|})^L$; it is easy to check that g is a homeomorphism between $(I^{\|})^L$ and its image in X. As for (b), if we take $g(y)$ to be the extension of the function $y : [0,1] \to [0,1]$ to the function which is constant on each of the intervals $]-\infty, 0]$ and $[1, \infty[$, then $g : Y \to X$ is a homeomorphism between Y and its image $g[Y]$.

Since both $(I^{\|})^{\mathbb{N}}$ and Y are compact, they are homeomorphic to closed subsets of X, and are K-analytic (422Gc) and hereditarily weakly θ-refinable (438La).

438R Proposition Assume that $\mathfrak{c}$ is measure-free. Then the space X of 438P, $(I^{\|})^{\mathbb{N}}$ and the Helly space (438Qb) are all Radon spaces.

proof By 438P and 438Q, they are K-analytic and hereditarily weakly θ-refinable, therefore pre-Radon and Borel-measure-complete (434Jf, 438M), therefore Radon (434Ka).

438S In 434R I described a construction of product measures. In accordance with my general practice of examining the measure algebra of any new measure, I give the following result.

Proposition Let X and Y be topological spaces with σ-finite Borel measures μ, ν respectively. Suppose that *either* X is first-countable *or* ν is τ-additive and effectively locally finite. Write λ for the Borel measure on $X \times Y$ defined by the formula

$$\lambda W = \int \nu W[\{x\}] \mu(dx) \text{ for every Borel set } W \subseteq X \times Y$$

as in 434R(ii). If *either* the weight of X *or* the Maharam type of ν is a measure-free cardinal, then for every Borel set $W \subseteq X \times Y$ there is a set $W' \in \mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y)$ such that $\lambda(W \triangle W') = 0$; consequently, the measure algebra of λ can be identified with the localizable measure algebra free product of the measure algebras of μ and ν.

proof (a) Write $(\mathfrak{B}, \bar{\nu})$ for the measure algebra of ν. With its measure-algebra topology, this is metrizable (323Gb). Let $\langle Y_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence of Borel sets of finite measure in Y with union Y.

(b) For the moment (down to the end of (e) below) fix on an open set $W \subseteq X \times Y$. For $x \in X$, set $f(x) = W[\{x\}]^\bullet$ in $\mathfrak{B}$. Then $f : X \to \mathfrak{B}$ is Borel measurable.

P (i) Let $H \subseteq \mathfrak{B}$ be an open set. For k, $n \in \mathbb{N}$ set

$$E_{nk} = \{x : x \in X, 2^{-n}k \le \nu(Y_n \cap W[\{x\}]) < 2^{-n}(k+1)\}.$$

Just as in part (a) of the proof of 434R, the function $x \mapsto \nu(Y_n \cap W[\{x\}])$ is lower semi-continuous, so E_{nk} is a Borel set. Set

$$G_{nk} = \bigcup \{G : G \subseteq X \text{ is open}, G \cap E_{nk} \subseteq f^{-1}[H]\};$$

then $E = \bigcup_{n,k \in \mathbb{N}} (G_{nk} \cap E_{nk})$ is a Borel set included in $f^{-1}[H]$.

(ii) The point is that $E = f^{-1}[H]$. To see this, take any x such that $f(x) \in H$. Then there are $b \in \mathfrak{B}$, $\epsilon > 0$ such that $\bar{\nu} b < \infty$ and $c \in H$ whenever $c \in \mathfrak{B}$ and $\bar{\nu}(b \cap (c \triangle f(x))) \le 5\epsilon$. Since $b = \sup_{n \in \mathbb{N}} b \cap Y_n^\bullet$,

there is an $n \in \mathbb{N}$ such that $\bar\nu(b \setminus Y_n^\bullet) \leq \epsilon$ and $2^{-n} \leq \epsilon$. In this case $c \in H$ whenever $c \in \mathfrak{B}$ and $\bar\nu(Y_n^\bullet \cap (c \triangle f(x))) \leq 4\epsilon$; thus

$$\{x' : \nu(Y_n \cap (W[\{x'\}] \triangle W[\{x\}])) \leq 4\epsilon\} \subseteq f^{-1}[H].$$

Let $k \in \mathbb{N}$ be such that $2^{-n}k \leq \nu(Y_n \cap W[\{x\}]) < 2^{-n}(k+1)$, that is, $x \in E_{nk}$. Again using the ideas of part (a) of the proof of 434R, there are an open set G containing x and an open set $V \subseteq Y$ such that $G \times V \subseteq W$ and $\nu(Y_n \cap V) \geq 2^{-n}(k-1)$. Now if $x' \in G \cap E_{nk}$, $V \subseteq W[\{x'\}] \cap W[\{x\}]$, so

$$\begin{aligned}\nu(Y_n \cap (W[\{x'\}] \triangle W[\{x\}])) &\leq \nu(Y_n \cap W[\{x'\}]) + \nu(Y_n \cap W[\{x\}]) - 2\nu(Y_n \cap V)\\ &\leq 2^{-n}((k+1) + (k+1) - 2(k-1)) = 4 \cdot 2^{-n} \leq 4\epsilon.\end{aligned}$$

But this means that $G \cap E_{nk} \subseteq f^{-1}[H]$, so $G \subseteq G_{nk}$ and $x \in G \cap E_{nk} \subseteq E$. As x is arbitrary, $f^{-1}[H] \subseteq E$ and $E = f^{-1}[H]$.

(iii) Thus $f^{-1}[H]$ is a Borel set. As H is arbitrary, f is Borel measurable. **Q**

(c) We need to know also that if $\mathcal{H}$ is a disjoint family of open subsets of $\mathfrak{B}$ all meeting $f[X]$, then

$$\#(\mathcal{H}) \leq \max(\omega, \min(w(X), \tau(\mathfrak{B})))$$

P Repeat the ideas of (b) above, setting

$$G_{nk}^{(H)} = \bigcup\{G : G \subseteq X \text{ is open}, G \cap E_{nk} \subseteq f^{-1}[H]\}$$

for $H \in \mathcal{H}$ and $k, n \in \mathbb{N}$, so that $f^{-1}[H] = \bigcup_{n,k\in\mathbb{N}} G_{nk}^{(H)} \cap E_{nk}$. For fixed n and k the family $\langle G_{nk}^{(H)} \cap E_{nk}\rangle_{H\in\mathcal{H}}$ is disjoint, so can have at most $w(E_{nk}) \leq w(X)$ non-empty members (4A2D). But this means that

$$\mathcal{H} = \bigcup_{n,k\in\mathbb{N}}\{H : G_{nk}^{(H)} \cap E_{nk} \neq \emptyset\}$$

has cardinal at most $\max(\omega, w(X))$.

On the other hand, there is a set $B \subseteq \mathfrak{B}$, of cardinal $\tau(\mathfrak{B})$, which τ-generates $\mathfrak{B}$. The algebra $\mathfrak{B}_0$ generated by B has cardinal at most $\max(\omega, \#(B))$ (331Gc), and $\mathfrak{B}_0$ is topologically dense in $\mathfrak{B}$ (323J), so every member of $\mathcal{H}$ meets $\mathfrak{B}_0$, and

$$\#(\mathcal{H}) \leq \#(\mathfrak{B}_0) \leq \max(\omega, \tau(\mathfrak{B})).$$

Putting these together, we have the result. **Q**

In particular, under the hypotheses above, $\#(\mathcal{H})$ is measure-free whenever $\mathcal{H}$ is a disjoint family of open subsets of $\mathfrak{B}$ all meeting $f[X]$.

(d) The next step is to observe that there is a conegligible Borel set $Z \subseteq X$ such that $f[Z]$ is separable. **P** Let $\mathcal{H}$ be a σ-disjoint base for the topology of $\mathfrak{B}$; express it as $\bigcup_{n\in\mathbb{N}} \mathcal{H}_n$ where each $\mathcal{H}_n$ is disjoint. Let $\langle X_m\rangle_{m\in\mathbb{N}}$ be a cover of X by Borel sets of finite measure. For $n \in \mathbb{N}$ consider $\mathcal{H}_n' = \{H : H \in \mathcal{H}_n, H \cap f[X] \neq \emptyset\}$. For $m \in \mathbb{N}$, we have a totally finite measure ν_{nm} with domain $\mathcal{P}\mathcal{H}_n'$ defined by saying

$$\nu_{nm}\mathcal{E} = \mu(X_m \cap f^{-1}(\bigcup \mathcal{E}))$$

for every $\mathcal{E} \subseteq \mathcal{H}_n'$. Since $\mathcal{H}_n'$ has measure-free cardinal, by (c), there must be a countable set $\mathcal{E}_{nm} \subseteq \mathcal{H}_n'$ such that $\nu_{nm}(\mathcal{H}_n' \setminus \mathcal{E}_{nm}) = 0$. Set

$$Z = X \setminus \bigcup_{m,n\in\mathbb{N}}(X_m \cap f^{-1}[\bigcup(\mathcal{H}_n' \setminus \mathcal{E}_{nm})]);$$

then Z is conegligible. If $x \in Z$ and $f(x) \in H \in \mathcal{H}_n$, then there is some $m \in \mathbb{N}$ such that $x \in X_m$, while $H \in \mathcal{H}_n'$, so H must belong to $\mathcal{E}_{nm}$. But this means that $\{f[Z] \cap H : H \in \mathcal{H}\}$, which is a base for the topology of $f[Z]$, is just $\{f[Z] \cap H : H \in \bigcup_{m,n\in\mathbb{N}} \mathcal{E}_{mn}\}$, and is countable. So $f[Z]$ is separable (4A2Oc), as required. **Q**

(e) 418T(a-ii) now tells us that there is a set $W' \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ such that $f(x) = W'[\{x\}]^\bullet$ for every $x \in Z$, so that $\nu(W[\{x\}] \triangle W'[\{x\}]) = 0$ for almost every x, that is, $\lambda(W \triangle W') = 0$. And this is true for every open set $W \subseteq X \times Y$.

(f) Now let $\mathcal{W}$ be the family of those Borel sets $W \subseteq X \times Y$ for which there is a $W' \in \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ such that $\lambda(W \triangle W') = 0$. This is a σ-algebra containing every open set, so is the whole Borel σ-algebra, as required.

Since the c.l.d. product measure λ_0 on $X \times Y$ is just the completion of its restriction to $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ (251K), and λ_0 and λ agree on $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$ (by Fubini's theorem), the embedding $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y) \subseteq \mathcal{B}(X \times Y)$ induces an isomorphism between the measure algebras of λ and λ_0. As remarked in 325Eb, because μ and ν are strictly localizable, the latter may be identified with 'the' localizable measure algebra free product of the measure algebras of μ and ν.

Remark The hypothesis on the weight of X can be slightly weakened; see 438Yg. To see that some restriction on (X, μ) and (Y, ν) is necessary, see 439L.

438X Basic exercises (a) Show that a cardinal κ is measure-free iff $M_\sigma = M_\tau$, where M_σ, M_τ are the spaces of countably additive and completely additive functionals on the algebra $\mathcal{P}\kappa$, as in §362.

(b) Let (X, Σ, μ) be a localizable measure space with magnitude (definition: 332Ga) which is either finite or a measure-free cardinal. Show that any absolutely continuous countably additive functional $\nu : \Sigma \to \mathbb{R}$ is truly continuous. (*Hint*: 363S.)

(c) Let $\mathfrak{A}$ be a Dedekind complete Boolean algebra with measure-free cellularity. Show that any countably additive functional $\nu : \mathfrak{A} \to \mathbb{R}$ is completely additive.

(d) Let U be a Dedekind complete Riesz space such that any disjoint order-bounded family in U^+ has measure-free cardinal. Show that $U_c^\sim = U^\times$.

>(e) Show that if κ is a measure-free cardinal, so is ω_κ. (*Hint*: show by induction on ordinals ξ that if $\#(\xi)$ is measure-free, then so is ω_ξ.)

>(f) Let (X, Σ, μ) be a complete locally determined measure space, (Y, ρ) a complete metric space, and $\langle f_n \rangle_{n \in \mathbb{N}}$ a sequence of measurable functions from X to Y. Suppose that $w(Y)$ is measure-free. Show that $\{x : \lim_{n \to \infty} f_n(x) \text{ is defined in } Y\}$ is measurable. (Cf. 418C.)

(g) Let (X, Σ, μ) and (Y, T, ν) be σ-finite measure spaces with c.l.d. product $(X \times Y, \Lambda, \lambda)$. Give $L^0(\nu)$ the topology of convergence in measure. Suppose that $f : X \to L^0(\nu)$ is measurable and there is a conegligible set $X_0 \subseteq X$ such that $w(f[X_0])$ is measure-free. Show that there is an $h \in \mathcal{L}^0(\lambda)$ such that $f(x) = h_x^\bullet$ for every $x \in X$. (Cf. 418S.)

(h) Let (Y, T, ν) be a σ-finite measure space, and $(\mathfrak{B}, \bar{\nu})$ its measure algebra, with its usual topology; assume that the Maharam type of $\mathfrak{B}$ is measure-free. Let (X, Σ, μ) be a σ-finite measure space and Λ the domain of the c.l.d. product measure λ on $X \times Y$. Show that if $f : X \to \mathfrak{B}$ is measurable, then there is a $W \in \Lambda$ such that $f(x) = W_x^\bullet$ for every $x \in X$. (Cf. 418T(b-ii).)

(i) Let (X, Σ, μ) is a complete locally determined measure space and V a normed space such that $w(V)$ is measure-free. (i) Show that the space $\mathcal{L}$ of measurable functions from X to V is a linear space, setting $(f + g)(x) = f(x) + g(x)$, etc. (ii) Show that if V is a Riesz space with a Riesz norm then $\mathcal{L}$ is a Riesz space under the natural operations.

>(j) Let X be a topological space and $\mathcal{G}$ a point-finite open cover of X such that $\#(\mathcal{G})$ is measure-free. Suppose that $E \subseteq X$ is such that $E \cap G$ is universally measurable for every $G \in \mathcal{G}$. Show that E is universally measurable. (Compare 434Xe(iv).)

>(k) Show that for a metrizable space X, the following are equiveridical: (i) X is Borel-measure-compact; (ii) X is Borel-measure-complete; (iii) X is measure-compact; (iv) $w(X)$ is measure-free.

(l) Let Y be the Helly space. (i) Show that Y is a compact convex subset of $\mathbb{R}^{[0,1]}$ with its usual topology. (ii) Show that there is a natural one-to-one correspondence between the split interval $I^{\|}$ and the set of extreme points of Y, matching $t^- \in I^{\|}$ with the function $\chi[0, t[$ and t^+ with $\chi[0, t]$. (iii) Let P be the set of Radon probability measures on $I^{\|}$ with its narrow topology (437J). Show that there is a natural homeomorphism $\phi : P \to Y$ defined by setting $\phi(\mu)(t) = \mu[0^-, t^-]$ for $\mu \in P$, $t \in [0, 1]$.

(m) Give $\mathbb{R}$ the right-facing Sorgenfrey topology (415Xc). Show that any countable power of $\mathbb{R}$ is hereditarily weakly θ-refinable.

>(n) Let $I^{\shortparallel}$ be the split interval. Show that $I^{\shortparallel} \times I^{\shortparallel}$ is a Radon space iff $\mathfrak{c}$ is measure-free. (*Hint*: $\{(\alpha^+, (1-\alpha)^+) : \alpha \in [0,1]\}$ is a discrete Borel subset of cardinal $\mathfrak{c}$.)

(o) Give $\mathbb{R}$ the right-facing Sorgenfrey topology. Show that the following are equiveridical: (i) $\mathfrak{c}$ is measure-free; (ii) $\mathbb{R}^{\mathbb{N}}$, with the corresponding product topology, is Borel-measure-complete; (iii) $\mathbb{R}^2$, with the product topology, is Borel-measure-compact. (Compare 439Q.)

(p) Suppose that $\mathfrak{c}$ is measure-free. Let $X \subseteq \mathbb{R}^{\mathbb{R}}$ be the set of functions of bounded variation on $\mathbb{R}$, with the topology of pointwise convergence inherited from the product topology of $\mathbb{R}^{\mathbb{R}}$. Show that X is a Radon space. (*Hint*: X is an F_σ subset of the space of 438P.)

438Y Further exercises (a) Let (X, Σ, μ) be a probability space and $\langle E_i \rangle_{i \in I}$ a point-finite family of measurable sets such that $\nu J = \mu(\bigcup_{i \in J} E_i)$ is defined for every $J \subseteq I$. Show directly that ν is a uniformly exhaustive Maharam submeasure on $\mathcal{P}I$ (definition: 392B), and use the Kalton-Roberts theorem (392J) to prove 438Ba.

(b) Suppose that $\mathfrak{c}$ is measure-free, but that $\kappa > \mathfrak{c}$ is not measure-free. Show that there is a non-principal ω_1-complete ultrafilter on κ. (*Hint*: part (b) of the proof of 451P.)

(c) Show that if X is a metrizable space and *either* $\mathfrak{c}$ *or* $w(X)$ is measure-free, then every σ-finite Borel measure on X has countable Maharam type.

(d) Let X be a metacompact T_1 space. Show that X is Borel-measure-compact iff every closed discrete subspace has measure-free cardinal.

(e) Let X be a topological space such that every subspace of X is metacompact and has measure-free cellularity. Show that X is Borel-measure-complete.

(f) Let X be a normal metacompact Hausdorff space. Show that it is measure-compact iff every closed discrete subspace has measure-free cardinal.

(g) In 438S, show that it would be enough to suppose that every discrete subset of X has cardinal of measure 0.

(h) Let Z be any Polish space, and let $\tilde{X} \subseteq Z^{\mathbb{R}}$ be the set of functions $x : \mathbb{R} \to Z$ such that $\lim_{s \uparrow t} x(s)$, $\lim_{s \downarrow t} x(s)$ are defined in Z for every $t \in \mathbb{R}$. Show that $\tilde{X}$, with the topology of pointwise convergence inherited from $Z^{\mathbb{R}}$, is a K-analytic hereditarily weakly θ-refinable space.

(i) Suppose that $\mathfrak{c}$ is measure-free. Let D be any subset of $\mathbb{R}$ and $X \subseteq \mathbb{R}^D$ the set of functions of bounded variation on D, with the topology of pointwise convergence inherited from the product topology of $\mathbb{R}^D$. Show that X is a Radon space.

(j) Let X be a totally ordered set with its order topology. Show that if $c(X)$ is measure-free then every σ-finite Borel measure on X has countable Maharam type. (Cf. 434Yl.)

438 Notes and comments Since the axiom 'every cardinal is measure-free' is admissible – that is, will not lead to a paradox unless one is already latent in the Zermelo-Fraenkel axioms for set theory – it is tempting, in the context of this section, to assume it; so that 'every complete metric space is Radon' becomes a theorem, along with 'every measurable function from a quasi-Radon measure space to a metrizable space is almost continuous' (438G), '$U_c^{\sim} = U^{\times}$ for every Dedekind complete Riesz space U' (438Xd), 'metacompact spaces are Borel-measure-compact' (438J), 'the sum of two measurable functions from a complete probability space to a normed space is measurable' (438Xi) and 'the Helly space is Radon' (438R). Undoubtedly the consequent mathematical universe is tidier. In my view, the tidiness is the tidiness of poverty. Apart from

anything else, it leads us to neglect such questions as 'is every measurable function from a Radon measure space to a metrizable space almost continuous?', which have answers in ZFC (451S).

From the point of view of measure theory, the really interesting question is whether $\mathfrak{c}$ is measure-free. It is not quite clear from the results above why this should be so; 438R is a very small part of the story. There is a larger hint in 438Ce-438Cf: if $\mathfrak{c}$ is measure-free, but $\kappa > \mathfrak{c}$ is not measure-free, then the witnessing measures will be purely atomic. I mean to return to this point in Volume 5. For a general exploration of universes in which $\mathfrak{c}$ is *not* measure-free, see FREMLIN 93. For fragments of what happens if we suppose that we have an atom for a measure which witnesses that κ is not measure-free, see 438Yb and the notes on normal filters in 4A1I-4A1L.

There are many further applications of 438P besides those in 438Q and 438Xm-438Xp. But the most obvious candidate, the space $C(\mathbb{R})$ of continuous real-valued functions on $\mathbb{R}$, although indeed it is a Borel subset of the potentially Radon space of 438P, is in fact Radon whether or not $\mathfrak{c}$ is measure-free (454Pa). As soon as we start using any such special axiom as '$\mathfrak{c} = \omega_1$' or '$\mathfrak{c}$ is measure-free', we must make a determined effort to check, through such examples as 438Xn, that our new theorems do indeed depend on something more than ZFC.

439 Examples

As in Chapter 41, I end this chapter with a number of examples, exhibiting some of the boundaries around the results in the rest of the chapter, and filling in a gap with basic facts about Lebesgue measure (439E). The first three examples (439A) are measures defined on σ-subalgebras of the Borel σ-algebra of $[0,1]$ which have no extensions to the whole Borel algebra. The next part of the section (439B-439G) deals with 'universally negligible' sets; I use properties of these to show that Hausdorff measures are generally not semi-finite (439H), closing some unfinished business from §§264, and that smooth linear functionals may fail to be representable by integrals in the absence of Stone's condition (439I). In 439J-439R I set out some examples relevant to §§434-435, filling out the classification schemes of 434A and 435A, with spaces which just miss being Radon (439K) or measure-compact (439N, 439P, 439Q). In 439S I present the canonical example of a non-Prokhorov topological space, answering an obvious question from §437.

439A Example Let $\mathcal{B}$ be the Borel σ-algebra of $[0,1]$. There is a probability measure ν defined on a σ-subalgebra T of $\mathcal{B}$ which has no extension to a measure on $\mathcal{B}$.

first construction Let $A \subseteq [0,1]$ be an analytic set which is not Borel (423L). Let $\mathcal{I}$ be the family of sets of the form $E \cup F$ where E, F are Borel sets, $E \subseteq A$ and $F \subseteq [0,1] \setminus A$. Then $\mathcal{I}$ is a σ-ideal of $\mathcal{B}$ not containing $[0,1]$. Set $\mathrm{T} = \mathcal{I} \cup \{[0,1] \setminus H : H \in \mathcal{I}\}$, and define $\nu : \mathrm{T} \to \{0,1\}$ by setting $\nu H = 0$, $\nu([0,1] \setminus H) = 1$ for every $H \in \mathcal{I}$; then ν is a probability measure (cf. countable-cocountable measures (211R) or Dieudonné's measure (411Q)).

? If $\mu : \mathcal{B} \to [0,1]$ is a measure extending ν, then its completion $\hat{\mu}$ measures A (432A). Also $\hat{\mu}$ is a Radon measure (433Cb). Now every compact subset of A belongs to $\mathcal{I}$, so

$$\hat{\mu}A = \sup\{\hat{\mu}K : K \subseteq A \text{ is compact}\} = \sup\{\nu K : K \subseteq A \text{ is compact}\} = 0.$$

Similarly $\hat{\mu}([0,1] \setminus A) = 0$, which is absurd. **X**

second construction This time, let $\mathcal{I}$ be the family of meager Borel sets in $[0,1]$. As before, let T be $\mathcal{I} \cup \{[0,1] \setminus E : E \in \mathcal{I}\}$, and set $\nu E = 0$, $\nu([0,1] \setminus E) = 1$ for $E \in \mathcal{I}$. **?** If μ is a Borel measure extending ν, then $\mu([0,1] \setminus \mathbb{Q}) = 1$, and μ is tight (that is, inner regular with respect to the compact sets), so there is a closed subset F of $[0,1] \setminus \mathbb{Q}$ such that $\mu F > 0$. But F is nowhere dense, so $\nu F = 0$. **X**

third construction[1] There is a function $f : [0,1] \to \{0,1\}^{\mathfrak{c}}$ which is $(\mathcal{B}, \mathcal{B}\mathfrak{a})$-measurable, where $\mathcal{B}\mathfrak{a}$ is the Baire σ-algebra of $\{0,1\}^{\mathfrak{c}}$, and such that $f[\,[0,1]\,]$ meets every non-empty member of $\mathcal{B}\mathfrak{a}$. **P** Set $X = C([0,1])^{\mathbb{N}}$ with the product of the norm topologies, so that X is an uncountable Polish space (4A2Pe, 4A2Qc), and $([0,1], \mathcal{B})$ is isomorphic to $(X, \mathcal{B}(X))$, where $\mathcal{B}(X)$ is the Borel σ-algebra of X (424Da). Define $g : X \to \{0,1\}^{[0,1]}$ by saying that $g(\langle u_i \rangle_{i \in \mathbb{N}})(t) = 1$ iff $\lim_{i \to \infty} u_i(t) = 1$. For each $t \in [0,1]$,

[1] I am grateful to M.Laczkovich and D.Preiss for showing this to me.

$$\{\langle u_i\rangle_{i\in\mathbb{N}} : \lim_{i\to\infty} u_i(t) = 1\} = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \{\langle u_i\rangle_{i\in\mathbb{N}} : |u_i(t) - 1| \leq 2^{-m} \text{ for every } i \geq n\}$$

is a Borel subset of X, so g is $(\mathcal{B}(X), \mathcal{Ba}(\{0,1\}^{[0,1]}))$-measurable, where $\mathcal{Ba}(\{0,1\}^{[0,1]})$ is the Baire σ-algebra of $\{0,1\}^{[0,1]}$ (4A3Ne). If $E \in \mathcal{Ba}(\{0,1\}^{[0,1]})$ is non-empty, there is a countable set $I \subseteq [0,1]$ such that E is determined by coordinates in I (4A3Nb), so that $E \supseteq \{w : w{\restriction}I = z\}$ for some $z \in \{0,1\}^I$. Now we can find a sequence $\langle u_i\rangle_{i\in\mathbb{N}}$ in $C([0,1])$ such that $\lim_{i\to\infty} u_i(t) = z(t)$ for every $t \in I$ (if $I \subseteq \{t_j : j \in \mathbb{N}\}$, take u_i such that $|u_i(t_j) - z(t_j)| \leq 2^{-i}$ whenever $j \leq i$), and in this case $g(\langle u_i\rangle_{i\in\mathbb{N}}) \in E$.

Because $(X, \mathcal{B}(X)) \cong ([0,1], \mathcal{B})$ and $(\{0,1\}^{[0,1]}, \mathcal{Ba}(\{0,1\}^{[0,1]})) \cong (\{0,1\}^{\mathfrak{c}}, \mathcal{Ba})$, we can copy g to a function f with the required properties. **Q**

Now set $\mathrm{T} = \{f^{-1}[H] : H \in \mathcal{Ba}\}$, and let λ be the usual measure on $\{0,1\}^{\mathfrak{c}}$. Then T is a σ-subalgebra of $\mathcal{B}$ and there is a probability measure ν with domain T defined by saying that $\nu f^{-1}[H] = \lambda H$ for every $H \in \mathcal{Ba}$. **P** We need to know that if H, $H' \in \mathcal{Ba}$ and $f^{-1}[H] = f^{-1}[H']$ then $\lambda H = \lambda H'$. But as $(H \triangle H') \cap f[\,[0,1]\,] = \emptyset$, $H = H'$, so this is trivial. Thus the given formula defines a function $\nu : \mathrm{T} \to [0,1]$. It is now elementary to check that ν is a probability measure. **Q**

In fact, because λ is completion regular (415E), $f[\,[0,1]\,]$ has full outer measure for λ, so the map $H^\bullet \mapsto f^{-1}[H]^\bullet$ from the measure algebra of λ to the measure algebra of ν is measure-preserving; since it is surely surjective, the measure algebras are isomorphic, and ν has Maharam type $\mathfrak{c}$.

However, any probability measure on the whole algebra $\mathcal{B}$ has countable Maharam type (433A), so cannot extend ν.

Remark Compare 433I-433J.

439B Definition Let X be a Hausdorff space. I will call X **universally negligible** if there is no Borel probability measure μ defined on X such that $\mu\{x\} = 0$ for every $x \in X$. A subset of X will be 'universally negligible' if it is universally negligible in its subspace topology.

439C Proposition Let X be a Hausdorff space.

(a) If A is a subset of X, the following are equiveridical:

(i) A is universally negligible;

(ii) $\mu^* A = 0$ whenever μ is a Borel probability measure on X such that $\mu\{x\} = 0$ for every $x \in X$;

(iii) $\mu^* A = 0$ whenever μ is a σ-finite topological measure on X such that $\mu\{x\} = 0$ for every $x \in A$;

(iv) for every σ-finite topological measure μ on X there is a countable set $B \subseteq A$ such that $\mu^* A = \mu B$;

(v) A is a Radon space and every compact subset of A is scattered.

In particular, countable subsets of X are universally negligible.

(b) The family of universally negligible subsets of X is a σ-ideal.

(c) Suppose that Y is a universally negligible Hausdorff space and $f : X \to Y$ a Borel measurable function such that $f^{-1}[\{y\}]$ is universally negligible for every $y \in Y$. Then X is universally negligible.

(d) If the topology on X is discrete, X is universally negligible iff $\#(X)$ is measure-free.

proof (a)(i)⇒(iii) If A is universally negligible and μ is a σ-finite topological measure on X such that $\mu\{x\} = 0$ for every $x \in A$, let μ_A be the subspace measure on A. **?** If $\mu^* A = \alpha > 0$, then (because μ is σ-finite) there is a measurable set $E \subseteq X$ such that $\gamma = \mu^*(E \cap A)$ is finite and non-zero. The subspace measure $\mu_{E\cap A}$ is a topological measure on $E \cap A$; set $\nu F = \gamma^{-1}\mu_{E\cap A}(E \cap F)$ for relatively Borel sets $F \subseteq A$; then ν is a Borel probability measure on A which is zero on singletons. **X** So $\mu^* A = 0$.

(iii)⇒(iv) If (iii) is true and μ is a σ-finite topological measure on X, set $B = \{x : x \in X, \mu\{x\} > 0\}$. Because μ is σ-finite, B must be countable, therefore measurable, and if we set $\nu E = \mu(E \setminus B)$ for every Borel set $E \subseteq X$, ν is a σ-finite Borel measure on X and $\nu\{x\} = 0$ for every $x \in X$. By (iii), $\nu^* A = 0$, that is, there is a Borel set $E \supseteq A$ such that $\mu(E \setminus B) = 0$; in which case

$$\mu^* A \leq \mu(E \setminus (B \setminus A)) = \mu(E \setminus B) + \mu(A \cap B) = \mu(A \cap B) \leq \mu^* A,$$

so $\mu^*A = \mu(A \cap B)$. As μ is arbitrary, (iv) is true.

(iv)⇒(ii) is trivial.

not-(i)⇒not-(ii) If A is not universally negligible, let μ be a Borel probability measure on A which is zero on singletons. Set $\nu E = \mu(E \cap A)$ for any Borel set $E \subseteq X$; then ν is a Borel probability measure on X which is zero on singletons, and $\nu^*A = 1$.

(i)⇒(v) Suppose that A is universally negligible. Let μ be a totally finite Borel measure on A. Applying (i)⇒(iv) with $X = A$, we see that there is a countable set $B \subseteq A$ such that $\mu B = \mu A$; but this means that μ is inner regular with respect to the finite subsets of B, which of course are compact. As μ is arbitrary, A is a Radon space.

? Suppose, if possible, that A has a compact set K which is not scattered. In this case there is a continuous surjection $f : K \to [0,1]$ (4A2G(i-iv)). Now there is a Radon probability measure ν on K such that f is inverse-measure-preserving for ν and Lebesgue measure on $[0,1]$ and induces an isomorphism of the measure algebras, so that ν is atomless (418L). Accordingly we have a Borel probability measure μ on A defined by setting $\mu E = \nu(K \cap E)$ for every relatively Borel set $E \subseteq A$, and $\mu\{x\} = 0$ for every $x \in A$, so A is not universally negligible. **X** Thus all compact subsets of A are scattered, and (v) is true.

(v)⇒(i) Now suppose that (v) is true and that that μ is a Borel probability measure on A. Then μ has an extension to a Radon measure $\tilde{\mu}$ (434F(a-iii)). Let $K \subseteq A$ be a non-empty compact set which is self-supporting for $\tilde{\mu}$ (416Dc). K is scattered, so has an isolated point $\{x\}$; because K is self-supporting, $\mu\{x\} = \tilde{\mu}\{x\} > 0$. As μ is arbitrary, A is universally negligible.

(b) This is immediate from (a-ii).

(c) Let μ be a Borel probability measure on X. Then $F \mapsto \nu f^{-1}[F]$ is a Borel probability measure on Y. Because Y is universally negligible, there must be a $y \in Y$ such that $\mu f^{-1}[\{y\}] > 0$. Set $E = f^{-1}[\{y\}]$ and let μ_E be the subspace measure on E. Then μ_E is a non-zero totally finite Borel measure on E. Since E is supposed to be universally negligible, there must be some $x \in E$ such that $0 < \mu_E\{x\} = \mu\{x\}$.

(d) This is just a re-phrasing of the definition in 438A.

439D Remarks (a) The following will be useful when interpreting the definition in 439B. Let X be a hereditarily Lindelöf Hausdorff space and μ a topological probability measure on X such that $\mu\{x\} = 0$ for every $x \in X$. Then μ is atomless.

P Suppose that $\mu H > 0$. Write

$$\mathcal{G} = \{G : G \subseteq X \text{ is open}, \mu(G \cap H) = 0\}.$$

Then there is a countable $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\bigcup \mathcal{G}_0 = \bigcup \mathcal{G}$ (4A2H(c-i)), so

$$\mu(H \cap \bigcup \mathcal{G}) = \mu(H \cap \bigcup \mathcal{G}_0) = 0,$$

and $\mu(H \setminus \bigcup \mathcal{G}) > 0$. Because μ is zero on singletons, $H \setminus \bigcup \mathcal{G}$ has at least two points x, y say. Now there are disjoint open sets G_0, G_1 containing x, y respectively, and neither belongs to $\mathcal{G}$, so $H \cap G_0$, $H \cap G_1$ are disjoint subsets of H of positive measure. Thus H is not an atom. As H is arbitrary, μ is atomless. **Q**

(b) The obvious applications of (a) are when X is separable and metrizable; but, more generally, we can use it on any Hausdorff space with a countable network, e.g., on any analytic space.

439E Lemma (a) Let E, $B \subseteq \mathbb{R}$ be such that E is measurable and $\mu_L E$, $\mu_L^* B$ are both greater than 0, where μ_L is Lebesgue measure. Then $E - B = \{x - y : x \in E, y \in B\}$ includes a non-trivial interval.

(b) If $A \subseteq \mathbb{R}$ and $\mu_L^* A > 0$, then $A + \mathbb{Q}$ is of full outer measure in $\mathbb{R}$.

proof (a) By 261Da, there are $a \in E$, $b \in B$ such that

$$\lim_{\delta \downarrow 0} \frac{1}{2\delta} \mu(E \cap [a - \delta, a + \delta]) = \lim_{\delta \downarrow 0} \frac{1}{2\delta} \mu^*(B \cap [b - \delta, b + \delta]) = 1.$$

Let $\gamma > 0$ be such that

$$\mu_L(E \cap [a-\delta, a+\delta]) > \tfrac{3}{2}\delta, \quad \mu_L^*(B \cap [b-\delta, b+\delta]) > \tfrac{3}{2}\delta$$

whenever $0 < \delta \leq \gamma$. Now suppose that $0 < \delta \leq \gamma$. Then

$$\begin{aligned}\mu_L((E+b) \cap [a+b, a+b+\delta]) &= \mu_L(E \cap [a, a+\delta]) \\ &\geq \mu_L(E \cap [a-\delta, a+\delta]) - \delta > \tfrac{1}{2}\delta,\end{aligned}$$

and similarly

$$\begin{aligned}\mu_L^*((B+a+\delta) \cap [a+b, a+b+\delta]) &= \mu_L^*(B \cap [b-\delta, b]) \\ &\geq \mu_L^*(B \cap [b-\delta, b+\delta]) - \delta > \tfrac{1}{2}\delta.\end{aligned}$$

But this means that $(E+b) \cap (B+a+\delta)$ cannot be empty. If $u \in (E+b) \cap (B+a+\delta)$, then $u-b \in E$, $u-a-\delta \in B$ so

$$a-b+\delta = (u-b) - (u-a-\delta) \in E - B.$$

As δ is arbitrary, $E-B$ includes the interval $]a-b, a-b+\gamma]$.

(b) ? Suppose, if possible, otherwise; that there is a measurable set $E \subseteq \mathbb{R}$ such that $\mu_L E > 0$ and $E \cap (A + \mathbb{Q}) = \emptyset$. Then $E - A$ does not meet $\mathbb{Q}$ and cannot include any non-trivial interval. **X**

Remark There will be a dramatic generalization of (a) in 443Db.

439F Lemma Let κ be the least cardinal of any set of non-zero Lebesgue outer measure in $\mathbb{R}$.

(a) There is a set $X \subseteq [0,1]$ of cardinal κ and full outer Lebesgue measure.

(b) If (Z, T, ν) is any atomless complete locally determined measure space and $A \subseteq Z$ has cardinal less than κ, then $\nu^* A = 0$.

(c) (Grzegorek 81) There is a universally negligible set $Y \subseteq [0,1]$ of cardinal κ.

proof (a) Take any set $A \subseteq \mathbb{R}$ such that $\#(A) = \kappa$ and $\mu_L^* A > 0$, where μ_L is Lebesgue measure. Set $B = A + \mathbb{Q}$. Then $(\mu_L)_*(\mathbb{R} \setminus B) = 0$, by 439Eb. Set $X = [0,1] \cap B$; then $\mu_L^* X = 1$ while $\#(X) \leq \#(B) = \kappa$. By the definition of κ, $\#(X)$ must be exactly κ.

(b) ? Otherwise, by 412Jc, there is a set $F \subseteq Z$ such that $\nu F < \infty$ and $\nu^*(F \cap A) > 0$. By 343Cc, there is a function $f : F \to [0, \nu Z_n]$ which is inverse-measure-preserving for the subspace measure ν_F and Lebesgue measure on $[0, \nu F]$. But $f[A \cap F]$ has cardinal less than κ, so $\mu_L f[A \cap F] = 0$ and

$$0 < \nu^*(A \cap F) \leq \nu f^{-1}[f[A \cap F]] = 0,$$

which is absurd. **X**

(c)(i) Enumerate X as $\langle x_\xi \rangle_{\xi < \kappa}$. For each $\xi < \kappa$, $A_\xi = \{x_\eta : \eta \leq \xi\}$ has cardinal less than κ, so is Lebesgue negligible; let $\langle I_{\xi n} \rangle_{n \in \mathbb{N}}$ be a sequence of intervals covering A_ξ with $\sum_{n=0}^\infty \mu_L I_{\xi n} < \frac{1}{2}$. Enlarging the intervals slightly if necessary, we may suppose that every $I_{\xi n}$ has rational endpoints; let $\langle J_m \rangle_{m \in \mathbb{N}}$ enumerate the family of intervals in $\mathbb{R}$ with rational endpoints.

Set

$$C_{mn} = \{\xi : \xi < \kappa, \, I_{\xi n} = J_m\}$$

for each m, $n \in \mathbb{N}$.

(ii) If ν is an atomless totally finite measure on κ which measures every C_{mn}, then $\nu\kappa = 0$. **P** Note first that (by (b), applied to the completion of ν) $\nu^* \xi = 0$ for every $\xi < \kappa$. Let λ be the (c.l.d.) product of μ_X, the subspace measure on X, with ν. Set

$$B = \bigcup_{m,n \in \mathbb{N}} ((X \cap J_m) \times C_{mn}) \subseteq X \times \kappa.$$

Then B is measured by λ, so, by Fubini's theorem,

$$\int \nu B[\{x\}] \mu_X(dx) = \int \mu_X B^{-1}[\{\xi\}] \nu(d\xi)$$

(252D).

Now look at the sectional measures $\nu B[\{x\}]$, $\mu_X B^{-1}[\{\xi\}]$. (Because B is actually a countable union of measurable rectangles, these are always defined.) For any $x \in X$, there is an $\eta < \kappa$ such that $x = x_\eta$, and now

$$\begin{aligned} B[\{x\}] &= \{\xi : \text{ there are } m, n \in \mathbb{N} \text{ such that } x \in J_m \text{ and } \xi \in C_{mn}\} \\ &= \{\xi : \text{ there are } m, n \in \mathbb{N} \text{ such that } x \in J_m \text{ and } I_{\xi n} = J_m\} \\ &= \{\xi : \text{ there is an } n \in \mathbb{N} \text{ such that } x \in I_{\xi n}\} \supseteq \kappa \setminus \eta \end{aligned}$$

by the choice of the $I_{\xi n}$. But as $\nu^*\eta = 0$, this means that $\nu B[\{x\}] = \nu\kappa$.

On the other hand, if $\xi < \kappa$, then

$$\begin{aligned} B^{-1}[\{\xi\}] &= \{x : \text{ there are } m, n \in \mathbb{N} \text{ such that } x \in J_m \text{ and } \xi \in C_{mn}\} \\ &= \{x : \text{ there are } m, n \in \mathbb{N} \text{ such that } x \in J_m \text{ and } I_{\xi n} = J_m\} \\ &= \{x : \text{ there is an } n \in \mathbb{N} \text{ such that } x \in I_{\xi n}\} = X \cap \bigcup_{n \in \mathbb{N}} I_{\xi n}, \end{aligned}$$

so that

$$\mu_X B^{-1}[\{\xi\}] \leq \textstyle\sum_{n=0}^{\infty} \mu_L I_{\xi n} \leq \frac{1}{2}.$$

Returning to the integrals, we have

$$\nu\kappa = \int \nu B[\{x\}]\mu_X(dx) = \int \mu_X B^{-1}[\{\xi\}]\nu(d\xi) \leq \tfrac{1}{2}\nu\kappa,$$

so that $\nu\kappa$ must be 0, as claimed. **Q**

(iii) Now there is an injective function $g : \kappa \to [0,1]$ such that $g[C_{mn}]$ is relatively Borel in $g[\kappa]$ for every m, $n \in \mathbb{N}$. **P** Define $h : \kappa \to \{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ by setting

$$\begin{aligned} h(\xi)(m,n,k) &= 1 \text{ if } \xi \in C_{mn} \text{ and } x_\xi \in J_k \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then h is injective (because if $\xi \neq \eta$ then $x_\xi \neq x_\eta$, so there is some k such that $x_\xi \in J_k$ and $x_\eta \notin J_k$), and

$$h[C_{mn}] = h[\kappa] \cap \{w : \text{ there is some } k \text{ such that } w(m,n,k) = 1\}$$

is relatively Borel in $h[\kappa]$ for every m, $n \in \mathbb{N}$. But now recall that $\{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}} \cong \{0,1\}^{\mathbb{N}}$ is homeomorphic to the Cantor set $C \subseteq [0,1]$ (4A2Uc). If $\phi : \{0,1\}^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}} \to C$ is any homeomorphism, then ϕh has the required properties. **Q**

(iv) Set $Y = g[\kappa]$. Because g is injective, $\#(Y) = \kappa$. Also Y is universally negligible. **P** Suppose that $\tilde{\nu}$ is a Borel measure on Y which is zero on singletons. Then it is atomless, because Y is separable and metrizable (439D). So its copy $\nu = \tilde{\nu}(g^{-1})^{-1}$ on κ is atomless. Because $g[C_{mn}]$ is a Borel subset of Y, ν measures C_{mn} for all m, $n \in \mathbb{N}$, so $\tilde{\nu}Y = \nu\kappa = 0$, by (ii) above. **Q**

439G Corollary A metrizable continuous image of a universally negligible metrizable space need not be universally negligible.

proof Take X and Y from 439Fa and 439Fb above, and let $f : X \to Y$ be any bijection. Let Γ be the graph of f. The projection map $(x,y) \mapsto y : \Gamma \to Y$ is continuous and injective, so Γ is universally negligible, by 439Cc. On the other hand, the projection map $(x,y) \mapsto x : \Gamma \to X$ is continuous and surjective, and X is surely not universally negligible, since it is not Lebesgue negligible.

439H Corollary One-dimensional Hausdorff measure on $\mathbb{R}^2$ is not semi-finite.

proof Let μ_{H1} be one-dimensional Hausdorff measure on $\mathbb{R}^2$. Let X, Γ be the sets described in 439F and the proof of 439G.

(a) $\mu^*_{H1}\Gamma > 0$. **P** The first-coordinate map $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is 1-Lipschitz, so, writing μ_L for Lebesgue measure on $\mathbb{R}$,

$$1 = \mu^*_L X = \mu^*_L \pi_1[\Gamma] \leq \mu^*_{H1}\Gamma$$

by 264G and 264I. **Q**

(b) If $E \subseteq \mathbb{R}^2$ and $\mu_{H1}E < \infty$, then $\mu_{H1}(E \cap \Gamma) = 0$, because $E \cap \Gamma$ is universally negligible (439Ca) and μ_{H1} is a topological measure (264E) which is zero on singletons.

(c) **?** Suppose, if possible, that Γ is not measured by μ_{H1}. Then there is a set $A \subseteq \mathbb{R}^2$ such that $\mu^*_{H1}A < \mu^*_{H1}(A \cap \Gamma) + \mu^*_{H1}(A \cap \Gamma)$ (264C, 264Fb). Let E be a Borel set including A such that $\mu_{H1}E = \mu^*_{H1}A$ (264Fa); then $\mu_{H1}(E \cap \Gamma) = 0$, so

$$\mu^*_{H1}(A \cap \Gamma) + \mu^*_{H1}(A \cap \Gamma) \leq \mu_{H1}(E \cap \Gamma) + \mu^*_{H1}A = \mu^*_{H1}A. \text{ } \blacksquare$$

(d) Since Γ is measurable, not negligible, and meets every measurable set of finite measure in a negligible set, it is purely infinite, and μ_{H1} is not semi-finite.

439I Example There are a set X, a Riesz subspace U of $\mathbb{R}^X$ and a smooth positive linear functional $h : U \to \mathbb{R}$ which is not expressible as an integral.

proof Take X and Y from 439F. Replacing Y by $Y \setminus \{0\}$ if need be, we may suppose that $0 \notin Y$. Let $f : X \to Y$ be any bijection.

Let U be the Riesz subspace $\{u \times f : u \in C_b\} \subseteq \mathbb{R}^X$, where C_b is the space of bounded continuous functions from X to $\mathbb{R}$. Because f is strictly positive, $u \mapsto u \times f : C_b \to U$ is a bijection, therefore a Riesz space isomorphism; moreover, for a non-empty set $A \subseteq C_b$, $\inf_{u \in A} u(x) = 0$ for every $x \in X$ iff $\inf_{u \in A} u(x)f(x) = 0$ for every $x \in X$. We therefore have a smooth linear functional $h : U \to \mathbb{R}$ defined by setting $h(u \times f) = \int u \, d\mu_X$ for every $u \in C_b$, where μ_X is the subspace measure on X induced by Lebesgue measure. (By 415B, μ_X is quasi-Radon, so the integral it defines on C_b is smooth, as noted in 436H.)

? But suppose, if possible, that h is the integral with respect to some measure ν on X. Since $f \in U$, it must be T-measurable, where T is the domain of the completion $\hat{\nu}$ of ν. Note that $\hat{\nu}\{x\} = 0$ for every $x \in X$. **P** Set $u_n(y) = \max(0, 1 - 2^n|y - x|)$ for $y \in X$. Then

$$\begin{aligned} f(x)\hat{\nu}\{x\} &= \lim_{n \to \infty} \int u_n \times f \, d\nu = \lim_{n \to \infty} h(u_n \times f) \\ &= \lim_{n \to \infty} \int u_n \, d\mu_X = \mu_X\{x\} = 0, \end{aligned}$$

so $\hat{\nu}\{x\} = 0$. **Q**

For Borel sets $E \subseteq [0,1]$ set $\lambda E = \hat{\nu} f^{-1}[E]$. Then the completion $\hat{\lambda}$ of λ is a Radon measure on $[0,1]$ (433Cb or 256C). If $t \in [0,1]$ then $f^{-1}[\{t\}]$ contains at most one point, so $\hat{\lambda}\{t\} = \lambda\{t\} = 0$. But Y is supposed to be universally negligible, so $\lambda^* Y = \hat{\lambda}^* Y = 0$ (439Ca), that is, there is a Borel set $E \supseteq Y$ with $\lambda E = 0$; in which case $\nu X = \hat{\nu} f^{-1}[E] = 0$, which is impossible. **X**

Thus h is not an integral, despite being a smooth linear functional on a Riesz subspace of $\mathbb{R}^X$.

Remark This example is adapted from Fremlin & Talagrand 78.

439J Example Assume that there is some cardinal κ which is not measure-free (definition: 438A). Give κ its discrete topology, and let μ be a probability measure with domain $\mathcal{P}\kappa$ such that $\mu\{\xi\} = 0$ for every $\xi < \kappa$. Now every subset of κ is open-and-closed, so μ is simultaneously a Baire probability measure and a completion regular Borel probability measure. Of course it is not τ-additive. In the classification schemes of 434A and 435A, we have a measure which is of type B_1 as a Borel measure and type E_3 as a Baire measure.

439K Example There is a first-countable compact Hausdorff space which is not Radon.

proof The construction starts from a compact metrizable space $(Z, \mathfrak{S})$ with an atomless Radon probability measure μ. The obvious candidate is $[0,1]$ with Lebesgue measure; but for technical convenience in a later application I will instead use $Z = \{0,1\}^{\mathbb{N}}$ with its usual product topology and measure (254J).

(a) There is a topology $\mathfrak{T}_{\mathfrak{c}}$ on Z such that

(α) $\mathfrak{S} \subseteq \mathfrak{T}_{\mathfrak{c}}$;

(β) every point of Z belongs to a countable set which is compact and open for $\mathfrak{T}_{\mathfrak{c}}$;

(γ) if $\langle F_n \rangle_{n\in\mathbb{N}}$ is a sequence of $\mathfrak{T}_{\mathfrak{c}}$-closed sets with empty intersection, then $\bigcap_{n\in\mathbb{N}} \overline{F}_n^{\mathfrak{S}}$ is countable,

where I write $\overline{F}^{\mathfrak{S}}$ for the $\mathfrak{S}$-closure of F.

P (i) $\mathfrak{T}_{\mathfrak{c}}$ will be the last in a family $\langle \mathfrak{T}_\xi \rangle_{\xi\leq\mathfrak{c}}$ of topologies. We must begin by enumerating Z as $\langle z_\xi \rangle_{\xi<\mathfrak{c}}$ and taking a family $\langle \langle I_{\xi n} \rangle_{n\in\mathbb{N}} \rangle_{\xi<\mathfrak{c}}$ running over $([Z]^{\leq\omega})^{\mathbb{N}}$ with cofinal repetitions, where $[Z]^{\leq\omega}$ is the family of countable subsets of Z. (This can be done because $\#([Z]^{\leq\omega}) = \mathfrak{c}$, by 2A1Hb.) Together with $\langle \mathfrak{T}_\xi \rangle_{\xi\leq\mathfrak{c}}$ we choose simultaneously families $\langle x_\xi \rangle_{\xi<\mathfrak{c}}$, $\langle y_\xi \rangle_{\xi<\mathfrak{c}}$ of points in Z, and the inductive hypothesis will be

$\mathfrak{T}_\xi$ is a topology on $X_\xi = \{x_\eta : \eta < \xi\} \cup \{y_\eta : \eta < \xi\}$;

if $\eta < \xi \leq \mathfrak{c}$, then $X_\eta \in \mathfrak{T}_\xi$ and $\mathfrak{T}_\eta$ is the subspace topology on X_η induced by $\mathfrak{T}_\xi$;

every point of X_ξ belongs to a countable set which is compact and open for $\mathfrak{T}_\xi$.

The induction starts with $X_0 = \emptyset$, $\mathfrak{T}_0 = \{\emptyset\}$.

(ii) *Inductive step to a successor ordinal* Suppose that we have found X_ξ and $\mathfrak{T}_\xi$ where $\xi < \mathfrak{c}$.

(α) Start by picking $y_\xi \in Z \setminus X_\xi$ such that $y_\xi = z_\xi$ if $z_\xi \notin X_\xi$. Examine the sequence $\langle I_{\xi n} \rangle_{n\in\mathbb{N}}$. If *either* $\bigcup_{n\in\mathbb{N}} I_{\xi n} \not\subseteq X_\xi$ *or* $\bigcap_{n\in\mathbb{N}} \overline{I}_{\xi n}^{\mathfrak{S}}$ is countable, take x_ξ to be any point of $Z \setminus (X_\xi \cup \{y_\xi\})$ and set $K_m = \emptyset$ for every m before proceeding to (γ) below.

(β) If $I_{\xi n} \subseteq X_\xi$ for every n and $\bigcap_{n\in\mathbb{N}} \overline{I}_{\xi n}^{\mathfrak{S}}$ is uncountable, it must have cardinal $\mathfrak{c}$, by 423K, so cannot be included in $X_\xi \cup \{y_\xi\}$. Take any $x_\xi \in \bigcap_{n\in\mathbb{N}} \overline{I}_{\xi n}^{\mathfrak{S}} \setminus (X_\xi \cup \{y_\xi\})$. Let $\langle t_m \rangle_{m\in\mathbb{N}}$ be a sequence in Z such that $t_m \restriction m = x_\xi \restriction m$ for every $m \in \mathbb{N}$ and $t_m \in I_{\xi n}$ whenever $r \in \mathbb{N}$, $n \leq 2r$ and $m = r^2 + n$. (Thus, for each n, $t_m \in I_{\xi n}$ for infinitely many m, while $\langle t_m \rangle_{m\in\mathbb{N}} \to x_\xi$ in the ordinary sense.) By the inductive hypothesis, we can find countable sets $K_m \subseteq X_\xi$, compact and open for $\mathfrak{T}_\xi$, such that $t_m \in K_m$ for each m. Because $\{t : t \in X_\xi, t \restriction m = x_\xi \restriction m\}$ is open-and-closed for $\mathfrak{T}_\xi$ and contains t_m, we may suppose that $t \restriction m = x_\xi \restriction m$ for every $t \in K_m$.

(γ) Let $\mathfrak{T}_{\xi+1}$ be the topology on $X_{\xi+1} = X_\xi \cup \{x_\xi, y_\xi\}$ generated by

$$\mathfrak{T}_\xi \cup \{\{y_\xi\}\} \cup \{L_n : n \in \mathbb{N}\},$$

where $L_n = \{x_\xi\} \cup \bigcup_{m\geq n} K_m$ for each n.

(δ) Because $\mathfrak{T}_\xi \subseteq \mathfrak{T}_{\xi+1}$, X_ξ will be open in $X_{\xi+1}$. Because the K_i are always $\mathfrak{T}_\xi$-open, and x_ξ, y_ξ are distinct points of $Z \setminus X_\xi$, the topology on X_ξ induced by $\mathfrak{T}_{\xi+1}$ is just $\mathfrak{T}_\xi$. Consequently (by the inductive hypothesis) the topology on X_η induced by $\mathfrak{T}_{\xi+1}$ is $\mathfrak{T}_\eta$ for every $\eta \leq \xi$. We have $t \restriction n = x_\xi \restriction n$ for every $t \in L_n$, so $\mathfrak{T}_{\xi+1}$ is finer than the usual topology on $X_{\xi+1}$.

If $x \in X_\xi$, then there is a countable $\mathfrak{T}_\xi$-open $\mathfrak{T}_\xi$-compact set containing x, which is still $\mathfrak{T}_{\xi+1}$-open and $\mathfrak{T}_{\xi+1}$-compact. Of course $\{y_\xi\}$ is a countable $\mathfrak{T}_{\xi+1}$-open $\mathfrak{T}_{\xi+1}$-compact set containing y_ξ. As for x_ξ, L_0 is surely countable and $\mathfrak{T}_{\xi+1}$-open. To see that it is $\mathfrak{T}_{\xi+1}$-compact, observe that any ultrafilter containing L_0 either contains every L_n, and converges to x_ξ, or contains some K_i and converges to a point of K_i.

Thus the induction proceeds at successor stages.

(iii) *Inductive step to a limit ordinal* If $\xi \leq \kappa$ is a non-zero limit ordinal, then we have $X_\xi = \bigcup_{\eta<\xi} X_\eta$, and can take $\mathfrak{T}_\xi$ to be the topology generated by $\bigcup_{\eta<\xi} \mathfrak{T}_\eta$. It is easy to check that this works (because the topologies $\mathfrak{T}_\eta$ are consistent).

(iv) At the end of the induction, we have $X_{\mathfrak{c}} = Z$ because $z_\xi \in X_{\xi+1} \subseteq X_{\mathfrak{c}}$ for every ξ. The final topology $\mathfrak{T}_{\mathfrak{c}}$ on Z will have the properties (α) and (β) required. **?** Now suppose, if possible, that $\langle F_n \rangle_{n\in\mathbb{N}}$ is a sequence of $\mathfrak{T}_{\mathfrak{c}}$-closed sets with empty intersection, and that $\bigcap_{n\in\mathbb{N}} \overline{F}_n^{\mathfrak{S}}$ is uncountable. For each $n \in \mathbb{N}$, let $J_n \subseteq F_n$ be a countable $\mathfrak{S}$-dense set. Then there is some $\zeta < \mathfrak{c}$ such that $\bigcup_{n\in\mathbb{N}} J_n \subseteq X_\zeta$ (because $\operatorname{cf} \mathfrak{c} > \omega$, see 4A1A(c-ii)). Let $\xi \geq \zeta$ be such that $J_n = I_{\xi n}$ for every $n \in \mathbb{N}$. Then in the construction of $\mathfrak{T}_{\xi+1}$ we must be in case (β) of (ii) above. Taking $\langle t_m \rangle_{m\in\mathbb{N}}$ as described there, we have $\langle t_m \rangle_{m\in\mathbb{N}} \to x_\xi$ for $\mathfrak{T}_{\xi+1}$, and therefore for $\mathfrak{T}_{\mathfrak{c}}$. But for any $n \in \mathbb{N}$, $t_m \in I_n \subseteq F_n$ for infinitely many m, so $x_\xi \in F_n$. Thus $x_\xi \in \bigcap_{n\in\mathbb{N}} F_n$; but this is impossible. **X**

So we have a topology of the type required. **Q**

(b) There is a probability measure ν on Z, extending the usual measure μ, such that with respect to $\mathfrak{T}_\mathfrak{c}$ ν is a topological measure inner regular with respect to the closed sets, but is not τ-additive.

P Let $\mathcal{K}$ be the family of $\mathfrak{T}_\mathfrak{c}$-closed subsets of Z. For $F \in \mathcal{K}$, set $\phi F = \mu^* F$.

(i) If E, F are disjoint $\mathfrak{T}_\mathfrak{c}$-closed sets, then $\overline{E}^{\mathfrak{S}} \cap \overline{F}^{\mathfrak{S}}$ must be countable (take $F_{2n} = E$, $F_{2n+1} = F$ in (a-γ)). So

$$\begin{aligned}\phi(E \cup F) = \mu^*(E \cup F) &= \mu^*((E \cup F) \cap \overline{E}^{\mathfrak{S}}) + \mu^*((E \cup F) \setminus \overline{E}^{\mathfrak{S}}) \\ &= \mu^* E + \mu^* F = \phi E + \phi F.\end{aligned}$$

(ii) If E, $F \in \mathcal{K}$, $E \subseteq F$ and $\epsilon > 0$, there is an $\mathfrak{S}$-open set $G \supseteq E$ such that $\mu G \leq \mu^* E + \epsilon$. Now $F \setminus G \in \mathcal{K}$ and

$$\phi F = \mu^*(F \cap G) + \mu^*(F \setminus G) \leq \mu G + \phi(F \setminus G) \leq \phi E + \phi(F \setminus G) + \epsilon.$$

Putting this together with (i), we see that

$$\phi F = \phi E + \sup\{\phi E' : E' \in \mathcal{K},\, E' \subseteq F \setminus E\}.$$

(iii) If $\langle F_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with empty intersection, then

$$\lim_{n\to\infty} \phi F_n \leq \lim_{n\to\infty} \mu \overline{F}_n^{\mathfrak{S}} = \mu(\textstyle\bigcap_{n\in\mathbb{N}} \overline{F}_n^{\mathfrak{S}}) = 0.$$

(iv) Thus $\mathcal{K}$ and ϕ satisfy all the conditions of 413I, and there is a measure ν, extending ϕ, which is defined on every member of $\mathcal{K}$ and inner regular with respect to $\mathcal{K}$, and therefore is (for $\mathfrak{T}_\mathfrak{c}$) a topological measure inner regular with respect to the closed sets.

If we write $\mathcal{V}$ for the family of $\mathfrak{T}_\mathfrak{c}$-compact $\mathfrak{T}_\mathfrak{c}$-open countable subsets of Z, then for any $K \in \mathcal{V}$

$$\nu K = \phi K = \mu K = 0,$$

while $\mathcal{V}$ is upwards-directed and has union Z; so that ν is not τ-additive. **Q**

(c) So far we seem to have very little more than is provided by ω_1 with the order topology and Dieudonné's measure. The point of doing all this work is the next step. Set $X = Z \times \{0, 1\}$ and give X the topology $\mathfrak{T}$ generated by

$$\{G \times \{0,1\} : G \in \mathfrak{S}\} \cup \{H \times \{1\} : H \in \mathfrak{T}_\mathfrak{c}\} \cup \{X \setminus (K \times \{1\}) : K \text{ is } \mathfrak{T}_\mathfrak{c}\text{-compact}\}.$$

(i) $\mathfrak{T}$ is Hausdorff. **P** If w, z are distinct points of X, then either their first coordinates differ and they are separated by sets of the form $G_0 \times \{0,1\}$, $G_1 \times \{0,1\}$ where G_0, G_1 belong to $\mathfrak{S}$, or they are of the form $(x, 1)$, $(x, 0)$ and are separated by open sets of the form $K \times \{1\}$, $X \setminus (K \times \{1\})$ for some set K which is compact and open for $\mathfrak{T}_\mathfrak{c}$. **Q**

(ii) $\mathfrak{T}$ is compact. **P** Let $\mathcal{F}$ be an ultrafilter on X. Writing $\pi_1(x, 0) = \pi_1(x, 1) = x$ for $x \in Z$, $\pi_1[[\mathcal{F}]]$ is $\mathfrak{S}$-convergent, to x_0 say. If $K \times \{1\} \in \mathcal{F}$ for some $\mathfrak{T}_\mathfrak{c}$-compact set K, then $\mathcal{F}$ is $\mathfrak{T}$-convergent to $(x, 1)$; otherwise, it is $\mathfrak{T}$-convergent to $(x, 0)$ (using 4A2B(a-v)). **Q**

(iii) $\mathfrak{T}$ is first-countable. **P** If $x \in Z$, then $\{(x,0),(x,1)\} = \pi_1^{-1}[\{x\}]$ is a G_δ set in X because $\{x\}$ is a G_δ set in Z and π_1 is continuous (4A2C(a-iii)). Now $\{(x,0)\}$ and $\{(x,1)\}$ are relatively open in $\{(x,0),(x,1)\}$, so are G_δ sets in X (4A2C(a-iv)). Thus singletons are G_δ sets. Because $\mathfrak{T}$ is compact and Hausdorff, it is first-countable (4A2Kf). **Q**

(iv) $(X, \mathfrak{T})$ is not a Radon space. **P** $Z \times \{1\}$ is an open subset of X, homeomorphic to Z with the topology $\mathfrak{T}_\mathfrak{c}$. But the measure ν of (b) above (or, if you prefer, its restriction to the $\mathfrak{T}_\mathfrak{c}$-Borel algebra) witnesses that $\mathfrak{T}_\mathfrak{c}$ is not a Radon topology, so $\mathfrak{T}$ also cannot be a Radon topology, by 434Fc. **Q**

Remark Aficionados will recognise $\mathfrak{T}_\mathfrak{c}$ as a kind of 'JKR-space', derived from the construction in Juhász Kunen & Rudin 76.

439L Example Suppose that κ is a cardinal which is not measure-free; let μ be a probability measure with domain $\mathcal{P}\kappa$ which is zero on singletons. Give κ the discrete topology, so that μ is a Borel measure and κ is first-countable. Let ν be the restriction of the usual measure on $Y = \{0,1\}^\kappa$ to the algebra $\mathcal{B}$ of Borel subsets of Y, so that ν is a τ-additive probability measure, and λ the product measure on $\kappa \times Y$ constructed by the method of 434R. Then

$$W = \{(\xi, y) : \xi < \kappa,\, y(\xi) = 1\} = \textstyle\bigcup_{\xi<\kappa} \{\xi\} \times \{y : y(\xi) = 1\}$$

is open in $\kappa \times Y$.

If $W' \in \mathcal{P}\kappa \widehat{\otimes} \mathcal{B}$ then $\lambda(W \triangle W') = \frac{1}{2}$. **P** There is a countable set $\mathcal{E} \subseteq \mathcal{B}$ such that W' belongs to the σ-algebra generated by $\{A \times E : A \subseteq \kappa,\, E \in \mathcal{E}\}$ (331Gd). For $J \subseteq \kappa$, write $\pi_J(y) = y \restriction J$ for $y \in Y$, let ν_J be the usual measure on $\{0,1\}^J$ and Λ_J its domain, and let Λ'_J be the family of sets $E \subseteq Y$ such that there are H, $H' \in \Lambda_J$ such that $\pi_J^{-1}[H] \subseteq E \subseteq \pi_J^{-1}[H']$ and $\nu_J(H' \setminus H) = 0$. Then $\Lambda'_J \subseteq \Lambda'_K$ whenever $J \subseteq K \subseteq \kappa$, and every set measured by ν belongs to Λ'_J for some countable J (254Ob). There is therefore a countable set $J \subseteq \kappa$ such that $\mathcal{E} \subseteq \Lambda'_J$. Also, of course, Λ'_J is a σ-algebra of subsets of Y.

The set

$$\{V : V \subseteq \kappa \times Y,\, V[\{\xi\}] \in \Lambda'_J \text{ for every } \xi < \kappa\}$$

is a σ-algebra of subsets of $\kappa \times Y$ containing $A \times E$ whenever $A \subseteq \kappa$ and $E \in \mathcal{E}$, so contains W'. But this means that if $\xi \in \kappa \setminus J$, $W[\{\xi\}]$ and $W'[\{\xi\}]$ are stochastically independent, and $\nu(W[\{\xi\}] \triangle W'[\{\xi\}]) = \frac{1}{2}$. Since $\mu(\kappa \setminus J) = 1$,

$$\lambda(W \triangle W') = \int \nu(W[\{\xi\}] \triangle W'[\{\xi\}]) \mu(d\xi) = \frac{1}{2},$$

as claimed. **Q**

In particular, $W^\bullet$ in the the measure algebra of λ cannot be represented by a member of $\mathcal{P}\kappa \widehat{\otimes} \mathcal{B}$.

439M Example There is a first-countable locally compact Hausdorff space X with a Baire probability measure μ which is not τ-additive and has no extension to a Borel measure. In the classification of 435A, μ is of type E_0.

proof Let Ω be the set of non-zero countable limit ordinals, and for each $\xi \in \Omega$ let $\langle \theta_\xi(i) \rangle_{i \in \mathbb{N}}$ be a strictly increasing sequence of ordinals with supremum ξ. Set $X = \omega_1 \times (\omega + 1)$, and define a topology $\mathfrak{T}$ on X by saying that $G \subseteq X$ is open iff

$\{\xi : (\xi, n) \in G\}$ is open in the order topology of ω_1 for every $n \in \mathbb{N}$,

whenever $\xi \in \Omega$ and $(\xi, \omega) \in G$ then there is some $n < \omega$ such that $(\eta, i) \in G$ whenever $n \leq i < \omega$ and $\theta_\xi(i) < \eta \leq \xi$.

This is finer than the product of the order topologies, so is Hausdorff. For every $\xi < \omega_1$, $n \in \mathbb{N}$, $(\xi+1) \times \{n\}$ is a countable compact open set containing (ξ, n); for every $\xi \in \omega_1 \setminus \Omega$, $\{(\xi, \omega)\}$ is a countable compact open set containg (ξ, ω); and for every $\xi \in \Omega$,

$$\{(\xi, \omega)\} \cup \{(\eta, i) : i < \omega,\, \theta_\xi(i) < \eta \leq \xi\}$$

is a countable compact open subset of X containing (ξ, ω). Thus $\mathfrak{T}$ is locally compact, and every singleton subset of X is G_δ, so $\mathfrak{T}$ is first-countable (4A2Kf).

If $f : X \to \mathbb{R}$ is continuous, then for every $n \in \mathbb{N}$ there is a $\zeta_n < \omega_1$ such that f is constant on $\{(\xi, n) : \zeta_n \leq \xi < \omega_1\}$ (4A2S(b-iii)). Setting $\zeta = \sup_{n \in \mathbb{N}} \zeta_n$, f must be constant on $\{(\xi, \omega) : \xi \in \Omega,\, \xi > \zeta\}$. **P** If ξ, $\eta \in \Omega \setminus (\zeta+1)$, then $f(\xi, \omega) = \lim_{i \to \infty} f(\theta_\xi(i)+1, i)$ and $f(\eta, \omega) = \lim_{i \to \infty} f(\theta_\eta(i)+1, i)$. But there is some n such that both $\theta_\xi(i)$ and $\theta_\eta(i)$ are greater than ζ for every $i \geq n$, so that $f(\theta_\xi(i)+1, i) = f(\theta_\eta(i)+1, i)$ for every $i \geq n$ and $f(\xi, \omega) = f(\eta, \omega)$. **Q**

Writing Σ for the family of subsets E of X such that $\{\xi : \xi \in \Omega,\, (\xi, \omega) \in E\}$ is either countable or cocountable in Ω, Σ is a σ-algebra of subsets of X such that every continuous function is Σ-measurable, so every Baire set belongs to Σ. We therefore have a Baire measure μ_0 on X defined by saying that $\mu_0 E = 0$ if $E \cap (\Omega \times \{\omega\})$ is countable, 1 otherwise. $\{(\xi + 1) \times (\omega + 1) : \xi < \omega_1\}$ is a cover of X by negligible open-and-closed sets, so μ_0 is not τ-additive.

? Suppose, if possible, that μ were a Borel measure on X extending μ_0. Then we must have $\mu(\omega_1 \times \{n\}) = \mu_0(\omega_1 \times \{n\}) = 0$ for every $n \in \mathbb{N}$, so $\mu(\omega_1 \times \{\omega\}) = 1$. Let λ be the subspace measure on $\omega_1 \times \{\omega\}$ induced

by μ. If $A \subseteq \omega_1$, $(A \times \{\omega\}) \cup (\omega_1 \times \omega)$ is an open set, so λ is defined on every subset of $\omega_1 \times \{\omega\}$; and if $\xi < \omega_1$, then $\mu_0((\xi+1) \times (\omega+1)) = 0$, so λ is zero on singletons. And this contradicts Ulam's theorem (419G, 438Cd). **X**

439N Example Give ω_1 its order topology.

(i) ω_1 is a normal Hausdorff space which is not measure-compact.

(ii) There is a Baire probability measure μ_0 on ω_1 which is not τ-additive and has a unique extension to a Borel measure, which is not completion regular; that is, μ_0 is of type E_2 in the classification of 435A.

proof (a) As noted in 4A2Rc, order topologies are always normal and Hausdorff.

(b) Let μ be Dieudonné's measure on ω_1 (411Q), and μ_0 its restriction to the Baire σ-algebra. Then μ is the only Borel measure extending μ_0. **P** Let ν be any Borel measure extending μ_0. Every set $[0,\xi] = [0,\xi+1[$, where $\xi < \omega_1$, is open-and-closed, so

$$\nu[0,\xi] = \mu_0[0,\xi] = \mu[0,\xi] = 0;$$

also, of course, $\nu X = 1$. Let $F \subseteq X$ be any closed set. If F is countable, then it is included in some initial segment $[0,\xi]$, so $\nu F = \mu F = 0$. Now suppose that F is uncountable. Set $G = \omega_1 \setminus F$. For each $\xi \in F$, set $\zeta_\xi = \min\{\eta : \xi < \eta \in F\}$ and $G_\xi =]\xi, \zeta_\xi[$. Then $\langle G_\xi \rangle_{\xi \in F}$ is a disjoint family of open sets. By 438Bb and 419G/438Cd,

$$\nu(\textstyle\bigcup_{\xi \in F} G_\xi) = \sum_{\xi \in F} \nu G_\xi = 0.$$

But now

$$1 = \nu\omega_1 = \nu F + \nu\,[0, \min F[\, + \nu(\textstyle\bigcup_{\xi \in F} G_\xi) = \nu F = \mu F.$$

Thus μ and ν agree on the family $\mathcal{E}$ of closed sets. By the Monotone Class Theorem (136C), they agree on the σ-algebra generated by $\mathcal{E}$, which is their common domain; so they are equal. **Q**

(c) I have already remarked in 411Q-411R that μ and μ_0 are not τ-additive and μ is not completion regular. So of course ω_1 is not measure-compact.

439O In 439M I described a Baire measure with no extension to a Borel measure. In view of Mařík's theorem (435C), it is natural to ask whether this can be done with a normal space. This leads us into relatively deep water, and the only examples known need special assumptions.

Example Assume ♣. Then there is a normal Hausdorff space with a Baire probability measure μ which is not τ-additive and not extendable to a Borel measure. (In the classification of 435A, μ is of type E_0.)

proof (a) ♣ implies that there is a family $\langle C_\xi \rangle_{\xi < \omega_1}$ of sets such that (i) $C_\xi \subseteq \xi$ for every $\xi < \omega_1$ (ii) $C_\xi \cap \eta$ is finite whenever $\eta < \xi < \omega_1$ (iii) for any uncountable sets A, $B \subseteq \omega_1$ there is a $\xi < \omega_1$ such that $A \cap C_\xi$ and $B \cap C_\xi$ are both infinite (4A1N). For $A \subseteq \omega_1$, set $A' = \{\xi : \xi < \omega_1, A \cap C_\xi \text{ is infinite}\}$; then $A' \cap B'$ is non-empty whenever A, $B \subseteq \omega_1$ are uncountable. But this means that $A' \cap B'$ is actually uncountable for uncountable A, B, since $A' \cap B' \setminus \gamma \supseteq (A \setminus \gamma)' \cap (B \setminus \gamma)'$ is non-empty for every $\gamma < \omega_1$.

Set $X = \omega_1 \times \mathbb{N}$. For $x = (\xi, n) \in X$, say that

$$\begin{aligned} I_x &= C_\xi \times \{n-1\} \text{ if } n \geq 1, \\ &= \emptyset \text{ otherwise .} \end{aligned}$$

(b) Define a topology $\mathfrak{T}$ on X by saying that a set $G \subseteq X$ is open iff $I_x \setminus G$ is finite for every $x \in G$.

The form of the construction ensures that $\mathfrak{T}$ is T_1. In fact, $I_x \cap I_y$ is finite whenever $x \neq y$ in X. **P** Express x as (ξ, m) and y as (η, n) where $\eta \leq \xi$. If either $m = 0$ or $n = 0$ or $m \neq n$, $I_x \cap I_y = \emptyset$. If $n \geq 1$ and $\eta < \xi$, then

$$I_x \cap I_y \subseteq (C_\xi \cap \eta) \times \{m-1\}$$

is finite. Similarly, $I_x \cap I_y$ is finite if $m \geq 1$ and $\xi < \eta$. **Q** Consequently $\{x\} \cup J$ is closed for every $x \in X$, $J \subseteq I_x$.

Observe that $(\xi+1)\times\mathbb{N}$ is open and closed for every $\xi<\omega_1$, again because $C_\eta\cap(\xi+1)$ is finite whenever $\eta\in\Omega$ and $\eta>\xi$, while $C_\xi\subseteq\xi$ for every ξ.

(c) The next step is to understand the uncountable closed subsets of X. First, if $F\subseteq X$ is closed and $n\in\mathbb{N}$, then $F^{-1}[\{n\}]'$, as defined in (a), is a subset of $F^{-1}[\{n+1\}]$, since if $\xi\in F^{-1}[\{n\}]'$ then $I_{(\xi,n+1)}\cap F$ is infinite. If F is uncountable, there is some $n\in\mathbb{N}$ such that $F^{-1}[\{n\}]$ is uncountable, so that (inducing on m) $F^{-1}[\{m\}]$ is uncountable for every $m\geq n$. Finally, this means that if E, $F\subseteq X$ are uncountable closed sets, there is an $m\in\mathbb{N}$ such that $E^{-1}[\{m\}]$ and $F^{-1}[\{m\}]$ are both uncountable, so that $E^{-1}[\{m\}]'\cap F^{-1}[\{m\}]'$ is non-empty and $E\cap F$ is non-empty.

(d) It follows that X is normal. **P** Let E and F be disjoint closed sets in X. By (c), at least one of them is countable; let us take it that $E\subseteq\zeta\times\mathbb{N}$ where $\zeta<\omega_1$. Enumerate the open-and-closed set $W=(\zeta+1)\times\mathbb{N}$ as $\langle x_n\rangle_{n\in\mathbb{N}}$. Choose $\langle U_n\rangle_{n\in\mathbb{N}}$, $\langle V_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $U_0=E$, $V_0=F\cap W$. If $x_n\in U_n$, then $U_{n+1}=U_n\cup(I_{x_n}\setminus V_n)$ and $V_{n+1}=V_n$; if $x_n\notin U_n$, then $U_{n+1}=U_n$ and $V_{n+1}=V_n\cup\{x_n\}\cup(I_{x_n}\setminus U_n)$. An easy induction shows that, for every n, (α) $U_n\cap V_n=\emptyset$ (β) $U_n\cup V_n\subseteq W$ (γ) $I_x\cap(U_n\cup V_n)$ is finite for every $x\in X\setminus(U_n\cup V_n)$ (δ) $I_x\cap V_n$ is finite for every $x\in U_n$ (ϵ) $I_x\cap U_n$ is finite for every $x\in V_n$.

At the end of the induction, set $G=\bigcup_{n\in\mathbb{N}}U_n$, $H=\bigcup_{n\in\mathbb{N}}V_n\cup(X\setminus W)$. Then $E\subseteq G$, $F\subseteq H$ and $G\cap H=\emptyset$. If $x\in G$, it is of the form x_n for some n, in which case $x_n\in U_n$ (because $x_n\notin V_{n+1}$) and $I_x\setminus U_{n+1}=I_x\cap V_n$ is finite; thus G is open. If $x\in H\cap W$, again it is of the form x_n where this time $x_n\notin U_n$, so that $I_x\setminus V_{n+1}=I_x\cap U_n$ is finite; so H is open.

Thus E and F are separated by open sets; since E and F are arbitrary, X is normal. **Q**

Being T_1 (see (b)), X is also Hausdorff.

(e) Because disjoint closed sets in X cannot both be uncountable ((c) above), any bounded continuous function on X must be constant on a cocountable set. (Compare 4A2S(b-iii).) The countable-cocountable measure μ_0 is therefore a Baire measure on X (cf. 411R). But it has no extension to a Borel measure. **P** The point is that if A is any subset of ω_1, and $n\in\mathbb{N}$, then

$$(A\times\{n\})\cup(\omega_1\times n),\quad \omega_1\times n$$

are both open, so $A\times\{n\}$ is Borel; accordingly every subset of X is a Borel set. But ω_1 is measure-free (419G, 438Cd), so there can be no Borel probability measure on X which is zero on singletons. **Q**

Of course μ_0 is not τ-additive, because $\{(\xi+1)\times\mathbb{N}:\xi<\omega_1\}$ is a cover of X by open-and-closed negligible sets.

Remark Thus in Mařík's theorem we really do need 'countably paracompact' as well as 'normal', at least if we want a theorem valid in ZFC.

Observe that any example of this phenomenon must involve a **Dowker space**, that is, a normal Hausdorff space which is not countably paracompact. The one here is based on DE CAUX 76. Such spaces are hard to come by in ZFC if we do not allow ourselves to use special principles like ♣. 'Real' Dowker spaces have been described by RUDIN 71 and BALOGH 96; for a survey, see RUDIN 84. I do not know if either of these can be adapted to provide a ZFC example to replace the one above.

439P Example (cf. MORAN 68) $\mathbb{N}^\mathfrak{c}$ is not Borel-measure-compact, therefore not Borel-measure-complete, measure-compact or Radon.

proof Consider the topology $\mathfrak{T}_\mathfrak{c}$ on $Z=\{0,1\}^\mathbb{N}$, as constructed in 439K. Then $(Z,\mathfrak{T}_\mathfrak{c})$ is homeomorphic to a closed subset of $\mathbb{N}^Z\times\{0,1\}^\mathbb{N}$, where in this product the second factor $\{0,1\}^\mathbb{N}$ is given its usual topology $\mathfrak{S}$. **P** For each $x\in Z$, let L_x be a $\mathfrak{T}_\mathfrak{c}$-open $\mathfrak{T}_\mathfrak{c}$-compact subset of Z. The first thing to observe is that if $x\in Z$, and we write $V_{xm}=\{y:y\in Z,\,y{\restriction}m=x{\restriction}m\}$ for each $m\in\mathbb{N}$, then $\mathcal{U}_x=\{L_x\cap V_{xm}:m\in\mathbb{N}\}$ is a downwards-directed family of compact open neighbourhoods of x with intersection $\{x\}$, so is a base of neighbourhoods of x (4A2Gd); thus $\mathcal{U}=\{L_x:x\in Z\}\cup\mathfrak{S}$ generates $\mathfrak{T}_\mathfrak{c}$. Now, for $x\in Z$, define $\phi_x:Z\to\mathbb{N}$ by setting

$$\begin{aligned}\phi_x(y)&=0 \text{ if } y\in L_x,\\ &=m+1 \text{ if } y\in V_{xm}\setminus(L_x\cup V_{x,m+1}).\end{aligned}$$

Then every ϕ_x is $\mathfrak{T}_\mathfrak{c}$-continuous, so we have a $\mathfrak{T}_\mathfrak{c}$-continuous function $\phi : Z \to \mathbb{N}^Z \times \{0,1\}^{\mathbb{N}}$ defined by setting $\phi(y) = (\langle \phi_x(y)\rangle_{x\in C}, y)$ for $y \in Z$. Because every element of $\mathcal{U}$ is of the form $\phi^{-1}[H]$ for some open set $H \subseteq \mathbb{N}^Z \times \{0,1\}^{\mathbb{N}}$, Z is homeomorphic to its image $\phi[Z]$.

Now suppose that $(w,z) \in \overline{\phi[Z]}$. In this case, there is an ultrafilter containing $\phi[Z]$ which converges to (w,z) (4A2Bc), and this must be of the form $\phi[[\mathcal{F}]]$ for some ultrafilter on Z (2A1Ib). Of course $z = \lim \mathcal{F}$ for $\mathfrak{S}$. **?** If z is not the $\mathfrak{T}_\mathfrak{c}$-limit of $\mathcal{F}$, then $\mathcal{F}$ can have no $\mathfrak{T}_\mathfrak{c}$-limit, and can contain no $\mathfrak{T}_\mathfrak{c}$-compact set (2A3R). In particular, $L_z \notin \mathcal{F}$; but in this case $V_{zm} \setminus L_z \in \mathcal{F}$ for every m, so that $\{(v,y) : v(x) > m\} \in \phi[[\mathcal{F}]]$ for every m, and $w(x) > m$ for every m, which is impossible. **X** Thus $\mathcal{F} \to z$, and (as ϕ is continuous) $(w,z) = \phi(z)$.

This shows that $\phi[Z]$ is closed, so we have the required homeomorphism between Z and a closed subset of $\mathbb{N}^Z \times \{0,1\}^{\mathbb{N}}$. **Q**

Of course $\mathbb{N}^Z \times \{0,1\}^{\mathbb{N}}$ is a closed subset of $\mathbb{N}^Z \times \mathbb{N}^{\mathbb{N}} \cong \mathbb{N}^\mathfrak{c}$. So Z is homeomorphic to a closed subset of $\mathbb{N}^\mathfrak{c}$. But Z, with $\mathfrak{T}_c$, carries a Borel probability measure ν which is inner regular with respect to the closed sets and is not τ-additive (439Kb). So $(Z, \mathfrak{T}_\mathfrak{c})$ is not Borel-measure-compact. By 434Hc, $\mathbb{N}^\mathfrak{c}$ is not Borel-measure-compact. By 434Ic, $\mathbb{N}^\mathfrak{c}$ is not Borel-measure-complete; by 434K, it is not Radon; by 435Fd, it is not measure-compact.

439Q Example Let X be the real line with the **right-facing Sorgenfrey topology**, generated by sets of the form $[a,b[$ where $a < b$ in $\mathbb{R}$. Then X is measure-compact but X^2 is not.

proof (a) Note that every set $[a,b[$ is open-and-closed in X, so that the topology is zero-dimensional, therefore completely regular; and it is finer than the usual topology of $\mathbb{R}$, so is Hausdorff.

X is Lindelöf. **P** Let $\mathcal{G}$ be an open cover of X. For each $q \in \mathbb{Q}$, set

$$A_q = \{a : a \in]-\infty, q[\text{ and there is some } G \in \mathcal{G} \text{ such that } [a,q[\subseteq G\}.$$

Then $\bigcup_{q\in\mathbb{Q}} A_q = \mathbb{R}$. For each $q \in \mathbb{Q}$, there is a countable set $A'_q \subseteq A_q$ such that $\inf A'_q = \inf A_q$ in $[-\infty,\infty]$ and A'_q contains $\min A_q$ if A_q has a least element. Now, for each pair (a,q) where $q \in \mathbb{Q}$ and $a \in A'_q$, choose $G_{aq} \in \mathcal{G}$ such that $[a,q[\subseteq G_{aq}$. It is easy to see that $\bigcup\{G_{aq} : a \in A'_q\} \supseteq A_q$, so that the countable family $\{G_{aq} : q \in \mathbb{Q}, a \in A'_q\}$ covers X. As $\mathcal{G}$ is arbitrary, X is Lindelöf. **Q**

It follows that X is measure-compact (435Fb).

(b) Let $\mathfrak{S}$ be the usual topology on $\mathbb{R}^2$, and $\mathfrak{T}$ the product topology on X^2.

(i) Whenever G, H are disjoint $\mathfrak{T}$-open sets, there is an $\mathfrak{S}$-Borel set E such that $G \subseteq E \subseteq X^2 \setminus H$. **P** For $n \in \mathbb{N}$, set

$$A_n = \{(a,b) : [a, a+2^{-n}[\times [b, b+2^{-n}[\subseteq G\}.$$

? Suppose, if possible, that there is a point $(x,y) \in \overline{A_n}^{\mathfrak{S}} \cap H$, where I write $\overline{\ }^{\mathfrak{S}}$ to denote closure for the topology $\mathfrak{S}$. Let $\delta > 0$ be such that $[x, x+2\delta[\times [y, y+2\delta[\subseteq H$ and $2\delta < 2^{-n}$. Then there must be $(a,b) \in A_n$ such that $|a-x| \le \delta$ and $|b-y| \le \delta$. In this case, $a \le x+\delta < a+2^{-n}$ and $b \le y+\delta < b+2^{-n}$, so $(x+\delta, y+\delta) \in G$; while δ was chosen so that $(x+\delta, y+\delta)$ would belong to H. **X**

Accordingly $E = \bigcup_{n\in\mathbb{N}} \overline{A_n}^{\mathfrak{S}}$ is an $\mathfrak{S}$-Borel set disjoint from H. But $G = \bigcup_{n\in\mathbb{N}} A_n$, so $G \subseteq E$. **Q**

(ii) Consequently every $\mathfrak{T}$-continuous real-valued function is $\mathfrak{S}$-Borel measurable. **P** If $f : X^2 \to \mathbb{R}$ is $\mathfrak{T}$-continuous and $\alpha \in \mathbb{R}$, then there is an $\mathfrak{S}$-Borel set E_α such that

$$\{(x,y) : f(x,y) < \alpha\} \subseteq E_\alpha \subseteq \{(x,y) : f(x,y) \le \alpha\}.$$

But this means that $\{(x,y) : f(x,y) < \alpha\} = \bigcup_{n\in\mathbb{N}} E_{\alpha - 2^{-n}}$ is $\mathfrak{S}$-Borel. **Q**

(iii) It follows that every $\mathfrak{T}$-Baire set is $\mathfrak{S}$-Borel. We therefore have a $\mathfrak{T}$-Baire probability measure ν on X defined by setting

$$\nu E = \mu_L\{t : t \in [0,1], (t, 1-t) \in E\}$$

for every $\mathfrak{T}$-Baire subset of X^2, where μ_L is Lebesgue measure on $\mathbb{R}$. In this case every point (x,y) of X^2 belongs to a $\mathfrak{T}$-open set of zero measure for ν. **P** Set $K = \{(t, 1-t) : t \in [0,1]\}$. Then K is $\mathfrak{S}$-closed, therefore $\mathfrak{T}$-closed, and $\nu(X^2 \setminus K) = 0$, so if $(x,y) \notin K$ then we can stop. If $(x,y) \in K$, then $[x, x+1[\times [y, y+1[$ is a $\mathfrak{T}$-open $\mathfrak{T}$-closed set meeting K in the single point (x,y), so is a negligible $\mathfrak{T}$-neighbourhood of (x,y). **Q**

Thus ν is not τ-additive and X^2 is not measure-compact.

Remark Contrast this with 438Xo.

439R Example There are first-countable completely regular Hausdorff spaces X, Y with Baire probability measures μ, ν such that the Baire measures λ, λ' on $X \times Y$ defined by the formulae

$$\int f d\lambda W = \iint f(x,y)\nu(dy)\mu(dx), \quad \int f d\lambda' = \iint f(x,y)\mu(dx)\nu(dy)$$

(436F) are different.

proof Let X, Y be disjoint stationary subsets of ω_1 (4A1Cd). Give each the topology induced by the order topology of ω_1. Let $\tilde{\mu}$ be Dieudonné's measure on ω_1 (411Q), and let $\tilde{\mu}_X$, $\tilde{\mu}_Y$ be the subspace measures induced on X and Y by $\tilde{\mu}$; let μ and ν be the restrictions of $\tilde{\mu}_X$, $\tilde{\nu}_Y$ to the Baire σ-algebras of X, Y respectively. Then

$$\mu X = \tilde{\mu}_X X = \tilde{\mu}^* X = 1$$

because X meets every cofinal closed set in ω_1; similarly, $\nu Y = 1$.

Set

$$W = \{(x,y) : x \in X,\, y \in Y,\, x < y\} = \{(x,y) : x \in X,\, y \in Y,\, x \leq y\}.$$

Then W is open-and-closed in $X \times Y$ (use 4A2Rl), so that $f = \chi W$ is continuous. But

$$\iint f(x,y)\nu(dy)\mu(dx) = \int \nu\{y : y \in Y,\, x < y\}\mu(dx) = 1,$$

$$\iint f(x,y)\mu(dx)\nu(dy) = \int \mu\{x : x \in X,\, x < y\}\nu(dy) = 0.$$

Remark Contrast this with 434Xx and 439Yh.

439S The results of 437W leave open the question of which familiar spaces, beyond Čech-complete spaces, can be Prokhorov. In fact rather few are. The basis of any further investigation must be the following result.

Theorem (Preiss 73) $\mathbb{Q}$ is not a Prokhorov space.

proof (a) There is a non-decreasing sequence $\langle X_k\rangle_{k\in\mathbb{N}}$ of non-empty compact subsets of $X = \mathbb{Q} \cap [0,1]$, with union X, such that whenever $k \in \mathbb{N}$, $x \in X_k$ and $\delta > 0$, then $X_{k+1} \cap [x-\delta, x+\delta]$ is infinite. **P** Start by enumerating X as $\langle q_k\rangle_{k\in\mathbb{N}}$. Set $X_0 = \{q_0\}$. Given that $X_k \subseteq X$ is compact, then for each $m \in \mathbb{N}$ let $\mathcal{E}_m$ be a finite cover of X_k by open intervals of length at most 2^{-m} all meeting X_k, and let I_{km} be a finite subset of $X \setminus X_k$ meeting every member of $\mathcal{E}_m$; set $X_{k+1} = X_k \cup \{q_{k+1}\} \cup \bigcup_{m\in\mathbb{N}} I_{km}$. If $\mathcal{H}$ is any cover of X_{k+1} by open sets in $\mathbb{R}$, then there is a finite $\mathcal{H}_0 \subseteq \mathcal{H}$ covering X_k. There must be an $m \in \mathbb{N}$ such that $[x-2^{-m}, x+2^{-m}] \subseteq \bigcup \mathcal{H}_0$ for every $x \in X_k$ (2A2Ed), so that $I_{kl} \subseteq \bigcup \mathcal{H}_0$ for every $l \geq m$, and $X_{k+1} \setminus \bigcup \mathcal{H}_0$ is finite; accordingly there is a finite $\mathcal{H}_1 \subseteq \mathcal{H}$ covering X_{k+1}. As $\mathcal{H}$ is arbitrary, X_{k+1} is compact, and the induction can proceed. If $x \in X_k$ and $\delta > 0$, then for every $m \in \mathbb{N}$ there is an $x' \in X_{k+1} \setminus X_k$ such that $|x' - x| \leq 2^{-m}$, so that $[x-\delta, x+\delta] \cap X_{k+1}$ must be infinite. **Q**

(b) If $\langle\epsilon_k\rangle_{k\in\mathbb{N}}$ is any sequence in $]0,\infty[$, and $F \subseteq [0,1]$ is a countable closed set, then there is an $x^* \in X \setminus F$ such that $\rho(x^*, X_k) < \epsilon_k$ for every $k \in \mathbb{N}$. **P** We can suppose that $\lim_{k\to\infty} \epsilon_k = 0$. Define $\langle H_k\rangle_{k\in\mathbb{N}}$ inductively, as follows. $H_0 = \mathbb{R}$. Given H_k, set $H_{k+1} = H_k \cap \{x : \rho(x, X_k \cap H_k) < \epsilon_k\}$, where $\rho(x,A) = \inf_{y\in A} |x-y|$ for $x \in \mathbb{R}$, $A \subseteq \mathbb{R}$. Observe that every H_k is an open subset of $\mathbb{R}$ and that $X_k \cap H_k \subseteq H_{k+1} \subseteq H_k$ for every k; consequently, setting $E = \bigcap_{k\in\mathbb{N}} H_k$, E is a G_δ subset of $\mathbb{R}$ and $X_k \cap H_k \subseteq E$ for every k. In particular, $E \cap X$ contains q_0 and is not empty. Next, for each k, $\rho(x, E\cap X_k) < \epsilon_k$ for every $x \in H_{k+1}$ and therefore for every $x \in E$; accordingly $E \cap X$ is dense in E. Moreover, if $x \in E \cap X$, there is a $k \in \mathbb{N}$ such that $x \in X_k$; we must have $x \in H_k$, and in this case H_{k+1} is a neighbourhood of x. So every neighbourhood of x contains infinitely many points of $H_{k+1} \cap X_{k+1} \subseteq E \cap X$. Thus $E \cap X$ has no isolated points; it follows that E has no isolated points. By 4A2Mc and 4A2Me, E is uncountable.

There is therefore a point $z \in E \setminus F$. Let $m \in \mathbb{N}$ be such that $\rho(z,F) \geq \epsilon_m$. As $z \in H_{m+1}$, there is an $x^* \in H_m \cap X_m$ such that $|z - x^*| < \epsilon_m$ and $x^* \notin F$. Let $k \in \mathbb{N}$. If $k \geq m$ then certainly $\rho(x^*, X_k) = 0 < \epsilon_k$. If $k < m$ then $x^* \in H_{k+1}$ so $\rho(x^*, X_k) \leq \rho(x^*, H_k \cap X_k) < \epsilon_k$. So we have a suitable x^*. **Q**

(c) For n, $k \in \mathbb{N}$ set

$$G_{kn} = \{x : x \in \mathbb{R} \setminus X_k,\ \rho(x, X_n) > 2^{-k}\}.$$

Then G_{kn} is an open subset of $\mathbb{R}$. Let A be the set of Radon probability measures μ on X such that $\mu(G_{kn} \cap X) \leq 2^{-n}$ for all n, $k \in \mathbb{N}$.

(d) Write $\tilde{A}$ for the set of Radon probability measures μ on $[0,1]$ such that $\mu(G_{kn} \cap [0,1]) \leq 2^{-n}$ for all k, $n \in \mathbb{N}$. Then $\tilde{A}$ is a narrowly closed subset of the set of Radon probability measures on $[0,1]$, which is itself narrowly compact (437O). Also $\mu([0,1] \setminus X) = 0$ for every $\mu \in \tilde{A}$. **P** Let set $K \subseteq [0,1] \setminus X$ be compact, and $n \in \mathbb{N}$. Then K and X_n are disjoint compact sets, so there is some $k \in \mathbb{N}$ such that $|x - y| > 2^{-k}$ for every $x \in X_n$ and $y \in K$. In this case $K \subseteq G_{kn}$ so $\mu K \leq 2^{-n}$. As n is arbitrary, $\mu K = 0$; as K is arbitrary, $\mu([0,1] \setminus X) = 0$. **Q**

A is compact in the narrow topology. **P** The identity map $\phi : X \to [0,1]$ induces a map $\tilde{\phi} : M^+_{\mathrm{P}}(X) \to M^+_{\mathrm{P}}([0,1])$ which is a homeomorphism between $M^+_{\mathrm{P}}(X)$ and $\{\mu : \mu \in M^+_{\mathrm{P}}([0,1]),\ \mu([0,1] \setminus X) = 0\}$ (437Qb). The definition of A makes it plain that it is $\tilde{\phi}^{-1}[\tilde{A}]$; since $\tilde{A} \subseteq \{\mu : \mu \in M^+_{\mathrm{P}}([0,1]),\ \mu([0,1] \setminus X) = 0\}$, $\tilde{\phi}{\restriction}A$ is a homeomorphism between A and $\tilde{A}$, and A is compact. **Q**

(e) A, regarded as a subset of $M^+_{\mathrm{P}}(X)$, is not uniformly tight. **P** Let $K \subseteq X$ be compact. Consider the set C of those $w \in [0,1]^X$ such that $w(x) = 0$ for every $x \in K$, $\sum_{x \in X} w(x) \leq 1$ and $\sum_{x \in G_{kn} \cap X} w(x) \leq 2^{-n}$ for all k, $n \in \mathbb{N}$. Then C is a compact subset of $[0,1]^X$. If $D \subseteq C$ is any non-empty upwards-directed set, then $\sup D$, taken in $[0,1]^X$, belongs to C. By Zorn's Lemma, C has a maximal member w say. **?** Suppose, if possible, that $\sum_{x \in X} w(x) = \gamma < 1$. For each $n \in \mathbb{N}$, let $L_n \subseteq X$ be a finite set such that $\sum_{x \in L_n} w(x) \geq \gamma - 2^{-n-1}$, and $m_n \in \mathbb{N}$ such that $L_n \subseteq X_{m_n}$. By (b), there is an $x^* \in X \setminus K$ such that $\rho(x^*, X_n) < 2^{-m_n}$ for every $n \in \mathbb{N}$. Let $r \in \mathbb{N}$ be such that $x^* \in X_r$ and $\gamma + 2^{-r} \leq 1$, and set $w'(x^*) = w(x^*) + 2^{-r}$, $w'(x) = w(x)$ for every $x \in X \setminus \{x^*\}$. Then certainly $w' \in [0,1]^X$ and $\sum_{x \in X} w'(x) \leq 1$. If k, $n \in \mathbb{N}$ and $x^* \notin G_{kn}$, then $\sum_{x \in G_{kn} \cap X} w'(x) = \sum_{x \in G_{kn} \cap X} w(x) \leq 2^{-n}$. If $x^* \in G_{kn}$, then $n < r$ and $2^{-k} < \rho(x^*, X_n) < 2^{-m_n}$, so $m_n < k$ and $L_n \subseteq X_k$ and

$$\textstyle\sum_{x \in G_{kn} \cap X} w(x) \leq \sum_{x \in X \setminus X_k} w(x) \leq \sum_{x \in X \setminus L_n} w(x) \leq 2^{-n-1},$$

$$\textstyle\sum_{x \in G_{kn} \cap X} w'(x) \leq 2^{-n-1} + 2^{-r} \leq 2^{-n}.$$

Thus $w' \in C$ and w was not maximal. **X**

Accordingly $\sum_{x \in X} w(x) = 1$ and the point-supported measure μ defined by w is a probability measure on X. By the definition of C, $\mu \in A$ and $\mu(X \setminus K) = 1$. As K is arbitrary, A cannot be uniformly tight. **Q**

(f) Thus A witnesses that $X = \mathbb{Q} \cap [0,1]$ is not a Prokhorov space. Since X is a closed subset of $\mathbb{Q}$, 437Wb tells us that $\mathbb{Q}$ is not a Prokhorov space.

439X Basic exercises (a) (i) Show that there is a set $A \subseteq [0,1]$ such that $\mu_L^* A = 1$, where μ_L is Lebesgue measure, and every member of $[0,1]$ is uniquely expressible as $a + q$ where $a \in A$, $q \in \mathbb{Q}$. (*Hint*: use an idea from 419J.) (ii) Define $f : [0,1] \to A$ by setting $f(x) = a$ when $x \in a + \mathbb{Q}$. Show that the image measure $\mu_L f^{-1}$ takes only the values 0 and 1. (Aldaz 95. Compare 342Xg.)

(b) Let X be a Radon Hausdorff space and A a subset of X. Show that A is universally negligible iff $\mu A = 0$ for every atomless Radon measure on X.

>(c) Let X be a Hausdorff space. Show that a set $A \subseteq X$ is universally negligible iff $\mu A = 0$ whenever μ is a complete locally determined topological measure on X such that $\mu\{x\} = 0$ for every $x \in X$.

(d) Let X be a Hausdorff space. Show that any universally negligible subset of X is universally measurable in the sense of 434D.

(e) (i) Show that there is an analytic set $A \subseteq \mathbb{R}$ such that for any Borel subset E of $\mathbb{R} \setminus A$ there is an uncountable Borel subset of $\mathbb{R} \setminus (A \cup E)$. (*Hint*: 423Qb, part (c) of the proof of 423L.) (ii) Show that A is universally measurable, but there is no Borel set E such that $A \triangle E$ is universally negligible.

(f) Show that a first-countable compact Hausdorff space is universally negligible iff it is scattered iff it is countable.

(g) Show that the product of two universally negligible Hausdorff spaces is universally negligible.

(h) Let us say that a Hausdorff space X is **universally τ-negligible** if there is no τ-additive Borel probability measure on X which is zero on singletons. (i) Show that if X is a Hausdorff space and $A \subseteq X$, then A is universally τ-negligible iff $\mu^* A = 0$ for every τ-additive Borel probability measure on X such that $\mu\{x\} = 0$ for every $x \in X$. (ii) Show that if X is a regular Hausdorff space, then a subset A of X is universally τ-negligible iff $\mu A = 0$ for every atomless quasi-Radon measure on X. (iii) Show that if X is a completely regular Hausdorff space, it is universally τ-negligible iff whenever μ is an atomless Radon measure on a space Z, and $X' \subseteq Z$ is homeomorphic to X, then $\mu X' = 0$. (iv) Show that a Hausdorff space X is universally negligible iff it is Borel-measure-complete and universally τ-negligible. (v) Show that if X is a Hausdorff space, Y is a universally τ-negligible Hausdorff space, and $f : X \to Y$ is a continuous function such that $f^{-1}[\{y\}]$ is universally τ-negligible for every $y \in Y$, then X is universally τ-negligible. (vi) Show that the product of two universally τ-negligible Hausdorff spaces is universally τ-negligible. (vi) Show that a scattered Hausdorff space (in particular, any discrete space) is universally τ-negligible. (vii) Show that a compact Hausdorff space is universally τ-negligible iff it is scattered.

>(i) Let κ be the smallest cardinal of any subset of $\mathbb{R}$ which is not Lebesgue negligible. Show that if (Z, T, ν) is any complete locally determined atomless measure space and $A \subseteq Z$ has cardinal less than κ, then $\nu A = 0$.

(j) Let $(X, \leq)$ be any well-ordered set and μ a non-zero σ-finite measure on X such that every singleton has measure 0. Show that $\{(x, y) : x \leq y\}$ is not measurable for the (c.l.d.) product measure on $X \times X$. (*Hint*: Reduce to the case in which μ is complete and totally finite, $X = \zeta$ is an ordinal and $\mu\xi = 0$ for every $\xi < \zeta$. You will probably need 251P.)

(k) Show that 1-dimensional Hausdorff measure on $\mathbb{R}^2$ is not inner regular with respect to the closed sets. (*Hint*: 439H, 471R.)

(l) Show that $\mathbb{N}^I$ is not pre-Radon for any uncountable set I. (*Hint*: 417Xq.)

(m) (i) Suppose that X is a completely regular Hausdorff space and there is a continuous function f from X to a separable metrizable space Z such that $f^{-1}[\{z\}]$ is Lindelöf for every $z \in Z$. Show that X is realcompact (definition: 436Xg). (ii) Show that the spaces X of 439K and X^2 of 439Q are realcompact. (iii) Show that $\mathfrak{c}$ with the discrete topology, and $\mathbb{N}^{\mathfrak{c}}$ with the product topology, are realcompact.

439Y Further exercises (a) (i) Show that a subset A of $\mathbb{R}$ is universally negligible iff $f[A]$ is Lebesgue negligible for every continuous injective function $f : \mathbb{R} \to \mathbb{R}$. (*Hint*: if ν is an atomless Borel probability measure on $\mathbb{R}$, set $f(x) = x + \nu[0, x]$ for $x \geq 0$, and show that $\mu_L f[E] \geq \nu E$ for every Borel set $E \subseteq [0, \infty[$.) (ii) Let X be a separable metrizable space. Show that X is universally negligible iff $f[X]$ is Lebesgue negligible for every injective Borel measurable function $f : X \to \mathbb{R}$.

(b) For this exercise only, let us say that a 'universally negligible measurable space' is a pair (X, Σ) where X is a set and Σ a σ-algebra of subsets of X containing every countable subset of X such that there is no probability measure μ with domain Σ such that $\mu\{x\} = 0$ for every $x \in X$. (i) Let X be a set, Σ a σ-algebra of subsets of X containing all countable subsets of X, $A \subseteq X$ and Σ_A the subspace σ-algebra. Show that (A, Σ_A) is universally negligible iff $\mu^* A = 0$ whenever μ is a probability measure with domain Σ which is zero on singletons. Show that if (X, Σ) is universally negligible so is (A, Σ_A). (ii) Let X and Y be sets, Σ and T σ-algebras of subsets of X and Y containing all appropriate countable sets, and $f : X \to Y$ a (Σ, T)-measurable function. Suppose that (Y, T) and $(f^{-1}[\{y\}], \Sigma_{f^{-1}[\{y\}]})$ are universally negligible for every $y \in Y$. Show that (X, Σ) is universally negligible. (iii) Let X be a set and Σ a σ-algebra of subsets of X containing all countable subsets of X. Show that the set of those $A \subseteq X$ such that (A, Σ_A) is universally negligible is a σ-ideal of subsets of X.

(c) Let X be an analytic space and A an analytic subset of X. Show that $X \setminus A$ is universally negligible iff all the constituents of $X \setminus A$ (for any Souslin scheme defining A) are countable.

(d) (i) Let X be a metrizable space such that $f[X]$ is Lebesgue negligible for every continuous function $f : X \to \mathbb{R}$. Show that X is universally negligible. (ii) Let X be a completely regular Hausdorff space such that $f[X]$ is Lebesgue negligible for every continuous function $f : X \to \mathbb{R}$. Show that X is universally τ-negligible.

(e) Let $\mathfrak{T}_{\mathfrak{c}}$ be the topology on $\{0,1\}^{\mathbb{N}}$ constructed in the proof of 439K. (i) Show that it is normal. (ii) Show that any $\mathfrak{T}_{\mathfrak{c}}$-zero set is an $\mathfrak{S}$-Borel set.

(f) Show that there is no atomless Borel probability measure on ω_1 endowed with its order topology. (*Hint*: 411R, 439N.)

(g) Show that the Sorgenfrey right-facing topology on $\mathbb{R}$ is hereditarily Lindelöf, but that its square is not Lindelöf.

(h) Show that if ω_1 is given its order topology, and $f : \omega_1^2 \to \mathbb{R}$ is continuous, then there is a $\zeta < \omega_1$ such that f is constant on $(\omega_1 \setminus \zeta)^2$. (*Hint*: 4A2S(b-iii).) Show that if μ and ν are Baire probability measures on ω_1, then the Baire probability measures $\mu \times \nu$, $\nu \times \mu$ on ω_1^2 defined by the formulae of 436F coincide.

(i) Let $X \subseteq [0,1]$ be a dense set with no uncountable compact subset. Show that X is not a Prokhorov space.

439 Notes and comments I give three separate constructions in 439A because the phenomenon here is particularly important. For two chapters I have, piecemeal, been offering theorems on the extension of measures. The principal ones to date seem to be 413O, 415L, 416M, 417C, 417E and 435C, and I have used methods reflecting my belief that the essential feature on which each such theorem depends is inner regularity of an appropriate kind. I think we should simultaneously seek to develop an intuition for measures which do *not* extend, and those in 439A are especially significant because they refer to the Borel algebra of the unit interval, which in so many other contexts is comfortably clear of the obstacles which beset more exotic structures.

Outside the context of Polish spaces, the terms 'universally measurable' and 'universally negligible' are not properly settled. I have tried to select definitions which lead to a reasonable pattern. At least a universally negligible subset of a Hausdorff space is universally measurable (439Xd), and both concepts can be expressed in terms of sets with σ-algebras, as in 439Yb. It is important to notice, in 439B, that I write '$\mu\{x\} = 0$ for every $x \in X$', not 'μ is atomless'. For instance, Dieudonné's measure shows that ω_1, with its order topology, is not universally negligible on the definition here; but it is easy to show that there is no atomless Borel probability measure on ω_1 (439Yf). In many cases, of course, we do not need to make this distinction (439D).

The cardinal κ of 439F (the 'uniformity' of the Lebesgue null ideal) is one of a large family of cardinals which will be examined in Chapter 52 in the next volume.

In some of the arguments above (439J, 439L, 439O) I appeal to (different) principles ('there is a cardinal which is not measure-free', ♣) which are not theorems according to the rules I follow in this book. Such examples would in some ways fit better into Volume 5, where I mean to investigate such principles properly. I include the examples here because they do at least exhibit bounds on what can be proved in ZFC. I should not want anyone to waste her time trying to show, for instance, that all completion regular Borel measures are τ-additive. Nevertheless, the absence of a 'real' counter-example (obviously we want a probability measure on a completely regular Hausdorff space) remains in my view a significant gap. It remains conceivable that there is a mathematical world in which no such space exists. Clearly the discovery of such a world is likely to require familiarity with the many worlds already known, and I am not going to embark on any such exploration in this volume. On the other hand, it is also very possible that all we need is a bit of extra ingenuity to construct a counter-example in ZFC. In this section we have two examples of successes of this kind. In 439F-439H, for instance, we have results which were long known as consequences of the continuum hypothesis; the particular insight of Grzegorek 81 was the observation that they depended on

determinate properties of the cardinal κ of 439F, and that its indeterminate position between ω_1 and $\mathfrak{c}$ was unimportant. In 439K I show how a re-working of ideas in JUHÁSZ KUNEN & RUDIN 76, where a similar example was constructed (for an entirely different purpose) assuming the continuum hypothesis, provides us with an interesting space (a first-countable non-Radon compact Hausdorff space) in ZFC. Let me emphasize that these ideas were originally set out in a framework supported by an extra axiom, where some technical details were easier and the prize aimed at (a non-Lindelöf hereditarily separable space) more important.

The examples in 439K-439R are mostly based on constructions more or less familiar from general topology. I have already mentioned the origins of 439K. 439M is related to the Tychonoff and Dieudonné planks (STEEN & SEEBACH 78, §§86-89). 439N and 439Q revisit yet again ω_1 and the Sorgenfrey line. 439O is adapted from one of the standard constructions of Dowker spaces. Products of disjoint stationary sets (439R) have also been used elsewhere.

Chapter 44

Topological groups

Measure theory begins on the real line, which is of course a group; and one of the most fundamental properties of Lebesgue measure is its translation-invariance (134A). Later we come to the standard measure on the unit circle (255M), and counting measure on the integers is also translation-invariant, if we care to notice; moreover, Fourier series and transforms clearly depend utterly on the fact that shift operators don't disturb the measure-theoretic structures we are building. Yet another example appears in the usual measure on $\{0,1\}^I$, which is translation-invariant if we identify $\{0,1\}^I$ with the group $\mathbb{Z}_2^I$ (345Ab). Each of these examples is special in many other ways. But it turns out that a particular combination of properties which they share, all being locally compact Hausdorff spaces with group operations for which multiplication and inversion are continuous, is the basis of an extraordinarily powerful theory of invariant measures.

As usual, I have no choice but to move rather briskly through a wealth of ideas. The first step is to set out a suitably general existence theorem, assuring us that every locally compact Hausdorff topological group has non-trivial invariant Radon measures, that is, 'Haar measures' (441E). As remarkable as the existence of Haar measures is their (essential) uniqueness (442B); the algebra, topology and measure theory of a topological group are linked in so many ways that they form a peculiarly solid structure.

I have already mentioned that Fourier analysis depends on the translation-invariance of Lebesgue measure. It turns out that substantial parts of the abstract theory of Fourier series and transforms can be generalized to arbitrary locally compact groups. In particular, convolutions (§255) appear again, even in non-commutative groups (§444). But for the main part of the theory, a transform relating functions on a group X to functions on its 'dual' group $\mathcal{X}$, we do need the group to be abelian. Actually I give only the foundation of this theory: if X is an abelian locally compact Hausdorff group, it is the dual of its dual (445U). (In 'ordinary' Fourier theory, where we are dealing with the cases $X = \mathcal{X} = \mathbb{R}$ and $X = S^1$, $\mathcal{X} = \mathbb{Z}$, this duality is so straightforward that one hardly notices it.) But on the way to the duality theorem we necessarily see many of the themes of Chapter 28 in more abstract guises.

A further remarkable fact is that any Haar measure has a translation-invariant lifting (447J). The proof demands a union between the ideas of the ordinary Lifting Theorem (§341) and some of the elaborate structure theory which has been developed for locally compact groups (§446).

For the last two sections of the chapter, I look at groups which are not locally compact. For the most important such groups, the Polish groups, we have a natural necessary and sufficient condition for the existence of a translation-invariant measure (448P); as in the locally compact case, this is best expressed as a theorem about measures invariant under group actions. (In fact it is the case that any topological group carrying a non-trivial translation-invariant quasi-Radon measure is very nearly locally compact (443L).) In a slightly different direction, we can look at those groups, the 'amenable' groups, for which all actions (on compact Hausdorff spaces) have invariant measures. This again leads to some very remarkable ideas, which I sketch in §449.

441 Invariant measures on locally compact spaces

I begin this chapter with the most important theorem on the existence of invariant measures: every locally compact Hausdorff group has left and right Haar measures (441E). I derive this as a corollary of a general result concerning invariant measures on locally compact spaces (441C), which has other interesting consequences (441H).

441A Group actions I recall a fundamental definition from group theory.

(a) If G is a group and X is a set, an **action** of G on X is a function $(a,x) \mapsto a\bullet x : G \times X \to X$ such that

$$(ab)\bullet x = a\bullet(b\bullet x) \text{ for all } a, b \in G, x \in X,$$

$$e\bullet x = x \text{ for every } x \in X$$

where e is the identity of G (4A5B). In this context I write

$$a\bullet A = \{a\bullet x : x \in A\}$$

for $a \in G$, $A \subseteq X$.

(b) If a group G acts on a set X, a measure μ on X is **G-invariant** if $\mu(a^{-1}\bullet E)$ is defined and equal to μE whenever $a \in G$ and μ measures E.

(Of course this is the same thing as saying that $\mu(a\bullet E) = \mu E$ for every $a \in G$ and measurable set E; I use the formula with a^{-1} so as to match my standard practice when a is actually a function from X to X.)

(c) It will be convenient if I immediately introduce the following notation. Let $\bullet$ be an action of a group G on a set X. If f is any function defined on a subset of X, and $a \in G$, write $a\bullet f$ for the function defined by saying that $(a\bullet f)(x) = f(a^{-1}\bullet x)$ whenever $x \in X$ and $a^{-1}\bullet x \in \operatorname{dom} f$. It is easy to check that this defines an action of G on the class of all functions with domains included in X.

If $X = G$, we have left, right and conjgacy actions $\bullet_l$, $\bullet_r$ and $\bullet_c$ of G on itself (4A5Ca). So we have corresponding actions $\bullet_l$, $\bullet_r$ and $\bullet_c$ of G on the class of functions with domains included in G;

$$(a\bullet_l f)(x) = f(a^{-1}x), \quad (a\bullet_r f)(x) = f(xa), \quad (a\bullet_c f)(x) = f(a^{-1}xa)$$

whenever these are defined.

(d) If $\bullet$ is an action of a group G on a set X, and $Y \subseteq X$ is G-invariant in the sense that $a\bullet y \in Y$ whenever $a \in G$ and $y \in Y$, then $\bullet\restriction G \times Y$ is an action of G on Y. In the context of (c) above, this means that if V is any set of functions with domains included in X such that $a\bullet f \in V$ whenever $a \in G$ and $f \in V$, then we have an action of G on V.

441B It will be useful later to be able to quote the following elementary results.

Lemma Let X be a topological space, G a group, and $\bullet$ an action of G on X such that $x \mapsto a\bullet x$ is continuous for every $a \in G$.

(a) If μ is a quasi-Radon measure on X such that $\mu(a\bullet U) \leq \mu U$ for every open set $U \subseteq X$ and every $a \in G$, then μ is G-invariant.

(b) If μ is a Radon measure on X such that $\mu(a\bullet K) \leq \mu K$ for every compact set $K \subseteq X$ and every $a \in G$, then μ is G-invariant.

proof Note first that the maps $x \mapsto a\bullet x$ are actually homeomorphisms (4A5Bd), so that $a\bullet U$ and $a\bullet K$ will be open, or compact, as U and K are. Next, the inequality $\leq$ in the hypotheses is an insignificant refinement; since we must also have

$$\mu(a\bullet U) \leq \mu(a^{-1}\bullet a\bullet U) = \mu U$$

in (a),

$$\mu(a\bullet K) \leq \mu(a^{-1}\bullet a\bullet K) = \mu K$$

in (b), we always have equality here.

Now fix $a \in G$, and set $T_a(x) = a\bullet x$ for $x \in X$. Then T_a is a homeomorphism, so the image measure μT_a^{-1} will be quasi-Radon, or Radon, if μ is. In (a), we are told that μT_a^{-1} agrees with μ on the open sets, while in (b) we are told that they agree on the compact sets; so in both cases we have $\mu = \mu T_a^{-1}$, by 415H(iii) or 416E(b-ii). Consequently we have $\mu T_a^{-1}[E] = \mu E$ whenever μ measures E. As a is arbitrary, μ is G-invariant.

441C Theorem (Steinlage 75) Let X be a non-empty locally compact Hausdorff space and G a group acting on X. Suppose that

(i) $x \mapsto a\bullet x$ is continuous for every $a \in G$;

(ii) every orbit $\{a\bullet x : a \in G\}$ is dense;

(iii) whenever K and L are disjoint compact subsets of X there is a non-empty open subset U of X such that, for every $a \in G$, at most one of K, L meets $a\bullet U$.

Then there is a non-zero G-invariant Radon measure μ on X.

proof (a) $\bigcup_{a\in G} a\bullet U = X$ for every non-empty open $U \subseteq X$. **P** If $x \in X$, then the orbit of x must meet U, so there is a $a \in G$ such that $a\bullet x \in U$; but this means that $x \in a^{-1}\bullet U$. **Q**

Fix some point z_0 of X and write $\mathcal{V}$ for the set of open sets containing z_0. Then if K, L are disjoint compact subsets of X there is a $U \in \mathcal{V}$ such that, for every $a \in G$, at most one of K, L meets $a\bullet U$. **P** By hypothesis, there is a non-empty open set V such that, for every $a \in G$, at most one of K, L meets $a\bullet V$. Now there is an $b \in G$ such that $b\bullet z_0 \in V$; set $U = b^{-1}\bullet V$. Because b^{-1} acts on X as a homeomorphism, $U \in \mathcal{V}$; and if $a \in G$, then $a\bullet U = (ab^{-1})\bullet V$ can meet at most one of K and L. **Q**

(b) If $U \in \mathcal{V}$ and $A \subseteq X$ is any relatively compact set, then $\{a\bullet U : a \in G\}$ is an open cover of X, so there is a finite set $I \subseteq G$ such that $\overline{A} \subseteq \bigcup_{a\in I} a\bullet U$. Write $\lceil A : U\rceil$ for $\min\{\#(I) : I \subseteq G, A \subseteq \bigcup_{a\in I} a\bullet U\}$.

(c) The following facts are now elementary.

(i) If $U \in \mathcal{V}$ and A, $B \subseteq X$ are relatively compact, then

$$0 \leq \lceil A : U\rceil \leq \lceil A \cup B : U\rceil \leq \lceil A : U\rceil + \lceil B : U\rceil,$$

and $\lceil A : U\rceil = 0$ iff $A = \emptyset$.

(ii) If U, $V \in \mathcal{V}$ and V is relatively compact, and $A \subseteq X$ also is relatively compact, then

$$\lceil A : U\rceil \leq \lceil A : V\rceil \lceil V : U\rceil.$$

P If $A \subseteq \bigcup_{a\in I} a\bullet V$ and $V \subseteq \bigcup_{b\in J} b\bullet U$, then $A \subseteq \bigcup_{a\in I, b\in J} (ab)\bullet U$. **Q**

(iii) If $U \in \mathcal{V}$, $A \subseteq X$ is relatively compact and $b \in G$, then $\lceil b\bullet A : U\rceil = \lceil A : U\rceil$. **P** If $I \subseteq G$ and $A \subseteq \bigcup_{a\in I} a\bullet U$, then $b\bullet A \subseteq \bigcup_{a\in I} (ba)\bullet U$, so $\lceil b\bullet A : U\rceil \leq \#(I)$; as I is arbitrary, $\lceil b\bullet A : U\rceil \leq \lceil A : U\rceil$. On the other hand, the same argument shows that

$$\lceil A : U\rceil \leq \lceil b^{-1}\bullet b\bullet A : U\rceil = \lceil A : U\rceil,$$

so we must have equality. **Q**

(d) Fix a relatively compact $V_0 \in \mathcal{V}$. (This is the first place where we use the hypothesis that X is locally compact.) For every $U \in \mathcal{V}$ and every relatively compact set $A \subseteq X$ write

$$\lambda_U A = \frac{\lceil A : U\rceil}{\lceil V_0 : U\rceil}.$$

Then (c) tells us immediately that

(i) if A, $B \subseteq X$ are relatively compact,

$$0 \leq \lambda_U A \leq \lambda_U(A \cup B) \leq \lambda_U A + \lambda_U B;$$

(ii) $\lambda_U A \leq \lceil A : V_0\rceil$ for every relatively compact $A \subseteq X$;

(iii) $\lambda_U(b\bullet A) = \lambda_U A$ for every relatively compact $A \subseteq X$ and every $b \in G$;

(iv) $\lambda_U V_0 = 1$.

(e) Now for the point of the hypothesis (ii) of the theorem. If K, L are disjoint compact subsets of X, there is a $V \in \mathcal{V}$ such that $\lambda_U(K \cup L) = \lambda_U K + \lambda_U L$ whenever $U \in \mathcal{V}$ and $U \subseteq V$. **P** By (a), there is a $V \in \mathcal{V}$ such that any translate $a\bullet V$ can meet at most one of K and L. Take any $U \in \mathcal{V}$ included in V. Let $I \subseteq G$ be such that $\bigcup_{a\in I} a\bullet U \supseteq K \cup L$ and $\#(I) = \lceil K \cup L : U\rceil$. Then

$$I' = \{a : a \in I, K \cap a\bullet U \neq \emptyset\}, \quad I'' = \{a : a \in I, L \cap a\bullet U \neq \emptyset\}$$

are disjoint. $K \subseteq \bigcup_{a\in I'} a\bullet U$, so $\lceil K : U\rceil \leq \#(I')$, and similarly $\lceil L : U\rceil \leq \#(I'')$. But this means that

$$\lceil K : U\rceil + \lceil L : U\rceil \leq \#(I') + \#(I'') \leq \#(I) = \lceil K \cup L : U\rceil,$$

$$\lambda_U K + \lambda_U L \leq \lambda_U(K \cup L).$$

As we already know that $\lambda_U(K \cup L) \leq \lambda_U K + \lambda_U L$, we must have equality, as claimed. **Q**

(f) Now let $\mathcal{F}$ be an ultrafilter on $\mathcal{V}$ containing all sets of the form $\{U : U \in \mathcal{V}, U \subseteq V\}$ for $V \in \mathcal{V}$. If $A \subseteq X$ is relatively compact, $0 \leq \lambda_U A \leq \lceil A : V_0\rceil$ for every $U \in \mathcal{V}$, so $\lambda A = \lim_{U\to\mathcal{F}} \lambda_U A$ is defined in $[0, \lceil A : U_0\rceil]$. From (d-i) and (d-iii) we see that

$$0 \leq \lambda(b\bullet A) = \lambda A \leq \lambda(A \cup B) \leq \lambda A + \lambda B$$

for all relatively compact A, $B \subseteq X$ and $b \in G$. From (d-iv) we see that $\lambda V_0 = 1$. Moreover, from (e) we see that if K, $L \subseteq X$ are disjoint compact sets,

$$\{U : U \in \mathcal{V},\ \lambda_U(K \cup L) = \lambda_U K + \lambda_U L\} \in \mathcal{F},$$

so $\lambda(K \cup L) = \lambda K + \lambda L$.

(g) By 416L, there is a Radon measure μ on X such that

$$\mu K = \inf\{\lambda L : L \subseteq X \text{ is compact, } K \subseteq \operatorname{int} L\}$$

for every compact set $K \subseteq X$. Now μ is G-invariant. **P** Take $b \in G$. If K, $L \subseteq X$ are compact and $K \subseteq \operatorname{int} L$, then $b\bullet K \subseteq \operatorname{int} b\bullet L$, because $x \mapsto b\bullet x$ is a homeomorphism; so

$$\mu(b\bullet K) \leq \lambda(b\bullet L) = \lambda L.$$

As L is arbitrary, $\mu(b\bullet K) \leq \mu K$. As K is arbitrary, μ is G-invariant, by 441Bb. **Q**

(h) Finally, $\mu V_0 \geq \lambda V_0 \geq 1$, so μ is non-zero.

441D The hypotheses of 441C are deliberately drawn as widely as possible. The principal application is the one for which the chapter is named.

Definition If G is a topological group, a **left Haar measure** on G is a non-zero quasi-Radon measure μ on G which is invariant for the left action of G on itself, that is, $\mu(aE) = \mu E$ whenever μ measures E and $a \in G$.

Similarly, a **right Haar measure** is a non-zero quasi-Radon measure μ such that $\mu(Ea) = \mu E$ for every $E \in \operatorname{dom}\mu$, $a \in G$.

(My reasons for requiring 'quasi-Radon' here will appear in §§442 and 443.)

441E Theorem A locally compact Hausdorff topological group has left and right Haar measures, which are both Radon measures.

proof Both the left and right actions of G on itself satisfy the conditions of 441C. **P** In both cases, condition (i) is just the (separate) continuity of multiplication, and (ii) is trivial, as every orbit is the whole of G. As for (iii), let us take the left action first. Given disjoint compact subsets K, L of G, then $M = \{y^{-1}z : y \in K,\ z \in L\}$ is a compact subset of G not containing the identity e. Because the topology is Hausdorff, M is closed and $X \setminus M$ is a neighbourhood of e. Because multiplication and inversion are continuous, there are open neighbourhoods V, V' of e such that $uv^{-1} \in G \setminus M$ whenever $u \in V$ and $v \in V'$. Set $U = V \cap V'$; then U is a non-empty open set in G.

? Suppose, if possible, that there is a $a \in G$ such that aU meets both K and L. Take $y \in K \cap aU$ and $z \in L \cap aU$. Then $a^{-1}y \in U \subseteq V$ and $a^{-1}z \in U \subseteq V'$, so

$$y^{-1}z = (a^{-1}y)^{-1}a^{-1}z \in G \setminus M;$$

but also $y^{-1}z \in M$. **X**

Thus aU meets at most one of K, L for any $a \in G$. As K and L are arbitrary, condition (ii) of 441C is satisfied.

For the right action, we use the same ideas, but vary the formulae. Set $M = \{yz^{-1} : y \in K,\ z \in L\}$, and choose V and V' such that $uv^{-1} \in X \setminus M$ for $u \in V$, $v \in V'$. Then if $a \in G$, $y \in K$ and $z \in L$, $za(ya)^{-1} \in M$ and one of za, ya does not belong to U, that is, one of z, y does not belong to $Ua^{-1} = a\bullet U$. **Q**

Then 441C provides us with non-zero left and right Haar measures on G, and also tells us that they are Radon measures.

441F A different type of example is provided by locally compact metric spaces.

Definition If (X, ρ) is any metric space, its **isometry group** is the set of bijections $g : X \to X$ which are **isometries**, that is, $\rho(g(x), g(y)) = \rho(x, y)$ for all x, $y \in X$.

441G The topology of an isometry group Let (X, ρ) be a metric space and G the isometry group of X.

(a) Give G the topology of pointwise convergence inherited from the product topology of X^X. Then G is a Hausdorff topological group and the action of G on X is continuous. **P** If $x \in X$, g_0, $h_0 \in G$ and $\epsilon > 0$, then $V = \{g : \rho(gh_0(x), g_0h_0(x)) \leq \frac{1}{2}\epsilon\}$ is a neighbourhood of g_0 and $V' = \{h : \rho(h(x), h_0(x)) \leq \frac{1}{2}\epsilon\}$ is a neighbourhood of h_0. If $g \in V$ and $h \in V'$ then

$$\rho(gh(x), g_0h_0(x)) \leq \rho(gh(x), gh_0(x)) + \rho(gh_0(x), g_0h_0(x)) \leq \rho(h(x), h_0(x) + \frac{1}{2}\epsilon \leq \epsilon.$$

As g_0, h_0 and ϵ are arbitrary, the function $(g, h) \mapsto gh(x)$ is continuous; as x is arbitrary, multiplication on G is continuous. As for inversion, suppose that $g_0 \in G$, $\epsilon > 0$ and $x \in X$. Then $V = \{g : \rho(gg_0^{-1}(x), x) \leq \epsilon\}$ is a neighbourhood of g_0, and if $g \in V$ then

$$\rho(g^{-1}(x), g_0^{-1}(x)) = \rho(x, gg_0^{-1}(x)) \leq \epsilon.$$

Because g_0, ϵ and x are arbitrary, inversion on G is continuous, and G is a topological group. Because X is Hausdorff, so is G.

To see that the action is continuous, take $g_0 \in G$, $x_0 \in X$ and $\epsilon > 0$. Then $V = \{g : g \in G, \rho(g(x_0), g_0(x_0)) < \frac{1}{2}\epsilon\}$ is a neighbourhood of g_0. If $g \in V$ and $x \in U(x_0, \frac{1}{2}\epsilon)$, then

$$\rho(g(x), g_0(x_0)) \leq \rho(g(x), g(x_0)) + \rho(g(x_0), g_0(x_0)) \leq \rho(x, x_0) + \frac{1}{2}\epsilon \leq \epsilon.$$

As g_0, x_0 and ϵ are arbitrary, $(g, x) \mapsto g(x) : G \times X \to X$ is continuous. **Q**

(b) If X is compact, so is G. **P** By Tychonoff's theorem (3A3J), X^X is compact. Suppose that $g \in X^X$ belongs to the closure of G in X^X. For any x, $y \in X$, the set $\{f : f \in X^X, \rho(f(x), f(y)) = \rho(x, y)\}$ is closed and includes G, so contains g; thus g is an isometry. **?** If $g[X] \neq X$, take $x \in X \setminus g[X]$ and set $x_n = g^n(x)$ for every $n \in \mathbb{N}$. Because g is continuous and X is compact, $g[X]$ is closed and there is some $\delta > 0$ such that $U(x, \delta) \cap g[X] = \emptyset$. But this means that

$$\rho(x_m, x_n) = \rho(g^m(x), g^m(x_{n-m})) = \rho(x, x_{n-m}) \geq \delta$$

whenever $m < n$, so that $\langle x_n \rangle_{n \in \mathbb{N}}$ can have no cluster point in X; which is impossible, because X is supposed to be compact. **X** This shows that g is surjective and belongs to G. As g is arbitrary, G is closed in X^X, therefore compact. **Q**

441H Theorem If (X, ρ) is a non-empty locally compact metric space with isometry group G, then there is a non-zero G-invariant Radon measure on X.

proof (a) Fix any $x_0 \in X$, and set $Z = \overline{\{g(x_0) : g \in G\}}$; then Z is a closed subset of X, so is in itself locally compact. Let H be the isometry group of Z.

(b) We need to know that $g \restriction Z \in H$ for every $g \in G$. **P** Because $g : X \to X$ is a homeomorphism,

$$g[Z] = \overline{\{gg'(x_0) : g' \in G\}} = Z,$$

so $g \restriction Z$ is a bijection from Z to itself, and of course it is an isometry, that is, belongs to H. **Q**

(c) Now Z and H satisfy the conditions of 441C.

P (i) is true just because all isometries are continuous.

(ii) Take $z \in Z$ and let U be a non-empty relatively open subset of Z. Then $U = Z \cap V$ for some open set $V \subseteq X$; as $Z \cap V \neq \emptyset$, there must be a $g_0 \in G$ such that $g_0(x_0) \in V$. At the same time, there is a sequence $\langle h_n \rangle_{n \in \mathbb{N}}$ in G such that $z = \lim_{n \to \infty} h_n(x_0)$. Now

$$\rho(g_0(x_0), g_0h_n^{-1}(z)) = \rho(h_n(x_0), z) \to 0,$$

so there is some n such that $g_0h_n^{-1}(z) \in V$; of course $g_0h_n^{-1}(z)$ also belongs to U, while $g_0h_n^{-1} \restriction Z$ belongs to H, by (b) above. As U is arbitrary, the H-orbit of z is dense in Z; as z is arbitrary, H satisfies condition (ii) of 441C.

(iii) Given that K and L are disjoint compact subsets of Z, there must be a $\delta > 0$ such that $\rho(y,z) \geq \delta$ for every $y \in K$, $z \in L$. Let U be the relatively open ball $\{z : z \in Z, \rho(z,x_0) < \frac{1}{2}\delta\}$. Then for any $h \in H$, $\rho(y,z) < \delta$ for any y, $z \in b\bullet U = h[U]$, so $h[U]$ cannot meet both K and L. **Q**

(d) 441C therefore provides us with a non-zero H-invariant Radon measure ν on Z. Setting $\mu E = \nu(E \cap Z)$ whenever $E \subseteq X$ and $E \cap Z \in \operatorname{dom}\nu$, it is easy to check that μ is G-invariant (using (b) again) and is a Radon measure on Z.

441I Remarks (a) Of course there is a degree of overlap between the cases above. In an abelian group, for instance, the left and right group actions necessarily give rise to the same invariant measures. If we take $X = \mathbb{R}^2$, it has a group structure (addition) for which we have invariant measures (e.g., Lebesgue measure); these are just the translation-invariant measures. But 441H tells us that we also have measures which are invariant under all isometries (rotations and reflections as well as translations); from where we now stand, there is no surprise remaining in the fact that Lebesgue measure is invariant under this much larger group. (Though if you look back at Chapter 26, you will see that a bare-handed proof of this takes a certain amount of effort.) If we turn next to the unit sphere $\{x : \|x\| = 1\}$ in $\mathbb{R}^3$, we find that there is no useful group structure, but it is a compact metric space, so carries invariant measures, e.g., two-dimensional Hausdorff measure.

(b) The arguments of 441C leave open the question of how far the invariant measures constructed there are unique. Of course any scalar multiple of an invariant measure will again be invariant. It is natural to give a special place to invariant probability measures, and call them 'normalized'; whenever we have a non-zero totally finite invariant measure we shall have an invariant probability measure. Counting measure on any set will be invariant under any action of any group, and it is natural to say that these measures also are 'normalized'; when faced with a finite set with two or more elements, we have to choose which normalization seems most reasonable in the context.

(c) We shall see in 442B that Haar measures (with a given handedness) are necessarily scalar multiples of each other. If, in 441C, there are non-trivial G-invariant subsets of X, we do not expect such a result. But there are interesting cases in which the question seems to be open.

441J Of course we shall be much concerned with integration with respect to invariant measures. The results we need are elementary corollaries of theorems already dealt with at length, but it will be useful to have them spelt out.

Proposition Let X be a set, G a group acting on X, and μ a G-invariant measure on X. If f is a real-valued function defined on a subset of X, and $g \in G$, then $\int f(x)\mu(dx) = \int f(a\bullet x)\mu(dx)$ if either integral is defined in $[-\infty,\infty]$.

proof Apply 235I to the inverse-measure-preserving functions $x \mapsto a\bullet x$ and $x \mapsto a^{-1}\bullet x$.

441K Theorem Let X be a set, G a group acting on X, and μ a G-invariant measure on X with measure algebra $\mathfrak{A}$.

(a) We have an action of G on $\mathfrak{A}$ defined by setting $a\bullet E^\bullet = (a\bullet E)^\bullet$ whenever $a \in G$ and μ measures E.

(b) We have an action of G on $L^0 = L^0(\mu)$ defined by setting $a\bullet f^\bullet = (a\bullet f)^\bullet$ for every $a \in G$, $f \in \mathcal{L}^0(\mu)$.

(c) For $1 \leq p \leq \infty$ the formula of (b) defines actions of G on $L^p = L^p(\mu)$, and $\|a\bullet u\|_p = \|u\|_p$ for every $u \in L^p$, $a \in G$.

proof (a) If E, $F \in \operatorname{dom}\mu$ and $E^\bullet = F^\bullet$, then (because $x \mapsto a^{-1}\bullet x$ is inverse-measure-preserving) $(a\bullet E)^\bullet = (a\bullet F)^\bullet$. So the given formula does define a function from $G \times \mathfrak{A}$ to $\mathfrak{A}$. It is now easy to check that it is an action.

(b) Let $f \in \mathcal{L}^0 = \mathcal{L}^0(\mu)$, $a \in G$. Set $\phi_a(x) = a^{-1}\bullet x$ for $x \in X$, so that $\phi_a : X \to X$ is inverse-measure-preserving. Then $a\bullet f = f\phi_a$ belongs to $\mathcal{L}^0$. If f, $g \in \mathcal{L}^0$ and $f =_{\text{a.e.}} g$, then $f\phi_a =_{\text{a.e.}} g\phi_a$, so $(a\bullet f)^\bullet = (a\bullet g)^\bullet$. This shows that the given formula defines a function from $G \times L^0$ to L^0, and again it is easy to see that it is an action.

(c) If $f \in \mathcal{L}^p(\mu)$ then

$$\int |a\bullet f|^p d\mu = \int |f(a^{-1}\bullet x)|^p d\mu = \int |f|^p d\mu$$

by 441J. So $a\bullet f \in \mathcal{L}^p(\mu)$ and $\|a\bullet f^\bullet\|_p = \|f^\bullet\|_p$. Thus we have a function from $G \times L_p$ to L^p, and once more it must be an action.

441L Proposition Let X be a locally compact Hausdorff space and G a group acting on X in such a way that $x \mapsto a\bullet x$ is continuous for every $a \in G$. If μ is a Radon measure on X, then μ is G-invariant iff $\int f(x)\mu(dx) = \int f(a\bullet x)\mu(dx)$ for every $a \in G$ and every continuous function $f : X \to \mathbb{R}$ with compact support.

proof For $a \in G$, set $T_a(x) = a\bullet x$ for every $x \in X$. Then $\nu_a = \mu T_a^{-1}$ is a Radon measure on X. If $f \in C_k(X)$, then

$$\int f d\nu_a = \int f T_a d\mu$$

by 235I. Now

$$\begin{aligned}
\mu \text{ is } G\text{-invariant} &\iff \nu_a = \mu \text{ for every } a \in G \\
&\iff \int f d\nu_a = \int f d\mu \text{ for every } a \in G,\ f \in C_k(X) \\
&\qquad \text{(416E(b-v))} \\
&\iff \int f T_a d\mu = \int f d\mu \text{ for every } a \in G,\ f \in C_k(X)
\end{aligned}$$

as claimed.

441X Basic exercises >(a) Let X be a set. (i) Show that there is a one-to-one correspondence between actions $\bullet$ of the group $\mathbb{Z}$ on X and bijections $f : X \to X$ defined by the formula $n\bullet x = f^n(x)$. (ii) Show that if $f : X \to X$ is a bijection, a measure μ on X is $\mathbb{Z}$-invariant for the corresponding action iff f and f^{-1} are both inverse-measure-preserving. (iii) Show that if X is a compact Hausdorff space and $\bullet$ is a continuous action of $\mathbb{Z}$ on X, then there is a $\mathbb{Z}$-invariant Radon probability measure on X. (*Hint*: 437S.)

(b) Let (X, T, ν) be a measure space and G a group acting on X. Set $\Sigma = \{E : E \subseteq X,\ g\bullet E \in \mathrm{T} \text{ for every } g \in G\}$, and for $E \in \Sigma$ set

$$\mu E = \sup\{\sum_{i=0}^{n} \nu(g_i\bullet F_i) : n \in \mathbb{N}, F_0, \dots, F_n \text{ are disjoint subsets of } E \text{ belonging to } \Sigma,\ g_i \in G \text{ for each } i \le n\}$$

(cf. 112Ye). Show that μ is a G-invariant measure on X.

(c) Let X be a topological space and G a group acting on X such that (α) all the maps $x \mapsto a\bullet x$ are continuous (β) all the orbits of G are dense. Show that any non-zero G-invariant quasi-Radon measure on X is strictly positive.

>(d) Let G be a compact Hausdorff topological group. (i) Show that its conjugacy classes are closed. (ii) Show that if K, $L \subseteq G$ are disjoint compact sets then $\{ac^{-1}da^{-1} : a \in G,\ c \in K,\ d \in L\}$ is a compact set not containing e, so that there is a neighbourhood U of e such that whenever $c^{-1}d \in U$ and $a \in G$ then either $aca^{-1} \notin K$ or $ada^{-1} \notin L$. (iii) Show that every conjugacy class of G carries a Radon probability measure which is invariant under the conjugacy action of G (definition: 4A5Ca).

(e) Let $(G, \cdot)$ be a topological group. (i) On G define a binary operation $\diamond$ by saying that $x \diamond y = y \cdot x$ for all x, $y \in G$. Show that $(G, \diamond)$ is a topological group isomorphic to $(G, \cdot)$, and that any element of G has the same inverse for either group operation. (ii) Suppose that μ is a left Haar measure on $(G, \cdot)$. Show

that μ is a right Haar measure on $(G,\diamond)$. (iii) Set $\phi(a)=a^{-1}$ for $a\in G$. Show that $\mu\phi^{-1}$ is a right Haar measure on $(G,\cdot)$. (iv) Show that $(G,\cdot)$ has a left Haar measure iff it has a right Haar measure. (v) Show that $(G,\cdot)$ has a left Haar probability measure iff it has a totally finite left Haar measure iff it has a right Haar probability measure. (iv) Show that $(G,\cdot)$ has a σ-finite left Haar measure iff it has a σ-finite right Haar measure.

>(f) (i) For Lebesgue measurable $E\subseteq\mathbb{R}\setminus\{0\}$, set $\nu E=\int_E\frac{1}{|x|}dx$. Show that ν is a (two-sided) Haar measure if $\mathbb{R}\setminus\{0\}$ is given the group operation of multiplication. (ii) For Lebesgue measurable $E\subseteq\mathbb{C}\setminus\{0\}$, identified with $\mathbb{R}^2\setminus\{0\}$, set $\nu E=\int_E\frac{1}{|z|^2}\mu(dz)$, where μ is two-dimensional Lebesgue measure. Show that ν is a (two-sided) Haar measure on $\mathbb{C}\setminus\{0\}$ if we take complex multiplication for the group operation. (*Hint*: 263D.)

>(g) (i) Show that Lebesgue measure is a (two-sided) Haar measure on $\mathbb{R}^r$, for any $r\geq 1$, if we take addition for the group operation. (ii) Show that the usual measure on $\{0,1\}^I$ is a two-sided Haar measure on $\{0,1\}^I$, for any set I, if we give $\{0,1\}^I$ the group operation corresponding to its identification with $\mathbb{Z}_2^I$. (iii) Describe the corresponding Haar measure on $\mathcal{P}I$ when $\mathcal{P}I$ is given the group operation $\triangle$.

(h) Let G be a locally compact Hausdorff topological group. (i) Show that any (left) Haar measure on G must be strictly positive. (ii) Show that G has a totally finite (left) Haar measure iff it is compact. (iii) Show that G has a σ-finite (left) Haar measure iff it is σ-compact.

>(i) (i) Let G and H be topological groups with left Haar measures μ and ν. Show that the quasi-Radon product measure on $G\times H$ (417N) is a left Haar measure on $G\times H$. (ii) Let $\langle G_i\rangle_{i\in I}$ be a family of topological groups, and suppose that each G_i has a left Haar probability measure (as happens, for instance, if each G_i is compact). Show that the quasi-Radon product measure on $\prod_{i\in I}G_i$ (417O) is a left Haar measure on $\prod_{i\in I}G_i$.

(j) (i) Show that any (left) Haar measure on a topological group, as defined in 441D, must be locally finite. (ii) Show that any (left) Haar measure on a locally compact Hausdorff group must be a Radon measure.

(k) Let $r\geq 1$ be an integer, and set $X=\{x:x\in\mathbb{R}^r,\ \|x\|=1\}$. Let G be the group of orthogonal $r\times r$ real matrices, so that G acts transitively on X. Show that (when given its natural topology as a subset of $\mathbb{R}^{r^2}$) G is a compact Hausdorff topological group. Let μ be a left Haar measure on G, and x any point of X; set $\phi_x(T)=Tx$ for $T\in G$. Show that $\mu\phi_x^{-1}$ is a G-invariant measure on X, independent of the choice of x.

(l) Let X be a compact metric space, and $g:X\to X$ any isometry. Show that g is surjective. (*Hint*: if $x\in X$, then $\rho(g^mx,g^nx)\geq\rho(x,g[X])$ for any $m<n$.)

>(m) Let (X,ρ) be a metric space, and G its isometry group with the topology $\mathfrak{T}$ of pointwise convergence. (i) Show that if X is compact, $\mathfrak{T}$ can be defined by the metric $(g,h)\mapsto\max_{x\in X}\rho(g(x),h(x))$. (ii) Show that if $\{y:\rho(y,x)\leq\gamma\}$ is compact for every $x\in X$ and $\gamma>0$, then G is locally compact. (iii) Show that if X is separable then G is metrizable. (iv) Show that if X is separable and (X,ρ) is complete then G is Polish. (*Hint*: if $D\subseteq X$ is a countable dense set, then $f\mapsto(f\restriction D,f^{-1}\restriction D)$ is a homeomorphism between G and a closed subset of $D^X\times D^X$.)

(n) Give $\mathbb{N}$ the metric ρ which takes only the values 0 and 1. Let G be the isometry group of $\mathbb{N}$ (that is, the group of all permutations of $\mathbb{N}$) with its pointwise topology. (i) Show that G is a G_δ subset of $\mathbb{N}^{\mathbb{N}}$, so is a Polish group. (ii) Show that if we set $\Delta(g,h)=\min\{n:n\in\mathbb{N},\ g(n)\neq h(n)\}$ and $\sigma(g,h)=1/(1+\Delta(g^{-1},h^{-1}))$ for distinct g, $h\in G$, then σ is a right-translation-invariant metric on G inducing its topology. (iii) Show that there is no complete right-translation-invariant metric on G inducing its topology. (*Hint*: any such metric must have the same Cauchy sequences as σ.) (iv) Show that G is not locally compact.

>(o) Let $r\geq 1$ be an integer, and S_{r-1} the sphere $\{x:x\in\mathbb{R}^r,\ \|x\|=1\}$. (i) Show that every isometry ϕ from S_{r-1} to itself corresponds to an orthogonal $r\times r$ matrix T. (*Hint*: $T=\langle\phi(e_i)\boldsymbol{.}e_j\rangle_{i,j<r}$.) (ii) Show that the topology of pointwise convergence on the isometry group of S_{r-1} corresponds to the topology on the set of $r\times r$ matrices regarded as a subset of $\mathbb{R}^{r^2}$.

(p) Let X be a locally compact metric space and G a subgroup of the isometry group of X. Show that for every $x \in X$ there is a non-zero G-invariant Radon measure on $\overline{\{g(x) : g \in G\}}$.

441Y Further exercises (a) Let (X, ρ) be a metric space, and $\mathcal{C}$ the family of closed subsets of X. For E, $F \in \mathcal{C} \setminus \{\emptyset\}$, set

$$\tilde{\rho}(E, F) = \max(\sup_{x \in E} \rho(x, F), \sup_{y \in F} \rho(y, E)),$$

where $\rho(x, F) = \inf_{y \in F} \rho(x, y)$. (i) Show that $\tilde{\rho}$ is a metric on $\mathcal{C} \setminus \{\emptyset\}$ (the **Hausdorff metric**). (ii) Show that if X is complete under ρ, $\mathcal{C} \setminus \{\emptyset\}$ is complete under $\tilde{\rho}$. (iii) Show that if X is totally bounded under ρ, $\mathcal{C} \setminus \{\emptyset\}$ is totally bounded under $\tilde{\rho}$. (iv) Show that if X is compact under ρ, $\mathcal{C} \setminus \{\emptyset\}$ is compact under $\tilde{\rho}$. (v) Show that if G is the isometry group of X, $(a, F) \mapsto g[F]$ is an action of G on $\mathcal{C}$.

(b) Take $1 \leq s \leq r \in \mathbb{N}$, and in $\mathbb{R}^r$ let X be $\{x : \|x\| = 1\}$. Let $\mathcal{C}$ be the set of closed subsets of X, and $\tilde{\rho}$ the Hausdorff metric on $\mathcal{C} \setminus \{\emptyset\}$ defined from the usual metric of X, so that $\mathcal{C} \setminus \{\emptyset\}$ is compact (441Ya). Let $\mathcal{C}_s \subseteq \mathcal{C}$ be

$$\{X \cap V : V \subseteq \mathbb{R}^r \text{ is an } s\text{-dimensional linear subspace}\}.$$

Show that $\mathcal{C}_s$ is a closed subset of $\mathcal{C} \setminus \{\emptyset\}$, therefore a compact metric space in its own right, and that the isometry group G of X acts transitively on $\mathcal{C}_s$, so that there is a G-invariant Radon probability measure on $\mathcal{C}_s$.

(c) Let $(X, \mathcal{W})$ be a locally compact Hausdorff uniform space, and suppose that G is a group acting on X 'uniformly equicontinuously'; that is, for every $W \in \mathcal{W}$ there is a $V \in \mathcal{W}$ such that $(a \bullet x, a \bullet y) \in W$ whenever $(x, y) \in V$ and $a \in G$. Show that there is a non-zero G-invariant Radon measure on X.

(d) In 441Xk, show that $\mu\phi_x^{-1}$ is a scalar multiple of Hausdorff $(r-1)$-dimensional measure on X.

(e) For any topological spaces X and Y, and any set G of functions from X to Y, the **compact-open** topology on G is the topology generated by sets of the form $\{g : g \in G, g[K] \subseteq H\}$, where $K \subseteq X$ is compact and $H \subseteq Y$ is open. Show that if (X, ρ) is a metric space and G is the isometry group of X, then the topology of pointwise convergence on G is its compact-open topology.

(f) Let X be a compact Hausdorff space and G the group of all homeomorphisms from X to itself. (i) Let P be the family of all continuous pseudometrics on X (see 4A2Jf). For $\rho \in \mathrm{P}$ and g, $h \in G$, set $\tau_\rho(g, h) = \max_{x \in X} \rho(g(x), h(x))$. Show that every τ_ρ is a right-translation-invariant pseudometric on G, and that G with the topology generated by $\{\tau_\rho : \rho \in \mathrm{P}\}$ is a topological group. (ii) Show that this is the compact-open topology as defined in 441Ye. (iii) Show that if X is metrizable then G is Polish.

(g) Let X be a compact metric space and G the isometry group of X. Show that every orbit in X is closed.

(h) For w, $z \in \mathbb{C} \setminus \{0\}$ set $\rho(w, z) = |\operatorname{Ln}(\frac{w}{z})|$, where $\operatorname{Ln} v = \ln |v| + i \arg v$ for non-zero complex numbers v. (i) Show that ρ is a metric on $\mathbb{C} \setminus \{0\}$. (ii) Show that the 2-dimensional Hausdorff measure $\mu_{H2}^{(\rho)}$ derived from ρ (471A) is a Haar measure for the multiplicative group $\mathbb{C} \setminus \{0\}$. (iii) Show that $\mu_{H2}^{(\rho)} = \frac{4}{\pi}\nu$, where ν is the measure of 441Xf(ii). (*Hint*: 264I.)

(i) Let X be the group of real $r \times r$ orthogonal matrices, where $r \geq 2$ is an integer. Give X the Euclidean metric, regarding it as a subset of $\mathbb{R}^{r^2}$. (i) Show that the left and right actions of X on itself are distance-preserving. (ii) Show that $\frac{r(r-1)}{2}$-dimensional Hausdorff measure on X is a two-sided Haar measure.

(j) Let T be any set, and ρ the $\{0, 1\}$-valued metric on T. Let X be the set of partial orders on T, regarded as a subset of $\mathcal{P}(T \times T)$. Show that X is compact (cf. 418Xw). Let G be the group of isometries of T with its topology of pointwise convergence. Set $\pi \bullet x = \{(t, u) : (\pi^{-1}(t), \pi^{-1}(u)) \in x\}$ for $\pi \in G$ and $x \in X$. Show that $\bullet$ is a continuous action of G on X. Show that there is a strictly positive G-invariant Radon probability measure μ on X.

(k) Let X be a set, G a group acting on X, and μ a G-invariant measure on X with measure algebra $\mathfrak{A}$. Show that if τ is any rearrangement-invariant extended Fatou norm on $L^0(\mathfrak{A})$ then the formula of 441Kb defines a norm-preserving action of G on the Banach space L^τ.

441 Notes and comments The proof I give of 441C is essentially the same as the proof of 441E in HALMOS 50, §58.

In part (f) of the proof of 441C I use an ultrafilter, relying on a fairly strong consequence of the Axiom of Choice. In this volume, as in the last, I generally employ the axiom of choice without stopping to consider whether it is really needed. But Haar measure, at least, is so important that I point out now that it can be built with much weaker principles. For a construction of Haar measure not dependent on anything more than countable choice, see NAIMARK 70, §27.4. I should remark that the argument there, while closely paralleling that of 441C in its ideas, relies for its conclusion on the results of §436 above, so there is a lot of checking to do to see that these also can be established with countable choice alone. Moreover, in the absence of the full axiom of choice, we may find that we have fewer locally compact topological groups than we expect.

While Haar measure is surely the pre-eminent application of the theory here, I think that some of the other consequences of 441C (441H, 441Xd, 441Xk, 441Yb, 441Yc) are sufficiently striking to justify the trouble involved in the extra generality. I ought to remark that there are important examples of invariant measures which have nothing to do with 441C. Some of these will appear in §449; for the moment I note only 441Xa.

FEDERER 69, §2.7, develops a general theory of 'covariant' measures μ ('relatively invariant' in HALMOS 50) such that $\mu(a\bullet E) = \psi(a)\mu E$ for appropriate sets $E \subseteq X$ and $a \in G$, where $\psi : G \to]0,\infty[$ is a homomorphism; for instance, taking μ to be Lebesgue measure on $\mathbb{R}^r$, we have $\mu T[E] = |\det T|\mu E$ for every linear space isomorphism $T : \mathbb{R}^r \to \mathbb{R}^r$ and every measurable set E (263A). The theory I have described here can deal only with the subgroup of isometric linear isomorphisms (that is, the orthogonal group). Covariant measures arise naturally in many other contexts, such as 443S below.

Hausdorff measures, being defined in terms of metrics, are necessarily invariant under isometries, so appear naturally in this context, starting with 264I. There are interesting challenges both in finding suitable metrics and in establishing exact constants, as in 441Yh-441Yi.

It is worth pausing over the topology of an isometry group, as described in 441G. It is quite surprising that such an elementary idea should give us a topological group at all. I offer some exercises (441Xm-441Xo, 441Ye) to help you relate the construction to material which may be more familiar. It is of course a 'weak' topology, except when the underlying space is compact (441Xm(i)). See also 441Yf. These groups are rarely locally compact, and you may find them pushed out of your mind by the extraordinary theory which you will see developed in the next hundred pages; but in the last two sections of this chapter, and in §493, they will become leading examples.

442 Uniqueness of Haar measures

Haar measure has an extraordinary wealth of special properties, and it will be impossible for me to cover them all properly in this chapter. But surely the second thing to take on board, after the existence of Haar measures on locally compact Hausdorff groups (441E), is the fact that they are, up to scalar multiples, unique. This is the content of 442B. We find also that while left and right Haar measures can be different (442Xf), they are not only direct mirror images of each other (442C) – as is, I suppose, to be expected – but even more closely related (442F, 442H, 442L). Investigating this relation, we are led naturally to the 'modular function' of a group (442I).

442A Lemma Let X be a topological group and μ a left Haar measure on X.

(a) μ is strictly positive and locally finite.

(b) If $G \subseteq X$ is open and $\gamma < \mu G$, there are an open set H and an open neighbourhood U of the identity such that $HU \subseteq G$ (writing HU for $\{xy : x \in H, y \in U\}$) and $\mu H \geq \gamma$.

(c) If X is locally compact and Hausdorff, μ is a Radon measure.

proof (a)(i) ? If $G \subseteq X$ were a non-empty open set such that $\mu G = 0$, then we should have $\mu(xG) = 0$ for every $x \in X$, so that X would be covered by negligible open sets; but as μ is supposed to be τ-additive, $\mu X = 0$. **X**

(ii) Because μ is effectively locally finite, there is some non-empty open set G such that $\mu G < \infty$; but now $\{xG : x \in X\}$ is a cover of X by open sets of finite measure.

(b) Let $\mathcal{U}$ be the family of open sets containing the identity e, and $\mathcal{H}$ the family of open sets H such that $HU \subseteq G$ for some $U \in \mathcal{U}$. Because $\mathcal{U}$ is downwards-directed, $\mathcal{H}$ is upwards-directed; because $e \in U$ for every $U \in \mathcal{U}$, $\bigcup \mathcal{H} \subseteq G$. If $x \in G$, then $x^{-1}G \in \mathcal{U}$, and there is a $U \in \mathcal{U}$ such that $UU \subseteq x^{-1}G$; but now $xUU \subseteq G$, so $xU \in \mathcal{H}$. Thus $\bigcup \mathcal{H} = G$. Because μ is τ-additive, there is an $H \in \mathcal{H}$ such that $\mu H \geq \gamma$.

(c) Use (a) and 416G.

442B Theorem Let X be a topological group. If μ and ν are left Haar measures on X, they are multiples of each other.

proof (a) Let $\mathcal{G}$ be the family of non-empty open sets G such that μG and νG are both finite; because μ and ν are locally finite (442Aa), $\mathcal{G}$ is a base for the topology of X. Note that $G \cup H \in \mathcal{G}$ for all G, $H \in \mathcal{G}$. Set $\mathcal{U} = \{U : U \in \mathcal{G}, U = U^{-1}, e \in U\}$, where e is the identity of G; then $\mathcal{U}$ is a base of neighbourhoods of e (4A5Ec). Let $\mathcal{F}$ be the filter on $\mathcal{U}$ generated by the sets $\{U : U \in \mathcal{U}, U \subseteq V\}$ as V runs over $\mathcal{U}$.

(b) (The key.) If $G \in \mathcal{G}$ and $0 < \epsilon < 1$, there is a $V_1 \in \mathcal{U}$ such that

$$(1-\epsilon)\frac{\mu G}{\nu G} \leq \frac{\mu U}{\nu U}$$

whenever $U \in \mathcal{U}$ and $U \subseteq V_1$. **P** By 442Aa, μG and νG are both non-zero. By 442Ab, there are an open set H and a neighbourhood V_1 of e such that $HV_1 \subseteq G$ and $\mu H \geq (1-\epsilon)\mu G$; shrinking V_1 if need be, we may suppose that $V_1 \in \mathcal{U}$. Take any $U \in \mathcal{U}$ such that $U \subseteq V_1$, so that $HU \subseteq G$. Consider the product quasi-Radon measure λ of μ and ν on $X \times X$ (417R), and the set $W = \{(x,y) : x, y \in G, x^{-1}y \in U\}$. Because the function $(x,y) \mapsto x^{-1}y$ is continuous (4A5Eb), W is open. Consequently

$$\int \mu\{x : (x,y) \in W\}\nu(dy) = \lambda W = \int \nu\{y : (x,y) \in W\}\mu(dx)$$

(417C(iv)). But we see that if $x \in H$ then $xU \subseteq G$, so $(x,y) \in W$ whenever $y \in xU$, and

$$\int \nu\{y : (x,y) \in W\}\mu(dx) \geq \int_H \nu(xU)\mu(dx) = \mu H \cdot \nu U.$$

On the other hand, if $y \in X$, then

$$\begin{aligned}\int \mu\{x : (x,y) \in W\}\nu(dy) &\leq \int_G \mu\{x : x^{-1}y \in U\}\nu(dy) = \int_G \mu\{x : y^{-1}x \in U\}\nu(dy)\\ &= \int_G \mu(yU)\nu(dy) = \mu U \cdot \nu G,\end{aligned}$$

so that

$$(1-\epsilon)\mu G \cdot \nu U \leq \mu H \cdot \nu U \leq \lambda W \leq \mu U \cdot \nu G.$$

Dividing both sides by $\nu U \cdot \nu G$, we have the result. **Q**

(c) In the same way, there is a $V_2 \in \mathcal{V}$ such that

$$(1-\epsilon)\frac{\nu G}{\mu G} \leq \frac{\nu U}{\mu U}$$

whenever $U \in \mathcal{U}$ and $U \subseteq V_2$. So if $U \in \mathcal{U}$ and $U \subseteq V_1 \cap V_2$, we have

$$(1-\epsilon)\frac{\mu G}{\nu G} \leq \frac{\mu U}{\nu U} \leq \frac{1}{1-\epsilon}\frac{\mu G}{\nu G}.$$

As ϵ is arbitrary,

$$\lim_{U \to \mathcal{F}} \frac{\mu U}{\nu U} = \frac{\mu G}{\nu G}.$$

And this is true for every $G \in \mathcal{G}$.

(d) So if we set $\alpha = \lim_{U \to \mathcal{F}} \frac{\mu U}{\nu U}$, we shall have $\mu G = \alpha \nu G$ for every $G \in \mathcal{G}$. Now μ and $\alpha\nu$ are quasi-Radon measures agreeing on the base $\mathcal{G}$, which is closed under finite unions, so are identical, by 415H(iv).

442C Proposition Let X be a topological group and μ a left Haar measure on X. Setting $\nu E = \mu(E^{-1})$ whenever $E \subseteq X$ is such that $E^{-1} = \{x^{-1} : x \in E\}$ is measured by μ, ν is a right Haar measure on X.

proof Set $\phi(x) = x^{-1}$ for $x \in X$. Then ϕ is a homeomorphism, so the image measure $\nu = \mu\phi^{-1}$ is a quasi-Radon measure. It is non-zero because $\nu X = \mu X$. If $E \in \operatorname{dom}\nu$ and $x \in X$, then

$$\nu(Ex) = \mu(x^{-1}E^{-1}) = \mu E^{-1} = \nu E.$$

So ν is a right Haar measure.

442D Remark Clearly all the arguments of 442A-442C must be applicable to right Haar measures; that is, any right Haar measure must be locally finite and strictly positive; two right Haar measures on the same group must be multiples of each other; and if X carries a right Haar measure ν then $E \mapsto \nu E^{-1}$ will be a left Haar measure on X. (If you are unhappy with such a bold appeal to the symmetry between 'left' and 'right' in topological groups, write the reflected version of 442C out in full, and use it to reflect 442A-442B.)

Thus we may say that a topological group **carries Haar measures** if it has either a left or a right Haar measure. These can, of course, be the same; in fact it takes a certain amount of exploration to find a group in which they are different (e.g., 442Xf).

442E Lemma Let X be a topological group, μ a left Haar measure on X and ν a right Haar measure on X. If G, $H \subseteq X$ are open, then

$$\mu G \cdot \nu H = \int_H \nu(xG^{-1})\mu(dx).$$

proof Let λ be the quasi-Radon product measure of μ and ν on $X \times X$. The sets $W_1 = \{(x,y) : y^{-1}x \in G,\ x \in H\}$ and $W_2 = \{(x,y) : x \in G,\ yx \in H\}$ are both open, so 417C(iv) tells us that

$$\begin{aligned}
\int_H \nu(xG^{-1})\mu(dx) &= \int \nu\{y : x \in H,\ y^{-1}x \in G\}\mu(dx) = \lambda W_1 \\
&= \int \mu\{x : x \in H,\ y^{-1}x \in G\}\nu(dy) = \int \mu(H \cap yG)\nu(dy) \\
&= \int \mu(y^{-1}H \cap G)\nu(dy) = \int \mu\{x : x \in G,\ yx \in H\}\nu(dy) \\
&= \lambda W_2 = \int_G \nu\{y : yx \in H\}\mu(dx) \\
&= \int_G \nu(Hx^{-1})\mu(dx) = \mu G \cdot \nu H.
\end{aligned}$$

442F Domains of Haar measures 442B tells us, in part, that any two left Haar measures on a topological group must have the same domain and the same negligible sets; similarly, any two right Haar measures have the same domain and the same negligible sets. In fact left and right Haar measures agree on both.

Proposition Let X be a topological group which carries Haar measures. If μ is a left Haar measure and ν is a right Haar measure on X, then they have the same domains and the same null ideals.

proof (a) Suppose that $F \subseteq X$ is a closed set such that $\mu F = 0$. Then $\nu F = 0$. **P?** Otherwise, there is an open set H such that $\nu H < \infty$ and $\nu(F \cap H) > 0$. Let G be any open set such that $0 < \mu G < \infty$. By 442E,

$$\mu G \cdot \nu H = \int_H \nu(xG^{-1})\mu(dx).$$

But also

$$\mu G \cdot \nu(H \setminus F) = \int_{H\setminus F} \nu(xG^{-1})\mu(dx) = \int_H \nu(xG^{-1})\mu(dx) = \mu G \cdot \nu H,$$

so $\mu G \cdot \nu(H \cap F) = 0$. **XQ**

(b) It follows that

$$\nu E = \sup_{F\subseteq E \text{ is closed}} \nu F = 0$$

whenever E is a Borel set such that $\mu E = 0$. Now take any $E \in \operatorname{dom}\mu$. Set

$$\mathcal{G} = \{G : G \subseteq X \text{ is open}, \mu G < \infty, \nu G < \infty\}.$$

Because both μ and ν are locally finite, $\mathcal{G}$ covers X. If $G \in \mathcal{G}$, there are Borel sets E', E'' such that $E' \subseteq E \cap G \subseteq E''$ and $\mu(E'' \setminus E') = 0$. In this case $\nu(E'' \setminus E') = 0$ so $E \cap G \in \operatorname{dom}\nu$. Because ν is complete, locally determined and τ-additive, $E \in \operatorname{dom}\nu$ (414I). If $\mu E = 0$, it follows that

$$\nu E = \sup_{F\subseteq E \text{ is closed}} \nu F = 0$$

just as above.

(c) Thus ν measures E whenever μ measures E, and E is ν-negligible whenever it is μ-negligible. I am sure you will have no difficulty in believing that all the arguments above, in particular that of 442E, can be re-cast to show that $\operatorname{dom}\nu \subseteq \operatorname{dom}\mu$; alternatively, apply the result in the form just demonstrated to the left Haar measure ν' and the right Haar measure μ', where

$$\nu' E = \nu E^{-1}, \quad \mu' E = \mu E^{-1}$$

as in 442Cb.

442G Corollary Let X be a topological group and μ a left Haar measure on X with domain Σ. Then, for $E \subseteq X$ and $a \in X$,

$$E \in \Sigma \iff E^{-1} \in \Sigma \iff Ea \in \Sigma,$$

$$\mu E = 0 \iff \mu E^{-1} = 0 \iff \mu(Ea) = 0.$$

proof Apply 442F with $\nu E = \mu E^{-1}$.

442H Remark From 442F-442G we see that if X is any topological group which carries Haar measures, there is a distinguished σ-algebra Σ of subsets of X, which we may call the algebra of **Haar measurable sets**, which is the domain of any Haar measure on X. Similarly, there is a σ-ideal $\mathcal{N}$ of $\mathcal{P}X$, the ideal of **Haar negligible sets**, which is the null ideal for any Haar measure on X. Both Σ and $\mathcal{N}$ are translation-invariant and also invariant under the inversion operation $x \mapsto x^{-1}$. Note that if $A \subseteq X$ is such that $A \cap K \in \Sigma$ for every compact set $K \subseteq X$, then $A \in \Sigma$ (412Ja); similarly, if $A \subseteq X$ and $A \cap K \in \mathcal{N}$ for every compact $K \subseteq X$, then $A \in \mathcal{N}$ (412Jb).

If we form the quotient $\mathfrak{A} = \Sigma/\mathcal{N}$, then we have a fixed Dedekind complete Boolean algebra which is the **Haar measure algebra** of the group X in the sense that any Haar measure on X, whether left or right, has measure algebra based on $\mathfrak{A}$. If $a \in X$, the maps $x \mapsto ax$, $x \mapsto xa$, $x \mapsto x^{-1}$ give rise to Boolean automorphisms of $\mathfrak{A}$.

For a member of Σ, we have a notion of 'σ-finite' which is symmetric between left and right (442Xd). We do not in general have a corresponding two-sided notion of 'finite measure' (442Xg(i)); but of course we can if we wish speak of a set as having 'finite left Haar measure' or 'finite right Haar measure' without declaring which Haar measure we are thinking of. It is the case, however, that if the group X itself has finite left Haar measure, it also has finite right Haar measure; see 442Ic-d below.

442I The modular function Let X be a topological group which carries Haar measures.

(a) There is a group homomorphism $\Delta : X \to]0, \infty[$ defined by the formula

$$\mu(Ex) = \Delta(x)\mu E \text{ whenever } \mu \text{ is a left Haar measure on } X \text{ and } E \in \operatorname{dom}\mu.$$

P Fix on a left Haar measure $\tilde{\mu}$ on X. For $x \in X$, let μ_x be the function defined by saying

$$\mu_x E = \tilde{\mu}(Ex) \text{ whenever } E \subseteq X,\ Ex \in \operatorname{dom}\mu,$$

that is, for every Haar measurable set $E \subseteq X$. Because the function $\phi_x : X \to X$ defined by setting $\phi_x(y) = yx^{-1}$ is a homeomorphism, $\mu_x = \tilde{\mu}\phi_x^{-1}$ is a quasi-Radon measure on X; and

$$\mu_x(yE) = \tilde{\mu}(yEx) = \tilde{\mu}(Ex) = \mu_x E$$

whenever μ_x measures E, so μ_x is a left Haar measure on X. By 442B, there is a $\Delta(x) \in]0,\infty[$ such that $\mu_x = \Delta(x)\tilde{\mu}$; because $\tilde{\mu}$ surely takes at least one value in $]0,\infty[$, $\Delta(x)$ is uniquely defined.

If μ is any other left Haar measure on X, then $\mu = \alpha\tilde{\mu}$ for some $\alpha > 0$, so that

$$\mu(Ex) = \alpha\tilde{\mu}(Ex) = \alpha\Delta(x)\tilde{\mu}E = \Delta(x)\mu E.$$

Thus $\Delta : X \to]0,\infty[$ has the property asserted in the formula offered.

To see that Δ is a group homomorphism, take any x, $y \in X$ and a Haar measurable set E such that $0 < \tilde{\mu}E < \infty$, and observe that

$$\Delta(xy)\tilde{\mu}E = \tilde{\mu}(Exy) = \Delta(y)\tilde{\mu}(Ex) = \Delta(y)\Delta(x)\tilde{\mu}E,$$

so that $\Delta(xy) = \Delta(y)\Delta(x) = \Delta(x)\Delta(y)$. **Q**

Δ is called the **left modular function** of X.

(b) We find now that $\nu(xE) = \Delta(x^{-1})\nu E$ whenever ν is a right Haar measure on X, $x \in X$ and $E \subseteq X$ is Haar measurable. **P** Let μ be the left Haar measure derived from ν, so that $\mu E = \nu E^{-1}$ whenever E is Haar measurable. If $x \in X$ and $E \in \operatorname{dom}\nu$, then

$$\nu(xE) = \mu(E^{-1}x^{-1}) = \Delta(x^{-1})\mu E^{-1} = \Delta(x^{-1})\nu E. \text{ } \mathbf{Q}$$

Thus we may call $x \mapsto \Delta(x^{-1}) = \frac{1}{\Delta(x)}$ the **right modular function** of X.

(c) If X is abelian, then obviously $\Delta(x) = 1$ for every $x \in X$, because $\mu(Ex) = \mu(xE) = \mu E$ whenever $x \in X$, μ is a left Haar measure on X and E is Haar measurable. Equally, if any (therefore every) left (or right) Haar measure μ on X is totally finite, then $\mu(Xx) = \mu(xX) = \mu X$, so again $\Delta(x) = 1$ for every $x \in X$. This will be the case, in particular, for any compact Hausdorff topological group (recall that by 441E any such group carries Haar measures), because its Haar measures are locally finite, therefore totally finite.

Groups in which $\Delta(x) = 1$ for every x are called **unimodular**.

(d) From the definition of Δ, we see that a topological group carrying Haar measures is unimodular iff every left Haar measure is a right Haar measure.

(e) In particular, if a group has any totally finite (left or right) Haar measure, its left and right Haar measures are the same, and it has a unique Haar probability measure, which we may call its **normalized Haar measure**.

In the other direction, any group with the discrete topology is unimodular, since counting measure is a two-sided Haar measure.

442J Proposition For any topological group carrying Haar measures, its left modular function is continuous.

proof Let X be a topological group carrying Haar measures, with left modular function Δ.

(a) If $\epsilon > 0$, there is an open set U_ϵ containing the identity e of X such that $\Delta(x) \leq 1+\epsilon$ for every $x \in U$. **P** Take any left Haar measure μ on X, and an open set G such that $0 < \mu G < \infty$. By 442Ab, there are an open set H and a neighbourhood U_ϵ of the identity such that $HU_\epsilon \subseteq G$ and $\mu G \leq (1+\epsilon)\mu H$. If $x \in U_\epsilon$, then $Hx \subseteq G$, so $\Delta(x) = \frac{\mu(Hx)}{\mu H} \leq 1+\epsilon$. **Q**

(b) Now, given $x_0 \in X$ and $\epsilon > 0$, $V = \{x : x^{-1}x_0 \in U_\epsilon,\ x_0^{-1}x \in U_\epsilon\}$ is an open set containing x_0. If $x \in V$, then

$$\Delta(x) = \Delta(x_0)\Delta(x_0^{-1}x) \leq (1+\epsilon)\Delta(x_0),$$

$$\Delta(x_0) = \Delta(x)\Delta(x^{-1}x_0) \leq (1+\epsilon)\Delta(x),$$

so

$$\frac{1}{1+\epsilon}\Delta(x_0) \leq \Delta(x) \leq (1+\epsilon)\Delta(x).$$

As ϵ is arbitrary, Δ is continuous at x_0; as x_0 is arbitrary, Δ is continuous.

442K Theorem Let X be a topological group and μ a left Haar measure on X. Let Δ be the left modular function of X.

(a) $\mu(E^{-1}) = \int_E \Delta(x^{-1})\mu(dx)$ for every $E \in \operatorname{dom}\mu$.

(b)(i) $\int f(x^{-1})\mu(dx) = \int \Delta(x^{-1})f(x)\mu(dx)$ whenever f is a real-valued function such that either integral is defined in $[-\infty,\infty]$;

(ii) $\int f(x)\mu(dx) = \int \Delta(x^{-1})f(x^{-1})\mu(dx)$ whenever f is a real-valued function such that either integral is defined in $[-\infty,\infty]$.

(c) $\int f(xy)\mu(dx) = \Delta(y^{-1})\int f(x)\mu(dx)$ whenever $y \in X$ and f is a real-valued function such that either integral is defined in $[-\infty,\infty]$.

proof (a)(i) Setting $\nu_1 E = \mu E^{-1}$ for Haar measurable sets $E \subseteq X$, we know that ν_1 is a right Haar measure, so 442E tells us that

$$\mu G \cdot \nu_1 H = \int_H \nu_1(xG^{-1})\mu(dx) = \int_H \mu(Gx^{-1})\mu(dx) = \mu G \int_H \Delta(x^{-1})\mu(dx)$$

for all open sets G, $H \subseteq X$. Since there is an open set G such that $0 < \mu G < \infty$, $\mu H^{-1} = \int_H \Delta(x^{-1})\mu(dx)$ for every open set $H \subseteq X$.

(ii) Now let ν_2 be the indefinite-integral measure defined by setting $\nu_2 E = \int \Delta(x^{-1})\chi E(x)\mu(dx)$ whenever this is defined in $[0,\infty]$ (§234). Then ν_2 is effectively locally finite. **P** If $\nu_2 E > 0$, then $\mu E > 0$, so there is an $n \in \mathbb{N}$ such that $\mu(E \cap H) > 0$, where H is the open set $\{x : \Delta(x^{-1}) < n\}$. Now there is an open set $G \subseteq H$ such that $\mu G < \infty$ and $\mu(E \cap G) > 0$, in which case $\nu_2 G \leq n\mu G < \infty$ and $\nu_2(E \cap G) > 0$. **Q**

Accordingly ν_2 is a quasi-Radon measure (415O). Since it agrees with the quasi-Radon measure ν_1 on open sets, by (i), the two are equal; that is, $\mu E^{-1} = \int_E \Delta(x^{-1})\mu(dx)$ for every $E \in \operatorname{dom}\mu$.

(b)(i) Apply 235E with $X = Y$, $\Sigma = \mathrm{T} = \operatorname{dom}\mu$, $\mu = \nu$ and $\phi(x) = x^{-1}$, $J(x) = \Delta(x^{-1})$, $g(x) = \Delta(x^{-1})f(x)$ for $x \in X$. From (a) we have

$$\int J \times \chi(\phi^{-1}[F])d\mu = \int_{F^{-1}} \Delta(x^{-1})\mu(dx) = \mu F = \nu F$$

for every $F \in \mathrm{T}$ (using 442G to see that $F^{-1} \in \Sigma$). So we get

$$\begin{aligned}\int f(x^{-1})\mu(dx) &= \int \Delta(x^{-1})g(x^{-1})\mu(dx) = \int J \times g\phi \, d\mu \\ &= \int g \, d\nu = \int \Delta(x^{-1})f(x)\mu(dx)\end{aligned}$$

if any of the integrals is defined in $[-\infty,\infty]$.

(ii) Set $\tilde{f}(x) = f(x^{-1})$ whenever this is defined, and apply (i) to $\tilde{f}$.

(c) Similarly, apply 235E with $\mu = \nu$, $\phi(x) = xy$, $J(x) = \Delta(y)$ for every $x \in X$; then

$$\int J \times \chi(\phi^{-1}[F])d\mu = \Delta(y)\mu(Fy^{-1}) = \mu F$$

for every $F \in \operatorname{dom}\mu$, so

$$\int f(xy)\mu(dx) = \Delta(y^{-1})\int J \times f\phi \, d\mu = \Delta(y^{-1})\int f(x)\mu(dx).$$

442L Corollary Let X be a group carrying Haar measures. If μ is a left Haar measure on X and ν is a right Haar measure, then each is an indefinite-integral measure over the other.

proof Let $\tilde{\mu}$ be the right Haar measure defined by setting $\tilde{\mu}E = \mu E^{-1}$ for every Haar measurable $E \subseteq X$. Then $\tilde{\mu}E = \int_E \Delta(x^{-1})\mu(dx)$ for every $E \in \operatorname{dom}\mu = \operatorname{dom}\tilde{\mu}$, so $\tilde{\mu}$ is an indefinite-integral measure over μ; because ν is a multiple of $\tilde{\mu}$, it also is an indefinite-integral measure over μ. Similarly, or because Δ is strictly positive, μ is an indefinite-integral measure over ν.

442X Basic exercises >(a) Let X and Y be topological groups with (left) Haar probability measures μ and ν, and $\phi : X \to Y$ a continuous surjective group homomorphism. Show that ϕ is inverse-measure-preserving for μ and ν.

(b) (i) Let X and Y be two topological groups carrying Haar measures. Show that the product topological group $X \times Y$ (4A5G) carries Haar measures. (ii) Let $\langle X_i \rangle_{i \in I}$ be any family of topological groups carrying totally finite Haar measures. Show that the product group $\prod_{i \in I} X_i$ carries a totally finite Haar measure. (*Hint*: 417O.)

(c) Let X be a subgroup of the group $(\mathbb{R}, +)$. Show that X carries Haar measures iff it is either discrete (so that counting measure is a Haar measure on X) or of full outer Lebesgue measure (so that the subspace measure on X is a Haar measure). (*Hint*: if G has a Haar measure ν and is not discrete, then $\nu(G \cap [\alpha, \beta]) = (\beta - \alpha)\nu(G \cap [0,1])$ whenever $\alpha \leq \beta$.) In particular, $\mathbb{Q}$ does not carry Haar measures.

(d) Let X be a topological group carrying Haar measures; let Σ be the algebra of Haar measurable subsets of X. Let μ and ν be any Haar measures on X (either left or right). Show that a set $E \in \Sigma$ can be covered by a sequence of sets of finite measure for μ iff it can be covered by a sequence of sets of finite measure for ν.

(e) Let X be a topological group carrying Haar measures and $\mathfrak{A}$ its Haar measure algebra (in the sense of 442H). Show that we have left, right and conjugacy actions of X on $\mathfrak{A}$ given by the formulae $z \bullet E^\bullet = (zE)^\bullet$, $z \bullet E^\bullet = (Ez^{-1})^\bullet$ and $z \bullet E^\bullet = (zEz^{-1})^\bullet$ for every Haar measurable $E \subseteq X$ and every $z \in X$.

>(f) On $\mathbb{R}^2$ define a binary operation $*$ by setting $(\xi_1, \xi_2) * (\eta_1, \eta_2) = (\xi_1 + \eta_1, \xi_2 + e^{\xi_1}\eta_2)$. (i) Show that $*$ is a group operation under which $\mathbb{R}^2$ is a locally compact topological group. (ii) Show that Lebesgue measure μ is a right Haar measure for $*$. (iii) Let ν be the indefinite-integral measure on $\mathbb{R}^2$ defined by setting $\nu E = \int_E e^{-\xi_1} d\xi_1 d\xi_2$ for Lebesgue measurable sets $E \subseteq \mathbb{R}^2$. Show that ν is a left Haar measure for $*$. (*Hint*: 263D.) (iv) Thus $(\mathbb{R}^2, *)$ is not unimodular. (v) Show that the left modular function of $(\mathbb{R}^2, *)$ is $(\xi_1, \xi_2) \mapsto e^{-\xi_1}$.

>(g) Let X be any topological group which is not unimodular. (i) Show that there is an open subset of X which is of finite measure for all left Haar measures on X and of infinite measure for all right Haar measures. (*Hint*: the modular function is unbounded.) (ii) Let μ be a left Haar measure on X and ν a right Haar measure. Show that $L^0(\mu) = L^0(\nu)$ and $L^\infty(\mu) = L^\infty(\mu)$, but that $L^p(\mu) \neq L^p(\nu)$ for any $p \in [1, \infty[$.

(h) Let X and Y be topological groups carrying Haar measures, with left modular functions Δ_X and Δ_Y respectively. Show that the left modular function of $X \times Y$ is $(x, y) \mapsto \Delta_X(x)\Delta_Y(y)$.

(i) Let X be any topological group and $\Delta : X \to]0, \infty[$ a group homomorphism such that $\{x : \Delta(x) \leq 1 + \epsilon\}$ is a neighbourhood of the identity in X for every $\epsilon > 0$. Show that Δ is continuous.

(j) Let X be a topological group with a right Haar measure ν and left modular function Δ. Show that $\nu E^{-1} = \int_E \Delta(x)\nu(dx)$ for every Haar measurable set $E \subseteq X$.

442Y Further exercises (a) In 441Yb, show that the only G-invariant Radon measures on $\mathcal{C}_s$ are multiples of Hausdorff $s(r-s)$-dimensional measure on $\mathcal{C}_s$. (*Hint*: G itself is $\frac{r(r-1)}{2}$-dimensional (cf. 441Yi), and for any $C \in \mathcal{C}_s$ the stabilizer of C is $\frac{s(s-1)}{2} + \frac{(r-s)(r-s-1)}{2}$-dimensional. See FEDERER 69, 3.2.28.)

(b) Let $r \geq 1$, and let X be the group of non-singular $r \times r$ real matrices. Regarding X as an open subset of $\mathbb{R}^{r^2}$, show that a two-sided Haar measure μ can be defined on X by setting $\mu E = \int_E \frac{1}{|\det A|^r} \mu_L(dA)$, where μ_L is Lebesgue measure on $\mathbb{R}^{r^2}$; so that X is unimodular.

(c) Show that there is a set $A \subseteq [0,1]$, of Lebesgue outer measure 1, such that no countable set of translates of A covers any set of Lebesgue measure greater than 0. (*Hint*: let $\langle F_\xi \rangle_{\xi < \mathfrak{c}}$ run over the uncountable closed subsets of $[0,1]$ with cofinal repetitions (419Ib), and enumerate the countable subsets of $\mathbb{R}$ as $\langle I_\xi \rangle_{\xi < \mathfrak{c}}$. Choose inductively x_ξ, $x'_\xi \in F_\xi$ such that $x_\xi \notin \bigcup_{\eta,\zeta < \xi} x'_\eta - I_\zeta$, $x'_\xi \notin \bigcup_{\eta,\zeta \leq \xi} x_\eta + I_\zeta$; set $A = \{x_\xi : \xi < \mathfrak{c}\}$.) Show that we can extend Lebesgue measure on $\mathbb{R}$ to a translation-invariant measure for which A is negligible. (*Hint*: 417A.)

442Z Problem Let X be a compact Hausdorff space, and G the group of autohomeomorphisms of Z. Suppose that G acts transitively on X. Does it follow that there is at most one G-invariant Radon probability measure on X?

442 Notes and comments Haar measure dominates the theory of locally compact topological groups for two reasons: it is ubiquitous (the existence theorem, 441E) and essentially defined by the group structure (the uniqueness theorem, 442B). I have tried to show that these are rather different results by setting the theorems out with different hypotheses. I presented the existence theorem as a special case of 441C, which demands a locally compact space and a group, but allows them to be different. In the uniqueness theorem (roughly following HALMOS 50) I demand a group with an invariant quasi-Radon measure, but do not (at this point) ask for any hypothesis of compactness. In fact it will become apparent in the next section that this is a somewhat spurious generality; 442B and 442I here can be deduced from the traditional forms in which the group is assumed to be locally compact and Hausdorff. From the point of view of the topological measure theory to which this volume is devoted, however, I think the small extra labour involved in tracing through the arguments without relying on the Riesz Representation Theorem is instructive. For instance, it emphasizes interesting features of the domains and null ideals of Haar measures (442H).

There is however a more serious question concerning the uniqueness theorem. I do not know whether it really belongs to the theory of topological groups, as described here, or to the theory of group actions along with 441C. The trouble is that I know of no example of a Hausdorff space X and a transitive group G of homeomorphisms of X such that X carries G-invariant Radon measures which are not multiples of each other (see 442Z). 443T and 443Xz below eliminate the simplest possibilities. We do need to put some restriction on the measures; for instance, counting measure on $\mathbb{R}$ is translation-invariant, but has nothing to do with Lebesgue measure. There are also proper translation-invariant extensions of Lebesgue measure (442Yc); for far-reaching elaborations of this idea see HEWITT & ROSS 63, §16.

443 Further properties of Haar measure

I devote a section to filling in some details of the general theory of Haar measures before turning to the special topics dealt with in the rest of the chapter. The first question concerns the left and right shift operators acting on sets, on elements of the measure algebra, on measurable functions and on function spaces. All these operations can be regarded as group actions, and, if appropriate topologies are assigned, they are continuous actions (443C, 443G). As an immediate consequence of this I give an important result about product sets $\{ab : a \in A, b \in B\}$ in a topological group carrying Haar measures (443D).

The second part of the section revolves around a basic structure theorem: all the Haar measures considered here can be reduced to Haar measures on locally compact Hausdorff groups (443L). The argument involves two steps: the reduction to the Hausdorff case, which is elementary, and the completion of a Hausdorff topological group. Since a group carries more than one natural uniform structure we must take care to use the correct one, which in this context is the 'bilateral' uniformity (443H-443I, 443K). On the way I pick up an essential fact about the approximation of Haar measurable sets by Borel sets (443J). Finally, I give Halmos' theorem that Haar measures are completion regular (443M).

In the third part of the section I turn to the special properties of quotient groups of locally compact groups and the corresponding actions, following A.Weil. If X is a locally compact Hausdorff group and Y is a closed subgroup of X, then Y is again a locally compact Hausdorff group, so has Haar measures and a modular function; at the same time, we have a natural action of X on the set of left cosets of Y. It turns out that there is an invariant Radon measure for this action if and only if the modular function of Y matches that of X (443Q). In this case we can express a left Haar measure of X as an integral of measures supported by the cosets of Y (443P). When Y is a normal subgroup, so that X/Y is itself a locally compact Hausdorff group, we can relate the modular functions of X and X/Y (443S). We can apply these results whenever we have a continuous transitive action of a compact group on a compact space (443T).

443A Haar measurability I recall and expand on some facts already noted in 442H. Let X be a topological group carrying Haar measures.

(a) All Haar measures on X, whether left or right, have the same domain Σ, which I call the algebra of 'Haar measurable' sets, and the same null ideal $\mathcal{N}$, which I call the ideal of 'Haar negligible' sets. The corresponding quotient algebra $\mathfrak{A} = \Sigma/\mathcal{N}$, the 'Haar measure algebra', is the Boolean algebra underlying the measure algebra of any Haar measure. Because Haar measures are (by the definition in 441D) quasi-Radon, therefore strictly localizable (415A), $\mathfrak{A}$ is Dedekind complete (322Be). Recall that any semi-finite measure on $\mathfrak{A}$, and in particular any Haar measure on X, gives rise to the same topology and uniformity on $\mathfrak{A}$ (324H), so we may speak of 'the' topology and uniformity of $\mathfrak{A}$.

Because Σ is the domain of a left Haar measure, $xE \in \Sigma$ for every $E \in \Sigma$, $x \in X$; because Σ is the domain of a right Haar measure, $Ex \in \Sigma$ for every $E \in \Sigma$, $x \in X$. Similarly, xE and Ex are Haar negligible whenever E is Haar negligible and $x \in X$. Moreover, $E^{-1} = \{x^{-1} : x \in E\}$ is Haar measurable or Haar negligible whenever E is.

(b) We even have a symmetric notion of 'measurable envelope' in X: for any $A \subseteq X$, there is a Haar measurable set $E \supseteq A$ such that $\mu(E \cap F) = \mu^*(A \cap F)$ for any Haar measurable $F \subseteq X$ and any Haar measure μ on X. **P** Start with a fixed Haar measure μ_0. Then A has a measurable envelope E for μ_0, by 213L. Now to say that 'E is a measurable envelope for A' means just that (i) $A \subseteq E \in \Sigma$ (ii) if $F \in \Sigma$ and $F \subseteq E \setminus A$ then $F \in \mathcal{N}$, so E is also a measurable envelope for A for any other Haar measure on X. **Q**

In this context I will call E a **Haar measurable envelope** of A.

(c) Similarly, we have a notion of **full outer Haar measure**: a subset A of X is of full outer Haar measure if X is a Haar measurable envelope of A, that is, $A \cap E \neq \emptyset$ whenever E is a Haar measurable set which is not Haar negligible, that is, A is of full outer measure for any (left or right) Haar measure on X.

(d) For any Haar measure μ on X, we can identify $L^\infty(\mu)$ with $L^\infty(\mathfrak{A})$ (363I) and $L^0(\mu)$ with $L^0(\mathfrak{A})$ (364Jc). Thus these constructions are independent of μ. The topology of convergence in measure of L^0 is determined by its Riesz space structure (367T); so that this also is independent of the particular Haar measure we may select. Of course the same is true of the norm of L^∞. Note however that the spaces L^p, for $1 \leq p < \infty$, are different for left and right Haar measures on any group which is not unimodular (442Xg), and that even for left Haar measures μ the norm on $L^p(\mu)$ changes if μ is re-normalized.

(e) When it seems appropriate, I will use the phrases **Haar measurable function**, meaning a function measurable with respect to the σ-algebra of Haar measurable sets, and **Haar almost everywhere**, meaning 'on the complement of a Haar negligible set'. Note that we can identify $L^0(\mathfrak{A})$ with the set of equivalence classes in the space $\mathcal{L}^0$, where $\mathcal{L}^0$ is the space of Haar measurable real-valued functions defined Haar-a.e. in X, and $f \sim g$ if $f = g$ Haar-a.e. In the language of §241, $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ for any Haar measure μ on X.

(f) Because $E^{-1} \in \Sigma$ for every $E \in \Sigma$, and $E^{-1} \in \mathcal{N}$ whenever $E \in \mathcal{N}$, we have a canonical automorphism $a \mapsto \tilde{a} : \mathfrak{A} \to \mathfrak{A}$ defined by writing $(E^\bullet)^\sim = (E^{-1})^\bullet$ for every $E \in \Sigma$. Being an automorphism, this must be a homeomorphism for the measure-algebra topology of $\mathfrak{A}$ (324G). In the same way, if $f \in \mathcal{L}^0$ then $\tilde{f} \in \mathcal{L}^0$, where $\tilde{f}(x) = f(x^{-1})$ whenever this is defined, since $\{x : x \in \operatorname{dom} \tilde{f},\ \tilde{f}(x) > \alpha\} = \{x : x \in \operatorname{dom} f,\ f(x) > \alpha\}^{-1} \in \Sigma$ for every α; and we can define an f-algebra automorphism $u \mapsto \tilde{u} : L^0 \to L^0$ by saying that $(f^\bullet)^\sim = (\tilde{f})^\bullet$ for $f \in \mathcal{L}^0$. If we identify L^0 with $L^0(\mathfrak{A})$ rather than with a set of equivalence classes in $\mathcal{L}^0$, then we can define the map $u \mapsto \tilde{u}$ as the Riesz homomorphism associated with the Boolean homomorphism $a \mapsto \tilde{a} : \mathfrak{A} \to \mathfrak{A}$, as in 364R. Note that $\|\tilde{u}\|_\infty = \|u\|_\infty$ for every $u \in L^\infty$, but that (unless X is unimodular) other L^p spaces are not invariant under the involution $\sim$ (442Xg).

(g) If X carries any totally finite (left or right) Haar measure, it is unimodular, and has a unique, two-sided, Haar probability measure (442Ie). (In particular, this is the case whenever X is compact.) For such groups we have L^p-spaces, for $1 \leq p \leq \infty$, defined by the group structure, with canonical norms.

443B Lemma Let X be a topological group and μ a left Haar measure on X. If $E \subseteq X$ is measurable and $\mu E < \infty$, then for any $\epsilon > 0$ there is a neighbourhood U of the identity e such that $\mu(E \triangle xEy) \leq \epsilon$ whenever x, $y \in U$.

proof Set $\delta = \min(1, \epsilon)/(10 + 3\mu E) > 0$. Write $\mathcal{U}$ for the family of open neighbourhoods of e. Because μ is effectively locally finite, there is an open set G_0 of finite measure such that $\mu(E \setminus G_0) \leq \delta$. Let $F \subseteq G_0 \setminus E$

be a closed set such that $\mu F \geq \mu(G_0 \setminus E) - \delta$, and set $G = G_0 \setminus F$, so that $\mu(G \setminus E) \leq \delta$. For $U \in \mathcal{U}$ set $H_U = \text{int}\{x : UxU \subseteq G\}$. Then $\mathcal{H} = \{H_U : U \in \mathcal{U}\}$ is upwards-directed, and has union G, because if $x \in G$ there is a $U \in \mathcal{U}$ such that $UxUU \subseteq G$, so that $x \in H_U$. So there is a $V \in \mathcal{U}$ such that $\mu(G \setminus H_V) \leq \delta$. Recall that the left modular function Δ of X is continuous (442J). So there is a $U \in \mathcal{U}$ such that $U \subseteq V$ and $|\Delta(y) - 1| \leq \delta$ for every $y \in U$.

Now suppose that x, $y \in U$. Set $E_1 = E \cap H_V$. Then $xE_1y \subseteq G$, so

$$\mu(E_1 \cup xE_1y) \leq \mu G \leq \mu E + \delta.$$

On the other hand,

$$\mu E_1 \geq \mu E - \mu(E \setminus G) - \mu(G \setminus H_V) \geq \mu E - 2\delta,$$

$$\mu(xE_1y) = \Delta(y)\mu E_1 \geq (1-\delta)(\mu E - 2\delta) \geq \mu E - (2 + \mu E)\delta.$$

So

$$\mu(E \cap xEy) \geq \mu(E_1 \cap xE_1y) = \mu E_1 + \mu(xE_1y) - \mu(E_1 \cup xE_1y) \geq \mu E - (5 + \mu E)\delta.$$

At the same time,

$$\mu(xEy) = \Delta(y)\mu E \leq (1+\delta)\mu E.$$

So

$$\mu(E \triangle xEy) = \mu E + \mu(xEy) - 2\mu(E \cap xEy) \leq (10 + 3\mu E)\delta \leq \epsilon,$$

as required.

443C Theorem Let X be a topological group carrying Haar measures, and $\mathfrak{A}$ its Haar measure algebra. Then we have continuous actions of X on $\mathfrak{A}$ defined by writing

$$x \bullet_l E^\bullet = (xE)^\bullet,$$

$$x \bullet_r E^\bullet = (Ex^{-1})^\bullet,$$

$$x \bullet_c E^\bullet = (xEx^{-1})^\bullet$$

for Haar measurable sets $E \subseteq X$ and $x \in X$.

proof (a) The functions $\bullet_l$, $\bullet_r$ and $\bullet_c$ are all well defined because the maps $E \mapsto xE$, $E \mapsto Ex$ and $E \mapsto xEx^{-1}$ are all Boolean automorphisms of the algebra Σ of Haar measurable sets preserving the ideal of Haar negligible sets (442G). It is elementary to check that they are actions of X on $\mathfrak{A}$.

Fix a left Haar measure μ on X and let $\bar{\mu}$ be the corresponding measure on $\mathfrak{A}$. Then the topology of $\mathfrak{A}$ is defined by the pseudometrics ρ_a, for $\bar{\mu}a < \infty$, where $\rho_a(b,c) = \bar{\mu}(a \cap (b \triangle c))$.

(b) Now suppose that $x_0 \in X$, $b_0 \in \mathfrak{A}$, $\bar{\mu}a < \infty$ and $\epsilon > 0$. Let E, $F_0 \in \Sigma$ be such that $E^\bullet = a$ and $F_0^\bullet = b_0$; set $\delta = \frac{1}{4}\epsilon > 0$. Note that

$$\mu(x_0^{-1}E \cap F_0) \leq \mu(x_0^{-1}E) = \mu E < \infty.$$

Let U be a neighbourhood of the identity e such that

$$\mu(E \triangle yE) \leq \delta, \quad \mu((x_0^{-1}E \cap F_0) \triangle y(x_0^{-1}E \cap F_0)) \leq \delta$$

whenever $y \in U$ (443B). Set

$$a' = x_0^{-1} \bullet_l a = (x_0^{-1}E)^\bullet.$$

Now suppose that $x \in Ux_0 \cap x_0U^{-1}$ and that $\rho_{a'}(b, b_0) \leq \delta$. Then $\rho_a(x \bullet_l b, x_0 \bullet_l b_0) \leq \epsilon$. **P** Let $F \in \Sigma$ be such that $F^\bullet = b$. Then xx_0^{-1} and $x^{-1}x_0$ both belong to U, so

$$\begin{aligned}\rho_a(x\bullet_l b, x_0\bullet_l b_0) &= \mu(E\cap(xF\triangle x_0F_0))\\ &= \mu(E\cap xF)+\mu(E\cap x_0F_0)-2\mu(E\cap xF\cap x_0F_0)\\ &= \mu(x^{-1}E\cap F)+\mu(E\cap x_0F_0)-2\mu(x^{-1}x_0(x_0^{-1}E\cap F_0)\cap F)\\ &\le \mu(x_0^{-1}E\cap F)+\mu(x^{-1}E\triangle x_0^{-1}E)+\mu(x_0^{-1}E\cap F_0)\\ &\qquad -2\mu(x_0^{-1}E\cap F_0\cap F)+2\mu(x^{-1}x_0(x_0^{-1}E\cap F_0)\triangle(x_0^{-1}E\cap F_0))\\ &\le \mu(x_0^{-1}E\cap(F\triangle F_0))+\mu(E\triangle xx_0^{-1}E)+2\delta\\ &\le \delta+\delta+2\delta=4\delta=\epsilon.\ \blacksquare\end{aligned}$$

As x_0, b_0, a and ϵ are arbitrary, $\bullet_l$ is continuous.

(c) The same arguments, using a right Haar measure to provide pseudometrics defining the topology of $\mathfrak{A}$, show that $\bullet_r$ is continuous. (Or use the method of 443Xe.)

(d) Accordingly the map $(x,y,a)\mapsto x\bullet_l(y\bullet_r a)$ is continuous. So

$$(x,a)\mapsto x\bullet_l(x\bullet_r a)=x\bullet_c a$$

is continuous.

443D Proposition Let X be a topological group carrying Haar measures. If $E\subseteq X$ is Haar measurable but not Haar negligible, and $A\subseteq X$ is not Haar negligible, then

(a) there are x, $y\in X$ such that $A\cap xE$, $A\cap Ey$ are not Haar negligible,

(b) EA and AE both have non-empty interior.

proof (a)(i) Let μ be any left Haar measure on X, and for Borel sets $F\subseteq X$ set

$$\nu F=\sup\{\mu(F\cap IE): I\subseteq X \text{ is finite}\}.$$

It is easy to check that ν is an effectively locally finite τ-additive Borel measure, inner regular with respect to the closed sets, because $\{IE: I\in[X]^{<\omega}\}$ is upwards-directed and μ is quasi-Radon. Moreover,

$$\nu(xF)=\sup_{I\in[X]^{<\omega}}\mu(xF\cap IE)=\sup_{I\in[X]^{<\omega}}\mu(F\cap x^{-1}IE)=\nu F$$

for every Borel set $F\subseteq X$, $x\in X$. Accordingly the c.l.d. version $\tilde\nu$ of ν is a left-translation-invariant quasi-Radon measure on X (415Cb); since $\nu E>0$, $\tilde\nu$ is non-zero and is itself a left Haar measure. Consequently A is not $\tilde\nu$-negligible. Let H be a measurable envelope of A for Haar measure (443Ab). Then H is not Haar negligible, so there is a closed set $F\subseteq H$ which is not Haar negligible, and $\nu F=\tilde\nu F>0$. Thus there is an $x\in X$ such that

$$0<\mu(F\cap xE)\le\mu(H\cap xE)=\mu^*(A\cap xE),$$

and $A\cap xE$ is not Haar negligible.

(ii) Applying the same arguments, but starting with a right Haar measure μ, we see that there is a $y\in X$ such that $A\cap Ey$ is not Haar negligible.

(b) Let μ be a left Haar measure on X, and F a Haar measurable envelope of A. The function $x\mapsto(xE^{-1})^\bullet: X\to\mathfrak{A}$ is continuous, where $\mathfrak{A}$ is the Haar measure algebra of X (443C), so

$$H=\{x:\mu^*(A\cap xE^{-1})>0\}=\{x:F^\bullet\cap(xE^{-1})^\bullet\ne 0\}$$

is open. Now

$$H\subseteq\{x:A\cap xE^{-1}\ne\emptyset\}=AE,$$

so $H\subseteq\operatorname{int}AE$; and E^{-1} is Haar measurable and not Haar negligible, so $H\ne\emptyset$, by (a). Thus $\operatorname{int}AE\ne\emptyset$.

Similarly, using a right Haar measure (or observing that $EA=(A^{-1}E^{-1})^{-1}$), we see that EA has non-empty interior.

443E Corollary Let X be a Hausdorff topological group carrying Haar measures. Then the following are equiveridical:

(i) X is locally compact;

(ii) every Haar measure on X is a Radon measure;

(iii) there is some compact subset of X which is not Haar negligible.

proof (i)⇒(ii) Haar measures are locally finite quasi-Radon measures (441D, 442Aa), so on locally compact Hausdorff spaces must be Radon measures (416G).

(ii)⇒(iii) is obvious, just because any Radon measure is tight (that is, inner regular with respect to the compact sets).

(iii)⇒(i) If $K \subseteq X$ is a compact set which is not Haar negligible, then KK is a compact set with non-empty interior, so X is locally compact (4A5Eg).

443F Later in the chapter we shall need the following straightforward fact.

Lemma Let X be a topological group carrying Haar measures, and Y an open subgroup of X. If μ is a left Haar measure on X, then the subspace measure μ_Y is a left Haar measure on Y. Consequently a subset of Y is Haar measurable or Haar negligible, when regarded as a subset of the topological group Y, iff it is Haar measurable or Haar negligible when regarded as a subset of the topological group X.

proof By 415B, μ_Y is a quasi-Radon measure; because μ is strictly positive, μ_Y is non-zero, and of course it is left-translation-invariant. So it is a Haar measure on Y. The rest follows at once from 442H/443A.

443G We can repeat the ideas of 443C in terms of function spaces, as follows.

Theorem Let X be a topological group with a left Haar measure μ. Let Σ be the domain of μ, $\mathcal{L}^0 = \mathcal{L}^0(\mu)$ the space of Σ-measurable real-valued functions defined almost everywhere on X, and $L^0 = L^0(\mu)$ the corresponding space of equivalence classes (§241).

(a) $a\bullet_l f$, $a\bullet_r f$ and $a\bullet_c f$ (definitions: 441Ac) belong to $\mathcal{L}^0$ for every $f \in \mathcal{L}^0$ and $a \in X$.

(b) If $a \in X$, then $\operatorname{ess\,sup} |a\bullet_l f| = \operatorname{ess\,sup} |a\bullet_r f| = \operatorname{ess\,sup} |f|$ for every $f \in \mathcal{L}^\infty = \mathcal{L}^\infty(\mu)$, where $\operatorname{ess\,sup} |f|$ is the essential supremum of $|f|$ (243Da). For $1 \leq p < \infty$, $\|a\bullet_l f\|_p = \|f\|_p$ and $\|a\bullet_r f\|_p = \Delta(a)^{-1/p}\|f\|_p$ for every $f \in \mathcal{L}^p = \mathcal{L}^p(\mu)$, where Δ is the left modular function of X.

(c) We have actions of X on L^0 defined by setting

$$a\bullet_l f^\bullet = (a\bullet_l f)^\bullet,$$

$$a\bullet_r f^\bullet = (a\bullet_r f)^\bullet,$$

$$a\bullet_c f^\bullet = (a\bullet_c f)^\bullet$$

for $a \in X$ and $f \in \mathcal{L}^0$. If we give L^0 its topology of convergence in measure these actions are continuous.

(d) For $1 \leq p \leq \infty$ the formulae of (c) define actions of X on $L^p = L^p(\mu)$, and $\|a\bullet_l u\|_p = \|u\|_p$ for every $u \in L^p$, $a \in X$.

(e) For $1 \leq p < \infty$ these actions are continuous.

proof (a) Let $f \in \mathcal{L}^0$. Then $F = \operatorname{dom} f$ is conegligible, so $aF = \operatorname{dom} a\bullet_l f$ and $Fa^{-1} = \operatorname{dom} a\bullet_r f$ are conegligible (442G). For any $\alpha \in \mathbb{R}$, set $E_\alpha = \{x : x \in F, f(x) < \alpha\}$; then $\{x : (a\bullet_l f)(x) < \alpha\} = aE_\alpha$ and $\{x : (a\bullet_r f)(x) < \alpha\} = E_\alpha a^{-1}$ are measurable, so $a\bullet_l f$ and $a\bullet_r f$ are measurable. Thus $a\bullet_l f$ and $a\bullet_r f$ belong to $\mathcal{L}^0$. It follows at once that $a\bullet_c f = a\bullet_l(a\bullet_r f)$ belongs to $\mathcal{L}^0$.

(b)(i) For $\alpha \geq 0$,

$$\begin{aligned}\operatorname{ess\,sup} |f| \leq \alpha &\iff |f(x)| \leq \alpha \text{ for almost all } x\\ &\iff |(a\bullet_l f)(x)| \leq \alpha \text{ for almost all } x\\ &\iff |(a\bullet_r f)(x)| \leq \alpha \text{ for almost all } x\end{aligned}$$

because the null ideal is invariant under both left and right translations. So $\operatorname{ess\,sup} |f| = \operatorname{ess\,sup} |a\bullet_l f| = \operatorname{ess\,sup} |a\bullet_r f|$.

(ii) For $1 \leq p < \infty$,

$$\|a\bullet_l f\|_p^p = \int |(a\bullet_l f)(x)|^p \mu(dx) = \int |f(a^{-1}x)|^p \mu(dx) = \int |f(x)|^p \mu(dx)$$

(441J)

$$= \|f\|_p^p,$$

$$\|a\bullet_r f\|_p^p = \int |(a\bullet_r f)(x)|^p \mu(dx) = \int |f(xa)|^p \mu(dx) = \Delta(a^{-1}) \int |f(x)|^p \mu(dx)$$

(442Kc)

$$= (\Delta(a)^{-1/p}\|f\|_p)^p.$$

(c)(i) I have already checked that $a\bullet_l f$, $a\bullet_r f$ and $a\bullet_c f$ belong to $\mathcal{L}^0$ whenever $f \in \mathcal{L}^0$ and $a \in X$. If f, $g \in \mathcal{L}^0$ and $f =_{\text{a.e.}} g$, let E be the conegligible set $\{x : x \in \operatorname{dom} f \cap \operatorname{dom} g,\ f(x) = g(x)\}$; then aE and Ea^{-1} and aEa^{-1} are conegligible and

$$(a\bullet_l f)(x) = (a\bullet_l g)(x) \text{ for every } x \in aE, \quad (a\bullet_r f)(x) = (a\bullet_r g)(x) \text{ for every } x \in Ea^{-1},$$

$$(a\bullet_c f)(x) = (a\bullet_c g)(x) \text{ for every } x \in aEa^{-1},$$

so $a\bullet_l f =_{\text{a.e.}} a\bullet_l g$, $a\bullet_r f =_{\text{a.e.}} a\bullet_r g$ and $a\bullet_c f =_{\text{a.e.}} a\bullet_c g$. Accordingly the formulae given define functions $\bullet_l$, $\bullet_r$ and $\bullet_c$ from $X \times L^0$ to L^0. They are actions just because the original $\bullet_l$, $\bullet_r$ and $\bullet_c$ are actions of X on $\mathcal{L}^0$ (441Ac-441Ad).

(ii) For the next step, it will be more convenient to work with the space $\mathcal{L}^0_{\text{strict}}$ of Haar measurable real-valued functions defined on the whole of X. I will use a characterization of convergence in measure from 245F: a subset W of L^0 is open iff whenever $f_0^\bullet \in W$ there are a set E of finite measure and an $\epsilon > 0$ such that $f^\bullet \in W$ whenever $\mu\{x : x \in E,\ |f(x) - f_0(x)| > \epsilon\} \leq \epsilon$. Now if E is a measurable set of finite measure, $f \in \mathcal{L}^0_{\text{strict}}$ and $\epsilon > 0$, there is a neighbourhood U of the identity e of X such that $\mu\{x : x \in E,\ |f(axb) - f(x)| > \epsilon\} \leq \epsilon$ for all a, $b \in U$. **P** Let $m \geq 1$ be such that $\mu\{x : x \in E,\ |f(x)| \geq m\epsilon\} \leq \frac{1}{2}\epsilon$. For $-m \leq k < m$, set $E_k = \{x : x \in E,\ k\epsilon \leq f(x) < (k+1)\epsilon\}$. By 443B, there is a neighbourhood U of e such that

$$\mu(E_k \triangle a^{-1}E_k b^{-1}) \leq \frac{\epsilon}{4m}$$

whenever a, $b \in U$ and $-m \leq k < m$. Now, for a, $b \in U$,

$$\{x : x \in E,\ |f(axb) - f(x)| > \epsilon\} \subseteq \{x : x \in E,\ |f(x)| \geq m\epsilon\} \cup \textstyle\bigcup_{k=-m}^{m-1}(E_k \triangle a^{-1}E_k b^{-1})$$

has measure at most

$$\frac{\epsilon}{2} + 2m\frac{\epsilon}{4m} = \epsilon. \ \textbf{Q}$$

(iii) Let E be a measurable set of finite measure, a_0, $b_0 \in X$, $f_0 \in \mathcal{L}^0_{\text{strict}}$ and $\epsilon > 0$. Set $\delta = \epsilon/(2 + 3\Delta(b_0^{-1})) > 0$. Note that $\mu(a_0^{-1}Eb_0) = \Delta(b_0)\mu E$ is finite. Let U be a neighbourhood of e such that

$$\mu\{x : x \in a_0^{-1}Eb_0,\ |f_0(yxz) - f_0(x)| \geq \delta\} \leq \delta,$$

$$\mu(yEz \triangle E) \leq \delta, \quad \Delta(y) \leq 2$$

whenever y, $z \in U$.

Now suppose that $a \in Ua_0 \cap a_0U^{-1}$ and $b \in U^{-1}b_0 \cap b_0U$ and that $f \in \mathcal{L}^0_{\text{strict}}$ is such that $\mu\{x : x \in a_0^{-1}Eb_0,\ |f(x) - f_0(x)| \geq \delta\} \leq \delta$. In this case,

$$\begin{aligned}\{x : x \in E,\ |f(a^{-1}xb) - f_0(a_0^{-1}xb_0)| \geq \epsilon\}&\\ \subseteq \{x : x \in E,\ |f(a^{-1}xb) - f_0(a^{-1}xb)| \geq \delta\}&\\ \cup \{x : x \in E,\ |f_0(a^{-1}xb) - f_0(a_0^{-1}xb_0)| \geq \delta\}&\\ \subseteq (E \triangle aa_0^{-1}Eb_0b^{-1}) \cup \{x : x \in aa_0^{-1}Eb_0b^{-1},\ |f(a^{-1}xb) - f_0(a^{-1}xb)| \geq \delta\}&\\ \cup a_0\{w : w \in a_0^{-1}Eb_0,\ |f_0(a^{-1}a_0wb_0^{-1}b) - f_0(w)| \geq \delta\}b_0^{-1}&\\ \subseteq (E \triangle aa_0^{-1}Eb_0b^{-1}) \cup a\{w : w \in a_0^{-1}Eb_0,\ |f(w) - f_0(w)| \geq \delta\}b^{-1}&\\ \cup a_0\{w : w \in a_0^{-1}Eb_0,\ |f_0(a^{-1}a_0wb_0^{-1}b) - f_0(w)| \geq \delta\}b_0^{-1}&\end{aligned}$$

has measure at most

$$\begin{aligned}\delta + \delta\Delta(b^{-1}) + \delta\Delta(b_0^{-1}) &\leq \delta(1 + \Delta(b_0b^{-1})\Delta(b_0^{-1}) + \Delta(b_0^{-1}))\\ &\leq \delta(1 + 3\Delta(b_0^{-1})) \leq \epsilon\end{aligned}$$

because aa_0^{-1}, $a^{-1}a_0$, b_0b^{-1} and $b_0^{-1}b$ all belong to U. Because E and ϵ are arbitrary, the function $(a, b, u) \mapsto a\bullet_l(b\bullet_r u)$ is continuous at $(a_0, b_0, f_0^\bullet)$; as a_0, b_0 and f_0 are arbitrary, $\bullet_l$ and $\bullet_r$ are continuous. It follows at once that $\bullet_c$ is continuous, since $a\bullet_c u = a\bullet_l(a\bullet_r u)$.

(d) follows at once from (b) and (c).

(e) Fix $p \in [1, \infty[$.

(i) If $u \in L^p$ and $\epsilon > 0$, there is a neighbourhood U of e such that $\|u - y\bullet_l(z\bullet_r u)\|_p \leq \epsilon$ whenever y, $z \in U$. **P** When u is of the form $(\chi E)^\bullet$, where $\mu E < \infty$, we have

$$y\bullet_l(z\bullet_r u) = \chi(yEz^{-1})^\bullet, \quad \|u - y\bullet_l(z\bullet_r u)\|_p = \mu(E \triangle yEz^{-1})^{1/p},$$

so the result is immediate from 443B. If $u = \sum_{i=0}^n \alpha_i(\chi E_i)^\bullet$, where every E_i is of finite measure, then, setting $u_i = (\chi E_i)^\bullet$ for each i,

$$\|u - y\bullet_l(z\bullet_r u)\|_p \leq \sum_{i=0}^n |\alpha_i| \|u_i - y\bullet_l(z\bullet_r u_i)\|_p \leq \epsilon$$

whenever y and z are close enough to e. In general, there is a v of this form such that $\|u - v\|_p \leq \frac{1}{4}\epsilon$. If we take a neighbourhood U of e such that $\|v - y\bullet_l(z\bullet_r v)\|_p \leq \frac{1}{4}\epsilon$ and $\Delta(z)^{-1/p} \leq 2$ whenever y, $z \in U$, then

$$\|y\bullet_l(z\bullet_r u) - y\bullet_l(z\bullet_r v)\|_p = \Delta(z)^{-1/p}\|u - v\|_p \leq \frac{1}{2}\epsilon$$

whenever $z \in U$, so

$$\begin{aligned}\|u - y\bullet_l(z\bullet_r u)\|_p &\leq \|u - v\|_p + \|v - y\bullet_l(z\bullet_r v)\|_p + \|y\bullet_l(z\bullet_r v) - y\bullet_l(z\bullet_r u)\|_p\\ &\leq \epsilon\end{aligned}$$

whenever y, $z \in U$. **Q**

(ii) Now suppose that $u_0 \in L^p$, a_0, $b_0 \in X$ and $\epsilon > 0$. Set $v_0 = a_0\bullet_l(b_0\bullet_r u_0)$ and $\delta = \epsilon/(1 + 2\Delta(b_0)^{-1/p}) > 0$. Let U be a neighbourhood of e such that $\Delta(y)^{-1/p} \leq 2$ and $\|v_0 - y\bullet_l(z\bullet_r v_0)\|_p \leq \delta$ whenever y, $z \in U$. If $a \in Ua_0$, $b \in Ub_0$ and $\|u - u_0\|_p \leq \delta$, then

$$\begin{aligned}\|a\bullet_l(b\bullet_r u) - v_0\|_p &\leq \|a\bullet_l(b\bullet_r u) - a\bullet_l(b\bullet_r u_0)\|_p + \|a\bullet_l(b\bullet_r u_0) - v_0\|_p\\ &= \Delta(b)^{-1/p}\|u - u_0\|_p + \|aa_0^{-1}\bullet_l(bb_0^{-1}\bullet_r v_0) - v_0\|_p\\ &\leq \Delta(bb_0^{-1})^{-1/p}\Delta(b_0)^{-1/p}\delta + \delta \leq \delta(1 + 2\Delta(b_0)^{-1/p}) = \epsilon.\end{aligned}$$

As ϵ is arbitrary, $(a, b, u) \mapsto a\bullet_l(b\bullet_r u)$ is continuous at (a_0, b_0, u_0).

As in (c), this is enough to show that $\bullet_l$, $\bullet_r$ and $\bullet_c$ are all continuous actions.

Remark I have written this out for a left Haar measure μ, since the spaces $L^p(\mu)$ depend on this; if ν is a right Haar measure, and X is not unimodular, then $L^p(\mu) \neq L^p(\nu)$ for $1 \leq p < \infty$. But recall that the topology of convergence in measure on L^0 depends only on its Riesz space structure (367T), while $L^0(\mu)$ can be identified, as Riesz space, with $L^0(\nu)$ (443Ad), so (c) above, and the case $p = \infty$ of (b) and (d), are two-sided; they belong to the theory of the Haar measure algebra.

443H Theorem Let X be a topological group carrying Haar measures. Then there is a neighbourhood of the identity which is totally bounded for the bilateral uniformity on X.

proof Let μ be a left Haar measure on X. Let V_0 be a neighbourhood of the identity e such that $\mu V_0 < \infty$ (442Aa). Let V be a neighbourhood of e such that $VV \subseteq V_0$ and $V^{-1} = V$.

? Suppose, if possible, that V is not totally bounded for the bilateral uniformity on X. By 4A5Oa, one of the following must occur:

case 1 There is an open neighbourhood U of e such that $V \not\subseteq IU$ for any finite set $I \subseteq X$. In this case, we may choose a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in V inductively such that $x_n \notin x_i U$ whenever $i < n$. Let U_1 be an open neighbourhood of e such that $U_1 \subseteq V$ and $U_1 U_1^{-1} \subseteq U$; then $\langle x_n U_1 \rangle_{n\in\mathbb{N}}$ is disjoint. Since $\mu(x_n U_1) = \mu U_1 > 0$ for every n (by the other clause in 442Aa), $\mu(\bigcup_{n\in\mathbb{N}} x_n U_1) = \infty$; but $x_n U_1 \subseteq V_0$ for every n, so this is impossible.

case 2 There is an open neighbourhood U of e such that $V \not\subseteq UI$ for any finite set $I \subseteq X$. So we may choose a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in V inductively such that $x_n \notin U x_i$ whenever $i < n$. Let U_1 be an open neighbourhood of e such that $U_1 \subseteq V$ and $U_1 U_1^{-1} \subseteq U$; then $\langle U_1^{-1} x_n \rangle_{n\in\mathbb{N}}$ is disjoint, so $\langle x_n^{-1} U_1 \rangle_{n\in\mathbb{N}}$ is also disjoint. Since $\mu(x_n^{-1} U_1) = \mu U_1 > 0$ for every n, $\mu(\bigcup_{n\in\mathbb{N}} x_n^{-1} U_1) = \infty$; but $x_n^{-1} U_1 \subseteq V_0$ for every n, so this also is impossible. **X**

Thus V is totally bounded for the bilateral uniformity on X, and we have the required totally bounded neighbourhood of e.

443I Lemma Let X be a topological group. If $A \subseteq X$ is totally bounded for the bilateral uniformity of X, it has finite outer measure for any (left or right) Haar measure on X.

proof If μ is a Haar measure on X, let U be an open neighbourhood of the identity e of finite measure. There is a finite set I such that $A \subseteq UI$, so that $\mu^* A \leq \mu U \sum_{x\in I} \Delta(x)$ (if μ is a left Haar measure) and $\mu^* A \leq \#(I)\mu U$ (if μ is a right Haar measure); in either case, $\mu^* A < \infty$.

443J Proposition Let X be a topological group carrying Haar measures.

(a) There is an open-and-closed subgroup Y of X such that, for any Haar measure μ on X, Y can be covered by countably many open sets of finite measure.

(b)(i) If $E \subseteq X$ is any Haar measurable set, there are an F_σ set $E' \subseteq E$ and a G_δ set $E'' \supseteq E$ such that $E'' \setminus E'$ is Haar negligible.

(ii) Every Haar negligible set is included in a Haar negligible Borel set, and for every Haar measurable set E there is a Borel set F such that $E \triangle F$ is Haar negligible.

(iii) The Haar measure algebra $\mathfrak{A}$ of X may be identified with $\mathcal{B}/\mathcal{I}$, where $\mathcal{B}$ is the Borel σ-algebra of X and $\mathcal{I}$ is the ideal of Haar negligible Borel sets.

(iv) Every member of $L^0(\mathfrak{A})$ is the equivalence class of some Borel measurable function from X to $\mathbb{R}$. Every member of $L^\infty(\mathfrak{A})$ is the equivalence class of a bounded Borel measurable function from X to $\mathbb{R}$.

proof (a) Let V be an open neighbourhood of the identity which is totally bounded for the bilateral uniformity of X (443H); we may suppose that $V^{-1} = V$. Set $Y = V \cup VV \cup VVV \cup VVVV \cup \ldots$. Then Y is an open subgroup of X, therefore also closed (4A5Ek). By 4A5Ob, every power of V is totally bounded, so Y is a countable union of totally bounded sets. If μ is any left Haar measure on X, then any totally bounded set has finite outer measure for μ (443I). Thus Y is a countable union of sets of finite measure for μ. The same argument applies to right Haar measures, so Y is a subgroup of the required form.

(b)(i) Let $E \subseteq X$ be a Haar measurable set, and fix a left Haar measure μ on X. Take the open subgroup Y of (a), and index the set of its left cosets as $\langle Y_i \rangle_{i\in I}$; because any translate of a totally bounded set is totally bounded (4A5Ob), each Y_i is an open set expressible as $\bigcup_{n\in\mathbb{N}} H_{in}$, where every H_{in} is a totally bounded open set, so that μH_{in} is finite.

For $i \in I$ and $m, n \in \mathbb{N}$ there is a closed set $F_{imn} \subseteq E \cap H_{im}$ such that $\mu F_{imn} \geq \mu(E \cap H_{im}) - 2^{-n}$. Set $F_{mn} = \bigcup_{i\in I} F_{imn}$ for each $m, n \in \mathbb{N}$; then F_{mn} is closed (4A2Bb). So $E' = \bigcup_{m,n\in\mathbb{N}} F_{mn}$ is F_σ. For each $i \in I$,

$$(E \setminus E') \cap Y_i \subseteq \bigcup_{m\in\mathbb{N}} \big(E \cap H_{in} \setminus \bigcup_{n\in\mathbb{N}} F_{imn}\big)$$

is negligible; thus $\{G : G \subseteq X \text{ is open}, \mu(G \cap E \setminus E') = 0\}$ covers X and $E \setminus E'$ must be negligible (414Ea).

In the same way, there is an F_σ set $F^* \subseteq X \setminus E$ such that $(X \setminus E) \setminus F^*$ is negligible; now $E'' = X \setminus F^*$ is G_δ and $E'' \setminus E$ is negligible, so $E'' \setminus E'$ also is. (I am speaking here as if 'negligible' meant 'μ-negligible'. But of course this is the same thing as the 'Haar negligible' of the statement of the proposition.)

(ii), (iii), (iv) follow at once.

443K Theorem Let X be a Hausdorff topological group carrying Haar measures. Then the completion $\widehat{X}$ of X under its bilateral uniformity is a locally compact Hausdorff group, and X is of full outer Haar measure in $\widehat{X}$. Any (left or right) Haar measure on X is the subspace measure corresponding to a Haar measure (of the same type) on $\widehat{X}$.

proof (a) By 4A5N, $\widehat{X}$ is a locally compact Hausdorff group in which X is embedded as a dense subgroup.

(b) Let μ be a left Haar measure on X. Then there is a Radon measure $\tilde{\mu}$ on $\widehat{X}$ such that μ is the subspace measure $\tilde{\mu}_X$. **P** For Borel sets $E \subseteq \widehat{X}$, set $\nu E = \mu(X \cap E)$. Then ν is a Borel measure, and it is τ-additive because μ is. Because X is locally compact, it follows that

$$\nu G = \sup\{\nu K : K \subseteq G \text{ is compact}\}$$

for every open set $G \subseteq X$. If $z \in \widehat{X}$, it has a compact neighbourhood V in $\widehat{X}$; now V must be totally bounded for the bilateral uniformity of $\widehat{X}$ (4A2Jd), so $V \cap X$ is totally bounded for the bilateral uniformity of X (4A5Ma), and $\nu V = \mu(V \cap X)$ is finite (443I). Thus ν is locally finite. If $\nu E > 0$, there is an open set $H \subseteq X$ such that $\mu H < \infty$ and $\mu(H \cap X \cap E) > 0$, because μ is effectively locally finite; now there is an open set $G \subseteq \widehat{X}$ such that $H = X \cap G$, so that $\nu G < \infty$ and $\nu(G \cap E) > 0$. Thus ν is effectively locally finite.

By 416F(iii), ν has an extension to a Radon measure $\tilde{\mu}$ on $\widehat{X}$. Since $\tilde{\nu} K = \nu K = 0$ whenever $K \subseteq \widehat{X} \setminus X$ is compact, X is of full outer measure for $\tilde{\mu}$. Accordingly

$$\tilde{\mu}_X(G \cap X) = \tilde{\mu} G = \nu G = \mu(G \cap X)$$

for every open set $G \subseteq \widehat{X}$, and $\tilde{\mu}_X = \mu$, because they are quasi-Radon measures agreeing on the open sets (415B, 415H(iii)). **Q**

(c) Continuing the argument of (b), $\tilde{\mu}$ is a left Haar measure on $\widehat{X}$. **P** Let $G \subseteq \widehat{X}$ be open, and $z \in \widehat{X}$. If $K \subseteq zG$ is compact, then $z^{-1}K \subseteq G$, and $\{w : w \in \widehat{X}, wK \subseteq G\}$ is a non-empty open set (4A5Ei), so meets X. Take $x \in X$ such that $xK \subseteq G$; then

$$\tilde{\mu} G = \mu(X \cap G) \geq \mu(X \cap xK) = \mu(x(X \cap K)) = \mu(X \cap K) = \tilde{\mu} K.$$

As K is arbitrary, $\tilde{\mu}(zG) \leq \tilde{\mu} G$. By 441Ba, $\tilde{\mu}$ is invariant under the left action of $\widehat{X}$ on itself, that is, is a left Haar measure. **Q**

We know that X is of full outer measure for $\tilde{\mu}$, so this shows that it is of full outer Haar measure in $\widehat{X}$.

(d) The same arguments, looking at Gz and Kz^{-1} in (c), show that if μ is a right Haar measure on X it is the subspace measure $\tilde{\mu}_X$ for a right Haar measure $\tilde{\mu}$ on $\widehat{X}$.

443L Corollary Let X be any topological group with a Haar measure μ. Then we can find a locally compact Hausdorff topological group Z, with a Haar measure $\tilde{\nu}$, and a continuous homomorphism $\phi : X \to Z$, inverse-measure-preserving for μ and $\tilde{\nu}$, such that μ is inner regular with respect to $\{\phi^{-1}[K] : K \subseteq Z \text{ is compact}\}$. If $E \subseteq X$ is Haar measurable, we can find a Haar measurable set $F \subseteq Z$ such that $\phi^{-1}[F] \subseteq E$ and $E \setminus \phi^{-1}[F]$ is Haar negligible.

proof (a) Let $Y \subseteq X$ be the closure of $\{e\}$, where e is the identity of X. Then Y is a closed normal subgroup of X, and if $\phi : X \to X/Y$ is the quotient map, every open (or closed) subset of X is of the form $\phi^{-1}[H]$ for some open (or closed) set $H \subseteq Y$ (4A5Kb).

Consider the image measure $\nu = \mu\phi^{-1}$ on X/Y. This is quasi-Radon. **P** Because μ is a complete τ-additive topological measure, so is ν. If $F \in \operatorname{dom}\nu$ and $\nu F > 0$, there is an open set $G \subseteq X$ such that $\mu G < \infty$ and $\mu(G \cap \phi^{-1}[F]) > 0$; now $G = \phi^{-1}[H]$ for some open set $H \subseteq X/Y$, and $\nu H = \mu G$ is finite,

while $\nu(H \cap F) = \mu(G \cap \phi^{-1}[F]) > 0$. Thus ν is effectively locally finite (therefore semi-finite). Again, if $F \in \operatorname{dom}\nu$ and $\nu F > \gamma$, there is a closed set $E \subseteq \phi^{-1}[F]$ such that $\mu E \geq \gamma$; now E is expressible as $\phi^{-1}[H]$ for some closed set $H \subseteq X/Y$; because ϕ is surjective, $H \subseteq F$, and $\nu H = \mu F \geq \gamma$. Thus ν is inner regular with respect to the closed sets. Finally, suppose that $F \subseteq X/Y$ is such that $F \cap F' \in \operatorname{dom}\nu$ whenever $\nu F' < \infty$. If $E \subseteq X$ is a closed set of finite measure, it is of the form $\phi^{-1}[F']$ where $\nu F' = \mu E < \infty$, so $F' \cap F \in \operatorname{dom}\nu$ and $E \cap f^{-1}[F] \in \operatorname{dom}\mu$; by 412Ja, we can conclude that $\phi^{-1}[F] \in \operatorname{dom}\mu$ and $F \in \operatorname{dom}\nu$. Thus ν is locally determined and is a quasi-Radon measure. **Q**

We find also that ν is a left Haar measure. **P** If $z \in X/Y$ and $F \in \operatorname{dom}\nu$, express z as $\phi(x)$ where $x \in X$; then $\phi^{-1}[zF] = x\phi^{-1}[F]$, so

$$\nu(zF) = \mu(x\phi^{-1}[F]) = \mu\phi^{-1}[F] = \nu F. \quad \textbf{Q}$$

(b) Thus X/Y is a topological group with a left Haar measure ν. Because Y is closed, X/Y is Hausdorff (4A5Jb). We can therefore form its completion $Z = \widehat{X/Y}$, a locally compact Hausdorff group, and find a left Haar measure $\tilde{\nu}$ on Z such that ν is the corresponding subspace measure on X/Y, which is of full outer measure in Z (443K). The embedding $X/Y \subseteq Z$ is therefore inverse-measure-preserving for ν and $\tilde{\nu}$, so that ϕ, regarded as a map from X to Z, is inverse-measure-preserving for μ and $\tilde{\nu}$. Also, of course, $\phi : X \to Z$ is a continuous homomorphism.

If $E \subseteq X$ and $\mu E > \gamma$, there is a closed set $E' \subseteq E$ such that $\mu E' > \gamma$. Now E' is of the form $\phi^{-1}[F]$ where $F \subseteq X/Y$ is closed and $\nu F = \mu E' > \gamma$. Next, F is of the form $(X/Y) \cap F'$ where $F' \subseteq Z$ is closed and $\tilde{\nu}F' = \nu F > \gamma$. So there is a compact set $K \subseteq F'$ such that $\tilde{\nu}K \geq \gamma$, and we have

$$\phi^{-1}[K] \subseteq \phi^{-1}[F'] = \phi^{-1}[F] \subseteq E, \quad \mu\phi^{-1}[K] = \nu(K \cap (X/Y)) = \tilde{\nu}K \geq \gamma.$$

As E and γ are arbitrary, μ is inner regular with respect to $\{\phi^{-1}[K] : K \subseteq Z \text{ is compact}\}$.

(c) If $E \subseteq X$ is Haar measurable, then by 443J(b-i) there is an F_σ set $E' \subseteq E$ such that $E \setminus E'$ is Haar negligible. Now there are an F_σ set $H \subseteq X/Y$ such that $E' = \phi^{-1}[H]$ and an F_σ set $F \subseteq Z$ such that $H = (X/Y) \cap F$, in which case $E' = \phi^{-1}[F]$, and $E \setminus \phi^{-1}[F]$ is Haar negligible.

443M Theorem (HALMOS 50) Let X be a topological group and μ a Haar measure on X. Then μ is completion regular.

proof (a) Suppose first that μ is a left Haar measure and that X is locally compact and Hausdorff. In this case any self-supporting compact set $K \subseteq X$ is a zero set. **P** For each $n \in \mathbb{N}$, there is an open neighbourhood U_n of the identity such that $\mu(K \triangle xK) \leq 2^{-n}$ for every $x \in U_n$ (443B); we may suppose that $\overline{U}_{n+1} \subseteq U_n$ for each n. Each set U_nK is open (4A5Ed), so $\bigcap_{n\in\mathbb{N}} U_nK$ is a G_δ set. **?** If $K \neq \bigcap_{n\in\mathbb{N}} U_nK$, there is an $x \in \bigcap_{n\in\mathbb{N}} U_nK \setminus K$. For each $n \in \mathbb{N}$, there are $y_n \in U_n$, $z_n \in K$ such that $x = y_nz_n$. Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. Then $z = \lim_{n\to\mathcal{F}} z_n$ is defined in K, so

$$xz^{-1} = \lim_{n\to\mathcal{F}} xz_n^{-1} = \lim_{n\to\mathcal{F}} y_n$$

is defined in X; because $y_n \in \overline{U}_i$ for every $i \leq n$,

$$xz^{-1} \in \textstyle\bigcap_{i\in\mathbb{N}} \overline{U}_i = \bigcap_{i\in\mathbb{N}} U_i.$$

Consequently $\mu(xz^{-1}K \triangle K) = 0$; because μ is left-translation-invariant, $\mu(K \setminus zx^{-1}K) = 0$. But as $x \notin K$, $z \in K \setminus zx^{-1}K$ and $K \cap (X \setminus zx^{-1}K)$ is non-empty. And $zx^{-1}K$ is closed, so $X \setminus zx^{-1}K$ is open and K is not self-supporting, contrary to hypothesis. **X**

Thus $K = \bigcap_{n\in\mathbb{N}} U_nK$ is a G_δ set. Being a compact G_δ set in a completely regular Hausdorff space, it is a zero set (4A2F(h-v)). **Q**

Since μ is surely inner regular with respect to the compact self-supporting sets (414F), it is inner regular with respect to the zero sets, and is completion regular.

(b) Now suppose that μ is a left Haar measure on an arbitrary topological group X. By 443L, we can find a locally compact Hausdorff topological group Z, a continuous homomorphism $\phi : X \to Z$ and a left Haar measure $\tilde{\nu}$ on Z such that ϕ is inverse-measure-preserving for μ and $\tilde{\nu}$ and μ is inner regular with respect to $\{\phi^{-1}[K] : K \subseteq Z \text{ is compact}\}$. Now if $E \in \operatorname{dom}\mu$ and $\gamma < \mu E$, there is a compact set $K \subseteq Z$ such that $\phi^{-1}[K] \subseteq E$ and $\nu K > \gamma$. Next, there is a zero set $L \subseteq K$ such that $\nu L \geq \gamma$; in which case $\phi^{-1}[L] \subseteq E$ is a zero set and $\mu\phi^{-1}[L] \geq \gamma$. Thus μ is inner regular with respect to the zero sets and is completion regular.

(c) Finally, if μ is a right Haar measure on a topological group X, let μ' be the corresponding left Haar measure, setting $\mu' E = \mu E^{-1}$ for Haar measurable sets E. Then μ' is inner regular with respect to the zero sets; because $x \mapsto x^{-1} : X \to X$ is a homeomorphism, so is μ.

443N I give a simple result showing how the measure-theoretic properties of groups carrying Haar measures have consequences of unexpected kinds.

Proposition Let be X is a topological group carrying Haar measures (for instance, X might be any locally compact Hausdorff group).

(i) Let G be a regular open subset of X. Then G is a cozero set.

(ii) Let F be a nowhere dense subset of X. Then F is included in a nowhere dense zero set.

proof (a) Suppose to begin with that X is locally compact, σ-compact and Hausdorff. Let μ be a left Haar measure on X; then μ is σ-finite, because X is covered by a sequence of compact sets, which must all have finite measure.

(i) Write $\mathcal{U}$ for the family of open neighbourhoods of the identity e of X. For each $U \in \mathcal{U}$, set $H_U = \operatorname{int}\{x : xU \subseteq G\}$; then $\{H_U : U \in \mathcal{U}\}$ is an upwards-directed family of open sets with union G, as in the proofs of 442Ab and 442B, so $G^\bullet = \sup_{U\in\mathcal{U}} H_U^\bullet$ in the measure algebra $\mathfrak{A}$ of μ. Because μ is σ-finite, $\mathfrak{A}$ is ccc (322G) and there is a sequence $\langle U_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{U}$ such that $G^\bullet = \sup_{n\in\mathbb{N}} H_{U_n}^\bullet$ (316E). In this case, $G \setminus \bigcup_{n\in\mathbb{N}} H_{U_n}$ is negligible, so must have empty interior.

By 4A5S, there is a closed normal subgroup Y of X, included in $\bigcap_{n\in\mathbb{N}} U_n$, such that X/Y is metrizable. Let $\pi : X \to X/Y$ be the canonical map.

For each $n \in \mathbb{N}$, $Q_n = \pi[H_{U_n}]$ is open (4A5Ja), and

$$H_{U_n} \subseteq \pi^{-1}[Q_n] = H_{U_n}Y \subseteq H_{U_n}U_n \subseteq G.$$

So

$$\overline{G} = \overline{\bigcup_{n\in\mathbb{N}} H_{U_n}} = \overline{\bigcup_{n\in\mathbb{N}} \pi^{-1}[Q_n]}.$$

Setting $Q = \operatorname{int}\overline{\bigcup_{n\in\mathbb{N}} Q_n}$, and using 4A5Ja and 4A2B(f-ii), we see that

$$\pi^{-1}[Q] = \operatorname{int}\overline{\bigcup_{n\in\mathbb{N}} \pi^{-1}[Q_n]} = \operatorname{int}\overline{G} = G$$

(this is where I use the hypothesis that G is a regular open set). But Q, being an open set in a metrizable space, is a cozero set (4A2Lc), so $G = \pi^{-1}[Q]$ is a cozero set (4A2C(b-iv)), as required by (i).

(ii) Now consider the nowhere dense set $F \subseteq X$. This time, let $\mathcal{G}$ be a maximal disjoint family of cozero subsets of $X \setminus F$. Then $\mathcal{G}$ is countable, again because μ is σ-finite, and $\bigcup\mathcal{G}$ is dense, because the topology of X is completely regular. So $X \setminus \bigcup\mathcal{G}$ is a nowhere dense zero set including F.

(b) Next, suppose that X is any locally compact Hausdorff topological group. Then X has a σ-compact open subgroup X_0 (4A5El). By (a), any regular open set in X_0 is a cozero set in X_0. The same applies to all the (left) cosets of X_0, because these are homeomorphic to X_0.

If C is any coset of X_0, then $G \cap C$ is a regular open set in C, so is a cozero set in C. But as the left cosets of X_0 form a partition of X into open sets, G is also a cozero set in X (4A2C(b-vii)).

Similarly, $F \cap C$ is nowhere dense in C for every left coset C of X_0, so is included in a nowhere dense zero set in C, and the union of these will be a nowhere dense zero set in X including F.

(c) Now suppose that X is a Hausdorff group carrying Haar measures. Then its completion $\widehat{X}$ under the bilateral uniformity is locally compact (443K). $H = \operatorname{int}_{\widehat{X}} \overline{G}^{\widehat{X}}$ is a regular open set in $\widehat{X}$, therefore a cozero set in $\widehat{X}$, by (b). But it is easy to check that $G = X \cap H$, so that G is a cozero set in X (4A2C(b-v)).

Similarly, $\overline{F}^{\widehat{X}}$ is a nowhere dense set in $\widehat{X}$, so is included in a nowhere dense zero set $F' \subseteq \widehat{X}$, and $F' \cap X$ is a zero set in X including F.

(d) Finally, let X be any topological group carrying Haar measures. Set $Y = \overline{\{e\}}$, where e is the identity of X, and let $\pi : X \to X/Y$ be the canonical map. $G = \pi^{-1}[\pi[G]]$ and $\pi[G]$ is a regular open set in X/Y (4A5Kb). X/Y is a Hausdorff topological group carrying Haar measures (see the proof of 443L), so $\pi[G]$ is a cozero set, by (c), and $G = \pi^{-1}[\pi[G]]$ is a cozero set in X, as claimed.

Similarly, $\pi[F]$ is nowhere dense (4A5K(b-iv)), so is included in a nowhere dense zero set $V \subseteq X/Y$ (by (c) here), and $\pi^{-1}[V]$ is a nowhere dense zero set (4A5K(b-v), 4A2C(b-iv)) including F. This completes the proof.

443O Quotient spaces I come now to the relationship between the modular functions of §442, normal subgroups and quotient spaces.

Lemma Let X be a locally compact Hausdorff topological group and Y a closed subgroup of X. Let $Z = X/Y$ be the set of left cosets of Y in X with the quotient topology and $\pi : X \mapsto Z$ the canonical map, so that Z is a locally compact Hausdorff space and we have a continuous action of X on Z defined by writing $a \bullet \pi x = \pi(ax)$ for a, $x \in X$ (4A5Jc). Let ν be a left Haar measure on Y and write $C_k(X)$, $C_k(Z)$ for the spaces of continuous real-valued functions with compact supports on X, Z respectively.

(a) We have a positive linear operator $T : C_k(X) \to C_k(Z)$ defined by writing

$$(Tf)(\pi x) = \int_Y f(xy)\nu(dy)$$

for every $f \in C_k(X)$ and $x \in X$. If $f > 0$ in $C_k(X)$ then $Tf > 0$ in $C_k(Z)$. If $h \geq 0$ in $C_k(Z)$ then there is an $f \geq 0$ in $C_k(X)$ such that $Tf = h$.

(b) If $a \in X$ and $f \in C_k(X)$, then $T(a \bullet_l f)(z) = (Tf)(a^{-1} \bullet z)$ for every $z \in Z$.

(c) Now suppose that a belongs to the normalizer of Y (that is, $aYa^{-1} = Y$). In this case, we can define $\psi(a) \in]0, \infty[$ by the formula

$$\nu(aFa^{-1}) = \psi(a)\nu F \text{ for every } F \in \operatorname{dom}\nu,$$

and

$$(T(a \bullet_r f)(\pi x) = \psi(a) \cdot (Tf)(\pi(xa))$$

for every $x \in X$ and $f \in C_k(X)$.

proof I should begin by remarking that because Y is a closed subgroup of a locally compact Hausdorff group, it is itself a locally compact Hausdorff group, so does have a left Haar measure, which is a Radon measure.

(a)(i) The first thing to check is that if $f \in C_k(X)$ then Tf is well-defined as a member of $\mathbb{R}^Z$. **P** (α) If $x \in X$, then $y \mapsto f(xy) : Y \to \mathbb{R}$ is a continuous function with compact support, so $\int f(xy)\nu(dy)$ is defined in $\mathbb{R}$. (β) If x_1, $x_2 \in X$ and $\pi x_1 = \pi x_2$, then $x_1^{-1}x_2 \in Y$, and

$$\int f(x_2y)\nu(dy) = \int f(x_1(x_1^{-1}x_2y))\nu(dy) = \int f(x_1y)\nu(dy),$$

applying 441J to the function $y \mapsto f(x_1y)$ and the left action of Y on itself. Thus we can safely write $(Tf)(\pi x) = \int f(xy)\nu(dy)$ for every $x \in X$, and Tf will be a real-valued function on Z. **Q**

(ii) Now Tf is continuous for every $f \in C_k(X)$. **P** Given $z_0 \in Z$, take $x_0 \in X$ such that $z = \pi x_0$. We have an $h \in C_k(X)^+$ such that for every $\epsilon > 0$ there is an open set U_ϵ containing x_0 such that $|f(x_0y) - f(xy)| \leq \epsilon h(y)$ whenever $x \in U_\epsilon$ and $y \in X$ (4A5Pb). In this case,

$$|(Tf)(\pi x) - (Tf)(x_0)| = |\int f(xy) - f(x_0y)\nu(dy)| \leq \epsilon \int_Y h(y)\nu(dy)$$

for every $x \in U_\epsilon$, so that $|(Tf)(z) - (Tf)(z_0)| \leq \epsilon \int_Y h\, d\nu$ for every $z \in \pi[U_\epsilon]$. Since each $\pi[U_\epsilon]$ is an open neighbourhood of z_0 (4A5Ja), Tf is continuous at z_0; as z_0 is arbitrary, Tf is continuous. **Q**

(iii) Since

$$\{z : (Tf)(z) \neq 0\} = \{\pi x : \int f(xy)\nu(dy) \neq 0\} \subseteq \{\pi x : f(xy) \neq 0 \text{ for some } y \in Y\}$$
$$= \{\pi x : f(x) \neq 0\} \subseteq \pi[\overline{\{x : f(x) \neq 0\}}]$$

is relatively compact, $Tf \in C_k(Z)$ for every $f \in C_k(X)$.

(iv) The formula for Tf makes it plain that $T : C_k(X) \to C_k(Z)$ is a positive linear operator.

(v) If $f \in C_k(X)^+$, $x \in X$ are such that $f(x) > 0$, then $\{y : y \in Y, f(xy) > 0\}$ is a non-empty open set in Y; because ν is strictly positive,

$$(Tf)(\pi x) = \int f(xy)\nu(dy) > 0.$$

In particular, $Tf > 0$ if $f > 0$. Moreover, if $z \in Z$ there is an $f \in C_k(X)^+$ such that $(Tf)(z) > 0$. Now

$$\{\{z : (Tf)(z) > 0\} : f \in C_k(X)^+\}$$

is an upwards-directed family of open subsets of Z, so if $L \subseteq Z$ is any compact set there is an $f \in C_k(X)^+$ such that $(Tf)(z) > 0$ for every $z \in L$.

(vi) Now suppose that $h \in C_k(Z)^+$. By (v), there is an $f_0 \in C_k(X)^+$ such that $(Tf_0)(z) > 0$ whenever $z \in \overline{\{w : h(w) \neq 0\}}$. Setting $h'(z) = h(z)/Tf_0(z)$ when $h(z) \neq 0$, 0 for other $z \in Z$, we have $h'(z) \in C_k(Z)$ and $h = h' \times Tf_0$. Set

$$f(x) = f_0(x)h'(\pi x) \geq 0$$

for every $x \in X$. Because h' and π are continuous, $f \in C_k(X)$. For any $x \in X$,

$$\begin{aligned}(Tf)(\pi x) &= \int f_0(xy)h'(\pi(xy))\nu(dy) = h'(\pi x)\int f_0(xy)\nu(dy)\\ &= h'(\pi x)Tf_0(\pi x) = h(\pi x).\end{aligned}$$

Thus $Tf = h$.

(b) If $z = \pi x$, then $a^{-1}\bullet z = \pi(a^{-1}x)$, so

$$(Tf)(a^{-1}\bullet z) = \int f(a^{-1}xy)\nu(dy) = \int (a\bullet_l f)(xy)\nu(dy) = T(a\bullet_l f)(z).$$

(c) Define $\phi : Y \to Y$ by writing $\phi(y) = a^{-1}ya$ for $y \in Y$. Because ϕ is a homeomorphism, the image measure $\nu\phi^{-1}$ is a Radon measure on Y; because ϕ is a group automorphism, $\nu\phi^{-1}$ is a left Haar measure. (If $F \in \operatorname{dom}\nu\phi^{-1}$ and $y \in Y$, then

$$\nu\phi^{-1}(yF) = \nu(ayFa^{-1}) = \nu(Fa^{-1}) = \nu(aFa^{-1}) = \nu\phi^{-1}(F).)$$

It must therefore be a multiple of ν; say $\nu\phi^{-1} = \psi(a)\nu$.

If $g \in C_k(Y)$, then

$$\int g(a^{-1}ya)\nu(dy) = \int g\phi\, d\nu = \int g\, d(\nu\phi^{-1}) = \psi(a)\int g\, d\nu.$$

Now take $f \in C_k(X)$. Then

$$\begin{aligned}T(a\bullet_r f)(\pi x) &= \int (a\bullet_r f)(xy)\nu(dy) = \int f(xya)\nu(dy)\\ &= \int f(xa(a^{-1}ya))\nu(dy) = \psi(a)\int f(xay)\nu(dy)\end{aligned}$$

(using the remark above with $g(y) = f(xay)$)

$$= \psi(a)(Tf)(\pi(xa))$$

for every $x \in X$, as claimed.

443P Theorem Let X be a locally compact Hausdorff topological group and Y a closed subgroup of X. Let $Z = X/Y$ be the set of left cosets of Y in X with the quotient topology, and $\pi : X \to Z$ the canonical map, so that Z is a locally compact Hausdorff space and we have a continuous action of X on Z defined by writing $a\bullet\pi x = \pi(ax)$ for a, $x \in X$. Let ν be a left Haar measure on Y. Suppose that λ is a non-zero X-invariant Radon measure on Z.

(a) For each $z \in Z$, we have a Radon measure ν_z on X defined by the formula

$$\nu_z E = \nu(Y \cap x^{-1}E)$$

whenever $\pi x = z$ and the right-hand side is defined. In this case, for a real-valued function f defined on a subset of X,

$$\int f\, d\nu_z = \int f(xy)\nu(dy)$$

whenever either side is defined in $[-\infty,\infty]$.

(b) We have a left Haar measure μ on X defined by the formulae

$$\int f\,d\mu = \iint f\,d\nu_z\lambda(dz)$$

for every $f \in C_k(X)$, and

$$\mu G = \int \nu_z(G)\lambda(dz)$$

for every open set $G \subseteq X$.

(c) If $D \subseteq Z$, then $D \in \operatorname{dom}\lambda$ iff $\pi^{-1}[D] \subseteq X$ is Haar measurable, and $\lambda D = 0$ iff $\pi^{-1}[D]$ is Haar negligible.

(d) If $\nu Y = 1$, then λ is the image measure $\mu\pi^{-1}$.

(e) Suppose now that X is σ-compact. Then $\mu E = \int \nu_z(E)\lambda(dz)$ for every Haar measurable set $E \subseteq X$. If $f \in \mathcal{L}^1(\mu)$, then $\int f\,d\mu = \int(\int f\,d\nu_z)\lambda(dz)$.

(f) Still supposing that X is σ-compact, take $f \in \mathcal{L}^1(\mu)$, and for $a \in X$ set $f_a(y) = f(ay)$ whenever $y \in Y$ and $ay \in \operatorname{dom} f$. Then $Q_f = \{a : a \in X,\ f_a \in \mathcal{L}^1(\nu)\}$ is μ-conegligible, and the function $a \mapsto f_a^\bullet : Q_f \to L^1(\nu)$ is almost continuous.

proof (a) First, we do have a function ν_z depending only on z, because if $z = \pi x_1 = \pi x_2$ then $x_2^{-1}x_1 \in Y$, so

$$\nu(Y \cap x_1^{-1}E) = \nu(x_2^{-1}x_1(Y \cap x_1^{-1}E)) = \nu(Y \cap x_2^{-1}E)$$

whenever either side is defined. Of course ν_z, being the image of the Radon measure ν under the continuous map $y \mapsto xy : Y \to X$ whenever $\pi x = z$, is always a Radon measure on X (418I). We also have

$$\int_Y f(xy)\nu(dy) = \int_X f\,d\nu_{\pi x}$$

whenever $x \in X$ and f is a real-valued function such that either side is defined in $[-\infty,\infty]$, by 235L.

I remark here that if $z \in Z$ then the coset $C = \pi^{-1}[\{z\}]$ is ν_z-conegligible, because if $\pi x = z$ then $Y = Y \cap x^{-1}C$.

(b)(i) Let $T : C_k(X) \to C_k(Z)$ be the positive linear operator of 443O; that is,

$$(Tf)(z) = \int f(xy)\nu(dy) = \int f\,d\nu_z$$

whenever $f \in C_k(X)$, $x \in X$ and $z = \pi x$. Then we have a positive linear functional $\theta : C_k(X) \to \mathbb{R}$ defined by setting $\theta(f) = \int Tf\,d\lambda$ for every $f \in C_k(X)$. By the Riesz Representation Theorem (436J), there is a Radon measure μ on X defined by saying that $\int f\,d\mu = \theta(f)$ for every $f \in C_k(X)$. Note that μ is non-zero. **P** Because λ is non-zero, there is some $h \in C_k(Z)^+$ such that $\int h\,d\lambda \neq 0$; now there is some $f \in C_k(X)$ such that $Tf = h$, by 443Oa, and $\int f\,d\mu \neq 0$. **Q**

(ii) μ is a left Haar measure. **P** If $f \in C_k(X)$ and $a \in X$, then we have $T(a\bullet_l f)(z) = (Tf)(a^{-1}\bullet z)$ for every $z \in Z$, by 443Ob. So

$$\int a\bullet_l f\,d\mu = \int T(a\bullet_l f)d\lambda = \int Tf(a^{-1}\bullet z)\lambda(dz) = \int Tf(z)\lambda(dz)$$

(by 441J or 441L, because λ is X-invariant)

$$= \int f\,d\mu.$$

By 441L in the other direction, μ is invariant under the left action of X on itself, that is, is a left Haar measure. **Q**

(iii) If $G \subseteq X$ is open then $\mu G = \int \nu_z(G)\lambda(dz)$. **P** Set $A = \{f : f \in C_k(X),\ 0 \leq f \leq \chi G\}$. Then $\mu G = \sup_{f\in A}\int f\,d\mu$ and

$$\nu_z G = \sup_{f\in A}\int f\,d\nu_z = \sup_{f\in A}(Tf)(z)$$

for every $z \in Z$, by 414Ba, because ν_z is τ-additive. But as Tf is continuous for every $f \in A$, and λ is τ-additive, we also have

$$\int \nu_z(G)\lambda(dz) = \sup_{f\in A}\int Tf\,d\lambda = \sup_{f\in A}\int f\,d\mu = \mu G. \ \blacksquare$$

(c)(i) Let $\mathcal{A}$ be the family of those sets $A \subseteq X$ such that μA and $\int \nu_z(A)\lambda(dz)$ are defined in $[0,\infty]$ and equal. Then $\bigcup_{n\in\mathbb{N}} A_n$ belongs to $\mathcal{A}$ whenever $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{A}$, and $A \setminus B \in \mathcal{A}$ whenever A, $B \in \mathcal{A}$, $B \subseteq A$ and $\mu A < \infty$. Moreover, if $A \in \mathcal{A}$ and $\mu A = 0$, then every subset of A belongs to $\mathcal{A}$, since A must be ν_z-negligible for λ-almost every z. We also know from (b) that every open set belongs to $\mathcal{A}$.

Applying the Monotone Class Theorem (136B) to $\{A : A \in \mathcal{A}, A \subseteq G\}$, we see that if $E \subseteq X$ is a Borel set included in an open set G of finite measure, then $E \in \mathcal{A}$. So if E is a relatively compact Haar measurable set, $E \in \mathcal{A}$ (using 443J(b-i), or otherwise).

(ii) If $D \in \operatorname{dom}\lambda$ then $\pi^{-1}[D] \in \operatorname{dom}\mu$. **P** Let $K \subseteq X$ be compact. Then $\pi[K] \subseteq Z$ is compact, so there are Borel sets F_1, $F_2 \subseteq Z$ such that $F_1 \subseteq D \cap \pi[K] \subseteq F_2$ and $F_2 \setminus F_1$ is λ-negligible. Now $\nu_z(K \cap \pi^{-1}[F_2 \setminus F_1]) = 0$ whenever $z \notin F_2 \setminus F_1$, by the remark added to the proof of (a) above, so

$$\mu(K \cap \pi^{-1}[F_2 \setminus F_1]) = \int \nu_z(K \cap \pi^{-1}[F_2 \setminus F_1])\lambda(dz) = 0.$$

Since

$$K \cap \pi^{-1}[F_1] \subseteq K \cap \pi^{-1}[D] \subseteq K \cap \pi^{-1}[F_2],$$

$K \cap \pi^{-1}[D] \in \operatorname{dom}\mu$. As K is arbitrary, $\pi^{-1}[D]$ is measured by μ, so is Haar measurable. $\blacksquare$

If $\lambda D = 0$ then the same arguments show that $\mu(K \cap \pi^{-1}[D]) = 0$ for every compact $K \subseteq X$, so that $\mu\pi^{-1}[D] = 0$ and $\pi^{-1}[D]$ is Haar negligible.

(iii) Now suppose that $D \subseteq Z$ is such that $\pi^{-1}[D] \in \operatorname{dom}\mu$. Let $L \subseteq Z$ be compact. Then there is a relatively compact open set $G \subseteq X$ such that $\pi[G] \supseteq L$ (because $\{\pi[G] : G \subseteq X$ is open and relatively compact$\}$ is an upwards-directed family of open sets covering Z). In this case,

$$\int \nu_z(G \cap \pi^{-1}[D \cap L])\lambda(dz) = \mu(G \cap \pi^{-1}[D] \cap \pi^{-1}[L])$$

is well-defined, by (i). But if $z = \pi x$ then

$$\begin{aligned}\nu_z(G \cap \pi^{-1}[D \cap L]) &= 0 \text{ if } z \notin D \cap L\\ &= \nu_z G = \nu(Y \cap x^{-1}G) > 0 \text{ if } z \in D \cap L,\end{aligned}$$

because if $z \in L$ then $\pi x \in \pi[G]$ and $Y \cap x^{-1}G \neq \emptyset$. So

$$D \cap L = \{z : \nu_z(G \cap \pi^{-1}[D \cap L]) > 0\}$$

is measured by λ. As L is arbitrary, and λ is a Radon measure, $D \in \operatorname{dom}\lambda$.

(d) If $\nu Y = 1$, then, for any open set $H \subseteq Z$, $\nu_z\pi^{-1}[H] = 1$ if $z \in H$, 0 otherwise. So

$$\mu\pi^{-1}[H] = \int \nu_z(\pi^{-1}[H])\lambda(dz) = \lambda H.$$

Thus λ and the image measure $\mu\pi^{-1}$ agree on the open sets and, being Radon measures (418I), must be equal (416E(b-iii)).

(e) If X is actually σ-compact, then (c)(i) of this proof tells us that $\mu E = \int \nu_z(E)\lambda(dz)$ for every Haar measurable set $E \subseteq X$, since E is the union of an increasing sequence of relatively compact measurable sets. Consequently $\int f\,d\mu = \iint f\,d\nu_z\lambda(dz)$ for every μ-simple function f. Now suppose that f is a non-negative μ-integrable function. Then there is a non-decreasing sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of non-negative μ-simple functions converging to f everywhere on $\operatorname{dom} f$. If we set $A = \{z : \nu_z^*(X \setminus \operatorname{dom} f) > 0\}$, then

$$\int \nu_z(X \setminus \operatorname{dom} f)\lambda(dz) = \mu(X \setminus \operatorname{dom} f) = 0,$$

so $\lambda A = 0$. Since $\int f\,d\nu_z = \lim_{n\to\infty}\int f_n d\nu_z$ for every $z \in Z \setminus A$,

$$\begin{aligned}\int f\,d\mu &= \lim_{n\to\infty}\int f_n\,d\mu\\ &= \lim_{n\to\infty}\iint f_n d\nu_z\lambda(dz) = \iint f\,d\nu_z\lambda(dz).\end{aligned}$$

Applying this to the positive and negative parts of f, we see that the same formula is valid for any μ-integrable function f.

(f)(i) If $f \in \mathcal{L}^1(\mu)$ and $a \in X$, then

$$\int f_a d\nu = \int f(ay)\nu(dy) = \int f \, d\nu_{\pi a}$$

if any of these are defined. So if f, $g \in \mathcal{L}^1(\mu)$, $\|f_a - g_a\|_1 = \int |f - g| d\nu_{\pi a}$ if either is defined.

(ii) Let Φ be the set of all almost continuous functions from μ-conegligible subsets of X to $L^1(\nu)$, where $L^1(\nu)$ is given its norm topology. (In terms of the definition in 411M, a member ϕ of Φ is to be almost continuous with respect to the subspace measure on $\operatorname{dom}\phi$.) If $\phi \in \Phi$ and ψ is a function from a conegligible subset of X to $L^1(\nu)$ which is equal almost everywhere to ϕ, then $\psi \in \Phi$. If $\langle \phi_n \rangle_{n\in\mathbb{N}}$ is a sequence in Φ converging μ-almost everywhere to ψ, then $\psi \in \Phi$ (418F).

(iii) For $f \in \mathcal{L}^1(\mu)$, set $\phi_f(a) = f_a^\bullet$ whenever this is defined in $L^1(\nu)$. Set $M = \{f : f \in \mathcal{L}^1(\mu), \phi_f \in \Phi\}$. If $\langle f^{(n)} \rangle_{n\in\mathbb{N}}$ is a sequence in M, $f \in \mathcal{L}^1(\mu)$ and $\|f^{(n)} - f\|_1 \leq 4^{-n}$ for every n, then $f \in M$. **P** Set $g = \sum_{n=0}^\infty 2^n |f^{(n)} - f|$, defined on

$$\{x : x \in \operatorname{dom} f \cap \textstyle\bigcap_{n\in\mathbb{N}} \operatorname{dom} f^{(n)}, \sum_{n=0}^\infty 2^n |f^{(n)}(x) - f(x)| < \infty\};$$

then $g \in \mathcal{L}^1(\mu)$. Now

$$D = \{z : z \in Z, \, g \text{ is } \nu_z\text{-integrable}\}$$

is λ-conegligible, by (e), and $E = \{a : a \in X, \pi a \in D\}$ is μ-conegligible, by (c).

If $a \in E$, then

$$|f_a^{(n)} - f_a| \leq 2^{-n} g \ \nu_{\pi a}\text{-a.e.}$$

for every $n \geq 1$. So

$$\begin{aligned}\|\phi_f(a) - \phi_{f^{(n)}}(a)\|_1 &= \int_Y |f_a - f_a^{(n)}| d\nu = \int_X |f - f^{(n)}| d\nu_{\pi a} \\ &\leq 2^{-n} \int g \, d\nu_{\pi a} \to 0\end{aligned}$$

as $n \to \infty$. Thus $\phi_f = \lim_{n\to\infty} \phi_{f^{(n)}}$ almost everywhere, and $\phi \in \Phi$, by (ii). **Q**

(iv) $C_k(X) \subseteq M$. **P** If $f \in C_k(X)$ and $a_0 \in X$, then there is an $h \in C_k(X)^+$ such that for every $\epsilon > 0$ there is an open set G_ϵ containing a_0 such that $|f(a_0 y) - f(ay)| \leq \epsilon h(y)$ whenever $a \in G_\epsilon$ and $y \in X$ (4A5Pb). In this case,

$$\|\phi_f(a_0) - \phi_f(a)\|_1 = \int |f(a_0 y) - f(ay)| \nu(dy) \leq \epsilon \textstyle\int_Y h \, d\nu$$

whenever $a \in G_\epsilon$. As ϵ is arbitrary, ϕ_f is continuous at a_0; as a_0 is arbitrary, ϕ_f is continuous, and $f \in M$. **Q**

(v) Now take any $f \in \mathcal{L}^1(\mu)$. Then for each $n \in \mathbb{N}$ we can find $f^{(n)} \in C_k(X)$ such that $\|f^{(n)} - f\|_1 \leq 4^{-n}$ (416I), so $f \in M$, by (iii). This completes the proof.

443Q Theorem Let X be a locally compact Hausdorff topological group and Y a closed subgroup of X. Let $Z = X/Y$ be the set of left cosets of Y in X with the quotient topology, so that Z is a locally compact Hausdorff space and we have a continuous action of X on Z defined by writing $a\bullet(xY) = axY$ for a, $x \in X$. Let Δ_X be the left modular function of X and Δ_Y the left modular function of Y. Then the following are equiveridical:

(i) there is a non-zero X-invariant Radon measure λ on Z;

(ii) Δ_Y is the restriction of Δ_X to Y.

proof Fix a left Haar measure ν on Y, and let $T : C_k(X) \to C_k(Z)$ be the corresponding linear operator as defined in 443Oa.

(a)(i)⇒(ii) Suppose that λ is a non-zero X-invariant Radon measure on Z. Construct a left Haar measure μ on X as in 443P.

Suppose that $a \in Y$. In this case, a surely belongs to the normalizer of Y, and, in the language of 443Oc, we have $\nu(aFa^{-1}) = \psi(a)\nu F$ for every $F \in \operatorname{dom}\nu$. But as

$$\nu(aFa^{-1}) = \nu(Fa^{-1}) = \Delta_Y(a^{-1})\nu F,$$

we must have $\psi(a) = \Delta_Y(a^{-1})$.

Fix some $f > 0$ in $C_k(X)$. We have

$$T(a\bullet_r f)(\pi x) = \psi(a) \cdot (Tf)(\pi(xa)) = \psi(a) \cdot (Tf)(\pi x)$$

for every x, so that (using 442Kc) we have

$$\begin{aligned}\Delta_X(a^{-1}) \int f\,d\mu &= \int a\bullet_r f d\mu = \int T(a\bullet_r f) d\lambda \\ &= \psi(a) \int Tf\,d\lambda = \psi(a) \int f\,d\mu = \Delta_Y(a^{-1}) \int f\,d\mu.\end{aligned}$$

As $\int f\,d\mu > 0$, we must have $\Delta_X(a^{-1}) = \Delta_Y(a^{-1})$; as a is arbitrary, $\Delta_Y = \Delta_X \restriction Y$, as required by (ii).

(b)(ii)⇒(i) Now suppose that $\Delta_Y = \Delta_X \restriction Y$. This time, start with a left Haar measure μ on X.

(α) (The key.) If $f \in C_k(X)$ is such that $Tf \geq 0$ in $C_k(Z)$, then $\int f\,d\mu \geq 0$. **P** There is an $h \in C_k(Z)^+$ such that $h(z) = 1$ whenever $z \in Z$ and $(Tf)(z) \neq 0$; by 443Oa, we can find a $g \in C_k(X)^+$ such that $Tg = h$. Now observe that $x \mapsto (Tf)(\pi x)$ is a non-negative continuous real-valued function on X, so

$$\begin{aligned}0 &\leq \int_X g(x)(Tf)(\pi x)\mu(dx) \\ &= \int_X g(x) \int_Y f(xy)\nu(dy)\mu(dx) = \int_Y \int_X g(x)f(xy)\mu(dx)\nu(dy)\end{aligned}$$

(by 417Ha or 417Hb, because $(x, y) \mapsto g(x)f(xy) : X \times Y \to \mathbb{R}$ is a continuous function with compact support)

$$= \int_Y \Delta_X(y^{-1}) \int_X g(xy^{-1})f(x)\mu(dx)\nu(dy)$$

(applying 442Kc to the function $x \mapsto g(xy^{-1})f(x)$)

$$= \int_X f(x) \int_Y \Delta_X(y^{-1})g(xy^{-1})\nu(dy)\mu(dx)$$

(because $(x, y) \mapsto \Delta_X(y^{-1})g(xy^{-1})f(x)$ is continuous and has compact support)

$$= \int_X f(x) \int_Y \Delta_Y(y^{-1})g(xy^{-1})\nu(dy)\mu(dx)$$

(because $\Delta_X \restriction Y = \Delta_Y$, by hypothesis)

$$= \int_X f(x) \int_Y g(xy)\nu(dy)\mu(dx)$$

(applying 442K(b-ii) to the function $y \mapsto g(xy)$)

$$= \int_X f(x)(Tg)(\pi x)\mu(dx) = \int_X f(x)\mu(dx)$$

because $(Tg)(\pi x) = 1$ whenever $f(x) \neq 0$. **Q**

(β) Applying this to f and $-f$, we see that $\int f\,d\mu = 0$ whenever $Tf = 0$, so that $\int f\,d\mu = \int g\,d\mu$ whenever f, $g \in C_k(X)$ and $Tf = Tg$. Accordingly (because T is surjective) we have a functional $\theta : C_k(Z) \to \mathbb{R}$ defined by saying that $\theta(Tf) = \int f\,d\mu$ whenever $f \in C_k(X)$, and θ is positive and linear. By the Riesz Representation Theorem again, there is a Radon measure λ on Z such that $\theta(h) = \int h\,d\lambda$ for every $h \in C_k(Z)$.

If $a \in X$ and $h \in C_k(Z)$ take $f \in C_k(X)$ such that $Tf = h$. Then, for any $x \in X$,

$$(Tf)(a\bullet\pi x) = (Tf)(\pi(ax)) = \int f(axy)\nu(dy) = \int (a^{-1}\bullet_l f)(xy)\nu(dy) = T(a^{-1}\bullet_l f)(\pi x).$$

So

$$\int h(a\bullet z)\lambda(dz) = \int (Tf)(a\bullet z)\lambda(dz) = \int T(a^{-1}\bullet_l f)(z)\lambda(dz)$$
$$= \int a^{-1}\bullet_l f\, d\mu = \int f\, d\mu = \int h(z)\lambda(dz).$$

By 441L, λ is X-invariant. Also λ is non-zero because there is surely some f such that $\int f\, d\mu \neq 0$. So we have the required non-zero X-invariant Radon measure on Z.

443R Applications This theorem applies in a variety of cases. Let X be a locally compact Hausdorff topological group and Y a closed subgroup of X.

(a) If Y is a normal subgroup of X, then $\Delta_Y = \Delta_X \restriction Y$. **P** X/Y has a group structure under which it is a locally compact Hausdorff group (4A5Jb). It therefore has a left Haar measure, which is surely X-invariant in the sense of 443Q. **Q**

(b) If Y is compact, then $\Delta_Y = \Delta_X \restriction Y$. **P** Both Δ_Y and $\Delta_X \restriction Y$ are continuous homomorphisms from Y to $]0,\infty[$; since the only compact subgroup of $]0,\infty[$ is $\{1\}$, they are both constant with value 1. **Q** So 443Q tells us that we have an X-invariant Radon measure λ on X/Y. Since Y has a Haar probability measure, we can take λ to be the image of a left Haar measure under the canonical map (443Pd).

(c) If Y is a compact normal subgroup, then we find that, for $W \in \operatorname{dom}\lambda$ and $x \in X$,

$$\lambda(W \cdot \pi x) = \mu(\pi^{-1}[W] \cdot x) = \Delta_X(x)\lambda W,$$

so that $\Delta_{X/Y}(\pi x) = \Delta_X(x)$, writing $\pi : X \to X/Y$ for the canonical map. This is a special case of 443S below, because (in the terminology there) $\psi(a) = \nu Y/\nu Y = 1$ for every $a \in X$.

(d) If Y is open, $\Delta_Y = \Delta_X \restriction Y$. **P** If μ is a left Haar measure on X, then the subspace measure μ_Y is a left Haar measure on Y (443F). There is an open set $G \subseteq Y$ such that $0 < \mu G < \infty$, and now $\mu(Gy) = \Delta_X(y)\mu G = \Delta_Y(y)\mu G$ for every $y \in Y$. **Q** This time, X/Y is discrete, so counting measure is an X-invariant Radon measure on X/Y.

443S Theorem Let X be a locally compact Hausdorff topological group and Y a closed normal subgroup of X; let $Z = X/Y$ be the quotient group, and $\pi : X \to Z$ the canonical map. Write Δ_X, Δ_Z for the left modular functions of X, Z respectively. Define $\psi : X \to]0,\infty[$ by the formula

$$\nu(aFa^{-1}) = \psi(a)\nu F \text{ whenever } F \in \operatorname{dom}\nu \text{ and } a \in X,$$

where ν is a left Haar measure on Y (cf. 443Oc). Then

$$\Delta_Z(\pi a) = \psi(a)\Delta_X(a)$$

for every $a \in X$.

proof Let $T : C_k(X) \to C_k(Z)$ be the map defined in 443O, and λ a left Haar measure on Z; then, as in 443Pb, we have a left Haar measure μ on X defined by the formula $\int f\, d\mu = \int Tf\, d\lambda$ for every $f \in C_k(X)$. Fix on some $f > 0$ in $C_k(X)$ and $a \in X$, and set $w = \pi a$. By 443Oc, we have

$$T(a\bullet_r f)(z) = \psi(a)(Tf)(\pi(xa)) = \psi(a)(Tf)(zw)$$

whenever $\pi x = z$. So

$$\Delta_X(a^{-1}) \int f\, d\mu = \int a\bullet_r f\, d\mu$$

(442Kc)

$$= \int T(a\bullet_r f)d\lambda = \psi(a) \int (Tf)(zw)\lambda(dz)$$
$$= \psi(a)\Delta_Z(w^{-1}) \int Tf(z)\lambda(dz) = \psi(a)\Delta_Z(w^{-1}) \int f\, d\mu.$$

Thus $\Delta_X(a^{-1}) = \psi(a)\Delta_Z(w^{-1})$; because both Δ_X and Δ_Z are multiplicative,

$$\Delta_Z(\pi a) = \Delta_Z(w) = \psi(a)\Delta_X(a).$$

443T Transitive actions All the results from 443O onwards have been expressed in terms of groups acting on quotient groups. But the same structures can appear if we start from a group action. To simplify the hypotheses, I give the following result for compact groups only.

Theorem Let X be a compact Hausdorff topological group, Z a non-empty compact Hausdorff space, and $\bullet$ a transitive continuous action of X on Z. For $z \in Z$, write $\pi_z(x) = x\bullet z$ for each $x \in X$.

(a) For every $z \in Z$, $Y_z = \{x : x \in X, x\bullet z = z\}$ is a compact subgroup of X. If we give the set X/Y_z of left cosets of Y_z in X its quotient topology, we have a homeomorphism $\phi_z : X/Y_z \to Z$ defined by the formula $\phi_z(xY_z) = x\bullet z$ for every $x \in X$.

(b) Let μ be a Haar probability measure on X. Then the image measure $\mu\pi_z^{-1}$ is an X-invariant Radon probability measure on Z, and $\mu\pi_w^{-1} = \mu\pi_z^{-1}$ for all w, $z \in Z$.

(c) Every non-zero X-invariant Radon measure on Z is of the form $\mu\pi_z^{-1}$ for a Haar measure μ on X and some (therefore any) $z \in Z$.

(d) There is a strictly positive X-invariant Radon probability measure on Z, and any two non-zero X-invariant Radon measures on Z are scalar multiples of each other.

(e) Take any $z \in Z$, and let ν be the Haar probability measure of Y_z. If μ is a Haar measure on X, then

$$\mu E = \int \nu(Y_z \cap x^{-1}E)\mu(dx)$$

whenever $E \subseteq X$ is Haar measurable.

proof (a) Given $z \in Z$, then for x, $y \in X$ we have

$$x\bullet z = y\bullet z \iff x^{-1}y \in Y_z \iff xY_z = yY_z.$$

So the formula given for ϕ_z defines a bijection between X/Y_z and Z. To see that ϕ_z is continuous, take any open set $H \subseteq Z$. Then

$$\{x : xY_z \in \phi_z^{-1}[H]\} = \{x : x\bullet z \in H\} = \pi_z^{-1}[H]$$

is open in X (because $\bullet$ is continuous), so $\phi_z^{-1}[H]$ is open in X/Y_z. Because X/Y_z is compact and ϕ_z is a bijection, it is a homeomorphism (3A3Dd).

(b) Because X is compact, therefore unimodular (442Ic), we can speak of 'Haar measures' on X without specifying 'left' or 'right'. If μ is a Haar probability measure on X, then the image measure $\mu\phi_z^{-1}$ is a Radon probability measure on Z (418I). To see that the measures $\mu\phi_z^{-1}$ are X-invariant, take any Borel set $H \subseteq Z$ and $y \in X$, and consider

$$\begin{aligned}(\mu\phi_z^{-1})(y^{-1}\bullet H) &= \mu\{x : x\bullet z \in y^{-1}\bullet H\} = \mu\{x : yx\bullet z \in H\}\\ &= \mu(y^{-1}\phi_z^{-1}[H]) = \mu(\phi_z^{-1}[H]) = (\mu\phi_z^{-1})(H).\end{aligned}$$

By 441B, this is enough to ensure that $\mu\phi_z^{-1}$ is invariant.

If w, $z \in Z$ and $H \subseteq Z$ is a Borel set, then there is a $y \in X$ such that $y\bullet w = z$ (because the action is transitive), and now

$$\phi_w^{-1}[H] = \{x : x\bullet w \in H\} = \{x : (xy^{-1})\bullet z \in H\} = (\phi_z^{-1}[H])y.$$

But μ is a two-sided Haar measure, so

$$(\mu\phi_w^{-1})(H) = \mu(\phi_w^{-1}[H]) = \mu((\phi_z^{-1}[H])y) = \mu(\phi_z^{-1}[H]) = (\mu\phi_z^{-1})(H).$$

Thus $\mu\phi_w^{-1}$ and $\mu\phi_z^{-1}$ agree on the Borel sets and must be equal (416Eb).

(c) Now we come to the interesting bit. Suppose that λ is a non-zero X-invariant Radon measure on Z. Take any $z \in Z$ and consider the Radon measure λ' on X/Y_z got by setting $\lambda' H = \lambda\phi_z[H]$ whenever $H \subseteq X/Y_z$ and $\phi_z[H]$ is measured by λ. In this case, if $x \in X$ and $H \subseteq X/Y_z$ is measured by λ,

$$\begin{aligned}\lambda'(x\bullet H) &= \lambda\{\phi_z(x\bullet w) : w \in H\} = \lambda\{\phi_z(x\bullet yY_z) : y \in X,\, yY_z \in H\}\\ &= \lambda\{\phi_z(xyY_z) : y \in X,\, yY_z \in H\} = \lambda\{xy\bullet z : y \in X,\, yY_z \in H\}\\ &= \lambda\{x\bullet(y\bullet z) : y \in X,\, yY_z \in H\} = \lambda(x\bullet\phi_z[H]) = \lambda\phi_z[H]\end{aligned}$$

(because λ is X-invariant)

$$= \lambda' H.$$

So λ' is X-invariant.

Now let ν be the Haar probability measure on the compact Hausdorff group Y_z. By 443Pb, we have a left Haar measure μ on X defined by the formula $\mu G = \int \nu_w(G)\lambda'(dw)$ for every open $G \subseteq X$, where $\nu_{xY_z}(G) = \nu(Y \cap x^{-1}G)$ for every $y \in X$ and every open $G \subseteq X$. Let $H \subseteq Z$ be an open set. Then for any $x \in X$,

$$\begin{aligned}\nu_{xY_z}(\pi_z^{-1}[H]) &= \nu\{y : y \in Y_z,\, xy \in \pi_z^{-1}[H]\} = \nu\{y : y \in Y_z,\, xy\bullet z \in H\}\\ &= \nu\{y : y \in Y_z,\, x\bullet(y\bullet z) \in H\} = \nu Y_z\\ &= 1 \text{ if } x\bullet z \in H,\\ &= \nu\emptyset = 0 \text{ otherwise.}\end{aligned}$$

So

$$\mu(\pi_z^{-1}[H]) = \lambda'\{xY_z : x \in X,\, x\bullet z \in H\} = \lambda'\phi_z^{-1}[H] = \lambda H.$$

As H is arbitrary, the image measure $\mu\pi_z^{-1}$ agrees with λ on the open subsets of Z; as they are both Radon measures, $\mu\pi_z^{-1} = \lambda$, as required.

(d) This is now easy. X carries a non-zero Haar measure, so by (b) there is an X-invariant Radon probability measure on Z. If λ_1 and λ_2 are non-zero X-invariant Radon measures on Z, then they are of the form $\mu_1\pi_w^{-1}$ and $\mu_2\pi_z^{-1}$ where μ_1 and μ_2 are Haar measures on X and w, $z \in Z$. By (b) again, $\mu_1\pi_w^{-1} = \mu_1\pi_z^{-1}$, and since μ_1 and μ_2 are multiples of each other (442B), so are λ_1 and λ_2.

To see that the invariant probability measure λ on X is strictly positive, take any non-empty open set $H \subseteq Z$. Then Z is covered by the open sets $x\bullet H$, as x runs through X. Because Z is compact, it is covered by finitely many of these, so at least one of them has non-zero measure. But they all have the same measure as H, so $\lambda H > 0$.

(e) Let $\theta : X \to X/Y_z$ be the canonical map. For $w \in X/Y_z$ we have a Radon measure ν_w on X defined by setting $\nu_w E = \nu(Y_z \cap x^{-1}E)$ whenever $\theta(x) = w$ and the right-hand side is defined (443Pa). By 443Q/443Rb, or otherwise, there is a non-zero X-invariant Radon measure λ on X/Y_z; re-scaling if necessary, we may suppose that $\lambda(X/Y_z) = \mu X$. By 443Pe, we have a Haar measure μ' on X defined by setting $\mu' E = \int \nu_w E\,\lambda(dw)$ for every Haar measurable E; since

$$\mu' X = \int \nu_w X\,\lambda(dw) = \int \nu Y_z\,\lambda(dw) = \lambda(X/Y_z) = \mu X,$$

$\mu' = \mu$. Moreover, $\lambda = \mu'\theta^{-1}$ (443Pd). So

$$\begin{aligned}\mu E = \mu' E &= \int \nu_w E\,\lambda(dw)\\ &= \int \nu_{\theta(x)} E\,\mu(dx) = \int \nu(Y_z \cap x^{-1}E)\mu(dx)\end{aligned}$$

for every Haar measurable $E \subseteq X$.

443X Basic exercises (a) Let X be a topological group carrying Haar measures. (i) Show that a set $A \subseteq X$ is self-supporting (definition: 411Na) for one Haar measure on X iff it is self-supporting for every Haar measure on X. (ii) Show that a set $A \subseteq X$ has non-zero inner measure for one Haar measure on X iff it has non-zero inner measure for every Haar measure on X.

>(b) Let X be a topological group and μ, ν two left Haar measures on X. (i) Show that the quasi-Radon product measure λ on $X \times X$ is a left Haar measure for the product group operation on $X \times X$. (ii) Show that the maps $(x,y) \mapsto (x,xy)$, $(x,y) \mapsto (y^{-1}x,y)$ are automorphisms of the measure space $(X \times X, \lambda)$. (*Hint*: use 417C(iv) to show that they preserve the measures of open sets.) (iii) Show that the maps $(x,y) \mapsto (y^{-1},xy)$, $(x,y) \mapsto (yx,x^{-1})$ are automorphisms of $(X \times X, \lambda)$. (*Hint*: express them as compositions of maps of the forms in (ii).) (iv) Show that the maps $(x,y) \mapsto (y,x)$ and $(x,y) \mapsto (y,y^{-1}x)$ are automorphisms of $(X \times X, \lambda)$.

(c) Let X be a topological group, μ a Haar measure on X, and E, F measurable subsets of X. Show that $(x,y,w,z) \mapsto \mu(xEy \cap wFz) : X^4 \to [0,\infty]$ is lower semi-continuous.

(d) Let X be a topological group carrying Haar measures and $\mathfrak{A}$ its Haar measure algebra. Show that we have a continuous action of $X \times X$ on $\mathfrak{A}$ defined by the formula $(x,y)\bullet E^\bullet = (xEy^{-1})^\bullet$ for x, $y \in X$ and Haar measurable sets $E \subseteq X$.

(e) Let X be a topological group carrying Haar measures and $\mathfrak{A}$ its Haar measure algebra. Show that if $x \in X$, $a \in \mathfrak{A}$ then $x\bullet_r a = (x\bullet_l \tilde{a})^\sim$, where $\tilde{a}$ is as defined in 443Af.

(f) Let X be a topological group carrying Haar measures, and $F \subseteq X$ a Haar measurable set such that $F \cap U$ is not Haar negligible for any neighbourhood U of the identity. Show that for any $A \subseteq X$ the set $A' = \{x : x \in A,\ A \cap xF \text{ is Haar negligible}\}$ is Haar negligible. (*Hint*: otherwise, take a closed non-negligible set H of finite measure, a Haar measurable envelope of $H \cap A'$, and show that $H \cap xF$ is negligible for every $x \in H$. Consider the set $\{(x,y) : x \in H,\ y \in F,\ xy \in H\} = \{(x,y) : y \in F,\ x \in H \cap Hy^{-1}\}$; show that its vertical sections are negligible but some of its horizontal sections are not.)

(g) Let X be a topological group with a left Haar measure μ. (i) Show that if Y is a subgroup of X such that $\mu_* Y > 0$, then Y is open. In particular, any non-negligible closed subgroup of X is open. (ii) Let Y be any subgroup of X which is not Haar negligible. Show that the subspace measure μ_Y is a left Haar measure on Y. Show that $\overline{Y}$ is a Haar measurable envelope of Y. (*Hint*: apply 443Db inside the topological group $\overline{Y}$.)

(h) Write out a version of 443G for right Haar measures.

(i) Let X be a topological group carrying Haar measures, and L^0 the space of equivalence classes of Haar measurable functions, as in 443A; let $u \mapsto \tilde{u} : L^0 \to L^0$ be the operator of 443Af. (i) Show that if μ is a left Haar measure on X and ν is a right Haar measure, $p \in [1,\infty]$ and $u \in L^0$, then $\tilde{u} \in L^p(\nu)$ iff $u \in L^p(\mu)$. (ii) Show that $x\bullet_r u = (x\bullet_l \tilde{u})^\sim$ for any $x \in X$, $u \in L^0$.

(j) Let X be a topological group carrying Haar measures, and $\mathfrak{A}$ its Haar measure algebra. Show that, in the language of 443C and 443G, $\chi(x\bullet_l a) = x\bullet_l \chi a$ and $\chi(x\bullet_r a) = x\bullet_r \chi a$ for every $x \in X$ and $a \in \mathfrak{A}$.

(k) Let X be a topological group carrying Haar measures. Show that X is totally bounded for its bilateral uniformity iff its Haar measures are totally finite.

(l) Let X be a topological group, μ a left Haar measure on X, and $A \subseteq X$ a set which is self-supporting for μ. Show that the following are equiveridical: (i) for every neighbourhood U of the identity e, there is a countable set $I \subseteq X$ such that $A \subseteq UI$; (ii) for every neighbourhood U of e, there is a countable set $I \subseteq X$ such that $A \subseteq IU$; (iii) for every neighbourhood U of e, there is a countable set $I \subseteq X$ such that $A \subseteq IUI$; (iv) A can be covered by countably many sets of finite measure for μ; (v) A can be covered by countably many open sets of finite measure for μ; (vi) A can be covered by countably many sets which are totally bounded for the bilateral uniformity on X.

>(m) Let X be a topological group carrying Haar measures. (i) Show that the following are equiveridical: (α) X is ccc; (β) X has a σ-finite Haar measure; (γ) every Haar measure on X is σ-finite. (ii) Show that if X is locally compact and Hausdorff, we can add (δ) X is σ-compact.

(n) Let X be a topological group carrying Haar measures. Show that every subset of X has a Haar measurable envelope which is a Borel set.

(o) In 443L, show that $\Delta_X = \Delta_Z \phi$, where Δ_X, Δ_Z are the left modular functions of X, Z respectively.

>(p) Let X and Y be topological groups with left Haar measures μ and ν. Show that the c.l.d. and quasi-Radon product measures of μ and ν on $X \times Y$ coincide. (*Hint*: start with locally compact Hausdorff spaces, and show that a compact G_δ set in $X \times Y$ belongs to $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$, where $\mathcal{B}(X)$ and $\mathcal{B}(Y)$ are the Borel σ-algebras of X and Y; now use 443L.)

(q) Let $\langle X_i \rangle_{i \in I}$ be a family of topological groups and $X = \prod_{i \in I} X_i$ their product. Suppose that each X_i has a Haar probability measure μ_i. Show that the ordinary and quasi-Radon product measures on X coincide.

(r) Let X be a topological group and μ a left Haar measure on X. Show that μ is inner regular with respect to the family of closed sets $F \subseteq X$ such that $F = \bigcap_{n \in \mathbb{N}} FU_n$, for some sequence $\langle U_n \rangle_{n \in \mathbb{N}}$ of neighbourhoods of the identity. (*Hint*: use 443M for the locally compact case, and 443L to extend the result to general Haar measures.)

(s) Let X be a locally compact Hausdorff group and Y a closed subgroup of X; write X/Y for the space of left cosets of Y in X, with its quotient topology. Show that if λ_1 and λ_2 are non-zero X-invariant Radon measures on X/Y, then each is a multiple of the other. (*Hint*: look at the Haar measures they define on X.)

>(t) Write S^1 for the circle group $\{s : s \in \mathbb{C}, |s| = 1\}$, and set $X = S^1 \times S^1$, where the first copy of S^1 is given its usual topology and the second copy is given the discrete topology, so that X is an abelian locally compact Hausdorff group. Set $E = \{(s, s) : s \in S^1\}$. Show that E is a closed Haar negligible subset of X. Set $Y = \{(1, s) : s \in S^1\}$; check that Y is a closed normal subgroup of X, and that the quotient group X/Y can be identified with S^1 with its usual topology; let λ be the Haar probability measure of X/Y. Let ν be counting measure on Y. Show that, in the language of 443P, $\nu_z E = 1$ for every $z \in X/Y$, so that $\mu E \neq \int \nu_z(E)\lambda(dz)$. Setting $f = \chi E$, show that the map $a \mapsto f^\bullet_a$ described in 443Pf is not almost continuous.

(u) (i) In 443O, suppose that $G \subseteq X$ is an open set such that $GY = X$. Show that for every $h \in C_k(Z)^+$ there is an $f \in C_k(X)^+$ such that $Tf = h$ and $\overline{\{x : f(x) > 0\}} \subseteq G$. (ii) In 443Q, suppose that there is an open set $G \subseteq X$ such that $GY = X$ and G has finite measure for the left Haar measures of X. Show that Z has an X-invariant Radon probability measure.

>(v) Let $X = \mathbb{R}^2$ be the example of 442Xf. (i) Let Y_1 be the subgroup $\{(\xi, 0) : \xi \in \mathbb{R}\}$. Describe the left cosets of Y_1 in X. Show that there is no non-trivial X-invariant Radon measure on the set X/Y of these left cosets. Find a base $\mathcal{U}$ for the topology of X/Y such that you can identify the sets $x^\bullet U$, where $x \in X$ and $U \in \mathcal{U}$, with sufficient precision to explain why the hypothesis (iii) of 441C is not satisfied. (ii) Let Y_2 be the normal subgroup $\{(0, \xi) : \xi \in \mathbb{R}\}$. Find the associated function $\psi : X \to]0, \infty[$ as described in 443Oc and 443S.

(w) Let X be a locally compact Hausdorff group and Y a compact normal subgroup of X. Show that X is unimodular iff the quotient group X/Y is unimodular. (*Hint*: the function ψ of 443S must be constant.)

(x) Take any integer $r \geq 1$, and let G be the isometry group of $\mathbb{R}^r$ with its topology of pointwise convergence (441G). (i) Show that G is metrizable and locally compact. (*Hint*: 441Xm.) (ii) Let $H \subseteq G$ be the set of translations. Show that H is an abelian closed normal subgroup of G, and that Lebesgue measure on $\mathbb{R}^r$ can be regarded as a Haar measure on H. (iii) Show that the quotient group G/H is compact. (iv) Show that G is unimodular. (*Hint*: the function ψ of 443S is constant.)

>(y) Set $X = \mathbb{R}^3$ with the operation

$$(\xi_1, \xi_2, \xi_3) * (\eta_1, \eta_2, \eta_3) = (\xi_1 + \eta_1, \xi_2 + e^{\xi_1}\eta_2, \xi_3 + e^{-\xi_1}\eta_3).$$

(i) Show that (with the usual topology of $\mathbb{R}^3$) X is a topological group. (ii) Show that it is unimodular. (*Hint*: set $Y = \{(\xi_1, \xi_2, \xi_3) : \xi_1 = 0\} \lhd X$, show that Lebesgue 2-dimensional measure on Y is a Haar measure, and that the conjugacy action of X on Y is measure-preserving, while $X/Y \cong \mathbb{R}$ is unimodular.) (iii) Show that X has both a closed subgroup and a Hausdorff quotient group which are not unimodular.

>(z) Let (X, ρ) be a non-empty compact metric space such that the group G of isometries of X is transitive. Show that any two non-zero G-invariant Radon measures on X must be multiples of each other. (*Hint*: 441Gb, 443T.)

443Y Further exercises (a) Let X be a topological group carrying Haar measures and $\mathfrak{A}$ its Haar measure algebra. Show that two principal ideals of $\mathfrak{A}$ are isomorphic (as Boolean algebras) iff they have the same cellularity.

(b) Let X be a locally compact Hausdorff group. Show that we have continuous actions $\bullet_l$, $\bullet_r$ and $\bullet_c$ of X on the Banach space $C_0(X)$ defined by formulae corresponding to those of 443G.

(c) Give an example of a locally compact Hausdorff group, with left Haar measure μ, such that no open normal subgroup can be covered by a sequence of sets of finite measure for μ.

(d) Let X be a topological group with a left Haar measure μ, and $A \subseteq X$. Show that the following are equiveridical: (i) A is totally bounded for the bilateral uniformity of X (ii) there are non-empty open sets G, $H \subseteq X$ such that $\mu(AG)$, $\mu(A^{-1}H)$ are both finite.

(e) Let X be a topological group carrying Haar measures. Let $E \subseteq X$ be a Haar measurable set such that $E \cap U$ is not Haar negligible for any neighbourhood U of the identity. Show that there is a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in E such that $x_{i_0} x_{i_1} \dots x_{i_n} \in E$ whenever $n \in \mathbb{N}$ and $i_0 < i_1 < \dots < i_n$ in $\mathbb{N}$. (See Plewik & Voigt 91.)

(f) Let X be a topological group and μ a Haar measure on X. Show that any closed self-supporting subset of X is a zero set. (*Hint*: start with the case in which X is locally compact and σ-compact.)

(g) Let X be a topological group. Let Σ be the family of subsets of X expressible in the form $\phi^{-1}[F]$ for some Borel subset F of a separable metrizable topological group Y and some continuous homomorphism $\phi : X \to Y$. Show that Σ is a σ-algebra of subsets of X and that multiplication, regarded as a function from $X \times X$ to X, is $(\Sigma \widehat{\otimes} \Sigma, \Sigma)$-measurable. Show that any compact G_δ set belongs to Σ. Show that if X is σ-compact, then Σ is the Baire σ-algebra of X.

(h) Let X be any Hausdorff topological group of cardinal greater than $\mathfrak{c}$. Let $\mathcal{B}$ be the Borel σ-algebra of X. Show that $(x, y) \mapsto xy$ is not $(\mathcal{P}X \widehat{\otimes} \mathcal{P}X, \mathcal{B})$-measurable. (*Hint*: show that for any sequence $\langle A_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{P}X$ there are distinct x, $y \in X$ such that $\{n : x \in A_n\} = \{n : y \in A_n\}$; consequently $\{(x, y) : xy^{-1} = e\} \notin \mathcal{P}X \widehat{\otimes} \mathcal{P}X$.)

(i) Find a compact Hausdorff space X with a strictly positive Radon measure such that there is a regular open set $G \subseteq X$ which is not a cozero set. (*Hint*: 419A.)

(j) Show that there is a subgroup X of the additive group $\mathbb{R}^2$ such that X has full outer Lebesgue measure but $\{\xi : (\xi, 0) \in X\} = \mathbb{Q}$. (*Hint*: enumerate the family $\mathcal{K}$ of compact subsets of $\mathbb{R}^2$ of positive Lebesgue measure as $\langle K_\xi \rangle_{\xi < \mathfrak{c}}$ (cf. 419Ib). Show that $\#(\pi_2[K]) = \mathfrak{c}$ for every $K \in \mathcal{K}$, where $\pi_2(\xi, \eta) = \eta$ (Fubini's theorem and 419Ia). Choose $x_\xi \in K_\xi$, for $\xi < \mathfrak{c}$, such that $\pi_2 x_\xi$ is never a rational linear combination of $\{\pi_2 x_\eta : \eta < \xi\}$. Let X be the group generated by $\{x_\xi : \xi < \mathfrak{c}\} \cup \{(q, 0) : q \in \mathbb{Q}\}$.) Show that X carries Haar measures, but that its closed subgroup $X \cap (\mathbb{R} \times \{0\})$ does not.

(k) Let X be a locally compact Hausdorff group. Show that it has a closed normal subgroup Y such that Y and X/Y are both unimodular. (*Hint*: take $Y = \{x : \Delta(x) = 1\}$.)

(l) Show that if X is a locally compact Hausdorff group, Y a compact subgroup of X, $Z = X/Y$ the set of left cosets of Y and $\pi : X \to Z$ the canonical map, and Z is given its quotient topology, then $R = \{(\pi x, x) : x \in X\}$ is an usco-compact relation in $Z \times X$ and $R[L] = \pi^{-1}[L]$ is compact for every compact $L \subseteq Z$.

(m) Let G be the isometry group of 443Xx. (i) Show that if we set $\rho(g,h) = \sup_{\|x\|\leq 1} \|g(x) - h(x)\|$, then ρ is a metric on G defining its topology. (ii) Describe Haar measures on X (α) in terms of Lebesgue measure on a suitable Euclidean space (β) in terms of Hausdorff measure of an appropriate dimension for the metric ρ.

(n) Let $r \geq 1$ be an integer, and X the group of invertible affine transformations of $\mathbb{R}^r$, with the topology of pointwise convergence inherited from $(\mathbb{R}^r)^{\mathbb{R}^r}$. (i) Show that X is a topological group. (ii) Show that X is not unimodular, and find its modular functions. (*Hint*: use the method of 443Xx, supported by 442Yb.)

(o) Let X be a topological group with a left Haar measure μ and left modular function Δ; suppose that X is not unimodular. Show that $\mu\{x : \alpha < \Delta(x) < \beta\} = \infty$ whenever $\alpha < \beta$. (*Hint*: begin by supposing that X is locally compact and Hausdorff. Set $Y = \{x : \Delta(x) = 1\}$. By 443S, $\nu Y = \infty$, where ν is any left Haar measure on Y, so Y cannot be totally bounded for its own bilateral uniformity nor for the bilateral uniformity of X. Consequently $\mu(YG) = \infty$ for any non-empty open set $G \subseteq X$, by 443Yd. For general X use 443L.)

443 Notes and comments Most of us, by the time we come to study measures on general topological groups, have come to trust our intuition concerning the behaviour of Lebesgue measure on $\mathbb{R}$, and the principal discipline imposed by the subject is the search for the true path between hopelessness and over-confidence when extending this intuition to the general setting. The biggest step is the loss of commutativity, especially as the non-abelian groups of elementary courses in group theory are mostly finite, and are therefore untrustworthy guides to the concerns of this chapter. Accordingly we find ourselves going rather slowly and carefully through the calculations in such results as 443C. Note, for instance, that in an abelian group the actions I call $\bullet_l$ and $\bullet_r$ are still different; $x \bullet_l E$ corresponds to $x + E = E + x$, but $x \bullet_r E$ corresponds to $E - x$. (The action $\bullet_c$ becomes trivial, of course.) When we come to translate the formulae of Fourier analysis in the next section, manoeuvres of this kind will often be necessary. In the present section, I have done my best to give results in 'symmetric' forms; you may therefore take it that when the words 'left' or 'right' appear in the statement of a proposition, there is some real need to break the symmetry. Subject to these remarks, such results as 443C and 443G are just a matter of careful conventional analysis. I see that I have used slightly different techniques in the two cases. Of course 443C can be thought of as a special case of 443Gc, if we remember that $\chi : \mathfrak{A} \to L^0(\mathfrak{A})$ embeds $\mathfrak{A}$ topologically as a subspace of L^0 (367R).

443B, 443D and 443E belong to a different family; they deal with actual sets rather than with members of a measure algebra or a function space. I suppose it is 443D which will most often be quoted. Its corollary 443E deals with the obvious question of when Haar measures are tight.

Readers who have previously encountered the theory of Haar measures on locally compact groups will have been struck by how closely the more general theory here is able to follow it. The explanation lies in 443K-443L; my general Haar measures are really just subspace measures on subgroups of full Haar measure in locally compact groups. Knowing this, it is easy to derive all the results above from the locally compact theory. I use this method only in 443M, because (following Halmos 50) I feel that from the point of view of pure measure theory the methods show themselves more clearly if we do not use ideas depending on compactness, but instead rely directly on τ-additivity. But I note that 443Xp and 443Yd also go faster with the aid of 443L.

Halmos 50 goes a little farther, with a theory of groups carrying translation-invariant measures for which the operation $(x,y) \mapsto xy$ is $(\Sigma \widehat{\otimes} \Sigma, \Sigma)$-measurable, where Σ is the domain of the measure. The essence of the theory here, and my reason for insisting on τ-additive measures, is that for these we have a suitable theory of product measures. If we start with quasi-Radon measures and use the quasi-Radon product measure, then multiplication is measurable just because it is continuous. In order to achieve similar results without either assuming metrizability or using the theory of τ-additive product measures, we have to restrict the measure to something smaller than the Borel σ-algebra, as in 443Yd. (See 443Yh.) 443J and 443Xp show

that certain disconcerting features of general Radon measures (419C, 419E) cannot arise in the case of Haar measures.

When I come to look at subgroups and quotient groups, I do specialize to the locally compact case. One obstacle to generalization is the fact that a closed subgroup of a group carrying Haar measures need not itself carry Haar measures (443Yj). 443O and 443Q are taken from FEDERER 69, who goes farther, with many other applications. I give a fairly detailed analysis of the relationship between a Haar measure on a group X and a corresponding X-invariant measure on a family of left cosets (443P) for the sake of applications in §447. One of the challenges here is to distinguish clearly those results which apply to all locally compact groups from those which rely on σ-compactness or some such limitation. There is a standard example (443Xt) which provides a useful test in such questions. You may recognise this as a version of a fundamental example related to Fubini's theorem, given in 252K.

Most of the results of this section begin with a topological group X. But starting from §441 it is equally natural to start with a group action. If we have a topological group X acting continuously on a topological space Z, and X carries Haar measures, then we have a good chance of finding an invariant measure on Z. In the simplest case, in which X and Z are both compact and the action is transitive, there is a unique invariant Radon probability measure on Z (443T).

444 Convolutions

In this section, I look again at the ideas of §§255 and 257, seeking the appropriate generalizations to topological groups other than $\mathbb{R}$. Following HEWITT & ROSS 63, I begin with convolutions of measures (444A-444E) before proceeding to convolutions of functions (444O-444V); in between, I mention the convolution of a function and a measure (444G-444M) and a general result concerning continuous group actions on quasi-Radon measure spaces (444F).

444A Convolution of measures: Proposition If X is a topological group and λ and ν are two totally finite quasi-Radon measures on X, we have a quasi-Radon measure $\lambda * \nu$ on X defined by saying that

$$\begin{aligned}(\lambda * \nu)(E) &= (\lambda \times \nu)\{(x,y) : xy \in E\} \\ &= \int \nu(x^{-1}E)\lambda(dx) = \int \lambda(Ey^{-1})\nu(dy)\end{aligned}$$

for every $E \in \operatorname{dom}(\lambda * \nu)$, where $\lambda \times \nu$ is the quasi-Radon product measure on $X \times X$.

proof Because multiplication is continuous, $Q(E) = \{(x,y) : xy \in E\}$ is a Borel set in $X \times X$ for every Borel set $E \subseteq X$, so the formula $\theta E = (\lambda \times \nu)Q(E)$ defines a functional on the Borel σ-algebra of X. Of course θ is non-negative and countably additive, and it is τ-additive because $\lambda \times \nu$ is. Because X, being a topological group, is regular (4A5Ha), the completion of θ is a quasi-Radon measure (415Cb); call this measure $\lambda * \nu$.

If $E \in \operatorname{dom}(\lambda * \nu)$, we have Borel sets E', E'' such that $E' \subseteq E \subseteq E''$ and $(\lambda * \nu)(E') = (\lambda * \nu)(E'') = (\lambda * \nu)(E)$, so that

$$(\lambda \times \nu)Q(E') = (\lambda \times \nu)Q(E'') = (\lambda * \nu)(E),$$

and $(\lambda \times \nu)Q(E)$ must be defined and equal to $(\lambda * \nu)(E)$.

As for the other formulae, Fubini's theorem (417Ha) tells us that

$$\begin{aligned}(\lambda * \nu)(E) &= (\lambda \times \nu)Q(E) \\ &= \int \nu Q(E)[\{x\}]\lambda(dx) = \int \nu(x^{-1}E)\lambda(dx) \\ &= \int \lambda Q(E)^{-1}[\{y\}]\nu(dy) = \int \lambda(Ey^{-1})\nu(dy)\end{aligned}$$

for any $E \in \operatorname{dom}(\lambda * \nu)$.

444B Proposition If X is a topological group, $\lambda_1 * (\lambda_2 * \lambda_3) = (\lambda_1 * \lambda_2) * \lambda_3$ for all totally finite quasi-Radon measures λ_1, λ_2 and λ_3 on X.

proof If $E \subseteq X$ is Borel, then

$$\begin{aligned}(\lambda_1 * (\lambda_2 * \lambda_3))(E) &= \int (\lambda_2 * \lambda_3)(x^{-1}E)\lambda_1(dx)\\ &= \int (\lambda_2 \times \lambda_3)\{(y,z) : xyz \in E\}\lambda_1(dx)\\ &= (\lambda_1 \times (\lambda_2 \times \lambda_3))\{(x,(y,z)) : xyz \in E\}\\ &= ((\lambda_1 \times \lambda_2) \times \lambda_3)\{((x,y),z) : xyz \in E\}\\ &= \int (\lambda_1 \times \lambda_2)\{(x,y) : xyz \in E\}\lambda_3(dz)\\ &= ((\lambda_1 * \lambda_2) * \lambda_3)(E).\end{aligned}$$

(For the central identification between $\lambda_1 \times (\lambda_2 \times \lambda_3)$ and $(\lambda_1 \times \lambda_2) \times \lambda_3$, observe that as both are quasi-Radon measures it is enough to check that they agree on sets of the type $G_1 \times (G_2 \times G_3) \cong (G_1 \times G_2) \times G_3$ where G_1, G_2 and G_3 are open, as in 417J.)

So $\lambda_1 * (\lambda_2 * \lambda_3)$ and $(\lambda_1 * \lambda_2) * \lambda_3$ agree on the Borel sets and must be identical.

444C Proposition Let X be a topological group and λ, ν two totally finite quasi-Radon measures on X. Then

$$\int f d(\lambda * \nu) = \int f(xy)(\lambda \times \nu)d(x,y) = \iint f(xy)\lambda(dx)\nu(dy) = \iint f(xy)\nu(dy)\lambda(dx)$$

for any $(\lambda * \nu)$-integrable real-valued function f. In particular, $(\lambda * \nu)(X) = \lambda X \cdot \nu X$.

proof If f is of the form χE, so that

$$\int f d(\lambda * \nu) = (\lambda * \nu)(E), \quad \int f(xy)(\lambda \times \nu)d(x,y) = (\lambda \times \nu)\{(x,y) : xy \in E\},$$

$$\iint f(xy)\lambda(dx)\nu(dy) = \int \lambda(Ey^{-1})\nu(dy), \quad \iint f(xy)\nu(dy)\lambda(dx) = \int \nu(x^{-1}E)\lambda(dx)$$

this is covered by the result in 444A. Now it is easy to run through the standard progression to the cases of (i) simple functions (ii) non-negative Borel measurable functions defined everywhere (iii) functions defined, and zero, almost everywhere (iv) non-negative integrable functions and (v) arbitrary integrable functions.

444D Proposition Let X be an abelian topological group. Then $\lambda * \nu = \nu * \lambda$ for all totally finite quasi-Radon measures λ, μ on X.

proof For any Borel set $E \subseteq X$,

$$\begin{aligned}(\lambda * \nu)(E) &= (\lambda \times \nu)\{(x,y) : xy \in E\} = (\nu \times \lambda)\{(y,x) : xy \in E\}\\ &= (\nu \times \lambda)\{(y,x) : yx \in E\} = (\nu * \lambda)(E).\end{aligned}$$

444E The Banach algebra of τ-additive measures (a) Let X be a topological group. Recall from 437Ab that we have a band $C_b(X)^\sim_\tau$ in the L-space $C_b(X)^\sim$ consisting of those order-bounded linear functionals $f : C_b(X) \to \mathbb{R}$ such that $|f|$ is smooth (equivalently, f^+ and f^- are both smooth); that is, such that $|f|$, f^+ and f^- can be represented by totally finite quasi-Radon measures on X. Because X is completely regular (4A5Ha), $C_b(X)^\sim_\tau$ can be identified with the band M_τ of signed τ-additive Borel measures on X, that is, the set of those countably additive functionals ν defined on the Borel σ-algebra $\mathcal{B}$ of X such that $|\nu|$ is τ-additive (437Ha).

(b) For any τ-additive totally finite Borel measures λ, ν on X we can define their convolution $\lambda * \nu$ by the formulae of 444A, that is,

$$(\lambda * \nu)(E) = \int \nu(x^{-1}E)\lambda(dx) = \int \lambda(Ey^{-1})\nu(dy)$$

for any Borel set $E \subseteq X$, if we note that the completions $\hat{\lambda}$, $\hat{\nu}$ of λ and ν are quasi-Radon measures (415Cb), so that $\lambda * \nu$, as defined by these formulae, is just the restriction of $\hat{\lambda} * \hat{\nu}$, as defined in 444A, to the Borel σ-algebra. Now the formulae make it obvious that the map $*$ is bilinear in the sense that

$$(\lambda_1 + \lambda_2) * \nu = \lambda_1 * \nu + \lambda_2 * \nu,$$

$$\lambda * (\nu_1 + \nu_2) = \lambda * \nu_1 + \lambda * \nu_2,$$

$$(\alpha\lambda) * \nu = \lambda * (\alpha\nu) = \alpha(\lambda * \nu)$$

for all totally finite τ-additive Borel measures λ, λ_1, λ_2, ν, ν_1, ν_2 and all $\alpha \geq 0$. Consequently, regarding elements of M_τ as functionals on the Borel σ-algebra, we have a bilinear map $* : M_\tau \times M_\tau \to M_\tau$ defined by saying that

$$(\lambda_1 - \lambda_2) * (\nu_1 - \nu_2) = \lambda_1 * \nu_1 - \lambda_1 * \nu_2 - \lambda_2 * \nu_1 + \lambda_2 * \nu_2$$

for all λ_1, λ_2, ν_1, $\nu_2 \in M_\tau^+$.

(c) We see from 444B that $*$ is associative on M_τ^+, so it will be associative on the whole of M_τ. Observe that $*$ is positive in the sense that $\lambda * \nu \geq 0$ if λ, $\nu \geq 0$; so that

$$\begin{aligned} |\lambda * \nu| &= |\lambda^+ * \nu^+ - \lambda^+ * \nu^- - \lambda^- * \nu^+ + \lambda^- * \nu^-| \\ &\leq \lambda^+ * \nu^+ + \lambda^+ * \nu^- + \lambda^- * \nu^+ + \lambda^- * \nu^- \\ &= |\lambda| * |\nu| \end{aligned}$$

for any λ, $\nu \in M_\tau$.

(d) If λ, $\nu \in M_\tau^+$ then $\|\lambda\| = \lambda X$ and $\|\nu\| = \nu X$ (362Ba), so

$$\|\lambda * \nu\| = (\lambda * \nu)(X) = (\lambda \times \nu)(X \times X) = \lambda X \cdot \nu X = \|\lambda\|\|\nu\|.$$

Generally, for any λ, $\nu \in M_\tau$,

$$\|\lambda * \nu\| = \||\lambda * \nu|\| \leq \||\lambda| * |\nu|\| = \||\lambda|\|\|\nu|\| = \|\lambda\|\|\nu\|.$$

Thus M_τ is a Banach algebra under the operation $*$, as well as being an L-space. If X is abelian then M_τ will be a commutative algebra, by 444D.

444F In preparation for the next construction I give a general result extending ideas already touched on in 443C and 443G.

Theorem Let X be a topological space, G a topological group and $\bullet$ a continuous action of G on X. For $A \subseteq X$, $a \in G$ write $a\bullet A = \{a\bullet x : x \in A\}$. Let ν be a quasi-Radon measure on X.

(a) If $f : X \to [0, \infty]$ is lower semi-continuous, then $a \mapsto \int a\bullet f d\nu : G \to [0, \infty]$ is lower semi-continuous. (See 441Ac for the definition of $a\bullet f$.) In particular, if $V \subseteq X$ is open, then $a \mapsto \nu(a\bullet V) : G \to [0, \infty]$ is lower semi-continuous.

(b) If $f : X \to \mathbb{R}$ is continuous, then $a \mapsto (a\bullet f)^\bullet : G \to L^0$ is continuous, if $L^0 = L^0(\nu)$ is given the topology of convergence in measure.

(c) If ν is σ-finite and $E \subseteq X$ is a Borel set, then $a \mapsto (a\bullet E)^\bullet : G \to \mathfrak{A}$ is Borel measurable, if the measure algebra $\mathfrak{A}$ of ν is given its measure-algebra topology.

(d) If ν is σ-finite and $f : X \to \mathbb{R}$ is Borel measurable, then $a \mapsto (a\bullet f)^\bullet : G \to L^0$ is Borel measurable.

(e) If ν is σ-finite, then (i) $a \mapsto \nu(a\bullet E) : G \to [0, \infty]$ is Borel measurable for any Borel set $E \subseteq X$ (ii) if $f : X \to \mathbb{R}$ is Borel measurable, then $Q = \{a : \int a\bullet f d\nu \text{ is defined in } [-\infty, \infty]\}$ is a Borel set, and $a \mapsto \int a\bullet f d\nu : Q \to [-\infty, \infty]$ is Borel measurable.

proof (a)(i) Note first that if $f : X \to [0, \infty]$ is Borel measurable, then, for each $a \in G$, $a\bullet f$ is the composition of f with the continuous function $x \mapsto a^{-1}\bullet x$, so is Borel measurable, and if f is finite-valued then $(a\bullet f)^\bullet$ is defined in $L^0 = L^0(\nu)$.

(ii) Suppose that $f : X \to [0, \infty]$ is lower semi-continuous, that $a \in G$ and that $\int a\bullet f > \gamma$. Let $\mathcal{U}$ be the set of open neighbourhoods of a in G. For $U \in \mathcal{U}$, $x \in X$ set

$$\phi_U(x) = \sup\{\inf_{b\in U, y\in V}(b\bullet f)(y) : V \text{ is an open neighbourhood of } x \text{ in } X\}.$$

Then ϕ_U is lower semi-continuous. **P** If $\phi_U(x) > \alpha$, there is an open neighbourhood V of x such that $\inf_{b\in U,y\in V}(b\bullet f)(y) > \alpha$; now $\phi_U(y) > \alpha$ for every $y \in V$. **Q** If $U' \subseteq U$ then $\phi_{U'} \geq \phi_U$, so $\{\phi_U : U \in \mathcal{U}\}$ is upwards-directed. Also $\sup_{U\in\mathcal{U}} \phi_U = a\bullet f$ in $[0,\infty]^X$. **P** Of course $\phi_U(x) \leq (a\bullet f)(x)$ for every x. If $x \in X$ and $(a\bullet f)(x) > \alpha$, then $\{y : f(y) > \alpha\}$ is an open set containing $a^{-1}\bullet x$, so (because $\bullet$ is continuous) there are a $U \in \mathcal{U}$ and an open neighbourhood V of x such that $f(b^{-1}\bullet y) > \alpha$ whenever $b \in U$, $y \in V$; in which case $\phi_U(x) \geq \alpha$. As α is arbitrary, $\sup_{U\in\mathcal{U}} \phi_U(x) = (a\bullet f)(x)$. **Q**

By 414Ba, $\int a\bullet f d\nu = \sup_{U\in\mathcal{U}} \int \phi_U d\nu$, and there is a $U \in \mathcal{U}$ such that $\int \phi_U d\nu > \gamma$. Now suppose that $b \in U$; then $\phi_U(x) \leq (b\bullet f)(x)$ for every x, so $\int b\bullet f d\nu > \gamma$. This shows that $\{a : \int a\bullet f d\nu > \gamma\}$ is an open set in G, so that $a \mapsto \int a\bullet f d\nu$ is lower semi-continuous.

(iii) If $V \subseteq X$ is open, then χV is lower semi-continuous, and $\chi(a\bullet V) = a\bullet(\chi V)$ for every $a \in G$. So $a \mapsto \nu(a\bullet V) = \int a\bullet(\chi V)d\nu$ is lower semi-continuous.

Thus (a) is true.

(b) Take any $a \in G$, $E \in \operatorname{dom}\nu$ such that $\nu E < \infty$ and $\epsilon > 0$. Let $\mathcal{U}$ be the family of open neighbourhoods of a in G, and for $U \in \mathcal{U}$ set

$$H_U = \operatorname{int}\{x : |(b\bullet f)(x) - (a\bullet f)(x)| \leq \epsilon \text{ whenever } b \in U\}.$$

Then $\mathcal{H} = \{H_U : U \in \mathcal{U}\}$ is upwards-directed. Also, it has union X. **P** If $x \in X$ then, because $(b, y) \mapsto f(b^{-1}\bullet y)$ is continuous, there are a $U \in \mathcal{U}$ and an open neighbourhood V of x such that $|(b\bullet f)(y)-(a\bullet f)(x)| \leq \frac{1}{2}\epsilon$ whenever $b \in U$ and $y \in V$. But now $|(b\bullet f)(y) - (a\bullet f)(y)| \leq \epsilon$ whenever $b \in U$ and $y \in V$, so that H_U includes V, which contains x. **Q**

So there is a $U \in \mathcal{U}$ such that $\nu(E \setminus H_U) \leq \epsilon$ (414Ea). In this case, for any $b \in U$, we must have

$$\int_E \min(1, |b\bullet f - a\bullet f|)d\nu \leq \epsilon(1 + \nu E).$$

As E and ϵ are arbitrary, $b \mapsto (b\bullet f)^\bullet$ is continuous at a; as a is arbitrary, it is continuous everywhere. Thus (ii) is true.

(c)(i) Let us start by supposing that E is an open set and that ν is totally finite. In this case the function $a \mapsto \nu(a\bullet E)$ is lower semi-continuous, by (a) above, therefore Borel measurable. Now let $W \subseteq \mathfrak{A}$ be an open set, and write $H = \{a : a \in G, (a\bullet E)^\bullet \in W\}$. For m, $k \in \mathbb{N}$ set $H_{mk} = \{a : 2^{-m}k \leq \nu(a\bullet E) < 2^{-m}(k+1)\}$, so that H_{mk} is Borel, and $U_{mk} = G\setminus\overline{H_{mk} \setminus H}$, so that U_{mk} is open. Let H' be $\bigcup_{m,k\in\mathbb{N}} H_{mk}\cap U_{mk}$; then H' is Borel and $H' \subseteq H$. In fact $H' = H$. **P** If $a \in H$, then W is an open set containing $(a\bullet E)^\bullet$. Let $\delta > 0$ be such that $a \in W$ whenever $\bar\nu(a \triangle (a\bullet E)^\bullet) \leq \delta$, where $\bar\nu$ is the measure on $\mathfrak{A}$; let m, $k \in \mathbb{N}$ be such that $2^{-m} \leq \frac{1}{4}\delta$ and $2^{-m}k \leq \nu(a\bullet E) < 2^{-m}(k+1)$. If we take ν' to be the indefinite-integral measure over ν defined by $\chi(a\bullet E)$, then ν' is a quasi-Radon measure (415O), so (by (a) again) $U = \{b : \nu'(b\bullet E) > 2^{-m}(k-1)\}$ is an open set, and of course it contains a. If $b \in U \cap H_{mk}$, then

$$\begin{aligned}\nu((b\bullet E)\triangle(a\bullet E)) &= \nu(b\bullet E) + \nu(a\bullet E) - 2\nu((b\bullet E)\cap(a\bullet E))\\ &\leq 2^{-m}(k+1) + 2^{-m}(k+1) - 2\cdot 2^{-m}(k-1) \leq 4\cdot 2^{-m} \leq \delta,\end{aligned}$$

so $(b\bullet E)^\bullet \in W$ and $b \in H$. This shows that $U \cap (H_{mk} \setminus H) = \emptyset$ and $U \subseteq U_{mk}$ and $a \in U \cap H_{mk} \subseteq H'$. As a is arbitrary, $H = H'$. **Q** Thus H is a Borel subset of G. As W is arbitrary, the map $a \mapsto (a\bullet E)^\bullet$ is Borel measurable.

(ii) To extend this to a general σ-finite quasi-Radon measure ν, still supposing that E is open, let $h : X \to \mathbb{R}$ be a strictly positive integrable function (215B(viii)) and ν' the corresponding indefinite-integral measure. As in (ii), this ν' also is a quasi-Radon measure. Since ν and ν' have the same domains and the same null ideals, the Boolean algebra $\mathfrak{A}$ is still the underlying algebra of the measure algebra of ν'; by 324H, the topologies on $\mathfrak{A}$ induced by the measures $\bar\nu$, $\bar\nu'$ are the same. So we can apply (i) to the measure ν' to see that $a \mapsto (a\bullet E)^\bullet : G \to \mathfrak{A}$ is still Borel measurable.

(iii) Next, suppose that E is expressible as $\bigcup_{i\leq n} V_{2i} \setminus V_{2i+1}$ where each V_i is open. Then $a \mapsto (a\bullet E)^\bullet$ is Borel measurable. **P** Set $X' = X \times \{0, \dots, 2n+1\}$, with the product topology (giving $\{0, \dots, 2n+1\}$ the discrete topology) and define a measure ν' on X and an action of G on X' by setting

$$\nu'F = \sum_{i=0}^{2n+1} \nu\{x : (x,i) \in F\}$$

whenever $F \subseteq X'$ is such that $\{x : (x,i) \in F\} \in \operatorname{dom}\nu$ for every $i \leq 2n+1$,

$$a\bullet(x,i) = (a\bullet x, i)$$

whenever $a \in G$, $x \in X$ and $i \leq 2n+1$. Then $V = \{(x,i) : i \leq 2n+1, x \in V_i\}$ is an open set in X', while ν' is a σ-finite quasi-Radon measure, as is easily checked; so, by (ii), the map $a \mapsto (a\bullet V)^\bullet : G \to \mathfrak{A}'$ is Borel measurable, where $\mathfrak{A}'$ is the measure algebra of ν'. On the other hand, we can identify $\mathfrak{A}'$ with the simple product $\mathfrak{A}^{2n+2}$ (322Kb), and the map

$$\langle c_i\rangle_{i\leq 2n+1} \mapsto \sup_{i\leq n} c_{2i} \setminus c_{2i+1} : \mathfrak{A}^{2n+1} \to \mathfrak{A}$$

is continuous, by 323B. So the map

$$a \mapsto (a\bullet E)^\bullet = \sup_{i\leq n}(a\bullet V_{2i})^\bullet \setminus (a\bullet V_{2i+1})^\bullet$$

is the composition of a Borel measurable function with a continuous function, and is Borel measurable. **Q**

(iv) Now the family $\mathcal{E}$ of all those Borel sets $E \subseteq X$ such that $a \mapsto (a\bullet E)^\bullet$ is Borel measurable is closed under unions and intersections of monotonic sequences. **P** (α) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{E}$ with union E, then

$$(a\bullet E)^\bullet = \sup_{n\in\mathbb{N}}(a\bullet E_n)^\bullet = \lim_{n\to\infty}(a\bullet E)^\bullet$$

(323Ea) for every $a \in G$. So $a \mapsto (a\bullet E)^\bullet$ is the pointwise limit of a sequence of Borel measurable functions into a metrizable space (323Gb, because $(\mathfrak{A}, \bar\nu)$ is σ-finite), and is Borel measurable, by 418Ba. Thus $E \in \mathcal{E}$. (β) If $\langle E_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{E}$ the same argument applies, since this time

$$(a\bullet E)^\bullet = \inf_{n\in\mathbb{N}}(a\bullet E_n)^\bullet = \lim_{n\to\infty}(a\bullet E)^\bullet$$

(323Eb) for every $a \in G$. **Q**

Since $\mathcal{E}$ contains all sets of the form $\bigcup_{i\leq n} V_i \cap F_i$ where every V_i is open and every F_i is closed, by (iii), $\mathcal{E}$ must be the whole Borel σ-algebra, by 4A3Cg.

This completes the proof of (c).

(d)(i) We need the following extension of (c): if $\langle E_n\rangle_{n\in\mathbb{N}}$ is any sequence of Borel sets in X, then $a \mapsto \langle (a\bullet E_n)^\bullet\rangle_{n\in\mathbb{N}} : G \mapsto \mathfrak{A}^{\mathbb{N}}$ is Borel measurable. **P** I repeat the idea of (c-iii) above. On $X' = X \times \mathbb{N}$ define a measure ν' by setting

$$\nu'F = \sum_{n=0}^{\infty} \nu\{x : (x,n) \in F\}$$

whenever $F \subseteq X'$ is such that $\{x : (x,n) \in F\} \in \operatorname{dom}\nu$ for every $n \in \mathbb{N}$. As before, it is easy to check that ν' is a σ-finite quasi-Radon measure, if we give $\mathbb{N}$ the discrete topology and X' the product topology. As before, set $a\bullet(x,n) = (a\bullet x, n)$ for $a \in G$, $x \in X$ and $n \in \mathbb{N}$, to obtain a continuous action of G on X'. Applying (c) to this action, the map $a \mapsto (a\bullet E)^\bullet : G \to \mathfrak{A}'$ is Borel measurable, where $\mathfrak{A}'$ is the measure algebra of ν' and $E = \{(x,n) : n \in \mathbb{N}, x \in E_n\}$. But we can identify $\mathfrak{A}'$ (as Boolean algebra) with $\mathfrak{A}^{\mathbb{N}}$, by 322K, as before; so that if we re-interpret $a \mapsto (a\bullet E)^\bullet : G \to \mathfrak{A}'$ as $a \mapsto \langle (a\bullet E_n)^\bullet\rangle_{n\in\mathbb{N}} : G \to \mathfrak{A}^{\mathbb{N}}$ it is still Borel measurable. (As in (c-ii), this time using 323L, the measure-algebra topology of $\mathfrak{A}'$ matches the product topology on $\mathfrak{A}^{\mathbb{N}}$.) **Q**

(ii) Now suppose that $f : X \to [0,1]$ is Borel measurable. Define $\langle E_n\rangle_{n\in\mathbb{N}}$ inductively by the formula

$$E_n = \{x : x \in X, (f - \textstyle\sum_{i<n} 2^{-i-1}\chi E_i)(x) \geq 2^{-n-1}\}.$$

Then every E_n is a Borel set and $f = \sum_{n=0}^{\infty} 2^{-n-1}\chi E_n$. Next, observe that the function

$$\langle c_n\rangle_{n\in\mathbb{N}} \mapsto \textstyle\sum_{n=0}^{\infty} 2^{-n-1}\chi c_i : \mathfrak{A}^{\mathbb{N}} \to L^0$$

is continuous, because each of the maps $c \mapsto 2^{-n-1}\chi c$ is continuous (367R), addition is continuous (245Da, 367Na) and the series is uniformly summable. Accordingly we may think of the map $a \mapsto (a\bullet f)^\bullet$ as the composition of the continuous function $\langle c_n\rangle_{n\in\mathbb{N}} \mapsto \sum_{n=0}^{\infty} 2^{-n-1}\chi c_i$ with the Borel measurable function $a \mapsto \langle a\bullet E_n)^\bullet\rangle_{n\in\mathbb{N}}$, and it is Borel measurable.

(iii) For general Borel measurable $f : X \to \mathbb{R}$, set $q(t) = \frac{1}{2}(1 + \frac{t}{|t|+1})$, so that $q : \mathbb{R} \to]0,1[$ is a homeomorphism, and set $p = q^{-1} :]0,1[\to \mathbb{R}$. Then the function $\bar p : P \to L^0$ is continuous, where

$P = \{u : u \in L^0, [\![u \in]0,1[\,]\!] = 1\}$ (367S). But now

$$a\bullet f = p\circ q\circ a\bullet f = p\circ a\bullet(q\circ f)$$

for every a, so that $a \mapsto (a\bullet f)^\bullet$ is the composition of the Borel map $a \mapsto (a\bullet(q\circ f))^\bullet$ with the continuous map $\bar{p}$, and is Borel measurable.

(e)(i) We need only recall that $\bar{\nu} : \mathfrak{A} \to \mathbb{R}$ is lower semi-continuous (323Cb), so that (applying (c) above)

$$a \mapsto \nu(a\bullet E) = \bar{\nu}(a\bullet E)^\bullet$$

is a composition of Borel measurable functions and is Borel measurable. (Of course there are much more direct arguments, using fragments of the proof above.)

(ii) The point is that the maps $u \mapsto u^+$, $u \mapsto u^- : L^0 \to (L^0)^+$ are continuous (245Db, 367S), while $u \mapsto \int u : (L^0)^+ \to [0,\infty]$ is lower semi-continuous (369Mb), therefore Borel measurable. Accordingly $a \mapsto \int (a\bullet f)^+ = \int ((a\bullet f)^\bullet)^+$ and $a \mapsto \int (a\bullet f)^-$ are Borel measurable functions from G to $[0,\infty]$, so that

$$Q = \{a : \min(\textstyle\int (a\bullet f)^+, \int (a\bullet f)^-) < \infty\}$$

is a Borel set, and

$$a \mapsto \textstyle\int a\bullet f = \int (a\bullet f)^+ - \int (a\bullet f)^- : Q \to [-\infty,\infty]$$

is Borel measurable.

444G Corollary Let X be a topological group and ν a σ-finite quasi-Radon measure on X.

(a) If $f : X \to \mathbb{R}$ is a Borel measurable function, then $\{x : \int f(y^{-1}x)\nu(dy)$ is defined in $[-\infty,\infty]\}$ is a Borel set in X and $x \mapsto \int f(y^{-1}x)\nu(dy)$ is Borel measurable.

(b) If f, $g : X \to \mathbb{R}$ are Borel measurable functions, then $\{x : \int f(xy^{-1})g(y)\nu(dy)$ is defined in $[-\infty,\infty]\}$ is a Borel set and $x \mapsto \int f(xy^{-1})g(y)\nu(dy)$ is Borel measurable.

(c) If ν is totally finite and $f : X \to \mathbb{R}$ is a bounded continuous function, then $x \mapsto \int f(y^{-1}x)\nu(dy) : X \to \mathbb{R}$ is continuous.

proof (a) Set $\tilde{f}(x) = f(x^{-1})$; then $\tilde{f}$ is Borel measurable. Let $\bullet_l$ be the left action of X on itself. Then, in the language of 444Fe, $Q = \{x : \int x\bullet_l\tilde{f}d\nu$ is defined in $[-\infty,\infty]\}$ is a Borel set, and $x \mapsto \int x\bullet_l\tilde{f}d\nu$ is Borel measurable. But

$$(x\bullet_l\tilde{f})(y) = \tilde{f}(x^{-1}y) = f(y^{-1}x)$$

for all x, y, so $\int (x\bullet_l\tilde{f})d\nu = \int f(y^{-1}x)\nu(dy)$ if either integral is defined.

(b)(i) Set $\tilde{\nu}E = \nu E^{-1}$ when this is defined, writing $E^{-1} = \{x^{-1} : x \in E\}$ for $E \subseteq X$; that is, $\tilde{\nu}$ is the image measure $\nu\phi^{-1}$, where $\phi(x) = x^{-1}$ for $x \in X$. Because ϕ is a homeomorphism, $\tilde{\nu}$ is a quasi-Radon measure. By 235L, $\int h\,d\tilde{\nu} = \int h(x^{-1})\nu(dx)$ for any real-valued function h for which either integral is defined in $[-\infty,\infty]$.

(ii) Now suppose that f and g are non-negative Borel measurable functions from X to $\mathbb{R}$. Set $\tilde{g}(y) = g(y^{-1})$ for every $y \in X$, so that $\tilde{g}$ is also a non-negative Borel measurable function. We know from 444Fd that $x \mapsto (x^{-1}\bullet_l f)^\bullet : X \to L^0(\tilde{\nu})$ is a Borel measurable function; now multiplication in L^0 is continuous (245Dc), so the map $x \mapsto ((x^{-1}\bullet_l f)\times\tilde{g})^\bullet$ is Borel measurable; since integration is lower semi-continuous on $(L^0)^+$, $x \mapsto \int (x^{-1}\bullet_l f)\times\tilde{g}\,d\tilde{\nu} : X \to [0,\infty]$ is Borel measurable. But

$$\textstyle\int (x^{-1}\bullet_l f)\times\tilde{g}\,d\tilde{\nu} = \int f(xy)g(y^{-1})\tilde{\nu}(dy) = \int f(xy^{-1})g(y)\nu(dy)$$

whenever any of the integrals is defined in $[-\infty,\infty]$, so this is the function we needed to know about.

(iii) For general Borel measurable functions f and g, we have

$$\begin{aligned}\int f(xy^{-1})g(y)\nu(dy) = &\big(\int f^+(xy^{-1})g^+(y)\nu(dy) + \int f^-(xy^{-1})g^-(y)\nu(dy)\big)\\ &- \big(\int f^+(xy^{-1})g^-(y)\nu(dy) + \int f^-(xy^{-1})g^+(y)\nu(dy)\big)\end{aligned}$$

exactly when the subtraction can be done in $[-\infty,\infty]$, so that $x \mapsto \int f(xy^{-1})g(y)\nu(dy)$ is a difference of Borel measurable functions and is Borel measurable, with Borel measurable domain.

(c) Continuing the argument of (a), if f is bounded and continuous and ν is totally finite then all the functions $x\bullet_l\tilde{f}$) are bounded and continuous, so $\int x\bullet_l\tilde{f}d\nu$ is defined and finite for every x. Next, the function $x \mapsto (x\bullet_l\tilde{f})^\bullet : X \to L^0(\nu)$ is continuous for the topology of convergence in measure (444Fb), which agrees with the norm topology of $L^1(\nu)$ on $\| \|_\infty$-bounded sets (246Jb). It follows that

$$x \mapsto \int (x\bullet_l\tilde{f})^\bullet = \int f(y^{-1}x)\nu(dy)$$

is continuous.

444H Convolutions of measures and functions I introduce some notation which I shall use for the rest of the section. Let X be a topological group. If f is a real-valued function defined on a subset of X, and ν is a measure on X, set

$$(\nu * f)(x) = \int f(y^{-1}x)\nu(dy)$$

whenever the integral is defined in $\mathbb{R}$.

444I Proposition Let X be a topological group and λ, ν two totally finite quasi-Radon measures on X.

(a) For any Borel measurable function $f : X \to \mathbb{R}$, $\nu * f$ is a Borel measurable function with a Borel domain.

(b) $\nu * f \in C_b(X)$ for every $f \in C_b(X)$.

(c) For any real-valued function f defined on a subset of X, $(\lambda * (\nu * f))(x) = ((\lambda * \nu) * f)(x)$ whenever the right-hand side is defined.

proof (a) This follows at once from 444Ga.

(b) This is just a restatement of 444Gc.

(c) If $((\lambda * \nu) * f)(x)$ is defined, then

$$((\lambda * \nu) * f)(x) = \int f(t^{-1}x)(\lambda * \nu)(dt) = \iint f((yz)^{-1}x)\nu(dz)\lambda(dy)$$

(444C)

$$= \iint f(z^{-1}y^{-1}x)\nu(dz)\lambda(dy)$$

$$= \int (\nu * f)(y^{-1}x)\lambda(dy) = (\lambda * (\nu * f))(x).$$

444J Convolutions of functions and measures Let X be a topological group carrying Haar measures; let Δ be its left modular function (442I). If f is a real-valued function defined on a subset of X, and ν is a measure on X, set

$$(f * \nu)(x) = \int f(xy^{-1})\Delta(y^{-1})\nu(dy)$$

whenever the integral is defined in $\mathbb{R}$. From 444Gb we see that if ν is a σ-finite quasi-Radon measure and f is Borel measurable, then $f * \nu$ is a Borel measurable function with a Borel domain. If f is non-negative and ν-integrable, write $f\nu$ for the corresponding indefinite-integral measure over ν (§234).

444K Proposition Let X be a topological group with a left Haar measure μ. Let ν be a totally finite quasi-Radon measure on X. Then for any non-negative μ-integrable real-valued function f, $f\mu$ is a quasi-Radon measure; moreover, $\nu * f$ and $f * \nu$ are μ-integrable, and we have

$$(\nu * f)\mu = \nu * f\mu, \quad (f * \nu)\mu = f\mu * \nu.$$

In particular, $\int(\nu * f)d\mu = \int(f * \nu)d\mu = \nu X \cdot \int f d\mu$.

proof (a) $f\mu$ is a quasi-Radon measure by 415O.

(b) Suppose first that f is bounded, Borel measurable and defined everywhere on X, as well as being non-negative and μ-integrable.

(i) In this case $\nu * f$ is defined everywhere, and is Borel measurable, by 444Ga, so we can be sure that $(\nu * f)\mu$ is well-defined. Let $G \subseteq X$ be any open set such that $\mu G < \infty$. The function $(x,y) \mapsto f(x^{-1}y)\chi G(y) : X \times X \to [0,\infty[$ is Borel measurable, so

$$(\nu * f\mu)(G) = \int (f\mu)(x^{-1}G)\nu(dx)$$

(444A)

$$= \iint f(y)\chi(x^{-1}G)(y)\mu(dy)\nu(dx)$$

(by the definition of $f\mu$, 234A)

$$= \iint f(x^{-1}y)\chi(x^{-1}G)(x^{-1}y)\mu(dy)\nu(dx)$$

(441J)

$$= \iint f(x^{-1}y)\chi G(y)\mu(dy)\nu(dx)$$

$$= \iint f(x^{-1}y)\chi G(y)\nu(dx)\mu(dy)$$

(by Fubini's theorem, 417Ha, because $(x,y) \mapsto f(x^{-1}y)\chi G(y)$ is non-negative and Borel measurable and zero outside $X \times G$)

$$= \int_G (\nu * f)(y)\mu(dy) = ((\nu * f)\mu)(G).$$

Thus the quasi-Radon measures $(\nu * f)\mu$ and $\nu * f\mu$ agree at least on open sets of finite measure for μ. By 415H(iv) (because μ is locally finite, by 442Aa), they are identical.

(ii) Similarly, $(f * \nu)(x)$ is defined in $\mathbb{R}$ for every x, and $f * \nu$ is Borel measurable, by 444Gb. This time, if $G \subseteq X$ is open and $\mu G < \infty$,

$$(f\mu * \nu)(G) = \int (f\mu)(Gy^{-1})\nu(dy) = \iint_{Gy^{-1}} f(x)\mu(dx)\nu(dy)$$

$$= \int \Delta(y^{-1}) \int_G f(xy^{-1})\mu(dx)\nu(dy)$$

(by 442Kc)

$$= \int_G \int \Delta(y^{-1}) f(xy^{-1})\nu(dy)\mu(dx)$$

$$= \int_G (f * \nu)(x)\mu(dx) = ((f * \nu)\mu)(G).$$

So once again the quasi-Radon measures $f\mu * \nu$ and $(f * \nu)\mu$ must be equal.

(c) Now suppose that f is Borel measurable and non-negative and defined everywhere, but not necessarily bounded. In this case set $f_n(x) = \min(n, f(x))$ for every $n \in \mathbb{N}$, $x \in X$, and observe that

$$(\nu * f)(x) = \lim_{n\to\infty}(\nu * f_n)(x),$$

$$(f * \nu)(x) = \lim_{n\to\infty}(f_n * \nu)(x)$$

whenever either side is defined in $\mathbb{R}$. So whenever $E \subseteq X$ is a Borel set, we must (using B.Levi's theorem twice) have

$$\begin{aligned}((\nu * f)\mu)(E) &= \int_E (\nu * f)(x)\mu(dx) = \lim_{n\to\infty} \int_E (\nu * f_n)\mu(dx)\\ &= \lim_{n\to\infty} ((\nu * f_n)\mu)(E) = \lim_{n\to\infty} (\nu * f_n\mu)(E)\\ &= \lim_{n\to\infty} \int \nu(x^{-1}E) f_n(x)\mu(dx) = \int \nu(x^{-1}E) f(x)\mu(dx) = (\nu * f\mu)(E)\end{aligned}$$

if any of these is finite, and $(\nu * f)\mu = \nu * f\mu$. Similarly,

$$\begin{aligned}((f * \nu)\mu)(E) &= \int_E (f * \nu)(x)\mu(dx) = \lim_{n\to\infty} \int_E (f_n * \nu)(x)\mu(dx)\\ &= \lim_{n\to\infty} ((f_n * \nu)\mu)(E) = \lim_{n\to\infty} (f_n\mu * \nu)(E)\\ &= \lim_{n\to\infty} \int \nu(Ey^{-1}) f_n(y)\mu(dy) = \int \nu(Ey^{-1}) f(y)\mu(dy) = (f\mu * \nu)(E)\end{aligned}$$

for any Borel set E such that either $((f * \nu)\mu)(E)$ or $(f\mu * \nu)(E)$ is finite, so that $f\mu * \nu = (f * \nu)\mu$.

(d) Next, suppose that f is defined and zero μ-a.e. In this case there is a Borel set E such that $\mu E = 0$ and $f(x)$ is defined and equal to zero for every $x \in X \setminus E$ (443J(b-ii)). Set $g = \chi E$. Then $g\mu$ is the zero measure, so $(\nu * g)\mu = \nu * g\mu$, $(g * \nu)\mu = g\mu * \nu$ are all zero; that is, there is some μ-conegligible set F such that

$$0 = (\nu * g)(x) = \int \chi E(y^{-1}x)\nu(dy) = \nu(xE^{-1}),$$

$$0 = (g * \nu)(x) = \int \chi E(xy^{-1})\Delta(y^{-1})\nu(dy) = \int_{E^{-1}x} \Delta(y^{-1})\nu(dy)$$

for every $x \in F$. But now, for $x \in F$, we must have $\nu(xE^{-1}) = \nu(E^{-1}x) = 0$ (since Δ is strictly positive), so that

$$(\nu * f)(x) = \int f(y^{-1}x)\nu(dy) = \int_{xE^{-1}} f(y^{-1}x)\nu(dy) = 0,$$

because if $y \notin xE^{-1}$ then $y^{-1}x \notin E$ and $f(y^{-1}x) = 0$. Similarly,

$$(f * \nu)(x) = \int f(xy^{-1})\Delta(y^{-1})\nu(dy) = \int_{E^{-1}x} f(xy^{-1})\Delta(y^{-1})\nu(dy) = 0.$$

Thus $\nu * f$ and $f * \nu$ are defined, and zero, μ-almost everywhere.

(e) For an arbitrary non-negative μ-integrable function f, we can express it in the form $g + h$ where g is a non-negative μ-integrable Borel measurable function defined everywhere, and h is zero almost everywhere. In this case, $\nu * h^+$, $\nu * h^-$, $h^+ * \nu$ and $h^- * \nu$ are defined, and zero, μ-a.e., so $\nu * f = \nu * g$ and $f * \nu = g * \nu$ μ-a.e., and $(\nu * f)\mu = (\nu * g)\mu$, $(f * \nu)\mu = (g * \nu)\mu$. We therefore have

$$(\nu * f)\mu = (\nu * g)\mu = \nu * g\mu = \nu * f\mu, \quad (f * \nu)\mu = (g * \nu)\mu = g\mu * \nu = f\mu * \nu,$$

as required.

(f) Finally, we have

$$\int \nu * f d\mu = ((\nu * f)\mu)(X) = (\nu * f\mu)(X) = \nu X \cdot (f\mu)(X) = \nu X \cdot \int f d\mu,$$

$$\int f * \nu\, d\mu = ((f * \nu)\mu)(X) = (f\mu * \nu)(X) = (f\mu)(X) \cdot \nu X = \nu X \cdot \int f d\mu.$$

444L Corollary Let X be a topological group carrying Haar measures. Suppose that ν is a non-zero quasi-Radon measure on X and $E \subseteq X$ is a Borel set such that $\nu(xE) = 0$ for every $x \in X$. Then E is Haar negligible.

proof Let μ be a left Haar measure on X. There is a non-zero totally finite quasi-Radon measure ν' on X such that $\nu'(xE) = 0$ for every $x \in X$. **P** Take any F such that $0 < \nu F < \infty$, and set $\nu' H = \nu(H \cap F)$ whenever this is defined. **Q** Let G be any Borel set such that $\mu G < \infty$, and set $f = \chi(G \cap E^{-1})$. Then

$$(\nu' * f)(x) = \int \chi(G \cap E^{-1})(y^{-1}x)\nu'(dy) = \nu'(xG^{-1} \cap xE) = 0$$

for every $x \in X$, so $\nu' * f\mu = (\nu' * f)\mu$ is the zero measure, and $(f\mu)(X) = 0$, that is, $\mu(G \cap E^{-1}) = 0$. As G is arbitrary, $\mu E^{-1} = 0$ and E is Haar negligible (442H).

444M Proposition Let X be a topological group and μ a left Haar measure on X. Let ν be a totally finite quasi-Radon measure on X and $p \in [1, \infty]$.

(a) We have a bounded positive linear operator $u \mapsto \nu * u : L^p(\mu) \to L^p(\mu)$, of norm at most νX, defined by saying that $\nu * f^\bullet = (\nu * f)^\bullet$ for every $f \in \mathcal{L}^p(\mu)$.

(b) Set $\gamma = \int \Delta(y)^{(1-p)/p} \nu(dy)$ if $p < \infty$, $\int \Delta(y)^{-1} \nu(dy)$ if $p = \infty$, where Δ is the left modular function of X. Suppose that $\gamma < \infty$. Then we have a bounded positive linear operator $u \mapsto u * \nu : L^p(\mu) \to L^p(\mu)$, of norm at most γ, defined by saying that $f^\bullet * \nu = (f * \nu)^\bullet$ for every $f \in \mathcal{L}^p(\mu)$.

proof I begin with some straightforward remarks. If $\nu X = 0$ the whole result is trivial, so we may suppose that $\nu X > 0$. Multiplying ν by a positive scalar does not affect the inequalities we need, so we may suppose that $\nu X = 1$. Note also that 444K tells us that changing a function f on a negligible set will change $\nu * f$ and $f * \nu$ on negligible sets.

I will write $\mathcal{L}^p$, L^p for $\mathcal{L}^p(\mu)$, $L^p(\mu)$.

(a) If $f \geq 0$ is μ-integrable, then 444K tells us that $\nu * f$ is μ-integrable and that

$$\begin{aligned}\|\nu * f\|_1 &= \int_X (\nu * f)(x)\mu(dx) = ((\nu * f)\mu)(X) \\ &= (\nu * f\mu)(X) = \nu X \cdot (f\mu)(X) = \|f\|_1,\end{aligned}$$

using 444C or 444A for the penultimate equality. Since evidently $\nu * (f + g) = \nu * f + \nu * g$, $\nu * (\alpha f) = \alpha \nu * f$ whenever the right-hand sides of the equations are defined, we have a positive linear operator $T_1 : L^1 \to L^1$ defined by saying that $T_1 g^\bullet = (\nu * g)^\bullet$ for every μ-integrable Borel measurable function g, with $\|T_1\| \leq 1$.

Similarly, if $h : X \to \mathbb{R}$ is a bounded Borel measurable function, then $\nu * h$ is also a Borel measurable function, by 444Ga. Of course it is bounded, since

$$|(\nu * h)(x)| = |\textstyle\int h(y^{-1}x)\nu(dy)| \leq \sup_{y \in X} |h(y)|$$

for every x. So we have a positive linear operator $T_\infty : L^\infty \to L^\infty$ defined by saying that $T_\infty h^\bullet = (\nu * h)^\bullet$ for every bounded Borel measurable function h. Moreover, if $u \in L^\infty$, there is a Borel measurable h such that $h^\bullet = u$ and $\sup_{y \in X} |h(y)| = \|u\|_\infty$, so that

$$\|T_\infty u\|_\infty \leq \sup_{x \in X} |(\nu * h)(x)| \leq \|u\|_\infty;$$

thus $\|T_\infty\| \leq 1$.

Since T_1 and T_∞ agree on $L^1 \cap L^\infty$, they have a common extension to a linear operator $T : L^1 + L^\infty \to L^1 + L^\infty$. By 371Gd, $\|Tu\|_p \leq \|u\|_p$ for every $p \in [1, \infty]$, $u \in L^p$. (Strictly speaking, I am relying on the standard identifications of L^1, L^∞ and L^p with the corresponding subspaces of $L^0(\mathfrak{A})$, where $(\mathfrak{A}, \bar{\mu})$ is the measure algebra of μ. Of course the argument for 371Gd applies equally well in $L^0(\mu)$.) Now suppose that $f \in \mathcal{L}^p$. Then it is expressible as $g + h$ where $g \in \mathcal{L}^1$ and $h : X \to \mathbb{R}$ is a bounded Borel measurable function, so we shall have

$$\nu * f = \nu * g + \nu * h \text{ wherever the right-hand side is defined;}$$

accordingly $\nu * f$ is defined a.e. and is measurable, and

$$\begin{aligned}\|\nu * f\|_p &= \|(\nu * g)^\bullet + (\nu * h)^\bullet\|_p = \|T_1 g^\bullet + T_\infty h^\bullet\|_p \\ &= \|T f^\bullet\|_p \leq \|f^\bullet\|_p = \|f\|_p,\end{aligned}$$

as required.

(b)(i) If $p = 1$, then $\gamma = \nu X = 1$. If $f \in \mathcal{L}^1$ is non-negative, then, by 444K, as in (a) above,

$$\|f * \nu\|_1 = ((f * \nu)\mu)(X) = (f\mu * \nu)(X) = (f\mu)(X) \cdot \nu X = \|f\|_1.$$

For general μ-integrable f,

$$\|f * \nu\|_1 \leq \|f^+ * \nu\|_1 + \|f^- * \nu\|_1 = \|f^+\|_1 + \|f^-\|_1 = \|f\|_1.$$

(ii) If $p = \infty$, then directly from the formula $(f * \nu)(x) = \int f(xy^{-1})\Delta(y^{-1})\nu(dy)$ we see that if $f : X \to \mathbb{R}$ is a bounded Borel measurable function then

$$|(f * \nu)(x)| \leq \int \Delta(y)^{-1}\nu(dy) \sup_{y\in X} |f(y)| = \gamma \sup_{y\in X} |f(y)|$$

for every x. Since changing f on a μ-negligible set changes $f * \nu$ on a μ-negligible set, we can argue as in (a) above to see that $f^\bullet \mapsto (f * \nu)^\bullet$ defines a linear operator from L^∞ to itself of norm at most γ.

(iii) Now suppose that $1 < p < \infty$. Set $q = \frac{p}{p-1}$, so that $\gamma = \int \Delta(y^{-1})^{1/q}\nu(dy)$. Suppose for the moment that $f \in \mathcal{L}^p$ is a non-negative Borel measurable function, and let $h : X \to \mathbb{R}$ be another non-negative Borel measurable function such that $\int h^q d\mu \leq 1$. In this case

$$\begin{aligned}\int (f * \nu) \times h \, d\mu &= \iint h(x)f(xy^{-1})\Delta(y^{-1})\nu(dy)\mu(dx)\\ &= \iint h(x)f(xy^{-1})\Delta(y^{-1})\mu(dx)\nu(dy)\end{aligned}$$

(by 417Ha, because $(x, y) \mapsto h(x)f(xy^{-1})\Delta(y^{-1})$ is Borel measurable and $\{x : h(x) \neq 0\}$ is a countable union of sets of finite measure for μ, while ν is totally finite)

$$= \iint h(xy)f(x)\mu(dx)\nu(dy)$$

by 442Kc, as usual, at least if the last integral is finite. But, for any $y \in X$,

$$\begin{aligned}\int h(xy)f(x)\mu(dx) &\leq \|f\|_p \big(\int |h(xy)|^q \mu(dx)\big)^{1/q}\\ &= \|f\|_p \big(\Delta(y^{-1}) \int |h(x)|^q \mu(dx)\big)^{1/q} \leq \|f\|_p \Delta(y^{-1})^{1/q}.\end{aligned}$$

So

$$\int (f * \nu) \times h \, d\mu = \iint h(xy)f(x)\mu(dx)\nu(dy) \leq \int \|f\|_p \Delta(y^{-1})^{1/q}\nu(dy) = \gamma\|f\|_p.$$

Because μ (being a quasi-Radon measure) is semi-finite, this means that $f * \nu \in \mathcal{L}^p$ and that $\|f * \nu\|_p \leq \gamma\|f\|_p$ (366D-366E, or 244Xf and 244Fa). (Once again, we need to know that every member of L^q can be represented by a Borel measurable function; this is a consequence of 443J or 412Xc.)

For general Borel measurable $f : X \to \mathbb{R}$ such that $\int |f|^p d\mu < \infty$, we know that from 444G that $f * \nu$ is Borel measurable, while $|f * \nu| \leq |f| * \nu$ (and $f * \nu$ is defined wherever $|f| * \nu$ is finite), so that

$$\|f * \nu\|_p \leq \||f| * \nu\|_p \leq \gamma\||f|\|_p = \gamma\|f\|_p.$$

Finally, if $f \in \mathcal{L}^p$ is arbitrary, then there is a Borel measurable $g : X \to \mathbb{R}$ such that $f =_{\text{a.e.}} g$, so that $f * \nu =_{\text{a.e.}} g * \nu$ and

$$\|f * \nu\|_p = \|g * \nu\|_p \leq \gamma\|g\|_p = \gamma\|f\|_p.$$

It follows at once that we have a bounded linear operator $f^\bullet \mapsto (f * \nu)^\bullet : L^p \to L^p$, of norm at most γ.

444N The following lemma on exchanging the order of repeated integrals will be fundamental to the formulae in the rest of the section.

Lemma Let X be a topological group and μ a left Haar measure on X. Suppose that f, g, $h \in \mathcal{L}^0(\mu)$ (the space of measurable real-valued functions defined μ-a.e. in X) are non-negative. Then, writing $\int d(x, y)$ to denote integration with respect to the quasi-Radon product measure $\mu \times \mu$,

$$\iint f(x)g(y)h(xy)dxdy = \iint f(x)g(y)h(xy)dydx = \int f(x)g(y)h(xy)d(x, y)$$

in $[0, \infty]$.

proof Following the standard pattern in results of this type, I deal with successively more complicated functions f, g and h. Evidently the situation is symmetric, so that it will be enough if I can show that $\iint f(x)g(y)h(xy)dxdy = \int f(x)g(y)h(xy)d(x, y)$.

(a) Suppose first that $f = \chi F$, $g = \chi G$ and $h = \chi H$, where F, G, H are Borel subsets of X. In this case

$$\iint f(x)g(y)h(xy)dxdy = \sup_{U,V\in\Sigma^f} \int_V \int_U f(x)g(y)h(xy)dxdy,$$

where Σ^f is the ideal of measurable sets of finite measure for μ. **P** For $y \in X$, $n \in \mathbb{N}$ and $U \in \Sigma^f$ write

$$q(y) = \int f(x)h(xy)dx = \mu(F \cap Hy^{-1}), \quad q_U(y) = \int_U f(x)h(xy)dx = \mu(U \cap F \cap Hy^{-1}),$$

$$q^{(n)}(y) = \min(n, q(y)), \quad q_U^{(n)}(y) = \min(n, q_U(y)).$$

Then every q_U is continuous, by 443C (with a little help from 323Cc), while $\sup_{U\in\Sigma^f} q_U(y) = q(y)$ for every y, because μ is semi-finite. Because μ is τ-additive and effectively locally finite, $(q^{(n)})^\bullet = \sup_{U\in\Sigma^f}(q_U^{(n)})^\bullet$ in $L^0(\mu)$ for every n (414Ab); because Σ^f is upwards-directed,

$$\begin{aligned}\int q(y)g(y)dy &= \sup_{n\in\mathbb{N}} \int q^{(n)}(y)g(y)dy \\ &= \sup_{n\in\mathbb{N},U\in\Sigma^f} \int q_U^{(n)}(y)g(y)dy = \sup_{U\in\Sigma^f} \int q_U(y)g(y)dy,\end{aligned}$$

that is,

$$\iint f(x)g(y)h(xy)dxdy = \sup_{U\in\Sigma_f} \int \int_U f(x)g(y)h(xy)dxdy.$$

On the other hand, for any $U \in \Sigma^f$, we surely have

$$\int \int_U f(x)g(y)h(xy)dxdy = \sup_{V\in\Sigma^f} \int_V \int_U f(x)g(y)h(xy)dxdy,$$

again because μ is semi-finite. Putting these together, we have the result. **Q**

Looking at the other side of the equation, $\int f(x)g(y)h(xy)d(x,y) = (\mu\times\mu)W$, where $W = (F \times G) \cap \{(x,y) : xy \in H\}$ is a Borel set; so that

$$\begin{aligned}\iint f(x)g(y)h(xy)dxdy &= \sup_{U,V\in\Sigma^f} (\mu\times\mu)((U\times V)\cap W) \\ &= \sup_{U,V\in\Sigma^f} \int_{U\times V} f(x)g(y)h(xy)d(x,y)\end{aligned}$$

(417C(iii)). But now we can apply 417Ha to see that, for any U, $V \in \Sigma^f$,

$$\int_{U\times V} f(x)g(y)h(xy)d(x,y) = \int_V \int_U f(x)g(y)h(xy)dxdy.$$

Taking the supremum over U and V, we get

$$\int f(x)g(y)h(xy)d(x,y) = \iint f(x)g(y)h(xy)dxdy.$$

(b) Clearly both sides of our equation

$$\int f(x)g(y)h(xy)d(x,y) = \iint f(x)g(y)h(xy)dxdy$$

are additive in f, g and h separately (subtraction, of course, will be another matter, as I am allowing ∞ to appear without restriction); and also behave identically if f or g or h is multiplied by a non-negative scalar. So the identity will be valid if f, g and h are all finite sums of non-negative multiples of characteristic functions of Borel sets. Moreover, by repeated use of B.Levi's theorem, we see that if $\langle f_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ and $\langle h_n\rangle_{n\in\mathbb{N}}$ are non-decreasing sequences of such functions with suprema f, g and h, then

$$\begin{aligned}\int f(x)g(y)h(xy)d(x,y) &= \sup_{n\in\mathbb{N}} \int f_n(x)g_n(y)h_n(xy)d(x,y) \\ &= \sup_{n\in\mathbb{N}} \iint f_n(x)g_n(y)h_n(xy)dxdy = \iint f(x)g(y)h(xy)dxdy.\end{aligned}$$

So the identity is valid for all non-negative Borel functions f, g and h.

(c) Finally, suppose only that f, g and h are non-negative, measurable and defined almost everywhere. In this case, by 443J(b-iv), there are Borel measurable functions f_0, g_0 and h_0, non-negative, defined everywhere on X and equal almost everywhere to f, g and h respectively. Let E be the conegligible set

$$\{x : x \in \operatorname{dom} f \cap \operatorname{dom} h \cap \operatorname{dom} g,\ f(x) = f_0(x),\ g(x) = g_0(x),\ h(x) = h_0(x)\}.$$

We find that $\int f(x)h(xy)dx = \int f_0(x)h_0(xy)dx$ for every $y \in X$. **P** $E \cap Ey^{-1}$ is conegligible (see 443A), and $f(x)h(xy) = f_0(x)h_0(xy)$ for every $x \in E \cap Ey^{-1}$. **Q** Consequently

$$\iint f(x)g(y)h(xy)dxdy = \iint f_0(x)g_0(y)h_0(xy)dxdy.$$

Secondly, $f(x)g(y)h(xy) = f_0(x)g_0(y)h_0(xy)$ $(\mu \times \mu)$-a.e. **P** Set $W = \{(x, y) : x \in E,\ y \in E,\ xy \in E\}$. Fubini's theorem, applied to $(U \times V) \setminus W$ where U, $V \in \Sigma^f$, shows that W is conegligible; but of course $f(x)g(y)h(xy) = f_0(x)g_0(y)h_0(xy)$ whenever $(x, y) \in W$. **Q** Accordingly

$$\int f(x)g(y)h(xy)d(x, y) = \int f_0(x)g_0(y)h_0(xy)d(x, y).$$

Combining this with the result of (b), applied to f_0, g_0 and h_0, we see that once again

$$\int f(x)g(y)h(xy)d(x, y) = \iint f(x)g(y)h(xy)dxdy,$$

as required.

444O Convolutions of functions: Theorem Let X be a topological group and μ a left Haar measure on X. For f, $g \in \mathcal{L}^0 = \mathcal{L}^0(\mu)$, write $(f * g)(x) = \int f(y)g(y^{-1}x)dy$ whenever this is defined in $\mathbb{R}$, taking the integral with respect to μ.

(a) Writing Δ for the left modular function of X,

$$\begin{aligned}(f * g)(x) &= \int f(y)g(y^{-1}x)dy = \int f(xy)g(y^{-1})dy \\ &= \int \Delta(y^{-1})f(xy^{-1})g(y)dy = \int \Delta(y^{-1})f(y^{-1})g(yx)dy\end{aligned}$$

whenever any of these integrals is defined in $\mathbb{R}$.

(b) If $f =_{\text{a.e.}} f_1$ (that is, $f = f_1$ a.e.) and $g =_{\text{a.e.}} g_1$, then $f * g = f_1 * g_1$.

(c)(i) $|(f * g)(x)| \leq (|f| * |g|)(x)$ whenever either is defined.

(ii)

$$((f_1 + f_2) * g)(x) = (f_1 * g)(x) + (f_2 * g)(x),$$

$$(f * (g_1 + g_2))(x) = (f * g_1)(x) + (f * g_2)(x),$$

$$((\alpha f) * g)(x) = (f * (\alpha g))(x) = \alpha(f * g)(x)$$

whenever the right-hand expressions are defined.

(d) If f, g and h belong to $\mathcal{L}^0$ and any of

$$\int (|f| * |g|)(x)|h|(x)dx, \quad \iint |f(x)g(y)h(xy)|dxdy,$$

$$\iint |f(x)g(y)h(xy)|dydx, \quad \int |f(x)g(y)h(xy)|d(x, y)$$

is defined in $[0, \infty[$ (writing $\int d(x, y)$ for integration with respect to the quasi-Radon product measure $\mu \times \mu$ on $X \times X$), then

$$\int (f * g)(x)h(x)dx, \quad \iint f(x)g(y)h(xy)dxdy,$$

$$\iint f(x)g(y)h(xy)dydx, \quad \int f(x)g(y)h(xy)d(x, y)$$

are all defined, finite and equal, provided that in the expression $(f * g)(x)h(x)$ we interpret the product as 0 when $h(x) = 0$ and $(f * g)(x)$ is undefined.

(e) If f, g and h belong to $\mathcal{L}^0$, $f*g$ and $g*h$ are defined a.e. and $x \in X$ is such that either $(|f|*(|g|*|h|))(x)$ or $((|f| * |g|) * |h|)(x)$ is defined in $\mathbb{R}$, then $(f * (g * h))(x)$ and $((f * g) * h)(x)$ are defined and equal.

(f) If $a \in X$ and f, $g \in \mathcal{L}^0$,

$$a\bullet_l(f*g) = (a\bullet_l f)*g, \quad a\bullet_r(f*g) = f*(a\bullet_r g),$$

$$(a\bullet_r f)*g = \Delta(a^{-1})f*(a^{-1}\bullet_l g).$$

(g) If X is abelian then $f*g = g*f$ for all f and g.

proof (a) Use 441J and 442Kc to see that the four formulae for $f*g$ coincide.

(b) Setting

$$E = \{y : y \in \operatorname{dom} f \cap \operatorname{dom} f_1 \cap \operatorname{dom} g \cap \operatorname{dom} g_1,\ f(y) = f_1(y),\ g(y) = g_1(y)\},$$

E is conegligible. If $x \in X$, then $f(y)g(y^{-1}x) = f_1(y)g_1(y^{-1}x)$ for every $y \in E \cap xE^{-1}$, which is also conegligible, by 443A; so $(f*g)(x) = (f_1*g_1)(x)$ if either is defined.

(c) These are all elementary.

(d) First consider non-negative f, g and h. The point is that, if any of the integrals is defined and finite,

$$\int (f*g)(x)h(x)dx = \iint f(x)g(x^{-1}y)h(y)dxdy = \iint \Delta(x^{-1})f(x^{-1})g(xy)h(y)dxdy$$

(by 442Kb)

$$= \iint \Delta(x^{-1})f(x^{-1})g(xy)h(y)dydx$$

(by 444N, recalling that $x \mapsto \Delta(x^{-1})f(x^{-1})$ will belong to $\mathcal{L}^0$ if f does, by 442J and 442H)

$$= \iint f(x)g(x^{-1}y)h(y)dydx = \iint f(x)g(y)h(xy)dydx$$

(substituting xy for y in the inner integral, as permitted by 441J). (The 'and finite' at the beginning of the last sentence is there because I have changed the rules since the last paragraph, and $f*g$ is not allowed to take the value ∞. So we have to take care that

$$\{y : h(y) > 0,\ \textstyle\int f(x)g(x^{-1}y)dx = \infty\}$$

is negligible.) Now applying 444N again, we get

$$\int (f*g)(x)h(x)dx = \iint f(x)g(y)h(xy)dxdy$$

$$= \iint f(x)g(y)h(xy)dydx = \int f(x)g(y)h(xy)d(x,y)$$

if any of these integrals is finite.

For the general case, the hypothesis on $|f|$, $|g|$ and $|h|$ is sufficient to ensure that the four expressions are equal for any combination of $f^{\pm}$, $g^{\pm}$ and $h^{\pm}$; adding and subtracting these combinations appropriately, we get the result.

(e) The point is that, for non-negative f, g and h,

$$((f*g)*h)(x) = \int (f*g)(z)h(z^{-1}x)dz = \int (f*g)(z)h'(z)dz$$

(setting $h'(z) = h(z^{-1}x)$)

$$= \iint f(y)g(z)h'(yz)dzdy$$

(using (d); to see that h' is measurable, refer to 443A as usual)

$$= \iint f(y)g(z)h(z^{-1}y^{-1}x)dzdy$$

$$= \int f(y)(g*h)(y^{-1}x)dy = (f*(g*h))(x)$$

at least as long as one of the expressions here is finite. (Note that, as in 255J, we need to suppose that $f*g$ and $g*h$ are defined a.e. when moving from $\int(f*g)(z)h(z^{-1}x)dz$ to $\iint f(y)g(z)h(z^{-1}y^{-1})dydz$ and from $\iint f(y)g(z)h(z^{-1}y^{-1}x)dzdy$ to $\int f(y)(g*h)(y^{-1}x)dy$, since in part (d) I am more tolerant of infinities in the repeated integrals than I was in the definition of $f*g$.) Once again, subject to the inner integrals implicit in the formulae $f*(g*h)$ and $(f*g)*h$ being adequately defined, we can use addition and subtraction to obtain the result for general f, g and h.

(f) These are immediate from the formulae in (a), using 442K if necessary.

(g) If X is abelian, then $\Delta(y)=1$ for every y, so

$$(g*f)(x)=\int g(y)f(y^{-1}x)dy=\int g(y)\Delta(y^{-1})f(xy^{-1})dy=(f*g)(x)$$

if either $(f*g)(x)$ or $(g*f)(x)$ is defined.

444P Proposition Let X be a topological group and μ a left Haar measure on X.
(a) If $f\in\mathcal{L}^1(\mu)^+$ and $g\in\mathcal{L}^0(\mu)$ then $f*g$, as defined in 444O, is equal to $(f\mu)*g$ as defined in 444H.
(b) If $f\in\mathcal{L}^0(\mu)$ and $g\in\mathcal{L}^1(\mu)^+$ then $f*g=f*(g\mu)$.

proof Again, these are immediate from the formulae above:

$$(f*g)(x)=\int g(y^{-1}x)f(y)\mu(dy)=\int g(y^{-1}x)(f\mu)(dy)=(f\mu*g)(x),$$

$$(f*g)(x)=\int f(xy^{-1})\Delta(y^{-1})g(y)\mu(dy)=\int f(xy^{-1})\Delta(y^{-1})(g\mu)(dy)=(f*g\mu)(x)$$

whenever these are defined, using 235M, as usual, to calculate $\int d(f\mu)$, $\int d(g\mu)$. (Note that as we assume throughout that f and g are defined μ-almost everywhere, all the functions $y\mapsto g(y^{-1}x)$, $y\mapsto f(xy^{-1})$ are also defined μ-a.e., by the results set out in 443A.)

444Q Proposition Let X be a topological group and μ a left Haar measure on X.
(a) Let f, g be non-negative μ-integrable functions. Then, defining $f*g$ as in 444O, we have $f*g\in\mathcal{L}^1=\mathcal{L}^1(\mu)$ and

$$(f\mu)*(g\mu)=(f*g)\mu$$

for all f, $g\in\mathcal{L}^1$.
(b) For any f, $g\in\mathcal{L}^1$, $f*g\in\mathcal{L}^1$ and

$$\int f*g\,d\mu=\int f d\mu\int g\,d\mu,\quad \|f*g\|_1\le\|f\|_1\|g\|_1.$$

proof (a) Putting 444K and 444P together, $f\mu*g\mu=(f\mu*g)\mu$, so that $f*g=f\mu*g$ is μ-integrable, and

$$(f*g)\mu=(f\mu*g)\mu=f\mu*g\mu.$$

(b) Taking $h=\chi X$ in 444Od, we get $\int f*g\,d\mu=\int f d\mu\int g\,d\mu$. Now

$$\|f*g\|_1=\int|f*g|\le\int|f|*|g|=\int|f|\int|g|=\|f\|_1\|g\|_1.$$

444R Proposition Let X be a topological group and μ a left Haar measure on X. Take any $p\in[1,\infty]$.
(a) $\|f*g\|_p\le\|f\|_1\|g\|_p$ whenever $f\in\mathcal{L}^1(\mu)$, $g\in\mathcal{L}^p(\mu)$.
(b) For a real-valued function f defined on a subset of X, set $\tilde f(x)=f(x^{-1})$ whenever this is defined. Then $f*\tilde g=(g*\tilde f)^\sim$ for all f, $g\in\mathcal{L}^0$. If X is unimodular then $\|\tilde f\|_p=\|f\|_p$ for every $f\in\mathcal{L}^0$.
(c) Set $q=\infty$ if $p=1$, $p/(p-1)$ if $1<p<\infty$, 1 if $p=\infty$. If $f\in\mathcal{L}^p(\mu)$ and $g\in\mathcal{L}^q(\mu)$, then $f*\tilde g$ is defined everywhere in X and is continuous, and $\|f*\tilde g\|_\infty\le\|f\|_p\|g\|_q$. If X is unimodular, then $f*g\in C_b(X)$ and $\|f*g\|_\infty\le\|f\|_p\|g\|_q$ for every $f\in\mathcal{L}^p(\mu)$, $g\in\mathcal{L}^q(\mu)$.

Remark In the formulae above, interpret $\|g\|_\infty$ as $\|g^\bullet\|_\infty=\operatorname{ess\,sup}|g|$ for $g\in\mathcal{L}^\infty=\mathcal{L}^\infty(\mu)$, and as ∞ for $g\in\mathcal{L}^0\setminus\mathcal{L}^\infty$. Because μ is strictly positive, this agrees with the usual definition $\|g\|_\infty=\sup_{x\in X}|g(x)|$ when g is continuous.

proof (a) If $f\ge0$, then

$$\|f*g\|_p = \|f\mu*g\|_p \leq (f\mu)(X)\|g\|_p = \|f\|_1\|g\|_p$$

by 444Pa and 444Ma. Generally,

$$\|f*g\|_p \leq \|f^+*g\|_p + \|f^-*g\|_p \leq (\|f^+\|_1 + \|f^-\|_1)\|g\|_p = \|f\|_1\|g\|_p.$$

(b)(i) By 443A, as usual, $\tilde{f} \in \mathcal{L}^0$ whenever $f \in \mathcal{L}^0$. For any $x \in X$,

$$(f*\tilde{g})(x) = \int f(y)\tilde{g}(y^{-1}x)dy = \int \tilde{f}(y^{-1})g(x^{-1}y)dy = (g*\tilde{f})(x^{-1}) = (g*\tilde{f})^\sim(x)$$

if any of these are defined.

(ii) If X is unimodular, then, for any $f \in \mathcal{L}^0$ and $p \in [1,\infty[$,

$$\|\tilde{f}\|_p^p = \int |f(x^{-1})|^p dx = \int \Delta(x^{-1})|f(x^{-1})|^p dx = \|f\|_p^p;$$

while ess sup $|\tilde{f}|$ = ess sup $|f|$ because E^{-1} is conegligible whenever $E \subseteq X$ is conegligible.

(c)(i) For any $x \in X$,

$$(f*\tilde{g})(x) = \int f(y)g(x^{-1}y)dy = \int f \times (x\bullet_l g)$$

in the language of 443G and 444O. By 443Gb, $x\bullet_l g \in \mathcal{L}^q$, so $(f*\tilde{g})(x) = \int f \times (x\bullet_l g)$ is defined.

(ii) If $p > 1$, so that $q < \infty$, then $x \mapsto (x\bullet_l g)^\bullet : X \to L^q$ is continuous (443Ge), so

$$x \mapsto (f*\tilde{g})(x) = \int f^\bullet \times (x\bullet_l g)^\bullet$$

is continuous, because $f^\bullet \in L^p \cong (L^q)^*$. If $p = 1$, then

$$(f*\tilde{g})(x) = \int f(xy)g(y)dy = \int (x^{-1}\bullet_l f) \times g$$

for every x; since $x \mapsto (x^{-1}\bullet_l f)^\bullet : X \to L^1$ is continuous, so is $f*\tilde{g}$.

(iii) If X is unimodular then $f*g = f*\tilde{\tilde{g}}$ is continuous, because $\tilde{g} \in \mathcal{L}^q$ by (b), and $\|f*g\|_\infty \leq \|f\|_p\|\tilde{g}\|_q = \|f\|_p\|g\|_q$.

444S Remarks (a) From 444Ob we see that $*$, regarded as a function from $\mathcal{L}^1 \times \mathcal{L}^p$ to $\mathcal{L}^p$ (444Ra), not merely descends to a bilinear operator from $L^1 \times L^p$ to L^p but can actually be regarded as a function from $L^1 \times L^p$ to $\mathcal{L}^p$. Similarly, if the group is unimodular, and $\frac{1}{p}+\frac{1}{q} = 1$, the map $* : \mathcal{L}^p \times \mathcal{L}^q \to C_b(X)$ (444Rc) descends to a map from $L^p \times L^q$ to $C_b(X)$.

(b) 444Qb tells us that if μ is a left Haar measure on a topological group then $L^1(\mu)$, with the operation $f^\bullet * g^\bullet = (f*g)^\bullet$, is a normed algebra (therefore, of course, a Banach algebra); it is commutative if X is abelian (444Og). If, for $f \in \mathcal{L}^1(\mu)$, we write $f\mu$ for the countably additive functional defined by setting $(f\mu)(E) = \int_E f d\mu$ for every Borel set E, then the formula $f^\bullet \mapsto f\mu$ defines an embedding of $L^1(\mu)$ as a subalgebra of the Banach algebra of τ-additive Borel measures described in 444E; 444Qa tells us that the embedding respects the multiplication.

444T Proposition Let X be a topological group and μ a left Haar measure on X. Then for any $p \in [1,\infty[$, $f \in \mathcal{L}^p(\mu)$ and $\epsilon > 0$ there is a neighbourhood U of the identity e in X such that $\|\nu*f - f\|_p \leq \epsilon$ and $\|f*\nu - f\|_p \leq \epsilon$ whenever ν is a quasi-Radon measure on X such that $\nu U = \nu X = 1$.

proof (a) To begin with, suppose that f is non-negative, continuous and bounded, and that $G = \{x : f(x) > 0\}$ has finite measure; set $M = \sup_{x\in X} f(x)$. Write $\mathcal{U}$ for the family of neighbourhoods of e. Take $\delta > 0$, $\eta \in]0,1]$ such that

$$(2\delta + (1+\delta)^p - 1)^{1/p}\|f\|_p \leq \epsilon,$$

$$(1-\eta)^p \int ((f - \eta\chi X)^+)^p d\mu - M^p\eta^p \geq (1-\delta)\int f^p d\mu, \quad (1-\eta)^{(1-p)/p} \leq 1+\delta.$$

For each $U \in \mathcal{U}$, set

$$H_U = \text{int}\{x : f(y) \geq f(x) - \eta \text{ for every } y \in xU^{-1} \cup U^{-1}x\}.$$

Then H_U is open and for every $x \in X$ there is a $U \in \mathcal{U}$ such that $|f(y) - f(x)| \leq \frac{1}{2}\eta$ whenever $y \in xUU^{-1} \cup U^{-1}xU$, so that $x \in \operatorname{int} xU \subseteq H_U$. Thus $\{H_U : U \in \mathcal{U}\}$ is an upwards-directed family of open sets with union X, and there is a $U \in \mathcal{U}$ such that $\mu(G \setminus H_U) \leq \eta$; moreover, because Δ is continuous, we can suppose that $\Delta(y^{-1}) \geq 1 - \eta$ for every $y \in U$.

Now suppose that ν is a quasi-Radon measure on X such that $\nu U = \nu X = 1$. Then, for any $x \in G \cap H_U$,

$$(\nu * f)(x) = \int f(y^{-1}x)\nu(dy) = \int_U f(y^{-1}x)\nu(dy) \geq f(x) - \eta$$

because $x \in H_U$ and $y^{-1}x \in U^{-1}x$ whenever $y \in U$. Similarly,

$$(f * \nu)(x) = \int_U f(xy^{-1})\Delta(y^{-1})\nu(dy) \geq (f(x) - \eta)(1 - \eta)$$

for every $x \in H_U$. Now this means that, setting $h_1 = \nu * f$, $h_2 = f * \nu$ we have $(f \wedge h_i)(x) \geq (f(x) - \eta)(1 - \eta)$ for every $x \in G \cap H_U$, both i. Accordingly

$$\begin{aligned}\int (f \wedge h_i)^p d\mu &\geq (1 - \eta)^p \int_{G \cap H_U} ((f - \eta\chi X)^+)^p d\mu \\ &\geq (1 - \eta)^p \int ((f - \eta\chi X)^+)^p d\mu - \int_{G \setminus H_U} f^p \\ &\geq (1 - \eta)^p \int ((f - \eta\chi X)^+)^p d\mu - M^p\eta^p \geq (1 - \delta) \int f^p d\mu.\end{aligned}$$

Now, just because f and h_i are non-negative, and $p \geq 1$,

$$|f - h_i|^p + 2(f \wedge h_i)^p \leq f^p + h_i^p.$$

Also, writing

$$\gamma = \int \Delta(y)^{(1-p)/p}\nu(dy) = \int_U \Delta(y)^{(1-p)/p}\nu(dy) \leq (1 - \eta)^{(1-p)/p} \leq 1 + \delta,$$

we have

$$\|h_1\|_p = \|\nu * f\|_p \leq \|f\|_p, \quad \|h_2\|_p = \|f * \nu\|_p \leq \gamma\|f\|_p$$

(444M), so that $\int h_i^p d\nu \leq (1 + \delta)^p \int f^p d\mu$ for both i, and

$$\int |f - h_i|^p d\mu \leq \int f^p d\mu + \int h_i^p d\mu - 2\int (f \wedge h_i)^p d\mu \leq (2\delta + (1 + \delta)^p - 1) \int f^p d\mu$$

for both i. But this means that

$$\max(\|f - f * \nu\|_p, \|f - \nu * f\|_p) \leq (2\delta + (1 + \delta)^p - 1)^{1/p}\|f\|_p \leq \epsilon.$$

As ν is arbitrary, we have found a suitable U.

(b) For any continuous bounded function f such that $\mu\{x : f(x) \neq 0\} < \infty$, we can find neighbourhoods U_1, U_2 of e such that

$$\|f^+ - \nu * f^+\|_p \leq \tfrac{1}{2}\epsilon, \quad \|f^+ - f^+ * \nu\|_p \leq \tfrac{1}{2}\epsilon$$

whenever $\nu U_1 = \nu X = 1$,

$$\|f^- - \nu * f^-\|_p \leq \tfrac{1}{2}\epsilon, \quad \|f^- - f^- * \nu\|_p \leq \tfrac{1}{2}\epsilon$$

whenever $\nu U_2 = \nu X = 1$. So we shall have

$$\|f - \nu * f\|_p \leq \epsilon, \quad \|f - f * \nu\|_p \leq \epsilon$$

whenever $\nu(U_1 \cap U_2) = \nu X = 1$.

(c) For general $f \in \mathcal{L}^p(\mu)$, there is a bounded continuous function $g : X \to \mathbb{R}$ such that $\mu\{x : g(x) \neq 0\} < \infty$ and $\|f - g\|_p \leq \frac{1}{4}\epsilon$ (415Pa). Now there is a neighbourhood U_1 of e such that

$$\|g - \nu * g\|_p \leq \tfrac{1}{4}\epsilon, \quad \|g - g * \nu\|_p \leq \tfrac{1}{4}\epsilon$$

whenever $\nu U_1 = \nu X = 1$. There is also a neighbourhood U_2 of e such that $\Delta(y^{-1})^{(1-p)/p} \leq 2$ for every $y \in U_2$, so that

$$\|g * \nu - f * \nu\|_p \leq 2\|g - f\|_p \leq \tfrac{1}{2}\epsilon$$

whenever $\nu U_2 = \nu X = 1$. Since we have

$$\|\nu * g - \nu * f\|_p \leq \|g - f\|_p \leq \tfrac{1}{4}\epsilon$$

whenever $\nu X = 1$, we get $\|f - \nu * f\|_p \leq \epsilon$, $\|f = f * \nu\|_p \leq \epsilon$ whenever $\nu(U_1 \cap U_2) = \nu X = 1$.

This completes the proof.

444U Corollary Let X be a topological group and μ a left Haar measure on X. For any Haar measurable $E \subseteq X$ such that $0 < \mu E < \infty$, and any $f \in \bigcup_{1 \leq p < \infty} \mathcal{L}^p(\mu)$, write

$$f_E(x) = \frac{1}{\mu E} \int_{xE} f d\mu, \quad f'_E(x) = \frac{1}{\mu(Ex)} \int_{Ex} f d\mu$$

for $x \in X$. Then, for any $p \in [1, \infty[$, $f \in \mathcal{L}^p$ and $\epsilon > 0$, there is a neighbourhood U of the identity in X such that $\|f_E - f\|_p \leq \epsilon$ and $\|f'_E - f\|_p \leq \epsilon$ whenever $E \subseteq U$ is a non-negligible Haar measurable set.

proof Take $\delta \in]0, 1[$ such that $\delta(1-\delta)^{(1-p)/p}\|f\|_p \leq \frac{1}{2}\epsilon$. By 444T, there is a neighbourhood U of the identity such that $\|f - f * \nu\|_p \leq \frac{1}{2}\epsilon$, $\|f - \nu * f\|_p \leq \epsilon$ whenever ν is a quasi-Radon measure such that $\nu U = \nu X = 1$. Shrinking U if necessary, we may suppose also that $U = U^{-1}$, that $\mu U < \infty$ and that $|\Delta(x) - 1| \leq \delta$ for every $x \in U$, where Δ is the left modular function of X. If $E \subseteq U$ and $\mu E > 0$, consider the quasi-Radon measures ν, ν', $\tilde{\nu}$ and $\tilde{\nu}'$ on X defined by setting

$$\nu F = \frac{1}{\mu E^{-1}} \int_{E \cap F} \Delta(x^{-1}) \mu(dx), \quad \nu' F = \frac{\mu(E \cap F)}{\mu E}, \quad \tilde{\nu} F = \nu F^{-1}, \quad \tilde{\nu}' F = \nu' F^{-1}$$

whenever these are defined. (They are quasi-Radon measures because ν and ν' are totally finite indefinite-integral measures over μ and the map $x \mapsto x^{-1}$ is a homeomorphism.) Because $E \subseteq U = U^{-1}$, we have

$$\tilde{\nu} U = \nu U^{-1} = \frac{1}{\mu E^{-1}} \int_E \Delta(x^{-1}) \mu(dx) = 1 = \tilde{\nu} X$$

by 442Ka, while

$$\tilde{\nu}' X = \tilde{\nu}' U = \nu' U^{-1} = 1.$$

Now consider $f * \tilde{\nu}$ and $\tilde{\nu}' * f$. For any $x \in X$,

$$(f * \tilde{\nu})(x) = \int f(xy^{-1}) \Delta(y^{-1}) \tilde{\nu}(dy) = \int f(xy) \Delta(y) \nu(dy)$$

(because $\tilde{\nu}$ is the image of ν under the map $y \mapsto y^{-1}$)

$$= \frac{1}{\mu E^{-1}} \int \chi E(y) \Delta(y^{-1}) f(xy) \Delta(y) \mu(dy)$$

(noting that ν is an indefinite-integral measure over μ, and using 235M)

$$= \frac{1}{\mu E^{-1}} \int \chi E(x^{-1}y) f(y) \mu(dy)$$

$$= \frac{1}{\mu E^{-1}} \int_{xE} f(y) \mu(dy) = \frac{\mu E}{\mu E^{-1}} f_E(x),$$

$$(\tilde{\nu}' * f)(x) = \int f(y^{-1}x) \tilde{\nu}'(dy) = \int f(yx) \nu'(dy)$$

(because $\tilde{\nu}'$ is the image of ν' under the map $y \mapsto y^{-1}$)

$$= \frac{1}{\mu E} \int \chi E(y) f(yx) \mu(dy) = \frac{\Delta(x^{-1})}{\mu E} \int \chi E(yx^{-1}) f(y) \mu(dy)$$

(by 442Kc)

$$= \frac{1}{\mu(Ex)} \int_{Ex} f(y) \mu(dy) = f'_E(x).$$

So

$$\|f - f_E\|_p \leq \|f - f * \tilde{\nu}\|_p + \left|\frac{\mu E^{-1}}{\mu E} - 1\right| \|f * \tilde{\nu}\|_p.$$

Now $\mu E^{-1} = \int_E \Delta(y^{-1})\mu(dy)$, so that

$$|\mu E - \mu E^{-1}| \leq \int_E |\Delta(y^{-1}) - 1|\mu(dy) \leq \delta\mu E, \quad |\frac{\mu E^{-1}}{\mu E} - 1| \leq \delta;$$

also

$$\|f * \tilde{\nu}\|_p \leq \|f\|_p \int \Delta(y)^{(1-p)/p}\nu(dy) \leq \|f\|_p(1-\delta)^{(1-p)/p}$$

by 444Mb, so

$$\big|\frac{\mu E^{-1}}{\mu E} - 1\big|\|f * \tilde{\nu}\|_p \leq \delta(1-\delta)^{(1-p)/p}\|f\|_p \leq \tfrac{1}{2}\epsilon,$$

and $\|f - f_E\|_p \leq \epsilon$.

On the other hand,

$$\|f - f'_E\|_p = \|f - \tilde{\nu}' * f\|_p \leq \epsilon;$$

as E is arbitrary, we have found a suitable U.

444V So far I have not emphasized the special properties of compact groups. But of course they are the centre of the subject, and for the sake of a fundamental theorem in §446 I give the following result.

Theorem Let X be a compact topological group and μ a left Haar measure on X.

(a) For any u, $v \in L^2 = L^2(\mu)$ we can interpret their convolution $u * v$ either as a member of the space $C(X)$ of continuous real-valued functions on X, or as a member of the space L^2.

(b) If $w \in L^2$, then $u \mapsto u * w$ is a compact linear operator whether regarded as a map from L^2 to $C(X)$ or as a map from L^2 to itself.

(c) If $w \in L^2$ and $w = \tilde{w}$ (as defined in 443Af), then $u \mapsto u * w : L^2 \to L^2$ is a self-adjoint operator.

proof (a) Being compact, X is unimodular (442Ic). As noted in 444Sa, $*$ can be regarded as a bilinear map from $L^2 \times L^2$ to $C_b(X) = C(X)$. Because μX must be finite, we now have a natural map $f \mapsto f^\bullet$ from $C(X)$ to L^2, so that we can think of $u * v$ as a member of L^2 for u, $v \in L^2$.

(b)(i) Evidently $(u, v) \mapsto u * v : L^2 \times L^2 \to C(X)$ is bilinear. So $u \mapsto u * w : L^2 \to C(X)$ is linear, for any $w \in L^2$.

(ii) Let B be the unit ball of L^2, and give it the topology induced by the weak topology $\mathfrak{T}_s(L^2, L^2)$, so that B is compact (4A4Ka). Let $\bullet_l$ be the left action of G on L^2 as in 443G, so that $a\bullet_l f^\bullet = (a\bullet_l f)^\bullet$ for every $a \in G$, $f \in \mathcal{L}^2 = \mathcal{L}^2(\mu)$.

If f, $g \in \mathcal{L}^2$ and $a \in X$, then

$$(f * g)(a) = \int f(x)g(x^{-1}a)dx = \int f(x)\tilde{g}(a^{-1}x)dx = \int f \times a\bullet_l\tilde{g},$$

where $\tilde{g}(x) = g(x^{-1})$ whenever this is defined. So if u, $w \in L^2$ and $a \in X$, $(u*w)(a) = (u|a\bullet_l\tilde{w})$, where $\tilde{w} = \tilde{g}^\bullet$ whenever $w = g^\bullet$, as in 443Af. It follows that, for any $w \in L^2$, the function $(a, u) \mapsto (u * w)(a) : X \times B \to \mathbb{R}$ is continuous. **P** We know that $\bullet_l : X \times L^2 \to L^2$ is continuous when L^2 is given its norm topology (443Ge). Given $a_0 \in X$ and $u_0 \in B$ and $\epsilon > 0$, set $G = \{a : a \in X, \|a\bullet_l\tilde{w} - a_0\bullet_l\tilde{w}\|_2 < \frac{1}{2}\epsilon\}$ and $H = \{u : u \in B, |(u|a_0\bullet_l\tilde{w}) - (u_0|a_0\bullet_l\tilde{w})| < \frac{1}{2}\epsilon\}$, so that G and H are open sets (for the topologies under consideration) containing a_0 and u_0 respectively. Then, for any $a \in G$ and $u \in H$,

$$\begin{aligned}|(u * w)(a) - (u_0 * w)(a_0)| &= |(u|a\bullet_l\tilde{w}) - (u_0|a_0\bullet_l\tilde{w})| \\ &\leq |(u|a\bullet_l\tilde{w}) - (u|a_0\bullet_l\tilde{w})| + |(u|a_0\bullet_l\tilde{w}) - (u_0|a_0\bullet_l\tilde{w})| \\ &\leq \|a\bullet_l\tilde{w} - a_0\bullet_l\tilde{w}\|_2 + \frac{1}{2}\epsilon \leq \epsilon.\end{aligned}$$

As a_0, u_0 and ϵ are arbitrary, we have the result. **Q**

Because X is compact, this means that $u \mapsto u * w : B \to C(X)$ is continuous when $C(X)$ is given its norm topology and B is given the weak topology (4A2G(g-ii)). Because B is compact in the weak topology, $\{u*w : u \in B\}$ is compact in $C(X)$. But this implies that $u \mapsto u*w$ is a compact linear operator (definition: 3A5Ka).

(iii) Because X is compact, μ is totally finite, so, for $f \in C(X)$, $\|f\|_2 \leq \|f\|_\infty \sqrt{\mu X}$, and the natural map $f \mapsto f^\bullet : C(X) \to L^2$ is a bounded linear operator. Consequently the map $u \mapsto (u * w)^\bullet : L^2 \to L^2$ is a compact operator, by 4A4La.

(c) Now suppose that $w = \tilde{w}$. In this case $(u * w|v) = (u|v * w)$ for all u, $v \in L^2$. **P** Express u, v and w as $f^\bullet$, $g^\bullet$ and $h^\bullet$ where f, g and h are square-integrable Borel measurable functions defined everywhere on X. We have

$$\begin{aligned}(u * w|v) &= \int (f * h)(x)g(x)dx = \iint f(y)h(y^{-1}x)g(x)dydx \\ &= \iint f(y)h(y^{-1}x)g(x)dxdy\end{aligned}$$

(because $(x, y) \mapsto f(y)h(y^{-1}x)g(x)$ is Borel measurable, μ is totally finite and $\iint |f(y)h(y^{-1}x)g(x)|dydx = (|u| * |w||v|)$ is finite)

$$\begin{aligned}&= \iint f(y)\tilde{h}(x^{-1}y)g(x)dxdy = \int f(y)(g * \tilde{h})(y)dy \\ &= (u|v * \tilde{w}) = (u|v * w). \textbf{ Q}\end{aligned}$$

As u and v are arbitrary, this shows that $u \mapsto u * w : L^2 \to L^2$ is self-adjoint.

444X Basic exercises >(a) Let X be a Hausdorff topological group. Show that if λ and ν are totally finite Radon measures on X then $\lambda * \nu$ is the image measure $(\lambda \times \nu)\phi^{-1}$, where $\phi(x, y) = xy$ for x, $y \in X$, and in particular is a Radon measure.

>(b) Let X be a topological group and λ, ν two totally finite quasi-Radon measures on X. Writing $\operatorname{supp}\lambda$ for the support of λ, show that $\operatorname{supp}(\lambda * \nu) = \overline{(\operatorname{supp}\lambda)(\operatorname{supp}\nu)}$.

(c) Let X be a topological group and M^+_{qR} the family of totally finite quasi-Radon measures on X. Show that $(\lambda, \nu) \mapsto \lambda * \nu : M^+_{\mathrm{qR}} \times M^+_{\mathrm{qR}} \to M^+_{\mathrm{qR}}$ is continuous for the narrow topology on M^+_{qR}. (*Hint*: 437Ma.)

(d) Let X be a Hausdorff topological group. Show that X is abelian iff its Banach algebra of signed τ-additive Borel measures is commutative.

(e) Let X be a topological group, and M_τ its Banach algebra of signed τ-additive Borel measures. (i) Show that we have actions $\bullet_l$, $\bullet_r$ of X on M_τ defined by writing $(a\bullet_l\nu)(E) = \nu(aE)$, $(a\bullet_r\nu)(E) = \nu(Ea^{-1})$. (ii) Show that $(a\bullet_l\lambda) * \nu = a\bullet_l(\lambda * \nu)$, $\lambda * (a\bullet_r\nu) = a\bullet_r(\lambda * \nu)$ for all $a \in X$ and λ, $\nu \in M_\tau$.

(f) Let X be a compact topological group, and B a norm-bounded subset of the Banach algebra M_τ of signed τ-additive Borel measures on X. Show that $(\lambda, \nu) \mapsto \lambda * \nu : B \times B \to M_\tau$ is continuous for the vague topology on M_τ.

(g) Let X be a topological group, and ν a totally finite quasi-Radon measure on X. Show that for any Borel sets E, $F \subseteq X$, the function $(g, h) \mapsto \nu(gE \cap Fh)$ is Borel measurable. (*Hint*: for Borel sets $W \subseteq X \times X$, set $\nu' W = \nu\{x : (x, x) \in W\}$. Consider the action of $X \times X$ on itself defined by writing $(g, h)\bullet(x, y) = (gx, yh^{-1})$.)

(h) Let X be a topological group and f a real-valued function defined on a subset of X. (i) Show that $a\bullet_r(\nu * f) = \nu * (a\bullet_r f)$ (definition: 441Ac) whenever $a \in X$ and ν is a measure on X. (ii) Show that if X carries Haar measures, then $a\bullet_l(f * \nu) = (a\bullet_l f) * \nu$ whenever $a \in X$ and ν is a measure on X.

(i) Let X be a topological group carrying Haar measures, $f : X \to \mathbb{R}$ a bounded continuous function and ν a totally finite quasi-Radon measure on X. Show that $f * \nu$ is continuous.

(j) Let X be a topological group carrying Haar measures, f a real-valued function defined on a subset of X, and λ, ν totally finite quasi-Radon measures on X. Show that $((f*\nu)*\lambda)(x) = (f*(\nu*\lambda))(x)$ whenever the right-hand side is defined. (See also 444Yi.)

(k) Let X be an abelian topological group carrying Haar measures. Show that $f*\nu = \nu*f$ for every measure ν on X and every real-valued function f defined on a subset of X.

>(l) Let X be a topological group and μ a left Haar measure on X. (i) Let ν be a totally finite quasi-Radon measure on X such that $x \mapsto \nu(xF)$ is continuous for every closed set $F \subseteq X$. Show that ν is truly continuous with respect to μ. (*Hint*: if $\mu F = 0$, apply 444K to $\nu * \chi F^{-1}$ to see that $\nu(xF) = 0$ for μ-almost every x.) (ii) Let ν be a totally finite Radon measure on X such that $x \mapsto \nu(xK)$ is continuous for every compact set $K \subseteq X$. Show that ν is truly continuous with respect to μ.

(m) Let X be a topological group carrying Haar measures, and ν a non-zero totally finite quasi-Radon measure on X such that $\nu(xE) = 0$ whenever $x \in X$ and $\nu E = 0$. (i) Show that ν is strictly positive, so that X is ccc. (ii) Show that a subset of X is ν-negligible iff it is Haar negligible. (*Hint*: if μ is a left Haar measure on X, consider the convolution $\chi G * \nu$ for a suitable set G.)

(n) Use the method of part (b) of the proof of 444M to prove part (a) there.

>(o) Let X be the group $S^1 \times S^1$, with the topology defined by giving the first coordinate the usual topology of S^1 and the second coordinate the discrete topology, so that X is a locally compact abelian group. Let μ be a Haar measure on X. (i) Find a Borel measurable function $f : X \times X \to \{0,1\}$ such that $\iint f(x,y)\mu(dx)\mu(dy) \neq \iint f(x,y)\mu(dy)\mu(dx)$. (ii) Let ν be the Radon measure on X defined by setting $\nu E = \#(\{s : (s,s^{-1}) \in E\})$ if this is finite, ∞ otherwise. Define $g : X \to \{0,1\}$ by setting $g(s,t) = 1$ if $s = t$, 0 otherwise. Show that $\iint g(xy)\nu(dy)\mu(dx) = \infty$, $\iint g(xy)\mu(dx)\nu(dy) = 0$. (iii) Find a closed set $F \subseteq X$ such that $x \mapsto \nu(xF)$ is not Haar measurable.

>(p) Let X be a Hausdorff topological group and for $a \in X$ write δ_a for the Radon measure on X defined by saying that $\delta_a(E) = (\chi E)(a)$ whenever $E \subseteq X$. (i) Show that $\delta_a * \delta_b = \delta_{ab}$ for all a, $b \in X$. (ii) Show that, in the notation of 444Xe, $\delta_a * \hat{\nu}$ is the completion of $a^{-1}\bullet_l\nu$ and $\hat{\nu} * \delta_a$ is the completion of $a\bullet_r\nu$ for every $a \in X$ and every totally finite τ-additive Borel measure ν on X with completion $\hat{\nu}$. (iii) Show that $\delta_a * f = a\bullet_l f$ for every $a \in X$ and every real-valued function f defined on a subset of X. (iv) Show that if X carries Haar measures, and has left modular function Δ, $f * \delta_a = \Delta(a^{-1})S'_{a^{-1}}(f)$ for every $a \in X$ and every real-valued function f defined on a subset of X. (v) Use these formulae to relate 444Of to 444B.

(q) Let X be a topological group and μ a left Haar measure on X. For a real-valued function f defined on a subset of X set $\tilde{f}(x) = f(x^{-1})$ whenever $x \in X$ and $x^{-1} \in \operatorname{dom} f$. Show that if f, $g \in \mathcal{L}^0(X)$ then $(f*g)^\sim = \tilde{g} * \tilde{f}$.

(r) Let X be a locally compact Hausdorff topological group and μ a left Haar measure on X. Show that if f, $g : X \to \mathbb{R}$ are continuous functions with compact support, then $f * g$ is a continuous function with compact support.

(s) In 444Rc, show that $f * \tilde{g}$ is uniformly continuous for the bilateral uniformity. (*Hint*: in 443Ge, $x \mapsto x\bullet_l u$ is uniformly continuous.)

(t) Let X be a topological group with a totally finite Haar measure μ. Show that (i) $(u*w|v) = (u|v*\tilde{w})$ for any u, v, $w \in L^2 = L^2(\mu)$, where $\tilde{w}$ and $u*v$ are defined as in 443Af and 444V (ii) the map $u \mapsto u*w : L^2 \to L^2$ is a compact linear operator for any $w \in L^2$. (*Hint*: for (ii), use 443L.)

(u) Let X be a topological group with a Haar probability measure μ. Show that $L^2(\mu)$ with convolution is a Banach algebra.

444Y Further exercises (a) Find a subgroup X of $\{0,1\}^{\mathbb{N}}$ and quasi-Radon probability measures λ, ν on X and a set $A \subseteq X$ such that $(\lambda*\nu)^*(A) = 1$ but $(\lambda \times \nu)\{(x,y) : x, y \in X, x+y \in A\} = 0$.

(b) Let X be a **topological semigroup**, that is, a semigroup with a topology such that multiplication is continuous. (i) For totally finite τ-additive Borel measures λ, ν on X, show that there is a τ-additive Borel measure $\lambda * \nu$ defined by saying that $(\lambda * \nu)(E) = (\lambda \times \nu)\{(x,y) : xy \in E\}$ for every Borel set $E \subseteq X$. (ii) Show that in this context $(\lambda_1 * \lambda_2) * \lambda_3 = \lambda_1 * (\lambda_2 * \lambda_3)$. (iii) Show that $\int f d(\lambda * \nu) = \int f(xy)\lambda(dx)\nu(dy)$ whenever f is $(\lambda * \nu)$-integrable. (iv) Show that if the topology is Hausdorff and λ and ν are tight (that is, inner regular with respect to the compact sets) so is $\lambda * \nu$. (v) Show that we have a Banach algebra of signed τ-additive Borel measures on X, as in 444E.

(c) Let X be a topological group, and write $M_\tau^{(\mathbb{C})}$ for the complexification of the L-space M_τ of 444E, as described in 354Yk. Show that $M_\tau^{(\mathbb{C})}$, with the natural extension of the convolution operator of 444E, is a complex Banach algebra, and that we still have $|\lambda * \nu| \leq |\lambda| * |\nu|$ for λ, $\nu \in M_\tau^{(\mathbb{C})}$.

(d) Find a locally compact Hausdorff topological group X, a Radon probability measure ν on X and an open set $G \subseteq X$ such that $\{(xGx^{-1})^\bullet : x \in X\}$ is not a separable subset of the measure algebra of ν.

(e) Let X be a Polish group. We say that a subset A of X is **Haar null** if there are a universally measurable set $E \supseteq A$ and a non-zero Radon measure ν on X such that $\nu(xEy) = 0$ for every x, $y \in X$. (i) Show that the family of Haar null sets is a translation-invariant σ-ideal of subsets of X. (*Hint*: if $\langle E_n \rangle_{n\in\mathbb{N}}$ is a sequence of universally measurable Haar null sets, we can find Radon probability measures ν_n concentrated near the identity such that $\nu_n(xE_ny) = 0$ for every x, y and n; now construct an infinite convolution product $\nu = \nu_0 * \nu_1 * \dots$ and show that $\nu(xE_ny) = 0$ for every x, y and n.) (ii) Show that if Y is another Polish group, $\phi : X \to Y$ is a surjective continuous homomorphism and $B \subseteq Y$ is Haar null, then $\phi^{-1}[B]$ is Haar null in X. (iii) Show that if X is a locally compact Polish group then a subset of X is Haar null iff it is Haar negligible in the sense of 442H. (See Solecki 01.)

(f) Let X be a topological group and μ a left Haar measure on X. Let τ be a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mu)$ (§374). Show that if ν is any totally finite quasi-Radon measure on X, then we have a linear operator $f^\bullet \mapsto (\nu * f)^\bullet$ from L^τ to itself, of norm at most νX.

(g) Let X be a topological group with a left Haar measure μ, M_τ the Banach algebra of signed τ-additive Borel measures on X, and $p \in [1,\infty]$. (i) Show that we have a multiplicative linear operator T from M_τ to the Banach algebra $\mathrm{B}(L^p(\mu); L^p(\mu))$ defined by writing $(T\nu)(f^\bullet) = (\hat{\nu} * f)^\bullet$ whenever ν is a totally finite τ-additive Borel measure on X with completion $\hat{\nu}$ and $f \in \mathcal{L}^p(\mu)$. (*Hint*: Use 444K and 444B to show that $(\lambda * \nu) * f =_{\text{a.e.}} \lambda * (\nu * f)$ for enough λ, ν and f. See also 444Yi.) (ii) Show that $\|T\nu\| = \|\nu\|$ for every $\nu \in M_\tau^+$.

(h) Let X be a unimodular topological group with left Haar measure μ. Suppose that p, q, $r \in [1,\infty]$ are such that $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, interpreting $\frac{1}{\infty}$ as 0. Show that if $f \in \mathcal{L}^p(\mu)$ and $g \in \mathcal{L}^q(\mu)$ then $f * g \in \mathcal{L}^r(\mu)$ and $\|f * g\|_r \leq \|f\|_p \|g\|_q$. (*Hint*: 255Ym. Take care to justify any changes in order of integration.)

(i) Let X be a topological group carrying Haar measures. Investigate conditions under which the associative laws

$$\lambda * (\nu * f) = (\lambda * \nu) * f, \quad \lambda * (f * \nu) = (\lambda * f) * \nu, \quad f * (\lambda * \nu) = (f * \lambda) * \nu,$$

$$f * (g * \nu) = (f * g) * \nu, \quad f * (\nu * g) = (f * \nu) * g, \quad \nu * (f * g) = (\nu * f) * g$$

will be valid, where λ and ν are quasi-Radon measures on X and f, g are real-valued functions. Relate your results to 444Xp.

(j) Let X be a topological group and μ a left Haar measure on X. Let τ be a $\mathcal{T}$-invariant extended Fatou norm on $L^0(\mu)$ such that $\tau{\restriction}L^\tau$ is an order-continuous norm. For a totally finite quasi-Radon measure ν on X, let $T_\nu : L^\tau \to L^\tau$ be the corresponding linear operator (444Yf). Show that for any $u \in L^\tau$ and $\epsilon > 0$ there is a neighbourhood U of the identity in X such that $\tau(T_\nu u - u) \leq \epsilon$ whenever $\nu U = \nu X = 1$.

(k) Let X be a topological group with a left Haar measure μ. For $u \in L^2 = L^2(\mu)$, set $A = \{a \bullet_l u : a \in X\}$ (443G) in L^2, and $D = \{v * u : v \in L^1(\mu), v \geq 0, \int v = 1\}$. (i) Show that the closed convex hull of A in

L^2 is the closure of D. (*Hint*: (α) use 444Od to show that if $w \in L^2$ and $(w'|w) \geq \gamma$ for every $w' \in A$, then $(w'|w) \geq \gamma$ for every $w' \in D$ (β) use 444U to show that $A \subseteq \overline{D}$.) (ii) Show that the closed linear subspace W_u generated by A is the closure of $\{v * u : v \in L^1\}$. (iii) Show that if $w \in L^2$ and $w \in A^\perp$, that is, $(u'|w) = 0$ for every $u' \in A$, then $W_w \subseteq W_u^\perp$. (iv) Show that if X is σ-compact, then W_u is separable. (*Hint*: A is σ-compact, by 443Ge.) (v) Set $C = \{f^\bullet : f \in \mathcal{L}^2 \cap C(X)\}$. Show that $C \cap W_u$ is dense in W_u. (*Hint*: $v * u \in C$ for many v, by 444Rc.) (vi) Show that if X is σ-compact, then W_u has an orthonormal basis in C. (vii) Show that L^2 has an orthonormal basis in C. (*Hint*: if X is σ-compact, take a maximal orthogonal family of subspaces W_u, find a suitable orthonormal basis of each, and use (iii) to see that these assemble to form a basis of L^2. For a general locally compact Hausdorff group, start with a σ-compact open subgroup, and then deal with its cosets. For a general topological group with a Haar measure, use 443L.) (Compare 416Yg.)

(l) Let X be a topological group with a left Haar measure μ. Let λ be the quasi-Radon product measure on $X \times X$. Let $\mathcal{U}$ be the set of those $h \in \mathcal{L}^1(\lambda)$ such that $(X \times X) \setminus \{(x, y) : (x, y) \in \operatorname{dom} h,\ h(x, y) = 0\}$ can be covered by a sequence of open sets of finite measure. (i) Show that if $h \in \mathcal{U}$, then $(x, y) \mapsto h(y, y^{-1}x)$ belongs to $\mathcal{U}$. (*Hint*: 443Xb.) (ii) Show that if $h \in \mathcal{U}$, then $(Th)(x) = \int h(y, y^{-1}x)\mu(dy)$ is defined for almost every $x \in X$ and Th is μ-integrable, with $\|Th\|_1 \leq \|h\|_1$. (*Hint*: 255Xd.) (ii) Show that if h_1, $h_2 \in \mathcal{U}$ are equal λ-a.e. then $Th_1 = Th_2$ μ-a.e. (iii) Show that every member of $L^1(\lambda)$ can be represented by a member of $\mathcal{U}$. (*Hint*: 443Xl.) (iv) Show that if f, $g \in \mathcal{L}^1(\mu)$ and both are zero outside some countable union of open sets of finite measure, then $T(f \otimes g) = f * g$, where $(f \otimes g)(x, y) = f(x)g(y)$. (v) Show that if we set $\tilde{T}(h^\bullet) = (Th)^\bullet$ for $h \in \mathcal{U}$, then $\tilde{T} : L^1(\lambda) \to L^1(\mu)$ is the unique continuous linear operator such that $\tilde{T}(u \otimes v) = u * v$ for all u, $v \in L^1(\mu)$, where $u * v$ is defined in 444S and $\otimes : L^1(\mu) \times L^1(\mu) \to L^1(\lambda)$ is the canonical bilinear map (253E).

(m) In 444Yl, suppose that $\mu X = 1$. (i) Show that the map $\tilde{T}$ belongs to the class $\mathcal{T}_{\bar{\lambda},\bar{\mu}}$ of §373. (ii) Show that if $p \in [1, \infty]$ then $\|Th\|_p \leq \|h\|_p$ whenever $h \in \mathcal{U} \cap \mathcal{L}^p(\lambda)$.

(n) Rewrite this section in terms of right Haar measures instead of left Haar measures.

444 Notes and comments The aim of this section and the next is to work through ideas from the second half of Chapter 25, and Chapter 28, in forms natural in the context of general topological groups. (It is of course possible to go farther; see 444Yb. It is the glory and confusion of twentieth-century mathematics that it has no firm stopping points.) The move from $\mathbb{R}$ to an arbitrary topological group is a large one, and I think it is worth examining the various aspects of this leap as they affect the theorems here. The most conspicuous change, and the one which most greatly affects the forms of the results, is the loss of commutativity. We are forced to re-examine every formula to determine exactly which manipulations can still be justified. Multiplications must be written the right way round, and inversions especially must be watched. But while there are undoubtedly some surprises, we find that in fact (provided we take care over the definitions) the most important results survive. Of course I wrote the earlier results out with a view to what I expected to do here, but no dramatic manoeuvers are needed to turn the fundamental results 255G, 255H, 255J, 257B, 257E, 257F into the new versions 444Od, 444Qb, 444Oe, 444C, 444B, 444Qa. (The changed order of presentation is an indication of the high connectivity of the web here, not of any new pattern.) In fact what makes the biggest difference is not commutativity, as such, but unimodularity. In groups which are not unimodular we do have new phenomena, as in 444Mb and 444Of, and these lead to complications in the proofs of such results as 444U, even though the result there is exactly what one would expect.

In this section I ignore right Haar measures entirely. I do not even put them in the exercises. If you wish to take this theory farther, you may some day have to work out the formulae appropriate to right Haar measures. (You can check your results in Hewitt & Ross 63, 20.32.) But for the moment, I think that they are likely to be just a source of confusion. There is one point which you may have noticed. The theory of groups is essentially symmetric. In the definition of 'group' there is no distinction between left and right. In the formulae defining group actions, we do have such a distinction, because they must reflect the fact that we write $g \bullet x$ rather than $x \bullet g$. With $\bullet_l$ and $\bullet_r$, for instance (444Of), if we want them to be actions in the standard sense we have to put an $^{-1}$ into the definition of $\bullet_l$ but not into the definition of $\bullet_r$. But we

still expect that, for instance, $\lambda * \nu$ and $\nu * \lambda$ will be related in some transparent way. However there is an exception to this rule in the definitions of $\nu * f$ and $f * \nu$ (444H, 444J). The modular function appears in the latter, so in fact the definition applies only in a more restricted class of groups. In abelian groups we assume that $f * \nu$ and $\nu * f$ will be equal, and they are (444Xk), but strictly speaking, on the definitions here, we can write $f * \nu = \nu * f$ only for abelian topological groups carrying Haar measures.

From the point of view of the proofs in this section, the principal change is that the Haar measures here are no longer assumed to be σ-finite. I am well aware that non-σ-finite measures are a minority interest, especially in harmonic analysis, but I do think it interesting that σ-finiteness is not relevant to the main results, and the techniques required to demonstrate this are very much in the spirit of this treatise (see, in particular, the proof of 444N, and the repeated applications of 443Jb). The basic difficulty is that we can no longer exchange repeated integrals, even of non-negative Borel measurable functions, quite automatically. Let me emphasize that the result in 444N is really rather special. If we try to generalize it to other measures or other types of function we encounter the usual obstacles (444Xo).

A difficulty of a different kind arises in the proof of 444Fc. Here I wish to show that the function $g \mapsto (g{\bullet}E)^\bullet : G \to \mathfrak{A}$ is Borel measurable for every Borel measurable set E. The first step is to deal with open sets E, and it would be nice if we could then apply the Monotone Class Theorem. But the difficulty is that even though the map $(a, b) \mapsto a \setminus b : \mathfrak{A} \times \mathfrak{A} \to \mathfrak{A}$ is continuous, it does not quite follow that the map

$$g \mapsto (g{\bullet}(E \setminus F))^\bullet = (g{\bullet}E)^\bullet \setminus (g{\bullet}F)^\bullet$$

is Borel measurable whenever $g \mapsto (g{\bullet}E)^\bullet$ and $g \mapsto (g{\bullet}F)^\bullet$ are, because the map $g \mapsto ((g{\bullet}E)^\bullet, (g{\bullet}F)^\bullet) : G \to \mathfrak{A} \times \mathfrak{A}$ might conceivably fail to be Borel measurable, if the metric space $\mathfrak{A}$ is not separable, that is, if the Maharam type of the measure ν is uncountable. Of course the difficulty is easily resolved by an extra twist in the argument.

I use different techniques for the two parts of 444M as an excuse to recall the ideas of §371; in fact part (a) is slightly easier than part (b) when proved by the method of the latter (444Xn).

444U is a kind of density theorem. Compared with the density theorems in §§223 and 261, it is a 'mean' rather than 'pointwise' density theorem; if E is concentrated near the identity, then $f^\bullet_E$ approximates $f^\bullet$ in L^p, but there is no suggestion that we can be sure that $f_E(x) \simeq f(x)$ for any particular xs unless we know much more about the set E. In fact this is to be expected from the form of the results concerning Lebesgue measure. The sets E considered there are generally intervals or balls, and even in such a general form as 223Ya we need a notion of scalar multiplication separate from the group operation.

445 The duality theorem

In this section I present a proof of the Pontryagin-van Kampen duality theorem (445U). As in Chapter 28, and for the same reasons, we need to use complex-valued functions; the relevant formulae in §444 apply unchanged, and I shall not repeat them here, but you may wish to re-read that section taking all functions to be complex- rather than real-valued. (It *is* possible to avoid complex-valued measures, which I relegate to the exercises.) The duality theorem itself applies only to abelian locally compact Hausdorff groups, and it would be reasonable, on first reading, to take it for granted that all groups here are of this type, which simplifies some of the proofs a little.

My exposition is based on that of RUDIN 67. I start with the definition of 'dual group', including a description of a topology on the dual (445A), and the simplest examples (445B), with a mention of Fourier-Stieltjes transforms of measures (445C-445D). The elementary special properties of dual groups of groups carrying Haar measures are in 445E-445G; in particular, in these cases, the bidual of a group begins to make sense, and we can start talking about Fourier transforms of functions.

Serious harmonic analysis begins with the identification of the dual group with the maximal ideal space of L^1 (445H-445K). The next idea is that of 'positive definite' function (445L-445M). Putting these together, we get the first result here which asserts that the dual group of an abelian group X carrying Haar measures is sufficiently large to effectively describe functions on X (Bochner's theorem, 445N). It is now easy to establish that X can be faithfully embedded in its bidual (445O). We also have most of the machinery necessary to describe the correctly normalized Haar measure of the dual group, with a first step towards identifying functions whose Fourier transforms will have inverse Fourier transforms (the Inversion Theorem, 445P). This leads directly to the Plancherel Theorem, identifying the L^2 spaces of X and its dual (445R).

At this point it is clear that the bidual $\mathfrak{X}$ cannot be substantially larger than X, since they must have essentially the same L^2 spaces. A little manipulation of shifts and convolutions in L^2 (445S-445T) shows that X must be dense in $\mathfrak{X}$, and a final appeal to local compactness shows that X is closed in $\mathfrak{X}$.

445A Dual groups Let X be any topological group.

(a) A **character** on X is a continuous group homomorphism from X to $S^1 = \{z : z \in \mathbb{C}, |z| = 1\}$. It is easy to see that the set $\mathcal{X}$ of all characters on X is a subgroup of the group $(S^1)^X$, just because S^1 is an abelian topological group. (If χ, $\theta \in \mathcal{X}$, then $x \mapsto \chi(x)\theta(x)$ is continuous, and

$$(\chi\theta)(xy) = \chi(xy)\theta(xy) = \chi(x)\chi(y)\theta(x)\theta(y) = \chi(x)\theta(x)\chi(y)\theta(y) = (\chi\theta)(x)(\chi\theta)(y).)$$

So $\mathcal{X}$ itself is an abelian group.

(b) Give $\mathcal{X}$ the topology of uniform convergence on subsets of X which are totally bounded for the bilateral uniformity on X (4A5Hb, 4A5O). (If $\mathcal{E}$ is the set of totally bounded subsets of X, then the topology of $\mathcal{X}$ is generated by the pseudometrics ρ_E, where $\rho_E(\chi, \theta) = \sup_{x\in E} |\chi(x) - \theta(x)|$ for $E \in \mathcal{E}$ and χ, $\theta \in \mathcal{X}$. It will be useful, in this formula, to interpret $\sup \emptyset = 0$, so that $\rho_\emptyset$ is the zero pseudometric. Note that $\mathcal{E}$ is closed under finite unions, so $\{\rho_E : E \in \mathcal{E}\}$ is upwards-directed, as in 2A3Fe.) Then $\mathcal{X}$ is a Hausdorff topological group. (If $x \in E \in \mathcal{E}$ and χ, χ_0, θ, $\theta_0 \in \mathcal{X}$,

$$\begin{aligned}|(\chi\theta)(x) - (\chi_0\theta_0)(x)| &= |\chi(x)(\theta(x) - \theta_0(x)) + \theta_0(x)(\chi(x) - \chi_0(x))| \\ &\leq |\theta(x) - \theta_0(x)| + |\chi(x) - \chi_0(x)|,\end{aligned}$$

$$|\chi^{-1}(x) - \chi_0^{-1}(x)| = |\chi(x) - \chi_0(x)|,$$

so

$$\rho_E(\chi\theta, \chi_0\theta_0) \leq \rho_E(\chi, \chi_0) + \rho_E(\theta, \theta_0), \quad \rho_E(\chi^{-1}, \chi_0^{-1}) = \rho_E(\chi, \chi_0).$$

If $\chi \neq \theta$ then there is an $x \in X$ such that $\chi(x) \neq \theta(x)$, and now $\{x\} \in \mathcal{E}$ and $\rho_{\{x\}}(\chi, \theta) > 0$.)

(c) Note that if X is locally compact, then its totally bounded sets are just its relatively compact sets (4A5Oe), so the topology of $\mathcal{X}$ is the topology of uniform convergence on compact subsets of X.

(d) If X is compact, then $\mathcal{X}$ is discrete. **P** X itself is totally bounded, so $U = \{\chi : |\chi(x) - 1| \leq 1$ for every $x \in X\}$ is a neighbourhood of the identity ι in $\mathcal{X}$. But if $\chi \in U$, $x \in X$ then $|\chi(x)^n - 1| \leq 1$ for every $n \in \mathbb{N}$, so $\chi(x) = 1$. Thus $U = \{\iota\}$ and ι is an isolated point of $\mathcal{X}$; it follows that every point of $\mathcal{X}$ is isolated. **Q**

(e) If X is discrete then $\mathcal{X}$ is compact. **P** The only totally bounded sets in X are the finite sets, so the topology of $\mathcal{X}$ is just that induced by its embedding in $(S^1)^X$. On the other hand, every homomorphism from X to S^1 is continuous, so $\mathcal{X}$ is a closed set in $(S^1)^X$, which is compact by Tychonoff's theorem. **Q**

445B Examples (a) If $X = \mathbb{R}$ with addition, then $\mathcal{X}$ can also be identified with the additive group $\mathbb{R}$, if we write $\chi_y(x) = e^{iyx}$ for x, $y \in \mathbb{R}$.

P It is easy to check that every χ_y, so defined, is a character on $\mathbb{R}$, and that $y \mapsto \chi_y : \mathbb{R} \to \mathcal{X}$ is a homomorphism. On the other hand, if χ is a character, then (because it is continuous) there is a $\delta \geq 0$ such that $|\chi(x) - 1| \leq 1$ whenever $|x| \leq \delta$. $\chi(\delta)$ is uniquely expressible as $e^{i\alpha}$ where $|\alpha| \leq \frac{\pi}{2}$. Set $y = \alpha/\delta$, so that $\chi(\delta) = \chi_y(\delta)$. Now $\chi(\frac{1}{2}\delta)$ must be one of the square roots of $\chi(\delta)$, so is $\pm\chi_y(\frac{1}{2}\delta)$; but as $|\chi(\frac{1}{2}\delta) - 1| \leq 1$, it must be $+\chi_y(\frac{1}{2}\delta)$. Inducing on n, we see that $\chi(2^{-n}\delta) = \chi_y(2^{-n}\delta)$ for every $n \in \mathbb{N}$, so that $\chi(2^{-n}k\delta) = \chi_y(2^{-n}k\delta)$ for every $k \in \mathbb{Z}$, $n \in \mathbb{N}$; as χ and χ_y are continuous, $\chi = \chi_y$. Thus the map $y \mapsto \chi_y$ is surjective and is a group isomorphism between $\mathbb{R}$ and $\mathcal{X}$.

As for the topology of $\mathcal{X}$, $\mathbb{R}$ is a locally compact topological group, so the totally bounded sets are just the relatively compact sets (4A5Oe), that is, the bounded sets in the usual sense (2A2F). Now a straightforward calculation shows that for any $\alpha \geq 0$ in $\mathbb{R}$ and $\epsilon \in]0, 2[$,

$$\rho_{[-\alpha,\alpha]}(\chi_y, \chi_z) \leq \epsilon \iff \alpha|y - z| \leq 2 \arcsin \frac{\epsilon}{2},$$

so that the topology of $\mathcal{X}$ agrees with that of $\mathbb{R}$. **Q**

(b) Let X be the group $\mathbb{Z}$ with its discrete topology. Then we may identify its dual group $\mathcal{X}$ with S^1 itself, writing $\chi_\zeta(n) = \zeta^n$ for $\zeta \in S^1$, $n \in \mathbb{Z}$. **P** Once again, it is elementary to check that every χ_ζ is a character, and that $\zeta \mapsto \chi_\zeta$ is an injective group homomorphism from S^1 to $\mathcal{X}$. If $\chi \in \mathcal{X}$, set $\zeta = \chi(1)$; then $\chi = \chi_\zeta$. So $\mathcal{X} \cong S^1$. And because the only totally bounded sets in X are finite, $\chi \mapsto \chi_\zeta$ is continuous, therefore a homeomorphism. **Q**

(c) On the other hand, if $X = S^1$ with its usual topology, then we may identify its dual group $\mathcal{X}$ with $\mathbb{Z}$, writing $\chi_n(\zeta) = \zeta^n$ for $n \in \mathbb{Z}$, $\zeta \in S^1$. **P** The verification follows the same lines as in (a) and (b). As usual, the key step is to show that the map $n \mapsto \chi_n : \mathbb{Z} \to \mathcal{X}$ is surjective. We can do this by applying (a). If $\chi \in \mathcal{X}$, then $x \mapsto \chi(e^{ix})$ is a character of $\mathbb{R}$, so there is a $y \in \mathbb{R}$ such that $\chi(e^{ix}) = e^{iyx}$ for every $x \in \mathbb{R}$. In particular, $e^{2iy\pi} = \chi(1) = 1$, so $y \in \mathbb{Z}$, and $\chi = \chi_y$. Concerning the topology of $\mathcal{X}$, we know from 445Ad that it must be discrete, so that also matches the usual topology of $\mathbb{Z}$. **Q**

(d) Let $\langle X_j \rangle_{j \in J}$ be any family of topological groups, and X their product (4A5G). For each $j \in J$ let $\mathcal{X}_j$ be the dual group of X_j. Then the dual group of X can be identified with the subgroup $\mathcal{X}$ of $\prod_{j \in I} \mathcal{X}_j$ consisting of those $\chi \in \prod_{j \in J} \mathcal{X}_j$ such that $\{j : \chi(j) \text{ is not the identity}\}$ is finite; the action of $\mathcal{X}$ on X is defined by the formula

$$\chi \bullet x = \prod_{j \in J} \chi(j)(x(j)).$$

(This is well-defined because only finitely many terms in the product are not equal to 1.) If I is finite, so that $\mathcal{X} = \prod_{j \in I} \mathcal{X}_j$, the topology of $\mathcal{X}$ is the product topology.

P As usual, it is easy to check that $\bullet$, as defined above, defines an injective homomorphism from $\mathcal{X}$ to the dual group of X. If θ is any character on X, then for each $j \in I$ we have a continuous group homomorphism $\varepsilon_j : X_j \to X$ defined by setting $\varepsilon_j(\xi)(j) = \xi$, $\varepsilon_j(\xi)(k) = e_k$, the identity of X_k, for every $k \neq j$. Setting $\chi(j) = \theta\varepsilon_j$ for each j, we obtain $\chi \in \prod_{j \in I} \mathcal{X}_j$. Now there is a neighbourhood U of the identity of X such that $|\theta(x) - 1| \leq 1$ for every $x \in U$, and we may suppose that U is of the form $\{x : x(j) \in G_j \text{ for every } j \in J\}$, where $J \subseteq I$ is finite and G_j is a neighbourhood of e_j for every $j \in J$. If $k \in I \setminus J$, $\varepsilon_k(\xi) \in U$ for every $\xi \in X_k$, so that $|\chi(k)(\xi) - 1| \leq 1$ for every ξ, and $\chi(k)$ must be the identity character on X_k; this shows that $\chi \in \mathcal{X}$. If $x \in X$ and $x(j) = e_j$ for $j \in J$, then again $|\theta(x^n) - 1| \leq 1$ for every $n \in \mathbb{N}$, so $\theta(x) = \chi \bullet x = 1$. For any $x \in X$, we can express it as a finite product $y \prod_{j \in J} \varepsilon_j(x(j))$ where $y(j) = e_j$ for every $j \in J$, so that

$$\theta(x) = \theta(y) \prod_{j \in J} \theta\varepsilon_j(x(j)) = \prod_{j \in J} \chi(j)(x(j)) = \chi \bullet x.$$

Thus $\bullet$ defines an isomorphism between $\mathcal{X}$ and the dual group of X.

As for the topology of $\mathcal{X}$, a subset of X is totally bounded iff it is included in a product of totally bounded sets (4A5Od). If $E = \prod_{j \in I} E_j$ is such a product (and not empty), then for χ, $\theta \in \mathcal{X}$

$$\sup_{j \in I} \rho_{E_j}(\chi(j), \theta(j)) \leq \rho_E(\chi, \theta) \leq \sum_{j \in I} \rho_{E_j}(\chi(j), \theta(j)),$$

so (if I is finite) the topology on $\mathcal{X}$ is just the product topology. **Q**

445C Fourier-Stieltjes transforms Let X be a topological group, and $\mathcal{X}$ its dual group. For any totally finite topological measure ν on X, we can form its 'characteristic function' or **Fourier-Stieltjes transform** $\hat{\nu} : \mathcal{X} \to \mathbb{C}$ by writing $\hat{\nu}(\chi) = \int \chi(x)\nu(dx)$ (cf. 285A).

445D Theorem Let X be a topological group, and $\mathcal{X}$ its dual group. If λ and ν are totally finite quasi-Radon measures on X, then $(\lambda * \nu)\hat{} = \hat{\lambda} \times \hat{\nu}$.

proof If $\chi \in \mathcal{X}$, then, by 444C,

$$\begin{aligned}(\lambda * \nu)\hat{}(\chi) &= \int \chi \, d(\lambda * \nu) = \iint \chi(xy)\lambda(dx)\nu(dy) \\ &= \iint \chi(x)\chi(y)\lambda(dx)\nu(dy) = \int \chi(x)\lambda(dx) \cdot \int \chi(y)\nu(dy) = \hat{\lambda}(\chi)\hat{\nu}(\chi).\end{aligned}$$

445E Let us turn now to groups carrying Haar measures. I start with three welcome properties.

Proposition Let X be a topological group with a neighbourhood of the identity which is totally bounded for the bilateral uniformity on X, and $\mathcal{X}$ its dual group, with its dual group topology.

(a) The map $(\chi, x) \mapsto \chi(x) : \mathcal{X} \times X \to S^1$ is continuous.

(b) Let $\mathfrak{X}$ be the dual group of $\mathcal{X}$, again with its dual group topology, the topology of uniform convergence on totally bounded subsets of $\mathcal{X}$. Then we have a continuous homomorphism $x \mapsto \hat{x} : X \to \mathfrak{X}$ defined by setting $\hat{x}(\chi) = \chi(x)$ for $x \in X$, $\chi \in \mathcal{X}$.

(c) For any totally finite quasi-Radon measure ν on X, its Fourier-Stieltjes transform $\hat{\nu} : \mathcal{X} \to \mathbb{C}$ is uniformly continuous.

Remark Note that the condition here is satisfied by any topological group X carrying Haar measures (443H).

proof Fix an open totally bounded set U_0 containing the identity.

(a) Let $\chi_0 \in \mathcal{X}$, $x_0 \in X$ and $\epsilon > 0$. Then x_0U_0 is totally bounded, so

$$V = \{\chi : |\chi(y) - \chi_0(y)| \leq \tfrac{1}{2}\epsilon \text{ for every } y \in x_0U_0\}$$

is a neighbourhood of χ_0. Also

$$U = \{x : x \in x_0U_0, |\chi_0(x) - \chi_0(x_0)| \leq \tfrac{1}{2}\epsilon\}$$

is a neighbourhood of x_0. And if $\chi \in V$, $x \in U$ we have

$$|\chi(x) - \chi_0(x_0)| \leq |\chi(x) - \chi_0(x)| + |\chi_0(x) - \chi_0(x_0)| \leq \epsilon.$$

As χ_0, x_0 and ϵ are arbitrary, $(\chi, x) \mapsto \chi(x)$ is continuous.

(b)(i) It is easy to check that $\hat{x}$, as defined above, is always a homomorphism from $\mathcal{X}$ to S^1, and that $x \mapsto \hat{x} : X \to (S^1)^{\mathcal{X}}$ is a homomorphism. Because $\rho_{\{x\}}$ is always one of the defining pseudometrics for the topology of $\mathcal{X}$ (445Ab), $\hat{x}$ is always continuous, so belongs to $\mathfrak{X}$.

(ii) To see that $\widehat{\ }$ is continuous, I argue as follows. Take an open set $H \subseteq \mathfrak{X}$ and $x_0 \in X$ such that $\hat{x}_0 \in H$. Then there are a totally bounded set $F \subseteq \mathcal{X}$ and an $\epsilon > 0$ such that $\mathfrak{x} \in H$ whenever $\mathfrak{x} \in \mathfrak{X}$ and $\rho_F(\mathfrak{x}, \hat{x}_0) \leq \epsilon$. Now x_0U_0 is a totally bounded neighbourhood of x_0, so

$$V = \{\theta : \theta \in \mathcal{X}, |\theta(y) - 1| \leq \tfrac{1}{2}\epsilon \text{ for every } y \in x_0U_0\}$$

is a neighbourhood of the identity in $\mathcal{X}$. There are therefore $\chi_0, \dots, \chi_n \in \mathcal{X}$ such that $F \subseteq \bigcup_{k \leq n} \chi_k V$. Set

$$U = \{x : x \in x_0U_0, |\chi_k(x) - \chi_k(x_0)| < \tfrac{1}{2}\epsilon \text{ for every } k \leq n\}.$$

Then U is an open neighbourhood of x_0 in X.

If $x \in U$ and $\chi \in F$ then there is a $k \leq n$ such that $\theta = \chi_k^{-1}\chi \in V$, so that

$$\begin{aligned} |\chi(x) - \chi(x_0)| &= |\chi_k(x)\theta(x) - \chi_k(x_0)\theta(x_0)| \\ &\leq |\chi_k(x) - \chi_k(x_0)| + |\theta(x) - \theta(x_0)| \leq \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon. \end{aligned}$$

But this shows that $\rho_F(\hat{x}, \hat{x}_0) \leq \epsilon$, so $\hat{x} \in H$.

So we have $x_0 \in U \subseteq \{x : \hat{x} \in H\}$. As x_0 is arbitrary, $\{x : \hat{x} \in H\}$ is open; as H is arbitrary, $x \mapsto \hat{x}$ is continuous.

(c) Let $\epsilon > 0$. Because ν is τ-additive, there are $x_0, \dots, x_n \in X$ such that $\nu(X \setminus \bigcup_{k \leq n} x_kU_0) \leq \frac{1}{3}\epsilon$. Set $E = \bigcup_{k \leq n} x_kU_0$; then E is totally bounded. So

$$V = \{\theta : |\theta(x) - 1| \leq \tfrac{\epsilon}{1 + 3\nu X} \text{ for every } x \in E\}$$

is a neighbourhood of the identity in $\mathcal{X}$. If χ, $\chi' \in \mathcal{X}$ are such that $\chi^{-1}\chi' \in V$, then

$$|\hat{\nu}(\chi) - \hat{\nu}(\chi')| \leq \int |\chi(x) - \chi'(x)|\nu(dx)$$
$$\leq 2\nu(X \setminus E) + \frac{\epsilon\nu E}{1+3\nu X} \leq \epsilon.$$

Since ϵ is arbitrary (and $\mathcal{X}$ is abelian), this is enough to show that $\hat{\nu}$ is uniformly continuous.

445F Fourier transforms of functions Let X be a topological group with a left Haar measure μ. For any μ-integrable complex-valued function f, define its **Fourier transform** $\hat{f} : \mathcal{X} \to \mathbb{C}$ by setting $\hat{f}(\chi) = \int f(x)\chi(x)\mu(dx)$ for every character χ of X. (Compare 283A. If f is non-negative, then $\hat{f} = (f\mu)^\wedge$ as defined in 445C, where $f\mu$ is the indefinite-integral measure, as in 444K.) Note that $\hat{f} = \hat{g}$ whenever $f =_{\text{a.e.}} g$, so we can equally well write $\hat{u}(\chi) = \hat{f}(\chi)$ whenever $u = f^\bullet$ in $L^1_{\mathbb{C}}(\mu)$.

445G Proposition Let X be a topological group with a left Haar measure μ. Then for any μ-integrable complex-valued functions f and g, $(f * g)^\wedge = \hat{f} \times \hat{g}$; that is, $(u * v)^\wedge = \hat{u} \times \hat{v}$ for all u, $v \in L^1_{\mathbb{C}}(\mu)$.

proof For any character χ on X,

$$\int \chi(x)(f * g)(x)dx = \iint \chi(xy)f(x)g(y)dxdy$$
$$= \iint \chi(x)\chi(y)f(x)g(y)dxdy = \hat{f}(\chi)\hat{g}(\chi)$$

(using 444Od).

445H Theorem Let X be a topological group with a left Haar measure μ; let $\mathcal{X}$ be its dual group and let Φ be the set of non-zero multiplicative linear functionals on the complex Banach algebra $L^1_{\mathbb{C}} = L^1_{\mathbb{C}}(\mu)$. Then there is a one-to-one correspondence between $\mathcal{X}$ and Φ, defined by the formulae

$$\phi(f^\bullet) = \int f \times \chi \, d\mu = \hat{f}(\chi) \text{ for every } f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu),$$

$$\phi(a\bullet_l u) = \chi(a)\phi(u) \text{ for every } u \in L^1_{\mathbb{C}},\ a \in X,$$

for $\chi \in \mathcal{X}$ and $\phi \in \Phi$.

Remark I follow 443G in writing $a\bullet_l f^\bullet = (a\bullet_l f)^\bullet$, where $(a\bullet_l f)(x) = f(a^{-1}x)$ for $f \in \mathcal{L}^1_{\mathbb{C}}$ and a, $x \in X$ as in 441Ac.

proof (a) If $\chi \in \mathcal{X}$ then we can think of its equivalence class $\chi^\bullet$ as a member of $L^\infty_{\mathbb{C}} = L^\infty_{\mathbb{C}}(\mu)$, so that we can define $\phi_\chi \in (L^1_{\mathbb{C}})^*$ by writing $\phi_\chi(u) = \int \chi^\bullet \times u$ for every $u \in L^1_{\mathbb{C}}$; that is, $\phi_\chi(f^\bullet) = \int f \times \chi = \hat{f}(\chi)$ for every $f \in \mathcal{L}^1_{\mathbb{C}}$. 445G tells us that ϕ_χ is multiplicative. To see that it is non-zero, recall that μ is strictly positive (442Aa) and that χ is continuous. Let G be an open set containing the identity e of X such that $|\chi(x) - 1| \leq \frac{1}{2}$ for every $x \in G$; then $\mathcal{R}\mathrm{e}(\chi(x)) \geq \frac{1}{2}$ for every $x \in G$, so

$$|\textstyle\int_G \chi(x)dx| \geq \mathcal{R}\mathrm{e} \int_G \chi(x)dx = \int_G \mathcal{R}\mathrm{e}(\chi(x))dx \geq \frac{1}{2}\mu G > 0.$$

Accordingly $\phi_\chi(\chi G)^\bullet \neq 0$ and $\phi_\chi \neq 0$. (I hope that no confusion will arise if I continue occasionally to write χE for the characteristic function of a set E, even if the symbol χ is already active in the sentence.)

(b) Now suppose that ϕ is a non-zero multiplicative linear functional on $L^1_{\mathbb{C}}$. Fix on some $g_0 \in \mathcal{L}^1_{\mathbb{C}}$ such that $\phi(g_0^\bullet) = 1$. Let Δ be the left modular function of X. (If you are reading this proof on the assumption that X is abelian, then $a\bullet_r f = a^{-1}\bullet_l f$ and $\Delta \equiv 1$, so the argument below simplifies usefully.)

(i) For any $f \in \mathcal{L}^1_{\mathbb{C}}$ and $a \in X$, $\phi(a^{-1}\bullet_l f)^\bullet = \Delta(a)\phi(a\bullet_r f)^\bullet$. **P** Let $\epsilon > 0$. Then for any sufficiently small open neighbourhood U of the identity, if we set $h = \frac{1}{\mu U}\chi U$, we shall have

$$\|(a\bullet_r f) * h - a\bullet_r f\|_1 \leq \epsilon, \quad \|h * f - f\|_1 \leq \epsilon$$

(444T, with 444P). Now $\Delta(a)(a\bullet_r f)*h = f*(a^{-1}\bullet_l h)$, by 444Of, so

$$\begin{aligned}
|\phi(a^{-1}\bullet_l f)^\bullet - \Delta(a)\phi(a\bullet_r f)^\bullet| &\le |\phi(a^{-1}\bullet_l f)^\bullet - \phi(f*(a^{-1}\bullet_l h))^\bullet| \\
&\qquad + |\Delta(a)\phi((a\bullet_r f)*h)^\bullet - \Delta(a)\phi(a\bullet_r f)^\bullet| \\
&= |\phi(a^{-1}\bullet_l f)^\bullet - \phi(f)^\bullet\phi(a^{-1}\bullet_l h)^\bullet| \\
&\qquad + \Delta(a)|\phi((a\bullet_r f)*h)^\bullet - \phi(a\bullet_r f)^\bullet| \\
&\le |\phi(a^{-1}\bullet_l f)^\bullet - \phi(a^{-1}\bullet_l h)^\bullet\phi(f)^\bullet| \\
&\qquad + \Delta(a)\|(a\bullet_r f)*h - a\bullet_r f\|_1 \\
\text{(because } \|\phi\|\le 1 \text{ in } (L^1_{\mathbb{C}})^*\text{, by 4A6F)}& \\
&\le |\phi(a^{-1}\bullet_l f)^\bullet - \phi((a^{-1}\bullet_l h)*f)^\bullet| + \epsilon\Delta(a) \\
&\le \|a^{-1}\bullet_l f - (a^{-1}\bullet_l h)*f\|_1 + \epsilon\Delta(a) \\
&= \|a^{-1}\bullet_l(f - h*f)\|_1 + \epsilon\Delta(a) \\
\text{(444Of)}& \\
&= \|f - h*f\|_1 + \epsilon\Delta(a) \\
\text{(443Gb)}& \\
&\le (1+\Delta(a))\epsilon.
\end{aligned}$$

As ϵ is arbitrary, we have the result. **Q**

(ii) For any f, $g\in\mathcal{L}^1_{\mathbb{C}}$ and $a\in X$, $\phi(a\bullet_l f)^\bullet\phi(g^\bullet) = \phi(f^\bullet)\phi(a\bullet_l g)^\bullet$. **P**

$$\begin{aligned}
\phi(a\bullet_l f)^\bullet\phi(g^\bullet) &= \phi((a\bullet_l f)*g)^\bullet = \phi(a\bullet_l(f*g))^\bullet = \Delta(a^{-1})\phi(a^{-1}\bullet_r(f*g))^\bullet \\
&= \Delta(a^{-1})\phi(f*(a^{-1}\bullet_r g))^\bullet = \phi(f^\bullet*(a\bullet_l g)^\bullet) = \phi(f^\bullet)\phi(a\bullet_l g)^\bullet,
\end{aligned}$$

using the other two formulae in 444Of for the second and fourth equalities. **Q**

(iii) Translating this into the language of $L^1_{\mathbb{C}}$, we have

$$\phi(a\bullet_l u)\phi(v) = \phi(a\bullet_l f)^\bullet\phi(g^\bullet) = \phi(f^\bullet)\phi(a\bullet_l g)^\bullet = \phi(u)\phi(a\bullet_l v)$$

whenever $a\in X$ and $u = f^\bullet$, $v = g^\bullet\in L^1_{\mathbb{C}}$.

(iv) Let v_0 be $g_0^\bullet$, so that $\phi(v_0)=1$, and set $\chi(a) = \phi(a\bullet_l v_0) = \phi(a\bullet_l g_0)^\bullet$ for every $a\in X$. Then if a, $b\in X$,

$$\begin{aligned}
\chi(ab) &= \phi(ab\bullet_l v_0) = \phi(a\bullet_l(b\bullet_l v_0)) \\
\text{(because } \bullet_l \text{ is an action of } X \text{ on } L^1_{\mathbb{C}}\text{, as noted in 443Gd)}& \\
&= \phi(a\bullet_l(b\bullet_l v_0))\phi(v_0) = \phi(b\bullet_l v_0)\phi(a\bullet_l v_0) \\
\text{(by (iii) above)}& \\
&= \chi(a)\chi(b).
\end{aligned}$$

So $\chi: X\to\mathbb{C}$ is a group homomorphism. Moreover, because $\bullet_l$ is continuous (443Ge), and ϕ also is continuous (indeed, of norm at most 1), χ is continuous. Finally,

$$|\chi(a)| \le \|a\bullet_l v_0\|_1 = \|v_0\|_1$$

for every $a\in X$, by 443Gb; it follows at once that $\{\chi(a)^n : n\in\mathbb{Z}\}$ is bounded, so that $|\chi(a)|=1$, for every $a\in X$. Thus $\chi\in\mathcal{X}$. Moreover, for any $u\in L^1_{\mathbb{C}}$,

$$\phi(a\bullet_l u) = \phi(a\bullet_l u)\phi(v_0) = \phi(u)\phi(a\bullet_l v_0) = \chi(a)\phi(u).$$

(v) Now $\phi = \phi_\chi$. **P** Because $\phi\in(L^1_{\mathbb{C}})^*$, there is some $h\in\mathcal{L}^\infty_{\mathbb{C}}(\mu)$ such that $\phi(f^\bullet) = \int h(x)f(x)dx$ for every $f\in\mathcal{L}^1_{\mathbb{C}}$ (243Gb/243K; recall that by the rules of 441D, μ is suppose to be a quasi-Radon measure, therefore strictly localizable, by 415A). In this case, for any $f\in\mathcal{L}^1_{\mathbb{C}}$,

$$\phi(f^\bullet) = \phi(f * g_0)^\bullet = \int h(x)(f * g_0)(x)dx = \iint h(xy)f(x)g_0(y)dydx$$

(444Od)

$$= \iint h(y)f(x)g_0(x^{-1}y)dydx = \int \phi(x^\bullet{}_l g_0)^\bullet f(x)dx$$

$$= \int \chi(x)f(x)dx = \phi_\chi(f^\bullet). \blacksquare$$

(c) Thus we see that the formulae announced do define a surjection from $\mathcal{X}$ onto Φ. We have still to confirm that it is injective. But if χ, θ are distinct members of $\mathcal{X}$, then $\{x : \chi(x) \neq \theta(x)\}$ is a non-empty open set, so has positive measure, because μ is strictly positive; because μ is semi-finite, they represent different linear functionals on $L^1_{\mathbb{C}}$, and $\phi_\chi \neq \phi_\theta$.

This completes the proof.

445I The topology of the dual group: Proposition Let X be a topological group with a left Haar measure μ, and $\mathcal{X}$ its dual group. For $\chi \in \mathcal{X}$, let $\chi^\bullet$ be its equivalence class in $L^0_{\mathbb{C}} = L^0_{\mathbb{C}}(\mu)$, and $\phi_\chi \in (L^1_{\mathbb{C}})^* = (L^1_{\mathbb{C}}(\mu))^*$ the multiplicative linear functional corresponding to χ, as in 445H. Then the maps $\chi \mapsto \chi^\bullet$ and $\chi \mapsto \phi_\chi$ are homeomorphisms between $\mathcal{X}$ and its images in $L^0_{\mathbb{C}}$ and $(L^1_{\mathbb{C}})^*$, if we give $L^0_{\mathbb{C}}$ the topology of convergence in measure (245A/245M) and $(L^1_{\mathbb{C}})^*$ the weak* topology (2A5Ig).

proof (a) Note that $\chi \mapsto \chi^\bullet$ is injective because μ is strictly positive, so that if χ, θ are distinct members of $\mathcal{X}$ then the non-empty open set $\{x : \chi(x) \neq \theta(x)\}$ has non-zero measure; and that $\chi \mapsto \phi_\chi$ is injective by 445H. So we have one-to-one correspondences between $\mathcal{X}$ and its images in $L^0_{\mathbb{C}}$ and $(L^1_{\mathbb{C}})^*$.

Write $\mathfrak{T}$ for the topology of $\mathcal{X}$ as defined in 445Ab, $\mathfrak{T}_m$ for the topology induced by its identification with its image in $L^0_{\mathbb{C}}$, and $\mathfrak{T}_w$ for the topology induced by its identification with its image in $(L^1_{\mathbb{C}})^*$. Let $\mathcal{E}$ be the family of non-empty totally bounded subsets of X and Σ^f the set of measurable sets of finite measure; for $E \in \mathcal{E}$, $F \in \Sigma^f$ and $f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu)$ set

$$\rho_E(\chi,\theta) = \sup_{x\in E} |\chi(x) - \theta(x)|,$$

$$\rho'_F(\chi,\theta) = \int_F \min(1, |\chi(x) - \theta(x)|)\mu(dx),$$

$$\rho''_f(\chi,\theta) = |\int f(x)\chi(x)\mu(dx) - \int f(x)\theta(x)\mu(dx)|$$

for χ, $\theta \in \mathcal{X}$. Then $\mathfrak{T}$ is generated by the pseudometrics $\{\rho_E : E \in \mathcal{E}\}$, $\mathfrak{T}_m$ is generated by $\{\rho'_F : F \in \Sigma^f\}$ and $\mathfrak{T}_w$ is generated by $\{\rho''_f : f \in \mathcal{L}^1_{\mathbb{C}}\}$.

(b) $\mathfrak{T}_m \subseteq \mathfrak{T}$. **P** Suppose that $F \subseteq X$ is a measurable set of finite measure, and $\epsilon > 0$. There is a non-empty totally bounded open set $U \subseteq X$ (443H). Since $\{xU : x \in X\}$ is an open cover of X and μ is τ-additive, there are $y_0, \dots, y_n \in X$ such that $\mu(F \setminus \bigcup_{j\leq n} y_j U) \leq \frac{1}{3}\epsilon$; set $E = \bigcup_{j\leq n} y_j U$. Then E is totally bounded, and $\rho'_F(\chi,\theta) \leq \epsilon$ whenever $\rho_E(\chi,\theta) \leq \frac{\epsilon}{1+3\mu E}$. As F and ϵ are arbitrary, the identity map $(\mathcal{X},\mathfrak{T}) \to (\mathcal{X},\mathfrak{T}_m)$ is continuous (2A3H), that is, $\mathfrak{T}_m \subseteq \mathfrak{T}$. $\blacksquare$

(c) $\mathfrak{T}_w \subseteq \mathfrak{T}_m$. **P** If $f \in \mathcal{L}^1_{\mathbb{C}}$ and $\epsilon > 0$ let $F \in \Sigma^f$, $M > 0$ be such that $\int (|f| - M\chi F)^+ d\mu \leq \frac{1}{4}\epsilon$. If χ, $\theta \in \mathcal{X}$ and $\rho'_F(\chi,\theta) \leq \frac{\epsilon}{4M}$, then

$$\rho''_f(\chi,\theta) = \big|\int (\chi - \theta) \times f\big| \leq \int |\chi - \theta| \times |f|$$

$$\leq M\int_F |\chi - \theta| + 2\int (|f| - M\chi F)^+$$

$$\leq \tfrac{1}{2}\epsilon + 2M\int_F \min(1, |\chi - \theta|) = \tfrac{1}{2}\epsilon + 2M\rho'_F(\chi,\theta) \leq \epsilon.$$

As f and ϵ are arbitrary, this shows that $\mathfrak{T}_w \subseteq \mathfrak{T}_m$. **Q**

(d) Finally, $\mathfrak{T} \subseteq \mathfrak{T}_w$. **P** Fix $\chi \in \mathcal{X}$, $E \in \mathcal{E}$ and $\epsilon > 0$. Let $u \in L^1_{\mathbb{C}}$ be such that $\phi_\chi(u) = 1$, and represent u as $f^\bullet$ where $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$. Set $U = \{a : a \in X, \|a\bullet_l u - u\|_1 < \frac{1}{4}\epsilon\}$; then U is an open neighbourhood of the identity e of X, because $a \mapsto a\bullet_l u$ is continuous (443Ge). Because E is totally bounded, there are $y_0, \dots, y_n \in X$ such that $E \subseteq \bigcup_{k \le n} y_k U$. Set $f_k = y_k^{-1}\bullet_l f$, so that $f_k^\bullet = y_k\bullet_l u$ for each $k \le n$.

Now suppose that $\theta \in \mathcal{X}$ is such that

$$\rho''_f(\theta,\chi) \le \tfrac{\epsilon}{4}, \quad \rho''_{f_k}(\theta,\chi) \le \tfrac{\epsilon}{4} \text{ for every } k \le n.$$

Take any $x \in E$. Then there is a $k \le n$ such that $x \in y_k U$, so that $y_k^{-1}x \in U$ and

$$\|x\bullet_l u - y_k\bullet_l u\|_1 = \|y_k\bullet_l(y_k^{-1}x\bullet_l u - u)\|_1 = \|y_k^{-1}x\bullet_l u - u\|_1 \le \tfrac{\epsilon}{4}$$

(using 443Gd for the second equality). Now $\phi_\chi(x\bullet_l u) = \chi(x)$ (445H), so

$$\begin{aligned}|\phi_\theta(x\bullet_l u) - \chi(x)| &\le |\phi_\theta(x\bullet_l u - y_k\bullet_l u)| + |\phi_\theta(y_k\bullet_l u) - \phi_\chi(y_k\bullet_l u)| + |\phi_\chi(y_k\bullet_l u - x\bullet_l u)| \\ &\le 2\|x\bullet_l u - y_k\bullet_l u\|_1 + \rho''_{f_k}(\theta,\chi) \le \tfrac{3}{4}\epsilon.\end{aligned}$$

On the other hand,

$$|\theta(x) - \phi_\theta(x\bullet_l u)| = |\theta(x)||1 - \phi_\theta(u)| = \rho''_f(\theta,\chi) \le \tfrac{\epsilon}{4}.$$

So $|\theta(x) - \phi(x)| \le \epsilon$. As x is arbitrary, $\rho_E(\theta,\chi) \le \epsilon$.

As χ, E and ϵ are arbitrary, this shows that $\mathfrak{T} \subseteq \mathfrak{T}_w$. **Q**

445J Corollary For any topological group X carrying Haar measures, its dual group $\mathcal{X}$ is locally compact and Hausdorff.

proof Let Φ be the set of non-zero multiplicative linear functionals on $L^1_{\mathbb{C}} = L^1_{\mathbb{C}}(\mu)$, for some left Haar measure μ on X, and give it its weak* topology. Then $\Phi \cup \{0\} \subseteq (L^1_{\mathbb{C}})^*$ is the set of all multiplicative linear functionals on $L^1_{\mathbb{C}}$, and is closed for the weak* topology, because

$$\{\phi : \phi \in (L^1_{\mathbb{C}})^*, \, \phi(u * v) = \phi(u)\phi(v)\}$$

is closed for all u, $v \in L^1_{\mathbb{C}}$. Because the unit ball of $(L^1_{\mathbb{C}})^*$ is a compact Hausdorff space for the weak* topology (3A5F), so is $\Phi \cup \{0\}$. So Φ itself is an open subset of a compact Hausdorff space and is a locally compact Hausdorff space (3A3Bg). Since the topology on $\mathcal{X}$ can be identified with the weak* topology on Φ (445I), $\mathcal{X}$ also is locally compact and Hausdorff.

445K Proposition Let X be a topological group and μ a left Haar measure on X. Let $\mathcal{X}$ be the dual group of X, and write $C_0 = C_0(\mathcal{X};\mathbb{C})$ for the Banach algebra of continuous functions $h : \mathcal{X} \to \mathbb{C}$ such that $\{\chi : |h(\chi)| \ge \epsilon\}$ is compact for every $\epsilon > 0$.

(a) For any $u \in L^1_{\mathbb{C}} = L^1_{\mathbb{C}}(\mu)$, its Fourier transform $\hat{u}$ belongs to C_0.

(b) The map $u \mapsto \hat{u} : L^1_{\mathbb{C}} \to C_0$ is a multiplicative linear operator, of norm at most 1.

(c) Suppose that X is abelian. For $f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu)$, set $\tilde{f}(x) = \overline{f(x^{-1})}$ whenever this is defined. Then $\tilde{f} \in \mathcal{L}^1_{\mathbb{C}}$ and $\|\tilde{f}\|_1 = \|f\|_1$. For $u \in L^1_{\mathbb{C}}$, we may define $\tilde{u} \in L^1_{\mathbb{C}}$ by setting $\tilde{u} = \tilde{f}^\bullet$ whenever $u = f^\bullet$. Now $\hat{\tilde{u}}$ is the complex conjugate of $\hat{u}$, so $(u * \tilde{u})\hat{} = |\hat{u}|^2$.

(d) Still supposing that X is abelian, $\{\hat{u} : u \in L^1_{\mathbb{C}}\}$ is a norm-dense subalgebra of C_0, and $\|\hat{u}\|_\infty = r(u)$, the spectral radius of u (4A6K), for every $u \in L^1_{\mathbb{C}}$.

proof (a) As in 445H and 445J, let Φ be the set of non-zero multiplicative linear functionals on $L^1_{\mathbb{C}}$, so that $\Phi \cup \{0\}$ is compact for the weak* topology of $(L^1_{\mathbb{C}})^*$, and $\hat{u}(\chi) = \phi_\chi(u)$ for every $\chi \in \mathcal{X}$. By the definition of the weak* topology, $\phi \mapsto \phi(u)$ is continuous; since we can identify the weak* topology on Φ with the dual group topology of $\mathcal{X}$ (445I), $\hat{u}$ is continuous. Also, for any $\epsilon > 0$,

$$\{\chi : \chi \in \mathcal{X}, \, |\hat{u}(\chi)| \ge \epsilon\} \cong \{\phi : \phi \in \Phi, \, |\phi(u)| \ge \epsilon\},$$

which is a closed subset of $\Phi \cup \{0\}$, therefore compact.

(b) It is immediate from the definition of $\hat{}$ that it is a linear operator from $L^1_{\mathbb{C}}$ to C_0; it is multiplicative by 445G, and of norm at most 1 because all the multiplicative linear functionals $u \mapsto \hat{u}(\chi)$ must be of norm at most 1 (4A6F).

(c) Now suppose that X is abelian. If $f \in \mathcal{L}^1_{\mathbb{C}}$, then

$$\int \tilde{f}(x)dx = \overline{\int f(x^{-1})dx} = \overline{\int f(x)dx}$$

by 442Kb, so $\tilde{f} \in \mathcal{L}^1_{\mathbb{C}}$; the same formulae tell us that $\|\tilde{f}\|_1 = \|f\|_1$. If $f =_{\text{a.e.}} g$ then $\tilde{f} =_{\text{a.e.}} \tilde{g}$ (442G, or otherwise), so $\tilde{u}$ is well-defined. If $\chi \in \mathcal{X}$, and $u = f^\bullet$, then

$$\begin{aligned}\hat{\tilde{u}}(\chi) &= \int \tilde{f}(x)\chi(x)dx = \int \overline{f(x^{-1})}\chi(x)dx = \int \overline{f(x)}\chi(x^{-1})dx \\ &= \int \overline{f(x)\chi(x)}dx = \overline{\int f(x)\chi(x)dx} = \overline{\hat{u}(\chi)},\end{aligned}$$

so $\hat{\tilde{u}}$ is the complex conjugate of $\hat{u}$, and

$$(u * \tilde{u})\hat{} = \hat{u} \times \hat{\tilde{u}} = |\hat{u}|^2.$$

(d) To see that $A = \{\hat{u} : u \in L^1_{\mathbb{C}}\}$ is dense in C_0, we may use the Stone-Weierstrass theorem in the form 4A6B. A is a subalgebra of C_0; it separates the points (because the canonical map from $\mathcal{X}$ to $(L^1_{\mathbb{C}})^*$ is injective); if $\chi \in \mathcal{X}$, there is an $h \in A$ such that $h(\chi) \neq 0$ (because elements of $\mathcal{X}$ act on $L^1_{\mathbb{C}}$ as non-zero functionals); and the complex conjugate of any function in A belongs to A, by (c) above.

Accordingly A is dense in C_0, by 4A6B.

The calculation of $\|\hat{u}\|_\infty$ is an immediate consequence of the characterization of $r(u)$ as $\sup\{|\phi(u)| : \phi \in \Phi\}$ (4A6K) and the identification of Φ with $\mathcal{X}$.

Remark This is the first point in this section where we really need to know whether or not our group is abelian.

445L Positive definite functions Let X be a group.

(a) A function $h : X \to \mathbb{C}$ is called **positive definite** if

$$\sum_{j,k=0}^n \zeta_j \bar{\zeta}_k h(x_k^{-1}x_j) \geq 0$$

for all $\zeta_0, \dots, \zeta_n \in \mathbb{C}$ and $x_0, \dots, x_n \in X$.

(b) Suppose that $h : X \to \mathbb{C}$ is positive definite. Then, writing e for the identity of X,
(i) $|h(x)| \leq h(e)$ for every $x \in X$;
(ii) $h(x^{-1}) = \overline{h(x)}$ for every $x \in X$.

P If $\zeta \in \mathbb{C}$ and $x \in X$, take $n = 1$, $x_0 = e$, $x_1 = x$, $\zeta_0 = 1$ and $\zeta_1 = \zeta$ in the definition in (a) above, and observe that

$$(1 + |\zeta|^2)h(e) + \zeta h(x) + \bar{\zeta}h(x^{-1}) = h(e^{-1}e) + \zeta h(e^{-1}x) + \bar{\zeta}h(x^{-1}e) + \zeta\bar{\zeta}h(x^{-1}x) \geq 0.$$

Taking $\zeta = 0$, $x = e$ we get $h(e) \geq 0$. Taking $\zeta = 1$ we see that $h(x) + h(x^{-1})$ is real, and taking $\zeta = i$, we see that $h(x) - h(x^{-1})$ is purely imaginary; that is, $h(x^{-1}) = \overline{h(x)}$, for any x. Taking ζ such that $|\zeta| = 1$, $\zeta h(x) = -|h(x)|$ we get $2h(e) - 2|h(x)| \geq 0$, that is, $|h(x)| \leq h(e)$ for every $x \in X$. **Q**

(c) If $h : X \to \mathbb{C}$ is positive definite and $\chi : X \to S^1$ is a homomorphism, then $h \times \chi$ is positive definite.
P If $\zeta_0, \dots, \zeta_n \in \mathbb{C}$ and $x_0, \dots, x_n \in X$ then

$$\sum_{j,k=0}^n \zeta_j\bar{\zeta}_k(h \times \chi)(x_k^{-1}x_j) = \sum_{j,k=0}^n \zeta_j\chi(x_j)\overline{\zeta_k\chi(x_k)}h(x_k^{-1}x_j) \geq 0. \ \textbf{Q}$$

(d) If X is an abelian topological group and μ a Haar measure on X, then for any $f \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ the convolution $f * \tilde{f} : X \to \mathbb{C}$ is continuous and positive definite, where $\tilde{f}(x) = \overline{f(x^{-1})}$ whenever this is defined.

P As in 444Rc, $f * \tilde{f}$ is defined everywhere on X and is continuous. (The definition of $\tilde{}$ has shifted since §444, but the argument there applies unchanged to the present situation.) Now, if $x_0, \dots, x_n \in X$ and $\zeta_0, \dots, \zeta_n \in \mathbb{C}$,

$$\begin{aligned}\sum_{j,k=0}^{n} \zeta_j \bar{\zeta}_k (f * \tilde{f})(x_k^{-1} x_j) &= \sum_{j,k=0}^{n} \zeta_j \bar{\zeta}_k \int f(y) \tilde{f}(y^{-1} x_k^{-1} x_j) dy \\ &= \int \sum_{j,k=0}^{n} \zeta_j \bar{\zeta}_k f(y) \overline{f(x_j^{-1} x_k y)} dy \\ &= \int \sum_{j,k=0}^{n} \zeta_j \bar{\zeta}_k f(x_j y) \overline{f(x_j^{-1} x_k x_j y)} dy \\ &= \int \sum_{j,k=0}^{n} \zeta_j \bar{\zeta}_k f(x_j y) \overline{f(x_k y)} dy \\ &= \int |\sum_{j=0}^{n} \zeta_j f(x_j y)|^2 dy \geq 0.\end{aligned}$$

So $f * \tilde{f}$ is positive definite. **Q**

445M Proposition Let X be a topological group and ν a quasi-Radon measure on X. If $h : X \to \mathbb{C}$ is a continuous positive definite function, then $\iint h(y^{-1}x) f(x) \overline{f(y)} \nu(dx) \nu(dy) \geq 0$ for every ν-integrable function f.

proof (a) Extend f, if necessary, to the whole of X; since the hypothesis implies that $\operatorname{dom} f$ is conegligible, this does not affect the integrals. Let λ be the product quasi-Radon measure on $X \times X$; because h is continuous (by hypothesis) and bounded (by 445L(b-i)), the function $(x, y) \mapsto h(y^{-1}x) f(x) \overline{f(y)}$ is λ-integrable, and (because $\{x : f(x) \neq 0\}$ can be covered by a sequence of sets of finite measure)

$$I = \iint h(y^{-1}x) f(x) \overline{f(y)} \nu(dx) \nu(dy) = \int h(y^{-1}x) f(x) \overline{f(y)} \lambda(d(x, y))$$

(417H).

(b) Let $\epsilon > 0$. Set $\gamma = \sup_{x \in X} |h(x)| = h(e)$ (445L(b-i)). Let $F \subseteq X$ be a non-empty measurable set of finite measure for ν such that $\gamma \int_{(X \times X) \setminus (F \times F)} |f(x) \overline{f(y)}| \lambda(d(x, y)) \leq \frac{1}{2}\epsilon$ and f is bounded on F; say $|f(x)| \leq M$ for every $x \in F$. Let $\delta > 0$ be such that

$$\delta(M^2 + 2M\gamma)(\nu F)^2 + 2M^2\gamma\delta \leq \tfrac{1}{2}\epsilon.$$

Let $\mathcal{G}$ be the set

$$\{G \times H : G, H \subseteq X \text{ are open}, |h(y^{-1}x) - h(y_1^{-1}x_1)| \leq \delta \text{ whenever } x, x_1 \in G, y, y_1 \in H\}.$$

Because h is continuous, $\mathcal{G}$ is a cover of $X \times X$. Because λ is τ-additive, there is a finite set $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\lambda((F \times F) \setminus \bigcup \mathcal{G}_0) \leq \delta$; we may suppose that $\mathcal{G}_0$ is non-empty. Set $W = (F \times F) \cap \bigcup \mathcal{G}_0$. Enumerate $\mathcal{G}_0$ as $\langle G_i \times H_i \rangle_{i \leq n}$.

Let $\mathcal{F}$ be a finite partition of F into measurable sets such that $|f(x) - f(x')| \leq \delta$ whenever x, x' belong to the same member of $\mathcal{F}$. Let $\mathcal{E}$ be the partition of F generated by $\mathcal{F} \cup \{F \cap G_j : j \leq n\} \cup \{F \cap H_j : j \leq n\}$. Enumerate $\mathcal{E}$ as $\langle E_k \rangle_{k \leq m}$; for each $k \leq m$ choose $x_k \in E_k$. Set $J = \{(j, k) : j \leq m, k \leq m, E_j \times E_k \subseteq W\}$; then $W = \bigcup_{(j,k) \in J} E_j \times E_k$.

(c) If $(j, k) \in J$, $x \in E_j$ and $y \in E_k$ then

$$|h(y^{-1}x) f(x) \overline{f(y)} - h(x_k^{-1}x_j) f(x_j) \overline{f(x_k)}| \leq \delta(M^2 + 2M\gamma),$$

because there must be some $r \leq n$ such that $E_j \times E_k \subseteq G_r \times H_r$, so that $|h(y^{-1}x) - h(x_k^{-1}x_j)| \leq \delta$, while there are members of $\mathcal{F}$ including E_j and E_k, so that $|f(x) - f(x_j)| \leq \delta$ and $|\overline{f(y)} - \overline{f(x_k)}| \leq \delta$; at the same time,

$$|f(x)\overline{f(y)}| \le M^2, \quad |h(x_k^{-1}x_j)||\overline{f(y)}| \le M\gamma, \quad |h(x_k^{-1}x_j)||f(x_j)| \le M\gamma$$

because x, y, x_j and x_k all belong to F.

(d) Set $\zeta_j = f(x_j)\nu E_j$ for $j \le m$, so that $\bar\zeta_j = \overline{f(x_j)}\nu E_j$. Now consider

$$\begin{aligned}
&\Big|\int_W h(y^{-1}x)f(x)\overline{f(y)}\lambda(d(x,y)) - \sum_{(j,k)\in J}\zeta_j\bar\zeta_k h(x_k^{-1}x_j)\Big| \\
&\quad = \Big|\sum_{(j,k)\in J}\int_{E_j\times E_k} h(y^{-1}x)f(x)\overline{f(y)} - h(x_k^{-1}x_j)f(x_j)\overline{f(x_k)}\lambda(d(x,y))\Big| \\
&\quad \le \sum_{(j,k)\in J}\int_{E_j\times E_k} |h(y^{-1}x)f(x)\overline{f(y)} - h(x_k^{-1}x_j)f(x_j)\overline{f(x_k)}|\lambda(d(x,y)) \\
&\quad \le \sum_{(j,k)\in J}\delta(M^2+2M\gamma)\nu E_j\nu E_k \\
&\quad \le \delta(M^2+2M\gamma)\lambda W \le \delta(M^2+2M\gamma)(\nu F)^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Big|\int_{(X\times X)\setminus W} h(y^{-1}x)f(x)\overline{f(y)}\lambda(d(x,y))\Big| &\le \gamma\int_{(X\times X)\setminus(F\times F)} |f(x)\overline{f(y)}|\lambda(d(x,y)) \\
&\quad + \gamma\int_{(F\times F)\setminus W} |f(x)\overline{f(y)}|\lambda(d(x,y)) \\
&\le \tfrac12\epsilon + \gamma\delta M^2,
\end{aligned}$$

and

$$\begin{aligned}
\Big|\sum_{j\le m,k\le m,(j,k)\notin J} h(x_k^{-1}x_j)f(x_j)\tilde f(x_k)\Big| &\le \gamma M^2 \sum_{j\le m,k\le m,(j,k)\notin J}\nu E_j\nu E_k \\
&= \gamma M^2\lambda((F\times F)\setminus W) \le \gamma M^2\delta.
\end{aligned}$$

Putting these together,

$$|I - \sum_{j,k=0}^m \zeta_j\bar\zeta_k h(x_k^{-1}x_j)| \le \delta(M^2+2M\gamma)(\nu F)^2 + \tfrac12\epsilon + \gamma\delta M^2 + \gamma M^2\delta \le \epsilon.$$

But $\sum_{j,k=0}^m \zeta_j\bar\zeta_k h(x_k^{-1}x_j) \ge 0$, because h is positive definite. As ϵ is arbitrary, $I \ge 0$, as required.

445N Bochner's Theorem (Herglotz 11, Bochner 33, Weil 40) Let X be an abelian topological group with a Haar measure μ, and $\mathcal{X}$ its dual group. Then for any continuous positive definite function $h : X \to \mathbb{C}$ there is a unique totally finite Radon measure ν on $\mathcal{X}$ such that

$$\textstyle\int h(x)f(x)\mu(dx) = \int \hat f(\chi)\nu(d\chi) \text{ for every } f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu),$$

$$h(x) = \textstyle\int \chi(x)\nu(d\chi) \text{ for every } x \in X.$$

proof (a) If $h(e) = 0$, where e is the identity in X, then $h = 0$, by 445L(b-i), and the result is trivial. Otherwise, since multiplying h by a positive scalar leaves h positive definite and does not affect the result, we may suppose that $h(e) = 1$. For f, $g \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu)$ set

$$(f|g) = \iint f(x)\overline{g(y)}h(y^{-1}x)\mu(dx)\mu(dy) = \iint f(x)\overline{g(y^{-1})}h(yx)\mu(dx)\mu(dy)$$

(by 442Kb, since X is unimodular). Then, by 445M, $(f|f) \ge 0$ for every $f \in \mathcal{L}^1_{\mathbb{C}}$. Also $(f_1+f_2|g) = (f_1|g)+(f_2|g)$, $(\zeta f|g) = \zeta(f|g)$ and $(g|f) = \overline{(f|g)}$ for all f, g, f_1, $f_2 \in \mathcal{L}^1_{\mathbb{C}}$ and $\zeta \in \mathbb{C}$. **P** Only the last is anything but trivial, and for this we have

$$(g|f) = \iint g(x)\overline{f(y)}h(y^{-1}x)\mu(dx)\mu(dy)$$
$$= \iint g(x)\overline{f(y)}h(y^{-1}x)\mu(dy)\mu(dx)$$

(by 417Ha, because $(x,y) \mapsto g(x)\overline{f(y)}h(y^{-1}x)$ is integrable for the product measure and zero off the square of a countable union of sets of finite measure)

$$= \iint g(x)\overline{f(y)h(x^{-1}y)}\mu(dy)\mu(dx)$$

(using 445L(b-ii))

$$= \overline{\iint f(y)\overline{g(x)}h(x^{-1}y)\mu(dy)\mu(dx)}$$
$$= \overline{(f|g)}. \blacksquare$$

(b) If f, $g \in \mathcal{L}^1_{\mathbb{C}}$, $|(f|g)|^2 \leq (f|f)(g|g)$. ▶ (Really this is just Cauchy's inequality.) For any α, $\beta \in \mathbb{C}$,

$$|\alpha|^2(f|f) + 2\,\mathcal{R}\mathrm{e}(\alpha\bar{\beta}(f|g)) + |\beta|^2(g|g) = (\alpha f + \beta g|\alpha f + \beta g) \geq 0.$$

If $(f|f) = 0$ we have $2\,\mathcal{R}\mathrm{e}(\alpha(f|g)) + (g|g) \geq 0$ for every $\alpha \in \mathbb{C}$ so in this case $(f|g) = 0$; similarly $(f|g) = 0$ if $(g|g) = 0$; otherwise we can find non-zero α, β such that $|\alpha|^2 = (g|g)$, $|\beta|^2 = (f|f)$ and $\alpha\bar{\beta}(f|g) = -|\alpha\beta(f|g)|$, in which case the inequality simplifies to $|(f|g)| \leq |\alpha\beta|$ and $|(f|g)|^2 \leq (f|f)(g|g)$, as required. ■

(c) Now consider the functional $\psi \in (L^1_{\mathbb{C}})^* = (L^1_{\mathbb{C}}(\mu))^*$ corresponding to h, so that $\psi(f^\bullet) = \int h \times f d\mu$ for every $f \in \mathcal{L}^1_{\mathbb{C}}$. Then $|\psi(f^\bullet)|^2 \leq (f|f)$ for every $f \in \mathcal{L}^1_{\mathbb{C}}$. ▶ Let $\epsilon > 0$. Then there is an open neighbourhood U of e such that $U = U^{-1}$ and

$$|h(y^{-1}x) - h(e)| \leq \epsilon \text{ whenever } x,\, y \in U,$$

$$\|a\bullet_l f - f\|_1 \leq \epsilon \text{ for every } a \in U$$

where $(a\bullet_l f)(x) = f(a^{-1}x)$ whenever this is defined, as usual (443Ge). Shrinking U if need be, we may suppose that $\mu U < \infty$, and of course $\mu U > 0$. Set $g = \frac{1}{\mu U}\chi U \in \mathcal{L}^1_{\mathbb{C}}$. Then

$$|(f|g) - \psi(f^\bullet)| = \tfrac{1}{\mu U}\big|\int_U \int_X f(x)h(y^{-1}x)\mu(dx)\mu(dy) - \int_U \int_X f(x)h(x)\mu(dx)\mu(dy)\big|$$
$$= \tfrac{1}{\mu U}\big|\int_U \int_X f(yx)h(x)\mu(dx)\mu(dy) - \int_U \int_X f(x)h(x)\mu(dx)\mu(dy)\big|$$
$$= \tfrac{1}{\mu U}\big|\int_U \int_X ((y^{-1}\bullet_l f)(x) - f(x))h(x)\mu(dx)\mu(dy)\big|$$
$$\leq \tfrac{1}{\mu U}\int_U \int_X |(y^{-1}\bullet_l f)(x) - f(x)||h(x)|\mu(dx)\mu(dy)$$
$$\leq \tfrac{1}{\mu U}\int_U \|y^{-1}\bullet_l f - f\|_1 \mu(dy) \leq \epsilon.$$

Also

$$|(g|g) - 1| = \big|\tfrac{1}{\mu U^2}\int_U \int_U (h(y^{-1}x) - 1)\mu(dx)\mu(dy)\big|$$
$$\leq \tfrac{1}{\mu U^2}\int_U \int_U |h(y^{-1}x) - 1|\mu(dx)\mu(dy) \leq \epsilon.$$

So

$$\max(0, |\psi(f^\bullet)| - \epsilon)^2 \leq |(f|g)|^2 \leq (f|f)(g|g) \leq (1+\epsilon)(f|f).$$

Letting $\epsilon \downarrow 0$ we have the result. ■

(d) If we look at $(f|f)$, however, and apply 444Od, we see that

$$\begin{aligned}(f|f) &= \iint f(x)\overline{f(y)}h(y^{-1}x)\mu(dx)\mu(dy)\\ &= \iint f(x)\overline{f(y^{-1})}h(yx)\mu(dx)\mu(dy) = \int h(x)(f*\tilde{f})(x)\mu(dx),\end{aligned}$$

where $\tilde{f}(x) = \overline{f(x^{-1})}$ whenever this is defined; that is,

$$(f|f) = \psi(f*\tilde{f})^\bullet.$$

(Note that $\tilde{f} \in \mathcal{L}^1_{\mathbb{C}}$ because X is unimodular, as in part (c) of the proof of 445K.) So (c) tells us that

$$|\psi(f^\bullet)|^2 \leq \psi(f*\tilde{f})^\bullet$$

for every $f \in \mathcal{L}^1_{\mathbb{C}}$, that is, $|\psi(u)|^2 \leq \psi(u*\tilde{u})$ for every $u \in L^1_{\mathbb{C}}$, defining $\tilde{u}$ as in 445Kc.

(e) In fact

$$|\psi(u)| \leq \|\hat{u}\|_\infty$$

for every $u \in L^1_{\mathbb{C}}$. **P** Set $u_0 = u$ and $u_{k+1} = u_k * \tilde{u}_k$ for every $k \in \mathbb{N}$. We need to know that $u_k = \tilde{u}_k$ for $k \geq 1$. To see this, represent u_{k-1} as $f^\bullet$ where $f \in \mathcal{L}^1_{\mathbb{C}}$, so that $u_k = (f*\tilde{f})^\bullet$. Now

$$\begin{aligned}(f*\tilde{f})^\sim(x) &= \overline{(f*\tilde{f})(x^{-1})} = \overline{\int f(x^{-1}y)\tilde{f}(y^{-1})dy}\\ &= \int \overline{f(x^{-1}y)}f(y)dy = \int \tilde{f}(y^{-1}x)f(y)dy = (f*\tilde{f})(x)\end{aligned}$$

for every x, so $(f*\tilde{f})^\sim = f*\tilde{f}$ and $\tilde{u}_k = u_k$. Accordingly $u_{k+1} = u_k * u_k$ for $k \geq 1$ and we have $u_k = (u_1)^{2^{k-1}}$ for every $k \geq 1$.

At the same time, we have $|\psi(u_k)|^2 \leq \psi(u_{k+1})$ for every k, by (d), so that, for $k \geq 1$,

$$|\psi(u)|^{2^k} \leq \psi(u_k) \leq \|u_k\|_1 = \|u_1^{2^{k-1}}\|_1,$$

$$|\psi(u)| \leq \|u_1^{2^{k-1}}\|_1^{1/2^k}.$$

Letting $k \to \infty$, $|\psi(u)| \leq \sqrt{r(u_1)}$, where $r(u_1)$ is the spectral radius of u_1.

At this point, recall that $r(u_1) = \|\hat{u}_1\|_\infty$ (445Kd), while $|\hat{u}|^2 = \hat{u}_1$ (445Kc), so $r(u_1) = \|\hat{u}\|^2_\infty$ and $|\psi(u)| \leq \|\hat{u}\|_\infty$. **Q**

(f) Now consider $\hat{}$ as a linear operator from $L^1_{\mathbb{C}}$ to $C_0 = C_0(\mathcal{X};\mathbb{C})$, as in 445K. If $\hat{u} = 0$ then $\psi(u) = 0$, by (e), so setting $A = \{\hat{u} : u \in L^1_{\mathbb{C}}\}$ we have a linear functional $\psi_0 : A \to \mathbb{C}$ defined by saying that $\psi_0(\hat{u}) = \psi(u)$ for every $u \in L^1_{\mathbb{C}}$. By (e), $\|\psi_0\| \leq 1$. Because A is norm-dense in C_0 (445Kd), ψ_0 has a unique extension to a bounded linear operator ψ_1, still of norm at most 1, from C_0 to $\mathbb{C}$ (2A4I).

(g) Suppose that $0 \leq q \leq \mathbf{1}$ in C_0, writing $\mathbf{1}$ for the constant function with value 1, and set $\alpha = \psi_1(q)$. Then for any $\zeta \in \mathbb{C}$, $\gamma \geq 0$ we have $|\zeta - \gamma\alpha| \leq \max(|\zeta|, |\gamma|, |\zeta-\gamma|)$. **P** Let $\epsilon > 0$. Set $V = \{x : |1-h(x)| < \epsilon\}$; then V is an open neighbourhood of e; set $f = \frac{1}{\mu V}\chi V$ and $u = f^\bullet$, so that

$$\|\hat{u}\|_\infty = r(u) \leq \|u\|_1 = 1,$$

$$|1 - \psi(u)| = |\tfrac{1}{\mu V}\textstyle\int_V (1-h(x))dx| \leq \epsilon.$$

Set $v = u * \tilde{u}$; then

$$\psi_1(\hat{v}) = \psi(v) \geq |\psi(u)|^2 \geq (1-\epsilon)^2,$$

using part (d) for the central inequality. But $\hat{v} = |\hat{u}|^2$, so that $0 \leq \hat{v} \leq \mathbf{1}$ and $\psi_1(\hat{v}) \leq 1$.

Now consider $\|\zeta\hat{v} - \gamma q\|_\infty$. If $\chi \in \mathcal{X}$, then $\zeta\hat{v}(\chi)$ and $\gamma q(\chi)$ both lie in the triangle with vertices 0, ζ and γ, because $0 \leq \hat{v} \leq \mathbf{1}$, $0 \leq q \leq \mathbf{1}$. So

$$|\zeta\hat{v}(\chi) - \gamma q(\chi)| \leq \max(|\gamma|, |\zeta|, |\gamma - \zeta|).$$

As χ is arbitrary,

$$\|\zeta\hat{v} - \gamma q\|_\infty \leq \max(|\gamma|, |\zeta|, |\gamma - \zeta|).$$

Accordingly

$$\begin{aligned}|\zeta - \gamma\alpha| &\leq |\zeta - \zeta\psi_1(\hat{v})| + |\psi_1(\zeta\hat{v} - \gamma q)| \\ &\leq |\zeta|(1 - (1-\epsilon)^2) + \|\zeta\hat{v} - \gamma q\|_\infty \leq 2\epsilon|\zeta| + \max(|\zeta|, |\gamma|, |\zeta - \gamma|).\end{aligned}$$

As ϵ is arbitrary, we have the result. **Q**

Taking $\zeta = \gamma = 1$ we see that $|1-\alpha| \leq 1$, so that $\mathcal{R}\mathrm{e}\,\alpha \geq 0$. Taking $\zeta = \pm i$, we see that $|i \pm \gamma\alpha| \leq \sqrt{1+\gamma^2}$ for every $\gamma \geq 0$, so that $\mathcal{I}\mathrm{m}\,\alpha = 0$. Thus $\psi_1(q) \geq 0$; and this is true whenever $0 \leq q \leq \mathbf{1}$ in C_0.

(h) It follows at once that $\psi_1(q) \geq 0$ whenever $q \geq 0$ in C_0. Applying the Riesz Representation Theorem, in the form 436K, to the restriction of ψ_1 to $C_0(\mathcal{X}; \mathbb{R})$, we see that there is a unique totally finite Radon measure ν on $\mathcal{X}$ such that $\psi_1(q) = \int q\,d\nu$ for every real-valued $q \in C_0$; of course it follows that $\psi_1(q) = \int q\,d\nu$ for every $q \in C_0$. Unwrapping the definition of ψ_1, we see that

$$\int h(x)f(x)\mu(dx) = \psi(f^\bullet) = \psi_1(\hat{f}) = \int \hat{f}(\chi)\nu(d\chi)$$

for every $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, and that this uniquely defines ν.

(i) For the second formula, argue as follows. Given $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, consider the function $(x, \chi) \mapsto f(x)\chi(x) : X \times \mathcal{X} \to \mathbb{C}$. Because $(\chi, x) \mapsto \chi(x)$ is continuous (445Ea), this is measurable for the product quasi-Radon measure $\mu \times \nu$ on $X \times \mathcal{X}$. It is integrable because $\nu\mathcal{X} < \infty$ and $|\chi(x)| = 1$ for every χ, x; moreover, it is zero off the set $\{x : f(x) \neq 0\} \times \mathcal{X}$, which is a countable union of products of sets of finite measure. Note also that because $\chi \mapsto \chi(x)$ is continuous and bounded for every $x \in X$, $h'(x) = \int \chi(x)\nu(d\chi)$ is defined, and $|h'(x)| \leq \nu X$, for every $x \in X$. What is more, h' is continuous. **P** Let $\mathfrak{X}$ be the dual group of $\mathcal{X}$, and for $x \in X$ let $\hat{x}$ be the corresponding member of $\mathfrak{X}$. Then, in the language of 445C, applied to the topological group $\mathcal{X}$,

$$h'(x) = \int \hat{x}\,d\nu = \hat{\nu}(\hat{x})$$

for every $x \in X$. But $\hat{\nu} : \mathfrak{X} \to \mathbb{C}$ is continuous, by 445Ec, and $x \mapsto \hat{x} : X \to \mathfrak{X}$ is continuous, by 445Eb; so h' also is continuous. **Q**

We may therefore apply Fubini's theorem (417H) to see that

$$\begin{aligned}\int f(x)h'(x)\mu(dx) &= \iint f(x)\chi(x)\nu(d\chi)\mu(dx) = \iint f(x)\chi(x)\mu(dx)\nu(d\chi) \\ &= \int \hat{f}(\chi)\nu(d\chi) = \int f(x)h(x)\mu(dx).\end{aligned}$$

Since this is true for every $f \in \mathcal{L}^1_{\mathbb{C}}$, $h' =_{\text{a.e.}} h$; since both are continuous, $h' = h$, as required.

445O Proposition Let X be a Hausdorff abelian topological group carrying Haar measures. Then the map $x \mapsto \hat{x}$ from X to its bidual group $\mathfrak{X}$ is a homeomorphism between X and its image in $\mathfrak{X}$. In particular, the dual group $\mathcal{X}$ of X separates the points of X.

proof We already know that $\hat{}$ is continuous (445Eb). Now let U be any neighbourhood of the identity e of X. Let $V \subseteq U$ be an open neighbourhood of e such that $VV^{-1} \subseteq U$ and $\mu V < \infty$. Then $f = \chi V \in \mathcal{L}^2_{\mathbb{C}}(\mu)$, so $f * \tilde{f}$ is positive definite (445Ld) and there is a totally finite Radon measure ν on $\mathcal{X}$ such that $(f * \tilde{f})(x) = \int \chi(x)\nu(d\chi)$ for every $x \in X$ (445N). Note that, writing $\mathfrak{e}$ for the identity of $\mathfrak{X}$ and $\hat{\nu} : \mathfrak{X} \to \mathbb{C}$ for the Fourier transform of ν,

$$\hat{\nu}(\mathfrak{e}) = \int \mathfrak{e}(\chi)\nu(d\chi) = \int \chi(e)\nu(d\chi) = (f * \tilde{f})(e) = \int f(y)\tilde{f}(y^{-1})\mu(dy) = \int |f(y)|^2\mu(dy) \neq 0.$$

Now $\hat{\nu}$ is continuous (445Ec), so $W = \{\mathfrak{x} : \hat{\nu}(\mathfrak{x}) \neq 0\}$ is a neighbourhood of $\mathfrak{e}$. If $x \in X$ and $\hat{x} \in W$, then

$$(f * \tilde{f})(x) = \int \chi(x)\nu(d\chi) = \int \hat{x}(\chi)\nu(d\chi) = \hat{\nu}(\hat{x}) \neq 0,$$

so there is some $y \in X$ such that $f(y)\tilde{f}(y^{-1}x) \neq 0$, that is, $f(y) \neq 0$ and $f(x^{-1}y) \neq 0$, that is, y and $x^{-1}y$ both belong to V; in which case $x \in VV^{-1} \subseteq U$.

Thus $U \supseteq \{x : \hat{x} \in W\}$. This means that, writing $\mathfrak{S}$ for $\{\{x : \hat{x} \in H\} : H \subseteq \mathfrak{X} \text{ is open}\}$, every neighbourhood of e for the original topology $\mathfrak{T}$ of X is a neighbourhood of e for $\mathfrak{S}$. But (it is easy to check) $(X, \mathfrak{S})$ is a topological group because $\mathfrak{X}$ is a topological group and $\hat{}$ is a homomorphism. So $\mathfrak{T} \subseteq \mathfrak{S}$ (4A5Fb). As we know already that $\mathfrak{S} \subseteq \mathfrak{T}$, the two topologies are equal.

It follows at once that if $\mathfrak{T}$ is Hausdorff, then (because $\mathfrak{S}$ is Hausdorff) the map $\hat{}$ is an injection and is a homeomorphism between X and its image in $\mathfrak{X}$.

445P The Inversion Theorem Let X be an abelian topological group and μ a Haar measure on X. Then there is a unique Haar measure λ on the dual group $\mathcal{X}$ of X such that whenever $f : X \to \mathbb{C}$ is continuous, μ-integrable and positive definite, then $\hat{f} : \mathcal{X} \to \mathbb{C}$ is λ-integrable and

$$f(x) = \int \hat{f}(\chi)\overline{\chi(x)}\lambda(d\chi)$$

for every $x \in X$.

proof (a) Write P for the set of μ-integrable positive definite continuous functions $h : X \to \mathbb{C}$. For $h \in P$, let ν_h be the corresponding totally finite Radon measure on $\mathcal{X}$ defined in 445N, so that

$$\int f(x)h(x)\mu(dx) = \int \hat{f}(\chi)\nu_h(d\chi)$$

for every $f \in \mathcal{L}^1_{\mathbb{C}} = \mathcal{L}^1_{\mathbb{C}}(\mu)$.

(b) The basis of the argument is the following fact. If $f \in \mathcal{L}^1_{\mathbb{C}}$ and h_1, $h_2 \in P$, then

$$\int \hat{f} \times \hat{h}_2 d\nu_{h_1} = \int \hat{f} \times \hat{h}_1 d\nu_{h_2}.$$

P
$$\int \hat{f} \times \hat{h}_1 d\nu_{h_2} = \int (f * \bar{h}_1)\hat{} \, d\nu_{h_2}$$

(445G)
$$= \int h_2(x)(f * \bar{h}_1)(x)\mu(dx) = \int \bar{h}_2(x^{-1})(f * \bar{h}_1)(x)\mu(dx)$$

(by 445Lb)
$$= ((f * \bar{h}_1) * \bar{h}_2)(e) = ((f * \bar{h}_2) * \bar{h}_1)(e)$$

(because $*$ is associative and commutative, by 444Oe and 444Og)
$$= \int \hat{f} \times \hat{h}_2 d\nu_{h_1}. \quad \textbf{Q}$$

Now because $\hat{h}_1$ and $\hat{h}_2$ are both bounded (by $\int |h_1| d\mu$ and $\int |h_2| d\mu$ respectively), and ν_{h_1} and ν_{h_2} are both totally finite measures, and $\{\hat{f} : f \in \mathcal{L}^1_{\mathbb{C}}(\mu)\}$ is $\| \|_\infty$-dense in $C_0 = C_0(\mathcal{X}; \mathbb{C})$ (445Kd), we must have

$$\int p \times \hat{h}_2 d\nu_{h_1} = \int p \times \hat{h}_1 d\nu_{h_2}$$

for every $p \in C_0$.

(c) Let $\mathcal{K}$ be the family of compact subsets of $\mathcal{X}$. For $K \in \mathcal{K}$ set

$$P_K = \{h : h \in P, \hat{h}(\chi) > 0 \text{ for every } \chi \in K\}.$$

Then P_K is non-empty. **P** Set

$$U = \{x : x \in X, |1 - \chi(x)| \leq \tfrac{1}{2} \text{ for every } \chi \in K\}.$$

Then U is a neighbourhood of the identity e of X, by 445Eb. Let V be an open neighbourhood of e, of finite measure, such that $VV^{-1} \subseteq U$, set $g = \frac{1}{\mu V}\chi V$, and try $h = g * \tilde{g}$. Then h is real-valued and non-negative

(because g and $\tilde{g}$ are), continuous and positive definite (445Ld), zero outside U (because $VV^{-1} \subseteq U$, as in the proof of 445O), and

$$\int h\,d\mu = \int g\,d\mu \cdot \int \tilde{g}\,d\mu = 1$$

(444Qb). Next,

$$\hat{\hat{h}} = \hat{h} = |\hat{g}|^2$$

(445Kc) is non-negative, and if $\chi \in K$ then

$$|1 - \hat{h}(\chi)| = |\int h(x) - h(x)\chi(x)\mu(dx)| \leq \int h(x)|1 - \chi(x)|\mu(dx) \leq \tfrac{1}{2}$$

because $|1 - \chi(x)| \leq \frac{1}{2}$ if $x \in U$ and $h(x) = 0$ if $x \in X \setminus U$. So

$$\hat{\hat{h}}(\chi) = \hat{h}(\chi) \geq \tfrac{1}{2}$$

for every $\chi \in K$, and $h \in P_K$. **Q**

(d) Because $\mathcal{K}$ is upwards-directed, $\{P_K : K \in \mathcal{K}\}$ is downwards-directed and generates a filter $\mathcal{F}$ on P. Let $C_k = C_k(\mathcal{X}; \mathbb{C})$ be the space of continuous complex-valued functions on $\mathcal{X}$ with compact support. If $q \in C_k$, then

$$\phi(q) = \lim_{h \to \mathcal{F}} \int \frac{q}{\hat{\hat{h}}}\,d\nu_h$$

is defined in $\mathbb{C}$, where in the division $q/\hat{\hat{h}}$ we interpret $0/0$ as 0 if necessary. **P** Setting $K = \overline{\{\chi : q(\chi) \neq 0\}}$, we see in fact that for any h_1, $h_2 \in P_K$ we may define a function $p \in C_k$ by setting

$$\begin{aligned} p(\chi) &= \frac{q(\chi)}{\hat{\hat{h}}_1(\chi)\hat{\hat{h}}_2(\chi)} \text{ if } \chi \in K, \\ &= 0 \text{ if } q(\chi) = 0, \end{aligned}$$

so that

$$\int \frac{q}{\hat{\hat{h}}_1}\,d\nu_{h_1} = \int p \times \hat{\hat{h}}_2\,d\nu_{h_1} = \int p \times \hat{\hat{h}}_1\,d\nu_{h_2}$$

(by (b) above)

$$= \int \frac{q}{\hat{\hat{h}}_2}\,d\nu_{h_2}.$$

So this common value must be $\phi(q)$. **Q**

If q, $q' \in C_k$ and $\alpha \in \mathbb{C}$, then

$$\int \frac{q+q'}{\hat{\hat{h}}}\,d\nu_h = \int \frac{q}{\hat{\hat{h}}}\,d\nu_h + \int \frac{q'}{\hat{\hat{h}}}\,d\nu_h,$$

$$\int \frac{\alpha q}{\hat{\hat{h}}}\,d\nu_h = \alpha \int \frac{q}{\hat{\hat{h}}}\,d\nu_h$$

whenever $h \in P_K$, where $K = \overline{\{\chi : |q(\chi)| + |q'(\chi)| > 0\}}$; so $\phi(q+q') = \phi(q) + \phi(q')$ and $\phi(\alpha q) = \alpha\phi(q)$. Moreover, if $q \geq 0$, then $q/\hat{\hat{h}} \geq 0$ for every $h \in P_K$, so $\phi(q) \geq 0$.

(e) By the Riesz Representation Theorem (in the form 436J) there is a Radon measure λ on $\mathcal{X}$ such that $\int q\,d\lambda = \phi(q)$ for any continuous function q of compact support. (As in part (h) of the proof of 445M, the shift from real-valued q to complex-valued q is elementary.)

(f) Now λ is translation-invariant. **P** Take $\theta \in \mathcal{X}$ and $q \in C_k$. Set $K = \overline{\{\chi : q(\chi) \neq 0\}}$ and $L = \theta^{-1}K$, and take any $h \in P_K$. Set $h_1(x) = h(x)\theta^{-1}(x)$ for $x \in X$. Then h_1 is positive definite (445Lc); of course it is continuous and μ-integrable; and for any $\chi \in L$,

$$\hat{\overline{h}}_1(\chi) = \int \overline{h(x)}\theta(x)\chi(x)\mu(dx) = \hat{\overline{h}}(\theta\chi) > 0.$$

So $h_1 \in P_L$.

To relate ν_{h_1} to ν_h, observe that if $f \in \mathfrak{L}^1_{\mathbb{C}}$ then

$$\hat{f}(\theta\chi) = \int f(x)\theta(x)\chi(x)\mu(dx) = (f \times \theta)\hat{}(\chi),$$

so

$$\int \hat{f}(\theta\chi)\nu_{h_1}(d\chi) = \int (f \times \theta)(x)h_1(x)\mu(dx) = \int f(x)h(x)\mu(dx) = \int \hat{f}(\chi)\nu_h(d\chi).$$

So we see that the equation

$$\int p(\theta\chi)\nu_{h_1}(d\chi) = \int p(\chi)\nu_h(d\chi)$$

is valid whenever p is of the form $\hat{f}$, for some $f \in \mathfrak{L}^1_{\mathbb{C}}$, and therefore for every $p \in C_0$.

Set $q_1(\chi) = q(\theta\chi)$ for every $\chi \in \mathcal{X}$, so that $q_1 \in C_k$ and $L = \overline{\{\chi : q_1(\chi) \neq 0\}}$. Accordingly

$$\begin{aligned}\phi(q_1) &= \int \frac{q_1(\chi)}{\hat{\overline{h}}_1(\chi)}\nu_{h_1}(d\chi) = \int \frac{q(\theta\chi)}{\hat{\overline{h}}(\theta\chi)}\nu_{h_1}(d\chi)\\ &= \int \frac{q(\chi)}{\hat{\overline{h}}(\chi)}\nu_h(d\chi) = \phi(q).\end{aligned}$$

So

$$\int q(\theta\chi)\lambda(d\chi) = \int q(\chi)\lambda(d\chi).$$

As q and θ are arbitrary, λ is translation-invariant (441L). **Q**

(g) Thus λ is a Haar measure on $\mathcal{X}$. I have still to confirm that

$$f(x) = \int \hat{f}(\chi)\overline{\chi(x)}\lambda(d\chi)$$

whenever f is continuous, positive definite and μ-integrable, and $x \in X$. But recall the formula from (b) above. If $q \in C_k$, $K = \overline{\{\chi : q(\chi) \neq 0\}}$ and $h \in P_K$, then we must have

$$\int q \times \hat{\overline{f}}\, d\lambda = \int \frac{q \times \hat{\overline{f}}}{\hat{\overline{h}}} d\nu_h = \int \frac{q \times \hat{\overline{h}}}{\hat{\overline{h}}} d\nu_f = \int q\, d\nu_f.$$

Thus we have $\int q \times \hat{\overline{f}}\, d\lambda = \int q\, d\nu_f$ for every $q \in C_k$.

In particular, $\int q \times \hat{\overline{f}}\, d\lambda \geq 0$ whenever $q \geq 0$; since $\hat{\overline{f}}$ is continuous (445Ka), and λ, being a Radon measure, is strictly positive, this shows that $\hat{\overline{f}} \geq 0$. Also

$$\int \hat{\overline{f}}\, d\lambda = \sup\{\int q \times \hat{\overline{f}}\, d\lambda : q \in C_k,\ 0 \leq q \leq \mathbf{1}\} = \nu_f(\mathcal{X}) < \infty,$$

so we have an indefinite-integral measure $\hat{\overline{f}}\lambda$; since this is a Radon measure (416S), and acts on C_k in the same way as ν_f, it is actually equal to ν_f (436J). In particular, for any $x \in X$,

$$f(x) = \int \chi(x)\nu_f(d\chi) = \int \chi(x)\hat{\overline{f}}(\chi)\lambda(d\chi)$$

by the second formula in 445N, and 235M. But

$$\hat{\overline{f}}(\chi) = \int \overline{f(x)}\chi(x)\mu(dx) = \int f(x^{-1})\chi(x)\mu(dx)$$

(by 445Lb)

$$= \int f(x)\chi(x^{-1})\mu(dx)$$

(because X is abelian, therefore unimodular)

$$= \hat{f}(\chi^{-1})$$

for every $\chi \in \mathcal{X}$. So

$$f(x) = \int \chi(x)\hat{f}(\chi^{-1})\lambda(d\chi) = \int \chi^{-1}(x)\hat{f}(\chi)\lambda(d\chi)$$

(because $\mathcal{X}$ is abelian)

$$= \int \hat{f}(\chi)\overline{\chi(x)}\lambda(d\chi).$$

(h) We should check that λ is unique. Since all Haar measures on $\mathcal{X}$ are multiples of each other (442B), it is enough to observe that the formula $f(e) = \int \hat{f}\, d\lambda$ can be valid for at most one such measure, if we can find any $f \in P$ such that $f(e) \neq 0$; and such functions are available in abundance, by the construction in part (c) of the proof.

445Q Remark We can extract the following useful fact from part (g) of the proof above. If $h : X \to \mathbb{C}$ is μ-integrable, continuous and positive definite, then $\hat{\bar{h}}$ is non-negative and λ-integrable, and the Radon measure ν_h of 445N is just the indefinite-integral measure $\hat{\bar{h}}\lambda$.

Note also that λ is actually a Radon measure; of course it has to be, because $\mathcal{X}$ is locally compact and Hausdorff (445J).

445R The Plancherel Theorem Let X be an abelian topological group with a Haar measure μ, and $\mathcal{X}$ its dual group. Let λ be the Haar measure on $\mathcal{X}$ corresponding to μ (445P). Then there is a normed space isomorphism $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ defined by setting $T(f^\bullet) = \hat{f}^\bullet$ whenever $f \in \mathcal{L}^1_{\mathbb{C}}(\mu) \cap \mathcal{L}^2_{\mathbb{C}}(\mu)$.

proof (a) Since $\hat{f} = \hat{g}$ whenever $f =_{\text{a.e.}} g$, the formula certainly defines an operator T from $L^1_{\mathbb{C}}(\mu) \cap L^2_{\mathbb{C}}(\mu)$ to $L^0_{\mathbb{C}}(\lambda)$, and of course it is linear.

If $f \in \mathcal{L}^1_{\mathbb{C}}(\mu) \cap \mathcal{L}^2_{\mathbb{C}}(\mu)$, set $\tilde{f}(x) = \overline{f(x^{-1})}$ whenever this is defined; then $h = f * \tilde{f}$ is integrable, continuous and positive definite (445Ld), and $\hat{h} = |\hat{f}|^2$ (445Kc). Now

$$\int |\hat{f}|^2\, d\lambda = \int \hat{h}\, d\lambda = h(e)$$

(445P)

$$= \int f(x)\tilde{f}(x^{-1})\mu(dx) = \int |f(x)|^2 \mu(dx).$$

Thus $\|Tu\|_2 = \|u\|_2$ whenever $u \in L^1_{\mathbb{C}}(\mu) \cap L^2_{\mathbb{C}}(\mu)$; since $L^1_{\mathbb{C}}(\mu) \cap L^2_{\mathbb{C}}(\mu)$ is $\| \ \|_2$-dense in $L^2_{\mathbb{C}}(\mu)$ (244Ha/244Ob, or otherwise), we have a unique isometry $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ agreeing with the given formula on $L^1_{\mathbb{C}}(\mu) \cap L^2_{\mathbb{C}}(\mu)$ (2A4I).

(b) The meat of the theorem is of course the proof that T is surjective. **P?** Suppose, if possible, that $W = T[L^2_{\mathbb{C}}(\mu)]$ is not equal to $L^2_{\mathbb{C}}(\lambda)$. Because T is linear, W is a linear subspace of $L^2_{\mathbb{C}}(\lambda)$; because T is an isometry, W is isometric to $L^2_{\mathbb{C}}(\mu)$, and in particular is complete, therefore closed in $L^2_{\mathbb{C}}(\lambda)$ (3A4Fd). There is therefore a non-zero continuous linear functional on $L^2_{\mathbb{C}}(\lambda)$ which is zero on W (3A5Ad), and this is of the form $u \mapsto \int u \times v$ for some $v \in L^2_{\mathbb{C}}(\lambda)$ (244J/244Oc). What this means is that there is a $g \in \mathcal{L}^2_{\mathbb{C}}(\lambda)$ such that $g^\bullet \neq 0$ in $L^2_{\mathbb{C}}(\lambda)$ but $\int \hat{f} \times g\, d\lambda = 0$ for every $f \in \mathcal{L}^1_{\mathbb{C}}(\mu) \cap \mathcal{L}^2_{\mathbb{C}}(\mu)$.

Suppose that $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$ and that h is any μ-integrable continuous positive definite function on X. Then h is bounded (445Lb), so $|h|^2$ is also μ-integrable, and $f * \bar{h} \in \mathcal{L}^1_{\mathbb{C}}(\mu) \cap \mathcal{L}^2_{\mathbb{C}}(\mu)$ (444Ra). Accordingly

$$\int g \times \hat{f} \times \hat{\bar{h}}\, d\lambda = \int g \times (f * \bar{h})\hat{}\, d\lambda = 0.$$

Thus $\int g \times \hat{f}\, d\nu_h = 0$, where $\nu_h = \hat{\bar{h}}\lambda$ is the Radon measure on $\mathcal{X}$ corresponding to h constructed in 445N (see 445Q). And this is true for every $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$. But as $\{\hat{f} : f \in \mathcal{L}^1_{\mathbb{C}}(\mu)\}$ is $\| \|_\infty$-dense in $C_0(\mathcal{X};\mathbb{C})$ (445Kd), and g is ν_h-integrable (because $\hat{\bar{h}} \in \mathcal{L}^2_{\mathbb{C}}(\lambda)$, by (a), so $\int |g \times \hat{\bar{h}}| d\lambda < \infty$), g must be zero ν_h-a.e., that is, $g \times \hat{\bar{h}} = 0$ λ-a.e. Now recall (from part (c) of the proof of 445P, for instance) that for every compact set $K \subseteq \mathcal{X}$ there is a μ-integrable continuous positive definite h such that $\hat{\bar{h}}(\chi) \neq 0$ for every $\chi \in K$, so $g = 0$ a.e. on K. Since λ is tight (that is, inner regular with respect to the compact sets), $g = 0$ a.e. (412Jc), which is impossible. **XQ**

Thus T is surjective and we have the result.

445S While we do not have a direct definition of $\hat{f}$ for $f \in \mathcal{L}^2_{\mathbb{C}} \setminus \mathcal{L}^1_{\mathbb{C}}$, the map $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ does correspond to the map $f \mapsto \hat{f}$ in many ways. In particular, we have the following useful properties.

Proposition Let X be an abelian topological group with a Haar measure μ, $\mathcal{X}$ its dual group, λ the associated Haar measure on $\mathcal{X}$ and $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ the standard isometry described in 445R. Suppose that $f_0, f_1 \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ and $g_0, g_1 \in \mathcal{L}^2_{\mathbb{C}}(\nu)$ are such that $Tf_0^\bullet = g_0^\bullet$ and $Tf_1^\bullet = g_1^\bullet$, and take any $\theta \in \mathcal{X}$. Then

(a) setting $f_2 = \bar{f}_0$, $g_2(\chi) = \overline{g_0(\chi^{-1})}$ whenever this is defined, $Tf_2^\bullet = g_2^\bullet$;

(b) setting $f_3 = f_1 \times \theta$, $g_3(\chi) = g_1(\theta\chi)$ whenever this is defined, $Tf_3^\bullet = g_3^\bullet$;

(c) setting $f_4 = f_0 \times f_1 \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, $\hat{f}_4(\theta) = (g_0 * g_1)(\theta)$.

proof (a) We have isometries $R_1 : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\mu)$ and $R_2 : L^2_{\mathbb{C}}(\lambda) \to L^2_{\mathbb{C}}(\lambda)$ defined by setting $R_1 f^\bullet = (\bar{f})^\bullet$ for every $f \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ and $R_2 g^\bullet = (\tilde{g})^\bullet$ for every $g \in \mathcal{L}^2_{\mathbb{C}}(\lambda)$, where $\tilde{g}(\chi) = \overline{g(\chi^{-1})}$ whenever this is defined. Now if $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, then

$$\hat{\bar{f}}(\chi) = \int \overline{f(x)}\chi(x)\mu(dx) = \overline{\int f(x)\chi^{-1}(x)\mu(dx)} = \tilde{\hat{f}}(\chi)$$

for every $\chi \in \mathcal{X}$. So $TR_1 u = R_2 Tu$ for every $u \in L^1_{\mathbb{C}}(\mu) \cap L^2_{\mathbb{C}}(\mu)$; as $L^1_{\mathbb{C}} \cap L^2_{\mathbb{C}}$ is dense in $L^2_{\mathbb{C}}$, $TR_1 = R_2 T$, which is what we need to know.

(b) This time, set $R_1 f^\bullet = (f \times \theta)^\bullet$ for every $f \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ and $R_2 g^\bullet = (\theta^{-1} \bullet_l g)^\bullet$ for $g \in \mathcal{L}^2_{\mathbb{C}}(\lambda)$, where $(\theta \bullet_l g)(\chi) = g(\theta^{-1}\chi)$ whenever this is defined. Once again, $R_1 : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\mu)$ and $R_2 : L^2_{\mathbb{C}}(\lambda) \to L^2_{\mathbb{C}}(\lambda)$ are isometries. If $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$, then

$$(f \times \theta)\hat{}(\chi) = \int f(x)\theta(x)\chi(x)\mu(dx) = \hat{f}(\theta\chi)$$

for every χ, so $TR_1 f^\bullet = R_2 Tf^\bullet$; once again, this is enough to prove that $TR_1 = R_2 T$, as required.

(c) We have

$$(g_0 * g_1)(\theta) = \int g_0(\chi^{-1}) g_1(\theta\chi)\lambda(d\chi) = \int \overline{g_2(\chi)} g_3(\chi)\lambda(d\chi) = (g_3^\bullet | g_2^\bullet)$$

(where $(\ |\ )$ is the standard inner product of $L^2_{\mathbb{C}}(\lambda)$)

$$= (Tf_3^\bullet | Tf_2^\bullet)$$

(using (a) and (b))

$$= (f_3^\bullet | f_2^\bullet)$$

(because linear isometries of Hilbert space preserve inner products, see 4A4Jb)

$$= \int f_3(x)\overline{f_2(x)}\mu(dx) = \int f_1(x)\theta(x)f_0(x)\mu(dx) = (f_0 \times f_1)\hat{}(\theta).$$

445T Corollary Let X be an abelian topological group with a Haar measure μ, and λ the corresponding Radon measure on the dual group $\mathcal{X}$ of X (445P). Then for any non-empty open set $H \subseteq \mathcal{X}$, there is an $f \in \mathcal{L}^1_{\mathbb{C}}(\mu)$ such that $\hat{f} \neq 0$ and $\hat{f}(\chi) = 0$ for $\chi \in \mathcal{X} \setminus H$.

proof Let V_1 and V_2 be non-empty open sets of finite measure such that $V_1V_2 \subseteq H$, and let g_1, g_2 be their characteristic functions. Then there are f_1, $f_2 \in \mathcal{L}^2_{\mathbb{C}}(\mu)$ such that $Tf_j^\bullet = g_j^\bullet$ for both j, where $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ is the isometry of 445R. In this case, by 445Sc, $(f_1 \times f_2)^\wedge = g_1 * g_2$. But it is easy to check that $g_1 * g_2$ is non-zero and zero outside H.

445U The Duality Theorem (Pontryagin 34, Kampen 35) Let X be a locally compact Hausdorff abelian topological group. Then the canonical map $x \mapsto \hat{x}$ from X to its bidual $\mathfrak{X}$ (445E) is an isomorphism between X and $\mathfrak{X}$ as topological groups.

proof By 445O, $\hat{}$ is a homeomorphism between X and its image $\hat{X} \subseteq \mathfrak{X}$. Accordingly $\hat{X}$ is itself, with its subspace topology, a locally compact topological group, and is closed in $\mathfrak{X}$ (4A5Mc).

? Suppose, if possible, that $\hat{X} \neq \mathfrak{X}$. Let μ be a Haar measure on X (441E) and λ the associated Radon measure on the dual $\mathcal{X}$ of X (445P). By 445T, there is a $g \in \mathcal{L}^1_{\mathbb{C}}(\lambda)$ such that $\hat{g}$ is zero on $\hat{X}$ but not zero everywhere, so that g is not zero a.e. We may suppose that g is defined everywhere on $\mathcal{X}$. But in this case we have

$$0 = \hat{g}(\hat{x}) = \int g(\chi)\hat{x}(\chi)\lambda(d\chi) = \int g(\chi)\chi(x)\lambda(d\chi)$$

for every $x \in X$.

By 418J, $g : \mathcal{X} \to \mathbb{C}$ is almost continuous. Let $K \subseteq \{\chi : \chi \in \mathcal{X}, g(\chi) \neq 0\}$ be a compact set such that $\int_K |g|d\lambda \geq \frac{3}{4}\int_{\mathcal{X}} |g|d\lambda$ and $g\restriction K$ is continuous. Set $q(\chi) = \overline{g(\chi)}/|g(\chi)|$ for $\chi \in K$. Now consider the linear span A of $\{\hat{x} : x \in X\}$ as a linear space of complex-valued functions on $\mathcal{X}$. Since $\widehat{xy} = \hat{x} \times \hat{y}$ for all x, $y \in X$, A is a subalgebra of $C_b = C_b(\mathcal{X}; \mathbb{C})$; since $\widehat{x^{-1}} = \bar{\hat{x}}$ for every $x \in X$, $\bar{h} \in A$ for every $h \in A$; the constant function $\hat{e}$ belongs to A; and A separates the points of $\mathcal{X}$. By the Stone-Weierstrass theorem, in the form 281G, there is an $h \in A$ such that $|h(\chi) - q(\chi)| \leq \frac{1}{2}$ for every $\chi \in K$ and $|h(\chi)| \leq 1$ for every $\chi \in \mathcal{X}$. Of course $\int g \times h\, d\lambda = 0$ for every $h \in A$ because $\int g \times \hat{x}\, d\lambda = 0$ for every $x \in X$.

Now, however,

$$\begin{aligned}\int_K |g|d\lambda = \int_K g \times q\, d\lambda &\leq \big|\int_K g \times h\, d\lambda\big| + \int_K |g| \times |h - q|d\lambda \\ &\leq \big|\int_{\mathcal{X}\setminus K} g \times h\, d\lambda\big| + \tfrac{1}{2}\int_K |g|d\lambda \leq \int_{\mathcal{X}\setminus K} |g|d\lambda + \tfrac{1}{2}\int_K |g|d\lambda < \int_K |g|d\lambda,\end{aligned}$$

which is impossible. **X**

Thus $\hat{X} = \mathfrak{X}$ and the proof is complete.

445X Basic exercises (a) Consider the additive group $\mathbb{Q}$ with its usual topology. Show that its dual group can be identified with the additive group $\mathbb{R}$ with its usual topology.

(b) Let X be any topological group, and $\mathcal{X}$ its dual group. Show that if ν is any totally finite Radon measure on X, then $\hat{\nu} : \mathcal{X} \to \mathbb{C}$ is continuous.

(c) Let X be a topological group carrying Haar measures and $\mathcal{X}$ its dual group. For a totally finite quasi-Radon measure ν on $\mathcal{X}$ set $\hat{\nu}(x) = \int \chi(x)\nu(d\chi)$ for every $x \in X$. (i) Show that $\hat{\nu}$ is continuous. (ii) Show that for any totally finite quasi-Radon measure μ on X, $\int \hat{\mu}\, d\nu = \int \hat{\nu}\, d\mu$.

>(d) Let X be a group and h_1, $h_2 : X \to \mathbb{C}$ positive definite functions. Show that $h_1 + h_2$, αh_1 and $\bar{h}_1$ are also positive definite for any $\alpha \geq 0$.

(e) (i) Let X be a group, Y a subgroup of X and $h : Y \to \mathbb{C}$ a positive definite function. Set $h_1(x) = h(x)$ if $x \in Y$, $h_1(x) = 0$ if $x \in X \setminus Y$. Show that h_1 is positive definite. (ii) Let X and Y be groups, $\phi : X \to Y$ a group homomorphism and $h : Y \to \mathbb{C}$ a positive definite function. Show that $h\phi : X \to \mathbb{C}$ is positive definite.

(f) Let X be a topological group and $\mathcal{X}$ its dual group. (i) Let ν be any totally finite topological measure on $\mathcal{X}$ and set $h(x) = \int \chi(x)\nu(d\chi)$ for $x \in X$. Show that $h : X \to \mathbb{C}$ is positive definite. (ii) Let ν be any totally finite topological measure on X. Show that $\chi \mapsto \int \chi\, d\nu : \mathcal{X} \to \mathbb{C}$ is positive definite.

>(g) Let X be a topological group with a left Haar measure, and $h : X \to \mathbb{C}$ a bounded continuous function. Show that h is positive definite iff $\int h(x^{-1}y)f(y)\overline{f(x)}dxdy \geq 0$ for every integrable function f.

>(h) Let $\phi : \mathbb{R}^r \to \mathbb{C}$ be a function. Show that it is the characteristic function of a probability distribution on $\mathbb{R}^r$ iff it is continuous and positive definite and $\phi(0) = 1$.

>(i) Let X be a compact Hausdorff abelian topological group, and μ the Haar probability measure on X. Show that the corresponding Haar measure on the dual group $\mathcal{X}$ of X is just counting measure on $\mathcal{X}$.

>(j) Let X be an abelian group with its discrete topology, and μ counting measure on X. Let $\mathcal{X}$ be the dual group and λ the corresponding Haar measure on $\mathcal{X}$. Show that $\lambda\mathcal{X} = 1$.

(k) Let X be the topological group $\mathbb{R}$, and $\mu = \frac{1}{\sqrt{2\pi}}\mu_L$, where μ_L is Lebesgue measure. (i) Show that if we identify the dual group $\mathcal{X}$ of X with $\mathbb{R}$, writing $\chi(x) = e^{i\chi x}$ for x, $\chi \in \mathbb{R}$, then the Haar measure on $\mathcal{X}$ corresponding to the Haar measure μ on X is μ itself. (ii) Show that if we change the action of $\mathbb{R}$ on itself by setting $\chi(x) = e^{2\pi i\chi x}$, then the Haar measure on $\mathcal{X}$ corresponding to μ_L is μ_L.

(l) Let $X_0, \ldots, X_n$ be abelian topological groups with Haar measures $\mu_0, \ldots, \mu_n$, and let $X = X_0 \times \ldots \times X_n$ be the product group with its Haar measure $\mu = \mu_0 \times \ldots \times \mu_n$. For each $k \leq n$ let $\mathcal{X}_k$ be the dual group of X_k and λ_k the Haar measure on $\mathcal{X}_k$ corresponding to μ_k. Show that if we identify $\mathcal{X} = \mathcal{X}_0 \times \ldots \times \mathcal{X}_n$ with the dual group of X, then the Haar measure on $\mathcal{X}$ corresponding to μ is just the product measure $\lambda_0 \times \ldots \times \lambda_n$.

>(m) Let X be a compact Hausdorff abelian topological group, with dual group $\mathcal{X}$, and μ the Haar probability measure on X. (i) Show that $\int \chi(x)\mu(dx) = 0$ for every $\chi \in \mathcal{X}$ except the identity. (ii) Show that $\{\chi^\bullet : \chi \in \mathcal{X}\}$ is an orthonormal basis of the Hilbert space $L^2_{\mathbb{C}}(\mu)$. (*Hint*: $(u|\chi^\bullet) = (Tu|\hat{\chi}^\bullet)$ where T is the operator of 445R.)

(n) Let X be an abelian topological group with a Haar measure μ, λ the associated Haar measure on the dual group $\mathcal{X}$ and $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ the standard isometry. Suppose that u, $v \in L^2_{\mathbb{C}}(\mu)$. Suppose that f_0, $f_1 \in \mathcal{L}^2_{\mathbb{C}}(\mu)$, g_0, $g_1 \in \mathcal{L}^2_{\mathbb{C}}(\nu)$ are such that $Tf_0^\bullet = g_0^\bullet$ and $Tf_1^\bullet = g_1^\bullet$. Show that $(f_0 * f_1)(x) = \int g_0(\chi)g_1(\chi)\overline{\chi(x)}\lambda(d\chi)$ for any $x \in X$.

(o) Let X be a locally compact Hausdorff abelian topological group and $\mathcal{X}$ its dual group. Show that a function $h : \mathcal{X} \to \mathbb{C}$ is the Fourier-Stieltjes transform of a totally finite Radon measure on X iff it is continuous and positive definite.

(p) (i) Show that we can define a binary operation $+_{2\text{adic}}$ on $X = \{0,1\}^{\mathbb{N}}$ by setting $x +_{2\text{adic}} y = z$ whenever x, y, $z \in X$ and $\sum_{i=0}^{k} 2^i(x(i) + y(i) - z(i))$ is divisible by 2^{k+1} for every k. (ii) Show that if we give X its usual topology then $(X, +_{2\text{adic}})$ is an abelian topological group. (iii) Show that the usual measure on X is the Haar probability measure for this group operation. (iv) Show that $G = \{\zeta : \zeta \in \mathbb{C}, \exists\, n \in \mathbb{N}, \zeta^{2^n} = 1\}$ is a subgroup of $\mathbb{C}$. (v) Show that the dual of $(X, +_{2\text{adic}})$ is $\{\chi_\zeta : \zeta \in G\}$ where $\chi_\zeta(x) = \prod_{i=0}^{\infty} \zeta^{2^i x(i)}$ for $\zeta \in G$ and $x \in X$. (vi) Show that the functions f, $g : X \to X$ described in 388E are of the form $x \mapsto x \pm_{2\text{adic}} x_0$ for a certain $x_0 \in X$.

445Y Further exercises (a) Let X be any Hausdorff topological group. Let $\widehat{X}$ be its completion under its bilateral uniformity (4A5N). Show that the dual groups of X and $\widehat{X}$ can be identified as groups. Show that they can be identified as topological groups if *either* X is metrizable *or* X has a totally bounded neighbourhood of the identity.

(b) Let $\langle X_j \rangle_{j \in I}$ be a countable family of topological groups, with product X; let $\mathcal{X}_j$ be the dual group of each X_j, and $\mathcal{X}$ the dual group of X. Show that the topology of $\mathcal{X}$ is generated by sets of the form $\mathcal{X} \cap \prod_{j \in I} H_j$ where $H_j \subseteq \mathcal{X}_j$ is open for each j.

(c) Let X be a real linear topological space, with addition as its group operation. Show that its dual group is just the set of functionals $x \mapsto e^{if(x)}$ where $f : X \to \mathbb{R}$ is a continuous linear functional. Hence show that there are abelian groups with trivial duals.

(d) Let X be the group of rotations of $\mathbb{R}^3$, that is, the group of orthogonal real 3×3 matrices with determinant 1, and give X its usual topology, corresponding to its embedding as a subspace of $\mathbb{R}^9$. Show that the only character on X is the constant function 1. (*Hint*: (i) show that two rotations through the same angle are conjugate in X; (ii) show that if $0 < \theta \leq \frac{\pi}{2}$ then the product of two rotations through the angle θ about orthogonal axes is not a rotation through an angle 2θ.)

(e) Let X be a finite abelian group, endowed with its discrete topology. Show that its dual is isomorphic to X. (*Hint*: X is a product of cyclic groups.)

(f) Show that if I is any uncountable set, then there is a quasi-Radon probability measure ν on the topological group $\mathbb{R}^I$ such that $\hat{\nu}$ is not continuous. (*Hint*: take ν to be a power of a suitably widely spread probability distribution on $\mathbb{R}$.)

(g) Let X be an abelian topological group with a Haar measure μ, and λ the associated Haar measure on the dual group $\mathcal{X}$ of X. Let $T : L^2_{\mathbb{C}}(\mu) \to L^2_{\mathbb{C}}(\lambda)$ be the standard isomorphism. Suppose that $f \in \mathcal{L}^2_{\mathbb{C}}(\mu)$, $g \in \mathcal{L}^1_{\mathbb{C}}(\lambda) \cap \mathcal{L}^2_{\mathbb{C}}(\lambda)$ are such that $Tf^\bullet = g^\bullet$. Show that $f(x) = \int g(\chi)\overline{\chi(x)}\lambda(d\chi)$ for almost every $x \in X$. (*Hint*: look first at locally compact Hausdorff X.)

(h) Let X be a locally compact Hausdorff abelian topological group with dual group $\mathcal{X}$, $\langle \nu_n \rangle_{n\in\mathbb{N}}$ a sequence of Radon probability measures on X and ν another Radon probability measure on X. Show that the following are equiveridical: (i) $\lim_{n\to\infty} \int f d\nu_n = \int f d\nu$ for every bounded continuous real-valued function on X; (ii) $\lim_{n\to\infty} \hat{\nu}_n(\chi) = \hat{\nu}(\chi)$ for every $\chi \in \mathcal{X}$. (*Hint*: compare 285L. For the critical step, showing that $\{\nu_n : n \in \mathbb{N}\}$ is uniformly tight, use the formulae in 445N to show that there is an integrable $f : \mathcal{X} \to \mathbb{C}$ such that $0 \leq \check{f} \leq \mathbf{1}$ and $\int \check{f}(x)\nu(dx) \geq 1 - \frac{1}{2}\epsilon$.)

(i) Let X be a topological group carrying Haar measures and $\mathcal{X}$ its dual group. Let M_τ be the complex Banach space of signed totally finite τ-additive Borel measures on $\mathcal{X}$ (put the ideas of 437F and 437Yb together). Show that X separates the points of M_τ in the sense that if $\nu \in M_\tau$ is non-zero, there is an $x \in X$ such that $\int \chi(x)\nu(d\chi) \neq 0$, if the integral is appropriately interpreted. (*Hint*: use the method in the proof of 445U.)

(j) Let X be an abelian topological group carrying Haar measures. Let M_τ be the complex Banach space of signed totally finite τ-additive Borel measures on X. Show that the dual $\mathcal{X}$ of X separates the points of M_τ in the sense that if $\nu \in M_\tau$ is non-zero, there is a $\chi \in \mathcal{X}$ such that $\int \chi(x)\nu(dx) \neq 0$. (*Hint*: use 443L to reduce to the case in which X is locally compact and Hausdorff; now use 445U and 445Yi.)

(k) Let X be an abelian topological group and μ a Haar measure on X. Show that the spectral radius of any non-zero element of $L^1_{\mathbb{C}}(\mu)$ is non-zero. (*Hint*: 445Yj, 445Kd.)

(l) Show that for any integer $p \geq 2$ there is an operation $+_{p\text{adic}}$ on $\{0, \ldots, p-1\}^{\mathbb{N}}$ with properties similar to those of the operation $+_{2\text{adic}}$ of 445Xp.

445 Notes and comments I repeat that this section is intended to be a more or less direct attack on the duality theorem. At every point the clause 'let X be a locally compact Hausdorff abelian topological group' is present in spirit. The actual statement of each theorem involves some subset of these properties, purely in accordance with the principle of omission of irrelevant hypotheses, not because I expect to employ the results in any more general setting.

In 445Ab I describe a topology on the dual group in a context so abstract that we have rather a lot of choice. For groups carrying Haar measures, the alternative descriptions of the topology on the dual (445I) make it plain that this must be the first topology to study. By 445E it is already becoming fairly convincing. But it is not clear that there is any such pre-eminent topology in the general case.

Fourier-Stieltjes transforms hardly enter into the arguments of this section; I mention them mostly because they form the obvious generalization of the ideas in §285. But I note that the principal theorem of §285 (that sequential convergence of characteristic functions corresponds to sequential convergence of distributions, 285L) generalizes directly to the context here (445Yh).

I have tried to lay this treatise out in such a way that we periodically return to themes from past chapters at a higher level of sophistication. There seem to be four really important differences between this section and the previous treatment in Chapter 28. (i) The first is the obvious one; we are dealing with general locally compact Hausdorff abelian groups, rather than with $\mathbb{R}$ and S^1 and $\mathbb{Z}$. Of course this puts much heavier demands on our technique, and, to begin with, leaves our imaginations unfocused. (ii) The second concerns differentiation, or rather its absence; since we no longer have any differential structure on our groups, a substantial part of the theory evaporates, and we are forced to employ new tactics in the rest. (iii) The third concerns the normalization of the measure on the dual group. As soon as we know that $\mathcal{X}$ is a locally compact group (445J) we know that it carries Haar measures. The problem is to describe the particular one we need in appropriate terms. In the case of the dual pairs $(\mathbb{R},\mathbb{R})$ or $(S^1,\mathbb{Z})$, we have measures already presented (counting measure on $\mathbb{Z}$, Lebesgue measure on $]-\pi,\pi]$ and $\mathbb{R}$). (They are not in fact dual in the sense we need here, at least not if we use the simplest formulae for the duality, and have to be corrected in each case by a factor of 2π. See 445Xk.) But since we do have dual pairs already to hand, we can simultaneously develop theories of Fourier transforms and inverse Fourier transforms (for the pair $(S^1,\mathbb{Z})$ the inverse Fourier transform is just summation of trigonometric series), and the problem is to successfully match operations which have independent existences. (iv) The final change concerns an interesting feature of $\mathbb{Z}$ and $\mathbb{R}$. Repeatedly, in §§282-283, the formulae invoked symmetric limits $\lim_{n\to\infty}\sum_{k=-n}^{n}$ or $\lim_{a\to\infty}\int_{-a}^{a}$ to approach some conditionally convergent sum or integral. Elsewhere one sometimes deals with singularities by examining 'Cauchy principal values'; if $\int_{-1}^{1} f$ is undefined, try $\lim_{a\downarrow 0}(\int_{-1}^{-a} f+\int_{a}^{1} f)$. This particular method seems to disappear in the general context. But the general challenge of the subject remains the same: to develop a theory of the transform $u\mapsto\hat{u}$ which will apply to the largest possible family of objects u and will enable us to justify, in the widest possible contexts, the manipulations listed in the notes to §284. The calculations in 445S and 445Xn, treating 'shift' and 'convolution' in L^2, are typical.

In terms of the actual proofs of the results here, 'test functions' (284A) have gone, and in their place we take a lot more trouble over the Banach algebra L^1. This algebra is the key to one of the magic bits, which turns up in rather undignified corners in 445Kd and part (e) of the proof of 445N. Down to 445O, the dominating problem is that we do not know that the dual group $\mathcal{X}$ of a group X is large enough to tell us anything interesting about X. (After that, the problem reverses; we have to show that $\mathcal{X}$ is not too big.) We find that (under rather specially arranged circumstances) we are able to say something useful about the spectral radius of a member of L^1, and we use this to guarantee that it has a non-trivial Fourier transform. If we identify $\mathcal{X}$ with the maximal ideal space of L^1 (445H), then the Fourier transform on L^1 becomes the 'Gelfand map', a general construction of great power in the theory of commutative Banach algebras.

There is one similarity between the methods of this section and those of §284. In both cases we have isomorphisms between $L^2_{\mathbb{C}}(\mu)$ and $L^2_{\mathbb{C}}(\lambda)$ (the Plancherel Theorem), but cannot define the Fourier transform of a function in $\mathcal{L}^2_{\mathbb{C}}$ in any direct way; indeed, while the Fourier transform of a function in $\mathcal{L}^1_{\mathbb{C}}$, or even of a (totally finite) measure, can really be thought of as a (continuous) function, the transform of a function in L^2 is at best a member of L^2, not a function at all. We manoeuvre around this difficulty by establishing that our (genuine) Fourier transforms match dense subspaces isometrically. In §284 I used test functions, and in the present section I use $\mathcal{L}^1\cap\mathcal{L}^2$. Test functions are easier partly because the Fourier transform of a test function is again a test function, and all the formulae we need are easy to establish for such functions.

Searching for classes of functions which will be readily manageable in general locally compact abelian groups, we come to the 'positive definite' functions. The phrase is unsettling, since the functions themselves are in no obvious sense positive (nor even, as a rule, real-valued). Also their natural analogues in the theory of bilinear forms are commonly called 'positive semi-definite'. However, their Fourier transforms, whether regarded as measures (445N) or as functions (445Q), are positive, and, as a bonus, we get a characterization of the Fourier transforms of measures (445Xf, 445Xh), answering a question left hanging in 285Xr.

446 The structure of locally compact groups

I develop those fragments of the structure theory of locally compact Hausdorff topological groups which are needed for the main theorem of the next section. Theorem 446B here is of independent interest, being both itself important and with a proof which uses the measure theory of this chapter in an interesting way; but the rest of the section, from 446D on, is starred. Note that in this section, unlike the last, groups are not expected to be abelian.

446A Finite-dimensional representations (a) Definitions (i) For any $r \in \mathbb{N}$, write $M_r = M_r(\mathbb{R})$ for the space of $r \times r$ real matrices. If we identify it with the space $\mathrm{B}(\mathbb{R}^r;\mathbb{R}^r)$, where $\mathbb{R}^r$ is given its Euclidean norm, then M_r becomes a unital Banach algebra (4A6C), with identity I, the $r \times r$ identity matrix. Write $GL(r,\mathbb{R})$ for the group of invertible elements of M_r.

(ii) Let X be a topological group. A **finite-dimensional representation** of X is a continuous homomorphism from X to a group of the form $GL(r,\mathbb{R})$ for some $r \in \mathbb{N}$. If the homomorphism is injective the representation is called **faithful** (cf. 4A5Be).

(b) Observe that if X is any topological group and ϕ is a finite-dimensional representation with kernel Y, then X/Y has a faithful finite-dimensional representation ψ defined by writing $\psi(x^\bullet) = \phi(x)$ for every $x \in X$ (4A5La).

446B Theorem Let X be a compact Hausdorff topological group. Then for any $a \in X$, other than the identity, there is a finite-dimensional representation ϕ of X such that $\phi(a) \neq I$.

proof (a) Let U be a symmetric neighbourhood of the identity e in X such that $a \notin UU$. Because X is completely regular (3A3Bb), there is a non-zero continuous function $h : X \to [0,\infty[$ such that $h(x) = 0$ for every $x \in X \setminus U$; replacing h by $x \mapsto h(x) + h(x^{-1})$ if necessary, we may suppose that $h(x) = h(x^{-1})$ for every x. Let μ be a (left) Haar measure on X (441E), and set $w = h^\bullet$ in $L^0(\mu)$.

(b) Define an operator T from $L^2 = L^2(\mu)$ to itself by setting

$$Tu = u * w \text{ for every } u \in L^2,$$

where $*$ is convolution; that is, $Tf^\bullet = (f * h)^\bullet$ for every $f \in \mathcal{L}^2 = \mathcal{L}^2(\mu)$. Then T is a compact self-adjoint operator on the real Hilbert space L^2 (444V).

(c) For any $z \in X$, define $S_z : L^2 \to L^2$ by setting $S_z u = z \bullet_l u$ for $u \in L^2$, where $\bullet_l$ is defined as in 443G. Then S_z is a norm-preserving linear operator. Also S_z commutes with T. **P** If $f \in \mathcal{L}^2$, then

$$S_z T f^\bullet = (z \bullet_l (f * h))^\bullet = ((z \bullet_l f) * h)^\bullet$$

(444Of)

$$= T S_z f^\bullet. \ \mathbf{Q}$$

(d) Now $S_a T w \neq T w$. **P** Set $g = h * h$, so that $Tw = g^\bullet$ and g, $a \bullet_l g$ are both continuous functions (444Rc). Then

$$(a \bullet_l g)(e) = g(a^{-1}) = \int h(y) h(y^{-1}a^{-1}) dy = 0$$

because if $y \in U$ then $y^{-1}a^{-1} \notin U$, while

$$g(e) = \int h(y) h(y^{-1}) dy = \int h(y)^2 dy > 0.$$

So the open set $\{x : g(x) \neq (a \bullet_l g)(x)\}$ is non-empty; because μ is strictly positive (442Aa), it is non-negligible, and

$$S_a T w = (a \bullet_l g)^\bullet \neq g^\bullet = T w. \ \mathbf{Q}$$

(e) The closed linear subspace $\{u : S_a u = u\}$ therefore does not include $T[L^2]$. But $T[L^2]$ is the closed linear span of $\{Tv : v \text{ is an eigenvector of } T\}$ (4A4M), so there is an eigenvector v^* of T such that

$S_aTv^* \neq Tv^*$. Let $\gamma \in \mathbb{R}$ be such that $Tv^* = \gamma v^*$; since $Tv^* \neq 0$, $\gamma \neq 0$, and $V = \{u : Tu = \gamma u\}$ is finite-dimensional (4A4Lb).

(f) $S_z[V] \subseteq V$ for every $z \in X$. **P**

$$TS_zu = S_zTu = S_z(\gamma u) = \gamma S_z u$$

for every $u \in V$. **Q**

We therefore have a map $z \mapsto S_z\restriction V : X \to \mathrm{B}(V;V)$. As observed in 443Gc, this is actually a semigroup homomorphism, and of course $S_e\restriction V$ is the identity of $\mathrm{B}(V;V)$, so $S_z\restriction V$ is always invertible, and we have a group homomorphism from X to the group of invertible elements of $\mathrm{B}(V;V)$. Taking an orthonormal basis $(v_1, \dots, v_r)$ of V, we have a homomorphism ϕ from X to $GL(r,\mathbb{R})$, defined by setting $\phi(z) = \langle (S_zv_i|v_j)\rangle_{1\leq i,j\leq r}$ for every $z \in X$. Moreover, ϕ is continuous. **P** For any $u \in L^2$, $z \mapsto S_zu : X \to L^2$ is continuous, by 443Ge. But this means that all the maps $z \mapsto (S_zv_i|v_j)$ are continuous; since the topology of $GL(r,\mathbb{R})$ can be defined in terms of these functionals (see the formulae in 262H), ϕ is continuous. **Q**

Thus ϕ is a finite-dimensional representation of X. But V was chosen to contain v^*; of course $Tv^* \in V$, while $S_aTv^* \neq Tv^*$; so that $\phi(a)$ is not the identity, as required.

446C Corollary Let X be a compact Hausdorff topological group. Then for any neighbourhood U of the identity of X there is a finite-dimensional representation of X with kernel included in U.

proof Let Φ be the set of finite-dimensional representations of X. By 446B, $\bigcap_{\phi\in\Phi} \ker(\phi) = \{e\}$, where e is the identity of X. Because $X \setminus \operatorname{int} U$ is compact and disjoint from $\bigcap_{\phi\in\Phi} \ker(\phi)$ (and $\ker(\phi)$ is closed for every ϕ), there must be $\phi_0, \dots, \phi_n \in \Phi$ such that $\bigcap_{i\leq n} \ker(\phi_i) \subseteq U$. For each $i \leq n$, let $r_i \in \mathbb{N}$ be the integer such that ϕ_i is a continuous homomorphism from X to $GL(r_i,\mathbb{R})$. Set $r = \sum_{i=0}^{n} r_i$. Then we have a map $\phi : X \to GL(r,\mathbb{R})$ given by the formula

$$\phi(x) = \begin{pmatrix} \phi_0(x) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \phi_1(x) & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \dots & \phi_n(x) \end{pmatrix}$$

for every $x \in X$. It is easy to check that ϕ is a continuous homomorphism, and $\ker(\phi) = \bigcap_{i\leq n} \ker(\phi_i) \subseteq U$. So we have an appropriate representation of X.

***446D Notation (a)** It will help to be clear on an elementary point of notation. If X is a group and $A \subseteq X$ I will write $A^0 = \{e\}$, where e is the identity of X, and $A^{n+1} = AA^n$ for $n \in \mathbb{N}$, so that $A^3 = \{x_1x_2x_3 : x_1, x_2, x_3 \in A\}$, etc. Now we find that $A^{m+n} = A^mA^n$ and $A^{mn} = (A^m)^n$ for all $m, n \in \mathbb{N}$. Writing $A^{-1} = \{x^{-1} : x \in A\}$ as usual, we have $(A^r)^{-1} = (A^{-1})^r$. But note that if we also continue to write $A^{-1} = \{x^{-1} : x \in A\}$, then AA^{-1} is not in general equal to A^0; and that there is no simple relation between A^r, B^r and $(AB)^r$, unless X is abelian.

(b) In the rest of this section, I shall make extensive use of the following device. If X is a group with identity e, $e \in A \subseteq X$ and $n \in \mathbb{N}$, write $D_n(A) = \{x : x \in X, x^i \in A \text{ for every } i \leq n\}$. Then

(i) $D_0(A) = X$.

(ii) $D_1(A) = A$.

(iii) $D_n(A) \subseteq D_m(A)$ whenever $m \leq n$.

(iv) $D_{mn}(A) \subseteq D_m(D_n(A))$ for all $m, n \in \mathbb{N}$.

P If $x \in D_{mn}(A)$ then $(x^i)^j \in A$ whenever $j \leq n$, $i \leq m$. **Q**

(v) If $r \in \mathbb{N}$ and $A^r \subseteq B$, then $D_n(A) \subseteq D_{nr}(B)$ for every $n \in \mathbb{N}$; in particular, $A \subseteq D_r(B)$.

P For $r = 0$ this is trivial. Otherwise, take $x \in D_n(A)$ and $i \leq nr$. Then we can express i as $i_1 + \dots + i_r$ where $i_j \leq n$ for each j, so that

$$x^i = \textstyle\prod_{j=1}^{r} x^{i_j} \in A^r \subseteq B. \quad \mathbf{Q}$$

(vi) If $A = A^{-1}$ then $D_n(A) = D_n(A)^{-1}$ for every $n \in \mathbb{N}$.

(vii) If $D_m(A) \subseteq B$ for some $m \in \mathbb{N}$, then $D_{mn}(A) \subseteq D_n(B)$ for every $n \in \mathbb{N}$.
P If $x \in D_{mn}(A)$ and $i \leq n$, then $x^{ij} \in A$ for every $j \leq m$, so $x^i \in D_m(A) \subseteq B$. **Q**

(c) In (b), if X is a topological group and A is closed, then every $D_n(A)$ is closed; if moreover A is compact, then $D_n(A)$ is compact for every $n \geq 1$. If A is a neighbourhood of e, then so is every $D_n(A)$, because the map $x \mapsto x^i$ is continuous for every $i \leq n$.

***446E 'Groups with no small subgroups' (a) Definition** Let X be a topological group. We say that X **has no small subgroups** if there is a neighbourhood U of the identity e of X such that the only subgroup of X included in U is $\{e\}$.

(b) If X is a Hausdorff topological group and U is a compact symmetric neighbourhood of the identity e such that the only subgroup of X included in U is $\{e\}$, then $\{D_n(U) : n \in \mathbb{N}\}$ is a base of neighbourhoods of e, where $D_n(U) = \{x : x \in X, x^i \in U \text{ for every } i \leq n\}$. **P** By 446Dc, $\langle D_n(U)\rangle_{n\geq 1}$ is a non-increasing sequence of compact neighbourhoods of e, and if $x \in \bigcap_{n\in\mathbb{N}} D_n(U)$ then $x^i \in U$ for every $i \in \mathbb{N}$; as $U^{-1} = U$, U includes the subgroup $\{x^i : i \in \mathbb{Z}\}$, so $x = e$. Thus $\bigcap_{n\in\mathbb{N}} D_n(U) = \{e\}$ and $\{D_n(U) : n \in \mathbb{N}\}$ is a base of neighbourhoods of e (4A2Gd). **Q**

(c) In particular, a locally compact Hausdorff topological group with no small subgroups is metrizable (4A5Q).

***446F Lemma** Let X be a group, and $U \subseteq X$. Let $f : X \to [0, \infty[$ be a bounded function such that $f(x) = 0$ for $x \in X \setminus U$; set $\alpha = \sup_{x\in X} f(x)$. Let $A \subseteq X$ be a symmetric set containing e, and K a set including A^k, where $k \geq 1$. Define $g : X \to [0, \infty[$ by setting

$$g(x) = \tfrac{1}{k} \textstyle\sum_{i=0}^{k-1} \sup\{f(yx) : y \in A^i\}$$

for $x \in X$. Then

(a) $f(x) \leq g(x) \leq \alpha$ for every $x \in X$, and $g(x) = 0$ if $x \notin K^{-1}U$.

(b) $|g(ax) - g(x)| \leq \frac{j\alpha}{k}$ if $j \in \mathbb{N}$, $a \in A^j$ and $x \in X$.

(c) For any x, $z \in X$, $|g(x) - g(z)| \leq \sup_{y\in K} |f(yx) - f(yz)|$.

proof (a) Of course

$$f(x) = \tfrac{1}{k} \textstyle\sum_{i=0}^{k-1} f(ex) \leq g(x) \leq \tfrac{1}{k} \sum_{i=0}^{k-1} \alpha = \alpha$$

for every x. Suppose that $g(x) \neq 0$. Then there must be an $i < k$ and a $y \in A^i$ such that $f(yx) \neq 0$, so that $yx \in U$. But also, because $e \in A$, $y \in A^k \subseteq K$, so $x \in y^{-1}U \subseteq K^{-1}U$.

(b) Suppose first that $j = 1$, so that $a \in A$. If $\epsilon > 0$ there are $y_i \in A^i$, for $i < k$, such that $g(ax) \leq \frac{1}{k} \sum_{i=0}^{k-1} f(y_i ax) + \epsilon$. Now $y_i a \in A^{i+1}$ for each i, so

$$g(x) \geq \tfrac{1}{k} \textstyle\sum_{i=0}^{k-2} f(y_i ax) \geq g(ax) - \epsilon - \frac{\alpha}{k}.$$

As ϵ is arbitrary, $g(x) \geq g(ax) - \frac{\alpha}{k}$. Similarly, as $a^{-1} \in A$ (because A is symmetric), $g(ax) \geq g(x) - \frac{\alpha}{k}$ and $|g(ax) - g(x)| \leq \frac{\alpha}{k}$.

For the general case, induce on j. (If $j = 0$, then $a = e$ and the result is trivial.)

(c) Set $\gamma = \sup_{y\in K} |f(yx) - f(yz)|$. If $\epsilon > 0$, there are $y_i \in A^i$, for $i < k$, such that $g(ax) \leq \frac{1}{k} \sum_{i=0}^{k-1} f(y_i x) + \epsilon$. Now every y_i belongs to K, so

$$g(z) \geq \tfrac{1}{k} \textstyle\sum_{i=0}^{k-1} f(y_i z) \geq \tfrac{1}{k} \sum_{i=0}^{k-1} (f(y_i x) - \gamma) \geq g(x) - \epsilon - \gamma.$$

As ϵ is arbitrary, $g(z) \geq g(x) - \gamma$; similarly, $g(x) \geq g(z) - \gamma$.

***446G Lemma** Let X be a locally compact Hausdorff topological group and $\langle A_n\rangle_{n\in\mathbb{N}}$ a sequence of symmetric subsets of X all containing the identity e of X. Suppose that for every neighbourhood W of e there is an $n_0 \in \mathbb{N}$ such that $A_n \subseteq W$ for every $n \geq n_0$. Let U be a compact neighbourhood of e and suppose that for each $n \in \mathbb{N}$ we have $k(n) \in \mathbb{N}$ such that $A_n^{k(n)} \subseteq U$, $A_n^{k(n)+1} \not\subseteq U$. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ and set $Q = \limsup_{n\to\mathcal{F}} A_n^{k(n)}$ (definition: 4A2A).

(i) If $Q^2 = Q$ then Q is a compact subgroup of X included in U and meeting the boundary of U.

(ii) If $Q^2 \neq Q$ then there are a neighbourhood W of e and an infinite set $I \subseteq \mathbb{N}$ such that for every $n \in I$ there are an $x \in A_n$ and an $i \leq k(n)$ such that $x^i \notin W$.

proof (a) Because every $A_n^{k(n)}$ is included in U, so is Q, and Q, being closed (4A2Ta), is compact. Because $x \mapsto x^{-1}$ is a homeomorphism,

$$Q^{-1} = \limsup_{n\to\mathcal{F}}(A_n^{k(n)})^{-1} = \limsup_{n\to\mathcal{F}}(A_n^{-1})^{k(n)} = \limsup_{n\to\mathcal{F}} A_n^{k(n)} = Q.$$

And of course $e \in Q$ because $e \in A_n^{k(n)}$ for every n.

For each $n \in \mathbb{N}$, we have an $a_n \in A_n^{k(n)}$ and an $x_n \in A_n$ such that $a_n x_n \notin U$. Now $a = \lim_{n\to\mathcal{F}} a_n$ is defined (because U is compact), and belongs to Q (4A2Ta). Also $\lim_{n\to\infty} x_n = e$, because every neighbourhood of e includes all but finitely many of the A_n, so $a = \lim_{n\to\mathcal{F}} a_n x_n \notin \operatorname{int} U$, and a belongs to the boundary of U. Thus Q meets the boundary of U.

(b) From (a) we see that if $Q^2 = Q$ then Q is a compact subgroup of X, included in U and meeting the boundary of U. So henceforth let us suppose that $Q^2 \neq Q$ and seek to prove (ii).

Let $w \in Q^2 \setminus Q$. Let $W_0 \subseteq U$ be an open neighbourhood of e such that $W_0 w W_0^2 \cap Q W_0^2 = \emptyset$ (4A5Ee).

(c) Fix a left Haar measure μ on X. Let $f : X \to [0,\infty[$ be a continuous function such that $\{x : f(x) > 0\} \subseteq W_0$ and $\int f(x)dx = 1$. Set $\alpha = \sup_{x\in X} f(x)$, $\beta = \int f(x)^2 dx$, so that α is finite and $\beta > 0$. $W_0 U^2 W_0 \subseteq U^4$ is compact, so has finite measure, and there is an $\eta > 0$ be such that

$$2\eta(1 + \alpha\mu(W_0 U^2 W_0)) < \beta.$$

Let W be a neighbourhood of e such that $W \subseteq W_0$ and $|f(yax) - f(ybx)| \leq \eta$ whenever $y \in (U^{-1})^2 \cup U$, $x \in X$ and $ab^{-1} \in W$ (4A5Pa).

(d) Express w as $w'w''$ where w', $w'' \in Q$. Then

$$\{n : A_n^{k(n)} \cap W_0 w' \neq \emptyset\}, \quad \{n : A_n^{k(n)} \cap W_0 w'' \neq \emptyset\},$$

$$\{n : A_n^{k(n)} \subseteq Q W_0\} = \{n : A_n^{k(n)} \cap (U \setminus Q W_0) = \emptyset\}$$

all belong to $\mathcal{F}$, by 4A2Tb and 4A2Tc. Also

$$\{n : k(n) \geq 1\}$$

is cofinite in $\mathbb{N}$, because $A_n \subseteq U$ for all n large enough. Let I be the intersection of these four sets, so that $I \in \mathcal{F}$ and must be infinite.

(e) Let $n \in I$. **?** Suppose, if possible, that $x^i \in W$ for every $x \in A_n$, $i \leq k(n)$. (The rest of the proof will be a search for a contradiction.) Note that $k(n) \geq 1$.

Choose $x_j \in A_n$, for $j < 2k(n)$, such that the products $x_{2k(n)-1}x_{2k(n)-2}\dots x_{k(n)}$, $x_{k(n)-1}\dots x_0$ belong to $W_0 w'$, $W_0 w''$ respectively; set $\tilde{w} = x_{2k(n)-1}\dots x_0$, so that

$$\tilde{w} \in A_n^{2k(n)} \cap W_0 w' w'' W_0 \subseteq W_0 w W_0.$$

Since $A_n^{k(n)} \subseteq Q W_0$ and $W_0 w W_0^2 \cap Q W_0^2$ is empty, $\tilde{w} W_0$ does not meet $A_n^{k(n)} W_0$.

(f) Define $g : X \to [0,\infty[$ by setting

$$g(x) = \frac{1}{k(n)} \sum_{i=0}^{k(n)-1} \sup\{f(yx) : y \in A_n^i\}.$$

Then $g(\tilde{w}x)f(x) = 0$ for every $x \in X$. **P** If $f(x) \neq 0$, then $x \in W_0$, so $\tilde{w}x \notin A_n^{k(n)} W_0$, and $g(\tilde{w}x) = 0$, by 446Fa. **Q** Accordingly

$$\int (g(x) - g(\tilde{w}x))f(x)dx = \int g(x)f(x)dx \geq \beta$$

since $g \geq f$ (446Fa).

Set $y_0 = e$, $y_{i+1} = x_i y_i$ for $i < 2k(n)$, so that $y_{2k(n)} = \tilde{w}$ and

$$y_i \in A_n^i \subseteq A_n^{2k(n)} = (A_n^{k(n)})^2 \subseteq U^2$$

for every $i \leq 2k(n)$. Then

$$g(x) - g(\tilde{w}x) = \sum_{i=0}^{2k(n)-1} g(y_i x) - g(y_{i+1}x)$$

for every $x \in X$. Let $i < 2k(n)$ be such that

$$\int (g(y_i x) - g(y_{i+1}x))f(x)dx \geq \frac{1}{2k(n)} \int (g(x) - g(\tilde{w}x))f(x)dx \geq \frac{\beta}{2k(n)}.$$

Set $u = x_i$, $v = y_i$, so that $u \in A_n$, $v \in U^2$ and

$$\begin{aligned}\int (g(x) - g(ux))f(v^{-1}x)dx &= \int (g(vx) - g(uvx))f(x)dx \\ &= \int (g(y_i x) - g(y_{i+1}x))f(x)dx \geq \frac{\beta}{2k(n)}.\end{aligned}$$

(g) We have

$$\begin{aligned}&\int (g(x) - g(u^{k(n)}x))f(v^{-1}x)dx \\ &= \sum_{j=0}^{k(n)-1} \int (g(u^j x) - g(u^{j+1}x))f(v^{-1}x)dx \\ &= \sum_{j=0}^{k(n)-1} \int (g(x) - g(ux))f(v^{-1}u^{-j}x)dx \\ &= k(n) \int (g(x) - g(ux))f(v^{-1}x)dx \\ &\qquad + \sum_{j=0}^{k(n)-1} \int (g(x) - g(ux))(f(v^{-1}u^{-j}x) - f(v^{-1}x))dx,\end{aligned}$$

that is,

$$\begin{aligned}&k(n) \int (g(x) - g(ux))f(v^{-1}x)dx \\ &= \int (g(x) - g(u^{k(n)}x))f(v^{-1}x)dx \\ &\qquad - \sum_{j=0}^{k(n)-1} \int (g(x) - g(ux))(f(v^{-1}u^{-j}x) - f(v^{-1}x))dx.\end{aligned}$$

Set

$$\beta_1 = \sum_{j=0}^{k(n)-1} \int (g(x) - g(ux))(f(v^{-1}u^{-j}x) - f(v^{-1}x))dx,$$

$$\beta_2 = \int (g(x) - g(u^{k(n)}x))f(v^{-1}x)dx;$$

then

$$\beta_2 - \beta_1 = k(n) \int (g(x) - g(ux))f(v^{-1}x)dx \geq \tfrac{1}{2}\beta.$$

(h)(i) Examine β_1. We know that, because $u \in A_n$, $|g(x) - g(ux)| \leq \frac{\alpha}{k(n)}$ for every x (see 446Fb). On the other hand, we are supposing that $x^j \in W$ for every $j \leq k(n)$ and every $x \in A_n$, so, in particular,

$u^j \in W \subseteq W_0$ for every $j \leq k(n)$. Also, as noted in (f), $v \in U^2$. So for any $j < k(n)$ we must have $|f(v^{-1}u^{-j}x) - f(v^{-1}x)| \leq \eta$ for every $x \in X$, by the choice of W, while $f(v^{-1}u^{-j}x) - f(v^{-1}x) = 0$ unless $x \in W_0U^2W_0$. So

$$\begin{aligned}|\beta_1| &\leq \sum_{j=0}^{k(n)-1} \int |g(x) - g(ux)||f(v^{-1}u^{-j}x) - f(v^{-1}x)|dx \\ &\leq \sum_{j=0}^{k(n)-1} \frac{\alpha}{k(n)}\eta\mu(W_0U^2W_0) = \alpha\eta\mu(W_0U^2W_0).\end{aligned}$$

(ii) Now consider β_2. As $u^{k(n)} \in W$, $|f(zu^{k(n)}x) - f(zx)| \leq \eta$ for every $z \in U$, by the choice of W, so (because $A_n^{k(n)} \subseteq U$) $|g(u^{k(n)}x) - g(x)| \leq \eta$ for every x (446Fc). Accordingly

$$|\beta_2| \leq \eta \int f(v^{-1}x)dx = \eta \int f(x)dx = \eta.$$

(i) But this means that

$$\beta \leq 2(|\beta_1| + |\beta_2|) \leq 2\eta(1 + \alpha\mu(W_0U^2W_0)) < \beta,$$

which is absurd. **X**

(j) Thus for every $n \in I$ there are an $x \in A_n$ and an $i \leq k(n)$ such that $x^i \notin W$, and (ii) is true. This completes the proof.

***446H Lemma** Let X be a locally compact Hausdorff topological group. For $A \subseteq X$, $n \in \mathbb{N}$ set $D_n(A) = \{x : x^i \in A \text{ for every } i \leq n\}$. If $U \subseteq X$ is a compact symmetric neighbourhood of the identity such that there is no subgroup of X included in U other than $\{e\}$, then there is an $r \geq 1$ such that $D_{rn}(U)^n \subseteq U$ for every $n \in \mathbb{N}$.

proof ? Suppose, if possible, that for every $r \geq 1$ there is an $n_r \geq 1$ such that $D_{rn_r}(U)^{n_r} \not\subseteq U$. Of course $n_r \geq 1$. Set $A_r = D_{rn_r}(U)$ for each $r \geq 1$, $A_0 = U$, and apply 446G to the sequence $\langle A_r\rangle_{r\in\mathbb{N}}$. Note that the $k(r)$ there are such that $A_r^{k(r)} \subseteq U$, so $k(r) < n_r$ for $r \geq 1$; also $k(r) \geq 1$, because $A_r \subseteq U$. Since U includes no non-trivial subgroup, (i) of 446G is impossible, and we are left with (ii). Let W, I be as declared there, so that $A_r \not\subseteq D_{k(r)}(W)$ for every $r \in I$. There must be some $m \geq 1$ such that $D_m(U) \subseteq W$ (446Eb). Take $r \in I$ such that $r \geq m$; then $rn_r \geq mk(r)$, so

$$\begin{aligned}A_r = D_{rn_r}(U) &\subseteq D_{mk(r)}(U) \subseteq D_{k(r)}(D_m(U)) \\ &\text{(446D(b-iv))} \\ &\subseteq D_{k(r)}(W),\end{aligned}$$

which is impossible. **X**

***446I Lemma** Let X be a locally compact Hausdorff topological group and U a compact symmetric neighbourhood of the identity in X such that U does not include any subgroup of X other than $\{e\}$. For $n \in \mathbb{N}$, set $D_n(U) = \{x : x^i \in U \text{ for every } i \leq n\}$, and let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. Suppose that $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in X such that $x_n \in D_n(U)$ for every $n \in \mathbb{N}$. Then we have a continuous homomorphism $q : \mathbb{R} \to X$ defined by setting $q(t) = \lim_{n\to\mathcal{F}} x_n^{i(n)}$ whenever $\langle i(n)\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} i(n)/n = t$ in $\mathbb{R}$.

Remark See 2A3S for the elementary properties of $\lim_{n\to\mathcal{F}}$.

proof (a) If $\langle i(n)\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathbb{N}$ such that $\lim_{n\to\mathcal{F}} i(n)/n$ is defined in $\mathbb{R}$, then $\lim_{n\to\mathcal{F}} x_n^{i(n)}$ is defined in X. **P** There is some $m \in \mathbb{N}$ such that $m > \lim_{n\to\mathcal{F}} i(n)/n$, so that $J = \{n : i(n) \leq mn\} \in \mathcal{F}$; but if $n \in J$, then

$$x_n \in D_n(U) \subseteq D_{mn}(U^m) \subseteq D_{i(n)}(U^m),$$

by 446D(b-v), and $x_n^{i(n)} \in U^m$. But this means that $\mathcal{F}$ contains $\{n : x_n^{i(n)} \in U^m\}$; as U^m is compact, $\lim_{n\to\mathcal{F}} x_n^{i(n)}$ is defined in X. **Q**

More generally, if $\langle i(n)\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} i(n)/n$ is defined in $\mathbb{R}$, then $\lim_{n\to\mathcal{F}} x_n^{i(n)}$ is defined in X. **P** At least one of $\{n : i(n) \geq 0\}$, $\{n : i(n) \leq 0\}$ belongs to $\mathcal{F}$. In the former case, $\lim_{n\to\mathcal{F}} x_n^{i(n)} = \lim_{n\to\mathcal{F}} x_n^{\max(0,i(n))}$ is defined; in the latter case,

$$\lim_{n\to\mathcal{F}} x_n^{i(n)} = \lim_{n\to\mathcal{F}} (x_n^{\max(0,-i(n))})^{-1} = (\lim_{n\to\mathcal{F}} x_n^{\max(0,-i(n))})^{-1}$$

is defined. **Q**

(b) If V is any neighbourhood of e, there is a $\delta > 0$ such that $\lim_{n\to\mathcal{F}} x_n^{i(n)} \in V$ whenever $\langle i(n)\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} |i(n)/n| \leq \delta$. **P** By 446Eb, there is an $m \geq 1$ such that $D_m(U) \subseteq V$. Take $\delta < \frac{1}{m}$. If $\lim_{n\to\mathcal{F}} |i(n)/n| \leq \delta$, then $J = \{n : m|i(n)| \leq n\} \in \mathcal{F}$. But if $n \in J$, then

$$x_n \in D_n(U) \subseteq D_{m|i(n)|}(U) \subseteq D_{|i(n)|}(D_m(U))$$

and $x_n^{|i(n)|} \in D_m(U)$; since $D_m(U)$, like U, is symmetric (446D(b-vi)), $x_n^{i(n)} \in D_m(U)$. This is true for every $n \in J$, so $\lim_{n\to\mathcal{F}} x_n^{i(n)} \in D_m(U) \subseteq V$, as required. **Q**

(c) It follows at once that if $\langle i(n)\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} i(n)/n = 0$, then $\lim_{n\to\mathcal{F}} x_n^{i(n)} = e$. Consequently, if $\langle i(n)\rangle_{n\in\mathbb{N}}$, $\langle j(n)\rangle_{n\in\mathbb{N}}$ are sequences in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} i(n)/n$ and $\lim_{n\to\mathcal{F}} j(n)/n$ both exist in $\mathbb{R}$ and are equal, then $\lim_{n\to\mathcal{F}} x_n^{i(n)} = \lim_{n\to\mathcal{F}} x_n^{j(n)}$. **P** Set $k(n) = i(n) - j(n)$. Then $\lim_{n\to\mathcal{F}} x_n^{k(n)} = e$ because $\lim_{n\to\mathcal{F}} k(n)/n = 0$. But now

$$\lim_{n\to\mathcal{F}} x_n^{i(n)} = \lim_{n\to\mathcal{F}} x_n^{j(n)} x_n^{k(n)} = \lim_{n\to\mathcal{F}} x_n^{j(n)}. \ \textbf{Q}$$

(d) We do therefore have a function $q : \mathbb{R} \to X$ defined by the given formula. Now $q(s+t) = q(s)q(t)$ for all s, $t \in \mathbb{R}$. **P** Take sequences $\langle i(n)\rangle_{n\in\mathbb{N}}$, $\langle j(n)\rangle_{n\in\mathbb{N}}$ such that $s = \lim_{n\to\infty} i(n)/n$, $t = \lim_{n\to\infty} j(n)/n$; then $s + t = \lim_{n\to\infty}(i(n) + j(n))/n$, so

$$q(s+t) = \lim_{n\to\mathcal{F}} x_n^{i(n)+j(n)} = \lim_{n\to\mathcal{F}} x_n^{i(n)} x_n^{j(n)} = q(s)q(t). \ \textbf{Q}$$

(e) Thus q is a homomorphism. Finally, (b) shows that it is continuous at 0, so it must be continuous.

***446J Lemma** Let X be a locally compact Hausdorff topological group with no small subgroups. Then there is a neighbourhood V of the identity e such that $x = y$ whenever x, $y \in V$ and $x^2 = y^2$.

proof Let U be a symmetric compact neighbourhood of e not including any subgroup of X except $\{e\}$. Let $\langle U_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence of neighbourhoods of e comprising a base of neighbourhoods of e (446Ec), and with $U_0 = U$.

? Suppose, if possible, that for each $n \in \mathbb{N}$ there are distinct x_n, $y_n \in U_n$ such that $x_n^2 = y_n^2$. Set $a_n = x_n^{-1}y_n$; then

$$x_n^{-1}a_n x_n = x_n^{-2}y_n x_n = y_n^{-1}x_n = a_n^{-1}.$$

Accordingly

$$x_n^{-1}a_n^m x_n = (x_n^{-1}a_n x_n)^m = a_n^{-m}$$

for every $m \in \mathbb{N}$.

Since U includes no non-trivial subgroup, and $a_n \neq e$, there is a $k(n) \in \mathbb{N}$ such that $a_n^i \in U$ for $i \leq k(n)$ and $a_n^{k(n)+1} \notin U$. Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$; then $a = \lim_{n\to\mathcal{F}} a_n^{k(n)}$ is defined and belongs to U. Also, because $\langle x_n\rangle_{n\in\mathbb{N}}$ and $\langle y_n\rangle_{n\in\mathbb{N}}$ both converge to e, so does $\langle a_n\rangle_{n\in\mathbb{N}}$, and

$$a = \lim_{n\to\mathcal{F}} a_n^{k(n)+1} \in X \setminus \operatorname{int} U;$$

thus a cannot be e.

However, $x_n^{-1}a_n^{k(n)}x_n = a_n^{-k(n)}$ for each n, so

$$e^{-1}ae = \lim_{n\to\mathcal{F}} x_n^{-1}a_n^{k(n)}x_n = \lim_{n\to\mathcal{F}}(a_n^{k(n)})^{-1} = a^{-1}.$$

So $a = a^{-1}$ and $\{e, a\}$ is a non-trivial subgroup of X included in U, which is supposed to be impossible. **X**

Thus some U_n serves for V.

***446K Lemma** Let X be a locally compact Hausdorff topological group with no small subgroups. For $A \subseteq X$ set $D_n(A) = \{x : x^i \in A \text{ for every } i \le n\}$. Then there is a compact symmetric neighbourhood U of the identity e such that whenever V is a neighbourhood of e there are an $n_0 \in \mathbb{N}$ and a neighbourhood W of e such that whenever $n \ge n_0$, $x \in D_n(U)$, $y \in D_n(U)$ and $x^n y^n \in W$, then $xy \in D_n(U)$.

proof (a) Let U_0 be a compact symmetric neighbourhood of e such that (α) U_0^3 includes no subgroup of X other than $\{e\}$ (β) whenever x, $y \in U_0$ and $x^2 = y^2$, then $x = y$; such a neighbourhood exists by 446J. Let $r \ge 1$ be such that $D_{rn}(U_0)^n \subseteq U_0$ for every $n \in \mathbb{N}$ (446H). Let U be a compact symmetric neighbourhood of e such that $U^r \subseteq U_0$. In this case $D_n(U) \subseteq D_{rn}(U_0)$ for every n, by 446D(b-v). So $D_n(U)^n \subseteq U_0$ for every n.

(b) Fix a left Haar measure μ on X. Let $f : X \to [0, \infty[$ be a continuous function such that $\int f(x)dx = 1$ and $f(x) = 0$ for $x \in X \setminus U_0$. Set $\alpha = \sup_{x\in X} |f(x)|$, $\beta = \int f(x)^2 dx$, so that α is finite and $\beta > 0$. For $n \ge 1$, set

$$f_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} \sup\{f(yx) : y \in D_n(U)^i\}$$

for $x \in X$. Because $D_n(U)^n \subseteq U_0$, we can apply 446F to see that, for each n,

(i) $f_n \ge f$,

(ii) $f_n(x) = 0$ if $x \notin U_0^2$,

(iii) $|f_n(ax) - f_n(x)| \le \frac{j\alpha}{n}$ if $j \in \mathbb{N}$, $a \in D_n(U)^j$ and $x \in X$,

(iv) for any x, $z \in X$, $|f_n(x) - f_n(z)| \le \sup_{y\in U_0} |f(yx) - f(yz)|$.

It follows that

(v) for any $\epsilon > 0$ there is a neighbourhood W of e such that $|f_n(ax) - f_n(bx)| \le \epsilon$ whenever a, b, $x \in X$, $n \in \mathbb{N}$ and $ab^{-1} \in W$

(4A5Pa).

(c) It will help to have the following fact available. Suppose we are given sequences $\langle x_n\rangle_{n\in\mathbb{N}}$, $\langle y_n\rangle_{n\in\mathbb{N}}$ such that x_n and y_n belong to $D_n(U)$ for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n^n y_n^n = e$. Write

$$\gamma_n = \sup\{|f_n(y_n^j x) - f_n(x_n^{-j}x)| : j \le n, x \in X\}$$

for each $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \gamma_n = 0$. **P?** Otherwise, there is an $\eta > 0$ such that $J = \{n : \gamma_n > \eta\}$ is infinite. Let W be a neighbourhood of e such that $|f_n(ax) - f_n(bx)| \le \eta$ whenever $n \in \mathbb{N}$, x, a, $b \in X$ and $ab^{-1} \in W$ ((b-v) above); let W' be a neighbourhood of e such that $ab \in W$ whenever a, $b \in U$ and $ba \in W'$ (4A5Ej). Then for each $n \in J$ there must be a $j(n) \le n$ such that $y_n^{j(n)}x_n^{j(n)} \notin W$, while $x_n^{j(n)}$ and $y_n^{j(n)}$ both belong to U, so that $x^{j(n)}y^{j(n)} \notin W'$. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ containing J. By 446I, there are continuous homomorphisms q, $\tilde{q}$ from $\mathbb{R}$ to X such that $q(t) = \lim_{n\to\mathcal{F}} x_n^{-i(n)}$, $\tilde{q}(t) = \lim_{n\to\mathcal{F}} y_n^{i(n)}$ whenever $\langle i(n)\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{Z}$ such that $\lim_{n\to\mathcal{F}} i(n)/n = t$ in $\mathbb{R}$. (Of course $x_n^{-1} \in D_n(U)$ for every n, by 446D(b-vi).) Setting $t_0 = \lim_{n\to\mathcal{F}} j(n)/n \in [0, 1]$,

$$q(-t_0)\tilde{q}(t_0) = \lim_{n\to\mathcal{F}} x_n^{j(n)}y_n^{j(n)} \notin \operatorname{int} W',$$

so $q(t_0) \ne \tilde{q}(t_0)$ and $q \ne \tilde{q}$. But

$$q(-1)\tilde{q}(1) = \lim_{n\to\mathcal{F}} x_n^n y_n^n = e,$$

so $q(1) = \tilde{q}(1)$. Now if $0 \le i(n) \le n$, then $x_n^{-i(n)} \in D_n(U)^n \subseteq U_0$; so if $0 \le t \le 1$, $q(t) \in U_0$. Similarly, $\tilde{q}(t) \in U_0$ whenever $t \in [0, 1]$. But recall that U_0 was chosen so that if x, $y \in U_0$ and $x^2 = y^2$ then $x = y$. In particular, since $q(\frac{1}{2})$ and $\tilde{q}(\frac{1}{2})$ both belong to U_0, and their squares $q(1)$, $\tilde{q}(1)$ are equal, $q(\frac{1}{2}) = \tilde{q}(\frac{1}{2})$. Repeating this argument, we see that $q(2^{-k}) = \tilde{q}(2^{-k})$ for every $k \in \mathbb{N}$, so that $q(2^{-k}i) = \tilde{q}(2^{-k}i)$ for every $k \in \mathbb{N}$, $i \in \mathbb{Z}$; since q and $\tilde{q}$ are supposed to be continuous, they must be equal; but $q(t_0) \ne \tilde{q}(t_0)$. **XQ**

(d) Now let V be any neighbourhood of e.

? Suppose, if possible, that for every neighbourhood W of e and $n_0 \in \mathbb{N}$ there are $n \geq n_0$, x, $y \in D_n(U)$ such that $x^n y^n \in W$ but $xy \notin D_n(V)$. For $k \in \mathbb{N}$ choose $n_k \in \mathbb{N}$ and $\tilde{x}_k$, $\tilde{y}_k \in D_{n_k}(U)$ such that $\tilde{x}_k^{n_k} \tilde{y}_k^{n_k} \in D_k(U)$ but $\tilde{x}_k \tilde{y}_k \notin D_{n_k}(V)$, and $n_k > n_{k-1}$ if $k \geq 1$. Now we know that $\langle D_k(U)\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence constituting a base of neighbourhoods of e (446Eb), so $\lim_{k\to\infty} \tilde{x}_k^{n_k} \tilde{y}_k^{n_k} = e$. Set $J = \{n_k : k \in \mathbb{N}\}$. For $n = n_k$, set $x_n = \tilde{x}_k$, $y_n = \tilde{y}_k$; for $n \in \mathbb{N} \setminus J$, set $x_n = y_n = e$. Then x_n, $y_n \in D_n(U)$ for every $n \in \mathbb{N}$ and $\langle x_n^n y_n^n\rangle_{n\in\mathbb{N}} \to e$ as $n \to \infty$, while $x_n y_n \notin D_n(V)$ for $n \in J$.

We know from (d) that

$$\gamma_n = \sup\{|f_n(y_n^j x) - f_n(x_n^{-j} x)| : j \leq n,\ x \in X\} \to 0$$

as $n \to \infty$; for future reference, take a sequence $\langle j(n)\rangle_{n\geq 1}$ such that $1 \leq j(n) \leq n$ for every $n \geq 1$, $\lim_{n\to\infty} j(n)/n = 0$ and $\lim_{n\to\infty} \frac{n\gamma_n}{j(n)} = 0$.

(e) Let $l \geq 1$ be such that $D_l(U_0^3) \subseteq V$ (446Eb again), and set $K = U_0^{2l+1}$. Then $D_{nl}(U_0^3) \subseteq D_n(V)$ for every n (446D(b-vii)). So $x_n y_n \notin D_{nl}(U_0^3)$ for $n \in J$. For each $n \in J$ choose $m(n) \leq nl$ such that $(x_n y_n)^{m(n)} \notin U_0^3$; for $n \in \mathbb{N} \setminus J$, set $m(n) = 1$. Then $(x_n y_n)^{-m(n)} x \notin U_0^2$ for any $x \in U_0$, $n \in J$, and $f_n((x_n y_n)^{-m(n)} x) f(x) = 0$ for any $x \in X$, $n \in J$. So

$$|\int (f_n((x_n y_n)^{-m(n)} x) - f_n(x)) f(x) dx| = \int f_n(x) f(x) dx \geq \beta$$

for $n \in J$, because $f_n \geq f$ ((b-i) above).

(f) Of course $m(n) > 0$ for every $n \in J$, therefore for every n. So we can set

$$g_n(x) = \frac{1}{m(n)} \textstyle\sum_{i=0}^{m(n)-1} f((x_n y_n)^i x)$$

for $x \in X$, $n \in \mathbb{N}$. Note that g_n, like f, is non-negative, and also that $\int g_n(x)dx = \int f(x)dx = 1$. We need to know that $g_n(x) = 0$ if $x \notin K$; this is because $(x_n y_n)^i \in D_n(U)^{2nl} \subseteq U_0^{2l}$ for every $i \leq m(n)$, while $f(x) = 0$ if $x \notin U_0$, and U_0 is symmetric.

We also have

$$|g_n(ax) - g_n(x)| \leq \sup\{|f(wax) - f(wx)| : w \in U_0^{2l}\}$$

for every $n \in \mathbb{N}$, a, $x \in X$ (cf. 446Fc), so for every $\eta > 0$ there must be a neighbourhood W of e such that $|g_n(ax) - g(x)| \leq \eta$ whenever $n \in \mathbb{N}$, $a \in W$ and $x \in X$, by 4A5Pb. Since also $g_n(ax) = g_n(x) = 0$ if $a \in U_0$ and $x \notin U_0^{-1} K$, and $U_0^{-1} K$ has finite measure, we see that for every $\eta > 0$ there is a neighbourhood W of e such that $\int |g_n(ax) - g(x)| dx \leq \eta$ for every $a \in W$, $n \in \mathbb{N}$.

(g) Returning to the formula in (e), we see that, for any $n \in J$,

$$\begin{aligned}
&\Big|\int \big(f_n((x_n y_n)^{-m(n)} x) - f_n(x)\big) f(x) dx\Big| \\
&\quad = \Big|\sum_{i=0}^{m(n)-1} \int \big(f_n((x_n y_n)^{-i-1} x) - f_n((x_n y_n)^{-i} x)\big) f(x) dx\Big| \\
&\quad = \Big|\sum_{i=0}^{m(n)-1} \int \big(f_n((x_n y_n)^{-1} x) - f_n(x)\big) f((x_n y_n)^i x) dx\Big| \\
&\quad = m(n)\Big|\int \big(f_n((x_n y_n)^{-1} x) - f_n(x)\big) g_n(x) dx\Big| \\
&\quad \leq ln\Big|\int \big(f_n(y_n^{-1} x_n^{-1} x) - f_n(x)\big) g_n(x) dx\Big| \\
&\quad \leq ln\Big|\int \big(f_n(y_n^{-1} x) - f_n(x) - f_n(y_n^{-1} x_n^{-1} x) + f_n(x_n^{-1} x)\big) g_n(x) dx\Big| \\
&\qquad\qquad + ln\Big|\int \big(2f_n(x) - f_n(x_n^{-1} x) - f_n(y_n^{-1} x)\big) g_n(x) dx\Big|.
\end{aligned}$$

Next,

$$\begin{aligned}j(n)\big(2f_n(x) - f_n(x_n^{-1}x) - f_n(y_n^{-1}x)\big)&\\ = j(n)\big(f_n(x) - f_n(x_n^{-1}x)\big) - f_n(x) + f_n(x_n^{-j(n)}x)&\\ + j(n)\big(f_n(x) - f_n(y_n^{-1}x)\big) - f_n(x) + f_n(y_n^{-j(n)}x)&\\ + 2f_n(x) - f_n(x_n^{-j(n)}x) - f_n(y_n^{-j(n)}x)&\end{aligned}$$

for every x, and finally

$$\begin{aligned}\int \big(2f_n(x) - f_n(x_n^{-j(n)}x) - f_n(y_n^{-j(n)}x)\big)g_n(x)dx&\\ = \int \big(f_n(x) - f_n(y_n^{-j(n)}x)\big)g_n(x)dx - \int \big(f_n(y_n^{j(n)}x) - f_n(x)\big)g_n(x)dx&\\ + \int \big(f_n(y_n^{j(n)}x) - f_n(x_n^{-j(n)}x)\big)g_n(x)dx&\\ = \int \big(f_n(y_n^{j(n)}x) - f_n(x)\big)g_n(y_n^{j(n)}x)dx - \int \big(f_n(y_n^{j(n)}x) - f(x)\big)g_n(x)dx&\\ + \int \big(f_n(y_n^{j(n)}x) - f_n(x_n^{-j(n)}x)\big)g_n(x)dx&\\ = \int \big(f_n(y_n^{j(n)}x) - f_n(x)\big)\big(g_n(y_n^{j(n)}x) - g_n(x)\big)dx&\\ + \int \big(f_n(y_n^{j(n)}x) - f_n(x_n^{-j(n)}x)\big)g_n(x)dx.&\end{aligned}$$

So if we write

$$\beta_{1n} = ln \int \big(f_n(y_n^{-1}x) - f_n(x) - f_n(y_n^{-1}x_n^{-1}x) + f_n(x_n^{-1}x)\big)g_n(x)dx,$$

$$\beta_{2n} = \frac{ln}{j(n)} \int \big(j(n)(f_n(x) - f_n(x_n^{-1}x)) - f_n(x) + f_n(x_n^{-j(n)}x)\big)g_n(x)dx,$$

$$\beta_{3n} = \frac{ln}{j(n)} \int \big(j(n)(f_n(x) - f_n(y_n^{-1}x)) - f_n(x) + f_n(y_n^{-j(n)}x)\big)g_n(x)dx,$$

$$\beta_{4n} = \frac{ln}{j(n)} \int \big(f_n(y_n^{j(n)}x) - f_n(x)\big)\big(g_n(y_n^{j(n)}x) - g_n(x)\big)dx,$$

$$\beta_{5n} = \frac{ln}{j(n)} \int \big(f_n(y_n^{j(n)}x) - f_n(x_n^{-j(n)}x)\big)g_n(x)dx$$

for $n \geq 1$, we have

$$|\beta_{1n}| + |\beta_{2n}| + |\beta_{3n}| + |\beta_{4n}| + |\beta_{5n}| \geq \beta$$

for every $n \in J$.

(h) Now $\beta_{1n} \to 0$ as $n \to \infty$. **P**

$$\begin{aligned}\Big|\int \big(f_n(y_n^{-1}x) - f_n(x) - f_n(y_n^{-1}x_n^{-1}x) + f_n(x_n^{-1}x)\big)g_n(x)dx\Big|&\\ = \Big|\int \big(f_n(y_n^{-1}x) - f_n(x)\big)g_n(x)dx - \int \big(f_n(y_n^{-1}x_n^{-1}x) - f_n(x_n^{-1}x)\big)g_n(x)dx\Big|&\\ = \Big|\int \big(f_n(y_n^{-1}x) - f_n(x)\big)g_n(x)dx - \int \big(f_n(y_n^{-1}x) - f_n(x)\big)g_n(x_nx)dx\Big|&\\ = \Big|\int \big(f_n(y_n^{-1}x) - f_n(x)\big)\big(g_n(x) - g_n(x_nx)\big)dx\Big|&\\ \leq \frac{\alpha}{n}\int |g_n(x) - g_n(x_nx)|dx&\end{aligned}$$

by (b-iii). So

$$|\beta_{1n}| \leq \alpha l \int |g_n(x) - g_n(x_n x)|dx \to 0$$

as $n \to \infty$, by (f), since surely $\langle x_n \rangle_{n\in\mathbb{N}} \to e$. **Q**

(i) Now look at β_{2n}. If we set

$$\gamma'_n = \sup\{\int |g_n(ax) - g(x)|dx : a \in D_n(U)^{j(n)}\},$$

then $\gamma'_n \to 0$ as $n \to \infty$. **P** Given $\epsilon > 0$, there is a neighbourhood W of e such that $\int |g_n(ax) - g_n(x)|dx \leq \epsilon$ whenever $n \geq 1$ and $a \in W$, as noted at the end of (f). Let p be such that $D_p(U_0) \subseteq W$. Then, for all n large enough, $pj(n) \leq n$, so that $D_n(U)^{pj(n)} \subseteq U_0$ and $D_n(U)^{j(n)} \subseteq D_p(U_0) \subseteq W$ (446D(b-v)) and $\int |g_n(ax) - g(x)|dx \leq \epsilon$ for every $a \in D_n(U)^{j(n)}$. **Q**

We have

$$\begin{aligned}
&\Big|\int \big(j(n)(f_n(x) - f_n(x_n^{-1}x)) - f_n(x) + f_n(x_n^{-j(n)}x)\big)g_n(x)dx\Big| \\
&\quad = \Big|\sum_{i=0}^{j(n)-1} \int \big(f_n(x) - f_n(x_n^{-1}x) - f_n(x_n^{-i}x) + f_n(x_n^{-i-1}x)\big)g_n(x)dx\Big| \\
&\quad = \Big|\sum_{i=0}^{j(n)-1} \int \big(f_n(x) - f_n(x_n^{-1}x)\big)g_n(x)dx - \int \big(f_n(x_n^{-i}x) - f_n(x_n^{-i-1}x)\big)g_n(x)dx\Big| \\
&\quad = \Big|\sum_{i=0}^{j(n)-1} \int \big(f_n(x) - f_n(x_n^{-1}x)\big)g_n(x)dx - \int \big(f_n(x) - f_n(x_n^{-1}x)\big)g_n(x_n^i x)dx\Big| \\
&\quad = \Big|\sum_{i=0}^{j(n)-1} \int \big(f_n(x) - f_n(x_n^{-1}x)\big)\big(g_n(x) - g_n(x_n^i x)\big)dx\Big| \\
&\quad \leq \sum_{i=0}^{j(n)-1} \int |f_n(x) - f_n(x_n^{-1}x)||g_n(x) - g_n(x_n^i x)|dx \leq j(n)\frac{\alpha}{n}\gamma'_n.
\end{aligned}$$

So

$$|\beta_{2n}| \leq l\alpha\gamma'_n \to 0$$

as $n \to \infty$. Similarly, $\langle \beta_{3n} \rangle_{n\geq 1} \to 0$.

(j) As for β_{4n}, we have

$$\int |f_n(y_n^{j(n)}x) - f_n(x)||g_n(y_n^{j(n)}x) - g_n(x)|dx \leq \frac{\alpha j(n)}{n}\gamma'_n,$$

putting the last remark in (c) and the definition of γ'_n together. So

$$|\beta_{4n}| \leq l\alpha\gamma'_n \to 0$$

as $n \to \infty$.

(k) We come at last to β_{5n}. Here, for every $n \geq 1$,

$$\begin{aligned}
\Big|\int \big(f_n(y_n^{j(n)}x) - f_n(x_n^{-j(n)}x)\big)g_n(x)dx\Big| &\leq \int \big|f_n(y_n^{j(n)}x) - f_n(x_n^{-j(n)}x)\big|g_n(x)dx \\
&\leq \gamma_n \int g_n(x)dx = \gamma_n
\end{aligned}$$

by the definition of γ_n in (c) above. So

$$|\beta_{5n}| \leq l\frac{n}{j(n)}\gamma_n \to 0$$

by the choice of the $j(n)$.

(l) Thus $\beta_{in} \to 0$ as $n \to \infty$ for every i. But this is impossible, because $0 < \beta \leq \sum_{i=1}^{5} |\beta_{in}|$ for every $n \in J$. **X**

This contradiction shows that we must be able to find a neighbourhood W of e and an $n_0 \in \mathbb{N}$ such that $xy \in D_n(V)$ whenever $n \geq n_0$, $x, y \in D_n(U)$ and $x^n y^n \in W$; as V is arbitrary, U has the property required.

***446L Definition** Let X be a topological group. A **B-sequence** in X is a non-increasing sequence $\langle V_n \rangle_{n \in \mathbb{N}}$ of closed neighbourhoods of the identity, constituting a base of neighbourhoods of the identity, such that there is some M such that for every $n \in \mathbb{N}$ the set $V_n V_n^{-1}$ can be covered by at most M left translates of V_n.

***446M Proposition** Let X be a locally compact Hausdorff topological group with no small subgroups. Then it has a B-sequence.

proof (a) For $A \subseteq X$ set $D_n(A) = \{x : x^i \in A \text{ for every } i \leq n\}$. We know from 446K that there is a compact symmetric neighbourhood U of the identity e such that whenever V is a neighbourhood of e there are an $n_0 \in \mathbb{N}$ and a neighbourhood W of e such that whenever $n \geq n_0$, $x \in D_n(U)$, $y \in D_n(U)$ and $x^n y^n \in W$, then $xy \in D_n(V)$. Shrinking U if necessary, we may suppose that U includes no subgroup of X other than e, so that there is an $r \geq 1$ such that $D_{rn}(U)^n \subseteq U$ for every $n \in \mathbb{N}$ (446H).

Let V be a closed symmetric neighbourhood of e such that $V^{2r} \subseteq U$. Then $D_n(V)^2 \subseteq D_n(U)$ for every $n \in \mathbb{N}$. **P** $D_n(V) \subseteq D_{2rn}(U)$, by 446D(b-v), so

$$(D_n(V)^2)^n \subseteq D_{2rn}(U)^{2n} \subseteq U,$$

and $D_n(V)^2 \subseteq D_n(U)$ (446D(b-v) again). **Q** Take $n_0 \in \mathbb{N}$ and a neighbourhood W of e such that whenever $n \geq n_0$, $x, y \in D_n(U)$ and $x^n y^n \in W^{-1}W$, then $xy \in D_n(V)$.

(b) Let M be so large that U can be covered by M left translates of W. Then for any $n \geq n_0$, $D_n(V)D_n(V)^{-1} = D_n(V)^2$ can be covered by M left translates of $D_n(V)$.

P Let $z_0, \ldots, z_{M-1}$ be such that $U \subseteq \bigcup_{i \leq M} z_i W$. For each $i < M$, set $A_i = \{x : x \in D_n(U), x^n \in z_i W\}$; if $A_i \neq \emptyset$ choose $x_i \in A_i$; otherwise, set $x_i = e$.

For any $y \in D_n(V)^2$, $y \in D_n(U)$, so $y^n \in U$ and there is some $i < M$ such that $y \in A_i$. In this case x_i also belongs to A_i. Now $z_i^{-1} y^n$ and $z_i^{-1} x_i^n$ both belong to W, so $x_i^{-n} y^n$ belongs to $W^{-1}W$, and $x_i^{-1} y \in D_n(V)$, by the choice of W and n_0. But this means that $y \in x_i D_n(V)$. As y is arbitrary, $D_n(V)^2 \subseteq \bigcup_{i<M} x_i D_n(V)$ is covered by M left translates of $D_n(V)$. **Q**

(d) But this means that $\langle D_{n+n_0}(V) \rangle_{n \in \mathbb{N}}$ is a B-sequence in X. (It constitutes a base of neighbourhoods of e by 446Eb, as usual.)

***446N Proposition** Let X be a locally compact Hausdorff topological group with a faithful finite-dimensional representation. Then it has a B-sequence.

proof (a) Let $\phi : X \to GL(r, \mathbb{R})$ be a faithful finite-dimensional representation. Identifying M_r with the Banach algebra $\mathsf{B} = \mathsf{B}(\mathbb{R}^r; \mathbb{R}^r)$, where $\mathbb{R}^r$ is given the Euclidean norm (4A6C), we see that $GL(r, \mathbb{R})$ is an open subset of B (4A6H). Note also that the operator norm of B is equivalent to its 'Euclidean' norm corresponding to an identification with $\mathbb{R}^{r^2}$, that is, writing $\|T\|_{HS} = \sum_{i=1}^{r} \sum_{j=1}^{r} \tau_{ij}^2$ if $T = \langle \tau_{ij} \rangle_{1 \leq i,j \leq r}$, $\| \ \|_{HS}$ is equivalent to $\| \ \|$. (See the inequalities in 262H.) In particular, all the balls $B(T, \delta) = \{S : \|S - T\| \leq \delta\}$ are closed for the Euclidean norm (4A2Lj). If we write μ_L for Lebesgue measure on B, identified with $\mathbb{R}^{r^2}$, and set $\gamma = \mu_L B(\mathbf{0}, 1)$, then $0 < \gamma < \infty$ (because $B(\mathbf{0}, 1)$ includes, and is included in, non-trivial Euclidean balls) and $\mu_L B(T, \delta) = \delta^{r^2} \gamma$ for every $T \in \mathsf{B}$, $\delta \geq 0$ (using 263A, or otherwise).

(b) We need to recall a basic inequality concerning inversion in Banach algebras. If $T \in \mathsf{B}$ and $\|T - I\| \leq \frac{1}{2}$, then T is invertible and

$$\|T^{-1} - I\| \leq \frac{\|T - I\|}{1 - \|T - I\|} \leq 1$$

(4A6H), so $\|T^{-1}\| \leq 2$.

(c) Now let V be a compact neighbourhood of the identity e of X. Let V_1 be a neighbourhood of e such that $(V_1V_1^{-1})^{-1}V_1V_1^{-1} \subseteq V$. For $\delta > 0$, set $U_\delta = \{x : x \in V, \|\phi(x) - I\| \leq \delta\}$. Then each U_δ is a compact neighbourhood of e, because ϕ is continuous. Also

$$\bigcap_{\delta>0} U_\delta = \{x : x \in V, \phi(x) = I\} = \{e\}.$$

So $\{U_\delta : \delta > 0\}$ is a base of neighbourhoods of e (4A2Gd), and there is a $\delta_1 > 0$ such that $U_{\delta_1} \subseteq V_1$; of course we may suppose that $\delta_1 \leq \frac{1}{8}$.

(d) If $\delta \leq \frac{1}{2}$ and $x \in U_\delta U_\delta^{-1}$, then $\|\phi(x) - I\| \leq 4\delta$. **P** Express x as yz^{-1} where y, $z \in U_\delta$. Then $\|\phi(z) - I\| \leq \frac{1}{2}$, so $\|\phi(z^{-1})\| \leq 2$ and

$$\|\phi(x) - I\| = \|(\phi(y) - \phi(z))\phi(z^{-1})\| \leq 2\|\phi(y) - \phi(z)\| \leq 4\delta. \textbf{ Q}$$

(e) Now if $\delta \leq \delta_1$, $U_\delta U_\delta^{-1}$ can be covered by at most $m = 17^{r^2}$ left translates of U_δ. **P?** Suppose, if possible, otherwise. Then we can choose $x_0, \dots, x_m \in U_\delta U_\delta^{-1}$ such that $x_j \notin x_i U_\delta$ whenever $i < j \leq m$. If $i < j \leq m$, then

$$x_i^{-1}x_j \in (U_\delta U_\delta^{-1})^{-1}U_\delta U_\delta^{-1}) \subseteq (V_1V_1^{-1})^{-1}V_1V_1^{-1} \subseteq V,$$

and $x_i^{-1}x_j \notin U_\delta$, so $\|\phi(x_i^{-1}x_j) - I\| > \delta$. Set $T_i = \phi(x_i)$ for each $i \leq m$; then

$$\|T_i - I\| \leq 4\delta \leq \tfrac{1}{2}$$

for each i, by (d), while

$$\delta < \|\phi(x_i^{-1}x_j) - I\| = \|T_i^{-1}T_j - I\| \leq \|T_i^{-1}\|\|T_j - T_i\| \leq 2\|T_j - T_i\|$$

whenever $i < j \leq m$. Write $B_i = B(T_i, \frac{\delta}{4})$ for each i; then all the B_i are disjoint. But also they are all included in $B(I, \frac{17\delta}{4})$, so we have

$$(17^{r^2} + 1)\big(\tfrac{\delta}{4}\big)^{r^2}\gamma \leq \big(\tfrac{17\delta}{4}\big)^{r^2}\gamma,$$

which is impossible. **XQ**

(f) Accordingly, setting $W_n = U_{2^{-n}\delta_1}$, $\langle W_n\rangle_{n\in\mathbb{N}}$ is a B-sequence in X.

***446O Theorem** Let X be a locally compact Hausdorff topological group. Then it has an open subgroup Y which has a compact normal subgroup Z such that Y/Z has no small subgroups.

proof (a) Let U be a compact neighbourhood of the identity e of X. Then there is a subgroup Y_0 of X, included in U, and a neighbourhood W_0 of e such that every subgroup of X included in W_0 is also included in Y_0.

P(i) To begin with (down to the end of (iii)) let us suppose that X is metrizable. Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a non-increasing sequence of symmetric neighbourhoods of e running over a base of neighbourhoods of e, and such that $V_1^2 \subseteq V_0 \subseteq U$. For each $n \in \mathbb{N}$, set $A_n = \{x : x^i \in V_n \text{ for every } i \in \mathbb{N}\}$.

(ii)? Suppose, if possible, that $\bigcup_{k\in\mathbb{N}} A_n^k \not\subseteq U$ for every $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let $k(n) \in \mathbb{N}$ be such that $A_n^{k(n)} \subseteq U$, $A_n^{k(n)+1} \not\subseteq U$. Then $\langle A_n\rangle_{n\in\mathbb{N}}$ and U satisfy the conditions of 446G (because $A_m \subseteq V_n$ whenever $m \geq n$, and $\{V_n : n \in \mathbb{N}\}$ is a base of neighbourhoods of e). Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ and set $Q = \limsup_{n\to\mathcal{F}} A_n^{k(n)}$.

If W is any neighbourhood of e, there is an $n \in \mathbb{N}$ such that $V_n \subseteq W$, so that $x^i \in W$ whenever $x \in \bigcup_{m\geq n} A_m$, $i \in \mathbb{N}$. Thus (ii) of 446G is not true, and Q must be a closed subgroup of X included in U and meeting the boundary of U.

By 446C, there are an $r \in \mathbb{N}$ and a continuous homomorphism $\phi : Q \to GL(r, \mathbb{R})$ such that the kernel Z of ϕ is included in $\operatorname{int} V_1$. Let $G \subseteq X$ be an open set including Z and with closure disjoint from $(X \setminus \operatorname{int} V_1) \cup \{x : x \in Q, \|\phi(x) - I\| \geq \frac{1}{6}\}$; such can be found because Z is compact and Hausdorff, therefore normal (4A2Gb). Then $Z \subseteq G$, and any subgroup Z' of Q included in $\overline{G}$ has $\|\phi(x)^i - I\| \leq \frac{1}{6}$ for every $x \in Z'$ and $i \in \mathbb{N}$, so that $Z' \subseteq Z$, by 4A6N. Set $V = \overline{G}$.

Since $V \subseteq U$ and $A_n^{k(n)+1} \not\subseteq U$ for every n, we can find $j(n) \leq k(n)$ such that $A_n^{j(n)} \subseteq V$, $A_n^{j(n)+1} \not\subseteq V$ for every n. Set $Q' = \limsup_{n\to\mathcal{F}} A_n^{j(n)}$. As before, (ii) of 446G cannot be true of Q', and Q' must be a closed subgroup of X meeting the boundary of V. Because $e \in A_n$, $A_n^{j(n)} \subseteq A_n^{k(n)}$ for every n, and $Q' \subseteq Q$ (4A2Ta); also $Q' \subseteq V$, so $Q' \subseteq Z$. But Z does not meet the boundary of V. **X**

(iii) So there is some $n \in \mathbb{N}$ such that $A_n^k \subseteq U$ for every $k \in \mathbb{N}$. Because $A_n^{-1} = A_n$, $Y_0 = \bigcup_{k\in\mathbb{N}} A_n^k$ is a subgroup of X. Any subgroup of X included in V_n is a subset of A_n so is included in Y. Thus we have a pair Y_0, $W_0 = V_n$ of the kind required, at least when X is metrizable.

(iv) Now suppose that X is σ-compact. Let U_1 be a neighbourhood of e such that $U_1^2 \subseteq U$. Then there is a closed normal subgroup X_0 of X such that $X_0 \subseteq U_1$ and $X' = X/X_0$ is metrizable (4A5S). By (i)-(iii), there are a subgroup Y_0' of X', included in the image of U_1 in X', and a neighbourhood W_0' of the identity in X' such that any subgroup of X' included in W_0' must also be included in Y_0'. Write $\pi : X \to X'$ for the canonical homomorphism and consider $Y_0 = \pi^{-1}[Y_0']$, $W_0 = \pi^{-1}[W_0']$. Then W_0 is a neighbourhood of e and Y_0 is a subgroup of X included in

$$\pi^{-1}[\pi[U_1]] = U_1X_0 \subseteq U_1^2 \subseteq U.$$

And if Z is any subgroup of X included in W_0, then $\pi[Z] \subseteq W_0'$ so $\pi[Z] \subseteq Y_0'$ and $Z \subseteq Y_0$. Thus in this case also we have the result.

(v) Finally, for the general case, observe that X has a σ-compact open subgroup X_1 (4A5El). So we can find a subgroup Y_0 of X_1, included in $U \cap X_1$, and a neighbourhood W_0 of the identity in X_1 such that any subgroup of X_1 included in W_0 is also included in Y_0. But of course Y_0 and W_0 also serve for X and U.

This completes the proof of (a). **Q**

(b) By 446C, there is a finite-dimensional representation $\phi : Y_0 \to GL(r, \mathbb{R})$, for some $r \in \mathbb{N}$, such that the kernel Z of ϕ is included in $\operatorname{int} W_0$. Let W_1 be a neighbourhood of e in X such that $\|\phi(x) - I\| \leq \frac{1}{6}$ for every $x \in W_1 \cap Y_0$, and set $W = W_1Z \cap W_0$. Note that if $x \in W \cap Y_0$, there is a $z \in Z$ such that $xz \in W_1 \cap Y_0$, so that $\|\phi(x) - I\| = \|\phi(xz) - I\| \leq \frac{1}{6}$. Of course $Z \subseteq \operatorname{int} W_1Z$, so $Z \subseteq \operatorname{int} W$.

If Y' is a subgroup of Y_0 included in W, then $\|\phi(x)^i - I\| \leq \frac{1}{6}$ for every $i \in \mathbb{N}$ and $x \in Y'$, so $Y' \subseteq Z$. Consequently any subgroup of X included in W is a subgroup of Z, since by the choice of Y_0 and W_0 it is a subgroup of Y_0.

Now let Y be the normalizer of Z in X. Z is compact, so $G = \{x : xZx^{-1} \subseteq \operatorname{int} W\}$ is open, and contains e; but also $G \subseteq Y$, because if $x \in G$ then xZx^{-1} is a subgroup of X included in W, and must be included in Z. Accordingly $Y = GY$ is open.

Since any subgroup of Y included in W is a subgroup of Z, we see that any subgroup of Y/Z included in the image of W is the trivial subgroup, and Y/Z has no small subgroups, as required.

***446P Corollary** Let X be a locally compact Hausdorff topological group. Then it has a chain $\langle X_\xi\rangle_{\xi\leq\kappa}$ of closed subgroups, where κ is an infinite cardinal, such that

(i) X_0 is open,
(ii) $X_{\xi+1}$ is a normal subgroup of X_ξ for every $\xi < \kappa$,
(iii) X_ξ is compact for $\xi \geq 1$,
(iv) $X_\xi = \bigcap_{\eta<\xi} X_\eta$ for non-zero limit ordinals $\xi \leq \kappa$,
(v) $X_\xi/X_{\xi+1}$ has a B-sequence for every $\xi < \kappa$,
(vi) $X_\kappa = \{e\}$, where e is the identity of X.

proof (a) By 446O, X has an open subgroup X_0 with a compact normal subgroup X_1 such that X_0/X_1 has no small subgroups. By 446M, X_0/X_1 has a B-sequence.

(b) Let Φ be the set of finite-dimensional representations of X_1; if we distinguish the trivial homomorphisms from X_1 to each $GL(r, \mathbb{R})$, Φ is infinite. Set $\kappa = \#(\Phi)$ and let $\langle\phi_\xi\rangle_{1\leq\xi<\kappa}$ run over Φ. For $1 \leq \xi \leq \kappa$, set

$$X_\xi = \{x : x \in X_1, \phi_\eta(x) = I \text{ for } 1 \leq \eta < \xi\}.$$

Then $\langle X_\xi\rangle_{\xi\leq\kappa}$ satisfies conditions (i)-(iv). As for (v), I have already checked the case $\xi = 0$, and if $1 \leq \xi < \kappa$, then $\phi_\xi \restriction X_\xi$ is a finite-dimensional representation of X_ξ with kernel $X_{\xi+1}$, so $X_\xi/X_{\xi+1}$ has a faithful finite-dimensional representation (446Ab), and therefore has a B-sequence (446N).

Finally, $X_\kappa = \{e\}$ by 446B; if $x \in X_1$ and $x \neq e$, there is a $\phi \in \Phi$ such that $\phi(x) \neq \phi(e)$, so that $x \notin X_\kappa$.

446X Basic exercises (a) Show that the constructions of 446B-446C produce representations which are homomorphisms from X into orthogonal groups.

(b) Let X be a locally compact Hausdorff abelian topological group. Show that for every element a of X, other than the identity, there is a finite-dimensional representation ϕ of X such that $\phi(a) \neq I$. (*Hint*: 445O.)

(c) Let $\langle X_n \rangle_{n\in\mathbb{N}}$ be any sequence of groups with their discrete topologies, and X the product topological group $\prod_{n\in\mathbb{N}} X_n$. Show that X has a B-sequence. (*Hint*: set $V_n = \{x : x(i) = e(i) \text{ for } i < n\}$.)

446Y Further exercises (a) Let X be the countable group of all bijections from $\mathbb{N}$ to itself which are products of an even number of transpositions. Give X the discrete topology, so that it is a locally compact topological group. Show that any finite-dimensional representation of X is trivial. (*Hint*: X is simple.)

(b) Let X be a compact Hausdorff topological group, $f \in C(X)$ and $\epsilon > 0$. Show that there are a finite-dimensional representation $\phi : X \to GL(r, \mathbb{R})$ and a, $b \in \mathbb{R}^r$ such that $|f(x) - (\phi(x)(a)|b)| \leq \epsilon$ for every $x \in X$.

446 Notes and comments The ideas above are extracted from the structure theory for locally compact groups, as described in Montgomery & Zippin 55. (A brisker and sometimes neater, but less complete, exposition can be found in Kaplansky 71.) The full theory goes very much deeper into the analysis of groups with no small subgroups. One of the most important ideas, hidden away in 446I and part (c) of the proof of 446K, is that of 'one-parameter subgroup'; if X is a group with no small subgroups, there are enough continuous homomorphisms from $\mathbb{R}$ to X not only to provide a great deal of information on the topological group structure of X, but even to set up a differential structure (Kaplansky 71, §II.3). For our purposes here, however, all we need to know is that groups with no small subgroups have 'B-sequences' (446L-446M), which can form the basis of a theory corresponding to Vitali's theorem and Lebesgue's Density Theorem in $\mathbb{R}^r$ (447C-447D below).

There are four essential elements in the argument here. Working from the outside, the first step is 446O: starting from a locally compact Hausdorff group X, we can find an open subgroup X_0 of X and a compact normal subgroup X_1 of X_0 such that X_0/X_1 has no small subgroups. This depends on a subtle argument based on the first key lemma, the dichotomy in 446G, which in turn uses the 'smoothing' construction in 446F and a careful analysis of inequalities involving integrals. (Naturally enough, the translation-invariance of the Haar integral is a leitmotiv of this investigation.) Note the remarkable transition in 446H. The sets $D_n(U)$ are defined solely in terms of powers, while the sets $D_n(U)^n$ involve products. We are able to obtain information about products $x_1 \dots x_n$ from information about the powers x_j^i for i, $j \leq n$.

Next, we need to find a chain of closed subnormal subgroups of X_1, decreasing to $\{e\}$, such that the quotients all have faithful finite-dimensional representations (in this context, this means that they are isomorphic to compact subgroups of $GL(r, \mathbb{R})$). This step depends on the older ideas in 446B-446C, where we use the theory of compact operators on Hilbert spaces to show that a compact group has many representations as actions on finite-dimensional subspaces of its L^2 space. (Observe that in this section I revert to real-valued, rather than complex-valued, functions.) This can be thought of as a development of the result of 445O. If X is a locally compact abelian group, its characters separate its points (cf. 446Xb); if X is compact but not necessarily abelian, its finite-dimensional representations separate its points. (But if X is neither compact nor abelian, there are further difficulties; see 446Ya.)

The other two necessary facts are that both groups with no small subgroups, and groups with faithful finite-dimensional representations, have B-sequences. The latter is reasonably straightforward (446N); any complications are due entirely to the fact that the natural measure on $GL(r, \mathbb{R})$, inherited from $\mathbb{R}^{r^2}$, is not quite invariant under multiplication, so we have to manipulate some inequalities. For groups with no small subgroups (446M) we have much more to do. The proof I give here depends on a second key lemma, 446K, refining the methods of 446G; a slightly stronger version of this result is the basis of the analysis of one-parameter subgroups in the general theory (compare Montgomery & Zippin 55, §3.8).

447 Translation-invariant liftings

I devote a section to the main theorem of Ionescu Tulcea & Ionescu Tulcea 67: a group carrying Haar measures has a translation-invariant lifting (447J). The argument uses an inductive construction of the same type as that used in §341 for the ordinary Lifting Theorem. It depends on the structure theory for locally compact groups described in §446. On the way I describe a Vitali theorem for certain metrizable groups (447C), with a corresponding density theorem (447D).

447A Liftings and lower densities Let X be a group carrying Haar measures, Σ its algebra of Haar measurable sets and $\mathfrak{A}$ its Haar measure algebra (442H, 443A).

(a) Recall that a **lifting** of $\mathfrak{A}$ is either a Boolean homomorphism $\theta : \mathfrak{A} \to \Sigma$ such that $(\theta a)^\bullet = a$ for every $a \in \mathfrak{A}$, or a Boolean homomorphism $\phi : \Sigma \to \Sigma$ such that $E \triangle \phi E$ is Haar negligible for every $E \in \Sigma$ and $\phi E = \emptyset$ whenever E is Haar negligible (341A). Such a lifting θ or ϕ is **left-translation-invariant** if $\theta((xE)^\bullet) = x(\theta E^\bullet)$ or $\phi(xE) = x(\phi E)$ for every $E \in \Sigma$ and $x \in X$. (In the notation of 443C, a lifting $\theta : \mathfrak{A} \to \Sigma$ is left-translation-invariant if $\theta(x \bullet_l a) = x\theta a$ for every $x \in X$, $a \in \mathfrak{A}$.)

The language of 341A demanded a named measure; I spoke there of a lifting of a measure space (X, Σ, μ) or of a measure μ. But (as noted in 341Lh) what the concept really depends on is a triple $(X, \Sigma, \mathcal{I})$, where Σ is an algebra of subsets of X and $\mathcal{I}$ is an ideal of Σ. Variations in the measure which do not affect the algebra of measurable sets or the null ideal are irrelevant. So, in the present context, we can speak of a 'lifting of Haar measure' without declaring which Haar measure we are using, nor even whether it is a left or right Haar measure.

(b) Now suppose that Σ_0 is a σ-subalgebra of Σ. In this case, a **partial lower density** on Σ_0 is a function $\underline{\phi} : \Sigma_0 \to \Sigma$ such that $\underline{\phi}E = \underline{\phi}F$ whenever E, $F \in \Sigma_0$ and $E \triangle F$ is negligible, $E \triangle \underline{\phi}E$ is negligible for every $E \in \Sigma_0$, $\underline{\phi}\emptyset = \emptyset$ and $\underline{\phi}(E \cap F) = \underline{\phi}E \cap \underline{\phi}F$ for all E, $F \in \Sigma_0$. (See 341C-341D.) As in (a), such a function is **left-translation-invariant** if $xE \in \Sigma_0$ and $\underline{\phi}(xE) = x(\underline{\phi}E)$ for every $x \in X$, $E \in \Sigma_0$.

447B Lemma Let X be a group carrying Haar measures and Y a subgroup of X. Write Σ_Y for the algebra of Haar measurable subsets E of X such that $EY = E$, and suppose that $\underline{\phi} : \Sigma_Y \to \Sigma_Y$ is a left-translation-invariant partial lower density. Then $G \subseteq \underline{\phi}(GY)$ for every open set $G \subseteq X$.

proof Of course GY is open (4A5Ed), so belongs to Σ_Y. Let $a \in G$ and let U be an open neighbourhood of the identity in X such that $U^{-1}Ua \subseteq G$. Then UaY is a non-empty open set, therefore not negligible (442Aa), and there is an $x \in UaY \cap \underline{\phi}(UaY)$. Express x as uay where $u \in U$ and $y \in Y$; then $ua = xy^{-1} \in Ua \cap \underline{\phi}(UaY)$, because $\underline{\phi}(UaY) \in \Sigma_Y$. So

$$a = u^{-1}ua \in u^{-1}\underline{\phi}(UaY) = \underline{\phi}(u^{-1}UaY) \subseteq \underline{\phi}(GY).$$

As a is arbitrary, $G \subseteq \underline{\phi}(GY)$.

447C Vitali's Theorem Let X be a topological group with a left Haar measure μ, and $\langle V_n \rangle_{n \in \mathbb{N}}$ a B-sequence in X (definition: 446L). If $A \subseteq X$ is any set and K_x is an infinite subset of $\mathbb{N}$ for every $x \in A$, then there is a disjoint family $\mathcal{V}$ of sets such that $A \setminus \bigcup \mathcal{V}$ is negligible and every member of $\mathcal{V}$ is of the form xV_n for some $x \in A$, $n \in K_x$.

proof (a) There is surely some r such that V_r is totally bounded for the bilateral uniformity on X (443H); replacing V_i by V_r for $i < r$ and K_x by $K_x \setminus r$ for each x, we may suppose that V_0 is totally bounded.

(b) Choose $\langle I_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. Given $I_j \subseteq X$ for $j < n$, choose a set $I_n \subseteq A$ which is maximal subject to the conditions

$$n \in K_x \text{ for every } x \in I_n,$$

$$xV_n \cap yV_j = \emptyset \text{ whenever } x \in I_n,\ j < n \text{ and } y \in I_j,$$

$$xV_n \cap yV_n = \emptyset \text{ whenever } x,\ y \in I_n \text{ are distinct.}$$

On completing the induction, set $\mathcal{V} = \{xV_n : n \in \mathbb{N},\ x \in I_n\}$; this is a disjoint family.

(c) ? Suppose, if possible, that $A \setminus \bigcup \mathcal{V}$ is not negligible. By 415B, the subspace measure on $A \setminus \bigcup \mathcal{V}$ is τ-additive and has a non-empty support. Take any a belonging to this support and set $G = \operatorname{int}(aV_0)$; then G is totally bounded and $\delta = \mu^*(G \cap A \setminus \bigcup \mathcal{V})$ is non-zero. Let M be such that every $V_n V_n^{-1}$ can be covered by M left translates of V_n, so that $\mu(V_n V_n^{-1}) \leq M\mu V_n$ for every n. Set

$$J_n = I_n \cap GV_0V_0^{-1}, \quad E_n = J_nV_n, \quad \tilde{E}_n = E_nV_n^{-1}$$

for each n.

If $n \in \mathbb{N}$ and $x \in J_n$, then $xV_n \subseteq GV_0V_0^{-1}V_0$. Accordingly (because $\langle xV_n\rangle_{x\in J_n}$ is disjoint)

$$\#(J_n)\mu V_n = \textstyle\sum_{x\in J_n} \mu(xV_n) \leq \mu(GV_0V_0^{-1}V_0) < \infty$$

because $GV_0V_0^{-1}V_0$ is totally bounded (4A5Ob). So J_n is finite and E_n is closed. Note that if $x \in I_n$ and $G \cap xV_n \neq \emptyset$, then $x \in GV_n^{-1} \subseteq GV_0V_0^{-1}$ and $x \in J_n$; so $G \cap \bigcup \mathcal{V} = G \cap \bigcup_{n\in\mathbb{N}} E_n$. Also, $\langle E_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of subsets of $GV_0V_0^{-1}V_0$; accordingly $\sum_{n=0}^{\infty} \mu E_n$ is finite, and there is an $m \in \mathbb{N}$ such that $M\sum_{n=m}^{\infty} \mu E_n < \delta$.

Observe next that, for any $x \in X$ and $n \in \mathbb{N}$,

$$\mu(xV_nV_n^{-1}) = \mu(V_nV_n^{-1}) \leq M\mu V_n = M\mu(xV_n).$$

So

$$\mu\tilde{E}_n \leq \textstyle\sum_{x\in J_n} \mu(xV_nV_n^{-1}) \leq M\sum_{x\in J_n} \mu(xV_n) = M\mu E_n$$

for each n, and $\mu(\bigcup_{n\geq m}\tilde{E}_n < \delta$.

This means that $\bigcup_{n\geq m} \tilde{E}_n$ cannot include $A \cap G \setminus \bigcup \mathcal{V}$, and there is a z belonging to

$$\begin{aligned} A \cap G \setminus (\bigcup \mathcal{V} \cup \bigcup_{n\geq m} \tilde{E}_n) &= A \cap G \setminus (\bigcup_{n\in\mathbb{N}} E_n \cup \bigcup_{n\geq m} \tilde{E}_n) \\ &= A \cap G \setminus (\bigcup_{n<m} E_n \cup \bigcup_{n\geq m} \tilde{E}_n). \end{aligned}$$

Now there must be a first $k \geq m$ such that $k \in K_z$ and $zV_k \subseteq G \setminus \bigcup_{n<m} E_n$. (This is where we use the hypothesis that $\{V_n : n \in \mathbb{N}\}$ is a base of neighbourhoods of the identity.) Since $z \in G \setminus J_k$, $z \notin I_k$, and there are $j \leq k$, $x \in I_j$ such that $zV_k \cap xV_j \neq \emptyset$. In this case, $x \in GV_0V_0^{-1}$, so $x \in J_j$. Accordingly

$$z \in xV_jV_k^{-1} \subseteq xV_jV_j^{-1} \subseteq \tilde{E}_j$$

and $j < m$; but this means that $zV_k \cap xV_j \subseteq zV_k \cap E_j$ must be empty, which is impossible. **X**

(d) Thus $\mu(A \setminus \bigcup \mathcal{V}) = 0$, and $\mathcal{V}$ is an appropriate family.

447D Theorem Let X be a topological group with a left Haar measure μ, and $\langle V_n\rangle_{n\in\mathbb{N}}$ a B-sequence in X. Then for any Haar measurable set $E \subseteq X$,

$$\lim_{n\to\infty} \frac{\mu(E \cap xV_n)}{\mu V_n} = \chi E(x)$$

for almost every $x \in X$.

proof (a) Let $\alpha < 1$, and set

$$A = \{x : x \in E, \liminf_{n\to\infty} \frac{\mu(E \cap xV_n)}{\mu V_n} < \alpha\}.$$

? Suppose, if possible, that A is not negligible. Then there is an open set G of finite measure such that $\mu^*(G \cap A) = \gamma > 0$. Let $\delta > 0$ be such that $\gamma > \alpha(\gamma + \delta) + \delta$. Take a Borel set F which is a measurable envelope of $G \cap A$ and a closed set $F_1 \subseteq G \setminus F$ such that $\mu F_1 \geq \mu(G \setminus F) - \delta$. Writing $H = G \setminus F_1$, we see that $H \cap A = G \cap A$ and

$$\mu H \leq \mu^*(H \cap A) + \delta = \gamma + \delta.$$

For each $x \in H \cap A$, set

$$K_x = \{n : xV_n \subseteq H,\ \mu(E \cap xV_n) \leq \alpha\mu V_n\}.$$

Then K_x is infinite. By Vitali's theorem in the form 447C, there is a disjoint family $\mathcal{V} \subseteq \{xV_n : x \in H \cap A,\ n \in K_x\}$ such that $(H \cap A) \setminus \bigcup \mathcal{V}$ is negligible. Since every member of $\mathcal{V}$ has non-zero measure, while μH is finite, $\mathcal{V}$ is countable. Now $\mu(\bigcup \mathcal{V}) \geq \mu^*(H \cap A)$, so $\mu(H \setminus \bigcup \mathcal{V}) \leq \delta$; also, because $\mu(E \cap V) \leq \alpha\mu V$ for every $V \in \mathcal{V}$, and $\mathcal{V}$ is disjoint,

$$\mu(E \cap \bigcup \mathcal{V}) \leq \alpha\mu(\bigcup \mathcal{V}) \leq \alpha\mu H$$

and

$$\gamma = \mu^*(A \cap H) \leq \mu(E \cap H) \leq \alpha\mu H + \delta \leq \alpha(\gamma + \delta) + \delta,$$

which is impossible, by the choice of δ. **X**

(b) As α is arbitrary,

$$\liminf_{n\to\infty} \frac{\mu(E \cap xV_n)}{\mu V_n} = 1$$

for almost every $x \in E$. Similarly,

$$\liminf_{n\to\infty} \frac{\mu(xV_n \setminus E)}{\mu V_n} = 1, \quad \limsup_{n\to\infty} \frac{\mu(E \cap xV_n)}{\mu V_n} = 0$$

for almost every $x \in X \setminus E$, so

$$\lim_{n\to\infty} \frac{\mu(E \cap xV_n)}{\mu V_n} = \chi E(x)$$

for almost every $x \in X$.

447E We need to recall some results from 443O-443Q. Let X be a locally compact Hausdorff topological group, and Y a closed subgroup of X such that the modular function of Y is the restriction to Y of the modular function of X. Let μ be a left Haar measure on X and μ_Y a left Haar measure on Y.

(a) Writing $C_k(X)$ for the space of continuous real-valued functions on X with compact support, and X/Y for the set of left cosets of Y in X with the quotient topology, we have a linear operator $T : C_k(X) \to C_k(X/Y)$ defined by writing $(Tf)(x^\bullet) = \int_Y f(xy)\mu_Y(dy)$ for every $x \in X$, $f \in C_k(X)$ (443O); moreover, $T[C_k(X)^+] = C_k(X/Y)^+$ (443Oa), and we have an invariant Radon measure λ on X/Y such that $\int Tf d\lambda = \int f d\mu$ for every $f \in C_k(X)$ (see part (b) of the proof of 443Q). If Y is a normal subgroup of X, so that X/Y is the quotient group, λ is a left Haar measure. If Y is compact and μ_Y is the Haar probability measure on Y, then λ is the image measure $\mu\pi^{-1}$, where $\pi(x) = x^\bullet = xY$ for every $x \in X$ (443Pd).

(b) If $E \subseteq X$ and $EY = Y$, then E is Haar measurable iff $\tilde{E} = \{x^\bullet : x \in E\}$ belongs to the domain of λ, and E is Haar negligible iff $\tilde{E}$ is λ-negligible (443Pc).

(c) Now suppose that X is σ-compact. Then for any Haar measurable $E \subseteq X$, $\mu E = \int g\, d\lambda$ in $[0, \infty]$, where $g(x^\bullet) = \mu_Y(Y \cap x^{-1}E)$ is defined for almost every $x \in X$ (443Pe). In particular, E is Haar negligible iff $\mu_Y(Y \cap x^{-1}E) = 0$ for almost every $x \in X$.

(d) Again suppose that X is σ-compact. Then we can extend the operator T of part (a) to an operator from $\mathcal{L}^1(\mu)$ to $\mathcal{L}^1(\lambda)$ by writing $(Tf)(x^\bullet) = \int f(xy)\mu_Y(dy)$ whenever $f \in \mathcal{L}^1(\mu)$, $x \in X$ and the integral is defined, and $\int Tf d\lambda = \int f d\mu$ for every $f \in \mathcal{L}^1(\mu)$ (443Pe). If $f \in \mathcal{L}^1(\mu)$, and we set $f_x(y) = f(xy)$ for all those $x \in X$, $y \in Y$ for which $xy \in \operatorname{dom} f$, then $Q = \{x : f_x \in \mathcal{L}^1(\mu_Y)\}$ is μ-conegligible, and $x \mapsto f_x^\bullet : Q \to L^1(\mu_Y)$ is almost continuous (443Pf).

(e) If X is σ-compact, Y is compact and μ_Y is the Haar probability measure on Y, so that λ is the image measure $\mu\pi^{-1}$, then we can apply 235I to the formula in (d) to see that

$$\iint f(xy)\mu_Y(dy)\mu(dx) = \int (Tf)(x^\bullet)\mu(dx) = \int Tf\, d\lambda = \int f d\mu$$

for every μ-integrable function f, and therefore (because μ is σ-finite) for every function f such that $\int f d\mu$ is defined in $[-\infty, \infty]$. In particular, $\mu E = \int \nu(Y \cap x^{-1}E)\mu(dx)$ for every Haar measurable set $E \subseteq X$.

447F Lemma Let X be a σ-compact locally compact Hausdorff topological group and Y a closed subgroup of X such that the modular function of Y is the restriction to Y of the modular function of X. Let Z be a compact normal subgroup of Y such that the quotient group Y/Z has a B-sequence. Let Σ_Y be the σ-algebra of those Haar measurable subsets E of X such that $EY = E$, and Σ_Z the algebra of Haar measurable sets $E \subseteq X$ such that $EZ = E$. Let $\underline{\phi} : \Sigma_Y \to \Sigma_Y$ be a left-translation-invariant partial lower density. Then there is a left-translation-invariant partial lower density $\underline{\psi} : \Sigma_Z \to \Sigma_Z$ extending $\underline{\phi}$.

proof (a) Let μ be a left Haar measure on X, μ_Y a left Haar measure on Y and μ_Z the Haar probability measure on Z; then there is a left Haar measure ν on Y/Z such that $\int g(y)\mu_Y(dy) = \int (Tg)(u)\nu(du)$ for every $g \in C_k(Y)$, where $(Tg)(y^\bullet) = \int g(yz)\mu_Z(dz)$ for every $y \in Y$ (447Ea). We are supposing that Y/Z has a B-sequence $\langle V_n\rangle_{n\in\mathbb{N}}$. It follows that there is a sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ in $C_k(Y)^+$ such that (i) $\int h_n(y)\mu_Y(dy) = 1$ for every n (ii) whenever $F \subseteq Y$ is Haar measurable (regarded as a subset of Y, that is), and $FZ = Z$, then

$$\lim_{n\to\infty} \int \chi F(by)h_n(y)\mu_Y(dy) = \chi F(b)$$

for μ_Y-almost every $b \in Y$.

P Since any subsequence of $\langle V_n\rangle_{n\in\mathbb{N}}$ is a B-sequence, and Y/Z is locally compact, we may suppose that every V_n is compact. For each $n \in \mathbb{N}$, choose a non-negative $h'_n \in C_k(Y/Z)$ such that

$$\int h'_n d\nu = 1, \quad \int |h'_n - \frac{1}{\nu V_n}\chi V_n| d\nu \leq 2^{-n}.$$

(This is possible by 416I, or otherwise.) Let $h_n \in C_k(Y)^+$ be such that $Th_n = h'_n$; then $\int h_n d\mu_Y = \int h'_n d\nu = 1$. Now if $F \subseteq Y$ is Haar measurable and $FZ = Z$, there is a Haar measurable $\tilde{F} \subseteq Y/Z$ such that $F = \{y : y^\bullet \in \tilde{F}\}$ (447Eb). Take $b \in Y$ and $n \in \mathbb{N}$. Because

$$\int \chi F(byz)h_n(yz)\mu_Z(dz) = \int \chi\tilde{F}(b^\bullet y^\bullet)h_n(yz)\mu_Z(dz) = \chi\tilde{F}(b^\bullet y^\bullet)h'_n(y^\bullet)$$

for every $y \in Y$,

$$\int \chi F(by)h_n(y)\mu_Y(dy) = \iint \chi F(byz)h_n(yz)\mu_Z(dz)\mu_Y(dy)$$

(447Ed)

$$= \int \chi\tilde{F}(b^\bullet y^\bullet)h'_n(y^\bullet)\mu_Y(dy) = \int \chi\tilde{F}(b^\bullet u)h'_n(u)\nu(du)$$

because $y \mapsto y^\bullet$ is inverse-measure-preserving for μ_Y and ν (447Ee). So

$$\begin{aligned}|\chi F(b) - \int \chi F(by)h_n(y)\mu_Y(dy)| &= |\chi\tilde{F}(b^\bullet) - \int \chi\tilde{F}(b^\bullet u)h'_n(u)\nu(du)| \\ &\leq |\chi\tilde{F}(b^\bullet) - \frac{1}{\nu V_n}\int \chi\tilde{F}(b^\bullet u)\chi V_n(u)\nu(du)| \\ &\qquad + \int |h'_n(u) - \frac{1}{\nu V_n}\chi V_n(u)|\nu(du) \\ &\leq |\chi\tilde{F}(b^\bullet) - \frac{\nu(\tilde{F}\cap b^\bullet V_n)}{\nu V_n}| + 2^{-n}.\end{aligned}$$

Since

$$\{v : v \in Y/Z, \lim_{n\to\infty} \frac{\nu(\tilde{F}\cap vV_n)}{\nu V_n} \neq \chi\tilde{F}(v)\}$$

is ν-negligible (447D), its inverse image in Y is μ_Y-negligible, so $\chi F(b) = \lim_{n\to\infty}\int \chi F(by)h_n(y)\mu_Y(dy)$ for almost every b, as claimed. **Q**

(b) We find now that if $E \in \Sigma_Z$, then $\lim_{n\to\infty}\int \chi E(xy)h_n(y)\mu_Y(dy) = \chi E(x)$ for μ-almost every $x \in X$. **P** Set $E_x = Y \cap x^{-1}E$ for $x \in X$. Because X is σ-compact, we can express E as the union of a non-decreasing sequence $\langle F^{(k)}\rangle_{k\in\mathbb{N}}$ where each $F^{(k)}$ is Haar measurable and relatively compact; set $F_x^{(k)} = Y \cap x^{-1}F^{(k)}$ for each x. In this case, for any $k \in \mathbb{N}$, $Q_k = \{x : F_x^{(k)} \in \mathrm{dom}(\mu_Y)\}$ is conegligible, and $x \mapsto (\chi F_x^{(k)})^\bullet : Q_k \to$

$L^1(\mu_Y)$ is almost continuous (447Ed), so that $x \mapsto \int \chi F_x^{(k)}(y)h_n(y)\mu_Y(dy) : Q_k \to [0,1]$ is measurable, for each n. But this means that, setting $Q = \bigcap_{k\in\mathbb{N}} Q_k$,

$$x \mapsto \int \chi E_x(y)h_n(y)\mu_Y(dy) = \lim_{k\to\infty} \int \chi F_x^{(k)}(y)h_n(y)\mu_Y(dy) : Q \mapsto [0,1]$$

is measurable, for every $n \in \mathbb{N}$. Note that if $x \in X$ and $y \in Y$ then $F_{xy}^{(k)} = y^{-1}F_x^{(k)}$, so that $Q_kY = Q_k$ for every $k \in \mathbb{N}$, and $QY = Q$.

Now consider $\tilde{Q} = \{x : x \in Q, \lim_{n\to\infty} \int \chi E(xy)h_n(y)\mu_Y(dy) = \chi E(x)\}$. This is a Haar measurable subset of X. If $a \in Q$, then

$$Y \cap a^{-1}\tilde{Q} = \{y : y \in Y, \lim_{n\to\infty} \chi E(ays)h_n(s)\mu_Y(ds) = \chi E(ay)\}$$

is μ_Y-conegligible, by the choice of the h_n in (a) above. Because Q is μ-conegligible, $Q \setminus \tilde{Q}$ is μ-negligible (447Ec) and $\tilde{Q}$ is conegligible, as required. **Q**

(c) We are now ready for the formulae at the centre of this proof. For any Haar measurable set $E \subseteq X$, $n \in \mathbb{N}$ and $\gamma < 1$, set

$$\psi_{n\gamma}(E) = \bigcup\{G \cap \underline{\phi}F : G \subseteq X \text{ is open, } F \in \Sigma_Y, \int \chi E(xy)h_n(y)\mu_Y(dy) \text{ is defined and at least } \gamma \text{ for every } x \in G \cap F\},$$

$$\underline{\psi}E = \bigcap_{\gamma<1} \bigcup_{n\in\mathbb{N}} \bigcap_{m\geq n} \psi_{m\gamma}(E).$$

The rest of the proof is devoted to checking that $\underline{\psi}\restriction\Sigma_Z$ is a left-translation-invariant partial lower density extending $\underline{\phi}$.

(d) I had better make one remark straight away. If $G \subseteq X$ is open, then $G \subseteq \underline{\phi}(GY)$ (447B). It follows that if $G \subseteq X$ is open, $F \in \Sigma_Y$ and $G \cap \underline{\phi}F \neq \emptyset$, then $G \cap F \neq \emptyset$. **P** If $a \in G \cap \underline{\phi}F$, then $a \in \underline{\phi}(GY) \cap \underline{\phi}F = \underline{\phi}(GY \cap F)$, so $GY \cap F \neq \emptyset$, that is, $G \cap F = G \cap FY^{-1}$ is non-empty. **Q** I mention this now because we need to know that the condition

$$\int \chi E(xy)h_n(y)\mu_Y(dy) \text{ is defined and at least } \gamma \text{ for every } x \in G \cap F$$

in the definition of $\psi_{n\gamma}(E)$ is never vacuously satisfied if $G \cap \underline{\phi}F \neq \emptyset$. In particular, $\psi_{n\gamma}(\emptyset) = \emptyset$ whenever $n \in \mathbb{N}$ and $0 < \gamma < 1$, so that $\underline{\psi}\emptyset = \emptyset$.

(e) If $E \subseteq X$ is Haar measurable, $n \in \mathbb{N}$ and $\epsilon > 0$, then for almost every $a \in X$ there are an open set $G \subseteq X$ and an $F \in \Sigma_Y$ such that $a \in G \cap \underline{\phi}F$ and $\int |\chi E(xy) - \chi E(ay)|h_n(y)\mu_Y(dy) \leq \epsilon$ whenever $x \in G \cap F$. **P** Take $\delta > 0$ such that $\delta(1 + 2\|h_n\|_\infty) \leq \epsilon$. Because X is σ-compact, locally compact and Hausdorff, therefore Lindelöf and regular (4A2Hd, 3A3Bb), there is a sequence $\langle f_r\rangle_{r\in\mathbb{N}}$ of continuous functions from X to $[0,1]$ such that $\int |\chi E(x) - f_r(x)|\mu(dx) \leq 2^{-r}$ for every r (415Pb). Set $g_r = |\chi E - f_r|$ for each r. For $f \in \mathcal{L}^1(\mu)$, define $\tilde{T}f \in \mathcal{L}^1(\tilde{\lambda})$ by writing $(\tilde{T}f)(x^\bullet) = \int f(xy)\mu_Y(dy)$ whenever this is defined (447Ed); we have $\int \tilde{T}f d\tilde{\lambda} = \int f d\mu$.

Set $Q = \{x : Y \cap x^{-1}E \in \operatorname{dom}\mu_Y\}$, so that $Q \in \Sigma_Y$ is conegligible (447Ec). For each r, set $\tilde{F}_r = \{u : u \in X/Y, \tilde{T}g_r(u) \text{ is defined and at least } \delta\}$; then

$$\tilde{\lambda}\tilde{F}_r \leq \tfrac{1}{\delta}\int Tg_r d\tilde{\lambda} = \tfrac{1}{\delta}\int g_r d\mu \leq 2^{-r}/\delta.$$

So $\bigcap_{r\in\mathbb{N}} \tilde{F}_r$ is $\tilde{\lambda}$-negligible. Set $F_r = \{x : x \in X, x^\bullet \in \tilde{F}_r\}$ for each r; then $F_rY = F_r$, F_r is Haar measurable (447Eb) and $\bigcap_{r\in\mathbb{N}} F_r$ is μ-negligible (also by 447Eb). Since $(X \setminus F_r)\triangle\underline{\phi}(X \setminus F_r)$ is negligible for each r, $Q_1 = Q \cap \bigcup_{r\in\mathbb{N}} \underline{\phi}(X \setminus F_r) \setminus F_r$ is conegligible. Note that $Q_1Y = Q_1$.

Suppose that $a \in Q_1$. Then there is an $r \in \mathbb{N}$ such that

$$a \in Q_1 \cap \underline{\phi}(X \setminus F_r) \setminus F_r = (Q_1 \setminus F_r) \cap \underline{\phi}(Q_1 \setminus F_r).$$

Set $F = Q_1 \setminus F_r \in \Sigma_Y$. Consider the function $x \mapsto \int f_r(xy)h_n(y)\mu_Y(dy)$. We chose h_n with compact support $L \subseteq Y$ say. If V is a compact neighbourhood of a in X, then f_r is uniformly continuous on VL for the right uniformity on X (4A5Ha, 4A2Je). There is therefore an open neighbourhood U of the identity of X such

that $|f_r(x') - f_r(x)| \leq \delta$ whenever x, $x' \in VL$ and $x'x^{-1} \in U$; of course we may suppose that $G = Ua$ is a subset of V.

Take any $x \in G \cap F$. Then if $y \in L$, we have ay, xy both in VL, while $xy(ay)^{-1} \in U$, so that $|f_r(xy) - f_r(ay)| \leq \delta$. Accordingly $|f_r(xy) - f_r(ay)|h_n(y) \leq \delta h_n(y)$ for every $y \in Y$, and

$$\int |f_r(ay) - f_r(xy)|h_n(y)\mu_Y(dy) \leq \delta.$$

At the same time, because both x and a belong to $F = Q_1 \setminus F_r$,

$$\int |\chi E(ay) - f_r(ay)|h_n(y)\mu_Y(dy) \leq \delta\|h_n\|_\infty,$$

$$\int |\chi E(xy) - f_r(xy)|h_n(y)\mu_Y(dy) \leq \delta\|h_n\|_\infty.$$

Putting these together,

$$\int |\chi E(ay) - \chi E(xy)|h_n(y)\mu_Y(dy) \leq \delta(1 + 2\|h_n\|_\infty) \leq \epsilon.$$

Thus G and F witness that a has the property required; as a is any member of the conegligible set Q_1, we have the result. **Q**

(f) If $E \in \Sigma_Z$ then $E \triangle \underline{\psi} E$ is negligible. **P** By (e), applied in turn to every n and every ϵ of the form 2^{-i}, there is a conegligible set $Q_1 \subseteq X$ such that whenever $a \in Q_1$, $n \in \mathbb{N}$ and $\epsilon > 0$ there are an open set G containing a and an $F \in \Sigma_Y$ such that $a \in \underline{\phi} F$ and $\int |\chi E(xy) - \chi E(ay)|h_n(y)\mu_Y(dy) \leq \epsilon$ for every $x \in G \cap F$. By (b), there is a conegligible set $Q_2 \subseteq X$ such that $\lim_{n\to\infty} \int \chi E(ay)h_n(y)\mu_Y(dy) = \chi E(a)$ for every $a \in Q_2$.

Suppose that $a \in Q_1 \cap Q_2 \cap E$. Let $\gamma < 1$; set $\epsilon = \frac{1}{2}(1 - \gamma)$. Because $a \in Q_2$, there is an $n \in \mathbb{N}$ such that $\int \chi E(ay)h_m(y)\mu_Y(dy) \geq 1 - \epsilon$ for every $m \geq n$. Take any $m \geq n$. Because $a \in Q_1$, there are an open set G and an $F \in \Sigma_Y$ such that $a \in G \cap \underline{\phi} F$ and $\int |\chi E(ay) - \chi E(xy)|h_m(y)\mu_Y(dy) \leq \epsilon$ whenever $x \in G \cap F$. But now

$$\int \chi E(xy)h_m(y)\mu_Y(dy) \geq 1 - 2\epsilon = \gamma$$

for every $x \in G \cap F$, so $a \in \psi_{m\gamma}(E)$. This is true for every $m \geq n$; as γ is arbitrary, $a \in \underline{\psi} E$. As a is arbitrary, $Q_1 \cap Q_2 \cap E \subseteq \underline{\psi} E$.

Now suppose that $a \in Q_1 \cap Q_2 \cap \underline{\psi} E$. Then there is an $n \in \mathbb{N}$ such that $a \in \psi_{m,3/4}(E)$ for every $m \geq n$. There is an $m \geq n$ such that $|\int \chi E(ay)h_m(y)\mu_Y(dy) - \chi E(a)| \leq \frac{1}{4}$. There are an open set G_1 and an $F_1 \in \Sigma_Y$ such that $a \in G_1 \cap \underline{\phi} F_1$ and $\int \chi E(xy)h_m(y)\mu_Y(dy) \geq \frac{3}{4}$ for every $x \in G_1 \cap F_1$. There are also an open set G_2 and an $F_2 \in \Sigma_Y$ such that $a \in G_2 \cap \underline{\phi} F_2$ and $\int |\chi E(ay) - \chi E(xy)|h_m(y)\mu_Y(dy) \leq \frac{1}{4}$ for every $x \in G_2 \cap F_2$. Set $G = G_1 \cap G_2$, $F = F_1 \cap F_2$; then $a \in G \cap \underline{\phi} F$, so $G \cap F$ is not empty ((d) above). Take $x \in G \cap F$. Then

$$\int \chi E(xy)h_m(y)\mu_Y(dy) \geq \frac{3}{4},$$

$$\int |\chi E(ay) - \chi E(xy)|h_m(y)\mu_Y(dy) \leq \frac{1}{4},$$

$$|\int \chi E(ay)h_m(y)\mu_Y(dy) - \chi E(a)| \leq \frac{1}{4},$$

so $\chi E(a) \geq \frac{1}{4}$ and $a \in E$. This shows that $Q_1 \cap Q_2 \cap \underline{\psi} E \subseteq E$.

Accordingly $E \triangle \underline{\psi} E \subseteq X \setminus (Q_1 \cap Q_2)$ is negligible, as required. **Q**

In particular, $\underline{\psi} E$ is Haar measurable for every $E \in \Sigma_Z$.

(g) If $E \in \Sigma_Z$ then $\underline{\psi} E \in \Sigma_Z$. **P** We have just seen that $\underline{\psi} E$ is Haar measurable. Take $z \in Z$, $n \in \mathbb{N}$, $\gamma < 1$ and $a \in \psi_{n\gamma}(E)$. Then there are an open $G \subseteq X$ and an $F \in \Sigma_Y$ such that $a \in G \cap \underline{\phi} F$ and $\int \chi E(xy)h_n(y)\mu_Y(dy) \geq \gamma$ for every $x \in G \cap F$. Because F and $\underline{\phi} F$ belong to Σ_Y, $az \in \underline{\phi} F$. Of course Gz is an open set containing az. If $x \in Gz \cap F$, then $xz^{-1} \in G \cap F$ and $\int \chi E(xz^{-1}y)h_n(y)\mu_Y(dy) \geq \gamma$. But

$$\chi E(xz^{-1}y) = \chi E(xy \cdot y^{-1}z^{-1}y) = \chi E(xy)$$

for every $y \in Y$, because $Z \lhd Y$ (so $z' = y^{-1}z^{-1}y \in Z$) and we are supposing that $E \in \Sigma_Z$ (so $xyz' \in E$ iff $xy \in E$). So

$$\int \chi E(xy)h_n(y)\mu_Y(dy) = \int \chi E(xz^{-1}y)h_n(y)\mu_Y(dy) \geq \gamma.$$

As x is arbitrary, Gz and F witness that $az \in \psi_{n\gamma}E$.

This shows that, for any n and γ, $az \in \psi_{n\gamma}(E)$ whenever $a \in \psi_{n\gamma}(E)$ and $z \in Z$. It follows at once that $az \in \underline{\psi}E$ whenever $a \in \underline{\psi}E$ and $z \in Z$, as claimed. **Q**

(h) For any Haar measurable $E \subseteq X$ and $c \in X$, $\underline{\psi}(cE) = c\underline{\psi}E$. **P** Suppose that $n \in \mathbb{N}$, $\gamma < 1$ and $a \in \psi_{n\gamma}(E)$. Then there are an open set $G \subseteq X$ and an $F \in \Sigma_Y$ such that $a \in G \cap \underline{\phi}F$ and $\int \chi E(xy)h_n(y)\mu_Y(dy) \geq \gamma$ for every $x \in G \cap F$. Now cG is an open set containing ca, $cF \in \Sigma_Y$ and $\underline{\phi}(cF) = c\underline{\phi}F$ contains ca, and if $x \in cG \cap cF$ we have

$$\int \chi(cE)(xy)h_n(xy)\mu_Y(dy) = \int \chi E(c^{-1}xy)h_n(xy)\mu_Y(dy) \geq \gamma$$

because $c^{-1}x \in G \cap F$. But this means that cG, cF witness that $ca \in \psi_{n\gamma}(cE)$. Since a is arbitrary, $c\psi_{n\gamma}(E) \subseteq \psi_{n\gamma}(cE)$; as n and γ are arbitrary, $c\underline{\psi}E \subseteq \underline{\psi}(cE)$. Similarly, of course, $c^{-1}\underline{\psi}(cE) \subseteq \underline{\psi}E$, so in fact $\underline{\psi}(cE) = c\underline{\psi}E$, as claimed. **Q**

(i) If E_1, $E_2 \subseteq X$ are Haar measurable and $E_1 \setminus E_2$ is μ-negligible, $\underline{\psi}E_1 \subseteq \underline{\psi}E_2$. **P** Take $n \in \mathbb{N}$, $\gamma < 1$ and $a \in \psi_{n\gamma}(E_1)$. Then there are an open set $G \subseteq X$ and an $F \in \Sigma_Y$ such that $a \in G \cap \underline{\phi}F$ and $\int \chi E_1(xy)h_n(y)\mu_Y(dy) \geq \gamma$ for every $x \in G \cap F$. Let Q be the set of those $x \in X$ such that μ_Y measures $Y \cap x^{-1}E$ and $\int \chi(E_1 \setminus E_2)(xy)\mu_Y(dy) = 0$; then $QY = Q$ is conegligible (447Ec). Now $Q \cap F \in \Sigma_Y$, and $\underline{\phi}(Q \cap F) = \underline{\phi}F$ contains a. But if $x \in G \cap Q \cap F$, $\chi E_1(xy) \leq \chi E_2(xy)$ for μ_Y-almost every y. So

$$\int \chi E_2(xy)h_n(y)\mu_Y(dy) \geq \int \chi E_1(xy)h_n(y)\mu_Y(dy) \geq \gamma.$$

Thus G and $Q \cap F$ witness that $a \in \psi_{n\gamma}(E_2)$. As a is arbitrary, $\psi_{n\gamma}(E_1) \subseteq \psi_{n\gamma}(E_2)$; as n and γ are arbitrary, $\underline{\psi}E_1 \subseteq \underline{\psi}E_2$. **Q**

In particular, (i) $\underline{\psi}E_1 = \underline{\psi}E_2$ whenever $E_1 \triangle E_2$ is negligible (ii) $\underline{\psi}E_1 \subseteq \underline{\psi}E_2$ whenever $E_1 \subseteq E_2$.

(j) If E_1, $E_2 \subseteq X$ are Haar measurable, $\underline{\psi}(E_1 \cap E_2) = \underline{\psi}E_1 \cap \underline{\psi}E_2$. **P** By (i), $\underline{\psi}(E_1 \cap E_2) \subseteq \underline{\psi}E_1 \cap \underline{\psi}E_2$. So take $a \in \underline{\psi}E_1 \cap \underline{\psi}E_2$. Let $\gamma < 1$. Set $\delta = \frac{1}{2}(1+\gamma) < 1$. Then there are n_1, $n_2 \in \mathbb{N}$ such that $a \in \psi_{m\delta}(E_1)$ for every $m \geq n_1$ and $a \in \psi_{m\delta}(E_2)$ for every $m \geq n_2$. Set $n = \max(n_1, n_2)$ and take any $m \geq n$. Then there are open sets G_1, $G_2 \subseteq X$ and F_1, $F_2 \in \Sigma_Y$ such that $a \in G_1 \cap G_2 \cap \underline{\phi}F_1 \cap \underline{\phi}F_2$ and $\int \chi E_1(xy)h_m(y)\mu_Y(dy) \geq \delta$ for every $x \in G_1 \cap F_1$, $\int \chi E_2(xy)h_m(y)\mu_Y(dy) \geq \delta$ for every $x \in G_2 \cap F_2$. Let $Q \in \Sigma_Y$ be the conegligible set of those $x \in X$ such that $\int \chi(E_1 \cap E_2)(xy)h_m(y)\mu_Y(dy)$ is defined. Set $G = G_1 \cap G_2$, $F = F_1 \cap F_2 \cap Q$; then G is open, $F \in \Sigma_Y$ and

$$\underline{\phi}F = \underline{\phi}(F_1 \cap F_2) = \underline{\phi}F_1 \cap \underline{\phi}F_2,$$

so that $a \in G \cap \underline{\phi}F$. Now take any $x \in G \cap F$. We have

$$\begin{aligned}\int (1 - \chi(E_1 \cap E_2))(xy)h_m(y)\mu_Y(dy)&\\ &\leq \int (1 - \chi E_1(xy))h_m(y)\mu(dy) + \int (1 - \chi E_2(xy))h_m(y)\mu(dy)\\ &\leq 2(1-\delta) = 1 - \gamma,\end{aligned}$$

and $\int \chi(E_1 \cap E_2)(xy)h_m(y)\mu_Y(dy) \geq \gamma$, because $\int h_m(y)\mu_Y(dy) = 1$.

As x is arbitrary, G and F witness that $a \in \psi_{m\gamma}(E_1 \cap E_2)$. And this is true for every $m \geq n$. As γ is arbitrary, $a \in \underline{\psi}(E_1 \cap E_2)$. As a is arbitrary, $\underline{\psi}E_1 \cap \underline{\psi}E_2 \subseteq \underline{\psi}(E_1 \cap E_2)$ and the two are equal. **Q**

(i) If $E \in \Sigma_Y$, $\underline{\psi}E = \underline{\phi}E$. **P** (i) Suppose $a \in \underline{\phi}E$, $n \in \mathbb{N}$ and $\gamma < 1$. Set $F = E \cap \underline{\phi}E \in \Sigma_Y$, $G = X$. Then $G \cap \underline{\phi}F = \underline{\phi}(E \cap \underline{\phi}E) = \underline{\phi}E$ contains a. Take any $x \in F$. Then

$$\int \chi E(xy)h_n(y)\mu_Y(dy) = \int h_n(y)\mu_Y(dy) = 1;$$

as x is arbitrary, so $a \in \psi_{n\gamma}(E)$. As n and γ are arbitrary, and $a \in \underline{\psi}E$; as a is arbitrary, $\underline{\phi}E \subseteq \underline{\psi}E$. (ii) Suppose $a \in \underline{\psi}E$. Then there must be some open $G \subseteq X$ and $F \in \Sigma_Y$ and $n \in \mathbb{N}$ such that $a \in G \cap \underline{\phi}F$

and $\int \chi E(xy)h_n(y)\mu_Y(dy) > 0$ for every $x \in G \cap F$. This surely implies that $G \cap F \subseteq EY = E$, so that $GY \cap F \subseteq E$. But $a \in G \subseteq \underline{\phi}(GY)$, by 447B, so

$$a \in \underline{\phi}(GY) \cap \underline{\phi}F = \underline{\phi}(GY \cap F) \subseteq \underline{\phi}E.$$

This shows that $\underline{\psi}E \subseteq \underline{\phi}E$. **Q**

(l) Thus we have assembled all the facts required to establish that $\underline{\psi}\restriction\Sigma_Z$ is a left-translation-invariant partial lower density extending $\underline{\phi}$.

447G Lemma Let X be a σ-compact locally compact Hausdorff topological group, and $\langle Y_n\rangle_{n\in\mathbb{N}}$ a non-increasing sequence of compact subgroups of X with intersection Y. Let Σ be the algebra of Haar measurable subsets of X; set $\Sigma_{Y_n} = \{E : E \in \Sigma, EY_n = E\}$ for each n, and $\Sigma_Y = \{E : E \in \Sigma, EY = E\}$. Suppose that for each $n \in \mathbb{N}$ we are given a left-translation-invariant partial lower density $\underline{\phi}_n : \Sigma_{Y_n} \to \Sigma_{Y_n}$, and that $\underline{\phi}_{n+1}$ extends $\underline{\phi}_n$ for every n. Then there is a left-translation-invariant partial lower density $\underline{\phi} : \Sigma_Y \to \Sigma_Y$ extending every $\underline{\phi}_n$.

proof (a) Fix a left Haar measure μ on X, and for each $n \in \mathbb{N}$ let ν_n be the Haar probability measure on Y_n (442Ie). As noted in 443Rb, the modular function of X must be equal to 1, and equal to the modular function of Y_n, everywhere on every Y_n.

(b) We need to know that for any $E \in \Sigma_Y$ there is an F in the σ-algebra Λ generated by $\bigcup_{n\in\mathbb{N}} \Sigma_{Y_n}$ such that $E\triangle F$ is negligible. **P** Because X is σ-compact and μ is a Radon measure (442Ac), there is a sequence $\langle K_i\rangle_{i\in\mathbb{N}}$ of compact sets such that $K_i \subseteq E$ for every i and $E \setminus \bigcup_{i\in\mathbb{N}} K_i$ is negligible. For each $i \in \mathbb{N}$, $\bigcap_{n\in\mathbb{N}} K_iY_n = K_iY$ (4A5Eh), so is included in E. Set $F = \bigcup_{i\in\mathbb{N}}\bigcap_{n\in\mathbb{N}} K_iY_n$; then F belongs to the σ-algebra generated by $\bigcup_{n\in\mathbb{N}} \Sigma_{Y_n}$, and $K_i \subseteq F \subseteq E$ for every i, so $E\triangle F$ is negligible. **Q**

(c) For each $E \in \Sigma_Y$, $n \in \mathbb{N}$ set

$$g_{En}(x) = \nu_n(Y_n \cap x^{-1}E) \text{ whenever this is defined.}$$

By 447Ee, g_{En} is defined μ-almost everywhere and is Σ-measurable. In fact g_{En} is Σ_Y-measurable, because $g_{En}(xy) = g_{En}(x)$ whenever $x \in X$, $y \in Y$ and either is defined. If $F \in \Sigma_{Y_n}$ then $g_{En}(x)\chi F(x) = \nu_n(Y_n \cap x^{-1}(E \cap F))$ whenever this is defined, which is almost everywhere; so $\int_F g_{En}d\mu = \mu(E \cap F)$, by 447Ee. If E, $E' \in \Sigma_Y$ and $E\triangle E'$ is negligible, then $g_{En} =_{\text{a.e.}} g_{E'n}$, because $g_{E\triangle E',n} = 0$ a.e.

It follows that $\langle g_{En}\rangle_{n\in\mathbb{N}} \to g_E$ μ-a.e. for every $E \in \Sigma_Y$. **P** Let $G \subseteq X$ be any non-empty relatively compact open set, and set $U = GY_0$, so that U is also a non-empty relatively compact open set and $UY_n = U$ for every n, $UY = U$. Set $\mu_U(F) = (\mu U)^{-1}\mu F$ whenever $F \in \Sigma$ and $F \subseteq U$, so that μ_U is a probability measure on U.

Writing $\Sigma^{(U)}_{Y_n}$, $\Sigma^{(U)}_Y$ and $\Lambda^{(U)}$ for the subspace σ-algebras on U generated by Σ_{Y_n}, Σ_Y and Λ, we see that if $F \in \Sigma^{(U)}_{Y_n}$ then

$$\int_F g_{En}d\mu_U = (\mu U)^{-1}\int_F g_{En}d\mu = \mu_U(E \cap F).$$

So $g_{En}\restriction U$ is a conditional expectation of $\chi(E\cap U)$ on $\Sigma^{(U)}_{Y_n}$. By Lévy's martingale theorem (275I), $\langle g_{En}\rangle_{n\in\mathbb{N}}$ converges almost everywhere on U to a conditional expectation g of $\chi(E \cap U)$ on $\Lambda^{(U)}$, because of course $\Lambda^{(U)}$ is the σ-algebra of subsets of U generated by $\bigcup_{n\in\mathbb{N}} \Sigma^{(U)}_{Y_n}$. But as there is an $F \in \Lambda$ such that $E\triangle F$ is negligible, by (b) above, g must be equal to χE almost everywhere on U.

Thus $g_{En} \to \chi E$ almost everywhere on U and therefore almost everywhere on G. As G is arbitrary, $g_{En} \to \chi E$ almost everywhere on X, by 412Jb (applied to the family $\mathcal{K}$ of subsets of relatively compact open sets). **Q**

(d) Now we can use the method of 341G, as follows. For $E \in \Sigma_Y$, $k \geq 1$ and $n \in \mathbb{N}$ set

$$H_{kn}(E) = \{x : x \in \text{dom}(g_{En}),\ g_{En}(x) \geq 1 - 2^{-k}\} \in \Sigma_{Y_n},$$

$$\tilde{H}_{kn}(E) = \underline{\phi}_n(H_{kn}(E)),$$

$$\underline{\phi}E = \bigcap_{k\geq 1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\geq n} \tilde{H}_{km}(E).$$

By the arguments of parts (e)-(i) of the proof of 341G, $\underline{\phi}$ is a lower density on Σ_Y extending every $\underline{\phi}_n$.

(e) To see that $\underline{\phi}$ is left-translation-invariant, we may argue as follows. Let $E \in \Sigma_Y$ and $a \in X$. Then, for any n,

$$g_{aE,n}(x) = \nu_n(Y_n \cap x^{-1}aE) = g_{En}(a^{-1}x)$$

for almost every x, so $aH_{kn}(E) \triangle H_{kn}(aE)$ is negligible, for every k, and

$$\tilde{H}_{kn}(aE) = \underline{\phi}_n(H_{kn}(aE)) = \underline{\phi}_n(aH_{kn}(E)) = a\underline{\phi}_n(H_{kn}(E)) = a\tilde{H}_{kn}(E).$$

Accordingly

$$\begin{aligned}\underline{\phi}(aE) &= \bigcap_{k\geq 1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\geq n}\tilde{H}_{km}(aE)\\ &= a(\bigcap_{k\geq 1}\bigcup_{n\in\mathbb{N}}\bigcap_{m\geq n}\tilde{H}_{km}(E)) = a\underline{\phi}(E),\end{aligned}$$

as required.

447H Lemma Let X be a locally compact Hausdorff topological group, and Σ the algebra of Haar measurable sets in X. Then there is a left-translation-invariant lower density $\underline{\phi} : \Sigma \to \Sigma$.

proof (a) To begin with (down to the end of (c) below) let us suppose that X is σ-compact. By 446P, there is a family $\langle X_\xi\rangle_{\xi\leq\kappa}$ of closed subgroups of X, where κ is an infinite cardinal, such that

X_0 is an open subgroup of X,

for every $\xi < \kappa$, $X_{\xi+1}$ is a normal subgroup of X_ξ and $X_\xi/X_{\xi+1}$ has a B-sequence,

for every non-zero limit ordinal $\xi \leq \kappa$, $X_\xi = \bigcap_{\eta<\xi} X_\eta$,

X_1 is compact,

$X_\kappa = \{e\}$, where e is the identity of X.

Note that for every $\xi \leq \kappa$, the modular function Δ_ξ of X_ξ is just the restriction to X_ξ of the modular function Δ of X. **P** For $\xi = 0$ this is because X_0 is an open subgroup of X (443Rd). For $\xi \geq 1$, X_ξ is compact, so Δ_ξ and $\Delta\restriction X_\xi$ are both constant with value 1, as noted in 443Rb. **Q**

(b) For each $\xi \leq \kappa$, write Σ_ξ for the σ-algebra $\{E : E \in \Sigma,\ EX_\xi = E\}$. I seek to choose inductively a family $\langle\underline{\phi}_\xi\rangle_{\xi\leq\kappa}$ such that each $\underline{\phi}_\xi : \Sigma_\xi \to \Sigma_\xi$ is a left-translation-invariant partial lower density, and $\underline{\phi}_\xi$ extends $\underline{\phi}_\eta$ whenever $\eta < \xi$.

(i) *Start* Since X_0 is an open subgroup of X, every member of Σ_0 is open, and we can start the induction by setting $\underline{\phi}_0 E = E$ for every $E \in \Sigma_0$.

(ii) *Inductive step to a successor ordinal* If we have defined $\langle\underline{\phi}_\eta\rangle_{\eta\leq\xi}$, where $\xi < \kappa$, then $X_{\xi+1} \lhd X_\xi$ is compact and $X_\xi/X_{\xi+1}$ has a B-sequence. So the conditions of 447F are satisfied and $\underline{\phi}_\xi$ has an extension to a left-translation-invariant partial lower density $\underline{\phi}_{\xi+1} : \Sigma_{\xi+1} \to \Sigma_{\xi+1}$. Of course $\underline{\phi}_{\xi+1}$ extends $\underline{\phi}_\eta$ for every $\eta \leq \xi$ because $\underline{\phi}_\xi$ does.

(iii) *Inductive step to a limit ordinal of countable cofinality* If we have defined $\langle\underline{\phi}_\eta\rangle_{\eta<\xi}$, where $\xi \leq \kappa$ is a non-zero limit ordinal of countable cofinality, let $\langle\xi_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence with supremum ξ; we may suppose that $\xi_0 \geq 1$, so that every X_{ξ_n} is compact. Then $X_\xi = \bigcap_{n\in\mathbb{N}} X_{\xi_n}$, so by 447G there is a left-translation-invariant partial lower density $\underline{\phi} : \Sigma_\xi \to \Sigma_\xi$ extending every $\underline{\phi}_{\xi_n}$, and therefore extending $\underline{\phi}_\eta$ whenever $\eta < \xi$.

(iv) *Inductive step to a limit ordinal of uncountable cofinality* Suppose we have defined $\langle\underline{\phi}_\eta\rangle_{\eta<\xi}$, where $\xi \leq \kappa$ is a limit ordinal of uncountable cofinality. Then for every $E \in \Sigma_\xi$ there are an $\eta < \xi$ and an $F \in \Sigma_\eta$ such that $E \triangle F$ is negligible. **P** (Cf. part (b) of the proof of 447G.) Because X is σ-compact, there are non-decreasing sequences $\langle K_i\rangle_{i\in\mathbb{N}}$, $\langle L_i\rangle_{i\in\mathbb{N}}$ of compact subsets of E, $X\setminus E$ respectively such that $E\setminus\bigcup_{i\in I} K_i$ and $(X\setminus E)\setminus\bigcup_{i\in\mathbb{N}} L_i$ are negligible. For each $i \in \mathbb{N}$, $K_iX_\xi \cap L_i \subseteq EX_\xi \setminus E$ is empty; by 4A5Eh, there is an $\eta_i < \xi$ such that $K_iX_{\eta_i} \cap L_i$ is empty. Set $\eta = \sup_{i\in\mathbb{N}} \eta_i$, $F = \bigcup_{i\in\mathbb{N}} K_iX_\eta$; this works. **Q**

Accordingly we have a function $\underline{\phi}_\xi : \Sigma_\xi \to \Sigma_\xi$ defined by writing $\underline{\phi}_\xi(E) = \underline{\phi}_\eta(F)$ whenever $E \in \Sigma_\xi$, $\eta < \xi$, $F \in \Sigma_\eta$ and $E \triangle F$ is negligible. **P** If $\eta \leq \eta' < \xi$ and $F \in \Sigma_\eta$, $F' \in \Sigma_{\eta'}$ are such that $E \triangle F$ and $E \triangle F'$ are both negligible, then $F \triangle F'$ is negligible so $\underline{\phi}_\eta(F) = \underline{\phi}_{\eta'}(F) = \underline{\phi}_{\eta'}(F')$. **Q** It is easy to check that $\underline{\phi}_\xi$ is a left-translation-invariant partial lower density (cf. part (d) of the proof of 341H), and of course it extends $\underline{\phi}_\eta$ for every $\eta < \xi$.

(c) On completing the induction, we see that $\Sigma_\kappa = \Sigma$, so that $\underline{\phi}_\kappa : \Sigma \to \Sigma$ is a left-translation-invariant lower density.

(d) For the general case, recall that X certainly has an open σ-compact subgroup Y say (4A5El). If Σ is the algebra of Haar measurable subsets of X, and T is the algebra of Haar measurable subsets of Y, then T is just $\Sigma \cap \mathcal{P}Y = \{E \cap Y : E \in \Sigma\}$, and the Haar negligible subsets of Y are just sets of the form $E \cap Y$ where E is a Haar negligible subset of X (443F).

Let $\underline{\psi} : \mathrm{T} \to \mathrm{T}$ be a left-translation-invariant lower density on Y. For $E \in \Sigma$ set

$$\underline{\phi}E = \{x : x \in X, e \in \underline{\psi}(Y \cap x^{-1}E)\},$$

where e is the identity of X. It is easy to check that

$$\underline{\phi}\emptyset = \emptyset,$$

$$\underline{\phi}E = \underline{\phi}F \text{ if } E \triangle F \text{ is negligible},$$

$$\underline{\phi}(E \cap F) = \underline{\phi}E \cap \underline{\phi}F \text{ for all } E, F \in \Sigma,$$

directly from the corresponding properties of $\underline{\psi}$. If $E \in \Sigma$ and $a \in X$, then

$$x \in \underline{\phi}E \iff e \in \underline{\psi}(Y \cap x^{-1}E) \iff e \in \underline{\psi}(Y \cap (ax)^{-1}aE) \iff ax \in \underline{\phi}(aE),$$

so $\underline{\phi}(aE) = a\underline{\phi}E$.

I have not yet checked that $E \triangle \underline{\phi}E$ is always negligible. But if $E \in \Sigma$, then

$$E \triangle \underline{\phi}E = \{x : e \in \underline{\psi}(Y \cap x^{-1}E) \triangle (Y \cap x^{-1}E)\},$$

so

$$\begin{aligned}(E \triangle \underline{\phi}E) \cap Y &= \{x : x \in Y, e \in \underline{\psi}(x^{-1}(E \cap Y)) \triangle x^{-1}(E \cap Y)\} \\ &= \{x : x \in Y, e \in x^{-1}\underline{\psi}(E \cap Y) \triangle x^{-1}(E \cap Y)\} = \underline{\psi}(E \cap Y) \triangle (E \cap Y)\end{aligned}$$

is negligible. Moreover, for any $a \in X$,

$$(E \triangle \underline{\phi}E) \cap aY = a((a^{-1}E \triangle \underline{\phi}(a^{-1}E)) \cap Y)$$

because $\underline{\phi}$ is translation-invariant, so $(E \triangle \underline{\phi}E) \cap aY$ is negligible. Since $\{aY : a \in X\}$ is an open cover of X, $E \triangle \underline{\phi}E$ is negligible (412Jc). In particular, $\underline{\phi}E \in \Sigma$. So $\underline{\phi} : \Sigma \to \Sigma$ is a left-translation-invariant lower density, as required.

447I Theorem (Ionescu Tulcea & Ionescu Tulcea 67) Let X be a locally compact Hausdorff topological group. Then it has a left-translation-invariant lifting for its Haar measures.

proof (Cf. 345B-345C.) Write Σ for the algebra of Haar measurable subsets of X, and let $\underline{\phi} : \Sigma \to \Sigma$ be a left-translation-invariant lower density (447H). Let $\phi_0 : \Sigma \to \Sigma$ be any lifting such that $\phi_0 E \supseteq \underline{\phi}E$ for every $E \in \Sigma$ (341Jb). For $E \in \Sigma$, set

$$\phi E = \{x : e \in \phi_0(x^{-1}E)\},$$

where e is the identity of X. It is easy to check that $\phi : \Sigma \to \mathcal{P}X$ is a Boolean homomorphism. Also

$$x \in \underline{\phi}E \implies e \in \underline{\phi}(x^{-1}E) \implies e \in \phi_0(x^{-1}E) \implies x \in \phi E.$$

So ϕ is a lifting (341I). Finally, ϕ is left-translation-invariant by the argument used in (d) of the proof of 447H (and also in (e) of the proof of 345B).

447J Corollary Let X be any topological group carrying Haar measures. Then it has a left-translation-invariant lifting for its Haar measures.

proof Let μ be a left Haar measure on X. By 443L, we have a locally compact Hausdorff topological group Z and a continuous homomorphism $f : X \to Z$, inverse-measure-preserving for μ and an appropriate left Haar measure ν on Z, such that for every E in the domain Σ of μ there is an F in the domain T of ν such that $f^{-1}[F] \subseteq E$ and $E \setminus f^{-1}[F]$ is negligible. Let ψ be a left-translation-invariant lifting for ν. For $E \in \Sigma$, we may define $\phi E \in \Sigma$ by saying that $\phi E = f^{-1}[\psi F]$ whenever $F \in \mathrm{T}$ and $E \triangle f^{-1}[F]$ is negligible. **P** By the construction of Z and f, there is always an $F \in \mathrm{T}$ such that $E \triangle f^{-1}[F]$ is negligible, so that there is a candidate for ϕE. If F, $F' \in \mathrm{T}$ are such that $E \triangle f^{-1}[F]$ and $E \triangle f^{-1}[F]$ are both negligible, then $f^{-1}[F \triangle F']$ is negligible, so $F \triangle F'$ is negligible (because f is inverse-measure-preserving), $\psi F = \psi F'$ and $f^{-1}[\psi F] = f^{-1}[\psi F']$. Thus ϕE is uniquely defined for every $E \in \Sigma$. **Q**

It is now easy to check that ϕ is a lifting for μ (cf. 341Xd), and ϕ is left-translation-invariant because f is a group homomorphism and ψ is left-translation-invariant.

447X Basic exercises >(a) Let $X = \mathbb{R} \times \{-1, 1\}$, with the usual topology, and define a multiplication on X by setting $(x, \delta)(y, \epsilon) = (x + \delta y, \delta\epsilon)$. Show that X is a locally compact topological group. Show that there is no lifting of the Haar measure algebra of X which is both left- and right-translation-invariant. (*Hint*: 345Xc.)

(b) Let X be a topological group carrying Haar measures which has a B-sequence. Show that it has a B-sequence $\langle V_n \rangle_{n \in \mathbb{N}}$ such that $\sup_{n \in \mathbb{N}} \mu(V_n V_n^{-1})/\mu V_n$ is finite for any Haar measure μ on X, whether left or right.

(c) Let X be a topological group with a left Haar measure μ, and $\langle V_n \rangle_{n \in \mathbb{N}}$ a B-sequence for X. Show that if $f \in \mathcal{L}^0(\mu)$ is locally integrable, then $f(x) = \lim_{n \to \infty} \frac{1}{\mu V_n} \int_{xV_n} f d\mu$ for almost every x.

447Y Further exercises (a) Describe a compact Hausdorff topological group such that its Haar measure has no lifting which is both left- and right-translation-invariant.

(b) Let (X, Σ, μ) be a measure space, with measure algebra $\mathfrak{A}$, and $\underline{\theta} : \mathfrak{A} \to \Sigma$ a lower density. Show that we have a function $q : L^\infty(\mathfrak{A})^+ \to L^\infty(\Sigma)$ such that $\{x : q(u)(x) > \alpha\} = \bigcup_{\beta > \alpha} \underline{\theta}[\![u > \beta]\!]$ for every $\alpha \geq 0$ and $u \in L^\infty(\mathfrak{A})^+$. Show that $q(u)^\bullet = u$, $q(\alpha u) = \alpha q(u)$, $q(u \wedge v) = q(u) \wedge q(v)$ and $q(\chi a) = \chi(\underline{\theta} a)$ for every u, $v \in L^\infty(\mathfrak{A})^+$, $\alpha \geq 0$ and $a \in \mathfrak{A}$.

447 Notes and comments The structure of the proof of 447I is exactly that of the proof of the ordinary Lifting Theorem in §341; the lifting is built from a lower density which is constructed inductively on a family of sub-σ-algebras. To get a translation-invariant lifting it is natural to look for a translation-invariant lower density, and a simple trick (already used in §345) ensures that this is indeed enough. The refinements we need here are dramatic but natural. To make the final lower density $\underline{\phi}$ (in 447H) translation-invariant, it is clearly sensible (if we can do it) to keep all the partial lower densities $\underline{\phi}_\xi$ translation-invariant. This means that their domains Σ_ξ should be translation-invariant. It does not quite follow that they have to be of the form $\Sigma_\xi = \{E : EX_\xi = X_\xi\}$ for closed subgroups X_ξ, but if we look at the leading example of $\{0, 1\}^I$ (345C) this is also a reasonable thing to try first. So now we have to consider what extra hypotheses will be needed to make the induction work. The inductive step to limit ordinals of uncountable cardinality remains elementary, at least if the X_ξ are compact (part (b-iv) of the proof of 447H). The inductive step to limit ordinals of countable cofinality (447G) is harder, but can be managed with ideas already presented. Indeed, compared with the version in 341G, we have the advantage of a formula for the auxiliary functions g_{En}, which is very helpful when we come to translation-invariance. We have to do something about the fact that we are no longer working with a probability space – that is the point of the μ_U in part (c) of the proof of 447G. (Another expression of the manoeuvre here is in 369Xq.)

Where we do need a new idea is in the inductive step to a successor ordinal. If $\Sigma_{\xi+1}$ is to be translation-invariant, it must be much bigger than the σ-algebra generated by $\Sigma_\xi \cup \{E\}$, as discussed in 341F. To make the step a small one (and therefore presumably easier), we want $X_{\xi+1}$ to be a large subgroup of X_ξ in some

sense; as it turns out, a helpful approach is to ask for $X_{\xi+1}$ to be a normal subgroup of X_ξ and for $X_\xi/X_{\xi+1}$ to be small. At this point we have to know something of the structure theory of locally compact topological groups. The right place to start is surely the theory of compact Hausdorff groups. Such a group X actually has a continuous decreasing chain $\langle X_\xi\rangle_{\xi\leq\kappa}$ of closed normal subgroups, from $X_0 = X$ to $X_\kappa = \{e\}$, such that all the quotients $X_\xi/X_{\xi+1}$ are Lie groups. I do not define 'Lie group' here, because for our purposes it is enough to know that the quotients have faithful finite-dimensional representations, and therefore have 'B-sequences' in the sense of 446L. Having identified this as a relevant property, it is not hard to repeat arguments from §221 and §261 to prove versions of Vitali's theorem and Lebesgue's Density Theorem in such groups (447C-447D). This will evidently provide translation-invariant lower densities for groups of this special type, just as Lebesgue lower density is a translation-invariant lower density on $\mathbb{R}^r$ (345B).

Of course we still have to find a way of combining this construction with a translation-invariant lower density on Σ_ξ to produce a translation-invariant lower density on $\Sigma_{\xi+1}$, and this is what I do in 447F. The argument I offer is essentially that of Ionescu Tulcea & Ionescu Tulcea 67, §7, and is the deepest part of this section.

For compact groups, these ideas are all we need, and indeed the step to a limit ordinal of countable cofinality is a little easier, since we have a Haar probability measure on the whole group. The next step, to general σ-compact locally compact groups, demands much deeper ideas from the structure theory, but from the point of view of the present section the modifications are minor. The subgroups X_ξ are now not always normal subgroups of X, which means that we have to be more careful in the description of the quotient spaces X/X_ξ (they must consist of *left* cosets), and we have to watch the modular functions of the X_ξ in order to be sure that there are invariant measures on the quotients. An extra obstacle at the beginning is that we may have to start the chain with a proper subgroup X_0 of X, but since X_0 can be taken to be open, it is pretty clear that this will not be serious, and in fact it gives no trouble (part (b-i) of the proof of 447H). For $\xi \geq 1$, the X_ξ are compact, so the inductive steps to limit ordinals are nearly the same.

The step to a general locally compact Hausdorff topological group (part (d) of the proof of 447H) is essentially elementary. And finally I note that the whole thing applies to general topological groups with Haar measures (447J), for the usual reasons. There is an implicit challenge here: find expressions of the arguments used in this section which will be valid in the more general context. The measure-theoretic part of such a programme might be achievable, but I do not see any hope of a workable structure theory to match that of §446 which does not use 443L or something like it.

448 Invariant measures on Polish spaces

In this section I set out an interesting result concerning measures on Polish spaces which are invariant under actions of Polish groups. In contrast to §441, we no longer have a strong general existence theorem for such measures, but instead have a natural necessary and sufficient condition in terms of countable dissections: there is an invariant probability measure on X iff there is no countable dissection of X into Borel sets which can be rearranged, by the action of the group, into two copies of X (448P).

The principal ideas needed here have already been set out in §394, and in many of the proofs I allow myself to direct you to the corresponding arguments there rather than write the formulae out again. I do not think you need read through §394 before embarking on this section; I will try to give sufficiently detailed references so that you can take them one paragraph at a time, and many of the arguments referred to are in any case elementary. But unless you are already familiar with this topic, you will need a copy of §394 to hand to fully follow the proofs below.

448A Definition (Compare 394A.) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. For $a, b \in \mathfrak{A}$ I will say that an isomorphism $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ between the corresponding principal ideals belongs to the **countably full local semigroup generated by** G if there are a countable partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}_a$ and a family $\langle\pi_i\rangle_{i\in I}$ in G such that $\phi c = \pi_i c$ whenever $i \in I$ and $c \subseteq a_i$. If such an isomorphism exists I will say that a and b are **G-σ-equidecomposable**.

I write $a \preccurlyeq^\sigma_G b$ to mean that there is a $b' \subseteq b$ such that a and b' are G-σ-equidecomposable.

As in §394, I will say that a function f with domain $\mathfrak{A}$ is **G-invariant** if $f(\pi a) = f(a)$ for every $a \in \mathfrak{A}$, $\pi \in G$.

I have expressed these definitions, and most of the work below, in terms of abstract Dedekind σ-complete Boolean algebras. The applications I have in mind for this section are to σ-algebras of sets. If you have already worked through §394, the version here should come very easily; but even if you have not, I think that the extra abstraction clarifies some of the ideas.

448B I begin with results corresponding to 394B-394D; there is hardly any difference, except that we must now occasionally pause to check that a partition of unity is countable.

Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write G^*_σ for the countably full local semigroup generated by G.

(a) If $a, b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ belongs to G^*_σ, then $\phi^{-1} : \mathfrak{A}_b \to \mathfrak{A}_a$ also belongs to G^*_σ.

(b) Suppose that $a, b, a', b' \in \mathfrak{A}$ and that $\phi : \mathfrak{A}_a \to \mathfrak{A}_{a'}$, $\psi : \mathfrak{A}_b \to \mathfrak{A}_{b'}$ belong to G^*_σ. Then $\psi\phi \in G^*_\sigma$; its domain is $\mathfrak{A}_c$ where $c = \phi^{-1}(b \cap a')$, and its set of values is $\mathfrak{A}_{c'}$ where $c' = \psi(b \cap a')$.

(c) If $a, b \in \mathfrak{A}$ and $\phi : \mathfrak{A}_a \to \mathfrak{A}_b$ belongs to G^*_σ, then $\phi \restriction \mathfrak{A}_c \in G^*_\sigma$ for any $c \subseteq a$.

(d) Suppose that $a, b \in \mathfrak{A}$ and that $\psi : \mathfrak{A}_a \to \mathfrak{A}_b$ is an isomorphism such that there are a countable partition of unity $\langle a_i\rangle_{i\in I}$ in $\mathfrak{A}_a$ and a family $\langle\phi_i\rangle_{i\in I}$ in G^*_σ such that $\psi c = \phi_i c$ whenever $i \in I$ and $c \subseteq a_i$. Then $\psi \in G^*_\sigma$.

proof (a) As 394Bb.

(b) As 394Bc.

(c) As 394Bd.

(d) For each $i \in I$, let $\langle a_{ij}\rangle_{j\in J(i)}$, $\langle\pi_{ij}\rangle_{j\in J(i)}$ witness that $\phi_i \in G^*_\sigma$; then $\langle a_i \cap a_{ij}\rangle_{i\in I, j\in J(i)}$ and $\langle\pi_{ij}\rangle_{i\in I,j\in J(i)}$ witness that $\psi \in G^*_\sigma$.

448C Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Write G^*_σ for the countably full local semigroup generated by G.

(a) For $a, b \in \mathfrak{A}$, $a \preccurlyeq^\sigma_G b$ iff there is a $\phi \in G^*_\sigma$ such that $a \in \operatorname{dom}\phi$ and $\phi a \subseteq b$.

(b)(i) $\preccurlyeq^\sigma_G$ is transitive and reflexive;

(ii) if $a \preccurlyeq^\sigma_G b$ and $b \preccurlyeq^\sigma_G a$ then a and b are G-σ-equidecomposable.

(c) G-σ-equidecomposability is an equivalence relation on $\mathfrak{A}$.

(d) If $\langle a_i\rangle_{i\in I}$ and $\langle b_i\rangle_{i\in I}$ are countable families in $\mathfrak{A}$, of which $\langle b_i\rangle_{i\in I}$ is disjoint, and $a_i \preccurlyeq^\sigma_G b_i$ for every $i \in I$, then $\sup_{i\in I} a_i \preccurlyeq^\sigma_G \sup_{i\in I} b_i$.

proof The arguments of 394C apply unchanged, calling on 448B in place of 394B.

448D Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. Then the following are equiveridical:

(i) there is an $a \neq 1$ such that a is G-σ-equidecomposable with 1;

(ii) there is a disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all G-σ-equidecomposable;

(iii) there are non-zero G-σ-equidecomposable $a, b, c \in \mathfrak{A}$ such that $a \cap b = 0$ and $a \cup b \subseteq c$;

(iv) there are G-σ-equidecomposable $a, b \in \mathfrak{A}$ such that $a \subset b$.

proof As 394D.

448E Definition (Compare 394E.) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$. I will say that G is **countably non-paradoxical** if the statements of 448D are false; that is, if one of the following equiveridical statements is true:

(i) if a is G-σ-equidecomposable with 1 then $a = 1$;

(ii) there is no disjoint sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ of non-zero elements of $\mathfrak{A}$ which are all G-σ-equidecomposable;

(iii) there are no non-zero G-σ-equidecomposable $a, b, c \in \mathfrak{A}$ such that $a \cap b = 0$ and $a \cup b \subseteq c$;

(iv) if $a \subseteq b \in \mathfrak{A}$ and a, b are G-σ-equidecomposable then $a = b$.

448F We now come to one of the points where we need to find a new path because we are looking at algebras which need not be Dedekind complete. Provided the original group G is *countable*, we can still follow the general line of §394, as follows.

Lemma (Compare 394G.) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a countable subgroup of $\operatorname{Aut}\mathfrak{A}$. Let $\mathfrak{C}$ be the fixed-point subalgebra of G.

(a) For any $a \in \mathfrak{A}$, $\operatorname{upr}(a, \mathfrak{C})$ (314V) is defined, and is given by the formula

$$\operatorname{upr}(a, \mathfrak{C}) = \sup\{\pi a : \pi \in G\}.$$

(b) If G^*_σ is the countably full local semigroup generated by G, then $\phi(a \cap c) = \phi a \cap c$ whenever $\phi \in G^*_\sigma$, $a \in \operatorname{dom}\phi$ and $c \in \mathfrak{C}$.

(c) $\operatorname{upr}(\phi a, \mathfrak{C}) = \operatorname{upr}(a, \mathfrak{C})$ whenever $\phi \in G^*_\sigma$ and $a \in \operatorname{dom}\phi$; consequently, $\operatorname{upr}(a, \mathfrak{C}) \subseteq \operatorname{upr}(b, \mathfrak{C})$ whenever $a \preccurlyeq^\sigma_G b$.

(d) If $a \preccurlyeq^\sigma_G b$ and $c \in \mathfrak{C}$ then $a \cap c \preccurlyeq^\sigma_G b \cap c$. So $a \cap c$ and $b \cap c$ are G-σ-equidecomposable whenever a and b are G-σ-equidecomposable and $c \in \mathfrak{C}$.

proof (a) As remarked in 394Ga, $\mathfrak{C}$ is order-closed. Because G is countable and $\mathfrak{A}$ is Dedekind σ-complete, $c^* = \sup\{\pi a : \pi \in G\}$ is defined in $\mathfrak{A}$. If $c \in \mathfrak{C}$, then

$$\begin{aligned} a \subseteq c &\iff \pi a \subseteq \pi c \text{ for every } \pi \in G \\ &\iff \pi a \subseteq c \text{ for every } \pi \in G \\ &\iff c^* \subseteq c, \end{aligned}$$

so $c^* = \inf\{c : a \subseteq c \in \mathfrak{C}\}$, taking the infimum in $\mathfrak{A}$, as required by the definition of $\operatorname{upr}(a, \mathfrak{C})$.

(b) Suppose that $\langle a_i\rangle_{i\in I}$, $\langle \pi_i\rangle_{i\in I}$ witness that $\phi \in G^*_\sigma$. Then

$$\phi(a \cap c) = \sup_{i\in I} \pi_i(a_i \cap a \cap c) = \sup_{i\in I} \pi_i(a_i \cap a) \cap c = \phi a \cap c.$$

(c) For $c \in \mathfrak{C}$,

$$a \subseteq c \iff a \cap c = a \iff \phi(a \cap c) = \phi a \iff \phi a \cap c = \phi a \iff \phi a \subseteq c.$$

(d) There is a $\phi \in G^*_\sigma$ such that $\phi a \subseteq b$; now

$$a \cap c \preccurlyeq^\sigma_G \phi(a \cap c) = \phi a \cap c \subseteq b \cap c.$$

448G With this support, we can now continue with the ideas of 394H-394L, adding at each step the hypothesis 'G is countable' to compensate for the weakening of the hypotheses '$\mathfrak{A}$ is Dedekind complete, G is fully non-paradoxical' to '$\mathfrak{A}$ is Dedekind σ-complete, G is countably non-paradoxical'.

Lemma (Compare 394H.) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Write $\mathfrak{C}$ for the fixed-point subalgebra of G. Take any $a, b \in \mathfrak{A}$. Then $c_0 = \sup\{c : c \in \mathfrak{C}, a \cap c \preccurlyeq^\sigma_G b\}$ is defined in $\mathfrak{A}$ and belongs to $\mathfrak{C}$; $a \cap c_0 \preccurlyeq^\sigma_G b$ and $b \setminus c_0 \preccurlyeq^\sigma_G a$.

proof Let $\langle \pi_n\rangle_{n\in\mathbb{N}}$ be a sequence running over G. Define $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively, setting

$$a_n = (a \setminus \sup_{i<n} a_i) \cap \pi_n^{-1}(b \setminus \sup_{i<n} b_i), \quad b_n = \pi_n a_n.$$

Then $\langle a_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}_a$ and $\langle b_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathfrak{A}_b$, and $\sup_{n\in\mathbb{N}} a_n$ is G-σ-equidecomposable with $\sup_{n\in\mathbb{N}} b_n$. Set $a' = a \setminus \sup_{n\in\mathbb{N}} a_n$, $b' = b \setminus \sup_{n\in\mathbb{N}} b_n$, $c_0 = 1 \setminus \operatorname{upr}(a', \mathfrak{C})$. Then

$$a \cap c_0 \subseteq \sup_{n\in\mathbb{N}} a_n \preccurlyeq^\sigma_G b.$$

Now $b' \subseteq c_0$. **P?** Otherwise, because $c_0 = 1 \setminus \sup_{n\in\mathbb{N}} \pi_n a'$, there must be an $n \in \mathbb{N}$ such that $\pi_n a' \cap b' \neq 0$. But in this case $d = a' \cap \pi_n^{-1} b' \neq 0$, and we have

$$d \subseteq (a \setminus \sup_{i<n} a_i) \cap \pi_n^{-1}(b \setminus \sup_{i<n} b_i),$$

so that $d \subseteq a_n$, which is absurd. **XQ** Consequently

$$b \setminus c_0 \subseteq b \setminus b' = \sup_{n\in\mathbb{N}} b_n \preccurlyeq^\sigma_G a.$$

Now take any $c \in \mathfrak{C}$ such that $a \cap c \preccurlyeq^\sigma_G b$, and consider $c' = c \setminus c_0$. Then $b' \cap c' = 0$, that is, $b \cap c' = \sup_{n\in\mathbb{N}} b_n \cap c'$, which is G-σ-equidecomposable with $\sup_{n\in\mathbb{N}} a_n \cap c' = (a \setminus a') \cap c'$. But now

$$a \cap c' \preccurlyeq^\sigma_G b \cap c' \preccurlyeq^\sigma_G (a \cap c') \setminus (a' \cap c');$$

because G is countably non-paradoxical, $a' \cap c'$ must be 0, that is, $c' \subseteq c_0$ and $c' = 0$. So c_0 has the required properties.

448H Lemma (Compare 394I.) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, not $\{0\}$, and G a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$. Let $\mathfrak{C}$ be the fixed-point subalgebra of G. Suppose that a, $b \in \mathfrak{A}$ and that $\operatorname{upr}(a, \mathfrak{C}) = 1$. Then there are non-negative u, $v \in L^0 = L^0(\mathfrak{C})$ such that

$$\begin{aligned}[\![u \geq n]\!] = \max\{c : c \in \mathfrak{C},\ &\text{there is a disjoint family } \langle d_i\rangle_{i<n} \\ &\text{such that } a \cap c \preccurlyeq^\sigma_G d_i \subseteq b \text{ for every } i < n\},\end{aligned}$$

$$\begin{aligned}[\![v \leq n]\!] = \max\{c : c \in \mathfrak{C},\ &\text{there is a family } \langle d_i\rangle_{i<n} \\ &\text{such that } d_i \preccurlyeq^\sigma_G a \text{ for every } i < n \text{ and } b \cap c \subseteq \sup_{i<n} d_i\}\end{aligned}$$

for every $n \in \mathbb{N}$. Moreover, we have
(i) $[\![u \in \mathbb{N}]\!] = [\![v \in \mathbb{N}]\!] = 1$,
(ii) $[\![v > 0]\!] = \operatorname{upr}(b, \mathfrak{C})$,
(iii) $u \leq v \leq u + \chi 1$.

proof The argument of 394I applies unchanged, except that every $\preccurlyeq^\tau_G$ must be replaced with a $\preccurlyeq^\sigma_G$, and we use 448F and 448G in place of 394G and 394H. $\mathfrak{C}$ is Dedekind σ-complete because it is order-closed in the Dedekind σ-complete algebra $\mathfrak{A}$ (314Eb).

448I Notation (Compare 394J.) In the context of 448H, I will write $\lfloor b : a \rfloor$ for u, $\lceil b : a \rceil$ for v.

448J Lemma (Compare 394K-394L.) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, not $\{0\}$, and G a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point subalgebra $\mathfrak{C}$. Suppose that a, a_1, a_2, b, b_1, $b_2 \in \mathfrak{A}$ and that

$$\operatorname{upr}(a, \mathfrak{C}) = \operatorname{upr}(a_1, \mathfrak{C}) = \operatorname{upr}(a_2, \mathfrak{C}) = 1.$$

Then
(a) $\lfloor 0 : a \rfloor = \lceil 0 : a \rceil = 0$ and $\lfloor 1 : a \rfloor \geq \chi 1$.
(b) If $b_1 \preccurlyeq^\sigma_G b_2$ then $\lfloor b_1 : a \rfloor \leq \lfloor b_2 : a \rfloor$ and $\lceil b_1 : a \rceil \leq \lceil b_2 : a \rceil$.
(c) $\lceil b_1 \cup b_2 : a \rceil \leq \lceil b_1 : a \rceil + \lceil b_2 : a \rceil$.
(d) If $b_1 \cap b_2 = 0$, then $\lfloor b_1 : a \rfloor + \lfloor b_2 : a \rfloor \leq \lfloor b_1 \cup b_2 : a \rfloor$.
(e) $\lfloor b : a_2 \rfloor \geq \lfloor b : a_1 \rfloor \times \lfloor a_1 : a_2 \rfloor$, $\lceil b : a_2 \rceil \leq \lceil b : a_1 \rceil \times \lceil a_1 : a_2 \rceil$.

proof As in 394K-394L.

448K For the result corresponding to 394Mb, we again need to find a new approach; I deal with it by adding a further hypothesis to the list which has already accreted.

Definition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point algebra $\mathfrak{C}$. I will say that G has the **σ-refinement property** if for every $a \in \mathfrak{A}$ there is a $d \subseteq a$ such that $d \preccurlyeq^\sigma_G a \setminus d$ and $a' = a \setminus \operatorname{upr}(d, \mathfrak{C})$ is a relative atom over $\mathfrak{C}$, that is, every $b \subseteq a'$ is expressible as $a' \cap c$ for some $c \in \mathfrak{C}$.

(If we replace $\preccurlyeq^\sigma_G$ with $\preccurlyeq^\tau_G$, as used in §394, we see that 394Ma could be read as 'if $\mathfrak{A}$ is a Dedekind complete Boolean algebra, then any subgroup of $\operatorname{Aut}\mathfrak{A}$ has the τ-refinement property'.)

448L I give the principal case in which the 'σ-refinement property' just defined arises.

Proposition Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra with countable Maharam type (definition: 331F). Then any countable subgroup of $\operatorname{Aut}\mathfrak{A}$ has the σ-refinement property.

proof (a) Let E be a countable subset of $\mathfrak{A}$ which τ-generates $\mathfrak{A}$, and $\mathfrak{E}$ the subalgebra of $\mathfrak{A}$ generated by E; then $\mathfrak{E}$ is countable (331Gc), and the smallest order-closed subset of $\mathfrak{A}$ including $\mathfrak{E}$ is a subalgebra of $\mathfrak{A}$ (313Fc), so must be $\mathfrak{A}$ itself.

(b) Suppose that $b \in \mathfrak{A} \setminus \{0\}$ and $\pi \in \operatorname{Aut}\mathfrak{A}$ are such that $b \cap \pi b = 0$. Then there is an $e \in \mathfrak{E}$ such that $b \cap e \setminus \pi e \neq 0$. **P?** Otherwise, set

$$D = \{d : d \in \mathfrak{A},\ b \cap d \setminus \pi d = 0\}.$$

Then $\mathfrak{E} \subseteq D$, but $b \notin D$. So D cannot be order-closed. **case 1** If $D_0 \subseteq D$ is a non-empty upwards-directed set with supremum $d_0 \notin D$, then $b \cap d_0 \setminus \pi d_0 \neq 0$, so there is a $d \in D_0$ such that $b \cap d \setminus \pi d_0 \neq 0$; but now $d \notin D$, which is impossible. **case 2** If $D_0 \subseteq D$ is a non-empty downwards-directed subset of D with infimum $d_0 \notin D$, then $b \cap d_0 \setminus \pi d_0 \neq 0$. But π is order-continuous, so there is a $d \in D_0$ such that $b \cap d_0 \setminus \pi d \neq 0$; and now $d \notin D$, which is impossible. Thus in either case we have a contradiction. **XQ**

(c) Now let G be a countable subgroup of $\operatorname{Aut}\mathfrak{A}$, with fixed-point algebra $\mathfrak{C}$, and let $\langle(\pi_n, e_n)\rangle_{n\in\mathbb{N}}$ be a sequence running over $G \times \mathfrak{E}$. Take any $a \in \mathfrak{A}$. For $k \in \mathbb{N}$ set

$$a_k = a \cap e_k \cap \pi_k^{-1}(a \setminus e_k),$$

$$a'_k = a_k \setminus \sup_{j<k} \operatorname{upr}(a'_j, \mathfrak{C}).$$

Then

$$a'_k \cap \pi_k a'_k = a_k \cap \pi_k a_k = 0$$

for every $k \in \mathbb{N}$, and whenever $j < k$ in $\mathbb{N}$ we have

$$a'_j \cap a'_k = 0, \quad \pi_j a'_j \cap a'_k = 0,$$

$$a'_j \cap \pi_k a'_k = \pi_k(\pi_k^{-1} a'_j \cap a'_k) = 0, \quad \pi_j a'_j \cap \pi_k a'_k = \pi_k(a'_k \cap \pi_k^{-1}\pi_j a'_j) = 0.$$

So, setting $d = \sup_{k\in\mathbb{N}} a'_k$ and $d' = \sup_{k\in\mathbb{N}} \pi_k a'_k$, d and d' are disjoint and G-σ-equidecomposable and included in a, and $d \preccurlyeq^\sigma_G a \setminus d$.

Consider $a' = a \setminus \operatorname{upr}(d, \mathfrak{C})$. Since $a'_k = a_k \setminus \sup_{j<k} \operatorname{upr}(a'_j, \mathfrak{C})$ for each k,

$$\operatorname{upr}(d, \mathfrak{C}) = \sup_{k\in\mathbb{N}} \operatorname{upr}(a'_k, \mathfrak{C}) = \sup_{k\in\mathbb{N}} \operatorname{upr}(a_k, \mathfrak{C}).$$

? Suppose, if possible, that a' is not a relative atom over $\mathfrak{C}$; that is, that there is a $b \subseteq a'$ such that $b \neq a' \cap c$ for any $c \in \mathfrak{C}$. Then, in particular, $b \neq a' \cap \operatorname{upr}(b, \mathfrak{C})$, and there is a $\pi \in G$ such that $b' = a' \cap \pi b \setminus b \neq 0$. Then $b' \cup \pi^{-1}b' \subseteq a$, while $b' \cap \pi^{-1}b' = 0$, so $\pi b' \cap b' = 0$. By (b), there is an $e \in \mathfrak{E}$ such that $b'' = b' \cap e \setminus \pi e \neq 0$. Let k be such that $\pi^{-1} = \pi_k$ and $e = e_k$, so that

$$b'' = b' \cap e_k \setminus \pi_k^{-1} e_k \subseteq a \cap e_k \cap \pi_k^{-1}(a \setminus e_k) = a_k.$$

(Because $\pi^{-1}b' \subseteq a$, $b' \subseteq \pi_k^{-1}a$.) Since also

$$b'' \cap \operatorname{upr}(a'_j, \mathfrak{C}) \subseteq a' \cap \operatorname{upr}(d, \mathfrak{C}) = 0$$

for every j, $b'' \subseteq a'_k \subseteq d$, which is absurd. **X**

Thus a' is a relative atom over $\mathfrak{C}$, as required.

448M Lemma Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, not $\{0\}$, and G a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point algebra $\mathfrak{C}$. If G has the σ-refinement property, then for any $\epsilon > 0$ there is an $a^* \in \mathfrak{A}$ such that $\operatorname{upr}(a^*, \mathfrak{C}) = 1$ and $\lceil b : a^* \rceil \leq \lfloor b : a^* \rfloor + \epsilon \lfloor 1 : a^* \rfloor$ for every $b \in \mathfrak{A}$.

proof As part (b) of the proof of 394M.

448N Theorem (Compare 394N.) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra and G a countable countably non-paradoxical subgroup of $\operatorname{Aut}\mathfrak{A}$ with fixed-point algebra $\mathfrak{C}$. Suppose that G has the σ-refinement property of 448K. Then there is a function $\theta : \mathfrak{A} \to L^\infty(\mathfrak{C})$ such that

(i) θ is additive, non-negative and sequentially order-continuous;

(ii) $\theta a = 0$ iff $a = 0$, $\theta 1 = \chi 1$;

(iii) $\theta(a \cap c) = \theta a \times \chi c$ for every $a \in \mathfrak{A}$, $c \in \mathfrak{C}$; in particular, $\theta c = \chi c$ for every $c \in \mathfrak{C}$;

(iv) If a, $b \in \mathfrak{A}$ are G-σ-equidecomposable, then $\theta a = \theta b$; in particular, θ is G-invariant.

proof The arguments of the proof of 394N apply here also, though we have to take things in a slightly different order. As in 394N, set

$$\theta_a(b) = \frac{\lceil b : a \rceil}{\lfloor 1 : a \rfloor} \in L^0(\mathfrak{C})$$

whenever $\operatorname{upr}(a, \mathfrak{C}) = 1$ and $b \in \mathfrak{A}$. This time, turn immediately to part (c) of the proof to see that if e_n is chosen (using 448M) such that $\operatorname{upr}(e_n, \mathfrak{C}) = 1$ and $\lceil b : e_n \rceil \leq \lfloor b : e_n \rfloor + 2^{-n} \lfloor 1 : e_n \rfloor$ for every $b \in \mathfrak{A}$, then $\theta_{e_n} b \leq \theta_a b + 2^{-n} \lceil 1 : a \rceil$ whenever $\operatorname{upr}(a, \mathfrak{C}) = 1$ and $b \in \mathfrak{A}$. So we can write

$$\theta b = \inf_{n \in \mathbb{N}} \theta_{e_n} b = \inf_{\operatorname{upr}(a,\mathfrak{C})=1} \theta_a b$$

for every $b \in \mathfrak{A}$, and we have a function $\theta : \mathfrak{A} \to L^0$ as before. The rest of the proof is unchanged, except that we have a simplification in (h), since we need consider only the case $\kappa = \omega$.

448O This concludes the adaptations we need from §394. I now return to the specific problem addressed in the present section. The first step is a version of 448N with weaker hypotheses.

Theorem Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, not $\{0\}$, and G a countable subgroup of $\operatorname{Aut}\mathfrak{A}$ with the σ-refinement property. Let $\mathfrak{C}$ be the fixed-point algebra of G. Then the following are equiveridical:

(i) there are a Dedekind σ-complete Boolean algebra $\mathfrak{D}$, not $\{0\}$, and a G-invariant sequentially order-continuous non-negative additive function $\theta : \mathfrak{A} \to L^\infty(\mathfrak{D})$ such that $\theta 1 = \chi 1$;

(ii) if $a \in \mathfrak{A}$ and $1 \preccurlyeq^\sigma_G a$, then $\operatorname{upr}(1 \setminus a, \mathfrak{C}) \neq 1$;

(iii) if $a \in \mathfrak{A}$ and $1 \preccurlyeq^\sigma_G a$, then $1 \not\preccurlyeq^\sigma_G 1 \setminus a$.

proof (a)(i)⇒(iii) Take $\theta : \mathfrak{A} \to L^\infty(\mathfrak{D})$ as in (i). If $1 \preccurlyeq^\sigma_G a$, then there is a $b \subseteq a$ which is G-σ-equidecomposable with 1, so that $\theta b = \chi 1$, just as in 394N(v)/448N(iv). But this means that $\theta a = \chi 1$; so that $\theta(1 \setminus a) \neq \chi 1$ and $1 \not\preccurlyeq^\sigma_G 1 \setminus a$.

(b) not-(ii)⇒not-(iii) Suppose that (ii) is false; that there is an $a \in \mathfrak{A}$ such that $1 \preccurlyeq^\sigma_G a$ and $\operatorname{upr}(1 \setminus a, \mathfrak{C}) = 1$. Let G^*_σ be the countably full local semigroup generated by G; then there is a $\psi \in G^*_\sigma$ such that $\psi 1 \subseteq a$. Set $b_0 = 1 \setminus \psi 1$ and $b_n = \psi^n b_0$ for every $n \geq 1$; then

$$b_0 \cap b_n \subseteq b_0 \cap \psi 1 = 0$$

for every $n \geq 1$, so

$$b_m \cap b_n = \psi^m(b_0 \cap b_{n-m}) = 0$$

whenever $m < n$.

Let $\langle \pi_i \rangle_{i \in \mathbb{N}}$ be a sequence running over G; then

$$\sup_{i \in \mathbb{N}} \pi_i b_0 = \operatorname{upr}(b_0, \mathfrak{C}) \supseteq \operatorname{upr}(1 \setminus a, \mathfrak{C}) = 1.$$

Set

$$a_j = \pi_j b_0 \setminus \sup_{i<j} \pi_i b_0$$

for every $j \in \mathbb{N}$, so that $\langle a_j \rangle_{j \in \mathbb{N}}$ is a partition of unity in $\mathfrak{A}$. Define ψ_1, $\psi_2 : \mathfrak{A} \to \mathfrak{A}$ by setting

$$\psi_1 d = \sup_{i \in \mathbb{N}} \psi^{2i} \pi_i^{-1}(d \cap a_i), \quad \psi_2 d = \sup_{i \in \mathbb{N}} \psi^{2i+1} \pi_i^{-1}(d \cap a_i)$$

for every $d \in \mathfrak{A}$. Because $\psi^{2i} \pi_i^{-1} a_i \subseteq b_{2i}$ for every i, $\langle \psi^{2i} \pi_i^{-1} a_i \rangle_{i \in \mathbb{N}}$ is disjoint, so $\psi_1 \in G^*_\sigma$; similarly, $\psi_2 \in G^*_\sigma$. Thus

$$1 \preccurlyeq^\sigma_G \psi_1 1 \subseteq \sup_{i \in \mathbb{N}} b_{2i},$$

$$1 \preccurlyeq^\sigma_G \psi_2 1 \subseteq \sup_{i \in \mathbb{N}} b_{2i+1} \subseteq 1 \setminus \psi_1 1$$

and (iii) is false.

(c) For the rest of this proof, therefore, I suppose that (ii) is true and seek to prove (i).

Let $\mathcal{I}$ be the σ-ideal of $\mathfrak{A}$ generated by $\{1 \setminus a : a \in \mathfrak{A}, 1 \preccurlyeq^\sigma_G a\}$. Then $1 \notin \mathcal{I}$. **P?** Otherwise, there is a sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ such that $1 \preccurlyeq^\sigma_G a_n$ for every n and $\sup_{n \in \mathbb{N}} 1 \setminus a_n = 1$. Choose $\psi_n \in G^*_\sigma$ such that

$\psi_n 1 \subseteq a_n$, and set $c_n = \operatorname{upr}(1 \setminus \psi_n 1, \mathfrak{C})$ for each n, so that $\sup_{n\in\mathbb{N}} c_n = 1$. Set $c'_n = c_n \setminus \sup_{i<n} c_i$ for each n, and write

$$\psi d = \sup_{n\in\mathbb{N}} \psi_n(d \cap c'_n)$$

for each d. Because every c'_n belongs to $\mathfrak{C}$, $\langle \psi_n c'_n\rangle_{n\in\mathbb{N}} = \langle c'_n\rangle_{n\in\mathbb{N}}$ is disjoint, and $\psi \in G^*_\sigma$ (448Bd).

By hypothesis, $c = \operatorname{upr}(1 \setminus \psi 1, \mathfrak{C})$ is not 1; let n be such that $c' = c'_n \setminus c \neq 0$. Because $c' \subseteq c'_n$, $c' \setminus \psi_n 1 = c' \setminus \psi 1$; because $c' \in \mathfrak{C}$,

$$\begin{aligned} 0 \neq c' \cap c'_n &\subseteq c' \cap \operatorname{upr}(1 \setminus \psi_n 1, \mathfrak{C}) = \operatorname{upr}(c' \setminus \psi_n 1, \mathfrak{C}) \\ &= \operatorname{upr}(c' \setminus \psi 1, \mathfrak{C}) = c' \cap \operatorname{upr}(1 \setminus \psi 1, \mathfrak{C}) = 0 \end{aligned}$$

which is absurd. **XQ**

Let $\mathfrak{B}$ be the quotient Boolean algebra $\mathfrak{A}/\mathcal{I}$; then $\mathfrak{B}$ is Dedekind σ-complete and the canonical homomorphism $a \mapsto a^\bullet : \mathfrak{A} \to \mathfrak{B}$ is sequentially order-continuous (314C, 313Qb).

(d) Next, $\pi b \in \mathcal{I}$ for every $b \in \mathcal{I}$, $\pi \in G$. **P** The sets $\{a : 1 \preccurlyeq^\sigma_G a\}$ and $\{1 \setminus a : 1 \preccurlyeq^\sigma_G a\}$ are both invariant under the action of G, so $\mathcal{I}$ must also be. **Q** We can therefore define, for each $\pi \in G$, a Boolean automorphism $\tilde\pi : \mathfrak{B} \to \mathfrak{B}$, setting $\tilde\pi a^\bullet = (\pi a)^\bullet$ for every $a \in \mathfrak{A}$. Because $(\pi\phi)^\sim = \tilde\pi\tilde\phi$ for all π, $\phi \in G$, $\tilde G = \{\tilde\pi : \pi \in G\}$ is a subgroup of $\operatorname{Aut}\mathfrak{B}$; of course it is countable. Let $\mathfrak{D}$ be the fixed-point subalgebra of $\tilde G$ in $\mathfrak{B}$. Because $\mathfrak{B}$ is not $\{0\}$, nor is $\mathfrak{D}$.

(e) $\tilde G$ is countably non-paradoxical. **P** Suppose that b is $\tilde G$-σ-equidecomposable with 1 in $\mathfrak{B}$. Let $\langle b_n\rangle_{n\in\mathbb{N}}$ be a partition of unity in $\mathfrak{B}$ and $\langle\pi_n\rangle_{n\in\mathbb{N}}$ a sequence in G such that $\langle\tilde\pi b_n\rangle_{n\in\mathbb{N}}$ is disjoint and has supremum b. For each $n \in \mathbb{N}$, let $a_n \in \mathfrak{A}$ be such that $a_n^\bullet = b_n$. We have

$$(a_m \cap a_n)^\bullet = (\pi_m a_m \cap \pi_n a_n)^\bullet = 0$$

whenever $m \neq n$, so

$$d = \sup_{m\neq n}(a_m \cap a_n) \cup \sup_{m\neq n}(a_m \cap \pi_m^{-1}\pi_n a_n)$$

belongs to $\mathcal{I}$, while $\langle a_n \setminus d\rangle_{n\in\mathbb{N}}$, $\langle\pi_n(a_n \setminus d)\rangle_{n\in\mathbb{N}}$ are disjoint. Because $\mathcal{I}$ is a σ-ideal and G is countable,

$$d^* = \operatorname{upr}(d, \mathfrak{C}) = \sup_{\pi\in G} \pi d \in \mathcal{I}.$$

Define $\psi \in G^*_\sigma$ by setting

$$\psi a = \sup_{n\in\mathbb{N}} \pi_n(a \cap a_n \setminus d^*) \cup (a \cap d^*)$$

for every $a \in \mathfrak{A}$. Then $1 \setminus \psi 1 \in \mathcal{I}$, by the definition of $\mathcal{I}$, so $(\psi 1)^\bullet = 1$ in $\mathfrak{B}$. But

$$\begin{aligned} (\psi 1)^\bullet &= \sup_{n\in\mathbb{N}} (\pi_n(a_n \setminus d^*))^\bullet = \sup_{n\in\mathbb{N}} (\pi_n a_n)^\bullet \\ &= \sup_{n\in\mathbb{N}} \tilde\pi_n b_n = b. \end{aligned}$$

So $b = 1$. As b is arbitrary, $\tilde G$ is countably non-paradoxical. **Q**

(f) $\tilde G$ has the σ-refinement property. **P** Let $b \in \mathfrak{B}$. Then there is an $a \in \mathfrak{A}$ such that $a^\bullet = b$. Because G is supposed to have the σ-refinement property, there is a $d \subseteq a$ such that $d \preccurlyeq^\sigma_G a \setminus d$ and $a \setminus \operatorname{upr}(d, \mathfrak{C})$ is a relative atom over $\mathfrak{C}$. Set $e = d^\bullet \subseteq b$.

We know that there are a partition of unity $\langle d_n\rangle_{n\in\mathbb{N}}$ in $\mathfrak{A}_d$ and a sequence $\langle\pi_n\rangle_{n\in\mathbb{N}}$ in G such that $\pi_n d_n \subseteq a \setminus d$ for every n and $\langle\pi_n d_n\rangle_{n\in\mathbb{N}}$ is disjoint. Now $\langle d_n^\bullet\rangle_{n\in\mathbb{N}}$ is a partition of unity in $\mathfrak{B}_e$, $\tilde\pi_n d_n^\bullet \subseteq b \setminus e$ for every n, and $\langle\tilde\pi_n d_n^\bullet\rangle_{n\in\mathbb{N}}$ is disjoint; so $e \preccurlyeq^\sigma_{\tilde G} b \setminus e$.

Suppose that $b_0 \subseteq b \setminus \operatorname{upr}(e, \mathfrak{D})$. Then it is expressible as $a_0^\bullet$ where $a_0 \subseteq a$ and

$$(a_0 \cap \pi d)^\bullet = b_0 \cap \tilde\pi e = 0$$

for every $\pi \in G$. So if we set $a_1 = a_0 \setminus \sup_{\pi\in G} \pi d$, we shall have $a_1 \subseteq a \setminus \operatorname{upr}(d, \mathfrak{C})$ and $a_1^\bullet = b_0$. Now $a \setminus \operatorname{upr}(d, \mathfrak{C})$ is supposed to be a relative atom over $\mathfrak{C}$, so $a_1 = a \cap c$ for some $c \in \mathfrak{C}$. In this case,

$$\tilde\pi c^\bullet = (\pi c)^\bullet = c^\bullet$$

for every $\pi \in G$, so $c^\bullet \in \mathfrak{D}$, while $b_0 = b \cap c^\bullet$. As b_0 is arbitrary, $b \setminus \operatorname{upr}(e, \mathfrak{D})$ is a relative atom over $\mathfrak{D}$.

Thus e has both the properties required by the definition 448K. As b is arbitrary, $\tilde{G}$ has the σ-refinement property. **Q**

(g) 448N now tells us that there is a sequentially order-continuous non-negative additive functional $\theta_0 : \mathfrak{B} \to L^\infty(\mathfrak{D})$ such that $\theta_0 1 = \chi 1$ and $\theta_0(\tilde{\pi} b) = \theta_0 b$ for every $b \in \mathfrak{B}$, $\pi \in G$. If we set $\theta a = \theta_0 a^\bullet$, it is easy to see that θ has all the properties required by (i) of this theorem. Thus (ii)⇒(i), and the proof is complete.

448P At last we come to Polish spaces.

Theorem (Nadkarni 90, Becker & Kechris 96) Let G be a Polish group acting on a non-empty Polish space $(X, \mathfrak{T})$ with a Borel measurable action $\bullet$. For Borel sets E, $F \subseteq X$ say that $E \preccurlyeq^\sigma_G F$ if there are a countable partition $\langle E_i \rangle_{i \in I}$ of E into Borel sets, and a family $\langle g_i \rangle_{i \in I}$ in G, such that $g_i \bullet E_i \subseteq F$ for every i and $\langle g_i \bullet E_i \rangle_{i \in I}$ is disjoint. Then the following are equiveridical:

(i) there is a G-invariant Radon probability measure μ on X;

(ii) if $F \subseteq X$ is a Borel set such that $X \preccurlyeq^\sigma_G F$, then $\bigcap_{n \in \mathbb{N}} g_n \bullet F \neq \emptyset$ for any sequence $\langle g_n \rangle_{n \in \mathbb{N}}$ in G;

(iii) there are no disjoint Borel sets E, $F \subseteq X$ such that $X \preccurlyeq^\sigma_G E$ and $X \preccurlyeq^\sigma_G F$.

proof (a) Let us start with the easy parts.

(i)⇒(ii) Let μ be a G-invariant Radon probability measure on X, and suppose that $X \preccurlyeq^\sigma_G F$. Let $\langle E_i \rangle_{i \in I}$ be a countable partition of X into Borel sets and $\langle h_i \rangle_{i \in I}$ a family in G such that $\langle h_i \bullet E_i \rangle_{i \in I}$ is disjoint and $h_i \bullet E_i \subseteq F$ for every i. Then

$$\mu F \geq \sum_{i \in I} \mu(h_i \bullet E_i) = \sum_{i \in I} \mu E_i = \mu X,$$

so F is conegligible. Consequently $\bigcap_{n \in \mathbb{N}} g_n \bullet F$ must be conegligible and cannot be empty, for any sequence $\langle g_n \rangle_{n \in \mathbb{N}}$ in G. As F and $\langle g_n \rangle_{n \in \mathbb{N}}$ are arbitrary, (ii) is true.

(ii)⇒(iii) Assume (ii). **?** If there are disjoint E, F such that $X \preccurlyeq^\sigma_G E$ and $X \preccurlyeq^\sigma_G F$, then we have a countable partition $\langle E_i \rangle_{i \in I}$ of X into Borel sets and a family $\langle g_i \rangle_{i \in I}$ in G such that $g_i \bullet E_i \subseteq E$ for every $i \in I$. But there is an $x \in \bigcap_{i \in I} g_i^{-1} \bullet F$, by (ii). In this case there is a $j \in I$ such that $x \in E_j$ and $g_j \bullet x \in E \cap F$, which is impossible. **X** So (iii) must be true.

(b) For the rest of the proof, therefore, I shall assume (iii) and seek to prove (i).

Let $\mathfrak{T}$ be the topology of X, and $\mathcal{B}$ its Borel σ-algebra. For $g \in G$ define $\pi_g : \mathcal{B} \to \mathcal{B}$ by writing $\pi_g E = g \bullet E$ for every $E \in \mathcal{B}$. Then $\pi_{gh} = \pi_g \pi_h$ for all g, $h \in G$, so $\tilde{G} = \{\pi_g : g \in G\}$ is a subgroup of $\operatorname{Aut} \mathcal{B}$. Observe that for E, $F \in \mathcal{B}$, $E \preccurlyeq^\sigma_G F$, in the sense here, iff $E \preccurlyeq^\sigma_{\tilde{G}} F$ in the sense of 448A.

By the Becker-Kechris theorem (424H), there is a Polish topology $\mathfrak{T}_1$ on X, giving rise to the same Borel σ-algebra $\mathcal{B}$ as the original topology, for which the action of G is continuous. Let $\mathcal{U}$ be a countable base for $\mathfrak{T}_1$; replacing $\mathcal{U}$ by $\{U_0 \cup \ldots \cup U_n : U_0, \ldots, U_n \in \mathcal{U}\}$ if necessary, we may suppose that $U \cup U' \in \mathcal{U}$ for all U, $U' \in \mathcal{U}$. (We are going to have three Polish topologies on X in this proof, so watch carefully.)

(c) For the time being (down to the end of (f) below) let us suppose that G, and therefore $\tilde{G}$, are countable. In this case $\tilde{G}$ has the σ-refinement property, by 448L. We can therefore apply 448O to see that (iii) implies that

> (i)′ there are a Dedekind σ-complete Boolean algebra $\mathfrak{D}$, not $\{0\}$, and a $\tilde{G}$-invariant sequentially order-continuous non-negative additive functional $\theta : \mathcal{B} \to L^\infty(\mathfrak{D})$ such that $\theta X = \chi 1$.

To see that (i)′⇒(i), argue as follows. Express $\mathfrak{D}$ as $\Sigma/\mathcal{J}$ where Σ is a σ-algebra of subsets of a set Z and $\mathcal{J}$ is a σ-ideal of Σ (314N). Then we can identify $L^\infty(\mathfrak{D})$ with the quotient $\mathcal{L}^\infty/W$, where $\mathcal{L}^\infty$ is the space of bounded Σ-measurable real-valued functions on Z and W is the set $\{f : f \in \mathcal{L}^\infty, \{z : f(z) \neq 0\} \in \mathcal{J}\}$ (363H). For each $E \in \mathcal{B}$, let $f_E \in \mathcal{L}^\infty$ be a representative of $\theta E \in L^\infty(\mathfrak{D})$; because $\theta(\pi E) = \theta E$ whenever $E \in \mathcal{B}$ and $\pi \in \tilde{G}$, we may suppose that $f_{\pi E} = f_E$ whenever $E \in \mathcal{B}$ and $\pi \in \tilde{G}$.

Let $\mathfrak{B}$ be the subalgebra of $\mathcal{B}$ generated by $\{\pi U : U \in \mathcal{U}, \pi \in \tilde{G}\}$. Then $\mathfrak{B}$ is countable and $\pi E \in \mathfrak{B}$ for every $E \in \mathfrak{B}$, $\pi \in \tilde{G}$. By 4A3I, there is yet another Polish topology $\mathfrak{S}$ on X which is zero-dimensional and such that every member of $\mathfrak{B}$ is open-and-closed for $\mathfrak{S}$. Of course $\mathfrak{S} \supseteq \mathcal{U}$, so $\mathcal{B}$ is still the algebra of $\mathfrak{S}$-Borel sets (423Fb). Let $\mathcal{V}$ be a countable base for $\mathfrak{S}$ consisting of sets which are open-and-closed for $\mathfrak{S}$, and let $\mathfrak{B}_1$ be the subalgebra of $\mathcal{B}$ generated by $\mathcal{V} \cup \mathfrak{B}$; then $\mathfrak{B}_1$ is countable and consists of open-and-closed sets

for $\mathfrak{S}$. Let $\langle W_n\rangle_{n\in\mathbb{N}}$ be a sequence running over $\mathcal{V}$. Let ρ be a complete metric on X defining the topology $\mathfrak{S}$, and for m, $n \in \mathbb{N}$ set

$$W_{mn} = \bigcup\{W_i : i \le n,\ \operatorname{diam}_\rho(W_i) \le 2^{-m}\};$$

then for each $m \in \mathbb{N}$, $\langle W_{mn}\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{B}_1$ with union X.

(d) Consider the subsets of Z of the following types:

$$P_E = \{z : f_E(z) < 0\}, \text{ where } E \in \mathfrak{B}_1,$$

$$Q_{EF} = \{z : f_{E\cup F}(z) \ne f_E(z) + f_F(z)\}, \text{ where } E,\ F \in \mathfrak{B}_1 \text{ and } E \cap F = \emptyset,$$

$$R = \{z : f_X(z) \ne 1\},$$

$$S_m = \{z : \sup_{n\in\mathbb{N}} f_{W_{mn}}(z) \ne 1\}, \text{ where } m \in \mathbb{N}.$$

Because

$$f^\bullet_E = \theta E \ge 0 \text{ for every } E \in \mathcal{B},$$

$$f^\bullet_{E\cup F} = \theta(E\cup F) = \theta E + \theta F = f^\bullet_E + f^\bullet_F \text{ whenever } E \cap F = \emptyset,$$

$$f^\bullet_X = \theta X = \chi 1,$$

$$\sup_{n\in\mathbb{N}} f^\bullet_{W_{mn}} = \sup_{n\in\mathbb{N}} \theta W_{mn} = \theta(\textstyle\bigcup_{n\in\mathbb{N}} W_{mn}) = \theta X = \chi 1$$

for every $m \in \mathbb{N}$, all the sets P_E, Q_{EF}, R and S_m belong to $\mathcal{J}$. Since $\mathfrak{D} \ne \{0\}$, $Z \notin \mathcal{J}$; so there is a $z_0 \in Z$ not belonging to R or P_E or Q_{EF} or S_m whenever $m \in \mathbb{N}$ and E, $F \in \mathfrak{B}_1$ are disjoint.

Set $\nu E = f_E(z_0)$ for every $E \in \mathfrak{B}_1$. If E, $F \in \mathfrak{B}_1$ are disjoint, then $\nu(E\cup F) = \nu E + \nu F$ because $z_0 \notin Q_{EF}$; thus $\nu : \mathfrak{B}_1 \to \mathbb{R}$ is additive. If $E \in \mathfrak{B}_1$ then $\nu E \ge 0$ because $z_0 \notin P_E$, so ν is non-negative. $\nu X = 1$ because $z_0 \notin R$. For each $m \in \mathbb{N}$, $\sup_{n\in\mathbb{N}} \nu W_{mn} = 1$ because $z_0 \notin S_m$.

(e) For any $\epsilon > 0$ there is an $\mathfrak{S}$-compact set $K \subseteq X$ such that $\nu E \ge 1 - \epsilon$ whenever $E \in \mathfrak{B}_1$ and $E \supseteq K$. **P** For each $m \in \mathbb{N}$ we have a $k(m) \in \mathbb{N}$ such that $\nu W_{m,k(m)} \ge 1 - 2^{-m-1}\epsilon$. Set $K = \bigcap_{m\in\mathbb{N}} W_{m,k(m)}$. Because every $W_{m,k(m)}$ is $\mathfrak{S}$-closed, K is $\mathfrak{S}$-closed, therefore ρ-complete; because every $W_{m,k(m)}$ is a finite union of sets of diameter at most 2^{-m}, K is ρ-totally bounded, therefore $\mathfrak{S}$-compact (4A2Jd). **?** Suppose, if possible, that $E \in \mathfrak{B}_1$ is such that $K \subseteq E$ and $\nu E < 1 - \epsilon$. For every $m \in \mathbb{N}$,

$$\nu(\textstyle\bigcap_{i\le m} W_{i,k(i)}) \ge 1 - \sum_{i=0}^m \nu(X \setminus W_{i,k(i)}) \ge 1 - \sum_{i=0}^m 2^{-i-1}\epsilon > 1 - \epsilon > \nu E$$

because ν is non-negative and finitely additive. So $\bigcap_{i\le m} W_{i,k(i)} \setminus E$ must be non-empty. There is therefore an ultrafilter $\mathcal{F}$ on X containing $W_{i,k(i)} \setminus E$ for every $i \in \mathbb{N}$. Now for each i there must be a $j \le k(i)$ such that $\operatorname{diam} W_j \le 2^{-i}$ and $W_j \in \mathcal{F}$, so $\mathcal{F}$ is a ρ-Cauchy filter, and $\mathfrak{S}$-converges to x say. Because every $W_{i,k(i)}$ is $\mathfrak{S}$-closed, $x \in \bigcap_{i\in\mathbb{N}} W_{i,k(i)} = K$; because $E \in \mathfrak{S}$, $x \notin E$; but K is supposed to be included in E. **X**

Thus $\inf\{\nu E : K \subseteq E \in \mathfrak{B}_1\} \ge 1 - \epsilon$. As ϵ is arbitrary, we have the result. **Q**

(f) By 416O, there is an $\mathfrak{S}$-Radon measure μ on X extending ν. Because μ is just the completion of its restriction to $\mathcal{B}$, it is also $\mathfrak{T}$-Radon and $\mathfrak{T}_1$-Radon (433Cb).

Now μ is G-invariant. **P** Take any $g \in G$. Set $\mu_g E = \mu(g\bullet E)$ whenever $E \subseteq X$ and $g\bullet E \in \operatorname{dom}\mu$. The map $x \mapsto g\bullet x$ is a homeomorphism for $\mathfrak{T}_1$, so μ_g is also a $\mathfrak{T}_1$-Radon measure. (Setting $\phi(x) = g^{-1}\bullet x$, μ_g is the image measure $\mu\phi^{-1}$.) Again because $\mathfrak{T}$ and $\mathfrak{T}_1$ have the same Borel σ-algebras, μ_g is $\mathfrak{T}$-Radon. If $E \in \mathfrak{B}$, then E and $g\bullet E$ belong to $\mathfrak{B} \subseteq \mathfrak{B}_1$, so

$$\mu_g E = \mu(g\bullet E) = \nu(g\bullet E) = \nu(\pi_g E) = f_{\pi_g E}(z_0) = f_E(z_0)$$

(because $f_{\pi_g E} = f_E$, as declared in (c) above)

$$= \nu E = \mu E.$$

In particular, $\mu_g E = \mu E$ for every E in the algebra generated by $\mathcal{U}$. But μ_g and μ are both $\mathfrak{T}$-Radon measures, and $\mathcal{U}$ is a base for $\mathfrak{T}$ closed under finite unions, so $\mu_g = \mu$ (415H(iv)). As g is arbitrary, μ is G-invariant. **Q**

Thus we have found a G-invariant Radon probability measure, and (i) is true.

(g) Thus (iii)⇒(i) if G is countable. Now let us consider the general case. Because G is a Polish group, it has a countable dense subgroup H. (Take H to be the subgroup generated by any countable dense subset of G.) Of course there can be no disjoint E, $F \in \mathcal{B}$ such that $X \preccurlyeq_H^\sigma E$ and $X \preccurlyeq_H^\sigma F$, so there must be an H-invariant Radon measure μ on X, by the arguments of (b)-(f). (H need not be a Polish group in its subspace topology. But if we give it its discrete topology, then $x \mapsto h\bullet x$ is still a $\mathfrak{T}_1$-homeomorphism for every $h \in H$, so the action of H on X is still continuous if H is given the discrete topology and X is given $\mathfrak{T}_1$.)

Now μ is G-invariant. **P** For any $g \in G$, let μ_g be the Radon probability measure defined by setting $\mu_g E = \mu(g\bullet E)$ whenever this is defined. (As in (f) above, this formula does define a probability measure which is Radon for either $\mathfrak{T}$ or $\mathfrak{T}_1$.) Let $f : X \to \mathbb{R}$ be any bounded $\mathfrak{T}_1$-continuous function. Then

$$\int f d\mu_g = \int f(g^{-1}\bullet x)\mu(dx)$$

(applying 235I with $\phi(x) = g^{-1}\bullet x$). Now there is a sequence $\langle h_n\rangle_{n\in\mathbb{N}}$ in H converging to g. In this case, because G is a topological group, $g^{-1} = \lim_{n\to\infty} h_n^{-1}$. Because the action of G on X is $\mathfrak{T}_1$-continuous, $g^{-1}\bullet x = \lim_{n\to\infty} h_n^{-1}\bullet x$, for $\mathfrak{T}_1$, for every $x \in X$. Because f is $\mathfrak{T}_1$-continuous, $f(g^{-1}\bullet x) = \lim_{n\to\infty} f(h_n^{-1}\bullet x)$ in $\mathbb{R}$ for every $x \in X$. By Lebesgue's Dominated Convergence Theorem,

$$\int f d\mu_g = \int f(g^{-1}\bullet x)\mu(dx) = \lim_{n\to\infty} f(h_n^{-1}\bullet x)\mu(dx) = \lim_{n\to\infty} \int f d\mu_{h_n} = \int f d\mu$$

because μ is H-invariant, so $\mu_{h_n} = \mu$ for every n. As f is arbitrary, $\mu_g = \mu$, by 415I. As g is arbitrary, μ is G-invariant. **Q**

Thus (iii)⇒(i) in all cases, and the proof is complete.

448X Basic exercises (a) Show that the results in 448Fb and 448Fd remain true if G is not assumed to be countable.

(b) In part (c) of the proof of 448O, show that $\mathcal{I}$ is just the set of those $d \in \mathfrak{A}$ such that $d \subseteq \operatorname{upr}(1 \setminus a, \mathfrak{C})$ for some a such that $1 \preccurlyeq_G^\sigma a$.

(c) Show that, in part (c) of the proof of 448P, we can if we wish take $Z = X$ and $\Sigma = \mathfrak{C}$.

>(d) Let (X, Σ) be a standard Borel space and Σ_0 a countable subalgebra of Σ. Show that there is a sequence $\langle\langle E_{ni}\rangle_{i\in\mathbb{N}}\rangle_{n\in\mathbb{N}}$ of partitions of unity in Σ such that whenever $\nu : \Sigma \to \mathbb{R}$ is a finitely additive functional and $\nu X = \sum_{i=0}^\infty \nu E_{ni}$ for every $n \in \mathbb{N}$, then $\nu\restriction\Sigma_0$ is countably additive.

(e) Let (X, Σ) be a standard Borel space, Y any set, T a σ-algebra of subsets of Y and $\mathcal{J}$ a σ-ideal of subsets of T. Suppose that $(Y, \mathrm{T}, \mathcal{J})$ is complete in the sense that $H \in \mathrm{T}$ whenever $H \subseteq F \in \mathcal{J}$. Let $\theta : \Sigma \to L^\infty(\mathrm{T}/\mathcal{J})$ be a non-negative, sequentially order-continuous additive function. Show that there is a family $\langle\mu_y\rangle_{y\in Y}$ of probability measures with domain Σ such that, for every $E \in \Sigma$, the function $y \mapsto \mu_y E$ is T-measurable and represents θE.

>(f) Set $X = [0,1] \setminus \mathbb{Q}$, $G = \mathbb{Q}$ and define $\bullet : G \times X \to X$ by requiring that $g\bullet x - g - x \in \mathbb{Z}$ for $g \in G$, $x \in X$. Show that this is a Borel measurable action and that Lebesgue measure on X is G-invariant. Find a metric on X, inducing its topology, for which all the maps $x \mapsto g\bullet x$ are isometries.

>(g) Show that a Polish group carries Haar measures iff it is locally compact. (*Hint*: 443E.)

(h) Give $\mathbb{Z}^\mathbb{N}$ its usual (product) topology and abelian group structure. Show that it is a Polish group, and has no Haar measure.

>(i) Let (X, ρ) be a metric space, G a group and $\bullet$ an action of G on X such that $x \mapsto g\bullet x$ is an isometry for every $g \in G$. (i) Show that if μ is a G-invariant quasi-Radon probability measure on X then $\{g\bullet x : g \in G\}$ is totally bounded for every x in the support of μ. (ii) Show that if the action is transitive and there is a non-zero G-invariant quasi-Radon measure on X, then X is covered by totally bounded open sets. (iii) Suppose that X has measure-free weight (see §438; for instance, X could be separable). Show that if the action is transitive and there is a G-invariant Borel probability measure on X then X is totally bounded.

(j) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra, with the operation $\triangle$ and the measure-algebra topology. (i) Show that $\mathfrak{A}$ is a topological group. (ii) Show that if $\bar{\mu}$ is σ-finite and $\mathfrak{A}$ has countable Maharam type, it is a Polish group. (iii) Show that if $(\mathfrak{A}, \bar{\mu})$ is semi-finite and not purely atomic, then $\mathfrak{A}$ has no Haar measures.

(k) Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure on $[0,1]$, with its measure metric, and $G = \mathrm{Aut}_{\bar{\mu}} \mathfrak{A}$ the group of measure-preserving automorphisms on $\mathfrak{A}$. (i) Show that if we give G the topology induced by the pointwise topology of the isometry group of $\mathfrak{A}$, then it is a Polish group. (*Hint*: 441Xm(iv).) (ii) Show that if ν is a G-invariant Borel probability measure on $\mathfrak{A}$, then $\nu\{0,1\} = 1$. (*Hint*: 448Xi(i).)

448Y Further exercises (a) Let $\mathfrak{A}$ be a Dedekind σ-complete Boolean algebra, and G a subgroup of $\mathrm{Aut}\,\mathfrak{A}$; let G^*_σ be the countably full local semigroup generated by G, and write H for the union of all the full local semigroups generated by countable subgroups of G (following the definition in 394A as written, without troubling about whether $\mathfrak{A}$ is Dedekind complete). (i) Show that $G^*_\sigma \subseteq H$. (ii) Find an example in which $H \neq G^*_\sigma$. (iii) Show that if $\mathfrak{A}$ is Dedekind complete then $G^*_\sigma = H$. (iv) Show that if $\mathfrak{A}$ is ccc then $G^*_\sigma = H$ is the full local semigroup generated by G.

(b) In 448N, show that θ is uniquely defined.

448 Notes and comments The keys to the work here are in 448F, 448G and 448L. Even though we no longer have a Dedekind complete algebra, the fact that we are working with countable groups means that the suprema we actually need are defined. The final step, however, uses yet another idea. In a standard Borel space, given a finitely additive functional on the σ-algebra, we can sometimes confirm an adequate approximation to countable additivity by looking at only countably many sequences (448Xd). This enables us to pass from a G-invariant map $\theta : \mathfrak{A} \to L^\infty(\mathfrak{D})$ to a G-invariant Radon measure (448Xe, parts (d)-(f) of the proof of 448P), without needing to know anything about the algebra $\mathfrak{D}$ except that it is Dedekind σ-complete. In particular (and in contrast to the corresponding step in 394P) we do not need to suppose that $\mathfrak{D}$ is a measurable algebra. I do not know whether there is a useful ergodicity condition which could be added to the hypotheses of 448O to ensure that $\mathfrak{D}$ there becomes $\{0,1\}$.

448P was proved in the case $G = \mathbb{Z}$ by Nadkarni 90; the extension to general Borel actions by Polish groups is due to Becker & Kechris 96. (See Nadkarni 90 for notes on the history of the problem, and Kechris 95 for the basic general theory of Polish groups and Borel actions.) It is a remarkable result, but its application is limited by the difficulty of determining whether either condition (ii) or condition (iii) is satisfied. Much commoner situations are those of 448Xf-448Xk, where either there is no invariant measure or we can find one easily.

449 Amenable groups

I end this chapter with a brief introduction to 'amenable' topological groups. I start with the definition (449A) and straightforward results assuring us that there are many amenable groups (449C). At a slightly deeper level we have a condition for a group to be amenable in terms of a universal object constructible from the group, not invoking 'all compact Hausdorff spaces' (449E). I give some notes on amenable locally compact groups, concentrating on a list of properties equivalent to amenability (449I), and end with a version of Tarski's theorem characterizing amenable discrete groups (449K).

449A Definition A topological group G is **amenable** if whenever X is a non-empty compact Hausdorff space and $\bullet$ is a continuous action of G on X, then there is a G-invariant Radon probability measure on X.

Warning: other definitions have been used, commonly based on conditions equivalent to amenability for locally compact Hausdorff groups, such as those listed in 449I(ii)-449I(x). In addition, many authors use the phrase 'amenable group' to mean a group which is amenable in its discrete topology. The danger of this to the non-specialist is that many theorems concerning amenable discrete groups do not generalize in the ways one might expect.

449B Lemma Let G be a topological group, X a locally compact Hausdorff space, and $\bullet$ a continuous action of G on X.

(a) Writing C_0 for the Banach space of continuous real-valued functions on X vanishing at ∞ (436I), the map $a \mapsto a^{-1}\bullet f : C_0 \to C_0$ (definition: 441Ac) is uniformly continuous for the right uniformity on G and the norm uniformity of C_0, for any $f \in C_0$.

(b) If μ is a G-invariant Radon measure on X and $1 \leq p < \infty$, then $a \mapsto a^{-1}\bullet u : G \to L^p$ (definition: 441K) is uniformly continuous for the right uniformity on G and the norm uniformity of $L^p = L^p(\mu)$, for any $u \in L^p(\mu)$.

proof (a)(i) Note first that if $a \in G$ and $f \in C_0$, then $x \mapsto a\bullet x : X \to X$ is a homeomorphism (4A5Bd), so $x \mapsto f(a\bullet x)$ belongs to C_0; but this is just the function $a^{-1}\bullet f$.

(ii) For any $\epsilon > 0$ and $f \in C_0$ there is a neighbourhood V of the identity e of G such that $\|f - a^{-1}\bullet_l f\|_\infty \leq \epsilon$ for every $a \in V$. **P?** Suppose, if possible, otherwise. For each symmetric neighbourhood V of e set

$$Q_V = \{(a,x) : a \in V,\, x \in X,\, |f(x)| \geq \tfrac{\epsilon}{2},\, |f(x) - f(a\bullet x)| \geq \epsilon\}.$$

We are supposing that there are $a \in V$ and $x \in X$ such that $|f(x) - f(a\bullet x)| \geq \epsilon$. If $|f(x)| \geq \frac{1}{2}\epsilon$ then $(a,x) \in Q_V$. Otherwise, $|f(a\bullet x)| \geq \frac{1}{2}\epsilon$, $a^{-1} \in V$ and $|f(a\bullet x) - f(a^{-1}a\bullet x)| \geq \epsilon$, so $(a^{-1}, a\bullet x) \in Q_V$. Thus Q_V is never empty. Since $Q_V \subseteq Q_{V'}$ whenever $V \subseteq V'$, there is an ultrafilter $\mathcal{F}$ on $G \times X$ such that $Q_V \in \mathcal{F}$ for every neighbourhood V of e. Setting $\pi_1(a,x) = a$ and $\pi_2(a,x) = x$ for $(a,x) \in G \times X$, we see that the image filter $\pi_1[[\mathcal{F}]]$ contains every neighbourhood of e, so converges to e, while $\pi_2[[\mathcal{F}]]$ contains the compact set $\{x : |f(x)| \geq \frac{1}{2}\epsilon\}$, so must have a limit x_0 in X. So $\mathcal{F} \to (e, x_0)$ in $G \times X$. Next, because the action is continuous, $\bullet[[\mathcal{F}]] \to e\bullet x_0 = x_0$, and there must be an $F \in \mathcal{F}$ such that $|f(x_0) - f(a\bullet x)| \leq \frac{1}{3}\epsilon$ for every $(a,x) \in F$. Also, of course, there is an $F' \in \mathcal{F}$ such that $|f(x_0) - f(x)| \leq \frac{1}{3}\epsilon$ whenever $(a,x) \in F'$. But now there is an $(a,x) \in Q_G \cap F \cap F'$, and we have

$$|f(x) - f(a\bullet x)| \geq \epsilon, \quad |f(x_0) - f(a\bullet x)| \leq \tfrac{1}{3}\epsilon, \quad |f(x_0) - f(x)| \leq \tfrac{1}{3}\epsilon$$

simultaneously, which is impossible. **XQ**

Now we find that if a, $b \in G$, $ab^{-1} \in V$ and $x \in X$, then

$$|(a^{-1}\bullet f)(x) - (b^{-1}\bullet f)(x)| = |f(a\bullet x) - f(b\bullet x)| = |f(ab^{-1}\bullet(b\bullet x) - f(b\bullet x)| \leq \epsilon.$$

As x is arbitrary, $\|a^{-1}\bullet f - b^{-1}\bullet f\|_\infty \leq \epsilon$; as ϵ is arbitrary, $a \mapsto a^{-1}\bullet f$ is uniformly continuous for the right uniformity.

(b)(i) Suppose that $f : X \to \mathbb{R}$ is continuous and has compact support K. Let $H \supseteq K$ be an open set of finite measure. Then $V_0 = \{a : a \in G,\, a\bullet x \in H \text{ for every } x \in K\}$ is a neighbourhood of e. **P** If we take a continuous function f_0 with compact support such that $\chi K \leq f_0 \leq \chi H$ (4A2G(e-i)), then $V_0 \supseteq \{a : \|f_0 - a^{-1}\bullet f_0\|_\infty < 1\}$, which is a neighbourhood of e by (a). **Q** Let $\epsilon > 0$. By (a) again, there is a symmetric neighbourhood V_1 of e such that $(\|f - a^{-1}\bullet f\|_\infty)^p \mu H \leq \epsilon^p$ for every $a \in V_1$; we may suppose that $V_1 \subseteq V_0$. If $a \in V_1$, $f(x) = f(a\bullet x) = 0$ for every $x \in X \setminus H$, so that

$$\|f - a^{-1}\bullet f\|_p^p = \int_H |f - a^{-1}\bullet f|^p d\mu \leq (\|f - a^{-1}\bullet f\|_\infty)^p \mu H \leq \epsilon^p.$$

Now suppose that a, $b \in G$ and that $ab^{-1} \in V_1$. Then

$$\begin{aligned}\|a^{-1}\bullet f - b^{-1}\bullet f\|_p^p &= \int |f(a\bullet x) - f(b\bullet x)|^p \mu(dx)\\ &= \int |f(a\bullet(b^{-1}\bullet x)) - f(b\bullet(b^{-1}\bullet x))|^p \mu(dx)\end{aligned}$$

(because μ is G-invariant, see 441L)

$$= \int |ba^{-1}\bullet f - f)|^p d\mu \leq \epsilon^p,$$

and $\|a^{-1}\bullet f - b^{-1}\bullet f\|_p \leq \epsilon$. As ϵ is arbitrary, $a \mapsto (a^{-1}\bullet f)^\bullet$ is uniformly continuous for the right uniformity.

(ii) In general, given $u \in L^p(\mu)$ and $\epsilon > 0$, there is an $f \in C_k(X)$ such that $\|u - f^\bullet\|_p \leq \epsilon$ (416I). Let V be a neighbourhood of e such that $\|a^{-1}\bullet f - a^{-1}\bullet f\| \leq \epsilon$ whenever $ab^{-1} \in V$; then $\|a^{-1}\bullet u - (a^{-1}\bullet f)^\bullet\|_p = \|u - f^\bullet\|_p$ (because μ is G-invariant), so

$$\|a^{-1}\bullet u - b^{-1}\bullet u\|_p \leq \|a^{-1}\bullet u - a^{-1}\bullet f^\bullet\|_p + \|a^{-1}\bullet f - b^{-1}\bullet f\|_p + \|b^{-1}\bullet f^\bullet - b^{-1}\bullet u\|_p \leq 3\epsilon$$

whenever $ab^{-1} \in V$ (using 441Kc). As ϵ is arbitrary, $a \mapsto a^{-1}\bullet u$ is uniformly continuous for the right uniformity. This completes the proof.

449C Theorem (a) Let G and H be topological groups such that there is a continuous surjective homomorphism from G onto H. If G is amenable, so is H.

(b) Let G be a topological group and suppose that there is a dense subset A of G such that every finite subset of A is included in an amenable subgroup of G. Then G is amenable.

(c) Let G be a topological group and H a normal subgroup of G. If H and G/H are both amenable, so is G.

(d) Let G be a topological group with two amenable subgroups H_0 and H_1 such that H_0 is normal and $H_0H_1 = G$. Then G is amenable.

(e) The product of any family of amenable topological groups is amenable.

(f) Any abelian topological group is amenable.

(g) Any compact Hausdorff topological group is amenable.

proof (a) Let $\phi : G \to H$ be a continuous surjective homomorphism. Let X be a non-empty compact Hausdorff space and $\bullet : H \times X \to X$ a continuous action. For $a \in G$ and $x \in X$, set $a\bullet_1 x = \phi(a)\bullet x$. Then $\bullet_1$ is a continuous action of G on X, so there is a G-invariant Radon probability measure μ on X. Because $\phi[G] = H$, μ is also H-invariant; as X and $\bullet$ are arbitrary, H is amenable.

(b)(i) Let X be a non-empty compact Hausdorff space and $\bullet$ a continuous action of G on X. Let P be the set of Radon probability measures on X with the narrow topology, so that P is a compact Hausdorff space (437O). For $a \in G$ and $x \in X$, set $T_a(x) = a\bullet x$, so that $T_a : X \to X$ is a homeomorphism. For $a \in G$ and $\mu \in P$ write $a\bullet\mu$ for the image measure μT_a^{-1}, so that $a\bullet\mu \in P$ (418I); it is easy to check that $(a, \mu) \mapsto a\bullet\mu : G \times P \to P$ is an action. The point is that it is continuous. **P** Let $f \in C(X)$, $a_0 \in G$, $\mu_0 \in P$ and $\epsilon > 0$. By 449Ba, there is a neighbourhood V of a_0 in G such that $\|a^{-1}\bullet f - a_0^{-1}\bullet f\|_\infty \leq \frac{1}{2}\epsilon$ for every $a \in V$. Next, there is a neighbourhood W of μ_0 in P such that $|\int a_0^{-1}\bullet f d\mu - \int a_0^{-1}\bullet f d\mu_0| \leq \frac{1}{2}\epsilon$ for every $\mu \in W$. But now, if $a \in V$ and $\mu \in W$,

$$\begin{aligned}
|\int f d(a\bullet\mu) - \int f d(a_0\bullet\mu_0)| &= |\int f T_a d\mu - \int f T_{a_0} d\mu_0| \\
&= |\int a^{-1}\bullet f d\mu - \int a_0^{-1}\bullet f d\mu_0| \\
&\leq |\int a^{-1}\bullet f d\mu - \int a_0^{-1}\bullet f d\mu| + |\int a_0^{-1}\bullet f d\mu - \int a_0^{-1}\bullet f d\mu_0| \\
&\leq \|a^{-1}\bullet f - a_0^{-1}\bullet f\|_\infty + \frac{1}{2}\epsilon \leq \epsilon.
\end{aligned}$$

As ϵ, a_0 and μ_0 are arbitrary, $(a, \mu) \mapsto \int f d(a\bullet\mu)$ is continuous; as f is arbitrary, $\bullet$ is continuous. **Q**

(ii) Because the topology of P is Hausdorff, it follows that $Q_a = \{\mu : \mu \in P, a\bullet\mu = \mu\}$ is closed in P for any $a \in G$, and that $G_\mu = \{a : a \in G, a\bullet\mu = \mu\}$ is closed in G for any $\mu \in P$. Now for any finite subset I of A there is an amenable subgroup H_I of G including I. The restriction of the action to $H_I \times X$ is a continuous action of H_I on X, so has an H_I-invariant Radon probability measure, and $\bigcap_{a\in I} Q_a \supseteq \bigcap_{a\in H_I} Q_a$ is non-empty. Because P is compact, there is a $\mu \in \bigcap_{a\in A} Q_a$. Since G_μ includes the dense set A, it is the whole of G, and μ is G-invariant. As X and $\bullet$ are arbitrary, G is amenable.

(c) Let X be a compact Hausdorff space and $\bullet$ a continuous action of G on X. Let P be the space of Radon probability measures on X with its vague topology. Define $a\bullet\mu$, for $a \in G$ and $\mu \in P$, as in (b-i) above, so that this is a continuous action of G on P. Set $Q = \{\mu : \mu \in P, a\bullet\mu = \mu \text{ for every } a \in H\}$; then Q is a closed subset of P and, because H is amenable, is non-empty, since it is the set of H-invariant Radon probability measures on X. Next, $b\bullet\mu \in Q$ for every $\mu \in Q$ and $b \in G$. **P** If $a \in H$, then

$$a\bullet(b\bullet\mu) = (ab)\bullet\mu = (bb^{-1}ab)\bullet\mu = b\bullet((b^{-1}ab)\bullet\mu) = b\bullet\mu,$$

because H is normal, so $b^{-1}ab \in H$. As a is arbitrary, $b\bullet\mu \in Q$. **Q** Accordingly we have a continuous action of G on the compact Hausdorff space Q.

If $a \in H$ and $b \in G$, then $b\bullet\mu = (ba)\bullet\mu$ for every $\mu \in Q$. We therefore have a map $\bullet : G/H \times Q \to Q$ defined by setting $b^\bullet\bullet\mu = b\bullet\mu$ for every $b \in G$, $\mu \in Q$. Moreover, this is a continuous action, because if $W \subseteq Q$ is relatively open then $\{(b,\mu) : b\bullet\mu \in W\}$ is open in $G \times Q$, so its image $\{(b^\bullet,\mu) : b^\bullet\bullet\mu \in W\}$ is open in $(G/H) \times Q$. Because G/H is amenable, there is a (G/H)-invariant Radon probability measure λ on Q.

Now consider the formula $p(f) = \int_Q (\int_X f(x)\mu(dx))\lambda(d\mu)$. If $f \in C(X)$, then $\mu \mapsto \int_X f(x)\mu(dx)$ is continuous for the vague topology on Q, so $p(f)$ is well-defined. Clearly p is a linear functional, $p(f) \geq 0$ if $f \geq 0$, and $p(\chi X) = 1$; so there is a Radon probability measure ν on X such that $p(f) = \int f d\nu$ for every $f \in C(X)$ (436J/436K). If $b \in G$, then

$$\begin{aligned}\int f d(b\bullet\nu) &= \int fT_b d\nu = p(fT_b) = \int_Q \Big(\int_X fT_b d\mu\Big)\lambda(d\mu)\\ &= \int_Q \int_X f d(b\bullet\mu)\lambda(d\mu) = \int_Q \int_X f d(b^\bullet\bullet\mu)\lambda(d\mu) = \int_Q \int_X f d\mu\lambda(d\mu)\end{aligned}$$

(because λ is G/H-invariant)

$$= \int f d\nu$$

for every $f \in C(X)$, so that $b\bullet\nu = \nu$. Thus ν is G-invariant. As X and $\bullet$ are arbitrary, G is amenable.

(d) The canonical map from H_1 to G/H_0 is a continuous surjective homomorphism. By (a), G/H_0 is amenable; by (c), G is amenable.

(e) By (c) or (d), the product of two amenable topological groups is amenable, since each can be regarded as a normal subgroup of the product. It follows that the product of finitely many amenable topological groups is amenable. Now let $\langle G_i\rangle_{i\in I}$ be any family of amenable topological groups with product G. For finite $J \subseteq I$ let H_J be the set of those $a \in G$ such that $a(i)$ is the identity in G_i for every $i \in I \setminus J$. Then H_J is isomorphic (as topological group) to $\prod_{i\in J} G_i$, so is amenable. Since $\{H_J : J \in [I]^{<\omega}\}$ is an upwards-directed family of subgroups of G with dense union, (b) tells us that G is amenable.

(f)(i) The first step is to observe that the group $\mathbb{Z}$, with its discrete topology, is amenable. **P** Let X be a compact Hausdorff space and $\bullet$ a continuous action of $\mathbb{Z}$ on X. Set $\phi(x) = 1\bullet x$ for $x \in X$. Then $\phi : X \to X$ is continuous, so by 437S there is a Radon probability measure μ on X such that μ is equal to the image measure $\mu\phi^{-1}$. Because ϕ is bijective, we see that, for $E \subseteq X$, $E \in \operatorname{dom}\mu$ iff $\phi[E] \in \operatorname{dom}(\mu\phi^{-1}) = \operatorname{dom}\mu$, and in this case $\mu\phi[E] = \mu E$; that is, ϕ^{-1}, like ϕ, is inverse-measure-preserving. Now we can induce on n to see that $\mu(\phi^n)^{-1}$ and $\mu(\phi^{-n})^{-1}$ are equal to μ for every n. Since $n\bullet x = \phi^n(x)$ for every $x \in X$ and $n \in \mathbb{Z}$, μ is $\mathbb{Z}$-invariant. As X and $\bullet$ are arbitrary, $\mathbb{Z}$ is amenable. **Q**

(ii) Now let G be any abelian topological group. For each finite set $I \subseteq G$ let $\phi_I : \mathbb{Z}^I \to G$ be the continuous homomorphism defined by setting $\phi_I(z) = \prod_{a\in I} a^{z(a)}$ for $z \in \mathbb{Z}^I$. By (i) just above and (e), we know that $\mathbb{Z}^I$ (with its discrete topology) is amenable, so (a) tells us that the subgroup $G_I = \phi_I[\mathbb{Z}^I]$ is amenable. But now $\{G_I : I \in [G]^{<\omega}\}$ is an upwards-directed family of amenable subgroups of G with union G, so from (b) we see that G is amenable.

(g) Let G be a compact Hausdorff topological group, X a compact Hausdorff space and $\bullet$ a continuous action of G on X. Fix $x_0 \in X$ and set $\phi(a) = a\bullet x_0$ for $a \in G$; then ϕ is continuous. Let μ be the Haar probability measure on G, and ν the Radon probability measure $\mu\phi^{-1}$ on X. If $F \in \operatorname{dom}\nu$ and $a \in G$, then

$$\begin{aligned}\nu(a\bullet F) &= \mu\{b : \phi(b) \in a\bullet F\} = \mu\{b : b\bullet x_0 \in a\bullet F\} = \mu\{b : a^{-1}\bullet b\bullet x_0 \in F\}\\ &= \mu\{b : a^{-1}b \in \phi^{-1}[F]\} = \mu\phi^{-1}[F] = \nu F.\end{aligned}$$

So ν is G-invariant. As X and $\bullet$ are arbitrary, G is amenable.

449D Theorem Let G be a topological group.

(a) Write U for the set of bounded real-valued functions on G which are uniformly continuous for the right uniformity of G. Then U is an M-space, and we have an action $\bullet_l$ of G on U defined by the formula $(a\bullet_l f)(y) = f(a^{-1}y)$ for a, $y \in G$ and $f \in U$.

(b) Let $Z \subseteq \mathbb{R}^U$ be the set of Riesz homomorphisms $z : U \to \mathbb{R}$ such that $z(\chi G) = 1$. Then Z is a compact Hausdorff space, and we have a continuous action of G on Z defined by the formula $(a\bullet_l z)(f) = z(a^{-1}\bullet_l f)$ for $a \in G$, $z \in Z$ and $f \in U$.

(c) Setting $\hat{a}(f) = f(a)$ for $a \in G$ and $f \in U$, the map $a \mapsto \hat{a} : G \to Z$ is a continuous function from G onto a dense subset of Z. If a, $b \in G$ then $a\bullet_l \hat{b} = \widehat{ab}$.

(d) Now suppose that X is a compact Hausdorff space, $(a, x) \mapsto a\bullet x$ is a continuous action of G on X, and $x_0 \in X$. Then there is a continuous function $\phi : Z \to X$ such that $\phi(\hat{e}) = x_0$ and $\phi(a\bullet_l z) = a\bullet\phi(z)$ for every $a \in G$ and $z \in Z$.

(e) If G is Hausdorff then the action of G on Z is faithful and the map $a \mapsto \hat{a}$ is a homeomorphism between G and its image in Z.

proof (a) Because U is a norm-closed Riesz subspace of $C_b(G)$ containing the constant functions (4A2Jg), it is an M-space. To see that the given formula defines an action, we need to check that $a\bullet_l f$ belongs to U whenever $a \in G$ and $f \in U$. Of course $a\bullet_l f$ is continuous and $\|a\bullet_l f\|_\infty = \|f\|_\infty$ is finite. If $\epsilon > 0$ there is a neighbourhood V of the identity e in G such that $|f(b) - f(c)| \leq \epsilon$ whenever b, $c \in G$ and $bc^{-1} \in V$; now $a^{-1}Va$ is a neighbourhood of e, and if $bc^{-1} \in aVa^{-1}$ then $(a^{-1}b)(a^{-1}c)^{-1} \in V$, so $|(a\bullet_l f)(b) - (a\bullet_l f)(c)| = |f(a^{-1}b) - f(a^{-1}c)| \leq \epsilon$. As ϵ is arbitrary, $a\bullet_l f$ is uniformly continuous with respect to the right uniformity. Now $\bullet_l$ is an action, just as in 441Ac.

(b)(i) Because U is an M-space with standard order unit χG, Z is a compact Hausdorff space and U can be identified, as normed Riesz space, with $C(Z)$ (354L). For any $a \in G$, the map $f \mapsto a\bullet_l f : U \to U$ is a Riesz homomorphism leaving the constant functions fixed. So we can define $a\bullet_l z$, for $z \in Z$, by saying that $(a\bullet_l z)(f) = z(a^{-1}\bullet_l f)$ for any $f \in U$, and $a\bullet_l z$ will belong to Z for any $a \in G$ and $z \in Z$. As usual, it is easy to check that this formula defines an action of G on Z.

(ii) $(a, z) \mapsto a\bullet_l z$ is continuous. **P** Take $a_0 \in G$, $z_0 \in Z$ and any neighbourhood W of $a_0\bullet_l z_0$ in Z. Because U corresponds to the whole of $C(Z)$, and Z is completely regular, there is an $f \in U^+$ such that $(a_0\bullet_l z_0)(f) = 1$ and $z(f) = 0$ for every $z \in Z \setminus W$. Set $W_0 = \{z : z(a_0^{-1}\bullet_l f) > \frac{1}{2}\}$. Observe that $z_0(a_0^{-1}\bullet_l f) = 1$, so W_0 is an open subset of Z containing z_0. Next, set $V_0 = \{a : a \in G, \|a^{-1}\bullet_l f - a_0^{-1}\bullet_l f\|_\infty \leq \frac{1}{2}\}$. There is a neighbourhood V of e such that $|f(b) - f(c)| \leq \frac{1}{2}$ whenever b, $c \in G$ and $bc^{-1} \in V$. If $a \in Va_0$ then $ab(a_0b)^{-1} = aa_0^{-1} \in V$, so

$$|(a^{-1}\bullet_l f)(b) - (a_0^{-1}\bullet_l f)(b)| = |f(ab) - f(a_0b)| \leq \frac{1}{2}$$

for every $b \in G$, and $a \in V_0$. Thus $V_0 \supseteq Va_0$ is a neighbourhood of a_0.

Now if $a \in V_0$ and $z \in W_0$ we shall have

$$(a\bullet_l z)(f) = z(a^{-1}\bullet_l f) \geq z(a_0^{-1}\bullet_l f) - \frac{1}{2} > 0$$

and $a\bullet_l z \in W$. As a_0, z_0 and W are arbitrary, the action of G on Z is continuous. **Q**

(c) Of course $\hat{a}$, as defined, is a Riesz homomorphism taking the correct value at χG, so belongs to Z. Because the topology on G is compatible with the right uniformity (4A5Ha), the map $a \mapsto \hat{a}$ is continuous. **?** If $\{\hat{a} : a \in G\}$ is not dense in Z, there is a non-zero $h \in C(Z)$ such that $h(\hat{a}) = 0$ for every $a \in G$; but as U is identified with $C(Z)$, there is an $f \in U$ such that $z(f) = h(z)$ for every $z \in Z$. In this case, f cannot be the zero function, but $f(a) = \hat{a}(f) = h(\hat{a}) = 0$ for every $a \in G$. **X** Thus the image of G is dense, as claimed.

If a, $b \in G$ and $f \in U$ then

$$(a\bullet_l \hat{b})(f) = \hat{b}(a^{-1}\bullet_l f) = (a^{-1}\bullet_l f)(b) = f(ab) = \widehat{ab}(f),$$

so $a\bullet_l \hat{b} = \widehat{ab}$.

(d) We have a Riesz homomorphism $T : C(X) \to \mathbb{R}^G$ defined by setting $(Tg)(a) = g(a\bullet x_0)$ for every $g \in C(X)$ and $a \in G$. Now $Tg \in U$ for every $g \in C(X)$. **P** $Tg(a) = (a^{-1}\bullet g)(x_0)$; since the map $a \mapsto a^{-1}\bullet g$ is uniformly continuous (449Ba), so is Tg. **Q**

Of course $T(\chi X) = \chi G$. So if $z \in Z$, $zT : C(X) \to \mathbb{R}$ is a Riesz homomorphism such that $(zT)(\chi X) = 1$. There is therefore a unique $\phi(z) \in X$ such that $(zT)(g) = g(\phi(z))$ for every $g \in C(X)$ (354L). Since the function $z \mapsto g(\phi(z)) = z(Tg)$ is continuous for every $g \in C(X)$, ϕ is continuous.

Now suppose that $a \in G$. Then $\phi(\hat{a}) = a\bullet x_0$. **P** If $g \in C(X)$, then

$$g(\phi(\hat{a})) = \hat{a}(Tg) = (Tg)(a) = g(a\bullet x_0). \quad \textbf{Q}$$

So if a, $b \in G$, then

$$\phi(a\bullet_l \hat{b}) = \phi(\widehat{ab}) = (ab)\bullet x_0 = a\bullet(b\bullet x_0) = a\bullet\phi(b).$$

Since $\{\hat{b} : y \in G\}$ is dense in Z, and all the functions here are continuous, $\phi(a\bullet_l z) = a\bullet\phi(z)$ for all $a \in G$ and $z \in Z$.

(e) Now suppose that the topology of G is Hausdorff. Then it is defined by the bounded uniformly continuous functions (4A2Ja); the map $a \mapsto \hat{a}$ is therefore injective and is a homeomorphism between G and its image in Z. If a, $b \in G$ are distinct, then $a\bullet\hat{e} = \hat{a} \neq \hat{b} = b\bullet\hat{e}$, so the action is faithful.

Remark Following Brook 70, the space Z, together with the canonical action of G on it and the map $a \mapsto \hat{a} : G \to Z$, is called the **greatest ambit** of the topological group G.

449E Corollary Let G be a topological group. Then the following are equiveridical:
(i) G is amenable;
(ii) there is a G-invariant Radon probability measure on the greatest ambit of G;
(iii) writing U for the space of bounded real-valued functions on G which are uniformly continuous for the right uniformity, there is a positive linear functional $p : U \to \mathbb{R}$ such that $p(\chi G) = 1$ and $p(a\bullet_l f) = p(f)$ for every $f \in U$ and $a \in G$.

proof Let Z be the greatest ambit of G.

(i)⇒(ii) As soon as we know that Z is a compact Hausdorff space and the action of G on Z is continuous (449Db), this becomes a special case of the definition of 'amenable topological group'.

(ii)⇒(i) Let μ be a G-invariant Radon probability measure on Z. Given any continuous action of G on a non-empty compact Hausdorff space X, fix $x_0 \in X$ and let $\phi : Z \to X$ be the corresponding continuous function such that $\phi(a\bullet_l z) = a\bullet\phi(z)$ for every $a \in G$ and $z \in Z$, as in 449Dd. Let ν be the image measure $\mu\phi^{-1}$. Then ν is a Radon probability measure on X (418I). If $F \in \operatorname{dom}\nu$ and $a \in G$, then

$$\nu(a\bullet F) = \mu\phi^{-1}[a\bullet F] = \mu(a\bullet_l \phi^{-1}[F]) = \mu\phi^{-1}[F] = \nu F.$$

As a and F are arbitrary, ν is G-invariant; as X and $\bullet$ are arbitrary, G is amenable.

(ii) ⟺ (iii) The identification of U with $C(Z)$ (see (b-i) of the proof of 449D) means that we have a one-to-one correspondence between Radon probability measures μ on Z and positive linear functionals p on U such that $p(\chi G) = 1$, given by the formula $p(f) = \int z(f)\mu(dz)$ for $f \in U$ (436J/436K). Now

$$\begin{aligned}
&\mu \text{ is } G\text{-invariant}\\
&\iff \int (a\bullet_l z)(f)\mu(dz) = \int z(f)\mu(dz) \text{ for every } f \in U,\ a \in G\\
&\iff \int z(a^{-1}\bullet_l f)\mu(dz) = \int z(f)\mu(dz) \text{ for every } f \in U,\ a \in G\\
&\iff \int z(a\bullet_l f)\mu(dz) = \int z(f)\mu(dz) \text{ for every } f \in U,\ a \in G\\
&\iff p(a\bullet_l f) = p(f) \text{ for every } f \in U,\ a \in G.
\end{aligned}$$

(441L)

So there is a G-invariant μ, as required by (ii), iff there is a G-invariant p as required by (iii).

449F Corollary If G is an amenable topological group, then every open subgroup of G is amenable.

proof (a) Let H be an open subgroup of G. Write U for the set of bounded real-valued functions on G which are uniformly continuous for the right uniformity of G, and V for the set of bounded real-valued functions on H which are uniformly continuous for the right uniformity of H. Let $A \subseteq G$ be a set meeting each right coset of H in just one point, so that each member of G is uniquely expressible as ya where $y \in H$ and $a \in A$. Define $T : V \to \mathbb{R}^G$ by setting $(Tf)(ya) = f(y)$ whenever $f \in V$, $y \in H$ and $a \in A$. Then T is a positive linear operator. Also $T[V] \subseteq U$. **P** Let $f \in V$. Of course Tf is bounded. If $\epsilon > 0$, there is a neighbourhood W of the identity in H such that $|f(x) - f(y)| \leq \epsilon$ whenever $x, y \in H$ and $xy^{-1} \in W$. Because H is open, W is also a neighbourhood of the identity in G. Now suppose that $x, y \in G$ and $xy^{-1} \in W$. Express x as x_0a and y as y_0b where $x_0, y_0 \in H$ and $a, b \in A$. Then

$$x_0ab^{-1}y_0^{-1} \in W \subseteq H,$$

so $ab^{-1} \in H$ and $a \in Hb$ and $a = b$ and $x_0y_0^{-1} \in W$ and

$$|(Tf)(x) - (Tf)(y)| = |f(x_0) - f(y_0)| \leq \epsilon.$$

As ϵ is arbitrary, Tf is uniformly continuous and belongs to U. **Q**

(b) Next, $b\bullet_l(Tf) = T(b\bullet_l f)$ whenever $f \in V$ and $b \in H$. **P** If $x \in G$, express it as ya where $y \in H$ and $a \in H$. Then

$$(b\bullet_l Tf)(x) = (Tf)(b^{-1}x) = (Tf)(b^{-1}ya) = f(b^{-1}y) = (b\bullet f)(y) = T(b\bullet_l f)(x). \ \textbf{Q}$$

(c) By 449E, there is a positive linear functional $p : U \to \mathbb{R}$ such that $p(\chi G) = 1$ and $p(a\bullet_l f) = p(f)$ whenever $f \in U$ and $a \in G$. Set $q(f) = p(Tf)$ for $f \in V$; then q is a positive linear operator, $q(\chi H) = 1$ and q is H-invariant, by (b). So by 449E in the other direction, H is amenable.

449G Example Let F_2 be the free group on two generators, with the discrete topology. Then F_2 is a σ-compact unimodular locally compact Polish group. But it is not amenable. **P** Let a and b be the generators of F_2. Then every element of F_2 is uniquely expressible as a word (possibly empty) in the letters a, b, a^{-1}, b^{-1} in which the letters a, a^{-1} are never adjacent and the letters b, b^{-1} are never adjacent. Write A for the set of elements of F_2 for which the canonical word does not begin with either b or b^{-1}, and B for the set of elements of F_2 for which the canonical word does not begin with either a or a^{-1}. Then $A \cup B = F_2$ and $A \cap B = \{e\}$. **?** Suppose, if possible, that F_2 is amenable. Every member of $\ell^\infty(F_2)$ is uniformly continuous with respect to the right uniformity. So there is an F_2-invariant positive linear functional $p : \ell^\infty(F_2) \to \mathbb{R}$ such that $p(\chi F_2) = 1$. Let ν be the corresponding non-negative additive functional on $\mathcal{P}F_2$, so that $\nu C = p(\chi C)$ for every $C \subseteq F_2$. For $c \in F_2$ and $C \subseteq F_2$, $c\bullet_l \chi C = \chi(cC)$, so $\nu(cC) = \nu C$ for every $C \subseteq F_2$ and $c \in F_2$. In particular, $\nu(b^nA) = \nu A$ for every $n \in \mathbb{Z}$; but as all the b^nA, for $n \in \mathbb{Z}$, are disjoint, $\nu A = 0$. Similarly $\nu B = 0$ and

$$0 = \nu(A \cup B) = \nu F_2 = p(\chi F_2) = 1,$$

which is absurd. **X** Thus F_2 is not amenable, as claimed. **Q**

449H In this section so far, I have taken care to avoid assuming that groups are locally compact. Some of the most interesting amenable groups are very far from being locally compact (e.g., 449Xc). But of course a great deal of work has been done on amenable locally compact groups. In particular, there is a remarkable list of equivalent properties, some of which I will now present.

Lemma Let G be a locally compact Hausdorff topological group, and U the space of bounded real-valued functions on G which are uniformly continuous for the right uniformity, as in 449D-449E. Let μ be a left Haar measure on G, and $*$ the corresponding convolution on $\mathcal{L}^0(\mu)$ (444O).

(a) If $h \in \mathcal{L}^1(\mu)$ and $f \in \mathcal{L}^\infty(\mu)$ then $h * f \in U$.

(b) Let $p : U \to \mathbb{R}$ be a positive linear functional such that $p(a\bullet_l f) = p(f)$ for every $f \in U$ and every $a \in G$. Then $p(h * f) = p(f) \int h \, d\mu$ for every $h \in \mathcal{L}^1(\mu)$ and every $f \in U$.

proof (a) For any $x \in G$,

$$(h * f)(x) = \int h(xy)f(y^{-1})\mu(dy) = \int (x^{-1}\bullet_l h) \times \tilde{f},$$

where $\tilde{f}(y) = f(y^{-1})$ whenever $y^{-1} \in \operatorname{dom} f$. By 449Bb, applied to the left action of G on itself, $x \mapsto (x^{-1}\bullet_l h)^\bullet : G \to L^1(\mu)$ is uniformly continuous for the right uniformity of G and the norm uniformity of $L^1(\mu)$. Since $u \mapsto \int u \times v : L^1(\mu) \to \mathbb{R}$ is uniformly continuous for every $v \in L^\infty(\mu)$, $x \mapsto (h * f)(x) = \int (x^{-1}\bullet_l h)^\bullet \times \tilde{f}^\bullet$ is uniformly continuous for the right uniformity (3A4Cb). Of course $\sup_{x\in G} |(h * f)(x)| \leq \|h\|_1 \|f\|_\infty$ is finite, so $h * f \in U$.

(b) Let $\epsilon > 0$. Then there are a compact set $K \subseteq G$ such that $\int_{G\setminus K} |h| d\mu \leq \epsilon$ (412Jd) and a symmetric neighbourhood V_0 of e such that $|f(x) - f(y)| \leq \epsilon$ whenever $xy^{-1} \in V_0$. Let $a_0, \ldots, a_n \in G$ be such that $K \subseteq \bigcup_{i\leq n} a_i V_0$, and set $E_i = a_i V_0 \setminus \bigcup_{j<i} a_j V_0$, $\alpha_i = \int_{E_i} h \, d\mu$ for each $i \leq n$, $F = G \setminus \bigcup_{j\leq n} a_j V_0$. If $x \in G$ and $y \in E_i$, then $y^{-1}x(a_i^{-1}x)^{-1} = (a_i^{-1}y)^{-1} \in V_0$, so $|f(y^{-1}x) - f(a_i^{-1}x)| \leq \epsilon$. So, for any $x \in G$,

$$\begin{aligned}
&\big|(h * f)(x) - \sum_{i=0}^n \alpha_i f(a_i^{-1}x)\big| \\
&\quad = \big| \int h(y) f(y^{-1}x) \mu(dy) - \sum_{i=0}^n \alpha_i f(a_i^{-1}x)\big| \\
&\quad = \big| \int_F h(y) f(y^{-1}x) \mu(dy) + \sum_{i=0}^n (\int_{E_i} h(y) f(y^{-1}x) \mu(dy) - \alpha_i f(a_i^{-1}x))\big| \\
&\quad \leq \|f\|_\infty \int_F |h| d\mu + \sum_{i=0}^n \big| \int_{E_i} h(y)(f(y^{-1}x) - f(a_i^{-1}x)) \mu(dy)\big| \\
&\quad \leq \|f\|_\infty \int_{X\setminus K} |h| d\mu + \epsilon \sum_{i=0}^n \int_{E_i} |h| d\mu \leq \epsilon(\|f\|_\infty + \|h\|_1).
\end{aligned}$$

Thus

$$\|h * f - \textstyle\sum_{i=0}^n \alpha_i a_i \bullet_l f\|_\infty \leq \epsilon(\|f\|_\infty + \|h\|_1).$$

Since $p(a_i \bullet_l f) = p(f)$ for every i, it follows that

$$\begin{aligned}
|p(h * f) - p(f) \int h \, d\mu| &\leq \epsilon(\|f\|_\infty + \|h\|_1) p(\chi G) + |\sum_{i=0}^n \alpha_i - \int h \, d\mu| |p(f)| \\
&\leq \epsilon(\|f\|_\infty + \|h\|_1) p(\chi G) + |\int_F h \, d\mu| \|f\|_\infty p(\chi G) \\
&\leq \epsilon(2\|f\|_\infty + \|h\|_1) p(\chi G).
\end{aligned}$$

As ϵ is arbitrary, $p(h * f) = p(f) \int h \, d\mu$, as claimed.

449I Theorem Let G be a locally compact Hausdorff group, and μ a left Haar measure on G. Let C_{k1}^+ be the set of continuous functions $h : G \to [0, \infty[$ with compact supports such that $\int h \, d\mu = 1$. Then the following are equiveridical:

(i) G is amenable;

(ii) there is a positive linear functional $p : C_b(X) \to \mathbb{R}$ such that $p(\chi G) = 1$ and $p(a \bullet_l f) = p(f)$ for every $u \in C_b(X)$ and every $a \in G$, where $a \bullet_l f$ is defined as in 441Ac;

(iii) there is a positive linear functional $\tilde{p} : L^\infty(\mu) \to \mathbb{R}$ such that $\tilde{p}(\chi G^\bullet) = 1$ and $\tilde{p}(a \bullet_l u) = \tilde{p}(u)$ for every $u \in L^\infty(\mu)$ and every $a \in G$, where $a \bullet_l u$ is defined as in 441K and 443G;

(iv) there is a positive linear functional $\tilde{p} : L^\infty(\mu) \to \mathbb{R}$ such that $\tilde{p}(\chi G^\bullet) = 1$ and $\tilde{p}(g * f)^\bullet = \tilde{p}(f^\bullet) \int g \, d\mu$ for every $g \in \mathcal{L}^1(\mu)$ and $f \in \mathcal{L}^\infty(\mu)$;

(v) there is a positive linear functional $\tilde{p} : L^\infty(\mu) \to \mathbb{R}$ such that $\tilde{p}(\chi G^\bullet) = 1$ and $\tilde{p}(\nu * u) = \nu G \cdot \tilde{p}(u)$ for every $u \in L^\infty(\mu)$ and every totally finite Radon measure ν on G, where $\nu * u$ is defined as in 444Ma;

(vi) for every finite set $I \subseteq G$, finite set $J \subseteq \mathcal{L}^\infty(\mu)$ and $\epsilon > 0$, there is an $h \in C_{k1}^+$ such that $|\int f(ax) h(x) \mu(dx) - \int f(x) h(x) \mu(dx)| \leq \epsilon$ for every $a \in I$ and $f \in J$;

(vii) for every finite set $J \subseteq \mathcal{L}^1(\mu)$ and $\epsilon > 0$, there is an $h \in C_{k1}^+$ such that $\|g * h - (\int g\,d\mu)h\|_1 \leq \epsilon$ for every $g \in J$;

(viii) for every compact set $K \subseteq G$ and $\epsilon > 0$, there is an $h \in C_{k1}^+$ such that $\|a\bullet_l h - h\|_1 \leq \epsilon$ for every $a \in K$, where $(a\bullet_l h)(x) = h(a^{-1}x)$ for every $x \in G$;

(ix) for every compact set $K \subseteq G$ and $\epsilon > 0$, there is a compact set $L \subseteq G$ with non-zero measure such that $\mu(L \triangle aL) \leq \epsilon\mu L$ for every $a \in K$;

(x) (Emerson & Greenleaf 67) for every compact set $K \subseteq G$ and $\epsilon > 0$, there is a compact set $L \subseteq G$ with non-zero measure such that $\mu(KL) \leq (1+\epsilon)\mu L$.

proof (a)(i)⇒(v) Write U for the space of bounded real-valued functions on G which are uniformly continuous for the right uniformity. Then we have a positive linear functional $p : U \to \mathbb{R}$ such that $p(\chi G) = 1$ and $p(a\bullet_l f) = p(f)$ for every $f \in U$ and $a \in G$. Now if $f \in \mathcal{L}^\infty = \mathcal{L}^\infty(\mu)$, h_1, $h_2 \in \mathcal{L}^1 = \mathcal{L}^1(\mu)$ and $\int h_1 d\mu = \int h_2 d\mu$, then $p(h_1 * f) = p(h_2 * f)$. **P** By 449Ha, both $h_1 * f$ and $h_2 * f$ belong to U. Set $h = h_1 - h_2$. By 444T, there is a neighbourhood V of the identity e of G such that $\|h * \nu - h\|_1 \leq \epsilon$ whenever ν is a quasi-Radon measure on G such that $\nu V = \nu G = 1$. In particular, taking ν to be the indefinite-integral measure over μ defined from $g = \frac{1}{\mu V}\chi V$, $\|h * g - h\|_1 \leq \epsilon$ (using 444Pb). Now

$$\begin{aligned} |p(h_1 * f) - p(h_2 * f)| &= |p(h * f)| \leq |p((h * g) * f)| + |p((h * g - h) * f)| \\ &\leq |p(h * (g * f))| + \|(h * g - h) * f\|_\infty \end{aligned}$$

(because $*$ is associative, 444Oe)

$$\leq |p(g * f)\int h\,d\mu| + \|h * g - h\|_1 \|f\|_\infty \leq \epsilon\|f\|_\infty$$

by 449Hb. As ϵ is arbitrary, $p(h_1 * f) = p(h_2 * f)$, as claimed. **Q**

Of course $p(h * f) = 0$ whenever $h \in \mathcal{L}^1$, $f \in \mathcal{L}^\infty$ and $f = 0$ a.e. We can therefore define a functional $\tilde{p} : L^\infty = L^\infty(\mu) \to \mathbb{R}$ by saying that $\tilde{p}(f^\bullet) = p(h * f)$ whenever $f \in \mathcal{L}^\infty$, $h \in \mathcal{L}^1$ and $\int h\,d\mu = 1$. $\tilde{p}$ is positive and linear because p is. It follows that $p(h * f) = \tilde{p}(f^\bullet)\int h\,d\mu$ whenever $h \in \mathcal{L}^1$ and $f \in \mathcal{L}^\infty$. Also $\tilde{p}(\chi G^\bullet) = p(\chi G) = 1$ because $h * \chi G = (\int h)\chi G$ for every $h \in \mathcal{L}^1$.

If $u \in L^\infty$ and ν is a totally finite Radon measure on G, express u as $f^\bullet$ where $f \in \mathcal{L}^\infty$, so that $\nu * u = (\nu * f)^\bullet$. Taking any non-negative $h \in \mathcal{L}^1$ such that $\int h\,d\mu = 1$, we have

$$\begin{aligned} h * (\nu * f) &= h\mu * (\nu * f) \\ &\text{(444Pa; here } h\mu \text{ is the indefinite-integral measure, as in 444J)} \\ &= (h\mu * \nu) * f \\ &\text{(444Ic)} \\ &= (h * \nu)\mu * f \\ &\text{(444K)} \\ &= (h * \nu) * f \end{aligned}$$

(444Pa again). So

$$\begin{aligned} \tilde{p}(\nu * u) = \tilde{p}((\nu * f)^\bullet) &= p(h * (\nu * f)) \\ = p((h * \nu) * f) &= \int (h * \nu)d\mu \cdot \tilde{p}(u) = \nu G \cdot \tilde{p}(u) \end{aligned}$$

(444K). As ν and u are arbitrary, $\tilde{p}$ has the required properties.

(b)(v)⇒(iii) Take $\tilde{p}$ from (v), $a \in G$ and $u \in L^\infty$. Let δ_a be the point-supported probability measure concentrated at a, and $f : G \to \mathbb{R}$ a function such that $f^\bullet = u$. For any $x \in G$,

$$(\delta_a * f)(x) = \int f(y^{-1}x)\delta_a(dy) = f(a^{-1}x) = (a\bullet_l f)(x),$$

so $\delta_a * f = a\bullet_l f$. Accordingly $\delta_a * u = a\bullet_l u$. Now, of course,

$$\tilde{p}(a\bullet_l u) = \tilde{p}(\delta_a * u) = \delta_a(G)\tilde{p}(u) = \tilde{p}(u),$$

as required by (iii).

(c)(iii)⇒(vi) Suppose that $\tilde{p}$ has the properties described in (iii). If $I \subseteq G$ and $J \subseteq \mathcal{L}^\infty(\mu)$ are finite, and $\epsilon > 0$, let $J^* \subseteq L^\infty$ be the finite set $\{a^{-1}\bullet_l f^\bullet : a \in I \cup \{e\}, f \in J \cup \{\chi G\}\}$. Let $\eta \in \left]0, \frac{1}{2}\right[$ be such that $2\eta(1 + 5\sup_{v\in J^*} \|v\|_\infty) \leq \epsilon$. By 243Gb, we can identify L^∞ with $(L^1)^*$, because Haar measures are (by definition) quasi-Radon, and quasi-Radon measure spaces are (strictly) localizable (415A). Because $\tilde{p} \in (L^\infty)^* \cong (L^1)^{**}$ has norm 1, there is a $u_0 \in L^1$ such that $\|u_0\| \leq 1$ and $|\tilde{p}(v) - \int v \times u_0| \leq \eta$ for every $v \in J^*$ (4A4If). In particular, taking $v = \chi G^\bullet$, $\int u_0 \geq 1 - \eta$. By 416I, there is a continuous function $h_0 : G \to \mathbb{R}$ with compact support such that $\|u_0 - h_0^\bullet\|_1 \leq \eta$. Then $\int h_0 d\mu \geq 1 - 2\eta$ and $\int |h_0| d\mu \leq 1 + \eta$. So if we set $h_0^+ = h_0 \vee 0$, $\gamma = \int h_0^+ d\mu$, $h = \frac{1}{\gamma} h_0^+$ we have

$$\|h_0^+ - h_0\|_1 = \frac{1}{2}\int |h_0| - h_0 \leq 2\eta$$

and $\|h - h_0^+\|_1 = |\gamma - 1| \leq 2\eta$, so $\|u_0 - h^\bullet\|_1 \leq 5\eta$, while $h \in C_{k1}^+$. This means that

$$|\tilde{p}(v) - \int v \times h^\bullet\| \leq \eta + 5\eta\|v\|_\infty \leq \frac{1}{2}\epsilon$$

for every $v \in J^*$.

Now suppose that $f \in J$ and $a \in I$. Then

$$\begin{aligned}
\Big|\int f(ax)h(x)\mu(dx) - \int f(x)h(x)\mu(dx)\Big| &\\
&= \Big|\int (a^{-1}\bullet_l f^\bullet) \times h^\bullet - \int f^\bullet \times h^\bullet\Big| \\
&\leq \Big|\int (a^{-1}\bullet_l f^\bullet) \times h^\bullet - \tilde{p}(a^{-1}\bullet_l f^\bullet)\Big| + \Big|\tilde{p}(f^\bullet) - \int f^\bullet \times h^\bullet\Big| \\
&\text{(because } \tilde{p} \text{ is } G\text{-invariant)} \\
&\leq \epsilon
\end{aligned}$$

because $f^\bullet$ and $a^{-1}\bullet_l f^\bullet$ belong to J^*. So h has the property required in (vi).

(d)(vi)⇒(ii) If $I \subseteq G$ and $J \subseteq C_b(X)$ are finite, and $\epsilon > 0$, then the set $A(I, J, \epsilon)$ of those $h \in C_{k1}^+$ such that $|\int f(ax)h(x)\mu(dx) - \int f(x)h(x)\mu(dx)| \leq \epsilon$ for every $a \in I$ and $f \in J$ is non-empty. So there is a filter $\mathcal{F}$ on C_{k1}^+ containing every $A(I, J, \epsilon)$. If we set $p(f) = \lim_{h\to\mathcal{F}} \int f(x)h(x)\mu(dx)$ for $f \in C_b(X)$, then p witnesses that (ii) is true.

(e)(ii)⇒(i) Given p as in (ii), its restriction to the space of bounded right-uniformly-continuous functions is positive, linear and G-invariant, so G is amenable.

(f)(v)⇒(iv) Take $\tilde{p}$ as in (v). If $g \in \mathcal{L}^1$, $f \in \mathcal{L}^\infty$ and $g \geq 0$, then

$$\begin{aligned}
\tilde{p}(g * f)^\bullet &= \tilde{p}(g\mu * f)^\bullet = \tilde{p}(g\mu * f^\bullet) \\
&= (g\mu)(G)\tilde{p}(f^\bullet) = \int g\, d\mu \cdot \tilde{p}(f^\bullet)
\end{aligned}$$

as required by (iv).

(g)(iv)⇒(vii) Suppose that (iv) is true.

(α) As in (d), just because $\tilde{p} \in (L^\infty)^*$ is a positive linear functional and $\tilde{p}(\chi G)^\bullet = 1$, we see that if $J \subseteq \mathcal{L}^\infty$ is finite and $\epsilon > 0$, then

$$A(J, \epsilon) = \{h : h \in C_{k1}^+, |\int f \times h\, d\mu - \tilde{p}(f^\bullet)| \leq \epsilon \text{ for every } f \in J\}$$

is non-empty. So we have a filter $\mathcal{F}$ on C^+_{k1} containing every $A(J,\epsilon)$, and $\tilde{p}(f^\bullet) = \lim_{h\to\mathcal{F}} \int f\times h\,d\mu$ for every $f \in \mathcal{L}^\infty$.

(β) Now $0 = \lim_{h\to\mathcal{F}} (g*h)^\bullet - (\int g\,d\mu)h^\bullet$ for the weak topology of L^1, for every $g \in \mathcal{L}^1$. **P** Set $\gamma = \int g\,d\mu$. Let $f \in \mathcal{L}^\infty$. Define g' by setting $g'(x) = \Delta(x^{-1})g(x^{-1})$ whenever this is defined, where Δ is the left modular function of G; then $g' \in \mathcal{L}^1$ and $\int g'd\mu = \gamma$ (442K(b-ii)). If $h \in C^+_{k1}$, then

$$\int f\times(g*h)d\mu = \iint f(xy)g(x)h(y)\mu(dx)\mu(dy)$$

(444Od)

$$= \int \big(\int \Delta(x^{-1})g'(x^{-1})f(xy)\mu(dx)\big)h(y)\mu(dy)$$

$$= \int (g'*f)(y)h(y)\mu(dy)$$

(444Oa)

$$= \int (g'*f)\times h\,d\mu.$$

As we have

$$\tilde{p}(g'*f)^\bullet = \tilde{p}(f^\bullet)\int g'd\mu = \gamma\tilde{p}(f^\bullet),$$

we get

$$\lim_{h\to\mathcal{F}} \int f\times(g*h-\gamma h)d\mu = \lim_{h\to\mathcal{F}} \int (g'*f-\gamma f)\times h\,d\mu$$
$$= \tilde{p}(g'*f)^\bullet - \gamma\tilde{p}(f^\bullet) = 0.$$

As f is arbitrary, and $(L^1)^*$ can be identified with L^∞, this is all we need. **Q**

(γ) Now take any finite set $J \subseteq \mathcal{L}^1$ and $\epsilon > 0$. On $(L^1)^J$ let $\mathfrak{T}$ be the locally convex linear space topology which is the product topology when each copy of L^1 is given the norm topology, and $\mathfrak{S}$ the corresponding weak topology. Define $T : C_k(G) \to (L^1)^J$ by setting

$$Th = \big\langle (g*h)^\bullet - (\textstyle\int g\,d\mu)h^\bullet \big\rangle_{g\in J}$$

for $h \in C_k(G)$, where $C_k(G)$ is the linear space of continuous real-valued functions on G with compact support. Then T is linear. Moreover, by (β), $\lim_{h\to\mathcal{F}} Th = 0$ in $(L^1)^J$ for the product topology, if each copy of L^1 is given its weak topology. But this is just $\mathfrak{S}$, by 4A4Be. In particular, 0 belongs to the $\mathfrak{S}$-closure of $T[C^+_{k1}]$. But C^+_{k1} is convex and T is linear, so $T[C^+_{k1}]$ is convex; by 4A4Ed, 0 belongs to the $\mathfrak{T}$-closure of $T[C^+_{k1}]$. There is therefore an $h \in C^+_{k1}$ such that $\|(g*h)^\bullet - (\int g\,d\mu)h^\bullet\|_1 \le \epsilon$ for every $g \in J$. As J and ϵ are arbitrary, (vii) is true.

(h)(vii)⇒(viii) Suppose that (vii) is true and that we have a compact set $K \subseteq G$ and $\epsilon > 0$. Set $\eta = \frac{1}{3}\epsilon$. Fix any $h_0 \in C^+_{k1}$. Let V be a neighbourhood of e such that $\|c\bullet_l h_0 - h_0\|_1 \le \eta$ whenever $c \in V$ (443Ge). Let $I \subseteq G$ be a finite set such that $K \subseteq IV$. By (vii), there is an $h_1 \in C^+_{k1}$ such that

$$\|b\bullet_l h_0 * h_1 - h_1\|_1 \le \eta \text{ for every } b \in I, \quad \|h_0 * h_1 - h_1\|_1 \le \eta.$$

(I omit brackets because $(b\bullet_l h_0)*h_1 = b\bullet_l(h_0*h_1)$, see 444Of.) Set $h = h_0*h_1$. Then $h \in C^+_{k1}$. **P** h is continuous, by 444Rc (or otherwise). If we set $M_i = \overline{\{x : h_i(x) \ne 0\}}$ for both i, then $h(x) = 0$ for every $x \in G \setminus M_0M_1$, so h has compact support. Of course $h \ge 0$, and $\int h\,d\mu = \int h_0d\mu \int h_1d\mu = 1$ (444Qb), so $h \in C^+_{k1}$. **Q**

If $a \in K$, there are $b \in I$, $c \in V$ such that $a = bc$, so that

$$\|a\bullet_l h - h\|_1 = \|b\bullet_l(c\bullet_l h_0 * h_1) - h_0 * h_1\|_1$$
$$\le \|b\bullet_l(c\bullet_l h_0 - h_0)*h_1\|_1 + \|b\bullet_l h_0 * h_1 - h_1\|_1 + \|h_1 - h_0*h_1\|_1$$
$$\le \|c\bullet_l h_0 - h_0\|_1 + \eta + \eta \le 3\eta = \epsilon.$$

So this h will serve.

(i)(viii)⇒(ix) Suppose that (viii) is true.

(α) I show first that for any ϵ, $\delta > 0$ and any compact set $K \subseteq G$ there are Borel sets E, $F \subseteq G$ such that $0 < \mu E < \infty$, $\mu F \leq \delta$ and $\mu(aE \triangle E) \leq \epsilon \mu E$ whenever $a \in K \setminus F$. **P** Of course it is enough to deal with the case in which $\mu K > 0$. Set $\eta = \epsilon\delta/\mu K$. Let $h \in C^+_{k1}$ be such that $\|a\bullet_l h - h\|_1 < \eta$ for every $a \in K$.

Set $K_0 = \overline{\{x : h(x) \neq 0\}}$ and $K^* = K_0 \cup KK_0$, so that $K^* \subseteq G$ is compact. Set

$$Q = \{(a, x, t) : a \in K,\ x \in G,\ t \in \mathbb{R},\ \text{either } h(x) \leq t < h(a^{-1}x) \text{ or } h(a^{-1}x) \leq t < h(x)\}.$$

Then Q is a Borel subset of $G \times G \times \mathbb{R}$ included in the compact set $K \times K^* \times [0, \|h\|_\infty]$. Let μ_L be Lebesgue measure on $\mathbb{R}$, and let $\mu \times \mu \times \mu_L$ be the τ-additive product measure on $G \times G \times \mathbb{R}$ (417D). (Of course this is actually a Radon measure.) For $t \in \mathbb{R}$ let E_t be the open set $\{x : h(x) > t\}$. Now 417H tells us that

$$(\mu \times \mu \times \mu_L)(Q) = \int_{G \times G} \mu_L\{t : (a, x, t) \in Q\}(\mu \times \mu)(d(a, x))$$

(where $\mu \times \mu$ is the τ-additive product measure on $G \times G$, so that we can identify $\mu \times \mu \times \mu_L$ with $(\mu \times \mu) \times \mu_L$), as in 417Db)

$$= \int_K \int_G |h(a^{-1}x) - h(x)| \mu(dx) \mu(da)$$

(we can use 417H again because $\{x : h(a^{-1}x) \neq h(x)\} \subseteq K^*$ if $a \in K$, and μK^* is finite)

$$= \int_K \|a\bullet_l h - h\|_1 \mu(da) < \eta \mu K$$

(by the choice of h)

$$= \eta \mu K \int h = \eta \mu K (\mu \times \mu_L)\{(x, t) : 0 \leq t < h(x)\}$$

$$= \eta \mu K \int_0^\infty \mu E_t \mu_L(dt)$$

as in 252N. (The c.l.d. and τ-additive product measures on $G \times \mathbb{R}$ coincide, by 417T.) On the other hand,

$$(\mu \times \mu \times \mu_L)(Q) = \int_{K \times \mathbb{R}} \mu\{x : (a, x, t) \in Q\}(\mu \times \mu_L)(d(a, t))$$

(again, we can use 417H because $x \in K^*$ whenever $(a, x, t) \in Q$)

$$= \int_{K \times \mathbb{R}} \mu(E_t \triangle aE_t)(\mu \times \mu_L)(d(a, t))$$

$$= \int_0^\infty \int_K \mu(E_t \triangle aE_t) \mu(da) \mu_L(dt)$$

because $E_t = aE_t = G$ whenever $t < 0$ and $a \in G$. So there must be some $t > 0$ such that

$$\textstyle\int_K \mu(E_t \triangle aE_t) \mu(da) < \eta \mu K \mu E_t = \epsilon\delta\mu E_t,$$

in which case $\mu E_t > 0$ and $F = \{a : a \in K,\ \mu(E_t \triangle aE_t) \geq \epsilon \mu E_t\}$ has measure at most δ. So we can set $E = E_t$. **Q**

(β) Now let $K \subseteq G$ be a compact set and $\epsilon > 0$, as in the statement of (ix); enlarging K and lowering ϵ if necessary, we may suppose that $\mu K > 0$ and $\epsilon \leq 1$. Set $K_1 = K \cup KK$, so that K_1 is still compact. By (α), we have a Borel set $E \subseteq G$ of finite measure such that $F = \{a : a \in K_1,\ \mu(aE \triangle E) > \frac{1}{3}\epsilon\mu E\}$ has measure less than $\frac{1}{2}\mu K$. If $a \in K$, then $F \cup a^{-1}F$ cannot cover K, so there is a $b \in K \setminus F$ such that $ab \notin F$; thus b and ab both belong to $K_1 \setminus F$, and

$$\mu(aE \triangle E) \leq \mu(aE \triangle abE) + \mu(abE \triangle E) \leq \mu(E \triangle bE) + \frac{1}{3}\epsilon\mu E \leq \frac{2}{3}\epsilon\mu E.$$

If we now take a compact set $L \subseteq E$ such that $8\mu(E \setminus L) \leq \epsilon\mu L$, we shall have

$$\begin{aligned}\mu(aL \triangle L) &\leq \mu(aE \setminus aL) + \mu(aE \triangle E) + \mu(E \setminus L)\\ &\leq 2\mu(E \setminus L) + \frac{2}{3}\epsilon(\mu L + \mu(E \setminus L)) \leq \epsilon\mu L\end{aligned}$$

for every $a \in K$, as required.

(j)(ix)⇒(x) Suppose that (ix) is true, and that $K \subseteq G$ is compact and $\epsilon > 0$. Enlarging K if necessary, we may suppose that it includes a neighbourhood of e. Of course we may also suppose that $\epsilon \leq 1$.

(α) The first thing to note is that there is a set $I \subseteq G$ such that $KI = G$ and $m = \sup_{y \in G} \#(\{x : x \in I, y \in Kx\})$ is finite. **P** Let V be a neighbourhood of e such that $VV^{-1} \subseteq K$. Let $I \subseteq G$ be maximal subject to $x^{-1}V \cap y^{-1}V = \emptyset$ for all distinct $x, y \in I$. If $x \in G$, there must be a $y \in I$ such that $x^{-1}V \cap y^{-1}V \neq \emptyset$, so that $x^{-1} \in y^{-1}VV^{-1}$ and $x \in VV^{-1}y \subseteq Ky \subseteq KI$; as x is arbitrary, $G \subseteq KI$. If $y \in G$ and $I_y = \{x : x \in I, y \in Kx\}$, then $I_y \subseteq K^{-1}y$, so that $\{x^{-1}V : x \in I_y\}$ is a disjoint family of subsets of $y^{-1}KV$. But this means that $\#(I_y)\mu V \leq \mu(y^{-1}KV) = \mu(KV)$. Accordingly $\sup_{y \in G} \#(I_y) \leq \frac{\mu(KV)}{\mu V}$ is finite. **Q**

(β) Set $\gamma = \sup_{a \in K} \Delta(a)$, $K^* = KKK^{-1}$. Let $\delta > 0$ be such that

$$\delta\gamma m < \epsilon(\mu K - \delta\gamma),$$

and let $\eta > 0$ be such that

$$1 + \frac{\delta\gamma m}{\mu K - \delta\gamma} \leq (1 + \epsilon)(1 - \frac{\eta}{\delta}\mu K^*).$$

By (ix), there is a compact set $L^* \subseteq G$ such that $\mu(aL^* \triangle L^*) \leq \eta\mu L^*$ for every $a \in K^*$. Set $L = \{x : x \in L^*, \mu(K^* \setminus L^*x^{-1}) \leq \delta\}$; note that L is closed (because $x \mapsto \mu(K^* \cap L^*x^{-1})$ is continuous, see 443C), therefore compact.

(γ) $\mu L \geq (1 - \frac{\eta}{\delta}\mu K^*)\mu L^*$. **P** Set $W = \{(x, y) : x \in K^*, y \in L^* \setminus L, xy \notin L^*\}$. Then W is a relatively compact Borel subset of $G \times G$, so we may apply Fubini's theorem (in the form 417H) to see that

$$\begin{aligned}\delta\mu(L^* \setminus L) &\leq \int_{L^* \setminus L} \mu(K^* \setminus L^*y^{-1})\mu(dy) = \int \mu W^{-1}[\{y\}]\mu(dy)\\ &= \int \mu W[\{x\}]\mu(dx) = \int_{K^*} \mu((L^* \setminus L) \setminus x^{-1}L^*)\mu(dx)\\ &\leq \int_{K^*} \mu(L^* \setminus x^{-1}L^*)\mu(dx) = \int_{K^*} \mu(xL^* \setminus L^*)\mu(dx) \leq \eta\mu K^*\mu L^*\end{aligned}$$

by the choice of L^*. Accordingly

$$\mu L = \mu L^* - \mu(L^* \setminus L) \geq (1 - \frac{\eta}{\delta}\mu K^*)\mu L^*. \ \textbf{Q}$$

In particular, $\mu L > 0$.

(δ) Set $J = \{x : x \in I, L \cap Kx \neq \emptyset\}$. Then $\Delta(x)(\mu K - \gamma\delta) \leq \mu(Kx \cap L^*)$ for every $x \in J$. **P** Take $y \in L \cap Kx$. Then $yx^{-1} \in K$ so $\Delta(y) = \Delta(yx^{-1})\Delta(x) \leq \gamma\Delta(x)$. Next,

$$\mu(K^*y \setminus L^*) = \Delta(y)\mu(K^* \setminus L^*y^{-1}) \leq \delta\Delta(y).$$

Since $x \in K^{-1}y$, $Kx \subseteq KK^{-1}y \subseteq K^*y$ and

$$\mu(Kx \setminus L^*) \leq \mu(K^*y \setminus L^*) \leq \delta\Delta(y) \leq \delta\gamma\Delta(x)$$

and

$$\mu(Kx \cap L^*) = \mu(Kx) - \mu(Kx \setminus L^*) \geq \Delta(x)\mu K - \delta\gamma\Delta(x) = \Delta(x)(\mu K - \delta\gamma). \ \textbf{Q}$$

(ε) Now recall that $\sum_{x \in I} \chi(Kx) \leq m\chi G$, by the definition of m, so that

$$(\mu K - \delta\gamma)\textstyle\sum_{x\in J}\Delta(x) \le \sum_{x\in J}\mu(Kx\cap L^*) \le m\mu L^*.$$

For each $x \in J$, choose $z_x \in Kx \cap L$. Since $KI = G$, $L \subseteq KJ$ and

$$KL \subseteq \bigcup_{x\in J} KKx \subseteq \bigcup_{x\in J} KKK^{-1}z_x = \bigcup_{x\in J} K^*z_x,$$

$$\begin{aligned}\mu(KL\setminus L^*) &\le \sum_{x\in J}\mu(K^*z_x\setminus L^*) = \sum_{x\in J}\Delta(z_x)\mu(K^*\setminus L^*z_x^{-1})\\ &\le \delta\sum_{x\in J}\Delta(z_x) \le \delta\gamma\sum_{x\in J}\Delta(x) \le \tfrac{\delta\gamma m}{\mu K-\delta\gamma}\mu L^*.\end{aligned}$$

Accordingly

$$\mu(KL) \le \mu L^*(1+\tfrac{\delta\gamma m}{\mu K-\delta\gamma}) \le \mu L\cdot\frac{1+\frac{\delta\gamma m}{\mu K-\delta\gamma}}{1-\frac{\eta}{\delta}\mu K^*}$$

(by (γ) above)

$$\le (1+\epsilon)\mu L,$$

by the choice of δ and η. Thus we have found an appropriate set L.

(k)(x)⇒(ii) Write $\mathcal{L}$ for the family of all compact subsets of G with non-zero measure. For $L \in \mathcal{L}$, define $p_L : C_b(X) \to \mathbb{R}$ by setting $p_L(f) = \frac{1}{\mu L}\int_L f d\mu$ for $f \in C_b(X)$. For finite $I \subseteq G$, $\epsilon > 0$ set

$$\mathcal{A}(I,\epsilon) = \{L : L \in \mathcal{L},\ \mu(aL\setminus L) \le \tfrac{1}{2}\epsilon\mu L \text{ for every } a \in I\}.$$

By (x), no $\mathcal{A}(I,\epsilon)$ is empty, since there is always an $L \in \mathcal{L}$ such that $\mu((I\cup\{e\})L) \le (1+\frac{1}{2}\epsilon)\mu L$. So we have an ultrafilter $\mathcal{F}$ on $\mathcal{L}$ containing every $\mathcal{A}(I,\epsilon)$. Set $p(f) = \lim_{L\to\mathcal{F}} p_L(f)$ for $f \in C_b(X)$; then $p : C_b(X) \to \mathbb{R}$ is a positive linear functional and $p(\chi G) = 1$.

If $a \in G$, $f \in C_b(X)$ and $L \in \mathcal{L}$, then

$$\begin{aligned}|p_L(a\bullet_l f) - p_L(f)| &= \tfrac{1}{\mu L}\big|\int_L f(a^{-1}x)\mu(dx) - \int_L f(x)\mu(dx)\big|\\ &= \tfrac{1}{\mu L}\big|\int_{aL} f d\mu - \int_L f d\mu\big|\\ &\le \tfrac{1}{\mu L}\|f\|_\infty\mu(aL\triangle L) = \tfrac{1}{\mu L}\|f\|_\infty(\mu(aL\setminus L)+\mu(a^{-1}L\setminus L)).\end{aligned}$$

Since $\mathcal{F}$ contains $\mathcal{A}(\{a,a^{-1}\},\epsilon)$ for every $\epsilon > 0$,

$$|p(a\bullet_l f) - p(f)| = \lim_{L\to\mathcal{F}}|p_L(a\bullet_l f) - p_L(f)| = 0.$$

So p witnesses that (ii) is true.

449J If we make a further step back towards the origin of this topic, and suppose that our group is discrete, then we have a striking further condition to add to the list above. I give this as a corollary of a general result on group actions recalling the main theorems of §§394 and 448.

Tarski's theorem Let G be a group acting on a non-empty set X. Then the following are equiveridical:

(i) there is an additive functional $\nu : \mathcal{P}X \to [0,1]$ such that $\nu X = 1$ and $\nu(a\bullet A) = \nu A$ whenever $A \subseteq X$ and $a \in G$;

(ii) there are no $A_0,\dots,A_n$, $a_0,\dots,a_n$, $b_0,\dots,b_n$ such that $A_0,\dots,A_n$ are subsets of X covering X, $a_0,\dots,a_n,b_0,\dots,b_n$ all belong to G, and $a_0\bullet A_0$, $b_0\bullet A_0$, $a_1\bullet A_1$, $b_1\bullet A_1,\dots,b_n\bullet A_n$ are all disjoint.

proof (a)(i)⇒(ii) This is elementary, for if $\nu : \mathcal{P}X \to [0,1]$ is a non-zero additive functional and $A_0, \dots, A_n$ cover X and $a_0, \dots, b_n \in G$, then

$$\sum_{i=0}^n \nu(a_i{\bullet}A_i) + \sum_{i=0}^n \nu(b_i{\bullet}A_i) \geq 2\nu X > \nu X,$$

and $a_0{\bullet}A_0, \dots, b_n{\bullet}A_n$ cannot all be disjoint.

(b)(ii)⇒(i) Now suppose that (ii) is true.

(α) Suppose that $c_0, \dots, c_n \in G$. Then there is a finite set $I \subseteq X$ such that $\#(\{c_i{\bullet}x : i \leq n, x \in I\}) < 2\#(I)$. **P?** Otherwise, by the Marriage Lemma in the form 4A1H, applied to the set

$$R = \{((x,j), c_i{\bullet}x) : x \in X, j \in \{0,1\}, i \leq n\} \subseteq (X \times \{0,1\}) \times X,$$

there is an injective function $\phi : X \times \{0,1\} \to X$ such that $\phi(x,j) \in \{c_i{\bullet}x : i \leq n\}$ for every $x \in X$ and $j \in \{0,1\}$. Now set $B_{ij} = \{x : \phi(x,0) = c_i{\bullet}x, \phi(x,1) = c_j{\bullet}x\}$ for $i, j \leq n$, so that $X = \bigcup_{i,j \leq n} B_{ij}$. Let $A_{ij} \subseteq B_{ij}$ be such that $\langle A_{ij}\rangle_{i,j\leq n}$ is a partition of X, and set $a_{ij} = c_i$, $b_{ij} = c_j$ for $i, j \leq n$; then $a_{ij}{\bullet}A_{ij} \subseteq \phi[A_{ij} \times \{0\}]$, $b_{ij}{\bullet}A_{ij} \subseteq \phi[A_{ij} \times \{1\}]$ are all disjoint, which is supposed to be impossible. **XQ**

(β) Suppose that $J \subseteq G$ is finite and $\epsilon > 0$. Then there is a non-empty finite set $I \subseteq X$ such that $\#(I \triangle c{\bullet}I) \leq \epsilon\#(I)$ for every $c \in J$. **P** It is enough to consider the case in which the identity e of G belongs to J. **?** Suppose, if possible, that there is no such set I. Let $m \geq 1$ be such that $(1 + \frac{1}{2}\epsilon)^m \geq 2$. Set $K = J^m = \{c_1c_2\dots c_m : c_1, \dots, c_m \in J\}$. By (α), there is a finite set $I_0 \subseteq X$ such that $\#(I_0^*) < 2\#(I_0)$, where $I_0^* = \{a{\bullet}x : a \in K, x \in I_0\}$. Choose $c_1, \dots, c_m$ and $I_1, \dots, I_m$ inductively such that

given that I_k is a non-empty finite subset of X, where $k < m$, take $c_{k+1} \in J$ such that $\#(I_k \triangle c_{k+1}{\bullet}I_k) > \epsilon\#(I_k)$ and set $I_{k+1} = I_k \cup c_{k+1}{\bullet}I_k$.

Then $I_k \subseteq \{a{\bullet}x : a \in J^k, x \in I_0\}$ for each $k \leq m$, and in particular $I_m \subseteq I_0^*$. But also

$$\begin{aligned}\#(I_{k+1}) &= \#(I_k) + \#((c_{k+1}{\bullet}I_k) \setminus I_k)\\ &= \#(I_k) + \tfrac{1}{2}\#((c_{k+1}{\bullet}I_k) \triangle I_k) \geq (1 + \tfrac{1}{2}\epsilon)\#(I_k)\end{aligned}$$

for every $k < m$, so

$$\#(I_0^*) \geq \#(I_m) \geq (1 + \tfrac{1}{2}\epsilon)^m \#(I_0) \geq 2\#(I_0),$$

contrary to the choice of I_0. **XQ**

(γ) There is therefore an ultrafilter $\mathcal{F}$ on $[X]^{<\omega} \setminus \{\emptyset\}$ such that

$$\mathcal{A}_{c\epsilon} = \{I : I \in [X]^{<\omega} \setminus \{\emptyset\}, \#(I \triangle c{\bullet}I) \leq \epsilon\#(I)\}$$

belongs to $\mathcal{F}$ for every $c \in G$ and $\epsilon > 0$. For $I \in [X]^{<\omega} \setminus \{\emptyset\}$ and $A \subseteq X$ set $\nu_I(A) = \#(A \cap I)/\#(I)$, and set $\nu A = \lim_{I \to \mathcal{F}} \nu_I A$ for every $A \subseteq X$, so that $\nu : \mathcal{P}X \to [0,1]$ is an additive functional and $\nu X = 1$.

Now ν is G-invariant. **P** If $A \subseteq X$ and $c \in G$ and $\epsilon > 0$, then $\mathcal{A}_{c^{-1},\epsilon} \in \mathcal{F}$. If $I \in \mathcal{A}_{c^{-1},\epsilon}$, then

$$\begin{aligned}|\nu_I(c{\bullet}A) - \nu_I(A)| &= \frac{1}{\#(I)}|\#(I \cap (c{\bullet}A)) - \#(I \cap A)|\\ &= \frac{1}{\#(I)}|\#((c^{-1}{\bullet}I) \cap A) - \#(I \cap A)|\\ &\leq \frac{1}{\#(I)}|\#((c^{-1}{\bullet}I) \triangle I| \leq \epsilon.\end{aligned}$$

As ϵ is arbitrary, $\lim_{I \to \mathcal{F}} \nu_I(c{\bullet}A) - \nu_I(A) = 0$, and $\nu(c{\bullet}A) = \nu A$. As A and c are arbitrary, ν is G-invariant. **Q**

So ν witnesses that (i) is true, and the proof is complete.

449K Corollary Let G be a group with its discrete topology. Then the following are equiveridical:

(i) G is amenable;

(ii) there are no $A_0, \dots, A_n$, $a_0, \dots, a_n$, $b_0, \dots, b_n$ such that $A_0, \dots, A_n$ are subsets of G covering G, $a_0, \dots, a_n, b_0, \dots, b_n$ all belong to G, and a_0A_0, b_0A_0, a_1A_1, $b_1A_1, \dots, b_nA_n$ are all disjoint.

proof All we have to observe is that (α) every function from G to $\mathbb{R}$ is uniformly continuous for the right uniformity of G, so that G is amenable iff there is an invariant positive linear functional $p : \ell^\infty(G) \to \mathbb{R}$ such that $p(\chi G) = 1$ (β) that a positive linear functional on $\ell^\infty(G)$ is G-invariant iff the corresponding additive functional on $\mathcal{P}G$ is G-invariant. So (i) of 449J is equivalent to amenability of G as defined in 449A.

449L Theorem Let G be a group which is amenable in its discrete topology, X a set, and $\bullet$ an action of G on X. Let Σ be a subring of $\mathcal{P}X$ and $\nu : \Sigma \to [0, \infty[$ a finitely additive functional which is G-invariant in the sense that $g\bullet E \in \Sigma$ and $\nu(g\bullet E) = \nu E$ whenever $E \in \Sigma$ and $g \in G$. Then there is an extension of ν to a G-invariant non-negative finitely additive functional $\tilde{\nu}$ defined on the ideal $\mathcal{I}$ of subsets of X generated by Σ.

proof (a) There is a non-negative finitely additive functional $\theta : \mathcal{I} \to \mathbb{R}$ extending ν. **P** Let V be the linear subspace of $\ell^\infty(X)$ generated by $\{\chi E : E \in \Sigma\}$, so that V can be identified with the Riesz space $S(\Sigma)$ (361L). Let U be the solid linear subspace of $\ell^\infty(X)$ generated by V. For $u \in U$ set $q(u) = \inf\{\int v\,d\nu : v \in V, |u| \leq v\}$. Then q is a seminorm, and $|\int v\,d\nu| \leq q(v)$ for every $v \in V$. So there is a linear functional $f : U \to \mathbb{R}$ such that $f(v) = \int v\,d\nu$ for every $v \in V$ and $|f(u)| \leq q(u)$ for every $u \in U$ (4A4D(a-i)). Set $\theta A = f(\chi A)$ for $A \in \mathcal{I}$. Then $\theta : \mathcal{I} \to \mathbb{R}$ is additive and extends ν. If $A \in \mathcal{I}$, there is an $E \in \Sigma$ including A. Now we have

$$\theta(E \setminus A) = f(\chi E - \chi A) \leq q(\chi E - \chi A) \leq \int \chi E\,d\nu = \nu E = \theta E,$$

so $\theta A \geq 0$. So θ is non-negative. **Q**

(b) As in the proof of 449K, we have a positive G-invariant linear functional $p : \ell^\infty(G) \to \mathbb{R}$ such that $p(\chi G) = 1$. For $A \in \mathcal{I}$, set $f_A(a) = \theta(a^{-1}\bullet A)$ for $a \in G$, and $\tilde{\nu}A = p(f_A)$. Then $\tilde{\nu} : \mathcal{I} \to [0, \infty[$ is additive. If $E \in \Sigma$ then $f_A(a) = \nu E$ for every a, so $\tilde{\nu}$ extends ν. If $A \in \mathcal{I}$ and a, $b \in G$, then $f_{aA}(b) = \theta(b^{-1}aA) = f_A(a^{-1}b)$, so $f_{aA} = a\bullet_l f_A$ and

$$\tilde{\nu}(aA) = p(f_{aA}) = p(f_A) = \tilde{\nu}A.$$

Thus $\tilde{\nu}$ is G-invariant, as required.

449M Corollary (Banach 23) If $r = 1$ or $r = 2$, there is a functional $\tilde{\mu} : \mathcal{P}\mathbb{R}^r \to [0, \infty]$ such that (i) $\tilde{\mu}(A \cup B) = \tilde{\mu}A + \tilde{\mu}B$ whenever A, $B \subseteq \mathbb{R}^r$ are disjoint (ii) $\tilde{\mu}E$ is the Lebesgue measure of E whenever $E \subseteq \mathbb{R}^r$ is Lebesgue measurable (iii) $\tilde{\mu}(g[A]) = \tilde{\mu}A$ whenever $A \subseteq \mathbb{R}^r$ and $g : \mathbb{R}^r \to \mathbb{R}^r$ is an isometry.

proof (a) The point is that the group G of all isometries of $\mathbb{R}^r$, with the discrete topology, is amenable. **P** Let $G_0 \subseteq G$ be the subgroup consisting of rotations about $\mathbf{0}$; because $r \leq 2$, this is abelian, therefore amenable (449Cf). Let $G_1 \subseteq G$ be the subgroup consisting of isometries keeping $\mathbf{0}$ fixed; then G_0 is a normal subgroup of G_1, and G_1/G_0 is abelian, so G_1 is amenable (449Cc). Let $G_2 \subseteq G$ be the normal subgroup consisting of translations; then G_2 is abelian, therefore amenable. Now $G = G_1G_2$, so G is amenable (449Cd). **Q**

(b) Let Σ be the ring of subsets of $\mathbb{R}^r$ with finite Lebesgue measure, and let ν be the restriction of Lebesgue measure to Σ. Then ν is G-invariant. By 449L, there is a G-invariant extension $\tilde{\nu}$ of ν to the ideal $\mathcal{I}$ generated by Σ. Setting $\tilde{\mu}A = \tilde{\nu}A$ for $A \in \mathcal{I}$, ∞ for $A \in \mathcal{P}\mathbb{R}^r \setminus \mathcal{I}$, we have a suitable functional $\tilde{\mu}$.

449X Basic exercises >(a) Let G be a topological group. On G define a binary operation $\diamond$ by saying that $x \diamond y = yx$ for all x, $y \in G$. Show that $(G, \diamond)$ is a topological group isomorphic to G, so is amenable iff G is.

(b) Show that any finite topological group is amenable.

>(c) Let X be a set and G the group of all permutations of X. (i) Give X the metric ρ such that $\rho(x, y) = 1$ for all distinct x, $y \in X$, so that G is the isometry group of X. Show that G, with the topology of pointwise convergence (441G), is amenable. (*Hint*: for any $I \in [X]^{<\omega}$, $\{a : a \in G, a(x) = x$ for every $x \in X \setminus I\}$ is amenable.) (ii) Show that if X is infinite then G, with its discrete topology, is not amenable. (*Hint*: the left action of F_2 on itself can be regarded as an embedding of F_2 in G; use 449F.)

>(d) Show that, for any $r \in \mathbb{N}$, the isometry group of $\mathbb{R}^r$, with the topology of pointwise convergence, is amenable. (*Hint*: 443Xx, 449Cd.)

(e) Find a metrizable σ-compact locally compact Hausdorff topological group which is amenable but not unimodular. (*Hint*: 442Xf, 449Cd.)

(f) Let G be a topological group and U the space of bounded real-valued functions on G which are uniformly continuous for the right uniformity of G. Show that $\bullet_r$, as defined in 441Ac, gives an action of G on U.

(g) Show that a dense subgroup of an amenable topological group is amenable. (*Hint*: 449E(iii).)

(h) Let G be a Hausdorff topological group, and $\hat{G}$ its completion with respect to its bilateral uniformity (definition: 4A5Hb). Show that G is amenable iff $\hat{G}$ is.

(i) Let $\bullet$ be an action of a group G on a set X, and U a Riesz subspace of $\ell^\infty(X)$, containing the constant functions, such that $a\bullet f \in U$ for every $f \in U$, $a \in G$. Show that the following are equiveridical: (i) there is a G-invariant positive linear functional $p : U \to \mathbb{R}$ such that $p(\chi X) = 1$; (ii) $\sup_{x\in X} \sum_{i=0}^n f(x_i) - f(a_i \bullet x_i) \geq 0$ whenever $f_0, \ldots, f_n \in U$ and $a_0, \ldots, a_n \in G$. (*Hint*: if (ii) is true, let V be the linear subspace generated by $\{f - a\bullet f : f \in U, a \in G\}$ and show that $\inf_{g\in V} \|g - \chi X\|_\infty = 1$.)

(j) Let G be the group with generators a, b and relations $a^2 = b^3 = e$; that is, the quotient of the free group on two generators a and b by the normal subgroup generated by $\{a^2, b^3\}$. Show that G, with its discrete topology, is not amenable.

>(k) Let G be a locally compact Hausdorff group, and μ a left Haar measure on G. Show that the following are equiveridical: (i) G is amenable; (ii) there is a positive linear functional $p^\# : L^\infty(\mu) \to \mathbb{R}$ such that $p^\#(\chi G^\bullet) = 1$ and $p^\#(a\bullet_r u) = p^\#(u)$ for every $u \in L^\infty(\mu)$ and every $a \in G$, where $\bullet_r$ is defined as in 443G; (iii) for every finite set $I \subseteq G$, finite set $J \subseteq \mathcal{L}^\infty(\mu)$ and $\epsilon > 0$, there is an $h \in C^+_{k1}$ (definition: 449I) such that $|\int f(xa)h(x)\mu(dx) - \int f(x)h(x)\mu(dx)| \leq \epsilon$ for every $a \in I$ and $f \in J$.

>(l) Let G be a unimodular locally compact Hausdorff group, and μ a Haar measure on G. Show that G is amenable iff whenever $K \subseteq G$ is compact and $\epsilon > 0$ there is a compact set $L \subseteq G$ with non-zero measure such that $\mu(LK) \leq (1 + \epsilon)\mu L$.

(m) Let G be a locally compact Hausdorff group, and μ a left Haar measure on G. Show that the following are equiveridical: (α) G is amenable; (β) for every finite set $I \subseteq G$ and $\epsilon > 0$ there is a $u \in L^2(\mu)$ such that $\|u\|_2 = 1$ and $\|a\bullet_l u - u\|_2 \leq \epsilon$ for every $a \in I$; (γ) for every $I \in [G]^{<\omega}$ and $\epsilon > 0$ there is an $h \in C_k(G)$ such that $\|h\|_2 = 1$ and $\|a\bullet_l h - h\|_2 \leq \epsilon$ for every $a \in I$; (δ) for every $I \in [G]^{<\omega}$ and $\epsilon > 0$ there is an $h \in C_k(G)^+$ such that $\|h\|_2 = 1$ and $\|a\bullet_l h - h\|_2 \leq \epsilon$ for every $a \in I$. (*Hint*: 449I(vii)$\Rightarrow$ (β) $\Rightarrow$ (γ) $\Rightarrow$ (δ) $\Rightarrow$449I(iv).)

(n) Let G be a locally compact Hausdorff topological group with left Haar measure μ. Show that the following are equiveridical: (i) G is amenable; (viii)$'$ for every compact set $K \subseteq G$ and $\epsilon > 0$, there is an $h \in C^+_{k1}$ (definition: 449I) such that $h(x^{-1}) = h(x)$ for every $x \in G$ and $\|a\bullet_l h - h\|_1 \leq \epsilon$ for every $a \in K$; (ix)$'$ for every compact set $K \subseteq G$ and $\epsilon > 0$ there is a symmetric compact set $L \subseteq G$ with non-empty interior such that $\mu L > 0$ and $\mu(L \triangle aL) \leq \epsilon\mu L$ for every $a \in K$.

(o) Let G be a unimodular locally compact Hausdorff topological group with Haar measure μ. Show that the following are equiveridical: (i) G is amenable; (iii)$'$ there is a positive linear functional $\tilde{p} : L^\infty(\mu) \to \mathbb{R}$ such that $\tilde{p}(\chi G^\bullet) = 1$ and $\tilde{p}(a\bullet_l u) = \tilde{p}(a\bullet_r u) = \tilde{p}(u)$ for every $u \in L^\infty(\mu)$ and $a \in G$.

(p) Let G be a locally compact Hausdorff group and μ a left Haar measure on G. Suppose that G is **exponentially bounded**, that is, $\lim_{n\to\infty}(\mu(K^n))^{1/n} \leq 1$ for every compact set $K \subseteq G$. (i) Show that G is amenable. (ii) Show that $(G, \diamond)$, as defined in 449Xa, is exponentially bounded.

(q) Let G be a locally compact Hausdorff group and $\mathcal{B}(G)$ its Borel σ-algebra. Show that G is amenable iff there is a non-zero finitely additive $\nu : \mathcal{B}(G) \to [0, 1]$ such that $\nu(aE) = \nu E$ for every $a \in G$ and $E \in \mathcal{B}(G)$.

(r) Let X be a non-empty set, G a group and $\bullet$ an action of G on X. Suppose that G is an amenable group when given its discrete topology. Show that there is an additive functional $\nu : \mathcal{P}X \to [0,1]$ such that $\nu X = 1$ and $\nu(a\bullet A) = \nu A$ for every $A \subseteq X$ and every $a \in G$.

>(s) Let G be a group which is amenable when given its discrete topology. Let $\nu : \mathcal{P}G \to [0,1]$ be an additive functional such that $\nu G = 1$ and $\nu(aE) = \nu E$ for every $E \subseteq G$, $a \in G$. For $E \subseteq G$ set $\tilde{\nu}E = \int \nu(Ex)\nu(dx)$ (notation: 363Lf). Show that $\tilde{\nu} : \mathcal{P}G \to [0,1]$ is an additive functional, that $\tilde{\nu}G = 1$ and that $\tilde{\nu}(aE) = \tilde{\nu}(Ea) = \tilde{\nu}E$ for every $E \subseteq G$ and $a \in G$.

(t) Let G be a locally compact Hausdorff group and μ a left Haar measure on G. Show that if G, with its discrete topology, is amenable, then there is a functional $\lambda : \mathcal{P}G \to [0,\infty]$, extending μ, such that $\lambda(A \cup B) = \lambda A + \lambda B$ whenever A, $B \subseteq G$ are disjoint and $\lambda(xA) = \lambda A$ for every $x \in G$ and $A \subseteq G$.

(u) Let X be a compact metrizable space, $\phi : X \to X$ a continuous function and μ a Radon probability measure on X such that $\mu\phi^{-1} = \mu$. Show that for μ-almost every $x \in X$, $\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} f(\phi^i(x))$ is defined for every $f \in C(X)$. (*Hint*: 4A2Pe, 372K.)

449Y Further exercises (a) If S is a semigroup with identity e and X is a set, an **action** of S on X is a map $(s,x) \mapsto s\bullet x : S \times X \to X$ such that $s\bullet(t\bullet x) = (st)\bullet x$ and $e\bullet x = x$ for every s, $t \in S$ and $x \in X$. A topological semigroup S with identity is **amenable** if for every non-empty compact Hausdorff space X and every continuous action of S on X there is a Radon probability measure μ on X such that $\int f(s\bullet x)\mu(dx) = \int f(x)\mu(dx)$ for every $s \in S$ and $f \in C(X)$. Show that

(i) $(\mathbb{N}, +)$, with the discrete topology, is amenable;

(ii) if S is a topological semigroup and $\mathcal{S}$ is an upwards-directed family of amenable sub-semigroups of S with dense union in S, then S is amenable;

(iii) if $\langle S_i \rangle_{i\in I}$ is a family of amenable topological semigroups with product S then S is amenable;

(iv) if S is an amenable topological semigroup, S' is a topological semigroup, and there is a continuous multiplicative surjection from S onto S', then S' is amenable;

(v) if S is an abelian topological semigroup, then it is amenable.

(b) Let G be the group with generators a, b and relations $a^2 = b^2 = e$. Show that G, with its discrete topology, is amenable.

(c) Let G be a topological group and U the space of bounded real-valued functions on G which are uniformly continuous for the right uniformity. Let M^+_{qR} be the space of totally finite quasi-Radon measures on G. (i) Show that if $\nu \in M^+_{\mathrm{qR}}$ then $\nu * f$ (definition: 444H) belongs to U for every $f \in U$. (ii) Show that $(\nu, f) \mapsto \nu * f : M^+_{\mathrm{qR}} \times U \to U$ is continuous if M^+_{qR} is given its narrow topology (437Jd) and U is given its norm topology. (iii) Show that if $p : U \to \mathbb{R}$ is a continuous linear functional such that $p(a\bullet_l f) = p(f)$ for every $f \in U$ and $a \in G$, then $p(\nu * f) = \nu G \cdot p(f)$ for every $f \in U$ and every totally finite quasi-Radon measure ν on G.

(d) Re-work 449I for general groups carrying Haar measures.

(e) Let X be a set, A a subset of X, and G a group acting on X. Show that the following are equiveridical: (i) there is a functional $\theta : \mathcal{P}X \to [0,\infty]$ such that $\theta A = 1$, $\theta(B \cup C) = \theta B + \theta C$ and $\theta(a\bullet B) = \theta B$ for all disjoint B, $C \subseteq X$ and $a \in G$; (ii) there are no $A_0, \dots, A_n$, $a_0, \dots, a_n$, $b_0, \dots, b_n$ such that $A_0, \dots, A_n$ are disjoint subsets of A covering A, $a_0, \dots, b_n$ all belong to G, and $a_0\bullet A_0$, $b_0\bullet A_0$, $a_1\bullet A_1$, $b_1\bullet A_1, \dots, b_n\bullet A_n$ are all disjoint subsets of A.

(f) (Swierczkowski 58) Let G be the group of orthogonal 3×3 matrices. Set $S = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} & 0 \\ -\frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{3}{5} & \frac{4}{5} \\ 0 & -\frac{4}{5} & \frac{3}{5} \end{pmatrix}$. Show that S and T are free in G (that is, no non-trivial product of the form

$S^{n_0}T^{n_1}S^{n_2}T^{n_3}\dots S^{n_{2k}}T^{n_{2k+1}}$ can be the identity), so that G is not amenable in its discrete topology. (*Hint*: let R be the ring of 3×3 matrices over the field $\mathbb{Z}_5$. In R set $\sigma = \begin{pmatrix} 3 & 4 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\tau = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & -4 & 3 \end{pmatrix}$. Show that $\sigma^2 = \sigma$. Now suppose $\rho \in R$ is defined as a non-trivial product of the elements σ, τ and their transposes σ', τ' in which σ and σ' are never adjacent, τ and τ' are never adjacent, and the last term is σ. Set $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \rho \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Show that if the first term in the product is σ or σ', then $c = 0$ and $b \neq 0$, and otherwise $a = 0$ and $b \neq 0$.)

(g) Let F_2 be the free group on two generators a, b. (i) Show that there is a set $W \subseteq F_2$ such that $aW = bW \cup b^2W = F_2 \setminus W$. (ii) Let S_2 be the unit sphere in $\mathbb{R}^3$. Show that if S, T are the matrices of 449Yf, there is a set $A \subseteq S_2$ such that $(A, S[A])$ and $(A, T[A], T^2[A])$ are both partitions of a cocountable subset of S_2. (iii) Show that there is no non-zero rotation-invariant additive functional from $\mathcal{P}S_2$ to $[0,1]$. (iv) Show that there is no rotation-invariant additive functional on the ball $\{x : x \in \mathbb{R}^3, \|x\| \leq 1\}$ which extends Lebesgue measure.

(h) Give an example of a topological semigroup S with identity such that S is amenable in the sense of 449Ya but $(S, \diamond)$ is not, where $a \diamond b = ba$ for a, $b \in S$.

449 Notes and comments The general theory of amenable groups is outside the scope of this book. Here I have tried only to indicate some of the specifically measure-theoretic arguments which are used in the theory. Primarily we have the Riesz representation theorem, enabling us to move between linear functionals and measures. Since the invariant measures considered in the definition of 'amenable group' are all Radon measures on compact Hausdorff spaces, they can equally well be thought of as linear functionals on spaces of continuous functions. What is striking is that the definition in terms of continuous actions on arbitrary compact Hausdorff spaces can be reduced to a question concerning an invariant mean on a single space easily constructed from the group (449E).

The first part of this section deals with general topological groups. It is a remarkable fact that some of the most important non-locally-compact topological groups are amenable. For most of these we shall have to wait until we have done 'concentration of measure' (§§476, 492) and can approach 'extremely amenable' groups (§493). But there is an easy example in 449Xc which already indicates one of the basic methods.

449I is taken from Greenleaf 69, where you will find many references to its development. Historically the subject was dominated by the case of discrete groups, in which combinatorial rather than measure-theoretic formulations seem more appropriate; we also have an extra method of attack, suggested in 449Yb. In 449I, conditions (i)-(iii) relate to 'invariant means' of one kind or another. Note that the invariant means p and $\tilde{p}$ of 449E(iii) and 449I(ii) are normalized by the condition $p(\chi G) = \tilde{p}(\chi G^\bullet) = 1$, while the left Haar measure μ of 449I has a degree of freedom; so that when they come together in 449I(iii) the two sides of the equation $\tilde{p}(g * f)^\bullet = \tilde{p}(f^\bullet) \int g\, d\mu$ must move together if we change μ by a scalar factor; of course this happens through the hidden dependence of the convolution operation on μ. Between 449I(i) and 449I(iii) there is a double step. First we note that a convolution $g * f$ is a kind of weighted average of left translates of f, so that if we have a mean which is invariant under translations we can hope that it will be invariant under convolutions (449H, 449Yc). What is more remarkable is that an invariant mean on the space U of bounded uniformly continuous functions should necessarily have an extension to an invariant mean on the space L^∞ of (equivalence classes of) bounded Haar measurable functions.

Condition (vi) in 449I looks at a different aspect of the phenomenon. In effect, it amounts to saying that not only is there an invariant mean, but that there is an invariant mean defined by the formula $p(f) = \lim_{L\to\mathcal{F}} \frac{1}{\mu L}\int_L f$ for a suitable filter $\mathcal{F}$ on the family of sets of non-zero finite measure. This may be called a 'Følner condition', following Følner 55. Condition (iv) looks for an invariant mean of the form $p(f) = \lim_{h\to\mathcal{F}} \int f \times h\, d\mu$, where $\mathcal{F}$ is a suitable filter on $L^1(\mu)$.

The techniques developed in §444 to handle Haar measures on groups which are not locally compact can also be used in 449H-449I, using 'totally bounded for the bilateral uniformity' in place of 'compact' when appropriate (449Yd). 443L provides another route to the same generalization.

The words 'right' and 'left' appear repeatedly in this section, and it is not perhaps immediately clear which of the ordinary symmetries we can expect to find. The fact that the operation $x \mapsto x^{-1} : G \to G$ always gives us an isomorphism between a group and the same set with the multiplication reversed (449Xa) means that we do not have to distinguish between 'left amenable' and 'right amenable' groups, at least if we start from the definition in 449A. In 449C also there is nothing to break the symmetry between left and right. In 449B and 449D-449E, however, we must commit ourselves to the *left* action of the group on the space of functions which are uniformly continuous with respect to the *right* uniformity. If we wish to change one, we must also change the other. In the list of conditions in 449I, some can be reflected, but in groups which are not unimodular many cannot; see 449Xk-449Xl. For semigroups we do have a difference between 'left' and 'right' amenability (449Yh).

In the case of discrete groups, in which all considerations of measurability and continuity evaporate, we have a completely different technique available, as in 449J. Here we can go directly from a non-paradoxicality condition, a weaker version of conditions already introduced in 394E and 448E, to a Følner condition ((β) in part (b) of the proof of 449J) which easily implies amenability. I remind you that I still do not know whether these ideas can be applied to other algebras than $\mathcal{P}X$ (394Z). The difficulty is that the unscrupulous use of the axiom of choice in the infinitary Marriage Lemma (4A1H) seems to give us no control over the nature of the sets A_{ij} described in (b-α) of the proof of 449J; moreover, the structure of the proof depends on having a suitable invariant measure (counting measure on X) to begin with. For more on amenable discrete groups and their connexions with measure theory see Laczkovich 02.

Chapter 45

Perfect measures and disintegrations

One of the most remarkable features of countably additive measures is that they provide us with a framework for probability theory, as described in Chapter 27. The extraordinary achievements of probability theory since Kolmogorov are to a large extent possible because of the rich variety of probability measures which can be constructed. We have already seen image measures (112E) and product measures (§254). The former are elementary, as the paragraph number declares, but a glance at the index will confirm that they have many surprises to offer; the latter are obviously fundamental to any idea of what probability theory means. In this chapter I will look at some further constructions. The most important are those associated with 'disintegrations' or 'regular conditional probabilities' (§§452-453) and methods for confirming the existence of measures on product spaces with given images on subproducts (§§454-455). We find that these constructions have to be based on measure spaces of special types; the measures involved in the principal results are the Radon measures of Chapter 41 (of course), the compact and perfect measures of Chapter 34, and an intermediate class, the 'countably compact' measures of Marczewski 53 (451B). So the first section of this chapter is a systematic discussion of compact, countably compact and perfect measures.

A 'disintegration', when present, is likely to provide us with a particularly effective instrument for studying a measure, analogous to Fubini's theorem for product measures (see 452F). §§452-453 therefore concentrate on theorems guaranteeing the existence of disintegrations compatible with some pre-existing structure, typically an inverse-measure-preserving function (452I, 452O, 453K) or a product structure (452B-452D, 452M). Both depend on the existence of suitable liftings, and for the topological version in §453 we need a 'strong' lifting, so much of that section is devoted to the study of such liftings.

One of the central concerns of probability theory is to understand 'stochastic processes', that is, models of systems evolving randomly over time. If we think of our state space as consisting of functions, so that a whole possible history is described by a random function of time, it is natural to think of our functions as members of some set $\prod_{n\in\mathbb{N}} Z_n$ (if we think of observations as being taken at discrete time intervals) or $\prod_{t\in[0,\infty[} Z_t$ (if we regard our system as evolving continuously), where Z_t represents the set of possible states of the system at time t. We are therefore led to consider measures on such product spaces, and the new idea is that we may have some definite intuition concerning the joint distribution of *finite* strings $(f(t_0),\dots,f(t_n))$ of values of our random function, that is to say, we may think we know something about the image measures on finite products $\prod_{i\le n} Z_{t_i}$. So we come immediately to a fundamental question: given a (probability) measure μ_J on $\prod_{i\in J} Z_i$ for each finite $J\subseteq T$, when will there be a measure on $\prod_{i\in T} Z_i$ compatible with every μ_J? In §454 I give the most important generally applicable existence theorems for such measures, and in §455 I show how they can be applied to a general construction for models of Markov processes.

The abstract theory of §454 yields measures on product spaces which, from the point of view of a probabilist, are unnaturally large, often much larger than intuition suggests. General techniques for determining which such measures can be transferred to more amenable spaces are beyond the scope of this book, but I show how the most important example, Brownian motion, can be interpreted as a Radon measure on $C([0,\infty[)$ rather than as a Baire measure on $\mathbb{R}^{[0,\infty[}$ (455D).

One of the defining characteristics of Brownian motion is the fact that all its finite-dimensional marginals are Gaussian distributions. Stochastic processes with this property form a particularly interesting class, which I examine in §456. From the point of view of this volume, one of their most striking properties is Talagrand's theorem that, regarded as measures on powers $\mathbb{R}^I$, they are τ-additive (456K).

The next two sections look again at some of the ideas of the previous sections when interpreted as answers to questions of the form 'can all the measures in such-and-such a family be simultaneously extended to a single measure?' If we seek only a *finitely* additive common extension, there is a reasonably convincing general result (457A); but countably additive measures remain puzzling even in apparently simple circumstances (457Z). In §458 I introduce 'relatively independent' families of σ-algebras, with the associated concept of 'relative product' of measures. Finally, in §459, I give some basic results on symmetric measures and exchangeable random variables, with de Finetti's theorem (459C) and corresponding theorems on representing permutation-invariant measures on products as mixtures of product measures (459E, 459G).

451 Perfect, compact and countably compact measures

In §§342-343 I introduced 'compact' and 'perfect' measures as part of a study of the representation of homomorphisms of measure algebras by functions between measure spaces. An intermediate class of 'countably compact' measures (the 'compact' measures of Marczewski 53) has appeared in the exercises. It is now time to collect these ideas together in a more systematic way. In this section I run through the standard properties of compact, countably compact and perfect measures (451A-451J), with a couple of simple examples of their interaction with topologies (451L-451O). An example of a perfect measure space which is not countably compact is in 451T. Some new ideas, involving non-trivial set theory, show that measurable functions from compact totally finite measure spaces to metrizable spaces have 'essentially separable ranges' (451Q); consequently, any measurable function from a Radon measure space to a metrizable space is almost continuous (451S).

451A Let me begin by recapitulating the principal facts already covered.

(a) A family $\mathcal{K}$ of sets is a **compact class** if $\bigcap \mathcal{K}' \neq \emptyset$ whenever $\mathcal{K}' \subseteq \mathcal{K}$ is a non-empty downwards-directed family. If $\mathcal{K} \subseteq \mathcal{P}X$, then $\mathcal{K}$ is a compact class iff there is a compact topology on X for which every member of $\mathcal{K}$ is closed (342D). A subfamily of a compact class is compact (342Ab).

(b) A measure on a set X is **compact** if it is inner regular with respect to some compact class of sets; equivalently, if it is inner regular with respect to the closed sets for some compact topology on X (342F). All Radon measures are compact measures (416Wa). If (X, Σ, μ) is a semi-finite compact measure space with measure algebra $\mathfrak{A}$, (Y, T, ν) is a complete strictly localizable measure space with measure algebra $\mathfrak{B}$, and $\pi : \mathfrak{A} \to \mathfrak{B}$ is an order-continuous Boolean homomorphism, there is a function $g : Y \to X$ such that $g^{-1}[E] \in \mathrm{T}$ and $g^{-1}[E]^\bullet = \pi(E^\bullet)$ for every $E \in \Sigma$ (343B).

(c) A family $\mathcal{K}$ of sets is a **countably compact class** if $\bigcap_{n\in\mathbb{N}} K_n \neq \emptyset$ whenever $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $\bigcap_{i\leq n} K_i \neq \emptyset$ for every $n \in \mathbb{N}$. Any subfamily of a countably compact class is countably compact. If $\mathcal{K}$ is a countably compact class, then there is a countably compact class $\mathcal{K}^* \supseteq \mathcal{K}$ which is closed under finite unions and countable intersections (413R).

(d) A measure space (X, Σ, μ) is **perfect** if whenever $f : X \to \mathbb{R}$ is measurable, $E \in \Sigma$ and $\mu E > 0$, there is a compact set $K \subseteq f[E]$ such that $\mu f^{-1}[K] > 0$. A countably separated semi-finite measure space is compact iff it is perfect (343K). A measure space (X, Σ, μ) is isomorphic to the unit interval with Lebesgue measure iff it is an atomless complete countably separated perfect probability space (344K).

451B Now for the new class of measures.

Definition Let (X, Σ, μ) be a measure space. Then (X, Σ, μ), or μ, is **countably compact** if μ is inner regular with respect to some countably compact class of sets.

Evidently compact measures are also countably compact. A simple example of a countably compact measure which is not compact is the countable-cocountable measure on an uncountable set (342M). For an example of a perfect measure which is not countably compact, see 451T.

Note that if μ is inner regular with respect to a countably compact class $\mathcal{K}$, then it is also inner regular with respect to $\mathcal{K} \cap \Sigma$ (411B), and $\mathcal{K} \cap \Sigma$ is still countably compact.

451C Proposition (Ryll-Nardzewski 53) Any semi-finite countably compact measure is perfect.

proof The central idea is the same as in 342L, but we need to refine the second half of the argument.

(a) Let (X, Σ, μ) be a countably compact measure space, $f : X \to \mathbb{R}$ a measurable function, and $E \in \Sigma$ a set of positive measure. Let $\mathcal{K}$ be a countably compact class such that μ is inner regular with respect to $\mathcal{K}$; by 451Ac, we may suppose that $\mathcal{K}$ is closed under finite unions and countable intersections.

Because μ is semi-finite, there is a measurable set $F \subseteq E$ such that $0 < \mu F < \infty$; replacing F by a set of the form $F \cap f^{-1}[[-n, n]]$ if necessary, we may suppose that $f[F]$ is bounded; finally, we may suppose that $F \in \mathcal{K}$. Let $\langle \epsilon_q\rangle_{q\in\mathbb{Q}}$ be a family of strictly positive real numbers such that $\sum_{q\in\mathbb{Q}} \epsilon_q < \frac{1}{2}\mu F$. For each $q \in \mathbb{Q}$,

set $E_q = \{x : x \in F, f(x) \leq q\}$, $E'_q = \{x : x \in F, f(x) > q\}$, and choose K_q, $K'_q \in \mathcal{K} \cap \Sigma$ such that $K_q \subseteq E_q$, $K'_q \subseteq E'_q$ and $\mu(E_q \setminus K_q) \leq \epsilon_q$, $\mu(E'_q \setminus K'_q) \leq \epsilon_q$. Then $K = \bigcap_{q \in \mathbb{Q}} (K_q \cup K'_q) \in \mathcal{K} \cap \Sigma$, $K \subseteq F$ and

$$\mu(F \setminus K) \leq \sum_{q \in \mathbb{Q}} \mu(E_q \setminus K_q) + \mu(E'_q \setminus K'_q) < \mu F,$$

so $\mu K > 0$.

(b) Take any $t \in \overline{f[K]}$. Enumerate $\mathbb{Q}$ as $\langle q_n \rangle_{n \in \mathbb{N}}$ and choose $\langle L_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{K}$ by the rule

$$\begin{aligned} L_n &= K_{q_n} \text{ if } t < q_n, \\ &= K'_{q_n} \text{ if } t > q_n, \\ &= F \text{ if } t = q_n. \end{aligned}$$

Now $\bigcap_{i \leq n} L_i \neq \emptyset$ for every $n \in \mathbb{N}$. **P** Because $t \in \overline{f[K]}$, there must be some $s \in f[K]$ such that $s < q_i$ whenever $i \leq n$ and $t < q_i$, while $s > q_i$ whenever $i \leq n$ and $t > q_i$. Let $x \in K$ be such that $f(x) = s$. Then, for any $i \leq n$,

either $t < q_i$, $f(x) < q_i$ so $x \notin K'_{q_i}$ and $x \in K_{q_i} = L_i$

or $t > q_i$, $f(x) > q_i$ so $x \notin K_{q_i}$ and $x \in K'_{q_i} = L_i$,

or $t = q_i$ and $x \in F = L_i$.

So $x \in \bigcap_{i \leq n} L_i$. **Q**

As $\mathcal{K}$ is a countably compact class, there must be some $x \in \bigcap_{n \in \mathbb{N}} L_n$. But this means that, for any $n \in \mathbb{N}$,

if $t > q_n$ then $x \in K'_{q_n}$ and $f(x) > q_n$,

if $t < q_n$ then $x \in K_{q_n}$ and $f(x) < q_n$.

So in fact $f(x) = t$. Accordingly $t \in f[K]$.

(c) What this shows is that $\overline{f[K]} \subseteq f[K]$ and $f[K]$ is closed. Because (by the choice of F) it is also bounded, it is compact (2A2F). Of course we now have $f[K] \subseteq f[E]$, while $\mu f^{-1}[f[K]] \geq \mu K > 0$. As f and E are arbitrary, μ is perfect.

451D Proposition Let (X, Σ, μ) be a measure space, and $E \in \Sigma$; let μ_E be the subspace measure on E.
(a) If μ is compact, so is μ_E.
(b) If μ is countably compact, so is μ_E.
(c) If μ is perfect, so is μ_E.

proof (a)-(b) Let $\mathcal{K}$ be a (countably) compact class such that μ is inner regular with respect to $\mathcal{K}$. Then μ_E is inner regular with respect to $\mathcal{K}$ (412Oa), so is (countably) compact.

(c) Suppose that $f : E \to \mathbb{R}$ is Σ_E-measurable, where $\Sigma_E = \Sigma \cap \mathcal{P}E$ is the subspace σ-algebra, and $F \subseteq E$ is such that $\mu F > 0$. Set

$$\begin{aligned} g(x) &= \arctan f(x) \text{ if } x \in E, \\ &= 2 \text{ if } x \in X \setminus E. \end{aligned}$$

Then g is Σ-measurable, so there is a compact set $K \subseteq g[F]$ such that $\mu g^{-1}[K] > 0$. Set $L = \{\tan t : t \in K\}$; then $L \subseteq f[F]$ is compact and $f^{-1}[L] = g^{-1}[K]$ has non-zero measure. As f and F are arbitrary, μ_E is perfect.

451E Proposition Let (X, Σ, μ) be a perfect measure space.
(a) If (Y, T, ν) is another measure space and $f : X \to Y$ is an inverse-measure-preserving function, then ν is perfect.
(b) In particular, $\mu \restriction \mathrm{T}$ is perfect for any σ-subalgebra T of Σ.

proof (a) Suppose that $g : Y \to \mathbb{R}$ is T-measurable and $F \in \mathrm{T}$ is such that $\nu F > 0$. Then $gf : X \to \mathbb{R}$ is Σ-measurable and $\mu f^{-1}[F] > 0$. So there is a compact set $K \subseteq (gf)[f^{-1}[F]]$ such that $\mu(gf)^{-1}[K] > 0$. But now $K \subseteq g[F]$ and $\nu g^{-1}[K] > 0$. As g and F are arbitrary, ν is perfect.

(b) Apply (a) to $Y = X$, $\nu = \mu \restriction \mathrm{T}$ and f the identity function.

Remark We shall see in 452R that there is a similar result for countably compact measures; but for compact measures, there is not (342Xf, 451Xg).

451F Lemma (SAZONOV 66) Let (X, Σ, μ) be a semi-finite measure space. Then the following are equiveridical:

(i) μ is perfect;

(ii) $\mu\restriction\mathrm{T}$ is compact for every countably generated σ-subalgebra T of Σ;

(iii) $\mu\restriction\mathrm{T}$ is perfect for every countably generated σ-subalgebra T of Σ;

(iv) for every countable set $\mathcal{E} \subseteq \Sigma$ there is a σ-algebra $\mathrm{T} \supseteq \mathcal{E}$ such that $\mu\restriction\mathrm{T}$ is perfect.

proof (a)(i)⇒(ii) Suppose that μ is perfect, and that T is a countably generated σ-subalgebra of Σ. Let $\langle E_n\rangle_{n\in\mathbb{N}}$ be a sequence in T which σ-generates it, and define $f : X \to \mathbb{R}$ by setting $f(x) = \sum_{n=0}^{\infty} 3^{-n}\chi E_n(x)$ for every $x \in X$. Then f is measurable. Set $\mathcal{K} = \{f^{-1}[L] : L \subseteq f[X] \text{ is compact}\}$. Then $\mathcal{K}$ is a compact class. **P** If $\mathcal{K}' \subseteq \mathcal{K}$ is non-empty and has the finite intersection property, then $\mathcal{L}' = \{L : L \subseteq f[X] \text{ is compact, } f^{-1}[L] \in \mathcal{K}'\}$ is also a non-empty family with the finite intersection property. So there is an $\alpha \in \bigcap\mathcal{L}'$; since $\alpha \in f[X]$, there is an x such that $f(x) = \alpha$, and now $x \in \bigcap\mathcal{K}'$. As $\mathcal{K}'$ is arbitrary, $\mathcal{K}$ is a compact class. **Q**

Observe next that, for any $n \in \mathbb{N}$,

$$E_n = \{x : \exists\, I \subseteq n, \textstyle\sum_{i\in I} 3^{-i} + 3^{-n} \le f(x) < \sum_{i\in I} 3^{-i} + 3^{-n+1}\}.$$

So $\mathrm{T}' = \{f^{-1}[F] : F \subseteq \mathbb{R}\}$ contains every E_n; as it is a σ-algebra of subsets of X, it includes T.

Now $\mu\restriction\mathrm{T}$ is inner regular with respect to $\mathcal{K}$. **P** If $E \in \mathrm{T}$ and $\mu E > 0$, there is a set $F \subseteq \mathbb{R}$ such that $E = f^{-1}[F]$. Because f is Σ-measurable and μ is perfect, there is a compact set $L \subseteq f[E]$ such that $\mu f^{-1}[L] > 0$. But now $f^{-1}[L] \in \mathcal{K} \cap \mathrm{T}$, and $f^{-1}[L] \subseteq E$ because $L \subseteq F$. Because $\mathcal{K}$ is closed under finite unions, this is enough to show that $\mu\restriction\mathrm{T}$ is inner regular with respect to $\mathcal{K}$. **Q**

Thus $\mathcal{K}$ witnesses that $\mu\restriction\mathrm{T}$ is a compact measure.

(b)(ii)⇒(i) Now suppose that $\mu\restriction\mathrm{T}$ is compact for every countably generated σ-algebra $\mathrm{T} \subseteq \Sigma$, that $f : X \to \mathbb{R}$ is a measurable function, and that $\mu E > 0$. Let $F \subseteq E$ be a measurable set of non-zero finite measure, and T the σ-algebra generated by $\{F\} \cup \{f^{-1}[\,]-\infty, q[\,] : q \in \mathbb{Q}\}$, so that T is countably generated and f is T-measurable. Because $\mu\restriction\mathrm{T}$ is compact, so is the subspace measure $(\mu\restriction\mathrm{T})_F$ (451Da); but this is now perfect (342L or 451C), while $F \in \mathrm{T}$ and $\mu F > 0$, so there is a compact set $L \subseteq f[F] \subseteq f[E]$ such that $\mu f^{-1}[L] > 0$. As f and E are arbitrary, μ is perfect.

(c)(i)⇒(iv) is trivial.

(d)(iv)⇒(iii) If (iv) is true, and T is a countably generated σ-subalgebra of Σ, let $\mathcal{E}$ be a countable set generating it. Then there is a σ-algebra $\mathrm{T}_1 \supseteq \mathcal{E}$ such that $\mu\restriction\mathrm{T}_1$ is perfect. By 451Eb, $\mu\restriction\mathrm{T} = (\mu\restriction\mathrm{T}_1)\restriction\mathrm{T}$ is compact, therefore perfect.

(e)(iii)⇒(ii) If (iii) is true, and T is a countably generated σ-subalgebra of Σ, then $\mu\restriction\mathrm{T}$ is perfect; but as (i)⇒(ii), and T is a countably generated σ-subalgebra of itself, $\mu\restriction\mathrm{T}$ is compact.

451G Proposition Let (X, Σ, μ) be a measure space. Let $(X, \hat{\Sigma}, \hat{\mu})$ be its completion and $(X, \tilde{\Sigma}, \tilde{\mu})$ its c.l.d. version. Then

(a)(i) if μ is compact, so are $\hat{\mu}$ and $\tilde{\mu}$;

(ii) if μ is semi-finite and either $\hat{\mu}$ or $\tilde{\mu}$ is compact, then μ is compact.

(b)(i) If μ is countably compact, so are $\hat{\mu}$ and $\tilde{\mu}$;

(ii) if μ is semi-finite and either $\hat{\mu}$ or $\tilde{\mu}$ is countably compact, then μ is countably compact.

(c)(i) If μ is perfect, so are $\hat{\mu}$ and $\tilde{\mu}$;

(ii) if $\hat{\mu}$ is perfect, then μ is perfect;

(iii) if μ is semi-finite and $\tilde{\mu}$ is perfect, then μ is perfect.

proof (a)-(b) The arguments for $\hat{\mu}$ and $\tilde{\mu}$ run very closely together. Write $\check{\mu}$ for either of them, and $\check{\Sigma}$ for its domain.

(i) If μ is inner regular with respect to $\mathcal{K}$, so is $\check{\mu}$ (412Ha). So if μ is (countably) compact, so is $\check{\mu}$.

(ii) Now suppose that μ is semi-finite. The point is that if $\mathcal{K}$ is closed under countable intersections and $\check{\mu}$ is inner regular with respect to $\mathcal{K}$, so is μ. **P** Suppose that $E \in \Sigma$ and that $\mu E > \gamma$. Choose sequences $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ and $\mathcal{K}_n$ in $\mathcal{K}$ inductively, as follows. E_0 is to be such that $E_0 \subseteq E$ and $\gamma < \mu E_0 < \infty$.

Given that $\gamma < \mu E_n < \infty$, let $K_n \in \mathcal{K} \cap \check{\Sigma}$ be such that $K_n \subseteq E_n$ and $\check{\mu}K_n > \gamma$; now take $E_{n+1} \in \Sigma$ such that $E_{n+1} \subseteq K_n$ and $\mu E_{n+1} = \check{\mu}K_n$ (212Cb or 213Fc), and continue. At the end of the induction, $\bigcap_{n\in\mathbb{N}} K_n = \bigcap_{n\in\mathbb{N}} E_n$ is a member of $\Sigma \cap \mathcal{K}$ included in E and of measure at least γ. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$. **Q**

It follows that if $\check{\mu}$ is compact or countably compact, so is μ. **P** Let $\mathcal{K}$ be a (countably) compact class such that $\check{\mu}$ is inner regular with respect to $\mathcal{K}$; by 451Aa or 451Ac, there is a (countably) compact class $\mathcal{K}^*$, including $\mathcal{K}$, which is closed under countable intersections, so that μ is inner regular with respect to $\mathcal{K}^*$, and is itself (countably) compact. **Q**

(c)(i)(α) Let $f : X \to \mathbb{R}$ be $\hat{\Sigma}$-measurable, and $E \in \hat{\Sigma}$ such that $\hat{\mu}E > 0$. Then there are a μ-conegligible set $F_0 \in \Sigma$ such that $f\restriction F_0$ is Σ-measurable (212Fa), and an $F_1 \in \Sigma$ such that $F_1 \subseteq E$ and $\hat{\mu}(E \setminus F_1) = 0$. Set $F = F_0 \cap F_1$. By 451Dc, the subspace measure μ_F is perfect, while $f\restriction F$ is Σ_F-measurable; so there is a compact set $K \subseteq f[F]$ such that $\mu(F \cap f^{-1}[K]) > 0$. But now $K \subseteq f[E]$ and $\hat{\mu}f^{-1}[K] > 0$. As f and E are arbitrary, $\hat{\mu}$ is perfect.

(β) Let $f : X \to \mathbb{R}$ be $\tilde{\Sigma}$-measurable, where $\tilde{\Sigma}$ is the domain of $\tilde{\mu}$, and $E \in \tilde{\Sigma}$ such that $\tilde{\mu}E > 0$. Then there is a set $F \in \Sigma$ such that $\mu F < \infty$ and $\hat{\mu}(F \cap E)$ is defined and greater than 0 (213D). In this case, $\hat{\mu}$ and $\tilde{\mu}$ induce the same subspace measure $\hat{\mu}_F$ on F. Accordingly $f\restriction F$ is $\hat{\Sigma}$-measurable. Because $\hat{\mu}$ is perfect (by (α) just above), so is $\hat{\mu}_F$ (451Dc), and there is a compact set $K \subseteq f[F \cap E]$ such that $\hat{\mu}_F(f\restriction F)^{-1}[K] > 0$. But now, of course, $K \subseteq f[E]$ and $\tilde{\mu}f^{-1}[K] > 0$. As f and E are arbitrary, $\tilde{\mu}$ is perfect.

(ii) Suppose that $\hat{\mu}$ is perfect. Since $\mu = \hat{\mu}\restriction\Sigma$, μ is perfect, by 451Eb.

(iii) Similarly, if $\tilde{\mu}$ is perfect and μ is semi-finite, then $\mu = \tilde{\mu}\restriction\Sigma$, by 213Hc, so μ is perfect.

451H Lemma Let $\langle X_i\rangle_{i\in I}$ be a family of sets with product X. Suppose that $\mathcal{K}_i \subseteq \mathcal{P}X_i$ for each $i \in I$, and set $\mathcal{K} = \{\pi_i^{-1}[K] : i \in I, K \in \mathcal{K}_i\}$, where $\pi_i : X \to X_i$ is the coordinate map for each $i \in I$. Then
(a) if every $\mathcal{K}_i$ is a compact class, so is $\mathcal{K}$;
(b) if every $\mathcal{K}_i$ is a countably compact class, so is $\mathcal{K}$.

proof (a) For each $i \in I$, let $\mathfrak{T}_i$ be a compact topology on X_i such that every member of $\mathcal{K}_i$ is closed. Then the product topology $\mathfrak{T}$ on X is compact (3A3J), and every member of $\mathcal{K}$ is $\mathfrak{T}$-closed, so $\mathcal{K}$ is a compact class.

(b) If $\langle K_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}$ such that $\bigcap_{k\le n} K_k \neq \emptyset$ for every $n \in \mathbb{N}$, then we must be able to express each K_n as $\pi_{j_n}^{-1}[L_n]$, where $j_n \in I$, $L_n \in \mathcal{K}_{j_n}$ for every n. Now, for $i \in I$, $\mathcal{L}_i = \{K_{j_n} : n \in \mathbb{N}, j_n = i\}$ is a countable subset of $\mathcal{K}_i$, and any finite subfamily of $\mathcal{L}_i$ has non-empty intersection. Since $K_0 \neq \emptyset$, $X_i \neq \emptyset$; so, whether $\mathcal{L}_i$ is empty or not, $X_i \cap \bigcap \mathcal{L}_i$ is non-empty. So

$$\textstyle\bigcap_{k\in\mathbb{N}} K_k = \prod_{i\in I}(X_i \cap \bigcap \mathcal{L}_i)$$

is not empty. As $\langle K_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{K}$ is countably compact.

451I Theorem Let (X, Σ, μ) and (Y, T, ν) be measure spaces, with c.l.d. product $(X \times Y, \Lambda, \lambda)$.
(a) If μ and ν are compact, so is λ.
(b) If μ and ν are countably compact, so is λ.
(c) If μ and ν are perfect, so is λ.

proof (a)-(b) Let $\mathcal{K} \subseteq \mathcal{P}X$, $\mathcal{L} \subseteq \mathcal{P}Y$ be (countably) compact classes such that μ is inner regular with respect to $\mathcal{K}$ and ν is inner regular with respect to $\mathcal{L}$. Set $\mathcal{M}_0 = \{K \times Y : K \in \mathcal{K}\} \cup \{X \times L : L \in \mathcal{L}\}$. Then $\mathcal{M}_0$ is (countably) compact, by 451H. By 451Aa/451Ac, there is a (countably) compact class $\mathcal{M} \supseteq \mathcal{M}_0$ which is closed under finite unions and countable intersections. By 412R, λ is inner regular with respect to $\mathcal{M}$, so is (countably) compact.

(c)(i) Let $f : X \times Y \to \mathbb{R}$ be Λ-measurable, and $V \in \Lambda$ a set of positive measure. Then there are $G \in \Sigma$, $H \in \mathrm{T}$ such that μG, νH are both finite and $\lambda(V \cap (G \times H)) > 0$. Recall that the subspace measure $\lambda_{G\times H}$ on $G \times H$ is just the product of the subspace measures μ_G and μ_H (251P(ii-α)), and is the completion of its restriction θ to the σ-algebra $\Sigma_G \widehat{\otimes} \mathrm{T}_H$ generated by $\{E \times F : E \in \Sigma_G, F \in \mathrm{T}_H\}$, where Σ_G and T_H are the subspace σ-algebras on G, H respectively, the domains of μ_G and μ_H (251K). Next, for any $W \in \Sigma_G \widehat{\otimes} \mathrm{T}_H$,

there are countable families $\mathcal{E} \subseteq \Sigma_G$, $\mathcal{F} \subseteq \mathrm{T}_H$ such that W belongs to the σ-algebra of subsets of $G \times H$ generated by $\{E \times F : E \in \mathcal{E}, F \in \mathcal{F}\}$ (331Gd).

(ii) The point is that θ is perfect. **P** Let Λ' be any countably generated σ-subalgebra of $\Sigma_G \widehat{\otimes} \mathrm{T}_H$; let $\langle W_n \rangle_{n \in \mathbb{N}}$ be a sequence in Λ' generating it. Then there are countable families $\mathcal{E} \subseteq \Sigma_G$, $\mathcal{F} \subseteq \mathrm{T}_H$ such that every W_n belongs to the σ-algebra generated by $\{E \times F : E \in \mathcal{E}, F \in \mathcal{F}\}$. Let Σ', T' be the σ-algebras of subsets of G and H generated by $\mathcal{E}$ and $\mathcal{F}$ respectively; then every W_n belongs to $\Sigma' \widehat{\otimes} \mathrm{T}'$, so $\Lambda' \subseteq \Sigma' \widehat{\otimes} \mathrm{T}'$. Let λ' be the product of the measures $\mu \restriction \Sigma' = \mu_G \restriction \Sigma'$ and $\nu \restriction \mathrm{T}'$. Then λ' is the completion of its restriction to $\Sigma' \widehat{\otimes} \mathrm{T}'$.

Now trace through the results above. μ_G and ν_H are perfect (451Dc), so $\mu_G \restriction \Sigma'$ and $\nu_H \restriction \mathrm{T}'$ are compact (451F), so λ' is compact ((a) of this theorem), so λ' is perfect (342L or 451C). But θ must agree with λ' on Λ', by Fubini's theorem (252D), or otherwise, so $\theta \restriction \Lambda'$ is a restriction of λ', and is perfect (451Eb).

Thus $\theta \restriction \Lambda'$ is perfect for every countably generated σ-subalgebra Λ' of $\operatorname{dom} \theta$. By 451F, θ is perfect. **Q**

(iii) By 451G(c-i), $\lambda_{G \times H}$ is perfect. Now $f \restriction G \times H$ is measurable, and $\lambda_{G \times H}(V \cap (G \times H)) > 0$, so there is a compact set $K \subseteq f[V \cap (G \times H)]$ such that $\lambda_{G \times H}((G \times H) \cap f^{-1}[K]) > 0$; in which case $K \subseteq f[V]$ and $\lambda f^{-1}[K] > 0$.

As f and V are arbitary, λ is perfect.

451J Theorem Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of probability spaces with product (X, Σ, μ).
(a) If every μ_i is compact, so is μ.
(b) (Marczewski 53) If every μ_i is countably compact, so is μ.
(c) If every μ_i is perfect, so is μ.

proof The same strategy as in 451I is again effective.

(a)-(b) For each $i \in I$, let $\mathcal{K}_i \subseteq \mathcal{P}X_i$ be a (countably) compact class such that μ_i is inner regular with respect to $\mathcal{K}_i$. Set $\mathcal{M}_0 = \{\pi_i^{-1}[K] : i \in I, K \in \mathcal{K}_i\}$, so that $\mathcal{M}_0$ is (countably) compact. Let $\mathcal{M} \supseteq \mathcal{M}_0$ be a (countably) compact class which is closed under finite unions and countable intersections. By 412T, μ is inner regular with respect to $\mathcal{M}$, so is (countably) compact.

(c) Let Λ' be a countably generated σ-subalgebra of $\widehat{\bigotimes}_{i \in I} \Sigma_i$, the σ-algebra of subsets of X generated by the sets $\{x : x(i) \in E\}$ for $i \in I$ and $E \in \Sigma_i$. Then $\lambda \restriction \Lambda'$ is perfect. **P** For every $W \in \widehat{\bigotimes}_{i \in I} \Sigma_i$, we must be able to find countable subsets $\mathcal{E}_i$ of Σ_i such that W is in the σ-algebra generated by $\{\pi_i^{-1}[E] : i \in I, E \in \Sigma_i\}$; so there are in fact countable sets $\mathcal{E}_i \subseteq \Sigma_i$ such that the σ-algebra generated by $\{\pi_i^{-1}[E] : i \in I, E \in \Sigma_i\}$ includes Λ'. Let T_i be the σ-subalgebra of Σ_i generated by $\mathcal{E}_i$, so that $\mu_i \restriction \mathrm{T}_i$ is compact. Let λ' be the product of $\langle \mu_i \restriction \mathrm{T}_i \rangle_{i \in I}$; then λ' is compact, by (a) above, therefore perfect. Now λ is an extension of λ', by 254G or otherwise, so λ' is an extension of $\lambda \restriction \Lambda'$, and $\lambda \restriction \Lambda'$ is perfect. **Q** As Λ' is arbitrary, $\lambda \restriction \widehat{\bigotimes}_{i \in I} \Sigma_i$ is perfect, and its completion λ (254Ff) is also perfect.

Remark This theorem is generalized in 454Ab.

***451K** The following result is interesting because it can be reached from an unexpectedly weak hypothesis, and has occasional applications.

Proposition Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of perfect probability spaces, and (X, Λ, λ) their product. For $J \subseteq I$, $x \in X$ set $\pi_J(x) = x \restriction J$. Let $\mathcal{K}$ be the set $\{V : V \subseteq X, \pi_J[V] \in \widehat{\bigotimes}_{i \in J} \Sigma_i \text{ for every } J \subseteq I\}$. Then λ is inner regular with respect to $\mathcal{K}$.

proof (a) Take any $W \in \Lambda$ and $\gamma < \lambda W$. Then there is a family $\langle E_{ni} \rangle_{n \in \mathbb{N}, i \in I}$ such that

$$E_{ni} \in \Sigma_i \text{ for every } n \in \mathbb{N}, i \in I,$$

$$J_n = \{i : i \in I, E_{ni} \neq X_i\} \text{ is finite for every } n \in \mathbb{N},$$

$$X \setminus W \subseteq \bigcup_{n \in \mathbb{N}} \prod_{i \in I} E_{ni},$$

$$\sum_{n=0}^{\infty} \prod_{i \in I} \mu_i E_{ni} = \sum_{n=0}^{\infty} \prod_{i \in J_n} \mu_i E_{ni} < 1 - \gamma$$

(254A). For each $i \in I$, let Σ_i' be the σ-subalgebra of Σ_i generated by $\{E_{ni} : n \in \mathbb{N}\}$. Then $\mu_i \restriction \Sigma_i'$ is a compact measure (451F); let $\mathcal{K}_i$ be a compact class such that $\mu_i \restriction \Sigma_i'$ is inner regular with respect to $\mathcal{K}_i$.

Let $\langle \epsilon_{ni} \rangle_{n \in \mathbb{N}, i \in J_n}$ be a family of strictly positive real numbers such that $\sum_{n=0}^{\infty} \prod_{i \in J_n} (\mu_i E_{ni} + \epsilon_{ni}) \leq 1 - \gamma$. For $n \in \mathbb{N}$, $i \in I$ choose $K_{ni} \in \mathcal{K}_i \cap \Sigma_i'$ such that $K_{ni} \subseteq X_i \setminus E_{ni}$ and $\mu_i K_{ni} \geq \mu_i(X_i \setminus E_{ni}) - \epsilon_{ni}$, that is, $\mu_i(X_i \setminus K_{ni}) \leq \mu_i E_{ni} + \epsilon_{ni}$.

(b) Set $V = \bigcap_{n \in \mathbb{N}} \bigcup_{i \in J_n} \{x : x \in X, x(i) \in K_{ni}\}$. Then

$$\begin{aligned} X \setminus V &= \bigcup_{n \in \mathbb{N}} \{x : x(i) \in X_i \setminus K_{ni} \text{ for every } i \in J_n\} \\ &\supseteq \bigcup_{n \in \mathbb{N}} \{x : x(i) \in E_{ni} \text{ for every } i \in J_n\} = \bigcup_{n \in \mathbb{N}} \prod_{i \in I} E_{ni} \supseteq X \setminus W, \end{aligned}$$

so $V \subseteq W$. Next,

$$\mu(X \setminus V) \leq \sum_{n=0}^{\infty} \prod_{i \in J_n} \mu_i(X_i \setminus K_{ni}) \leq \sum_{n=0}^{\infty} \prod_{i \in J_n} (\mu_i E_{ni} + \epsilon_{ni}) \leq 1 - \gamma,$$

so $\mu V \geq \gamma$.

(c) Take any $J \subseteq I$. For $m \in \mathbb{N}$, set $V_m = \bigcap_{n \leq m} \bigcup_{i \in J_n} \{x : x \in X, x(i) \in K_{ni}\}$. Then $\pi_J[V] = \bigcap_{m \in \mathbb{N}} \pi_J[V_m]$. **P** Of course $\pi_J[V] \subseteq \pi_J[V_m]$ for every m, just because $V \subseteq V_m$. On the other hand, suppose that $y \in \bigcap_{m \in \mathbb{N}} \pi_J[V_m]$. Then for every $m \in \mathbb{N}$ there is a $z_m \in \prod_{i \in I \setminus J} X_i$ such that $y \cup z_m \in V_m$, writing $y \cup z_m \in X$ for the common extension of the functions y and z_m. Now for $n \leq m \in \mathbb{N}$ there is an $i(m,n) \in J_n$ such that $(y \cup z_m)(i(m,n)) \in K_{n,i(m,n)}$. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$. For every $n \in \mathbb{N}$ there is an $i_n \in J_n$ such that $\{m : m \geq n, i(m,n) = i_n\} \in \mathcal{F}$, because J_n is finite. Fix $i \in I \setminus J$ for a moment. If we set $\mathcal{K}_i' = \{K_{ni} : n \in \mathbb{N}, i_n = i\}$, then for any $m \in \mathbb{N}$ there is an $m' \geq m$ such that $i(m',n) = i_n$ for every $n \leq m$, so that $\bigcap_{n \leq m, i_n = i} K_{ni}$ contains $z_{m'}(i)$ and is therefore non-empty. Thus $\mathcal{K}_i'$ has the finite intersection property; because $\mathcal{K}$ is a compact class, we can choose a point $z(i) \in \bigcap \mathcal{K}_i'$. Doing this for each $i \in I \setminus J$, we have $z \in \prod_{i \in I \setminus J} X_i$.

Consider $y \cup z$. Let $n \in \mathbb{N}$. If $i_n \in J$, then there is an $m \geq n$ such that $i(m,n) = i_n$, in which case

$$(y \cup z)(i_n) = y(i_n) = (y \cup z_m)(i(m,n)) \in K_{n,i(m,n)} = K_{ni_n}.$$

If $i_n \in I \setminus J$, then $K_{ni_n} \in \mathcal{K}_{i_n}'$ so $(y \cup z)(i_n) = z(i_n) \in K_{ni_n}$. Thus $y \cup z \in \bigcup_{i \in J_n} \{x : x(i) \in K_{ni}\}$. As n is arbitrary, $y \cup z \in V$ and $y \in \pi_J[V]$. As y is arbitrary, $\pi_J[V] = \bigcap_{m \in \mathbb{N}} \pi_J[V_m]$. **Q**

Now observe that every V_m is in the algebra of subsets of X generated by sets of the form $\{x : x(i) \in H\}$ where $i \in I$ and $H \in \Sigma_i$, which we can identify with the free product $\bigotimes_{i \in I} \Sigma_i$ (315La). This means that V_m can be expressed as a finite union of cylinder sets of the form $C = \prod_{i \in I} H_i$ where $H_i \in \Sigma_i$ for every i and $\{i : H_i \neq X_i\}$ is finite (315Jb). But in this case $\pi_J[C]$ is either empty or $\prod_{i \in J} H_i$, and in either case belongs to $\widehat{\bigotimes}_{i \in J} \Sigma_i$. So $\pi_J[V_m]$, being a finite union of such sets, also belongs to $\widehat{\bigotimes}_{i \in J} \Sigma_i$. As this is true for every $m \in \mathbb{N}$, $\pi_J[V] = \bigcap_{m \in \mathbb{N}} \pi_J[V_m]$ belongs to $\widehat{\bigotimes}_{i \in J} \Sigma_i$.

(d) This is true for every $J \subseteq I$, so $V \in \mathcal{K}$. As W and γ are arbitrary, λ is inner regular with respect to $\mathcal{K}$.

451L Proposition Let (X, Σ) be a standard Borel space. Then any semi-finite measure μ with domain Σ is compact, therefore perfect.

proof If $\mathfrak{T}$ is a Polish topology on X with respect to which Σ is the Borel σ-algebra, then μ is inner regular with respect to the family $\mathcal{K}$ of $\mathfrak{T}$-compact sets (433Ca), which is a compact class.

451M Proposition Let (X, Σ, μ) be a perfect measure space and $\mathfrak{T}$ a T_0 topology on X with a countable network consisting of measurable sets. (For instance, μ might be a topological measure on a regular space with a countable network (4A2Ng), or a second-countable space. In particular, X might be a separable metrizable space.) Then μ is inner regular with respect to the compact sets.

proof This is a refinement of 343K. Let $\langle E_n \rangle_{n \in \mathbb{N}}$ be a sequence in Σ running over a network for $\mathfrak{T}$. Define $g : X \to \mathbb{R}$ by setting $g = \sum_{n=0}^{\infty} 3^{-n} \chi H_n$ (cf. 343E). Then g is measurable, because every χE_n is. Writing $\alpha_I = \sum_{i \in I} 3^{-i}$ for $I \subseteq \mathbb{N}$, and

$$H_n = \bigcup_{I \subseteq n}]\alpha_I + \tfrac{1}{2}3^{-n}, \alpha_I + 3^{-n+1}[,$$

we see that $E_n = g^{-1}[H_n]$ for each $n \in \mathbb{N}$. This shows that g is injective, because if x, y are distinct points in X there is an open set containing one but not the other, and now there is an $n \in \mathbb{N}$ such that E_n contains that one and not the other, so that just one of $g(x)$, $g(y)$ belongs to H_n. Also $g^{-1} : g[X] \to X$ is continuous, since $(g^{-1})^{-1}[E_n] = g[E_n] = H_n \cap g[X]$ is relatively open in $g[X]$ for every $n \in \mathbb{N}$ (4A2B(a-ii)).

Now suppose that $E \in \Sigma$ and $\mu E > 0$. Then there is a compact set $K \subseteq g[E]$ such that $\mu g^{-1}[K] > 0$. But as g is injective, $g^{-1}[K] \subseteq E$, and as g^{-1} is continuous, $g^{-1}[K]$ is compact. By 412B, this is enough to show that μ is inner regular with respect to the compact sets.

451N Corollary Let (X, Σ, μ) be a complete perfect measure space, Y a Hausdorff space with a countable network consisting of Borel sets and $f : X \to Y$ a measurable function. If the image measure μf^{-1} is locally finite, it is a Radon measure.

proof Because f is measurable, μf^{-1} is a topological measure; by 451Ea, it is perfect; by 451M, it is tight; and it is complete because μ is. Because Y has a countable network, it is Lindelöf (4A2Nb), and μf^{-1} is σ-finite (411Ge), therefore locally determined. So it is a Radon measure.

451O Corollary Let (X, Σ, μ) be a perfect measure space, Y a separable metrizable space, and $f : X \to Y$ a measurable function.

(a) If $E \in \Sigma$ and $\gamma < \mu E$, there is a compact set $K \subseteq f[E]$ such that $\mu(E \cap f^{-1}[K]) \geq \gamma$.

(b) If $\nu = \mu\phi^{-1}$ is the image measure, then $\mu_* f^{-1}[B] = \nu_* B$ for every $B \subseteq Y$.

(c) If moreover μ is σ-finite, then $\mu^* f^{-1}[B] = \nu^* B$ for every $B \subseteq Y$.

proof (a) Consider the subspace measure μ_E, the measurable function $f \restriction E$ from E to the separable metrizable space $f[E]$, and the image measure $\nu' = \mu_E(f \restriction E)^{-1}$ on $f[E]$. By 451Ea and 451M, this is tight, while $\nu' f[E] = \mu E$; so there is a compact set $K \subseteq f[E]$ such that $\nu' K \geq \gamma$, and this serves.

(b)(i) If $F \in \operatorname{dom} \nu$ and $F \subseteq B$, then

$$\nu F = \mu f^{-1}[F] \leq \mu_* f^{-1}[B];$$

as F is arbitrary, $\mu_* f^{-1}[B] \geq \nu_* B$. (ii) If $E \in \Sigma$ and $E \subseteq f^{-1}[B]$ and $\gamma < \mu E$, then (a) tells us that there is a compact set $K \subseteq f[E]$ such that $\mu(E \cap f^{-1}[K]) > \gamma$, in which case

$$\nu_* B \geq \nu K > \gamma.$$

As E and γ are arbitrary, $\nu_* B \geq \mu_* f^{-1}[B]$.

(c)(i) If $F \in \operatorname{dom} \nu$ and $F \supseteq B$, then

$$\nu F = \mu f^{-1}[F] \geq \mu^* f^{-1}[B];$$

as F is arbitrary, $\mu^* f^{-1}[B] \leq \nu^* B$. (ii) If $\mu^* f^{-1}[B] = \infty$, then of course $\mu^* f^{-1}[B] = \nu^* B$. Otherwise, because μ is σ-finite, we can find a disjoint sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ of subsets of X of finite measure, covering X, such that $E_0 \supseteq f^{-1}[B]$ and $\mu E_0 = \mu^* f^{-1}[B]$. Let $\epsilon > 0$. For each $n \geq 1$, (a) tells us that there is a compact set $K_n \subseteq f[E_n]$ such that $\mu f^{-1}[E_n \setminus K_n] \leq 2^{-n}\epsilon$. Set $H = Y \setminus \bigcup_{n \geq 1} K_n$; then $\nu H \leq \mu E + \epsilon$, and $B \subseteq H$. So

$$\nu^* B \leq \nu H \leq \mu E + \epsilon = \mu^* f^{-1}[B] + \epsilon.$$

As ϵ is arbitrary, $\nu^* B \leq \mu^* f^{-1}[B]$.

451P I turn now to a remarkable extension of the idea above to general metric spaces Y.

Lemma Let (X, Σ, μ) be a semi-finite compact measure space, and $\langle E_i \rangle_{i \in I}$ a disjoint family of subsets of X such that $\bigcup_{i \in J} E_i \in \Sigma$ for every $J \subseteq I$. Then $\mu(\bigcup_{i \in I} E_i) = \sum_{i \in I} \mu E_i$.

proof (a) To begin with (down to the end of part (d) of the proof) assume that μ is totally finite and that every E_i is negligible. Set $X_0 = \bigcup_{i \in I} E_i$, and let μ_0 be the subspace measure on X_0. Define $f : X_0 \to I$ by setting $f(x) = i$ if $i \in I$, $x \in E_i$, and let ν be the image measure $\mu_0 f^{-1}$, so that $\nu J = \mu(\bigcup_{i \in J} E_i)$ for $J \subseteq I$; then $(I, \mathcal{P}I, \nu)$ is a totally finite measure space.

(b) ν is purely atomic. **P?** Suppose, if possible, otherwise; that there is a $K \subseteq I$ such that $\nu K > 0$ and the subspace measure $\nu\restriction\mathcal{P}K$ is atomless. In this case there is an inverse-measure-preserving function $g : K \to [0,\gamma]$, where $[0,\gamma]$ is given Lebesgue measure (343Cc); write λ for Lebesgue measure on $[0,\gamma]$. Set $X_1 = f^{-1}[K] = \bigcup_{i\in K} E_i$ and let μ_1 be the subspace measure on X_1. Now $gf : X_1 \to [0,\gamma]$ is inverse-measure-preserving for μ_1 and λ. Because μ is compact, so is μ_1 (451Da), so μ_1 is perfect (342L or 451C). By 451N, the image measure $\lambda_1 = \mu_1(gf)^{-1}$ is a Radon measure. But λ_1 must be an extension of Lebesgue measure λ, because gf is inverse-measure-preserving for μ_1 and λ, and λ_1 and λ must agree on all compact sets. By 416E(b-ii), λ_1 and λ are identical, and, in particular, have the same domains. Now for any set $A \subseteq [0,\gamma]$, $(gf)^{-1}[A] = \bigcup_{i\in J} E_i \in \Sigma$, where $J = g^{-1}[A] \subseteq I$; so $A \in \operatorname{dom}\lambda_1 = \operatorname{dom}\lambda$. But we know from 134D or 419J that not every subset of $[0,\gamma]$ can be Lebesgue measurable. **XQ**

(c) But ν is also atomless. **P?** Suppose, if possible, that $M \subseteq I$ is an atom for ν. Set $\gamma = \nu M = \mu(\bigcup_{i\in M} E_i)$,

$$\mathcal{F} = \{F : F \subseteq M,\ \nu(M \setminus F) = 0\}.$$

Because νF is defined for every $F \subseteq M$, and M is an atom, $\mathcal{F}$ is an ultrafilter on M; and because ν is countably additive, the intersection of any sequence in $\mathcal{F}$ belongs to $\mathcal{F}$, that is, $\mathcal{F}$ is ω_1-complete (definition: 4A1Ib). Also $\mathcal{F}$ must be non-principal, because we are supposing that $\nu\{i\} = 0$ for every $i \in M$. By 4A1K, there are a regular uncountable cardinal κ and a function $h : M \to \kappa$ such that the image filter $\mathcal{H} = h[[\mathcal{F}]]$ is normal.

For each $\xi < \kappa$, $\kappa \setminus \xi \in \mathcal{H}$, so

$$G_\xi = (hf)^{-1}[\kappa \setminus \xi] = \bigcup\{E_i : h(i) \geq \xi\} \in \Sigma, \quad \mu G_\xi = \nu h^{-1}[\kappa \setminus \xi] = \gamma > 0.$$

At this point I apply the full strength of the hypothesis that μ is a compact measure. Let $\mathcal{K} \subseteq \Sigma$ be a compact class such that μ is inner regular with respect to $\mathcal{K}$, and for each $\xi < \kappa$ choose $K_\xi \in \mathcal{K}_\xi$ such that $K_\xi \subseteq G_\xi$ and $\mu K_\xi \geq \frac{1}{2}\gamma$. Let $S \subseteq [\kappa]^{<\omega}$ be the family of those finite sets $L \subseteq \kappa$ such that $\bigcap_{\xi\in L} K_\xi = \emptyset$. Because $\mathcal{H}$ is a normal filter, there is an $H \in \mathcal{H}$ such that, for every $n \in \mathbb{N}$, $[H]^n$ is either a subset of S or disjoint from S (4A1L).

If we look at $\{G_\xi : \xi \in H\}$, we see that it has empty intersection, because $h(f(x)) \geq \xi$ for every $x \in G_\xi$, and $\sup H = \kappa$. So $\bigcap_{\xi\in H} K_\xi = \emptyset$. Because all the K_ξ belong to the compact class $\mathcal{K}$, there must be a finite set $L_0 \subseteq H$ such that $\bigcap_{\xi\in L_0} K_\xi = \emptyset$, that is, $L_0 \in S$. But this means that $[H]^n \cap S \neq \emptyset$, where $n = \#(L_0)$, so that $[H]^n \subseteq S$, by the choice of H. However, H is surely infinite, so we can find distinct $\xi_0, \ldots, \xi_{2n}$ in H. If we now look at $K_{\xi_0}, \ldots, K_{\xi_{2n}}$, we see that $\#(\{i : i \leq 2n,\ x \in K_{\xi_i}\}) < n$ for every $x \in X$, so

$$\textstyle\sum_{i=0}^{2n} \chi K_{\xi_i} \leq (n-1)\chi G_0, \quad \sum_{i=0}^{2n} \int \chi K_{\xi_i} \geq \frac{1}{2}\gamma(2n+1),$$

which is impossible, because $\mu G_0 = \gamma$. **XQ**

(d) Thus ν is simultaneously atomless and purely atomic, which means that $\nu I = 0$, that is, that $\mu(\bigcup_{i\in I} E_i) = 0 = \sum_{i\in I} \mu E_i$.

(e) Now let us return to the general case. Of course

$$\textstyle\sum_{i\in I} \mu E_i = \sup_{J\subseteq I \text{ is finite}} \sum_{i\in J} \mu E_i \leq \mu(\bigcup_{i\in I} E_i).$$

? Suppose, if possible, that $\sum_{i\in I} \mu E_i < \mu(\bigcup_{i\in I} E_i)$. Because μ is semi-finite, there is a set $F \subseteq \bigcup_{i\in I} E_i$ such that $\sum_{i\in I} \mu E_i < \mu F < \infty$. Set $L = \{i : i \in I,\ \mu E_i > 0\}$; then L must be countable, so $\mu(\bigcup_{i\in J} E_i) = \sum_{i\in J} \mu E_i < \mu F$, and $\mu G > 0$, where $G = F \setminus \bigcup_{i\in L} E_i$. Set $E_i' = G \cap E_i$ for every $i \in I$, and let μ_G be the subspace measure on G. Then μ_G is compact (451Da) and totally finite, $\mu_G E_i' = 0$ for every $i \in I$, $\bigcup_{i\in J} E_i' = G \cap \bigcup_{i\in J} E_i$ is measured by μ_G for every $J \subseteq I$, every E_i' is μ_G-negligible, but $\mu_G(\bigcup_{i\in I} E_i') = \mu G$ is not zero; which contradicts the result of (a)-(d) above. **X**

So $\sum_{i\in I} \mu E_i = \mu(\bigcup_{i\in I} E_i)$, as required.

451Q Lemma Let (X,Σ,μ) be a totally finite compact measure space, Y a metrizable space, and $f : X \to Y$ a measurable function. Then there is a closed separable subspace Y_0 of Y such that $f^{-1}[Y \setminus Y_0]$ is negligible.

proof (a) (Cf. 438D.) By 4A2L(h-ii), there is a σ-disjoint base $\mathcal{U}$ for the topology of Y. Express $\mathcal{U}$ as $\bigcup_{n\in\mathbb{N}}\mathcal{U}_n$ where $\mathcal{U}_n$ is disjoint for each n. Then $\langle f^{-1}[U]\rangle_{U\in\mathcal{U}_n}$ is disjoint, so $\sum_{U\in\mathcal{U}_n}\mu f^{-1}[U]\leq\mu X$ is finite, and $\mathcal{V}_n=\{V:V\in\mathcal{U}_n,\,\mu f^{-1}[V]>0\}$ is countable for each n.

If $\mathcal{W}\subseteq\mathcal{U}_n\setminus\mathcal{V}_n$, then

$$\mu(\textstyle\bigcup_{U\in\mathcal{W}}f^{-1}[U])=f^{-1}[\bigcup\mathcal{W}]$$

is measurable. By 451P,

$$\mu f^{-1}[\textstyle\bigcup(\mathcal{U}_n\setminus\mathcal{V}_n)]=\mu(\bigcup_{U\in\mathcal{U}_n\setminus\mathcal{V}_n}f^{-1}[U])=\sum_{U\in\mathcal{U}_n\setminus\mathcal{V}_n}\mu f^{-1}[U]=0.$$

Set

$$\mathcal{V}=\textstyle\bigcup_{n\in\mathbb{N}}\mathcal{V}_n,\quad Y_0=Y\setminus\bigcup(\mathcal{U}\setminus\mathcal{V}).$$

Then Y_0 is closed, and

$$f^{-1}[Y\setminus Y_0]\subseteq\textstyle\bigcup_{n\in\mathbb{N}}f^{-1}[\bigcup(\mathcal{U}_n\setminus\mathcal{V}_n)]$$

is negligible, so $f^{-1}[Y_0]$ is conegligible. On the other hand, Y_0 is separable. **P** Because $\mathcal{U}$ is a base for the topology of X, $\{Y\cap U:U\in\mathcal{U}\}$ is a base for the topology of Y (4A2B(a-iv)). But this is included in the countable family $\{Y\cap V:V\in\mathcal{V}\}\cup\{\emptyset\}$, so Y is second-countable, therefore separable (4A2Oc). **Q**

So we have found an appropriate Y_0.

451R Proposition Let (X,Σ,μ) be a semi-finite compact measure space, Y a metrizable space and $f:X\to Y$ a measurable function.

(a) The image measure $\nu=\mu f^{-1}$ is tight.

(b) If ν is locally finite and μ is complete and locally determined, ν is a Radon measure.

proof (a) Take $F\subseteq Y$ such that $\nu F>0$. Then $\mu f^{-1}[F]>0$. Because μ is semi-finite, there is an $E\in\Sigma$ such that $E\subseteq f^{-1}[F]$ and $0<\mu E<\infty$.

Consider the subspace measure μ_E and the restriction $f\restriction E$. μ_E is a totally finite compact measure and $f\restriction E$ is measurable, so 451Q tells us that there is a closed separable subspace $Y_0\subseteq Y$ such that $\mu(E\setminus f^{-1}[Y_0])=0$. Set $E_1=E\cap f^{-1}[Y_0]$, so that $\mu E_1>0$. Again, the subspace measure μ_{E_1} is a totally finite compact measure, therefore perfect, while $f[E_1]\subseteq Y_0$. So the image measure $\mu_{E_1}(f\restriction E_1)^{-1}$ on Y_0 is perfect (451Ea), therefore tight (451M), and there is a compact set $K\subseteq Y_0\cap F$ such that $\nu K=\mu f^{-1}[K]>0$. By 412B, this is enough to show that ν is tight.

(b) ν is complete because μ is. Now suppose that $H\subseteq Y$ is such that $H\cap F$ belongs to the domain T of ν whenever $\mu F<\infty$. In this case μ is inner regular with respect to $\mathcal{E}=\{E:E\in\Sigma,\,E\cap f^{-1}[H]\in\Sigma\}$. **P** Suppose that $E\in\Sigma$ and that $\mu E>0$. Applying (a) to μ_E and $f\restriction E$, there is a compact set $K\subseteq f[E]$ such that $\mu f^{-1}[K]>0$. Now $\nu K<\infty$, because ν is locally finite, so $K\cap H\in\mathrm{T}$ and $f^{-1}[K]\cap f^{-1}[H]\in\Sigma$. Thus $f^{-1}[K]$ is a non-negligible member of $\mathcal{E}$ included in E. Since $\mathcal{E}$ is closed under finite unions, this is enough to show that μ is inner regular with respect to $\mathcal{E}$. **Q**

Accordingly $f^{-1}[H]\in\Sigma$, by 412Ja. As H is arbitrary, ν is locally determined, therefore a Radon measure.

451S Theorem (Fremlin 81, Koumoullis & Prikry 83) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a Radon measure space and Y a metrizable space. Then a function $f:X\to Y$ is measurable iff it is almost continuous.

proof If f is almost continuous it is surely measurable, by 418E. Now suppose that f is measurable and that $E\in\Sigma$ and $\gamma<\mu E$. Let $E_0\subseteq E$ be such that $E_0\in\Sigma$ and $\gamma<\mu E_0<\infty$. Applying 451Q to the subspace measure μ_{E_0} and the restricted function $f\restriction E_0$, we see that there is a closed separable subspace Y_0 of Y such that $\mu(E_0\setminus f^{-1}[Y_0])=0$. Set $E_1=E_0\cap f^{-1}[Y_0]$; then $\mu E_1>\gamma$. Applying 418J to μ_{E_1} and $f\restriction E_1:E_1\to Y_0$, we can find a measurable set $F\subseteq E_1$ such that $f\restriction F$ is continuous and $\mu F\geq\gamma$. As E and γ are arbitrary, f is almost continuous.

451T Example (Vinokurov & Makhkamov 73[2], Musiał 76) There is a perfect completion regular quasi-Radon probability space which is not countably compact.

[2] I am indebted to J.Pachl for the reference.

proof (a) Let Ω be the set of non-zero countable limit ordinals. For each $\xi \in \Omega$, let $\langle \theta_\xi(n)\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in ξ with supremum ξ, and set

$$Q_\xi = \{x : x \in \{0,1\}^{\omega_1},\ x(\theta_\xi(n)) = 0 \text{ for every } n \in \mathbb{N}\}.$$

Write

$$X = \{0,1\}^{\omega_1} \setminus \textstyle\bigcup_{\xi\in\Omega} Q_\xi.$$

Let ν be the usual measure on $\{0,1\}^{\omega_1}$, and T its domain; let μ be the subspace measure on X, and $\Sigma = \operatorname{dom}\mu$.

(b) It is convenient to note immediately the following fact: for every countable set $J \subseteq \omega_1$, the set $\pi_J[X]$ is conegligible in $\{0,1\}^J$, where $\pi_J(x) = x{\restriction}J$ for $x \in \{0,1\}^{\omega_1}$. **P** Set

$$A = \{\xi : \xi \in \Omega,\ \theta_\xi(n) \in J \text{ for every } n \in \mathbb{N}\}.$$

Then A is countable, because $\xi \le \sup J$ for every $\xi \in A$. So

$$D = \textstyle\bigcup_{\xi\in A}\{y : y \in \{0,1\}^J,\ y(\theta_\xi(n)) = 0 \text{ for every } n \in \mathbb{N}\}$$

is negligible in $\{0,1\}^J$, being a countable union of negligible sets. If $y \in \{0,1\}^J \setminus D$, define $x \in \{0,1\}^{\omega_1}$ by setting $x(\eta) = y(\eta)$ for $\eta \in J$, $x(\eta) = 1$ for $\eta \in \omega_1 \setminus J$. Then $x \notin Q_\xi$ for any $\xi \in A$, because $x{\restriction}J = y{\restriction}J$, while $x \notin Q_\xi$ for any $\xi \in \Omega \setminus A$ by the definition of A. So $x \in X$. As y is arbitrary, $\pi_J[X] \supseteq \{0,1\}^J \setminus D$ is conegligible. **Q**

(c) μ is a completion regular quasi-Radon measure because ν is (415E, 415B, 412Pd). Also $\mu X = 1$. **P** Let $F \in \mathrm{T}$ be a measurable envelope for X. Then there is a countable $J \subseteq \omega_1$ such that $\nu_J\pi_J[F]$ is defined and equal to νF (254Od), where ν_J is the usual measure on $\{0,1\}^J$. But we know that $\nu_J\pi_J[X] = 1$, so

$$\mu X = \nu^* X = \nu F = \nu_J\pi_J F = 1. \ \textbf{Q}$$

(d) μ is perfect. **P** Take $E \in \Sigma$ such that $\mu E > 0$, and a measurable function $f : E \to \mathbb{R}$. Set $f_1(x) = \frac{f(x)}{1+|f(x)|}$ for $x \in E$, 1 for $x \in X \setminus E$; then $f_1 : X \to \mathbb{R}$ is measurable. Let $g : \{0,1\}^{\omega_1} \to \mathbb{R}$ be a measurable function extending f_1. By 254Pb, there are a countable set $J \subseteq \omega_1$, a conegligible set $W \subseteq \{0,1\}^J$, and a measurable $h : W \to \mathbb{R}$ such that g extends $h\pi_J$. By (b), $W' = W \cap \pi_J[X]$ is conegligible, while $W'' = W' \cap \pi_J[E] = \{z : z \in W',\ h_1(z) < 1\}$ is measurable and not negligible. Because W'' is a non-negligible measurable subset of the perfect measure space $\{0,1\}^J$, there is a compact set $K_1 \subseteq h[W'']$ such that $\nu_J h^{-1}[K_1] > 0$. Set $K = \{\frac{t}{1-|t|} : t \in K_1\}$; then K is compact, and we have

$$K_1 \subseteq h[W''] = h\pi_J[X \cap \pi_J^{-1}[W]] = g[E \cap \pi_J^{-1}[W]] \subseteq f_1[E],$$

$$K \subseteq f[E],$$

while f_1, g and $h\pi_J$ all agree on the μ-conegligible set $X \cap \pi_J^{-1}[W]$, so

$$\begin{aligned}\mu f^{-1}[K] &= \mu f_1^{-1}[K_1] = \mu(X \cap (h\pi_J)^{-1}[K_1])\\ &= \nu^*(X \cap (h\pi_J)^{-1}[K_1]) = \nu(h\pi_J)^{-1}[K_1]\end{aligned}$$

(because $\nu^* X = 1$ and $(h\pi_J)^{-1}[K_1]$ is measurable)

$$= \nu_J h^{-1}[K_1] > 0.$$

As f is arbitrary, μ is perfect. **Q**

(e) **?** Suppose, if possible, that μ is countably compact. Let $\mathcal{K}$ be a countably compact class of sets such that μ is inner regular with respect to $\mathcal{K}$; we may suppose that $\mathcal{K} \subseteq \Sigma$.

(i) For $I \subseteq \omega_1$ set

$$U(I) = \{x : x \in X,\ x(\eta) = 0 \text{ for every } \eta \in I\}.$$

It will be helpful to know that if $E \in \Sigma$ and $\mu E > 0$, there is a $\gamma < \omega_1$ such that $\mu(E \cap U(I)) > 0$ for every finite $I \subseteq \omega_1 \setminus \gamma$. **P** Express E as $X \cap F$ where $F \in \mathrm{T}$. Let $J \subseteq \omega_1$ be a countable set such that $\nu(F' \setminus F) = 0$, where $F' = \pi_J^{-1}[\pi_J[F]]$ (254Od), and $\gamma < \omega_1$ such that $J \subseteq \gamma$. If $I \subseteq \omega_1 \setminus \gamma$ is finite, then $I \cap J = \emptyset$, while $U(I)$ is determined by coordinates in I and F' is determined by coordinates in J; so

$$\begin{aligned}\mu(E \cap U(I)) &= \nu^*(X \cap F \cap U(I)) = \nu(F \cap U(I)) \\ &= \nu(F' \cap U(I)) = \nu F' \cdot \nu U(I) = \mu E \cdot \nu U(I) > 0.\end{aligned}$$

Thus this γ serves. **Q**

(ii) Let $\mathcal{M}$ be the family of countable subsets M of $\omega_1 \cup \mathcal{K}$ such that

(α) if $I \subseteq M \cap \omega_1$ is finite there is a $K \in M \cap \mathcal{K}$ such that $K \subseteq U(I)$ and $\mu K > 0$;

(β) if $K \in M \cap \mathcal{K}$, $I \subseteq M \cap \omega_1$ is finite and $\mu(K \cap U(I)) > 0$, then there is a $K' \in M \cap \mathcal{K}$ such that $K' \subseteq K \cap U(I)$ and $\mu K' > 0$;

(γ) if $\gamma \in M \cap \omega_1$ then $\gamma \subseteq M$;

(δ) if $K \in M \cap \mathcal{K}$ and $\mu K > 0$ then there is a $\gamma \in M \cap \omega_1$ such that $\mu(K \cap U(I)) > 0$ whenever $I \subseteq \omega_1 \setminus \gamma$ is finite.

Then every countable $M \subseteq \omega_1 \cup \mathcal{K}$ is included in some member of $\mathcal{M}$.

P Choose $\langle N_n \rangle_{n \in \mathbb{N}}$ as follows. $N_0 = M$. Given that N_n is a countable subset of $\omega_1 \cup \mathcal{K}$ then let $N_{n+1} \subseteq \omega_1 \cup \mathcal{K}$ be a countable set such that

(α) if $I \subseteq N_n \cap \omega_1$ is finite there is a $K \in N_{n+1} \cap \mathcal{K}$ such that $K \subseteq U(I)$ and $\mu K > 0$;

(β) if $K \in N_n \cap \mathcal{K}$, $I \subseteq N_n \cap \omega_1$ is finite and $\mu(K \cap U(I)) > 0$, then there is a $K' \in N_{n+1} \cap \mathcal{K}$ such that $K' \subseteq K \cap U(I)$ and $\mu K' > 0$;

(γ) if $\gamma \in N_n \cap \omega_1$ then $\gamma \subseteq N_{n+1}$;

(δ) if $K \in N_n \cap \mathcal{K}$ and $\mu K > 0$ then there is a $\gamma \in N_{n+1} \cap \omega_1$ such that $\mu(K \cap U(I)) > 0$ whenever $I \subseteq \omega_1 \setminus \gamma$ is finite;

(ϵ) $N_n \subseteq N_{n+1}$.

On completing the induction, set $M' = \bigcup_{n \in \mathbb{N}} N_n$; this serves (because every finite subset of M' is a subset of some N_n). **Q**

(iii) Choose a sequence $\langle M_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{M}$ such that, for each n, $M_n \cup \{\sup(M_n \cap \omega_1) + 1\} \subseteq M_{n+1}$. Set $\gamma_n = \sup(M_n \cap \omega_1)$ for each n. Note that $\gamma_n \subseteq M_n$, because if $\eta < \gamma_n$ then there is some $\xi \in M_n$ such that $\eta < \xi$; now $\xi \subseteq M_n$ because $M_n \in \mathcal{M}$, so $\eta \in M_n$. Also $\gamma_n + 1 \in M_{n+1}$ for each n, so $\langle \gamma_n \rangle_{n \in \mathbb{N}}$ is strictly increasing, and $\xi = \sup_{n \in \mathbb{N}} \gamma_n$ belongs to Ω.

Set $J = \{\theta_\xi(n) : n \in \mathbb{N}\}$. Then $J \cap \eta$ is finite for every $\eta < \xi$, and in particular $J \cap \gamma_n$ is finite for every n. Set $I_0 = J \cap \gamma_0$ and $I_n = J \cap \gamma_n \setminus \gamma_{n-1}$ for $n \geq 1$. Then $\bigcap_{n \in \mathbb{N}} U(I_n) = Q_\xi$ is disjoint from X.

Choose a sequence $\langle K_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{K}$ as follows. Because I_0 is a finite subset of $M_0 \cap \omega_1$, there is a $K_0 \in M_0 \cap \mathcal{K}$ such that $K_0 \subseteq U(I_0)$ and $\mu K_0 > 0$. Given that $K_n \in M_n \cap \mathcal{K}$ and $\mu K_n > 0$, then there is a $\beta \in M_n \cap \omega_1$ such that $\mu(K_n \cap U(I)) > 0$ for every finite $I \subseteq \omega_1 \setminus \beta$; now $\beta \leq \gamma_n$ and $I_{n+1} \cap \gamma_n = \emptyset$, so $\mu(K_n \cap U(I_{n+1})) > 0$. But $K_n \in M_{n+1} \cap \mathcal{K}$ and I_{n+1} is a finite subset of $M_{n+1} \cap \omega_1$, so there is a $K_{n+1} \in M_{n+1} \cap \mathcal{K}$ such that $K_{n+1} \subseteq K_n \cap U(I_{n+1})$ and $\mu K_{n+1} > 0$. Continue.

In this way we find a non-increasing sequence $\langle K_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{K}$ such that $K_n \subseteq U(I_n)$ for every n and no K_n is empty. But in this case $\bigcap_{i \leq n} K_i = K_n$ is non-empty for every n, while $\bigcap_{n \in \mathbb{N}} K_n \subseteq X \cap \bigcap_{n \in \mathbb{N}} U(I_n)$ is empty. So $\mathcal{K}$ is not a countably compact class. **X**

(f) Thus μ is not countably compact, and has all the properties claimed.

***451U Weakly α-favourable spaces** There is an interesting variation on the concept of 'countably compact' measure space, as follows. For any measure space (X, Σ, μ) we can imagine an infinite game for two players, whom I will call 'Empty' and 'Nonemepty'. Empty chooses a non-negligible measurable set E_0; Nonempty chooses a non-negligible measurable set $F_0 \subseteq E_0$; Empty chooses a non-negligible measurable set $E_1 \subseteq F_0$; Nonempty chooses a non-negligible measurable set $F_1 \subseteq E_1$, and so on. At the end of the game, Empty wins if $\bigcap_{n \in \mathbb{N}} E_n = \bigcap_{n \in \mathbb{N}} F_n$ is empty; otherwise Nonempty wins. If you have seen 'Banach-Mazur'

games, you will recognise this as a similar construction, in which open sets are replaced by non-negligible measurable sets.

A **strategy** for Nonempty is a rule to determine his moves in terms of the preceding moves for Empty; that is, a function $\sigma : \bigcup_{n\in\mathbb{N}}(\Sigma \setminus \mathcal{N})^{n+1} \to \Sigma \setminus \mathcal{N}$, where $\mathcal{N}$ is the ideal of negligible sets, such that $\sigma(E_0, E_1, \dots, E_n) \subseteq E_n$, at least whenever $E_0, \dots, E_n \in \Sigma \setminus \mathcal{N}$ are such that $E_{k+1} \subseteq \sigma(E_0, \dots, E_k)$ for every $k < n$; since it never matters what Nonempty does if Empty has already broken the rules, we usually just demand that $\sigma(E_0, \dots, E_n) \subseteq E_n$ for all $E_0, \dots, E_n \in \Sigma\setminus\mathcal{N}$. σ is a **winning strategy** if $\bigcap_{n\in\mathbb{N}} E_n \neq \emptyset$ whenever $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\Sigma \setminus \mathcal{N}$ such that $E_{n+1} \subseteq \sigma(E_0, \dots, E_n)$ for every $n \in \mathbb{N}$. In terms of the game, we interpret this as saying that Nonempty will win if he plays $F_n = \sigma(E_0, \dots, E_n)$ whenever faced with the position $(E_0, F_0, E_1, F_1, \dots, F_{n-1}, E_n)$. (Since it is supposed that Nonempty will use the same strategy throughout the game, the moves $F_0, \dots, F_{n-1}$ are determined by $E_0, \dots, E_{n-1}$ and there is no advantage in taking them separately into account when choosing F_n.)

Now we say that the measure space (X, Σ, μ) is **weakly α-favourable** if there is such a winning strategy for Nonempty.

It turns out that the class of weakly α-favourable spaces behaves in much the same way as the class of countably compact spaces. For the moment, however, I leave the details to the exercises (451Yg-451Yq). See Fremlin 00.

451X Basic exercises (a) (i) Show that any purely atomic measure space is perfect. (ii) Show that any strictly localizable purely atomic measure space is countably compact. (iii) Show that the space of 216E/342N is not countably compact.

>(b) Show that a compact measure space in which singleton sets are negligible is atomless.

>(c) Let (X, Σ, μ) be a measure space, and ν an indefinite-integral measure over μ (§234). Show that ν is compact, or countably compact, or perfect if μ is.

(d) In 413Xn, show that μ is a countably compact measure. (*Hint*: show that the algebra Σ in 413Xn is a countably compact class.)

(e) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of measure spaces, with direct sum (X, Σ, μ). Show that μ is compact, or countably compact, or perfect iff every μ_i is.

(f) Let (X, Σ, μ) be a measure space. For $A \subseteq X$, write μ_A for the subspace measure on A. Suppose that whenever $E \in \Sigma$ and $\mu E > 0$ there is a set $A \subseteq X$ such that μ_A is perfect and $\mu^*(A \cap E) > 0$. Show that μ is perfect.

(g) (i) Give an example of a compact probability space (X, Σ, μ) and a σ-subalgebra T of Σ such that $\mu\restriction\mathrm{T}$ is not compact. (ii) Give an example of a compact probability space (X, Σ, μ), a set Y and a function $f : X \to Y$ such that the image measure μf^{-1} is not compact. (*Hint*: 342M, 342Xf, 439Xa.)

(h) Let $\langle X_i\rangle_{i\in I}$ be a family of sets, with product X. Suppose that $\mathcal{K}_i \subseteq \mathcal{P}X_i$ for each i, and set $\mathcal{K} = \{\prod_{i\in I} K_i : K_i \in \mathcal{K}_i \text{ for each } i\}$. (i) Show that if $\mathcal{K}_i$ is a compact class for each i, so is $\mathcal{K}$. (ii) Show that if $\mathcal{K}_i$ is a countably compact class for each i, so is $\mathcal{K}$.

(i) Let X be a Polish space and E a subset of X. Show that the following are equiveridical: (i) E is universally measurable; (ii) every Borel probability measure on E is perfect; (iii) every σ-finite Borel measure on E is compact; (iv) $f[E]$ is universally measurable in $\mathbb{R}$ for every Borel measurable function $f : X \to \mathbb{R}$.

(j) In 451M, show that μ is a compact measure.

(k) Find a Radon measure space $(X, \mathfrak{T}, \Sigma, \mu)$, a continuous function $f : X \to [0, 1]$ and a set $B \subseteq [0, 1]$ such that $\mu^*(f^{-1}[B]) < (\mu f^{-1})^* B$.

(l) Let (X, Σ, μ) be a σ-finite measure space. Show that it is perfect iff whenever $f : X \to \mathbb{R}$ is measurable there is a K_σ set $H \subseteq f[X]$ such that $f^{-1}[H]$ is conegligible.

(m) Let X be a metrizable space, and μ a semi-finite topological measure on X which (regarded as a measure) is compact. Show that μ is τ-additive.

(n) Let (X, Σ, μ) be a compact strictly localizable measure space (e.g., any Radon measure space), (Y, T, ν) a σ-finite measure space, and $f : X \to L^0(\nu)$ a function. Show that the following are equiveridical: (i) f is measurable, when $L^0(\nu)$ is given its topology of convergence in measure; (ii) there is a function $h \in \mathcal{L}^0(\lambda)$, where λ is the c.l.d. product measure on $X \times Y$, such that $f(x) = h_x^\bullet$ for almost every $x \in X$, where $h_x(y) = h(x, y)$. (*Hint*: 418R.)

>(o) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space. Show that $\Sigma = \mathcal{P}X$ iff μ is purely atomic. (*Hint*: if $\Sigma = \mathcal{P}X$, apply 451S with $Y = X$, the discrete topology on Y and the identity function from X to Y.)

(p) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space and U a normed space. Show that if f, $g : X \to U$ are measurable functions, then $f + g$ is measurable. (Cf. 418Xj.)

451Y Further exercises (a) Show that for any probability space (X, Σ, μ), there is a compact probability space (Y, T, ν) with a subspace isomorphic to (X, Σ, μ).

(b) Let $\langle X_i \rangle_{i \in I}$ be a family of sets, and Σ_i a σ-algebra of subsets of X_i for each i. Suppose that for each finite $J \subseteq I$ we are given a finitely additive functional ν_J on $X_J = \prod_{i \in J} X_i$, with domain the algebra $\mathrm{T}_J = \bigotimes_{i \in J} \Sigma_i$ generated by sets of the form $\{x : x \in X_J, x(i) \in E\}$ for $i \in J$, $E \in \Sigma_i$, and that (α) $\nu_K\{x : x \in X_K, x{\restriction}J \in W\} = \nu_J W$ whenever $J \subseteq K \in [I]^{<\omega}$ and $W \in \mathrm{T}_J$ (β) $\mu_i = \nu_i$ is a countably compact probability measure for every $i \in I$. Show that there is a countably compact measure μ on $X = X_I$ such that $\mu\{x : x \in X, x{\restriction}J \in W\} = \nu_J W$ whenever $J \in [I]^{<\omega}$ and $W \in \mathrm{T}_J$. (*Hint*: 454D.) (Compare 418M.)

(c) Describe μ in the case of 451Yb in which $I = [0, 1]$, $X_i = [0, 1] \setminus \{i\}$, Σ_i is the algebra of Lebesgue measurable subsets of X_i, and $\nu_J E = \mu_L\{t : z_{Jt} \in E\}$ for every $E \in \bigotimes_{i \in J} \Sigma_i$, where $z_{Jt}(i) = t$ for $i \in J$, $t \in [0, 1]$. Contrast this with the difficulty encountered in 418Xv.

(d) Let (X, Σ, μ) be a semi-finite compact measure space, and $\langle E_i \rangle_{i \in I}$ a point-finite family of measurable subsets of X such that $\bigcup_{i \in J} E_i \in \Sigma$ for every $J \subseteq I$. Show that $\mu(\bigcup_{i \in I} E_i) = \sup_{J \subseteq I \text{ is finite}} \mu(\bigcup_{i \in J} E_i)$. (*Hint*: 438Ya.)

(e) Let X be a metacompact space, and μ a semi-finite topological measure on X which (regarded as a measure) is compact. Show that μ is τ-additive.

(f) Let (X, Σ, μ) be a compact measure space, V a Banach space and $f : X \to V$ a measurable function such that $\|f\| : X \to [0, \infty[$ is integrable. Show that f is Bochner integrable (253Yf).

(g) Show that any purely atomic measure space is weakly α-favourable, so that the space of 216E/342N is weakly α-favourable but not countably compact.

(h) Show that the direct sum of a family of weakly α-favourable measure spaces is weakly α-favourable.

(i) Show that an indefinite-integral measure over a weakly α-favourable measure is weakly α-favourable.

(j) (i) Show that a countably compact measure space is weakly α-favourable. (ii) Show that a semi-finite weakly α-favourable measure space is perfect.

(k) Show that any measurable subspace of a weakly α-favourable measure space is weakly α-favourable.

(l) Let (X, Σ, μ) be a weakly α-favourable measure space, (Y, T, ν) a semi-finite measure space, and $f : X \to Y$ a (Σ, T)-measurable function such that $f^{-1}[F]$ is negligible whenever $F \subseteq Y$ is negligible. Show that (Y, T, ν) is weakly α-favourable.

(m) (i) Show that a measure space is weakly α-favourable iff its completion is weakly α-favourable. (ii) Show that a semi-finite measure space is weakly α-favourable iff its c.l.d. version is weakly α-favourable.

(n) Show that the c.l.d. product of two weakly α-favourable measure spaces is weakly α-favourable.

(o) Show that the product of any family of weakly α-favourable probability measures is weakly α-favourable.

(p) Show that the space of 451T is not weakly α-favourable.

(q) Let (X, Σ, μ) be a complete locally determined measure space and $\underline{\phi}$ a lower density on X such that $\underline{\phi}X = X$; let $\mathfrak{T}$ be the corresponding density topology (414P). Show that (X, Σ, μ) is weakly α-favourable iff $(X, \mathfrak{T})$ is weakly α-favourable (definition: 4A2A).

451 Notes and comments For a useful survey of results on countably compact and perfect measures, with historical notes, see RAMACHANDRAN 02.

The concepts of 'compact', 'countably compact' and 'perfect' measure space can all be regarded as attempts to understand and classify the special properties of Lebesgue measure on $[0, 1]$, regarded as a measure space. Because a countably separated perfect probability space is very nearly isomorphic to Lebesgue measure (344K), we can think of a perfect measure space as one in which the countably-generated σ-subalgebras look like Lebesgue measure (451F). The arguments of 451Ic and 451Jc already hint at the kind of results we can hope for. When we form a product measure, each measurable set in the product will depend, in effect, on sequences of measurable sets in the factors, and therefore can be studied in terms of countably generated subalgebras; so that many results about products of perfect measures will be derivable, if we wish to take that route, from results about products of copies of Lebesgue measure. Of course my normal approach in this treatise is to go straight for the general result; but like anyone else I often start from a picture based on the familiar special case. In the next section we shall have some theorems for which countable compactness, rather than perfectness, seems to be the relevant property.

The first half of the section (down to 451O) is essentially a matter of tidying up the theory of compact and perfect measures, and showing that the same ideas will cover the new class of countably compact measures. (You may like to go back to 342G, in which I worked through the basic properties of compact measures, and contrast the arguments used there with the slightly more sophisticated ones above.) In 451P-451S I enter new territory, showing that for compact measures (and therefore for Radon measures) the theory of measurable functions into metric spaces is particularly simple, without making any assumptions about measure-free cardinals.

452 Integration and disintegration of measures

A standard method of defining measures is through a formula

$$\mu E = \int \mu_y E \, \nu(dy)$$

where (Y, T, ν) is a measure space and $\langle \mu_y \rangle_{y \in Y}$ is a family of measures on another set X. In practice these constructions commonly involve technical problems concerning the domain of μ (as in 452Xi), which is why I have hardly used them so far in this treatise. There are not-quite-trivial examples in 417Yb, 434R and 436F, and the indefinite-integral measures of §234 can also be expressed in this way (452Xf); for a case in which this approach is worked out fully, see 453N. But when a formula of this kind is valid, as in Fubini's theorem, it is likely to be so useful that it dominates further investigation of the topic. In this section I give one of the two most important theorems guaranteeing the existence of appropriate families $\langle \mu_y \rangle_{y \in Y}$ when μ and ν are given (452I); another will follow in the next section (453K). They both suppose that we are provided with a suitable function $f : X \to Y$, and rely heavily on the Lifting Theorem (§341) and on considerations of inner regularity from Chapter 41. A variant of the argument can be applied to certain measures on product spaces, with a projection onto a factor in place of the function f (452M).

The formal definition of a 'disintegration' (which is nearly the same thing as a 'regular conditional probability') is in 452E. The main theorem depends, for its full generality, on the concept of 'countably compact measure' (451B). It can be strengthened when μ is actually a Radon measure (452O).

The greater part of the section is concerned with general disintegrations, in which the measures μ_y are supposed to be measures on X and are not necessarily related to any particular structure on X. However a natural, and obviously important, class of applications has $X = Y \times Z$ and each μ_y based on the section $\{y\} \times Z$, so that it can be regarded as a measure on Z. I begin the section, therefore, with an outline of the elementary theory of measures defined by 'integrating' such a family (452A-452D). For such measures, we can derive the fundamental requirement, that $\int \mu_y E \, \nu(dy)$ should be defined for all measurable sets $E \subseteq X$, from conditions expressed in terms of the factors Y and Z.

452A Lemma Let (Y, T, ν) be a probability space, X a set, and $\langle \mu_y \rangle_{y \in Y}$ a family of probability measures on X. Let $\mathcal{A}$ be the family of subsets A of X such that $\theta E = \int \mu_y E \, \nu(dy)$ is defined.

(a) $\mathcal{A}$ is a Dynkin class.

(b) If Σ is any σ-subalgebra of $\mathcal{A}$ then $\mu = \theta \restriction \Sigma$ is a probability measure on X.

(c) Suppose now that every μ_y is complete. If, in (b), $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\hat{\Sigma} \subseteq \mathcal{A}$ and $\hat{\mu} = \theta \restriction \hat{\Sigma}$.

proof For (a) and (b), we have only to look at the definitions of 'Dynkin class' and 'measure' and apply the elementary properties of the integral. For (c), if $E \in \hat{\Sigma}$, then there are $E', E'' \in \Sigma$ such that $E' \subseteq E \subseteq E''$ and $\theta E' = \theta E''$. So $\mu_y E' = \mu_y E''$ for ν-almost every y; since all the μ_y are supposed to be complete, $\mu_y E$ is defined and equal to $\mu_y E'$ for almost every y, and θE is defined and equal to $\theta E' = \mu E' = \hat{\mu} E$.

452B Theorem (a) Let X be a set, (Y, T, ν) a probability space, and $\langle \mu_y \rangle_{y \in Y}$ a family of probability measures on X. Let $\mathcal{E}$ be a family of subsets of X, closed under finite intersections, such that $\int \mu_y E \, \nu(dy)$ is defined for every $E \in \mathcal{E}$.

(i) If Σ is the σ-algebra of subsets of X generated by $\mathcal{E}$, we have a probability measure μ on X, with domain Σ, given by the formula $\mu E = \int \mu_y E \, \nu(dy)$ for every $E \in \Sigma$.

(ii) If $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\int \hat{\mu}_y E \, \nu(dy)$ is defined and equal to $\hat{\mu} E$ for every $E \in \hat{\Sigma}$, where $\hat{\mu}_y$ is the completion of μ_y for each $y \in Y$.

(b) Let Z be a set, (Y, T, ν) a probability space, and $\langle \mu_y \rangle_{y \in Y}$ a family of probability measures on Z. Let $\mathcal{H}$ be a family of subsets of Z, closed under finite intersections, such that $\int \mu_y H \, \nu(dy)$ is defined for every $H \in \mathcal{H}$.

(i) If Υ is the σ-algebra of subsets of Z generated by $\mathcal{H}$, we have a probability measure μ on $Y \times Z$, with domain $\mathrm{T} \widehat{\otimes} \Upsilon$, defined by setting $\mu E = \int \mu_y E[\{y\}] \nu(dy)$ for every $E \in \mathrm{T} \widehat{\otimes} \Upsilon$.

(ii) If $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\int \hat{\mu}_y E[\{y\}] \nu(dy)$ is defined and equal to $\hat{\mu} E$ for every $E \in \hat{\Sigma}$, where $\hat{\mu}_y$ is the completion of μ_y for each $y \in Y$.

proof (a) Define $\mathcal{A} \subseteq \mathcal{P}(Y \times Z)$ as in 452A. Then $\mathcal{E} \subseteq \mathcal{A}$, so by the Monotone Class Theorem (136B) $\Sigma \subseteq \mathcal{A}$ and we have (i). Applying 452Ac to $\langle \hat{\mu}_y \rangle_{y \in Y}$ we have (ii).

(b) Set $X = Y \times Z$. For $y \in Y$, let μ'_y be the measure on X defined by setting $\mu'_y E = \mu_y E[\{y\}]$ whenever this is defined; that is, μ'_y is the image of μ_y under the function $z \mapsto (y, z) : Z \to X$. Set $\mathcal{E} = \{F \times H : F \in \mathrm{T}, H \in \mathcal{H}\}$. Then $\mathcal{E}$ is a family of subsets of X closed under finite intersections, and

$$\int \mu'_y (F \times H) \nu(dy) = \int \chi F(y) \mu_y H \, \nu(dy)$$

is defined for every $F \in \mathrm{T}$, $H \in \mathcal{H}$. By (a), we have a measure μ on X, with domain the σ-algebra Σ generated by $\mathcal{E}$, defined by writing

$$\mu E = \int \mu'_y E \, \nu(dy) = \int \mu_y E[\{y\}] \nu(dy)$$

for every $E \in \Sigma$. Of course Σ includes $\mathrm{T} \widehat{\otimes} \Upsilon$ (the set $\{H : Y \times H \in \Sigma\}$ is a σ-algebra of subsets of Z including $\mathcal{H}$, so includes Υ) and is therefore equal to $\mathrm{T} \widehat{\otimes} \Upsilon$.

This proves (i). If now $E \in \hat{\Sigma}$, (a-ii) tells us that

$$\hat{\mu} E = \int \hat{\mu}'_y E \, \nu(dy) = \int \hat{\mu}_y E[\{y\}] \nu(dy).$$

452C Theorem(a) Let Y be a topological space, ν a τ-additive topological probability on Y, $(X, \mathfrak{T})$ a topological space, and $\langle \mu_y \rangle_{y \in Y}$ a family of τ-additive topological probability measures on X. Suppose that there is a base $\mathcal{U}$ for $\mathfrak{T}$, closed under finite unions, such that $y \mapsto \mu_y U$ is lower semi-continuous for every $U \in \mathcal{U}$.

(i) We can define a τ-additive Borel probability measure μ on X by writing $\mu E = \int \mu_y E \, \nu(dy)$ for every Borel set $E \subseteq X$.

(ii) If $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\int \hat{\mu}_y E \, \nu(dy)$ is defined and equal to $\hat{\mu} E$ for every $E \in \hat{\Sigma}$, where $\hat{\mu}_y$ is the completion of μ_y for each $y \in Y$.

(b) Let Y be a topological space, ν a τ-additive topological probability on Y, $(Z, \mathfrak{U})$ a topological space, and $\langle \mu_y \rangle_{y \in Y}$ a family of τ-additive topological probability measures on Z. Suppose that there is a base $\mathcal{V}$ for $\mathfrak{U}$, closed under finite unions, such that $y \mapsto \mu_y V$ is lower semi-continuous for every $V \in \mathcal{V}$.

(i) We can define a τ-additive Borel probability measure μ on $Y \times Z$ by writing $\mu E = \int \mu_y E[\{y\}] \nu(dy)$ for every Borel set $E \subseteq Y \times Z$.

(ii) If $\hat{\mu}$ is the completion of μ and $\hat{\Sigma}$ its domain, then $\int \hat{\mu}_y E[\{y\}] \nu(dy)$ is defined and equal to $\hat{\mu} E$ for every $E \in \hat{\Sigma}$, where $\hat{\mu}_y$ is the completion of μ_y for each $y \in Y$.

proof (a) For $A \subseteq X$, set $f_A(y) = \mu_y A$ when this is defined. We may suppose that $\emptyset \in \mathcal{U}$. If $\mathcal{W} \subseteq \mathcal{U}$ is a non-empty upwards-directed set with union G, $\langle f_W \rangle_{W \in \mathcal{W}}$ is an upwards-directed family of lower semi-continuous functions with supremum f_G, because every μ_y is τ-additive. So f_G is lower semi-continuous, and also $\int f_G d\nu = \sup_{W \in \mathcal{W}} \int f_W d\nu$, by 414Ba. Taking $\mathcal{E}$ to be the family of open subsets of X in 452Ba, we see that we have a τ-additive Borel measure μ on X such that $\mu E = \int \mu_y E \, \nu(dy)$ for every Borel set $E \subseteq X$. Moreover, if $\mathcal{G}$ is a non-empty upwards-directed family of open subsets of X with union G^*, then $\mathcal{W} = \{W : W \in \mathcal{U}, W \subseteq G \text{ for some } G \in \mathcal{G}\}$ is an upwards-directed family with union G^*, so

$$\mu G^* = \int f_{G^*} d\nu = \sup_{W \in \mathcal{W}} \int f_W d\nu \leq \sup_{G \in \mathcal{G}} \mu G \leq \mu G^*.$$

As $\mathcal{G}$ is arbitrary, μ is τ-additive. This proves (i); (ii) follows immediately, as in 452Ba.

(b) Let $\mathcal{U}$ be the family of sets expressible as $\bigcup_{i \leq n} H_i \times V_i$ where $H_i \subseteq Y$ is open and $V_i \in \mathcal{V}$ for every $i \leq n$. Because $\mathcal{V}$ is a base for $\mathfrak{U}$, $\mathcal{U}$ is a base for the topology of $X = Y \times Z$. For $y \in Y$ let μ'_y be the measure on X defined by saying that $\mu'_y E = \mu_y E[\{y\}]$ whenever this is defined. Then μ'_y is a τ-additive topological probability measure on X, by 418Ha or otherwise. If $U \in \mathcal{U}$, $y \mapsto \mu'_y U$ is lower semi-continuous. **P** Express U as $\bigcup_{i \leq n} H_i \times V_i$ where $H_i \subseteq Y$ is open and $V_i \in \mathcal{V}$ for each i. Suppose that $y \in Y$ and $\gamma < \mu_y U$. Set $I = \{i : i \leq n, y \in H_i\}$, $H = Y \cap \bigcap_{i \in I} H_i$ and $V = \bigcup_{i \in I} V_i$. Then $U[\{y\}] = V \subseteq U[\{y'\}]$ for every $y' \in H$. Also $H' = \{y' : \mu_{y'} V > \gamma\}$ is a neighbourhood of y. So $H \cap H'$ is a neighbourhood of y, and $\mu'_{y'} G[\{y'\}] > \gamma$ for every $y' \in H \cap H'$. As y and γ are arbitrary, we have the result. **Q**

Now applying (a) to $\langle \mu'_y \rangle_{y \in Y}$ we see that (b) is true.

452D Theorem (a) Let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be a Radon probability space, $(X, \mathfrak{T})$ a topological space, and $\langle \mu_y \rangle_{y \in Y}$ a uniformly tight (definition: 437Tb) family of Radon probability measures on X. Suppose that there is a base $\mathcal{U}$ for $\mathfrak{T}$, closed under finite unions, such that $y \mapsto \mu_y U$ is lower semi-continuous for every $U \in \mathcal{U}$. Then we have a Radon probability measure $\tilde{\mu}$ on X such that $\tilde{\mu} E = \int \mu_y E \, \nu(dy)$ whenever $\tilde{\mu}$ measures E.

(b) Let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be a Radon probability space, $(Z, \mathfrak{U})$ a topological space, and $\langle \mu_y \rangle_{y \in Y}$ a uniformly tight family of Radon probability measures on Z. Suppose that there is a base $\mathcal{V}$ for $\mathfrak{U}$, closed under finite unions, such that $y \mapsto \mu_y V$ is lower semi-continuous for every $V \in \mathcal{V}$. Then we have a Radon probability measure $\tilde{\mu}$ on $X = Y \times Z$ such that $\tilde{\mu} E = \int \mu_y E[\{y\}] \nu(dy)$ whenever $\tilde{\mu}$ measures E.

proof I take the two parts together. By 452C we have a τ-additive Borel probability measure μ satisfying the appropriate formula. Now for any $\epsilon > 0$ there is a compact set $K \subseteq X$ such that $\mu K \geq 1 - 2\epsilon$. **P** In (a), we just have to take a K such that $\mu_y K \geq 1 - 2\epsilon$ for every $y \in Y$. In (b), start with a compact set $K_1 \subseteq Y$ such that $\nu K_1 \geq 1 - \epsilon$. Now there is a compact set $K_2 \subseteq Z$ such that $\mu_y K_2 \geq 1 - \epsilon$ for every $y \in Y$. So $K = K_1 \times K_2$ is compact and $\mu K \geq (1 - \epsilon)^2 \geq 1 - 2\epsilon$. **Q**

Since μ is a probability measure it is surely locally finite and effectively locally finite, so the conditions of 416F(iv) are satisfied and the c.l.d. version $\tilde{\mu}$ of μ is a Radon measure on X. But of course $\tilde{\mu}$ is just the completion of μ, so 452C(a-ii) or 452C(b-ii) tells us that the declared formula also applies to $\tilde{\mu}$.

452E All the constructions above can be thought of as special cases of the following.

Definition Let (X, Σ, μ) and (Y, T, ν) be measure spaces. A **disintegration** of μ over ν is a family $\langle \mu_y \rangle_{y \in Y}$ of measures on X such that $\int \mu_y E \, \nu(dy)$ is defined in $[0, \infty]$ and equal to μE for every $E \in \Sigma$. If $f : X \to Y$ is an inverse-measure-preserving function, a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν is **consistent** with f if, for each $F \in \mathrm{T}$, $\mu_y f^{-1}[F] = 1$ for ν-almost every $y \in Y$. $\langle \mu_y \rangle_{y \in Y}$ is **strongly consistent** with f if, for almost every $y \in Y$, μ_y is a probability measure for which $f^{-1}[\{y\}]$ is conegligible.

A trivial example of a disintegration is when ν is a probability measure and $\mu_y = \mu$ for every y. Of course this is of little interest. The archetypal disintegration is 452Bb when all the μ_y are the same, in which case Fubini's theorem tells us that we are looking at a product measure on $X = Y \times Z$. Clearly this disintegration is consistent with the function $(y, z) \mapsto y : X \to Y$, and if μ is a probability measure then it is strongly consistent.

The phrase **regular conditional probability** is used for special types of disintegration; typically, when μ and ν and every μ_y are probabilities, and sometimes supposing that every μ_y has the same domain as μ. I have seen the word **decomposition** used for what I call a disintegration.

452F Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces and $\langle \mu_y \rangle_{y \in Y}$ a disintegration of μ over ν. Then $\iint f(x) \mu_y(dx) \nu(dy)$ is defined and equal to $\int f d\mu$ for every $[-\infty, \infty]$-valued function f such that $\int f d\mu$ is defined in $[-\infty, \infty]$.

proof (a) Suppose first that f is non-negative. Let $H \in \Sigma$ be a conegligible set such that $f \restriction H$ is Σ-measurable. For $n \in \mathbb{N}$ set

$$E_{nk} = \{x : x \in H, \, 2^{-n} k \leq f(x)\} \text{ for } k \geq 1, \quad f_n = 2^{-n} \textstyle\sum_{k=1}^{4^n} \chi E_{nk}.$$

Then $\langle f_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of functions with $\lim_{n \to \infty} f_n(x) = f(x)$ for every $x \in H$. Now $\int \mu_y(X \setminus H) \nu(dy) = 0$, so $X \setminus H$ is μ_y-negligible for almost every y. Set

$$V = \{y : \mu_y(X \setminus H) = 0, \, E_{nk} \in \operatorname{dom} \mu_y \text{ for every } n \in \mathbb{N}, \, k \geq 1\};$$

then V is ν-conegligible. For $y \in V$,

$$\int f d\mu_y = \lim_{n \to \infty} \int f_n d\mu_y = \lim_{n \to \infty} 2^{-n} \textstyle\sum_{k=1}^{4^n} \mu_y E_{nk},$$

while each function $y \mapsto \mu_y E_{nk}$ is ν-virtually measurable, so $y \mapsto \int f d\mu_y$ is ν-virtually measurable and

$$\begin{aligned} \iint f d\mu_y \nu(dy) &= \lim_{n \to \infty} \iint f_n d\mu_y \nu(dy) = \lim_{n \to \infty} 2^{-n} \sum_{k=1}^{4^n} \int \mu_y E_{nk} \, \nu(dy) \\ &= \lim_{n \to \infty} 2^{-n} \sum_{k=1}^{\infty} \mu E_{nk} = \lim_{n \to \infty} \int f_n d\mu = \int f d\mu. \end{aligned}$$

(b) For general f we now have

$$\begin{aligned} \iint f(x) \mu_y(dx) \nu(dy) &= \iint f^+(x) \mu_y(dx) \nu(dy) - \iint f^-(x) \mu_y(dx) \nu(dy) \\ &= \int f^+ d\mu - \int f^- d\mu = \int f d\mu, \end{aligned}$$

where f^+, f^- are the positive and negative parts of f.

Remark When $X = Y \times Z$ and our disintegration is a family $\langle \mu'_y \rangle_{y \in Y}$ of measures on X defined from a family $\langle \mu_y \rangle_{y \in Y}$ of probability measures on Z, as in 452Bb, we can more naturally write $\int f(y, z) \mu_y(dz)$ in place of $\int f(x) \mu'_y(dx)$, and we get

$$\textstyle\iint f(y, z) \mu_y(dz) \nu(dy) = \int f d\mu \text{ whenever the latter is defined in } [-\infty, \infty]$$

as in 252B.

452G The most useful theorems about disintegrations of course involve some restrictions on their form, most commonly involving consistency with some kind of projection. I clear the path with statements of some elementary facts.

Proposition Let (X, Σ, μ) and (Y, T, ν) be measure spaces, $f : X \to Y$ an inverse-measure-preserving function, and $\langle \mu_y \rangle_{y \in Y}$ a disintegration of μ over ν.

(a) If $\langle \mu_y \rangle_{y \in Y}$ is consistent with f, and $F \in \mathrm{T}$, then $\mu_y f^{-1}[F] = \chi F(y)$ for ν-almost every $y \in Y$; in particular, almost every μ_y is a probability measure.

(b) If $\langle \mu_y \rangle_{y \in Y}$ is strongly consistent with f it is consistent with f.

(c) If ν is countably separated (definition: 343D) and $\langle \mu_y \rangle_{y \in Y}$ is consistent with f, then it is strongly consistent with f.

proof (a) We have $\mu_y f^{-1}[F] = 1$ for almost every $y \in F$. Since also

$$\mu_y(X \setminus f^{-1}[F]) = \mu_y f^{-1}[Y \setminus F] = 1, \quad \mu_y X = \mu_y f^{-1}[Y] = 1$$

for almost every $y \in Y \setminus F$, $\mu_y f^{-1}[F] = 0$ for almost every $y \in X \setminus F$.

(b) If $F \in \mathrm{T}$, then $f^{-1}[F] \supseteq f^{-1}[\{y\}]$ is μ_y-conegligible for almost every $y \in F$; since we are also told that $\mu_y X = 1$ for almost every y, $\mu_y f^{-1}[F] = 1$ for almost every $y \in F$.

(c) There is a countable $\mathcal{F} \subseteq \mathrm{T}$ separating the points of Y; we may suppose that $Y \setminus F \in \mathcal{F}$ for every $F \in \mathcal{F}$. Now

$$H_F = F \setminus \{y : \mu_y f^{-1}[F] \text{ is defined and equal to } 1\}$$

are negligible for every $F \in \mathcal{F}$, so that

$$Z = \{y : \mu_y X = 1\} \setminus \textstyle\bigcup_{F \in \mathcal{F}} H_F$$

is conegligible. For $y \in Z$, set $\mathcal{F}_y = \{F : y \in F \in \mathcal{F}\}$; then

$$\{y\} = \textstyle\bigcap \mathcal{F}_y, \quad f^{-1}[\{y\}] = \bigcap \{f^{-1}[F] : F \in \mathcal{F}_y\},$$

while $\mu_y f^{-1}[F] = 1$ for every $F \in \mathcal{F}_y$. Because $\mathcal{F}_y$ is countable, $\mu_y f^{-1}[\{y\}] = 1$. This is true for almost every y, so $\langle \mu_y \rangle_{y \in Y}$ is strongly consistent with f.

452H Lemma Let (X, Σ, μ) and (Y, T, ν) be probability spaces, and $T : L^\infty(\mu) \to L^\infty(\nu)$ a positive linear operator such that $T(\chi X^\bullet) = \chi Y^\bullet$ and $\int Tu = \int u$ whenever $u \in L^\infty(\mu)^+$. Let $\mathcal{K}$ be a countably compact class of subsets of X, closed under finite unions and countable intersections, such that μ is inner regular with respect to $\mathcal{K}$. Then there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν such that

(i) $\mu_y X = 1$, $\mathcal{K} \subseteq \operatorname{dom} \mu_y$ and μ_y is inner regular with respect to $\mathcal{K}$, for every $y \in Y$;

(ii) setting $h_g(y) = \int g \, d\mu_y$ whenever $g \in \mathcal{L}^\infty(\mu)$ and $y \in Y$ are such that the integral is defined, $h_g \in \mathcal{L}^\infty(\nu)$ and $T(g^\bullet) = h_g^\bullet$ for every $g \in \mathcal{L}^\infty(\mu)$.

proof (a) Completing ν does not change its $\mathcal{L}^\infty(\nu)$ or $L^\infty(\nu)$, nor does it change the families which are disintegrations over ν; so we may assume throughout that ν is complete. It therefore has a lifting $\theta : \mathfrak{B} \to \mathrm{T}$, where $\mathfrak{B}$ is the measure algebra of ν, which gives rise to a Riesz homomorphism S from $L^\infty(\nu) \cong L^\infty(\mathfrak{B})$ to the space $L^\infty(\mathrm{T})$ of bounded T-measurable real-valued functions on Y such that $(Sv)^\bullet = v$ for every $v \in L^\infty(\nu)$ (363I, 363F, 363Ha).

(b) For $y \in Y$ and $E \in \Sigma$, set $\psi_y E = (ST(\chi E^\bullet))(y)$. Because $0 \leq T(\chi E^\bullet) \leq \chi Y^\bullet$ in $L^\infty(\nu)$, $0 \leq \psi_y E \leq 1$. The maps

$$E \mapsto \chi E \mapsto \chi E^\bullet \mapsto T(\chi E^\bullet) \mapsto ST(\chi E^\bullet)$$

are all additive, so $\psi_y : \Sigma \to [0,1]$ is additive for each $y \in Y$. For fixed $E \in \Sigma$,

$$\mu E = \textstyle\int \chi E \, d\mu = \int (\chi E^\bullet) = \int T(\chi E^\bullet) = \int ST(\chi E^\bullet) = \int \psi_y E \, \nu(dy).$$

(c) At this point, recall that μ is supposed to be inner regular with respect to the countably compact class $\mathcal{K}$. By 413Sa, there is for every $y \in Y$ a complete measure μ'_y on X such that $\mu'_y X \leq \psi_y X \leq 1$, $\mathcal{K} \subseteq \operatorname{dom} \mu'_y$, and $\mu'_y K \geq \psi_y K$ for every $K \in \mathcal{K} \cap \Sigma$.

(d) Now, for any fixed $E \in \Sigma$, $\mu'_y E$ is defined and equal to $\psi_y E$ for almost every $y \in Y$. **P** Let $\langle K_n \rangle_{n\in\mathbb{N}}$, $\langle K'_n \rangle_{n\in\mathbb{N}}$ be sequences in $\mathcal{K} \cap \Sigma$ such that $K_n \subseteq E$ and $K'_n \subseteq X \setminus E$ for every n, while $\mu E = \sup_{n\in\mathbb{N}} \mu K_n$ and $\mu(X \setminus E) = \sup_{n\in\mathbb{N}} \mu K'_n$. Set $L = \bigcup_{n\in\mathbb{N}} K_n$, $L' = \bigcap_{n\in\mathbb{N}} (X \setminus K'_n)$. Then both L and L' belong to the domain of every μ'_y, and

$$\begin{aligned}\sup_{n\in\mathbb{N}} \psi_y K_n &\le \sup_{n\in\mathbb{N}} \mu'_y K_n \le \mu'_y L \le \mu'_y L' \\ &\le \inf_{n\in\mathbb{N}} \mu'_y (X \setminus K'_n) = \mu'_y X - \sup_{n\in\mathbb{N}} \mu'_y K'_n \le 1 - \sup_{n\in\mathbb{N}} \psi_y K'_n\end{aligned}$$

for every y. On the other hand,

$$\begin{aligned}\int (1 - \sup_{n\in\mathbb{N}} \psi_y K'_n)\nu(dy) &\le \nu Y - \sup_{n\in\mathbb{N}} \int \psi_y(K'_n)\nu(dy) = \mu X - \sup_{n\in\mathbb{N}} \mu K'_n = \mu E \\ &= \sup_{n\in\mathbb{N}} \mu K_n = \sup_{n\in\mathbb{N}} \int \psi_y K_n \nu(dy) \le \int \sup_{n\in\mathbb{N}} \psi_y K_n \nu(dy).\end{aligned}$$

So

$$\sup_{n\in\mathbb{N}} \psi_y K_n = \mu'_y L = \mu'_y L' = 1 - \sup_{n\in\mathbb{N}} \psi_y K'_n$$

for almost every y. Because $L \subseteq E \subseteq L'$ and μ'_y is complete, $E \in \operatorname{dom} \mu'_y$ and

$$\mu'_y E = 1 - \sup_{n\in\mathbb{N}} \psi_y K'_n \ge 1 - \psi_y(X \setminus E) \ge \psi_y E$$

for almost every $y \in Y$. Similarly, $\mu'_y(X \setminus E) \ge \psi_y(X \setminus E)$ for almost every y. But as

$$\mu'_y E + \mu'_y(X \setminus E) = \mu'_y X \le \psi_y X \le 1$$

whenever the left-hand side is defined, we must have $\mu'_y E = \psi_y E$ for almost every y, as claimed. **Q**

It follows at once that

$$\int \mu'_y E \, \nu(dy) = \int \psi_y E \, \nu(dy) = \mu E$$

for every $E \in \Sigma$, and $\langle \mu'_y \rangle_{y\in Y}$ is a disintegration of μ over ν.

(e) At this point observe that

$$\int \mu'_y X \, \nu(dy) = \mu X = \int \chi X^\bullet = \int T(\chi X^\bullet) = \nu Y,$$

so $F_0 = \{y : \mu'_y X < 1\}$ is negligible. Taking any $y_0 \in Y \setminus F_0$ and setting

$$\begin{aligned}\mu_y &= \mu'_{y_0} \text{ for } y \in F_0, \\ &= \mu_y \text{ for } y \in Y \setminus F_0,\end{aligned}$$

we find ourselves with a disintegration $\langle \mu_y \rangle_{y\in Y}$ of μ over ν with the same properties as $\langle \mu'_y \rangle_{y\in Y}$, but now consisting entirely of probability measures.

(f) For $g \in \mathcal{L}^\infty(\mu)$, set $h_g(y) = \int g \, d\mu_y$ whenever $y \in Y$ is such that the integral is defined. Consider the set V of those $g \in \mathcal{L}^\infty(\mu)$ such that $h_g \in \mathcal{L}^\infty(\nu)$ and $Tg^\bullet = h_g^\bullet$ in $L^\infty(\nu)$. If $E \in \Sigma$, then $h_{\chi E}(y) = \psi_y E$ for almost every y, so

$$h_{\chi E}^\bullet = (ST(\chi E^\bullet))^\bullet = T(\chi E^\bullet);$$

accordingly $\chi E \in V$. It is easy to check that V is closed under addition and scalar multiplication, so it contains all simple functions. Next, if $\langle g_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of simple functions with limit $g \in \mathcal{L}^\infty(\nu)$, then $h_g = \sup_{n\in\mathbb{N}} h_{g_n}$ wherever the right-hand side is defined. Also T is order-continuous, because it preserves integrals, so

$$Tg^\bullet = \sup_{n\in\mathbb{N}} Tg_n^\bullet = \sup_{n\in\mathbb{N}} h_{g_n}^\bullet = h_g^\bullet$$

and $g \in V$. Finally, if $g \in \mathcal{L}^\infty(\mu)$ is zero almost everywhere, there is a negligible $E \in \Sigma$ such that $g(x) = 0$ for every $x \in X \setminus E$; $\mu_y E = 0$ for almost every y, so $h_g(y) = \int g \, d\mu_y = 0$ for almost every y and again $g \in V$. Putting these together, we see that $V = \mathcal{L}^\infty(\nu)$, as required by (ii) as stated above.

452I Theorem (PACHL 78) Let (X, Σ, μ) be a countably compact measure space, (Y, T, ν) a σ-finite measure space, and $f : X \to Y$ an inverse-measure-preserving function. Then there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν, consistent with f, such that $\mu_y X \leq 1$ for every $y \in Y$. Moreover,

(i) if $\mathcal{K}$ is a countably compact class of subsets of X such that μ is inner regular with respect to $\mathcal{K}$, then we can arrange that $\mathcal{K} \subseteq \operatorname{dom} \mu_y$ for every $y \in Y$;

(ii) if, in (i), $\mathcal{K}$ is closed under finite unions and countable intersections, then we can arrange that $\mathcal{K} \subseteq \operatorname{dom} \mu_y$ and μ_y is inner regular with respect to $\mathcal{K}$ for every $y \in Y$.

proof (a) Consider first the case in which ν and μ are probability measures and we are provided with a class $\mathcal{K}$ as in (ii). In this case, for each $u \in L^\infty(\mu)$, $F \mapsto \int_{f^{-1}[F]} u$ is countably additive. So we have an operator $T : L^\infty(\mu) \to L^\infty(\nu)$ defined by saying that $\int_F Tu = \int_{f^{-1}[F]} u$ whenever $u \in L^\infty(\mu)$ and $F \in \mathrm{T}$. Of course T is linear and positive and $\int Tu = \int u$ whenever $u \in L^\infty(\mu)$.

By 452H, there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν such that

(α) for every $y \in Y$, $\mu_y X = 1$, $\mathcal{K} \subseteq \operatorname{dom} \mu_y$ and μ_y is inner regular with respect to $\mathcal{K}$;

(β) $T(g^\bullet) = h_g^\bullet$ whenever $g \in \mathcal{L}^\infty(\mu)$ and $h_g(y) = \int g \, d\mu_y$ when the integral is defined.

If now $F \in \mathrm{T}$, set $g = \chi f^{-1}[F]$ in (β); then $Tg^\bullet$ is defined by saying that

$$\int_H Tg^\bullet = \int_{f^{-1}[H]} g = \mu f^{-1}[F \cap H] = \nu(F \cap H)$$

for every $H \in \mathrm{T}$, so that $Tg^\bullet = \chi F^\bullet$ and we must have $\mu_y f^{-1}[F] = 1$ for almost every $y \in F$.

(b) The theorem is stated in a way that makes it quotable in parts without committing oneself to a particular class $\mathcal{K}$. But if we are given a class satisfying (i), we can extend it to one satisfying (ii), by 413Sb; and if we are told only that μ is countably compact, we know from the definition that we shall be able to choose a countably compact class satisfying (i).

(c) This proves the theorem on the assumption that μ and ν are probability measures. If $\mu X = \nu Y = 0$ then the result is trivial, as we can take every μ_y to be the zero measure. Otherwise, because ν is σ-finite, there is a partition $\langle Y_n \rangle_{n \in \mathbb{N}}$ of Y into measurable sets of finite measure. Let $\langle \gamma_n \rangle_{n \in \mathbb{N}}$ be a sequence of strictly positive real numbers such that $\sum_{n=0}^\infty \gamma_n \nu Y_n = 1$, and write

$$\nu' F = \textstyle\sum_{n=0}^\infty \gamma_n \nu(F \cap Y_n) \text{ for } F \in \mathrm{T},$$

$$\mu' E = \textstyle\sum_{n=0}^\infty \gamma_n \mu(E \cap X_n) \text{ for } E \in \Sigma.$$

It is easy to check (α) that ν' and μ' are probability measures (β) that f is inverse-measure-preserving for μ' and ν' (γ) that if μ is inner regular with respect to $\mathcal{K}$ so is μ'. Note that ν' and ν have the same negligible sets. By (a)-(b), μ' has a disintegration $\langle \mu_y \rangle_{y \in Y}$ over ν' which is consistent with f, and (if appropriate) has the properties demanded in (i) or (ii). Now, if $E \in \Sigma$,

$$\begin{aligned}
\mu E &= \sum_{n=0}^\infty \gamma_n^{-1} \mu'(E \cap X_n) = \sum_{n=0}^\infty \gamma_n^{-1} \int \mu_y(E \cap X_n) \nu'(dy) \\
&= \sum_{n=0}^\infty \gamma_n^{-1} \int_{Y_n} \mu_y E \, \nu'(dy) \\
&\text{(because } \mu_y X = 1, \ \mu_y X_n = (\chi Y_n)(y) \text{ for } \nu'\text{-almost every } y, \text{ every } n) \\
&= \sum_{n=0}^\infty \int_{Y_n} \mu_y E \, \nu(dy) = \sum_{n=0}^\infty \int \mu_y(E \cap X_n) \nu(dy) = \int \mu_y E \, \nu(dy).
\end{aligned}$$

So $\langle \mu_y \rangle_{y \in Y}$ is a disintegration of μ over ν. If $F \in \mathrm{T}$, then $\mu_y f^{-1}[F] = 1$ for ν'-almost every y, that is, for ν-almost every y, so $\langle \mu_y \rangle_{y \in Y}$ is still consistent with f with respect to the measure ν.

452J Remarks (a) In the theorem above, I have carefully avoided making any promises about the domains of the μ_y beyond that in (i). If Σ_0 is the σ-algebra generated by $\mathcal{K} \cap \Sigma$, then whenever $E \in \Sigma$ there are E', $E'' \in \Sigma_0$ such that $E' \subseteq E \subseteq E''$ and $\mu(E'' \setminus E') = 0$. (For μ, like ν, must be σ-finite, so

we can choose E' to be a countable union of members of $\mathcal{K} \cap \Sigma$, and E'' to be the complement of such a union.) Thus we shall have a σ-algebra on which every μ_y is defined and which will be adequate to describe nearly everything about μ. The example of Lebesgue measure on the square (452E) shows that we cannot ordinarily expect the μ_y to be defined on the whole of Σ itself. In many important cases, of course, we can say more (452Xl).

(b) Necessarily (as remarked in the course of the proof) $\mu_y X = 1$ for almost every y. In some applications it seems right to change μ_y for a negligible set of y's so that every μ_y is a probability measure. Of course this cannot be done if $X = \emptyset \neq Y$, but this case is trivial (we should have to have $\nu Y = 0$). In other cases, we can make sure that any new μ_y is equal to some old one, so that a property required by (i) or (ii) remains true of the new disintegration. If we want to have '$\mu_y f^{-1}[\{y\}] = \mu_y X = 1$ for every $y \in Y$', strengthening 'strongly consistent', we shall of course have to begin by checking that f is surjective.

(c) The question of whether 'σ-finite' can be weakened to 'strictly localizable' in the hypotheses of 452I is related to the Banach-Ulam problem (452Yb). See also 452O.

452K Example The hypothesis 'countably compact' in 452I is in fact essential (452Ye). To see at least that it cannot be omitted, we have the following elementary example. Set $Y = [0,1]$, and let ν be Lebesgue measure on Y, with domain T. Let $X \subseteq [0,1]$ have outer measure 1 and inner measure 0 (134D, 419J); let μ be the subspace measure on X. Set $f(x) = x$ for $x \in X$. Then there is no disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν which is consistent with f.

P? Suppose, if possible, that $\langle \mu_y \rangle_{y \in [0,1]}$ is such a disintegration. Then, in particular, the sets

$$H_q = [0,q] \setminus \{y : X \cap [0,q] \in \operatorname{dom} \mu_y,\ \mu_y(X \cap [0,q]) = 1\},$$

$$H'_q = [q,1] \setminus \{y : X \cap [q,1] \in \operatorname{dom} \mu_y,\ \mu_y(X \cap [q,1]) = 1\}$$

are negligible for every $q \in [0,1]$. Set $G = [0,1] \setminus \bigcup_{q \in \mathbb{Q} \cap [0,1]} (H_q \cup H'_q)$, so that G is ν-conegligible. Then there must be some $y \in G \setminus X$. Now $\mu_y(X \cap [0,q']) = \mu_y(X \cap [q,1]) = 1$ whenever $q, q' \in \mathbb{Q}$ and $0 \leq q < y < q' \leq 1$, so that $\mu_y(X \cap \{y\}) = 1$. But $X \cap \{y\} = \emptyset$. **XQ**

452L The same ideas as in 452I can be used to prove a result on the disintegration of measures on product spaces. It will help to have a definition.

Definition Let $\langle X_i \rangle_{i \in I}$ be a family of sets, and λ a measure on $X = \prod_{i \in I} X_i$. For each $i \in I$ set $\pi_i(x) = x(i)$ for $x \in X$. Then the image measure $\lambda \pi_i^{-1}$ is the **marginal** measure of λ on X_i.

452M I return to the context of 452B-452D.

Theorem Let Y and Z be sets and $\mathrm{T} \subseteq \mathcal{P}Y$, $\Upsilon \subseteq \mathcal{P}Z$ σ-algebras. Let μ be a non-zero totally finite measure with domain $\mathrm{T}\widehat{\otimes}\Upsilon$, and ν the marginal measure of μ on Y. Suppose that the marginal measure λ of μ on Z is inner regular with respect to a countably compact class $\mathcal{K} \subseteq \mathcal{P}Z$ which is closed under finite unions and countable intersections. Then there is a family $\langle \mu_y \rangle_{y \in Y}$ of complete probability measures on Z, all measuring every member of $\mathcal{K}$ and inner regular with respect to $\mathcal{K}$, such that

$$\mu E = \int \mu_y E[\{y\}] \nu(dy)$$

for every $E \in \mathrm{T}\widehat{\otimes}\Upsilon$, and

$$\int f d\mu = \iint f(y,z) \mu_y(dz) \nu(dy)$$

whenever f is a $[-\infty, \infty]$-valued function such that $\int f d\mu$ is defined in $[-\infty, \infty]$.

proof (a) To begin with, assume that μ is a probability measure and that ν is complete. Let $\mathfrak{B}$ be the measure algebra of ν and $\theta : \mathfrak{B} \to \mathrm{T}$ a lifting. For $H \in \Upsilon$, $F \in \mathrm{T}$ set $\nu_H F = \mu(F \times H)$; then $\nu_H : \mathrm{T} \to [0,1]$ is countably additive and $\nu_H F \leq \nu F$ for every $F \in \mathrm{T}$, so there is a $v_H \in L^1(\nu)$ such that $\int_F v_H = \nu_H F$ for every $F \in \mathrm{T}$ and $0 \leq v_H \leq \chi 1$. We can therefore think of v_H as a member of $L^\infty(\nu) \cong L^\infty(\mathfrak{B})$. Let $T : L^\infty(\mathfrak{B}) \to L^\infty(\mathrm{T})$ be the Riesz homomorphism associated with θ, and set $\psi_y H = (T v_H)(y)$ for every $y \in Y$.

Each $\psi_y : \Upsilon \to [0,\infty[$ is finitely additive. So we have a complete measure μ_y on Z such that $\mu_y Z \leq \psi_y Z = 1$, $\mathcal{K} \subseteq \operatorname{dom}\mu_y$, μ_y is inner regular with respect to $\mathcal{K}$ and $\mu_y K \geq \psi_y K$ for every $K \in \mathcal{K}$.

For $H \in \Upsilon$, $F \in \mathrm{T}$ we have

$$\int_F \psi_y H\,\nu(dy) = \int_F Tv_H = \int_F v_H = \mu(F \times H).$$

So

$$\underline{\int} \mu_y K \cdot \chi F(y)\nu(dy) \geq \int_F \psi_y K\,\nu(dy) = \mu(F \times K)$$

for every $K \in \mathcal{K}$. Now note that, for any $H \in \Upsilon$ and $F \in \mathrm{T}$,

$$\begin{aligned}\mu(F \times H) - \sup_{K\in\mathcal{K},K\subseteq H} \mu(F \times K) &= \inf_{K\in\mathcal{K},K\subseteq H} \mu(F \times (H \setminus K)) \\ &\leq \inf_{K\in\mathcal{K},K\subseteq H} \lambda(H \setminus K) = 0\end{aligned}$$

because λ is inner regular with respect to $\mathcal{K}$ (and, like μ, is a probability measure). So

$$\begin{aligned}\underline{\int} (\mu_y)_* H \cdot \chi F(y)\nu(dy) &\geq \sup_{K\in\mathcal{K},K\subseteq H} \underline{\int} \mu_y K \cdot \chi F(y)\nu(dy) \\ &\geq \sup_{K\in\mathcal{K},K\subseteq H} \mu(F \times K) = \mu(F \times H).\end{aligned}$$

In particular,

$$\underline{\int} (\mu_y)_* H\,\nu(dy) \geq \mu(Y \times H) = \lambda H,$$

and similarly $\underline{\int}(\mu_y)_*(Z \setminus H)\nu(dy) \geq \lambda(Z \setminus H)$.

Taking ν-integrable functions g_1, g_2 such that $g_1(y) \leq (\mu_y)_* H$ and $g_2(y) \leq (\mu_y)_*(Z \setminus H)$ for almost every y, $\int g_1 d\nu = \underline{\int}(\mu_y)_* H\,\nu(dy)$ and $\int g_2 d\nu = \underline{\int}(\mu_y)_*(Z \setminus H)\nu(dy)$ (133J(a-ii)), we must have

$$g_1(y) + g_2(y) \leq (\mu_y)_* H + (\mu_y)_*(Z \setminus H) \leq \mu_y Z \leq 1$$

for almost every y, while $\int g_1 + g_2 d\nu \geq 1$; so that, for almost all y,

$$g_1(y) + g_2(y) = (\mu_y)_* H + (\mu_y)_*(Z \setminus H) = \mu_y Z = 1,$$

and (because μ_y is complete) $\mu_y H$ is defined and equal to $g_1(y)$ (413Ef). It now follows that

$$\int_F \mu_y H\,\nu(dy) = \int_F g_1(y)\nu(dy) = \underline{\int}(\mu_y)_* H \cdot \chi F(y)\nu(dy) \geq \mu(F \times H)$$

for every $F \in \mathrm{T}$. But since also

$$\int_F \mu_y(Z \setminus H)\nu(dy) \geq \mu(F \times (Z \setminus H)),$$

$$\int_F \mu_y H + \mu_y(Z \setminus H)\nu(dy) \leq \nu F = \mu(F \times H) + \mu(F \times (Z \setminus H)),$$

we must actually have $\int_F \mu_y H\,\nu(dy) = \mu(F \times H)$.

All this is true for every $F \in \mathrm{T}$, $H \in \Upsilon$. But now, setting

$$\mathcal{E} = \{E : E \in \mathrm{T}\widehat{\otimes}\Upsilon,\ \mu E = \textstyle\int \mu_y E[\{y\}]\nu(dy)\},$$

we see that $\mathcal{E}$ is a Dynkin class and includes $\mathcal{I} = \{F \times H : F \in \mathrm{T},\ H \in \Upsilon\}$, which is closed under finite intersections; so that the Monotone Class Theorem tells us that $\mathcal{E}$ includes the σ-algebra generated by $\mathcal{I}$, and is the whole of $\mathrm{T}\widehat{\otimes}\Upsilon$.

(b) The rest is just tidying up. First I should note that the construction in (a) allows $\mu_y Z$ to be less than 1 for a ν-negligible set of y; but of course all we have to do, if that happens, is to amend μ_y arbitrarily on that set to any of the 'ordinary' values of μ_y. Next, if the original measure ν is not complete, just apply the argument of (a) to its completion. If the original measure μ is not a probability measure, apply the arguments so far to suitable scalar multiples of μ and ν.

(c) Thus we have the formula

$$\mu E = \textstyle\int \mu_y E[\{y\}]\nu(dy)$$

for every $E \in \mathrm{T}\widehat{\otimes}\Upsilon$. The second formula announced follows as in the remark following 452F.

452N Corollary Let Y and Z be sets and $\mathrm{T} \subseteq \mathcal{P}Y$, $\Upsilon \subseteq \mathcal{P}Z$ σ-algebras. Let μ be a probability measure with domain $\mathrm{T}\widehat{\otimes}\Upsilon$, and ν the marginal measure of μ on Y. Suppose that

either Υ is the Baire σ-algebra with respect to a compact Hausdorff topology on Z

or Υ is the Borel σ-algebra with respect to an analytic Hausdorff topology on Z

or (Z, Υ) is a standard Borel space.

Then there is a family $\langle \mu_y \rangle_{y \in Y}$ of probability measures on Z, all with domain Υ, such that

$$\mu E = \int \mu_y E[\{y\}] \nu(dy)$$

for every $E \in \mathrm{T}\widehat{\otimes}\Upsilon$, and

$$\int f d\mu = \iint f(y,z) \mu_y(dz) \nu(dy)$$

whenever f is a $[-\infty, \infty]$-valued function such that $\int f d\mu$ is defined in $[-\infty, \infty]$.

proof In each case, the marginal measure of μ on Z is tight, (that is, inner regular with respect to the compact sets) for a Hausdorff topology on Z. (Use 412D when Υ is the Baire σ-algebra on a compact Hausdorff space and 433Ca when it is the Borel σ-algebra on an analytic Hausdorff space; when (Z, Υ) is a standard Borel space, take any appropriate Polish topology on Z and use 423Ba.) So 452M tells us that we can achieve the formulae sought with Radon probability measures μ_y. Since (in all three cases) $\operatorname{dom} \mu_y$ will include Υ for every y, we can get the result as stated by replacing each μ_y by $\mu_y \restriction \Upsilon$.

452O Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, (Y, T, ν) a strictly localizable measure space, and $f : X \to Y$ an inverse-measure-preserving function. Then there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν, consistent with f, such that every μ_y is a Radon measure on X.

proof (a) Let $\langle Y_i \rangle_{i \in I}$ be a decomposition of Y. For each $i \in I$, let ν_i be the subspace measure on Y_i and λ_i the subspace measure on $X_i = f^{-1}[Y_i]$. Then $f_i = f \restriction X_i$ is inverse-measure-preserving for λ_i and ν_i. Let $\mathcal{K}_i$ be the family of compact subsets of X_i; of course $\mathcal{K}_i$ is a (countably) compact class and λ_i is inner regular with respect to $\mathcal{K}_i$ (412Oa). By 452I, we can choose, for each $i \in I$, a disintegration $\langle \tilde{\mu}_y \rangle_{y \in Y_i}$ of λ_i over ν_i, consistent with $f \restriction X_i$, such that (α) every $\tilde{\mu}_y$ is complete and totally finite (β) $\tilde{\mu}_y$ measures every compact subset of X_i and is inner regular with respect to $\mathcal{K}_i$ for every $y \in Y_i$. By 412Ja, $\tilde{\mu}_y$ measures every relatively closed subset of X_i for every $y \in Y_i$.

Now, for $i \in I$ and $y \in Y_i$, set

$$\mu_y E = \tilde{\mu}_y(E \cap X_i)$$

whenever $E \subseteq X$ and $E \cap X_i$ is measured by $\tilde{\mu}_y$. Then μ_y is a complete totally finite measure on X; it is inner regular with respect to $\mathcal{K}_i$ and measures every closed subset of X. It follows at once that it is tight and measures every Borel set, that is, is a Radon measure on X.

(b) Now $\mu E = \int \mu_y E \, \nu(dy)$ for every $E \in \Sigma$. **P** $\bigcup_{i \in J} E \cap X_i = E \cap f^{-1}[\bigcup_{i \in J} Y_i]$ belongs to Σ for every $J \subseteq I$. By 451P, $\mu E = \sum_{i \in I} \mu(E \cap X_i)$. For $i \in I$, we have $\int_{Y_i} \tilde{\mu}_y(E \cap X_i) \nu_i(dy) = \mu(E \cap X_i)$. So

$$\begin{aligned} \mu E &= \sum_{i \in I} \mu(E \cap X_i) = \sum_{i \in I} \int_{Y_i} \tilde{\mu}_y(E \cap X_i) \nu_i(dy) \\ &= \sum_{i \in I} \int_{Y_i} \mu_y E \, \nu(dy) = \int \mu_y E \, \nu(dy) \end{aligned}$$

by 214M. **Q**

Thus $\langle \mu_y \rangle_{y \in Y}$ is a disintegration of μ over ν.

(c) Finally, if $F \in \mathrm{T}$ and $i \in I$, then

$$\begin{aligned} Y_i \cap F \setminus \{y : \mu_y f^{-1}[F] &\text{ is defined and equal to } 1\} \\ &= (F \cap Y_i) \setminus \{y : y \in Y_i, \, \tilde{\mu}_y f^{-1}[F \cap Y_i] = 1\} \end{aligned}$$

is negligible for every i, so $\mu_y f^{-1}[F] = 1$ for almost every y. Thus $\langle \mu_y \rangle_{y \in Y}$ is consistent with f.

452P Corollary (cf. BLACKWELL 56) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ an analytic Radon measure space and $f : X \to Y$ an inverse-measure-preserving function. Then there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν, strongly consistent with f, such that every μ_y is a Radon measure on X.

proof By 433B, ν is countably separated; now put 452O and 452Gc together.

452Q Disintegrations and conditional expectations Fubini's theorem provides a relatively concrete description of the conditional expectation of a function on a product of probability spaces with respect to the σ-algebra defined by one of the factors, by means of the formula $g(x,y) = \int f(x,z)dz$ (253H). This generalizes straightforwardly to measures with disintegrations, as follows.

Proposition Let (X, Σ, μ) and (Y, T, ν) be probability spaces and $f : X \to Y$ an inverse-measure-preserving function. Suppose that $\langle \mu_y \rangle_{y \in Y}$ is a disintegration of μ over ν which is consistent with f, and that g is a μ-integrable real-valued function.

(a) Setting $h_0(y) = \int g \, d\mu_y$ whenever $y \in Y$ and the integral is defined in $\mathbb{R}$, h_0 is a Radon-Nikodým derivative of the functional $F \mapsto \int_{f^{-1}[F]} g \, d\mu : \mathrm{T} \to \mathbb{R}$.

(b) Now suppose that ν is complete. Setting $h_1(x) = \int g \, d\mu_{f(x)}$ whenever $x \in X$ and the integral is defined in $\mathbb{R}$, then h is a conditional expectation of g on the σ-algebra $\Sigma_0 = \{f^{-1}[F] : F \in \mathrm{T}\}$.

proof (a) If $F \in \mathrm{T}$, then $f^{-1}[F]$ is μ_y-conegligible for almost every $y \in F$, and μ_y-negligible for almost every $y \in Y \setminus F$, so $\int g \times \chi f^{-1}[F] d\mu_y = h_0(y) \times \chi F(y)$ for almost every y, and

$$\int_F h_0 d\nu = \iint g \times \chi f^{-1}[F] d\mu_y \nu(dy) = \int_{f^{-1}[F]} g \, d\mu$$

(452F). As F is arbitrary, we have the result.

(b) Of course Σ_0 is a σ-algebra (111Xd), and it is included in Σ because f is inverse-measure-preserving. By 452F, $Y_0 = \{y : g \text{ is } \mu_y\text{-integrable}\}$ is conegligible, so $\operatorname{dom} h_1 = f^{-1}[Y_0]$ is conegligible. If $\alpha \in \mathbb{R}$, then

$$F = \{y : y \in Y_0, \textstyle\int g \, d\mu_y \geq \alpha\} \in \mathrm{T}$$

because $y \mapsto \int g \, d\mu_y$ is ν-virtually measurable and ν is complete. So

$$\{x : x \in \operatorname{dom} h_1, \, h_1(x) \geq \alpha\} = f^{-1}[F]$$

belongs to Σ_0, and h_1 is Σ_0-measurable. If $F \in \mathrm{T}$, then

$$\int_{f^{-1}[F]} h_1 \, d\mu = \int_{f^{-1}[F]} \int g \, d\mu_{f(x)} \mu(dx) = \int_F \int g \, d\mu_y \nu(dy)$$

(235I)

$$= \int (g \times \chi(f^{-1}[F])) d\mu_y \nu(dy)$$

(because $\mu_y f^{-1}[F] = \mu_y X = 1$ for almost every $y \in F$, $\mu_y f^{-1}[F] = 0$ for almost every $y \in Y \setminus F$)

$$= \int g \times \chi(f^{-1}[F]) d\mu$$

(452F)

$$= \int_{f^{-1}[F]} g \, d\mu.$$

As F is arbitrary, h_1 is a conditional expectation of g on Σ_0, as claimed.

***452R** I take the opportunity to interpolate an interesting result about countably compact measures. It demonstrates the power of 452I to work in unexpected ways.

Theorem (PACHL 79) Let (X, Σ, μ) be a countably compact measure space, (Y, T, ν) a strictly localizable measure space, and $f : X \to Y$ an inverse-measure-preserving function. Then ν is countably compact.

proof (a) For most of the proof (down to the end of (b) below) I suppose that μ and ν are totally finite.

Let Z be the Stone space of the Boolean algebra T. (I am *not* using the measure algebra here!) For $F \in \mathrm{T}$, let F^* be the corresponding open-and-closed subset of Z. For each $y \in Y$, the map $F \mapsto \chi F(y)$ is a Boolean homomorphism from T to $\{0,1\}$, so belongs to Z; define $g : Y \to Z$ by saying that $g(y)(F) = \chi F(y)$ for $y \in Y$, $F \in \mathrm{T}$, that is, that $g^{-1}[F^*] = F$ for every $F \in \mathrm{T}$. Let $\mathcal{Z}$ be the family of zero sets in Z, and Λ the Baire σ-algebra of Z.

The set

$$\{W : W \subseteq Z,\ g^{-1}[W] \in \mathrm{T}\}$$

is a σ-algebra of subsets of Z containing all the open-and-closed sets, so contains every zero set (4A3Oe) and includes Λ. Set $\lambda W = \nu g^{-1}[W]$ for $W \in \Lambda$. Then λ is a Baire measure on Z, so is inner regular with respect to $\mathcal{Z}$ (412D).

Set $h = gf : X \to Z$. Then h is a composition of inverse-measure-preserving functions, so is inverse-measure-preserving. By 452I, there is a disintegration $\langle \mu_z \rangle_{z \in Z}$ of μ over λ which is consistent with h.

(b) Let $\mathcal{K}$ be the family of sets

$$\{g^{-1}[V] : V \in \mathcal{Z},\ \mu_z h^{-1}[V] = \mu_z X = 1 \text{ for every } z \in V\}.$$

(i) $\mathcal{K}$ is a countably compact class of sets. **P** Let $\langle K_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathcal{K}$ such that $\bigcap_{i \le n} K_i \neq \emptyset$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let $V_n \in \mathcal{Z}$ be such that $K_n = g^{-1}[V_n]$ and $\mu_z h^{-1} V_n = \mu_z X = 1$ for every $z \in V_n$. Then

$$g^{-1}[\textstyle\bigcap_{i \le n} V_i] = \bigcap_{i \le n} K_i \neq \emptyset$$

for every $n \in \mathbb{N}$, so $\{V_n : n \in \mathbb{N}\}$ has the finite intersection property and (because Z is compact) there is a $z \in \bigcap_{n \in \mathbb{N}} V_n$. Now

$$\mu_z h^{-1}[V_n] = \mu_z X = 1$$

for every $n \in \mathbb{N}$, so

$$\emptyset \neq \textstyle\bigcap_{n \in \mathbb{N}} h^{-1}[V_n] = f^{-1}[\bigcap_{n \in \mathbb{N}} K_n].$$

Thus $\bigcap_{n \in \mathbb{N}} K_n$ is non-empty. As $\langle K_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $\mathcal{K}$ is a countably compact class. **Q**

(ii) ν is inner regular with respect to $\mathcal{K}$. **P** Suppose that $F \in \mathrm{T}$ and $\gamma < \nu F$. Choose a sequence $\langle V_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{Z}$ as follows. Start with $V_0 = F^*$, so that

$$\lambda V_0 = \nu g^{-1}[V_0] = \nu F > \gamma.$$

Given that $V_n \in \mathcal{Z}$ and $\lambda V_n > \gamma$, then we know that $\mu_z h^{-1}[V_n] = \mu_z X = 1$ for λ-almost every $z \in V_n$; because λ is inner regular with respect to $\mathcal{Z}$, there is a $V_{n+1} \in \mathcal{Z}$ such that $V_{n+1} \subseteq V_n$, $\lambda V_{n+1} > \gamma$ and $\mu_z h^{-1}[V_n] = \mu_z X = 1$ for every $z \in V_{n+1}$. Continue.

At the end of the induction, set $V = \bigcap_{n \in \mathbb{N}} V_n$. Then $V \in \mathcal{Z}$. If $z \in V$, then

$$\mu_z h^{-1}[V] = \lim_{n \to \infty} \mu_z h^{-1}[V_n] = 1 = \mu_z X,$$

so $g^{-1}[V] \in \mathcal{K}$. Because $V \subseteq V_0 = F^*$, $g^{-1}[V] \subseteq F$, and

$$\nu g^{-1}[V] = \lambda V = \lim_{n \to \infty} \lambda V_n \geq \gamma.$$

As F and γ are arbitrary, ν is inner regular with respect to $\mathcal{K}$. **Q**

Thus $\mathcal{K}$ witnesses that ν is countably compact.

(c) For the general case, let $\langle Y_i \rangle_{i \in I}$ be a decomposition of Y. For each $i \in I$, set $X_i = f^{-1}[Y_i]$; let μ_i be the subspace measure on X_i and ν_i the subspace measure on Y_i. Then μ_i is countably compact (451Db) and $f \restriction X_i : X_i \to Y_i$ is inverse-measure-preserving for μ_i and ν_i, so ν_i is countably compact, by (a)-(b) above. Let $\mathcal{K}_i \subseteq \mathcal{P} Y_i$ be a countably compact class such that ν_i is inner regular with respect to $\mathcal{K}_i$. Then $\mathcal{K} = \bigcup_{i \in I} \mathcal{K}_i$ is a countably compact class (because any sequence in $\mathcal{K}$ with the finite intersection property must lie within a single $\mathcal{K}_i$). By 413R, there is a countably compact class $\mathcal{K}^* \supseteq \mathcal{K}$ which is closed under finite unions; by 412Aa, ν is inner regular with respect to $\mathcal{K}^*$, so is countably compact. This completes the proof.

***452S Corollary** (PACHL 78) If (X, Σ, μ) is a countably compact totally finite measure space, and T is any σ-subalgebra of Σ, then $\mu\restriction \mathrm{T}$ is countably compact.

452X Basic exercises (a) Let Y be a first-countable topological space, ν a topological probability measure on Y, Z a topological space, and $\langle \mu_y \rangle_{y\in Y}$ a family of topological probability measures on Z such that $y \mapsto \mu_y V$ is lower semi-continuous for every open set $V \subseteq Z$. Show that there is a Borel probability measure μ on $Y \times Z$ such that $\mu E = \int \mu_y E[\{y\}]\nu(dy)$ for every Borel set $E \subseteq Y \times Z$. (*Hint*: 434R.)

(b) Let (Y, T, ν) be a probability space, Z a topological space and P the set of topological probability measures on Z with its narrow topology (437Jd). Let $y \mapsto \mu_y : Y \to P$ be a function which is measurable in the sense of 411L. Show that, writing $\mathcal{B}(Z)$ for the Borel σ-algebra of Z, we have a probability measure μ defined on $\mathrm{T}\widehat{\otimes}\mathcal{B}(Z)$ such that $\mu E = \int \mu_y E[\{y\}]\nu(dy)$ for every $E \in \mathrm{T}\widehat{\otimes}\mathcal{B}(Z)$.

(c) Let (Y, T, ν) be a probability space, Z a topological space and $P_{\mathcal{B}\mathfrak{a}}$ the set of Baire probability measures on Z with its vague topology (437Jc). Let $y \mapsto \mu_y : Y \to P_{\mathcal{B}\mathfrak{a}}$ be a measurable function. Show that, writing $\mathcal{B}\mathfrak{a}(Z)$ for the Baire σ-algebra of Z, we have a probability measure μ defined on $\mathrm{T}\widehat{\otimes}\mathcal{B}\mathfrak{a}(Z)$ such that $\mu E = \int \mu_y E[\{y\}]\nu(dy)$ for every $E \in \mathrm{T}\widehat{\otimes}\mathcal{B}\mathfrak{a}(Z)$.

(d) Let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be a Radon probability space, $(X, \mathfrak{T})$ a topological space, and $\langle \mu_y \rangle_{y\in Y}$ a family of Radon probability measures on X. Suppose that (i) there is a base $\mathcal{U}$ for $\mathfrak{T}$, closed under finite unions, such that $y \mapsto \mu_y U$ is lower semi-continuous for every $U \in \mathcal{U}$ (ii) ν is inner regular with respect to the family $\{K : K \subseteq Y, \{\mu_y : y \in K\} \text{ is uniformly tight}\}$. Show that we have a Radon probability measure $\tilde{\mu}$ on $Y \times Z$ such that $\tilde{\mu}E = \int \mu_y E[\{y\}]\nu(dy)$ whenever $\tilde{\mu}$ measures E.

(e) Let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be a Radon probability space, $(Z, \mathfrak{U})$ a Prokhorov Hausdorff space (437V), and P the space of Radon probability measures on Z with its narrow topology. Suppose that $y \mapsto \mu_y : Y \to P$ is almost continuous. Show that we have a Radon probability measure $\tilde{\mu}$ on $Y \times Z$ such that $\tilde{\mu}E = \int \mu_y E[\{y\}]\nu(dy)$ whenever $\tilde{\mu}$ measures E.

(f) Let (X, T, ν) be a measure space, and μ an indefinite-integral measure over ν. Show that there is a disintegration $\langle \mu_x \rangle_{x\in X}$ of μ over ν such that $\mu_x\{x\} = \mu_x X$ for every $x \in X$.

(g) Let (X, Σ, μ) and (Y, T, ν) be measure spaces and $\langle \mu_y \rangle_{y\in Y}$ a disintegration of μ over ν. Show that $\langle \hat{\mu}_y \rangle_{y\in Y}$ is a disintegration of $\hat{\mu}$ over ν, where $\hat{\mu}_y$ and $\hat{\mu}$ are the completions of μ_y and μ respectively.

>(h) Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and ν' an indefinite-integral measure over ν, defined from a ν-virtually measurable function $g : Y \to [0, \infty[$ (234B). Suppose that $\langle \mu_y \rangle_{y\in Y}$ is a disintegration of μ over ν'. Show that $\langle g(y)\mu_y \rangle_{y\in Y}$ is a disintegration of μ over ν.

(i) Let (Y, T, ν) be a probability space, X a set and $\langle \mu_y \rangle_{y\in Y}$ a family of probability measures on X. Set $\theta A = \overline{\int} \mu_y^*(A)\, \nu(dy)$ for every $A \subseteq X$. (i) Show that θ is an outer measure on X. (ii) Let μ be the measure on X defined from θ by Carathéodory's construction. Show that $\langle \mu_y \rangle_{y\in Y}$ is a disintegration of μ over ν. (iii) Suppose that $X = [0,1]^2$, ν is Lebesgue measure on $[0,1] = Y$ and $\mu_y E = \nu\{x : (x,y) \in E\}$ whenever this is defined. Show that, for any E measured by μ, $\mu_y E \in \{0,1\}$ for ν-almost every y.

(j) Explore connexions between 452F and the formula $\int f d\mu = \iint f d\nu_z \lambda(dz)$ of 443Pe.

(k) Let (X, Σ, μ) be a countably compact σ-finite measure space, (Y, T, ν) a σ-finite measure space, and $f : X \to Y$ a (Σ, T)-measurable function such that $f^{-1}[F]$ is μ-negligible whenever $F \subseteq Y$ is ν-negligible. Show that there is a disintegration $\langle \mu_y \rangle_{y\in Y}$ of μ over ν such that, for each $F \in \mathrm{T}$, $\mu_y(X \setminus f^{-1}[F]) = 0$ for almost every $y \in F$. (*Hint*: Reduce to the case in which μ is totally finite, and disintegrate μ over $\nu' = (\mu f^{-1})\restriction \mathrm{T}$.)

>(l) Let (X, Σ, μ) be a non-empty countably compact measure space such that Σ is countably generated (as σ-algebra), (Y, T, ν) a σ-finite measure space, and $f : X \to Y$ an inverse-measure-preserving function. (i) Show that there is a disintegration $\langle \mu_y \rangle_{y\in Y}$ of μ over ν, consistent with f, such that every μ_y is a probability measure with domain Σ. (ii) Show that if $\langle \mu'_y \rangle_{y\in Y}$ is any other disintegration of μ over ν which is consistent with f, then $\mu_y = \mu'_y\restriction \Sigma$ for almost every y.

(m) Let (X, Σ) be a non-empty standard Borel space, μ a measure with domain Σ, (Y, T, ν) a σ-finite measure space, and $f : X \to Y$ an inverse-measure-preserving function. (i) Show that there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν, consistent with f, such that every μ_y is a probability measure with domain Σ. (ii) Show that if $\langle \mu'_y \rangle_{y \in Y}$ is any other disintegration of μ over ν which is consistent with f, then $\mu_y = \mu'_y \restriction \Sigma$ for almost every y.

(n) Let (X, Σ, μ) be a totally finite countably compact measure space and $\mathrm{T} \subseteq \Sigma$ a countably-generated σ-algebra; set $\nu = \mu \restriction \mathrm{T}$. Show that there is a disintegration $\langle \mu_x \rangle_{x \in X}$ of μ over ν such that $\mu_x H_x = \mu_x X = 1$ for every $x \in X$, where $H_x = \bigcap \{F : x \in F \in \mathrm{T}\}$ for every x. (*Hint*: apply 452I with $Y = \{H_x : x \in X\}$.)

(o) Show that 452I can be deduced from 452M. (*Hint*: start with the case $\nu Y = 1$; set $\lambda W = \mu\{x : (x, f(x)) \in W\}$ for $W \in \Sigma \widehat{\otimes} \mathrm{T}$.)

(p) Show that, in 452M, we shall have $\hat{\mu} E = \int \mu_y E^{-1}[\{y\}] \nu(dy)$ whenever the completion $\hat{\mu}$ of μ measures E.

>(q) Let T be the Borel σ-algebra of $[0, 1]$, ν the restriction of Lebesgue measure to T, $Z \subseteq [0, 1]$ a set with inner measure 0 and outer measure 1, and Υ the Borel σ-algebra of Z. Show that there is a probability measure μ on $[0, 1] \times Z$ defined by setting $\mu E = \nu^* \{y : (y, y) \in E\}$ for $E \in \mathrm{T} \widehat{\otimes} \Upsilon$. Show that there is no disintegration of μ over ν which is consistent with the projection $(y, z) \mapsto y$.

>(r) Let (X, Σ, μ) be a complete totally finite countably compact measure space and T a σ-subalgebra of Σ containing all negligible sets. Show that there is a family $\langle \mu_x \rangle_{x \in X}$ of probability measures on X such that (i) $x \mapsto \mu_x E$ is T-measurable and $\int \mu_x E \, \mu(dx) = \mu E$ for every $E \in \Sigma$ (ii) if $F \in \mathrm{T}$, then $\mu_x F = 1$ for almost every $x \in F$. Show that if g is any μ-integrable real-valued function, then g is μ_x-integrable for almost every x, and $x \mapsto \int g \, d\mu_x$ is a conditional expectation of g on T.

452Y Further exercises (a) Let Z be a set, (Y, T, ν) a measure space, and $\langle \mu_y \rangle_{y \in Y}$ a family of measures on Z. Let Υ be a σ-algebra of subsets of Z such that, for every $H \in \Upsilon$, $y \mapsto \mu_y H : Y \to [0, \infty]$ is defined ν-a.e. and is ν-virtually measurable. For $F \in \mathrm{T}$, set $\mathcal{H}_F = \{H : H \in \Upsilon, \mu_y H \text{ is defined for every } y \in F \text{ and } \sup_{y \in F} \mu_y H < \infty\}$. Show that there is a measure μ on $Y \times Z$, with domain $\mathrm{T} \widehat{\otimes} \Upsilon$, defined by setting

$$\mu E = \sup\{\sum_{i=0}^{n} \int_{F_i} \mu(E[\{y\}] \cap H_i) \nu(dy) : F_0, \dots, F_n \in \mathrm{T} \text{ are disjoint}, \\ \nu F_i < \infty \text{ and } H_i \in \mathcal{H}_{F_i} \text{ for every } i \leq n\}$$

for $E \in \mathrm{T} \widehat{\otimes} \Upsilon$.

(b) Let (X, Σ, μ) be a semi-finite countably compact measure space, (Y, T, ν) a strictly localizable measure space, and $f : X \to Y$ an inverse-measure-preserving function. Suppose that the magnitude of ν (definition: 332Ga) is finite or a measure-free cardinal (definition: 438A). Show that there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν which is consistent with f.

(c) Give an example to show that the phrase 'strictly localizable' in the statements of 452O and 452Yb cannot be dispensed with.

(d) Give an example to show that, in 452M, we cannot always arrange that $\Upsilon \subseteq \operatorname{dom} \mu_y$ for ν-almost every $y \in Y$.

(e) Let (X, Σ, μ) be a probability space such that whenever (Y, T, ν) is a probability space and $f : X \to Y$ is an inverse-measure-preserving function, there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν which is consistent with f. Show that μ is countably compact. (*Hint*: 452R, or Pachl 78.)

452 Notes and comments 452B and 452C correspond respectively to the ordinary and τ-additive product measures of §§251 and 417. I have not attempted to find a suitable general formulation for the constructions when the measures involved are not probability measures. In 452Ya I set out a possible version which at least agrees with the c.l.d. product measure when all the μ_y are the same. Any product measure which has an associated Fubini theorem can be expected to be generalizable in the same way; for instance, 434R becomes 452Xa.

The hypotheses in 452B are closely matched with the conclusion, and clearly cannot be relaxed substantially if the theorem is to remain true. 452C and 452D are a rather different matter. While the condition '$y \mapsto \mu_y V$ is lower semi-continuous' is a natural one, and plainly necessary for the argument given, the integrated measure μ can be τ-additive or Radon for other reasons. In particular, the most interesting specific example in this book of a Radon measure constructed through these formulae (453N below) does not satisfy the lower semi-continuity condition for the section measures.

Early theorems on disintegrations concentrated on cases in which all the measure spaces involved were 'standard' in that the measures were defined on standard Borel algebras, or were the completions of such measures. Theorem 452I here is the end (so far) of a long search for ways to escape from topological considerations. As usual, of course, the most important applications (in probability theory) are still rooted in the standard case. Being countably separated, such spaces automatically yield disintegrations which are concentrated on fibers, in the sense that $\mu_y f^{-1}[\{y\}] = \mu_y X = 1$ for almost every y (452P). The general question of when we can expect to find disintegrations of this type is an important one to which I will return in the next section.

452I and 452O, as stated, assume that the functions $f : X \to Y$ controlling the disintegrations are inverse-measure-preserving. In fact it is easy to weaken this assumption (452Xk). Note the constructions for conditional expectations in 452Q and 452Xr.

Obviously 452I and 452M are nearly the same theorem; but I write out formally independent proofs because the constructions needed to move between them are not quite trivial. In fact I think it is easier to deduce 452I from 452M than the other way about (452Xo). The point of 452N is that the spaces (X, Σ) there have the 'countably compact measure property', that is, any totally finite measure with domain Σ is countably compact. I will return to this in the exercises to §454 (454Xe *et seq.*).

The method of 452R, due to J.Pachl, may have inspired the proof of (vi)⇒(i) in 343B. In the general introduction to this work I wrote 'I have very little confidence in anything I have ever read concerning the history of ideas'. We have here a case indicating the difficulties a historian faces. I proved 343B in the winter of 1996-97, while a guest of the University of Wisconsin at Madison. Around that time I was renewing my acquaintance with Pachl 78. I know I ran my eye over the proof of 452R, without, I may say, understanding it, as became plain when I came to write the first draft of the present section in the summer of 1997; whether I had understood it twenty years earlier I do not know. It is entirely possible that a subterranean percolation of Pachl's idea was what dislodged an obstacle to my attempts to prove 343B, but I was not at the time conscious of any connexion.

453 Strong liftings

The next step involves the concept of 'strong' lifting on a topological measure space (453A); I devote a few pages to describing the principal cases in which strong liftings are known to exist (453B-453J). When we have *Radon* measures μ and ν, with an *almost continuous* inverse-measure-preserving function between them, and a *strong* lifting of ν, we can hope for a disintegration $\langle \mu_y \rangle_{y \in Y}$ such that (almost) every μ_y lives on the appropriate fiber. This is the content of 453K. I end the section with a note on the relation between strong liftings and Stone spaces (453M) and with V.Losert's example of a space with no strong lifting (453N).

Much of the work here is based on ideas in Ionescu Tulcea & Ionescu Tulcea 69.

453A The proof of the first disintegration theorem I presented, 452I, depended on two essential steps: the use of a lifting of (Y, T, ν) to define the finitely additive functionals λ_y, and the use of a countably compact class to convert these into countably additive functionals. In 452O I observed that if our countably compact class is the family of compact sets in a Hausdorff space, we can get Radon measures in our disintegration. Similarly, if we have a lifting of a special type, we can hope for special properties of the disintegration. A particularly important kind of lifting, in this context, is the following.

Definition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space. A lifting $\phi : \Sigma \to \Sigma$ is **strong** or **of local type** if $\phi G \supseteq G$ for every open set $G \subseteq X$, that is, if $\phi F \subseteq F$ for every closed set $F \subseteq X$. I will say that ϕ is **almost strong** if $\bigcup_{G \in \mathfrak{T}} G \setminus \phi G$ is negligible.

Similarly, if $\mathfrak{A}$ is the measure algebra of μ, a lifting $\theta : \mathfrak{A} \to \Sigma$ is **strong** if $\theta G^\bullet \supseteq G$ for every open set $G \subseteq X$, and **almost strong** if $\bigcup_{G \in \mathfrak{T}} G \setminus \theta G^\bullet$ is negligible.

Obviously a strong lifting is almost strong.

453B We already have the machinery to describe a particularly striking class of strong liftings.

Theorem Let X be a topological group with a Haar measure μ, and Σ its algebra of Haar measurable sets.

(a) If $\phi : \Sigma \to \Sigma$ is a left-translation-invariant lifting, in the sense of 447A, then ϕ is strong.

(b) μ has a strong lifting.

proof (a) Apply 447B with $Y = \{e\}$ and $\underline{\phi} = \phi$.

(b) For there is a left-translation-invariant lifting (447J).

Remark In particular, translation-invariant liftings on $\mathbb{R}^r$ or $\{0,1\}^I$ (§345) are strong.

453C Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space and $\phi : \Sigma \to \Sigma$ a lifting. Write $\mathcal{L}^\infty$ for the space of bounded Σ-measurable real-valued functions on X, so that $\mathcal{L}^\infty$ can be identified with $L^\infty(\Sigma)$ (363H) and the Boolean homomorphism $\phi : \Sigma \to \Sigma$ gives rise to a Riesz homomorphism $T : \mathcal{L}^\infty \to \mathcal{L}^\infty$ (363F).

(a) If ϕ is a strong lifting, then $Tf = f$ for every bounded continuous function $f : X \to \mathbb{R}$.

(b) If $(X, \mathfrak{T})$ is completely regular and $Tf = f$ for every $f \in C_b(X)$, then ϕ is strong.

proof (a) Suppose first that $f \geq 0$. For $\alpha \in \mathbb{R}$, set $G_\alpha = \{x : x \in X, f(x) > \alpha\}$; then G_α is open, so $\phi G_\alpha \supseteq G_\alpha$. We have $f \geq \alpha \chi G_\alpha$, so

$$Tf \geq \alpha T(\chi G_\alpha) = \alpha T(\chi(\phi G_\alpha)) \geq \alpha T(\chi G_\alpha),$$

that is, $(Tf)(x) \geq \alpha$ whenever $f(x) > \alpha$. As α is arbitrary, $Tf \geq f$. At the same time, setting $\gamma = \|f\|_\infty$, we have

$$T(\gamma\chi X - f) \geq \gamma\chi X - f, \quad T(\gamma\chi X) = \gamma\chi(\phi X) = \gamma\chi X,$$

so $Tf \leq f$ and $Tf = f$.

For general $f \in C_b(X)$,

$$Tf = T(f^+ - f^-) = Tf^+ - Tf^- = f^+ - f^- = f,$$

where f^+ and f^- are the positive and negative parts of f.

(b) Let $G \subseteq X$ be open and x any point of G. Then there is an $f \in C_b(X)$ such that $f \leq \chi G$ and $f(x) = 1$. In this case

$$f = Tf \leq T(\chi G) = \chi(\phi G),$$

so $x \in \phi G$. As x is arbitrary, $G \subseteq \phi G$; as G is arbitrary, ϕ is strong.

453D Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space.

(a) If μ has a strong lifting it is strictly positive (definition: 411Nf).

(b) If μ is strictly positive and complete, and has an almost strong lifting, it has a strong lifting.

(c) If μ has an almost strong lifting it is τ-additive, so has a support.

(d) If μ is complete and $\mu X > 0$ and the subspace measure μ_E has an almost strong lifting for some conegligible set $E \subseteq X$, then μ has an almost strong lifting.

proof (a) If $\phi : \Sigma \to \Sigma$ is a strong lifting, then $G \subseteq \phi G = \emptyset$ whenever G is a negligible open set, so μ is strictly positive.

(b) If μ is strictly positive and complete and $\phi : \Sigma \to \Sigma$ is an almost strong lifting, set $A = \bigcup_{G \in \mathfrak{T}} G \setminus \phi G$. For each $x \in A$, let $\mathcal{I}_x$ be the ideal of subsets of X generated by

$$\{F : F \subseteq X \text{ is closed}, x \notin F\} \cup \{B : B \subseteq X \text{ is negligible}\}.$$

Then $X \notin \mathcal{I}_x$, because μ is strictly positive, so a closed set not containing x cannot be conegligible. There is therefore a Boolean homomorphism $\psi_x : \mathcal{P}X \to \{0,1\}$ such that $\psi_x F = 0$ for every $F \in \mathcal{I}_x$ (311D). Set

$$\tilde{\phi}E = (\phi E \setminus A) \cup \{x : x \in A, \psi_x E = 1\}$$

for $E \in \Sigma$. It is easy to check that $\tilde{\phi} : \Sigma \to \mathcal{P}X$ is a Boolean homomorphism. (Compare the proof of 341J.) If $E \in \Sigma$, then

$$E \triangle \tilde{\phi}E \subseteq (E \triangle \phi E) \cup A$$

is negligible, so (because μ is complete) $\tilde{\phi}E \in \Sigma$. If E is negligible, then $E \in \mathcal{I}_x$ and $\psi_x E = 0$ for every $x \in A$, so $\tilde{\phi}E = \phi E = \emptyset$. Thus $\tilde{\phi}$ is a lifting. Now suppose that $x \in G \in \mathfrak{T}$. If $x \in A$, then $X \setminus G \in \mathcal{I}_x$, so $\psi_x(X \setminus G) = 0$, $\psi_x G = 1$ and $x \in \tilde{\phi}G$. If $x \notin A$, then $x \in \phi G$ and again $x \in \tilde{\phi}G$. As x and G are arbitrary, $\tilde{\phi}$ is a strong lifting.

(c) Suppose that $\phi : \Sigma \to \Sigma$ is an almost strong lifting. Let $\mathcal{G}$ be a non-empty upwards-directed family of open sets with union H. If $\sup_{G \in \mathcal{G}} \mu G = \infty$, this is surely equal to μH. Otherwise, there is a non-decreasing sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{G}$ such that $G \setminus H_0$ is negligible for every $G \in \mathcal{G}$, where $H_0 = \bigcup_{n \in \mathbb{N}} G_n$ (215Ab). Then $\phi G \subseteq \phi H_0$ for every $G \in \mathcal{G}$. This means that

$$H \setminus \phi H_0 \subseteq \bigcup_{G \in \mathcal{G}} G \setminus \phi G$$

is negligible, because ϕ is almost strong, and

$$\mu H \leq \mu(\phi H_0) = \mu H_0 = \lim_{n \to \infty} \mu G_n = \sup_{G \in \mathcal{G}} \mu G.$$

As $\mathcal{G}$ is arbitrary, μ is τ-additive. By 411Nd, it has a support.

(d) Now suppose that μ is complete, that $\mu X > 0$ and that there is a conegligible $E \subseteq X$ such that μ_E has an almost strong lifting ϕ. Let $\psi : \mathcal{P}X \to \{\emptyset, X\}$ be any Boolean homomorphism such that $\psi A = \emptyset$ whenever A is negligible. (This is where I use the hypothesis that X is not negligible.) Define $\tilde{\phi} : \Sigma \to \mathcal{P}X$ by setting

$$\tilde{\phi}F = \phi(E \cap F) \cup (\psi F \setminus E).$$

Then $\tilde{\phi}$ is a Boolean homomorphism because ϕ and ψ are;

$$F \triangle \tilde{\phi}F \subseteq ((E \cap F) \triangle \phi(E \cap F)) \cup (X \setminus E)$$

is negligible, so $\tilde{\phi}F \in \Sigma$, for every $F \in \Sigma$, because μ is complete; and if F is negligible, then $\phi(E \cap F) = \psi F = \emptyset$ so $\tilde{\phi}F = \emptyset$. Thus $\tilde{\phi}$ is a lifting. Finally,

$$\bigcup_{G \in \mathfrak{T}} G \setminus \tilde{\phi}G \subseteq (X \setminus E) \cup \bigcup_{G \in \mathfrak{T}} ((G \cap E) \setminus \phi(G \cap E))$$

is negligible because ϕ is almost strong and E is conegligible.

453E Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete strictly localizable topological measure space with an almost strong lifting, and $A \subseteq X$ a non-negligible set. Then the subspace measure μ_A has an almost strong lifting.

proof Let $\phi : \Sigma \to \Sigma$ be an almost strong lifting. Because μ is strictly localizable, A has a measurable envelope W say (213L). Write Σ_A for the subspace σ-algebra on A. Let $\psi : \Sigma_A \to \{\emptyset, A\}$ be any Boolean homomorphism such that $\psi H = \emptyset$ for every negligible set $H \subseteq A$.

If E, $F \in \Sigma$ and $E \cap A = F \cap A$, then $\phi E \cap \phi W = \phi F \cap \phi W$. **P**

$$\mu((E \triangle F) \cap W) = \mu^*((E \triangle F) \cap A) = 0,$$

so

$$(\phi E \cap \phi W) \triangle (\phi F \cap \phi W) = \phi((E \triangle F) \cap W) = \emptyset. \quad \textbf{Q}$$

We can therefore define a function $\tilde{\phi} : \Sigma_A \to \mathcal{P}A$ by setting

$$\tilde{\phi}H = (\phi E \cap \phi W \cap A) \cup (\psi H \setminus \phi W)$$

whenever $E \in \Sigma$ and $H = E \cap A$. It is easy to check that $\tilde{\phi}$ is a Boolean homomorphism. If $E \in \Sigma$ then

$$(E \cap A) \triangle \tilde{\phi}(E \cap A) \subseteq (E \triangle \phi E) \cup (A \setminus \phi W) \subseteq (E \triangle \phi E) \cup (W \setminus \phi W)$$

is negligible, so $\tilde{\phi}(E \cap A) \in \Sigma_A$ (because μ and μ_A are complete). If $H \in \Sigma_A$ is negligible, then

$$\tilde{\phi}H \subseteq \phi H \cup \psi H = \emptyset,$$

so $\tilde{\phi}$ is a lifting.

Now set $B = (A \setminus \phi W) \cup \bigcup_{G \in \mathfrak{T}} G \setminus \phi G$. Because ϕ is almost strong, B is negligible. If $H \subseteq A$ is relatively open, then $H \setminus \tilde{\phi}H \subseteq B$. **P** Take $x \in H \setminus \tilde{\phi}H$. Express H as $G \cap A$ where $G \subseteq X$ is open. If $x \in \phi W$, then $x \notin \phi G$ so $x \in B$; if $x \notin \phi W$, then of course $x \in B$. **Q** Thus

$$\bigcup \{H \setminus \tilde{\phi}H : H \subseteq A \text{ is relatively open}\} \subseteq B$$

is negligible and $\tilde{\phi}$ is almost strong.

453F Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete strictly localizable topological measure space.

(a) If $\mathfrak{T}$ has a countable network consisting of measurable sets, any lifting for μ is almost strong.

(b) Suppose that $\mu X > 0$ and μ is inner regular with respect to

$$\mathcal{K} = \{K : K \in \Sigma, \mu_K \text{ has an almost strong lifting}\},$$

where μ_K is the subspace measure on K. Then μ has an almost strong lifting.

proof (a) Let $\mathcal{E} \subseteq \Sigma$ be a countable network for $\mathfrak{T}$, and $\phi : \Sigma \to \Sigma$ a lifting. Then

$$\begin{aligned}\bigcup_{G \in \mathfrak{T}} G \setminus \phi G &= \bigcup_{G \in \mathfrak{T}, E \in \mathcal{E}, E \subseteq G} E \setminus \phi G \\ &\subseteq \bigcup_{G \in \mathfrak{T}, E \in \mathcal{E}, E \subseteq G} E \setminus \phi E = \bigcup_{E \in \mathcal{E}} E \setminus \phi E\end{aligned}$$

is negligible, so ϕ is almost strong.

(b) Let $\mathcal{L} \subseteq \mathcal{K}$ be a disjoint family such that $\mu^* A = \sum_{L \in \mathcal{L}} \mu^*(A \cap L)$ for every $A \subseteq X$ (412Ib). For each $L \in \mathcal{L}$, let Σ_L be the corresponding subspace σ-algebra and $\phi_L : \Sigma_L \to \Sigma_L$ an almost strong lifting. Set $E = \bigcup \mathcal{L}$; then

$$\mu^*(X \setminus E) = \sum_{L \in \mathcal{L}} \mu(L \setminus E) = 0,$$

so E is conegligible. For $F \in \Sigma_E$ set $\phi F = \bigcup_{L \in \mathcal{L}} \phi_L(F \cap L)$; then ϕ is a Boolean homomorphism from Σ_E to $\mathcal{P}E$. If $F \in \Sigma_E$, then

$$\mu^*(F \triangle \phi F) = \sum_{L \in \mathcal{L}} \mu^*(L \cap (F \triangle \phi F)) = \sum_{L \in \mathcal{L}} \mu^*((F \cap L) \triangle \phi_L(F \cap L)) = 0,$$

while if $\mu F = 0$ then $\phi_L(F \cap L) = \emptyset$ for every L, so $\phi F = \emptyset$. Thus ϕ is a lifting. Now set

$$A = \bigcup \{H \setminus \phi H : H \subseteq E \text{ is relatively open}\}.$$

If $L \in \mathcal{L}$, then

$$A \cap L = \bigcup \{(H \cap L) \setminus \phi_L(H \cap L) : H \subseteq E \text{ is relatively open}\}$$

is negligible, because ϕ_L is almost strong; thus ϕ is an almost strong lifting for μ_E. By 453Dd, μ also has an almost strong lifting.

453G Corollary (a) A non-zero quasi-Radon measure on a separable metrizable space has an almost strong lifting.

(b) A non-zero Radon measure μ on an analytic Hausdorff space X has an almost strong lifting.

proof (a) A quasi-Radon measure is complete and strictly localizable (415A), so, if non-zero, has a lifting (341K). A separable metrizable space has a countable network consisting of open sets (4A2P(a-i)), so this lifting must be almost strong.

(b) If $K \subseteq X$ is compact and non-negligible, it is metrizable (423Dc), so that the subspace measure μ_K has an almost strong lifting, by (a); as μ is tight (that is, inner regular with respect to the closed compact sets), it has an almost strong lifting, by 453Fb.

Remark In particular, Lebesgue measure on $\mathbb{R}^r$ has an almost strong lifting and therefore, by 453Db, a strong lifting, as already noted in 453B.

453H Lemma Let (X, Σ, μ) be a complete strictly localizable measure space and $\mathfrak{T}$ a topology on X generated by a family $\mathcal{U} \subseteq \Sigma$. Suppose that $\phi : \Sigma \to \Sigma$ is a lifting such that $\phi U \supseteq U$ for every $U \in \mathcal{U}$. Then μ is a τ-additive topological measure, and ϕ is a strong lifting.

proof Of course ϕ is a lower density, and $\phi X = X$, so by 414P we have a density topology

$$\mathfrak{T}_d = \{E : E \in \Sigma, E \subseteq \phi E\}$$

with respect to which μ is a τ-additive topological measure. But our hypothesis is that $\mathcal{U} \subseteq \mathfrak{T}_d$, so $\mathfrak{T} \subseteq \mathfrak{T}_d$ and μ is a τ-additive topological measure with respect to $\mathfrak{T}$. Also, of course, $\phi G \supseteq G$ for every $G \in \mathfrak{T}$, so ϕ is a strong lifting.

453I Proposition Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i \in I}$ be a family of topological probability spaces such that every $\mathfrak{T}_i$ has a countable network consisting of measurable sets and every μ_i is strictly positive. Let λ be the (ordinary) complete product measure on $X = \prod_{i \in I} X_i$. Then λ is a τ-additive topological measure and has a strong lifting.

proof (a) The strategy of the proof is as follows. We may suppose that $I = \kappa$ is a cardinal. Write Λ for the domain of λ, and for each $\xi \leq \kappa$ let Λ_ξ be the σ-algebra of members of Λ determined by coordinates in ξ; write $\pi_\xi : X \to X_\xi$ for the canonical map. I seek to define a lifting $\phi : \Lambda \to \Lambda$ such that $\phi W \supseteq W$ for every open set $W \in \Lambda$. This will be the last in a family $\langle \phi_\xi \rangle_{\xi \leq \kappa}$ of partial liftings, constructed inductively as in the proof of 341H, with $\operatorname{dom} \phi_\xi = \Lambda_\xi$ for each ξ. The inductive hypothesis will be that ϕ_ξ extends ϕ_η whenever $\eta \leq \xi$, and $\phi_\xi \pi_\eta^{-1}[G] \supseteq \pi_\eta^{-1}[G]$ for every $\eta < \xi$ and every open $G \subseteq X_\eta$.

The induction starts with $\Lambda_0 = \{\emptyset, X\}$, $\phi_0 \emptyset = \emptyset$, $\phi_0 X = X$. For $\xi \leq \kappa$, set $\mathfrak{B}_\xi = \{W^\bullet : W \in \Lambda_\xi\}$.

(b) *Inductive step to a successor ordinal* $\xi + 1$ Suppose that ϕ_ξ has been defined, where $\xi < \kappa$.

(i) By 341Nb, there is a lifting $\phi'_\xi : \Lambda \to \Lambda$ extending ϕ_ξ. Let $\mathcal{E}_\xi \subseteq \Sigma_\xi$ be a countable network for $\mathfrak{T}_\xi$, and $\pi_\xi : X \to X_\xi$ the canonical map. Set

$$Q = \bigcup \{\pi_\xi^{-1}[E] \setminus \phi'_\xi(\pi_\xi^{-1}[E]) : E \in \mathcal{E}_\xi\};$$

then Q is negligible.

(ii) For $x \in Q$, let $\mathcal{I}_x \subseteq \Lambda$ be the ideal generated by

$$\{W : W \in \Lambda_\xi, x \notin \phi_\xi W\} \cup \{\pi_\xi^{-1}[F] : F \subseteq X_\xi \text{ is closed}, \pi_\xi(x) \notin F\} \cup \{W : \lambda W = 0\}.$$

Then $X \notin \mathcal{I}_x$. **P?** Otherwise, there are a $W \in \Lambda_\xi$, a closed $F \subseteq X_\xi$ and a negligible W' such that $W \cup W' \cup \pi_\xi^{-1}[F] = X$ while $x \notin \phi_\xi W \cup \pi_\xi^{-1}[F]$. But in this case

$$\begin{aligned} 0 = \lambda W' &\geq \lambda((X \setminus W) \cap (X \setminus \pi_\xi^{-1}[F])) \\ &= \lambda(X \setminus W) \cdot \lambda(X \setminus \pi_\xi^{-1}[F]) = \lambda(X \setminus W) \cdot \mu_\xi(X_\xi \setminus F) > 0 \end{aligned}$$

because μ_ξ is strictly positive and $\phi_\xi W \neq X$. **XQ**

There is therefore a Boolean homomorphism $\psi_x : \Sigma \to \{0, 1\}$ which is zero on $\mathcal{I}_x$.

(iii) Set

$$\phi_{\xi+1} W = (\phi'_\xi W \setminus Q) \cup \{x : x \in Q, \psi_x W = 1\}$$

for every $W \in \Lambda_{\xi+1}$. Then $\phi_{\xi+1}$ is a Boolean homomorphism from $\Lambda_{\xi+1}$ to $\mathcal{P}X$. Because $\phi_{\xi+1} W \triangle \phi'_\xi W \subseteq Q$ is negligible, $\phi_{\xi+1} W \in \Lambda$ for every $W \in \Lambda_{\xi+1}$. If $\lambda W = 0$ then $\phi'_\xi W = \emptyset$ and $\psi_x W = 0$ for every $x \in Q$, so $\phi_{\xi+1} W = \emptyset$; thus $\phi_{\xi+1} : \Lambda_{\xi+1} \to \Lambda$ is a partial lifting. If $W \in \Lambda_\xi$, then, for $x \in Q$,

$$\begin{aligned} x \in \phi_\xi W &\implies x \notin \phi_\xi(X \setminus W) \implies X \setminus W \in \mathcal{I}_x \\ &\implies \psi_x(X \setminus W) = 0 \implies \psi_x W = 1 \iff x \in \phi_{\xi+1} W \\ &\implies W \notin \mathcal{I}_x \implies x \in \phi_\xi W, \end{aligned}$$

so $\phi_{\xi+1}W = \phi_\xi W$. Thus $\phi_{\xi+1}$ extends ϕ_ξ.

(iv) Suppose that $\eta \leq \xi$ and $G \subseteq X_\eta$ is open. If $\eta < \xi$ then

$$\phi_{\xi+1}(\pi_\eta^{-1}[G]) = \phi_\xi(\pi_\eta^{-1}[G]) \supseteq \pi_\eta^{-1}[G]$$

by the inductive hypothesis. If $\eta = \xi$, take any $x \in \pi_\xi^{-1}[G]$. If $x \in Q$, then $X \setminus \pi_\xi^{-1}[G] \in \mathcal{I}_x$, so $\psi_x(\pi_\xi^{-1}[G]) = 1$ and $x \in \phi_{\xi+1}(\pi_\xi^{-1}[G])$. If $x \notin Q$, there is an $E \in \mathcal{E}_\xi$ such that $x(\xi) \in E \subseteq G$. In this case,

$$x \in \pi_\xi^{-1}[E] \setminus Q \subseteq \phi'_\xi(\pi_\xi^{-1}[E]) \setminus Q \subseteq \phi_{\xi+1}(\pi_\xi^{-1}[E]) \subseteq \phi_{\xi+1}(\pi_\xi^{-1}[G]).$$

As x is arbitary, $\pi_\eta^{-1}[G] \subseteq \phi_{\xi+1}(\pi_\eta^{-1}[G])$ in this case also.

Thus the induction continues.

(c) *Inductive step to a non-zero limit ordinal ξ of countable cofinality* Suppose that $0 < \xi \leq \kappa$, that $\operatorname{cf}\xi = \omega$ and that ϕ_η has been defined for every $\eta < \xi$. Let $\langle \zeta_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in ξ with limit ξ. Then $\mathfrak{B}_\xi$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{B}_{\zeta_n}$ (using 254N and 254Fe, or otherwise). By 341G, there is a partial lower density $\underline{\phi} : \Lambda_\xi \to \Lambda$ extending every ϕ_{ζ_n}, and therefore extending ϕ_η for every $\eta < \xi$. By 341J (applied to $\lambda \restriction \widehat{\Lambda}_\xi$, where $\widehat{\Lambda}_\xi$ is the σ-subalgebra of Λ generated by $\Lambda_\xi \cup \{W : \lambda W = 0\}$), there is a partial lifting $\phi_\xi : \Lambda_\xi \to \Lambda$ such that $\underline{\phi} W \subseteq \phi_\xi W$ for every $W \in \Lambda_\xi$.

If $\eta < \xi$ and $W \in \Lambda_\eta$, then

$$\phi_\eta W = \underline{\phi} W \subseteq \phi_\xi W, \quad X \setminus \phi_\eta W = \phi_\eta(X \setminus W) \subseteq \phi_\xi(X \setminus W) = X \setminus \phi_\xi W,$$

so ϕ_ξ extends ϕ_η. If $\eta < \xi$ and $G \subseteq X_\eta$ is open,

$$\phi_\xi(\pi_\eta^{-1}[G]) = \phi_{\eta+1}(\pi_\eta^{-1}[G]) \supseteq \pi_\eta^{-1}[G].$$

So again the induction continues.

(d) *Inductive step to a limit ordinal ξ of uncountable cofinality* In this case, $\mathfrak{B}_\xi = \bigcup_{\eta<\xi} \mathfrak{B}_\eta$, as in the proof of 341H; so there will be a unique partial lifting $\phi_\xi : \Lambda_\xi \to \Lambda$ extending ϕ_η for every $\eta < \xi$ (set $\phi_\xi W = \phi_\eta W'$ whenever $W \in \Lambda_\xi$, $\eta < \xi$, $W' \in \Lambda_\eta$ and $W \triangle W'$ is negligible). As in (c), we again have

$$\phi_\xi(\pi_\eta^{-1}[G]) = \phi_{\eta+1}(\pi_\eta^{-1}[G]) \supseteq \pi_\eta^{-1}[G]$$

whenever $\eta < \xi$ and $G \subseteq X_\eta$ is open.

(e) At the end of the induction, we have a lifting $\phi = \phi_\kappa$ of Λ such that $\phi U \supseteq U$ for every $U \in \mathcal{U}$, where $\mathcal{U} = \{\pi_\xi^{-1}[G] : \xi < \kappa, G \in \mathfrak{T}_\xi\}$. By 453H, λ is a τ-additive topological measure and ϕ is a strong lifting.

453J Corollary Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i) \rangle_{i\in I}$ be a family of quasi-Radon probability spaces such that every $\mathfrak{T}_i$ has a countable network consisting of measurable sets and every μ_i is strictly positive. Then the ordinary product measure λ on $X = \prod_{i\in I} X_i$ is quasi-Radon and has a strong lifting. If every X_i is compact and Hausdorff, then λ is a Radon measure.

proof We have just seen that λ is a τ-additive topological measure; but also it is inner regular with respect to the closed sets, by 412Ua, so it is a quasi-Radon measure. If all the X_i are compact and Hausdorff, so is X, so λ is a Radon measure (416G).

453K We come now to the construction of disintegrations from strong liftings.

Theorem Let $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be Radon measure spaces and $f : X \to Y$ an almost continuous inverse-measure-preserving function. Suppose that ν has an almost strong lifting. Then there is a disintegration $\langle \mu_y \rangle_{y\in Y}$ of μ over ν such that every μ_y is a Radon measure and $\mu_y X = \mu_y f^{-1}[\{y\}] = 1$ for almost every $y \in Y$.

proof (a) Suppose first that X is compact and that f is continuous.

Turn back to the proof of 452I, parts (a)-(g). In that argument, suppose that $\theta : \mathfrak{B} \to \mathrm{T}$ corresponds to an almost strong lifting $\phi : \mathrm{T} \to \mathrm{T}$ (see 341Ba). Set $B = \bigcup_{H \in \mathfrak{S}} H \setminus \phi H$, so that B is negligible. Take $\mathcal{K}$ to be the family of compact subsets of X. Then all the μ_y will necessarily be Radon measures, and almost all of them will be probability measures, as in 452O. For every y, $f^{-1}[\{y\}]$ is a closed set, so is necessarily measured by μ_y. But also it is μ_y-conegligible for every $y \in Y \setminus B$. **P** Let $K \subseteq X \setminus f^{-1}[\{y\}]$ be a compact set. Then $f[K]$ is a compact set not containing y. Because Y is Hausdorff, there is an open set H containing y such that $\overline{H} \cap f[K] = \emptyset$ (4A2F(h-i)). Now

$$y \in H \setminus B \subseteq \phi H \subseteq \phi \overline{H}.$$

Set $E = f^{-1}[\overline{H}]$. Then $v_E = (\chi \overline{H})^\bullet$, as in (f) of the proof of 452I, and $Tv_E = \chi(\phi \overline{H})$, so

$$\lambda_y E = Tv_E(y) = 1.$$

Because $E \in \mathcal{K}$, $\mu_y E \geq \lambda_y E$; since we always have $\mu_y X \leq 1$, we must have $\mu_y E = \mu_y X = 1$. But $K \cap E = \emptyset$, so $\mu_y K = 0$. As K is arbitrary, $\mu_y(X \setminus f^{-1}[\{y\}]) = 0$. **Q**

Thus $\mu_y f^{-1}[\{y\}] = \mu_y X = 1$ whenever $y \in Y \setminus B$ and $\mu_y X = 1$, which is almost always.

(b) Now suppose that μ and ν are totally finite.

(i) Let $\mathcal{K}$ be the family of subsets $K \subseteq X$ such that
K is compact and $f \restriction K$ is continuous,
whenever $F \in \mathrm{T}$ and $\nu(F \cap f[K]) > 0$ then $\mu(K \cap f^{-1}[F]) > 0$,
either $K = \emptyset$ or $\mu K > 0$.

Take any $E \in \Sigma$ such that $\mu E > 0$. Then there is a $K \in \mathcal{K}$ such that $K \subseteq E$ and $\mu K > 0$. **P** Let $K_0 \subseteq E$ be a compact set such that $f \restriction K_0$ is continuous and $\mu K_0 > 0$. Let $\delta > 0$ be such that $\mu K_0 - \delta \nu Y > 0$. For compact sets $K \subseteq K_0$ set $q(K) = \mu K - \delta \nu f[K]$. Choose $\langle \alpha_n \rangle_{n \in \mathbb{N}}$, $\langle K_n \rangle_{n \geq 1}$ as follows. Given that K_n is a compact subset of K_0, where $n \in \mathbb{N}$, set

$$\alpha_n = \sup\{q(K) : K \subseteq K_n \text{ is compact}\},$$

and choose a compact subset K_{n+1} of K_n such that $q(K_{n+1}) \geq \max(q(K_n), \alpha_n - 2^{-n})$. Continue.

Set $K = \bigcap_{n \in \mathbb{N}} K_n$. We have

$$\begin{aligned} q(K) &= \mu K - \delta \nu f[K] \\ &\geq \lim_{n \to \infty} \mu K_n - \delta \inf_{n \in \mathbb{N}} \nu f[K_n] = \lim_{n \to \infty} q(K_n) = \sup_{n \in \mathbb{N}} q(K_n) \end{aligned}$$

because $\langle q(K_n) \rangle_{n \in \mathbb{N}}$ is non-decreasing. Of course $K \subseteq E$,

$$\mu K \geq q(K) \geq q(K_0) > 0,$$

and $f \restriction K$ is continuous because $K \subseteq K_0$.

? If there is an $F \in \mathrm{T}$ such that $\nu(F \cap f[K]) > 0$ but $\mu(K \cap f^{-1}[F]) = 0$, take a compact set $K' \subseteq K \setminus f^{-1}[F]$ such that $\mu K' > \mu K - \delta \nu(F \cap f[K])$. Then $f[K'] \subseteq f[K] \setminus F$, so

$$q(K') = \mu K' - \delta \nu f[K'] \geq \mu K' - \delta(\nu f[K] - \nu(F \cap f[K])) > \mu K - \delta \nu f[K] = q(K).$$

Let $n \in \mathbb{N}$ be such that $q(K') > q(K) + 2^{-n}$; then K' is a compact subset of K_n, so

$$\alpha_n \geq q(K') > q(K) + 2^{-n} \geq q(K_{n+1}) + 2^{-n} \geq \alpha_n,$$

which is impossible. **X** Thus K belongs to $\mathcal{K}$ and will serve. **Q**

(ii) By 342B, there is a countable disjoint set $\mathcal{K}_0 \subseteq \mathcal{K}$ such that $\mu(X \setminus \bigcup \mathcal{K}_0) = 0$. Enumerate $\mathcal{K}_0$ as $\langle K_n \rangle_{n < \#(\mathcal{K}_0)}$; for convenience of notation, if $\mathcal{K}_0$ is finite, set $K_n = \emptyset$ for $n \geq \#(\mathcal{K}_0)$, so that every K_n belongs to $\mathcal{K}$ and $\mu E = \sum_{n=0}^{\infty} \mu(E \cap K_n)$ for every $E \in \Sigma$.

(iii) For each $n \in \mathbb{N}$, define $\lambda_n : \mathrm{T} \to \mathbb{R}$ by setting $\lambda_n F = \mu(K_n \cap f^{-1}[F])$ for every $F \in \mathrm{T}$. Then λ_n is a measure dominated by ν, so there is a T-measurable $g_n : Y \to [0,1]$ such that $\lambda_n F = \int_F g_n$ for every $F \in \mathrm{T}$, by the Radon-Nikodým theorem. Because $\lambda_n(Y \setminus f[K_n]) = 0$, we may suppose that $g_n(y) = 0$ for $y \notin f[K_n]$. We have

$$\int_F \sum_{n=0}^\infty g_n = \sum_{n=0}^\infty \int_F g_n = \sum_{n=0}^\infty \mu(K_n \cap f^{-1}[F]) = \mu f^{-1}[F] = \nu F$$

for every $F \in \mathrm{T}$, so $\sum_{n=0}^\infty g_n(y) = 1$ for ν-almost every y. Reducing the g_n further on a set of measure zero, if need be, we may suppose that $\sum_{n=0}^\infty g_n(y) \leq 1$ for every y.

(iv) For each $n \in \mathbb{N}$, let $\tilde{\lambda}_n$ be the subspace measure on $f[K_n]$ induced by λ_n, and $\tilde{\mu}_n$ the subspace measure on K_n induced by μ. Then $f \restriction K_n$ is inverse-measure-preserving for $\tilde{\mu}_n$ and $\tilde{\lambda}_n$. Also, $\tilde{\lambda}_n$ has an almost strong lifting. **P** If $K_n = \emptyset$, this is trivial. Otherwise, $\nu f[K_n] \geq \mu K_n > 0$, so the subspace measure $\tilde{\nu}_n$ induced by ν on $f[K_n]$ has an almost strong lifting, by 453E. But $\tilde{\nu}_n$ and $\tilde{\lambda}_n$ have the same domain $\mathrm{T} \cap \mathcal{P}(f[K_n])$ and the same null ideal, because $K_n \in \mathcal{K}$; so an almost strong lifting for $\tilde{\nu}_n$ is an almost strong lifting for $\tilde{\lambda}_n$. **Q**

By (a) above, we can find a disintegration $\langle \mu_{ny} \rangle_{y \in f[K_n]}$ of $\tilde{\mu}_n$ over $\tilde{\lambda}_n$ such that every μ_{ny} is a Radon measure on K_n, $\mu_{ny} K_n \leq 1$ for every y and

$$\mu_{ny}\{x : x \in K_n,\ f(x) = y\} = \mu_{ny} K_n = 1$$

for $\tilde{\lambda}_n$-almost every $y \in f[K_n]$, that is, for ν-almost every $y \in f[K_n]$. For $y \in Y \setminus f[K_n]$, let μ_{ny} be the zero measure on K_n.

(v) Now, for $y \in Y$, set

$$\mu_y E = \sum_{n=0}^\infty g_n(y) \mu_{ny}(E \cap K_n)$$

for all those $E \subseteq X$ such that the sum is defined. Then μ_y is a Radon measure and $\mu_y X \leq 1$. **P** Because every μ_{ny} is a complete measure, so is μ_y. We have

$$\mu_y X = \sum_{n=0}^\infty g_n(y) \mu_{ny} K_n \leq \sum_{n=0}^\infty g_n(y) \leq 1$$

by the choice of the g_n. If $G \subseteq X$ is open then μ_{ny} measures $G \cap K_n$ for every n, so μ_y measures G; accordingly μ_y measures every compact set. If $\mu_y E > 0$, there is some $n \in \mathbb{N}$ such that $g_n(y) > 0$ and $\mu_{ny}(E \cap K_n) > 0$; now there is a compact set $K \subseteq E \cap K_n$ such that $\mu_{ny} K > 0$, in which case $\mu_y K > 0$. By 412B, μ_y is tight, and is a Radon measure. **Q**

(vi) $\langle \mu_y \rangle_{y \in Y}$ is a disintegration of μ over ν. **P** If $E \in \Sigma$ then

$$\mu E = \sum_{n=0}^\infty \mu(E \cap K_n)$$

(by the choice of the K_n in (ii) above)

$$= \sum_{n=0}^\infty \tilde{\mu}_n(E \cap K_n) = \sum_{n=0}^\infty \int_{f[K_n]} \mu_{ny}(E \cap K_n) \tilde{\lambda}_n(dy)$$

(because $\langle \mu_{ny} \rangle_{y \in f[K_n]}$ is a disintegration of $\tilde{\mu}_n$ over $\tilde{\lambda}_n$)

$$= \sum_{n=0}^\infty \int_{f[K_n]} \mu_{ny}(E \cap K_n) \lambda_n(dy) = \sum_{n=0}^\infty \int \mu_{ny}(E \cap K_n) \lambda_n(dy)$$

(because $\lambda_n(Y \setminus f[K_n]) = 0$)

$$= \sum_{n=0}^\infty \int g_n(y) \mu_{ny}(E \cap K_n) \nu(dy)$$

(235A)

$$= \int \sum_{n=0}^\infty g_n(y) \mu_{ny}(E \cap K_n) \nu(dy) = \int \mu_y E \, \nu(dy).$$ **Q**

(vii) In particular, $\int \mu_y X \, \nu(dy) = \nu Y$; since $\mu_y X \leq 1$ for every y, $\mu_y X = 1$ for almost every y. And $\mu_y f^{-1}[\{y\}] = 1$ for almost every y. **P**

$$\{y : \mu_y f^{-1}[\{y\}] \neq 1\} \subseteq \{y : \mu_y X \neq 1\} \cup \{y : \mu_y^*(X \setminus f^{-1}[\{y\}]) > 0\}$$
$$\subseteq \{y : \mu_y X \neq 1\} \cup \bigcup_{n\in\mathbb{N}} \{y : y \in f[K_n],\, \mu_{ny}(K_n \setminus f^{-1}[\{y\}]) > 0\}$$

is negligible. **Q**

(c) Now let us turn to the general case. This proceeds just as in 452O. Let $\langle Y_i\rangle_{i\in I}$ be a decomposition of Y. For each $i \in I$, take X_i, λ_i and ν_i as in the proof of 452O. Note that λ_i and ν_i are Radon measures, so that we can apply (b) above to find a disintegration $\langle\tilde\mu_y\rangle_{y\in Y_i}$ of λ_i over ν_i such that every $\tilde\mu_y$ is a Radon measure and $\tilde\mu_y X_i = \tilde\mu_y f^{-1}[\{y\}] = 1$ for ν_i-almost every $y \in Y_i$. Just as in 452O, we can set

$$\mu_y E = \tilde\mu_y(E \cap X_i)$$

whenever $y \in Y_i$ and μ_y measures $E \cap X_i$, to obtain a disintegration $\langle\mu_y\rangle_{y\in Y}$ of μ over ν in which every μ_y is a Radon measure and $\mu_y X = 1$ for almost every y; and this time

$$\{y : y \in Y,\, \mu_y f^{-1}[\{y\}] \neq 1\} = \textstyle\bigcup_{i\in I}\{y : y \in Y_i,\, \tilde\mu_y f^{-1}[\{y\}] \neq 1\}$$

is negligible. So we have a disintegration of the required type.

453L Remark If f is surjective, we can arrange that every μ_y is a Radon probability measure for which $f^{-1}[\{y\}]$ is μ_y-conegligible, just by changing some of the μ_y to point masses. If f is not surjective, then we can still (if X itself is not empty) arrange that every μ_y is a Radon probability measure; but it might be more appropriate to make some of the μ_y the zero measure, so that $f^{-1}[\{y\}]$ is always μ_y-conegligible.

I have continued to express this theorem in terms of measures μ_y on the whole space X. Of course, if we take it that $X_y = f^{-1}[\{y\}]$ is to be conegligible for every y, it will sometimes be easier to think of μ_y as a measure on X_y; this is very much what we do in the case of Fubini's theorem, where all the X_y are, in effect, the same.

453M Strong liftings and Stone spaces Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space, and $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ the Stone space of the measure algebra $(\mathfrak{A}, \bar\mu)$ of μ. For $E \in \Sigma$ let $E^* \subseteq Z$ be the open-and-closed set corresponding to the equivalence class $E^\bullet \in \mathfrak{A}$. Let R be the relation

$$\textstyle\bigcap_{F\subseteq X \text{ is closed}}\{(z, x) : z \in Z \setminus F^* \text{ or } x \in F\} \subseteq Z \times X$$

(415Q). For every lifting $\phi : \Sigma \to \Sigma$ we have a unique function $g_\phi : X \to Z$ such that $\phi E = g_\phi^{-1}[E^*]$ for every $E \in \Sigma$ (see 341P). Now we have the following easy facts.

(a) ϕ is strong iff $(g_\phi(x), x) \in R$ for every $x \in X$. **P**

$(g_\phi(x), x) \in R$ for every $x \in X$
$\iff$ $x \in F$ whenever F is closed and $g_\phi(x) \in F^*$
$\iff$ $F \subseteq g_\phi^{-1}[F^*]$ for every closed set $F \subseteq X$
$\iff$ $F \subseteq \phi F$ for every closed set $F \subseteq X$
$\iff$ ϕ is strong. **Q**

(b) If $\mathfrak{T}$ is Hausdorff, so that R is the graph of a function f (415R), then ϕ is strong iff $fg_\phi(x) = x$ for every $x \in X$. (For $(g_\phi(x), x) \in R$ iff $fg_\phi(x) = x$.)

453N Losert's example (Losert 79) There is a compact Hausdorff space with a strictly positive completion regular Radon probability measure which has no strong lifting.

proof (a) Let ν be the usual measure on $\{0, 1\}^{\mathbb{N}} = Y$. Let $M \subseteq Y$ be a closed nowhere dense set such that $\nu M > 0$ (cf. 419B), and ν_1 a Radon probability measure on Y such that $\nu_1 M = 1$ (e.g., a point mass at some point of M).

Let I be any set of cardinal at least ω_2 such that $I \cap (I \times I) = \emptyset$. Let λ be the product measure on Y^I, giving each factor the measure ν; of course λ can be identified with the usual measure on $\{0,1\}^{\mathbb{N}\times I}$ (254N). Note that λ and ν are both strictly positive. For $i \in I$ write $M_i = \{z : z \in Y^I, z(i) \in M\}$; then M_i is closed in Y^I.

Set $A = \{(i,j) : i, j \in I, i \neq j\}$. For $z \in Y^I$ and $(i,j) \in A$ let $\nu_{ij}^{(z)}$ be the Radon probability measure on Y given by setting

$$\begin{aligned}\nu_{ij}^{(z)} &= \nu_1 \text{ if } z \in M_i \cap M_j,\\ &= \nu \text{ otherwise.}\end{aligned}$$

Now, for $z \in Y^I$, let λ_z be the Radon product measure of $\langle \nu_{ij}^{(z)}\rangle_{(i,j)\in A}$ on Y^A.

(b) Let $\mathcal{U}$ be the family of sets $U \subseteq Y^A$ of the form $\{u : u(i,j) \in U_{ij} \text{ for } (i,j) \in B\}$, where $B \subseteq A$ is finite and $U_{ij} \subseteq Y$ is open-and-closed for every $(i,j) \in B$. Then the function $z \mapsto \lambda_z U : Y^I \to [0,1]$ is Borel measurable for every $U \in \mathcal{U}$. **P** Express U in the given form. For $C \subseteq B$ set

$$E_C = \{z : z \in Y^I, C = \{(i,j) : (i,j) \in B, z \in M_i \cap M_j\}\},$$

so that $\langle E_C\rangle_{C\subseteq B}$ is a partition of Y^I into Borel sets. For any $C \subseteq B$,

$$\lambda_z U = \prod_{(i,j)\in B} \nu_{ij}^{(z)}(U_{ij}) = \prod_{(i,j)\in C} \nu_1 U_{ij} \cdot \prod_{(i,j)\in B\setminus C} \nu U_{ij}$$

is constant for $z \in E_C$. **Q**

(c) There is a Radon measure on $X = Y^I \times Y^A$ given by the formula

$$\mu E = \int \lambda_z E[\{z\}]\lambda(dz)$$

for every Baire set $E \subseteq X$. **P** Let $\mathcal{E}$ be the class of those sets $E \subseteq X$ such that $\int \lambda_z E[\{z\}]\lambda(dz)$ is defined. Then $\mathcal{E}$ is closed under monotone limits of sequences, and $E \setminus E' \in \mathcal{E}$ whenever $E, E' \in \mathcal{E}$ and $E' \subseteq E$; also $\mathcal{E}$ contains all the basic open-and-closed sets in X of the form $V \times U$, where $V \subseteq Y^I$ is open-and-closed and $U \in \mathcal{U}$. By the Monotone Class Theorem (136B), $\mathcal{E}$ includes the σ-algebra generated by such sets, which is the Baire σ-algebra $\mathcal{B}\mathfrak{a}$ of X (4A3Of). Of course $E \mapsto \int \lambda_z E[\{z\}]\lambda(dz)$ is countably additive on $\mathcal{B}\mathfrak{a}$, so is a Baire measure on X, and has a unique extension to a Radon measure, by 432F. **Q**

μ is strictly positive. **P** Let $W \subseteq X$ be any non-empty open set. Then it includes an open set of the form $U \times V$ where $U = \{z : z \in Y^I, z(i) \in V_i \text{ for every } i \in J\}$, $V = \{u : u \in Y^A, u(j,k) \in U_{jk} \text{ for every } (j,k) \in B\}$, $J \subseteq I$ and $B \subseteq A$ are finite sets, and V_i, $U_{jk} \subseteq Y$ are non-empty open sets for every $i \in J$ and $(j,k) \in B$. Now ν is strictly positive, so $\lambda U' > 0$, where

$$U' = \{z : z \in U, z \notin M_j \text{ whenever } (j,k) \in B\}.$$

(This is where we need to know that the M_j are nowhere dense.) But if $z \in U'$ then $\nu_{jk}^{(z)} = \nu$ for every $(j,k) \in B$, so

$$\lambda_z V = \textstyle\prod_{(j,k)\in B} \nu U_{jk} > 0.$$

Accordingly

$$\mu W \geq \textstyle\int_{U'} \lambda_z V \lambda(dz) > 0.$$

As W is arbitrary, μ is strictly positive. **Q**

Write Σ for the domain of μ.

(d) Fix on a self-supporting compact set $K \subseteq X$. I seek to show that, regarded as a subset of $Y^{I\cup A}$, K is determined by coordinates in some countable set.

(i) There is a zero set $L \supseteq K$ such that $\mu L = \mu K$. **P** Let $\langle K_n\rangle_{n\in\mathbb{N}}$ be a sequence of compact subsets of $X \setminus K$ such that $\lim_{n\to\infty} \mu K_n = \mu(X \setminus K)$. For each $n \in \mathbb{N}$ there is a continuous function $f_n : X \to [0,1]$ which is zero on K and 1 on K_n; now $L = \{x : f_n(x) = 0 \text{ for every } n \in \mathbb{N}\}$ is a zero set including K and of the same measure as K. **Q**

(ii) By 4A3Nc, L is determined by coordinates in a countable subset of $I \cup A$, that is, there are countable sets $J_0 \subseteq I$, $B_0 \subseteq A$ such that whenever $(z,u) \in L$, $(z',u') \in X$, $z{\restriction}J_0 = z'{\restriction}J_0$ and $u{\restriction}B_0 = u'{\restriction}B_0$ we shall have $(z',u') \in L$. Set

$$J = J_0 \cup \{i : (i,j) \in B_0\} \cup \{j : (i,j) \in B_0\}, \quad B = A \cap (J \times J);$$

then $J \supseteq J_0$ and $B \supseteq B_0$ are still countable, and L is determined by coordinates in $J \cup B$.

(iii) Take any $(z_0,u_0) \in X \setminus K$. Because K is closed, we can find finite sets $J_1 \subseteq I$ and $B_1 \subseteq A$, open-and-closed sets $V_i \subseteq Y$ for $i \in J_1$, and open-and-closed sets $U_{ij} \subseteq Y$ for $(i,j) \in B_1$, such that

$$W = \{(z,u) : z(i) \in V_i \text{ for every } i \in J_1,\ u(i,j) \in U_{ij} \text{ for every } (i,j) \in B_1\}$$

contains (z_0,u_0) and is disjoint from K. Set

$$W_1 = \{(z,u) : (z,u) \in X, z(i) \in V_i \text{ for every } i \in J_1 \cap J, \\ u(i,j) \in U_{ij} \text{ for every } (i,j) \in B_1 \cap B\},$$

$$Q = \{z : z \in Y^I,\ \lambda_z((L \cap W_1)[\{z\}]) > 0\},$$

so that W_1 is an open-and-closed set in X and Q is a Borel set in Y^I ((b) above). Now Q is determined by coordinates in J. **P** Suppose that $z \in Q$, $z' \in Y^I$ and $z{\restriction}J = z'{\restriction}J$. Because both L and W_1 are determined by coordinates in $J \cup B$, $(L \cap W_1)[\{z\}] = (L \cap W_1)[\{z'\}] = H$ say, and H is determined by coordinates in B. At the same time, for any $(i,j) \in B$, $M_i \cap M_j$ is determined by coordinates in J, so contains z iff it contains z', and $\nu_{ij}^{(z)} = \nu_{ij}^{(z')}$. This means that, writing λ'_z and $\lambda'_{z'}$ for the products of $\langle \nu_{ij}^{(z)} \rangle_{(i,j)\in B}$ and $\langle \nu_{ij}^{(z')} \rangle_{(i,j)\in B}$ on Y^B, $\lambda'_z = \lambda'_{z'}$. So

$$\lambda_{z'}((L \cap W_1)[\{z'\}]) = \lambda_{z'}H = \lambda'_{z'}H' = \lambda'_z H' = \lambda_z H = \lambda_z((L \cap W_1)[\{z\}]) > 0,$$

where $H' = \{u{\restriction}B : u \in H\}$ (254Ob), and $z' \in Q$. **Q**

(iv) Set

$$J_2 = (\{i : (i,j) \in B_1 \setminus B\} \cup \{j : (i,j) \in B_1 \setminus B\}) \setminus J.$$

Then J_2 is a finite subset of $I \setminus J$, and $B_1 \subseteq (J \cup J_2) \times (J \cup J_2)$. Set

$$G = \{z : z \in Y^I,\ z(i) \notin M \text{ for every } i \in J_2\},$$

so that G is a dense open subset of Y^I. Set

$$G_1 = \{z : z \in Y^I,\ z(i) \in V_i \text{ for every } i \in J_1 \setminus J\}.$$

Then G_1 is a non-empty open set, so $G \cap G_1 \neq \emptyset$ and $\lambda(G \cap G_1) > 0$.

(v) Set

$$U = \{u : u \in Y^A,\ u(i,j) \in U_{ij} \text{ for every } (i,j) \in B_1 \setminus B\}.$$

If $z \in G$, then $z \notin M_i \cap M_j$ whenever $(i,j) \in B_1 \setminus B$, so $\nu_{ij}^{(z)} = \nu$ for every $(i,j) \in B_1 \setminus B$, and

$$\lambda_z U = \prod_{(i,j)\in B_1\setminus B} \nu U_{ij} > 0.$$

(vi) **?** Suppose, if possible, that $\lambda Q > 0$. Because Q is determined by coordinates in J and $G \cap G_1$ is determined by coordinates in $J_2 \cup (J_1 \setminus J)$,

$$\lambda(Q \cap G \cap G_1) = \lambda Q \cdot \lambda(G \cap G_1) > 0.$$

If $z \in Q \cap G \cap G_1$,

$$\lambda_z((L \cap W)[\{z\}]) = \lambda_z(U \cap (L \cap W_1)[\{z\}])$$

(because $W = W_1 \cap (Y^I \times U) \cap (G_1 \times Y^A)$, and $z \in G_1$)

$$= \lambda_z U \cdot \lambda_z((L \cap W_1)[\{z\}])$$

(because $(L \cap W_1)[\{z\}]$ is determined by coordinates in B, while U is determined by coordinates in $B_1 \setminus B$, and λ_z is a product measure)

$$> 0$$

because $z \in G \cap Q$. But this means that

$$0 < \int \lambda_z((L \cap W)[\{z\}])\lambda(dz) = \mu(L \cap W) = \mu(K \cap W) = \mu\emptyset,$$

which is absurd. **X**

Thus λQ must be zero.

(vii) Consequently

$$\mu(K \cap W_1) = \mu(L \cap W_1) = \int \lambda_z((L \cap W_1)[\{z\}])\lambda(dz) = 0;$$

because K is self-supporting, $K \cap W_1 = \emptyset$. And W_1 contains (z_0, u_0) and is determined by coordinates in $J \cup B$.

(viii) What this means is that there can be no $(z, u) \in K$ such that $z\restriction J = z_0\restriction J$ and $u\restriction B = u_0\restriction B$. At this point, recall that (z_0, u_0) was an arbitrary point of $X \setminus K$. So what must be happening is that K is determined by coordinates in the countable set $J \cup B$. By 4A3Nc again, in the other direction, K is a zero set.

(e) Part (d) shows that every self-supporting compact subset of X is a zero set. Since μ is certainly inner regular with respect to the self-supporting compact sets, it is inner regular with respect to the zero sets, that is, is completion regular.

It follows that whenever $E \in \Sigma$ there is an $E' \subseteq E$, determined by coordinates in a countable subset of $I \cup A$, such that $E \setminus E'$ is negligible. (Take E' to be a countable union of self-supporting compact sets.)

(f) **?** Now suppose, if possible, that we could find a strong lifting ϕ for μ. For each $i \in I$, take a set $E_i \subseteq \phi(M_i \times Y^A)$ such that $\mu E_i = \mu\phi(M_i \times Y^A)$ and E_i is determined by coordinates in $J_i \cup B_i$, where $J_i \subseteq I$ and $B_i \subseteq A$ are countable. Set

$$J_i^* = \{j : (j,k) \in B_i\} \cup \{k : (j,k) \in B_i\},$$

so that J_i^* is also countable. Because $\#(I) \geq \omega_2$, there are distinct i, $j \in I$ such that $i \notin J_j^*$ and $j \notin J_i^*$ (4A1Ea). So neither (i,j) nor (j,i) belongs to $B_i \cup B_j$.

Set

$$F = \{u : u \in Y^A,\ u(i,j) \in M,\ u(j,i) \in M\}.$$

Then $\mu((M_i \cap M_j) \times (Y^A \setminus F)) = 0$. **P** If $z \in M_i \cap M_j$, then

$$\lambda_z F = \nu_{ij}^{(z)} M \cdot \nu_{ji}^{(z)} M = (\nu_1 M)^2 = 1,$$

so $\lambda_z(Y^A \setminus F) = 0$. But $(M_i \cap M_j) \times (Y^A \setminus F)$ is a Baire set, so

$$\begin{aligned}\mu((M_i \cap M_j) \times (Y^A \setminus F)) &= \int \lambda_z((M_i \cap M_j) \times (Y^A \setminus F))[\{z\}]\lambda(dz) \\ &= \int_{M_i \cap M_j} \lambda_z(Y^A \setminus F)\lambda(dz) = 0.\ \textbf{Q}\end{aligned}$$

Accordingly

$$E_i \cap E_j \subseteq \phi(M_i \times Y^A) \cap \phi(M_j \times Y^A) = \phi((M_i \cap M_j) \times Y^A) \subseteq \phi(Y^I \times F)$$

(because $((M_i \cap M_j) \times Y^A) \setminus (Y^I \times F)$ is negligible)

$$\subseteq Y^I \times F$$

because $Y^I \times F$ is closed and ϕ is supposed to be strong. However, $E_i \cap E_j$ is determined by coordinates in $J_i \cup J_j \cup B_i \cup B_j$, while $Y^I \times F$ is determined by coordinates in $\{(i,j),(j,i)\}$, which does not meet $B_i \cup B_j$. So either $E_i \cap E_j$ is empty or $F = Y^A$. But $F \neq Y^A$ because $M \neq Y$, while

$$\begin{aligned}\mu(E_i \cap E_j) &= \mu(\phi(M_i \times Y^A) \cap \phi(M_j \times Y^A)) = \mu((M_i \cap M_j) \times Y^A)\\ &= \lambda(M_i \cap M_j) = (\nu M)^2 > 0,\end{aligned}$$

so $E_i \cap E_j \neq \emptyset$. **X**

(g) Thus μ has no strong lifting, as claimed.

453X Basic exercises >(a) Let $(\mathfrak{A}, \bar{\mu})$ be a measure algebra and $(Z, \mathfrak{T}, \Sigma, \mu)$ its Stone space. Show that the canonical lifting of μ (341O) is strong.

(b) Let $\mathfrak{S}$ be the right-facing Sorgenfrey topology on $\mathbb{R}$ (415Xc). Show that there is a lifting of Lebesgue measure on $\mathbb{R}$ which is strong with respect to $\mathfrak{S}$. (*Hint*: set $\underline{\phi}E = \{x : \lim_{\delta \downarrow 0} \frac{1}{\delta}\mu(E \cap [x, x+\delta]) = 1\}$, and use 341J.)

>(c) Let μ be the usual measure on the split interval (343J, 419L). Show that μ has a strong lifting.

(d) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined topological measure space such that μ is inner regular with respect to the closed sets, and $\phi : \Sigma \to \Sigma$ a strong lifting. Show that μ is a quasi-Radon measure with respect to the lifting topology $\mathfrak{T}_l$ (414Q). Show that if $\mathfrak{T}$ is regular then $\mathfrak{T}_l \supseteq \mathfrak{T}$.

(e) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space which has a strong lifting. Show that any non-zero indefinite-integral measure over μ (§234) has a strong lifting.

(f) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space such that (X, Σ, μ) is countably separated and $\mu X > 0$; for example, $(X, \mathfrak{T})$ could be an analytic space (433B). Show that μ is inner regular with respect to the compact metrizable subsets of X, so has an almost strong lifting. (*Hint*: there is an injective measurable $f : X \to \mathbb{R}$, which must be almost continuous.)

(g) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a complete locally determined topological measure space such that μ is effectively locally finite and inner regular with respect to the closed sets, and $\underline{\phi} : \Sigma \to \Sigma$ a lower density such that $\underline{\phi}G \supseteq G$ for every open $G \subseteq X$. Show that μ is a quasi-Radon measure with respect to both $\mathfrak{T}$ and the density topology associated with $\underline{\phi}$.

(h) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space and $\underline{\phi} : \Sigma \to \Sigma$ a lower density such that $\underline{\phi}G \supseteq G$ for every open $G \subseteq X$. Let $\langle G_x \rangle_{x \in X}$ be a family of open sets in X such that $x \notin \underline{\phi}(X \setminus G_x)$ for every $x \in X$. (i) Show that $A \setminus \bigcup_{x \in A}(G_x \cap U_x)$ is negligible whenever $A \subseteq X$ and U_x is a neighbourhood of x for every $x \in A$. (ii) Let $\mathfrak{S}$ be the topology on X generated by $\mathfrak{T} \cup \{\{x\} \cup G_x : x \in X\}$. Show that μ is quasi-Radon with respect to $\mathfrak{S}$.

(i) Let X and Y be Hausdorff spaces, and μ a Radon probability measure on $X \times Y$; set $\pi(x,y) = y$ for $x \in X$, $y \in Y$, and let ν be the image measure $\mu\pi^{-1}$. Suppose that ν has an almost strong lifting. Show that there is a family $\langle \mu_y \rangle_{y \in Y}$ of Radon probability measures on X such that $\mu E = \int \mu_y(E^{-1}[\{y\}])\nu(dy)$ for every $E \in \operatorname{dom}\mu$.

(j) Use 453Xe to simplify part (b) of the proof of 453K.

(k) In 453N, show that $\int \lambda_z E[\{z\}]\lambda(dz)$ is defined and equal to μE whenever μ measures E.

453Y Further exercises (a) Let $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ be a Radon measure space such that $\nu Y > 0$ and whenever $(X, \mathfrak{T}, \Sigma, \mu)$ is a Radon measure space and $f : X \to Y$ is an almost continuous inverse-measure-preserving function, then there is a disintegration $\langle \mu_y \rangle_{y \in Y}$ of μ over ν such that $\mu_y f^{-1}[\{y\}] = 1$ for almost every y. Show that ν has an almost strong lifting. (*Hint*: Start with the case in which Y is compact. Take $f : X \to Y$ to be the function described in 415R, 416V and 453Mb. Set $\underline{\phi}E = \{y : \mu_y E^* = \mu_y X = 1\}$.)

453Z Problems (a) If $(X, \mathfrak{T}, \Sigma, \mu)$ and $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ are compact Radon measure spaces with strong liftings, does their product necessarily have a strong lifting? What if they are both Stone spaces?

(b) If $(X, \mathfrak{T}, \Sigma, \mu)$ is a Radon probability space with countable Maharam type, must it have an almost strong lifting?

453 Notes and comments As I noted in §452, early theorems on disintegrations concentrated on cases in which all the measure spaces involved were 'standard' in that the measures were defined on standard Borel spaces (§424), or were the completions of such measures. Under these conditions the distinction between 452I and 453K becomes blurred; measures (when completed) have to be Radon measures (433Cb), liftings have to be almost strong (453F) and disintegrations have to be concentrated on fibers (452Gc). Theorem 453K provides disintegrations concentrated on fibers without any limitation on the size of the spaces involved, though making strong topological assumptions.

The strength of 453K derives from the remarkable variety of the (Radon) measure spaces which have strong liftings, as in 453F, 453G, 453I and 453J. For some ten years there were hopes that every self-supporting Radon measure had an strong lifting, which were finally dashed by Losert 79; I give a version of the example in 453N. This is a special construction, and it remains unclear whether some much more direct approach might yield another example (453Za). I should perhaps remark straight away that if the continuum hypothesis is true, then any self-supporting Radon measure space of Maharam type at most ω_1 has a strong lifting; I will present the argument in Volume 5. In particular, subject to the continuum hypothesis, $Z \times Z$ has a strong lifting, where Z is the Stone space of the Lebesgue measure algebra, and we have a positive answer to 453Zb.

454 Measures on product spaces

A central concern of probability theory is the study of 'processes', that is, families $\langle X_t \rangle_{t \in T}$ of random variables thought of as representing the evolution of a system in time. The representation of such processes as random variables in the modern sense, that is, measurable functions on an abstract probability space, was one of the first challenges faced by Kolmogorov. In this section I give a version of Kolmogorov's theorem on the extension of consistent families of measures on subproducts to a measure on the whole product (454D). It turns out that some restriction on the marginal measures is necessary, and 'perfectness' seems to be an appropriate hypothesis, necessarily satisfied if the factor spaces are standard Borel spaces or the marginal measures are Radon measures. If we have marginal measures with stronger properties then we shall be able to infer corresponding properties of the measure on the product space (454A, generalizing 451J).

The apparatus here makes it easy to describe joint distributions of arbitrary families of real-valued random variables (454J-454M), extending the ideas of §271. For the sake of the theorem that almost all Brownian paths are continuous (455D) I briefly investigate measures on $C(T)$, where T is a Polish space (454N-454P).

454A Theorem Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a non-empty family of totally finite measure spaces. Set $X = \prod_{i \in I} X_i$ and let μ be a measure on X which is inner regular with respect to the σ-algebra $\widehat{\bigotimes}_{i \in I} \Sigma_i$ generated by $\{\pi_i^{-1}[E] : i \in I, E \in \Sigma_i\}$, where $\pi_i : X \to X_i$ is the coordinate map for each $i \in I$. Suppose that every π_i is inverse-measure-preserving.

(a) If $\mathcal{K} \subseteq \mathcal{P}X$ is a family of sets which is closed under finite unions and countable intersections, and μ_i is inner regular with respect to $\mathcal{K}_i = \{K : K \subseteq X_i, \pi_i^{-1}[K] \in \mathcal{K}\}$ for every $i \in I$, then μ is inner regular with respect to $\mathcal{K}$.

(b)(i) If every μ_i is a compact measure, so is μ;

(ii) if every μ_i is a countably compact measure, so is μ;

(iii) if every μ_i is a perfect measure, so is μ.

proof If X is empty this is all trivial, so we may suppose that $X \neq \emptyset$.

(a) Set $\mathcal{A} = \{\pi_i^{-1}[E] : i \in I, E \in \Sigma_i\}$. If $A \in \mathcal{A}$, $V \in \Sigma$ and $\mu(A \cap V) > 0$, there is a $K \in \mathcal{K} \cap \mathcal{A}$ such that $K \subseteq A$ and $\mu(K \cap V) > 0$. **P** Express A as $\pi_i^{-1}[E]$, where $E \in \Sigma_i$; take $L \in \mathcal{K}_i$ such that $L \subseteq E$ and $\mu_i L > \mu_i E - \mu(A \cap V)$, and set $K = \pi_i^{-1}[L]$. **Q**

By 412C, $\mu\restriction\widehat{\bigotimes}_{i\in I}\Sigma_i$ is inner regular with respect to $\mathcal{K}$; by 412Ab, so is μ.

(b)(i)-(ii) Suppose that every μ_i is (countably) compact. Then for each $i \in I$ we can find a (countably) compact class $\mathcal{K}_i \subseteq \mathcal{P}X_i$ such that μ_i is inner regular with respect to $\mathcal{K}_i$. Set $\mathcal{L} = \{\pi_i^{-1}[K] : i \in I, K \in \mathcal{K}_i\}$. Then $\mathcal{L}$ is (countably) compact (451H). So there is a (countably) compact $\mathcal{K} \supseteq \mathcal{L}$ which is closed under finite unions and countable intersections (342D, 413R). Now μ is inner regular with respect to $\mathcal{K}$, by (a), and therefore (countably) compact.

(iii) Let T_0 be a countably generated σ-subalgebra of $\widehat{\bigotimes}_{i\in I}\Sigma_i$. Then there must be some countable subfamily $\mathcal{E}$ of $\{\pi_i^{-1}[E] : i \in I, E \in \Sigma_i\}$ such that T_0 is included in the σ-algebra generated by $\mathcal{E}$ (use 331Gd). Set $\mathcal{E}_i = \{E : E \in \Sigma_i, \pi_i^{-1}[E] \in \mathcal{E}\}$ for each i, so that $\mathcal{E}_i$ is countable, and let Σ_i' be the σ-algebra generated by $\mathcal{E}_i$. Then $\mu_i\restriction\Sigma_i'$ is compact (451F). Applying (i), we see that $\mu\restriction\widehat{\bigotimes}_{i\in I}\Sigma_i'$ is compact, therefore perfect; while $\mathrm{T}_0 \subseteq \widehat{\bigotimes}_{i\in I}\Sigma_i'$. As T_0 is arbitrary, $\mu\restriction\widehat{\bigotimes}_{i\in I}\Sigma_i$ is perfect (451F). But as the completion of μ is exactly the completion of $\mu\restriction\widehat{\bigotimes}_{i\in I}\Sigma_i$, μ also is perfect, by 451Gc.

454B Corollary Let $\langle X_i\rangle_{i\in I}$ be a family of Polish spaces with product X. Then any totally finite Baire measure on X is a compact measure.

proof If μ is a Baire measure on X, then its domain $\mathcal{B}\mathfrak{a}(X)$ is $\widehat{\bigotimes}_{i\in I}\mathcal{B}(X_i)$, where $\mathcal{B}(X_i)$ is the Borel σ-algebra of X_i for each $i \in I$ (4A3Na). So each image measure μ_i on X_i is a Borel measure, therefore tight (that is, inner regular with respect to the compact sets, 433Ca), and by 454A(b-i) μ is compact.

454C Theorem(Marczewski & Ryll-Nardzewski 53) Let (X, Σ, μ) be a perfect totally finite measure space and (Y, T, ν) any totally finite measure space. Let $\Sigma \otimes \mathrm{T}$ be the algebra of subsets of $X \times Y$ generated by $\{E \times F : E \in \Sigma, F \in \mathrm{T}\}$. If $\lambda_0 : \Sigma \otimes \mathrm{T} \to [0, \infty[$ is a non-negative finitely additive functional such that $\lambda_0(E\times Y) = \mu E$ and $\lambda_0(X \times F) = \nu F$ whenever $E \in \Sigma$ and $F \in \mathrm{T}$, then λ_0 has a unique extension to a measure defined on the σ-algebra $\Sigma\widehat{\otimes}\mathrm{T}$ generated by $\Sigma \otimes \mathrm{T}$.

proof (a) By 413K, it will be enough to show that $\lim_{n\to\infty}\lambda_0 W_n = 0$ for every non-increasing sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\Sigma \otimes \mathrm{T}$ with empty intersection. Take such a sequence. Each W_n must belong to the algebra generated by some finite subset of $\{E \times F : E \in \Sigma, F \in \mathrm{T}\}$, so there must be a countable set $\mathcal{E} \subseteq \Sigma$ such that every W_n belongs to the algebra generated by $\{E \times F : E \in \mathcal{E}, F \in \mathrm{T}\}$; let Σ_0 be the σ-subalgebra of Σ generated by $\mathcal{E}$, so that every W_n belongs to $\Sigma_0\widehat{\otimes}\mathrm{T}$.

(b) By 451F, $\mu\restriction\Sigma_0$ is a compact measure; let $\mathcal{K} \subseteq \mathcal{P}X$ be a compact class such that $\mu\restriction\Sigma_0$ is inner regular with respect to $\mathcal{K}$. We may suppose that $\mathcal{K}$ is the family of closed sets for a compact topology on X (342Da). Let $\mathcal{W}$ be the family of those elements W of $\Sigma_0 \otimes \mathrm{T}$ such that every horizontal section $W^{-1}[\{y\}]$ belongs to $\mathcal{K}$. Then $\mathcal{W}$ is closed under finite unions and intersections.

(c) If $W \in \Sigma_0 \otimes \mathrm{T}$ and $\epsilon > 0$, then there is a $W' \in \mathcal{W}$ such that $W' \subseteq W$ and $\lambda_0(W \setminus W') \leq \epsilon$. **P** Express W as $\bigcup_{i\leq n} E_i \times F_i$, where $E_i \in \Sigma_0$, $F_i \in \mathrm{T}$ for each $i \leq n$. (Cf. 315Jb.) For each $i \leq n$, take $K_i \in \mathcal{K} \cap \Sigma_0$ such that $\mu(E_i \setminus K_i) \leq \frac{1}{n+1}\epsilon$, and set $W' = \bigcup_{i\leq n} K_i \times F_i$. Then $W' \in \mathcal{W}$, $W' \subseteq W$ and

$$\begin{aligned}\lambda_0(W \setminus W') &\leq \sum_{i=0}^{n} \lambda_0((E_i \times F_i) \setminus (K_i \times F_i)) \leq \sum_{i=0}^{n} \lambda_0((E_i \setminus K_i) \times Y)\\ &= \sum_{i=0}^{n} \mu_0(E_i \setminus K_i) \leq \epsilon.\end{aligned}$$ **Q**

(d) Take any $\epsilon > 0$. Then for each $n \in \mathbb{N}$ we can find $W_n' \in \mathcal{W}$ such that $W_n' \subseteq W_n$ and $\lambda_0(W_n \setminus W_n') \leq 2^{-n}\epsilon$. Set $V_n = \bigcap_{i\leq n} W_i'$, so that $V_n \in \mathcal{W}$ and

$$\lambda_0(W_n \setminus V_n) \leq \textstyle\sum_{i=0}^{n} \lambda_0(W_i \setminus W_i') \leq 2\epsilon$$

for each n, and $\langle V_n\rangle_{n\in\mathbb{N}}$ is non-increasing, with empty intersection.

Because $V_n \in \Sigma_0 \otimes \mathrm{T}$, its projection $H_n = V_n[X]$ belongs to T, for each n. Of course $\langle H_n\rangle_{n\in\mathbb{N}}$ is non-increasing; also $\bigcap_{n\in\mathbb{N}} H_n = \emptyset$. **P** If $y \in Y$, then $\langle V_n^{-1}[\{y\}]\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with

empty intersection, because $\bigcap_{n\in\mathbb{N}} V_n \subseteq \bigcap_{n\in\mathbb{N}} W_n$ is empty. But $\mathcal{K}$ is a compact class, so there must be some n such that $V_n^{-1}[\{y\}]$ is empty, that is, $y \notin H_n$. **Q** Accordingly $\lim_{n\to\infty} \nu H_n = 0$. But as $V_n \subseteq X \times H_n$, $\lim_{n\to\infty} \lambda_0 V_n = 0$.

This means that $\lim_{n\to\infty} \lambda_0 W_n \leq 2\epsilon$. But as ϵ is arbitrary, $\lim_{n\to\infty} \lambda_0 W_n = 0$, as required.

454D Theorem (KOLMOGOROV 33, §III.4) Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of totally finite perfect measure spaces. Set $X = \prod_{i\in I} X_i$, and write $\bigotimes_{i\in I} \Sigma_i$ for the algebra of subsets of X generated by $\{\pi_i^{-1}[E] : i \in I, E \in \Sigma_i\}$, where $\pi_i : X \to X_i$ is the coordinate map for each $i \in I$. Suppose that $\lambda_0 : \bigotimes_{i\in I} \Sigma_i \to [0,\infty[$ is a non-negative finitely additive functional such that $\lambda_0 \pi_i^{-1}[E] = \mu_i E$ whenever $i \in I$ and $E \in \Sigma_i$. Then λ_0 has a unique extension to a measure λ with domain $\widehat{\bigotimes}_{i\in I}\Sigma_i$, and λ is perfect.

proof (a) The argument follows the same pattern as that of 454C. This time, take a non-increasing sequence $\langle W_n\rangle_{n\in\mathbb{N}}$ in $\bigotimes_{i\in I} \Sigma_i$ with empty intersection. Each W_n belongs to the algebra generated by some finite subset of $\{\pi_i^{-1}[E] : i \in I, E \in \Sigma_i\}$, so we can find countable sets $\mathcal{E}_i \subseteq \Sigma_i$ such that every W_n belongs to the subalgebra generated by $\{\pi_i^{-1}[E] : i \in I, E \in \mathcal{E}_i\}$. Let T_i be the σ-subalgebra of Σ_i generated by $\mathcal{E}_i$, so that every W_n belongs to $\bigotimes_{i\in I} \mathrm{T}_i$.

(b) For each $i \in I$, $\mu_i \restriction \mathrm{T}_i$ is compact (451F); let $\mathfrak{T}_i$ be a compact topology on X_i such that $\mu_i \restriction \mathrm{T}_i$ is inner regular with respect to the closed sets (342F). Let $\mathfrak{T}$ be the product topology on X, so that $\mathfrak{T}$ is compact (3A3J). Let $\mathcal{W}$ be the family of closed sets in X belonging to $\bigotimes_{i\in I} \mathrm{T}_i$.

(c) If $W \in \bigotimes_{i\in I} \mathrm{T}_i$ and $\epsilon > 0$, there is a $W' \in \mathcal{W}$ such that $W' \subseteq W$ and $\lambda_0(W \setminus W') \leq \epsilon$. **P** We can express W as $\bigcup_{k\leq n} \bigcap_{i\in J_k} \pi_i^{-1}[E_{ki}]$ where each J_k is a finite subset of I and $E_{ki} \in \Sigma_i$ for $k \leq n$, $i \in J_k$ (cf. 315Jb). Let $\langle \epsilon_{ki}\rangle_{k\leq n, i\in J_k}$ be a family of strictly positive numbers with sum at most ϵ. For each $k \leq n$, $i \in J_k$ take $K_{ki} \in \mathcal{K}_i \cap \mathrm{T}_i$ such that $K_{ki} \subseteq E_{ki}$ and $\mu_i(E_{ki} \setminus K_{ki}) \leq \epsilon_{ki}$, and set $W' = \bigcup_{k\leq n} \bigcap_{i\in J_k} \pi_i^{-1}[K_{ki}]$. **Q**

(d) Take any $\epsilon > 0$. Then for each $n \in \mathbb{N}$ we can find $W_n' \in \mathcal{W}$ such that $W_n' \subseteq W_n$ and $\lambda_0(W_n \setminus W_n') \leq 2^{-n}\epsilon$. Set $V_n = \bigcap_{i\leq n} W_i'$. Then $\langle V_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of closed sets in the compact space X, and has empty intersection, so there is some n such that V_n is empty, and

$$\lambda_0 W_n \leq \textstyle\sum_{i=0}^{n} \lambda_0(W_i \setminus W_i') \leq 2\epsilon.$$

As ϵ is arbitrary, $\lim_{n\to\infty} \lambda_0 W_n = 0$.

(e) As $\langle W_n\rangle_{n\in\mathbb{N}}$ is arbitrary, λ_0 has a unique countably additive extension to $\widehat{\bigotimes}_{i\in I}\Sigma_i$, by 413K, as before. Of course the extension is perfect, by 454A(b-iii).

454E Corollary Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of perfect measure spaces. Let $\mathcal{C}$ be the family of subsets of $X = \prod_{i\in I} X_i$ expressible in the form $X \cap \bigcap_{i\in J} \pi_i^{-1}[E_i]$ where $J \subseteq I$ is finite and $E_i \in \Sigma_i$ for every $i \in I$, writing $\pi_i(x) = x(i)$ for $x \in X$, $i \in I$. Suppose that $\lambda_0 : \mathcal{C} \to \mathbb{R}$ is a functional such that (i) $\lambda_0 \pi_i^{-1}[E] = \mu_i E$ whenever $i \in I$ and $E \in \Sigma_i$ (ii) $\lambda_0 C = \lambda_0(C \cap \pi_i^{-1}[E]) + \lambda_0(C \setminus \pi_i^{-1}[E])$ whenever $C \in \mathcal{C}$, $i \in I$ and $E \in \Sigma_i$. Then λ_0 has a unique extension to a measure on $\widehat{\bigotimes}_{i\in I}\Sigma_i$, which is necessarily perfect.

proof By 326Q, λ_0 has an extension to an additive functional on $\bigotimes_{i\in I} \Sigma_i$, so we can apply 454D.

454F Corollary Let $\langle (X_i, \Sigma_i)\rangle_{i\in I}$ be a family of standard Borel spaces. Set $X = \prod_{i\in I} X_i$, and let $\bigotimes_{i\in I} \Sigma_i$ be the algebra of subsets of X generated by $\{\pi_i^{-1}[E] : i \in I, E \in \Sigma_i\}$, where $\pi_i : X \to X_i$ is the coordinate map for each i. Let $\lambda_0 : \Sigma \to [0,\infty[$ be a non-negative finitely additive functional such that all the marginal functionals $E \mapsto \lambda_0 \pi_i^{-1}[E] : \Sigma_i \to [0,\infty[$ are countably additive. Then λ_0 has a unique extension to a measure defined on $\widehat{\bigotimes}_{i\in I}\Sigma_i$, which is a compact measure.

proof This follows immediately from 454D and 454A if we note that all the measures $\lambda_0 \pi_i^{-1}$ are necessarily compact, therefore perfect (451L).

454G Corollary Let $\langle X_i\rangle_{i\in I}$ be a family of sets, and Σ_i a σ-algebra of subsets of X_i for each $i \in I$. Suppose that for each finite set $J \subseteq I$ we are given a totally finite measure μ_J on $Z_J = \prod_{i\in J} X_i$ with domain $\widehat{\bigotimes}_{i\in J}\Sigma_i$ such that (i) whenever J, K are finite subsets of I and $J \subseteq K$, then the canonical projection from Z_K to Z_J is inverse-measure-preserving (ii) every marginal measure $\mu_{\{i\}}$ on $Z_{\{i\}} \cong X_i$ is perfect. Then there is a unique measure μ defined on $\widehat{\bigotimes}_{i\in I}\Sigma_i$ such that the canonical projection $\tilde\pi_J : \prod_{i\in I} X_i \to Z_J$ is inverse-measure-preserving for every finite $J \subseteq I$.

proof All we need to observe is that

$$\textstyle\bigotimes_{i\in I}\Sigma_i = \{\tilde\pi_J^{-1}[V] : J \in [I]^{<\omega},\ V \in \bigotimes_{i\in J}\Sigma_i\}.$$

Because all the canonical projections from X_K onto X_J are inverse-measure-preserving, we have $\mu_J V = \mu_K V'$ whenever J, K are finite subsets of I, $V \in \bigotimes_{i\in J}\Sigma_i$, $V' \in \bigotimes_{i\in K}\Sigma_i$ and $\tilde\pi_J^{-1}[V] = \tilde\pi_K^{-1}[V']$. So we have a functional $\lambda_0 : \bigotimes_{i\in I}\Sigma_i \to [0,\infty[$ such that $\lambda_0\tilde\pi_J^{-1}[V] = \mu_J V$ whenever $J \subseteq I$ is finite and $V \in \bigotimes_{i\in J}\Sigma_i$. It is easy to check that λ_0 is finitely additive and satisfies the conditions of 454D. So λ_0 can be extended to a measure μ defined on $\widehat{\bigotimes}_{i\in I}\Sigma_i$.

If $J \subseteq I$ is finite, then μ_J and $\mu\tilde\pi_J^{-1}$ agree on $\bigotimes_{i\in J}\Sigma_J$ and therefore (by the Monotone Class Theorem, 136C) on $\widehat{\bigotimes}_{i\in J}\Sigma_i$; that is, $\tilde\pi_J$ is inverse-measure-preserving. To see that μ itself is unique, observe that the conditions define its values on $\bigotimes_{i\in I}\Sigma_i$ and therefore on $\widehat{\bigotimes}_{i\in I}\Sigma_i$, by the Monotone Class Theorem once more.

454H Corollary Let $\langle(X_n,\Sigma_n)\rangle_{n\in\mathbb{N}}$ be a sequence of standard Borel spaces. For each $n \in \mathbb{N}$ set $Z_n = \prod_{i<n} X_i$, $\mathrm{T}_n = \widehat{\bigotimes}_{i<n}\Sigma_i$. (For $n = 0$, we have $Z_0 = \{\emptyset\}$, $\mathrm{T}_0 = \{\emptyset, Z_0\}$.) For $n \in \mathbb{N}$, $W \in \mathrm{T}_{n+1}$, $z \in Z_n$ write $W[\{z\}] = \{\xi : \xi \in X_n, (z,\xi) \in W\}$; set $X = \prod_{n\in\mathbb{N}} X_n$ and write $\tilde\pi_n$ for the canonical projection of X onto Z_n. Suppose that for each $n \in \mathbb{N}$ and $z \in Z_n$ we are given a probability measure ν_z on X_n with domain Σ_n such that $z \mapsto \nu_z(E)$ is T_n-measurable for every $E \in \Sigma_n$. Then there is a unique probability measure μ on $X = \prod_{n\in\mathbb{N}} X_n$, with domain $\Sigma = \widehat{\bigotimes}_{n\in\mathbb{N}}\Sigma_n$, such that, writing $\tilde\mu_n$ for the image measure $\mu\tilde\pi_n^{-1}$ on Z_n,

$$\tilde\mu_{n+1}(W) = \textstyle\int \nu_z W[\{z\}]\tilde\mu_n(dz)$$

for every $n \in \mathbb{N}$ and $W \in \mathrm{T}_{n+1}$, and

$$\textstyle\int f d\tilde\mu_{n+1} = \iint \dots \iint f(\xi_0,\dots,\xi_n)\nu_{(\xi_0,\dots,\xi_{n-1})}(d\xi_n)$$
$$\nu_{(\xi_0,\dots,\xi_{n-2})}(d\xi_{n-1})\dots\nu_{\xi_0}(d\xi_1)\nu_\emptyset(d\xi_0)$$

for every $n \in \mathbb{N}$ and $\tilde\mu_{n+1}$-integrable real-valued function f.

proof (a) We must check that the first formula given actually defines $\tilde\mu_{n+1}(W)$ for every $W \in \mathrm{T}_{n+1}$. Of course this is an induction on n. $\tilde\mu_0$ will be the unique probability measure on the singleton set Z_0. Given that $\operatorname{dom}\tilde\mu_n = \mathrm{T}_n$, the class $\mathcal{W}$ of sets $W \subseteq Z_{n+1}$ for which $\int \nu_z W[\{z\}]\tilde\mu_n(dz)$ is defined will contain all sets of the form $\prod_{i\le n} E_i$ where $E_i \in \Sigma_i$ for every $i \le n$, just because the function $z \mapsto \nu_z E_n$ is T_n-measurable. Since $\mathcal{W}$ is closed under increasing sequential unions and differences of comparable sets, the Monotone Class Theorem (136B) tells us that it includes the σ-algebra generated by the cylinder sets, which is T_{n+1}.

(b) Accordingly $\langle\nu'_z\rangle_{z\in Z_n}$ is a disintegration of $\tilde\mu_{n+1}$ over $\tilde\mu_n$, where ν'_z is the measure with domain T_{n+1} defined by writing $\nu'_z(W) = \nu_z W[\{z\}]$ for $W \in \mathrm{T}_{n+1}$, $z \in Z_n$. By 452F,

$$\begin{aligned}\int_{Z_{n+1}} f d\tilde\mu_{n+1} &= \int_{Z_n}\int_{Z_{n+1}} f(w)\nu'_z(dw)\tilde\mu_n(dz)\\ &= \int_{Z_n}\int_{X_n} f(z,\xi_{n+1})\nu_z(d\xi_{n+1})\tilde\mu_n(dz)\end{aligned}$$

for every $\tilde\mu_{n+1}$-integrable function f. (The second equality can be regarded as an application of the change-of-variable formula 235Ib applied to the (ν_z,ν'_z)-inverse-measure-preserving function $\xi \mapsto (z,\xi) : X_{n+1} \to Z_{n+1}$.) Now a direct induction yields the general formula for $\int f d\tilde\mu_{n+1}$ in the statement of this corollary.

(c) The canonical maps from Z_{n+1} to Z_n are all inverse-measure-preserving, just because every ν_z is a probability measure. We therefore have a well-defined functional $\lambda_0 : \bigotimes_{n\in\mathbb{N}} \Sigma_n \to [0,1]$ defined by setting $\lambda_0 \tilde{\pi}_n^{-1}[W] = \mu_n W$ whenever $n \in \mathbb{N}$ and $W \in \bigotimes_{i<n} \Sigma_i$, and this λ_0 is finitely additive; moreover, each marginal measure $\lambda_0 \pi_n^{-1}$, where $\pi_n : X \to X_n$ is the coordinate map, is countably additive, because it is expressible as an image measure of μ_{n+1} on X_n.

(d) Everything so far has been valid for any sequence $\langle (X_n, \Sigma_n)\rangle_{n\in\mathbb{N}}$ of sets with attached σ-algebras. But at this point we note that the marginal measures $\lambda_0 \pi_n^{-1}$ must be perfect, because (X_n, Σ_n) is a standard Borel space. So Theorem 454D gives the result.

454I Remarks In 454F and 454H the hypotheses call for 'standard Borel spaces' (X_i, Σ_i). As the proofs make clear, what is needed in each case is that 'every totally finite measure with domain Σ_i must be perfect'. We have already seen other ways in which this can be true: for instance, if X is any Radon Hausdorff space (§434), and Σ its Borel σ-algebra. Further examples are in 454Xd-454Xi and 454Yb-454Yc. Indeed, even weaker hypotheses can be fully adequate. In 454H, for instance, it will be quite enough if all the marginal measures on the factors X_n are perfect; in view of 454A and 451E, this will be so iff all the measures $\tilde{\mu}_n$ on the partial products Z_n are perfect. It may be difficult to be sure of this unless either we have some argument from the nature of the factor spaces (X_n, Σ_n), as suggested above, or a clear understanding of the marginal measures. In applications such as 455A below, however, there may be other approaches available.

454J Distributions of random processes For the next few paragraphs I shift to probabilists' notation.

Proposition Let (Ω, Σ, μ) be a probability space and $\langle X_i\rangle_{i\in I}$ a family of real-valued random variables on Ω (see §271).

(i) There is a unique complete probability measure ν on $\mathbb{R}^I$, measuring every Baire set and inner regular with respect to the zero sets, such that

$$\nu\{x : x \in \mathbb{R}^I, x(i_r) \le \alpha_r \text{ for every } r \le n) = \Pr(X_{i_r} \le \alpha_r \text{ for every } r \le n\}$$

whenever $i_0, \dots, i_n \in I$ and $\alpha_0, \dots, \alpha_n \in \mathbb{R}$.

(ii) If $i_0, \dots, i_n \in I$ and $\tilde{\pi}(x) = (x(i_0), \dots, x(i_n))$ for $x \in \mathbb{R}^I$, then the image measure $\nu\tilde{\pi}^{-1}$ on $\mathbb{R}^{n+1}$ is the joint distribution of $X_{i_0}, \dots, X_{i_n}$ as defined in 271C.

(iii) ν is a compact measure. If I is countable then ν is a Radon measure.

(iv) If every X_i is defined everywhere on Ω, then the function $\omega \mapsto \langle X_i(\omega)\rangle_{i\in I} : \Omega \to \mathbb{R}^I$ is inverse-measure-preserving for $\hat{\mu}$ and ν, where $\hat{\mu}$ is the completion of μ.

proof (a) Completing μ, and adjusting the X_i on negligible sets, does not change any of the joint distributions of families $X_{i_0}, \dots, X_{i_n}$ (271Ad), so we may suppose henceforth that μ is complete and that every X_i is defined on the whole of Ω. Set $\phi(\omega) = \langle X_i(\omega)\rangle_{i\in I}$ for $\omega \in \Omega$. Then $\{F : F \subseteq \mathbb{R}^I, \phi^{-1}[F] \in \Sigma\}$ is a σ-algebra of subsets of $\mathbb{R}^I$ containing $\{x : x(i) \le \alpha\}$ whenever $i \in I$ and $\alpha \in \mathbb{R}$, so includes the Baire σ-algebra $\mathcal{Ba}(\mathbb{R}^I)$ of $\mathbb{R}^I$ (4A3Na). If we define $\nu_0 F = \mu\phi^{-1}[F]$ for $F \in \mathcal{Ba}(\mathbb{R}^I)$, ν_0 is a Baire measure on $\mathbb{R}^I$ for which ϕ is inverse-measure-preserving. We are supposing that μ is complete, so ϕ is still inverse-measure-preserving for μ and the completion ν of ν_0 (235Hc). Since ν_0 is inner regular with respect to the zero sets (412D), so is ν (412Ha), and of course ν measures every Baire set. By 454B, ν_0 is compact, so ν also is (451Ga).

(b) If $i_0, \dots, i_n \in I$ and $\alpha_0, \dots, \alpha_n \in \mathbb{R}$, then

$$\begin{aligned}\Pr(X_{i_r} \le \alpha_r \text{ for every } r \le n) &= \mu\{\omega : X_{i_r}(\omega) \le \alpha_r \text{ for } r \le n\}\\ &= \mu\{\omega : \phi(x)(i_r) \le \alpha_r \text{ for } r \le n\}\\ &= \nu\{x : x(i_r) \le \alpha_r \text{ for } r \le n\}.\end{aligned}$$

(c) If $i_0, \dots, i_n \in I$ and we set $\tilde{\pi}(x) = (x(i_0), \dots, x(i_n))$ for $x \in \mathbb{R}^I$, then $\nu\tilde{\pi}^{-1}$ is a Radon measure (451N). Since it agrees with the distribution of $X_{i_0}, \dots, X_{i_n}$ on all sets of the form $\{z : z(r) \le \alpha_r \text{ for } r \le n\}$, it must be exactly the distribution of $X_{i_0}, \dots, X_{i_n}$ (271Ba).

(d) If I is countable, then $\mathbb{R}^I$ is Polish, so ν is a Radon measure (433Cb).

(e) The only point I have not covered is the uniqueness of ν. But suppose that ν' is another measure on $\mathbb{R}^I$ with the properties described in (i). If $i_0, \dots, i_n \in I$ and $\tilde{\pi}(x) = (x(i_0), \dots, x(i_n))$ for $x \in \mathbb{R}^I$, then the image measures $\nu\tilde{\pi}^{-1}$ and $\nu'\tilde{\pi}^{-1}$ on $\mathbb{R}^{n+1}$ are both the distribution of $X_{i_0}, \dots, X_{i_n}$, by the argument of (c) above. This means that ν and ν' agree on the algebra of subsets of $\mathbb{R}^I$ generated by sets of the form $\{x : x(i) \in E\}$ where $i \in I$ and $E \subseteq \mathbb{R}$ is Borel. By 454D, they agree on all zero sets, and must be equal (412L).

454K Definition In the context of 454J, I will call ν the (**joint**) **distribution** of the process $\langle X_i \rangle_{i \in I}$.

Note that if $I = n \in \mathbb{N} \setminus \{0\}$, then ν is a Radon measure on $\mathbb{R}^n$, so is the distribution of $\langle X_i \rangle_{i<n}$ in the sense of 271C.

454L Proposition Let (Ω, Σ, μ) be a probability space and $\langle X_i \rangle_{i \in I}$ a family of random variables on Ω, with distribution ν. If $\langle i_n \rangle_{n \in \mathbb{N}}$ is a sequence in I and $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a Borel measurable function, then the random variables

$$\omega \mapsto f(\langle X_{i_n}(\omega) \rangle_{n \in \mathbb{N}}) : \textstyle\bigcap_{n \in \mathbb{N}} \operatorname{dom} X_{i_n} \to \mathbb{R}, \quad x \mapsto f(\langle x(i_n) \rangle_{n \in \mathbb{N}}) : \mathbb{R}^I \to \mathbb{R}$$

have the same distribution.

proof Completing μ and extending each X_i to the whole of Ω, we may suppose that $\phi : \Omega \to \mathbb{R}^I$ is inverse-measure-preserving, where $\phi(\omega) = \langle X_i(\omega) \rangle_{i \in I}$ for $\omega \in \Omega$. Now, for any $\alpha \in \mathbb{R}$,

$$\begin{aligned} \nu\{x : f(\langle x_(i_n) \rangle_{n \in \mathbb{N}}) \leq \alpha\} &= \mu\{\omega : f(\langle \phi(\omega)(i_n) \rangle_{n \in \mathbb{N}}) \leq \alpha\} \\ &= \mu\{\omega : f(\langle X_{i_n}(\omega) \rangle_{n \in \mathbb{N}}) \leq \alpha\}, \end{aligned}$$

so the distributions are the same.

454M Theorem Let I be a set.

(a) Let ν and ν' be Baire probability measures on $\mathbb{R}^I$ such that $\int e^{if(x)}\nu(dx) = \int e^{if(x)}\nu'(dx)$ for every continuous linear functional $f : \mathbb{R}^I \to \mathbb{R}$. Then $\nu = \nu'$.

(b) Let $\langle X_j \rangle_{j \in I}$ and $\langle Y_j \rangle_{j \in I}$ be two families of random variables such that

$$\mathbb{E}(\exp(i \textstyle\sum_{r=0}^n \alpha_r X_{j_r})) = \mathbb{E}(\exp(i \sum_{r=0}^n \alpha_r Y_{j_r}))$$

whenever $j_0, \dots, j_n \in I$ and $\alpha_0, \dots, \alpha_n \in \mathbb{R}$. Then $\langle X_j \rangle_{j \in I}$ and $\langle Y_j \rangle_{j \in I}$ have the same distribution.

proof (a) For each finite set $J \subseteq I$, write $\tilde{\pi}_J(x) = x \restriction J$ for $x \in X$. Then we have Radon measures μ_J and μ'_J on $\mathbb{R}^J$ defined by saying that $\mu_J F = \nu\tilde{\pi}_J^{-1}[F]$, $\mu'_J F = \nu'\tilde{\pi}_J^{-1}[F]$ for Borel sets $F \subseteq \mathbb{R}^J$. If $\langle \alpha_j \rangle_{j \in J} \in \mathbb{R}^J$, then

$$\begin{aligned} \int \exp(i \sum_{j \in J} \alpha_j z(j)) \mu_J(dz) &= \int \exp(i \sum_{j \in J} \alpha_j x(j)) \nu(dx) \\ &= \int \exp(i \sum_{j \in J} \alpha_j x(j)) \nu'(dx) = \int \exp(i \sum_{j \in J} \alpha_j z(j)) \mu'_J(dz), \end{aligned}$$

so μ_J and μ'_J have the same characteristic function, therefore are equal. This is true for every J, so ν and ν' are equal, by 454D.

(b) Taking ν and ν' to be the two distributions, (a) (with 454L) tells us that their restrictions to the Baire σ-algebra of $\mathbb{R}^I$ are the same, so they must be identical.

454N Continuous processes The original, and still by far the most important, context for 454D is when every (X_i, Σ_i) is $\mathbb{R}$ with its Borel σ-algebra, so that $X = \prod_{i \in I} X_i$ can be identified with $\mathbb{R}^I$. In the discussion so far, the set I has been an abstract set, except in the very special case of 454H. But some of the most important applications (to which I shall come in the next section) involve index sets carrying a topological structure; for instance, I could be the unit interval $[0, 1]$ or the half-line $[0, \infty[$. In such a case, we have a wide variety of subspaces of $\mathbb{R}^I$ (for instance, the space of continuous functions) marked out as

special, and it is important to know when, and in what sense, our measures on the product space $\mathbb{R}^I$ can be regarded as, or replaced by, measures on the subspace of interest. In the next few paragraphs I look briefly at spaces of continuous functions on Polish spaces.

Lemma Let T be a separable metrizable space and (X, Σ, μ) a semi-finite measure space. Let $\mathfrak{T}$ be a topology on X such that μ is inner regular with respect to the closed sets.

(a) Let $\phi : X \times T \to \mathbb{R}$ be a function such that (i) for each $x \in X$, $t \mapsto \phi(x,t)$ is continuous (ii) for each $t \in T$, $x \mapsto \phi(x,t)$ is Σ-measurable. Then μ is inner regular with respect to $\mathcal{K} = \{K : K \subseteq X, \phi\restriction K \times T \text{ is continuous}\}$.

(b) Let $\theta : X \to C(T)$ be a function such that $x \mapsto \theta(x)(t)$ is Σ-measurable for every $t \in T$. Give $C(T)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact subsets of T. Then θ is almost continuous.

proof (a) Take $E \in \Sigma$ and $\gamma < \mu E$; take $F \in \Sigma$ such that $F \subseteq E$ and $\gamma < \mu F < \infty$. Let $\mathcal{U}$ be a countable base for the topology of T consisting of non-empty sets, D a countable dense subset of T and $\mathcal{V}$ a countable base for the topology of $\mathbb{R}$. For $U \in \mathcal{U}$, $V \in \mathcal{V}$ set

$$E_{UV} = \{x : \phi(x,t) \in V \text{ for every } t \in U \cap D\};$$

then $E_{UV} \in \Sigma$. Let $\langle \epsilon_{UV} \rangle_{U \in \mathcal{U}, V \in \mathcal{V}}$ be a family of strictly positive numbers with sum at most $\mu F - \gamma$. For each $U \in \mathcal{U}$, $V \in \mathcal{V}$ take a closed set $F_{UV} \subseteq F \setminus E_{UV}$ such that $\mu F_{UV} \geq \mu(F \setminus E_{UV}) - \epsilon_{UV}$. Consider

$$K = \bigcap_{U \in \mathcal{U}, V \in \mathcal{V}} F_{UV} \cup (F \cap E_{UV}).$$

Then $K \subseteq E$ and $\mu K \geq \gamma$.

If $x \in K$, $t \in T$ and $\phi(x,t) \in V_0 \in \mathcal{V}$, let $V \in \mathcal{V}$ be such that $\phi(x,t) \in V$ and $\overline{V} \subseteq V_0$. Then $\{t' : \phi(x,t') \in V\}$ is an open set containing t, so there is some $U \in \mathcal{U}$ such that $t \in U$ and $\phi(x,t') \in V$ for every $t' \in U$. This means that $x \in E_{UV}$, so that $(K \setminus F_{UV}) \times U$ contains (x,t), and is a relatively open set in $K \times T$. If $(x',t') \in (K \setminus F_{UV}) \times U$, then $x' \in E_{UV}$, so $\phi(x',t'') \in V$ whenever $t'' \in U \cap D$; as D is dense, $\phi(x',t'') \in \overline{V}$ whenever $t'' \in U$; in particular, $\phi(x',t') \in \overline{V} \subseteq V_0$. This shows that $(K \times T) \cap \phi^{-1}[V_0]$ is relatively open in $K \times T$; as V_0 is arbitrary, $\phi\restriction K \times T$ is continuous.

So $K \in \mathcal{K}$. As E and γ are arbitrary, μ is inner regular with respect to $\mathcal{K}$.

(b) Set $\phi(x,t) = \theta(x)(t)$ for $x \in X$, $t \in T$. Because $\theta(x) \in C(T)$ for every x, ϕ is continuous in the second variable; and the hypothesis on θ is just that ϕ is measurable in the first variable. So μ is inner regular with respect to $\mathcal{K}$ as described in (a). But $\theta\restriction K$ is continuous for every $K \in \mathcal{K}$, by 4A2G(g-ii). So θ is almost continuous.

454O Proposition Let T be an analytic metrizable space (e.g., a Polish space, or any Souslin-F subset of a Polish space), and μ a probability measure on $C(T)$ with domain the σ-algebra Σ generated by the evaluation functionals $f \mapsto f(t) : C(T) \to \mathbb{R}$ for $t \in T$. Give $C(T)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact subsets of T. Then the completion of μ is a $\mathfrak{T}_c$-Radon measure.

proof If T is empty this is trivial, so let us suppose henceforth that $T \neq \emptyset$.

(a) Let D be a countable dense subset of T. Let $\pi : C(T) \to \mathbb{R}^D$ be the restriction map. Set $X = \pi[C(T)] \subseteq \mathbb{R}^D$; then X, with the topology it inherits from $\mathbb{R}^D$, is a separable metrizable space. Note that, because D is dense, π is injective.

We need to know that π is an isomorphism between $(C(T), \Sigma)$ and $(X, \mathcal{B})$, where $\mathcal{B}$ is the Borel σ-algebra of X. **P** Since the Borel σ-algebra of $\mathbb{R}^D$ is just the σ-algebra generated by the functionals $g \mapsto g(t) : \mathbb{R}^D \to \mathbb{R}$ as t runs over D (4A3Dc/4A3E), $\mathcal{B}$ is the σ-algebra of subsets of X generated by the functionals $g \mapsto g(t) : X \to \mathbb{R}$ for $t \in D$. So π is surely $(\Sigma, \mathcal{B})$-measurable. On the other hand, if $t \in X$, there is a sequence $\langle t_n \rangle_{n \in \mathbb{N}}$ in D converging to t, so that $\pi^{-1}(g)(t) = \lim_{n \to \infty} g(t_n)$ for every $g \in X$, and $g \mapsto \pi^{-1}(g)(t) : X \to \mathbb{R}$ is $\mathcal{B}$-measurable. Accordingly π^{-1} is $(\mathcal{B}, \Sigma)$-measurable. **Q**

(b) Let $\langle U_n \rangle_{n \in \mathbb{N}}$ be a sequence running over a base for the topology of T, with no U_n empty. For each $n \in \mathbb{N}$, $g \in \mathbb{R}^D$ set

$$\omega_n(g) = \sup_{t,u \in U_n \cap D} \min(1, g(t) - g(u)),$$

so that $\omega_n : \mathbb{R}^D \to [0,1]$ is T-measurable, where T is the Borel (or Baire) algebra of $\mathbb{R}^D$. For $g \in \mathbb{R}^D$, $g \in X$ iff g has an extension to a continuous function on T, that is,

for every $t \in T$, $k \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that $t \in U_n$ and $\omega_n(g) \leq 2^{-k}$.

Turning this round, $\mathbb{R}^D \setminus X$ is the projection onto the first coordinate of the set

$$Q = \bigcup_{k\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \{(g,t)\text{: either } t \notin U_n \text{ or } \omega_n(g) > 2^{-k}\} \subseteq \mathbb{R}^D \times T.$$

But (because every U_n is an open set and every ω_n is Borel measurable) Q is a Borel set in the analytic space $\mathbb{R}^D \times T$. So Q is analytic (423Eb) and $\mathbb{R}^D \setminus X$ is analytic (423Bb). Since $\mathbb{R}^D$, being Polish (4A2Qc), is a Radon space (434Kb), X is a Radon space (434F).

(c) The image measure $\nu = \mu\pi^{-1}$ on X is a Borel probability measure. Because X is a Radon space, ν is tight, and its completion $\hat{\nu}$ is a Radon measure.

By 454Nb, $\pi^{-1} : X \to C(T)$ is almost continuous if we give $C(T)$ the topology $\mathfrak{T}_c$. So the image measure $\lambda = \hat{\nu}(\pi^{-1})^{-1}$ is a Radon measure for $\mathfrak{T}_c$ (418I). But of course λ is the completion of μ, just because π is a bijection and $\hat{\nu}$ is the completion of ν.

454P Corollary Let T be an analytic metrizable space.

(a) $C(T)$, with either the topology $\mathfrak{T}_p$ of uniform convergence on finite subsets of T or the topology $\mathfrak{T}_c$ of uniform convergence on compact subsets of T, is a measure-compact Radon space.

(b) Let μ be a Baire probability measure on $\mathbb{R}^T$ such that $\mu^* C(T) = 1$. Then μ is τ-additive, so has a unique extension $\tilde{\mu}$ which is a Radon measure on $\mathbb{R}^T$. $\tilde{\mu}C(T) = 1$, and the subspace measure on $C(T)$ induced by $\tilde{\mu}$ is a Radon measure if $C(T)$ is given the topology $\mathfrak{T}_c$.

proof (a) Let μ be a probability measure on $C(T)$ which is either a Baire measure or a Borel measure with respect to either $\mathfrak{T}_p$ or $\mathfrak{T}_c$. Let $\tilde{\mu}$ be the completion of $\mu\restriction\Sigma$, where Σ is the σ-algebra generated by the functionals $f \mapsto f(t)$; because all these are $\mathfrak{T}_p$-continuous, Σ is certainly included in the Baire σ-algebra for $\mathfrak{T}_p$, so that $\Sigma \subseteq \operatorname{dom}\mu$. 454O tells us that $\tilde{\mu}$ is a Radon measure for $\mathfrak{T}_c$. Because $\mathfrak{T}_p$ is a coarser Hausdorff topology, $\tilde{\mu}$ is also a Radon measure for $\mathfrak{T}_p$. Also $\tilde{\mu}$ must extend μ, because its domain includes that of μ and the completion of μ must extend $\tilde{\mu}$ (in fact, of course, this means that $\tilde{\mu}$ is actually the completion of μ). Now $\tilde{\mu}$ is τ-additive (for either topology), so μ also is; as μ is arbitrary, $C(T)$ is measure-compact (for either topology). On the other hand, if μ is a Borel measure (for either topology), it must be tight for that topology; so that $C(T)$ is a Radon space.

(b) Write μ_C for the subspace measure on $C(T)$. Recall that the domain Σ of μ is just the σ-algebra generated by the functionals $f \mapsto f(t) : \mathbb{R}^T \to \mathbb{R}$, as t runs over T (4A3Na), so that the domain Σ_C of μ_C is the σ-algebra of subsets of $C(T)$ generated by the functionals $f \mapsto f(t) : C(T) \to \mathbb{R}$. By 454O, the completion $\hat{\mu}_C$ of μ_C is a Radon measure on $C(T)$ if we give $C(T)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact subsets of T.

The embedding $C(T) \subseteq \mathbb{R}^T$ is of course continuous for $\mathfrak{T}_c$ and the product topology on $\mathbb{R}^T$, so we have a Radon image measure $\tilde{\mu}$ on $\mathbb{R}^T$ defined by saying that $\tilde{\mu}E = \hat{\mu}_C(E \cap C(T))$ whenever $E \cap C(T)$ is measured by $\hat{\mu}_C$. If $E \in \Sigma$, then

$$\tilde{\mu}E = \hat{\mu}_C(E \cap C(T)) = \mu_C(E \cap C(T)) = \mu^*(E \cap C(T)) = \mu E$$

because $\mu^* C(T) = 1$, so $\tilde{\mu}$ extends μ. Of course $\tilde{\mu}C(T) = 1$ and the subspace measure on $C(T)$ induced by $\tilde{\mu}$ is just $\hat{\mu}_C$, so is a Radon measure with respect to $\mathfrak{T}_c$.

Finally, because μ has an extension to a Radon measure, it must itself be τ-additive. Because Σ includes a base for the topology of $\mathbb{R}^T$, μ can have only one extension to a Radon measure on $\mathbb{R}^T$ (415H(iv)).

454X Basic exercises >(a) Let μ be Lebesgue measure on $[0,1]$. Let $X_0 \subseteq [0,1]$ be a set of inner measure zero and outer measure 1. Set $X_1 = [0,1] \setminus X_0$; for each i, let μ_i be the subspace measure on X_i and Σ_i its domain. Show that we have a finitely additive functional λ defined on $\Sigma_0 \otimes \Sigma_1$ by the formula

$$\lambda((E \cap X_0) \times (F \cap X_1)) = \mu(E \cap F) \text{ for all } E,\ F \in \operatorname{dom}\mu,$$

and that λ has no extension to a measure on $X_0 \times X_1$.

(b) Adapt the example of 419K to provide a counter-example for 454G if we omit the hypothesis that the marginal measures $\mu_{\{i\}}$ must be perfect.

(c) Adapt the example of 419K/454Xb to provide a counter-example for 454H if we omit the hypothesis that the (X_n, Σ_n) must be standard Borel spaces. (*Hint*: if $z \in \prod_{i\leq n} X_i$, try $\nu_z(E) = 1$ if $z(n) \in E$, 0 otherwise.)

>(d) Let X be a set and Σ a σ-algebra of subsets of X. Let us say that (X, Σ) has the **perfect measure property** if every totally finite measure with domain Σ is perfect. Show that (i) if (X, Σ) has the perfect measure property, so does (E, Σ_E) for any $E \in \Sigma$, where Σ_E is the subspace σ-algebra on E (ii) if $\langle (X_i, \Sigma_i)\rangle_{i\in I}$ is a family of spaces with the perfect measure property, then $(\prod_{i\in I} X_i, \widehat{\bigotimes}_{i\in I}\Sigma_i)$ has the perfect measure property.

(e) Let X be a set and Σ a σ-algebra of subsets of X. Let us say that (X, Σ) has the **countably compact measure property** if every totally finite measure with domain Σ is countably compact. Show that (i) if (X, Σ) has the countably compact measure property it has the perfect measure property (ii) if (X, Σ) has the countably compact measure property so does (E, Σ_E) for every $E \in \Sigma$, where Σ_E is the subspace σ-algebra on E (iii) if $\langle (X_i, \Sigma_i)\rangle_{i\in I}$ is a family of spaces with the countably compact measure property, then $(\prod_{i\in I} X_i, \widehat{\bigotimes}_{i\in I}\Sigma_i)$ has the countably compact measure property.

(f) Suppose that (X, Σ) has the countably compact measure property. (i) Let μ be a totally finite measure with domain Σ, (Y, T, ν) a measure space, and $f : X \to Y$ an inverse-measure-preserving function. Show that μ has a disintegration $\langle \mu_y\rangle_{y\in Y}$ over ν which is consistent with f. (ii) Let Y be any set, T any σ-algebra of subsets of Y, and λ a probability measure with domain $\Sigma\widehat{\otimes}\mathrm{T}$. Show that there is a family $\langle \mu_y\rangle_{y\in Y}$ of probability measures on X such that $\lambda W = \int \mu_y W^{-1}[\{y\}]\nu(dy)$ for every $W \in \Sigma\widehat{\otimes}\mathrm{T}$, where ν is the marginal measure of λ on Y. (*Hint*: 452M.)

(g) (i) Let X be any set, and Σ the countable-cocountable algebra on X. Show that (X, Σ) has the countably compact measure property. (ii) Show that any standard Borel space has the countably compact measure property.

(h) Let (X, Σ) be a space with the perfect measure property, and T the smallest σ-algebra including Σ and closed under Souslin's operation. Show that (X, T) has the perfect measure property.

(i) Let X be a Radon Hausdorff space, and Σ the algebra of universally measurable sets in X (434D). Show that (X, Σ) has the countably compact measure property.

>(j) Let $\langle X_i\rangle_{i\in I}$ be a family of real-valued random variables. Show that $\langle X_i\rangle_{i\in I}$ is independent iff its distribution is the product of the distributions of the X_i, and is then a quasi-Radon measure on $\mathbb{R}^I$. (*Hint*: 272G, 415E.)

(k) Let I be a set and ν, ν' two quasi-Radon measures on $\mathbb{R}^I$ such that $\int e^{if(x)}\nu(dx) = \int e^{if(x)}\nu'(dx)$ for every continuous linear functional $f : \mathbb{R}^I \to \mathbb{R}$. Show that $\nu = \nu'$.

>(l) Let Σ be the σ-algebra of subsets of $C([0, \infty[)$ generated by the functionals $f \mapsto f(t)$ for $t \geq 0$. Give $C([0, \infty[)$ the topology $\mathfrak{T}_c$ of uniform convergence on compact sets. (i) Show that $\mathfrak{T}_c$ is Polish, and that $\Sigma \cap \mathfrak{T}_c$ is a base for $\mathfrak{T}_c$ which generates Σ as σ-algebra. (ii) Use this to give a quick proof of 454O in this case.

(m) Let T be a Polish space, and $\mathfrak{T}_c$ the topology on $C(T)$ of uniform convergence on compact sets. Show that if $\mathfrak{T}$ is any Hausdorff topology on $C(T)$, coarser than $\mathfrak{T}_c$, such that all the functionals $f \mapsto f(t)$, for $t \in T$, are Baire measurable for $\mathfrak{T}$, then $(C(T), \mathfrak{T})$ is a measure-compact Radon space.

454Y Further exercises (a) In 454A, show that μ is weakly α-favourable (definition: 451U) if every μ_i is.

(b) Let Σ be the algebra of Lebesgue measurable subsets of $\mathbb{R}$. Show that $(\mathbb{R}, \Sigma)$ has the perfect measure property (454Xd) iff $\mathfrak{c}$ is measure-free.

(c) Let Σ be the Borel σ-algebra of ω_1 with its order topology. Show that (ω_1, Σ) has the perfect measure property. (*Hint*: 439Yf.)

(d) Let (X, Σ, μ) be a semi-finite measure space with a topology such that μ is inner regular with respect to the closed sets, T a second-countable space and Y a separable metrizable space. Suppose that $\phi : X \times T \to Y$ is continuous in the second variable and measurable in the first, as in 454N. Show that μ is inner regular with respect to $\mathcal{K} = \{K : K \subseteq X, \phi{\restriction}K \times T \text{ is continuous}\}$.

454 Notes and comments 454A generalizes Theorem 451J, which gave the same result (with essentially the same proof) for product measures. One of the themes of this section is the idea that we can deduce properties of measures on product spaces from properties of their marginal measures, that is, the image measures on the factors. The essence of 'compactness', 'countable compactness' and 'perfectness' is that we can find enough points in the measure space to do what we want. (See, for instance, the characterization of compactness in 343B, or Pachl's characterization of countable compactness in 452Ye.) Since the canonical feature of a product space is that we put in every point the Axiom of Choice provides us with, it's perhaps not surprising that such properties can be inherited by measures on product spaces.

Theorems 454C and 454D can be regarded as further variations on the same theme. A finitely additive non-negative functional on an algebra of sets will have an extension to a measure if, and only if, it is sequentially smooth in the sense that the measures of a decreasing sequence of sets with empty intersection converge to zero (413K). If we have a decreasing sequence of sets, with measures bounded away from zero, but with empty intersection, one interpretation of the phenomenon is that some points which ought to have been present got left out of the sets. What 454D tells us is that perfectness (and countable additivity) of the marginal measures is enough to ensure that there are enough points in the product to stop this happening. In effect, 454C tells us that it will be enough if every marginal but one is perfect.

These results are of course associated with the projective limit constructions in 418M-418Q. In the theorems there we had Radon measures, so that they were actually compact rather than perfect; in return for the stronger hypothesis on the measures, we could handle projective limits corresponding to rather small subsets of the product spaces (see the formulae in 418O-418Q). Just as in §418, the patterns change when we have countable rather than uncountable families to deal with (418P-418Q, 454H).

In 454J-454M, I insist rather arbitrarily that 'the' joint distribution of a family $\langle X_i \rangle_{i \in I}$ of real-valued random variables is the completion of a Baire measure on $\mathbb{R}^I$. Of course all the ideas can also be expressed in terms of the Baire measure itself, but I have sought a formulation which is consistent with the rules set out in §271. When I is countable, we get a Radon measure (454J(iii)), as in the finite-dimensional case. There are other cases in which the distribution is a quasi-Radon measure (454Xj). As always, we can ask whether the distribution is τ-additive; in this case it will have a canonical extension to a quasi-Radon measure (415N). Important examples of this phenomenon are described in 455D and 456K. Because $\mathbb{R}^I$ has a linear topological space structure, we have a notion of 'characteristic function' for any probability measure on $\mathbb{R}^I$ measuring the zero sets, and the characteristic function of a Baire measure determines that measure (454M, 454Xk).

In 454O, $C(T)$, with $\mathfrak{T}_c$, has a countable network (4A2Oe), so the subspace measure μ_C induced by μ on $C(T)$ must be a τ-additive topological measure with respect to $\mathfrak{T}_c$ (414Xc) and has a unique extension to a quasi-Radon measure on $C(T)$ (415M). The hard bit is the next step, showing that $C(T)$, under $\mathfrak{T}_c$, is a Radon space; this is the real point of 454N-454O. For the most important case, in which $T = [0, \infty[$, we have a useful simplification, because $\mathfrak{T}_c$ is actually Polish (454Xl). Even in this case, however, we need to observe that the measure we are seeking is a little more complicated than a simple completion of a measure on $\mathbb{R}^T$. We must complete the *subspace* measure on $C(T)$, and $C(T)$ is far from being a measurable set. The measure $\tilde{\mu}$ of 454P will not as a rule be completion regular, for instance. Spaces of continuous functions are so important that it is worth noticing that the results here will be valid for various topologies on $C(T)$ (454Xm).

455 Markov processes and Brownian motion

For a 'Markov' process, in which the evolution of the system after a time t depends only on the state at time t, the general theory of §454 leads to a straightforward existence theorem (at least for random variables taking values in standard Borel spaces) dependent only on a natural consistency condition on the conditional distributions (455A). I apply this to the most important example, Brownian motion, in 455C.

455A Theorem Let T be a totally ordered set with least element a, and for each $t \in T$ let (X_t, Σ_t) be a standard Borel space. Set $X = \prod_{t\in T} X_t$ and for each $t \in T$ let $\pi_t : X \to X_t$ be the coordinate map. Let μ_a be a probability measure with domain Σ_a. Suppose that we are given, for each pair $s < t$ in T, a family $\langle \nu_x^{(s,t)}\rangle_{x\in X_s}$ of probability measures on X_t all with domain Σ_t, and suppose that

(i) whenever $s < t$ in T and $E \in \Sigma_t$, $x \mapsto \nu_x^{(s,t)}(E)$ is Σ_s-measurable,

(ii) whenever $s < t < u$ in T, $x \in X_s$ and $E \in \Sigma_u$, then

$$\nu_x^{(s,u)}(E) = \int \nu_y^{(t,u)}(E)\nu_x^{(s,t)}(dy).$$

Then there is a unique probability measure μ on X, with domain $\widehat{\bigotimes}_{t\in T}\Sigma_t$, such that μ_a is the image measure $\mu\pi_a^{-1}$ and, writing μ_t for the image measure on X_t, $\mu_t(E) = \int \nu_x^{(s,t)}(E)\mu_s(dx)$ for every $E \in \Sigma_t$ whenever $s < t$ in T. Moreover, if $J \subseteq T$ is a finite set containing a, and we write $\tilde\pi_J$ for the canonical map from X onto $Z_J = \prod_{t\in J} X_t$ and $\tilde\mu_J$ for the image measure $\mu\tilde\pi_J^{-1}$, then for any $\tilde\mu_J$-integrable real-valued function f we have

$$\int f d\tilde\mu_J = \iint \dots \iint f(x_0, \dots, x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)$$
$$\nu_{x_{n-2}}^{(t_{n-2},t_{n-1})}(dx_{n-1}) \dots \nu_{x_0}^{(a,t_1)}(dx_1)\mu_a(dx_0)$$

where $t_0 = a, t_1, \dots, t_n$ are the elements of J written in ascending order.

proof (a) For each finite set $I \subseteq T$, write $\mathrm{T}_I = \widehat{\bigotimes}_{t\in I}\Sigma_t$. If $I = \{t_0, t_1, \dots, t_n\}$ is a finite subset of T with $a = t_0 < t_1 < \dots < t_n$, then we have a probability measure λ_I on Z_I with domain T_I such that

$$\int f d\lambda_I = \iint \dots \iint f(x_0, \dots, x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)$$
$$\nu_{x_{n-2}}^{(t_{n-2},t_{n-1})}(dx_{n-1}) \dots \nu_{x_0}^{(a,t_1)}(dx_1)\mu_a(dx_0)$$

for every λ_I-integrable function f. **P** Use 454H on the finite sequence $(X_{t_0}, \dots, X_{t_n})$. The measures ν_z required by 454H must be constructed by the rule

$$\nu_z = \nu_{z(t_m)}^{(t_m,t_{m+1})}$$

for $m < n$, $z \in \prod_{i\le m} X_{t_i}$, while of course $\nu_\emptyset = \mu_a$. (Having a finite sequence rather than an infinite one clearly makes things easier; we can stop the argument at the end of part (b) of the proof of 454H.) **Q**

(b) Of course the point of this is that these measures λ_I form a consistent family; if $a \in I \subseteq J \in [T]^{<\omega}$, then the canonical projection $\pi_{IJ} : Z_J \to Z_I$ is inverse-measure-preserving. **P** It is enough to consider the case in which J has just one more point than I, since then we can induce on $\#(J \setminus I)$. In this case, express J as $\{t_0, \dots, t_n\}$ where $a = t_0 < \dots < t_n$, and suppose that $I = J \setminus \{t_m\}$. If $W \in \mathrm{T}_I$, then

$$\begin{aligned}\lambda_J\pi_{IJ}^{-1}[W] &= \int \dots \iint \dots \int \chi W(x_0, \dots, x_{m-1}, x_{m+1}, \dots, x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\\ &\qquad \dots \nu_{x_m}^{(t_m,t_{m+1})}(dx_{m+1})\nu_{x_{m-1}}^{(t_{m-1},t_m)}(dx_m) \dots \mu_a(dx_0)\\ &= \int \dots \iint g_{(x_0,\dots,x_{m-1})}(x_{m+1})\nu_{x_m}^{(t_m,t_{m+1})}(dx_{m+1}) \qquad (*)\\ &\qquad \nu_{x_{m-1}}^{(t_{m-1},t_m)}(dx_m) \dots \mu_a(dx_0)\end{aligned}$$

where

$$g_{(x_0,\dots,x_{m-1})}(x_{m+1}) = \int \dots \int \chi W(x_0,\dots,x_{m-1},x_{m+1},\dots,x_n) \\ \nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\dots\nu_{x_{m+1}}^{(t_{m+1},t_{m+2})}(dx_{m+2}).$$

Here, of course, we use the hypothesis (ii); since

$$\nu_{x_{m-1}}^{(t_{m-1},t_{m+1})}(E) = \int \nu_{x_m}^{(t_m,t_{m+1})}(E)\nu_{x_{m-1}}^{(t_{m-1},t_m)}(dx_m)$$

for every $E \in \Sigma_{t_{m+1}}$, and since $g_{(x_0,\dots,x_{m-1})}$ is bounded and $\nu_{x_{m-1}}^{(t_{m-1},t_{m+1})}$-integrable,

$$\int g_{(x_0,\dots,x_{m-1})}(x_{m+1})\nu_{x_{m-1}}^{(t_{m-1},t_{m+1})}(dx_{m+1}) \\ = \iint g_{(x_0,\dots,x_{m-1})}(x_{m+1})\nu_{x_m}^{(t_m,t_{m+1})}(dx_{m+1})\nu_{x_{m-1}}^{(t_{m-1},t_m)}(dx_m)$$

(452F). Substituting this into (*) above,

$$\begin{aligned}\lambda_J\pi_{IJ}^{-1}[W] &= \int \dots \iint g_{(x_0,\dots,x_{m-1})}(x_{m+1}) \\ &\qquad \nu_{x_m}^{(t_m,t_{m+1})}(dx_{m+1})\nu_{x_{m-1}}^{(t_{m-1},t_m)}(dx_m)\dots\mu_a(dx_0) \\ &= \int \dots \int g_{(x_0,\dots,x_{m-1})}(x_{m+1})\nu_{x_{m-1}}^{(t_{m-1},t_{m+1})}(dx_{m+1})\dots\mu_a(dx_0) \\ &= \int \dots \int \dots \int \chi W(x_0,\dots,x_{m-1},x_{m+1},\dots,x_n) \\ &\qquad \nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\dots\nu_{x_{m-1}}^{(t_{m-1},t_{m+1})}(dx_{m+1})\dots\mu_a(dx_0) \\ &= \lambda_I W,\end{aligned}$$

applying the formula in (a) again. **Q**

(Some of the formulae here are inappropriate if $m = n$. In this case, of course,

$$\begin{aligned}\lambda_J\pi_{IJ}^{-1}[W] &= \int \dots \int \chi W(x_0,\dots,x_{n-1})\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\dots\mu_a(dx_0) \\ &= \int \dots \int \chi W(x_0,\dots,x_{n-1})\nu_{x_{n-2}}^{(t_{n-2},t_{n-1})}(dx_{n-1})\dots\mu_a(dx_0) \\ &= \lambda_I W.)\end{aligned}$$

(c) We also see from the formulae in (a) that if $J \subseteq T$ is of the form $\{t_0,\dots,t_n\}$ with $n \geq 1$ and $a = t_0 < \dots < t_n$, then for any λ_J-integrable function f

$$\int f d\lambda_J = \iint \dots \iint f(x_0,\dots,x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n) \\ \nu_{x_{n-2}}^{(t_{n-2},t_{n-1}}(dx_{n-1})\dots\nu_{x_0}^{(a,t_1)}(dx_1)\mu_a(dx_0) \\ = \iint f(x_0,\dots,x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\lambda_I(d(x_0,\dots,x_{n-1}))$$

where $I = \{t_0,\dots,t_{n-1}\}$.

(d) Part (b) tells us that we have a consistent family of measures on the finite products Z_J, and therefore have a functional λ on $\bigotimes_{t\in T}\Sigma_t$ defined by setting $\lambda\tilde{\pi}_J^{-1}[W] = \lambda_J W$ for every finite $J \subseteq T$ containing a and $W \in \bigotimes_{t\in J}\Sigma_t$, writing $\tilde{\pi}_J : X \to Z_J$ for the canonical maps. λ is finitely additive, and its images $\mu_t = \lambda\pi_t^{-1}$ are all countably additive because they are also images of measures $\lambda_{\{a,t\}}$. So we have a measure μ extending λ, by 454F. We have to check that each λ_J is the image measure $\mu\tilde{\pi}_J^{-1}$; but this is true because

they agree on $\bigotimes_{t\in J}\Sigma_t$ (using the Monotone Class Theorem in the form 136C, as always). So the integral formula sought for $\tilde{\mu}_J$ is just that obtained in part (a) for λ_J.

(e) Accordingly, if $s<t$, we can calculate $\mu_t E$, for $E\in\Sigma_t$, as follows: setting $J=\{a,s,t\}$, $I=J\setminus\{t\}$, we have

$$\mu_t E=\mu\pi_t^{-1}[E]=\mu\tilde{\pi}_J^{-1}[F]$$

(where $F=\{z : z\in Z_J,\ z(t)\in E\}$)

$$=\lambda_J F=\int \nu^{(s,t)}_{z(s)}(E)\lambda_I(dz)$$

(by (c))

$$=\int \nu^{(s,t)}_{w(s)}(E)\mu(dw)=\int \nu^{(s,t)}_x(E)\mu_s(dx).$$

455B Remarks (a) I noted in 454I that in theorems of this type we do not really need to know that the factors (X_t,Σ_t) are standard Borel spaces; what counts is that they should have the 'perfect measure property' (454Xd). In the present context, we may be able to do even better, since all that matters is that the marginal measures μ_t should be perfect, and we have a direct description of them from the formula $\mu_t=\int\nu^{(a,t)}_x\mu_a(dx)$. So if, for instance, we can show from this presentation that all the μ_t are tight (that is, inner regular with respect to compact sets), then we need no further control on the algebras Σ_t.

(b) The formula

$$\mu_t(E)=\int\nu^{(s,t)}_x(E)\mu_s(dx)\text{ for every }E\in\Sigma_t\text{ whenever }s<t\text{ in }T$$

in 455A is the defining property of a **Markov process**. In something closer to probabilists' language, it comes out as

$$\Pr(X_t\in E)=\mathbb{E}\big(\nu^{(s,t)}_{X_s}(E)\big)\text{ where }\nu^{(s,t)}_x(E)=\Pr(X_t\in E|X_s=x).$$

The idea is that $\langle X_t\rangle_{t\in T}$ is a family of random variables, and that we start from the assurance that 'history is irrelevant'; if, at time s, we wish to make guesses about the behaviour of X_t, the state of the system at a future time t, then we expect that it will be useful to look at the current state X_s, but once we know the value of X_s then any further information about $X_{s'}$ for $s'<s$ will tell us nothing more about X_t. We are given the distributions $\nu^{(s,t)}_x$, which can be thought of as the conditional distributions of X_t given that $X_s=x$, or (in the language of §452) as a disintegration of the joint distribution of (X_s,X_t) over the distribution of X_s. The condition (ii) of 455A is plainly necessary if the system is going to make sense at all; the content of the theorem is that this condition is also sufficient (at least when the random variables X_t are taking values in standard spaces) to ensure that the system as a whole can indeed be represented as a family of random variables, in Kolmogorov's sense, on a suitable probability space.

455C Brownian motion As an example of the use of 455A I give a first existence proof for **Brownian motion** or **Wiener measure**.

Theorem There are a probability space (Ω,Σ,μ) and a family $\langle X_t\rangle_{t\geq 0}$ of real-valued random variables on Ω such that

(i) $X_0=0$ almost everywhere;

(ii) whenever $0\leq s<t$ then X_t-X_s is normally distributed with expectation 0 and variance $t-s$;

(iii) whenever $0\leq t_1<\ldots<t_n$, then $X_{t_2}-X_{t_1},X_{t_3}-X_{t_2},\ldots,X_{t_n}-X_{t_{n-1}}$ are independent.

proof (a) In 455A, take $T=[0,\infty[$, so that $a=0$; take $\mathbb{R}$ for each of the factor spaces, so that $X=\langle X_t\rangle_{t\geq 0}$ becomes a name for the identity function on the product $\Omega=\mathbb{R}^T$ (instead of the name of the product itself!), and each Σ_t is the Borel σ-algebra $\mathcal{B}$ of $\mathbb{R}$. Let μ_0 be the Borel probability measure on $\mathbb{R}$ which gives measure 1 to $\{0\}$, and whenever $0\leq s<t$, $x\in\mathbb{R}$ let $\nu^{(s,t)}_x$ be the distribution of a normal random variable with expectation x and variance $t-s$. (Strictly speaking, since elsewhere I insist that a distribution is a Radon measure, and that a Radon measure is complete, $\nu^{(s,t)}_x$ must be the restriction of such a distribution to $\mathcal{B}$.)

(b) The condition

whenever $0 \leq s < t$ and $E \in \mathcal{B}$, $x \mapsto \nu_x^{(s,t)}(E)$ is $\mathcal{B}$-measurable,

in 455A is elementary;

$$\nu_x^{(s,t)}(E) = \frac{1}{\sqrt{2\pi(t-s)}} \int_E e^{-(y-x)^2/2(t-s)} dy$$

(274Ad) is actually a continuous function of x, because the integrand is continuous as a function of two variables (and is dominated by an integrable function as x varies over any bounded interval). As for the condition

whenever $0 \leq s < t < u$, $x \in \mathbb{R}$ and $E \in \mathcal{B}$, then $\nu_x^{(s,u)}(E) = \int \nu_y^{(t,u)}(E)\nu_x^{(s,t)}(dy)$,

this is really just the theorem that the sum of independent normal random variables is normal (274B). For if U, V are independent normal random variables, U has expectation x and variance $t-s$, and V has expectation 0 and variance $u-t$, then $U+V$ will be normal with expectation x and variance $u-s$, so that $\nu_x^{(s,u)}(E) = \Pr(U+V \in E)$. On the other side of the equation, $\nu_x^{(s,t)}$ is the distribution of U, while $\nu_y^{(t,u)}$ is the distribution of $V+y$; so that

$$\begin{aligned}
\nu_x^{(s,u)}(E) &= \Pr(U+V \in E) = (\nu_x^{(s,t)} * \nu_0^{(t,u)})(E) \\
&\text{(272S)} \\
&= \int \nu_0^{(t,u)}(E-y)\nu_x^{(s,t)}(dy) \\
&\text{(257Xb with 257D, or 444A)} \\
&= \int \nu_y^{(t,u)}(E)\nu_x^{(s,t)}(dy),
\end{aligned}$$

as required.

(c) Thus the conditions of 455A are satisfied, and we get a probability measure μ on $\Omega = \mathbb{R}^{[0,\infty[}$. The conclusion

μ_a is the image measure $\mu\pi_a^{-1}$

says just that $\Pr(X_0 = 0) = 1$. If $s < t$ and $E \in \mathcal{B}$, then

$$\begin{aligned}
\Pr(X_t - X_s \in E) &= \iiint \chi E(z-y)\nu_y^{(s,t)}(dz)\nu_x^{(0,s)}(dy)\mu_0(dx) \\
&= \int \nu_y^{(s,t)}(E+y)\nu_0^{(0,s)}(dy) \\
&= \int \nu_0^{(s,t)}(E)\nu_0^{(0,s)}(dy) = \nu_0^{(s,t)}(E),
\end{aligned}$$

so $X_t - X_s$ has distribution $\nu_0^{(s,t)}$, that is, is normally distributed with expectation 0 and variance $t-s$. Similarly, if $0 \leq t_1 < \ldots < t_n$ and $E_1, \ldots, E_{n-1} \in \mathcal{B}$, then

$$\begin{aligned}\Pr(X_{t_{i+1}} - X_{t_i} \in E_i \text{ for } 1 \le i < n)&\\
&= \iint \dots \int \chi E_1(x_2 - x_1)\chi E_2(x_3 - x_2) \dots \chi E_{n-1}(x_n - x_{n-1})\\
&\qquad \nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n) \dots \nu_{x_0}^{(0,t_1)}(dx_1)\mu_0(dx_0)\\
&= \iint \dots \int \chi(E_1 + x_1)(x_2)\chi(E_2 + x_2)(x_3) \dots \chi(E_{n-1} + x_{n-1})(x_n)\\
&\qquad \nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n) \dots \nu_{x_0}^{(0,t_1)}(dx_1)\mu_0(dx_0)\\
&= \iiint \dots \iint \chi(E_{n-1} + x_{n-1})(x_n)\nu_{x_{n-1}}^{(t_{n-1},t_n)}(dx_n)\\
&\qquad \chi(E_{n-2} + x_{n-2})(x_{n-1})\nu_{x_{n-2}}^{(t_{n-2},t_{n-1})}(dx_{n-1})\\
&\qquad \dots \chi(E_1 + x_1)(x_2)\nu_{x_1}^{(t_1,t_2)}(dx_2)\nu_{x_0}^{(0,t_1)}(dx_1)\mu_0(dx_0)\\
&= \nu_0^{(t_{n-1},t_n)}(E_{n-1})\nu_0^{(t_{n-2},t_{n-1})}(E_{n-2}) \dots \nu_0^{(t_1,t_2)}(E_1)\\
&= \prod_{i=1}^{n-1} \Pr(X_{i+1} - X_i \in E_i).\end{aligned}$$

So $X_{t_2} - X_{t_1}, \dots, X_{t_n} - X_{t_{n-1}}$ are independent.

455D This construction of Brownian motion is sufficient to show that there is a process, satisfying the defining conditions (i)-(iii), which can be studied with the tools of measure theory. It gives no hint of the most important property of Brownian motion, that 'sample paths are continuous'. With some simple inequalities from Chapter 27 and the ideas of 454N-454P, we can find a proof of this, as follows.

Theorem In 455C, we can take Ω to be $C([0,\infty[)$ and μ to be a Radon measure when $C([0,\infty[)$ is given the topology $\mathfrak{T}_c$ of uniform convergence on compact sets, while $X_t(f) = f(t)$ for $f \in C([0,\infty[)$.

proof (a) The construction in 455C in fact gives us $\Omega = \mathbb{R}^{[0,\infty[}$, $X_t(f) = f(t)$ for $f \in \mathbb{R}^{[0,\infty[}$, and a Baire measure μ. The main part of the argument here (down to the end of (e)) is devoted to showing that $\mu^*C([0,\infty[) = 1$; the result will then follow easily from 454Pb.

(b) ❓ Suppose, if possible, that $\mu^*C([0,\infty[) < 1$. Then there is a non-negligible Baire set $H \subseteq \mathbb{R}^{[0,\infty[} \setminus C([0,\infty[)$. There is a countable set $D \subseteq [0,\infty[$ such that H is determined by coordinates in D (4A3Nb); we may suppose that D includes $\mathbb{Q} \cap [0,\infty[$.

(c) (The key.) Let q, q' be rational numbers such that $0 \le q < q'$, and $\epsilon > 0$. Then

$$\Pr(\sup_{t \in D \cap [q,q']} |X_t - X_q| \ge \epsilon) \le \frac{18\sqrt{q'-q}}{\epsilon\sqrt{2\pi}} e^{-\epsilon^2/18(q'-q)}.$$

P If $q = t_0 < t_1 < \dots < t_n = q'$, set $Y_i = X_{t_i} - X_{t_{i-1}}$ for $1 \le i \le n$, so that $X_{t_m} - X_q = \sum_{i=1}^m Y_i$ for $1 \le m \le n$, and $Y_1, \dots, Y_n$ are independent. By Etemadi's lemma (272U),

$$\begin{aligned}\Pr(\sup_{i \le n} |X_{t_i} - X_q| \ge \epsilon) &\le 3 \max_{i \le n} \Pr(|X_{t_i} - X_q| \ge \tfrac{1}{3}\epsilon)\\
&= 3 \max_{1 \le i \le n} \Pr(\tfrac{1}{\sqrt{t_i - q}}|X_{t_i} - X_q| \ge \tfrac{\epsilon}{3\sqrt{t_i - q}})\\
&= 6 \max_{1 \le i \le n} \frac{1}{\sqrt{2\pi}} \int_{\epsilon/3\sqrt{t_i-q}}^\infty e^{-x^2/2} dx\end{aligned}$$

(because $\frac{1}{\sqrt{t_i - q}} X_{t_i} - X_q$ is standard normal)

$$\begin{aligned}&= \frac{6}{\sqrt{2\pi}} \int_{\epsilon/3\sqrt{q'-q}}^\infty e^{-x^2/2} dx\\
&\le \frac{18\sqrt{q'-q}}{\epsilon\sqrt{2\pi}} e^{-\epsilon^2/18(q'-q)}\end{aligned}$$

by 274Ma. Thus if $I \subseteq [q, q']$ is any finite set containing q and q',

$$\Pr(\sup_{t\in I} |X_t - X_q| \geq \epsilon) \leq \frac{18\sqrt{q'-q}}{\epsilon\sqrt{2\pi}} e^{-\epsilon^2/18(q'-q)}.$$

Taking $\langle I_n\rangle_{n\in\mathbb{N}}$ to be a non-decreasing sequence of finite sets with union $D\cap[q,q']$, starting from $I_0 = \{q, q'\}$, we get

$$\begin{aligned}\Pr(\sup_{t\in D\cap[q,q']} |X_t - X_q| \geq \epsilon) &= \lim_{n\to\infty} \Pr(\sup_{t\in I_n} |X_t - X_q| \geq \epsilon)\\ &\leq \frac{18\sqrt{q'-q}}{\epsilon\sqrt{2\pi}} e^{-\epsilon^2/18(q'-q)},\end{aligned}$$

as required. **Q**

(d) If $\epsilon > 0$ and $n \geq 1$, then

$$\begin{aligned}&\Pr(\text{there are } t,\, u \in D\cap[0,n] \text{ such that } |t-u| \leq \frac{1}{n^2} \text{ and } |X_t - X_u| \geq 3\epsilon)\\ &\qquad \leq \frac{18n^2}{\epsilon\sqrt{2\pi}} e^{-n^2\epsilon^2/18}.\end{aligned}$$

P Divide $[0,n]$ into n^3 intervals $[q_i, q_{i+1}]$ of length $1/n^2$. For each of these,

$$\Pr(\sup_{t\in D\cap[q_i,q_{i+1}]} |X_t - X_{q_i}| \geq \epsilon) \leq \frac{18}{n\epsilon\sqrt{2\pi}} e^{-n^2\epsilon^2/18}.$$

So

$$\Pr(\text{there are } i < n^3,\, t \in D\cap[q_i,q_{i+1}] \text{ such that } |X_t - X_{q_i}| \geq \epsilon)$$

is at most $\frac{18n^2}{\epsilon\sqrt{2\pi}} e^{-n^2\epsilon^2/18}$.

But if $t,\, u \in [0,n]$ and $|t-u| \leq 1/n^2$ and $|X_t - X_u| \geq 3\epsilon$, there must be an $i < n^3$ such that both t and u belong to $[q_i, q_{i+2}]$, so that either there is a $t' \in D\cap[q_i,q_{i+1}]$ such that $|X_{t'} - X_{q_i}| \geq \epsilon$ or there is a $t' \in D\cap[q_{i+1},q_{i+2}]$ such that $|X_{t'} - X_{q_{i+1}}| \geq \epsilon$. So

$$\begin{aligned}&\Pr(\text{there are } t,\, u \in D\cap[0,n] \text{ such that } |t-u| \leq \frac{1}{n^2} \text{ and } |X_t - X_u| \geq 3\epsilon)\\ &\quad \leq \Pr(\text{there are } i < n^3,\, t \in D\cap[q_i,q_{i+1}] \text{ such that } \text{ and } |X_t - X_{q_i}| \geq \epsilon)\\ &\quad \leq \frac{18n^2}{\epsilon\sqrt{2\pi}} e^{-n^2\epsilon^2/18},\end{aligned}$$

as required. **Q**

(e) So if we take $G_{\epsilon n}$ to be the Baire set

$$\begin{aligned}\{f : f \in \mathbb{R}^{[0,\infty[},\ &\text{there are } t,\, u \in D\cap[0,n] \text{ such that } |t-u| \leq \frac{1}{n^2}\\ &\text{and } |f(t) - f(u)| \geq 3\epsilon\},\end{aligned}$$

we have

$$\mu G_{\epsilon n} \leq \frac{18n^2}{\epsilon\sqrt{2\pi}} e^{-n^2\epsilon^2/18},$$

and $\lim_{n\to\infty} \mu G_{\epsilon n} = 0$. We can therefore find a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\sum_{k=1}^{\infty} \mu(G_{1/k,n_k}) < \mu H$, so that there is an $f \in H \setminus \bigcup_{k\geq 1} G_{1/k,n_k}$.

What this means is that if $k \geq 1$ and $t,\, u \in D\cap[0,n_k]$ are such that $|t-u| \leq \frac{1}{n_k^2}$, then $|f(t) - f(u)| < \frac{3}{k}$. Since $n_k \to \infty$ as $k \to \infty$, there is a continuous function f' such that $f'\restriction D = f\restriction D$. But H is determined by coordinates in D, so f' belongs to $H \cap C([0,\infty[)$, which is supposed to be empty. **X**

(f) Thus $\mu^* C([0,\infty[) = 1$. By 454Pb, we have an extension of μ to a measure $\tilde{\mu}$ such that $\tilde{\mu}C([0,\infty[) = 1$ and the subspace measure $\tilde{\mu}_C$ is a Radon measure for $\mathfrak{T}_c$. If we interpret the random variables $X_t(f) = f(t)$

as functions defined on the probability space $(\mathbb{R}^{[0,\infty[}, \tilde{\mu})$, they still have the properties (i)-(iii) of 455C, because $\tilde{\mu}$ extends μ; and since $C([0,1[)$ is $\tilde{\mu}$-conegligible, their restrictions to $C([0,\infty[)$, regarded as random variables on the probability space $(C([0,\infty[), \tilde{\mu}_C)$ will also have these properties, as required.

455X Basic exercises (a) In 455A, set $T = [0,\infty[$, let each (X_t, Σ_t) be $\mathbb{R}$ with its Borel σ-algebra, let μ_0 be any atomless Borel probability measure on $\mathbb{R}$, and for $x \in \mathbb{R}$, $0 \leq s < t$ let $\nu_x^{(s,t)}$ be the Borel measure on $\mathbb{R}$ giving measure 1 to $\{\psi(x, t-s)\}$, where

$$\begin{aligned}\psi(x,t) &= \tfrac{x}{1-xt} \text{ if } xt \neq 1 \text{ and } x \neq 0,\\ &= 0 \text{ if } xt = 1,\\ &= -\tfrac{1}{t} \text{ if } x = 0.\end{aligned}$$

Show that the conditions of 455A are satisfied, that the measure μ constructed in 455A is a Baire measure on $\mathbb{R}^T$, and that μ is not τ-additive. (*Hint*: setting $\phi(x)(t) = \psi(x,t)$, show that $\phi : \mathbb{R} \to \mathbb{R}^T$ is inverse-measure-preserving for μ_0 and μ, and that every point in $\mathbb{R}^T$ has a neighbourhood of zero measure.)

(b) Suppose, in 455A, that every (X_t, Σ_t) is $\mathbb{R}$ with its Borel σ-algebra, and that $\int y\nu_x^{(s,t)}(dy) = x$ whenever $s < t$ in T and $x \in \mathbb{R}$. Suppose also that $\int y\mu_0(dy) = \alpha$. Show that $\int \pi_t d\mu = \alpha$ for every $t \in T$ (that is, that all the random variables $\pi_t : X \to \mathbb{R}$ have the same expectation).

(c) Suppose, in 455C, we set $\nu_x^{(s,t)}(E) = \lambda_{t-s}(E-x)$, where λ_t is a Poisson distribution with expectation t, that is, $\lambda_t(E) = e^{-t}\sum_{m\in E\cap\mathbb{N}} t^m/m!$ (cf. 285Q, 285Xo). Show that we can again use 455A to build a measure on $\Omega = \mathbb{R}^{[0,\infty[}$, and that as in 455C the increments $X_t - X_s$ over non-overlapping intervals are independent.

(d) Show that, in 455A, we can set $\nu_x^{(s,t)}(E) = \lambda_{t-s}(E-x)$, where λ_t is a Cauchy distribution with centre 0 and scale parameter t (285Xm). Show that we obtain a measure on $\Omega = \mathbb{R}^{[0,\infty[}$, such that, as in 455C, the increments $X_t - X_s$ over non-overlapping intervals are independent.

(e) (i) The **standard gamma distribution** with parameter γ is the probability distribution λ_γ on $\mathbb{R}$ with probability density function $t \mapsto \frac{1}{\Gamma(\gamma)}t^{\gamma-1}e^{-t}$ for $t > 0$. Show that its expectation is γ. (*Hint*: 225Xj(iv).) Show that its variance is γ. (ii) Show that if X and Y are independent random variables with standard gamma distributions, then $X + Y$ has a standard gamma distribution. (*Hint*: 272T, 252Yk.) (iii) Show that $\lim_{\gamma\downarrow 0}\gamma\Gamma(\gamma) = 1$, so that $\frac{1}{\gamma}\lim_{\gamma\downarrow 0}\lambda_\gamma[1,\infty[= \int_1^\infty \frac{1}{t}e^{-t} > 0$. (iv) Show that, in 455A, we can set $\nu_x^{(s,t)}(E) = \lambda_{t-s}(E-x)$ to get a process in which increments over non-overlapping intervals are independent, the set of non-decreasing functions has full outer measure, and the set of continuous functions is negligible. (v) Show that this process can be represented by a Radon measure on $\mathbb{R}^{[0,\infty[}$. (*Hint*: 438P, 432D.)

(f) Use 272Ye to simplify the formulae in the proof of 455D.

455Y Further exercises (a) Suppose, in 455A, that $T = \mathbb{N}$ or $T = [0,\infty[$, that every (X_t, Σ_t) is $\mathbb{R}$ with its Borel σ-algebra, and that $\nu_x^{(s,t)}(E) = \nu_0^{(0,t-s)}(E-x)$ whenever $x \in \mathbb{R}$, $E \in \mathcal{B}$ and $s < t$. Take $t \in T$ and let $\tilde{\mu}_{[0,t]}$ be the image of μ on $\prod_{0\leq s\leq t} X_s$. Show that we have a disintegration of μ over $\tilde{\mu}_{[0,t]}$ defined by the formula

$$\mu W = \int \mu\{f : g^\frown f \in W\}\tilde{\mu}_{[0,t]}(dg),$$

where for $g \in \prod_{0\leq s\leq t} X_s$, $f \in \prod_{s\in T} X_s$ we write $(g^\frown f)(s) = g(s)$ if $s \leq t$, $g(t) + f(s-t)$ if $s \geq t$.

(b) Let $\langle X_n\rangle_{n\in\mathbb{Z}}$ be a double-ended sequence of real-valued random variables such that (i) for each $n \in \mathbb{Z}$, $Y_n = X_{n+1} - X_n$ is independent of $\{X_i : i \leq n\}$ (ii) $\langle Y_n\rangle_{n\in\mathbb{Z}}$ is identically distributed. Show that the Y_n are essentially constant. (*Hint*: 285Yc.)

(c) Write D_n for $\{2^{-n}i : i \in \mathbb{N}\}$ and $D = \bigcup_{n\in\mathbb{N}} D_n$, the set of dyadic rationals in $[0,\infty[$. For $d \in D$ define $f_d \in C([0,\infty[)$ as follows. If $n \in \mathbb{N}$, $f_n(t) = 0$ if $t \leq n$, $t-n$ if $n \leq t \leq n+1$, 1 if $t \geq n+1$. If $d = 2^{-n}k$ where $n \geq 1$ and $k \in \mathbb{N}$ is odd, $f_d(t) = \max(0, \frac{1}{\sqrt{2^{n+1}}}(1 - 2^n|t-d|))$ for $t \geq 0$. Now let $\langle Z_d\rangle_{d\in D}$ be an independent family of standard normal distributions, and set $G_n(t) = \sum_{d\in D_n} f_d(t)Z_d$ for $t \geq 0$, so that each G_n is a random continuous function on $[0,\infty[$. Show that for any $n \in \mathbb{N}$, $\epsilon > 0$,

$$\Pr(\sup_{t\in[0,n]} |G_{n+1}(t) - G_n(t)| \geq \epsilon) \leq 2^n n \cdot \frac{2}{\sqrt{2\pi}} \int_{2\epsilon\sqrt{2^n}}^{\infty} e^{-x^2/2} dx,$$

and hence that $\langle G_n\rangle_{n\in\mathbb{N}}$ converges almost surely to a continuous function. Explain how to interpret this as a construction of the measure on $C([0,\infty[)$ described in 455D.

(d) Let $\langle\lambda_t\rangle_{t\in]0,\infty[}$ be any family of probability distributions on $\mathbb{R}$ such that $\lambda_s * \lambda_t = \lambda_{s+t}$ for all s, $t > 0$. Show that we can set $\nu_x^{(s,t)}(E) = \lambda_{t-s}(E - x)$ in 455A to obtain a measure on $\mathbb{R}^{[0,\infty[}$ such that the increments over non-overlapping intervals are independent.

(e) In 455Xc, show that the measure μ obtained on $\mathbb{R}^{[0,\infty[}$ is a Baire measure, and has an extension to a Radon measure $\tilde{\mu}$ on $\mathbb{R}^{[0,\infty[}$ such that $\tilde{\mu}Q = 1$, where Q is the set of non-decreasing functions $f : [0,\infty[\to \mathbb{N}$ such that $\lim_{t\downarrow 0} f(t) = 0$, $\lim_{s\downarrow t} f(s) - \lim_{s\uparrow t} f(s) \leq 1$ for every $t \geq 0$, and $\lim_{t\to\infty} f(t) = \infty$. (*Hint*: show that Q is a K-analytic space with metrizable compact sets, so is a Radon space.)

(f) In 455Xd, show that the measure μ obtained on $\mathbb{R}^{[0,\infty[}$ is a Baire measure, and that $C([0,\infty[)$ is negligible. (*Hint*: estimate $\Pr(|X_{(i+1)/n} - X_{i/n}| \leq \epsilon$ for every $i < n)$.)

(g) Let μ be the Radon measure on $C([0,\infty[)$ representing Brownian motion, and $\gamma > \frac{1}{2}$. Show that $\lim_{t\to\infty} t^{-\gamma} f(t) = 0$ for μ-almost every $f \in C([0,\infty[)$.

455 Notes and comments Obviously the principal applications of 455A will be with the totally ordered sets $\mathbb{N}$ and $[0,\infty[$, as in 454H and 455C. I use the letters t, T to suggest the probabilistic intuitions behind these results; we think of the spaces X_t in 455A as being the sets of possible states of a system at 'time' t, so that the measures $\nu_x^{(s,t)}$ are descriptions of how we believe the system is likely to evolve between times s and t, given that it is in state x at time s. In the case of 'discrete time', when we observe the system only at clearly separated moments, it is easy to handle non-Markov processes, in which evolution between times $n-1$ and n can depend on the whole history up to time $n-1$; thus in 454H the measures $\nu_z = \nu_z^{(n-1,n)}$ are defined for every $z \in \prod_{i<n} X_i$, but we make no attempt to describe measures $\nu_z^{(n-1,m)}$ for any $m > n$. In 'continuous time' we do need to say something about arbitrary time steps, and it is hard to formulate a consistency condition to fill the place of (ii) in 455A without limiting the kind of process being examined. At the cost of an appalling increase in complexity, of course, the formulae of 455A can be adapted to general processes, if we replace the 'current' state space X_t by the 'historical' state space $\prod_{a\leq s\leq t} X_s$. (For this product will still have the 'perfect measure property' of 454Xd.) We should finish up with a measure on $\prod_{t\in T}(\prod_{a\leq s\leq t} X_s)$. But the important applications, even when not Markov, are open to more economical and more enlightening approaches.

We really do need a least element a of T; see 455Yb. I find that in the exposition here I have made a rather wrenching change in notation between the abstract measure theory of 455A and the probability theory of 455C. I hope that despite the conflicting use of the symbol X this will give no trouble.

I remarked in the introduction to Chapter 27 that probability spaces themselves are rather shadowy objects in probability theory. The manoeuvres of 455C-455D are a typical example of the shifts we find ourselves making. Brownian motion is essentially defined by the properties (i)-(iii) in 455C; these determine the joint distribution of the family $\langle X_t\rangle_{t\in J}$ for any finite set $J \subseteq [0,\infty[$. We have a great deal of choice about the measure space Ω we choose to realize the process on. In 455C we have a Baire measure on $\mathbb{R}^{[0,\infty[}$, and in 455D we have a Radon measure on $C([0,\infty[)$. In both cases the measure is uniquely defined. The second is the more useful representation; for instance, it makes every sample path continuous, while in the first formulation the continuous paths are only a set of full outer measure; more important, we have a Radon measure on a Polish space, which gives us a springboard for further investigation.

Perhaps it is worth recalling here that a Baire measure on a completely regular space has at most one quasi-Radon measure extending it (415G). The step from 455C to 455D amounts to showing that Brownian motion, interpreted as a Baire probability measure on $\mathbb{R}^{[0,\infty[}$, has an extension to a Radon measure $\tilde{\mu}$ on $\mathbb{R}^{[0,\infty[}$ for which $C([0,\infty[)$ is conegligible, so that we have the option of regarding $\tilde{\mu}$ as a Radon measure on $C([0\infty[)$; by 454P or 454Xl, we can give $C([0,\infty[)$ either the topology of pointwise convergence or the topology of uniform convergence on compact sets. The point I wish to make is that the step from μ to $\tilde{\mu}$ is a canonical one and worth performing as soon as we know that μ is τ-additive, whether or not we come to a measure living on the continuous functions. In 455Xe, for instance, $\tilde{\mu}$ gives measure 1 to the set of non-decreasing functions, and this is clearly an essential property of the process.

There are of course many routes to the fundamental theorems 455C-455D, and I do not claim that the one described here is the 'best'. I use it because it is an instructive example of the class of problems which led to the theorems of this section. The excursion through a measure on $\mathbb{R}^{[0,\infty[}$ is by no means necessary; in 455Yc I sketch an alternative method ('Lévy's construction') which avoids it.

Generally it is true that interesting stochastic processes have much stronger properties than those described in 455A. Any construction using these abstract methods should be regarded as provisional; it may be a useful, even necessary, first confirmation that there is something there to work on, but we are likely to need much more penetrating techniques to form a proper conception of what we have found. The general argument above does however provide a clear description of what is necessary for the 'independent increments' property of Brownian motion: the fact that if $t_1 < t_2 < \ldots < t_n$ then $X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent. We need the Markov property (so that, given $X_{t_{n-1}}$, X_{t_n} is independent of previous history) and the translation property of the $\nu_x^{(s,t)}$, encapsulated in the formula $\nu_x^{(s,t)}(E) = \nu_0^{(s,t)}(E-x)$, which assures us that the distribution of $X_t - X_s$ is $\nu_0^{(s,t)}$ (so that $X_s, X_t - X_s$ are independent). It also has a place for 'temporal homogeneity', that is, the fact that $\nu_x^{(s,t)} = \nu_x^{(s',t')}$ whenever $t' - s' = t - s$ (455Ya). And the method does at least make it clear which property of normal distributions we need (part (b) of the proof of 455C), so leads us to examine other important processes (455Xc-455Xe, 455Yd). These are all based on families of 'infinitely divisible' distributions; for the general theory of such distributions, see Loève 77 or Fristedt & Gray 97.

456 Gaussian distributions

Uncountable powers of $\mathbb{R}$ are not as a rule measure-compact (439P, 455Xa). Accordingly distributions, in the sense of 454K, need not be τ-additive. But, as we have seen in 455D, some, at least, of the distributions most important to us are indeed τ-additive, and therefore have interesting canonical extensions. This section is devoted to a remarkable result, taken from Talagrand 81, concerning a class of distributions which are of great importance in probability theory. It demands a combination of techniques from classical probability theory and from the topological measure theory of this volume. I begin with the definition and fundamental properties of what I call 'centered Gaussian distributions' (456A-456E). These are minor adaptations of the classical finite-dimensional theory. Another relatively easy idea is that of 'universal' Gaussian distribution (456F-456H). In 456I we come to a much deeper result, a step towards classifying the ways in which a Gaussian family of n-dimensional random variables can accumulate at 0. The ideas are combined in 456J-456K to complete the proof of Talagrand's theorem that Gaussian distributions on powers of $\mathbb{R}$ are τ-additive.

456A Definitions (a) Write μ_G for the Radon probability measure on $\mathbb{R}$ which is the distribution of a standard normal random variable, that is, the probability distribution with density function $x \mapsto \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ (274A). For any set I, write $\mu_G^{(I)}$ for the measure on $\mathbb{R}^I$ which is the product of copies of μ_G; this is always quasi-Radon (415E/453J); if I is countable, it is Radon (417Q); if $I = n \in \mathbb{N} \setminus \{0\}$, it is the probability distribution with density function $x \mapsto (2\pi)^{-n/2} e^{-x \cdot x/2}$ (272I); if $I = \emptyset$, it is the unique probability measure on the singleton set $\mathbb{R}^\emptyset$.

(b) I will use the phrase **centered Gaussian distribution** to mean a measure μ on a power $\mathbb{R}^I$ of $\mathbb{R}$ such that μ is the completion of a Baire measure (that is, is a distribution in the sense of 454K) and every

continuous linear functional $f : \mathbb{R}^I \to \mathbb{R}$ is either zero almost everywhere or is a normal random variable with zero expectation. (Note that I call the distribution concentrated on the point 0 in $\mathbb{R}^I$ a 'Gaussian distribution'.)

(c) If I is a set and μ is a centered Gaussian distribution on $\mathbb{R}^I$, its **covariance matrix** is the family $\langle \sigma_{ij} \rangle_{i,j \in I}$ where $\sigma_{ij} = \int x(i)x(j)\mu(dx)$ for $i, j \in I$. (The integral is always defined and finite because each function $x \mapsto x(i)$ is either essentially constant or normally distributed, and in either case is square-integrable.)

456B I start with some fundamental facts about Gaussian distributions.

Proposition (a) Suppose that I and J are sets, μ is a centered Gaussian distribution on $\mathbb{R}^I$, and $T : \mathbb{R}^I \to \mathbb{R}^J$ is a continuous linear operator. Then there is a unique centered Gaussian distribution on $\mathbb{R}^J$ for which T is inverse-measure-preserving; if J is countable, this is the image measure μT^{-1}.

(b) Let I be a set, and μ, ν two centered Gaussian distributions on $\mathbb{R}^I$. If they have the same covariance matrices they are equal.

(c) For any set I, $\mu_G^{(I)}$ is the centered Gaussian distribution on $\mathbb{R}^I$ with the identity matrix for its covariance matrix.

(d) Suppose that I is a countable set. Then a measure μ on $\mathbb{R}^I$ is a centered Gaussian distribution iff it is of the form $\mu_G^{(\mathbb{N})} T^{-1}$ where $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^I$ is a continuous linear operator.

(e) Suppose $\langle I_j \rangle_{j \in J}$ is a disjoint family of sets with union I, and that for each $j \in J$ we have a centered Gaussian distribution ν_j on $\mathbb{R}^{I_j}$. Then the product ν of the measures ν_j, regarded as a measure on $\mathbb{R}^I$, is a centered Gaussian distribution.

(f) Let I be any set, μ a centered Gaussian distribution on $\mathbb{R}^I$ and $E \subseteq \mathbb{R}^I$ a set such that μ measures E. Writing $-E = \{-x : x \in E\}$, $\mu(-E) = \mu E$.

proof (a)(i) For Baire sets $F \subseteq \mathbb{R}^J$, set $\nu F = \mu T^{-1}[F]$; this is always defined because T is continuous (4A3Kc). This makes ν a Baire probability measure on $\mathbb{R}^J$ for which T is inverse-measure-preserving. Because μ is complete, T is still inverse-measure-preserving for the completion $\hat{\nu}$ of ν (235Hc). If $g : \mathbb{R}^J \to \mathbb{R}$ is a continuous linear functional, so is $gT : \mathbb{R}^I \to \mathbb{R}$; now $\nu\{y : g(y) \leq \alpha\} = \mu\{x : gT(x) \leq \alpha\}$ for every α, so g and gT have the same distribution, and are both either zero a.e. or normal random variables. As g is arbitrary, $\hat{\nu}$ is a centered Gaussian distribution as defined in 456Ab. Of course it is the only such distribution on $\mathbb{R}^J$ for which T is inverse-measure-preserving.

(ii) Now suppose that J is countable. Then $\mathbb{R}^J$ is is Polish (4A2Qc), so ν is a Borel measure and $\hat{\nu}$ is a Radon measure (433Cb). $\mathbb{R}^J$ has a countable network consisting of Borel sets, μ is perfect (454A(b-iii)) and totally finite, and T is measurable (418Bd), so μT^{-1} is a Radon measure (451N). Thus $\hat{\nu}$ and μT^{-1} are Radon measures agreeing on the Borel sets and must be equal.

(b) The point is that $\mu f^{-1} = \nu f^{-1}$ for every continuous linear functional $f : \mathbb{R}^I \to \mathbb{R}$. **P** By (a), μf^{-1} and νf^{-1} are Radon measures on $\mathbb{R}$, and by the definition of 'Gaussian distribution' each is either a normal distribution with expectation zero, or is concentrated at 0. By 4A4Be, we can express f in the form $f(x) = \sum_{i \in I} \beta_i x(i)$ for every $x \in \mathbb{R}^I$, where $\{i : \beta_i \neq 0\}$ is finite. In this case

$$\begin{aligned}\int t^2 (\mu f^{-1})(dt) &= \int f(x)^2 \mu(dx) = \sum_{i,j \in I} \beta_i \beta_j \int x(i)x(j)\mu(dx) \\ &= \sum_{i,j \in I} \beta_i \beta_j \int x(i)x(j)\nu(dx) = \int t^2 (\nu f^{-1})(dt)\end{aligned}$$

because μ and ν have the same covariance matrices. But this means that μf^{-1} and νf^{-1} have the same variance; if this is zero, they both give measure 1 to $\{0\}$; otherwise, they are normal distributions with the same expectation and the same variance, so again are equal. **Q**

By 454M, $\mu = \nu$.

(c) Being a completion regular quasi-Radon probability measure (415E), $\mu_G^{(I)}$ is the completion of a Baire probability measure on $\mathbb{R}^I$. If $f : \mathbb{R}^I \to \mathbb{R}$ is a continuous linear functional, then it is expressible in the

form $f(z) = \sum_{i\in I} \beta_i z(i)$, where $J = \{i : \beta_i \neq 0\}$ is finite. I need to show that f is either zero a.e. or a normal random variable with expectation 0. If $J = \emptyset$ then $f = 0$ everywhere and we can stop. Otherwise, $f = \sum_{i\in J} \beta_i \pi_i$, where $\pi_i(x) = x(i)$ for $i \in I$, $x \in \mathbb{R}^I$. Now, with respect to the measure $\mu_G^{(I)}$, $\langle \pi_i \rangle_{i\in J}$ is an independent family of normal random variables with zero expectation (272G). So $\langle \beta_i \pi_i \rangle_{i\in J}$ is independent (272E), and $\beta_i \pi_i$ is normal for $i \in J$ (274Ae). By 274B, $f = \sum_{i\in J} \beta_i \pi_i$ is normal, and of course it has zero expectation. As f is arbitrary, $\mu_G^{(I)}$ is a centered Gaussian distribution.

We have

$$\int x(i)x(i)\mu_G^{(I)}(dx) = \int t^2 \mu_G(dt) = 1$$

for $i \in I$, and

$$\int x(i)x(j)\mu_G^{(I)}(dx) = \int t\mu_G(dt) \cdot \int t\mu_G(dt) = 0$$

if $i, j \in I$ are distinct. So the covariance matrix of $\mu_G^{(J)}$ is the identity matrix. By (b), it is the only centered Gaussian distribution with this covariance matrix.

(d)(i) It follows from (c) and (a) that if $\mu = \mu_G^{(\mathbb{N})} T^{-1}$, where $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^I$ is a continuous linear operator, then μ is a centered Gaussian distribution.

(ii) Now suppose that μ is a centered Gaussian distribution on $\mathbb{R}^I$. Set $\pi_i(x) = x(i)$ for $i \in I$ and $x \in \mathbb{R}^I$; for $i \in I$, set $u_i = \pi_i^\bullet$ in $L^2 = L^2(\mu)$. By 4A4Jg, there is a countable orthonormal family $\langle v_j \rangle_{j\in J}$ in L^2 such that every v_j is a linear combination of the u_i, and every u_i is a linear combination of the v_j. We may suppose that $J \subseteq \mathbb{N}$. For $i \in I$, express u_i as $\sum_{j\in J} \alpha_{ij} v_j$, where $\{j : \alpha_{ij} \neq 0\}$ is finite. Define $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^I$ by setting $(Tz)(i) = \sum_{j\in J} \alpha_{ij} z(j)$ for every $z \in \mathbb{R}^{\mathbb{N}}$, $i \in I$. Then T is a continuous linear functional. Set $\nu = \mu_G^{(\mathbb{N})} T^{-1}$, so that ν is a centered Gaussian distribution on $\mathbb{R}^I$, by (a). Because $\mathbb{R}^I$ is Polish, both μ and ν must be Radon measures.

(iii) Now μ and ν have the same covariance matrices. **P** If $i, i' \in I$ then

$$\begin{aligned}\int x(i)x(i')\mu(dx) &= (u_i|u_{i'}) = \sum_{j,j'\in J} \alpha_{ij}\alpha_{i'j'}(v_j|v_{j'}) \\ &= \sum_{j\in J} \alpha_{ij}\alpha_{i'j} = \sum_{j,j'\in J} \alpha_{ij}\alpha_{i'j'} \int z(j)z(j')\mu_G^{(\mathbb{N})}(dz) \\ &= \int (Tz)(i)(Tz)(i')\mu_G^{(\mathbb{N})}(dz) = \int x(i)x(i')\nu(dx). \ \textbf{Q}\end{aligned}$$

By (b), $\mu = \nu$ is of the required form.

(e) We must first confirm that ν is the completion of a Baire measure. **P** If we write $\mathcal{Ba}(\mathbb{R}^{I_j})$ for the Baire σ-algebra of $\mathbb{R}^{I_j}$, then each ν_j is the completion of its restriction $\nu_j \restriction \mathcal{Ba}(\mathbb{R}^{I_j})$, so ν is also the product of the measures $\nu_j \restriction \mathcal{Ba}(\mathbb{R}^{I_j})$ (254I), and is therefore the completion of its restriction to $\widehat{\bigotimes}_{j\in J} \mathcal{Ba}(\mathbb{R}^{I_j})$ (254Ff). But as $\mathcal{Ba}(\mathbb{R}^{I_j}) = \widehat{\bigotimes}_{i\in I_j} \mathcal{Ba}(\mathbb{R})$ for every j (4A3Na), $\widehat{\bigotimes}_{j\in J} \mathcal{Ba}(\mathbb{R}^{I_j})$ can be identified with $\widehat{\bigotimes}_{i\in I} \mathcal{Ba}(\mathbb{R}) = \mathcal{Ba}(\mathbb{R}^I)$, so that ν is indeed the completion of $\nu \restriction \mathcal{Ba}(\mathbb{R}^I)$. **Q**

Now suppose that $f : \mathbb{R}^I \to \mathbb{R}$ is a continuous linear functional. Then we can express f in the form $f(x) = \sum_{i\in K} \alpha_i x(i)$ for every $x \in \mathbb{R}^I$, where $K \subseteq I$ is finite. Set $L = \{j : K \cap I_j \neq \emptyset\}$ and $K_j = K \cap I_j$ for $j \in L$, so that L and every K_j are finite; for $j \in L$, $x \in \mathbb{R}^I$ set $f_j(x) = \sum_{i\in K_j} \alpha_i x(i)$. Now $f = \sum_{j\in L} f_j$.

If we set $g_j(y) = \sum_{i\in K_j} \alpha_i y(i)$ for $y \in \mathbb{R}^{I_j}$, then g_j is either zero a.e. or a normal random variable with respect to the probability measure ν_j. Since

$$\nu\{x : f_j(x) \leq \alpha\} = \nu\{x : g_j(x \restriction I_j) \leq \alpha\} = \nu_j\{y : g_j(y) \leq \alpha\}$$

for every $\alpha \in \mathbb{R}$, f_j (regarded as a random variable on $(\mathbb{R}^I, \nu)$) has the same distribution as g_j (regarded as a random variable on $(\mathbb{R}^{I_j}, \nu_j)$). This is true for every $j \in L$. Moreover, the different f_j, as j runs over L, are independent. So $f = \sum_{j\in L} f_j$ is the sum of independent random variables which are all either normal or essentially constant. By 274B again, f also is either normal or essentially constant. And of course its expectation is zero. As f is arbitrary, this shows that ν is a centered Gaussian distribution.

(f) Set $Tx = -x$ for $x \in \mathbb{R}^I$, so that T is a continuous linear operator and we have a unique centered Gaussian distribution ν on $\mathbb{R}^I$ such that T is inverse-measure-preserving for μ and ν, by (a). For any i, $j \in I$,

$$\int x(i)x(j)\nu(dx) = \int (Tx)(i)(Tx)(j)\mu(dx) = \int x(i)x(j)\mu(dx),$$

so μ and ν have the same covariance matrices and are equal, by (b). Accordingly

$$\mu(-E) = \mu T^{-1}[E] = \nu E = \mu E$$

whenever μ measures E.

456C Lemma Let I be a finite set and μ a centered Gaussian distribution on $\mathbb{R}^I$. Suppose that $\gamma \geq 0$ and $\alpha = \mu\{x : \sup_{i\in I} |x(i)| \geq \gamma\}$. Then $\mu\{x : \sup_{i\in I} |x(i)| \geq \frac{1}{2}\gamma\} \geq 2\alpha(1-\alpha)^3$.

proof The case $\gamma = 0$, $\alpha = 1$ is trivial; suppose that $\gamma > 0$. We may suppose that I has a total order $\leq$. Give $\mathbb{R}^{I\times 4} \cong (\mathbb{R}^I)^4$ the product λ of four copies of μ; then λ is a centered Gaussian distribution (456Be). Define $T : \mathbb{R}^{I\times 4} \to \mathbb{R}^I$ by setting $(Ty)(i) = \frac{1}{2}\sum_{r=0}^3 y(i,r)$ for $i \in I$, $y \in \mathbb{R}^{I\times 4}$; then λT^{-1} is a centered Gaussian distribution on $\mathbb{R}^I$ (456Ba). Now λT^{-1} has the same covariance matrix as μ. **P** If i, $j \in I$ then

$$\begin{aligned}\int x(i)x(j)(\lambda T^{-1})(dx) &= \int (Ty)(i)(Ty)(j)\lambda(dy) = \frac{1}{4}\sum_{r=0}^3\sum_{s=0}^3 \int y(i,r)y(j,s)\lambda(dy)\\ &= \frac{1}{4}\sum_{r=0}^3 \int y(i,r)y(j,r)\lambda(dy) = \frac{1}{4}\sum_{r=0}^3 \int x(i)x(j)\mu(dx)\end{aligned}$$

(because the map $y \mapsto \langle y(i,r)\rangle_{i\in I} : \mathbb{R}^{I\times 4} \to \mathbb{R}^I$ is inverse-measure-preserving for each r)

$$= \int x(i)x(j)\mu(dx).$$ **Q**

So $\lambda T^{-1} = \mu$, by 456Bb.

Define

$$E_{ir} = \{y : y \in \mathbb{R}^{I\times 4},\ |y(i,r)| \geq \gamma\}$$

for $r < 4$ and $i \in I$, and

$$E_r = \bigcup_{i\in I} E_{ir}$$

for $r < 4$, so that

$$\lambda E_r = \lambda\{y : \sup_{i\in I} |y(i,r)| \geq \gamma\} = \mu\{x : \sup_{i\in I} |x(i)| \geq \gamma\} = \alpha.$$

Set $E'_r = E_r \setminus \bigcup_{s\neq r} E_s$, so that

$$\begin{aligned}\lambda E'_r &= \lambda\{y : \sup_{i\in I} |y(i,r)| \geq \gamma,\ \sup_{i\in I} |y(i,s)| < \gamma \text{ for } s \neq r)\\ &= \lambda\{y : \sup_{i\in I} |y(i,r)| \geq \gamma\} \cdot \prod_{s\neq r} \lambda\{y : \sup_{i\in I} |y(i,s)| < \gamma\}\end{aligned}$$

(because these are independent events)

$$= \alpha(1-\alpha)^3.$$

Next, for $i \in I$ and $r < 4$, set $E'_{ir} = E_{ir} \setminus (\bigcup_{j<i} E_{jr} \cup \bigcup_{s\neq r} E_s)$, so that $E'_r = \bigcup_{i\in I} E'_{ir}$. Observe that $\langle E'_{ir}\rangle_{i\in I, r<4}$ is disjoint. Set

$$F_{ir} = \{y : y \in E'_{ir},\ y(i,r)\textstyle\sum_{s\neq r} y(i,s) \geq 0\}.$$

Then $\nu F_{ir} \geq \frac{1}{2}\nu E'_{ir}$. **P** We can think of $\mathbb{R}^{I\times 4}$ as a product $\mathbb{R}^J \times \mathbb{R}^K$, where $J = I \times \{r\}$ and $K = I \times (4 \setminus \{r\})$. In this case, λ becomes identified with a product $\lambda_J \times \lambda_K$, where λ_J and λ_K are centered Gaussian distributions on $\mathbb{R}^J$ and $\mathbb{R}^K$ respectively, and E'_{ir} is of the form $V \times W$, where

$$V = \{v : v \in \mathbb{R}^J, |v(i,r)| \geq \gamma, |v(j,r)| < \gamma \text{ for } j < i\},$$

$$W = \{w : w \in \mathbb{R}^K, |w(j,s)| < \gamma \text{ for every } j \in I, s \neq r\}.$$

In the same representation, F_{ir} becomes $(V^+ \times W^+) \cup (V^- \times W^-)$, where

$$V^+ = \{v : v \in V, v(i,r) \geq \gamma\}, \quad V^- = \{v : v \in V, v(i,r) \leq -\gamma\},$$

$$W^+ = \{w : w \in W, \textstyle\sum_{r \neq s} w(i,s) \geq 0\}, \quad W^- = \{w : w \in W, \textstyle\sum_{r \neq s} w(i,s) \leq 0\}.$$

By 456Bf, $\lambda_J V^+ = \lambda_J V^-$ and $\lambda_K W^+ = \lambda_K W^-$; since $V^- = V \setminus V^+$, while $W^+ \cup W^- = W$, we have

$$\lambda_J V^+ = \lambda_J V^- = \tfrac{1}{2}\lambda_J V, \quad \lambda_K W^+ = \lambda_K W^- \geq \tfrac{1}{2}\lambda_K W.$$

But this means that

$$\begin{aligned}\lambda F_{ir} &= \lambda(V^+ \times W^+) + \lambda(V^- \times W^-)\\ &= \lambda_J V^+ \cdot \lambda_K W^+ + \lambda_J V^- \cdot \lambda_K W^- \geq \tfrac{1}{2}\lambda_J V \cdot \lambda_K W = \tfrac{1}{2}\lambda E'_{ir},\end{aligned}$$

as claimed. **Q**

At this point, observe that if $y \in F_{ir}$ then $|\sum_{s=0}^4 y(i,s)| \geq |y(i,r)| \geq \gamma$. So

$$\begin{aligned}\mu\{x : \sup_{i\in I} |x(i)| \geq \tfrac{1}{2}\gamma\}) &= \lambda T^{-1}[\{x : \sup_{i\in I} |x(i)| \geq \tfrac{1}{2}\gamma\})]\\ &= \lambda\{y : \sup_{i\in I} |\frac{1}{2}\sum_{r=0}^3 y(i,r)| \geq \tfrac{1}{2}\gamma\})\\ &= \lambda\{y : \sup_{i\in I} |\sum_{r=0}^3 y(i,r)| \geq \gamma)\\ &\geq \lambda(\bigcup_{i\in I, r<4} F_{ir}) = \sum_{i\in I, r<4} \lambda F_{ir}\\ &\geq \frac{1}{2}\sum_{r<4}\sum_{i\in I} \lambda E'_{ir} \geq \frac{1}{2}\sum_{r<4} \lambda E'_r = 2\alpha(1-\alpha)^3,\end{aligned}$$

which is what we set out to prove.

456D The support of a Gaussian distribution: Proposition Let I be a set and μ a centered Gaussian distribution on $\mathbb{R}^I$. Write Z for the set of those $x \in \mathbb{R}^I$ such that $f(x) = 0$ whenever $f : \mathbb{R}^I \to \mathbb{R}$ is a continuous linear functional and $f = 0$ a.e. Then Z is a self-supporting closed linear subspace of $\mathbb{R}^I$ with full outer measure. If I is countable it is the support of μ.

proof (a) Being the intersection of a family of closed linear subspaces, of course Z is a closed linear subspace.

(b) Z is of full outer measure. **P** Let $F \subseteq \mathbb{R}^I$ be a non-negligible zero set. Let $J \subseteq I$ be a countable set such that F is determined by coordinates in J. For $i \in I$, $x \in X$ set $\pi_i(x) = x(i)$; then each π_i is either normally distributed or zero almost everywhere, so is square-integrable; set $u_i = \pi_i^\bullet$ in $L^2 = L^2(\mu)$. Let $\langle v_k\rangle_{k\in K}$ be a countable orthonormal family in L^2 such that every v_k is a linear combination of the u_i, for $i \in J$, and every u_i, for $i \in J$, is a linear combination of the v_k (4A4Jg). Extend $\langle v_k\rangle_{k\in K}$ to a Hamel basis $\langle v_l\rangle_{l\in L}$ of L^2. For every $i \in I$, we can express u_i as $\sum_{l\in L} \alpha_{il} v_l$, where $\{l : \alpha_{il} \neq 0\}$ is finite; and the construction ensures that $\alpha_{il} = 0$ if $i \in J$ and $l \in L \setminus K$.

Consider the linear operator $T_0 : \mathbb{R}^K \to \mathbb{R}^J$ defined by setting $(T_0 z)(i) = \sum_{k\in K} \alpha_{ik} z(k)$ for $z \in \mathbb{R}^K$ and $i \in J$. If we give $\mathbb{R}^K$ the product measure $\mu_G^{(K)}$, then the image measure $\mu_G^{(K)} T_0^{-1}$ is a Gaussian distribution (456Ba), with covariance matrix

$$\begin{aligned}\sigma_{ii'} &= \int x(i)x(i')(\mu_G^{(K)}T_0^{-1})dx = \int (T_0z)(i)(T_0z)(i')\mu_G^{(K)}(dz)\\ &= \sum_{k,k'\in K}\alpha_{ik}\alpha_{i'k'}z(k)z(k')\mu_G^{(K)}(dz) = \sum_{k\in K}\alpha_{ik}\alpha_{i'k}\\ &= \sum_{k,k'\in K}\alpha_{ik}\alpha_{i'k}(v_k|v_{k'}) = (u_i|u_{i'}) = \int x(i)x(i')\mu(dx).\end{aligned}$$

But this means that $\mu_G^{(K)}T_0^{-1}$ has the same covariance matrix as $\mu\tilde{\pi}_J^{-1}$, where $\tilde{\pi}_Jx = x\restriction J$ for $x \in \mathbb{R}^I$. Since this also is a centered Gaussian distribution, the two measures must be equal (456Bb). We know that $\tilde{\pi}_J[F]$ has non-zero measure, so there is a $z_0 \in \mathbb{R}^K$ such that $T_0z_0 \in \tilde{\pi}_J[F]$. Extend z_0 arbitrarily to $z_1 \in \mathbb{R}^L$.

Set $x_1(i) = \sum_{l\in L}\alpha_{il}z_1(l)$ for $i \in I$. Then $\tilde{\pi}_Jx_1 = T_0z_0 \in \tilde{\pi}_J[F]$, so $x_1 \in F$, because F is determined by coordinates in J. If a continuous linear functional $f : \mathbb{R}^I \to \mathbb{R}$ is zero a.e., it can be expressed in the form $f(x) = \sum_{i\in I}\beta_ix(i)$ where $\{i : \beta_i \neq 0\}$ is finite (4A4Be). In this case,

$$0 = f^\bullet = \textstyle\sum_{i\in I}\beta_iu_i = \sum_{l\in L}\sum_{i\in I}\beta_i\alpha_{il}v_l$$

in L^2. Since $\langle v_l\rangle_{l\in L}$ is linearly independent, $\sum_{i\in I}\beta_i\alpha_{il} = 0$ for every $l \in L$. But this means that

$$f(x_1) = \textstyle\sum_{i\in I,l\in L}\beta_i\alpha_{il}z_1(l) = 0.$$

As f is arbitrary, $x_1 \in Z$ and $Z \cap F \neq \emptyset$. As F is arbitrary, and μ is inner regular with respect to the zero sets, Z has full outer measure. **Q**

(c) Z is self-supporting. **P** If $W \subseteq \mathbb{R}^I$ is an open set meeting Z, there is an open set V, depending on coordinates in a finite set $J \subseteq I$, such that $V \subseteq W$ and $V \cap Z \neq \emptyset$. Write $\tilde{\pi}_J(x) = x\restriction J$ for $x \in \mathbb{R}^I$, and ν_J for the image measure $\mu\tilde{\pi}_J^{-1}$ on $\mathbb{R}^J$; by 456Ba, this is a centered Gaussian distribution. By 456Bd, there is a continuous linear operator $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^J$ such that $\nu_J = \mu_G^{(\mathbb{N})}T^{-1}$. Since the support of μ_G is $\mathbb{R}$, the support of $\mu_G^{(\mathbb{N})}$ is $\mathbb{R}^{\mathbb{N}}$ (417E(iv), or otherwise), and the support of ν_J is $Z_1 = \overline{T[\mathbb{R}^{\mathbb{N}}]}$ (411Ne).

Write Q for the set of linear functionals $g : \mathbb{R}^J \to \mathbb{R}$ (necessarily continuous) which are zero on Z_1. If $g \in Q$, then $gT = 0$, so $g = 0$ ν_J-a.e. and $g\tilde{\pi}_J = 0$ μ-a.e. This means that $g\tilde{\pi}_J(x) = 0$ for every $x \in Z$, that is, $g(y) = 0$ for every $y \in \tilde{\pi}_J[Z]$. Because Z_1 is a linear subspace of $\mathbb{R}^J$, this is enough to show that $\tilde{\pi}_J[Z] \subseteq Z_1$.

Now recall that $V \cap Z \neq \emptyset$ so $Z_1 \cap \tilde{\pi}_J[V] \neq \emptyset$, while $V = \tilde{\pi}_J^{-1}[\tilde{\pi}_J[V]]$. Since $\tilde{\pi}_J$ is an open map (4A2B(f-i)), $\tilde{\pi}_J[V]$ is open and

$$\mu^*(W \cap Z) = \mu W \geq \mu V = \nu_J\tilde{\pi}_J[V] > 0,$$

because Z_1 is the support of ν_J. **Q**

(d) If I is countable, μ is a topological measure so measures Z, and Z is the support of μ.

456E Remarks (a) In the context of 456D, I will call Z the **support** of the centered Gaussian distribution μ, even though μ need not be a topological measure, so the definition 411Nb is not immediately applicable. In 456L we shall see that Z really is the support of a canonical extension of μ.

(b) It is worth making one elementary point at once. If I and J are sets, μ and ν are centered Gaussian distributions on $\mathbb{R}^I$ and $\mathbb{R}^J$ respectively with supports Z and Z', and $T : \mathbb{R}^I \to \mathbb{R}^J$ is an inverse-measure-preserving continuous linear operator, then $Tz \in Z'$ for every $z \in Z$. **P** If $g : \mathbb{R}^J \to \mathbb{R}$ is a continuous linear functional which is zero ν-a.e., then $gT : \mathbb{R}^I \to \mathbb{R}$ is a continuous linear functional which is zero μ-a.e., so $g(Tz) = (gT)(z) = 0$. **Q**

456F Universal Gaussian distributions: Definition A centered Gaussian distribution on $\mathbb{R}^I$ is **universal** if its covariance matrix $\langle\sigma_{ij}\rangle_{i,j\in I}$ is the inner product for a Hilbert space structure on I. (See 456Xd.)

456G Proposition Let I be any set, and μ a centered Gaussian distribution on I. Then there are a set J, a universal centered Gaussian distribution ν on $\mathbb{R}^J$, and a continuous inverse-measure-preserving linear operator $T : \mathbb{R}^J \to \mathbb{R}^I$.

proof (a) Set $L^2 = L^2(\mu)$. Then for any finite $K \subseteq L^2$ there is a centered Gaussian distribution μ_K on $\mathbb{R}^K$ such that $\int x(u)x(v)\mu(dx) = (u|v)$ for all u, $v \in K$. **P** If $K = \emptyset$ or $K = \{0\}$ this is trivial, as we take μ to be the trivial distribution concentrated at 0. Otherwise, let $\langle w_i\rangle_{i<n}$ be an orthonormal basis for the linear subspace of L^2 generated by K. For each $u \in K$, express it as $\sum_{i=0}^{n-1} \alpha_{ui} w_i$. Define $T : \mathbb{R}^n \to \mathbb{R}^K$ by setting $(Tz)(u) = \sum_{i=0}^{n-1} \alpha_{ui} z(i)$ for $z \in \mathbb{R}^n$ and $u \in K$. Set $\mu_K = \mu_G^{(n)} T^{-1}$. Then μ_K is a centered Gaussian distribution, by 456Ba, and its covariance matrix is given by

$$\begin{aligned}\int x(u)x(v)\mu_K(dx) &= \int (Ty)(u)(Ty)(v)\mu_G^{(n)}(dy)\\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \alpha_{ui}\alpha_{vj} \int y(i)y(j)\mu_G^{(n)}(dy)\\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \alpha_{ui}\alpha_{vj}(w_i|w_j) = (u|v)\end{aligned}$$

for all u, $v \in K$. So μ_K is the distribution we seek. **Q**

(b) If $K \subseteq L^2$ is finite and $L \subseteq K$, then $\mu_L = \mu_K \pi_{KL}^{-1}$, where $\pi_{KL}(x) = x\restriction L$ for $x \in \mathbb{R}^K$. **P** Since $\mu_K \pi_{KL}^{-1}$ is a centered Gaussian distribution, all we have to do is to check its covariance matrix. But if u, $v \in L$ then

$$\begin{aligned}\int y(u)y(v)(\mu_K\pi_{KL}^{-1})(dy) &= \int (\pi_{KL}x)(u)(\pi_{KL}x)(v)\mu_K(dx)\\ &= \int x(u)x(v)\mu_K(dx) = (u|v) = \int y(u)y(v)\mu_L(dy).\end{aligned}$$

By 456Bb, $\mu_L = \mu_K\pi_{KL}^{-1}$. **Q**

(c) By 454G, there is a Baire measure ν' on $\mathbb{R}^{L^2}$ such that $\nu'\tilde{\pi}_K^{-1}[E] = \mu_K E$ for every finite $K \subseteq L^2$ and every Borel set $E \subseteq \mathbb{R}^K$, where $\tilde{\pi}_K(x) = x\restriction K$ for $x \in \mathbb{R}^{L^2}$. Take ν to be the completion of ν'. Then $\tilde{\pi}_K$ is inverse-measure-preserving for ν and μ_K, for every finite $K \subseteq L^2$. If $f : \mathbb{R}^{L^2} \to \mathbb{R}$ is a continuous linear functional, there are a finite $K \subseteq L^2$ and a linear functional $g : \mathbb{R}^K \to \mathbb{R}$ such that $f = \tilde{\pi}_K g$, so that

$$\nu\{x : f(x) \leq \alpha\} = \mu_K\{x : g(x) \leq \alpha\}$$

for every α, and f and g have the same distribution; as g is either normal with zero expectation or zero a.e., so is f. As f is arbitrary, ν is a centered Gaussian distribution.

(d) ν is universal. **P** If u, $v \in L^2$ set $K = \{u, v\}$. Then

$$\begin{aligned}\int x(u)x(v)\nu(dx) &= \int (\tilde{\pi}_K x)(u)(\tilde{\pi}_K x)(v)\nu(dx)\\ &= \int y(u)y(v)\mu_K(dy) = (u|v).\end{aligned}$$

Thus the covariance matrix of ν is just the inner product of the standard Hilbert space structure of L^2. **Q**

(e) For $i \in I$, let $u_i \in L^2$ be the equivalence class of the square-integrable function $x \mapsto x(i) : \mathbb{R}^I \to \mathbb{R}$. Define $T : \mathbb{R}^{L^2} \to \mathbb{R}^I$ by setting $(Ty)(i) = y(u_i)$ for every $i \in I$, $y \in \mathbb{R}^{L^2}$. Then there is a centered Gaussian distribution μ' on $\mathbb{R}^I$ such that T is inverse-measure-preserving for ν and μ'. Now the covariance matrix of μ' is defined by

$$\int x(i)x(j)\mu'(dx) = \int (Ty)(i)(Ty)(j)\nu(dy) = \int y(u_i)y(u_j)\nu(dy)$$
$$= (u_i|u_j) = \int x(i)x(j)\mu(dx)$$

for all i, $j \in I$. So μ and μ' are equal and T is inverse-measure-preserving for ν and μ.

456H Lemma Let μ be a universal centered Gaussian distribution on $\mathbb{R}^I$; give I a corresponding Hilbert space structure such that $\int x(i)x(j)\mu(dx) = (i|j)$ for all i, $j \in I$. Let $F \subseteq \mathbb{R}^I$ be a set determined by coordinates in J, where $J \subseteq I$ is a closed linear subspace for the Hilbert space structure of I. Let W be the union of all the open subsets of $\mathbb{R}^I$ which meet F in a negligible set, and W' the union of the open subsets of $\mathbb{R}^I$ which meet F in a negligible set and are determined by coordinates in J. If $F \subseteq W$ then $F \subseteq W'$.

proof (a) Let Z be the support of μ in the sense of 456D. We need to know that Z is just the set of all linear functionals from I to $\mathbb{R}$. **P** If $K \subseteq I$ is finite and $\langle \alpha_i \rangle_{i\in K} \in \mathbb{R}^K$ and $f(x) = \sum_{i\in I} \alpha_i x(i)$ for $x \in \mathbb{R}^I$, then

$$\|f\|_2^2 = \textstyle\sum_{i,j\in K} \alpha_i\alpha_j(i|j) = \|\sum_{i\in K} \alpha_i i\|^2.$$

So, for $x \in \mathbb{R}^I$,

$$\begin{aligned} x \in Z &\iff f(x) = 0 \text{ whenever } f \in (\mathbb{R}^I)^* \text{ and } \|f\|_2 = 0 \\ &\iff \sum_{i\in K} \alpha_i x(i) = 0 \text{ whenever } K \subseteq I \text{ is finite and } \sum_{i\in K} \alpha_i i = 0 \text{ in } I \\ &\iff x : I \to \mathbb{R} \text{ is linear. } \textbf{Q} \end{aligned}$$

(b) Let $J^\perp$ be the orthogonal complement of J in I, so that $I = J \oplus J^\perp$ (4A4Je). Give $\mathbb{R}^J$ and $\mathbb{R}^{J^\perp}$ the centered Gaussian distributions μ_J, $\mu_{J^\perp}$ induced by μ and the projections $x \mapsto x\restriction J$, $x \mapsto x\restriction J^\perp$. Then the product measure λ on $\mathbb{R}^J \times \mathbb{R}^{J^\perp}$ is also a centered Gaussian distribution (456Be). Define $T : \mathbb{R}^J \times \mathbb{R}^{J^\perp} \to \mathbb{R}^I$ by setting $T(u,v)(j+k) = u(j) + v(k)$ whenever $j \in J$, $k \in J^\perp$, $u \in \mathbb{R}^J$ and $v \in \mathbb{R}^{J^\perp}$. Then T is inverse-measure-preserving for λ and μ. **P** T is a continuous linear operator so we have a centered Gaussian distribution μ' on $\mathbb{R}^I$ such that T is inverse-measure-preserving for λ and μ'. If j, $j' \in J$ and k, $k' \in J^\perp$,

$$\begin{aligned} \int x(j+k)x(j'+k')\mu'(dx) &= \int T(u,v)(j+k)T(u,v)(j'+k')\lambda(d(u,v)) \\ &= \int (u(j)+v(k))(u(j')+v(k'))\lambda(d(u,v)) \\ &= \int u(j)u(j')\mu_J(du) + \int v(k)v(k')\mu_{J^\perp}(dv) \\ &= \int x(j)x(j')\mu(dx) + \int x(k)x(k')\mu(dx) \\ &= (j|j') + (k|k') = (j+k|j'+k') \\ &= \int x(j+k)x(j'+k')\mu(dx). \end{aligned}$$

Thus μ and μ' have the same covariance matrix and are equal, and T is inverse-measure-preserving for λ and μ. **Q**

(c) Take any $z \in W \cap Z$. Then there is an open set V, determined by coordinates in a finite set $K_0 \subseteq I$, such that $z \in V$ and $\mu(V \cap F) = 0$. Let $\epsilon > 0$ be such that $y \in V$ whenever $x \in \mathbb{R}^I$ and $|x(i) - z(i)| < 2\epsilon$ for every $i \in K_0$. Express each $k \in K_0$ as $k' + k''$ where $k' \in J$ and $k'' \in J^\perp$. Set

$$V' = \{x : x \in \mathbb{R}^I, |x(k') - z(k')| < \epsilon \text{ for every } k \in K_0\}.$$

Then V' is an open set, determined by coordinates in J, and contains z. Also $\mu(V' \cap F) = 0$. **P** Set $V'' = \{x : x \in \mathbb{R}^I, |x(k'') - z(k'')| < \epsilon \text{ for every } k \in K_0\}$. Then $V \supseteq V' \cap V'' \cap Z$. Since Z has full outer measure (456D),

$$\mu(V' \cap V'' \cap F) = \mu^*(V' \cap V'' \cap F \cap Z) \leq \mu(V \cap F) = 0.$$

Now

$$\begin{aligned} 0 = \mu(V' \cap V'' \cap F) &= \lambda T^{-1}[V' \cap F \cap V''] \\ &= \lambda\{(x\restriction J, x\restriction J^\perp) : x \in V' \cap F \cap V''\} \end{aligned}$$

(because $V' \cap F \cap V''$ is determined by coordinates in $J \cup J^\perp$)

$$= \mu_J\{x\restriction J : x \in V' \cap F\} \cdot \mu_{J^\perp}\{x\restriction J^\perp : x \in V''\}$$

because $V' \cap F$ is determined by coordinates in J, while V'' is determined by coordinates in $J^\perp$. However, $z \in V''$, and $z\restriction J^\perp$ belongs to the support Z' of $\mu_{J^\perp}$, by 456Eb; since Z' is self-supporting, and $\{x\restriction J^\perp : x \in V''\}$ is open, $\mu_{J^\perp}\{x\restriction J^\perp : x \in V''\} > 0$. We conclude that

$$0 = \mu_J\{x\restriction J : x \in V' \cap F\} = \mu(V' \cap F). \ \textbf{Q}$$

(d) This shows that $z \in W'$. As z is arbitrary, $W \cap Z \subseteq W'$.

? Suppose, if possible, that there is a point $z_0 \in F \setminus W'$. If i, $j \in J$ and $\alpha \in \mathbb{R}$, then

$$\{x : x(i+j) \neq x(i) + x(j)\}, \quad \{x : x(\alpha i) \neq \alpha x(i)\}$$

are negligible open sets determined by coordinates in J, so are included in W' and do not contain z_0. Thus $z_0\restriction J : J \to \mathbb{R}$ is linear. Let $z : I \to \mathbb{R}$ be a linear functional extending $z_0\restriction J$. Then $z \in Z$ and $z\restriction J = z_0\restriction J$; as both F and W' are determined by coordinates in J, $z \in F \setminus W'$. But this means that $z \in Z \cap W \setminus W'$, which is impossible. **X**

So $F \subseteq W'$, as claimed.

456I Cluster sets: Lemma Let I be a countable set, $n \geq 1$ an integer and μ a centered Gaussian distribution on $\mathbb{R}^{I \times n}$. For $\epsilon > 0$ set

$$I_\epsilon = \{i : i \in I, \int |x(i,r)|^2 \mu(dx) \leq \epsilon^2 \text{ for every } r < n\};$$

suppose that no I_ϵ is empty.

(a) There is a closed set $F \subseteq \mathbb{R}^n$ such that

$$F = \textstyle\bigcap_{\epsilon>0} \overline{\{\langle x(i,r)\rangle_{r<n} : i \in I_\epsilon\}}$$

for almost every $x \in \mathbb{R}^{I \times n}$.

(b) If $z \in F$ and $-1 \leq \alpha \leq 1$, then $\alpha z \in F$.

(c) If F is bounded, then there is some $\epsilon > 0$ such that $\sup_{i \in I_\epsilon, r<n} |x(i,r)| < \infty$ for almost every $x \in \mathbb{R}^{I \times n}$.

proof (a)(i) For $x \in \mathbb{R}^{I \times n}$, $i \in I$ set $S_i(x) = \langle x(i,r)\rangle_{r<n} \in \mathbb{R}^n$. For $x \in \mathbb{R}^{I \times n}$ set

$$F_x = \textstyle\bigcap_{\epsilon>0} \overline{\{S_i x : i \in I_\epsilon\}},$$

so that F_x is a closed subset of $\mathbb{R}^n$. For $A \subseteq \mathbb{R}^n$ set $E_A = \{x : x \in \mathbb{R}^{I \times n}, A \cap F_x \neq \emptyset\}$.

(ii) By 456Bd, there is a continuous linear operator $T : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{I \times n}$ such that $\mu = \mu_G^{(\mathbb{N})} T^{-1}$. Set $T_i = S_i T$ for $i \in I$; then $T_i : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^n$ is a continuous linear operator. For $y \in \mathbb{R}^{\mathbb{N}}$, set

$$\tilde{F}_y = F_{T(y)} = \textstyle\bigcap_{\epsilon>0} \overline{\{T_i(y) : i \in I_\epsilon\}}.$$

For $A \subseteq \mathbb{R}^n$ set

$$\tilde{E}_A = T^{-1}[E_A] = \{y : y \in \mathbb{R}^{\mathbb{N}}, A \cap \tilde{F}_y \neq \emptyset\}.$$

(iii) If $K \subseteq \mathbb{R}^n$ is compact, then $\tilde{E}_K$ is a Borel subset of $\mathbb{R}^{\mathbb{N}}$. **P** Let $\mathcal{V}$ be a countable base for the topology of $\mathbb{R}^n$, and for $k \geq 1$ let $\mathcal{V}_k$ be the set of members of $\mathcal{V}$ with diameter at most $1/k$ which meet K. Then $T_i^{-1}[V]$ is a Borel set for every $V \in \mathcal{V}$ and $i \in I$, so

$$E' = \bigcap_{k\geq 1} \bigcup_{V\in\mathcal{V}_k} \bigcup_{i\in I_{1/k}} T_i^{-1}[V]$$

is a Borel set.

If $y \in \tilde{E}_K$, take $k \geq 1$. There is a $z \in \tilde{F}_y \cap K$. Let $V \in \mathcal{V}$ be such that $z \in V$ and $\operatorname{diam} V \leq 1/k$; in this case $V \in \mathcal{V}_k$. Because $z \in \overline{\{T_i(y) : i \in I_{1/k}\}}$, there is an $i \in I_{1/k}$ such that $T_i(y) \in V$. As k is arbitrary, this shows that $y \in E'$; thus $\tilde{E}_K \subseteq E'$.

If $y \notin \tilde{E}_K$, then K is a compact set disjoint from the closed set $\tilde{F}_y$. There is therefore some $\epsilon > 0$ such that $K \cap \overline{\{T_i(y) : i \in I_\epsilon\}} = \emptyset$ (since these form a downwards-directed family of compact sets with empty intersection). Next, there is a $\delta > 0$ such that $B(z,\delta) \cap \overline{\{T_i(y) : i \in I_\epsilon\}} = \emptyset$ for every $z \in K$ (2A2Ed). Let $k \geq 1$ be such that $1/k \leq \min(\epsilon,\delta)$. If $V \in \mathcal{V}_k$ and $i \in I_{1/k}$, there is some $z \in K \cap V$ so $V \subseteq B(z,\delta)$ and $T_i(y) \notin V$. This shows that $y \notin E'$. As y is arbitrary, $E' \subseteq \tilde{E}_K$.

So $\tilde{E}_K = E'$ is a Borel set. **Q**

It follows at once that $\tilde{E}_H$ is a Borel set for every K_σ set H, in particular, for any open or closed set H.

(iv) We need a simple estimate on the coefficients of the linear operators T_i. Let α_{irj} be such that $T_i(y) = \langle \sum_{j=0}^\infty \alpha_{irj} y(j)\rangle_{r<n}$ for $i \in I$, $r < n$ and $y \in \mathbb{R}^{\mathbb{N}}$. (Of course $\{j : \alpha_{irj} \neq 0\}$ is finite for each i and r.) Then

$$\int x(i,r)^2 \mu(dx) = \int T_i(y)(r)^2 \mu_G^{(\mathbb{N})}(dy) = \sum_{j=0}^\infty \alpha_{irj}^2,$$

so $|\alpha_{irj}| \leq \epsilon$ whenever $i \in I_\epsilon$, $r < n$ and $j \in \mathbb{N}$.

(v) Now suppose that $H \subseteq \mathbb{R}^n$ is open, that $K \subseteq H$ is compact, and that $\mu_G^{(\mathbb{N})}\tilde{E}_K > 0$. Then $\mu_G^{(\mathbb{N})}\tilde{E}_H = 1$. **P** Let $\epsilon > 0$. Let $\langle \epsilon_j\rangle_{j\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $\sum_{j=0}^\infty \epsilon_j \leq \frac{1}{2}\min(\epsilon, \mu_G^{(\mathbb{N})}\tilde{E}_K)$, and for each $j \in \mathbb{N}$ let $\gamma_j \geq 0$ be such that $\mu_G[-\gamma_j,\gamma_j] \geq 1 - \epsilon_j$. Then

$$\tilde{E} = \{y : y \in \tilde{E}_K, |y(j)| \leq \gamma_j \text{ for every } j \in \mathbb{N}\}$$

has measure at least $\mu_G^{(\mathbb{N})}\tilde{E}_K - \sum_{j=0}^\infty \epsilon_j > 0$. By 254Sb, there are an $m \in \mathbb{N}$ and a set $\tilde{E}'$, of measure at least $1 - \frac{1}{2}\epsilon$, such that for every $y' \in \tilde{E}'$ there is a $y \in \tilde{E}$ such that $y(j) = y'(j)$ whenever $j > m$. Set $\tilde{E}'' = \{y : y \in \tilde{E}', |y(j)| \leq \gamma_j \text{ for every } j \in \mathbb{N}\}$, so that $\mu_G^{(\mathbb{N})}\tilde{E}'' \geq 1 - \epsilon$.

Let $\delta > 0$ be such that $z' \in H$ whenever $z \in K$ and $\|z - z'\| \leq 2\delta$. Let $\eta > 0$ be such that $2\eta\sqrt{n}\sum_{j=0}^m \gamma_j \leq \delta$. If $y' \in \tilde{E}''$ and $i \in I_\eta$, there is a $y \in \tilde{E}$ such that $y(j) = y'(j)$ for $j > m$. Also $|y(j) - y'(j)| \leq 2\gamma_j$ for $j \leq m$, so

$$|T_i(y)(r) - T_i(y')(r)| \leq \sum_{j=0}^\infty |\alpha_{irj}||y(j) - y'(j)| \leq \sum_{j=0}^m 2\eta\gamma_j \leq \frac{\delta}{\sqrt{n}}$$

for every $r < n$, and $\|T_i(y) - T_i(y')\| \leq \delta$.

Now $\tilde{F}_y \cap K \neq \emptyset$; take $z \in \tilde{F}_y \cap K$. For every $\zeta > 0$, there is an $i \in I_{\min(\eta,\zeta)}$ such that $\|z - T_i(y)\| \leq \delta$, so that $\|z - T_i(y')\| \leq 2\delta$. This means that $B(x,2\delta) \cap \{T_i(y') : i \in I_\zeta\}$ is not empty. As $B(x,2\delta)$ is compact, it must meet $\tilde{F}_{y'}$. But this means that $H \cap \tilde{F}_{y'} \neq \emptyset$, by the choice of δ. As y' is arbitrary, $\tilde{E}'' \subseteq \tilde{E}_H$, while $\mu_G^{(\mathbb{N})}\tilde{E}'' \geq 1 - \epsilon$.

This works for every $\epsilon > 0$. So $\tilde{E}_H$ is conegligible, as claimed. **Q**

(vi) If $H \subseteq \mathbb{R}^n$ is open and $\mu_G^{(\mathbb{N})}\tilde{E}_H > 0$, then (because H is σ-compact) there is a compact set $K \subseteq H$ such that $\mu_G^{(\mathbb{N})}\tilde{E}_K > 0$, and (e) tells us that $\mu_G^{(\mathbb{N})}\tilde{E}_H = 1$.

Set

$$\mathcal{V}_0 = \{V : V \in \mathcal{V}, \mu_G^{(\mathbb{N})}\tilde{E}_V = 0\} = \{V : V \in \mathcal{V}, \mu_G^{(\mathbb{N})}\tilde{E}_V < 1\}.$$

Then we see that

$$\mathcal{V}_0 = \{V : V \in \mathcal{V}, \tilde{F}_y \cap V = \emptyset\}$$

for almost every $y \in \mathbb{R}^{\mathbb{N}}$, that is,

$$\mathcal{V}_0 = \{V : V \in \mathcal{V}, F_x \cap V = \emptyset\}$$

for almost every $x \in \mathbb{R}^{I\times n}$. But as every F_x is closed, we have $F_x = \mathbb{R}^n \setminus \bigcup \mathcal{V}_0$ for almost every x. So we can set $F = \mathbb{R}^n \setminus \bigcup \mathcal{V}_0$.

(b)(i) Give $\mathbb{R}^{I\times 2n} \cong (\mathbb{R}^{I\times n})^2$ the measure λ corresponding to the product measure $\mu\times\mu$; by 456Be, this is a centered Gaussian distribution. For $(x_1, x_2) \in (\mathbb{R}^{I\times n})^2$, set

$$F'_{x_1x_2} = \textstyle\bigcap_{\epsilon>0} \overline{\{(S_ix_1, S_ix_2) : i \in I_\epsilon\}}.$$

By (a), we have a closed set $F' \subseteq \mathbb{R}^{2n}$ such that $F' = F'_{x_1x_2}$ for almost all x_1, x_2. (Of course $I_\epsilon = \{i : \int |x_j(i,r)|^2\lambda(dx) \leq \epsilon^2$ for every $j \in \{1,2\}$, $r < n\}$ whenever $\epsilon > 0$.)

(ii) Now $(z, 0) \in F'$. **P** Take $x_1 \in \mathbb{R}^{I\times n}$ such that $F = F_{x_1}$ and $E = \{x_2 : F' = F'_{x_1x_2}\}$ is conegligible. (Almost every point of $\mathbb{R}^{I\times n}$ has these properties.) For $k \in \mathbb{N}$, $z \in \overline{\{S_ix_1 : i \in I_{2^{-k}}\}}$; let $i_k \in I_{2^{-k}}$ be such that $\sum_{r<n} |z(r) - x_1(i_k, r)| \leq 2^{-k}$. Next,

$$\textstyle\sum_{k=0}^\infty \sum_{r=0}^{n-1} \int |x(i_k,r)|^2\mu(dx) \leq \sum_{k=0}^\infty 2^{-2k}n < \infty,$$

so $\sum_{k=0}^\infty \sum_{r=0}^{n-1} |x(i_k,r)|^2$ is finite for almost every $x \in \mathbb{R}^I$, and there must be an $x_2 \in E$ such that $\sum_{k=0}^\infty \sum_{r=0}^{n-1} |x_2(i_k,r)|^2$ is finite. But in this case $\lim_{k\to\infty} x_2(i_k, r) = 0$ for every r, while $\lim_{k\to\infty} x_1(i_k, r) = z(r)$. Accordingly $(z, 0) \in F'_{x_1x_2} = F'$. **Q**

(iii) Set $\beta = \sqrt{1-\alpha^2}$ and define $\tilde{T} : (\mathbb{R}^{I\times n})^2 \to \mathbb{R}^{I\times n}$ by setting $\tilde{T}(x_1, x_2) = \alpha x_1 + \beta x_2$ for x_1, $x_2 \in \mathbb{R}^{I\times n}$. Then $\tilde{T}$ is a continuous linear operator, so the image measure $\lambda\tilde{T}^{-1}$ is a centered Gaussian distribution on $\mathbb{R}^{I\times n}$ (456Ba). Moreover, it has the same covariance matrix as μ. **P** If i, $j \in I$ then

$$\begin{aligned}
\int x(i)x(j)(\lambda\tilde{T}^{-1})(dx) &= \int \tilde{T}(x_1,x_2)(i)\tilde{T}(x_1,x_2)(j)\lambda(d(x_1,x_2)) \\
&= \int (\alpha x_1(i) + \beta x_2(i))(\alpha x_1(j) + \beta x_2(j))\lambda(d(x_1,x_2)) \\
&= \alpha^2 \int x_1(i)x_1(j)\lambda(d(x_1,x_2)) + \beta^2 \int x_2(i)x_2(j)\lambda(d(x_1,x_2)) \\
&= (\alpha^2+\beta^2)\int x(i)x(j)\mu(dx) = \int x(i)x(j)\mu(dx). \ \textbf{Q}
\end{aligned}$$

So $\lambda\tilde{T}^{-1} = \mu$ (456Bb).

(iv) If x_1, $x_2 \in \mathbb{R}^I$ are such that $(z, 0) \in F'_{x_1x_2}$, then $\alpha z \in F_{\tilde{T}(x_1,x_2)}$. **P** For every $\epsilon > 0$ there is an $i \in I_\epsilon$ such that $|z(r) - x_1(i, r)| \leq \epsilon$ and $|x_2(i, r)| \leq \epsilon$ for every $r < n$. But now $|\alpha z(r) - \tilde{T}(x_1, x_2)(r)| \leq 2\epsilon$ for every $r < n$. **Q** So

$$\tilde{T}^{-1}[\{x : \alpha z \in F_x\}] = \{(x_1,x_2) : \alpha z \in F_{\tilde{T}(x_1,x_2)}\} \supseteq \{(x_1,x_2) : (z,0) \in F'_{x_1x_2}\}$$

is λ-conegligible, and $\alpha z \in F_x$ for μ-almost every x, that is, $\alpha z \in F$, as claimed.

(c) Suppose now that F is bounded.

(i) For $L \subseteq I$, $\alpha \geq 0$ set

$$Q(L,\alpha) = \textstyle\bigcup_{i\in L, r<n} \{x : |x(i,r)| \geq \alpha\}.$$

By 456C, applied to the image of μ under the map $x \mapsto x\restriction L : \mathbb{R}^I \to \mathbb{R}^L$,

$$\mu Q(L, \tfrac{1}{2}\alpha) \geq 2\mu Q(L,\alpha)(1 - \mu Q(L,\alpha))^3$$

for every finite $L \subseteq I$ and every $\alpha \geq 0$.

Let $\beta > 0$ be such that $\delta = 2\beta(1-(n+1)\beta)^3 - \beta > 0$, and let $\alpha_0 > 0$ be such that $\alpha_0^2\beta \geq 1$ and $\|z\| < \frac{1}{2}\alpha_0$ for every $z \in F$, so that $\mu\{x : |x(i,r)| \geq \alpha_0\} \leq \beta$ whenever $i \in I_1$ and $r < n$, and $\mu Q(\{i\}, \alpha_0) \leq n\beta$ for every $i \in I_1$. Set $K = \{z : z \in \mathbb{R}^n, \frac{1}{2}\alpha_0 \leq \max_{r<n} |z(r)| \leq \alpha_0\}$, so that K is a compact set disjoint from F. For almost every x,

$$\emptyset = K \cap F = K \cap F_x \supseteq \textstyle\bigcap_{k\geq 1} K \cap \overline{\{S_ix : i \in I_{1/k}\}},$$

so there is a $k \geq 1$ such that $S_ix \notin K$ for every $i \in I_{1/k}$. Since the sets

$$\{x : S_i(x) \in K \text{ for some } i \in I_{1/k}\}$$

form a non-increasing sequence of measurable sets with negligible intersection, there is a $k \geq 1$ such that

$$\mu\{x : S_i(x) \in K \text{ for some } i \in I_{1/k}\} < \delta.$$

(ii) ? Suppose, if possible, that

$$\mu Q(I_{1/k}, \alpha_0) > \beta.$$

Let $L \subseteq I_{1/k}$ be a finite set of minimal size such that $\gamma = \mu Q(L, \alpha_0) \geq \beta$. Since $\mu Q(\{i\}, \alpha_0) \leq n\beta$ for any $i \in L$, and L is minimal, we must have

$$\beta \leq \gamma \leq (n+1)\beta.$$

Now this means that

$$\mu Q(L, \tfrac{1}{2}\alpha_0) \geq 2\gamma(1-\gamma)^3$$

(see (i) above)

$$\geq 2\gamma(1-(n+1)\beta)^3 = \gamma(1+\frac{\delta}{\beta}) \geq \gamma + \delta,$$

so that $\mu(Q(L, \frac{1}{2}\alpha_0) \setminus Q(L, \alpha_0)) \geq \delta$. But if $x \in Q(L, \frac{1}{2}\alpha_0) \setminus Q(L, \alpha_0)$ there is some $i \in L$ such that $\max_{r<n} |x(i,r)| \geq \frac{1}{2}\alpha_0$ while $\max_{r<n} |x(i,r)| < \alpha_0$, in which case $S_i(x) \in K$. So we get

$$\delta \leq \mu(Q(L, \tfrac{1}{2}\alpha_0) \setminus Q(L, \alpha_0)) \leq \mu(\{x : S_i(x) \in K \text{ for some } i \in L\}$$
$$\leq \mu(\{x : S_i(x) \in K \text{ for some } i \in I_{1/k}\} < \delta$$

which is absurd. **X**

(iii) Thus $\mu Q(I_{1/k}, \alpha_0) \leq \beta$. For $\alpha \geq 0$, set $f(\alpha) = \mu Q(I_{1/k}, \alpha)$; then f is non-increasing. Also $f(\frac{1}{2}\alpha) \geq 2f(\alpha)(1-f(\alpha))^3$ for every α. **P?** Otherwise, because

$$f(\alpha) = \sup\{\mu Q(L, \alpha) : L \subseteq I_{1/k} \text{ is finite}\},$$

there is a finite $L \subseteq I_{1/k}$ such that $f(\frac{1}{2}\alpha) < 2\gamma(1-\gamma)^3$, where $\gamma = \mu Q(L, \alpha)$. But in this case $\mu Q(L, \frac{1}{2}\alpha) \leq f(\frac{1}{2}\alpha) < 2\gamma(1-\gamma)^3$, which is impossible, as remarked in (i). **XQ**

Set $\zeta = \lim_{\alpha\to\infty} f(\alpha)$. Then

$$\zeta = \lim_{\alpha\to\infty} f(\tfrac{1}{2}\alpha) \geq 2\zeta(1-\zeta)^3.$$

But we also know, from (ii), that $\zeta \leq f(\alpha_0) \leq \beta$. So $(1-\zeta)^3 \geq (1-\beta)^3 > \frac{1}{2}$ and ζ must be 0.

What this means is that if we set $\epsilon = \frac{1}{k}$ then

$$\lim_{\alpha\to\infty} \mu\{x : \sup_{i\in I_\epsilon, r<n} |x(i,r)| > \alpha\} = 0,$$

that is, $\sup_{i\in I_\epsilon, r<n} |x(i,r)|$ is finite for almost every $x \in \mathbb{R}^{I\times n}$, as claimed.

456J Lemma Let J be a set and μ a centered Gaussian distribution on $\mathbb{R}^J$. Let M be the linear subspace of $L^2(\mu)$ generated by $\{\pi_j^\bullet : j \in J\}$, where $\pi_j(x) = x(j)$ for $x \in \mathbb{R}^J$ and $j \in J$. If M is separable (for the norm topology) then μ is τ-additive.

proof Suppose, if possible, otherwise.

(a) There is an upwards-directed family $\mathcal{G}$ of open Baire sets in $\mathbb{R}^J$ such that $W_0 = \bigcup \mathcal{G}$ is a Baire set and $\mu W_0 > \sup_{G\in\mathcal{G}} \mu G$. Let $\mathcal{G}_0 \subseteq \mathcal{G}$ be a countable upwards-directed set such that $\sup_{G\in\mathcal{G}_0} \mu G = \sup_{G\in\mathcal{G}} \mu G$, and set $W_1 = W_0 \setminus \bigcup \mathcal{G}_0$; then $\mu W_1 > 0$ and $\mu(W_1 \cap G) = 0$ for every $G \in \mathcal{G}$. Let W be a non-negligible zero set included in W_1.

For each $n \in \mathbb{N}$, let $\mathcal{V}_n$ be a countable base for the topology of $\mathbb{R}^n$ consisting of open balls. Let $\mathcal{G}_n^*$ be the family of open sets of $\mathbb{R}^J$ of the form $T^{-1}[V]$, where $T : \mathbb{R}^J \to \mathbb{R}^n$ is a continuous linear operator, $V \subseteq \mathbb{R}^n$ is open and $\mu(W \cap T^{-1}[V]) = 0$. Of course $\mathcal{G}_0^* = \emptyset$.

(b) For $n \geq 1$ and $V \in \mathcal{V}_n$, let $\mathcal{T}_{nV}$ be the family of continuous linear operators $T : \mathbb{R}^J \to \mathbb{R}^n$ such that $W \cap T^{-1}[V]$ is negligible, but not included in $\bigcup \mathcal{G}^*_{n-1}$. Index $\mathcal{T}_{nv}$ as $\langle T_i \rangle_{i \in I(n,V)}$; it will be convenient to do this in such a way that all the $I(n,V)$ are disjoint. Define f_{ir}, for $i \in I(n,V)$ and $r < n$, by saying that $T_i(x) = \langle f_{ir}(x) \rangle_{r<n}$ for $x \in \mathbb{R}^J$. Define $\phi_n : \bigcup_{V \in \mathcal{V}_n} I(n,V) \to M^n$ by setting $\phi_n(i) = \langle f^\bullet_{ir} \rangle_{r<n}$ for each $i \in \bigcup_{V \in \mathcal{V}_n} I(n,V)$. Because M is separable in its norm topology, M^n is separable in its product topology (4A2Pa). Fix a countable set $I'(n,V) \subseteq I(n,V)$ such that $\{\phi_n(i) : i \in I'(n,V)\}$ is dense in $\{\phi_n(i) : i \in I(n,V)\}$. Set $\rho_n(\langle u_r \rangle_{r<n}, \langle v_r \rangle_{r<n}) = \max_{r<n} \|u_r - v_r\|_2$ for $\langle u_r \rangle_{r<n}$, $\langle v_r \rangle_{r<n} \in M^n$, so that ρ_n is a metric defining the product topology of M^n.

(c) If $j \in I(n,V)$, then there is a $\delta > 0$ such that

$$\{T_i(x) : i \in I'(n,V),\ \rho_n(\phi_n(i), \phi_n(j)) \leq \delta\}$$

is bounded for almost every $x \in \mathbb{R}^J$. **P** For $i \in I'(n,V)$ and $r < n$, set $f'_{ir} = f_{ir} - f_{jr}$, $T'_i = T_i - T_j$; for $\delta > 0$, write I'_δ for

$$\{i : i \in I'(n,V),\ \rho_n(\phi_n(i), \phi_n(j)) \leq \delta\} = \{i : i \in I'(n,V),\ \|f'_{ir}\|_2 \leq \delta \text{ for every } r < n\}.$$

Because $\phi_n(j) \in \overline{\{\phi_n(i) : i \in I'(n,V)\}}$, no I'_δ is empty. So 456Ia tells us that there is a closed set $F \subseteq \mathbb{R}^n$ such that

$$F = F_x = \bigcap_{\delta > 0} \overline{\{T'_i(x) : i \in I'_\delta\}}$$

for almost every $x \in \mathbb{R}^J$. By 456Ib, $\alpha z \in F$ whenever $z \in F$ and $|\alpha| \leq 1$. **?** If F is not bounded, then it must include a line L through 0. (The sets $\{\frac{1}{n} z : z \in F, \|z\| = n\}$, for $n \geq 1$, form a non-increasing sequence of non-empty compact sets, so there is a point z_0 belonging to them all; take L to be the set of multiples of z_0.) Let $D \subseteq L$ be a countable dense set. For $z \in D$ and $k \in \mathbb{N}$ we know that

for every $i \in I(n,V)$, $T_i(x) \notin V$ for almost every $x \in W$,

for almost every $x \in \mathbb{R}^J$ there is an $i \in I'(n,V)$ such that $\|T_i(x) - T_j(x) - z\| \leq 2^{-k}$

and therefore

for almost every $x \in W$, $T_i(x) \notin V$ for every $i \in I'(n,V)$, but there is an $i \in I'(n,V)$ such that $\|T_i(x) - T_j(x) - z\| \leq 2^{-k}$

so that

for almost every $x \in W$, the distance from $T_j(x) + z$ to the closed set $\mathbb{R}^n \setminus V$ is at most 2^{-k}.

This is true for every $k \in \mathbb{N}$, so we get

$T_j(x) + z \notin V$ for almost every $x \in W$.

And *this* is true for every $z \in D$, so we get

for almost every $x \in W$, $T_j(x) + z \notin V$ for every $z \in D$, so $T_j(x) \notin V + L$.

Let $S : \mathbb{R}^n \to \mathbb{R}^{n-1}$ be a linear operator with kernel L, and set $V' = S[V]$. Then $V' \subseteq \mathbb{R}^{n-1}$ is open, and $W \cap (ST_j)^{-1}[V'] = W \cap T_j^{-1}[V + L]$ is negligible. But this means that $T_j^{-1}[V] \subseteq (ST_j)^{-1}[V'] \in \mathcal{G}^*_{n-1}$ and $T_j^{-1}[V]$ is included in $\bigcup \mathcal{G}^*_{n-1}$; which contradicts the definition of $\mathcal{T}_{nV}$. **X**

So F is bounded. By 456Ic, there is some $\delta > 0$ such that $\sup_{i \in I'_\delta, r<n} |f'_{ir}(x)| < \infty$ for almost every $x \in \mathbb{R}^J$. Of course this means that $\{T_i(x) : i \in I(n,V), \rho_n(\phi_n(i), \phi_n(j)) \leq \delta\}$ is bounded for almost every $x \in \mathbb{R}^J$. **Q**

(d) Accordingly $\phi_n[I(n,V)]$ is covered by the family $\mathcal{U}_{nV}$ of open sets $U \subseteq M^n$ such that $\{T_i(x) : i \in I'(n,V),\ \phi_n(i) \in U\}$ is bounded for almost every x. Because M^n is separable and metrizable, it is hereditarily Lindelöf (4A2P(a-iii)), so there is a sequence $\langle U_{nVk} \rangle_{k \in \mathbb{N}}$ in $\mathcal{U}_{nV}$ covering $\phi_n[I(n,V)]$. For each k, set $I_{nVk} = I'(n,V) \cap \phi_n^{-1}[U_{nVk}]$. Then $\{T_i(x) : i \in I_{nVk}\}$ is bounded for almost every x. Because $\phi_n[I'(n,V)]$ is dense in $\phi_n[I(n,V)]$, $\phi_n[I_{nVk}] = \phi_n[I'(n,V)] \cap U_{nVk}$ is dense in $\phi_n[I(n,V)] \cap U_{nVk}$. So for every $i \in I(n,V)$ there is a $k \in \mathbb{N}$ such that $\phi_n(i) \in \overline{\phi_n[I_{nVk}]}$.

(e) Recall that $W \subseteq \mathbb{R}^J$ is a non-negligible zero set included in

$$\bigcup_{n \geq 1} \bigcup \mathcal{G}^*_n = \bigcup_{n \geq 1} \bigcup_{V \in \mathcal{V}_n} \bigcup_{i \in I(n,V)} T_i^{-1}[V].$$

Let $J_0 \subseteq J$ be a countable set such that W is determined by coordinates in J_0.

Let $\langle \epsilon_j \rangle_{j\in J_0}$ and $\langle \epsilon'_{nVk} \rangle_{n\geq 1, V\in\mathcal{V}_n, k\in\mathbb{N}}$ be families of strictly positive real numbers such that $\sum_{j\in J_0} \epsilon_j$ and $\sum_{n=1}^\infty \sum_{V\in\mathcal{V}_n} \sum_{k=0}^\infty \epsilon'_{nVk}$ are both at most $\frac{1}{3}\mu W$. Let $\langle \gamma_j \rangle_{j\in J_0}$ and $\langle \gamma'_{nVk} \rangle_{n\geq 1, V\in\mathcal{V}_n, k\in\mathbb{N}}$ be such that

$$\mu\{x : x \in \mathbb{R}^J, |x(j)| \geq \gamma_j\} \leq \epsilon_j \text{ for every } j \in J_0,$$

$$\mu\{x : x \in \mathbb{R}^J, \sup_{i\in I_{nVk}} \|T_i(x)\| \geq \gamma'_{nVk}\} \leq \epsilon'_{nVk} \text{ for every } n \geq 1,\ V \in \mathcal{V}_n,\ k \in \mathbb{N}.$$

Set $W' = \{x : x \in W, |x(j)| \leq \gamma_j \text{ for every } j \in J_0\}$; then $\mu W' \geq \frac{2}{3}\mu W$ and W' is of the form $C \times \mathbb{R}^{J\setminus J_0}$, where $C \subseteq \mathbb{R}^{J_0}$ is compact. Set

$$W'' = \{x : x \in W', \|T_i(x)\| \leq \gamma'_{nVk} \text{ whenever } n \geq 1,\ V \in \mathcal{V}_n,\ k \in \mathbb{N} \text{ and } i \in I_{nVk}\};$$

then $\mu W'' \geq \frac{1}{3}\mu W$.

(f) Set $I = \bigcup_{n\geq 1, V\in\mathcal{V}_n} I(n,V) \times n$. If $K \subseteq I$ is finite, then

$$W'_K = \{x : x \in W', \|f_{ir}(x)\| \leq \gamma'_{nVk} \text{ whenever } n \geq 1,\ V \in \mathcal{V}_n,\ k \in \mathbb{N},\ (i,r) \in K \text{ and } \phi_n(i) \in K \cap \overline{\phi_n[I_{nVk}]}\}$$

has measure at least $\frac{1}{3}\mu W$. **P** For each quintuple (i,r,n,V,k) with $n \geq 1$, $V \in \mathcal{V}_n$, $k \in \mathbb{N}$, $(i,r) \in K$ and $\phi_n(i) \in \overline{\phi_n[I_{nVk}]}$, there is a sequence $\langle i_m \rangle_{m\in\mathbb{N}}$ in I_{nVk} such that $\rho_n(\phi_n(i), \phi_n(i_m)) \leq 2^{-m}$ for every m; so that $\|f_{ir} - f_{i_m r}\|_2 \leq 2^{-m}$ for every m. But this means that $f_{ir}(x) = \lim_{m\to\infty} f_{i_m r}(x)$ for almost every $x \in \mathbb{R}^J$. Accordingly $|f_{ir}(x)| \leq \gamma'_{nVk}$ for almost every $x \in W''$. Since there are only countably many such quintuples (i,r,n,V,k), we see that $W'' \setminus W'_K$ is negligible, so $\mu W'_K \geq \mu W'' \geq \frac{1}{3}\mu W$. **Q**

(g) For $x \in \mathbb{R}^J$, define $Tx \in \mathbb{R}^I$ by setting $(Tx)(i,r) = f_{ir}(x)$ for $i \in I(n,V)$ and $r < n$. Then $T : \mathbb{R}^J \to \mathbb{R}^I$ is a continuous linear operator. By 4A4H, $T[W']$ is closed.

For finite $K \subseteq I$, let $\mathcal{H}_K$ be the family of open subsets H of $\mathbb{R}^K$ such that $\mu\{x : x \in W, Tx\restriction K \in H\} = 0$. Then $\mathcal{H}_K$ is closed under countable unions so has a largest member H_K. Now there is a $K \in [I]^{<\omega}$ such that $Tx\restriction K \in H_K$ for every $x \in W'_K$. **P?** Otherwise, choose for each $K \in [I]^{<\omega}$ an $x_K \in W'_K$ such that $Tx_K\restriction K \notin H_K$. Let $\mathcal{F}$ be an ultrafilter on $[I]^{<\omega}$ containing $\{K : L \subseteq K \in [I]^{<\omega}\}$ for every finite $L \subseteq I$. If $(i,r) \in I$, there are $n \geq 1$, $V \in \mathcal{V}_n$ and $k \in \mathbb{N}$ such that $r < n$ and $\phi_n(i) \in \overline{\phi_n[I_{nVk}]}$, in which case $|f_{ir}(x_K)| \leq \gamma'_{nVk}$ whenever $(i,r) \in K \in [I]^{<\omega}$. This means that $\lim_{K\to\mathcal{F}} f_{ir}(x_K)$ must be defined in $[-\gamma'_{nVk}, \gamma'_{nVk}]$; consequently $y^* = \lim_{K\to\mathcal{F}} Tx_K$ is defined in $\mathbb{R}^I$. Since $x_K \in W'$ for every K, $y^* \in \overline{T[W']} = T[W']$.

Let $x^* \in W'$ be such that $Tx^* = y^*$. Since $x^* \in W$, there are $n \geq 1$, $V \in \mathcal{V}_n$ and $i \in I(n,V)$ such that $T_i(x^*) \in V$. Set $L = \{(i,r) : r < n\}$, $H = \{z : z \in \mathbb{R}^L, \langle z(i,r)\rangle_{r<n} \in V\}$; then $\{x : Tx\restriction L \in H\} = T_i^{-1}[V]$. Since $y^*\restriction L = Tx^*\restriction L$ belongs to H, and H is open, there must be a $K \supseteq L$ such that $Tx_K\restriction L \in H$. But in this case $H' = \{z : z \in \mathbb{R}^K, z\restriction L \in H\}$ is an open subset of $\mathbb{R}^K$ and

$$\{x : Tx\restriction K \in H'\} = \{x : Tx\restriction L \in H\} = \{x : T_i(x) \in V\}$$

meets W in a negligible set, and $H' \subseteq H_K$. But this means that $Tx_K\restriction K \in H_K$, contrary to the choice of x_K. **XQ**

(h) Putting (f) and (g) together, we find ourselves trying to believe simultaneously that $\mu W'_K > 0$ and that $Tx\restriction K \in H_K$ for every $x \in W'_K$ and that $W'_K \subseteq W$ and that $\{x : x \in W, Tx\restriction K \in H_K\}$ is negligible. Faced with this we have to abandon the original supposition that μ is not τ-additive.

456K We now have all the ideas needed for the main theorem of this section.

Theorem (Talagrand 81) Every centered Gaussian distribution is τ-additive.

proof ? Suppose, if possible, that μ is a centered Gaussian distribution on a set $\mathbb{R}^I$ which is not τ-additive.

(a) By 456G, there are a set J and a universal centered Gaussian distribution ν on $\mathbb{R}^J$ and a continuous linear operator $T : \mathbb{R}^J \to \mathbb{R}^I$ which is inverse-measure-preserving for ν and μ. By 418Ha, ν is not τ-additive.

(b) As in part (a) of the proof of 456J, there are a non-negligible zero set $W \subseteq \mathbb{R}^J$ and a family $\mathcal{G}$ of open sets, covering W, such that $\nu(W \cap G) = \emptyset$ for every $G \in \mathcal{G}$. Give J a Hilbert space structure such

that $\int x(i)x(j)\nu(dx) = (i|j)$ for all $i, j \in J$. Let $K_0 \subseteq J$ be a countable set such that W is determined by coordinates in K_0, and let K be the closed linear subspace of J generated by K_0. Let $\mathcal{G}'$ be the family of open sets determined by coordinates in K which meet W in negligible sets. Then $W \subseteq \bigcup \mathcal{G}'$, by 456H.

Let λ be the centered Gaussian distribution on $\mathbb{R}^K$ for which the map $\tilde{\pi}_K = x \mapsto x\restriction K : \mathbb{R}^J \to \mathbb{R}^K$ is inverse-measure-preserving. Then $\tilde{\pi}_K[W]$ is a zero set in $\mathbb{R}^K$, $\lambda\tilde{\pi}_K[W] = \nu W > 0$, $\{\tilde{\pi}_K[G] : G \in \mathcal{G}'\}$ is a family of open sets in $\mathbb{R}^K$ covering $\tilde{\pi}_K[W]$, and $\lambda(\tilde{\pi}_K[W] \cap \tilde{\pi}_K[G]) = \nu(W \cap G) = 0$ for every $G \in \mathcal{G}'$; so λ is not τ-additive. However, K, regarded as a normed space, is separable (see 4A4Bi); and if we set $\pi_j(y) = y(j)$ for $y \in \mathbb{R}^K$ and $j \in K$, then $\|\pi_i^\bullet - \pi_j^\bullet\|_2 = \|i - j\|$ for all $i, j \in K$. So $\{\pi_j^\bullet : j \in K\}$ is separable in $L^2(\lambda)$. And this is impossible, by 456J. **X**

Thus every centered Gaussian distribution must be τ-additive.

456L Corollary If μ is a centered Gaussian distribution on $\mathbb{R}^I$, there is a unique quasi-Radon measure $\tilde{\mu}$ on $\mathbb{R}^I$ extending μ. The support of μ as defined in 456D is the support of $\tilde{\mu}$ as defined in 411N.

proof By 415L, μ has a unique extension to a quasi-Radon measure $\tilde{\mu}$. Now the support Z of μ is a closed set, so $\tilde{\mu}Z = \mu^*Z$ (415L(i)). Also Z is self-supporting for μ. If $G \subseteq \mathbb{R}^I$ is an open set meeting Z, then there is a cozero set $H \subseteq G$ which also meets Z, and $\mu^*(Z \cap H) > 0$. It follows that $\mu^*(Z \setminus H) < 1$; as $\tilde{\mu}$ extends μ, $\tilde{\mu}(Z \setminus H) < 1$ and $\tilde{\mu}(Z \cap G) > 0$. This shows that Z is self-supporting for $\tilde{\mu}$, so must be the support of $\tilde{\mu}$ in the standard sense.

456M Gaussian processes I take half a page to spell out the connexion between centered Gaussian distributions, as defined in 456A, and the corresponding processes considered in 454J-454K.

Proposition Let $\langle X_i\rangle_{i\in I}$ be a family of random variables on a probability space (Ω, Σ, μ). Then the following are equiveridical:

(i) the distribution of $\langle X_i\rangle_{i\in I}$, in the sense of 454K, is a centered Gaussian distribution;

(ii) whenever $i_0, \dots, i_n \in I$ and $\alpha_0, \dots, \alpha_n \in \mathbb{R}$ then $\sum_{r=0}^n \alpha_r X_{i_r}$ is either zero a.e. or a normal random variable with zero expectation;

(iii) whenever $i_0, \dots, i_n \in I$ then the joint distribution of $X_{i_0}, \dots, X_{i_n}$, in the sense of 271C, is a centered Gaussian distribution;

(iv) whenever $J \subseteq I$ is finite then there is an independent family $\langle Y_k\rangle_{k\in K}$ of standard normal random variables on Ω such that each X_i, for $i \in J$, is almost everywhere equal to a linear combination of the Y_k.

proof (ii) $\iff$ (i) is immediate from the definition in 456Ab and 454L.

(i) $\iff$ (iii) is also direct from 456Ab and the identification of the two concepts of 'distribution' (454K).

(iv)$\Rightarrow$(ii) is direct from 274A-274B.

(i)$\Rightarrow$(iv) For $i \in J$ set $u_i = X_i^\bullet$ in $L^2(\mu)$. By 4A4Jg there is an orthonormal family $\langle v_k\rangle_{k\in K}$ in $L^2(\mu)$ such that each v_k is a linear combination of the u_i and each u_i is a linear combination of the v_k. Take Y_k such that $Y_k^\bullet = v_k$ for each k; then each X_i is equal almost everywhere to a linear combination of the Y_k, while each Y_k is equal almost everywhere to a linear combination of the X_i. As $\#(K)$ must be the dimension of the linear span of $\{u_i : i \in J\}$, K is finite. Any linear combination of the Y_k is equal almost everywhere to a linear combination of the X_k, so is either zero a.e. or a normal random variable with zero expectation. Because (ii)$\Rightarrow$(iii), $\langle Y_k\rangle_{k\in K}$ has a centered Gaussian distribution ν say. Each Y_k has variance $(v_k|v_k) = 1$, so is a standard normal random variable.

The covariance matrix of ν is given by

$$\int y(j)y(k)\nu(dy) = \mathbb{E}(Y_j \times Y_k) = (v_j|v_k) = 1 \text{ if } j = k$$
$$= 0 \text{ otherwise.}$$

But this means that ν has the same covariance matrix as the product measure $\mu_G^{(K)}$, which is also a centered Gaussian distribution (456Bc). So $\nu = \mu_G^{(K)}$ (456Bb). But this means that $\langle Y_k\rangle_{k\in K}$ is independent (272G). Thus we have found a suitable family.

456N Definition A family of random variables satisfying the equivalent conditions in 456M is a **centered Gaussian process**.

456X Basic exercises >(a) Let I be a set and $\langle\sigma_{ij}\rangle_{i,j\in I}$ a family of real numbers. Show that the following are equiveridical: (i) $\langle\sigma_{ij}\rangle_{i,j\in I}$ is the covariance matrix of a centered Gaussian distribution on $\mathbb{R}^I$; (ii) there are a Hilbert space U and a family $\langle u_i\rangle_{i\in I}$ in U such that $(u_i|u_j)=\sigma_{ij}$ for all i, $j\in I$; (iii) $\langle\sigma_{ij}\rangle_{i,j\in J}$ is the covariance matrix of a centered Gaussian distribution on $\mathbb{R}^J$ for every finite $J\subseteq I$; (iv) $\sigma_{ij}=\sigma_{ji}$ for all i, $j\in I$ and $\sum_{i,j\in J}\alpha_i\alpha_j\sigma_{ij}\geq 0$ whenever $J\subseteq I$ is finite and $\langle\alpha_i\rangle_{i\in J}\in\mathbb{R}^J$. (*Hint*: 4A4Jg, for finite I, and 454G. For (iv)⇒(iii) use the finite-dimensional version of 4A4M.)

(b) Let $n\geq 1$ be an integer. (i) Show that $\mu_G^{(n)}T^{-1}=\mu_G^{(n)}$ for any orthogonal linear operator $T:\mathbb{R}^n\to\mathbb{R}^n$. (ii) Set $p(x)=\frac{1}{\|x\|}x$ for $x\in\mathbb{R}^n\setminus\{0\}$; take $p(0)$ to be any point of S^{n-1}. Show that $\mu_G^{(n)}p^{-1}$ is a multiple of $(n-1)$-dimensional Hausdorff measure on S^{n-1}. (*Hint*: 443T.)

(c) Let I be a countable set, μ a centered Gaussian distribution on $\mathbb{R}^I$, and $\gamma\geq 0$. Set $\alpha=\mu\{x:\sup_{i\in I}|x(i)|\geq\gamma)$. Show that $\mu\{x:\sup_{i\in I}|x(i)|\geq\frac{1}{2}\gamma)\geq 2\alpha(1-\alpha)^3$.

(d) (i) Let U and V be Hilbert spaces, $I\subseteq U$ a set and $f:I\to V$ a function such that $(f(u)|f(v))=(u|v)$ for all u, $v\in I$. Show that there is a linear operator $T:U\to V$, extending f, such that $(Tu|Tv)=(u|v)$ for all u, $v\in U$. (ii) Let I be a set and $\langle\sigma_{ij}\rangle_{i,j\in I}$ a family of real numbers. Show that there is at most one inner product space structure on I for which $\sigma_{ij}=(i|j)$ for all i, $j\in I$.

(e) Let $\langle X_n\rangle_{n\in\mathbb{N}}$ be an independent sequence of standard normal random variables. Show that for any square-summable sequence $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ and any bijection $\pi:\mathbb{N}\to\mathbb{N}$, $\sum_{n=0}^\infty\alpha_nX_n$ and $\sum_{n=0}^\infty\alpha_{\pi(n)}X_{\pi(n)}$ are finite and equal a.e. (*Hint*: 273B.)

>(f) For any set I, I will say that a **centered Gaussian quasi-Radon measure** on $\mathbb{R}^I$ is a quasi-Radon measure μ on $\mathbb{R}^I$ such that every continuous linear functional $f:\mathbb{R}^I\to\mathbb{R}$ is either zero a.e. or is normally distributed with zero expectation. Show that

(i) there is a one-to-one correspondence between centered Gaussian quasi-Radon measures μ on $\mathbb{R}^I$ and centered Gaussian distributions ν on $\mathbb{R}^I$ obtained by matching μ with ν iff they agree on the zero sets of $\mathbb{R}^I$;

(ii) if μ, ν are centered Gaussian quasi-Radon measures on $\mathbb{R}^I$ and $\int x(i)x(j)\mu(dx)=\int x(i)x(j)\nu(dx)$ for all i, $j\in I$, then $\mu=\nu$;

(iii) the support of a centered Gaussian quasi-Radon measure on $\mathbb{R}^I$ is a linear subspace of $\mathbb{R}^I$;

(iv) if $\langle I_j\rangle_{j\in J}$ is a disjoint family of sets with union I, and μ_j is a centered Gaussian quasi-Radon measure on $\mathbb{R}^{I_j}$ for each $j\in J$, then the quasi-Radon product of $\langle\mu_j\rangle_{j\in J}$, regarded as a measure on $\mathbb{R}^I$, is a centered Gaussian quasi-Radon measure.

>(g) Show that the completion of the measure on $\Omega=\mathbb{R}^{[0,\infty[}$ described in 455C is a centered Gaussian distribution with covariance matrix $\sigma=\langle\min(s,t)\rangle_{s,t\in[0,\infty[}$.

>(h) Let $\langle X_i\rangle_{i\in I}$ be a Gaussian process. Show that $\langle X_i\rangle_{i\in I}$ is independent iff $\mathbb{E}(X_i\times X_j)=0$ for all distinct i, $j\in I$.

456Y Further exercises (a) Let (Ω,Σ,μ) be a probability space with measure algebra $(\mathfrak{A},\bar\mu)$, and $\langle u_i\rangle_{i\in I}$ a family in $L^2(\mu)\cong L^2(\mathfrak{A},\bar\mu)$ which is a Gaussian process in the sense that whenever $X_i\in\mathcal{L}^2(\mu)$ is such that $X_i^\bullet=u_i$ for every i, then $\langle X_i\rangle_{i\in I}$ is a Gaussian process. Suppose that $\gamma\geq 0$ and that $\alpha=\bar\mu(\sup_{i\in I}[\![|u_i|\geq\gamma]\!])$. Show that $\bar\mu(\sup_{i\in I}[\![|u_i|\geq\frac{1}{2}\gamma]\!])\geq 2\alpha(1-\alpha)^3$.

(b) Let U be a Hilbert space with an orthonormal basis $\langle u_j\rangle_{j\in J}$, and μ the universal centered Gaussian distribution on $\mathbb{R}^U$ with covariance matrix defined by the inner product of U. Show that there is a function $T:\mathbb{R}^J\to\mathbb{R}^U$, inverse-measure-preserving for $\mu_G^{(J)}$ and μ, such that whenever $\langle j_n\rangle_{n\in\mathbb{N}}$ is a sequence of distinct elements of J and $\langle\alpha_n\rangle_{n\in\mathbb{N}}$ is a square-summable sequence in $\mathbb{R}$, then $(Tx)(\sum_{n=0}^\infty\alpha_nu_{j_n})=\sum_{n=0}^\infty\alpha_nx(j_n)$ for almost every $x\in\mathbb{R}^J$.

(c) Let U be an infinite-dimensional Hilbert space and μ the universal centered Gaussian distribution on $\mathbb{R}^U$ with covariance matrix defined by the inner product of U. Show that $\mu C = 0$ for every compact set $C \subseteq \mathbb{R}^U$. (*Hint*: there is an $m \in \mathbb{N}$ such that $\{u : |x(u)| \leq m \text{ for every } x \in C\}$ is somewhere dense in U.)

(d) Let μ be a centered Gaussian distribution on a set $\mathbb{R}^I$. Show that the following are equiveridical: (i) μ has countable Maharam type; (ii) $L^2(\mu)$ is separable; (iii) I is separable under the pseudometric $(i,j) \mapsto \sqrt{\int (x(i)-x(j))^2 \mu(dx)}$.

(e) Fix $p \in \left]-\frac{1}{2}, \frac{1}{2}\right[\setminus \{0\}$. (i) For $\alpha \in \mathbb{R}$ set $f_\alpha(t) = |t-\alpha|^p - |t|^p$ when this is defined. Show that $f_\alpha \in L^2(\mu_L)$, where μ_L is Lebesgue measure, that $\|f_\alpha\|_2^2 = |\alpha|^{2p+1}\|f_1\|_2^2$ and that $\|f_\alpha - f_\beta\|_2 = \|f_{\alpha-\beta}\|_2$ for all α, $\beta \in \mathbb{R}$. (ii) Show that there is a centered Gaussian process $\langle X_\alpha \rangle_{\alpha \in \mathbb{R}}$ such that $\mathbb{E}(X_\alpha \times X_\beta) = |\alpha|^{2p+1} + |\beta|^{2p+1} - |\alpha-\beta|^{2p+1}$ for all α, $\beta \in \mathbb{R}$. (iii) Show that such a process can be represented by a Radon measure on $C(\mathbb{R})$. (*Hint*: 455Yc.) (This is **fractional Brownian motion**.)

456 Notes and comments This section has taken a direct route to Talagrand's theorem 456K, leaving all the real reasons for studying Gaussian processes (see FERNIQUE 97) to one side. I hope that it will nevertheless be clear from such fragments as 252Xh, 456Bb, 456C and the exercises here that they are one of the many concepts of probability theory which are both important and delightful.

457 Simultaneous extension of measures

The questions addressed in §§451, 454 and 455 can all be regarded as special cases of a general class of problems: given a set X and a family $\langle \nu_i \rangle_{i \in I}$ of (probability) measures on X, when can we expect to find a measure on X extending every ν_i? An alternative formulation, superficially more general, is to ask: given a set X, a family $\langle (Y_i, \mathrm{T}_i, \nu_i) \rangle_{i \in I}$ of probability spaces, and functions $\phi_i : X \to Y_i$ for each i, when can we find a measure on X for which every ϕ_i is inverse-measure-preserving? Even the simplest non-trivial case, when $X = \prod_{i \in I} Y_i$ and every ϕ_i is the coordinate map, demands a significant construction (the product measures of Chapter 25). In this section I bring together a handful of important further cases which are accessible to the methods of this chapter. I begin with a discussion of extensions of finitely additive measures (457A-457D), which are much easier, before considering the problems associated with countably additive measures (457E-457H).

457A It is helpful to start with a widely applicable result on common extensions of *finitely* additive measures.

Lemma Let $\mathfrak{A}$ be a Boolean algebra and $\langle \mathfrak{B}_i \rangle_{i \in I}$ a non-empty family of subalgebras of $\mathfrak{A}$. For each $i \in I$, we may identify $L^\infty(\mathfrak{B}_i)$ with the closed linear subspace of $L^\infty(\mathfrak{A})$ generated by $\{\chi b : b \in \mathfrak{B}_i\}$ (363Ga). Suppose that for each $i \in I$ we are given a finitely additive functional $\nu_i : \mathfrak{B}_i \to [0,1]$ such that $\nu_i 1 = 1$. Write $\int \dots d\nu_i$ for the corresponding positive linear functional on $L^\infty(\mathfrak{B}_i)$ (363L). Then the following are equiveridical:

(i) there is an additive functional $\mu : \mathfrak{A} \to [0,1]$ extending every ν_i;

(ii) whenever $i_0, \dots, i_n \in I$, $a_k \in \mathfrak{B}_{i_k}$ for $k \leq n$, and $\sum_{k=0}^n \chi a_k \geq m\chi 1$ in $S(\mathfrak{A})$, where $m \in \mathbb{N}$, then $\sum_{k=0}^n \nu_{i_k} a_k \geq m$;

(iii) whenever $i_0, \dots, i_n \in I$, $a_k \in \mathfrak{B}_{i_k}$ for $k \leq n$, and $\sum_{k=0}^n \chi a_k \leq m\chi 1$, where $m \in \mathbb{N}$, then $\sum_{k=0}^n \nu_{i_k} a_k \leq m$;

(iv) whenever $i_0, \dots, i_n \in I$, $u_k \in L^\infty(\mathfrak{B}_{i_k})$ for every $k \leq n$, and $\sum_{k=0}^n u_k \geq \chi 1$, then $\sum_{i=0}^n \int u_k d\nu_{i_k} \geq 1$;

(v) whenever $i_0, \dots, i_n \in I$, $u_k \in L^\infty(\mathfrak{B}_{i_k})$ for every $k \leq n$, and $\sum_{k=0}^n u_k \leq \chi 1$, then $\sum_{i=0}^\infty \int u_k d\nu_{i_k} \leq 1$.

proof (a) It is elementary to check that if (i) is true then (ii)-(v) are all true, simply because we have a positive linear functional $\int d\mu$ extending all the functionals $\int d\nu_i$.

(b)(ii)⇒(iii) Given that $a_k \in \mathfrak{B}_{i_k}$ and $\sum_{k=0}^n \chi a_k \leq m\chi 1$, then

$$\sum_{k=0}^n \chi(1 \setminus a_k) = (n+1)\chi 1 - \sum_{k=0}^n \chi a_k \geq (n+1-m)\chi 1,$$

so

$$\sum_{k=0}^n \nu_{i_k} a_k = n+1-\sum_{k=0}^n \nu_{i_k}(1\setminus a_k) \le n+1-(n+1-m) = m,$$

as required by (iii).

(c)(iii)⇒(i) Assume (iii). Set $\psi a = \sup\{\nu_i a : i \in I,\ a \in \mathfrak{B}_i\}$ for $a \in \mathfrak{A}$ (interpreting $\sup\emptyset$ as 0, as usual in such contexts). Then (iii) tells us that ψ satisfies the condition (ii) of 391F. So there is a non-negative finitely additive functional μ such that $\mu 1 = 1$ and $\mu a \ge \psi a$ for every $a \in \mathfrak{A}$, that is, $\mu b \ge \nu_i b$ whenever $i \in I$ and $b \in \mathfrak{B}_i$. But observe now that, because $\mu 1 = \nu 1$ and $\mu(1\setminus b) \ge \nu_i(1\setminus b)$, we actually have $\mu b = \nu_i b$ for every $b \in \mathfrak{B}_i$, so that μ extends ν_i, for every $i \in I$.

(d)(iv)⇒(ii) & (v)⇒(iii) are elementary; if $m > 0$, apply (iv) or (v) to $u_k = \frac{1}{m}\chi a_k$.

457B Corollary Let X be a set and $\langle Y_i\rangle_{i\in I}$ a family of sets. Suppose that for each $i \in I$ we have an algebra $\mathcal{E}_i$ of subsets of Y_i, an additive functional $\nu_i : \mathcal{E}_i \to [0,1]$ such that $\nu_i Y_i = 1$, and a function $f_i : X \to Y_i$. Then the following are equiveridical:

(i) there is an additive functional $\mu : \mathcal{P}X \to [0,1]$ such that $\mu f_i^{-1}[E] = \nu_i E$ whenever $i \in I$ and $E \in \mathcal{E}_i$;

(ii) whenever $i_0,\dots,i_n \in I$ and $E_k \in \mathcal{E}_{i_k}$ for $k \le n$, then there is an $x \in X$ such that $\sum_{k=0}^n \nu_{i_k} E_k \le \#(\{k : k \le n,\ f_{i_k}(x) \in E_k\})$.

proof (i)⇒(ii) is elementary; if $m = \lceil \sum_{k=0}^n \nu_{i_k} E_k\rceil - 1$, then $\sum_{k=0}^n \mu f_{i_k}^{-1}[E_k] > m\mu X$, so $\sum_{k=0}^n \chi f_{i_k}^{-1} E_k \not\le m\chi X$, that is, there is an $x \in X$ such that

$$\#(\{k : f_{i_k}(x) \in E_k\}) = \sum_{k=0}^n (\chi f_{i_k}^{-1}[E_k])(x) \ge m+1 \ge \sum_{k=0}^n \nu_{i_k} E_k.$$

(ii)⇒(i) Now suppose that (ii) is true. For $i \in I$ set $\mathfrak{B}_i = \{f_i^{-1}[E] : E \in \mathcal{E}_i\}$. Note that if $E \in \mathcal{E}_i$ and $\nu_i E > 0$, then (applying (ii) with $n = 0$, $i_0 = i$ and $E_0 = E$) $f_i^{-1}[E]$ cannot be empty; accordingly we have an additive functional $\nu_i' : \mathfrak{B}_i \to [0,1]$ defined by setting $\nu_i' f^{-1}[E] = \nu_i E$ for every $E \in \mathcal{E}_i$, and $\nu_i' X = 1$. If $i_0,\dots,i_n \in I$, $H_0 \in \mathfrak{B}_{i_0},\dots,H_n \in \mathfrak{B}_{i_n}$ and $m \in \mathbb{N}$ are such that $\sum_{k=0}^n \chi H_k \le m\chi X$, express each H_k as $f_{i_k}^{-1}[E_k]$, where $E_k \in \mathcal{E}_{i_k}$; then there is an $x \in X$ such that

$$\sum_{k=0}^n \nu_{i_k}' H_k = \sum_{k=0}^n \nu_{i_k} E_k \le \#(\{k : f_k(x) \in E_k\}) = \sum_{k=0}^m \chi H_k(x) \le m.$$

But this means that the condition of 457A(iii) is satisfied, with $\mathfrak{A} = \mathcal{P}X$, so 457A(i) and (i) here are also true.

457C Corollary (a) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}_1$, $\mathfrak{B}_2$ two subalgebras of $\mathfrak{A}$ with finitely additive functionals $\nu_i : \mathfrak{B}_i \to [0,1]$ such that $\nu_1 1 = \nu_2 1 = 1$. Then the following are equiveridical:

(i) there is an additive functional $\mu : \mathfrak{A} \to [0,1]$ extending both the ν_i;

(ii) whenever $b_1 \in \mathfrak{B}_1$, $b_2 \in \mathfrak{B}_2$ and $b_1 \cup b_2 = 1$, then $\nu_1 b_1 + \nu_2 b_2 \ge 1$;

(iii) whenever $b_1 \in \mathfrak{B}_1$, $b_2 \in \mathfrak{B}_2$ and $b_1 \cap b_2 = 0$, then $\nu_1 b_1 + \nu_2 b_2 \le 1$.

(b) Let X, Y_1, Y_2 be sets, and for $i \in \{1,2\}$ let $\mathcal{E}_i$ be an algebra of subsets of Y_i, $\nu_i : \mathcal{E}_i \to [0,1]$ an additive functional such that $\nu_i Y_i = 1$, and $f_i : X \to Y_i$ a function. Then the following are equiveridical:

(i) there is an additive functional $\mu : \mathcal{P}X \to [0,1]$ such that $\mu f_i^{-1}[E] = \nu_i E$ whenever $i \in \{1,2\}$ and $E \in \mathcal{E}_i$;

(ii) $f_1^{-1}[E_1] \cap f_2^{-1}[E_2] \ne \emptyset$ whenever $E_1 \in \mathcal{E}_1$, $E_2 \in \mathcal{E}_2$ and $\nu_1 E_1 + \nu_2 E_2 > 1$;

(iii) $\nu E_1 \le \nu E_2$ whenever $E_1 \in \mathcal{E}_1$, $E_2 \in \mathcal{E}_2$ and $f_1^{-1}[E_1] \subseteq f_2^{-1}[E_2]$.

proof (a)(i)⇒(iii) is elementary (and is a special case of 457A(i)⇒457A(iii)).

(iii)⇒(ii) If (iii) is true, and $b_1 \in \mathfrak{B}_1$, $b_2 \in \mathfrak{B}_2$ are such that $b_1 \cup b_2 = 1$, then $(1\setminus b_1)\cap(1\setminus b_2) = 0$, so

$$\nu_1 b_1 + \nu_2 b_2 = 2 - \nu_1(1\setminus b_1) - \nu_2(1\setminus b_2) \ge 1.$$

(ii)⇒(i) The point is that (ii) here implies (ii) of 457A. **P** Suppose that $i_0,\dots,i_n \in \{1,2\}$, $a_k \in \mathfrak{B}_{i_k}$ for $k \le n$ and $\sum_{k=0}^n \chi a_k \ge m\chi 1$ in $S(\mathfrak{A})$, where $m \in \mathbb{N}$. Set $K_j = \{k : k \le n,\ i_k = j\}$ for each j, $u = \sum_{k\in K_1} \chi a_k \in S(\mathfrak{B}_1)$, $v = \sum_{k\in K_2} \chi a_k \in S(\mathfrak{B}_2)$. Then we can express u as $\sum_{j=0}^{m_1} \chi c_j$ where $c_j \in \mathfrak{B}_1$ for each $j \le m_1$ and $c_0 \supseteq c_1 \supseteq \dots \supseteq c_{m_1}$ (see the proof of 361Ec). Taking $c_j = 0$ for $m_1 < j \le m$ if necessary,

we may suppose that $m_1 \geq m$. Similarly, $v = \sum_{j=0}^{m_2} \chi d_j$ where $m_2 \geq m$, $d_j \in \mathfrak{B}_2$ for each $j \leq m_2$ and $d_0 \supseteq \ldots \supseteq d_{m_2}$.

For $j < m$, set $b_j = 1 \setminus (c_j \cup d_{m-j-1})$. Then, because $b_j \cap c_j = 0$,

$$u \times \chi b_j = \sum_{r=0}^{m_1} \chi(c_r \cap b_j) = \sum_{r=0}^{j-1} \chi(c_r \cap b_j) \leq j\chi b_j,$$

and similarly $v \times \chi b_j \leq (m-j-1)\chi b_j$, so

$$m\chi b_j \leq (u+v) \times \chi b_j = u \times \chi b_j + v \times \chi b_j \leq (m-1)\chi b_j,$$

and b_j must be 0.

Thus $c_j \cup d_{m-j-1} = 1$ for every $j < m$. But this means that $\nu_1 c_j + \nu_2 d_{m-j-1} \geq 1$ for every $j < m$, so that

$$\begin{aligned}\sum_{k=0}^{n} \nu_{i_k} a_k &= \sum_{k \in K_1} \nu_1 a_k + \sum_{k \in K_2} \nu_2 a_k = \int u \, d\nu_1 + \int v \, d\nu_2 \\ &= \sum_{j=0}^{m_1} \nu_1 c_j + \sum_{j=0}^{m_2} \nu_2 d_j \geq \sum_{j=0}^{m-1} \nu_1 c_j + \nu_2 d_{m-1-j} \geq m,\end{aligned}$$

as required. **Q**

Because 457A(ii) implies 457A(i), we have the result.

(b) We can convert (i) and (ii) here into (a-i) and (a-iii) just above by the same translation as in 457B. So (i) and (ii) are equiveridical. As for (iii), this corresponds exactly to replacing E_2 by $Y_2 \setminus E_2$ in (ii).

***457D** The proof of 457A is based, at some remove, on the Hahn-Banach theorem, as applied in the proof of 391E-391F. An alternative proof uses the max-flow min-cut theorem of graph theory. To show the power of this method I apply it to an elaboration of 457C, as follows.

Proposition (Strassen 65) Let $\mathfrak{A}$ be a Boolean algebra and $\mathfrak{B}_1$, $\mathfrak{B}_2$ two subalgebras of $\mathfrak{A}$. Suppose that $\nu_i : \mathfrak{B}_i \to [0,1]$ are finitely additive functionals such that $\nu_1 1 = \nu_2 1 = 1$, and $\theta : \mathfrak{A} \to [0,\infty[$ another additive functional. Then the following are equiveridical:

(i) there is an additive functional $\mu : \mathfrak{A} \to [0,\infty[$ extending both the ν_i, and such that $\mu a \leq \theta a$ for every $a \in \mathfrak{A}$;

(ii) $\nu_1 b_1 + \nu_2 b_2 \leq 1 + \theta(b_1 \cap b_2)$ for every $b_1 \in \mathfrak{B}_1$, $b_2 \in \mathfrak{B}_2$.

proof (a) As usual in this context, (i)⇒(ii) is elementary; if $\mu \leq \theta$ extends both ν_j, and $b_j \in \mathfrak{B}_j$ for both j, then

$$\nu_1 b_1 + \nu_2 b_2 = \mu b_1 + \mu b_2 = \mu(b_1 \cup b_2) + \mu(b_1 \cap b_2) \leq 1 + \theta(b_1 \cap b_2).$$

(b) For the reverse implication, suppose to begin with (down to the end of (d) below) that $\mathfrak{A}$ is finite. Let I, J and K be the sets of atoms of $\mathfrak{B}_1$, $\mathfrak{B}_2$ and $\mathfrak{A}$ respectively. Consider the transportation network (V, E, γ) where

$$V = \{(0,0)\} \cup \{(b,1) : b \in I\} \cup \{(d,2) : d \in K\} \cup \{(c,3) : c \in J\} \cup \{(1,4)\},$$

$$E = \{e_b^0 : b \in I\} \cup \{e_d^1 : d \in K\} \cup \{e_d^2 : d \in K\} \cup \{e_c^3 : c \in J\},$$

where

for $b \in I$, e_b^0 runs from $(0,0)$ to $(b,1)$,
for $d \in K$, e_d^1 runs from $(b,1)$ to $(d,2)$, where b is the member of I including d,
for $d \in K$, e_d^2 runs from $(d,2)$ to $(c,3)$, where c is the member of J including d,
for $c \in J$, e_c^3 runs from $(c,3)$ to $(1,4)$.

Define the capacity $\gamma(e)$ of each link by setting

$$\gamma(e_b^0) = \nu_1 b \text{ for } b \in I,$$

$$\gamma(e_d^1) = \gamma(e_d^2) = \theta d \text{ for } d \in K,$$

$$\gamma(e^3_c) = \nu_2 c \text{ for } c \in J.$$

By the max-flow min-cut theorem (4A4N), there are a flow ϕ and a cut X of the same value; that is, we have a function $\phi : E \to [0,\infty[$ and a set $X \subseteq E$ such that

$$\sum_{e \text{ starts from } v} \phi(e) = \sum_{e \text{ ends at } v} \phi(e)$$

for every $v \in V \setminus \{(0,0),(1,4)\}$,

$$\phi(e) \le \gamma(e)$$

for every $e \in E$,

$$\sum_{e \text{ starts from } (0,0)} \phi e = \sum_{e \text{ ends at } (1,4)} \phi e = \sum_{e \in X} \gamma(e),$$

and there is no path from $(0,0)$ to $(1,4)$ using only links in $E \setminus X$.

Now, for any $d \in K$, there is exactly one link e^1_d ending at d and exactly one link e^2_d starting from d. So $\phi(e^1_d) = \phi(e^2_d)$, and we may define an additive functional μ on $\mathfrak{A}$ by setting

$$\mu a = \sum_{d \in K, d \subseteq a} \phi(e^1_d) = \sum_{d \in K, d \subseteq a} \phi(e^2_d)$$

for every $a \in \mathfrak{A}$.

(c)(i) $\mu b \le \nu_1 b$ for every $b \in \mathfrak{B}_1$. **P** Because I is the set of atoms of the finite Boolean algebra $\mathfrak{B}_1$, it is enough to show that $\mu b \le \nu_1 b$ for every $b \in I$. Now, for such b,

$$\begin{aligned}\mu b &= \sum_{d \in K, d \subseteq b} \mu d = \sum_{d \in K, d \subseteq b} \phi(e^1_d) = \sum_{e \text{ starts from } (b,1)} \phi(e) \\ &= \sum_{e \text{ ends at } (b,1)} \phi(e) = \phi(e^0_b) \le \gamma(e^0_b) = \nu_1 b,\end{aligned}$$

because the only link ending at $(b,1)$ is e^0_b. **Q**

(ii) Similarly, because the only link starting at $(c,3)$ has capacity $\nu_2 c$, $\mu c \le \nu_2 c$ for every $c \in J$. But this means that $\mu c \le \nu_2 c$ for every $c \in \mathfrak{B}_2$.

(iii) In third place, because

$$\mu d = \phi(e^1_d) \le \gamma(e^1_d) = \theta d$$

for every $d \in K$, $\mu a \le \theta a$ for every $a \in \mathfrak{A}$.

(d) (The key.) $\mu 1 \ge 1$. **P** We have

$$\begin{aligned}\mu 1 &= \sum_{d \in K} \mu d = \sum_{d \in K} \phi(e^1_d) \\ &= \sum_{b \in I} \sum_{d \in K, d \subseteq b} \phi(e^1_d) = \sum_{b \in I} \sum_{e \text{ starts from } (b,1)} \phi(e) \\ &= \sum_{b \in I} \sum_{e \text{ ends at } (b,1)} \phi(e) = \sum_{e \text{ starts from } (0,0)} \phi(e) = \sum_{e \in X} \gamma(e).\end{aligned}$$

Set

$$\begin{aligned}b^* &= \sup\{b : b \in I,\, e^0_b \in X\} \in \mathfrak{B}_1, \\ a^*_1 &= \sup\{d : d \in K,\, e^1_d \in X\}, \\ a^*_2 &= \sup\{d : d \in K,\, e^2_d \in X\}, \\ c^* &= \sup\{c : c \in J,\, e^3_c \in X\} \in \mathfrak{B}_2.\end{aligned}$$

For any $d \in K$, we have a four-link path $e^0_b, e^1_d, e^2_d, e^3_c$ from $(0,0)$ to $(1,4)$, where $b \in I$, $c \in J$ are the atoms of $\mathfrak{B}_1$, $\mathfrak{B}_2$ including d. At least one of the links in this path must belong to X, so that d is included in $b^* \cup a^*_1 \cup a^*_2 \cup c^*$. Thus, writing $a = (1 \setminus b^*) \cap (1 \setminus c^*)$, $a \subseteq a^*_1 \cup a^*_2$ and $\theta a \le \theta a^*_1 + \theta a^*_2$. But this means that

$$\begin{aligned}\mu 1 &= \sum_{e\in X}\gamma(e)\\ &= \sum_{b\in I,e_b^0\in X}\gamma(e_b^0)+\sum_{d\in K,e_d^1\in X}\gamma(e_d^1)+\sum_{d\in K,e_d^2\in X}\gamma(e_d^2)+\sum_{c\in J,e_c^3\in X}\gamma(e_c^3)\\ &= \sum_{b\in I,e_b^0\in X}\nu_1 b+\sum_{d\in K,e_d^1\in X}\theta d+\sum_{d\in K,e_d^2\in X}\theta d+\sum_{c\in J,e_c^3\in X}\nu_2 c\\ &= \nu_1 b^*+\theta a_1^*+\theta a_2^*+\nu_2 c^*\geq \nu_1 b^*+\theta((1\setminus b^*)\cap(1\setminus c^*))+\nu_2 c^*\\ &\geq \nu_1 b^*+\nu_1(1\setminus b^*)+\nu_2(1\setminus c^*)-1+\nu_2 c^*\end{aligned}$$

(applying the hypothesis (ii))

$$= 1,$$

as claimed. **Q**

Since we already know that $\nu_1 1 = 1$ and that $\mu b \leq \nu_1 b$ for every $b \in \mathfrak{B}_1$, we must have $\mu 1 = 1$ and $\mu b = \nu_1 b$ for every $b \in \mathfrak{B}$, so that μ extends ν_1. Similarly, μ extends ν_2.

(e) Thus the proposition is proved in the case in which $\mathfrak{A}$ is finite. In the general case, for each finite subset K of $\mathfrak{A}$ write $\mathfrak{A}_K$ for the subalgebra of $\mathfrak{A}$ generated by K. Then (b)-(d) tell us that there is a non-negative additive functional μ_K on $\mathfrak{A}_K$, dominated by θ on $\mathfrak{A}_K$, agreeing with ν_1 on $\mathfrak{A}_K \cap \mathfrak{B}_1$ and agreeing with ν_2 on $\mathfrak{A}_K \cap \mathfrak{B}_2$. Let μ be any cluster point of the μ_K in $[0,1]^{\mathfrak{A}}$ as K increases through the finite subsets of $\mathfrak{A}$; then μ will be a non-negative additive functional on $\mathfrak{A}$, dominated by θ, and extending ν_1 and ν_2.

This proves the result.

457E Proposition Let X be a non-empty set and $\langle\nu_i\rangle_{i\in I}$ a family of probability measures on X satisfying the conditions of Lemma 457A, taking $\mathfrak{A} = \mathcal{P}X$ and $\mathfrak{B}_i = \operatorname{dom}\nu_i$ for each i. Suppose that there is a countably compact class $\mathcal{K} \subseteq \mathcal{P}X$ such that every ν_i is inner regular with respect to $\mathcal{K}$. Then there is a probability measure μ on X extending every ν_i.

proof If $I = \emptyset$ this is trivial. Otherwise, by 457A, there is a finitely additive functional ν on $\mathcal{P}X$ extending every ν_i. Now 413S tells us that there is a complete measure μ on X such that $\mu X \leq \nu X$ and $\mu K \geq \nu K$ for every $K \in \mathcal{K}$. In this case, for any $i \in I$ and $E \in \mathrm{T}_i = \operatorname{dom}\nu_i$, we must have

$$\begin{aligned}\mu_* E &\geq \sup_{K\in\mathcal{K},K\subseteq E}\mu K \geq \sup_{K\in\mathcal{K}\cap\operatorname{dom}\nu_i,K\subseteq E}\mu K\\ &\geq \sup_{K\in\mathcal{K}\cap\operatorname{dom}\nu_i,K\subseteq E}\nu K = \sup_{K\in\mathcal{K}\cap\operatorname{dom}\nu_i,K\subseteq E}\nu_i K = \nu_i E.\end{aligned}$$

In particular, $\mu X \geq \nu_i X = 1$. Now

$$\mu^* E = 1-\mu_*(X\setminus E)\leq 1-\nu_i(X\setminus E)=\nu_i E$$

for any $E \in \mathrm{T}_i$; as μ is complete, μE is defined and equal to $\nu_i E$ for every $E \in \mathrm{T}_i$, and μ extends ν_i, as required.

457F Proposition (a) Let (X,Σ,μ) be a perfect probability space and (Y,T,ν) any probability space. Write $\Sigma\otimes\mathrm{T}$ for the algebra of subsets of $X\times Y$ generated by $\{E\times F : E\in\Sigma, F\in\mathrm{T}\}$. Suppose that $Z\subseteq X\times Y$ is such that

(i) Z is expressible as the intersection of a sequence in $\Sigma\otimes\mathrm{T}$,

(ii) $Z\cap(E\times F)\neq\emptyset$ whenever $E\in\Sigma$, $F\in\mathrm{T}$ are such that $\mu E+\nu F>1$.

Then there is a probability measure λ on Z such that the maps $(x,y)\mapsto x : Z\to X$ and $(x,y)\mapsto y : Z\to Y$ are both inverse-measure-preserving.

(b) Let $\langle(X_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a family of perfect probability spaces. Write $\bigotimes_{i\in I}\Sigma_i$ for the algebra of subsets of $X=\prod_{i\in I}X_i$ generated by $\{\{x : x\in X, x(i)\in E\} : i\in I, E\in\Sigma_i\}$. Suppose that $Z\subseteq X$ is such that

(i) Z is expressible as the intersection of a sequence in $\bigotimes_{i\in I}\Sigma_i$,

(ii) whenever $i_0, \dots, i_n \in I$ and $E_k \in \Sigma_{i_k}$ for $k \leq n$, there is a $z \in Z$ such that $\#(\{k : k \leq n, z(i_k) \in E_k\}) \geq \sum_{k=0}^{n} \mu_{i_k} E_k$.

Then there is a perfect probability measure λ on Z such that $z \mapsto z(i) : Z \to X_i$ is inverse-measure-preserving for every $i \in I$.

proof (a) Apply 457Cb to the coordinate maps $f_1 : Z \to X$ and $f_2 : Z \to Y$. The condition (ii) here shows that 457C(b-ii) is satisfied, so there is an additive functional $\theta : \mathcal{P}Z \to [0,1]$ such that $\theta f_1^{-1}[E] = \mu E$ for every $E \in \Sigma$ and $\theta f_2^{-1}[F] = \nu F$ for every $F \in \mathrm{T}$.

Define $\theta' : \Sigma \otimes \mathrm{T} \to [0,1]$ by setting $\theta' W = \theta(Z \cap W)$ for every $W \in \Sigma \otimes \mathrm{T}$. Then $\theta'(E \times Y) = \mu E$ for every $E \in \Sigma$ and $\theta'(X \times F) = \nu F$ for every $F \in \mathrm{T}$. Because μ is perfect, θ' has an extension to a measure $\tilde{\lambda}$ defined on $\Sigma \widehat{\otimes} \mathrm{T}$ (454C). Now Z is supposed to be expressible as $\bigcap_{n \in \mathbb{N}} W_n$ where $W_n \in \Sigma \otimes \mathrm{T}$ for every n; since

$$\tilde{\lambda} W_n = \theta' W_n = \theta(Z \cap W_n) = \theta Z = 1$$

for every n, $\tilde{\lambda} Z = 1$. So if we take λ to be the subspace measure on Z induced by $\tilde{\lambda}$, λ will be a probability measure on Z. If $E \in \Sigma$, then

$$\begin{aligned} \lambda(Z \cap (E \times Y)) &= \tilde{\lambda}(Z \cap (E \times Y)) = \tilde{\lambda}(E \times Y) \\ &= \theta'(E \times Y) = \theta(Z \cap (E \times Y)) = \mu E. \end{aligned}$$

So $f_1 : Z \to X$ is inverse-measure-preserving for λ and μ. Similarly, $f_2 : Z \to Y$ is inverse-measure-preserving for λ and ν.

(b) We use the same ideas, but appealing to 457B and 454D instead of 457Cb and 454C. Taking $f_i : X \to X_i$ to be the coordinate map for each $i \in I$, (ii) here, with 457B, tells us that there is an additive functional $\theta : \mathcal{P}Z \to [0,1]$ such that $\theta f_i^{-1}[E] = \mu_i E$ whenever $i \in I$ and $E \in \Sigma_i$.

Define $\theta' : \bigotimes_{i \in I} \Sigma_i \to [0,1]$ by setting $\theta' W = \theta(Z \cap W)$ for every $W \in \bigotimes_{i \in I} \Sigma_i$. Then

$$\theta'\{x : x \in X, x(i) \in E\} = \theta\{z : z \in Z, z(i) \in E\} = \mu_i E$$

whenever $i \in I$ and $E \in \Sigma_i$. Because every μ_i is perfect, θ' has an extension to a perfect measure $\tilde{\lambda}$ defined on $\widehat{\bigotimes}_{i \in I} \Sigma_i$ (454D). Now Z is supposed to be expressible as $\bigcap_{n \in \mathbb{N}} W_n$ where $W_n \in \bigotimes_{i \in I} \Sigma_i$ for every n; since

$$\tilde{\lambda} W_n = \theta' W_n = \theta(Z \cap W_n) = \theta Z = 1$$

for every n, $\tilde{\lambda} Z = 1$. So if we take λ to be the subspace measure on Z induced by $\tilde{\lambda}$, λ will be a probability measure on Z; by 451Dc, λ is perfect. If $i \in I$ and $E \in \Sigma_i$, then

$$\begin{aligned} \lambda\{z : z \in Z, z(i) \in E\} &= \tilde{\lambda}\{x : x \in X, x(i) \in E\} \\ &= \theta'\{x : x \in X, x(i) \in E\} = \theta\{z : z \in Z, z(i) \in E\} = \mu_i E. \end{aligned}$$

So $z \mapsto z(i) : Z \to X_i$ is inverse-measure-preserving for λ and μ_i for every $i \in I$, as required.

457G Proposition Let X be a set and $\langle \mu_i \rangle_{i \in I}$ a family of probability measures on X which is upwards-directed in the sense that for any $i, j \in I$ there is a $k \in I$ such that μ_k extends both μ_i and μ_j. Suppose that for any countable $J \subseteq I$ there is a measure on X extending μ_i for every $i \in J$. Then there is a measure on X extending μ_i for every $i \in I$.

proof Set $\Sigma_i = \operatorname{dom} \mu_i$ for each $i \in I$. Because $\langle \mu_i \rangle_{i \in I}$ is upwards-directed, $\mathrm{T} = \bigcup_{i \in I} \Sigma_i$ is an algebra of subsets of X, and we have a finitely additive functional $\nu : \mathrm{T} \to [0,1]$ defined by saying that $\nu E = \mu_i E$ whenever $i \in I$ and $E \in \Sigma_i$. Now if $\langle E_n \rangle_{n \in \mathbb{N}}$ is any non-increasing sequence in T with empty intersection, there is a countable set $J \subseteq I$ such that $E_n \subseteq \bigcup_{i \in J} \Sigma_i$ for every $n \in \mathbb{N}$. We are told that there is a measure λ on X extending μ_i for every $i \in J$; now $\nu E_n = \lambda E_n$ for every $n \in \mathbb{N}$, so $\lim_{n \to \infty} \nu E_n = 0$. By 413K, ν has an extension to a measure on X, which of course extends every μ_i.

457H Example Set $X = \{(x, y) : 0 \leq x < y \leq 1\} \subseteq [0,1]^2$. Write $\pi_1, \pi_2 : X \to \mathbb{R}$ for the coordinate maps, and μ_L for Lebesgue measure on $[0,1]$, with Σ_L its domain.

(a) There is a finitely additive functional $\nu : \mathcal{P}X \to [0,1]$ such that $\nu\pi_i^{-1}[E] = \mu_L E$ whenever $i \in \{1,2\}$ and $E \in \Sigma_L$. **P** If E_1, $E_2 \in \Sigma_L$ and $\mu_L E_1 + \mu_L E_2 > 1$, then neither is empty and $\inf E_1 < \sup E_2$, so there are $x \in E_1$, $y \in E_2$ such that $x < y$, and $(x,y) \in \pi_1^{-1}[E_1] \cap \pi_2^{-1}[E_2]$. So the result follows by 457Cb. **Q**

(b) However, there is no measure μ on X for which both π_1 and π_2 are inverse-measure-preserving. **P?** If there were,

$$\int \pi_1(x,y)\mu(d(x,y)) = \int x\mu_L(dx) = \int y\mu_L(dy) = \int \pi_2(x,y)\mu(d(x,y))$$

by 235I; but $\pi_1(x,y) < \pi_2(x,y)$ for every $(x,y) \in X$, so this is impossible. **XQ**

(c) If we write $\mathrm{T}_i = \{\pi_i^{-1}[E] : E \subseteq [0,1] \text{ is Borel}\}$ for each i, then we have a measure ν_i with domain T_i defined by setting $\nu_i\pi_i^{-1}[E] = \mu_L E$ for each Borel set $E \subseteq [0,1]$. Now ν_1 and ν_2 have no common extension to a Borel measure on X, even though X is a Polish space and each ν_i is a compact measure, being inner regular with respect to the compact class $\mathcal{K}_i = \{\pi_i^{-1}[K] : K \subseteq]0,1[\text{ is compact}\}$. (The trouble is that $\mathcal{K}_1 \cup \mathcal{K}_2$ is *not* compact, so we cannot apply 457E.)

457I Example Let μ_L be Lebesgue measure on $[0,1]$ and Σ_L its domain. Set

$$X = \{(\xi_1,\xi_2,\xi_3) : 0 \le \xi_i \le 1 \text{ for each } i, \textstyle\sum_{i=1}^3 \xi_i \le \frac{3}{2}, \sum_{i=1}^3 \xi_i^2 \le 1\}.$$

For $1 \le i \le 3$ set $\pi_i(x) = \xi_i$ for $x = (\xi_1,\xi_2,\xi_3) \in X$.

(a) If $E_i \in \Sigma_L$ for $i \le 3$, then there is an $x \in X$ such that $\#(\{i : \pi_i(x) \in E_i\}) \ge \sum_{i=1}^3 \mu_L E_i$. **P** Set $\alpha_i = \inf(E_i \cup \{1\})$ for each i, and set

$$m = \lceil \textstyle\sum_{i=1}^3 \mu_L E_i \rceil \le \lceil \sum_{i=1}^3 1 - \alpha_i \rceil = 3 - \lfloor \sum_{i=1}^3 \alpha_i \rfloor,$$

so that $\sum_{i=1}^3 \alpha_i < 4 - m$. Take $\xi_i \in E_i \cup \{1\}$ such that $\sum_{i=1}^3 \xi_i < 4 - m$. It will be enough to consider the case in which $\xi_1 \le \xi_2 \le \xi_3$.

(i) If $m = 1$, then $\sum_{i=1}^3 \xi_i < 3$ so $\xi_1 < 1$ and $\xi_1 \in E_1$. Set $x = (\xi_1, 0, 0)$; then $x \in X$ and

$$\#(\{i : \pi_i(x) \in E_i\}) \ge 1 \ge \textstyle\sum_{i=1}^3 \mu_L E_i.$$

(ii) If $m = 2$, then $\sum_{i=1}^3 \xi_i < 2$ so $\xi_2 < 1$ and $\xi_1 \in E_1$, $\xi_2 \in E_2$. Set $x = (\xi_1,\xi_2,0)$. We have $\xi_1 + \xi_2 \le \frac{4}{3} \le \frac{3}{2}$. Also

$$\xi_2 \le \tfrac{1}{2}(\xi_2 + \xi_3) \le 1 - \tfrac{1}{2}\xi_1,$$

so

$$\xi_1^2 + \xi_2^2 \le \xi_1^2 + (1 - \tfrac{1}{2}\xi_1)^2 = 1 - \xi_1 + \tfrac{5}{4}\xi_1^2 \le 1$$

because $\xi_1 \le \frac{2}{3} \le \frac{4}{5}$. So $x \in X$ and

$$\#(\{i : \pi_i(x) \in E_i\}) \ge 2 \ge \textstyle\sum_{i=1}^3 \mu_L E_i.$$

(iii) If $m = 3$ then $\sum_{i=1}^3 \xi_i < 1$ so $\xi_i \in E_i$ for every i; set $x = (\xi_1,\xi_2,\xi_3)$. Since $\sum_{i=1}^3 \xi_i^2 \le \sum_{i=1}^3 \xi_i \le 1$, $x \in X$ and

$$\#(\{i : \pi_i(x) \in E_i\}) = 3 \ge \textstyle\sum_{i=1}^3 \mu_L E_i.$$

Putting these together, we have the result. **Q**

(b) There is no finitely additive functional ν on X such that $\nu\pi_i^{-1}[E] = \mu E$ for each i and every $E \in \Sigma_L$. **P?** Suppose there were. Set $\mathrm{T}_i = \{\pi_i^{-1}[E] : E \in \Sigma_L\}$ and $\nu_i = \nu \restriction \mathrm{T}_i$ for each i. Then ν_i is a probability measure on X; moreover, because X is compact, $\pi_i^{-1}[K]$ is compact for every compact $K \subseteq [0,1]$, so ν_i is inner regular with respect to the compact subsets of X. By 457E, the ν_i have a common extension to a countably additive measure μ. Now

$$\int_X \xi_1 + \xi_2 + \xi_3 \, \mu(dx) = 3 \int_0^1 t \, dt = \frac{3}{2},$$

so we must have $\xi_1 + \xi_2 + \xi_3 = \frac{3}{2}$ for μ-almost every x; similarly,

$$\int_X \xi_1^2 + \xi_2^2 + \xi_3^2 \, \mu(dx) = 3 \int_0^1 t^2 \, dt = 1,$$

so we must have $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$ for μ-almost every x. Since

$$(\tfrac{3}{2} - \xi_3)^2 = (\xi_1 + \xi_2)^2 \leq 2(\xi_1^2 + \xi_2^2) \leq 2(1 - \xi_3^2)$$

for almost every x, $\xi_3 - \xi_3^2 \geq \frac{1}{12}$ for almost every x, which is impossible, since $\mu\{x : \xi_3 \leq \frac{1}{2} - \frac{1}{\sqrt{6}}\} > 0$. **XQ**

457J Example There are a set X and a family $\langle \mu_i \rangle_{i \in I}$ of probability measures on X such that (i) for every countable set $J \subseteq I$ there is a measure on X extending μ_i for every $i \in J$ (ii) there is no measure on X extending μ_i for every $i \in I$.

proof By 439Fc, there is an uncountable universally negligible subset of $[0,1]$. Because $[0,1]$ and $\mathcal{P}\mathbb{N}$ are uncountable Polish spaces, they have isomorphic Borel structures (424Cb), so there is an uncountable universally negligible set $X_0 \subseteq \mathcal{P}\mathbb{N}$. The map $a \mapsto \mathbb{N} \setminus a$ is an autohomeomorphism of $\mathcal{P}\mathbb{N}$, so $X_1 = \{\mathbb{N} \setminus a : a \in X_0\}$ is universally negligible, and $X = X_0 \cup X_1$ is universally negligible (439Cb).

For $n \in \mathbb{N}$, set $E_n = \{a : n \in a \in X\}$ and $\Sigma_n = \{\emptyset, E_n, X \setminus E_n, X\}$; note that, because X is closed under complementation, neither E_n nor $X \setminus E_n$ is empty, and we have a probability measure μ_n with domain Σ_n defined by setting $\mu_n E_n = \mu_n(X \setminus E_n) = \frac{1}{2}$. Next, for $a \in X$, set $\Sigma'_a = \{\emptyset, \{a\}, X \setminus \{a\}, X\}$, and let μ'_a be the probability measure with domain Σ'_a defined by setting $\mu'_a\{a\} = 0$.

If $J \subseteq X$ is countable, then there is a probability measure on X extending μ_n for every $n \in \mathbb{N}$ and μ'_a for every $a \in J$. **P** Because X_0 is uncountable, there is a $b \in X_0$ such that neither b nor $b' = \mathcal{P}\mathbb{N} \setminus b$ belongs to J. Let μ be the probability measure with domain $\mathcal{P}X$ defined by setting $\mu\{b\} = \mu\{b'\} = \frac{1}{2}$; this extends all the μ_n and all the μ'_a for $a \in J$. **Q**

? Suppose, if possible, that μ is a measure on X extending every μ_n and every μ'_a. In this case, because μ extends every μ_n, its domain includes the Borel σ-algebra $\mathcal{B}$ of X, and $\mu \restriction \mathcal{B}$ is a Borel probability measure on X. Since X is universally negligible, there is a point $a \in X$ such that $\mu\{a\} > 0$; in which case μ cannot extend μ'_a. **X**

Thus the μ_n, μ'_a constitute a family of the kind required.

457X Basic exercises (a) Let X be a non-empty set and $\langle \nu_i \rangle_{i \in I}$ a family of probability measures on X satisfying the conditions of Lemma 457A, taking $\mathfrak{A} = \mathcal{P}X$ and $\mathfrak{B}_i = \operatorname{dom} \nu_i$ for each i. Suppose that there is a totally finite measure θ on X such that θE is defined and greater than or equal to $\nu_i E$ whenever $i \in I$ and ν_i measures E. Show that there is a measure on X extending every ν_i.

(b) Find a set X and non-negative additive functionals μ_1, μ_2 defined on subalgebras of $\mathcal{P}X$ which agree on $\operatorname{dom} \mu_1 \cap \operatorname{dom} \mu_2$ but have no common extension to a non-negative additive functional. (*Hint*: take $\#(X) = 3$.)

(c) Let $\mathfrak{A}$ be a Boolean algebra and $\langle \nu_i \rangle_{i \in I}$ a family of non-negative finitely additive functionals, each ν_i being defined on a subalgebra $\mathfrak{B}_i$ of $\mathfrak{A}$. Show that if any finite number of the ν_i have a common extension to an additive functional on a subalgebra of $\mathfrak{A}$, then the whole family has a common extension to an additive functional on the whole algebra $\mathfrak{A}$.

(d) Set $X = \{0,1,2\}$ and in the algebra $\mathcal{P}X$ let $\mathfrak{B}_i$ be the subalgebra $\{0, \{i\}, X \setminus \{i\}, X\}$ for each i. Let $\nu_i : \mathfrak{B}_i \to [0,1]$ be the additive functional such that $\nu_i\{i\} = \frac{1}{2}$, $\nu_i X = 1$. Show that any pair of ν_0, ν_1, ν_2 have a common extension to an additive functional on $\mathcal{P}X$, but that the three together have no such extension.

(e) Let $\mathfrak{A}$ be a Boolean algebra, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$, and $\nu : \mathfrak{B} \to [0, \infty[$, $\theta : \mathfrak{A} \to [0, \infty[$ additive functionals such that $\nu b \leq \theta b$ for every $b \in \mathfrak{B}$. Show directly, without using either 457D or 391F, that there

is an additive functional $\mu : \mathfrak{A} \to [0,\infty[$, extending ν, such that $\mu a \leq \theta a$ for every $a \in \mathfrak{A}$. (*Hint*: first consider the case in which $\mathfrak{A}$ is the algebra generated by $\mathfrak{B} \cup \{c\}$.)

>(f) Let $(Y_1, \mathfrak{S}_1, \mathrm{T}_1, \nu_1)$ and $(Y_2, \mathfrak{S}_2, \mathrm{T}_2, \nu_2)$ be Radon probability spaces and $X \subseteq Y_1 \times Y_2$ a closed set. Show that the following are equiveridical: (i) there is a measure on X such that the coordinate map from X to Y_i is inverse-measure-preserving for both i; (ii) there is a Radon measure on X such that the coordinate map from X to Y_i is inverse-measure-preserving for both i; (iii) for every compact $K \subseteq Y_1$, $\nu_1 K \leq \nu_2^*(X[K])$. (*Hint*: for (iii)⇒(ii), use 457C to show that there is a finitely additive functional ν on $\mathcal{P}X$ of the required type; now observe that ν must give large mass to compact subsets of X, and apply 413S.)

(g) Let $X \subseteq [0,1]^2$ be a Lebesgue measurable set such that $X \cap (E \times F)$ is not negligible for any non-negligible sets E, $F \subseteq [0,1]$. (For the construction of such sets, see the notes to §325.) Show that there is a Radon measure on X such that both the coordinate projections from X to $[0,1]$ are inverse-measure-preserving, where $[0,1]$ is given Lebesgue measure. (*Hint*: show that there is a measure-preserving bijection ϕ between conegligible subsets of $[0,1]$ which is covered by X; ϕ can be taken to be of the form $\phi(x) = x - \alpha_n$ for $x \in E_n$.)

(h) Set $X = \{(t, 2t) : 0 \leq t \leq \frac{1}{2}\} \cup \{(t, 2t-1) : \frac{1}{2} \leq t \leq 1\}$. Show that there is a Radon measure on X for which both the coordinate maps onto $[0,1]$ are inverse-measure-preserving, but that X does not include the graph of any measure-preserving bijection between conegligible subsets of $[0,1]$.

(i) Let X be the eighth-sphere $\{x : x \in [0,1]^3, \|x\| = 1\}$. Show that there is a measure on X such that all three coordinate maps from X onto $[0,1]$ are inverse-measure-preserving. (*Hint*: 265Xe.)

(j) Set $X = \{x : x \in [0,1]^3, \xi_1 + \xi_2 + \xi_3 = \frac{3}{2}\}$. Show that there is a measure on X such that all the coordinate maps from X onto $[0,1]$ are inverse-measure-preserving. (*Hint*: note that X is a regular hexagon; try one-dimensional Hausdorff measure on its boundary.)

(k) Explain how to adapt the example in 457J to provide a family $\langle \mu_i \rangle_{i \in I}$ of probability measures on a set X such that (i) $\langle \mu_i \rangle_{i \in I}$ is upwards-directed (iii) there is no measure on X extending μ_i for every $i \in I$.

457Y Further exercises (a) Show that for any $n \geq 2$ there are a finite set X and a family $\langle \mu_i \rangle_{i \leq n}$ of measures on X such that $\{\mu_i : i \leq n, i \neq j\}$ have a common extension to a measure on X for every $j \leq n$, but the whole family $\{\mu_i : i \leq n\}$ has no such extension.

(b) Show that the example in 457H has the property: if f_i is a ν_i-integrable real-valued function for each i, and $\int f_1 d\nu_1 + \int f_2 d\nu_2 < 1$, then there is an $(x,y) \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ such that $f_1(x,y) + f_2(x,y) < 1$.

(c) Suppose we replace the set X in 457H with $X' = X \cup \{(x,x) : x \in [0, \frac{1}{2}]\}$, and write ν_i' for the measures on X' defined by the coordinate projections. Show that (i) if f_i is a ν_i'-integrable real-valued function on X' for each i, and $\int f_1 d\nu_1' + \int f_2 d\nu_2' \leq 1$, then there is an $(x,y) \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ such that $f_1(x,y) + f_2(x,y) \leq 1$ (ii) there is no measure on X' extending both ν_i'.

(d) In 457Xj, show that there are many Radon measures on X such that all the coordinate maps from X onto $[0,1]$ are inverse-measure-preserving.

457Z Problems (a) Characterize the sets $X \subseteq [0,1]^2$ for which there is a measure on X such that both the projections from X to $[0,1]$ are inverse-measure-preserving.

(b) Set $X = \{x : x \in [0,1]^3, \|x\| = 1\}$. Is there more than one Radon measure on X for which all the coordinate maps from X onto $[0,1]$ are inverse-measure-preserving? (See 457Xi, 457Yd.)

457 Notes and comments In the context of this section, as elsewhere (compare 391E-391G and 391J), finitely additive extensions, as in 457A-457D, generally present easier problems than countably additive extensions. So techniques for turning additive functionals into measures (391D, 454C, 454D, 457E, 457G)

are very valuable. Note that 457D offers possibilities in this direction: if θ there is countably additive, μ also will be (457Xa).

457H-457J demonstrate obstacles which can arise when seeking countably additive extensions even when finitely additive extensions give no difficulty. For finitely additive extensions a problem can arise at any finite number of measures (see 457Ya), but there is no further obstruction with infinite families (457Xc). For countably additive measures we have a positive result (457G) only under very restricted circumstances; relaxing any of the hypotheses can lead to failure (457J, 457Xk). Even in the apparently concrete case in which we have an open or closed set $X \subseteq [0,1]^2$ and we are seeking a measure on X with prescribed image measures on each coordinate, there can be surprises (457H, 457Xg, 457Xh), and I know of no useful description of the sets for which such a measure can be found (457Za).

The two-dimensional case has a special feature: when verifying the conditions (ii) or (iii) in 457A, or the condition (ii) of 457B, it is enough to consider only one set associated with each coordinate (457C). Put another way, in conditions (iv) and (v) of 457A it is enough to examine characteristic functions. This is not the case as soon as we have three coordinates (457I).

458 Relative independence and relative products

Stochastic independence is one of the central concepts of probability theory, and pervades measure theory. We come now to a generalization of great importance. If X_1, X_2 and Y are random variables, we may find that X_1 and X_2 are 'relatively independent over Y', or 'independent when conditioned on Y', in the sense that if we know the value of Y, then we learn nothing further about one of the X_i if we are told the value of the other. For any stochastic process, where information comes to us piecemeal, this idea is likely to be fundamental. In this section I set out a general framework for discussion of relative independence (458A), introducing relative distributions (458C) and relative independence in measure algebras (458H). In the second half of the section I look at 'relative product measures' (458I, 458L), giving the basic existence theorems (458J, 458N, 458O).

458A Relative independence Let (X, Σ, μ) be a probability space and T a σ-subalgebra of Σ.

(a) I say that a family $\langle E_i\rangle_{i\in I}$ in Σ is **relatively (stochastically) independent over** T if whenever $J \subseteq I$ is finite and not empty, and g_i is a conditional expectation of χE_i on T for each $i \in J$, then $\mu(\bigcap_{i\in J} E_i) = \int \prod_{i\in J} g_i d\mu$. (Note that $\int \prod_{i\in J} g_i d\mu$ does not depend on which conditional expectation g_i we take.) A family $\langle \Sigma_i\rangle_{i\in I}$ of σ-subalgebras of Σ is **relatively independent** over T if $\langle E_i\rangle_{i\in I}$ is relatively independent over T whenever $E_i \in \Sigma_i$ for every $i \in I$.

(b) I say that a family $\langle f_i\rangle_{i\in I}$ in $\mathcal{L}^0(\mu)$ (the space of almost-everywhere-defined virtually measurable real-valued functions, or 'random variables') is **relatively independent over** T if $\langle \Sigma_i\rangle_{i\in I}$ is relatively independent over T (with respect to the completion of μ), where Σ_i is the σ-algebra defined by f_i in the sense of 272C, that is, the σ-algebra generated by $\{f_i^{-1}[F] : F \subseteq \mathbb{R} \text{ is a Borel set}\}$.

(c) I remark at once that a family of σ-algebras or random variables is relatively independent iff every finite subfamily is (cf. 272Bb).

458B It is sometimes useful to know that 'relative independence' can be defined without using the apparatus of conditional expectations.

Lemma Let (X, Σ, μ) be a probability space, T_0 a σ-subalgebra of Σ, and $\langle \Sigma_i\rangle_{i\in I}$ a family of σ-subalgebras of Σ. Let $\mathbb{T}$ be the family of finite subalgebras of T. For $\Lambda \in \mathbb{T}$ write $\mathcal{A}_\Lambda$ for the set of non-negligible atoms in Λ. For non-empty finite $J \subseteq I$ and $\langle E_i\rangle_{i\in J} \in \prod_{i\in J} \Sigma_i$, set

$$\phi_\Lambda(\langle E_i\rangle_{i\in J}) = \sum_{H\in\mathcal{A}_\Lambda} \mu H \cdot \prod_{i\in J} \frac{\mu(E_i\cap H)}{\mu H}.$$

Then $\langle \Sigma_i\rangle_{i\in I}$ is relatively independent over T iff $\lim_{\Lambda\in\mathbb{T},\Lambda\uparrow} \phi_\Lambda(\langle E_i\rangle_{i\in J}) = \mu(\bigcap_{i\in J} E_i)$ whenever $J \subseteq I$ is finite and not empty and $E_i \in \Sigma_i$ for every $i \in J$.

proof (a) The point is just that if $J \subseteq I$ is finite and not empty, $E_i \in \Sigma_i$ for $i \in J$, and g_i is a conditional expectation of χE_i on T for each i, then $\int \prod_{i\in J} g_i d\mu = \lim_{\Lambda\uparrow} \phi_\Lambda(\langle E_i\rangle_{i\in J})$. **P** Adjusting each g_i on a negligible set if necessary, we may suppose that it is T-measurable, defined everywhere on X and takes values between 0 and 1.

Fix $n \in \mathbb{N}$ for the moment. Let Λ_n be the finite subalgebra of T generated by sets of the form $\{x : g_i(x) \leq 2^{-n}k\}$ for $i \in J$ and $1 \leq k \leq 2^n + 1$, and Λ any finite subalgebra of Σ_0 including Λ_n. If H is an atom of Λ and $\mu H > 0$, then there are integers k_i, for $i \in J$, $2^{-n}k_i \leq g_i(x) < 2^{-n}(k_i + 1)$ for every $i \in J$ and $x \in H$. So

$$2^{-n}k_i \leq \frac{\mu_i(E\cap H)}{\mu H} < 2^{-n}(k_i + 1)$$

for each i. Accordingly $\phi_\Lambda(\langle E_i\rangle_{i\in J})$ lies between $\sum_{H\in\mathcal{A}_\Lambda} \mu H \cdot \prod_{i\in J} 2^{-n}k_i$ and $\sum_{H\in\mathcal{A}_\Lambda} \mu H \cdot \prod_{i\in J} 2^{-n}(k_i+1)$, that is, between $\int \prod_{i\in J} g'_{in} d\mu$ and $\int \prod_{i\in J} g''_{in} d\mu$, where $g'_{in}(x) = 2^{-n}k$, $g''_{in}(x) = \min(1, 2^{-n}(k+1))$ when $2^{-n}k \leq g_i(x) < 2^{-n}(k+1)$. But this means that

$$\begin{aligned}|\phi_\Lambda(\langle E_i\rangle_{i\in J}) - \int \prod_{i\in J} g_i d\mu| &\leq \max(\int |\prod_{i\in J} g'_{in} - \prod_{i\in J} g_i| d\mu, \int |\prod_{i\in J} g''_{in} - \prod_{i\in J} g_i| d\mu)\\ &\leq \max(\int \sum_{i\in J} |g'_{in} - g_i| d\mu, \int \sum_{i\in J} |g''_{in} - g_i| d\mu)\end{aligned}$$

(because all the g_i, g'_{in}, g''_{in} take values in $[0,1]$, see 285O)

$$\leq 2^{-n}\#(J).$$

Since this is true for every $\Lambda \supseteq \Lambda_n$ and every $n \in \mathbb{N}$, $\lim_{\Lambda\uparrow} \phi_\Lambda(\langle E_i\rangle_{i\in J}) = \int \prod_{i\in J} g_i d\mu$. **Q**

(b) Accordingly the condition given exactly matches the definition in 458A.

458C All the fundamental theorems concerning stochastic independence have relativized forms. For the first we need a concept of 'relative probability distribution', as follows.

Definition Let (X, Σ, μ) be a probability space, T a σ-subalgebra of subsets of X, and $f \in \mathcal{L}^0(\mu)$. Then a **relative distribution of f over** T will be a family $\langle \nu_x\rangle_{x\in X}$ of Radon probability measures on $\mathbb{R}$ such that $x \mapsto \nu_x(H) : X \to [0,1]$ is T-measurable and $\int_F \nu_x(H)\mu(dx) = \mu(F \cap f^{-1}[H])$ for every Borel set $H \subseteq \mathbb{R}$ and every $F \in \mathrm{T}$, that is, $x \mapsto \nu_x H$ is a conditional expectation of $\chi f^{-1}[H]$ on T.

458D Theorem Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $f \in \mathcal{L}^0(\mu)$. Then there is a relative distribution of f over T, which is essentially unique in the sense that if $\langle \nu_x\rangle_{x\in X}$ and $\langle \nu'_x\rangle_{x\in X}$ are two such families, then $\nu_x = \nu'_x$ for $\mu\restriction$T-almost every x.

proof (a) Write μ_0 for the restriction of μ to T, $\hat{\mu}$ for the completion of μ, $\hat{\Sigma}$ for the domain of $\hat{\mu}$, and $\mathcal{B}$ for the Borel σ-algebra of $\mathbb{R}$. Then the function $x \mapsto (x, f(x)) : \operatorname{dom} f \to X \times \mathbb{R}$ is $(\hat{\Sigma}, \mathrm{T}\widehat{\otimes}\mathcal{B})$-measurable, just because $F \cap f^{-1}[H] \in \hat{\Sigma}$ for every $F \in \mathrm{T}$ and $H \in \mathcal{B}$. So we have a probability measure λ on $X \times \mathbb{R}$ defined by setting $\lambda W = \mu\{x : (x, f(x)) \in W\}$ for every $W \in \mathrm{T}\widehat{\otimes}\mathcal{B}$. The marginal measure on $\mathbb{R}$ is tight (that is, inner regular with respect to the compact sets) just because it is a Borel probability measure (433Ca). By 452M, we have a family $\langle \lambda_x\rangle_{x\in X}$ of Radon probability measures on $\mathbb{R}$ such that $\lambda W = \int \lambda_x W^{-1}[\{x\}]\mu_0(dx)$ for every $W \in \mathrm{T}\widehat{\otimes}\mathcal{B}$.

(b) The functions $x \mapsto \lambda_x H$ need not be T-measurable. However, if we set

$$g_q(x) = \lambda_x(]-\infty, q])$$

for $q \in \mathbb{Q}$, then every g_q is μ_0-virtually measurable, so there is a μ_0-conegligible set G such that every $g_q\restriction G$ is T-measurable. By the Monotone Class Theorem (136B), the family

$$\{H : H \in \mathcal{B},\ x \mapsto \lambda_x H : G \to [0,1] \text{ is T-measurable}\}$$

is the whole of $\mathcal{B}$. So if we set $\nu_x = \lambda_x$ for $x \in G$, and take λ_x to be the point mass at 0 (for instance) for $x \in X \setminus G$, we shall have a relative distribution of f over T as defined in 458C.

(c) Now suppose that $\langle \nu'_x \rangle_{x\in X}$ is another relative distribution of f over T. Then for each $H \in \mathcal{B}$ we have $\int_F \nu_x H\mu(dx) = \int_F \nu'_x H\mu(dx)$ for every $F \in \mathrm{T}$, so that $\nu_x H = \nu'_x H$ for μ_0-almost every x. But this means that for μ_0-almost every x, we have $\nu_x H = \nu'_x H$ for every interval H with rational endpoints; and for such x we must have $\nu_x = \nu'_x$ (416Eb).

458E Now we can state and prove a result corresponding to 272G.

Theorem Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $\langle f_i \rangle_{i\in I}$ a family in $\mathcal{L}^0(\mu)$. For each $i \in I$, let $\langle \nu_{ix} \rangle_{x\in X}$ be a relative distribution of f_i over T, and $\tilde{f}_i : X \to \mathbb{R}$ an arbitrary extension of f_i to the whole of X. Then the following are equiveridical:

(i) $\langle f_i \rangle_{i\in I}$ is relatively independent over T;

(ii) for any Baire set $W \subseteq \mathbb{R}^I$ and any $F \in \mathrm{T}$,

$$\hat{\mu}(F \cap \boldsymbol{f}^{-1}[W]) = \int_F \lambda_x W\mu(dx),$$

where $\hat{\mu}$ is the completion of μ, $\boldsymbol{f}(x) = \langle \tilde{f}_i(x) \rangle_{i\in I}$ for $x \in X$, and λ_x is the product of $\langle \nu_{ix} \rangle_{i\in I}$ for each x.

proof (a) Note first that if $i \in I$ and $H \subseteq \mathbb{R}$ is a Borel set, then $\int_F \nu_{ix} H\mu(dx) = \hat{\mu}(F \cap f_i^{-1}[H])$ for every $F \in \mathrm{T}$, so $x \mapsto \nu_{ix} H$ is a conditional expectation of $\chi f_i^{-1}[H]$ on T.

Suppose that $\langle f_i \rangle_{i\in I}$ is relatively independent, and $F \in \mathrm{T}$. Let $\mathcal{C}$ be the family of subsets of $\mathbb{R}^I$ expressible in the form $C = \{z : z \in \mathbb{R}^I, z(i) \in H_i \text{ for every } i \in J\}$ where $J \subseteq I$ is finite and $H_i \subseteq \mathbb{R}$ is a Borel set for each $i \in J$. For such a set C,

$$\hat{\mu}(F \cap \boldsymbol{f}^{-1}[C]) = \hat{\mu}(F \cap \bigcap_{i\in J} \tilde{f}_i^{-1}[H_i]) = \int_F \prod_{i=0}^{n} \nu_{ix} H_i \mu(dx)$$

(interpreting an empty product as χX)

$$= \int_F \lambda_x C\mu(dx).$$

So

$$\mathcal{W} = \{W : W \subseteq \mathbb{R}^I, \mu(F \cap \boldsymbol{f}^{-1}[W]) = \textstyle\int \lambda_x W\mu(dx)\}$$

includes $\mathcal{C}$; since it is a Dynkin class, it contains every Baire subset of $\mathbb{R}^I$ (by the Monotone Class Theorem, 136B), and (ii) is true.

(b) Now suppose that (ii) is true. Let Σ_i be the σ-algebra defined by f_i for each i. If $J \subseteq I$ is finite and $E_i \in \Sigma_i$ for each $i \in J$, then there are Borel sets $H_i \subseteq \mathbb{R}$ such that $E_i \triangle f_i^{-1}[H_i]$ is negligible for each i, so that $x \mapsto \nu_{ix} H_i$ is a conditional expectation of χE_i on T. Now by the same equations as before, in the opposite direction,

$$\int_F \prod_{i\in J} \nu_{ix} H_i \mu(dx) = \int_F \lambda_x C\mu(dx)$$

(where $C = \{z : z(i) \in H_i \text{ for } i \in J\}$)

$$= \hat{\mu}(F \cap \boldsymbol{f}^{-1}[C]) = \hat{\mu}(F \cap \bigcap_{i\in J} f_i^{-1}[H_i]) = \hat{\mu}(F \cap \bigcap_{i\in J} E_i)$$

for every $F \in \mathrm{T}$. As $\langle E_i \rangle_{i\in J}$ is arbitrary, $\langle \Sigma_i \rangle_{i\in I}$ and $\langle f_i \rangle_{i\in I}$ are relatively independent.

Remarks Of course the ungainly shift to $\tilde{f}_i$ is unnecessary if I is countable; but for uncountable I the intersection $\bigcap_{i\in I} \operatorname{dom} f_i$, which is the only suitable domain for $\boldsymbol{f}$, may not be conegligible.

I said that λ_x should be 'the product of $\langle \nu_{ix} \rangle_{i\in I}$'. Since the ν_{ix} are Radon probability measures, we have two possible interpretations of this: either the 'ordinary' product measure of §254 or the 'quasi-Radon' product measure of §417. But as we are interested only in the values of $\lambda_x W$ for Baire sets W, it makes no difference which we choose.

458F Proposition (Compare 272K.) Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a family of σ-subalgebras of Σ which is relatively independent over T. Let $\langle I_j\rangle_{j\in J}$ be a partition of I, and for each $j\in J$ let $\tilde\Sigma_j$ be the σ-algebra generated by $\bigcup_{i\in I_j}\Sigma_i$. Then $\langle\tilde\Sigma_j\rangle_{j\in J}$ is relatively independent over T.

proof For each $E\in\Sigma$ let g_E be a conditional expectation of χE on T. Take any finite $K\subseteq J$, and let $\boldsymbol{W}$ be the set of families $\langle W_j\rangle_{j\in K}$ such that $W_j\in\tilde\Sigma_j$ for each $j\in K$ and $\mu(F\cap\bigcap_{j\in K}W_j)=\int_F\prod_{j\in K}f_{W_j}d\mu$ for every $F\in\mathrm{T}$. For each $j\in K$, let $\mathcal{C}_j$ be the family of sets expressible as $W=X\cap\bigcap_{i\in L}E_i$ where $L\subseteq I_j$ is finite and $E_i\in\Sigma_i$ for $i\in L$. Note that in this case

$$\mu(F\cap W)=\mu(F\cap\textstyle\bigcap_{i\in L}E_i)=\int\prod_{i\in L}f_{E_i}d\mu$$

for every $F\in\mathrm{T}$, so $f_W=_{\text{a.e.}}\prod_{i\in L}f_{E_i}$, taking the product to be χX if L is empty.

If $W_j\in\mathcal{C}_j$ for each $j\in K$, then $\langle W_j\rangle_{j\in J}\in\boldsymbol{W}$. **P** Express W_j as $X\cap\bigcap_{i\in L_j}E_i$ where $L_j\subseteq I_j$ is finite and $E_i\in\Sigma_i$ whenever $j\in K$ and $i\in L_j$. Then

$$\mu(F\cap\bigcap_{j\in K}W_j)=\mu(F\cap\bigcap_{i\in L}E_i)$$

(where $L=\bigcup_{j\in K}L_j$)

$$=\int_F\prod_{i\in L}f_{E_i}d\mu$$

(because $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent)

$$=\int_F\prod_{j\in K}\prod_{i\in L_j}f_{E_i}d\mu=\int_F\prod_{j\in K}f_{W_i}d\mu$$

for every $F\in\mathrm{T}$. **Q**

Observe next that if we fix $k\in K$, and a family $\langle W_j\rangle_{j\in K\setminus\{k\}}$, then the set of those $W_k\in\tilde\Sigma_k$ such that $\langle W_j\rangle_{j\in K}\in\boldsymbol{W}$ is a Dynkin class, so if it includes $\mathcal{C}_k$ it must include the σ-algebra generated by $\mathcal{C}_k$, viz., $\tilde\Sigma_k$. Now an easy induction on n shows that if $\langle W_j\rangle_{j\in K}\in\prod_{j\in K}\tilde\Sigma_j$ and $\#(\{j:W_j\notin\mathcal{C}_j\})=n$, then $\langle W_j\rangle_{j\in K}\in\boldsymbol{W}$. Taking $n=\#(K)$ we see that $\prod_{j\in K}\tilde\Sigma_j\subseteq\boldsymbol{W}$.

As this is true for every finite $K\subseteq J$, $\langle\tilde\Sigma_j\rangle_{j\in J}$ is relatively independent over T, as claimed.

458G Example The simplest examples of relatively independent σ-algebras arise as follows. Let (X,Σ,μ) be a probability space, $\langle\mathrm{T}_i\rangle_{i\in I}$ an independent family of σ-subalgebras of Σ, as in 272Ab, and T a σ-subalgebra of Σ which is independent of the σ-algebra generated by $\bigcup_{i\in I}\mathrm{T}_i$. For each $i\in I$, let Σ_i be the σ-algebra generated by $\mathrm{T}\cup\mathrm{T}_i$. Then $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T.

proof For $i\in I$, set $\Sigma_{i0}=\mathrm{T}_i$ and $\Sigma_{i1}=\mathrm{T}$. Then $\langle\Sigma_{i\epsilon}\rangle_{(i,\epsilon)\in I\times\{0,1\}}$ is relatively independent over T. **P** Let $J\subseteq I\times\{0,1\}$ be a non-empty finite set and $E_{i\epsilon}\in\Sigma_{i\epsilon}$ for each $(i,\epsilon)\in J$. Set $J_0=\{i:(i,0)\in J\}$, $J_1=\{i:(i,1)\in J\}$. Then $g_{i0}=\mu E_{i0}\chi X$ is a conditional expectation of χE_{i0} on T for each $i\in J_0$, and $g_{i1}=\chi E_{i1}$ is a conditional expectation of χE_{i1} on T for each $i\in J_1$. If $F\in\mathrm{T}$, then

$$\begin{aligned}\mu(F\cap\bigcap_{(i,\epsilon)\in J}E_{i\epsilon})&=\mu((F\cap\bigcap_{i\in J_1}E_{i\epsilon})\cap\bigcap_{i\in J_0}E_{i\epsilon})\\&=\mu(F\cap\bigcap_{i\in J_1}E_{i\epsilon})\cdot\prod_{i\in J_0}\mu E_{i\epsilon}\end{aligned}$$

(taking the product to be 1 if J_0 is empty)

$$=\int_F\prod_{(i,\epsilon)\in J}g_{i\epsilon}d\mu.$$

As F and $\langle E_{i\epsilon}\rangle_{(i,\epsilon)\in J}$ are arbitrary, $\langle\Sigma_{i\epsilon}\rangle_{(i,\epsilon)\in I\times\{0,1\}}$ is relatively independent over T. **Q**

Since each Σ_i is the σ-algebra generated by $\Sigma_{i0}\cup\Sigma_{i1}$, 458F tells us that $\langle\Sigma_i\rangle_{i\in I}$ is also relatively independent over T.

458H Measure algebras We can look at the same ideas in the context of measure algebras.

(a) Suppose that $(\mathfrak{A},\bar\mu)$ is a probability algebra, and that $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$. If $a\in\mathfrak{A}$, then we can say that $u\in L^\infty(\mathfrak{C})$ is the conditional expectation of χa on $\mathfrak{C}$ if $\int_c u=\bar\mu(c\cap a)$ for every $c\in\mathfrak{C}$ (365R). Now we can say that a family $\langle b_i\rangle_{i\in I}$ in $\mathfrak{A}$ is **relatively (stochastically) independent** over $\mathfrak{C}$ if $\bar\mu(c\cap\inf_{i\in J}b_i)=\int_c\prod_{i\in J}u_i$ whenever $J\subseteq I$ is a non-empty finite set and u_i is the conditional expectation of χb_i on $\mathfrak{C}$ for every $i\in J$; while a family $\langle\mathfrak{B}_i\rangle_{i\in I}$ of subalgebras of $\mathfrak{A}$ is **relatively (stochastically) independent over** $\mathfrak{C}$ if $\langle b_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$ whenever $b_i\in\mathfrak{B}_i$ for every $i\in I$.

Following 458F, we have the result that if $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$, and $\langle I_j\rangle_{j\in J}$ is a partition of I, and $\tilde{\mathfrak{B}}_j$ is the closed subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i\in I_j}\mathfrak{B}_i$ for every $j\in J$, then $\langle\tilde{\mathfrak{B}}_j\rangle_{j\in J}$ is relatively independent over $\mathfrak{C}$. The most natural proof, from where we are now standing, is to express $(\mathfrak{A},\bar\mu)$ as the measure algebra of a probability space (X,Σ,μ), set $\mathrm{T}=\{F:F^\bullet\in\mathfrak{C}\}$ and $\Sigma_i=\{E:E^\bullet\in\mathfrak{B}_i\}$ for each $i\in I$, and use 458F.)

(b) Note that if $a\in\mathfrak{A}$ and u is the conditional expectation of a on $\mathfrak{C}$, then $[\![u>0]\!]=\mathrm{upr}(a,\mathfrak{C})$, by 365Rc. So if $\langle\mathfrak{B}_i\rangle_{i\in I}$ is a family of subalgebras of $\mathfrak{A}$ which is relatively independent over $\mathfrak{C}$, and $J\subseteq I$ is finite, and $b_i\in\mathfrak{B}_i$ for each $i\in J$, then $\inf_{i\in J}b_i=0$ iff $\inf_{i\in J}\mathrm{upr}(b_i,\mathfrak{C})=0$. (If u_i is a conditional expectation of b_i on $\mathfrak{C}$ for each i, then

$$\inf_{i\in J}\mathrm{upr}(b_i,\mathfrak{C})=\inf_{i\in J}[\![u_i>0]\!]=[\![\textstyle\prod_{i\in J}u_i]\!]$$

is zero iff $\bar\mu(\inf_{i\in J}b_i)=\int\prod_{i\in J}u_i=0$.)

(c) We have a straightforward version of 458G, as follows. If $(\mathfrak{A},\bar\mu)$ is a probability algebra, $\langle\mathfrak{C}_i\rangle_{i\in I}$ is a (stochastically) independent family of closed subalgebras of $\mathfrak{A}$, $\mathfrak{C}$ is a closed subalgebra of $\mathfrak{A}$ independent of the algebra generated by $\bigcup_{i\in I}\mathfrak{C}_i$, and $\mathfrak{B}_i$ is the closed subalgebra of $\mathfrak{A}$ generated by $\mathfrak{C}\cup\mathfrak{C}_i$ for each i, then $\langle\mathfrak{B}_i\rangle_{i\in I}$ is relatively independent over $\mathfrak{C}$. (Either repeat the proof of 458G, looking at $\mathfrak{B}_{i0}=\mathfrak{C}_i$ and $\mathfrak{B}_{i1}=\mathfrak{C}$ for each i, or move to a measure space representing $\mathfrak{A}$ and quote 458G.)

458I Relative free products of probability algebras: Definition Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras and $(\mathfrak{C},\bar\nu)$ a probability algebra, and suppose that we are given a measure-preserving Boolean homomomorphism $\pi_i:\mathfrak{C}\to\mathfrak{A}_i$ for each $i\in I$. A **relative free product** of $\langle(\mathfrak{A}_i,\bar\mu_i,\pi_i)\rangle_{i\in I}$ over $(\mathfrak{C},\bar\nu)$ is a probability algebra $(\mathfrak{A},\bar\mu)$, together with a measure-preserving Boolean homomorphism $\phi_i:\mathfrak{A}_i\to\mathfrak{A}$ for each $i\in I$, such that

$\phi_i\pi_i=\phi_j\pi_j:\mathfrak{C}\to\mathfrak{A}$ for all $i,j\in I$,
writing $\mathfrak{D}$ for the common value of the $\phi_i[\pi_i[\mathfrak{C}]]$, $\langle\phi_i[\mathfrak{A}_i]\rangle_{i\in I}$ is relatively independent over $\mathfrak{D}$,
$\mathfrak{A}$ is the closed subalgebra of itself generated by $\bigcup_{i\in I}\phi_i[\mathfrak{A}_i]$.

458J Theorem Let $\langle(\mathfrak{A}_i,\bar\mu_i)\rangle_{i\in I}$ be a family of probability algebras, $(\mathfrak{C},\bar\nu)$ a probability algebra and $\pi_i:\mathfrak{C}\to\mathfrak{A}_i$ a measure-preserving Boolean homomomorphism for each $i\in I$. Then $\langle(\mathfrak{A}_i,\bar\mu_i,\pi_i)\rangle_{i\in I}$ has an essentially unique relative free product over $(\mathfrak{C},\bar\nu)$.

proof (a)(i) Let $\mathfrak{B}$ be the free product of $\langle\mathfrak{A}_i\rangle_{i\in I}$ (315H); write $\varepsilon_i:\mathfrak{A}_i\to\mathfrak{B}$ for the canonical embedding of $\mathfrak{A}_i$ in $\mathfrak{B}$. For each $i\in I$, $a\in\mathfrak{A}$ we have a completely additive functional $c\mapsto\bar\mu_i(a\cap\pi_ic):\mathfrak{C}\to[0,1]$; let $u_{ia}\in L^\infty(\mathfrak{C})$ be a corresponding Radon-Nikodým derivative, so that $\int_c u_{ia}d\bar\nu=\bar\mu_i(a\cap\pi_ic)$ for every $c\in\mathfrak{C}$ (365E). By 326Q, there is a unique additive functional $\lambda:\mathfrak{B}\to[0,1]$ such that

$$\lambda(\inf_{i\in J}\varepsilon_ia_i)=\int\prod_{i\in J}u_{i,a_i}d\bar\nu$$

whenever $J\subseteq I$ is a non-empty finite set and $a_i\in\mathfrak{A}_i$ for every $i\in J$. Of course $u_{i1}=\chi1$ in $L^\infty(\mathfrak{C})$ for every $i\in I$, interpreting the '1' in u_{i1} in the Boolean algebra $\mathfrak{A}_i$, and the '1' in $\chi1$ in the Boolean algebra $\mathfrak{C}$; so (this time interpreting '1' in $\mathfrak{B}$) $\lambda1=1$ (the final '1' being a real number, of course). Note also that $u_{i,a\cap\pi_ic}=\chi c\times u_{ia}$ for every $c\in\mathfrak{C}$ and $a\in\mathfrak{A}_i$, because

$$\int_{c'}u_{i,a\cap\pi_ic}d\bar\nu=\bar\mu(a\cap\pi_ic\cap\pi_ic')=\int_{c\cap c'}u_{ia}d\bar\nu=\int_{c'}\chi c\times u_{ia}d\bar\nu$$

for every $c'\in\mathfrak{C}$.

(ii) By 393C, there are a probability algebra $(\mathfrak{A},\bar\mu)$ and a Boolean homomorphism $\phi:\mathfrak{B}\to\mathfrak{A}$ such that $\lambda=\bar\mu\phi$. We can of course suppose that $\mathfrak{A}$ is the order-closed subalgebra of itself generated by $\phi[\mathfrak{B}]$ (which is in fact automatically the case if we use the construction in the proof of 393C).

For each $i\in I$, set $\phi_i=\phi\varepsilon_i:\mathfrak{A}_i\to\mathfrak{A}$. It is a Boolean homomorphism because ϕ and ε_i are. If $a\in\mathfrak{A}_i$, then

$$\bar\mu\phi_i a=\lambda\varepsilon_i a=\int u_{ia}\,d\bar\nu=\bar\mu_i(a\cap\pi_i 1)=\bar\mu_i a,$$

so ϕ_i is measure-preserving.

If i, $j\in I$ and $c\in\mathfrak{C}$ then

$$\begin{aligned}\bar\mu(\phi_i\pi_i c\triangle\phi_j\pi_j c)&=\lambda(\varepsilon_i\pi_i c\triangle\varepsilon_j\pi_j c)\\&=\lambda(\varepsilon_i\pi_i c)+\lambda(\varepsilon_j\pi_j c)-2\lambda(\varepsilon_i\pi_i c\cap\varepsilon_j\pi_j c)\\&=\int u_{i,\pi_i c}\,d\bar\nu+\int u_{j,\pi_j c}\,d\bar\nu-2\int u_{i,\pi_i c}\times u_{j,\pi_j c}\,d\bar\nu\\&=\int\chi c\,d\bar\nu+\int\chi c\,d\bar\nu-2\int\chi c\times\chi c\,d\bar\nu=0.\end{aligned}$$

So $\phi_i\pi_i=\phi_j\pi_j$; write π for this common value. Set $\mathfrak{D}=\pi[\mathfrak{C}]$, so that π is a measure-preserving Boolean isomorphism between $\mathfrak{C}$ and $\mathfrak{D}$. Let $T:L^0(\mathfrak{C})\to L^0(\mathfrak{D})$ be the corresponding f-algebra isomorphism.

(iii) If $i\in I$ and $a\in\mathfrak{A}_i$, then Tu_{ia} is the conditional expectation of $\chi(\phi_i a)$ on $\mathfrak{D}$. **P** For any $d\in\mathfrak{D}$, set $c=\pi^{-1}d$; then

$$\begin{aligned}\bar\mu(d\cap\phi_i a)&=\bar\mu(\phi_i(\pi_i c\cap a))=\lambda(\varepsilon_i(\pi_i c\cap a))\\&=\int u_{a\cap\pi_i c}\,d\bar\nu=\int_c u_{ia}\,d\bar\nu=\int_d Tu_{ia}\end{aligned}$$

by 365H. **Q**

(iv) Suppose that $J\subseteq I$ is finite and not empty and that $d_i\in\phi_i[\mathfrak{A}_i]$ for each $i\in J$. For each $i\in I$ let $v_i\in L^\infty(\mathfrak{D})$ be the conditional expectation of χd_i on $\mathfrak{D}$, and let $a_i\in\mathfrak{A}_i$ be such that $\phi_i a=d_i$. Set $c=\pi^{-1}d$. Then

$$\begin{aligned}\bar\mu(d\cap\inf_{i\in J}d_i)&=\bar\mu(\inf_{i\in J}(d_i\cap d))=\lambda(\inf_{i\in J}\varepsilon_i(a_i\cap\pi_i c))\\&=\int\prod_{i\in J}u_{i,a_i\cap\pi_i c}\,d\bar\nu=\int_c\prod_{i\in J}u_{i,a_i}\,d\bar\nu\\&=\int_{\pi c}T(\prod_{i\in J}u_{i,a_i})\,d\bar\mu=\int_d\prod_{i\in J}Tu_{i,a_i}\,d\bar\mu=\int_d\prod_{i\in J}v_i\,d\bar\mu.\end{aligned}$$

As $\langle d_i\rangle_{i\in J}$ and d are arbitrary, $\langle\phi_i[\mathfrak{A}_i]\rangle_{i\in I}$ is relatively independent over $\mathfrak{D}$. This shows that $(\mathfrak{A},\bar\mu,\langle\phi_i\rangle_{i\in I})$ is a relative free product of $\langle(\mathfrak{A}_i,\bar\mu_i,\pi_i)\rangle_{i\in I}$ over $(\mathfrak{C},\bar\nu)$.

(b) Now suppose that $(\mathfrak{A}',\bar\mu',\langle\phi'_i\rangle_{i\in I})$ is another relative free product of $\langle(\mathfrak{A}_i,\bar\mu_i,\pi_i)\rangle_{i\in I}$ over $(\mathfrak{C},\bar\nu)$. Then we have a Boolean homomorphism $\psi:\mathfrak{B}\to\mathfrak{A}'$ such that $\phi'_i=\psi\varepsilon_i$ for every $i\in I$ (315I). In this case, $\bar\mu'\psi=\lambda$. **P** Let π' be the common value of $\phi'_i\pi_i$ for $i\in I$, set $\mathfrak{D}'=\pi[\mathfrak{C}]$, and let $T':L^0(\mathfrak{C})\to L^0(\mathfrak{D}')$ be the isomorphism corresponding to $\pi':\mathfrak{C}\to\mathfrak{D}$. If $a\in\mathfrak{A}$ and $c\in\mathfrak{C}$, then

$$\bar\mu'(\phi'_i a\cap\pi' c)=\bar\mu(a\cap\pi_c)=\int_c u_{ia}\,d\bar\nu=\int_{\pi' c}T'u_{ia}\,d\bar\mu'$$

by 365H (compare (a-iii) above); it follows that $T'u_{ia}$ is the conditional expectation of $\chi(\phi'_i a)$ on $\mathfrak{D}'$.

If $J\subseteq I$ is finite and not empty, and $a_i\in\mathfrak{A}_i$ for $i\in J$, then

$$\begin{aligned}\bar\mu'(\psi(\inf_{i\in J}\varepsilon_i a_i))&=\bar\mu'(\inf_{i\in J}\phi'_i a_i)=\int\prod_{i\in J}T'u_{i,a_i}\,d\bar\mu'\\&=\int T'(\prod_{i\in J}u_{i,a_i})\,d\bar\mu'=\int\prod_{i\in J}u_{i,a_i}\,d\bar\nu=\lambda(\inf_{i\in J}\varepsilon_i a_i).\end{aligned}$$

Because λ is the only additive functional on $\mathfrak{B}$ taking the right values on elements of this form, $\bar{\mu}'\psi = \lambda$. **Q**

In particular, $\psi b = 0$ whenever $b \in \mathfrak{B}$ and $\lambda b = 0$. It follows that $\psi b = \psi b'$ whenever b, $b' \in \mathfrak{B}$ and $\phi b = \phi b'$, since in this case $\lambda(b \triangle b') = \bar{\mu}(\phi b \triangle \phi b')$ is zero. So we have a function $\theta : \phi[\mathfrak{B}] \to \mathfrak{A}'$ defined by setting $\theta(\phi b) = \psi b$ for every $b \in \mathfrak{B}$, and of course θ is a Boolean homomorphism; moreover,

$$\bar{\mu}'\theta(\phi b) = \bar{\mu}'\psi b = \lambda b = \bar{\mu}\phi b$$

so θ is measure-preserving and an isometry for the measure metrics of $\mathfrak{A}$ and $\mathfrak{A}'$. If $i \in I$ and $a \in \mathfrak{A}$, then

$$\theta\phi_i a = \theta\phi\varepsilon_i a = \psi\varepsilon_i a = \phi' a,$$

so $\theta\phi_i = \phi'_i$ for every i. Because $\mathfrak{A}$ and $\mathfrak{A}'$ are the closed subalgebras generated by $\bigcup_{i\in I} \phi_i[\mathfrak{A}_i]$ and $\bigcup_{i\in I} \phi'_i[\mathfrak{A}_i]$ respectively, $\phi[\mathfrak{B}]$ and $\psi[\mathfrak{B}]$ are dense (323J). The isometry θ therefore extends uniquely to a measure algebra isomorphism $\hat{\theta} : \mathfrak{A} \to \mathfrak{A}'$ which must be the unique isomorphism such that $\hat{\theta}\phi_i = \phi'_i$ for every i. Thus $(\mathfrak{A}, \bar{\mu}, \langle\phi_i\rangle_{i\in I})$ and $(\mathfrak{A}', \bar{\mu}', \langle\phi'_i\rangle_{i\in I})$ are isomorphic, and the relative free product is essentially unique.

458K Developing the argument of the last part of the proof of 458J, we have the following.

Theorem Let $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$, $\langle(\mathfrak{A}'_i, \bar{\mu}'_i)\rangle_{i\in I}$ be two families of probability algebras, and $\psi_i : \mathfrak{A}_i \to \mathfrak{A}'_i$ a measure-preserving Boolean homomorphism for each i. Let $(\mathfrak{C}, \bar{\nu})$, $(\mathfrak{C}', \bar{\nu}')$ be probability algebras and $\pi_i : \mathfrak{C} \to \mathfrak{A}_i$, $\pi'_i : \mathfrak{C}' \to \mathfrak{A}'_i$ measure-preserving Boolean homomomorphisms for each $i \in I$; suppose that we have a measure-preserving isomorphism $\psi : \mathfrak{C} \to \mathfrak{C}'$ such that $\pi'_i\psi = \psi_i\pi_i : \mathfrak{C} \to \mathfrak{A}'_i$ for each i. Let $(\mathfrak{A}, \bar{\mu}, \langle\phi_i\rangle_{i\in I})$ and $(\mathfrak{A}', \bar{\mu}', \langle\phi'_i\rangle_{i\in I})$ be relative free products of $\langle(\mathfrak{A}_i, \bar{\mu}_i, \pi_i)\rangle_{i\in I}$, $\langle(\mathfrak{A}'_i, \bar{\mu}'_i, \pi'_i)\rangle_{i\in I}$ over $(\mathfrak{C}, \bar{\nu})$, $(\mathfrak{C}', \bar{\nu}')$ respectively. Then there is a unique measure-preserving Boolean homomorphism $\hat{\psi} : \mathfrak{A} \to \mathfrak{A}'$ such that $\hat{\psi}\phi_i = \phi'_i\pi_i : \mathfrak{A}_i \to \mathfrak{A}'$ for every $i \in I$.

proof By the uniqueness assertion of 458J, we may suppose that $(\mathfrak{A}, \bar{\mu}, \langle\phi_i\rangle_{i\in I})$ has been constructed by the method in the proof of 458J.

(a) For $i \in A$, $a \in \mathfrak{A}$, $a' \in \mathfrak{A}'$ let $u_{ia} \in L^\infty(\mathfrak{C})$, $u'_{ia'} \in L^\infty(\mathfrak{C}')$ be defined as in the proof of 458J, so that

$$\int_c u_{ia}d\bar{\nu} = \bar{\mu}_i(a \cap \pi_i c), \quad \int_{c'} u'_{ia'}d\bar{\nu}' = \bar{\mu}'_i(a' \cap \pi'_i c')$$

whenever $c \in \mathfrak{C}$, $c' \in \mathfrak{C}'$. Let $T : L^0(\mathfrak{C}) \to L^0(\mathfrak{C}')$ be the sequentially order-continuous Riesz homomorphism such that $T(\chi c) = \chi(\psi c)$ for every $c \in \mathfrak{C}$ (364R). Now $u'_{i,\psi_i a} = Tu_{ia}$ whenever $i \in I$ and $a \in \mathfrak{A}_i$. **P** If $c \in \mathfrak{C}$, then

$$\int_{\psi c} Tu_{ia}d\bar{\nu}' = \int_c u_{ia}d\bar{\nu}$$

(365H)

$$= \bar{\mu}(a \cap \pi_i c) = \bar{\mu}'\psi_i(a \cap \pi_i c) = \bar{\mu}'(\psi_i a \cap \pi'_i\psi c) = \int_{\psi c} u_{i,\psi_i a}.$$

Because ψ is surjective, it follows that $Tu_{ia} = u_{i,\psi_i(a)}$. **Q**

(b) Let $\mathfrak{B}$ be the free product of $\langle\mathfrak{A}_i\rangle_{i\in I}$ $\varepsilon_i : \mathfrak{A}_i \to \mathfrak{B}$ the canonical embedding for each i; let λ be the functional on $\mathfrak{B}$ defined by the process of (a-i) in the proof of 458J. By 315I, there is a Boolean homomorphism $\theta : \mathfrak{B} \to \mathfrak{A}'$ such that $\theta\varepsilon_i = \phi'_i\psi_i : \mathfrak{A}_i \to \mathfrak{A}'$ for every i. Now $\bar{\mu}'\theta = \lambda$. **P** If $J \subseteq I$ is finite and $a_i \in \mathfrak{A}_i$ for every $i \in J$, then

$$\bar{\mu}'\theta(\inf_{i\in J} \varepsilon_i a_i) = \bar{\mu}'(\inf_{i\in J} \phi_i\psi_i a_i) = \int \prod_{i\in J} u'_{i,\psi_i a_i}d\bar{\nu}'$$

$$= \int \prod_{i\in J} Tu_{i,a_i}d\bar{\nu}' = \int T(\prod_{i\in J} u_{i,a_i})d\bar{\nu}'$$

(because $T\restriction L^\infty(\mathfrak{C})$ is multiplicative)

$$= \int \prod_{i\in J} u_{i,a_i})d\bar\nu = \lambda(\inf_{i\in J} \varepsilon_i a_i).$$

As λ, θ and $\bar\nu$ are all additive, $\lambda = \bar\nu\theta'$ (using 315Jb). **Q**

(c) Let $\phi : \mathfrak{B} \to \mathfrak{A}$ be the map described in (a-ii) of the proof of 458J. Then $\bar\mu(\phi b) = \lambda b = \bar\mu'(\theta b)$ for every $b \in \mathfrak{B}$; in particular,

$$\phi b = 0 \Rightarrow \bar\mu(\phi b) = 0 \Rightarrow \bar\mu'(\theta b) = 0 \Rightarrow \theta b = 0.$$

There is therefore a Boolean homomorphism $\tilde\theta : \phi[\mathfrak{B}] \to \mathfrak{A}'$ such that $\tilde\theta\phi = \theta$, and $\tilde\theta$ is measure-preserving on $\phi[\mathfrak{A}]$. Since $\phi[\mathfrak{B}]$ is topologically dense in $\mathfrak{A}$ (use 323H), $\tilde\theta$ has an extension to a measure-preserving Boolean homomorphism $\hat\psi : \mathfrak{A} \to \mathfrak{A}'$ (324O). Now, for $i \in I$ and $a \in \mathfrak{A}_i$,

$$\hat\psi\phi_i a = \hat\psi\phi\varepsilon_i a = \tilde\theta\phi\varepsilon_i a = \theta\varepsilon_i a = \phi_i'\psi_i a,$$

as required.

(d) To see that $\hat\psi$ is unique, we need observe only that the given formula defines in on the subalgebra $\phi[\mathfrak{B}]$ and that this is topologically dense in $\mathfrak{A}$, while $\hat\psi$, being measure-preserving, must be continuous.

458L Relative product measures: Definitions (a) Let $\langle X_i\rangle_{i\in I}$ be a family of sets, Y a set, and $\pi_i : X_i \to Y$ a function for each $i \in I$. The **fiber product** of $\langle (X_i, \pi_i)\rangle_{i\in I}$ is the set $\Delta = \{x : x \in \prod_{i\in I} X_i, \pi_i x(i) = \pi_j x(j) \text{ for all } i, j \in I\}$.

(b) Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces and (Y, T, ν) a probability space, and suppose that we are given an inverse-measure-preserving function $\pi_i : X_i \to Y$ for each $i \in I$; let Δ be the fiber product of $\langle X_i, \pi_i\rangle_{i\in I}$. A **relative product measure** on Δ is a probability measure μ on Δ such that

(†) whenever $J \subseteq I$ is finite and not empty and $E_i \in \Sigma_i$ for $i \in J$, and g_i is a Radon-Nikodým derivative of the functional $F \mapsto \mu_i(E \cap \pi_i^{-1}[F]) : \mathrm{T} \to [0, 1]$ for each $i \in J$, then $\mu\{x : x \in \Delta, x(i) \in E_i \text{ for every } i \in J\}$ is defined and equal to $\int \prod_{i\in J} g_i d\nu$;

(‡) for every $W \in \Sigma$ there is a W' in the σ-algebra generated by $\{\{x : x \in \Delta, x(i) \in E\} : i \in I, E \in \Sigma_i\}$ such that $\mu(W \triangle W') = 0$.

Remark If μ is a relative product measure of $\langle (\mu_i, \pi_i)\rangle_{i\in I}$ over ν, then all the functions $x \mapsto x(i) : \Delta \to X_i$ are inverse-measure-preserving. **P** The condition (†) tells us that if $E \in \Sigma_i$ and g is any Radon-Nikodým derivative of $F \mapsto \mu_i(E \cap \pi_i^{-1}[F])$, then

$$\mu\{x : x(i) \in E\} = \int g\, d\nu = \mu_i E. \ \mathbf{Q}$$

It follows that if I is not empty then we have an inverse-measure-preserving function $\pi : \Delta \to Y$ defined by setting $\pi x = \pi_i x(i)$ whenever $x \in \Delta$ and $i \in I$.

Note that when verifying (†) we need check the equality $\mu\{x : x \in \Delta, x(i) \in E_i \text{ for every } i \in J\} = \int \prod_{i\in J} g_i d\nu$ for only one representative family $\langle g_i\rangle_{i\in J}$ of Radon-Nikodým derivatives for any given $\langle E_i\rangle_{i\in J}$.

458M Proposition Suppose that $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ is a family of probability spaces, (Y, T, ν) a probability space, $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each $i \in I$, Δ the fiber product of $\langle X_i, \pi_i\rangle_{i\in I}$ and μ a relative product measure on Δ with domain Σ. Let $(\mathfrak{A}_i, \bar\mu_i)$, $(\mathfrak{C}, \bar\nu)$ and $(\mathfrak{A}, \bar\mu)$ be the measure algebras of (X_i, Σ_i, μ_i), (Y, T, ν) and (Δ, Σ, μ) respectively, and for $i \in I$ define $\bar\pi_i : \mathfrak{C} \to \mathfrak{A}_i$ and $\bar\phi_i : \mathfrak{A}_i \to \mathfrak{A}$ by setting $\bar\pi_i F^\bullet = \pi_i^{-1}[F]^\bullet$, $\bar\phi_i E^\bullet = \{x : x \in \Delta, x(i) \in E\}^\bullet$ for every $F \in \mathrm{T}$, $E \in \Sigma_i$. Then $(\mathfrak{A}, \bar\mu, \langle\bar\phi_i\rangle_{i\in I})$ is a relative free product of $\langle (\mathfrak{A}_i, \bar\mu_i, \bar\pi_i)\rangle_{i\in I}$ over $(\mathfrak{C}, \bar\nu)$.

proof The case $I = \emptyset$ is trivial (if you care to follow through the definitions to the letter, $\Delta = \prod_{i\in I} X_i = \{\emptyset\}$ and $\mathfrak{A}$ is the two-point algebra). So I will take it that I is not empty.

(a) Of course we have to check that all the $\bar\pi_i$ and $\bar\phi_i$ are measure-preserving Boolean homomorphisms between the appropriate algebras, but in view of the remark following the definition 458L, this is elementary. The condition that $\mathfrak{A}$ should be the closed subalgebra generated by $\bigcup_{i\in I} \bar\phi_i[\mathfrak{A}_i]$ is just a translation of the condition (‡).

(b) As I is not empty, we have a well-defined inverse-measure-preserving map $\pi : \Delta \to Y$ given by the formula $\pi(x) = \pi_i x(i)$ whenever $x \in \Delta$ and $i \in I$. Let $\bar{\pi} : \mathfrak{C} \to \mathfrak{A}$ be the corresponding measure-preserving homomorphism, so that $\bar{\pi} = \bar{\phi}_i \bar{\pi}_i$ for every i. Set $\mathfrak{D} = \bar{\pi}[\mathfrak{C}] \subseteq \mathfrak{A}$, and let $T : L^\infty(\mathfrak{C}) \to L^\infty(\mathfrak{D})$ be the f-algebra isomorphism corresponding to $\bar{\pi}$ (363F). For $i \in I$ and $E \in \Sigma_i$, set $u_{iE} = Tg^\bullet_{iE} \in L^\infty(\mathfrak{D})$. Then u_{iE} is the conditional expectation of $\chi\{x : x(i) \in E\}^\bullet$ on $\mathfrak{D}$. **P** If $d \in \mathfrak{D}$, it is of the form $\bar{\phi}_i \bar{\pi}_i F^\bullet$ where $F \in \mathrm{T}$, so that

$$\begin{aligned}\int_d u_{iE} d\bar{\mu} &= \int u_{iE} \times \chi d \, d\bar{\mu} = \int T(g^\bullet_{iE} \times \chi F^\bullet) d\bar{\mu} \\ &= \int g^\bullet_{iE} \times \chi F^\bullet d\bar{\nu} = \int_F g_{iE} d\nu \\ &= \mu_i(E \cap \pi_i^{-1}[F]) = \mu(\phi_i^{-1}[E] \cap \phi_i^{-1}[\pi_i^{-1}[F]]) = \bar{\mu}(d \cap \phi_i^{-1}[E]^\bullet).\end{aligned}$$

As d is arbitrary, we have the result. **Q**

(c) It follows that if $J \subseteq I$ is finite and not empty, and $a_i \in \bar{\phi}_i[\mathfrak{A}_i]$ and v_i is the conditional expectation of χa_i on $\mathfrak{D}$ for each $i \in J$, then $\bar{\mu}(\inf_{i \in J} a_i) = \int \prod_{i \in J} v_i d\bar{\mu}$. **P** Express a_i as $\phi_i^{-1}[E_i]^\bullet$, where $E_i \in \Sigma_i$, so that $v_i = u_{i,E_i}$ for each i. Then

$$\begin{aligned}\int \prod_{i \in J} v_i d\bar{\mu} &= \int \prod_{i \in J} Tg^\bullet_{i,E_i} d\bar{\mu} = \int T(\prod_{i \in J} g^\bullet_{i,E_i}) d\bar{\mu} = \int \prod_{i \in J} g^\bullet_{i,E_i} d\bar{\nu} \\ &= \int \prod_{i \in J} g_{i,E_i} d\nu = \mu(\bigcap_{i \in J} \phi_i^{-1}[E_i]) = \bar{\mu}(\inf_{i \in J} a_i). \ \mathbf{Q}\end{aligned}$$

But this is exactly what we need to know to see that $\langle \bar{\phi}_i[\mathfrak{A}_i] \rangle_{i \in I}$ is relatively independent over $\mathfrak{D}$, completing the proof that $(\mathfrak{A}, \bar{\mu}, \langle \bar{\phi}_i \rangle_{i \in I})$ is a relative free product of $(\langle (\mathfrak{A}_i, \bar{\mu}_i, \bar{\pi}_i) \rangle_{i \in I}$ over $\mathfrak{C}, \bar{\nu})$.

458N There is no general result on relative product measures to match 458J (see 458Xg, 458Xh). The general question of when we can expect relative product measures to exist seems interesting. Here I give a couple of sample results.

Proposition Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ be a family of probability spaces, (Y, T, ν) a probability space, and $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each i. Suppose that for each i we have a disintegration $\langle \mu_{iy} \rangle_{y \in Y}$ of μ_i such that $\mu^*_{iy} \pi_i^{-1}[\{y\}] = \mu_{iy} X_i = 1$ for every $y \in Y$. Let $\Delta \subseteq \prod_{i \in I} X_i$ be the fiber product of $\langle (X_i, \pi_i) \rangle_{i \in I}$, and Υ the subspace σ-algebra on Δ induced by $\widehat{\bigotimes}_{i \in I} \Sigma_i$. For $y \in Y$, let λ_y be the product of $\langle \mu_{iy} \rangle_{i \in I}$, $(\lambda_y)_\Delta$ the subspace measure on Δ and λ'_y its restriction to Υ. Then $\mu W = \int \lambda'_y W \nu(dy)$ is defined for every $W \in \Upsilon$, and μ is a relative product measure of $\langle (\mu_i, \pi_i) \rangle_{i \in I}$ over ν.

proof If $y \in Y$, then

$$\lambda'_y \Delta = \lambda^*_y \Delta == (\lambda_y)^* \textstyle\prod_{i \in I} \pi_i^{-1}[\{y\}] = 1$$

(254Lb). For $i \in I$ and $E \in \Sigma_i$ set $g_{iE}(y) = \mu_{iy} E$ when this is defined; then g_{iE} is a Radon-Nikodým derivative of $F \mapsto \mu_i(E \cap \pi_i^{-1}[F]) : \mathrm{T} \to [0,1]$ (452Qa). Write X for $\prod_{i \in I} X_i$; for $i \in I$ and $x \in X$ set $\phi_i(x) = x(i)$. If $J \subseteq I$ is finite and $E_i \in \Sigma_i$ for each $i \in J$, then

$$\begin{aligned}\int \lambda'_y(\Delta \cap \bigcap_{i \in J} \phi_i^{-1}[E_i]) \nu(dy) &= \int (\lambda_y)_\Delta(\Delta \cap \bigcap_{i \in J} \phi_i^{-1}[E_i]) \nu(dy) \\ &= \int \lambda_y(X \cap \bigcap_{i \in J} \phi_i^{-1}[E_i]) \nu(dy)\end{aligned}$$

(because $\lambda^*_y \Delta = 1$ and λ_y measures every $\phi_i^{-1}[E_i]$ for almost every y)

$$= \int \prod_{i \in J} \mu_{iy} E_i \nu(dy) = \int \prod_{i \in J} g_{i,E_i} d\nu.$$

In particular, $\int \lambda'_y(\Delta \cap \bigcap_{i\in J} \phi_i^{-1}[E_i])\nu(dy)$ is defined.

The set $\{W : W \subseteq X, \int \lambda'_y(W \cap \Delta)\nu(dy) \text{ is defined}\}$ is a Dynkin class of subsets of X containing $\bigcap_{i\in J} \phi_i^{-1}[E_i]$ whenever $J \subseteq I$ is finite and not empty and $E_i \in \Sigma_i$ for each $i \in J$; by the Monotone Class Theorem, it includes $\widehat{\bigotimes}_{i\in I}\Sigma_i$. So $\mu W = \int \lambda'_y W \nu(dy)$ is defined for every $W \in \Upsilon$. Moreover, the formula displayed above tells us that $\mu(\Delta \cap \bigcap_{i\in I} \phi_i^{-1}[E_i]) = \int \prod_{i\in J} g_{i,E_i} d\nu$ whenever $J \subseteq I$ is finite and $E_i \in \Sigma_i$ for each $i \in I$. Thus (†) of 458L is satisfied. And (‡) is true by the choice of Υ.

458O The latitude I have permitted in the definition of 'relative product' makes it possible to look for relative product measures with further properties, as in the following.

Proposition Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of compact Radon probability spaces, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ a Radon probability space, and $\pi_i : X_i \to Y$ a continuous inverse-measure-preserving function for each i. Then $\langle (\mu_i, \pi_i)\rangle_{i\in I}$ has a relative product measure μ over ν which is a Radon measure for the topology on the fiber product of $\langle (X_i, \pi_i)\rangle_{i\in I}$ induced by the product topology on $\prod_{i\in I} X_i$.

proof (a) For each $i \in I$, $E \in \Sigma_i$ let g_{iE} be a Radon-Nikodým derivative of $F \mapsto \mu_i(E \cap \pi_i^{-1}[F]) : \mathrm{T} \to [0, 1]$. Let $\mathcal{C}$ be the family of subsets of $X = \prod_{i\in I} X_i$ expressible in the form $C = X \cap \bigcap_{i\in J} \phi_i^{-1}[E_i]$ where $J \subseteq I$ is finite and $E_i \in \Sigma_i$ for each $i \in J$, writing $\phi_i(x) = x(i)$ for $x \in X$, $i \in I$. Because the expression here is unique unless one of the E_i is empty or X_i, we have a functional $\lambda_0 : \mathcal{C} \to [0, 1]$ defined by setting

$$\lambda_0(\textstyle\bigcap_{i\in J} \phi_i^{-1}[E_i]) = \int \prod_{i\in J} g_{i,E_i} d\nu$$

whenever $J \subseteq I$ is finite and not empty and $E_i \in \Sigma_i$ for every $i \in I$. It is easy to check that λ_0 is additive in the sense of 454E so (because every μ_i is perfect, by 416Wa and 343K) it has an extension to a measure λ on X with domain $\widehat{\bigotimes}_{i\in I}\Sigma_i$. By 454A, with $\mathcal{K}$ the family of compact subsets of X, λ is inner regular with respect to the compact sets. By 416O, there is a Radon measure $\tilde\lambda$ on X extending λ.

(b) Let Δ be the fiber product of $\langle (X_i, \pi_i)\rangle_{i\in I}$. Now the point is that Δ is $\tilde\lambda$-conegligible. **P** Because every π_i is continuous, Δ is closed. **?** If it is not conegligible, then, because $\tilde\lambda$ is τ-additive, there must be a basic open set of non-zero measure disjoint from Δ; express such a set as $W = \prod_{i\in J} \phi_i^{-1}[G_i]$ where $J \subseteq I$ is finite and $G_i \subseteq X_i$ is open for each $i \in J$. Because $\tilde\lambda$ is inner regular with respect to the compact sets, there is a compact set $K \subseteq W$ such that $\tilde\lambda K > 0$; setting $K_i = \phi_i[K]$, $K_i \subseteq G_i$ is compact for each i and $W' = \prod_{i\in J} \phi_i^{-1}[K_i]$ is non-negligible. Now we have

$$0 < \tilde\lambda(\textstyle\prod_{i\in J} \phi_i^{-1}[K_i]) = \lambda_0(\prod_{i\in J} \phi_i^{-1}[K_i]) = \int \prod_{i\in J} g_{i,K_i} d\nu,$$

so $F = \{y : y \in Y, g_{i,K_i}(y) > 0 \text{ for every } i \in J\}$ is non-negligible. On the other hand, for each $i \in J$ we have

$$\textstyle\int_{Y\setminus\pi_i[K_i]} g_{i,K_i} d\nu = \mu_i(K_i \cap \pi_i^{-1}[Y \setminus K_i]) = 0$$

so that $F \setminus \pi_i[K_i]$ is negligible. Accordingly $\bigcap_{i\in J} \pi_i[K_i]$ is non-negligible, and must meet the support Y_0 of Y; let y be any point of the intersection. For $i \in J$, choose $x(i) \in K_i$ such that $\pi_i x(i) = y$. For $i \in I \setminus J$, $\pi_i[X_i]$ is a compact subset of Y, and $\nu\pi_i[X_i] = \mu_i\pi_i^{-1}[\pi_i[X_i]] = 1$, so $Y_0 \subseteq \pi_i[X_i]$ and we can therefore choose $x(i) \in X_i$ with $\pi_i x(i) = y$. This defines $x \in \Delta$. But as $x(i) \in K_i$ for $i \in J$, we also have

$$x \in \textstyle\bigcap_{i\in J} \phi_i^{-1}[K_i] \subseteq \bigcap_{i\in J} \phi_i^{-1}[G_i] \subseteq X \setminus \Delta,$$

which is impossible. **X**

Thus Δ is $\tilde\lambda$-conegligible, as claimed. **Q**

(c) Let μ be the subspace measure on Δ induced by $\tilde\lambda$, and Σ its domain, so that μ is a Radon probability measure on Δ with its subspace topology (416Rb). Concerning (†) of 458L, if $J \subseteq I$ is finite and not empty and $E_i \in \Sigma_i$ for $i \in J$, then

$$\mu(\textstyle\bigcap_{i\in J} \phi_i^{-1}[E_i]) = \tilde\lambda(\bigcap_{i\in J} \phi_i^{-1}[E_i]) = \lambda_0(\bigcap_{i\in J} \phi_i^{-1}[E_i]) = \int \prod_{i\in J} g_{i,E_i} d\nu,$$

as required. Finally, for (‡), the σ-algebra Υ generated by $\{\phi_i^{-1}[E] : i \in I, E \in \Sigma_i\}$ is just the subspace σ-algebra induced by $\widehat{\bigotimes}_{i\in I}\Sigma_i$. Let $\mathfrak{A}$ be the measure algebra of $\tilde\lambda$ and $\mathfrak{B} \subseteq \mathfrak{A}$ the set $\{W^\bullet : W \in \widehat{\bigotimes}_{i\in I}\Sigma_i\}$.

Then $\mathfrak{B}$ is a closed subalgebra of $\mathfrak{A}$. If $W \subseteq X$ is open, then for every $\epsilon > 0$ there is a $W_0 \in \widehat{\bigotimes}_{i\in I}\Sigma_i$ such that $W_0 \subseteq W$ and $\tilde{\lambda}(W \setminus W_0) \leq \epsilon$, so $W^\bullet \in \mathfrak{B}$; accordingly $\{W : W^\bullet \in \mathfrak{B}\}$ contains every open set and every Borel set and must be the whole of $\operatorname{dom}\lambda$. Returning to the measure μ, we see that if $W \in \Sigma$ there must be a $W_0 \in \widehat{\bigotimes}_{i\in I}\Sigma_i$ such that $\tilde{\lambda}(W \triangle W_0) = 0$; now $W_0 \cap \Delta \in \Upsilon$ and $\mu(W \triangle (W_0 \cap \Delta)) = 0$. So (‡) also is true, and we have a relative product measure of the declared type.

458P We can of course make a general search through theorems about product measures, looking for ways of re-presenting them as theorems about relative product measures. There is an associative law, for instance (458Xn). To give an idea of what is to be expected, I offer a result corresponding to 253D.

Proposition Let (X_1, Σ_1, μ_1), (X_2, Σ_2, μ_2) and (Y, T, ν) be probability spaces, and $\pi_1 : X_1 \to Y$, $\pi_2 : X_2 \to Y$ inverse-measure-preserving functions. Let Δ be the fiber product of (X_1, π_1) and (X_2, π_2), and suppose that μ is a relative product measure of (μ_1, π_1) and (μ_2, π_2) over ν; set $\pi x = \pi_1 x(1) = \pi_2 x(2)$ for $x \in \Delta$. Take $f_1 \in \mathcal{L}^1(\mu_1)$ and $f_2 \in \mathcal{L}^2(\mu_2)$, and set $(f_1 \otimes f_2)(x) = f_1(x(1))f_2(x(2))$ when $x \in \Delta \cap (\operatorname{dom} f_1 \times \operatorname{dom} f_2)$. For $i = 1, 2$ let $g_i \in \mathcal{L}^1(\nu)$ be a Radon-Nikodým derivative of $F \mapsto \int_{\pi_i^{-1}[F]} f_i d\mu_i : \mathrm{T} \to \mathbb{R}$. Then $\int_F g_1 \times g_2 d\nu = \int_{\pi^{-1}[F]} f_1 \otimes f_2 d\mu$ for every $F \in \mathrm{T}$.

proof When f_1 and f_2 are characteristic functions of measurable sets, this is just the definition of 'relative product measure'. The formula for g_i corresponds to a linear operator from $L^1(\mu_i)$ to $L^1(\nu)$, so the result is true for simple functions f_1 and f_2. If f_1 and f_2 are almost everywhere limits of non-decreasing sequences $\langle f_{1n}\rangle_{n\in\mathbb{N}}$, $\langle f_{2n}\rangle_{n\in\mathbb{N}}$ of non-negative simple functions, then the corresponding sequences $\langle g_{1n}\rangle_{n\in\mathbb{N}}$, $\langle g_{2n}\rangle_{n\in\mathbb{N}}$ will also be non-decreasing and non-negative and convergent to g_1, g_2 ν-a.e.; moreover, because $x \mapsto x(1)$ and $x \mapsto x(2)$ are inverse-measure-preserving, $f_1 \otimes f_2 = \lim_{n\to\infty} f_{1n} \otimes f_{2n}$ μ-a.e. So in this case we shall have

$$\begin{aligned}\int_{\pi^{-1}[F]} f_1 \otimes f_2 d\mu &= \lim_{n\to\infty} \int_{\pi^{-1}[F]} f_{1n} \otimes f_{2n} d\mu \\ &= \lim_{n\to\infty} \int_F g_{1n} \times g_{2n} d\nu = \int_F g_1 \times g_2 d\nu\end{aligned}$$

for every $F \in \mathrm{T}$. Finally, considering positive and negative parts, we can extend the result to general integrable f_1 and f_2.

458X Basic exercises >(a) Find an example of a probability space (X, Σ, μ) with σ-subalgebras Σ_1, Σ_2 and T of Σ such that Σ_1 and Σ_2 are independent but are not relatively independent over T.

(b) In 458C, show that (writing $\hat{\mu}$ for the completion of μ) $\hat{\mu}(F \cap f^{-1}[H]) = \int_F \nu_x H \mu(dx)$ for every $F \in \mathrm{T}$ and every universally measurable $H \subseteq \mathbb{R}$.

(c) Let (X, Σ, μ) be a probability space, $\langle \mathrm{T}_i\rangle_{i\in I}$ an independent family of σ-subalgebras of Σ, as in 272Ab, and T a σ-subalgebra of Σ which is independent of the σ-algebra generated by $\bigcup_{i\in I} \mathrm{T}_i$. For each $i \in I$, let T_i' be a σ-subalgebra of T, and Σ_i the σ-algebra generated by $\mathrm{T}_i' \cup \mathrm{T}_i$. Show that $\langle \Sigma_i\rangle_{i\in I}$ is relatively independent over T.

(d) In 458H, prove the result concerning the $\tilde{\mathfrak{B}}_j$ directly from 313G, without appealing to 458F.

(e) Let $\langle(\mathfrak{A}_i, \bar{\mu}_i)\rangle_{i\in I}$ be a family of probability algebras. Show that their probability algebra free product (325K) can be identified with their relative free product over $(\mathfrak{C}, \bar{\nu})$ if $\mathfrak{C}$ is the two-element Boolean algebra, $\bar{\nu}$ its unique probability measure, and $\pi_i : \mathfrak{C} \to \mathfrak{A}_i$ the trivial Boolean homomorphism for every i.

>(f) Let Y be a set, $\langle Z_i\rangle_{i\in I}$ a family of sets, and $\pi_i : Y \times Z_i \to Y$ the canonical map for each i. Show that the fiber product of $\langle(Y \times Z_i, \pi_i)\rangle_{i\in I}$ can be identified with $Y \times \prod_{i\in I} Z_i$.

(g) Let ν be Lebesgue measure on $[0, 1]$, and X_1, $X_2 \subseteq [0, 1]$ disjoint sets with outer measure 1. For each $i \in \{1, 2\}$ let μ_i be the subspace measure on X_i and $\pi_i : X_i \to [0, 1]$ the identity map. Show that (μ_1, π_1) and (μ_2, π_2) have no relative product measure over ν.

(h) Let ν be the usual measure on the split interval $I^{\|}$ (343J), and μ Lebesgue measure on $[0,1]$. Set $\pi_1(t) = t^+$, $\pi_2(t) = t^-$ for $t \in [0,1]$. Show that (μ, π_1) and (μ, π_2) have no relative product measure over ν.

(i) Let ν be Lebesgue measure on $[0,1]$. For each $t \in [0,1]$, set $X_t = [0,1] \setminus \{t\}$; let μ_t be the subspace measure on X_t and $\pi_t : X_t \to [0,1]$ the identity map. Show that $\langle(\mu_t, \pi_t)\rangle_{t\in[0,1]}$ has no relative product measure over ν.

(j) Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $\langle\Sigma_i\rangle_{i\in I}$ a family of σ-subalgebras of Σ, all including T. Set $\pi_i(x) = x$ for every $x \in X$. Show that $\langle\Sigma_i\rangle_{i\in I}$ is relatively independent over T iff $\mu\restriction\Sigma^*$ is a relative product measure of $\langle(\mu\restriction\Sigma_i, \pi_i)\rangle_{i\in I}$ over $\mu\restriction\mathrm{T}$, where Σ^* is the σ-algebra generated by $\bigcup_{i\in I}\Sigma_i$.

(k) (i) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces and (X, Σ, μ) their ordinary probability space product. Show that μ is a relative product measure of $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ over ν where Y is a singleton set, ν its unique probability measure, and $\pi_i : X_i \to Y$ the unique function for each i. (ii) Let $\langle(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of quasi-Radon probability spaces and $(X, \mathfrak{T}, \Sigma, \mu)$ their quasi-Radon probability space product. Show that μ is a relative product measure of $\langle\mu_i\rangle_{i\in I}$ in the same sense as in (i).

(l) Suppose that $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ is a family of probability spaces and (Y, T, ν) is a probability space, and that for each $i \in I$ we are given an inverse-measure-preserving function $\pi_i : X_i \to Y$. Write $\hat\mu_i$ and $\hat\nu$ for the completions of μ_i, ν respectively. Show that $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ has a relative product measure over ν iff $\langle(\hat\mu_i, \pi_i)\rangle_{i\in I}$ has a relative product measure over $\hat\nu$.

(m) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, (Y, T, ν) a probability space, and $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each $i \in I$. Show that if $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ has a relative product measure over ν, so does $\langle(\mu_i, \pi_i)\rangle_{i\in J}$ for any $J \subseteq I$.

(n) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of probability spaces, (Y, T, ν) a probability space, and $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each $i \in I$. Let $\langle J_k\rangle_{k\in K}$ be a partition of I into non-empty sets. For each $k \in K$, let Δ_k be the fiber product of $\langle(X_i, \pi_i)\rangle_{i\in J_k}$; suppose that $\tilde\mu_k$ is a relative product measure of $\langle(\mu_i, \pi_i)\rangle_{i\in J_k}$ and $\tilde\Sigma_k$ its domain. Define $\tilde\pi_k : \Delta_k \to Y$ by setting $\tilde\pi_k(x) = \pi_i x(i)$ whenever $x \in \Delta_k$ and $i \in J_k$, so that $\tilde\pi_k$ is inverse-measure-preserving. Suppose that μ is a relative product measure of $\langle(\tilde\mu_k, \tilde\pi_k)\rangle_{k\in K}$ over ν. Show that μ can be regarded as a relative product measure of $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ over ν.

(o) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ and $\langle(X'_i, \Sigma'_i, \mu'_i)\rangle_{i\in I}$ be two families of probability spaces, (Y, T, ν) and (Y', T', ν') probability spaces, and $\pi_i : X_i \to Y$, $\pi'_i : X'_i \to Y'$ inverse-measure-preserving functions for each i. Suppose that we have a measure space isomorphism $g : Y \to Y'$ and inverse-measure-preserving functions $f_i : X_i \to X'_i$, for $i \in I$, such that $g\pi_i = \pi'_i f_i$ for every i. Show that if there is a relative product measure of $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ over ν, then there is a relative product measure of $\langle(\mu'_i, \pi'_i)\rangle_{i\in I}$ over ν'.

(p) Let $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a countable family of probability spaces, (Y, T, ν) a probability space, and $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each i. Suppose that for each i we have a disintegration of μ_i over ν which is strongly consistent with π_i. Show that $\langle(\mu_i, \pi_i)\rangle_{i\in I}$ has a relative product measure over ν.

(q) Let (X, Σ, μ), (X', Σ', μ') and (Y, T, ν) be probability spaces. Suppose that $\pi : X \to Y$ and $\pi' : X' \to Y$ are inverse-measure-preserving functions, and that μ' has a disintegration $\langle\mu'_y\rangle_{y\in Y}$ over (Y, T, ν) which is strongly consistent with π'. Show that (μ, π) and (μ', π') have a relative product measure over ν. (*Hint*: set $\lambda W = \int \mu'_{\pi x} W[\{x\}]\mu(dx)$ for every $W \in \Sigma\widehat{\otimes}\Sigma'$.)

>(r) Let Y be a Hausdorff space, $\langle Z_i\rangle_{i\in I}$ a family of Hausdorff spaces, μ_i a Radon probability measure on $Z_i \times Y$ and $\pi_i : Y \times Z_i \to Y$ the canonical map for each i. Suppose that all the image measures $\mu_i\pi_i^{-1}$ on Y are the same, and that all but countably many of the Z_i are compact. Show that there is a Radon probability measure μ on $Y \times \prod_{i\in I} Z_i$ such that $\mu_i = \mu\phi_i^{-1}$ for each i, where $\phi_i(y, z) = (y, z(i))$ for $y \in Y$, $z \in \prod_{j\in I} Z_j$.

(s) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a countable family of Radon probability spaces, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ a Radon probability space, and $\pi_i : X_i \to Y$ an almost continuous inverse-measure-preserving function for each i. Show that $\langle (\mu_i, \pi_i)\rangle_{i\in I}$ has a relative product measure over ν which is a Radon measure for the topology on the fiber product of $\langle (X_i, \pi_i)\rangle_{i\in I}$ induced by the product topology on $\prod_{i\in I} X_i$.

(t) Let (X, Σ, μ) be a probability space and T a σ-subalgbra of Σ. Let f_1, f_2 be μ-integrable real-valued functions which are relatively independent over T, and suppose that $f_1 \times f_2$ is integrable. Let g_1, g_2 be conditional expectations of f_1, f_2 on T. Show that $g_1 \times g_2$ is a conditional expectation of $f_1 \times f_2$ on T. (*Hint*: 458P.)

458Y Further exercises (a) Let (X, Σ, μ) be a complete probability space, T a σ-algebra of subsets of Σ, and $\langle f_i\rangle_{i\in I}$ a family in $\mathcal{L}^0(\mu)$ which is relatively independent over T. Show that there is a family $\langle \nu_x\rangle_{x\in X}$ of countably compact quasi-Radon probability measures on $\mathbb{R}^I$ such that whenever $F \in \mathrm{T}$ and $J \subseteq I$ is countable and $H \subseteq \mathbb{R}^J$ is a Borel set, then

$$\mu\{x : x \in F \cap \textstyle\bigcap_{i\in J} \operatorname{dom} f_i,\ \langle f_i(x)\rangle_{i\in J} \in H\} = \int_F \nu_z\{x : x\restriction J \in H\}\mu(dz).$$

(b) Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $\langle \Sigma_n\rangle_{n\in\mathbb{N}}$ a sequence of σ-subalgebras of Σ which is relatively independent over T. For each $n \in \mathbb{N}$ let Σ_n^* be the σ-algebra generated by $\bigcup_{m\geq n} \Sigma_m$, and set $\Sigma_\infty = \bigcap_{n\in\mathbb{N}} \Sigma_n^*$. Show that for every $E \in \Sigma_\infty$ there is an $F \in \mathrm{T}$ such that $E \triangle F$ is negligible. (Compare 272O.)

(c) Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and f, $g \in \mathcal{L}^0(\mu)$ relatively independent over T; suppose that $\langle \nu_x\rangle_{x\in X}$ and $\langle \nu'_x\rangle_{x\in X}$ are relative distributions of f and g over T. Show that $\langle \nu_x * \nu'_x\rangle_{x\in X}$ is a relative distribution of $f + g$ over T. (Compare 272S.)

(d) Let (X, Σ, μ) be a probability space, T a σ-subalgebra of Σ, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence in $\mathcal{L}^2(\mu)$ such that $\langle f_n\rangle_{n\in\mathbb{N}}$ is relatively independent over T and $\int_F f_n d\mu = 0$ for every $n \in \mathbb{N}$ and every $F \in \mathrm{T}$. (i) Suppose that $\langle \beta_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $]0, \infty[$, diverging to ∞, such that $\sum_{n=0}^\infty \frac{1}{\beta_n^2}\|f_n\|_2^2 < \infty$. Show that $\lim_{n\to\infty} \frac{1}{\beta_n}\sum_{i=0}^n f_i = 0$ a.e. (ii) Suppose that $\sup_{n\in\mathbb{N}} \|f_n\|_2 < \infty$. Show that $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f_i = 0$ a.e. (Compare 273D.)

(e) Let $\langle (X_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of measure spaces, (Y, T, ν) a measure space, and $\pi_i : X_i \to Y$ an inverse-measure-preserving function for each $i \in I$. For $i \in I$ and $E \in \Sigma_i$ let g_{iE} be a Radon-Nikodým derivative of the functional $F \mapsto \mu_i(E \cap \pi_i^{-1}[F])$. Let $\mathcal{C}$ be the family of subsets of $X = \prod_{i\in I} X_i$ of the form $X \cap \bigcap_{i\in J} \phi_i^{-1}[E_i]$ where $J \subseteq I$ is finite, $E_i \in \Sigma_i$ for $i \in J$ and $\phi_i(x) = x(i)$ for $x \in X$, $i \in I$. If $C = \bigcap_{i\in J} \phi_i^{-1}[E_i]$ where $J \subseteq I$ is finite and not empty and $E_i \in \Sigma_i$ for $i \in J$, set $\lambda_0 C = \int \prod_{i\in J} g_{i,E_i} d\nu$. Let $\Delta \subseteq X$ be the fiber product of $\langle (X_i, \pi_i)\rangle_{i\in I}$. Show that the following are equiveridical: (i) $\langle (\mu_i, \pi_i)\rangle_{i\in I}$ has a relative product measure over ν (ii) whenever $\langle C_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{C}$ covering Δ, $\sum_{n=0}^\infty \lambda_0 C_n \geq 1$ (iii) $\nu^*(\bigcap_{i\in I} \pi_i[X_i]) = 1$ and whenever T' is a countably generated σ-subalgebra of T, $J \subseteq I$ is countable, and Σ'_i is a countably generated σ-subalgebra of Σ_i including $\{\pi_i^{-1}[F] : F \in \mathrm{T}'\}$ for every $i \in J$, then $\langle (\mu_i\restriction\Sigma'_i, \pi_i)\rangle_{i\in J}$ has a relative product over $\nu\restriction\mathrm{T}'$.

(f) Let X be a Hausdorff space and P_{R} the set of Radon probability measures on X. For ν_1, $\nu_2 \in P_{\mathrm{R}}$, write $\mathrm{M}(\nu_1, \nu_2)$ for the set of Radon probability measures μ on $X \times X$ which have marginal measures ν_1 on the first copy of X, ν_2 on the second. (i) For a continuous pseudometric ρ on X, set $d_\rho(\nu_1, \nu_2) = \inf\{\int \rho(x, y)\mu(d(x, y)) : \mu \in \mathrm{M}(\nu_1, \nu_2)\}$. Show that d_ρ is a pseudometric on P_{R}. (ii) Show that if X is completely regular and P is a family of pseudometrics defining the topology of X, then $\{d_\rho : \rho \in \mathrm{P}\}$ defines the narrow topology of P_{R}. (Compare 274Ya, 437Yi.)

458 Notes and comments At a couple of points in Volume 3 (Dye's theorem, in §388, and Kawada's theorem, in §394) I took the trouble to generalize standard theorems to 'non-ergodic' forms. In both 388L and 394P the results are complicated by potentially non-trivial closed subalgebras of the probability algebra

we are studying. I remarked on both occasions that the generalization is only a matter of technique, but I do not suppose that it was obvious just why this must be so. It is however a fundamental theorem of the topic of 'random reals' in the theory of Forcing that *any* theorem about probability algebras must have a relativized form as a theorem about probability algebras with arbitrary closed subalgebras. The concept of 'relative Maharam type' from §333, for instance, is what matches 'Maharam type' for simple algebras; the concept of 'exchangeable' sequence is what matches 'independent identically distributed' sequence. (In probability theory, the keyword is 'mixture'.) In this section I present another example in the idea of 'relatively independent' closed subalgebras (458H). I should emphasize that the theory of random reals, when we eventually come to it in Volume 5, will not as a rule apply directly to measure spaces; it deals with measure algebras. But of course the ideas generated by this theory can often be profitably applied to constructions in measure spaces, and this is what I am seeking to do with relatively independent σ-algebras and relative product measures.

Just as independent σ-algebras are associated with product spaces (272J), relatively independent algebras are associated with relative products (458Xj). The archetype of a relative product measure is 458N; it is a kind of disintegrated product. It is frequently profitable to express the 'relative' concepts of measure theory in terms of disintegrations.

I introduce 'relative free products' of probability algebras before proceeding to measure spaces because the uniqueness property proved in 458J shows that we have an unambiguous definition. For measure spaces it seems for the moment better to leave ourselves a bit of freedom, not (for instance) favouring one product construction over another (458Xk). The requirement that a relative product measure be carried by the fiber product is seriously limiting (458Xg-458Xi, 458Ye), and forces us to seek strictly consistent disintegrations (458N), at least for uncountable products (see 458Xp). However, as we might hope, the special case of compact spaces with Radon measures and continuous functions is amenable to a different approach (458O); and we have a one-sided method for the product of two spaces (458Xq) which is reminiscent of 454C and 457F.

There are corresponding complications when we come to look at maps between different relative products. For measure algebras, we have a natural theorem (458K), based on the same algebraic considerations as the corresponding theorems in §§315 and 325; the only possibly surprising feature is the need to assume that $\psi : \mathfrak{C} \to \mathfrak{C}'$ is actually an isomorphism. For measure spaces there is a similar result (458Xo).

459 Symmetric measures and exchangeable random variables

Among the relatively independent families of random variables discussed in 458A-458E, it is natural to give extra attention to those which are 'relatively identically distributed'. It turns out that these have a particularly appealing characterization as the 'exchangeable' families (459C). In the same way, among the measures on a product space X^I there is a special place for those which are invariant under permutations of coordinates (459E, 459G).

459A The following elementary fact seems to have gone unmentioned so far.

Lemma Let (X, Σ, μ) and (Y, T, ν) be probability spaces and $f : X \to Y$ an inverse-measure-preserving function; set $\Sigma_0 = \{f^{-1}[F] : F \in \mathrm{T}\}$. Let T_1 be a σ-subalgebra of T and $\Sigma_1 = \{f^{-1}[F] : F \in \mathrm{T}_1\}$. If $g \in \mathcal{L}^1(\nu)$ and h is a conditional expectation of g on T_1, then hf is a conditional expectation of gf on Σ_1.

proof h is $\nu\restriction\mathrm{T}_1$-integrable and f is inverse-measure-preserving for $\mu\restriction\Sigma_1$ and $\nu\restriction\mathrm{T}_1$, so hf is $\mu\restriction\Sigma_1$-integrable. If $E \in \Sigma_1$ then there is an $F \in \mathrm{T}_1$ such that $E = f^{-1}[F]$, and now

$$\int_E gf\,d\mu = \int_{f^{-1}[F]} gf\,d\mu = \int_F g\,d\nu = \int_F h\,d\nu = \int_E hf\,d\mu.$$

As E is arbitrary, hf is a conditional expectation of hg on Σ_1.

459B Theorem Let (X, Σ, μ) be a probability space, Z a set, Υ a σ-algebra of subsets of Z and $\langle f_i \rangle_{i \in I}$ an infinite family of (Σ, Υ)-measurable functions from X to Z. For each $i \in I$, set $\Sigma_i = \{f_i^{-1}[H] : H \in \Upsilon\}$. Then the following are equiveridical:

(i) whenever $i_0, \ldots, i_r \in I$ are distinct, $j_0, \ldots, j_r \in I$ are distinct, and $H_k \in \Upsilon$ for each $k \leq r$, then $\mu(\bigcap_{k \leq r} f_{i_k}^{-1}[H_k]) = \mu(\bigcap_{k \leq r} f_{j_k}^{-1}[H_k])$;

(ii) there is a σ-subalgebra T of Σ such that

(α) $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over T,

(β) whenever $i, j \in I$, $H \in \Upsilon$ and $F \in \mathrm{T}$, then $\mu(F \cap f_i^{-1}[H]) = \mu(F \cap f_j^{-1}[H])$.

Moreover, if I is totally ordered by $\leq$, we can add

(iii) whenever $i_0 < \ldots < i_r \in I$, $j_0 < \ldots < j_r \in I$ and $H_k \in \Upsilon$ for each $k \leq r$, then $\mu(\bigcap_{k \leq r} f_{i_k}^{-1}[H_k]) = \mu(\bigcap_{k \leq r} f_{j_k}^{-1}[H_k])$.

proof (a) Since there is always some total ordering on I, we may assume that we have one from the start. Of course (i)⇒(iii). Also (ii)⇒(i). **P** Suppose that (ii) is true. Then (ii-β) tells us that for each $H \in \Upsilon$ there is a T-measurable function $g_H : X \to [0,1]$ which is a conditional expectation of $\chi(f_i^{-1}[H])$ on T for every $i \in I$. Now suppose that $i_0, \ldots, i_r \in I$ are distinct, $j_0, \ldots, j_r \in I$ are distinct, and $H_k \in \Upsilon$ for each $k \leq r$. Then

$$\mu(\textstyle\bigcap_{k \leq r} f_{i_k}^{-1}[H_k]) = \int (\prod_{k=0}^{r} g_{H_k}) d\mu = \mu(\bigcap_{k \leq r} f_{j_k}^{-1}[H_k])$$

because $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over T. So (i) is true. **Q**

So henceforth I will suppose that (iii) is true and seek to prove (ii).

(b) Suppose first that $I = \mathbb{N}$ with its usual ordering.

($\boldsymbol{\alpha}$) For each n, $r \in \mathbb{N}$, let Σ_{nr} be the σ-subalgebra of Σ generated by $\bigcup_{n \leq i \leq n+r} \Sigma_i$; let T_n be the σ-algebra generated by $\bigcup_{r \in \mathbb{N}} \Sigma_{nr}$, and $\mathrm{T} = \bigcap_{n \in \mathbb{N}} \mathrm{T}_n$. For $n \in \mathbb{N}$ and $H \in \Upsilon$, let $g_{nH} : X \to \mathbb{R}$ be a T-measurable function which is a conditional expectation of $\chi f_n^{-1}[H]$ on T.

($\boldsymbol{\beta}$) (The key.) For any $n \in \mathbb{N}$ and Borel set $H \subseteq \mathbb{R}$, g_{nH} is a conditional expectation of $\chi f_n^{-1}[H]$ on T_{n+1}. **P** For m, $r \in \mathbb{N}$, let $h_{mr} : X \to [0,1]$ be a Σ_{mr}-measurable function which is a conditional expectation of $\chi f_n^{-1}[H]$ on Σ_{mr}; for $m \in \mathbb{N}$, set $h_m = \lim_{r \to \infty} h_{mr}$ where this is defined. By Lévy's martingale theorem (275I) h_m is defined almost everywhere and is a conditional expectation of $\chi f_n^{-1}[H]$ on T_m.

For m, $r \in \mathbb{N}$, define $F_{mr} : X \to Z^{r+2}$ by setting $F_{mr}(x) = (f_n(x), f_m(x), f_{m+1}(x), \ldots, f_{m+r}(x))$ for $x \in X$. At this point, examine the hypothesis (iii). This implies that if $m > n$ and $r \in \mathbb{N}$ then

$$\begin{aligned}
\mu(F_{mr}^{-1}[H' \times H_0 \times \ldots \times H_r]) &= \mu(f_n^{-1}[H'] \cap \bigcap_{k \leq r} f_{m+k}^{-1}[H_k]) \\
&= \mu(f_n^{-1}[H'] \cap \bigcap_{k \leq r} f_{m+1+k}^{-1}[H_k]) \\
&= \mu(F_{m+1,r}^{-1}[H' \times H_0 \times \ldots \times H_r])
\end{aligned}$$

for all H', $H_0, \ldots, H_r \in \Upsilon$. By the Monotone Class Theorem (136C), the image measures μF_{mr}^{-1} and $\mu F_{m+1,r}^{-1}$ agree on the σ-algebra $\widehat{\bigotimes}_{r+2} \Upsilon$ of subsets of Z^{r+2} generated by measurable rectangles; set

$$\lambda = \mu F_{mr}^{-1} \restriction \widehat{\textstyle\bigotimes}_{r+2} \Upsilon = \mu F_{m+1,r}^{-1} \restriction \widehat{\textstyle\bigotimes}_{r+2} \Upsilon.$$

Let Λ be the σ-subalgebra of $\widehat{\bigotimes}_{r+2} \Upsilon$ generated by sets of the form $Z \times H_0 \times \ldots \times H_r$, and let h be a conditional expectation of $\chi(H \times Z^{r+1})$ on Λ with respect to λ. Then 459A tells us that hF_{mr} is a conditional expectation of $\chi(f_n^{-1}[H])$ on Σ_{mr}, and is therefore equal almost everywhere to h_{mr}. Similarly, $hF_{m+1,r} =_{\text{a.e.}} h_{m+1,r}$, and this is true for every $r \in \mathbb{N}$. But as F_{mr} and $F_{m+1,r}$ are both inverse-measure-preserving for μ and λ, this means that h_{mr}, h and $h_{m+1,r}$ all have the same distribution. In particular, $\int h_{mr}^2 d\mu = \int h_{m+1,r}^2 d\mu$. Now $\langle h_{mr} \rangle_{r \in \mathbb{N}}$ and $\langle h_{m+1,r} \rangle_{r \in \mathbb{N}}$ converge almost everywhere to h_m and h_{m+1} respectively, so

$$\textstyle\int h_m^2 d\mu = \lim_{r \to \infty} \int h_{mr}^2 d\mu = \lim_{r \to \infty} \int h_{m+1,r}^2 d\mu = \int h_{m+1}^2 d\mu.$$

On the other hand, h_{m+1} is a conditional expectation of h_m on T_{m+1} (233Eh). This means that

$$\int h_m \times h_{m+1} d\mu = \int h_{m+1} \times h_{m+1} d\mu$$

(233K). A direct calculation tells us that $\int (h_m - h_{m+1})^2 d\mu = 0$, so that $h_m =_{\text{a.e.}} h_{m+1}$. Inducing on r, we see that $h_m =_{\text{a.e.}} h_r$ whenever $n < m \leq r$.

Now the reverse martingale theorem (275K) tells us that $\lim_{m\to\infty} h_m$ is defined almost everywhere and is a conditional expectation of $\chi f_n^{-1}[H]$ on T, that is, is equal almost everywhere to g_{nH}. Since the h_m, for $m > n$, are equal almost everywhere, they are all equal to g_{nH} a.e. In particular, g_{nH} is equal a.e. to h_{n+1}, and is a conditional expectation of $\chi f_n^{-1}[H]$ on T_{n+1}. **Q**

(γ) If $n \in \mathbb{N}$ and $H_0, \ldots, H_r \in \Upsilon$, then $\prod_{i=0}^r g_{n+i,H_i}$ is a conditional expectation of $\chi(\bigcap_{i\leq r} f_{n+i}^{-1}[H_i])$ on T. **P** Induce on r. For $r = 0$ this is just the definition of g_{nH_0}. For the inductive step to $r \geq 1$, observe that $g_{nH_0} \times \prod_{i=1}^r \chi f_{n+i}^{-1}[H_i]$ is a conditional expectation of $\prod_{i=0}^r \chi f_{n+i}^{-1}[H_i]$ on T_{n+1}, by 233Eg or 233K, because g_{nH_0} is a conditional expectation of $\chi f_n^{-1}[H_0]$ on T_{n+1} and $\prod_{i=1}^r \chi f_{n+i}^{-1}[H_i]$ is T_{n+1}-measurable. But as (by the inductive hypothesis) $\prod_{i=1}^r g_{n+i,H_i}$ is a conditional expectation of $\prod_{i=1}^r \chi f_{n+i}^{-1}[H_i]$ on T, $\prod_{i=0}^r g_{n+i,H_i}$ is a conditional expectation of $\prod_{i=0}^r \chi f_{n+i}^{-1}[H_i]$ on T, by 233Eg/233K again. **Q**

(δ) In particular, $\prod_{i=0}^r g_{iH_i}$ is a conditional expectation of $\chi(\bigcap_{i\leq r} f_i^{-1}[H_i])$ on T for every $r \in \mathbb{N}$ and $H_0, \ldots, H_r \in \Upsilon$. This shows that $\langle \Sigma_n \rangle_{n\in\mathbb{N}}$ is relatively independent over T.

(ϵ) Now consider part (β) of the condition (ii). For this, observe that if $m > 0$, $H \in \Upsilon$ and $H_i \in \Upsilon$ for $i \leq r$, then

$$\mu(f_0^{-1}[H] \cap \textstyle\bigcap_{i\leq r} f_{m+i+1}^{-1}[H_i]) = \mu(f_m^{-1}[H] \cap \bigcap_{i\leq r} f_{m+i+1}^{-1}[H_i]).$$

By the Monotone Class Theorem,

$$\mu(F \cap f_0^{-1}[H]) = \mu(F \cap f_m^{-1}[H])$$

for any $F \in \mathrm{T}_{m+1}$ and in particular for any $F \in \mathrm{T}$. Thus (ii) is true.

(c) Now suppose that there is a strictly increasing sequence $\langle j_k \rangle_{k\in\mathbb{N}}$ in I. For each n, let T_n be the σ-algebra generated by $\bigcup_{k\geq n} \Sigma_{j_k}$, and set $\mathrm{T} = \bigcap_{n\in\mathbb{N}} \mathrm{T}_n$. Then (b), applied to $\langle f_{j_k} \rangle_{k\in\mathbb{N}}$, tells us that $\langle \Sigma_{j_k} \rangle_{k\in\mathbb{N}}$ is relatively independent over T and that for each $H \in \Upsilon$ there is a function g_H which is a conditional expectation of $\chi(f_{j_k}^{-1}[H])$ on T for every $k \in \mathbb{N}$.

(α) If $i_0, \ldots, i_r \in I$ are distinct and $H_0, \ldots, H_r \in \Upsilon$, then $\mu(\bigcap_{k\leq r} f_{i_k}^{-1}[H_k]) = \mu(\bigcap_{k\leq r} f_{j_k}^{-1}[H_k])$. **P** Let ρ be the permutation of $\{0, \ldots, r\}$ such that $i_{\rho(0)} < i_{\rho(1)} < \ldots < i_{\rho(r)}$. Then

$$\begin{aligned}\mu(\bigcap_{k\leq r} f_{i_k}^{-1}[H_k]) &= \mu(\bigcap_{k\leq r} f_{i_{\rho(k)}}^{-1}[H_{\rho(k)}]) = \mu(\bigcap_{k\leq r} f_{j_k}^{-1}[H_{\rho(k)}]) \\ &= \int (\prod_{k=0}^r g_{H_{\rho(k)}}) d\mu = \int (\prod_{k=0}^r g_{H_k}) d\mu = \mu(\bigcap_{k\leq r} f_{j_k}^{-1}[H_k]).\end{aligned}$$ **Q**

(β) Now suppose that $i_0, \ldots, i_r \in I$ are distinct. Then there is some $m \in \mathbb{N}$ such that $j_k \neq i_l$ for any $l \leq r$ and $k \geq m$. In this case, consider the sequence $f_{i_0}, \ldots, f_{i_r}, f_{j_m}, f_{j_{m+1}}, \ldots$. By ($\alpha$) here, this sequence satisfies the condition (iii). We can therefore apply the construction of (c). But observe that the tail σ-algebra obtained from $f_{i_0}, \ldots, f_{i_r}, f_{j_m}, \ldots$ is precisely T, as defined from $\langle f_{j_k} \rangle_{k\in\mathbb{N}}$ just above. So $\langle \Sigma_{i_k} \rangle_{k\leq r}$ is relatively independent over T. As $i_0, \ldots, i_r$ are arbitrary, $\langle \Sigma_i \rangle_{i\in I}$ is relatively independent over T. At the same time we see that if $H \in \Upsilon$ then all the $\chi(f_{i_k}^{-1}[H])$ have the same conditional expectations over T as $\chi(f_{j_m}^{-1}[H])$. So (ii-β) is satisfied.

(d) Finally, if there is no strictly increasing sequence in I, then $(I, \geq)$ is well-ordered; since I is infinite, the well-ordering starts with an initial segment of order type ω, that is, a sequence $\langle j_k \rangle_{k\in\mathbb{N}}$ such that $j_0 > j_1 > \ldots$. But note now that (iii) tells us that $\mu(\bigcap_{k\leq r} f_{i_k}^{-1}[H_k]) = \mu(\bigcap_{k\leq r} f_{j_k}^{-1}[H_k])$ whenever $i_0 > \ldots > i_r$ and $j_0 > \ldots > j_r$ and $H_k \in \Upsilon$ for every k. So we can apply (c) to $(I, \geq)$ to get the result in this case also.

459C Exchangeable random variables I spell out the leading special case of this theorem.

de Finetti's theorem Let (X, Σ, μ) be a probability space, and $\langle f_i \rangle_{i \in I}$ an infinite family in $\mathcal{L}^0(\mu)$. Then the following are equiveridical:

(i) the joint distribution of $(f_{i_0}, f_{i_1}, \ldots, f_{i_r})$ is the same as the joint distribution of $(f_{j_0}, f_{j_1}, \ldots, f_{j_r})$ whenever $i_0, \ldots, i_r \in I$ are distinct and $j_0, \ldots, j_r \in I$ are distinct;

(ii) there is a σ-subalgebra T of Σ such that $\langle f_i \rangle_{i \in I}$ is relatively independent over T and all the f_i have the same relative distributions over T.

Moreover, if I is totally ordered by $\leq$, we can add

(iii) the joint distribution of $(f_{i_0}, f_{i_1}, \ldots, f_{i_r})$ is the same as the joint distribution of $(f_{j_0}, f_{j_1}, \ldots, f_{j_r})$ whenever $i_0 < \ldots < i_r$ and $j_0 < \ldots < j_r$ in I.

Remark Families of random variables satisfying the condition in (i) are called **exchangeable**. The equivalence of (i) and (ii) can be expressed by saying that 'an exchangeable family of random variables is a mixture of independent identically distributed families'.

proof Changing each f_i on a negligible set will not change either their joint distributions (271De) or their relative distributions over T or their relative independence; so we may suppose that every f_i is a Σ-measurable function from X to $\mathbb{R}$. Now look at 459B, taking (Z, Υ) to be $\mathbb{R}$ with its Borel σ-algebra. The condition 459B(i) reads

whenever $i_0, \ldots, i_r \in I$ are distinct, $j_0, \ldots, j_r \in I$ are distinct, and $H_k \in \Upsilon$ for each $k \leq r$, then $\mu(\bigcap_{k \leq r} f_{i_k}^{-1}[H_k]) = \mu(\bigcap_{k \leq r} f_{j_k}^{-1}[H_k])$,

matching (i) here, by 271B; similarly, (iii) of 459B matches (iii) here. Equally, condition (ii) here is just a re-phrasing of 459B(ii) in the language of 458A. So 459B gives the result.

459D Specializing 459B in another direction, we have the case in which X is actually the product Z^I. In this case, the condition 459B(i) corresponds to a strong kind of symmetry in the measure μ. It now makes sense to look for subsets of $X = Z^I$ which are essentially invariant under permutations, and we have the following result.

Proposition Let Z be a set, Υ a σ-algebra of subsets of Z, I an infinite set and μ a measure on Z^I with domain the σ-algebra $\widehat{\bigotimes}_I \Upsilon$ generated by $\{\pi_i^{-1}[H] : i \in I, H \in \Upsilon\}$, taking $\pi_i(x) = x(i)$ for $x \in Z^I$ and $i \in I$. For each permutation ρ of I, define $\hat{\rho} : Z^I \to Z^I$ by setting $\hat{\rho}(x) = x\rho$ for $x \in Z^I$. Suppose that $\mu = \mu\hat{\rho}^{-1}$ for every ρ. Let $\mathcal{E}$ be the family of those sets $E \in \widehat{\bigotimes}_I \Upsilon$ such that $\mu(E \triangle \hat{\rho}^{-1}[E]) = 0$ for every permutation ρ of I, and $\mathcal{V}$ the family of those sets $E \in \widehat{\bigotimes}_I \Upsilon$ such that E is determined by coordinates in $I \setminus \{i\}$ for every $i \in I$.

(a) $\mathcal{E}$ is a σ-subalgebra of $\widehat{\bigotimes}_I \Upsilon$.

(b) $\mathcal{V}$ is a σ-subalgebra of $\mathcal{E}$.

(c) If $E \in \mathcal{E}$ and $J \subseteq I$ is infinite, then there is a $V \in \mathcal{V}$, determined by coordinates in J, such that $\mu(E \triangle V) = 0$.

(d) Setting $\Sigma_i = \{\pi_i^{-1}[H] : H \in \Upsilon\}$ for each $i \in I$,

(α) $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over $\mathcal{E}$,

(β) for every $H \in \Upsilon$ there is a $\mathcal{E}$-measurable function $g_H : Z^I \to [0,1]$ which is a conditional expectation of $\chi(\pi_i^{-1}[H])$ on $\mathcal{E}$ for every $i \in I$.

proof (a) is elementary.

(b) Let $V \in \mathcal{V}$. Suppose that $\rho : I \to I$ is a permutation, $i_0, \ldots, i_r \in I$ are distinct and $H_k \in \Upsilon$ for every $k \leq r$. Then there is a permutation $\sigma : I \to I$ such that $\sigma(i_k) = \rho(i_k)$ for every $k \leq r$ and $J = \{i : \sigma(i) \neq i\}$ is finite. By 254Ta, V is determined by coordinates in $I \setminus J$, so $\hat{\sigma}^{-1}[V] = V$. Set $j_k = \rho^{-1}(i_k)$ for each $k \leq r$. Now

$$\begin{aligned}\mu(\hat{\rho}^{-1}[V] \cap \bigcap_{k \leq r} \pi_{i_k}^{-1}[H_k]) &= \mu(V \cap \bigcap_{k \leq r} \pi_{j_k}^{-1}[H_k]) \\ &= \mu(\hat{\sigma}^{-1}[V] \cap \bigcap_{k \leq r} \pi_{i_k}^{-1}[H_k]) = \mu(V \cap \bigcap_{k \leq r} \pi_{i_k}^{-1}[H_k]).\end{aligned}$$

By the Monotone Class Theorem, as usual, $\mu(E \cap \hat{\rho}^{-1}[V]) = \mu(E \cap V)$ for every $E \in \widehat{\bigotimes}_I \Upsilon$. In particular, taking $E = V$ and $E = Z^I \setminus V$, we see that $V \triangle \hat{\rho}^{-1}[V]$ is negligible. As ρ is arbitrary, $V \in \mathcal{E}$.

This shows that $\mathcal{V} \subseteq \mathcal{E}$. Of course $\mathcal{V}$ is a σ-algebra, since it is just the intersection of the σ-algebras $\{V : V \in \widehat{\bigotimes}_I \Sigma, V \text{ is determined by coordinates in } I \setminus \{i\}\}$.

(c) For each $n \in \mathbb{N}$, there is a set $E_n \in \widehat{\bigotimes}_I \Sigma$, determined by a finite set J_n of coordinates, such that $\mu(E \triangle E_n) \leq 2^{-n}$. Choose permutations ρ_n of I such that $\langle \rho_n[J_n] \rangle_{n \in \mathbb{N}}$ is a disjoint sequence of subsets of J. Set $F_n = \hat{\rho}_n^{-1}[E_n]$; then F_n is determined by coordinates in $\rho_n[J_n]$ for each $n \in \mathbb{N}$, so $V = \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} F_m$ belongs to $\mathcal{V}$ and is determined by coordinates in J. Also

$$\mu(E \triangle F_n) = \mu(\hat{\rho}[E] \triangle E_n) = \mu(E \triangle E_n) \leq 2^{-n}$$

for each n, so $\mu(E \triangle V) = 0$, as required.

(d) Let $\langle j_n \rangle_{n \in \mathbb{N}}$ be any sequence of distinct points of I. Set $J = \{j_n : n \in \mathbb{N}\}$. For $n \in \mathbb{N}$ let T_n be the σ-algebra generated by $\bigcup_{k \geq n} \Sigma_{j_k}$, and set $\mathrm{T} = \bigcap_{n \in \mathbb{N}} \mathrm{T}_n$, so that $\mathrm{T} = \{V : V \in \mathcal{V}, V \text{ is determined by coordinates in } J\}$. **P** Of course $\mathrm{T} \subseteq \mathcal{V}$ and every member of T is determined by coordinates in J, because every member of T_0 is. On the other hand, if $V \in \mathcal{V}$ is determined by coordinates in J, then fix some $w \in Z^{I \setminus J}$. In this case, identifying Z^I with $Z^J \times Z^{I \setminus J}$, the set $V_1 = \{z : z \in Z^J, (z, w) \in V\}$ must belong to $\widehat{\bigotimes}_J \Upsilon$, so $V = V_1 \times Z^{I \setminus J}$ belongs to T_0. Applying the same idea to $J \setminus \{j_k : k < n\}$, we see that $V \in \mathrm{T}_n$ for every n, so that $V \in \mathrm{T}$. **Q**

Part (c) of the proof of 459B tells us that $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over T and that for every $H \in \Upsilon$ there is a T-measurable g_H which is a conditional expectation of $\chi(\pi_i^{-1}[H])$ on T for every $i \in I$. Now (c) here tells us that g_H is a conditional expectation of $\chi(\pi_i^{-1}[H])$ on $\mathcal{E}$; and examining the definition in 458Aa, we see that $\langle \Sigma_i \rangle_{i \in I}$ is relatively independent over $\mathcal{E}$, as claimed.

459E If μ is countably compact, we have a strong disintegration theorem, as follows.

Theorem Let Z be a set, Υ a σ-algebra of subsets of Z, I an infinite set, and μ a countably compact probability measure on Z^I with domain the σ-algebra $\widehat{\bigotimes}_I \Upsilon$ generated by $\{\pi_i^{-1}[H] : i \in I, H \in \Upsilon\}$, taking $\pi_i(x) = x(i)$ for $x \in Z^I$ and $i \in I$. Then the following are equiveridical:

(i) for every permutation ρ of I, $x \mapsto x\rho : Z^I \to Z^I$ is inverse-measure-preserving for μ;

(ii) for every transposition ρ of two elements of I, $x \mapsto x\rho : Z^I \to Z^I$ is inverse-measure-preserving for μ;

(iii) for each $n \in \mathbb{N}$ and any two injective functions $p, q : n \to I$ the maps $x \mapsto xp : Z^I \to Z^n$, $x \mapsto xq : Z^I \to Z^n$ induce the same measure on Z^n;

(iv) there are a probability space (Y, T, ν) and a family $\langle \lambda_y \rangle_{y \in Y}$ of probability measures on Z such that $\langle \lambda_y^I \rangle_{y \in Y}$ is a disintegration of μ over ν, writing λ_y^I for the product of copies of λ_y.

Moreover, if I is totally ordered, we can add

(v) for each $n \in \mathbb{N}$ and any two strictly increasing functions $p, q : n \to I$ the maps $x \mapsto xp : Z^I \to Z^n$, $x \mapsto xq : Z^I \to Z^n$ induce the same measure on Z^n.

If the conditions (i)-(v) are satisfied, then there is a countably compact measure λ, with domain Υ, which is the common marginal measure of μ on every coordinate; and if $\mathcal{K}$ is a countably compact class of subsets of Z, closed under finite unions and countable intersections, such that λ is inner regular with respect to $\mathcal{K}$, then

(iv)′ there are a probability space (Y, T, ν) and a family $\langle \lambda_y \rangle_{y \in Y}$ of complete probability measures on Z, all with domains including $\mathcal{K}$ and inner regular with respect to $\mathcal{K}$, such that $\langle \lambda_y^I \rangle_{y \in Y}$ is a disintegration of μ over ν.

proof (a) Since any set I can be totally ordered, we may suppose from the outset that we have been given a total ordering $\leq$ on I. I start with the easy bits.

(iv)′⇒(iv) is trivial, at least if there is a common countably compact marginal measure on Z.

(iv)⇒(i) If (iv) is true and $\rho : I \to I$ is a permutation, take any $E \in \widehat{\bigotimes}_I \Sigma$ and set $E' = \{x : x \in Z^I, x\rho \in E\}$. For any $y \in Y$, $x \mapsto x\rho$ is an isomorphism of the measure space (Z^I, λ_y^I), so

$$\mu E' = \int \lambda_y^I E' \, \nu(dy) = \int \lambda_y^I E \, \nu(dy) = \mu E.$$

As E is arbitrary, (i) is true.

(i)⇒(ii) is trivial.

(ii)⇒(iii) There is a permutation ρ of I such that $q = \rho p$ and ρ moves only finitely many points of I, that is, ρ is a product of transpositions. By (ii), $x \mapsto x\rho$ and $x \mapsto x\rho^{-1}$ are inverse-measure-preserving for μ, that is, are isomorphisms of (Z^I, μ). But this means that $x \mapsto xp$ and $x \mapsto x\rho p = xq$ must induce the same measure on Z^n.

(iii)⇒(v) is trivial.

(b) So for the rest of the proof I assume that (v) is true. Taking $n = 1$ in the statement of (v), we see that there is a common image measure $\lambda = \mu\pi_i^{-1}$ for every $i \in I$. By 452R, λ is countably compact. Let $\mathcal{K} \subseteq \mathcal{P}Z$ be a countably compact class, closed under finite unions and countable intersections, such that λ is inner regular with respect to $\mathcal{K}$.

In 459B, set $X = Z^I$ and $\Sigma = \widehat{\bigotimes}_I \Upsilon$ and $f_i = \pi_i : X \to Z$ for $i \in I$. Then (v) here corresponds to (iii) of 459B, so (translating (ii) of 459B) we have a σ-subalgebra T of $\widehat{\bigotimes}_I \Upsilon$ and a family $\langle g_H \rangle_{H \in \Upsilon}$ of T-measurable functions from Z^I to $[0,1]$ such that

$$\mu(\textstyle\bigcap_{i \in J} \pi_i^{-1}[H_i]) = \int (\prod_{i \in J} g_{H_i}) d\mu$$

whenever $J \subseteq I$ is finite and not empty and $H_i \in \Upsilon$ for $i \in J$. In particular, g_H is a conditional expectation of $\chi(\pi_i^{-1}[H])$ on T whenever $H \in \Upsilon$ and $i \in I$.

Fix $i^* \in I$ for the moment. Set $\nu = \mu\restriction\mathrm{T}$. The inverse-measure-preserving function π_{i^*} from (X, μ) to (Z, λ) gives us an integral-preserving Riesz homomorphism $T_0 : L^\infty(\lambda) \to L^\infty(\mu)$ defined by setting $T_0 h^\bullet = (h\pi_{i^*})^\bullet$ for every $h \in \mathcal{L}^\infty(\lambda)$. Let $P : L^1(\mu) \to L^1(\nu)$ be the conditional expectation operator; then $T = PT_0 : L^\infty(\lambda) \to L^\infty(\nu)$ is an integral-preserving positive linear operator, and $T(\chi Z^\bullet) = \chi X^\bullet$.

By 452H, we have a family $\langle \lambda_x \rangle_{x \in X}$ of complete probability measures on Z, all with domains including $\mathcal{K}$ and inner regular with respect to $\mathcal{K}$, such that $\int_F h\pi_{i^*} d\mu = \int_F \int_Z h \, d\lambda_x \nu(dx)$ for every $h \in \mathcal{L}^\infty(\lambda)$ and $F \in \mathrm{T}$. In particular, setting $g'_H(x) = \lambda_x H$ whenever $H \in \Upsilon$ and $x \in X$ are such that $H \in \operatorname{dom} \lambda_x$, then g'_H will be a conditional expectation of $\chi\pi_{i^*}^{-1}[H]$ on T, and will be equal ν-almost everywhere to g_H.

This means that if $J \subseteq I$ is finite and not empty and $H_i \in \Upsilon$ for $i \in J$,

$$\begin{aligned}\int_X \lambda_x^I(\bigcap_{i \in J} \pi_i^{-1}[H_i]) \nu(dx) &= \int_X \prod_{i \in J} \lambda_x H_i \nu(dx) = \int_X \prod_{i \in J} g'_{H_i} d\nu \\ &= \int_X \prod_{i \in J} g_{H_i} d\nu = \int_X \prod_{i \in J} g_{H_i} d\mu = \mu(\bigcap_{i \in J} \pi_i^{-1}[H_i]).\end{aligned}$$

Thus the family $\mathcal{W}$ of sets $E \subseteq X$ such that $\int \lambda_x^I E \, \nu(dx)$ and μE are defined and equal contains all measurable cylinders. As $\mathcal{W}$ is a Dynkin class it includes $\widehat{\bigotimes}_I \Upsilon$. But this says exactly that $\langle \lambda_x^I \rangle_{x \in X}$ is a disintegration of μ over ν, as required by (iv)′.

Thus (v)⇒(iv)′ and the proof is complete.

459F Lemma Let X be a topological space, $\tilde{M}_\tau^+(X)$ the set of τ-additive totally finite topological measures on X and $\mu \in \tilde{M}_\tau^+(X)$ a measure which is inner regular with respect to the Borel sets. Let Y be a topological space, ν a totally finite τ-additive topological measure on Y and $\langle \mu_y \rangle_{y \in Y}$ a family of complete measures in $\tilde{M}_\tau^+(X)$ such that $y \mapsto \mu_y$ is continuous for the given topology on Y and the narrow topology on $\tilde{M}_\tau^+(X)$ (definition: 437Jd).

(a) Suppose that $\mathcal{U}$ is a base for the topology of X, closed under finite intersections and containing X, such that $\mu U = \int \mu_y U \, \nu(dy)$ for every $U \in \mathcal{U}$. Then $\langle \mu_y \rangle_{y \in Y}$ is a disintegration of μ over ν.

(b) Suppose that X is completely regular and that $\int u \, d\mu = \iint u \, d\mu_y \nu(dy)$ for every $u \in C_b(X)$. Then $\langle \mu_y \rangle_{y \in Y}$ is a disintegration of μ over ν.

proof (a) Let $\mathcal{E}$ be the family of those sets $E \subseteq X$ such that μE and $\int \mu_y E \, \nu(dy)$ are defined and equal. Then $\mathcal{E}$ is a Dynkin class including $\mathcal{U}$, so includes the σ-algebra generated by $\mathcal{U}$; in particular, it includes

$\mathcal{U}^* = \{U_0 \cup \ldots \cup U_n : U_0, \ldots, U_n \in \mathcal{U}\}$. For $A \subseteq X$, set $f_A(y) = \mu_y A$ whenever this is defined; note that if $G \subseteq X$ is open, then f_G is lower semi-continuous, being the composition of the continuous function $y \mapsto \mu_y$ and the semi-continuous function $\theta \mapsto \theta G$. If $H \subseteq X$ is open, set $\mathcal{G} = \{G : G \in \mathcal{U}^*, G \subseteq H\}$; then $\mathcal{G}$ is upwards-directed and has union H, so $f_H(y) = \sup_{G\in\mathcal{G}} f_G(y)$ for every $y \in Y$, because every μ_y is τ-additive. Accordingly

$$\int \mu_y H \, \nu(dy) = \int f_H d\nu = \sup_{G\in\mathcal{G}} \int f_G d\nu$$

(414Ba)

$$= \sup_{G\in\mathcal{G}} \mu G = \mu H,$$

and $H \in \mathcal{E}$.

Again using the Monotone Class Theorem, we see that every Borel set belongs to $\mathcal{E}$. Now if $E \in \operatorname{dom}\mu$, there are Borel sets E', E'' such that $E' \subseteq E \subseteq E''$ and $\mu E = \mu E' = \mu E''$, because μ is inner regular with respect to the Borel sets. Accordingly

$$\int f_{E'} d\nu = \mu E' = \mu E = \mu E'' = \int f_{E''} d\nu,$$

and $F = \{y : f_{E'}(y) = f_{E''}(y)\}$ is conegligible. But we are supposing that every μ_y is complete, so $\mu_y E$ is defined and equal to $\mu_y E'$ for every $y \in F$, and

$$\int \mu_y E \, \nu(dy) = \int \mu_y E' \, \nu(dy) = \mu E' = \mu E.$$

As E is arbitrary, $\langle \mu_y \rangle_{y\in Y}$ is a disintegration of μ over ν.

(b) In this case, the set $\mathcal{U}$ of cozero sets is a base for the topology of X closed under finite intersections. If $U \in \mathcal{U}$, there is a continuous $u : X \to \mathbb{R}$ such that $U = \{x : u(x) \neq 0\}$; setting $u_n = \chi X \wedge n|u|$ for $n \in \mathbb{N}$, $\langle u_n \rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $C_b(X)$ with supremum χU, so

$$\begin{aligned}\mu U &= \sup_{n\in\mathbb{N}} \int u_n d\mu = \sup_{n\in\mathbb{N}} \iint u_n d\mu_y \nu(dy) \\ &= \iint (\sup_{n\in\mathbb{N}} u_n) d\mu_y \nu(dy) = \int \mu_y U \, \nu(dy).\end{aligned}$$

Thus the condition of (a) is satisfied and $\langle \mu_y \rangle_{y\in Y}$ is a disintegration of μ over ν.

459G Theorem Let Z be a Hausdorff space, I an infinite set, and $\tilde{\mu}$ a quasi-Radon probability measure on Z^I such that the marginal measures on each copy of Z are Radon measures. Write $P_{\mathrm{R}}(Z)$ for the set of Radon probability measures on Z with its narrow topology. Then the following are equiveridical:

(i) for every permutation ρ of I, $w \mapsto w\rho : Z^I \to Z^I$ is inverse-measure-preserving for $\tilde{\mu}$;

(ii) for every transposition ρ of two elements of I, $w \mapsto w\rho : Z^I \to Z^I$ is inverse-measure-preserving for $\tilde{\mu}$;

(iii) for each $n \in \mathbb{N}$ and any two injective functions $p, q : n \to I$ the maps $w \mapsto wp : Z^I \to Z^n$ and $w \mapsto wq : Z^I \to Z^n$ induce the same measure on Z^n;

(iv) there are a probability space (Y, T, ν) and a family $\langle \mu_y \rangle_{y\in Y}$ of τ-additive Borel probability measures on Z such that $\langle \tilde{\mu}_y^I \rangle_{y\in Y}$ is a disintegration of $\tilde{\mu}$ over ν, writing $\tilde{\mu}_y^I$ for the τ-additive product of copies of μ_y;

(v) there is a Radon probability measure $\tilde{\nu}$ on $P_{\mathrm{R}}(Z)$ such that $\langle \tilde{\theta}^I \rangle_{\theta\in P_{\mathrm{R}}(Z)}$ is disintegration of $\tilde{\mu}$ over $\tilde{\nu}$, writing $\tilde{\theta}^I$ for the quasi-Radon product of copies of θ.

Moreover, if I is totally ordered, we can add

(vi) for each $n \in \mathbb{N}$ and any two strictly increasing functions $p, q : n \to I$ the maps $w \mapsto wp : Z^I \to Z^n$ and $w \mapsto wq : Z^I \to Z^n$ induce the same measure on Z^n.

proof (a) As in 459E, we need consider only the case in which I is totally ordered, and the implications

$$\text{(v)} \Rightarrow \text{(iv)} \Rightarrow \text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(vi)}$$

are elementary. So henceforth I will suppose that (vi) is true and seek to prove (v).

(b) Write μ for $\tilde{\mu}\restriction\widehat{\bigotimes}_I\mathcal{B}(Z)$, where $\mathcal{B}(Z)$ is the Borel σ-algebra of Z. Then (vi) is also true of μ. (Strictly speaking, we ought to check that the different images of μ all have the same domain. But this is true, because the image of μ corresponding to a strictly increasing function $p : r \to I$ has domain $\widehat{\bigotimes}_r\mathcal{B}(Z)$.) The (unique) marginal measure λ of μ is the restriction to $\mathcal{B}(Z)$ of the Radon measure $\tilde{\lambda}$ which is the image of $\tilde{\mu}$, so is a tight Borel measure, therefore countably compact. By 454A, μ is countably compact. So 459E, with $\mathcal{K}$ the family of compact subsets of Z, tells us that there are a probability space (Y_0, T, ν_0) and a family $\langle\mu_y\rangle_{y\in Y_0}$ in $P_{\mathrm{R}}(Z)$ such that $\langle\mu_y^I\rangle_{y\in Y_0}$ is a disintegration of μ over ν_0, writing μ_y^I for the ordinary product of copies of μ_y. It follows that the family $\langle\tilde{\mu}_y^I\rangle_{y\in Y_0}$ of Radon product measures is a disintegration of μ over ν_0, just because $\tilde{\mu}_y^I$ extends μ_y^I. Note also that $\langle\mu_y\rangle_{y\in Y_0}$ is a disintegration of λ; this is clearly achieved by the proof of 459E, and it is necessarily true if $\langle\mu_y^I\rangle_{y\in Y_0}$ is to be a disintegration of μ. It follows that $\langle\mu_y\rangle_{y\in Y_0}$ is a disintegration of the completion $\tilde{\lambda}$ of λ over ν_0 (452B(a-ii)).

Let $\langle K_n\rangle_{n\in\mathbb{N}}$ be a disjoint sequence of compact subsets of Z such that $\sum_{n=0}^{\infty}\tilde{\lambda}K_n = 1$, and set $X_0 = \bigcup_{n\in\mathbb{N}}K_n$. Since

$$1 = \tilde{\lambda}X_0 = \lambda X_0 = \int_{Y_0}\mu_y(X_0)\nu_0(dy),$$

$\mu_y X_0 = 1$ for ν_0-almost every y, and $Y = \{y : y \in Y_0, \mu_y X_0 = 1\}$ is ν_0-conegligible. It follows that, writing ν for the subspace measure on Y, $\langle\tilde{\mu}_y^I\rangle_{y\in Y}$ is a disintegration of μ over ν, and $\langle\tilde{\mu}_y\rangle_{y\in Y}$ is a disintegration of $\tilde{\lambda}$ over ν.

(c) Give X_0 the disjoint union topology corresponding to the partition $\{K_n : n \in \mathbb{N}\}$, so that X_0 is σ-compact and locally compact; note that the identity map from X_0 to Z is continuous, and that the Borel σ-algebra $\mathcal{B}(X_0)$ is just $\{X_0 \cap E : E \in \mathcal{B}(Z)\}$. Consequently the subspace measure $\mu_y\restriction\mathcal{P}X_0$ induced on X_0 by μ_y is a Radon measure on X_0 for every $y \in Y$. Let $X = X_0 \cup \{x^*\}$ be the one-point compactification of X_0, and for $y \in Y$ let λ'_y be the Radon probability measure on X extending $\mu_y\restriction\mathcal{P}X_0$ (so that $\lambda'_y\{x^*\} = 0$). In the same way, let $\tilde{\lambda}'$ be the Radon probability measure on X extending $\tilde{\lambda}\restriction\mathcal{P}X_0$; of course $\tilde{\lambda}'X_0 = \tilde{\lambda}X_0 = 1$.

(d) Let ν_1 be the image measure on $P_{\mathrm{R}}(X)$ defined by setting $\nu_1 Q = \nu\{y : y \in Y, \lambda'_y \in Q\}$ whenever this is defined. Then ν_1 is a complete probability measure. For each $n \in \mathbb{N}$, K_n is an open-and-closed set in X, so $\theta \mapsto \theta K_n$ is continuous for the narrow topology on $P_{\mathrm{R}}(X)$, and

$$Q^* = \{\theta : \theta \in P_{\mathrm{R}}(X), \theta X_0 = 1\} = \{\theta : \theta \in P_{\mathrm{R}}(X), \textstyle\sum_{n=0}^{\infty}\theta K_n = 1\}$$

is a Baire set; of course $\nu_1 Q^* = 1$.

If $u \in C(X)$, then

$$\begin{aligned}\int_{P_{\mathrm{R}}(X)}\int_X u\,d\theta\,\nu_1(d\theta) &= \int_{Q^*}\int_X u\,d\theta\,\nu_1(d\theta)\\ &= \int_{Q^*}\int_{X_0} u\,d\theta\,\nu_1(d\theta) = \int_{P_{\mathrm{R}}(X)}\int_{X_0} u\,d\theta\,\nu_1(d\theta)\end{aligned}$$

is defined and equal to

$$\int_Y\int_{X_0} u\,d\mu_y\nu(dy) = \int_{X_0} u\,d\lambda$$

by 235L and 452F. In particular (because ν_1 is complete), $\theta \mapsto \int_X u\,d\theta$ is $\operatorname{dom}\nu_1$-measurable. So ν_1 measures every Baire subset of $P_{\mathrm{R}}(X)$. **P** Setting $\hat{u}(\theta) = \int_X u\,d\theta$ for $u \in C(X)$ and $\theta \in P_{\mathrm{R}}(X)$, the functions $\hat{u}$ are continuous and separate the points of $P_{\mathrm{R}}(X)$, which is a compact Hausdorff space (437O). By the Stone-Weierstrass theorem (281E), every continuous real-valued function on $P_{\mathrm{R}}(X)$ is uniformly approximated by linear combinations of products of such functions $\hat{u}$, and is therefore $\operatorname{dom}\nu_1$-measurable. It follows at once that all zero sets and all Baire sets are measured by ν_1. **Q**

By 432F, there is a Radon measure $\tilde{\nu}_1$ on $P_{\mathrm{R}}(X)$ agreeing with ν_1 on the Baire sets. In particular, Q^* is $\tilde{\nu}_1$-conegligible. Now

$$\int_{P_{\mathrm{R}}(X)}\int_X u\,d\theta\,\tilde{\nu}_1(d\theta) = \int_{P_{\mathrm{R}}(X)}\hat{u}\,d\nu_1 = \int_{X_0} u\,d\lambda = \int_{X_0} u\,d\tilde{\lambda} = \int_X u\,d\tilde{\lambda}'$$

for every $u \in C(X)$. By 459Fb, $\langle\theta\rangle_{\theta\in P_{\rm R}(X)}$ is a disintegration of $\tilde\lambda'$ over $\tilde\nu_1$.

(e) Because the identity map $\iota : X_0 \to Z$ is continuous, we have a map $\psi : Q^* \to P_{\rm R}(Z)$ defined by saying that $\psi(\theta) = (\theta\restriction\mathcal{P}X_0)\iota^{-1}$ (416Rb, 418I). If $G \subseteq Z$ is open and $\alpha > 0$, then

$$\{\theta : \psi(\theta)(G) > \alpha\} = \{\theta : \theta(G \cap X_0) > \alpha\}$$

is open in $P_{\rm R}(X)$, so ψ is continuous. We therefore have a Radon probability measure $\tilde\nu = (\tilde\nu_1\restriction\mathcal{P}Q^*)\psi^{-1}$ on $P_{\rm R}(Z)$. (Perhaps I should remark here that $P_{\rm R}(Z)$ is a Hausdorff space, by 437Je.)

(f) Suppose that $i_0, \dots, i_r \in I$ are distinct and $G_k \subseteq Z$ is open for each $k \le r$. For each $k \le r$, set $A_k = \{u : u \in C(X),\ 0 \le u \le \chi(X_0 \cap G_k)\}$. For $u \in C(X)$, define $\bar u : Z \to \mathbb{R}$ by setting $\bar u(z) = u(z)$ if $z \in X_0$, 0 if $z \in Z \setminus X_0$. Then $\bar u$ is Borel measurable. Now, writing $\pi_{i_k}(w) = w(i_k)$ for $w \in Z^I$ and $k \le r$,

$$\mu(\textstyle\bigcap_{k\le r}\pi_{i_k}^{-1}[G_k]) = \sup_{u_k\in A_k\forall k\le r}\int(\prod_{k=0}^r \bar u_k\pi_{i_k})d\mu.$$

P Of course

$$\mu(\textstyle\bigcap_{k\le r}\pi_{i_k}^{-1}[G_k]) \ge \sup_{u_k\in A_k\forall k\le r}\int(\prod_{k=0}^r \bar u_k\pi_{i_k})d\mu$$

because $\bar u_k \le \chi G_k$ whenever $u_k \in A_k$. On the other hand, given $\epsilon > 0$, then for each $k \le r$ we have $\tilde\lambda'(G_k \cap X_0) = \lambda G_k$, so there is a $u_k \in A_k$ such that

$$\lambda G_k \le \epsilon + \textstyle\int_X u_k d\tilde\lambda' = \epsilon + \int_{X_0} u_k d\tilde\lambda' = \epsilon + \int_{X_0} u_k d\tilde\lambda = \epsilon + \int_Z \bar u_k d\lambda.$$

Now

$$\textstyle\int \chi G_k\pi_{i_k} - \bar u_k\pi_{i_k} d\mu = \int \chi G_k - \bar u_k d\lambda \le \epsilon$$

for each k, so

$$\begin{aligned}\mu(\bigcap_{k\le r}\pi_{i_k}^{-1}[G_k]) - \int\prod_{k=0}^r \bar u_k\pi_{i_k}d\mu &= \int(\prod_{k=0}^r \chi G_k\pi_{i_k} - \prod_{k=0}^r \bar u_k\pi_{i_k})d\mu\\ &\le \int\sum_{k=0}^r(\chi G_k\pi_{i_k} - \bar u_k\pi_{i_k})d\mu \le (r+1)\epsilon.\end{aligned}$$

Thus

$$\mu(\textstyle\bigcap_{k\le r}\pi_{i_k}^{-1}[G_k]) \le (r+1)\epsilon + \int\prod_{k=0}^r \bar u_k\pi_{i_k}d\mu.$$

As ϵ is arbitrary,

$$\mu(\textstyle\bigcap_{k\le r}\pi_{i_k}^{-1}[G_k]) = \sup_{u_k\in A_k\forall k\le r}\int\prod_{k=0}^r \bar u_k\pi_{i_k}d\mu.$$ **Q**

Next observe that if $u_k \in A_k$ for $k \le r$, then

$$\textstyle\int_{Z^I}\prod_{k=0}^r \bar u_k\pi_{i_k}d\mu = \int_{P_{\rm R}(X)}\big(\prod_{k=0}^r\int u_k d\theta\big)\tilde\nu_1(d\theta).$$

P

$$\int_{Z^I}\prod_{k=0}^r \bar u_k\pi_{i_k}d\mu = \int_Y\int_{Z^I}\prod_{k=0}^r \bar u_k\pi_{i_k}d\mu_y^I\nu(dy)$$

(452F, because $\prod_{k=0}^r \bar u_k\pi_{i_k}$ is $\widehat{\bigotimes}_I\mathcal{B}(Z)$-measurable)

$$\begin{aligned}&= \int_Y(\prod_{k=0}^r\int_Z \bar u_k d\mu_y)\nu(dy) = \int_Y(\prod_{k=0}^r\int_{X_0} \bar u_k d\mu_y)\nu(dy)\\ &= \int_Y(\prod_{k=0}^r\int_{X_0} u_k d\lambda'_y)\nu(dy) = \int_Y(\prod_{k=0}^r\int_X u_k d\lambda'_y)\nu(dy)\\ &= \int_{P_{\rm R}(X)}(\prod_{k=0}^r\int_X u_k d\theta)\nu_1(d\theta)\end{aligned}$$

(235L)

$$= \int_{P_{\rm R}(X)} (\prod_{k=0}^{r} \int_Z u_k d\theta) \tilde{\nu}_1(d\theta)$$

because $\theta \mapsto \prod_{k=0}^r \int_X u_k d\theta$ is continuous, and $\tilde{\nu}_1$ agrees with ν_1 on Baire sets. **Q** Accordingly

$$\tilde{\mu}(\bigcap_{k\le r} \pi_{i_k}^{-1}[G_k]) = \mu(\bigcap_{k\le r} \pi_{i_k}^{-1}[G_k]) = \sup_{u_k \in A_k \forall k \le r} \int_{Z^I} \prod_{k=0}^r \bar{u}_k \pi_{i_k} d\mu$$

$$= \sup_{u_k \in A_k \forall k \le r} \int_{P_{\rm R}(X)} (\prod_{k=0}^r \int_X u_k d\theta) \tilde{\nu}_1(d\theta)$$

$$= \int_{P_{\rm R}(X)} \sup_{u_k \in A_k \forall k \le r} (\prod_{k=0}^r \int_X u_k d\theta) \tilde{\nu}_1(d\theta)$$

(because $\tilde{\nu}_1$ is τ-additive and the functions $\theta \mapsto \prod_{k=0}^r \int_X u_k d\theta$ are continuous and the sets A_k are upwards-directed)

$$= \int_{P_{\rm R}(X)} (\prod_{k=0}^r \sup_{u_k \in A_k} \int_X u_k d\theta) \tilde{\nu}_1(d\theta)$$

$$= \int_{P_{\rm R}(X)} \prod_{k=0}^r \theta(G_k \cap X_0) \tilde{\nu}_1(d\theta)$$

(because every θ is τ-additive, and $G_k \cap X_0$ is an open set in the completely regular space X)

$$= \int_{Q^*} \prod_{k=0}^r \theta(G_k \cap X_0) \tilde{\nu}_1(d\theta)$$

$$= \int_{Q^*} \prod_{k=0}^r \psi(\theta)(G_k) \tilde{\nu}_1(d\theta) = \int_{P_{\rm R}(Z)} \prod_{k=0}^r \theta(G_k) \tilde{\nu}(d\theta)$$

(235L again)

$$= \int_{P_{\rm R}(Z)} \tilde{\theta}^I (\bigcap_{k\le r} \pi_{i_k}^{-1}[G_k]) \tilde{\nu}(d\theta),$$

where I write $\tilde{\theta}^I$ to indicate that I intend to use the Radon product measure, even though in this formula it makes no difference which product measure is employed.

(g) What this means is that $\tilde{\mu}W = \int \tilde{\theta}^I W \, \tilde{\nu}(d\theta)$ for every open measurable cylinder W. Since the map $\theta \mapsto \tilde{\theta}^I : P_{\rm R}(Z) \to P_{\rm R}(Z^I)$ is continuous (437Mb), 459Fa tells us that $\langle \tilde{\theta}^I \rangle_{\theta \in P_{\rm R}(Z)}$ is a disintegration of $\tilde{\mu}$ over $\tilde{\nu}$, as required.

459H Following the results of §452 (especially 452Ye), we do not generally expect to find disintegrations of measures which are not countably compact. It may however illuminate the constructions here if I give a specific example related to the contexts of 459E and 459G.

Example (Dubins & Freedman 79) There are a separable metrizable space Z and a quasi-Radon measure on $Z^{\mathbb{N}}$, invariant under permutations of coordinates, which cannot be disintegrated into powers of measures on Z.

proof (a) Let λ be Lebesgue measure on $[0,1]$. $Q = [0,1] \times [0,1]^{\mathbb{N}}$, with its usual topology, is a compact metrizable space, so has just $\mathfrak{c}$ Borel sets (4A3F). Let $\langle W_\xi \rangle_{\xi < \mathfrak{c}}$ enumerate the Borel subsets of Q with non-zero measure for the product measure $\lambda \times \lambda^{\mathbb{N}}$. (Remember that $\lambda \times \lambda^{\mathbb{N}}$ is a Radon measure, by 416U.) For each ξ, we have $0 < (\lambda \times \lambda^{\mathbb{N}})(W_\xi) = \int \lambda^{\mathbb{N}}(W_\xi[\{t\}]) \lambda(dt)$, so $A_\xi = \{t : W_\xi[\{t\}] \neq \emptyset\}$ has cardinal $\mathfrak{c}$ (419Ia); we can therefore choose $\langle t_\xi \rangle_{\xi < \mathfrak{c}}$ in $[0,1]$ such that $t_\xi \in A_\xi \setminus \{t_\eta : \eta < \xi\}$ for every $\xi < \mathfrak{c}$. Now choose $t_{\xi n}$, for $\xi < \mathfrak{c}$ and $n \in \mathbb{N}$, such that $(t_\xi, \langle t_{\xi n} \rangle_{n \in \mathbb{N}}) \in W_\xi$. Set $Z = \{(t_\xi, t_{\xi n}) : \xi < \mathfrak{c}, n \in \mathbb{N}\} \subseteq [0,1]^2$.

(b) Set $X = ([0,1]^2)^{\mathbb{N}}$ and define $\phi : Q \to X$ by setting $\phi(t, \langle t_n\rangle_{n\in\mathbb{N}}) = \langle (t, t_n)\rangle_{n\in\mathbb{N}}$ for t, $t_n \in [0,1]$. Then ϕ is a homeomorphism between Q and $\phi[Q]$, so there is a unique Radon measure $\mu^\#$ on X such that ϕ is inverse-measure-preserving for $\lambda \times \lambda^{\mathbb{N}}$ and $\mu^\#$. Now $\mu^\#$ is invariant under permutations of coordinates, because if $\rho : \mathbb{N} \to \mathbb{N}$ is a permutation and $\hat\rho(x) = x\rho$ for $x \in X$, then $\hat\rho\phi = \phi\bar\rho$, where $\bar\rho(t, \langle t_n\rangle_{n\in\mathbb{N}}) = (t, \langle t_{\rho(n)}\rangle_{n\in\mathbb{N}})$; and as $\bar\rho : Q \to Q$ is inverse-measure-preserving, so is $\hat\rho : X \to X$.

Also $Z^{\mathbb{N}}$ has full outer measure for $\mu^\#$. **P** If $\mu^\# W > 0$, then $(\lambda \times \lambda^{\mathbb{N}})\phi^{-1}[W] > 0$, so there is some $\xi < \mathfrak{c}$ such that $W_\xi \subseteq \phi^{-1}[W]$. Now $\langle (t_\xi, t_{\xi n})\rangle_{n\in\mathbb{N}} \in Z^{\mathbb{N}} \cap W$. **Q** Accordingly the subspace measure $\tilde\mu$ on Z is a probability measure. Because $\mu^\#$ is invariant under permutations of coordinates, so is $\tilde\mu$; because $\mu^\#$ is a Radon measure, $\tilde\mu$ is a quasi-Radon measure (416Ra).

(c) **?** Suppose, if possible, that there are a probability space (Y, T, ν) and a family $\langle \mu_y\rangle_{y\in Y}$ of probability measures on Z such that $\tilde\mu E = \int \mu_y^{\mathbb{N}} E\, \nu(dy)$ for every Borel set $E \subseteq Z^{\mathbb{N}}$. (The argument to follow will not depend on which product measure is used in forming the $\mu_y^{\mathbb{N}}$.) Looking at sets of the form $(Z \cap H) \times Z \times Z \times \dots$, where $H \subseteq [0,1]^2$ is a Borel set, we see that $\mu_y(Z \cap H)$ must be defined for almost every y; as Z is second-countable, μ_y must be a topological measure for almost every y. Looking at sets of the form $(Z \cap (G_0 \times [0,1])) \times (Z \cap (G_1 \times [0,1])) \times Z \times \dots$, where G_0 and G_1 are disjoint Borel subsets of $[0,1]$, we see that $\mu_y(Z \cap (G_0 \times [0,1])) \cdot \mu_y(Z \cap (G_1 \times [0,1])) = 0$ for almost every y; as $[0,1]$ is second-countable and Hausdorff, there must be, for almost every $y \in Y$, an $s_y \in [0,1]$ such that $\mu_y(Z \cap (\{s_y\} \times [0,1])) = 1$.

Next, if $G \subseteq [0,1]$ is a Borel set, then $\mu_y(Z \cap ([0,1] \times G)) = \lambda G$ for almost every y. **P**

$$h(y) = \mu_y^{\mathbb{N}}((Z \cap ([0,1] \times G)) \times Z \times \dots) = \mu_y(Z \cap ([0,1] \times G))$$

is defined for almost every y, and h is ν-integrable, with

$$\int h\, d\nu = \tilde\mu((Z \cap ([0,1] \times G)) \times Z \times \dots) = \mu^\#(([0,1] \times G) \times [0,1]^2 \times \dots) = \lambda G.$$

At the same time,

$$\begin{aligned}\int h(y)(1-h(y))\nu(dy) &= \tilde\mu((Z \cap ([0,1] \times G)) \times (Z \cap ([0,1] \times ([0,1] \setminus G))) \times Z \times \dots)\\ &= \mu^\#(([0,1] \times G) \times ([0,1] \times ([0,1] \setminus G)) \times [0,1]^2 \times \dots)\\ &= \lambda G(1 - \lambda G).\end{aligned}$$

Rearranging, we see that $\int h^2 d\nu = (\int h)^2$. But this means that $\int (h(y) - \int h)^2 \nu(dy) = 0$ and $h(y) = \lambda G$ for almost every y. **Q**

It follows that, for at least some y, $\mu_y(Z \cap (\{s_y\} \times G)) = \lambda G$ for every interval $G \subseteq [0,1]$ with rational endpoints. But this is impossible, because all the vertical sections of Z are countable. **X**

Thus there is no such disintegration, as claimed.

459X Basic exercises (a) Let (X, Σ, μ) be a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ an exchangeable sequence of real-valued random variables on X all with finite expectation. Use 459C and 273I to show that $\langle \frac{1}{n+1}\sum_{i=0}^n f_i\rangle_{n\in\mathbb{N}}$ converges a.e. (Compare 276Xe.)

(b) Let (X, Σ, μ) be a probability space and $\langle f_n\rangle_{n\in\mathbb{N}}$ an exchangeable sequence of real-valued random variables on X all with finite variance, such that $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f_i = 0$ a.e. Show that $\langle \Pr(\sum_{i=0}^n f_i \geq \alpha\sqrt{n+1})\rangle_{n\in\mathbb{N}}$ is convergent for every $\alpha \in \mathbb{R}$. (*Hint*: 274I.)

>(c) (Diaconis & Freedman 80) Let X be a non-empty compact Hausdorff space and I an infinite set including $\mathbb{N}$. Let $\tilde\lambda$ be a Radon probability measure on X^I invariant under permutations of I. For $k \leq n$ let $D_{nk} \subseteq n^k$ be the set of injective functions from k to n and Ω_{nk} the set $X^I \times n^k \times D_{nk}$, endowed with the product ν_{nk} of $\tilde\lambda$ and the uniform probability measures on the finite sets n^k and D_{nk}. Define $f_{nk} : \Omega_{nk} \to X^k$ and $g_{nk} : \Omega_{nk} \to X^k$ by setting

$$\begin{aligned}f_{nk}(w,p,q) &= wp,\\ g_{nk}(w,p,q) &= wp \text{ if } p \in D_{nk}\\ &= wq \text{ otherwise.}\end{aligned}$$

(i) Show that there is a disintegration $\langle \mu^k_{nw} \rangle_{w \in X^I}$ of the image measure $\nu_{nk} f_{nk}^{-1}$ over $\tilde{\lambda}$ where each μ_{nw} is a suitable point-supported measure. (ii) Show that the image measure $\nu_{nk} g_{nk}^{-1}$ is the image measure $\tilde{\lambda}_k = \tilde{\lambda}\tilde{\pi}_k^{-1}$, where $\tilde{\pi}_k(w) = w \restriction k$ for $w \in X^I$. (iii) Show that $|\tilde{\lambda}_k W - \int \mu^k_{nw} W \, \tilde{\lambda}(dw)| \leq \frac{k(k-1)}{2n}$ for every Baire set $W \subseteq X^k$. (iv) Show that there is a Radon probability measure θ_n on $P_{\mathrm{R}}(X)$ for which $w \mapsto \mu_{nw}$ is inverse-measure-preserving. (v) Show that if θ is any cluster point of $\langle \theta_n \rangle_{n \in \mathbb{N}}$ in $P_{\mathrm{R}}(X)$ then $\langle \tilde{\mu}^I \rangle_{\mu \in P_{\mathrm{R}}(X)}$ is a disintegration of $\tilde{\lambda}$ over θ, writing $\tilde{\mu}^I$ for the Radon product of copies of any $\mu \in P_{\mathrm{R}}(X)$.

>(d) (Hewitt & Savage 55) Let X be a non-empty compact Hausdorff space and I an infinite set. Let Q be the set of Radon probability measures on X^I which are invariant under permutations of I. Show that (i) Q is a closed convex subset of the set $P_{\mathrm{R}}(X^I)$ of all Radon probability measures on X^I with its narrow topology; (ii) Q is isomorphic, as topological convex structure, to $P_{\mathrm{R}}(P_{\mathrm{R}}(X))$; (iii) the extreme points of Q are just the powers of Radon probability measures on X.

(e) Let X be a non-empty Hausdorff space. Let Q be the set of Radon probability measures on $X^{\mathbb{N}}$ which are invariant under permutations of $\mathbb{N}$. Show that (i) Q is a closed convex subset of the set $P_{\mathrm{R}}(X^{\mathbb{N}})$ of all Radon probability measures on $X^{\mathbb{N}}$ with its narrow topology; (ii) Q is isomorphic, as topological convex structure, to $P_{\mathrm{R}}(P_{\mathrm{R}}(X))$; (iii) the extreme points of Q are just the powers of Radon probability measures on X.

459Y Further exercises (a) Discuss the problems which arise in 459B, 459C, 459E and 459G if the index set I is finite.

459 Notes and comments As I have presented this material, the centre of the argument is the martingales in part (b) of the proof of 459B. We are trying to resolve the functions f_i into 'common' and 'independent' parts. The 'common' part is given by the conditional expectations of the f_i over an appropriate σ-algebra T, and we approach these by looking at the conditional expectations of each f_i on σ-algebras T_n generated by 'distant' f_j. All the most important ideas are already exhibited when the index set I is equal to $\mathbb{N}$. Note in particular that in the basic hypothesis that all finite strings $(f_{i_0}, \ldots, f_{i_r})$ have the same joint distribution, it is enough to look at increasing strings. But there is a striking phenomenon which appears in sharper relief with uncountable sets I: any sequence $\langle j_k \rangle_{k \in \mathbb{N}}$ of distinct elements of I can be used to generate an adequate σ-algebra, because while the tail σ-algebra of sets depends on the choice of the j_k, they all lead to the same closed subalgebra of the measure algebra (459D).

Perhaps I should emphasize at this point that I really does have to be infinite, though for large finite I there are approximations to the results here.

The proof of 459B is one of the standard proofs of de Finetti's theorem, with trifling modifications. In the case of real-valued random variables we have a notion of relative distribution (458C) which gives a quick way of saying that all the f_i have the same conditional expectations over T, as in 459C(ii). For variables taking values in other spaces the situation may be different (459H), unless (as in §452) we have a countably compact measure (459E).

Specializing to the case $X = Z^I$ in 459B, we find ourselves examining symmetric measures on infinite product spaces, which are of great interest in themselves. Note that while in the hypothesis of 459E I have asked for the measure μ on the product space Z^I to be countably compact, what is actually necessary is that the marginal measure on Z should be countably compact. By 454A, this comes to the same thing.

As in 452O, we can look for a disintegration consisting of Radon measures, provided of course that the marginal measure is a Radon measure. What we have to work harder for is a direct expression in terms of an integral $\int \tilde{\theta}^I \tilde{\nu}(d\theta)$ where $\tilde{\nu}$ is itself a Radon probability measure on the space of Radon probability measures θ (459G). But most of the extra work consists of finding the correct reduction to the case of compact spaces. For compact spaces we can approach by a completely different route (459Xc). I will not go farther with this idea here, but I note that the method can be used in a wide variety of problems involving symmetric structures.

Chapter 46

Pointwise compact sets of measurable functions

This chapter collects results inspired by problems in functional analysis. §§461 and 466 look directly at measures on linear topological spaces. The primary applications are of course to Banach spaces, but as usual we quickly find ourselves considering weak topologies. In §461 I look at 'barycenters', or centres of mass, of probability measures, with the basic theorems on existence and location of barycenters of given measures and the construction of measures with given barycenters. In §466 I examine topological measures on linear spaces in terms of the classification developed in Chapter 41. A special class of normed spaces, those with 'Kadec norms', is particularly important, and in §467 I sketch the theory of the most interesting Kadec norms, the 'locally uniformly rotund' norms.

In the middle sections of the chapter, I give an account of the theory of pointwise compact sets of measurable functions, as developed by A.Bellow, M.Talagrand and myself. The first step is to examine pointwise compact sets of continuous functions (§462); these have been extensively studied because they represent an effective tool for investigating weakly compact sets in Banach spaces, but here I give only results which are important in measure theory, with a little background material. In §463 I present results on the relationship between the two most important topologies on spaces of measurable functions, *not* identifying functions which are equal almost everywhere: the pointwise topology and the topology of convergence in measure. These topologies have very different natures but nevertheless interact in striking ways. In particular, we have important theorems giving conditions under which a pointwise compact set of measurable functions will be compact for the topology of convergence in measure (463F, 463K).

The remaining two sections are devoted to some remarkable ideas due to Talagrand. The first, 'Talagrand's measure' (§464), is a special measure on $\mathcal{P}I$ (or $\ell^\infty(I)$), extending the usual measure of $\mathcal{P}I$ in a canonical way. In §465 I turn to the theory of 'stable' sets of measurable functions, showing how a concept arising naturally in the theory of pointwise compact sets led to a characterization of Glivenko-Cantelli classes in the theory of empirical measures.

461 Barycenters and Choquet's theorem

One of the themes of this chapter will be the theory of measures on linear spaces, and the first fundamental concept is that of 'barycenter' of a measure, its centre of mass (461Aa). The elementary theory (461B-461D) uses non-trivial results from the theory of locally convex spaces (§4A4), but is otherwise natural and straightforward. It is not always easy to be sure whether a measure has a barycenter in a given space, and I give a representative pair of results in this direction (461E, 461G). Deeper questions concern the existence and nature of measures on a given compact set with a given barycenter. The Riesz representation theorem is enough to tell us just which points can be barycenters of measures on compact sets (461H). A new idea (461J) shows that the measures can be moved out towards the boundary of the compact set. We need a precise definition of 'boundary'; the set of extreme points seems to be the appropriate concept (461N).

461A Definitions (a) Let X be a Hausdorff locally convex linear topological space, and μ a probability measure on a subset A of X. Then $x^* \in X$ is the **barycenter** or **resultant** of μ if $\int_A f d\mu$ is defined and equal to $f(x^*)$ for every $f \in X^*$. Because X^* separates the points of X (4A4Ec), μ can have at most one barycenter.

(b) Let X be any real linear space, and $C \subseteq X$ a convex set (definition: 2A5E). Then a function $g : C \to \mathbb{R}$ is **convex** if $g(tx + (1-t)y) \leq tg(x) + (1-t)g(y)$ for all x, $y \in C$ and $t \in [0,1]$. (Compare 233G, 233Xd.)

(c) The following elementary remark is useful. Let X be a linear space over $\mathbb{R}$, $C \subseteq X$ a convex set, and $g : C \to \mathbb{R}$ a function. Then g is convex iff the set $\{(x,\alpha) : x \in C, \alpha \geq g(x)\}$ is convex in $X \times \mathbb{R}$ (cf. 233Xd).

461B Proposition Let X and Y be Hausdorff locally convex linear topological spaces, and $T : X \to Y$ a continuous linear operator. Suppose that $A \subseteq X$, $B \subseteq Y$ are such that $T[A] \subseteq B$, and let μ be a probability measure on A which has a barycenter x^* in X. Then Tx^* is the barycenter of the image measure μT^{-1} on B.

proof All we have to observe is that if $g \in Y^*$ then $gT \in X^*$ (4A4Bd), so that

$$g(Tx^*) = \int_A g(Tx)\mu(dx) = \int_B g(y)\nu(dy)$$

by 235I.

461C Theorem Let X be a Hausdorff locally convex linear topological space, $C \subseteq X$ a convex set and μ a topological probability measure on a subset A of C. Suppose that μ has a barycenter x^* in X which belongs to C. Then $g(x^*) \leq \int_A g\,d\mu$ for every lower semi-continuous convex function $g : C \to \mathbb{R}$.

proof Take any $\gamma < g(x^*)$. Let D be the convex set $\{(x,\alpha) : x \in C,\ \alpha \geq g(x)\}$ in $X \times \mathbb{R}$ (461Ac). Then the closure $\overline{D}$ of D is also convex (2A5Eb). Now $(x^*,\gamma) \notin \overline{D}$. **P** Set $\delta = \frac{1}{2}(g(x^*) - \gamma) > 0$. As g is lower semi-continuous, there is an open neighbourhood G of x^* such that $g(x) \geq g(x^*) - \delta$ for every $x \in C \cap G$, so that $G \times]-\infty, \gamma + \delta[$ is an open set containing (x^*,γ) and disjoint from D. **Q**

Consequently there is a continuous linear functional $h : X \times \mathbb{R} \to \mathbb{R}$ such that $h(x^*,\gamma) < \inf_{w \in \overline{D}} h(w)$ (4A4Eb). Now there are $f \in X^*$, $\beta \in \mathbb{R}$ such that $h(x,\alpha) = f(x) + \alpha\beta$ for every $x \in X$, $\alpha \in \mathbb{R}$ (4A4Be). So we have

$$f(x^*) + \beta\gamma = h(x^*,\gamma) < h(x, g(x)) = f(x) + \beta g(x)$$

for every $x \in C$. In particular, $f(x^*) + \beta\gamma < f(x^*) + \beta g(x^*)$, so $\beta > 0$. Integrating with respect to μ, we have

$$f(x^*) + \beta\gamma < \int_A f\,d\mu + \beta \int_A g\,d\mu = f(x^*) + \beta \int_A g\,d\mu,$$

and $\gamma < \int_A g\,d\mu$. As γ is arbitrary, $g(x^*) \leq \int_A g\,d\mu$.

Remark Of course $\int_A g\,d\mu$ might be infinite.

461D Theorem Let X be a Hausdorff locally convex linear topological space, and μ a probability measure on X such that (i) the domain of μ includes the cylindrical σ-algebra of X (ii) there is a compact convex set $K \subseteq X$ such that $\mu^* K = 1$. Then μ has a barycenter in X, which belongs to K.

proof If $f \in X^*$, then $\gamma_f = \sup_{x \in K} |f(x)|$ is finite, and $\{x : |f(x)| \leq \gamma_f\}$ is a measurable set including K, so must be conegligible, and $\phi(f) = \int f d\mu$ is defined and finite. Now $\phi : X^* \to \mathbb{R}$ is a linear functional and $\phi(f) \leq \sup_{x \in K} f(x)$ for every $f \in X^*$; because K is compact and convex, there is an $x_0 \in K$ such that $\phi(f) = f(x_0)$ for every $f \in X^*$ (4A4Ef), so that x_0 is a barycenter of μ in X.

461E Theorem Let X be a complete locally convex linear topological space, and $A \subseteq X$ a bounded set. Let μ be a τ-additive topological probability measure on A. Then μ has a barycenter in X.

proof (a) If $f \in X^*$, then $f \restriction A$ is continuous and bounded (4A4Bg), therefore μ-integrable. For each open neighbourhood G of 0 in X, set

$$F_G = \{y : y \in X,\ |f(y) - \int_A f d\mu| \leq 2\tau_G(f) \text{ for every } f \in X^*\}$$

where $\tau_G(f) = \sup_{x \in G} f(x)$ for $f \in X^*$. Then F_G is non-empty. **P** Set $H = \operatorname{int}(G \cap (-G))$, so that H is an open neighbourhood of 0. Because $\overline{A}$ is bounded, there is an $m \geq 1$ such that $A \subseteq mH$. The set $\{x + H : x \in A\}$ is an open cover of A, and μ is a τ-additive topological measure, so there are $x_0, \ldots, x_n \in A$ such that $\mu(A \setminus \bigcup_{i \leq n}(x_i + H)) \leq \frac{1}{m}$. Set

$$E_i = A \cap (x_i + H) \setminus \bigcup_{j<i}(x_j + H)$$

for $i \leq n$, and $y = \sum_{i=0}^n (\mu E_i) x_i$; set $E = A \setminus \bigcup_{i \leq n}(x_i + H)$. Then, for any $f \in X^*$,

$$|f(y) - \int_A f d\mu| \leq \sum_{i=0}^{n} |\mu E_i f(x_i) - \int_{E_i} f d\mu| + \int_E |f| d\mu$$
$$\leq \sum_{i=0}^{n} \int_{E_i} |f(x) - f(x_i)| \mu(dx) + m\tau_G(f)\mu E$$
(because $E \subseteq mH$, so $f(x) \leq m\tau_G(f)$ and $f(-x) \leq m\tau_G(f)$ for every $x \in E$)
$$\leq \sum_{i=0}^{n} \tau_G(f)\mu E_i + m\tau_G(f)\mu E$$
(because if $i \leq n$ and $x \in E_i$, then $x - x_i$ and $x_i - x$ belong to G, so $|f(x) - f(x_i)| = |f(x - x_i)| \leq \tau_G(f)$)
$$\leq 2\tau_G(f).$$

As f is arbitrary, $y \in F_G$ and $F_G \neq \emptyset$. **Q**

(c) Since $F_{G \cap H} \subseteq F_G \cap F_H$ for all neighbourhoods G and H of 0, $\{F_G : G \text{ is a neighbourhood of } 0\}$ is a filter base and generates a filter $\mathcal{F}$ on X. Now $\mathcal{F}$ is Cauchy. **P** If G is any neighbourhood of 0, let $G_1 \subseteq G$ be a closed convex neighbourhood of 0, and set $H = \frac{1}{4}G_1$. **?** If $y, y' \in F_H$ and $y - y' \notin G$, then there is an $f \in X^*$ such that $f(y - y') > \tau_{G_1}(f)$ (4A4Eb). But now

$$\tau_H(f) = \frac{1}{4}\tau_{G_1}(f) < \frac{1}{4}(|f(y) - \textstyle\int_A f d\mu| + |f(y') - \int_A f d\mu|) \leq \frac{1}{4}(2\tau_H(f) + 2\tau_H(f)).$$ **X**

This means that $F_H - F_H \subseteq G$; as G is arbitrary, $\mathcal{F}$ is Cauchy. **Q**

(d) Because X is complete, $\mathcal{F}$ has a limit x^* say. Take any $f \in X^*$. **?** If $f(x^*) \neq \int_A f d\mu$, set $G = \{x : |f(x)| \leq \frac{1}{3}|f(x^*) - \int_A f d\mu|\}$. Then G is a neighbourhood of 0 in X, and

$$|f(x^*) - \int_A f d\mu| = \lim_{x \to \mathcal{F}} |f(x) - \int_A f d\mu| \leq \sup_{x \in F_G} |f(x) - \int_A f d\mu|$$
$$\leq 2\tau_G(f) \leq \frac{2}{3}|f(x^*) - \int_A f d\mu|.$$ **X**

So $f(x^*) = \int_A f d\mu$; as f is arbitrary, x^* is a barycenter of μ.

461F Lemma Let X be a normed space, and μ a probability measure on X such that every member of the dual X^* of X is integrable. Then $f \mapsto \int f d\mu : X^* \to \mathbb{R}$ is a bounded linear functional on X^*.

proof Replacing μ by its completion if necessary, we may suppose that μ is complete, so that every member of X^* is Σ-measurable, where Σ is the domain of μ. (The point is that μ and its completion give rise to the same integrals, by 212Fb.) For each $n \in \mathbb{N}$ let E_n be a measurable envelope of $B_n = \{x : \|x\| \leq n\}$; replacing E_n by $\bigcap_{i \geq n} E_i$ if necessary, we may suppose that $\langle E_n \rangle_{n \in \mathbb{N}}$ is non-decreasing. If $n \in \mathbb{N}$, $f \in X^*$ then $\{x : x \in E_n, |f(x)| > n\|f\|\}$ is a measurable subset of E_n disjoint from B_n, so must be negligible, and $|\int_{E_n} f| \leq n\|f\|$. We therefore have an element ϕ_n of X^{**} defined by setting $\phi_n(f) = \int_{E_n} f$ for every $f \in X^*$. But also $\phi(f) = \lim_{n \to \infty} \phi_n(f)$ for every f, because $\langle E_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of measurable sets with union X. By the Uniform Boundedness Theorem (3A5Ha), $\{\phi_n : n \in \mathbb{N}\}$ is bounded in X^{**}, and $\phi \in X^{**}$, with $\|\phi\| \leq \sup_{n \in \mathbb{N}} \|\phi_n\|$.

461G Proposition Let X be a reflexive Banach space, and μ a probability measure on X such that every member of X^* is μ-integrable. Then μ has a barycenter in X.

proof By 461F, $f \mapsto \int f$ is a bounded linear functional on X^*; but this means that it is represented by a member of X, which is a barycenter of μ.

461H Theorem Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a compact set. Then the closed convex hull of K in X is just the set of barycenters of Radon probability measures on K.

proof (a) If μ is a Radon probability measure on K with barycenter x^*, then

$$f(x^*) = \int_K f(x)\mu(dx) \leq \sup_{x\in K} f(x) \leq \sup_{x\in\overline{\Gamma(K)}} f(x);$$

because $\overline{\Gamma(K)}$ is closed and convex, it must contain x^* (4A4Eb).

(b) Now suppose that $x^* \in \overline{\Gamma(K)}$. Let $W \subseteq C(X)$ be the set of functionals of the form $f + \alpha\chi X$, where $f \in X^*$ and $\alpha \in \mathbb{R}$. Set $U = \{f\restriction K : f \in W\}$, so that U is a linear subspace of $C(K)$ containing χK.

If f_1, $f_2 \in W$ and $f_1\restriction K = f_2\restriction K$, then $\{x : f_1(x) = f_2(x)\}$ is a closed convex set including K, so contains x^*, and $f_1(x^*) = f_2(x^*)$; accordingly we have a functional $\phi : U \to \mathbb{R}$ defined by setting $\phi(f\restriction K) = f(x^*)$ for every $f \in W$. Of course ϕ is linear; moreover, $\phi(g) \leq \sup_{x\in K} g(x)$ for every $g \in U$, by 4A4Eb. Applying this to $\pm g$, we see that $|\phi(g)| \leq \|g\|_\infty$ for every $g \in U$. We therefore have an extension of ϕ to a continuous linear functional ψ on $C(K)$ such that $\|\psi\| \leq 1$, by the Hahn-Banach theorem (3A5Ab). Now

$$\psi(\chi K) = \phi(\chi K) = \chi X(x^*) = 1;$$

so if $0 \leq h \leq \chi K$ then

$$|1 - \psi(h)| = |\psi(\chi K - h)| \leq \|\chi K - h\|_\infty \leq 1,$$

and $\psi(h) \geq 0$. It follows that $\psi(h) \geq 0$ for every $h \in C(K)^+$. But this means that there is a Radon probability measure μ on K such that $\psi(h) = \int h\,d\mu$ for every $h \in C(K)$ (436J/436K). As $\mu K = \psi(\chi K) = 1$, μ is a probability measure; and for any $f \in X^*$

$$\int_K f d\mu = \psi(f\restriction K) = \phi(f\restriction K) = f(x^*),$$

so x^* is the barycenter of μ, as required.

461I Corollary: Kreĭn's theorem Let X be a complete Hausdorff locally convex linear topological space, and $K \subseteq X$ a weakly compact set. Then the closed convex hull $\overline{\Gamma(K)}$ of K is weakly compact.

proof Give K the weak topology induced by $\mathfrak{T}_s(X, X^*)$. Let P be the set of Radon probability measures on K (or, if you prefer, the set of tight Borel probability measures) with the narrow topology, so that P is compact (437O). By 461E, every $\mu \in P$ has a barycenter $b(\mu)$ in K. If $f \in X^*$, $f(b(\mu)) = \int_K f d\mu$, while $f\restriction K$ is continuous; so $b : P \to X$ is continuous for the vague topology on P and the weak topology on X. Accordingly $b[P]$ is weakly compact. But $b[P]$ is the weakly closed convex hull of K, by 461H applied to the weak topology on X. By 4A4Ed, $\Gamma(K)$ has the same closure for the original topology of X as it has for the weak topology, and $\overline{\Gamma(K)} = b[P]$ is weakly compact.

461J Lemma Let X be a Hausdorff locally convex linear topological space, and K a compact convex subset of X. Then for any $x^* \in K$ there is a Radon probability measure μ on K such that x^* is the barycenter of μ and $\mu\{\frac{1}{2}(x + y) : x \in M_1, y \in M_2\} = 0$ for all disjoint closed convex subsets M_1, M_2 of K.

proof (a) Let $P \subseteq C(K)^*$ be the set of linear functionals ϕ such that (i) $\phi(g) \geq 0$ for every non-negative $g \in C(K)$ (ii) $\phi(\chi K) = 1$ (iii) $\phi(f\restriction K) = f(x^*)$ for every $f \in X^*$. Then P is a weak*-closed subset of the unit ball of $C(K)^*$, so is compact for the weak* topology on $C(K)^*$ (3A5F). In the proof of 461H we found a Radon probability measure on K with barycenter x^* precisely by finding a member of P, so P is not empty.

(b) For ϕ, $\psi \in P$, say that $\phi \preccurlyeq \psi$ if $\phi(g) \leq \psi(g)$ for every convex continuous function $g : K \to \mathbb{R}$. Then $\preccurlyeq$ is reflexive and transitive, and $R_\phi = \{\psi : \psi \in P, \phi \preccurlyeq \psi\}$ is closed for every $\phi \in P$. Let $Q \subseteq P$ be a maximal set such that $\{R_\phi : \phi \in Q\}$ has the finite intersection property; then $\bigcap_{\phi\in Q} R_\phi$ is not empty; fix $\tilde{\phi}$ in the intersection. Of course $\tilde{\phi} \in P$ (because Q is not empty). Note that if $\psi \in P$ and $\tilde{\phi} \preccurlyeq \psi$, then $\{R_\phi : \phi \in Q \cup \{\psi\}\}$ has non-empty intersection, so that $\psi \in Q$ and $\psi \preccurlyeq \tilde{\phi}$.

Let $\tilde{\mu}$ be the Radon probability measure on K corresponding to $\tilde{\phi}$, so that x^* is the barycenter of $\tilde{\mu}$.

(c) Now take disjoint closed convex sets M_1, $M_2 \subseteq K$, and set $M = \{\frac{1}{2}(x + y) : x \in M_1, y \in M_2\}$.

? Suppose, if possible, that $\tilde{\mu}M > 0$. Set $q(x, y) = \frac{1}{2}(x+y)$ for $x \in M_1$, $y \in M_2$, so that $q : M_1 \times M_2 \to M$ is continuous. Let $\tilde{\mu}_M$ be the subspace measure on M induced by $\tilde{\mu}$, so that $\tilde{\mu}_M$ is a Radon measure on M (416Rb). Let λ be a Radon measure on $M_1 \times M_2$ such that $\tilde{\mu}_M = \lambda q^{-1}$ (418L), and define $\psi : C(K) \to \mathbb{R}$ by writing

$$\psi(g) = \int_{K\setminus M} g\,d\tilde\mu + \int_{M_1\times M_2} \tfrac12(g(x)+g(y))\lambda(d(x,y)).$$

Then ψ is linear, $\psi(g) \geq 0$ whenever $g \in C(K)^+$, and

$$\psi(\chi K) = \tilde\mu(K\setminus M) + \lambda(M_1\times M_2) = \tilde\mu(K\setminus M) + \tilde\mu_M(M) = 1.$$

If $g : K \to \mathbb{R}$ is continuous and convex, then

$$\begin{aligned}
\psi(g) &= \int_{K\setminus M} g\,d\tilde\mu + \int_{M_1\times M_2} \tfrac12(g(x)+g(y))\lambda(d(x,y)) \\
&\geq \int_{K\setminus M} g\,d\tilde\mu + \int_{M_1\times M_2} g(\tfrac12(x+y))\lambda(d(x,y)) \\
&= \int_{K\setminus M} g\,d\tilde\mu + \int_{M_1\times M_2} gq\,d\lambda \\
&= \int_{K\setminus M} g\,d\tilde\mu + \int_M g\,d\tilde\mu_M \\
&\text{(235I)} \\
&= \int_K g\times\chi(K\setminus M)d\tilde\mu + \int_K g\times\chi M\,d\tilde\mu \\
&\text{(131Fa)} \\
&= \int_K g\,d\tilde\mu = \tilde\phi(g).
\end{aligned}$$

Applying this to $\pm f\restriction K$, we see that $\psi(f\restriction K) = \tilde\phi(f\restriction K) = f(x^*)$ for every $f \in X^*$, so that $\psi \in P$. Also $\tilde\phi \preccurlyeq \psi$, so (as remarked in (b)) $\psi \preccurlyeq \tilde\phi$.

Because M_1 and M_2 are disjoint non-empty compact convex sets in the Hausdorff locally convex space X, there are an $f \in X^*$ and an $\alpha \in \mathbb{R}$ such that $f(x) < \alpha < f(y)$ whenever $x \in M_1$, $y \in M_2$ (4A4Ee). Set $g(x) = |f(x) - \alpha|$ for $x \in K$; then g is a continuous convex function. If $x \in M_1$ and $y \in M_2$, then

$$\begin{aligned}
g(\tfrac12(x+y)) &= |f(\tfrac12(x+y)) - \alpha| = \tfrac12|(f(x)-\alpha) + (f(y)-\alpha)| \\
&< \tfrac12(|f(x)-\alpha| + |f(y)-\alpha|) = \tfrac12(g(x)+g(y)).
\end{aligned}$$

Looking at the formulae above for $\psi(g)$, we see that we have

$$\begin{aligned}
\psi(g) &= \int_{K\setminus M} g\,d\tilde\mu + \int_{M_1\times M_2} \tfrac12(g(x)+g(y))\lambda(d(x,y)) \\
&> \int_{K\setminus M} g\,d\tilde\mu + \int_{M_1\times M_2} g(\tfrac12(x+y))\lambda(d(x,y)) = \tilde\phi(g)
\end{aligned}$$

(because $\lambda(M_1\times M_2) = \tilde\mu M > 0$). But this means that $\psi \not\preccurlyeq \tilde\phi$, which is impossible. **X**

Thus $\tilde\mu M = 0$. As M_1 and M_2 are arbitrary, $\tilde\mu$ has the properties required.

461K Choquet's theorem Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a metrizable compact convex set. Let E be the set of extreme points of K. Then for any $x^* \in K$ there is a Radon probability measure μ on K such that x^* is the barycenter of μ and $\mu E = 1$.

proof By 461J, there is a Radon probability measure μ on K such that x^* is the barycenter of μ and $\mu(\frac12(M+M')) = 0$ whenever M, M' are disjoint closed convex subsets of K, writing $\frac12(M+M')$ for $\{\frac12(x+y) : x \in M,\ y \in M'\}$. Now $\mu E = 1$. **P** Let $\mathcal{U}$ be the set of convex open subsets of X. Because the topology of X is locally convex, $\mathcal{U}$ is a base for the topology of X, and $\mathcal{V} = \{K\cap U : U \in \mathcal{U}\}$ is a base for the topology of K (4A2B(a-iv)); because K is compact and metrizable, it is separable (4A2Od), and there

is a countable base $\mathcal{V}_0 \subseteq \mathcal{V}$ (4A2O(a-iii)). Set $\mathcal{M} = \{\overline{V} : V \in \mathcal{V}_0\}$, so that $\mathcal{M}$ is a countable family of closed convex subsets of K.

If $x \in K \setminus E$, then there are distinct y, $z \in K$ such that $x = \frac{1}{2}(y+z)$. Now there must be disjoint M, $M' \in \mathcal{M}$ such that $y \in M$ and $z \in M'$, so that $x \in \frac{1}{2}(M + M')$. Thus

$$K \setminus E \subseteq \bigcup\{\tfrac{1}{2}(M+M') : M,\, M' \in \mathcal{M},\, M \cap M' = \emptyset\}.$$

As $\mathcal{M}$ is countable, $\mu(K \setminus E) = 0$ and $\mu E = 1$, as required. **Q**

461L In the form above, Choquet's theorem really does depend on the metrizability of X (see 461Yb). But there is a version which applies more generally, as follows.

Lemma Let X be a Hausdorff locally convex linear topological space and K a compact convex subset of X. Let $T : X \to \mathbb{R}^{\mathbb{N}}$ be a continuous linear operator. Let E be the set of extreme points of K and x_0 a point of K such that $Tx_0 \notin T[E]$. Then x_0 is expressible as $\frac{1}{2}(y_1 + y_2)$ where y_1, $y_2 \in K$ and $Ty_1 \neq Ty_2$.

proof (a) For each $n \in \mathbb{N}$, set $\gamma_n = \sup_{x \in K} |(Tx)(n)|$, and consider

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{2^n(\gamma_n+1)} |(Tx)(n) - (Tx_0)(n)|$$

for $x \in K$. Then $g : K \to [0, \infty[$ is a continuous convex function and $g^{-1}[\{0\}] = \{x : x \in K,\, Tx = Tx_0\}$, so $g(x) > 0$ for every $x \in E$, while $g(x_0) = 0$. Let M be the closed convex hull of $\{(x, g(x)) : x \in K\}$ in $X \times \mathbb{R}$.

(b) **?** Suppose, if possible, that $(x_0, \alpha) \notin M$ for every $\alpha > 0$. Then for every $n \in \mathbb{N}$ there is a continuous linear functional $h_n : X \times \mathbb{R} \to \mathbb{R}$ such that $h_n(x_0, 2^{-n}) > \sup_{x \in K} h_n(x, g(x))$. Let $f_n \in X^*$, $\beta_n \in \mathbb{R}$ be such that $h_n(x, \alpha) = f_n(x) + \beta_n \alpha$ for every $x \in X$, $\alpha \in \mathbb{R}$; because $h_n(x_0, 2^{-n}) > h_n(x_0, 0)$, $\beta_n > 0$.

Set $Y = \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ and define $S : X \to Y$ by setting $Sx = (Tx, \langle f_n(x)\rangle_{n \in \mathbb{N}})$ for $x \in X$; then S is continuous and linear, so $S[K]$ is compact and convex in Y. Set

$$\tilde{g}(w, z) = \sum_{n=0}^{\infty} \frac{1}{2^n(\gamma_n+1)} |w(n) - (Tx_0)(n)|$$

for $(w, z) \in S[K]$, so that $\tilde{g}$ is a continuous convex function and $g = \tilde{g}S$. Let F be the set of extreme points of $S[K]$. By Choquet's theorem, there is a Radon probability measure ν on $S[K]$ such that Sx_0 is the barycenter of ν and $\nu F = 1$.

For each $n \in \mathbb{N}$, we have

$$f_n(x) + \beta_n g(x) = h_n(x, g(x)) \leq h_n(x_0, 2^{-n}) = f_n(x_0) + 2^{-n}\beta_n$$

for each $x \in K$. So for $(w, z) \in S[K]$ we get

$$z(n) + \beta_n \tilde{g}(w, z) \leq f_n(x_0) + 2^{-n}\beta_n.$$

Integrating with respect to ν, and recalling that $Sx_0 = (Tx_0, \langle f_n(x_0)\rangle_{n \in \mathbb{N}})$ is the barycenter of ν, we get

$$f_n(x_0) + \beta_n \int \tilde{g}\, d\nu \leq f_n(x_0) + 2^{-n}\beta_n,$$

and (because $\beta_n > 0$) $\int \tilde{g}\, d\nu \leq 2^{-n}$. As n is arbitrary, $\int \tilde{g}\, d\nu = 0$.

But $F \subseteq S[E]$ (4A4Gc), and $g(x) > 0$ for every $x \in E$, so $\tilde{g}(w, z) > 0$ for every $(w, z) \in F$. Since $\nu F = 1$, $\int \tilde{g}\, d\nu > 0$, which is impossible. **X**

(c) There is therefore some $\alpha_0 > 0$ such that $(x_0, \alpha_0) \in M$. Of course $L = \{(x, g(x)) : x \in K\}$ is compact (being a continuous image of K), so there is a Radon probability measure λ on L such that (x_0, α_0) is the barycenter of λ (461H). Let μ be the image measure $\lambda\pi_1^{-1}$ on K, where $\pi_1(x, \alpha) = x$; then μ is a Radon probability measure on K, and x_0 is the barycenter of μ (461B). At the same time,

$$\int_K g\, d\mu = \int_L g\pi_1(x, \alpha)\lambda(d(x, \alpha)) = \int_L \alpha\lambda(d(x, \alpha))$$

(because if $(x, \alpha) \in L$ then $\alpha = g(x)$)

$$= \alpha_0 > 0$$

(because $(x,\alpha)\mapsto\alpha$ is a continuous linear functional on $X\times\mathbb{R}$ and (x_0,α_0) is the barycenter of λ). Looking back at the construction of g, there must be some $n\in\mathbb{N}$ such that $\int_K|(Tx)(n)-(Tx_0)(n)|\mu(dx)$ is non-zero.

(d) Since $x\mapsto(Tx)(n)$ is a continuous linear functional, $\int_K(Tx)(n)\mu(dx)=(Tx_0)(n)=\beta$ say, and

$$M_1=\{x:(Tx)(n)\leq\beta\},\quad M_2=\{x:(Tx)(n)>\beta\}$$

must both have non-zero measure. We therefore have Radon probability measures μ_1, μ_2 on K defined by saying that $\mu_1H=\mu(H\cap M_1)/\mu M_1$, $\mu_2H=\mu(H\cap M_2)/\mu M_2$ whenever these are defined. For each i, we have a barycenter y_i' of μ_i; since

$$(Ty_1')(n)=\int_K(Tx)(n)\mu_1(dx)\leq\beta<\int_K(Tx)(n)\mu_2(dx)=(Ty_2')(n),$$

$Ty_1'\neq Ty_2'$. On the other hand, for any $f\in X^*$,

$$f((\mu M_1)y_1'+(\mu M_2)y_2')=\mu M_1\int_K f d\mu_1+\mu M_2\int_K f d\mu_2=\int_K f d\mu=f(x_0),$$

so $x_0=(\mu M_1)y_1'+(\mu M_2)y_2'$.

(e) Thus x_0 is a proper convex combination of y_1' and y_2', which are members of K distinguished by T. There must therefore be distinct y_1, y_2 in the line segment $\Gamma(\{y_1',y_2'\})$ such that $x_0=\frac{1}{2}(y_1+y_2)$, and in this case $Ty_1\neq Ty_2$, as required.

461M Theorem Let X be a Hausdorff locally convex linear topological space and K a compact convex subset of X. Let E be the set of extreme points of K. Then for any x^* in K there is a Baire probability measure μ on K such that x^* is the barycenter of μ and $\mu^*E=1$.

proof (a) By 461J, there is a Radon probability measure $\tilde\mu$ on K such that x^* is the barycenter of $\tilde\mu$ and $\tilde\mu(\frac{1}{2}(M+M'))=0$ whenever M, M' are disjoint closed convex subsets of K. Let μ be the restriction of $\tilde\mu$ to the Baire σ-algebra of K; since $f\restriction K$ is surely Baire measurable for every $f\in X^*$, we have $\int f d\mu=\int f d\tilde\mu=f(x^*)$ for every $f\in X^*$, and x^* is still the barycenter of μ.

(b) The point is that $\tilde\mu Z=0$ for every zero set Z in K which is disjoint from E. **P** Since $Z\subseteq K$ is a closed G_δ set in K, $K\setminus Z$ is expressible as a union $\bigcup_{n\in\mathbb{N}}F_n$ of compact sets. For any $n\in\mathbb{N}$, $z\in Z$ and $y\in F_n$, there is an $f\in X^*$ such that $f(z)\neq f(y)$; since Z and F_n are compact, there is a finite set $\Phi_n\subseteq X^*$ such that whenever $z\in Z$ and $y\in F_n$ there is an $f\in\Phi_n$ such that $f(z)\neq f(y)$. Set $\Phi=\bigcup_{n\in\mathbb{N}}\Phi_n\cup\{0\}$; then Φ is countable; let $\langle f_n\rangle_{n\in\mathbb{N}}$ be a sequence running over Φ, and define $T:X\to\mathbb{R}^{\mathbb{N}}$ by setting $(Tx)(n)=f_n(x)$ for $n\in\mathbb{N}$, $x\in X$. Then T is a continuous linear operator and $T[Z]$ is disjoint from $T[F_n]$ for every $n\in\mathbb{N}$, so $T[Z]$ is disjoint from $T[E]$.

Lemma 461L now tells us that for every $z\in Z$ there are x, $y\in K$ such that $z=\frac{1}{2}(x+y)$ and $Tx\neq Ty$. For $n\in\mathbb{N}$ and $q<q'$ in $\mathbb{Q}$ set $M_{nqq'}=\{x:x\in K,\,f_n(x)\leq q\}$, $M'_{nqq'}=\{x:x\in K,\,f_n(x)\geq q'\}$. Then $M_{nqq'}$ and $M'_{nqq'}$ are disjoint closed convex subsets of K, so $\tilde\mu(\frac{1}{2}(M_{nqq'}+M'_{nqq'}))=0$; but as $Z\subseteq\bigcup_{n\in\mathbb{N},q<q'}\frac{1}{2}(M_{nqq'}+M'_{nqq'})$, $\tilde\mu Z=0$. **Q**

(c) Since μ is inner regular with respect to the zero sets (412D), it follows that $\mu F=0$ whenever $F\in\operatorname{dom}\mu$ and $F\cap E=\emptyset$, so that $\mu^*E=1$, as required.

461N Corollary Let X be a Hausdorff locally convex linear topological space and K a compact convex subset of X, with E its set of extreme points. Then for any x^* in K there is a probability measure ν on E such that x^* is the barycenter of ν. If K is metrizable, we can take ν to be a Radon measure.

proof Take ν to be the subspace measure μ_E, where μ is the Baire measure of 461M or the Radon measure of 461K; then

$$f(x^*)=\int_K f d\mu=\int_E f d\nu$$

for every $f\in X^*$, by 214F.

461O It is a nearly universal rule that when we encounter a compact convex set we should try to identify its extreme points. I look now at some sets which arose naturally in §437.

Proposition Let X be a set, Σ a σ-algebra of subsets of X, and $\phi : X \to X$ a (Σ, Σ)-measurable function. Let M_σ be the L-space of countably additive real-valued functionals on Σ, and $Q \subseteq M_\sigma$ the set of probability measures with domain Σ for which ϕ is inverse-measure-preserving. If $\mu \in Q$, then μ is an extreme point of Q iff ϕ is ergodic with respect to μ.

proof (a) Suppose that ϕ is μ-ergodic, and that $\mu = \frac{1}{2}(\mu_1 + \mu_2)$ where μ_1, $\mu_2 \in Q$. Then $0 \leq \mu_1 E \leq 2\mu E$ for every $E \in \Sigma$, so there is a Σ-measurable $f : X \to [0, 2]$ such that $\mu_1 E = \int_E f d\mu$ for every $E \in \Sigma$ (232F). For $\alpha \geq 0$, set $G_\alpha = \{x : x \in X, f(x) > \alpha\}$. Then, for any $E \in \Sigma$,

$$\mu_1 E = \int f \times \chi E \, d\mu = \int_0^\infty \mu(E \cap G_\alpha) d\alpha$$

where the integral $\int d\alpha$ is taken with respect to Lebesgue measure (252O). We are supposing that $\mu_1 \phi^{-1}[E] = \mu_1 E$ for every $E \in \Sigma$. In particular, for any $\alpha \geq 0$,

$$\begin{aligned}
\alpha\mu G_\alpha + \int_\alpha^\infty \mu G_\beta d\beta &= \int_0^\infty \mu(G_\alpha \cap G_\beta) d\beta = \mu_1 G_\alpha \\
&= \mu_1 \phi^{-1}[G_\alpha] = \int_0^\infty \mu(\phi^{-1}[G_\alpha] \cap G_\beta) d\beta \\
&\leq \alpha\mu\phi^{-1}[G_\alpha] + \int_\alpha^\infty \mu(\phi^{-1}[G_\alpha] \cap G_\beta) d\beta \\
&= \alpha\mu G_\alpha + \int_\alpha^\infty \mu(\phi^{-1}[G_\alpha] \cap G_\beta) d\beta
\end{aligned}$$

because μ also belongs to Q. So

$$\int_\alpha^\infty \mu G_\beta d\beta \leq \int_\alpha^\infty \mu(\phi^{-1}[G_\alpha] \cap G_\beta) d\beta.$$

It follows that $\mu(\phi^{-1}[G_\alpha] \cap G_\beta) = \mu G_\beta$, that is, $G_\beta \setminus \phi^{-1}[G_\alpha]$ is μ-negligible, for almost every $\beta > \alpha$. Take a sequence $\langle \beta_n \rangle_{n \in \mathbb{N}}$ with infimum α such that $G_{\beta_n} \setminus \phi^{-1}[G_\alpha]$ is μ-negligible for every n; then $G_\alpha = \bigcup_{n \in \mathbb{N}} G_{\beta_n}$, so $G_\alpha \setminus \phi^{-1}[G_\alpha]$ is μ-negligible. As $\mu G_\alpha = \mu\phi^{-1}[G_\alpha]$, $G_\alpha \triangle \phi^{-1}[G_\alpha]$ is μ-negligible. Since ϕ is μ-ergodic, μG_α must be 0 or 1 (372J).

As this is true for every α, f must be μ-essentially constant, and $\mu_1 = \mu$. As μ_1 and μ_2 are arbitrary, μ is an extreme point of Q.

(b) Suppose that ϕ is not ergodic. Let $E \in \Sigma$ be such that $\phi^{-1}[E] = E$ and $0 < \mu E < 1$. Set $\alpha = \mu E$, $\beta = 1 - \mu E$ and let μ_1, μ_2 be the uncompleted indefinite-integral measures over μ defined by $\frac{1}{\alpha}\chi E$ and $\frac{1}{\beta}\chi(X \setminus E)$ (234Cc). Then both are probability measures with domain Σ. For any $F \in \Sigma$,

$$\begin{aligned}
\mu_1(\phi^{-1}[F]) &= \frac{1}{\alpha}\mu(E \cap \phi^{-1}[F]) = \frac{1}{\alpha}\mu(\phi^{-1}[E] \cap \phi^{-1}[F]) \\
&= \frac{1}{\alpha}\mu(\phi^{-1}[E \cap F]) = \frac{1}{\alpha}\mu(E \cap F) = \mu_1 F.
\end{aligned}$$

So μ_1 and $\mu_1\phi^{-1}$ agree on Σ and $\mu_1 \in Q$. Similarly, $\mu_2 \in Q$. As neither μ_1 nor μ_2 is equal to μ, μ is not extreme in Q.

461P Corollary Let X be a compact Hausdorff space and $\phi : X \to X$ a continuous function. Let Q_ϕ be the non-empty compact convex set of Radon probability measures μ on X such that ϕ is inverse-measure-preserving for μ, with its narrow topology and the convex structure inherited from its identifications with subsets of $C(X)^*$ and of the L-space of countably additive functionals on the Borel σ-algebra of X (437S). Then the extreme points of Q_ϕ are those for which ϕ is inverse-measure-preserving and ergodic.

proof (a) If $\mu_0 \in Q_\phi$ is not extreme, let $\mathcal{B} = \mathcal{B}(X)$ be the Borel σ-algebra of X, so that ϕ is $(\mathcal{B}, \mathcal{B})$-measurable, and write Q'_ϕ for the set of ϕ-invariant Borel probability measures on X. Then 461O tells us that the extreme points of Q'_ϕ are just the measures for which ϕ is ergodic. If $\mu \in Q_\phi$, then $\mu\restriction\mathcal{B} \in Q'_\phi$, and of course the function $\mu \mapsto \mu\restriction\mathcal{B}$ is injective (416Eb) and preserves convex combinations. So $\mu_0\restriction\mathcal{B}$ is not extreme in Q'_ϕ. By 461O, ϕ is not $\mu_0\restriction\mathcal{B}$-ergodic, and therefore not μ_0-ergodic.

(b) If $\mu_0 \in Q_\phi$ and ϕ is not μ_0-ergodic, let $E \in \operatorname{dom}\mu_0$ be such that $0 < \mu_0 E < 1$ and $\phi^{-1}[E] = E$. Set $\alpha = \mu_0 E$ and $\beta = 1 - \alpha$, and let μ_1, μ_2 be the indefinite-integral measures over μ_0 defined by $\frac{1}{\alpha}\chi E$ and $\frac{1}{\beta}\chi(X \setminus E)$. Then μ_1 is a Radon probability measures on X (416S), so the image measure $\mu_1\phi^{-1}$ is also a Radon measure (418I). The argument of part (b) of the proof of 461O tells us that $\mu_1\phi^{-1}$ agrees with μ_1 on Borel sets, so $\mu_1 = \mu_1\phi^{-1}$ (416Eb) and $\mu_1 \in Q_\phi$. Similarly, $\mu_2 \in Q_\phi$, and $\mu_0 = \alpha\mu_1 + \beta\mu_2$, so μ_0 is not extreme in Q_ϕ.

461X Basic exercises >(a) Let X be a Hausdorff locally convex linear topological space, $C \subseteq X$ a convex set, and $g : C \to \mathbb{R}$ a function. Show that the following are equiveridical: (i) g is convex and lower semi-continuous; (ii) there are a non-empty set $D \subseteq X^*$ and a family $\langle \beta_f \rangle_{f\in D}$ in $\mathbb{R}$ such that $g(x) = \sup_{f\in D} f(x) + \beta_f$ for every $x \in C$. (Compare 233Hb.)

>(b) Let X be a Hausdorff locally convex linear topological space, $K \subseteq X$ a compact convex set, and x an extreme point of K. Let μ be a probability measure on X such that $\mu^* K = 1$ and x is the barycenter of μ. (i) Show that $\{y : y \in K,\ f(y) \neq f(x)\}$ is μ-negligible for every $f \in X^*$. (*Hint*: 461D.) (ii) Show that if μ is a Radon measure then $\mu\{x\} = 1$.

(c) For each $n \in \mathbb{N}$, define $d_n \in \boldsymbol{c}_0$ by saying that $d_n(n) = 1$, $d_n(i) = 0$ if $i \neq n$. Let μ be the point-supported Radon probability measure on $\boldsymbol{c}_0$ defined by saying that $\mu E = \sum_{n=0}^{\infty} 2^{-n-1}\chi E(2^n d_n)$ for every $E \subseteq \boldsymbol{c}_0$. (i) Show that every member of $\boldsymbol{c}_0^*$ is μ-integrable. (*Hint*: $\boldsymbol{c}_0^*$ can be identified with ℓ^1.) (ii) Show that μ has no barycenter in $\boldsymbol{c}_0$.

(d) Let I be an uncountable set, and $X = \{x : x \in \ell^\infty(I), \{i : x(i) \neq 0\} \text{ is countable}\}$. Show that X is a closed linear subspace of $\ell^\infty(I)$. (i) Show that there is a probability measure μ on X such that (α) $\mu\{x : \|x\| \leq 1\}$ is defined and equal to 1 (β) $\mu\{x : x(i) = 1\} = 1$ for every $i \in I$. (ii) Show that $\int f d\mu$ is defined for every $f \in X^*$. (*Hint*: for any $f \in X^\sim$, there is a countable set $J \subseteq X$ such that $f(x) = 0$ whenever $x \restriction J = 0$, so that $f =_{\text{a.e.}} f(\chi J)$.) (iii) Show that μ has no barycenter in X.

>(e) Let X be a complete Hausdorff locally convex linear topological space, and $K \subseteq X$ a compact set. Show that every extreme point of $\overline{\Gamma(K)}$ belongs to K. (*Hint*: show that it cannot be the barycenter of any measure on K which is not supported by a single point.)

>(f) Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a metrizable compact set. Show that $\overline{\Gamma(K)}$ is metrizable. (*Hint*: we may suppose that X is complete, so that $\overline{\Gamma(K)}$ is compact. Show that $\overline{\Gamma(K)}$ is a continuous image of the space P of Radon probability measures on K, and use 437Ra.)

(g) Let X be a Hausdorff locally convex linear topological space and $K \subseteq X$ a compact set. Show that the Baire σ-algebra of K is just the subspace σ-algebra induced by the cylindrical σ-algebra of X. (*Hint*: disjoint closed subsets of K can be separated by finite subsets of X^*.)

>(h) Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a compact convex set; let E be the set of extreme points of K. Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence in X^* such that (i) $\sup_{x\in K, n\in\mathbb{N}} |f_n(x)|$ is finite (ii) $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in E$. Show that $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in K$.

>(i) Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a metrizable compact convex set. Show that the set E of extreme points of K is a G_δ set, therefore Polish. Show that the algebra of Borel subsets of K is just the subspace algebra of the cylindrical σ-algebra of X.

(j) Let X be a Hausdorff locally convex linear topological space, and $K \subseteq X$ a compact set. Let us say that a point x of K is **extreme** if the only Radon probability measure on K with barycenter x is the one for which $\{x\}$ has measure 1. (Cf. 461Xb.) (i) Show that if X is complete, then $x \in X$ is an extreme point of K iff it is an extreme point of $\overline{\Gamma(K)}$. (ii) Writing E for the set of extreme points of K, show that any point of K is the barycenter of some probability measure on E. (iii) Show that if K is metrizable then E is a G_δ subset of K and any point of K is the barycenter of some Radon probability measure on E.

(k) Let X be a set, Σ a σ-algebra of subsets of X, and P the set of probability measures with domain Σ, regarded as a convex subset of the linear space of countably additive functionals on Σ. Show that $\mu \in P$ is an extreme point in P iff it takes only the values 0 and 1.

(l) Let X be a set, Σ a σ-algebra of subsets of X, $\phi : X \to X$ a (Σ, Σ)-measurable function and M_σ the L-space of countably additive functionals with domain Σ. (i) Show that the set $V = \{\mu : \mu \in M_\sigma, \mu\phi^{-1}[E] = \mu E \text{ for every } E \in \Sigma\}$ is a norm-closed Riesz subspace of M_σ, so is an L-space in its own right. (ii) Let Q_ϕ be the set of probability measures in V. Show that if μ, ν are distinct extreme points of Q_ϕ then there is an $E \in \Sigma$ which is negligible for one and conegligible for the other. (*Hint*: show that $\sup_{n\in\mathbb{N}} \mu \wedge n\nu = 0$ and apply 326I to $\mu - \nu$.)

>(m) Set $S^1 = \{z : z \in \mathbb{C}, |z| = 1\}$, and let $w \in S^1$ be such that $w^n \neq 1$ for any integer 1. Define $\phi : S^1 \to S^1$ by setting $\phi(z) = wz$ for every $z \in S^1$. Show that the only Radon probability measure on S^1 for which ϕ is inverse-measure-preserving is the Haar probability measure μ of S^1. (*Hint*: use 281N to show that $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f(w^k z) = \int f d\mu$ for every $f \in C(S^1)$; now put 461P and 372I together.)

(n) Set $\phi(x) = 2\min(x, 1-x)$ for $x \in [0,1]$ (cf. 372Xm). Show that there are many point-supported Radon measures on $[0,1]$ for which ϕ is inverse-measure-preserving.

461Y Further exercises (a) Let X be a Hausdorff locally convex linear topological space. (i) Show that if $M_0, \ldots, M_n$ are non-empty compact convex subsets of X with empty intersection, then there is a continuous convex function $g : X \to \mathbb{R}$ such that $g(\sum_{i=0}^{n} \alpha_i x_i) < \sum_{i=0}^{n} \alpha_i g(x_i)$ whenever $x_i \in M_i$ and $\alpha_i > 0$ for every $i \leq n$. (ii) Show that if $K \subseteq X$ is compact and $x^* \in \overline{\Gamma(K)}$ then there is a Radon probability measure μ on K, with barycenter x^*, such that $\mu(\alpha_0 M_0 + \ldots + \alpha_n M_n) = 0$ whenever $M_0, \ldots, M_n$ are compact convex sets with empty intersection, $\alpha_i \geq 0$ for every $i \leq n$, and $\sum_{i=0}^{n} \alpha_i = 1$.

(b) In $\mathbb{R}^{[0,2[}$ let K be the set of those functions u such that (α) $0 \leq u(s) \leq u(t) \leq 1$ whenever $0 \leq s \leq t \leq 1$ (β) $|u(t+1)| \leq u(s') - u(s)$ whenever $0 \leq s < t < s' \leq 1$. (i) Show that K is a compact convex set. (ii) Show that the set E of extreme points of K is just the set of functions of the types $\mathbf{0}$, $\chi[0,1]$, $\chi[s,1] \pm \chi\{1+s\}$ and $\chi\,]s,1] \pm \chi\{1+s\}$ for $0 < s < 1$. (iii) Set $w(s) = s$ for $s \in [0,1]$, 0 for $s \in]1,2[$. Show that if μ is any Radon probability measure on K with barycenter w then $\mu E = 0$.

(c) Let G be a topological group. Show that it is amenable iff whenever X is a Hausdorff linear topological space, and $\bullet$ is a continuous action of G on X such that $x \mapsto a\bullet x$ is a linear operator for every $a \in G$, and $K \subseteq X$ is a compact convex set such that $a\bullet x \in K$ for every $a \in G$ and $x \in K$, then there is an $x \in K$ such that $a\bullet x = x$ for every $a \in G$. Use this to simplify parts of the proof of 449C. (*Hint*: 493B.)

(d) Let X be a set, Σ a σ-algebra of subsets of X, and Φ a set of (Σ, Σ)-measurable functions from X to itself. Let M_σ be the L-space of countably additive real-valued functionals on Σ, and $Q \subseteq M_\sigma$ the set of probability measures with domain Σ for which every member of Φ is inverse-measure-preserving. (i) Show that if $\mu \in Q$, then μ is an extreme point of Q iff $\mu E \in \{0,1\}$ whenever $E \in \Sigma$ and $\mu(E \triangle \phi^{-1}[E]) = 0$ for every $\phi \in \Phi$. (ii) Show that if $\mu \in Q$ and Φ is countable and commutative, then μ is an extreme point of Q iff $\mu E \in \{0,1\}$ whenever $E \in \Sigma$ and $E = \phi^{-1}[E]$ for every $\phi \in \Phi$.

(e) Let X be a non-empty Hausdorff space, and define $\phi : X^{\mathbb{N}} \to X^{\mathbb{N}}$ by setting $\phi(x)(n) = x(n+1)$ for $x \in X^{\mathbb{N}}$, $n \in \mathbb{N}$. Let Q_ϕ be the set of Radon measures on $X^{\mathbb{N}}$ for which ϕ is inverse-measure-preserving. Show that a Radon probability measure λ on $X^{\mathbb{N}}$ is an extreme point of Q_ϕ iff it is a Radon product measure $\mu^{\mathbb{N}}$ for some Radon probability measure μ on X.

461 Notes and comments The results above are a little unusual in that we have studied locally convex spaces for several pages without encountering two topologies on the same space more than once (461I). In fact some of the most interesting properties of measures on locally convex spaces concern their relationships with strong and weak topologies, but I defer these ideas to later parts of the chapter. For the moment, we just have the basic results (i) affirming that barycenters exist (461D, 461E, 461G) (ii) that points can be represented as barycenters (461H, 461K, 461M). The last two can be thought of as refinements of the

Kreĭn-Milman theorem. Any compact convex set K (in a locally compact Hausdorff space) is the closed convex hull of the set E of its extreme points. By 461N, given $x \in K$, we can actually find a measure on E with barycenter x; and if K is metrizable we can do this with a Radon measure. Of course the second part of 461N is a straightforward consequence of the first, because in the metrizable case E is a Borel set, the cylindrical σ-algebra (on K or E) is the Borel algebra, and any Borel probability measure on E must be tight (461Xi, 433Ca). But I do not know of any proof of 461N which does not pass through 461K.

Kreĭn's theorem (461I) is a fundamental result in the theory of linear topological spaces. The proof here, using the Riesz representation theorem and vague topologies, is a version of the standard one (e.g., Bourbaki 87, II.4.1), written out to be a little heavier in the measure theory and a little lighter in the topological linear space theory than is usual. There are of course proofs which do not use measure theory.

In §437 I have already looked at an archetypal special case of 461H and 461M. If X is a compact Hausdorff space and P the compact convex set of Radon probability measures on X with the vague topology, then the set of extreme points of P can be identified with $\Delta = \{\delta(x) : x \in X\}$, where $\delta(x)$ is the Radon probability measure concentrated at the point x (437P, 437Xo). If we think of P as a subset of $C(X)^*$ with the weak* topology, so that the dual of the linear topological space $C(X)^*$ can be identified with $C(X)$, then any $\mu \in P$ is the barycenter of a Baire measure ν on Δ. In fact (because Δ here is compact) μ is the barycentre of a Radon measure on Δ, and this is just the image measure $\mu\delta^{-1}$.

Another class of examples arising in §437 is explored in 461O-461P, 461Xk-461Xn and 461Yd-461Ye. It is when we have an explicit listing of the extreme points, as in 461Yb and 461Ye, that we begin to feel that we understand a compact convex set.

462 Pointwise compact sets of continuous functions

In preparation for the main work of this chapter, beginning in the next section, I offer a few pages on spaces of continuous functions under their 'pointwise' topologies (462Ab). There is an extensive general theory of such spaces, described in Arkhangel'skiĭ 92; here I present only those fragments which seem directly relevant to the theory of measures on normed spaces and spaces of functions. In particular, I star the paragraphs 462B-462D, which are functional analysis rather than measure theory. They are here because although this material is well known, and may be found in many places, I think that the ideas, as well as the results, are essential for any understanding of measures on linear topological spaces.

Measure theory enters the section in the proof of 462E, in the form of an application of the Riesz representation theorem, though 462E itself remains visibly part of functional analysis. In the rest of the section, however, we come to results which are pure measure theory. For (countably) compact spaces X, the Radon measures on $C(X)$ are the same for the pointwise and norm topologies (462G). This fact has extensive implications for the theory of separately continuous functions (462I) and for the theory of convex hulls in linear topological spaces (462J).

462A Definitions (a) A regular Hausdorff space X is **angelic** if whenever A is a subset of X which is relatively countably compact in X, then (i) its closure $\overline{A}$ is compact (ii) every point of $\overline{A}$ is the limit of a sequence in A.

(A **Fréchet-Urysohn** space is a topological space in which, for any set A, every point of the closure of A is a limit of a sequence in A. So (ii) here may be written as 'every compact subspace of X is a Fréchet-Urysohn space'.)

(b) If X is any set and A a subset of $\mathbb{R}^X$, then the topology of **pointwise convergence** on A is that inherited from the usual product topology of $\mathbb{R}^X$; that is, the coarsest topology on A for which the map $f \mapsto f(x) : A \to \mathbb{R}$ is continuous for every $x \in X$. I shall commonly use the symbol $\mathfrak{T}_p$ for such a topology. In this context, I will say that a sequence or filter is **pointwise convergent** if it is convergent for the topology of pointwise convergence. Note that if A is a linear subspace of $\mathbb{R}^X$ then $\mathfrak{T}_p$ is a linear space topology on A (4A4Ba).

***462B Theorem** (PRYCE 71) Let X be a topological space such that there is a sequence $\langle X_n\rangle_{n\in\mathbb{N}}$ of countably compact subsets of X, covering X, with the property that a function $f : X \to \mathbb{R}$ is continuous whenever $f\restriction X_n$ is continuous for every $n \in \mathbb{N}$. Then the space $C(X)$ of continuous real-valued functions on X is angelic in its topology of pointwise convergence.

proof Of course $C(X)$ is regular and Hausdorff under $\mathfrak{T}_p$, because $\mathbb{R}^X$ is, so we need attend only to the rest of the definition in 462Aa. Let $A \subseteq C(X)$ be relatively countably compact for $\mathfrak{T}_p$.

(a) Since $\{f(x) : f \in A\}$, being a continuous image of A, must be relatively countably compact in $\mathbb{R}$ (4A2G(f-iv)), therefore relatively compact (4A2Le), for every $x \in X$, the closure $\overline{A}$ of A in $\mathbb{R}^X$ is compact, by Tychonoff's theorem.

? Suppose, if possible, that $\overline{A} \not\subseteq C(X)$; let $g \in \overline{A}$ be a discontinuous function. By the hypothesis of the theorem, there is an $n \in \mathbb{N}$ such that $g\restriction X_n$ is not continuous; take $x^* \in X_n$ such that $g\restriction X_n$ is discontinuous at x^*. Let $\epsilon > 0$ be such that for every neighbourhood U of x^* in X_n there is a point $x \in U$ such that $|g(x) - g(x^*)| \geq \epsilon$.

Choose sequences $\langle f_i\rangle_{i\in\mathbb{N}}$ in A, $\langle x_i\rangle_{i\in\mathbb{N}}$ in X_n as follows. Given $\langle f_i\rangle_{i<m}$ and $\langle x_i\rangle_{i<m}$, choose $x_m \in X_n$ such that $|f_i(x_m) - f_i(x^*)| \leq 2^{-m}$ for every $i < m$ and $|g(x^*) - g(x_m)| \geq \epsilon$. Now choose $f_m \in A$ such that $|f_m(x^*) - g(x^*)| \leq 2^{-m}$ and $|f_m(x_i) - g(x_i)| \leq 2^{-m}$ for every $i \leq m$. Continue.

At the end of the induction, take a cluster point x of $\langle x_i\rangle_{i\in\mathbb{N}}$ in X_n and a cluster point f of $\langle f_i\rangle_{i\in\mathbb{N}}$ in $C(X)$. Because $|f_i(x_m) - f_i(x^*)| \leq 2^{-m}$ whenever $i < m$, $f_i(x) = f_i(x^*)$ for every i, and $f(x) = f(x^*)$. Because $|f_m(x^*) - g(x^*)| \leq 2^{-m}$ for every m, $f(x^*) = g(x^*)$. Because $|f_m(x_i) - g(x_i)| \leq 2^{-m}$ whenever $i \leq m$, $f(x_i) = g(x_i)$ for every i, $|g(x^*) - f(x_i)| \geq \epsilon$ for every i, and $|g(x^*) - f(x)| \geq \epsilon$; but this is impossible, because $f(x) = f(x^*) = g(x^*)$. **X**

Thus the compact set $\overline{A} \subseteq C(X)$ is the closure of A in $C(X)$, and A is relatively compact in $C(X)$.

(b) Now take any $g \in \overline{A}$. There are countable sets $D \subseteq X$, $B \subseteq A$ such that

whenever $I \subseteq B \cup \{g\}$ is finite, $n \in \mathbb{N}$, $\epsilon > 0$ and $x \in X_n$, there is a $y \in D \cap X_n$ such that $|f(y) - f(x)| \leq \epsilon$ for every $f \in I$;

whenever $J \subseteq D$ is finite and $\epsilon > 0$ there is an $f \in B$ such that $|f(x) - g(x)| \leq \epsilon$ for every $x \in J$.

P For any finite set $I \subseteq \mathbb{R}^X$ and $n \in \mathbb{N}$, the set $Q_{In} = \{\langle f(x)\rangle_{f\in I} : x \in X_n\}$ is a subset of the separable metric space $\mathbb{R}^I$, so is itself separable, and there is a countable dense set $D_{In} \subseteq X_n$ such that $Q'_{In} = \{\langle f(x)\rangle_{f\in I} : x \in D_{In}\}$ is dense in Q_{In}. Similarly, because $g \in \overline{A}$, we can choose for any finite set $J \subseteq X$ a sequence $\langle f_{Ji}\rangle_{i\in\mathbb{N}}$ in A such that $\lim_{i\to\infty} f_{Ji}(x) = g(x)$ for every $x \in J$.

Now construct $\langle D_m\rangle_{m\in\mathbb{N}}$, $\langle B_m\rangle_{m\in\mathbb{N}}$ inductively by setting

$$D_m = \bigcup\{D_{In} : n \in \mathbb{N},\ I \subseteq \{g\} \cup \textstyle\bigcup_{i<m} B_i \text{ is finite}\},$$

$$B_m = \{f_{Jk} : k \in \mathbb{N},\ J \subseteq \textstyle\bigcup_{i<m} D_i \text{ is finite}\}.$$

At the end of the induction, set $D = \bigcup_{m\in\mathbb{N}} D_m$ and $B = \bigcup_{m\in\mathbb{N}} B_m$. Since the construction clearly ensures that $\langle D_m\rangle_{m\in\mathbb{N}}$ and $\langle B_m\rangle_{m\in\mathbb{N}}$ are non-decreasing sequences of countable sets, D and B are countable, and we shall have $D_{In} \subseteq D$ whenever $n \in \mathbb{N}$ and $I \subseteq B \cup \{g\}$ is finite, while $f_{Ji} \in B$ whenever $i \in \mathbb{N}$ and $J \subseteq D$ is finite. Thus we have suitable sets D and B. **Q**

By the second condition on D and B, there must be a sequence $\langle f_i\rangle_{i\in\mathbb{N}}$ in B such that $g(x) = \lim_{i\to\infty} f_i(x)$ for every $x \in D$. In fact $g(y) = \lim_{i\to\infty} f_i(y)$ for every $y \in X$. **P?** Otherwise, there is an $\epsilon > 0$ such that $J = \{i : |g(y) - f_i(y)| \geq \epsilon\}$ is infinite. Let n be such that $y \in X_n$. For each $m \in \mathbb{N}$, $I_m = \{f_i : i \leq m\}$ is a finite subset of B, so there is an $x_m \in D_{I_m n}$ such that $|f(x_m) - f(y)| \leq 2^{-m}$ for every $f \in I_m \cup \{g\}$. Let $x^* \in X_n$ be a cluster point of $\langle x_m\rangle_{m\in\mathbb{N}}$, and $h \in C(X)$ a cluster point of $\langle f_i\rangle_{i\in J}$. Then

because $g(x) = \lim_{i\to\infty} f_i(x)$ for every $x \in D$, $g(x_m) = h(x_m)$ for every $m \in \mathbb{N}$, and $g(x^*) = h(x^*)$;

because $|g(y) - f_i(y)| \geq \epsilon$ for every $i \in J$, $|g(y) - h(y)| \geq \epsilon$;

because $|f_i(x_m) - f_i(y)| \leq 2^{-m}$ whenever $i \leq m$, $f_i(x^*) = f_i(y)$ for every $i \in \mathbb{N}$, and $h(x^*) = h(y)$;

because $|g(x_m) - g(y)| \leq 2^{-m}$ for every m, $g(x^*) = g(y)$;

but this means that $g(y) = g(x^*) = h(x^*) = h(y) \neq g(y)$, which is absurd. **XQ**

So $g = \lim_{i\to\infty} f_i$ for $\mathfrak{T}_p$. As g is arbitrary, A has both properties required in 462Aa; as A is arbitrary, $C(X)$ is angelic.

Remark For a slight strengthening of this result, see 462Ya.

***462C Proposition** (PRYCE 71) Let $(X, \mathfrak{T})$ be an angelic regular Hausdorff space.
(a) Any subspace of X is angelic.
(b) If $\mathfrak{S}$ is a regular topology on X finer than $\mathfrak{T}$, then $\mathfrak{S}$ is angelic.
(c) Any countably compact subset of X is compact and sequentially compact.

proof (a) Let Y be any subset of X. Then of course the subspace topology on Y is regular and Hausdorff. If $A \subseteq Y$ is relatively countably compact in Y, then A is relatively countably compact in X, so $\overline{A}$, the closure of A in X, is compact. Now if $x \in \overline{A}$, there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in A converging to x; but $\langle x_n\rangle_{n\in\mathbb{N}}$ must have a cluster point in Y, and (because $\mathfrak{T}$ is Hausdorff) this cluster point can only be x. Accordingly $\overline{A} \subseteq Y$ and is the closure of A in Y. Thus A is relatively compact in Y. Moreover, any point of $\overline{A}$ is the limit of a sequence in A. As A is arbitrary, Y is angelic.

(b) By hypothesis, $\mathfrak{S}$ is regular, and it is Hausdorff because it is finer than $\mathfrak{T}$. Now suppose that $A \subseteq X$ is $\mathfrak{S}$-relatively countably compact. Because the identity map from $(X, \mathfrak{S})$ to $(X, \mathfrak{T})$ is continuous, A is $\mathfrak{T}$-relatively countably compact (4A2G(f-iv)), and the $\mathfrak{T}$-closure $\overline{A}$ of A is $\mathfrak{T}$-compact.

Let $\mathcal{F}$ be any ultrafilter on X containing A. Then $\mathcal{F}$ has a $\mathfrak{T}$-limit $x \in X$. **?** If $\mathcal{F}$ is not $\mathfrak{S}$-convergent to x, there is an $H \in \mathfrak{S}$ such that $x \in H$ and $X \setminus H \in \mathcal{F}$, so that $A \setminus H \in \mathcal{F}$. Now x belongs to the $\mathfrak{T}$-closure of $A \setminus H$, because $A \setminus H \in \mathcal{F}$ and $\mathcal{F}$ is $\mathfrak{T}$-convergent to x; because $\mathfrak{T}$ is angelic, there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $A \setminus H$ which $\mathfrak{T}$-converges to x. But now $\langle x_n\rangle_{n\in\mathbb{N}}$ has a $\mathfrak{S}$-cluster point x'. x' must also be a $\mathfrak{T}$-cluster point of $\langle x_n\rangle_{n\in\mathbb{N}}$, so $x' = x$; but every x_n belongs to the $\mathfrak{S}$-closed set $X \setminus H$, so $x' \notin H$, which is impossible. **X**

Thus every ultrafilter on X containing A is $\mathfrak{S}$-convergent. Because $\mathfrak{S}$ is regular, the $\mathfrak{S}$-closure $\tilde{A}$ of A is $\mathfrak{S}$-compact (3A3De).

Again because $\mathfrak{S}$ is finer than $\mathfrak{T}$, and $\mathfrak{T}$ is Hausdorff, the two topologies must agree on $\tilde{A} = \overline{A}$. But now every point of $\overline{A}$ is the $\mathfrak{T}$-limit of a sequence in A, so every point of $\tilde{A}$ is the $\mathfrak{S}$-limit of a sequence in A. As A is arbitrary, $\mathfrak{S}$ is angelic.

(c) If $K \subseteq X$ is countably compact, then of course it is relatively countably compact in its subspace topology, so (being angelic) must be compact in its subspace topology. If $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in K, let x be any cluster point of $\langle x_n\rangle_{n\in\mathbb{N}}$. If $\{n : x_n = x\}$ is infinite, then this immediately provides us with a subsequence converging to x. Otherwise, take n such that $x \neq x_i$ for $i \geq n$. Since $x \in \overline{\{x_i : i \geq n\}}$, and $\{x_i : i \geq n\}$ is relatively countably compact, there is a sequence $\langle y_i\rangle_{i\in\mathbb{N}}$ in $\{x_i : i \geq n\}$ converging to x. The topology of X being Hausdorff, $\{y_i : i \in \mathbb{N}\}$ must be infinite, and $\langle y_i\rangle_{i\in\mathbb{N}}$, $\langle x_i\rangle_{i\in\mathbb{N}}$ have a common subsequence which converges to x. As $\langle x_i\rangle_{i\in\mathbb{N}}$ is arbitrary, K is sequentially compact.

***462D Theorem** Let U be any normed space. Then it is angelic in its weak topology.

proof Write X for the unit ball of the dual space U^*, with its weak* topology. Then X is compact (3A5F). We have a natural map $u \mapsto \hat{u} : U \to \mathbb{R}^X$ defined by setting $\hat{u}(x) = x(u)$ for $x \in X$, $u \in U$. By the definition of the weak* topology, $\hat{u} \in C(X)$ for every $u \in U$. The weak topology of U is normally defined in terms of all functionals $u \mapsto f(u)$, for $f \in U^*$; but as every member of U^* is a scalar multiple of some $x \in X$, we can equally regard the weak topology of U as defined just by the functionals $u \mapsto x(u) = \hat{u}(x)$, for $x \in X$. But this means that the map $u \mapsto \hat{u}$ is a homeomorphism between U, with its weak topology, and its image $\widehat{U}$ in $C(X)$, with the topology of pointwise convergence.

Since $C(X)$ is $\mathfrak{T}_p$-angelic (462B), so is $\widehat{U}$ (462Ca), and U is angelic in its weak topology.

462E Theorem Let X be a locally compact Hausdorff space, and $C_0(X)$ the Banach lattice of continuous real-valued functions on X which vanish at infinity (436I). Write $\mathfrak{T}_p$ for the topology of pointwise convergence on $C_0(X)$.

(i) A sequence $\langle u_n\rangle_{n\in\mathbb{N}}$ in $C_0(X)$ is weakly convergent to $u \in C_0(X)$ iff it is $\mathfrak{T}_p$-convergent and norm-bounded.

(ii) A subset K of $C_0(X)$ is weakly compact iff it is norm-bounded and $\mathfrak{T}_p$-compact.

proof (a) Since all the maps $f \mapsto f(x)$, where $x \in X$, are bounded linear functionals on $C_0(X)$, the $\mathfrak{T}_p$ is coarser than the weak topology $\mathfrak{T}_s$; so a $\mathfrak{T}_s$-convergent sequence is $\mathfrak{T}_p$-convergent, and a $\mathfrak{T}_s$-compact set is $\mathfrak{T}_p$-compact.

(b) If $K \subseteq C_0(X)$ is $\mathfrak{T}_s$-compact, then $f[K] \subseteq \mathbb{R}$ must be compact, therefore bounded, for every $f \in C_0(X)^*$; by the Uniform Boundedness Theorem (3A5Hb), K is norm-bounded. Applying this to $\{u\} \cup \{u_n : n \in \mathbb{N}\}$, we see that any $\mathfrak{T}_s$-convergent sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ with limit u is norm-bounded.

(c) Suppose that $\langle u_n \rangle_{n\in\mathbb{N}}$ is norm-bounded and $\mathfrak{T}_p$-convergent to u. By Lebesgue's Dominated Convergence Theorem, $\lim_{n\to\infty} \int u_n d\mu = \int u\, d\mu$ for every totally finite Radon measure μ on X. But by the Riesz Representation Theorem (in the form 436K), this says just that $\lim_{n\to\infty} f(u_n) = f(u)$ for every positive linear functional f on $C_0(X)$. Since every member of $C_0(X)^*$ is expressible as the difference of two positive linear functionals (356Dc), $\lim_{n\to\infty} f(u_n) = f(u)$ for every $f \in U^*$, that is, $\langle u_n \rangle_{n\in\mathbb{N}}$ is $\mathfrak{T}_s$-convergent to u.

Putting this together with (a) and (b), we see that (i) is true.

(d) Now suppose that $K \subseteq C_0(X)$ is norm-bounded and $\mathfrak{T}_p$-compact. Then it is $\mathfrak{T}_p$-closed, therefore $\mathfrak{T}_s$-closed. Because $\mathfrak{T}_p$ is angelic (462B), any sequence $\langle u_n \rangle_{n\in\mathbb{N}}$ in K has a $\mathfrak{T}_p$-convergent subsequence (462Cc), which is also $\mathfrak{T}_s$-convergent, by (c). This means that K is (relatively) countably compact in $C_0(X)$ for the topology $\mathfrak{T}_s$. Since $\mathfrak{T}_s$ is angelic (462D), the $\mathfrak{T}_s$-closure of K is $\mathfrak{T}_s$-compact; but this is just K, so K is $\mathfrak{T}_s$-compact, as claimed.

462F Proposition Let X be a countably compact topological space. Then a subset of $C_b(X)$ is weakly compact iff it is norm-bounded and compact for the topology $\mathfrak{T}_p$ of pointwise convergence.

proof I should perhaps begin by remarking that, because X is countably compact, every continuous real-valued function on X is bounded (4A2G(f-v)), so that we have a natural norm $\|\ \|_\infty$ on $C(X) = C_b(X)$ (354Hb). A weakly compact subset of $C(X)$ is norm-bounded and $\mathfrak{T}_p$-compact by the same arguments as in (a)-(b) of the proof of 462E.

Suppose that $K \subseteq C_b(X)$ is norm-bounded and $\mathfrak{T}_p$-compact. We have a natural map $x \mapsto \hat{x} : X \to \mathbb{R}^K$ defined by writing $\hat{x}(u) = u(x)$ for every $u \in K$, $x \in X$. By the definition of $\mathfrak{T}_p$, $\hat{x} \in C(K)$ for every $x \in X$, if we interpret $C(K)$ as the space of $\mathfrak{T}_p$-continuous real-valued functions on K; and $x \mapsto \hat{x} : X \to C(K)$ is continuous for the given topology on X and the topology of pointwise convergence on $C(K)$ because $K \subseteq C(X)$. It follows that $Z = \{\hat{x} : x \in X\}$ is countably compact for the topology of pointwise convergence on $C(K)$ (4A2G(f-vi)). But now Z must be actually compact for the topology of pointwise convergence on $C(K)$, by 462B.

Next, consider the natural map $u \mapsto \hat{u} : K \to \mathbb{R}^Z$ defined by setting $\hat{u}(f) = f(u)$ for $f \in Z$, $u \in K$. Just as in the last paragraph, this is a continuous function from K to $C(Z)$, if we give K, Z and $C(Z)$ their topologies of pointwise convergence. So $L = \{\hat{u} : u \in K\}$ is compact for the topology of pointwise convergence on $C(Z)$. Moreover, it is norm-bounded, because

$$\sup_{\phi\in L} \|\phi\|_\infty = \sup_{u\in K, f\in Z} |\hat{u}(f)| = \sup_{u\in K, x\in X} |u(x)| = \sup_{u\in K} \|u\|_\infty$$

is finite. So 462E tells us that L is weakly compact in $C(Z)$. (Note that $C(Z) = C_0(Z)$ because Z is compact.)

Now we have an operator $T : C(Z) \to \mathbb{R}^X$ defined by setting

$$T(\phi)(x) = \phi(\hat{x})$$

for $\phi \in C(Z)$, $x \in X$. Because $x \mapsto \hat{x} : X \to Z$ is continuous, $T\phi \in C(X)$ for every $\phi \in C(Z)$, and of course T is norm-preserving. So $T[L]$ is weakly compact in $C(X)$ (2A5If). But if $u \in K$ and $x \in X$,

$$(T\hat{u})(x) = \hat{u}(\hat{x}) = \hat{x}(u) = u(x),$$

so $T\hat{u} = u$. Accordingly $K = T[L]$, and K is weakly compact, as claimed.

462G Theorem Let X be a countably compact topological space. Then the totally finite Radon measures on $C(X)$ are the same for the topology of pointwise convergence and the norm topology.

proof (a) As in 462F, $C(X) = C_b(X)$ is a Banach space under its norm $\|\ \|_\infty$. Write $\mathfrak{T}_p$ for the topology of pointwise convergence on $C(X)$ and $\mathfrak{T}_\infty$ for the norm topology. Because $\mathfrak{T}_p \subseteq \mathfrak{T}_\infty$, every totally finite $\mathfrak{T}_\infty$-Radon measure is $\mathfrak{T}_p$-Radon (418I).

(b) For the rest of the proof, therefore, I take a totally finite $\mathfrak{T}_p$-Radon measure μ on $C(X)$ and seek to prove that it is $\mathfrak{T}_\infty$-Radon. By 418L, applied to the identity map from $(C(X), \mathfrak{T}_\infty)$ to $(C(X), \mathfrak{T}_p)$, it will be enough to show that whenever $E \in \operatorname{dom}\mu$ and $\mu E > 0$ there is some $\mathfrak{T}_\infty$-compact set K such that $\mu(E \cap K) > 0$.

(c) I show first that if $E \in \operatorname{dom}\mu$ and $\mu E > 0$, and $\epsilon > 0$, there is some $g \in C(X)$ such that $\mu(E \cap B(g,\epsilon)) > 0$, where $B(g,\epsilon) = \{f : \|f - g\|_\infty \leq \epsilon\}$. **P?** Suppose, if possible, otherwise. Let $K \subseteq E$ be a non-empty $\mathfrak{T}_p$-compact set such that $\mu K > 0$. Since all the balls $B(0,k)$ are $\mathfrak{T}_p$-closed, we may suppose that K is $\|\ \|_\infty$-bounded, therefore weakly compact (462F). Choose $\langle K_n\rangle_{n\in\mathbb{N}}$ and $\langle f_n\rangle_{n\in\mathbb{N}}$ as follows. $K_0 = K$. Given that $K_n \subseteq E$ is non-negligible and $\mathfrak{T}_p$-compact, and that $\langle f_i\rangle_{i<n}$ is a finite sequence in $C(X)$, then the convex hull Γ_n of $\{f_i : i < n\}$ is $\mathfrak{T}_\infty$-compact, so there is a finite set $D_n \subseteq \Gamma_n$ such that for every $g \in \Gamma_n$ there is a $g' \in D_n$ such that $\|g - g'\|_\infty \leq \frac{1}{2}\epsilon$. Now

$$H_n = \{f : \|f - g\|_\infty > \epsilon \text{ for every } g \in D_n\}$$

is a $\mathfrak{T}_p$-open set and $E \setminus H_n$ is negligible, so $\mu(K_n \cap H_n) > 0$ and we can find a non-negligible $\mathfrak{T}_p$-closed set $K_{n+1} \subseteq K_n \cap H_n$; choose $f_n \in K_{n+1}$. Continue.

At the end of the induction, let f^* be any cluster point of $\langle f_n\rangle_{n\in\mathbb{N}}$ for the weak topology on $C(X)$. The set $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ is convex, so its norm-closure $\overline{\Gamma}$ is also convex (2A5Eb), therefore closed for the weak topology (3A5Ee), and contains f^*. So there is a $g \in \Gamma$ such that $\|f^* - g\|_\infty \leq \frac{1}{2}\epsilon$. Now there is some n such that $g \in \Gamma_n$. Let $g' \in D_n$ be such that $\|g - g'\|_\infty \leq \frac{1}{2}\epsilon$, so that $\|f^* - g'\| \leq \epsilon$. Since $f_i \in K_{n+1}$ for every $i \geq n$, and K_{n+1} is $\mathfrak{T}_p$-closed, therefore closed for the weak topology, $f^* \in K_{n+1} \subseteq H_n$, and $\|f^* - g'\| > \epsilon$, which is impossible. **XQ**

(d) What this means is that if we take $\mathcal{K}_n$ to be the family of subsets of $C(X)$ which can be covered by finitely many balls of radius at most 2^{-n}, then μ is inner regular with respect to $\mathcal{K}_n$ (see 412Aa), and therefore with respect to $\mathcal{K} = \bigcap_{n\in\mathbb{N}} \mathcal{K}_n$ (412Ac). But $\mathcal{K}$ is just the set of subsets of $C(X)$ which are totally bounded for the norm-metric ρ on $C(X)$. By 412Ac again, μ must be inner regular with respect to the family of $\mathfrak{T}_p$-closed ρ-totally bounded sets, which are surely $\mathfrak{T}_\infty$-closed therefore ρ-complete and $\mathfrak{T}_\infty$-compact. By 418L, μ is $\mathfrak{T}_\infty$-Radon.

462H Corollary Let X be a countably compact Hausdorff space, and give $C(X)$ its topology of pointwise convergence. If μ is any Radon measure on $C(X)$, it is inner regular with respect to the family of compact metrizable subsets of $C(X)$.

proof In the language of 462G, μ is inner regular with respect to the family of $\mathfrak{T}_\infty$-compact sets; but as $\mathfrak{T}_p \subseteq \mathfrak{T}_\infty$, the two topologies agree on all such sets, and they are compact and metrizable for $\mathfrak{T}_p$.

462I Proposition Let X be a topological space, Y a Hausdorff space, $f : X \times Y \to \mathbb{R}$ a bounded separately continuous function, and ν a totally finite Radon measure on Y. Set $\phi(x) = \int f(x,y)\nu(dy)$ for every $x \in X$. Then $\phi\restriction C$ is continuous for any relatively countably compact set $C \subseteq X$.

proof (a) To begin with, suppose that X itself is countably compact. For $y \in Y$, set $u_y(x) = f(x,y)$ for every $x \in X$. Then every u_y is continuous, because f is continuous in the first variable, and $y \mapsto u_y : Y \to C(X)$ is continuous, if we give $C(X)$ the topology $\mathfrak{T}_p$ of pointwise convergence, because f is continuous in the second variable. We therefore have a $\mathfrak{T}_p$-Radon image measure μ on $C(X)$, by 418I.

By 462G, μ is a Radon measure for the norm topology of $C(X)$. Now recall that f is bounded. If $|f(x,y)| \leq M$ for all $x \in X$, $y \in Y$, then $\|u_y\|_\infty \leq M$ for every $y \in Y$, and the ball $B(0,M)$ in $C(X)$ is μ-conegligible. By 461E, applied to the subspace measure on $B(0,M)$, μ has a barycenter h in $C(X)$. Now we can compute h by the formulae

$$h(x) = \int g(x)\mu(dg)$$

(because $g \mapsto g(x)$ belongs to $C(X)^*$)

$$= \int u_y(x)\nu(dy)$$

(by 235I)

$$= \int f(x,y)\nu(dy) = \phi(x),$$

for every $x \in X$. So $\phi = h$ is continuous.

(b) For the general result, we have only to apply (a) to $f\restriction C$ for each countably compact $C \subseteq X$.

462J Corollary Let X be a topological space such that

whenever $h \in \mathbb{R}^X$ is such that $h\restriction C$ is continuous for every relatively countably compact $C \subseteq X$, then h is continuous.

Write $\mathfrak{T}_p$ for the topology of pointwise convergence on $C(X)$. Let $K \subseteq C(X)$ be a $\mathfrak{T}_p$-compact set such that $\sup_{h\in K, x\in C} |h(x)|$ is finite for any countably compact set $C \subseteq X$. Then the $\mathfrak{T}_p$-closed convex hull of K, taken in $C(X)$, is $\mathfrak{T}_p$-compact.

proof The closed convex hull $\overline{\Gamma(K)}$ of K, taken in $\mathbb{R}^X$, is closed and included in a product of closed bounded intervals, therefore compact. If $h \in \overline{\Gamma(K)}$, then there is a Radon probability measure μ on K such that h is the barycenter of μ (461H), so that $h(x) = \int f(x)\mu(df)$ for every $x \in X$. If $C \subseteq X$ is countably compact, the map $(x, f) \mapsto f(x) : C \times K \to \mathbb{R}$ is separately continuous, because $K \subseteq C(X)$ is being given the topology of pointwise convergence, and is bounded by hypothesis. So 462I tells us that $x \mapsto \int f(x)\mu(df) : C \to \mathbb{R}$ is continuous, that is, that $h\restriction C$ is continuous. By the hypothesis on X here, h is continuous.

Thus the $\mathfrak{T}_p$-compact set $\overline{\Gamma(K)}$ is included in $C(X)$, and must be the closed convex hull of K in $C(X)$.

Remark The hypothesis

whenever $h \in \mathbb{R}^X$ is such that $h\restriction C$ is continuous for every relatively countably compact $C \subseteq X$, then h is continuous

is clumsy, but seems the best way to cover the large number of potential applications of the ideas here. Besides the obvious case of countably compact spaces X, we have all first-countable spaces (for which, of course, the other hypotheses can be relaxed, as in 462Xc), and all k-spaces. (A **k-space** is a topological space X such that a set $G \subseteq X$ is open iff $G \cap K$ is relatively open in K for every compact set $K \subseteq X$; see Engelking 89, 3.3.18 *et seq.* In particular, all locally compact spaces are k-spaces.) Perhaps the most important case of all is when X is the the dual U^* of a complete locally convex Hausdorff linear topological space U, endowed with the topology $\mathfrak{T}_s(U^*, U)$, so that U, with its weak topology $\mathfrak{T}_s(U, U^*)$, can be regarded as a linear subspace of $C(X)$ with its topology of pointwise convergence (see 4A4Ch).

462X Basic exercises (a) (i) Show that $\mathbb{R}$, with the right-facing Sorgenfrey topology, is angelic. (ii) Show that any metrizable space is angelic. (iii) Show that the one-point compactification of an angelic locally compact Hausdorff space is angelic. (iv) Find a first-countable regular Hausdorff space which is not angelic.

>(b) Let X be any countably compact topological space. Show that a norm-bounded sequence in $C_b(X)$ which is pointwise convergent is weakly convergent.

(c) Let X be a first-countable topological space, (Y, T, ν) a totally finite measure space, and $f : X \times Y \to \mathbb{R}$ a bounded function such that $y \mapsto f(x, y)$ is measurable for every $x \in X$, and $x \mapsto f(x, y)$ is continuous for almost every $y \in Y$. Show that $x \mapsto \int f(x, y)\nu(dy)$ is continuous.

(d) Give an example of a $\mathfrak{T}_p$-compact subset K of $C([0, 1])$ such that the convex hull of K is not relatively compact in $C([0, 1])$.

462Y Further exercises (a) Let X be a topological space such that there is a sequence $\langle X_n\rangle_{n\in\mathbb{N}}$ of countably compact subsets of X, covering X, with the property that a function $f : X \to \mathbb{R}$ is continuous whenever $f\restriction X_n$ is continuous for every $n \in \mathbb{N}$. Let $\mathfrak{T}_p$ be the topology of pointwise convergence on $C(X)$. Show that, for a set $K \subseteq C(X)$, the following are equiveridical: (i) $\phi[K]$ is bounded for every $\mathfrak{T}_p$-continuous function $\phi : C(X) \to \mathbb{R}$; (ii) whenever $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in K and $A \subseteq X$ is countable, there is a cluster point of $\langle f_n\restriction A\rangle_{n\in\mathbb{N}}$ in $C(A)$ for the topology of pointwise convergence on $C(A)$; (iii) K is relatively compact in $C(X)$ for $\mathfrak{T}_p$. (See Asanov & Velichko 81.)

(b) Let U be a metrizable locally convex linear topological space. Show that it is angelic in its weak topology. (*Hint*: start with the case in which U is complete, using Grothendieck's theorem and the full strength of 462B, with $X = U^*$.)

(c) In 462I, show that the conclusion remains valid for any totally finite τ-additive topological measure ν on Y which is inner regular with respect to the relatively countably compact subsets of Y.

(d) Show that if X is any compact topological space (more generally, any topological space such that X^n is Lindelöf for every $n \in \mathbb{N}$), then $C(X)$, with its topology of pointwise convergence, is countably tight.

462Z Problem Let K be a compact Hausdorff space. Is $C(K)$, with the topology of pointwise convergence, necessarily a pre-Radon space?

462 Notes and comments The theory of pointwise convergence in spaces of continuous functions is intimately connected with the theory of separately continuous functions of two variables. For if X and Y are topological spaces, and $f : X \times Y \to \mathbb{R}$ is any separately continuous function, then we have natural maps $x \mapsto f_x : X \to C(Y)$ and $y \mapsto f^y : Y \to C(X)$, writing $f_x(y) = f^y(x) = f(x, y)$, which are continuous if $C(X)$ and $C(Y)$ are given their topologies of pointwise convergence; and if X is a topological space and Y is any subset of $C(X)$ with its topology of pointwise convergence, the map $(x, y) \mapsto y(x) : X \times Y \to \mathbb{R}$ is separately continuous. I include a back-and-forth shuffle between $C(X)$ and separately continuous functions in 462G-462I-462J largely as a demonstration of the principle that all the theorems here can be expressed in both languages.

462Yb is a compendium of Šmulian's theorem with part of Eberlein's theorem; 462E and 462J can be thought of as the centre of Krein's theorem. There are many alternative routes to these results, which may be found in KÖTHE 69 or GROTHENDIECK 92. In particular, 462E can be proved without using measure theory; see, for instance, FREMLIN 74, A2F.

Topological spaces homeomorphic to compact subsets of $C(X)$, where X is some compact space and $C(X)$ is given its topology of pointwise convergence, are called **Eberlein compacta**; see 467O-467P.

A positive answer to A.Bellow's problem (463Za below) would imply a positive answer to 462Z; so if the continuum hypothesis, for instance, is true, then $C(K)$ is pre-Radon in its topology of pointwise convergence for any compact space K.

463 $\mathfrak{T}_p$ and $\mathfrak{T}_m$

We are now ready to start on the central ideas of this chapter with an investigation of sets of measurable functions which are compact for the topology of pointwise convergence. Because 'measurability' is, from the point of view of this topology on $\mathbb{R}^X$, a rather arbitrary condition, we are looking at compact subsets of a topologically irregular subspace of $\mathbb{R}^X$; there are consequently relatively few of them, and (under a variety of special circumstances, to be examined later in the chapter and also in Volume 5) they have some striking special properties.

The presentation here is focused on the relationship between the two natural topologies on any space of measurable functions, the 'pointwise' topology $\mathfrak{T}_p$ and the topology $\mathfrak{T}_m$ of convergence in measure (463A). In this section I begin with results which apply to any σ-finite measure space (463B-463G) before turning to some which apply to perfect measure spaces (463H-463K) – in particular, to Lebesgue measure. These lead to some interesting properties of separately continuous functions (463L-463M).

463A Notation Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0 = \mathcal{L}^0(\Sigma)$ the space of all Σ-measurable functions from X to $\mathbb{R}$, so that $\mathcal{L}^0$ is a linear subspace of $\mathbb{R}^X$. On $\mathcal{L}^0$ we shall be concerned with two very different topologies. The first is the topology $\mathfrak{T}_p$ of pointwise convergence (462Ab); the second is the topology $\mathfrak{T}_m$ of (local) convergence in measure (245Ab). Both are linear space topologies. **P** For $\mathfrak{T}_p$ I have already noted this in 462Ab. For $\mathfrak{T}_m$, repeat the argument of 245D; $\mathfrak{T}_m$ is defined by the functionals $f \mapsto \int_F \min(1, |f|)d\mu$, where $\mu F < \infty$, and these satisfy the criterion of 2A5B. **Q** $\mathfrak{T}_p$ is Hausdorff (3A3Id) and locally convex (4A4Ce); only in exceptional circumstances is either true of $\mathfrak{T}_m$. However, $\mathfrak{T}_m$ can easily

be pseudometrizable (if, for instance, μ is σ-finite, as in 245Eb), while $\mathfrak{T}_p$ is not, except in nearly trivial cases.

Associated with the topology of pointwise convergence on $\mathbb{R}^X$ is the usual topology of $\mathcal{P}X$ (definition: 4A2A); the map $\chi : \mathcal{P}X \to \mathbb{R}^X$ is a homeomorphism between $\mathcal{P}X$ and its image $\{0,1\}^X \subseteq \mathbb{R}^X$.

$\mathfrak{T}_m$ is intimately associated with the topology of convergence in measure on $L^0 = L^0(\mu)$ (§245). A subset of $\mathcal{L}^0$ is open for $\mathfrak{T}_m$ iff it is of the form $\{f : f^\bullet \in G\}$ for some open set $G \subseteq L^0$; consequently, a subset K of $\mathcal{L}^0$ is compact, or separable, for $\mathfrak{T}_m$ iff $\{f^\bullet : f \in K\}$ is compact or separable for the topology of convergence in measure on L^0.

It turns out that the identity map from $(\mathcal{L}^0, \mathfrak{T}_p)$ to $(\mathcal{L}^0, \mathfrak{T}_m)$ is sequentially continuous (463B). Only in nearly trivial cases is it actually continuous (463Xa(i)), and it is similarly rare for the reverse map from $(\mathcal{L}^0, \mathfrak{T}_m)$ to $(\mathcal{L}^0, \mathfrak{T}_p)$ to be continuous (463Xa(ii)). If, however, we relativise both topologies to a $\mathfrak{T}_p$-compact subset of $\mathcal{L}^0$, the situation becomes very different, and there are many important cases in which the topologies are comparable.

463B Lemma Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Then every pointwise convergent sequence in $\mathcal{L}^0$ is convergent in measure to the same limit.

proof 245Ca.

463C Proposition (Ionescu Tulcea 73) Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Write $\mathfrak{T}_p$, $\mathfrak{T}_m$ for the topologies of pointwise convergence and convergence in measure on $\mathcal{L}^0$; for $A \subseteq \mathcal{L}^0$, write $\mathfrak{T}_{pA}$, $\mathfrak{T}_{mA}$ for the corresponding subspace topologies.

(a) If $A \subseteq \mathcal{L}^0$ and $\mathfrak{T}_{pA}$ is metrizable, then the identity map from A to itself is $(\mathfrak{T}_{pA}, \mathfrak{T}_{mA})$-continuous.

(b) Suppose that μ is semi-finite. Then, for any $A \subseteq \mathcal{L}^0$, $\mathfrak{T}_{mA}$ is Hausdorff iff whenever f, g are distinct members of A the set $\{x : f(x) \neq g(x)\}$ is non-negligible.

(c) Suppose that $K \subseteq \mathcal{L}^0$ is such that $\mathfrak{T}_{pK}$ is compact and metrizable. Then $\mathfrak{T}_{pK} = \mathfrak{T}_{mK}$ iff $\mathfrak{T}_{mK}$ is Hausdorff.

(d) Suppose that μ is σ-finite, and that $K \subseteq \mathcal{L}^0$ is $\mathfrak{T}_p$-sequentially compact. Then $\mathfrak{T}_{pK} = \mathfrak{T}_{mK}$ iff $\mathfrak{T}_{mK}$ is Hausdorff, and in this case $\mathfrak{T}_{pK}$ is compact and metrizable.

(e) Suppose that $K \subseteq \mathcal{L}^0$ is such that $\mathfrak{T}_{pK}$ is compact and metrizable. Then whenever $\epsilon > 0$ and $E \in \Sigma$ is a non-negligible measurable set, there is a non-negligible measurable set $F \subseteq E$ such that $|f(x) - f(y)| \leq \epsilon$ whenever $f \in K$ and $x, y \in F$.

proof (a) All we need is to remember that sequentially continuous functions from metrizable spaces are continuous (4A2Ld), and apply 463B.

(b) $\mathfrak{T}_{mA}$ is Hausdorff iff for any distinct f, $g \in A$ there is a measurable set F of finite measure such that $\int_F \min(1, |f-g|)d\mu > 0$, that is, $\mu\{x : x \in F, f(x) \neq g(x)\} > 0$; because μ is semi-finite, this happens iff $\mu\{x : f(x) \neq g(x)\} > 0$.

(c) If $\mathfrak{T}_{pK} = \mathfrak{T}_{mK}$ then of course $\mathfrak{T}_{mK}$ is Hausdorff, because $\mathfrak{T}_{pK}$ is. If $\mathfrak{T}_{mK}$ is Hausdorff then the identity map $(K, \mathfrak{T}_{pK}) \to (K, \mathfrak{T}_{mK})$ is an injective function from a compact space to a Hausdorff space and (by (a)) is continuous, therefore a homeomorphism, so the two topologies are equal.

(d) If $\mathfrak{T}_{pK} = \mathfrak{T}_{mK}$ then $\mathfrak{T}_{mK}$ must be Hausdorff, just as in (c). So let us suppose that $\mathfrak{T}_{mK}$ is Hausdorff. Note that, by 245Eb, the topology of convergence in measure on L^0 is metrizable; in terms of $\mathcal{L}^0$, this says just that the topology of convergence in measure on $\mathcal{L}^0$ is pseudometrizable. So $\mathfrak{T}_{mK}$ is Hausdorff and pseudometrizable, therefore metrizable (4A2La).

We are told that any sequence in K has a $\mathfrak{T}_{pK}$-convergent subsequence. But this subsequence is now $\mathfrak{T}_{mK}$-convergent (463B), so $\mathfrak{T}_{mK}$ is sequentially compact; being metrizable, it is compact (4A2Lf). Moreover, the same is true of any $\mathfrak{T}_{pK}$-closed subset of K, so every $\mathfrak{T}_{pK}$-closed set is $\mathfrak{T}_{mK}$-compact, therefore $\mathfrak{T}_{mK}$-closed. Thus the identity map from $(K, \mathfrak{T}_{mK})$ to $(K, \mathfrak{T}_{pK})$ is continuous. Since $\mathfrak{T}_{mK}$ is compact and $\mathfrak{T}_{pK}$ is Hausdorff, the two topologies are equal; and, in particular, $\mathfrak{T}_{pK}$ is compact and metrizable.

(e) Let ρ be a metric on K inducing the topology $\mathfrak{T}_{pK}$. Let $D \subseteq K$ be a countable dense set. For each $n \in \mathbb{N}$, set

$$G_n = \{x : |f(x) - g(x)| \leq \tfrac{1}{3}\epsilon \text{ whenever } f, g \in D \text{ and } \rho(f,g) \leq 2^{-n}\}.$$

Because D is countable, G_n is measurable. Now $\bigcup_{n\in\mathbb{N}} G_n = X$. **P?** If $x \in X \setminus \bigcup_{n\in\mathbb{N}} G_n$, then for each $n \in \mathbb{N}$ we can find f_n, $g_n \in D$ such that $\rho(f_n, g_n) \leq 2^{-n}$ and $|f_n(x) - g_n(x)| \geq \frac{1}{3}\epsilon$. Because K is compact, there is a strictly increasing sequence $\langle n_k\rangle_{k\in\mathbb{N}}$ such that $\langle f_{n_k}\rangle_{k\in\mathbb{N}}$ and $\langle g_{n_k}\rangle_{k\in\mathbb{N}}$ are both convergent to f, g say. Now

$$\rho(f,g) = \lim_{k\to\infty} \rho(f_{n_k}, g_{n_k}) = 0, \quad |f(x) - g(x)| = \lim_{k\to\infty} |f_{n_k}(x) - g_{n_k}(x)| \geq \tfrac{1}{3}\epsilon,$$

so $f = g$ while $f(x) \neq g(x)$, which is impossible. **XQ**

There is therefore some $n \in \mathbb{N}$ such that $\mu(E \cap G_n) > 0$. Since K, being compact, is totally bounded for ρ, there is a finite set $D' \subseteq D$ such that every member of D is within a distance of 2^{-n} of some member of D'. Now there is a measurable set $F \subseteq E \cap G_n$ such that $\mu F > 0$ and $|g(x) - g(y)| \leq \frac{1}{3}\epsilon$ whenever $g \in D'$ and x, $y \in F$. So $|f(x) - f(y)| \leq \epsilon$ whenever $f \in D$ and x, $y \in F$. But as D is dense in K, $|f(x) - f(y)| \leq \epsilon$ whenever $f \in K$ and x, $y \in F$, as required.

463D Proposition Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Write $\mathfrak{T}_p$, $\mathfrak{T}_m$ for the topologies of pointwise convergence and convergence in measure on $\mathcal{L}^0$. Suppose that $K \subseteq \mathcal{L}^0$ is $\mathfrak{T}_p$-compact and that there is no $\mathfrak{T}_p$-continuous surjection from any closed subset of K onto ω_1+1 with its order topology. Then the identity map from $(K, \mathfrak{T}_p)$ to $(K, \mathfrak{T}_m)$ is continuous.

proof (a) It is worth noting straight away that $\xi \mapsto \chi\xi : \omega_1 + 1 \to \{0,1\}^{\omega_1}$ is a homeomorphism between $\omega_1 + 1$ and a subspace of $\{0,1\}^{\omega_1}$. So our hypothesis tells us that there is no continuous surjection from any closed subset of K onto $\{0,1\}^{\omega_1}$, and therefore none onto $\{0,1\}^A$ for any uncountable A.

(b) For the moment, fix on a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in K and a set $E \in \Sigma$ of finite measure. Set $q(x) = \max(1, 2\sup_{n\in\mathbb{N}} |f_n(x)|)$ for each $x \in X$. For any infinite $I \subseteq \mathbb{N}$, set

$$g_I = \sup_{n\in\mathbb{N}} \inf_{i\in I, i\geq n} f_i, \quad h_I = \inf_{n\in\mathbb{N}} \sup_{i\in I, i\geq n} f_i;$$

because $\sup_{f\in K} |f(x)|$ is surely finite for each $x \in X$, g_I and h_I are defined in $\mathcal{L}^0$, and $g_I \leq h_I$. For $f \in \mathcal{L}^0$ set $\tau'(f) = \int_E \min(1, |f|/q)$, and for $I \in [\mathbb{N}]^\omega$ (the set of infinite subsets of $\mathbb{N}$) set $\Delta(I) = \tau'(h_I - g_I)$. Since $h_I - g_I \leq q$, $\Delta(I) = \int_E (h_I - g_I)/q$. If I, $J \in [\mathbb{N}]^\omega$ and $J \setminus I$ is finite, then $g_I \leq g_J \leq h_J \leq h_I$, so $\Delta(J) \leq \Delta(I)$, with equality only when $g_I = g_J$ a.e. on E and $h_I = h_J$ a.e. on E.

(i) There is a $J \in [\mathbb{N}]^\omega$ such that $\Delta(I) = \Delta(J)$ for every $I \in [J]^\omega$. **P** For $J \in [\mathbb{N}]^\omega$, set $\underline{\Delta}(J) = \inf\{\Delta(I) : I \in [J]^\omega\}$. Choose $\langle I_n\rangle_{n\in\mathbb{N}}$ in $[\mathbb{N}]^\omega$ inductively in such a way that $I_{n+1} \subseteq I_n$ and $\Delta(I_{n+1}) \leq \underline{\Delta}(I_n) + 2^{-n}$ for every n. If we now set

$$J = \{\min\{i : i \in I_n, i \geq n\} : n \in \mathbb{N}\},$$

$J \subseteq \mathbb{N}$ will be an infinite set and $J \setminus I_n$ is finite for every n. If $I \in [J]^\omega$ then, for every n,

$$\Delta(J) \leq \Delta(I_{n+1}) \leq \underline{\Delta}(I_n) + 2^{-n} \leq \Delta(I_n \cap I) + 2^{-n} = \Delta(I) + 2^{-n};$$

as n is arbitrary, $\underline{\Delta}(J) = \Delta(J)$, as required. **Q**

Now for any $I \in [J]^\omega$ we have $\Delta(I) = \Delta(J)$, so that $g_I = g_J$ a.e. on E and $h_I = h_J$ a.e. on E.

(ii) Now $\Delta(J) = 0$. **P?** Otherwise, $F = \{x : x \in E, g_J(x) < h_J(x)\}$ has positive measure. Set $K_0 = \bigcap_{n\in\mathbb{N}} \overline{\{f_i : i \in J, i \geq n\}}$, the closure being taken for $\mathfrak{T}_p$, so that K_0 is $\mathfrak{T}_p$-compact. Let $\mathcal{A}$ be the family of sets $A \subseteq F$ such that whenever L, $M \subseteq A$ are finite and disjoint there is an $f \in K_0$ such that $f(x) = g_J(x)$ for $x \in L$ and $f(x) = h_J(x)$ for $x \in M$. Then $\mathcal{A}$ has a maximal member A_0 say. If we define $\phi : \mathcal{L}^0 \to [0,1]^{A_0}$ by setting $\phi(f)(x) = \min(h_J(x), \max(f(x), g_J(x)))/(h_J(x) - g_J(x))$ for $x \in A_0$ and $f \in \mathcal{L}^0$, $\phi[K_0]$ is a compact subset of $[0,1]^{A_0}$, and whenever L, $M \subseteq A_0$ are finite there is a $g \in \phi[K_0]$ such that $g(x) = 0$ for $x \in L$ and $g(x) = 1$ for $x \in M$. This means that $\phi[K_0] \cap \{0,1\}^{A_0}$ is dense in $\{0,1\}^{A_0}$ and must therefore be the whole of $\{0,1\}^{A_0}$. So $\{0,1\}^{A_0}$ is a continuous image of a closed subset of K.

By (a), A_0 is countable. For each pair L, M of disjoint finite subsets of A_0, we have a cluster point f_{LM} of $\langle f_j\rangle_{j\in J}$ such that $f_{LM}(x) = g_J(x)$ for $x \in L$ and $f_{LM}(x) = h_J(x)$ for $x \in M$. Let $I(L,M)$ be an infinite subset of J such that $\lim_{i\to\infty, i\in I(L,M)} f_i(x) = f_{LM}(x)$ for every $x \in A_0$. Then $g_{I(L,M)} = g_J$ and $h_{I(L,M)} = h_J$ almost everywhere on E. Because $\mu F > 0$ and A_0 has only countably many finite subsets,

there is a $y \in F$ such that $g_{I(L,M)}(y) = g_J(y)$ and $h_{I(L,M)}(y) = h_J(y)$ whenever L and M are disjoint finite subsets of A_0.

What this means is that if L and M are disjoint finite subsets of A_0, then there are infinite sets I', $I'' \subseteq I(L,M)$ such that $\lim_{i\to\infty, i\in I'} f_i(y) = g_J(y)$ and $\lim_{i\to\infty, i\in I''} f_i(y) = h_J(y)$; so that there are f', $f'' \in K_0$ such that

$$f'(x) = g_J(x) \text{ for } x \in L \cup \{y\}, \quad f'(x) = h_J(x) \text{ for } x \in M,$$

$$f''(x) = g_J(x) \text{ for } x \in L, \quad f''(x) = h_J(x) \text{ for } x \in M \cup \{y\}.$$

But this means that $A_0 \cup \{y\} \in \mathcal{A}$; and A_0 was chosen to be maximal. **XQ**

(iii) So $\Delta(J) = 0$ and $g_J = h_J$ almost everywhere on E. But if we enumerate J in ascending order as $\langle n_i\rangle_{i\in\mathbb{N}}$, $g_J = \liminf_{i\to\infty} f_{n_i}$ and $h_J = \limsup_{i\to\infty} f_{n_i}$, so $\langle f_{n_i}\rangle_{i\in\mathbb{N}}$ converges almost everywhere on E.

Thus we see that for every sequence in K, and every set of finite measure, there is a subsequence converging almost everywhere on that set.

(c) ? Suppose, if possible, that the identity map from $(K, \mathfrak{T}_p)$ to $(K, \mathfrak{T}_m)$ is not continuous at $f_0 \in K$. Then there are an $E \in \Sigma$, of finite measure, and an $\epsilon > 0$ such that $C = \{f : f \in K, \tau_E(f - f_0) \geq \epsilon\}$ meets every $\mathfrak{T}_p$-neighbourhood of f, where $\tau_E(f) = \int_E \min(1, |f|)$ for every $f \in \mathcal{L}^0$, and there is an ultrafilter $\mathcal{F}$ on $\mathcal{L}^0$ which contains C and converges to f_0 for $\mathfrak{T}_p$. Consider the map $\psi : \mathcal{L}^0 \to L^0(\mu_E)$, where μ_E is the subspace measure on E, defined by setting $\psi(f) = (f\restriction E)^\bullet$ for $f \in \mathcal{L}^0$. We know from (b) that every sequence in K has a subsequence convergent almost everywhere on E, so every sequence in $\psi[K]$ has a subsequence which is convergent for the topology of convergence in measure on $L^0(\mu_E)$. Since this is metrizable, $\psi[K]$ is relatively compact in $L^0(\mu_E)$ (4A2Le), and the image filter $\psi[[\mathcal{F}]]$ has a limit $v \in L^0(\mu_E)$. Let $f_1 \in \mathcal{L}^0$ be such that $\psi(f_1) = v$.

For any countable set $A \subseteq X$ there is a $g \in C$ such that $g\restriction A = f_0\restriction A$ and $g = f_1$ almost everywhere on E. **P** If $X = \emptyset$ this is trivial, so we may, if necessary, enlarge A by one point so that it is not empty. Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be a sequence running over A. Then for each $n \in \mathbb{N}$ the set

$$\{g : g \in C, |g(x_i) - f_0(x_i)| \leq 2^{-n} \text{ for every } i \leq n, \tau_E(g, f_1) \leq 2^{-n}\}$$

belongs to $\mathcal{F}$, so is not empty; take g_n in this set. Let $g \in K$ be any cluster point of $\langle g_n\rangle_{n\in\mathbb{N}}$. Since $\langle g_n\rangle_{n\in\mathbb{N}}$ converges to f_1 almost everywhere on E, $g = f_1$ a.e. on E. Since $\tau_E(g_n - f_0) \geq \epsilon$ for every n, $\tau_E(g - f_0) \geq \epsilon$ and $g \in C$. Since $\langle g_n(x_i)\rangle_{n\in\mathbb{N}}$ converges to $f_0(x_i)$ for every i, $g\restriction A = f_0\restriction A$. So we have the result. **Q**

In particular, there is a $g \in C$ such that $g =_{\text{a.e.}} f_1$, so $F = \{x : x \in E, f_0(x) \neq f_1(x)\}$ has non-zero measure. Now choose $\langle g_\xi\rangle_{\xi<\omega_1}$ in K and $\langle x_\xi\rangle_{\xi<\omega_1}$ in F inductively so that

$$g_\xi \in C, \quad g_\xi = f_1 \text{ almost everywhere on } E, \; g_\xi(x_\eta) = f_0(x_\eta) \text{ for } \eta < \xi$$

(choosing g_ξ),

$$x_\xi \in F, \quad g_\eta(x_\xi) = f_1(x_\xi) \text{ for } \eta < \xi$$

(choosing x_ξ). If we now set $A = \{x_\xi : \xi < \omega_1\}$,

$$K_1 = \textstyle\bigcap_{\xi<\eta<\omega_1} \{f : f \in K, \text{ either } f(x_\xi) = f_0(x_\xi) \text{ or } f(x_\xi) = f(x_\eta) = f_1(x_\eta)\},$$

then K_1 is a closed subset of K containing every g_ξ and also f_0. But if we look at $\{f\restriction A : f \in K_1\}$, this is homeomorphic to $\omega_1 + 1$; which is supposed to be impossible. **X**

So we conclude that the identity map from $(K, \mathfrak{T}_p)$ to $(K, \mathfrak{T}_m)$ is continuous.

463E Corollary Let (X, Σ, μ) be a measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Write $\mathfrak{T}_p$, $\mathfrak{T}_m$ for the topologies of pointwise convergence and convergence in measure on $\mathcal{L}^0$. Suppose that $K \subseteq \mathcal{L}^0$ is compact and countably tight for $\mathfrak{T}_p$. Then the identity map from $(K, \mathfrak{T}_p)$ to $(K, \mathfrak{T}_m)$ is continuous. If $\mathfrak{T}_m$ is Hausdorff on K, the two topologies coincide on K.

proof Since $\omega_1 + 1$ is not countably tight (the top point ω_1 is not in the closure of any countable subset of ω_1), $\omega_1 + 1$ is not a continuous image of any closed subset of K (4A2Kb), and we can apply 463D to see that the identity map is continuous. It follows at once that if $\mathfrak{T}_m$ is Hausdorff on K, then the topologies coincide.

463F Theorem (IONESCU TULCEA 74) Let (X, Σ, μ) be a σ-finite measure space, and K a convex set of measurable functions from X to $\mathbb{R}$ such that (i) K is compact for the topology $\mathfrak{T}_p$ of pointwise convergence (ii) $\{x : f(x) \neq g(x)\}$ is not negligible for any distinct $f, g \in K$. Then K is metrizable for $\mathfrak{T}_p$, which agrees with the topology of convergence in measure on K.

proof (a) Let $\langle f_n \rangle_{n\in\mathbb{N}}$ be a sequence in K. Then it has a pointwise convergent subsequence. **P** Because K is compact, we surely have $\sup_{f\in K} |f(x)| < \infty$ for every $x \in X$, so that $q_0 = \chi X + \sup_{n\in\mathbb{N}} |f_n|$ is defined in $\mathbb{R}^X$, and is measurable. Let $\langle X_k \rangle_{k\in\mathbb{N}}$ be a sequence of measurable sets of finite measure covering X, and set

$$q_1 = \textstyle\sum_{k=0}^{\infty} \frac{1}{2^k(1+\mu X_k)} \chi X_k,$$

so that q_1 is a strictly positive measurable function and is square-integrable.

Set $K' = \{f \times q_1/q_0 : f \in K\}$, so that K' is another convex pointwise compact set of measurable functions, this time all dominated by q_1, so that $K' \subseteq \mathcal{L}^2(\mu)$. Setting $g_n = f_n \times q_1/q_0$, the sequence $\langle g_n^\bullet \rangle_{n\in\mathbb{N}}$ of equivalence classes is a norm-bounded sequence in the Hilbert space $L^2 = L^2(\mu)$. It therefore has a weakly convergent subsequence $\langle g^\bullet_{n(i)} \rangle_{i\in\mathbb{N}}$ say (4A4Kb), with limit v.

? Suppose, if possible, that $\langle f_{n(i)} \rangle_{i\in\mathbb{N}}$ is not pointwise convergent. Then there must be some $x_0 \in X$ such that $\liminf_{i\to\infty} f_{n(i)}(x_0) < \limsup_{i\to\infty} f_{n(i)}(x_0)$; let $\alpha < \beta$ in $\mathbb{R}$ be such that $I = \{i : f_{n(i)}(x_0) \leq \alpha\}$, $I' = \{i : f_{n(i)}(x_0) \geq \beta\}$ are both infinite. In this case v belongs to the weak closures of both $D = \{g^\bullet_{n(i)} : i \in I\}$ and $D' = \{g^\bullet_{n(i)} : i \in I'\}$. It must therefore belong to the *norm* closures of their convex hulls $\Gamma(D)$, $\Gamma(D')$ (4A4Ed). Accordingly we can find $v_n \in \Gamma(D)$, $v'_n \in \Gamma(D')$ such that $\|v - v_n\|_2 \leq 3^{-n}$, $\|v - v'_n\|_2 \leq 3^{-n}$ for every $n \in \mathbb{N}$.

Setting $A = \{f_{n(i)} : i \in I\}$, $A' = \{f_{n(i)} : i \in I'\}$, we see that there must be $h_n \in \Gamma(A)$, $h'_n \in \Gamma(A')$ such that $v_n = (h_n \times q_1/q_0)^\bullet$, $v'_n = (h'_n \times q_1/q_0)^\bullet$ for every $n \in \mathbb{N}$. Now if $g : X \to \mathbb{R}$ is a measurable function such that $g^\bullet = v$, and $\tilde{h} = g \times q_0/q_1$, we have

$$\textstyle\int |g - (h_n \times q_1/q_0)|^2 \leq 9^{-n},$$

$$\begin{aligned}\mu\{x : |\tilde{h}(x) - h_n(x)| \geq 2^{-n} q_0(x)/q_1(x)\} &= \mu\{x : |g(x) - (h_n \times q_1/q_0)(x)| \geq 2^{-n}\} \\ &\leq 4^n \|v - v_n\|_2^2 \leq 2^{-n}\end{aligned}$$

for every $n \in \mathbb{N}$, and $h_n \to \tilde{h}$ a.e. Similarly, $h'_n \to \tilde{h}$ a.e.

At this point, recall that K is supposed to be convex, so all the h_n, h'_n belong to K. Let $\mathcal{F}$ be any non-principal ultrafilter on $\mathbb{N}$. Because K is pointwise compact, $h = \lim_{n\to\mathcal{F}} h_n$ and $h' = \lim_{n\to\mathcal{F}} h'_n$ are both defined in K for the topology of pointwise convergence. For any x such that $\lim_{n\to\infty} h_n(x) = \tilde{h}(x)$, we surely have $h(x) = \tilde{h}(x)$; so $h =_{\text{a.e.}} \tilde{h}$. Similarly, $h' =_{\text{a.e.}} \tilde{h}$, and $h =_{\text{a.e.}} h'$.

Now at last we apply the hypothesis that distinct members of K are not equal almost everywhere, to see that $h = h'$. But if we look at what happens at the distinguished point x_0 above, we see that $f(x_0) \leq \alpha$ for every $f \in A$, so that $f(x_0) \leq \alpha$ for every $f \in \Gamma(A)$, $h_n(x_0) \leq \alpha$ for every $n \in \mathbb{N}$, and $h(x_0) \leq \alpha$; and similarly $h'(x_0) \geq \beta$. So $h \neq h'$, which is absurd. **X**

This contradiction shows that $\langle f_{n(i)} \rangle_{i\in\mathbb{N}}$ is pointwise convergent, and is an appropriate subsequence. **Q**

(b) Now 463Cd tells us that K is metrizable for $\mathfrak{T}_p$, and that $\mathfrak{T}_p$ agrees on K with the topology of convergence in measure.

463G Corollary Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a σ-finite topological measure space in which μ is strictly positive. Suppose that

> whenever $h \in \mathbb{R}^X$ is such that $h\restriction C$ is continuous for every relatively countably compact $C \subseteq X$, then h is continuous.

If $K \subseteq C_b(X)$ is a norm-bounded $\mathfrak{T}_p$-compact set, then it is $\mathfrak{T}_p$-metrizable.

proof By 462J, the $\mathfrak{T}_p$-closed convex hull $\overline{\Gamma(K)}$ of K in $C_b(X)$ is $\mathfrak{T}_p$-compact. Because μ is strictly positive, $\mu\{x : f(x) \neq g(x)\} > 0$ whenever f and g are distinct continuous real-valued functions on X. So the result is immediate from 463F.

463H Lemma Let (X, Σ, μ) be a perfect probability space, and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence in Σ. Suppose that there is an $\epsilon > 0$ such that

$$\epsilon\mu F \leq \liminf_{n\to\infty} \mu(F \cap E_n) \leq \limsup_{n\to\infty} \mu(F \cap E_n) \leq (1-\epsilon)\mu F$$

for every $F \in \Sigma$. Then $\langle E_n\rangle_{n\in\mathbb{N}}$ has a subsequence $\langle E_{n_k}\rangle_{k\in\mathbb{N}}$ such that $\mu_* A = 0$ and $\mu^* A = 1$ for any cluster point A of $\langle E_{n_k}\rangle_{k\in\mathbb{N}}$ in $\mathcal{P}X$; in particular, $\langle E_{n_k}\rangle_{k\in\mathbb{N}}$ has no measurable cluster point.

proof (a) If we replace μ by its completion, we do not change μ^* and μ_* (212Ea, 413Eg), so we may suppose that μ is already complete.

(b) Suppose that $\langle E_n\rangle_{n\in\mathbb{N}}$ is actually stochastically independent, with $\mu E_n = \frac{1}{k}$ for every n, where $k \geq 2$ is an integer. In this case $\mu^* A = 1$ for any cluster point A of $\langle E_n\rangle_{n\in\mathbb{N}}$.

P (i) There is a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that $A = \lim_{n\to\mathcal{F}} E_n$ in $\mathcal{P}X$ (4A2Bc); that is, $\chi A(x) = \lim_{n\to\mathcal{F}} \chi E_n(x)$ for every $x \in X$; that is, $A = \{x : x \in X, \{n : x \in E_n\} \in \mathcal{F}\}$.

(ii) We have a measurable function $\phi : X \to Y = \{0,1\}^{\mathbb{N}}$ defined by setting $\phi(x)(n) = (\chi E_n)(x)$ for every $x \in X$, $n \in \mathbb{N}$. Because μ is complete and perfect and totally finite, and $\{0,1\}^{\mathbb{N}}$ is compact and metrizable, the image measure $\nu = \mu\phi^{-1}$ is a Radon measure (451N). By 451Oc, $\mu^*\phi^{-1}[B] = \nu^* B$ for every $B \subseteq Y$. Moreover, for any basic open set of the form $H = \{y : y(i) = \epsilon_i \text{ for every } i \leq n\}$, $\mu\phi^{-1}[H] = \tilde{\nu}H$, where $\tilde{\nu}$ is the product measure corresponding to the measure ν_0 on $\{0,1\}$ for which $\nu_0\{1\} = \frac{1}{k}$, $\nu_0\{0\} = \frac{k-1}{k}$. Since $\tilde{\nu}$ is also a Radon measure (416U), $\tilde{\nu} = \nu$ (415H(v)).

Set $B = \{y : y \in Y, \{n : y(n) = 1\} \in \mathcal{F}\}$, so that $\phi^{-1}[B] = A$ and B is determined by coordinates in $\{n, n+1, \dots\}$ for every $n \in \mathbb{N}$. By 451Oc, $\mu^* A = \nu^* B$; by the zero-one law (254Sa), $\nu^* B$ must be either 0 or 1. So $\mu^* A$ is either 0 or 1.

(iii) To see that $\mu^* A$ cannot be 0, we repeat the arguments of (ii) from the other side, as follows. Let λ_0 be the uniform probability measure on $\{0, 1, \dots, k-1\}$, giving measure $\frac{1}{k}$ to each point; let λ be the corresponding product measure on $Z = \{0, \dots, k-1\}^{\mathbb{N}}$. Let $\psi : Z \to Y$ be defined by setting $\psi(z)(n) = 1$ if $z(n) = 0$, 1 otherwise; then ψ is inverse-measure-preserving (254G). Since λ is a Radon measure and ψ is continuous, $\lambda\psi^{-1}$ is a Radon measure on Y and must be equal to ν. Accordingly $\nu^* B = \lambda^*\psi^{-1}[B]$, by 451Oc yet again or otherwise.

(iv) We have a measure space automorphism $\theta : Z \to Z$ defined by setting $\theta(z)(n) = z(n) +_k 1$ for every $z \in Z$, $n \in \mathbb{N}$, where $+_k$ is addition mod k. So, writing $C = \psi^{-1}[B]$, $\lambda^* C = \lambda^*\theta^i[C]$ for every $i \in \mathbb{N}$. Now, for $z \in Z$,

$$\{n : z(n) = 0\} \in \mathcal{F} \iff \{n : \psi(z)(n) = 1\} \in \mathcal{F} \iff \psi(z) \in B \iff z \in C.$$

But for any $z \in Z$, there is some $i < k$ such that $\{n : z(n) = i\} \in \mathcal{F}$, so that $\theta^{k-i}(z) \in C$. Thus $\bigcup_{i<k} \theta^i[C] = Z$ and $\sum_{i=0}^{k-1} \lambda^*\theta^i[C] \geq 1$ and $\lambda^* C > 0$. But this means that $\mu^* A = \nu^* B = \lambda^* C$ is non-zero, and $\mu^* A$ must be 1. **Q**

(c) Now return to the general case considered in (a). Note first that μ is atomless, because if $\mu F > 0$ there is some $n \in \mathbb{N}$ such that $0 < \mu(F \cap E_n) < \mu F$.

Let $k \geq 2$ be such that $\frac{1}{k} < \epsilon$. Then there are a strictly increasing sequence $\langle m(i)\rangle_{i\in\mathbb{N}}$ in $\mathbb{N}$ and a stochastically independent sequence $\langle F_i\rangle_{i\in\mathbb{N}}$ in Σ such that $F_i \subseteq E_{m(i)}$ and $\mu F_i = \frac{1}{k}$ for every $i \in \mathbb{N}$. **P** Choose $\langle m(i)\rangle_{i\in\mathbb{N}}$, F_i inductively, as follows. Let Σ_i be the (finite) algebra generated by $\{F_j : j < i\}$. Choose $m(i)$ such that $m(i) > m(j)$ for any $j < i$ and $\mu(F \cap E_{m(i)}) \geq \frac{1}{k}\mu F$ for every $F \in \Sigma_i$. List the atoms of Σ_i as $G_{i0}, \dots, G_{ip_i}$, and choose $F_{ir} \subseteq E_{m(i)} \cap G_{ir}$ such that $\mu F_{ir} = \frac{1}{k}\mu G_{ir}$, for each $r \leq p_i$; this is possible by 215D. Set $F_i = \bigcup_{r\leq p_i} F_{ir}$; then $\mu(F_i \cap F) = \frac{1}{k}\mu F$ for every $F \in \Sigma_i$. Continue. It is easy to check that $\mu(F_{i_1} \cap \dots \cap F_{i_r}) = 1/k^r$ whenever $i_1 < \dots < i_r$, so that $\langle F_i\rangle_{i\in\mathbb{N}}$ is stochastically independent. **Q**

If A is a cluster point of $\langle E_{m(i)}\rangle_{i\in\mathbb{N}}$, then there is a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that $A = \lim_{i\to\mathcal{F}} E_{m(i)}$ in $\mathcal{P}X$. In this case, $A \supseteq A'$, where $A' = \lim_{i\to\mathcal{F}} F_i$. But (b) tells us that $\mu^* A'$ must be 1, so $\mu^* A = 1$.

(d) Thus we have a subsequence $\langle E_{m(i)}\rangle_{i\in\mathbb{N}}$ of $\langle E_n\rangle_{n\in\mathbb{N}}$ such that any cluster point of $\langle E_{m(i)}\rangle_{i\in\mathbb{N}}$ has outer measure 1. But the same argument applies to $\langle X \setminus E_{m(i)}\rangle_{i\in\mathbb{N}}$ to show that there is a strictly increasing sequence $\langle i_k\rangle_{k\in\mathbb{N}}$ such that every cluster point of $\langle X \setminus E_{m(i_k)}\rangle_{k\in\mathbb{N}}$ has outer measure 1. Since complementation

is a homeomorphism of $\mathcal{P}X$, $\mu^*(X \setminus A) = 1$, that is, $\mu_* A = 0$, for every cluster point A of $\langle E_{m(i_k)}\rangle_{k\in\mathbb{N}}$. So if we set $n(k) = m(i_k)$, any cluster point of $\langle E_{n_k}\rangle_{k\in\mathbb{N}}$ will have inner measure 0 and outer measure 1, as claimed.

463I Lemma Let (X, Σ, μ) be a perfect probability space, and $\langle E_n\rangle_{n\in\mathbb{N}}$ a sequence in Σ. Then
either $\langle \chi E_n\rangle_{n\in\mathbb{N}}$ has a subsequence which is convergent almost everywhere
or $\langle E_n\rangle_{n\in\mathbb{N}}$ has a subsequence with no measurable cluster point in $\mathcal{P}X$.

proof Consider the sequence $\langle \chi E_n^\bullet\rangle_{n\in\mathbb{N}}$ in the Hilbert space $L^2 = L^2(\mu)$. This is a norm-bounded sequence, so has a weakly convergent subsequence $\langle \chi E_{n_i}^\bullet\rangle_{i\in\mathbb{N}}$ with limit v say (4A4Kb). Express v as $g^\bullet$ where $g : X \to \mathbb{R}$ is Σ-measurable.

case 1 Suppose that $g(x) \in \{0, 1\}$ for almost every $x \in X$; set $F = \{x : g(x) = 1\}$. Then

$$\lim_{i\to\infty} \int_F \chi E_{n_i} = \int_F g = \mu F, \quad \lim_{i\to\infty} \int_{X\setminus F} \chi E_{n_i} = \int_{X\setminus F} g = 0.$$

So, replacing $\langle \chi E_{n_i}\rangle_{i\in\mathbb{N}}$ with a sub-subsequence if necessary, we may suppose that

$$|\mu F - \int_F \chi E_{n_i}| \le 2^{-i}, \quad |\int_{X\setminus F} \chi E_{n_i}| \le 2^{-i}$$

for every i. But as $0 \le \chi E_{n_i} \le 1$ everywhere, we have

$$\int |\chi F - \chi E_{n_i}| \le 2^{-i+1}$$

for every i, so that $\chi E_{n_i} \to \chi F$ a.e., and we have a subsequence of $\langle \chi E_n\rangle_{n\in\mathbb{N}}$ which is convergent almost everywhere.

case 2 Suppose that $\{x : g(x) \notin \{0, 1\}\}$ has positive measure. Note that because $\int_F g = \lim_{i\to\infty} \mu(F \cap E_{n_i})$ lies between 0 and μF for every $F \in \Sigma$, $0 \le g \le 1$ a.e., and $\mu\{x : 0 < g(x) < 1\} > 0$. There is therefore an $\epsilon > 0$ such that $\mu G > 0$, where $G = \{x : \epsilon \le g(x) \le 1 - \epsilon\}$.

Write μ_G for the subspace measure on G, and Σ_G for its domain; set $\nu = (\mu G)^{-1}\mu_G$, so that ν is a probability measure. We know that μ_G is perfect (451Dc), so ν also is (see the definition in 451Ad). Now if $F \in \Sigma_G$,

$$\lim_{i\to\infty} \nu(F \cap E_{n_i}) = (\mu G)^{-1} \int_F g$$

lies between $\epsilon\mu F/\mu G = \epsilon\nu F$ and $(1 - \epsilon)\nu F$.

By 463H, there is a strictly increasing sequence $\langle i(k)\rangle_{k\in\mathbb{N}}$ such that $B \notin \Sigma_G$ whenever B is a cluster point of $\langle G \cap E_{n_{i(k)}}\rangle_{k\in\mathbb{N}}$ in $\mathcal{P}G$. If A is any cluster point of $\langle E_{n_{i(k)}}\rangle_{k\in\mathbb{N}}$ in $\mathbb{R}^X$, then $A \cap G$ is a cluster point of $\langle G \cap E_{n_{i(k)}}\rangle_{k\in\mathbb{N}}$ in $\mathcal{P}G$, so cannot belong to Σ_G. Thus $A \notin \Sigma$.

So in this case we have a subsequence $\langle E_{n_{i(k)}}\rangle_{k\in\mathbb{N}}$ of $\langle E_n\rangle_{n\in\mathbb{N}}$ which has no measurable cluster point.

463J Fremlin's Alternative (FREMLIN 75A) Let (X, Σ, μ) be a perfect σ-finite measure space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a sequence of real-valued measurable functions on X. Then
either $\langle f_n\rangle_{n\in\mathbb{N}}$ has a subsequence which is convergent almost everywhere
or $\langle f_n\rangle_{n\in\mathbb{N}}$ has a subsequence with no measurable cluster point in $\mathbb{R}^X$.

proof (a) If $\mu X = 0$ then of course $\langle f_n\rangle_{n\in\mathbb{N}}$ itself is convergent a.e., so we may suppose that $\mu X > 0$. If there is any $x \in X$ such that $\sup_{n\in\mathbb{N}} |f_n(x)| = \infty$, then $\langle f_n\rangle_{n\in\mathbb{N}}$ has a subsequence with no cluster point in $\mathbb{R}^X$, measurable or otherwise; so we may suppose that $\langle f_n\rangle_{n\in\mathbb{N}}$ is bounded at each point of X.

(b) Let λ be the c.l.d. product of μ with Lebesgue measure on $\mathbb{R}$, and Λ its domain. Then λ is perfect (451Ic) and also σ-finite (251K). There is therefore a probability measure ν on Y with the same domain and the same negligible sets as λ (215B(vii)), so that ν also is perfect. For any function $h \in \mathbb{R}^X$, write

$$\Omega(h) = \{(x, \alpha) : x \in X, \alpha \le h(x)\} \subseteq X \times \mathbb{R}$$

(compare 252N).

(c) By 463I, applied to the measure space $(X \times \mathbb{R}, \Lambda, \nu)$ and the sequence $\langle \chi\Omega(f_n)\rangle_{n\in\mathbb{N}}$, we have a strictly increasing sequence $\langle n(i)\rangle_{i\in\mathbb{N}}$ such that either $\langle \chi\Omega(f_{n(i)})\rangle_{i\in\mathbb{N}}$ is convergent ν-a.e. or $\langle \Omega(f_{n(i)})\rangle_{i\in\mathbb{N}}$ has no cluster point in Λ.

case 1 Suppose that $\langle \chi\Omega(f_{n(i)})\rangle_{i\in\mathbb{N}}$ is convergent ν-a.e. Set

$$W = \{(x,\alpha) : \lim_{i\to\infty} \chi\Omega(f_{n(i)})(x,\alpha) \text{ is defined}\}.$$

Then W is λ-conegligible, so $W^{-1}[\{\alpha\}] = \{x : (x,\alpha)\in W\}$ is μ-conegligible for almost every α (252D). Set $D = \{\alpha : W^{-1}[\{\alpha\}] \text{ is } \mu\text{-conegligible}\}$, and let $Q\subseteq D$ be a countable dense set; then $G = \bigcap_{\alpha\in Q} W^{-1}[\{\alpha\}]$ is μ-conegligible. But if $x\in G$, then for any $\alpha\in Q$ the set $\{i : f_{n(i)}(x)\geq\alpha\} = \{i : \chi\Omega(f_{n(i)})(x,\alpha) = 1\}$ is either finite or has finite complement in $\mathbb{N}$, so $\langle f_{n(i)}(x)\rangle_{i\in\mathbb{N}}$ must be convergent in $[-\infty,\infty]$. Since $\langle f_n(x)\rangle_{n\in\mathbb{N}}$ is supposed to be bounded, $\langle f_{n(i)}(x)\rangle_{i\in\mathbb{N}}$ is convergent in $\mathbb{R}$. Thus in this case we have an almost-everywhere-convergent subsequence of $\langle f_n\rangle_{n\in\mathbb{N}}$.

case 2 Suppose that $\langle\Omega(f_{n(i)})\rangle_{i\in\mathbb{N}}$ has no cluster point in Λ. Let h be any cluster point of $\langle f_{n(i)}\rangle_{i\in\mathbb{N}}$ in $\mathbb{R}^X$. Then there is a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that $h = \lim_{i\to\mathcal{F}} f_{n(i)}$ in $\mathbb{R}^X$. Set $A = \lim_{i\to\mathcal{F}}\Omega(f_{n(i)})$, so that $A\notin\Lambda$. If $x\in X$ and $\alpha\in\mathbb{R}$, then

$$\alpha < h(x) \Longrightarrow \{i : \alpha < f_{n(i)}(x)\}\in\mathcal{F} \Longrightarrow (x,\alpha)\in A,$$

$$h(x) < \alpha \Longrightarrow \{i : \alpha < f_{n(i)}(x)\}\notin\mathcal{F} \Longrightarrow (x,\alpha)\notin A.$$

Thus $\Omega'(h)\subseteq A\subseteq\Omega(h)$, where $\Omega'(h) = \{(x,\alpha) : \alpha < h(x)\}$.

? If h is Σ-measurable, then

$$\Omega'(h) = \textstyle\bigcup_{q\in\mathbb{Q}}\{x : h(x) > q\}\times\,]-\infty,q],$$

$$\Omega(h) = (X\times\mathbb{R})\setminus\textstyle\bigcup_{q\in\mathbb{Q}}\{x : h(x) < q\}\times[q,\infty[$$

belong to Λ, and $\lambda(\Omega(h)\setminus\Omega'(h)) = 0$ (because every vertical section of $\Omega(h)\setminus\Omega'(h)$ is negligible). But as $\Omega'(h)\subseteq A\subseteq\Omega(h)$, $A\in\Lambda$ (remember that product measures in this book are complete), which is impossible. **X**

Thus h is not Σ-measurable. As h is arbitrary, $\langle f_{n(i)}\rangle_{i\in\mathbb{N}}$ has no measurable cluster point in $\mathbb{R}^X$.

So at least one of the envisaged alternatives must be true.

463K Corollary Let (X,Σ,μ) be a perfect σ-finite measure space. Write $\mathcal{L}^0\subseteq\mathbb{R}^X$ for the space of real-valued Σ-measurable functions on X.

(a) If $K\subseteq\mathcal{L}^0$ is relatively countably compact for the topology $\mathfrak{T}_p$ of pointwise convergence on $\mathcal{L}^0$, then every sequence in K has a subsequence which is convergent almost everywhere. Consequently K is relatively compact in $\mathcal{L}^0$ for the topology $\mathfrak{T}_m$ of convergence in measure.

(b) If $K\subseteq\mathcal{L}^0$ is countably compact for $\mathfrak{T}_p$, then it is compact for $\mathfrak{T}_m$.

(c) Suppose that $K\subseteq\mathcal{L}^0$ is countably compact for $\mathfrak{T}_p$ and that $\mu\{x : f(x)\neq g(x)\} > 0$ for any distinct $f, g\in K$. Then the topologies $\mathfrak{T}_m$, $\mathfrak{T}_p$ agree on K, so both are compact and metrizable.

proof (a) Since every sequence in K must have a $\mathfrak{T}_p$-cluster point in $\mathcal{L}^0$, 463J tells us that every sequence in K has a subsequence which is convergent almost everywhere, therefore $\mathfrak{T}_m$-convergent. Now K is relatively sequentially compact in the pseudometrizable space $(\mathcal{L}^0,\mathfrak{T}_m)$, therefore relatively compact (4A2Le).

(b) As in (a), every sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in K has a subsequence $\langle g_n\rangle_{n\in\mathbb{N}}$ which is convergent almost everywhere. But $\langle g_n\rangle_{n\in\mathbb{N}}$ has a $\mathfrak{T}_p$-cluster point g in K, and now $g(x) = \lim_{n\to\infty} g_n(x)$ for every x for which the limit is defined; accordingly $g_n\to g$ a.e., and g is a $\mathfrak{T}_m$-limit of $\langle g_n\rangle_{n\in\mathbb{N}}$ in K. Thus every sequence in K has a $\mathfrak{T}_m$-cluster point in K, and (because $\mathfrak{T}_m$ is pseudometrizable) K is $\mathfrak{T}_m$-compact.

(c) The point is that K is sequentially compact under $\mathfrak{T}_p$. **P** Note that as K is countably compact, $\sup_{f\in K}|f(x)|$ is finite for every $x\in K$. (I am passing over the trivial case $K = \emptyset$.) If $\langle f_n\rangle_{n\in\mathbb{N}}$ is a sequence in K, then, by (b), it has a subsequence $\langle g_n\rangle_{n\in\mathbb{N}}$ which is convergent for the pseudometrizable topology $\mathfrak{T}_m$. Now $\langle g_n\rangle_{n\in\mathbb{N}}$ has a further subsequence $\langle h_n\rangle_{n\in\mathbb{N}}$ which is convergent a.e., by 245K. **?** If $\langle h_n\rangle_{n\in\mathbb{N}}$ is not $\mathfrak{T}_p$-convergent, then there are a point $x_0\in X$ and two further subsequences $\langle h'_n\rangle_{n\in\mathbb{N}}$, $\langle h''_n\rangle_{n\in\mathbb{N}}$ of $\langle h_n\rangle_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} h'_n(x_0)$, $\lim_{n\to\infty} h''_n(x_0)$ exist and are different. Now $\langle h'_n\rangle_{n\in\mathbb{N}}$, $\langle h''_n\rangle_{n\in\mathbb{N}}$ must have cluster points h', $h''\in K$ with $h'(x_0)\neq h''(x_0)$.

However,

$$h'(x) = \lim_{n\to\infty} h_n(x) = h''(x)$$

whenever the limit is defined, which is almost everywhere; so $h' =_{\text{a.e.}} h''$. And this contradicts the hypothesis that if two elements of K are equal a.e., they are identical. **X** Thus $\langle h_n\rangle_{n\in\mathbb{N}}$ is a $\mathfrak{T}_p$-convergent subsequence of $\langle f_n\rangle_{n\in\mathbb{N}}$. As $\langle f_n\rangle_{n\in\mathbb{N}}$ is arbitrary, K is $\mathfrak{T}_p$-sequentially compact. **Q**

Now 463Cd gives the result.

463L Proposition Let $X_0, \dots, X_n$ be compact Hausdorff spaces, each carrying a strictly positive Radon measure. Let X be their product and $\mathcal{B}\mathfrak{a}(X)$ the Baire σ-algebra of X. Then any separately continuous function $f : X \to \mathbb{R}$ is $\mathcal{B}\mathfrak{a}(X)$-measurable.

proof For $i \le n$ let μ_i be a strictly positive Radon measure on X_i; let μ be the product Radon measure on X.

(a) The proof relies on the fact that

($*$) if $g, g' : X \to \mathbb{R}$ are distinct separately continuous functions, then $\mu\{x : g(x) \ne g'(x)\} > 0$;

I seek to prove this, together with the stated result, by induction on n. The induction starts easily with $n = 0$, so that X can be identified with X_0, a separately continuous function on X is just a continuous function on X_0, and ($*$) is true because $\mu = \mu_0$ is strictly positive.

(b) For the inductive step to $n+1$, given a separately continuous function $f : X_0 \times \dots \times X_{n+1} \to \mathbb{R}$, set $f_t(y) = f(y,t)$ for every $y \in Y = X_0 \times \dots \times X_n$ and $t \in X_{n+1}$. Then every f_t is separately continuous, therefore $\mathcal{B}\mathfrak{a}(Y)$-measurable, by the inductive hypothesis. Moreover, again because f is separately continuous, the function $t \mapsto f_t(y)$ is continuous for every y, that is, $t \mapsto f_t : X_{n+1} \to \mathbb{R}^Y$ is continuous; and finally, by the inductive hypothesis ($*$), $\nu\{y : f_t(y) \ne f_{t'}(y)\} > 0$ whenever $t, t' \in X_{n+1}$ and $f_t \ne f_{t'}$, where ν is the product Radon measure on Y.

Since ν, being a Radon measure, is perfect (416W), we can apply 463Kc to see that $K = \{f_t : t \in X_{n+1}\}$ is metrizable for the topology of pointwise convergence. Let ρ be a metric on K inducing its topology, and $\langle g_i\rangle_{i\in\mathbb{N}}$ a sequence running over a dense subset of K. (I am passing over the trivial case $K = \emptyset = X_{n+1}$.) For $m, i \in \mathbb{N}$ set

$$E_{mi} = \{t : \rho(f_t, g_i) \le 2^{-m},\ \rho(f_t, g_j) > 2^{-m} \text{ for } j < i\},$$

so that every E_{mi} belongs to $\mathcal{B}\mathfrak{a}(X_{n+1})$. Set $f^{(m)}(y,t) = g_i(y)$ for $t \in E_{mi}$. Then $f^{(m)} : X \to \mathbb{R}$ is $\mathcal{B}\mathfrak{a}(X)$-measurable because every g_i is $\mathcal{B}\mathfrak{a}(Y)$-measurable, every E_{mi} belongs to $\mathcal{B}\mathfrak{a}(X_{m+1})$, and $\mathcal{B}\mathfrak{a}(X) = \mathcal{B}\mathfrak{a}(Y)\widehat{\otimes}\mathcal{B}\mathfrak{a}(X_{n+1})$ (4A3Od). Also $\langle f^{(m)}\rangle_{m\in\mathbb{N}} \to f$ at every point, because $\rho(f_t^{(m)}, f_t) \le 2^{-m}$ for every $m \in \mathbb{N}$ and $t \in X_{n+1}$. So f is $\mathcal{B}\mathfrak{a}(X)$-measurable.

(c) We still have to check that ($*$) is true at the new level. But if $h, h' : X \to \mathbb{R}$ are distinct separately continuous functions, then there are $t_0 \in X_{n+1}$, $y_0 \in Y$ such that $h(y_0, t_0) \ne h'(y_0, t_0)$. Let G be an open set containing t_0 such that $h(y_0, t) \ne h'(y_0, t)$ whenever $t \in G$. Then $\nu\{y : h(y,t) \ne h'(y,t)\} > 0$ for every $t \in G$, by the inductive hypothesis, so

$$\mu\{(y,t) : h(y,t) \ne h'(y,t)\} = \int \nu\{y : h(y,t) \ne h'(y,t)\}\mu_{n+1}(dt) > 0$$

because μ_{n+1} is strictly positive. Thus the induction continues.

463M Corollary Let $X_0, \dots, X_n$ be Hausdorff spaces with product X, and μ any Radon measure on X; let Σ be the domain of μ. Then every separately continuous function $f : X \to \mathbb{R}$ is Σ-measurable.

proof (a) Suppose first that there is a conegligible compact set $C \subseteq X$. For each $i \le n$, let $\pi_i : X \to X_i$ be the coordinate projection, and $\mu_i = \mu\pi_i^{-1}$ the image Radon measure; let Z_i be the support of μ_i and $Z = \prod_{i\le n} Z_i$. Note that every Z_i is a closed subset of $\pi_i[C]$, so that Z_i is compact and $C \subseteq Z$. By 463L, $f\restriction Z$ is $\mathcal{B}\mathfrak{a}(Z)$-measurable; because Z is conegligible, f is Σ-measurable.

(b) In general, if $C \subseteq X$ is compact, then we can apply (a) to the measure $E \mapsto \mu(E \cap C)$ to see that $f\restriction C$ is measurable. As μ is complete and locally determined and tight (that is, inner regular with respect to the compact sets), f is measurable (see 412Ja).

***463N** There are important open questions concerning the possibility of extending the results of 463K to non-perfect measure spaces (see 463Za-463Zb). I give here one curious fact which is relevant to a question which arises naturally in the next section.

Proposition Let (X, Σ, μ) be a σ-finite measure space, and $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X. Suppose that $K \subseteq \mathcal{L}^0$ is an infinite set, compact for the topology $\mathfrak{T}_p$ of pointwise convergence, and that $\mu\{x : f(x) \neq g(x)\} > 0$ for any distinct f, $g \in K$. Then there is a sequence of distinct elements of K which is $\mathfrak{T}_p$-convergent.

proof (a) To begin with (down to the end of (c) below) I will suppose that $\mu X = 1$ and that $0 \leq f \leq \chi X$ for every $f \in K$.

For any set $A \subseteq K$, define $\Delta(A)$ as follows. If $A = \emptyset$, then $\Delta(A) = 0$. Otherwise, set $\Delta(A) = \underline{\int}(\sup A - \inf A)d\mu$, where $\sup A$ and $\inf A$ are taken in $\mathbb{R}^X$, and $\underline{\int} d\mu$ is the lower integral (133I). (There is no suggestion at this point that $\sup A$ or $\inf A$ should be measurable.) Clearly $\Delta(A) \leq \Delta(B)$ whenever $A \subseteq B \subseteq K$. If A has more than one member, then $\Delta(A) > 0$, because two elements of K which are equal a.e. are identical; this is where I use the hypothesis that K is separated by the topology of convergence in measure.

(b) ? Now suppose, if possible, that there is no non-trivial convergent sequence in K. (Here, and henceforth, all topological terms will refer to the topology $\mathfrak{T}_p$.) Then there is an infinite closed set $K_1 \subseteq K$ such that $\Delta(L) = \Delta(K_1)$ for every infinite closed set $L \subseteq K_1$. **P** Write $\mathcal{L}$ for the family of infinite closed subsets of K. For $L \in \mathcal{L}$, set $\underline{\Delta}(L) = \inf\{\Delta(L') : L' \in \mathcal{L}, L' \subseteq L\}$. Choose $\langle L_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{L}$ inductively, as follows. $L_0 = K$. Given L_n, let $L_{n+1} \in \mathcal{L}$ be such that $L_{n+1} \subseteq L_n$ and $\Delta(L_{n+1}) \leq \underline{\Delta}(L_n) + 2^{-n}$; continue. At the end of the induction, set $K_1 = \bigcap_{n\in\mathbb{N}} L_n$. Because there are supposed to be no convergent sequences in K, K_1 must be infinite (4A2G(h-i)). Now

$$\Delta(K_1) \leq \inf_{n\in\mathbb{N}} \Delta(L_n) \leq \sup_{n\in\mathbb{N}} \underline{\Delta}(L_n) \leq \underline{\Delta}(K_1) \leq \Delta(K_1),$$

so $\Delta(L) = \Delta(K_1)$ whenever $L \subseteq K_1$ and $L \in \mathcal{L}$, as required. **Q**

(c) Now there is no non-trivial convergent sequence in K_1, so K_1 cannot be scattered (4A2G(h-ii)), and it must have an infinite closed subset K_2 without isolated points which has a countable π-base $\mathcal{U}$ say (4A2G(i-iii)). We may suppose that $\emptyset \notin \mathcal{U}$. Set $g_0 = \inf K_2$, $g_1 = \sup K_2$ in $\mathbb{R}^X$. Then g_0 and g_1 are measurable. **P** For each non-empty $U \in \mathcal{U}$, choose $h_U \in U$. Then $g_0' = \inf_{U\in\mathcal{U}} h_U$ is measurable, and $g_0 \leq g_0'$. But, for any $t \in X$, $\{f : f \in K_2, f(t) < g_0'(t)\}$ is a relatively open set in K_2, not containing any h_U and therefore not including any member of $\mathcal{U}$, and must accordingly be empty. This means that $g_0' \leq f$ for every $f \in K_2$ and $g_0 = g_0'$ is measurable. Similarly, $g_1 = \sup_{U\in\mathcal{U}} h_U$ is measurable. **Q**

Since

$$\int g_1 - g_0 = \underline{\int} g_1 - g_0 = \Delta(K_2) > 0,$$

there is a rational number α such that $E = \{t : g_0(t) < \alpha < g_1(t)\}$ is non-negligible. For $t \in E$, the set $\{f : f \in K_2, f(t) < \alpha\}$ is a non-empty relatively open set in K_2, so includes some member U_t of $\mathcal{U}$. Because $\mathcal{U}$ is countable, there is a $U \in \mathcal{U}$ such that $D = \{t : t \in E, U_t = U\}$ is non-negligible. But now observe that $f(t) < \alpha$ whenever $f \in U$ and $t \in D$, so $f(t) \leq \alpha$ whenever $f \in \overline{U}$ and $t \in D$. Now there is a measurable function h such that $h \leq \sup\overline{U} - \inf\overline{U}$ and $\int h = \Delta(\overline{U})$ (133J(a-ii)). In this case $h \leq g_1 - g_0$ and also $\{t : h(t) < g_1(t) - g_0(t)\} \supseteq D$ is non-negligible, so that

$$\Delta(\overline{U}) = \int h < \int g_1 - g_0 = \Delta(K_2).$$

On the other hand, U is infinite, because K_2 has no isolated points, so $\overline{U}$ is an infinite closed subset of K_2 with $\Delta(\overline{U}) < \Delta(K_2)$; which contradicts the choice of K_2. **X**

(d) This shows that, at least in the case in which $\mu X = 1$ and $K \subseteq [0,1]^X$, K has a non-trivial convergent sequence. For the general case, observe that μX is certainly non-zero (since K contains functions which are not equal a.e.), so there is a probability measure ν with the same measurable sets and the same negligible sets (215B(vii)). Moreover, if we set $q(f)(t) = \frac{1}{2}(1 + \frac{f(t)}{1+|f(t)|})$ for $f \in \mathbb{R}^X$ and $t \in X$, then $K' = \{q(f) : f \in K\}$ is homeomorphic to K, and it is still true that $\nu\{t : f(t) \neq g(t)\} > 0$ for all distinct f, $g \in K'$, while K' is bounded by the constant functions $\mathbf{0}$ and χX. So, applying (a)-(c) to ν and K', we see that K' has a non-trivial convergent sequence and therefore K also does.

***463O Corollary** (Talagrand 80) Under the hypotheses of 463N, K cannot be homeomorphic to $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$.

proof For there are no non-trivial convergent sequences in $\beta\mathbb{N}$ (4A2I(b-iv)).

463X Basic exercises >(a) Let (X, Σ, μ) be a measure space, $\mathcal{L}^0$ the space of Σ-measurable real-valued functions on X, $\mathfrak{T}_p$ the topology of pointwise convergence on $\mathcal{L}^0$ and $\mathfrak{T}_m$ the topology of convergence in measure on $\mathcal{L}^0$. (i) Show that $\mathfrak{T}_m \subseteq \mathfrak{T}_p$ iff for every measurable set E of finite measure there is a countable set $D \subseteq E$ such that $\mu^* D = \mu E$. (ii) Show that $\mathfrak{T}_p \subseteq \mathfrak{T}_m$ iff $0 < \mu^*\{x\} < \infty$ for every $x \in X$.

(b) Let (X, Σ, μ) be a σ-finite measure space, and $K \subseteq \mathcal{L}^0(\Sigma)$ a $\mathfrak{T}_p$-countably compact set. Show that the following are equiveridical: (i) every sequence in K has a subsequence which converges almost everywhere; (ii) K is $\mathfrak{T}_m$-compact; (iii) K is totally bounded for the uniformity associated with the linear space topology $\mathfrak{T}_m$. Show that if moreover the topology on K induced by $\mathfrak{T}_m$ is Hausdorff, then K is $\mathfrak{T}_p$-metrizable.

(c) Let (X, Σ, μ) be a complete locally determined measure space, and $K \subseteq \mathcal{L}^0(\Sigma)$ a set such that whenever $E \in \Sigma$ is a non-negligible measurable set and $\epsilon > 0$, there is a non-negligible measurable set $F \subseteq E$ such that $|f(x) - f(y)| \leq \epsilon$ whenever $f \in K$ and $x, y \in F$. Show that $\mathfrak{T}_p$ and $\mathfrak{T}_m$ agree on K, and that if $\sup_{f \in K} |f(x)|$ is finite for every $x \in X$ then K is relatively compact in $\mathcal{L}^0$ for both topologies.

(d) (i) Show that there is a set of Borel measurable functions on $[0, 1]$ which countably tight, compact and non-metrizable for the topology of pointwise convergence. (ii) Show that there is a strictly localizable measure space (X, Σ, μ) with a set K of measurable functions which is countably tight, compact, Hausdorff and non-metrizable for both the topology of pointwise convergence and the topology of convergence in measure. (*Hint*: the one-point compactification of any discrete space is countably tight.)

(e) Let X be a topological space and $K \subseteq C(X)$ a convex $\mathfrak{T}_p$-compact set. Show that if there is a strictly positive σ-finite topological measure on X, then K is $\mathfrak{T}_p$-metrizable.

(f) Use Komlós' theorem (276H) to shorten the proof of 463F.

(g) Let (X, Σ, μ) be any complete σ-finite measure space. Show that if $K \subseteq \mathcal{L}^0(\Sigma)$ is $\mathfrak{T}_m$-relatively compact, and $\sup_{f \in K} |f(x)|$ is finite for every $x \in X$, then K is $\mathfrak{T}_p$-relatively countably compact in $\mathcal{L}^0(\Sigma)$.

(h) Let K be the set of non-decreasing functions from $[0, 1]$ to $\{0, 1\}$. Show that K, with its topology of pointwise convergence, is homeomorphic to the split interval (419L). Show that the identity map from $(K, \mathfrak{T}_p)$ to $(K, \mathfrak{T}_m)$ (for any Radon measure μ on $[0, 1]$) is continuous.

>(i) Let K be the set of non-decreasing functions from ω_1 to $\{0, 1\}$. Show that if μ is the countable-cocountable measure on ω_1 then K is a $\mathfrak{T}_p$-compact set of measurable functions and is also $\mathfrak{T}_m$-compact, but the identity map from $(K, \mathfrak{T}_p)$ to $(K, \mathfrak{T}_m)$ is not continuous.

>(j) Let K be the set of functions $f : [0, 1] \to \mathbb{R}$ such that $\max(\|f\|_\infty, \operatorname{Var}_{[0,1]} f) \leq 1$, where $\operatorname{Var}_{[0,1]} f$ is the variation of f (224A). Show that K is $\mathfrak{T}_p$-compact and that (for any Radon measure on $[0, 1]$) the identity map from $(K, \mathfrak{T}_p)$ to $(K, \mathfrak{T}_m)$ is continuous.

>(k) Let A be the set of functions $f : [0, 1] \to [0, 1]$ such that $\overline{\int} |f| d\mu \cdot \operatorname{Var}_{[0,1]} f \leq 1$, where μ is Lebesgue measure. Show that every member of A is measurable and that every sequence in A has a subsequence which converges almost everywhere to a member of A, but that $\int : A \to [0, 1]$ is not $\mathfrak{T}_p$-continuous, while A is $\mathfrak{T}_p$-dense in $[0, 1]^{[0,1]}$.

(l) Let X be a Hausdorff space and $K \subseteq C(X)$ a $\mathfrak{T}_p$-compact set. Show that if there is a strictly positive σ-finite Radon measure on X then K is $\mathfrak{T}_p$-metrizable.

>(m) Let (X, Σ, μ) be any perfect measure space and $K \subseteq \mathcal{L}^0(\Sigma)$ a $\mathfrak{T}_p$-compact set. Show that the identity map on K is $(\mathfrak{T}_p, \mathfrak{T}_m)$-continuous, so that K is $\mathfrak{T}_m$-compact.

(n) Let (X, Σ, μ) be a localizable measure space and $K \subseteq \mathcal{L}^0(\Sigma)$ a non-empty $\mathfrak{T}_p$-compact set. Show that $\sup\{f^\bullet : f \in K\}$ is defined in $L^0(\mu)$.

463Y Further exercises (a) Let (X, Σ, μ) be a probability space and V a Banach space. A function $\phi : X \to V$ is **scalarly measurable** (often called **weakly measurable**) if $h\phi : X \to \mathbb{R}$ is Σ-measurable for every $h \in V^*$. ϕ is **Pettis integrable**, with **indefinite Pettis integral** $\theta : \Sigma \to V$, if $\int_E h\phi \, d\mu$ is defined and equal to $h(\theta E)$ for every $E \in \Sigma$ and every $h \in V^*$. (i) Show that if ϕ is scalarly measurable, then $K = \{h\phi : h \in V^*, \|h\| \leq 1\}$ is a $\mathfrak{T}_p$-compact subset of $\mathcal{L}^0(\Sigma)$. (ii) Show that if ϕ is scalarly measurable, then it is Pettis integrable iff every function in K is integrable and $f \mapsto \int_E f : K \to \mathbb{R}$ is $\mathfrak{T}_p$-continuous for every $E \in \Sigma$. (*Hint*: 4A4Cg.) (iii) In particular, if ϕ is bounded and scalarly measurable and the identity map from $(K, \mathfrak{T}_p)$ to $(K, \mathfrak{T}_m)$ is continuous, then ϕ is Pettis integrable. (See Talagrand 84, chap. 4.)

(b) Show that any Bochner integrable function (253Yf) is Pettis integrable.

(c) Let μ be Lebesgue measure on $[0,1]$, and define $\phi : [0,1] \to L^\infty(\mu)$ by setting $\phi(t) = \chi[0,t]^\bullet$ for every $t \in [0,1]$. (i) Show that if $h \in L^\infty(\mu)^*$ and $\|h\| \leq 1$, then $h\phi$ has variation at most 1. (ii) Show that $K = \{h\phi : h \in L^\infty(\mu)^*, \|h\| \leq 1\}$ is a $\mathfrak{T}_p$-compact set of Lebesgue measurable functions, and that the identity map from $(K, \mathfrak{T}_p)$ to $(K, \mathfrak{T}_m)$ is continuous, so that ϕ is Pettis integrable. (iii) Show that ϕ is not Bochner integrable.

(d) Let (X, Σ, μ) be a σ-finite measure space and suppose that μ is inner regular with respect to some family $\mathcal{E} \subseteq \Sigma$ of cardinal at most ω_1. (Subject to the continuum hypothesis, this is true for any subset of $\mathbb{R}$, for instance.) Show that if $K \subseteq \mathcal{L}^0(\Sigma)$ is $\mathfrak{T}_p$-compact then it is $\mathfrak{T}_m$-compact. (See Talagrand 84, 9-3-3.)

(e) Assume that the continuum hypothesis is true; let $\preccurlyeq$ be a well-ordering of $[0,1]$ with order type ω_1 (4A1Ad). Let (Z, ν) be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$, and $q : Z \to [0,1]$ the canonical inverse-measure-preserving map (416V). Let $g : [0,1] \to [0, \infty[$ be any function. Show that there is a function $f : [0,1] \times Z \to [0, \infty[$ such that (α) f is continuous in the second variable (β) $f(t,z) = 0$ whenever $q(z) \preccurlyeq t$ (γ) $\int f(t,z)\nu(dz) = g(t)$ for every $t \in [0,1]$. Show that f is universally measurable in the first variable, but need not be measurable for the product Radon measure on $[0,1] \times Z$. Setting $f_z(t) = f(t,z)$, show that $K = \{f_z : z \in Z\}$ is a $\mathfrak{T}_p$-compact set of Lebesgue measurable functions and that g belongs to the $\mathfrak{T}_p$-closed convex hull of K in $\mathbb{R}^{[0,1]}$.

463Z Problems (a) A.Bellow's problem Let (X, Σ, μ) be a probability space, and $K \subseteq \mathcal{L}^0(\Sigma)$ a $\mathfrak{T}_p$-compact set such that $\{x : f(x) \neq g(x)\}$ is non-negligible for any distinct functions $f, g \in K$, as in 463F and 463Kc. Does it follow that K is metrizable for $\mathfrak{T}_p$?

A positive answer would displace several of the arguments of this section, and have other consequences (see 462Z, for instance). It is known that under any of a variety of special axioms (starting with the continuum hypothesis) there is indeed a positive answer; see Talagrand 84, chap. 12.

(b) Let $X \subseteq [0,1]$ be a set of outer Lebesgue measure 1, and μ the subspace measure on X, with Σ its domain. Let K be a $\mathfrak{T}_p$-compact subset of $\mathcal{L}^0(\Sigma)$. Must K be $\mathfrak{T}_m$-compact?

(c) Let $X_0, \ldots, X_n$ be compact Hausdorff spaces and $f : X_0 \times \ldots \times X_n \to \mathbb{R}$ a separately continuous function. Must f be universally measurable?

463 Notes and comments The relationship between the topologies $\mathfrak{T}_p$ and $\mathfrak{T}_m$ is complex, and I do not think that the results here are complete; in particular, we have a remarkable outstanding problem in 463Za. Much of the work presented here has been stimulated by problems concerning the integration of vector-valued functions. I am keeping this theory firmly in the 'further exercises' (463Ya-463Yc), but it is certainly the most important source of examples of pointwise compact sets of measurable functions. In particular, since the set $\{h\phi : h \in V^*, \|h\| \leq 1\}$ is necessarily convex whenever V is a Banach space and $\phi : X \to V$ is a function, we are led to look at the special properties of convex sets, as in 463F. There are obvious connexions with the theory of measures on linear topological spaces, which I will come to in §466.

The dichotomy in 463J shows that sets of measurable functions on perfect measure spaces are either 'good' (relatively countably compact for $\mathfrak{T}_p$, relatively compact for $\mathfrak{T}_m$) or 'bad' (with neither property). It is known that the result is not true for arbitrary σ-finite measure spaces (see §464 below), but it is not clear whether there are important non-perfect spaces in which it still applies in some form; see 463Zb.

Just as in §462, many questions concerning the topology $\mathfrak{T}_p$ on $\mathbb{R}^X$ can be re-phrased as questions about real-valued functions on products $X \times K$ which are continuous in the second variable. For the topology of pointwise convergence on sets of measurable functions, we find ourselves looking at functions which are measurable in the first variable. In this way we are led to such results as 463L-463M and 463Ye. Concerning 463L and 463Zc, R.Pol has recently (June 2000) proved that if X and Y are *any* compact Hausdorff spaces, and $f : X \times Y \to \mathbb{R}$ is separately continuous, then f is Borel measurable.

A substantial proportion of the questions which arise naturally in this topic are known to be undecidable without using special axioms. I am avoiding such questions in this volume, but it is worth noting that the continuum hypothesis, in particular, has many striking consequences here, of which 463Ye is a sample. It also decides 463Za and 463Zb (see 463Yd).

464 Talagrand's measure

An obvious question arising from 463H and its corollaries is, do we really need the hypothesis that the measure involved is perfect? A very remarkable construction of M.Talagrand (464C) shows that these results are certainly not true of all probability spaces (464D). Investigating the properties of this measure we are led to some surprising facts about additive functionals on algebras $\mathcal{P}I$ and the duals of ℓ^∞ spaces (464M, 464R).

464A The usual measure on $\mathcal{P}I$ Recall from 254J and 416U that for any set I we have a standard measure ν, a Radon measure for the usual topology on $\mathcal{P}I$, defined by saying that $\nu\{a : a \subseteq I,\, a \cap J = c\} = 2^{-\#(J)}$ whenever $J \subseteq I$ is a finite set and $c \subseteq J$, or by copying from the usual product measure on $\{0,1\}^I$ by means of the bijection $a \mapsto \chi a : \mathcal{P}I \to \{0,1\}^I$. We shall need a couple of simple facts about these measures.

(a) If $\langle I_j\rangle_{j\in J}$ is any partition of I, then ν can be identified with the product of the family $\langle \nu_j\rangle_{j\in J}$, where ν_j is the usual measure on $\mathcal{P}I_j$, and we identify $\mathcal{P}I$ with $\prod_{j\in J}\mathcal{P}I_j$ by matching $a \subseteq I$ with $\langle a\cap I_j\rangle_{j\in J}$; this is the 'associative law' 254N. It follows that if we have any family $\langle A_j\rangle_{j\in J}$ of subsets of $\mathcal{P}I$, and if for each j the set A_j is 'determined by coordinates in I_j' in the sense that, for $a \subseteq I$, $a \in A_j$ iff $a \cap I_j \in A_j$, then $\nu^*(\bigcap_{j\in J} A_j) = \prod_{j\in J}\nu^* A_j$ (use 254Lb).

(b) Similarly, if f_1, f_2 are non-negative real-valued functions on $\mathcal{P}I$, and if there are disjoint sets I_1, $I_2 \subseteq I$ such that $f_j(a) = f_j(a \cap I_j)$ for every $a \subseteq I$ and both j, then the upper integral $\overline{\int} f_1 + f_2 d\nu$ is $\overline{\int} f_1 d\nu + \overline{\int} f_2 d\nu$. **P** We may suppose that $I_2 = I \setminus I_1$. For each j, define $g_j : \mathcal{P}I_j \to [0,\infty[$ by setting $g_j = f_j \restriction \mathcal{P}I_j$, so that $f_j(a) = g_j(a \cap I_j)$ for every $a \subseteq I$. Let ν_j be the usual measure on $\mathcal{P}I_j$, so that we can identify ν with the produce measure $\nu_1 \times \nu_2$, if we identify $\mathcal{P}I$ with $\mathcal{P}I_1 \times \mathcal{P}I_2$; that is, we think of a subset of I as a pair (a_1, a_2) where $a_j \subseteq I_j$ for both j.

Now we have

$$\begin{aligned}\overline{\int} f_1 + f_2\, d\nu &= \overline{\int} g_1\, d\nu_1 + \overline{\int} g_2\, d\nu_2 \\ \text{(by 253K)}\quad &= \overline{\int} g_1\, d\nu_1 \cdot \overline{\int} \mathbf{1} d\nu_2 + \overline{\int} \mathbf{1} d\nu_1 \cdot \overline{\int} g_2\, d\nu_2 \\ &= \overline{\int} f_1\, d\nu + \overline{\int} f_2\, d\nu\end{aligned}$$

by 253J, because we can think of $f_1(a_1, a_2)$ as $g_1(a_1)\mathbf{1}(a_2)$ for all a_1, a_2. **Q**

(c) If $A \subseteq \mathcal{P}I$ is such that $b \in A$ whenever $a \in A$, $b \subseteq I$ and $a \triangle b$ is finite, then $\nu^* A$ must be either 0 or 1; this is the zero-one law 254Sa, applied to the set $\{\chi a : a \in A\} \subseteq \{0,1\}^I$ and the usual measure on $\{0,1\}^I$.

464B Lemma Let I be any set, and ν the usual measure on $\mathcal{P}I$.

(a)(i) There is a sequence $\langle m(n)\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $\prod_{n=0}^{\infty} 1-2^{-m(n)} = \frac{1}{2}$.

(ii) Given such a sequence, write X for $\prod_{n\in\mathbb{N}}(\mathcal{P}I)^{m(n)}$, and let λ be the product measure on X. We have a function $\phi : X \to \mathcal{P}I$ defined by setting

$$\phi(\langle\langle a_{ni}\rangle_{i<m(n)}\rangle_{n\in\mathbb{N}}) = \bigcup_{n\in\mathbb{N}}\bigcap_{i<m(n)} a_{ni}$$

whenever $\langle\langle a_{ni}\rangle_{i<m(n)}\rangle_{n\in\mathbb{N}} \in X$. Now ϕ is inverse-measure-preserving for λ and ν.

(b) The map

$$(a,b,c) \mapsto (a\cap b)\cup(a\cap c)\cup(b\cap c) : (\mathcal{P}I)^3 \to \mathcal{P}I$$

is inverse-measure-preserving for the product measure on $(\mathcal{P}I)^3$.

proof (a)(i) Choose $m(n)$ inductively so that, for each n in turn, $m(n)$ is minimal subject to the requirement $\prod_{k=0}^{n} 1-2^{-m(k)} > \frac{1}{2}$.

(ii) If $t \in I$, then

$$\{\mathfrak{x} : \mathfrak{x}\in X,\ t\notin\phi(\mathfrak{x})\} = \{\langle\langle a_{ni}\rangle_{i<m(n)}\rangle_{n\in\mathbb{N}} : t\notin\bigcup_{n\in\mathbb{N}}\bigcap_{i<m(n)} a_{ni}\}$$

has measure

$$\prod_{n=0}^{\infty}\lambda_n\{\langle a_i\rangle_{i<m(n)} : t\notin\bigcap_{i<m(n)} a_i\},$$

where λ_n is the product measure on $(\mathcal{P}I)^{m(n)}$ for each n. But this is just

$$\prod_{n=0}^{\infty} 1-2^{-m(n)} = \frac{1}{2},$$

by the choice of $\langle m(n)\rangle_{n\in\mathbb{N}}$. Accordingly $\lambda\{\mathfrak{x} : t\in\phi(\mathfrak{x})\} = \frac{1}{2}$ for every $t\in I$. Next, each set $E_t = \{\mathfrak{x} : t\in\phi(\mathfrak{x})\}$ is determined by coordinates in $J_t = \{(t,n,i) : n\in\mathbb{N},\ i<m(n)\}$. Since the sets J_t are disjoint, the sets E_t, for different t, are stochastically independent (464Aa), so if $J\subseteq I$ is finite,

$$\lambda\{\mathfrak{x} : J\subseteq\phi(\mathfrak{x})\} = \prod_{t\in J}\lambda E_t = 2^{-\#(J)} = \nu\{a : J\subseteq a\}.$$

This shows that $\lambda\phi^{-1}[F] = \nu F$ whenever F is of the form $\{a : J\subseteq a\}$ for some finite $J\subseteq I$. By the Monotone Class Theorem (136B), $\lambda\phi^{-1}[F] = \nu F$ for every F belonging to the σ-algebra generated by sets of this form. But this σ-algebra certainly contains all sets of the form $\{a : a\cap J = c\}$ where $J\subseteq I$ is finite and $c\subseteq J$, which are the sets corresponding to the basic cylinder sets in the product $\{0,1\}^I$. By 254G, ϕ is inverse-measure-preserving.

(b) This uses the same idea as (a-ii). Writing ν^3 for the product measure on $(\mathcal{P}I)^3$, then, for any $t\in I$,

$$\begin{aligned}\nu^3\{(a,b,c) : t\in(a\cap b)\cup(a\cap c)\cup(b\cap c)\}&\\ &= \nu^3\{(a,b,c) : t\in a\cap b\} + \nu^3\{(a,b,c) : t\in a\cap c\}\\ &\qquad + \nu^3\{(a,b,c) : t\in b\cap c\} - 2\nu^3\{(a,b,c) : t\in a\cap b\cap c\}\\ &= 3\cdot\frac{1}{4} - 2\cdot\frac{1}{8} = \frac{1}{2}.\end{aligned}$$

Once again, these sets are independent for different t, and this is all we need to know in order to be sure that the map is inverse-measure-preserving.

464C Lemma Let I be any set, and let ν be the usual measure on $\mathcal{P}I$.

(a) (see Sierpiński 45) If $\mathcal{F}\subseteq\mathcal{P}I$ is any filter containing every cofinite set, then $\nu_*\mathcal{F} = 0$ and $\nu^*\mathcal{F}$ is either 0 or 1. If $\mathcal{F}$ is a non-principal ultrafilter then $\nu^*\mathcal{F} = 1$.

(b) (Talagrand 80[3]) If $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ is a sequence of filters on I, all of outer measure 1, then $\bigcap_{n\in\mathbb{N}}\mathcal{F}_n$ also has outer measure 1.

[3]The date of this paper is misleading, as there was an unusual backlog in the journal; in reality it preceded Fremlin & Talagrand 79.

proof (a) That $\nu^*\mathcal{F} \in \{0,1\}$ is immediate from 464Ac. If $\mathcal{F}$ is an ultrafilter, then $\mathcal{P}I = \mathcal{F} \cup \{I \setminus a : a \in \mathcal{F}\}$; but as $a \mapsto I \setminus a$ is a measure space automorphism of $(\mathcal{P}I, \nu)$, $\nu^*\{I \setminus a : a \in \mathcal{F}\} = \nu^*\mathcal{F}$, and both must be at least $\frac{1}{2}$, so $\nu^*\mathcal{F} = 1$. Equally, $\nu^*(\mathcal{P}I \setminus \mathcal{F}) = \nu^*\{I \setminus a : a \in \mathcal{F}\} = 1$, so $\nu_*\mathcal{F} = 0$. Returning to a general filter containing every cofinite set, this is included in a non-principal ultrafilter, so also has inner measure 0.

(b) Let $\langle m(n)\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ such that $\prod_{n=0}^{\infty} 1 - 2^{-m(n)} = \frac{1}{2}$, and let X, λ and $\phi : X \to \mathcal{P}I$ be as in 464Ba. Consider the set $D = \prod_{n\in\mathbb{N}} \mathcal{F}_n^{m(n)}$ as a subset of X. By 254Lb, $\lambda^* D = 1$. If we set $\mathcal{F} = \bigcap_{n\in\mathbb{N}} \mathcal{F}_n$, then we see that whenever $\mathfrak{x} = \langle\langle a_{ni}\rangle_{i<m(n)}\rangle_{n\in\mathbb{N}}$ belongs to D,

$$\phi(\mathfrak{x}) \supseteq \textstyle\bigcap_{i<m(n)} a_{ni} \in \mathcal{F}_n$$

for every n, so that $\phi(\mathfrak{x}) \in \mathcal{F}$. Thus $D \subseteq \phi^{-1}[\mathcal{F}]$ and

$$\nu^*\mathcal{F} \geq \lambda^*\phi^{-1}[\mathcal{F}] \geq \lambda^* D = 1$$

because ϕ is inverse-measure-preserving (413Eh).

As $\langle\mathcal{F}_n\rangle_{n\in\mathbb{N}}$ is arbitrary, we have the result.

464D Construction (Talagrand 80) Let I be any set, and ν the usual Radon measure on $\mathcal{P}I$, with T its domain. Let Σ be the set

$$\{E : E \subseteq \mathcal{P}I, \text{ there are a set } F \in \mathrm{T} \text{ and a filter } \mathcal{F} \text{ on } I \text{ such that } \nu^*\mathcal{F} = 1 \text{ and } E \cap \mathcal{F} = F \cap \mathcal{F}\}.$$

Then there is a unique extension of ν to a complete probability measure μ, with domain Σ, defined by saying that $\mu E = \nu F$ whenever $E \in \Sigma$, $F \in \mathrm{T}$ and there is a filter $\mathcal{F}$ on I such that $\nu^*\mathcal{F} = 1$ and $E \cap \mathcal{F} = F \cap \mathcal{F}$. (Apply 417A with

$$\mathcal{A} = \{\mathcal{P}I \setminus \mathcal{F} : \mathcal{F} \text{ is a filter on } I \text{ such that } \nu^*\mathcal{F} = 1\}.)$$

Definition This measure μ is **Talagrand's measure** on $\mathcal{P}I$.

464E Example If μ is Talagrand's measure on $X = \mathcal{P}\mathbb{N}$, and Σ its domain, then there is a set $K \subseteq \mathbb{R}^X$, consisting of Σ-measurable functions and compact for the topology $\mathfrak{T}_p$ of pointwise convergence, such that K is not compact for the topology of convergence in measure, that is, there is a sequence in K with no subsequence which is convergent almost everywhere, even though every cluster point is Σ-measurable.

proof Take K to be $\{\chi\mathcal{F} : \mathcal{F} \text{ is an ultrafilter on } \mathbb{N}\}$. Then K is $\mathfrak{T}_p$-compact (in fact, is precisely the set of Boolean homomorphisms from $\mathcal{P}\mathbb{N}$ to $\{0,1\}$, identified with the Stone space of $\mathcal{P}\mathbb{N}$ in §311). By 464Ca, we see that any non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ belongs to Σ, and $\mu\mathcal{F} = 1$. On the other hand, all the principal ultrafilters $\mathcal{F}_n = \{a : n \in a \subseteq \mathbb{N}\}$ are measured by ν and therefore by μ, and form a stochastically independent sequence of sets of measure $\frac{1}{2}$. So K consists of Σ-measurable functions; but the sequence $\langle\chi F_n\rangle_{n\in\mathbb{N}}$ has no subsequence which is convergent almost everywhere, and K is not relatively compact for the topology of convergence in measure.

Remark In this example, a very large number of members of K are equal almost everywhere; indeed, all non-principal ultrafilters are equal a.e., and if we look at $\{f^\bullet : f \in K\}$ in $L^0(\mu)$, it is a countable discrete set. Given that K is a set of measurable functions homeomorphic to $\beta\mathbb{N}$, something like this has to happen (463L-463M). Looking at this from a different angle, if we wish to extend the usual measure ν on $\mathcal{P}\mathbb{N}$ to measure every ultrafilter, there will have to be two distinct ultrafilters $\mathcal{F}_1$, $\mathcal{F}_2$ such that $\mathcal{F}_1 \triangle \mathcal{F}_2$ is negligible for the extended measure.

464F The L-space $\ell^\infty(I)^*$ For the next step, we shall need to recall some facts from Volume 3. Let I be any set.

(a) The space $\ell^\infty(I)$ of bounded real-valued functions on I is an M-space (354Ha), so its dual $\ell^\infty(I)^\sim = \ell^\infty(I)^*$ is an L-space (356N), that is, a Banach lattice such that $\|f + g\| = \|f\| + \|g\|$ for all non-negative f, $g \in \ell^\infty(I)^*$. Since $\ell^\infty(I)$ can be identified with the space $L^\infty(\mathcal{P}I)$ as described in §363 (see 363Ha), we can identify $\ell^\infty(I)^*$ with the L-space M of bounded finitely additive functionals on $\mathcal{P}I$ (363K), matching any $f \in \ell^\infty(I)^*$ with the functional $a \mapsto f(\chi a) : \mathcal{P}I \to \mathbb{R}$ in M.

(b) In §§361-363 I examined some of the bands in M. The most significant ones for our present purposes are the band M_τ of completely additive functionals (362Bb) and its complement $M_\tau^\perp$ (352P); because M is Dedekind complete, we have $M = M_\tau \oplus M_\tau^\perp$ (353I). In fact M_τ is just the set of those $\theta \in M$ such that $\theta a = \sum_{t\in a} \theta\{t\}$ for every $a \subseteq I$, while $M_\tau^\perp$ is the set of those $\theta \in M$ such that $\theta\{t\} = 0$ for every $t \in I$. **P** For any $\theta \in M$, we can set $\alpha_t = \theta\{t\}$ for each $t \in I$; in this case,

$$|\textstyle\sum_{t\in J} \alpha_t| = |\theta J| \leq \|\theta\|$$

for every finite set $J \subseteq I$, so $\sum_{t\in I} |\alpha_t|$ is finite, and we have a functional θ_1 defined by setting $\theta_1 a = \sum_{t\in a} \alpha_t$ for every $a \subseteq I$. It is easy to check that $\theta_1 \in M_\tau$. If we write $\theta_2 = \theta - \theta_1$ then $\theta_2\{t\} = 0$ for every $t \in I$. So if $\phi \in M_\tau$ and $0 \leq \phi \leq |\theta_2|$, we still have $\phi\{t\} = 0$ for every $t \in I$, and $\phi a = 0$ for every finite $a \subseteq I$ and $\phi = 0$; thus $\theta_2 \in M_\tau^\perp$. Now $\theta \in M_\tau$ iff $\theta_2 = 0$, that is, iff $\theta = \theta_1$ and $\theta a = \sum_{t\in a} \theta\{t\}$ for every $a \subseteq I$; while $\theta \in M_\tau^\perp$ iff $\theta_1 = 0$, that is, $\theta\{t\} = 0$ for every $t \in I$. **Q**

Observe that if $\theta \in M_\tau^\perp$ and a, $b \subseteq I$ are such that $a \triangle b$ is finite, then $\theta a = \theta b$, because $\theta(a \setminus b) = \theta(b \setminus a) = 0$.

(c) It will be useful to have an elementary fact out in the open. If $\theta \in M^+$, then $\{a : a \subseteq I, \theta a = \theta I\}$ is a filter; this is because $\{a : \theta a = 0\}$ is an ideal in $\mathcal{P}I$.

464G We also need a new result not exactly covered by those in Chapters 35 and 36.

Lemma Let $\mathfrak{A}$ be any Boolean algebra. Write M for the L-space of bounded additive functionals on $\mathfrak{A}$, and M^+ for its positive cone, the set of non-negative additive functionals. Suppose that $\Delta : M^+ \to [0, \infty[$ is a functional such that

(α) Δ is non-decreasing,

(β) $\Delta(\alpha\theta) = \alpha\Delta(\theta)$ whenever $\theta \in M^+$, $\alpha \geq 0$,

(γ) $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$ whenever θ_1, $\theta_2 \in M^+$ are such that, for some $e \subseteq I$, $\theta_1(1 \setminus e) = \theta_2 e = 0$,

(δ) $|\Delta(\theta_1) - \Delta(\theta_2)| \leq \|\theta_1 - \theta_2\|$ for all θ_1, $\theta_2 \in M^+$.

Then there is a non-negative $h \in M^*$ extending Δ.

proof (a) If θ_1, $\theta_2 \in M^+$ are such that $\theta_1 \wedge \theta_2 = 0$ in M^+, then $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$. **P** Let $\epsilon > 0$. Then there is an $e \in \mathfrak{A}$ such that $\theta_1(1 \setminus e) + \theta_2 e \leq \epsilon$ (362Ba). Set $\theta_1' a = \theta_1(a \cap e)$, $\theta_2'(a) = \theta_2(a \setminus e)$ for $a \in \mathfrak{A}$; then θ_1' and θ_2' belong to M^+ and $\theta_1'(1 \setminus e) = \theta_2' e = 0$, so $\Delta(\theta_1' + \theta_2') = \Delta(\theta_1') + \Delta(\theta_2')$, by hypothesis ($\gamma$). On the other hand,

$$0 \leq \theta_1 a - \theta_1' a = \theta(a \setminus e) \leq \theta(1 \setminus e) \leq \epsilon$$

for every $a \in \mathfrak{A}$, so $\|\theta_1 - \theta_1'\| \leq \epsilon$. Similarly, $\|\theta_2 - \theta_2'\| \leq \epsilon$ and $\|(\theta_1 + \theta_2) - (\theta_1' - \theta_2')\| \leq 2\epsilon$. But this means that we can use the hypothesis (δ) to see that

$$|\Delta(\theta_1 + \theta_2) - \Delta(\theta_1) - \Delta(\theta_2)| \leq |\Delta(\theta_1' + \theta_2') - \Delta(\theta_1') - \Delta(\theta_2')| + 4\epsilon = 4\epsilon.$$

As ϵ is arbitrary, $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$. **Q**

(b) Now recall that M, being a Dedekind complete Riesz space, can be identified with an order-dense solid linear subspace of $L^0(\mathfrak{C})$ for some Boolean algebra $\mathfrak{C}$ (368H). Inside $L^0(\mathfrak{C})$ we have the order-dense Riesz subspace $S(\mathfrak{C})$ (364K). Write S_1 for $M \cap S(\mathfrak{C})$, so that S_1 is an order-dense Riesz subspace of M (352N, 353A).

If θ_1, $\theta_2 \in S_1^+ = S_1 \cap M^+$, then $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$. **P** We can express θ_1 as $\sum_{i=0}^m \alpha_i \chi c_i$, where $c_0, \ldots, c_m$ are disjoint members of $\mathfrak{C}$, and $\alpha_i \geq 0$ for each i (361Ec); adding a term $0 \cdot \chi(1 \setminus \sup_{i\leq m} c_i)$ if necessary, we may suppose that $\sup_{i\leq m} c_i = 1$. Similarly, we can express θ_2 as $\sum_{j=0}^n \beta_j \chi d_j$ where $d_0, \ldots, d_n \in \mathfrak{C}$ are disjoint, $\beta_j \geq 0$ for every j and $\sup_{j\leq n} d_j = 1$. In this case, $\theta_1 = \sum_{i\leq m, j\leq n} \alpha_i \chi(c_i \cap d_j)$ and $\theta_2 = \sum_{i\leq m, j\leq n} \beta_j \chi(c_i \cap d_j)$. Re-enumerating $\{c_i \cap d_j : i \leq m, j \leq n\}$ as $\langle e_i : i \leq k\}$ we have expressions of θ_1, θ_2 in the form $\sum_{i\leq k} \gamma_i \chi e_i$, $\sum_{i\leq k} \delta_i \chi e_i$, while $e_0, \ldots, e_k$ are disjoint.

Setting $\theta_1^{(r)} = \sum_{i=0}^r \gamma_i \chi e_i$ for $r \leq k$, we see that $\theta_1^{(r)} \wedge \gamma_{r+1} \chi e_{r+1} = 0$ for $r < k$, so (a) above, together with the hypothesis (β), tell us that

$$\Delta(\theta_1^{(r+1)}) = \Delta(\theta_1^{(r)}) + \Delta(\gamma_{r+1}\chi e_{r+1}) = \Delta(\theta_1^{(r)}) + \gamma_{r+1}\Delta(\chi e_{r+1})$$

for $r < k$. Accordingly $\Delta(\theta_1) = \sum_{i=0}^k \gamma_i \Delta(\chi e_i)$. Similarly, $\Delta(\theta_2) = \sum_{i=0}^k \delta_i \Delta(\chi e_i)$ and

$$\Delta(\theta_1 + \theta_2) = \sum_{i=0}^k (\gamma_i + \delta_i)\Delta(\chi e_i) = \Delta(\theta_1) + \Delta(\theta_2),$$

as claimed. **Q**

(c) Consequently, $\Delta(\theta_1+\theta_2) = \Delta(\theta_1)+\Delta(\theta_2)$ for all $\theta_1, \theta_2 \in M^+$. **P** Let $\epsilon > 0$. Because S_1 is order-dense in M, and the norm of M is order-continuous (354N), S_1 is norm-dense (354Ef), and there are $\theta_1', \theta_2' \in S_1^+$ such that $\|\theta_j - \theta_j'\| \leq \epsilon$ for both j (354Be). But now, just as in (a),

$$|\Delta(\theta_1 + \theta_2) - \Delta(\theta_1) - \Delta(\theta_2)| \leq |\Delta(\theta_1' + \theta_2') - \Delta(\theta_1') - \Delta(\theta_2')| + 4\epsilon = 4\epsilon.$$

As ϵ is arbitrary, we have the result. **Q**

(d) Now (c) and the hypothesis (β) are sufficient to ensure that Δ has an extension to a positive linear functional (355D).

464H The next lemma contains the key ideas needed for the rest of the section.

Lemma Let I be any set, and M the L-space of bounded additive functionals on $\mathcal{P}I$; let ν be the usual measure on $\mathcal{P}I$. For $\theta \in M^+$, set

$$\Delta(\theta) = \overline{\int} \theta \, d\nu.$$

(a) For every $\theta \in M^+$, $\frac{1}{2}\theta I \leq \Delta(\theta) \leq \theta I$.

(b) There is a non-negative $h \in M^*$ such that $h(\theta) = \Delta(\theta)$ for every $\theta \in M^+$.

(c) If $\theta \in (M_\tau^\perp)^+$, where $M_\tau \subseteq M$ is the band of completely additive functionals, then $\theta \leq \Delta(\theta)$ ν-a.e., and $\nu^*\{a : \alpha \leq \theta a \leq \Delta(\theta)\} = 1$ for every $\alpha < \Delta(\theta)$.

(d) Suppose that $\theta \in (M_\tau^\perp)^+$ and $\beta, \gamma \in [0,1]$ are such that $\theta I = 1$ and $\beta\theta' I \leq \Delta(\theta') \leq \gamma\theta' I$ whenever $\theta' \leq \theta$ in M^+. Then, for any $\alpha < \beta$,

(i) for any finite set $K \subseteq \mathcal{P}I$, the set

$$Q_K = \{a : a \subseteq I, \, \alpha\theta b \leq \theta(a \cap b) \leq \gamma\theta b \text{ for every } b \in K\}$$

has outer measure 1 in $\mathcal{P}I$;

(ii) if $\alpha \geq \frac{1}{2}$, the set

$$R = \{(a,b,c) : a, \, b, \, c \subseteq I, \, \theta((a \cap b) \cup (a \cap c) \cup (b \cap c)) \geq 2\alpha^2 + (1-2\alpha)\gamma^2\}$$

has outer measure 1 in $(\mathcal{P}I)^3$;

(iii) if $\alpha \geq \frac{1}{2}$, then $2\alpha^2 + (1-2\alpha)\gamma^2 \leq \gamma$.

(e) Any $\theta \in M^+$ can be expressed as $\theta_1 + \theta_2$ where $\Delta(\theta_1) = \frac{1}{2}\theta_1 I$ and $\Delta(\theta_2) = \theta_2 I$.

(f) Suppose that $0 \leq \theta' \leq \theta$ in M.

(i) If $\Delta(\theta) = \frac{1}{2}\theta I$, then $\Delta(\theta') = \frac{1}{2}\theta' I$.

(ii) If $\Delta(\theta) = \theta I$, then $\Delta(\theta') = \theta' I$.

proof (a) Of course

$$\Delta(\theta) \leq \int \theta I \, d\nu = \theta I.$$

On the other hand, because $a \mapsto I \setminus a : \mathcal{P}I \to \mathcal{P}I$ is an automorphism of the measure space $(\mathcal{P}I, \nu)$, $\overline{\int} f(I \setminus a)\nu(da) = \overline{\int} f(a)\nu(da)$ for any real-valued function f. In particular,

$$\Delta(\theta) = \overline{\int} \theta(a)\nu(da) = \overline{\int} \theta(I \setminus a)\nu(da).$$

So

$$2\Delta(\theta) = \overline{\int} \theta(a)\nu(da) + \overline{\int} \theta(I \setminus a)\nu(da) \geq \overline{\int} \theta(a) + \theta(I \setminus a)\nu(da)$$

(133J(b-ii))

$$= \overline{\int} \theta I \, \nu(da) = \theta I,$$

and $\Delta(\theta) \geq \frac{1}{2}\theta I$.

(b) I use 464G with $\mathfrak{A} = \mathcal{P}I$. Examine the conditions (α)-(δ) there.

(α) Of course Δ is non-decreasing (133Jc).

(β) $\Delta(\alpha\theta) = \alpha\Delta(\theta)$ for every $\theta \in M^+$ and every $\alpha \geq 0$, by 133J(b-iii).

(γ) If θ_1, $\theta_2 \in M^+$ then $\Delta(\theta_1 + \theta_2) \leq \Delta(\theta_1) + \Delta(\theta_2)$ by 133J(b-ii), as in (a). If θ_1, $\theta_2 \in M^+$ and $e \subseteq I$ are such that $\theta_1(I \setminus e) = \theta_2 e = 0$, then $\theta_1 a = \theta_1(a \cap e)$, $\theta_2(a) = \theta_2(a \setminus e)$ for every $a \subseteq I$. So $\Delta(\theta_1 + \theta_2) = \Delta(\theta_1) + \Delta(\theta_2)$ by 464Ab.

(δ) For every $a \subseteq I$, $\theta_2 a \leq \theta_1 a + \|\theta_2 - \theta_1\|$, so $\Delta(\theta_2) \leq \Delta(\theta_1) + \|\theta_2 - \theta_1\|$. Similarly, $\Delta(\theta_1) \leq \Delta(\theta_2) + \|\theta_1 - \theta_2\|$. So $|\Delta(\theta_1) - \Delta(\theta_2)| \leq \|\theta_1 - \theta_2\|$, and Δ satisfies condition (δ) of 464G.

(ϵ) Accordingly 464G tells us that Δ has an extension to a member of M^*.

(c) If $\theta \in M_\tau^\perp$, then $\theta a = \theta b$ whenever $a \triangle b$ is finite (464Fb), so all the sets $\{a : \theta a > \alpha\}$ have outer measure either 0 or 1, by 464Ac. But this means that if f is a measurable function and $\theta \leq_{\text{a.e.}} f$ and $\int f = \Delta(\theta)$, as in 133Ja, $\{a : f(a) \leq \Delta(\theta)\}$ has positive measure and meets $A = \{a : \theta a > \Delta(\theta)\}$ in a negligible set; so A cannot have full outer measure and is negligible.

On the other hand, if $\alpha < \Delta(\theta)$, then θ cannot be dominated a.e. by $\alpha\mathbf{1}$, so $\{a : \theta a > \alpha\}$ is not negligible and has outer measure 1.

(d)(i) The point is just that for any $b \subseteq I$, the functional θ_b, defined by saying that $\theta_b(a) = \theta(a \cap b)$ for every $a \subseteq I$, belongs to M^+ and is dominated by θ, so that $\beta\theta_b I \leq \Delta(\theta_b) \leq \gamma\theta_b I$, and $\{a : \alpha\theta_b I \leq \theta_b a \leq \gamma\theta_b I\}$ has outer measure 1, by (c). (If $\alpha\theta_b I = \Delta(\theta_b)$, this is because $\theta_b I = 0$, and the result is trivial; otherwise, $\alpha\theta_b I < \Delta(\theta_b) \leq \gamma\theta_b I$.) But this just says that, for any $b \subseteq I$,

$$\{a : \alpha\theta b \leq \theta(a \cap b) \leq \gamma\theta b\}$$

has outer measure 1.

Now, given any finite set $K \subseteq \mathcal{P}I$, let $\mathcal{B}$ be the subalgebra of $\mathcal{P}I$ generated by K, and $b_0, \ldots, b_n$ the atoms of $\mathcal{B}$. Then all the sets

$$A_i = \{a : \alpha\theta b_i \leq \theta(a \cap b_i) \leq \gamma\theta b_i\}$$

have outer measure 1; because each A_i is determined by coordinates in b_i, and $b_0, \ldots, b_n$ are disjoint, $A = \bigcap_{i \leq n} A_i$ still has outer measure 1, by 464Aa. But if $a \in A$ and $b \in K$, then $b = \bigcup_{i \in J} b_i$ for some $J \subseteq \{0, \ldots, n\}$ and

$$\begin{aligned}\alpha\theta b &= \sum_{i \in J} \alpha\theta b_i \leq \sum_{i \in J} \theta(a \cap b_i) \\ &= \theta(a \cap b) \leq \sum_{i \in J} \gamma\theta b_i = \gamma\theta b.\end{aligned}$$

So $\{a : \alpha\theta b \leq \theta(a \cap b) \leq \gamma\theta b \text{ for every } b \in K\}$ includes A and has outer measure 1.

(ii) ? Suppose, if possible, otherwise; then $\nu^3_*(\mathcal{P}I \setminus R) > 0$, where ν^3 is the product measure on $(\mathcal{P}I)^3$, and there is a measurable set $W \subseteq (\mathcal{P}I)^3 \setminus R$ such that $\nu^3 W > 0$. For $a \subseteq I$, set $W_a = \{(b, c) : (a, b, c) \in W\}$; then $\nu^3 W = \int \nu^2 W_a \nu(da)$, where ν^2 is the product measure on $(\mathcal{P}I)^2$, by Fubini's theorem (252D). Set $E = \{a : \nu^2 W_a \text{ is defined and not zero}\}$; then $\nu E > 0$. Since $\{a : \alpha \leq \theta a \leq \gamma\}$ has outer measure 1, there is an $a \in E$ such that $\alpha \leq \theta a \leq \gamma$.

For $b \subseteq I$, set $W_{ab} = \{c : (b, c) \in W_a\} = \{c : (a, b, c) \in W\}$. Then $0 < \nu^2 W_a = \int \nu W_{ab} \nu(db)$, so $F = \{b : \nu W_{ab} \text{ is defined and not } 0\}$ has non-zero measure. But also, by (i),

$$\{b : \theta b \geq \alpha, \, \theta(a \cap b) \leq \gamma\theta a\}$$

has outer measure 1, so we can find a $b \in F$ such that $\theta b \geq \alpha$ and $\theta(a \cap b) \leq \gamma\theta a$.

By (i) again,

$$\{c : \theta(c \cap (a \triangle b)) \geq \alpha\theta(a \triangle b)\}$$

has outer measure 1, so meets W_{ab}; accordingly we have a $c \subseteq I$ such that $(a, b, c) \in W$ while $\theta(c \cap (a \triangle b)) \geq \alpha\theta(a \triangle b)$.

Now calculate, for this triple (a, b, c),

$$\theta((a \cap b) \cup (a \cap c) \cup (b \cap c)) = \theta(a \cap b) + \theta(c \cap (a \triangle b)) \geq \theta(a \cap b) + \alpha\theta(a \triangle b)$$

(by the choice of c)

$$= \alpha(\theta a + \theta b) + (1 - 2\alpha)\theta(a \cap b)$$
$$\geq \alpha(\theta a + \alpha) + (1 - 2\alpha)\gamma\theta a$$

(by the choice of b, recalling that $1 - 2\alpha \leq 0$)

$$\geq 2\alpha^2 + (1 - 2\alpha)\gamma^2$$

by the choice of a. But this means that $(a, b, c) \in W \cap R$, which is supposed to be impossible. **X**

(iii) Now recall that the map $(a, b, c) \mapsto (a \cap b) \cup (a \cap c) \cup (b \cap c)$ is inverse-measure-preserving (464Ac). Since $\theta a \leq \Delta(\theta) \leq \gamma$ for ν-almost every a, we must have $\theta((a \cap b) \cup (a \cap c) \cup (b \cap c)) \leq \gamma$ for ν^3-almost every (a, b, c). But as R is not negligible, there must be some $(a, b, c) \in R$ such that $\theta((a \cap b) \cup (a \cap c) \cup (b \cap c)) \leq \gamma$, and $2\alpha^2 + (1 - 2\alpha)\gamma^2 \leq \gamma$.

(e)(i) M^*, being the dual of an L-space, is an M-space (356P), so can be represented as $C(Z)$ for some compact Hausdorff space Z (354L). The functional h of (b) above therefore corresponds to a function $w \in C(Z)$. Any $\theta \in M^+$ acts on M^* as a positive linear functional, so corresponds to a Radon measure μ_θ on Z (436J/436K); we have $\Delta(\theta) = h(\theta) = \int w \, d\mu_\theta$. The inequalities $\frac{1}{2}\theta I \leq \Delta(\theta) \leq \theta I$ become $\frac{1}{2}\mu_\theta Z \leq \int w \, d\mu_\theta \leq \mu_\theta Z$, because the constant function χZ corresponds to the standard order unit of M^* (356P), so that

$$\mu_\theta Z = \int \chi Z \, d\mu_\theta = \|\theta\| = \theta I$$

for every $\theta \geq 0$. Since $0 \leq h(\theta) \leq \|\theta\|$ for every $\theta \geq 0$, $\|w\|_\infty = \|h\| \leq 1$ and $0 \leq w \leq \chi Z$.

(ii) Now suppose that $\beta < \gamma$ and that $G = \{z : z \in Z, \beta < w(z) < \gamma\}$ is non-empty. In this case there is a non-zero $\theta_0 \in M^+$ such that $\beta\theta' I \leq \Delta(\theta') \leq \gamma\theta' I$ whenever $0 \leq \theta' \leq \theta_0$.

P(α) We have a solid linear subspace $V = \{v : v \in C(Z), v(z) = 0 \text{ for every } z \in G\}$ of $C(Z)$. Consider $U = \{\theta : \theta \in M, (\theta|v) = 0 \text{ for every } v \in V\}$, where I write $(\,|\,)$ for the duality between M and $C(Z)$ corresponding to the identification of $C(Z)$ with M^*.

(β) If $\theta \in U \cap M^+$, then $\beta\theta I \leq \Delta(\theta) \leq \gamma\theta I$. To see this, observe that $\int v \, d\mu_\theta = (\theta|v) = 0$ for every $v \in V$, so

$$\mu_\theta(Z \setminus \overline{G}) = \sup\{\mu_\theta K : K \subseteq Z \setminus \overline{G} \text{ is compact}\}$$
$$= \sup\{\int v \, d\mu_\theta : v \in C(Z), 0 \leq v \leq \chi(Z \setminus \overline{G})\}$$

(because Z is normal, so that whenever $K \subseteq Z \setminus \overline{G}$ is compact there is a $v \in C(Z)$ such that $\chi K \leq v \leq \chi(Z \setminus \overline{G})$, by Urysohn's Lemma)

$$= 0.$$

Accordingly $\beta < w < \gamma$ μ_θ-a.e. and $\int w \, d\mu_\theta$ must lie between $\beta\mu_\theta Z = \beta\theta I$ and $\gamma\mu_\theta Z = \gamma\theta I$.

(γ) Because M is an L-space, it is perfect (356P), so its dual $M^* = M^\times$ is also perfect, and $C(Z)$ is perfect; moreover, the duality $(\,|\,)$ identifies M with $C(Z)^\times$. Now $V^\perp$, taken in $C(Z)$, contains any continuous function zero on $Z \setminus G$, so is not $\{0\}$; since $V^\perp$, like $C(Z)$, must be perfect (356L), $(V^\perp)^\times$ is non-trivial. Take any $\psi > 0$ in $(V^\perp)^\times$. Being perfect, $C(Z)$ is Dedekind complete (356K), so there is a band projection $P : C(Z) \to V^\perp$ (353I). Now ψP is a positive element of $C(Z)^\times$ which is zero on V, and must correspond to a non-zero element θ_0 of $U \cap M^+$.

($\boldsymbol{\delta}$) If $0 \leq \theta' \leq \theta_0$ in M, then, for any $v \in V$,

$$|(\theta'|v)| \leq (\theta'||v|) \leq (\theta_0||v|) = 0,$$

because $|v| \in V$. So $\theta' \in U$ and $\beta\theta' I \leq \Delta(\theta') \leq \gamma\theta' I$, by ($\beta$). Thus θ_0 has the required property. **Q**

(iii) It follows at once that $w(z) \geq \frac{1}{2}$ for every $z \in Z$. **P?** If $w(z_0) < \frac{1}{2}$, then we can apply (ii) with $\beta = -1$, $\gamma \in \left]w(z_0), \frac{1}{2}\right[$ to see that there is a non-zero $\theta \in M^+$ such that $\Delta(\theta) \leq \gamma\theta I < \frac{1}{2}\theta I$, which is impossible, by (a). **XQ**

(iv) But we find also that $w(z) \notin \left]\frac{1}{2}, 1\right[$ for any $z \in Z$. **P?** If $w(z_0) = \delta \in \left]\frac{1}{2}, 1\right[$, then $2\delta^2 + (1-2\delta)\delta^2 > \delta$ (because $\delta(2\delta-1)(1-\delta) > 0$). We can therefore find α, β and γ such that $\frac{1}{2} \leq \alpha < \beta < \delta < \gamma$ and $2\alpha^2 + (1-2\alpha)\gamma^2 > \gamma$. But now $\{z : \beta < w(z) < \gamma\}$ is non-empty, so by (ii) there is a non-zero $\theta \in M^+$ such that $\beta\theta' I \leq \Delta(\theta') \leq \gamma\theta' I$ whenever $0 \leq \theta' \leq \theta$. Multiplying θ by a suitable scalar if necessary, we can arrange that θI should be 1. But this is impossible, by (d-iii). **XQ**

(v) Thus w takes only the values $\frac{1}{2}$ and 1; let H_1 and H_2 be the corresponding open subsets of Z.

Take $\theta \in M^+$. For $u \in C(Z)$, set $\phi(u) = \int u \, d\mu_\theta$ and $\phi_j(u) = \int_{H_j} u \, d\mu_\theta$ for each j. Then each ϕ_j is a positive linear functional on $C(Z)$ and $\phi_j \leq \phi$. But ϕ is the image of θ under the canonical isomorphism from M to $C(Z)^\times \cong M^{\times\times}$, and $C(Z)^\times$ is solid in $C(Z)^\sim$ (356B), so both ϕ_1 and ϕ_2 belong to the image of M, and correspond to θ_1, $\theta_2 \in M$. For any $u \in C(Z)$,

$$(\theta_1 + \theta_2|u) = \phi_1(u) + \phi_2(u) = (\theta|u),$$

so $\theta = \theta_1 + \theta_2$. We have

$$\begin{aligned}\Delta(\theta_j) = \phi_j(w) &= \int_{H_j} w \, d\mu_\theta \\ &= \tfrac{1}{2}\mu_\theta(H_1) = \tfrac{1}{2}\theta_1 I \text{ if } j = 1, \\ &= \mu_\theta(H_2) = \theta_2 I \text{ if } j = 2.\end{aligned}$$

So we have a suitable decomposition $\theta = \theta_1 + \theta_2$.

(f) This is easy. Set $\theta'' = \theta - \theta'$; then

$$\tfrac{1}{2}\theta' I \leq \Delta(\theta') \leq \theta' I,$$

$$\tfrac{1}{2}\theta'' I \leq \Delta(\theta'') \leq \theta'' I$$

by (a), while $\Delta(\theta') + \Delta(\theta'') = \Delta(\theta)$ by (b), and of course $\theta' I + \theta'' I = \theta I$. But this means that

$$\Delta(\theta') - \tfrac{1}{2}\theta' I \leq \Delta(\theta) - \tfrac{1}{2}\theta I, \quad \theta' I - \Delta(\theta') \leq \theta I - \Delta(\theta),$$

and the results follow.

464I Measurable and purely non-measurable functionals As before, let I be any set, ν the usual measure on $\mathcal{P}I$, T its domain, and M the L-space of bounded additive functionals on $\mathcal{P}I$. Following Fremlin & Talagrand 79, I say that $\theta \in M$ is **measurable** if it is T-measurable when regarded as a real-valued function on $\mathcal{P}I$, and **purely non-measurable** if $\{a : a \subseteq I, |\theta|(a) = |\theta|(I)\}$ has outer measure 1. (Of course the zero functional is both measurable and purely non-measurable.)

464J Examples Before going farther, I had better offer some examples of measurable and purely non-measurable functionals. Let I, ν and M be as in 464I.

(a) Any $\theta \in M_\tau$ is measurable, where M_τ is the space of completely additive functionals on $\mathcal{P}I$. **P** By 464Fb, θ can be expressed as a sum of point masses; say $\theta a = \sum_{t \in a} \alpha_t$ for some family $\langle \alpha_t \rangle_{t \in I}$ in $\mathbb{R}$. Since $\sum_{t \in I} |\alpha_t|$ must be finite, $\{t : \alpha_t \neq 0\}$ is countable, and we can express θ as the limit of a sequence of finite sums $\sum_{t \in K} \alpha_t \hat{t}$, where $\hat{t}(a) = 1$ if $t \in a$, 0 otherwise. But of course every $\hat{t}$ is a measurable function, so $\sum_{t \in K} \alpha_t \hat{t}$ is measurable for every finite set K, and θ is measurable. **Q**

(b) For a less elementary measurable functional, consider the following construction. Let $\langle t_n\rangle_{n\in\mathbb{N}}$ be any sequence of distinct points in I. Then $\lim_{n\to\infty}\frac{1}{n}\#(\{i : i<n,\, t_i\in a\})=\frac{1}{2}$ for ν-almost every $a\subseteq I$. **P** Set $f_n(a)=1$ if $t_n\in a$, 0 otherwise. Then $\langle f_n\rangle_{n\in\mathbb{N}}$ is an independent sequence of random variables. By any of the versions of the Strong Law of Large Numbers in §273 (273D, 273H, 273I), $\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}f_i=\frac{1}{2}$ a.e., which is what was claimed. **Q** So if we take any non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$, and set $\theta a=\lim_{n\to\mathcal{F}}\frac{1}{n}\#(\{i : i<n,\, t_i\in a\})$ for $a\subseteq I$, θ will be constant ν-almost everywhere, and measurable; and it is easy to check that θ is additive. Note that $\theta\{t\}=0$ for every t, so $\theta\in M_\tau^\perp$, by 464Fb.

(c) If $\mathcal{F}$ is any non-principal ultrafilter on $\mathcal{I}$, and we set $\theta a=1$ for $a\in\mathcal{F}$, 0 otherwise, then θ is an additive functional which is purely non-measurable, by 464Ca.

For further remarks on where to look for measurable and purely non-measurable functionals, see 464P-464Q below.

464K The space $M_{\rm m}$: Lemma Let I be any set, ν the usual measure on $\mathcal{P}I$, and M the L-space of bounded additive functionals on $\mathcal{P}I$. Write $M_{\rm m}$ for the set of measurable $\theta\in M$, and $\Delta(\theta)=\overline{\int}\theta\,d\nu$ for $\theta\in M^+$, as in 464H.

(a) $|\theta|\in M_{\rm m}$ for every $\theta\in M_{\rm m}$.

(b) A functional $\theta\in M^+$ is measurable iff $\Delta(\theta)=\frac{1}{2}\theta I$.

(c) $M_{\rm m}$ is a solid linear subspace of M.

proof (a)(i) Take any $\theta\in M_{\rm m}\cap M_\tau^\perp$, where M_τ is the space of completely additive functionals on $\mathcal{P}I$.

(α) $\theta=_{\rm a.e.}\frac{1}{2}\theta I$. **P** For any $\alpha\in\mathbb{R}$, $A_\alpha=\{a : \theta a<\alpha\}$ is measurable; but also $b\in A$ whenever $a\in A$ and $a\triangle b$ is finite, by 464Fb, so νA must be either 1 or 0, by 464Ac. Setting $\delta=\sup\{\alpha : \nu A_\alpha=0\}$, we see that $\nu A_\delta=0$, $\nu A_{\delta+2^{-n}}=1$ for every $n\in\mathbb{N}$, so that $\theta=\delta$ a.e. Also, because $a\mapsto I\setminus a$ is a measure space automorphism, $\theta(I\setminus a)=\delta$ for almost every a, so there is some a such that $\theta a=\theta(I\setminus a)=\delta$, and $\delta=\frac{1}{2}\theta I$. **Q**

(β) For any $b\subseteq I$, $\theta(a\cap b)=\frac{1}{2}\theta b$ for almost every a. **P** We know that $\theta a=\frac{1}{2}\theta I$ for almost every a, by (α). But $a\mapsto a\triangle b:\mathcal{P}I\to\mathcal{P}I$ is inverse-measure-preserving, so $\theta(a\triangle b)=\frac{1}{2}\theta I$ for almost every a. This means that $\theta a=\theta(a\triangle b)$ for almost every a, and

$$\theta(a\cap b)=\frac{1}{2}(\theta b+\theta a-\theta(a\triangle b))=\frac{1}{2}\theta b$$

for almost every a. **Q**

(γ) θ^+, taken in M, is measurable. **P** For any $n\in\mathbb{N}$ we can find $b_n\subseteq I$ such that $\theta^-b_n+\theta^+(I\setminus b_n)\le 2^{-n}$, so that

$$|\theta^+a-\theta(a\cap b_n)|=|\theta^+a-\theta^+(a\cap b_n)+\theta^-(a\cap b_n)|\le\theta^+(I\setminus b_n)+\theta^-b_n\le 2^{-n}$$

for every $a\subseteq I$. But as $\theta(a\cap b_n)$ is constant a.e. for every n, by (β), so is θ^+, and θ^+ is measurable. **Q**

Consequently $|\theta|=2\theta^+-\theta$ is measurable.

(ii) Now take an arbitrary $\theta\in M_{\rm m}$. Because M is Dedekind complete (354N, 354Ee), $M=M_\tau+M_\tau^\perp$ (353I), and we can express θ as $\theta_1+\theta_2$ where $\theta_1\in M_\tau$ and $\theta_2\in M_\tau^\perp$; moreover, $|\theta|=|\theta_1|+|\theta_2|$ (352Fb). Now θ_1 is measurable, by 464Ja, so $\theta_2=\theta-\theta_1$ is measurable; as $\theta_2\in M_\tau^\perp$, (i-γ) tells us that $|\theta_2|$ is measurable. On the other hand, $|\theta_1|$ belongs to M_τ and is measurable, so $|\theta|=|\theta_1|+|\theta_2|$ is measurable.

Thus (a) is true.

(b) Let f be a ν-integrable function such that $\Delta(\theta)\le f$ a.e. and $\int f\,d\nu=\Delta(\theta)$. Then

$$\theta I-f(a)\le\theta I-\theta a=\theta(I\setminus a)$$

for almost every a, so

$$\theta I-\Delta(\theta)\le\underline{\int}\theta(I\setminus a)\nu(da)=\underline{\int}\theta(a)\nu(da)$$

because $a \mapsto I \setminus a$ is a measure space automorphism, as in the proof of 464Ha. So if $\Delta(\theta) = \frac{1}{2}\theta I$ then $\underline{\int}\theta\,d\nu = \overline{\int}\theta\,d\nu$ and θ is ν-integrable (133Jd), therefore (because ν is complete) $(\operatorname{dom}\nu)$-measurable. On the other hand, if θ is measurable, then

$$\Delta(\theta) = \int \theta\,d\nu = \int \theta(I \setminus a)\nu(da) = \theta I - \int \theta\,d\nu = \theta I - \Delta(\theta),$$

so surely $\Delta(\theta) = \frac{1}{2}\theta I$.

(c) Of course $M_{\rm m}$ is a linear subspace. If $\theta_0 \in M_{\rm m}$ and $|\theta| \leq |\theta_0|$, then $|\theta_0| \in M_{\rm m}$, by (a), so $\Delta(|\theta_0|) = \frac{1}{2}|\theta_0|(I)$, by (b). Because $\theta^+ \leq |\theta| \leq |\theta_0|$, $\Delta(\theta^+) = \frac{1}{2}\theta^+ I$ (464H(f-i)), and θ^+ is measurable, by (b) in the reverse direction. Similarly, θ^- is measurable, and $\theta = \theta^+ - \theta^-$ is measurable. As θ and θ_0 are arbitrary, $M_{\rm m}$ is solid.

464L The space $M_{\rm pnm}$: Lemma Let I be any set, ν the usual measure on $\mathcal{P}I$, and M the L-space of bounded additive functionals on $\mathcal{P}I$. This time, write $M_{\rm pnm}$ for the set of those members of M which are purely non-measurable in the sense of 464I.

(a) If $\theta \in M^+$, then θ is purely non-measurable iff $\Delta(\theta) = \theta I$.

(b) $M_{\rm pnm}$ is a solid linear subspace of M.

proof (a)(i) If θ is purely non-measurable, and $f \geq \theta$ is integrable, then $\{a : f(a) \geq \theta I\}$ is a measurable set including $\{a : \theta a = \theta I\}$, so has measure 1, and $\int f \geq \theta I$; as f is arbitrary, $\Delta(\theta) = \theta I$.

(ii) If $\Delta(\theta) = \theta I$, then $\Delta(\theta') = \theta' I$ whenever $0 \leq \theta' \leq \theta$, by 464Hf. But this means that θ' cannot be measurable whenever $0 < \theta' \leq \theta$, by 464Kc above, so that $\theta' \notin M_\tau$ whenever $0 < \theta' \leq \theta$, by 464Ja. Thus $\theta \in M_\tau^\perp$.

By 464Hc, $\nu^*\{a : \alpha \leq \theta a \leq \Delta(\theta)\} = 1$ for every $\alpha < \Delta(\theta)$. Let $\langle m(n)\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N}$ such that $\prod_{n=0}^\infty 1 - 2^{-m(n)} = \frac{1}{2}$, and define X, λ and ϕ as in 464Ba. Set $\eta_n = 2^{-n}/m(n) > 0$ for each n. Consider the sets $A_n = \{a : \theta a \geq (1-\eta_n)\theta I\}$ for each $n \in \mathbb{N}$. Then $\nu^* A_n = 1$ for each n, and $\lambda^*(\prod_{n\in\mathbb{N}} A_n^{m(n)}) = 1$. Because ϕ is inverse-measure-preserving, $\nu^*(\phi[\prod_{n\in\mathbb{N}} A_n^{m(n)}]) = 1$ (413Eh). But if $\mathfrak{x} = \langle\langle a_{ni}\rangle_{i<m(n)}\rangle_{n\in\mathbb{N}}$ belongs to $\prod_{n\in\mathbb{N}} A_n^{m(n)}$, then

$$\theta(\textstyle\bigcap_{i<m(n)} a_{ni}) \geq \theta I - m(n)\eta_n\theta I = (1-2^{-n})\theta I$$

for each n, and

$$\theta(\phi(\mathfrak{x})) \geq \sup_{n\in\mathbb{N}} \theta(\textstyle\bigcap_{i<m(n)} a_{ni}) = \theta I.$$

Thus the filter $\{a : \theta a = \theta I\}$ includes $\phi[\prod_{n\in\mathbb{N}} A_n^{m(n)}]$ and has outer measure 1, so that θ is purely non-measurable.

(b)(i) If $\theta \in M_{\rm pnm}$ and $|\theta'| \leq |\theta|$, then $\Delta(|\theta|) = |\theta|(I)$, by the definition in 464I, so $\Delta(|\theta'|) = |\theta'|(I)$, by 464H(f-ii), and $\theta' \in M_{\rm pnm}$, by (a) above. Thus $M_{\rm pnm}$ is solid.

(ii) If θ_1, $\theta_2 \in M_{\rm pnm}$, then

$$\Delta(|\theta_1| + |\theta_2|) = \Delta(|\theta_1|) + \Delta(|\theta_2|) = |\theta_1|(I) + |\theta_2|(I) = (|\theta_1| + |\theta_2|)(I)$$

(using 464Hb), and $|\theta_1| + |\theta_2| \in M_{\rm pnm}$; as $|\theta_1 + \theta_2| \leq |\theta_1| + |\theta_2|$, $\theta_1 + \theta_2 \in M_{\rm pnm}$. Thus $M_{\rm pnm}$ is closed under addition.

(iii) It follows from (ii) that if $\theta \in M_{\rm pnm}$ then $n\theta \in M_{\rm pnm}$ for every integer $n \geq 1$, and then from (i) that $\alpha\theta \in M_{\rm pnm}$ for every $\alpha \in \mathbb{R}$; so that $M_{\rm pnm}$ is closed under scalar multiplication, and is a linear subspace.

464M Theorem (Fremlin & Talagrand 79) Let I be any set. Write M for the L-space of bounded finitely additive functionals on $\mathcal{P}I$, and $M_{\rm m}$, $M_{\rm pnm}$ for the spaces of measurable and purely non-measurable functionals, as in 464K-464L. Then $M_{\rm m}$ and $M_{\rm pnm}$ are complementary bands in M.

proof (a) We know from 464K and 464L that these are both solid linear subspaces of M. Next, $M_{\rm m} \cap M_{\rm pnm} = \{0\}$. **P** If θ belongs to the intersection, then $\Delta(|\theta|) = \frac{1}{2}|\theta|(I) = |\theta|(I)$, by 464Kb and 464La; so $\theta = 0$. **Q**

(b) Now recall that every element of M^+ is expressible in the form $\theta_1+\theta_2$ where $\theta_1 \in M_{\rm m}^+$ and $\theta_2 \in M_{\rm pnm}^+$; this is 464He, using 464Kb and 464La again. Because $M_{\rm m}$ and $M_{\rm pnm}$ are linear subspaces, with intersection $\{0\}$, $M = M_{\rm m} \oplus M_{\rm pnm}$. Now $M_{\rm pnm} \subseteq M_{\rm m}^\perp$, so $M_{\rm m} + M_{\rm m}^\perp = M$ and $M_{\rm m} = M_{\rm m}^{\perp\perp}$ is a complemented band (352R); similarly, $M_{\rm pnm}$ is a complemented band. Since $(M_{\rm m} + M_{\rm pnm})^\perp = \{0\}$, $M_{\rm m}$ and $M_{\rm pnm}$ are complementary bands (see 352S).

464N Corollary (FREMLIN & TALAGRAND 79) Let I be any set, and let μ be Talagrand's measure on $\mathcal{P}I$; write Σ for its domain. Then every bounded additive functional on $\mathcal{P}I$ is Σ-measurable.

proof Defining M, $M_{\rm m}$ and $M_{\rm pnm}$ as in 464K and 464L, we see that every functional in $M_{\rm m}$ is Σ-measurable because it is (by definition) $(\operatorname{dom}\nu)$-measurable, where ν is the usual measure on $\mathcal{P}I$. If $\theta \in M_{\rm pnm}^+$, then $\mathcal{F} = \{a : \theta a = \theta I\}$ is a non-measurable filter; but this means that $\mu\mathcal{F} = 1$, by the construction of μ, so that $\theta = \theta I$ μ-a.e. So if θ is any member of $M_{\rm pnm}$, both θ^+ and θ^- are Σ-measurable, and $\theta = \theta^+ - \theta^-$ also is.

464O Remark Note that we have a very simple description of the behaviour of additive functionals as seen by the measure μ. Since $M_\tau \subseteq M_{\rm m}$, we have a three-part band decomposition $M = M_\tau \oplus (M_{\rm m} \cap M_\tau^\perp) \oplus M_{\rm pnm}$.

(i) Functionals in M_τ are T-measurable, where T is the domain of ν, therefore Σ-measurable, just because they can be built up from the functionals $a \mapsto \chi a(t)$, as in 464Fb.

(ii) A functional in $M_{\rm m}$ is T-measurable, by definition; but a functional θ in $M_{\rm m} \cap M_\tau^\perp$ is actually constant, with value $\frac{1}{2}\theta I$, ν-almost everywhere. **P** For $\theta \geq 0$, this is 464Kb; the general case follows because $M_{\rm m} \cap M_\tau^\perp$ is solid in M. **Q** Thus the almost-constant nature of the functionals described in 464Jb is typical of measurable functionals in $M_\tau^\perp$.

(iii) Finally, a functional $\theta \in M_{\rm pnm}$ is equal to θI μ-almost everywhere; once again, this follows from the definition of 'purely non-measurable' and the construction of μ for $\theta \geq 0$, and from the fact that $M_{\rm pnm}$ is solid for other θ.

(iv) Thus we see that any $\theta \in M_\tau^\perp = (M_{\rm m} \cap M_\tau^\perp) \oplus M_{\rm pnm}$ is constant μ-a.e. We also have $\int \theta\,d\mu = \Delta(\theta)$ for every $\theta \geq 0$ (look at $\theta \in M_{\rm m}$ and $\theta \in M_{\rm pnm}$ separately, using 464La for the latter), so that if $h \in M^*$ is the linear functional of 464Hb, then $\int \theta\,d\mu = h(\theta)$ for every $\theta \in M$.

464P More on purely non-measurable functionals (a) We can discuss non-negative additive functionals on $\mathcal{P}I$ in terms of the Stone-Čech compactification βI of I, as follows. For any set $A \subseteq \beta I$ set $H_A = \{a : a \subseteq I,\ A \subseteq \widehat{a}\}$, where $\widehat{a} \subseteq \beta I$ is the open-and-closed set corresponding to $a \subseteq I$. If $A \neq \emptyset$, H_A is a filter on I. Write $\mathcal{A}$ for the family of those sets $A \subseteq \beta I$ such that $\nu^* H_A = 1$, where ν is the usual measure on $\mathcal{P}I$. Then $\mathcal{A}$ is a σ-ideal. **P** Of course $A \in \mathcal{A}$ whenever $A \subseteq B \in \mathcal{A}$, since then $H_A \supseteq H_B$. If $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{A}$ with union A, then $\nu^*(\bigcap_{n\in\mathbb{N}} H_{A_n}) = 1$, by 464C; but $H_A \supseteq \bigcap_{n\in\mathbb{N}} H_{A_n}$. **Q** Note that if $A \in \mathcal{A}$ then $\overline{A} \in \mathcal{A}$, because $H_A = H_{\overline{A}}$. We see also that $\{z\} \in \mathcal{A}$ for every $z \in \beta I \setminus I$ (since $H_{\{z\}}$ is a non-principal ultrafilter, as in 464Jb), while $\{t\} \notin \mathcal{A}$ for any $t \in I$ (since $H_{\{t\}} = \{a : t \in a\}$ has measure $\frac{1}{2}$).

Because βI can be identified with the Stone space of $\mathcal{P}I$ (4A2I(b-i)), we have a one-to-one correspondence between non-negative additive functionals θ on $\mathcal{P}I$ and Radon measures μ_θ on βI, defined by writing $\mu_\theta(\widehat{a}) = \theta a$ whenever $a \subseteq I$ and $\theta \in M^+$ (416Qb). (This is not the same as the measure μ_θ of part (e) of the proof of 464H, which is on a much larger space.) Now suppose that θ is a non-negative additive functional on $\mathcal{P}I$. Then $\mathcal{F}_\theta = \{a : \theta a = \theta I\}$ is either $\mathcal{P}I$ or a filter on I. If we set $F_\theta = \bigcap\{\widehat{a} : a \in \mathcal{F}_\theta\}$, then $\mathcal{F}_\theta = H_{F_\theta}$ (4A2I(b-iii)). Since $a \in \mathcal{F}_\theta$ iff $\theta a = \theta I$,

$$\begin{aligned} F_\theta &= \bigcap\{\widehat{a} : a \in \mathcal{F}_\theta\} = \bigcap\{\widehat{a} : \theta a = \theta I\} \\ &= \bigcap\{\widehat{a} : \mu_\theta \widehat{a} = \mu_\theta(\beta I)\} = \beta I \setminus \bigcup\{\widehat{a} : \mu_\theta \widehat{a} = 0\}. \end{aligned}$$

But this is just the support of μ_θ (411Nd), because $\{\widehat{a} : a \subseteq I\}$ is a base for the topology of βI.

Thus we see that $\theta \in M^+$ is purely non-measurable iff the support F_θ of the measure μ_θ belongs to $\mathcal{A}$. If you like, θ is purely non-measurable iff the support of μ_θ is 'small'.

(b) Yet another corollary of 464C is the following. Since M is a set of real-valued functions on $\mathcal{P}I$, it has the corresponding topology $\mathfrak{T}_p$ of pointwise convergence as a subspace of $\mathbb{R}^{\mathcal{P}I}$. Now if $C \subseteq M_{\rm pnm}$

is countable, its $\mathfrak{T}_p$-closure $\overline{C}$ is included in $M_{\rm pnm}$. **P** It is enough to consider the case in which C is non-empty and $0 \notin C$. For each $\theta \in C$, $\mathcal{F}_{|\theta|} = \{a : |\theta|(a) = |\theta|(I)\}$ is a filter with outer measure 1, so $\mathcal{F} = \{a : |\theta|(a) = |\theta|(I) \text{ for every } \theta \in C\}$ also has outer measure 1, by 464Cb. Now suppose that $\theta_0 \in \overline{C}$. If $a \in \mathcal{F}$, then $|\theta|(I \setminus a) = 0$ for every $\theta \in C$, that is, $\theta b = 0$ whenever $\theta \in C$ and $b \subseteq I \setminus a$ (362Ba). But this means that $\theta_0 b = 0$ whenever $b \subseteq I \setminus a$, so $|\theta_0|(I \setminus a) = 0$ and $|\theta_0|(a) = |\theta_0|(I)$. Thus $\mathcal{F}_{|\theta_0|} \supseteq \mathcal{F}$, so has outer measure 1, and θ_0 is also purely non-measurable. **Q**

(c) If $\theta \in M$ is such that $\theta a = 0$ for every countable set $a \subseteq I$, then $\theta \in M_{\rm pnm}$. **P** ν is inner regular with respect to the family $\mathcal{W}$ of sets which are determined by coordinates in countable subsets of I, by 254Ob. But if $W \in \mathcal{W}$ and $\nu W > 0$, let $J \subseteq I$ be a countable set such that W is determined by coordinates in J; then $|\theta|(J) = 0$, so if a is any member of W we shall have $a \cup (I \setminus J) \in W \cap \mathcal{F}_{|\theta|}$. As W is arbitrary, $\nu^* \mathcal{F}_{|\theta|} = 1$ and θ is purely non-measurable. **Q**

In particular, if $\theta \in M_\sigma \cap M_\tau^\perp$, where M_σ is the space of countably additive functionals on $\mathcal{P}I$ (362B), then $\theta \in M_{\rm pnm}$. (For 'ordinary' sets I, $M_\sigma = M_\tau$; see 438Xa. But this observation is peripheral to the concerns of the present section.)

In the language of (a) above, we have a closed set in βI, being $F = \beta I \setminus \bigcup\{\widehat{a} : a \in [I]^{\leq\omega}\}$; and if θ is such that the support of μ_θ is included in F, then θ is purely non-measurable.

464Q More on measurable functionals (a) We know that $M_{\rm m}$ is a band in M, and that it includes the band M_τ. So it is natural to look at the band $M_{\rm m} \cap M_\tau^\perp$. I showed in part (a) of the proof of 464K that if $\theta \in M_{\rm m} \cap M_\tau^\perp$, and $b \subseteq I$, we have $\theta(a \cap b) = \frac{1}{2}\theta b$ for ν-almost every a.

(b) It follows that if θ is any non-zero non-negative functional in $M_{\rm m} \cap M_\tau^\perp$, we can find a family $\langle a_\xi \rangle_{\xi<\omega_1}$ in $\mathcal{P}I$ which is independent in the sense that $\theta(\bigcap_{\xi\in K} a_\xi) = 2^{-\#(K)}\theta I$ for every non-empty finite $K \subseteq I$. **P** Choose the a_ξ inductively, observing that at the inductive step we have to satisfy only countably many conditions of the form $\theta(a_\xi \cap b) = \frac{1}{2}\theta b$, where b runs over the subalgebra generated by $\{a_\eta : \eta < \xi\}$; so that ν-almost any a will serve for a_ξ. **Q**

In terms of the associated measure μ_θ on βI, this means that $\langle (\widehat{a}_\xi)^\bullet \rangle_{\xi<\omega_1}$ is an uncountable independent family in the measure algebra of μ_θ, and μ_θ has Maharam type at least ω_1.

Turning this round, we see that if λ is a Radon measure on βI, of countable Maharam type, and $\lambda I = 0$, then the corresponding functional on $\mathcal{P}I$ is purely non-measurable.

[For a stronger result in this direction, see §531 in Volume 5.]

(c) Another striking property of measurable additive functionals is the following. If $\theta \in M_{\rm m} \cap M_\tau^\perp$, and $n \in \mathbb{N}$, then $\theta(a_0 \cap a_1 \cap \ldots \cap a_n) = 2^{-n-1}\theta I$ for ν^{n+1}-almost every $a_0, \ldots, a_n \subseteq I$, where ν^{n+1} is the product measure on $(\mathcal{P}I)^{n+1}$. **P** For $K \subseteq \{0, \ldots, n\}$ write $\psi_K(a_0, \ldots, a_n) = I \cap \bigcap_{i\in K} a_i \setminus \bigcup_{i\leq n, i\notin K} a_i$ for $a_0, \ldots, a_n \subseteq I$; for $\mathcal{S} \subseteq \mathcal{P}\{0, \ldots, n\}$ write $\phi_{\mathcal{S}}(\mathfrak{x}) = \bigcup_{K\in\mathcal{S}} \psi_K(\mathfrak{x})$ for $\mathfrak{x} \in (\mathcal{P}I)^{n+1}$. Then, for any $t \in I$,

$$\nu^{n+1}\{\mathfrak{x} : t \in \phi_{\mathcal{S}}(\mathfrak{x})\} = \sum_{K\in\mathcal{S}} \nu^{n+1}\{\mathfrak{x} : t \in \phi_K(\mathfrak{x})\} = 2^{-n-1}\#(\mathcal{S}).$$

Moreover, for different t, these sets are independent. So if $\#(\mathcal{S}) = 2^n$, that is, if $\mathcal{S}$ is just half of $\mathcal{P}\{0, \ldots, n\}$, then $\phi_{\mathcal{S}}$ will be inverse-measure-preserving, by the arguments in 464B. (In fact 464Bb is the special case $n = 2$, $\mathcal{S} = \{K : \#(K) \geq 2\}$.)

Accordingly we shall have $\theta(\phi_{\mathcal{S}}(\mathfrak{x})) = \frac{1}{2}\theta I$ for almost every $\mathfrak{x}$ whenever $\mathcal{S} \subseteq \mathcal{P}\{0, \ldots, n\}$ has 2^n members. Since there are only finitely many sets $\mathcal{S}$, the set E is ν^{n+1}-conegligible, where

$$E = \{\mathfrak{x} : \mathfrak{x} \in (\mathcal{P}I)^{n+1}, \theta(\phi_{\mathcal{S}}(\mathfrak{x})) = \frac{1}{2}\theta I \text{ whenever } \mathcal{S} \subseteq \mathcal{P}\{0, \ldots, n\} \text{ and } \#(\mathcal{S}) = 2^n\}.$$

But given $\mathfrak{x} = (a_0, \ldots, a_n) \in E$, let $K_1, \ldots, K_{2^{n+1}}$ be a listing of $\mathcal{P}\{0, \ldots, n\}$ in such an order that $\theta(\psi_{K_i}(\mathfrak{x})) \leq \theta(\psi_{K_j}(\mathfrak{x}))$ whenever $i \leq j$, and consider $\mathcal{S} = \{K_1, \ldots, K_{2^n}\}$; since

$$\sum_{i=1}^{2^n} \theta(\psi_{K_i}(\mathfrak{x})) = \theta(\phi_{\mathcal{S}}(\mathfrak{x})) = \frac{1}{2}\theta I = \frac{1}{2}\sum_{i=0}^{2^{n+1}} \theta(\psi_{K_i}(\mathfrak{x})),$$

we must have $\theta(\psi_{K_i}(\mathfrak{x})) = 2^{-n-1}\theta I$ for every i. In particular, $\theta(\psi_{\{0,\ldots,n\}}(\mathfrak{x})) = 2^{-n-1}\theta I$, that is, $\theta(\bigcap_{i\leq n} a_i) = 2^{-n-1}\theta I$. As this is true whenever $a_0, \ldots, a_n \in E$, we have the result. **Q**

464R A note on $\ell^\infty(I)$ As already noted in 464F, we have, for any set I, a natural additive map $\chi : \mathcal{P}I \to \ell^\infty(I) \cong L^\infty(\mathcal{P}I)$, giving rise to an isomorphism between the L-space M of bounded additive functionals on $\mathcal{P}I$ and $\ell^\infty(I)^*$. If we write $\tilde{\mu}$ for the image measure $\mu\chi^{-1}$ on $\ell^\infty(I)$, where μ is Talagrand's measure on $\mathcal{P}I$, and $\tilde{\Sigma}$ for the domain of $\tilde{\mu}$, then every member of $\ell^\infty(I)^*$ is $\tilde{\Sigma}$-measurable, by 464N. Thus $\tilde{\Sigma}$ includes the cylindrical σ-algebra of $\ell^\infty(I)$ (4A3T). We also have a band decomposition $\ell^\infty(I)^* = \ell^\infty(I)^*_{\rm m} \oplus \ell^\infty(I)^*_{\rm pnm}$ corresponding to the decomposition $M = M_{\rm m} \oplus M_{\rm pnm}$.

In this context, M_τ corresponds to $\ell^\infty(I)^\times$ (363K). Since we can identify $\ell^\infty(I)$ with $\ell^1(I)^* = \ell^1(I)^\times$ (243Xl), and $\ell^1(I)$, like any L-space, is perfect, $\ell^\infty(I)^\times$ is the canonical image of $\ell^1(I)$ in $\ell^\infty(I)^*$. Because any functional in $M_\tau^\perp$ is μ-almost constant (464O), any functional in $(\ell^\infty(I)^\times)^\perp$ will be $\tilde{\mu}$-almost constant.

464X Basic exercises (a) Let I be any set and λ a Radon measure on βI. Show that if the support of λ is a separable subset of $\beta I \setminus I$, then the corresponding additive functional on $\mathcal{P}I$ is purely non-measurable.

(b) Let I be any set, and $\tilde{\mu}$ the image of Talagrand's measure on $\ell^\infty(I)$, as in 464R. Show that $\tilde{\mu}$ has a barycenter in $\ell^\infty(I)$ iff I is finite.

464Y Further exercises (a) Show that there is a sequence $\langle \mathcal{F}_n \rangle_{n\in\mathbb{N}}$ of distinct non-principal ultrafilters on $\mathbb{N}$ with the following property: if we define $h(a) = \{n : a \in \mathcal{F}_n\}$ for $a \subseteq \mathbb{N}$, then $\{h(a) : a \subseteq \mathbb{N}\}$ is ν-negligible.

(b) Let μ be Talagrand's measure on $\mathcal{P}\mathbb{N}$, and let λ be the corresponding product measure on $X = \mathcal{P}\mathbb{N} \times \mathcal{P}\mathbb{N}$. Define $\Phi : X \to \ell^\infty$ by setting $\Phi(a,b) = \chi a - \chi b$ for all $a, b \subseteq \mathbb{N}$. Show that Φ is Pettis integrable (463Ya) with indefinite Pettis integral Θ defined by setting $(\Theta E)(n) = \int_E f_n d\lambda$, where $f_n(a,b) = \chi a(n) - \chi b(n)$. Show that $K = \{h\Phi : h \in \ell^{\infty *}, \|h\| \leq 1\}$ contains every f_n, and in particular is not $\mathfrak{T}_m$-compact, so the identity map from $(K, \mathfrak{T}_p)$ to $(K, \mathfrak{T}_m)$ is not continuous.

464Z Problem Let I be an infinite set, and $\tilde{\mu}$ the image in $\ell^\infty(I)$ of Talagrand's measure on $\mathcal{P}I$ (464R). Is $\tilde{\mu}$ a topological measure for the weak topology of $\ell^\infty(I)$?

464 Notes and comments The central idea of this section appears in 464B: the algebraic structure of $\mathcal{P}I$ leads to a variety of inverse-measure-preserving functions ϕ from powers $(\mathcal{P}I)^K$ to $\mathcal{P}I$. The simplest of these is the measure space automorphism $a \mapsto I \setminus a$, as used in the proofs of 464Ca, 464Hf, 464Ka and 464Kb. Then we have the map $(a,b,c) \mapsto (a \cap b) \cup (a \cap c) \cup (b \cap c)$, as in 464Bb, and the generalization of this in the argument of 464Qc; and, most important of all, the map $\langle \langle a_{ni} \rangle_{i<m(n)} \rangle_{n\in\mathbb{N}} \mapsto \bigcup_{n\in\mathbb{N}} \bigcap_{i<m(n)} a_{ni}$ of 464Ba. In each case we can use probabilistic intuitions to guide us to appropriate formulae, since the events $t \in \phi(\mathfrak{x})$ are always independent, so if they have probability $\frac{1}{2}$ for every $t \in I$, the function ϕ will be inverse-measure-preserving. Of course this depends on the analysis of product measures in §254. It means also that we must use the 'ordinary' product measure defined there; but happily this coincides with the 'Radon' product measure of §417 (416U).

Talagrand devised his measure when seeking an example of a pointwise compact set of measurable functions which is not compact for the topology of convergence in measure, as in 464E. The remarkable fact that it is already, in effect, a measure on the cylindrical σ-algebra of ℓ^∞ (464R) became apparent later, and requires a much more detailed analysis. An alternative argument not explicitly involving the Riesz space structure of the space M of bounded additive functionals may be found in Fremlin & Talagrand 79. The proof I give here depends on the surprising fact that, for non-negative additive functionals, the upper integral $\overline{\int} d\nu$ is additive (464Hb), even though the functionals may be very far from being measurable. Once we know this, we can apply the theory of Banach lattices to investigate the corresponding linear functional on the L-space M. There is a further key step in 464Hd. We there have a non-negative $\theta \in M_\tau^\perp$ such that $\theta I = 1$ and $\beta\theta' I \leq \overline{\int} \theta' d\nu \leq \gamma\theta' I$ whenever $0 \leq \theta' \leq \theta$. It is easy to deduce that $\{a : \alpha\nu b \leq \theta(a \cap b) \leq \gamma\theta b$ for every $b \in K\}$ has outer measure 1 for every finite $K \subseteq \mathcal{P}I$ and $\alpha < \beta$. If α and γ are very close, this means that there will be many families (a,b,c) in $\mathcal{P}I$ which, as measured by θ, look like independent sets of measure γ, so that $\theta((a\cap b)\cup(a\cap c)\cup(b\cap c)) \eqsim 3\gamma^2 - 2\gamma^3$. But since we must also have $\theta((a\cap b)\cup(a\cap c)\cup(b\cap c)) \leq \gamma$ almost everywhere, we can get information about the possible values of γ.

Once having noted this remarkable dichotomy between 'measurable' and 'purely non-measurable' functionals, it is natural to look for other ways in which they differ. The position seems to be that 'simple' Radon measures on $\beta I \setminus I$ (e.g., all measures with separable support (464Xa) or countable Maharam type (464Qb)) have to correspond to purely non-measurable functionals. Of course the simplest possible measures on βI are those concentrated on I, which give rise to functionals which belong to M_τ and are therefore measurable; it is functionals in $M_{\rm m} \cap M_\tau^\perp$ which have to give rise to 'complicated' Radon measures.

The fact that every element of $M_\tau^\perp$ is almost constant for Talagrand's measure leads to an interesting Pettis integrable function (464Yb). The suggestion that Talagrand's measure on ℓ^∞ might be a topological measure for the weak topology (464Z) is a bold one, but no more outrageous than the suggestion that it might measure every continuous linear functional once seemed. Talagrand's measure does of course measure every Baire set for the weak topology (4A3U). I note here that the usual measure on $\{0,1\}^I$, when transferred to $\ell^\infty(I)$, is actually a Radon measure for the weak* topology $\mathfrak{T}_s(\ell^\infty, \ell^1)$, because on $\{0,1\}^I$ this is just the ordinary topology.

465 Stable sets

The structure of general pointwise compact sets of measurable functions is complex and elusive. One particular class of such sets, however, is relatively easy to describe, and has a variety of remarkable properties, some of them relevant to important questions arising in the theory of empirical measures. In this section I outline the theory of 'stable' sets of measurable functions from Talagrand 84 and Talagrand 87.

The first steps are straightforward enough. The definition of stable set (465B) is not obvious, but given this the basic properties of stable sets listed in 465C are natural and easy to check, and we come quickly to the fact that (for complete locally determined spaces) pointwise bounded stable sets are relatively pointwise compact sets of measurable functions (465D). A less transparent, but still fairly elementary, argument leads to the next reason for looking at stable sets: the topology of pointwise convervence on a stable set is finer than the topology of convergence in measure (465G).

At this point we come to a remarkable fact: a uniformly bounded set A of functions on a complete probability space is stable if and only if certain laws of large numbers apply 'nearly uniformly' on A. These laws are expressed in conditions (ii), (iv) and (v) of 465M. For singleton sets A, they can be thought of as versions of the strong law of large numbers described in §273. To get the full strength of 465M a further idea in this direction needs to be added, described in 465H here.

The theory of stable sets applies in the first place to sets of true functions. There is however a corresponding notion applicable in function spaces, which I explore briefly in 465O-465R. Finally, I mention the idea of 'R-stable' set (465S-465U), obtained by using τ-additive product measures instead of c.l.d. product measures in the definition.

465A Notation Throughout this section, I will use the following notation.

(a) If X is a set and Σ a σ-algebra of subsets of X, I will write $\mathcal{L}^0(\Sigma)$ for the space of Σ-measurable functions from X to $\mathbb{R}$, as in §463.

(b) I will identify $\mathbb{N}$ with the set of finite ordinals, so that each $n \in \mathbb{N}$ is the set of its predecessors, and a power X^n becomes identified with the set of functions from $\{0, \dots, n-1\}$ to X.

(c) If (X, Σ, μ) is any measure space, then for finite sets I (in particular, if $I = k \in \mathbb{N}$) I write μ^I for the c.l.d. product measure on X^I, as defined in 251W. (For definiteness, let us take μ^0 to be the unique probability measure on $X^0 = \{\emptyset\}$.) If (X, Σ, μ) is a probability space, then for any set I μ^I is to be the product probability measure on X^I, as defined in §254.

(d) We shall have occasion to look at free powers of algebras of sets. If X is a set and Σ is an algebra of subsets of X, then for any set I write $\bigotimes_I \Sigma$ for the algebra of subsets of X^I generated by sets of the form $\{w : w(i) \in E\}$ where $i \in I$ and $E \in \Sigma$, and $\widehat{\bigotimes}_I \Sigma$ for the σ-algebra generated by $\bigotimes_I \Sigma$.

(e) Now for a new idea, which will be used in almost every paragraph of the section. If X is a set, $A \subseteq \mathbb{R}^X$ a set of real-valued functions defined on X, $E \subseteq X$, $\alpha < \beta$ in $\mathbb{R}$ and $k \geq 1$, write

$$D_k(A,E,\alpha,\beta) = \bigcup_{f\in A}\{w : w\in E^{2k}, f(w(2i))\le\alpha, \\ f(w(2i+1))\ge\beta \text{ for } i<k\}.$$

We are now ready for the main definition.

465B Definition Let (X,Σ,μ) be a semi-finite measure space. Following TALAGRAND 84, I say that a set $A\subseteq\mathbb{R}^X$ is **stable** if whenever $E\in\Sigma$, $0<\mu E<\infty$ and $\alpha<\beta$ in $\mathbb{R}$, there is some $k\ge 1$ such that $(\mu^{2k})^*D_k(A,E,\alpha,\beta)<(\mu E)^{2k}$.

Remark I hope that the next few results will show why this concept is important. It is worth noting at once that these sets D_k need not be measurable, and that some of the power of the definition derives precisely from the fact that quite naturally arising sets A can give rise to non-measurable sets $D_k(A,E,\alpha,\beta)$. If, however, the set A is countable, then all the corresponding D_k will be measurable; this will be important in the results following 465R.

465C I start with a list of the 'easy' properties of stable sets, derivable more or less directly from the definition.

Proposition Let (X,Σ,μ) be a semi-finite measure space.

(a) If $A\subseteq\mathbb{R}^X$ is stable, then any subset of A is stable.

(b) If $A\subseteq\mathbb{R}^X$ is stable, then $\overline{A}$, the closure of A in $\mathbb{R}^X$ for the topology of pointwise convergence, is stable.

(c) Suppose that $A\subseteq\mathbb{R}^X$, $E\in\Sigma$, $n\ge 1$ and $\alpha<\beta$ are such that $0<\mu E<\infty$ and $(\mu^{2n})^*D_n(A,E,\alpha,\beta)<(\mu E)^{2n}$. Then

$$\frac{1}{(\mu E)^{2k}}(\mu^{2k})^*D_k(A,E,\alpha,\beta)\to 0$$

as $k\to\infty$.

(d) If A, $B\subseteq\mathbb{R}^X$ are stable, then $A\cup B$ is stable.

(e) If $A\subseteq\mathbb{R}^X$ is stable, then $\gamma A=\{\gamma f : f\in A\}$ is stable, for any $\gamma\in\mathbb{R}$.

(f) If $A\subseteq\mathbb{R}^X$ is stable and $g\in\mathcal{L}^0=\mathcal{L}^0(\Sigma)$, then $A+g=\{f+g : f\in A\}$ is stable.

(g) If $A\subseteq\mathcal{L}^0$ is finite it is stable.

(h) If $A\subseteq\mathbb{R}^X$ is stable so is $\{|f| : f\in A\}$.

(i) If $A\subseteq\mathbb{R}^X$ is stable and $g\in\mathcal{L}^0$, then $A\times g=\{f\times g : f\in A\}$ is stable.

(j) If $\hat{\mu}$, $\tilde{\mu}$ are the completion and c.l.d. version of μ, then $A\subseteq\mathbb{R}^X$ is stable with respect to one of the measures μ, $\hat{\mu}$, $\tilde{\mu}$ iff it is stable with respect to the others.

(k) Let ν be an indefinite-integral measure over μ (§234). If $A\subseteq\mathbb{R}^X$ is stable with respect to μ, it is stable with respect to ν and with respect to $\nu\restriction\Sigma$.

(l) Let $h:\mathbb{R}\to\mathbb{R}$ be a continuous non-decreasing function. If $A\subseteq\mathbb{R}^X$ is stable, so is $\{hf : f\in A\}$.

(m) In particular, if $A\subseteq\mathbb{R}^X$ is stable, so is $\{f^+ : f\in A\}$.

proof (a) This is immediate from the definition in 465B, since $D_k(B,E,\alpha,\beta)\subseteq D_k(A,E,\alpha,\beta)$ for all k, E, α and β.

(b) Given E such that $0<\mu E<\infty$, and $\alpha<\beta$, take α', β' such that $\alpha<\alpha'<\beta'<\beta$. Then it is easy to see that $D_k(\overline{A},E,\alpha,\beta)\subseteq D_k(A,E,\alpha',\beta')$, so

$$(\mu^{2k})^*D_k(\overline{A},E,\alpha,\beta)\le(\mu^{2k})^*D_k(A,E,\alpha',\beta')<(\mu E)^{2k}$$

for some $k\ge 1$. As E, α and β are arbitrary, $\overline{A}$ is stable.

(c) For any $m\ge 1$ and $l<n$, if we identify $X^{2(mn+l)}$ with $(X^{2n})^m\times X^{2l}$, we see that $D_{mn+l}(A,E,\alpha,\beta)$ becomes identified with a subset of $D_n(A,E,\alpha,\beta)^m\times E^{2l}$. (If $w\in D_{mn+l}(A,E,\alpha,\beta)$, there is an $f\in A$ such that $f(w(2i))\le\alpha$, $f(w(2i+1))\ge\beta$ for $i<mn+l$. Now $(w(2rn),w(2rn+1),\dots,w(2rn+2n-1))\in D_n(A,E,\alpha,\beta)$ for $r<m$.) So

$$\begin{aligned}\tfrac{1}{(\mu E)^{2(mn+l)}}(\mu^{2(mn+l)})^*D_{mn+l}(A,E,\alpha,\beta)&\\ \leq \tfrac{1}{(\mu E)^{2(mn+l)}}\big((\mu^{2n})^*D_n(A,E,\alpha,\beta)\big)^m(\mu E)^{2l}&\\ = \big(\tfrac{1}{(\mu E)^{2n}}(\mu^{2n})^*D_n(A,E,\alpha,\beta)\big)^m \to 0&\end{aligned} \tag{251Wm}$$

as $m \to \infty$.

(d) Note that $D_k(A \cup B, E, \alpha, \beta) = D_k(A, E, \alpha, \beta) \cup D_k(B, E, \alpha, \beta)$ for all E, k, α and β. Now, given that $0 < \mu E < \infty$ and $\alpha < \beta$, there are m, $n \geq 1$ such that $(\mu^{2m})^*D_m(A, E, \alpha, \beta) < (\mu E)^{2m}$ and $(\mu^{2n})^*D_n(A, E, \alpha, \beta) < (\mu E)^{2n}$. So, by (c) above,

$$\begin{aligned}\tfrac{1}{(\mu E)^{2k}}(\mu^{2k})^*D_k(A \cup B, E, \alpha, \beta)&\\ \leq \tfrac{1}{(\mu E)^{2k}}\big((\mu^{2k})^*D_k(A, E, \alpha, \beta) + (\mu^{2k})^*D_k(B, E, \alpha, \beta)\big) \to 0&\end{aligned}$$

as $k \to \infty$, and there is some k such that $(\mu^{2k})^*D_k(A \cup B, E, \alpha, \beta) < (\mu E)^{2k}$. As E, α and β are arbitrary, $A \cup B$ is stable.

(e)(i) If $\gamma > 0$, $D_k(\gamma A, E, \alpha, \beta) = D_k(A, E, \alpha/\gamma, \beta/\gamma)$ for all k, E, α and β, so the result is elementary. Similarly, if $\gamma = 0$, then $D_k(\gamma A, E, \alpha, \beta) = \emptyset$ whenever $k \geq 1$, $E \in \Sigma$ and $\alpha < \beta$, so again we see that γA is stable.

(ii) If $\gamma = -1$ then, for any k, E, α and β,

$$\begin{aligned}D_k(-A, E, \alpha, \beta) &= \bigcup_{f \in A}\{w : w \in E^{2k},\ f(w(2i)) \geq -\alpha,\\ &\qquad f(w(2i+1)) \leq -\beta\ \forall\ i < k\}\\ &= \phi[D_k(A, E, -\beta, -\alpha)],\end{aligned}$$

where $\phi : X^{2k} \to X^{2k}$ is the measure space automorphism defined by setting

$$\phi(w) = (w(1), w(0), w(3), w(2), \dots, w(2k-1), w(2k-2))$$

for $w \in X^{2k}$. So, given $E \in \Sigma$ and $\alpha < \beta$, there is a $k \geq 1$ such that

$$(\mu^{2k})^*D_k(-A, E, \alpha, \beta) = (\mu^{2k})^*D_k(A, E, -\beta, -\alpha) < (\mu E)^{2k}.$$

So $-A$ is stable.

(iii) Together with (i) this shows that γA is stable for every $\gamma \in \mathbb{R}$.

(f) Take E such that $0 < \mu E < \infty$, and $\alpha < \beta$. Set $\eta = \frac{1}{2}(\beta - \alpha) > 0$. Then there is a $\gamma \in \mathbb{R}$ such that $F = \{x : x \in E, \gamma \leq g(x) \leq \gamma + \eta\}$ has non-zero measure. Set $\alpha' = \alpha - \gamma$, $\beta' = \beta - \gamma - \eta$. Then $D_k(A + g, F, \alpha, \beta) \subseteq D_k(A, F, \alpha', \beta')$, while $\alpha' < \beta'$. So if we take $k \geq 1$ such that $(\mu^{2k})^*D_k(A, F, \alpha', \beta') < (\mu F)^{2k}$, then

$$(\mu^{2k})^*D_k(A + g, E, \alpha, \beta) \leq (\mu^{2k})^*D_k(A + g, F, \alpha, \beta) + \mu^{2k}(E^{2k} \setminus F^{2k}) < (\mu E)^{2k}.$$

(g) If A is empty, or contains only the constant function $\mathbf{0}$ with value 0, this is trivial. Now (f) tells us that $\{g\} = \{\mathbf{0}\} + g$ is stable for every $g \in \mathcal{L}^0$, and from (d) it follows that any finite subset of $\mathcal{L}^0$ is stable.

(h) Writing $|A|$ for $\{|f| : f \in A\}$, then

$$D_k(|A|, E, \alpha, \beta) \subseteq D_k(A, E, \alpha, \beta) \cup D_k(-A, E, \alpha, \beta) = D_k(A \cup (-A), E, \alpha, \beta)$$

whenever $0 \leq \alpha < \beta$. So (d) and (e) tell us that in this case there is some k such that $D_k(|A|, E, \alpha, \beta)$ has outer measure less than $(\mu E)^{2k}$.

(i) Let $E \in \Sigma$, $\alpha < \beta$ be such that $0 < \mu E < \infty$. Set

$$E_0 = \{x : x \in E,\, g(x) = 0\}, \quad E_1 = \{x : x \in E,\, g(x) > 0\},$$

$$E_2 = \{x : x \in E,\, g(x) < 0\}.$$

(i) Suppose that $\mu E_0 > 0$. Then $D_1(A \times g, E, \alpha, \beta)$ does not meet E_0, so $\mu^* D_1(A \times g, E, \alpha, \beta) < \mu E$.

(ii) Suppose that $\mu E_1 > 0$. Let $\eta > 0$ be such that

$$\max(\alpha, \tfrac{\alpha}{1+\eta}) = \alpha' < \beta' = \min(\beta, \tfrac{\beta}{1+\eta}).$$

Let $\gamma > 0$ be such that $\mu F > 0$, where $F = \{x : x \in E,\, \gamma \leq g(x) \leq \gamma(1+\eta)\}$. If $x \in F$ then

$$f(x)g(x) \leq \alpha \implies f(x) \leq \frac{\alpha}{g(x)} \leq \frac{\alpha'}{\gamma},$$

$$f(x)g(x) \geq \beta \implies f(x) \geq \frac{\beta}{g(x)} \geq \frac{\beta'}{\gamma}.$$

So

$$D_k(A \times g, F, \alpha, \beta) \subseteq D_k(A, F, \frac{\alpha'}{\gamma}, \frac{\beta'}{\gamma})$$

for every k. Now, because A is stable, there is some $k \geq 1$ such that

$$(\mu^{2k})^* D_k(A, F, \frac{\alpha'}{\gamma}, \frac{\beta'}{\gamma}) < (\mu F)^{2k},$$

and in this case $(\mu^{2k})^* D_k(A \times g, E, \alpha, \beta) < (\mu E)^{2k}$, just as in the argument for (f) above.

(iii) If $\mu E_2 > 0$, then we know from (e) that $-A$ is stable, so (ii) tells us that there is a $k \geq 1$ such that $(\mu^{2k})^* D_k((-A) \times (-g), E, \alpha, \beta) < (\mu E)^{2k}$.

Since one of these three cases must occur, and since E, α and β are arbitrary, $A \times g$ is stable.

(j) The product measures μ^{2k}, $\hat{\mu}^{2k}$ and $\tilde{\mu}^{2k}$ are all the same (251Wn), so this follows immediately from the definition in 465B.

(k) Let h be a Radon-Nikodým derivative of ν with respect to μ (234B). Suppose that $0 < \nu E < \infty$ and $\alpha < \beta$. Then there is an $F \in \Sigma$ such that $F \subseteq E \cap \operatorname{dom} h$, $h(x) > 0$ for every $x \in F$, and $0 < \mu F < \infty$. There is a $k \geq 1$ such that $(\mu^{2k})^* D_k(A, F, \alpha, \beta) < (\mu F)^{2k}$, that is, there is a $W \subseteq F^{2k} \setminus D_k(A, F, \alpha, \beta)$ such that $\mu^{2k} W > 0$. In this case, $\nu^{2k} W > 0$. **P** Set $\tilde{h}(w) = \prod_{i=0}^{2k-1} h(w(i))$ for $w \in (\operatorname{dom} h)^{2k}$. Then ν^{2k} is the indefinite integral of $\tilde{h}$ with respect to μ^{2k} (253I), and $\tilde{h}(w) > 0$ for every $w \in F^{2k}$. **Q** Since $W \subseteq E^{2k} \setminus D_k(A, E, \alpha, \beta)$, $(\nu^{2k})^* D_k(A, E, \alpha, \beta) < (\nu E)^{2k}$; as E, α and β are arbitrary, A is stable with respect to ν.

Since ν is the completion of its restriction $\nu{\restriction}\Sigma$ (234Db), A is also stable with respect to $\nu{\restriction}\Sigma$, by (j).

(l) Write $B = \{hf : f \in A\}$. Suppose that $0 < \mu E < \infty$ and $\alpha < \beta$. If either $\alpha < h(\gamma)$ for every $\gamma \in \mathbb{R}$ or $h(\gamma) < \beta$ for every $\gamma \in \mathbb{R}$,

$$\mu^* D_1(B, E, \alpha, \beta) = \mu\emptyset = 0 < \mu E.$$

Otherwise, because h is continuous, the Intermediate Value Theorem tells us that there are $\alpha' < \beta'$ such that $\alpha < h(\alpha') < h(\beta') < \beta$. In this case $D_k(B, E, \alpha, \beta) \subseteq D_k(A, E, \alpha', \beta')$ for every k. Because A is stable, there is some $k \geq 1$ such that $(\mu^{2k})^* D_k(A, E, \alpha', \beta') < (\mu E)^{2k}$, so that $(\mu^{2k})^* D_k(B, E, \alpha, \beta) < (\mu E)^{2k}$. As E, α and β are arbitrary, B is stable.

(m) follows at once.

465D Now for the first result connecting the notion of 'stable' set with the concerns of this chapter.

Proposition Let (X,Σ,μ) be a complete locally determined measure space, and $A\subseteq\mathbb{R}^X$ a stable set.

(a) $A\subseteq\mathcal{L}^0=\mathcal{L}^0(\Sigma)$ (that is, every member of A is Σ-measurable).

(b) If $\{f(x):f\in A\}$ is bounded for each $x\in X$, then A is relatively compact in $\mathcal{L}^0$ for the topology of pointwise convergence.

proof (a) **?** Suppose, if possible, that there is a non-measurable $f\in A$. Then there is an $\alpha\in\mathbb{R}$ such that $D_0=\{x:f(x)>\alpha\}\notin\Sigma$. Because μ is locally determined, there is an $F_0\in\Sigma$ such that $\mu F_0<\infty$ and $D_0\cap F_0\notin\Sigma$. Let $F_1\subseteq F_0$ be a measurable envelope of $D_0\cap F_0$ (132Ee). Then $D_0\cap F_1=D_0\cap F_0$ is not measurable; because μ is complete, $F_1\setminus D_0$ cannot be negligible. Now $D_0=\bigcup_{n\in\mathbb{N}}\{x:f(x)\geq\alpha+2^{-n}\}$, so there is some $\beta>\alpha$ such that $D_1=F_1\cap\{x:f(x)\geq\beta\}$ is not negligible. Let E be a measurable envelope of D_1. Then, setting $P=\{x:x\in E,\,f(x)\leq\alpha\}$, $Q=\{x:x\in E,\,f(x)\geq\beta\}$ we have $\mu^*P=\mu^*Q=\mu E>0$.

Now suppose that $k\geq 1$. Then $D_k(\{f\},E,\alpha,\beta)\supseteq(P\times Q)^k$, so

$$(\mu^{2k})^*D_k(\{f\},E,\alpha,\beta)=(\mu^*P\cdot\mu^*Q)^k=(\mu E)^{2k}$$

(251Wm again). Since this is true for every k, $\{f\}$ is not stable, and (by 465Ca) A cannot be stable; which contradicts our hypothesis. **X**

(b) Because $\{f(x):x\in A\}$ is bounded for each x, $\overline{A}$, the closure of A in $\mathbb{R}^X$, is compact for the topology of pointwise convergence. But $\overline{A}$ is stable, by 465Cb, so is included in $\mathcal{L}^0$, by (a).

465E The topology $\mathfrak{T}_s(\mathcal{L}^2,\mathcal{L}^2)$ Some of the arguments below will rely on ideas of compactness in function spaces. There are of course many ways of expressing the method, but a reasonably accessible one uses the Hilbert space L^2, as follows. Let (X,Σ,μ) be any measure space. Then $L^2=L^2(\mu)$ is a Hilbert space with a corresponding weak topology $\mathfrak{T}_s(L^2,L^2)$ defined by the functionals $u\mapsto(u|v)$ for $v\in L^2$. In the present section it will be more convenient to regard this as a topology $\mathfrak{T}_s(\mathcal{L}^2,\mathcal{L}^2)$ on the space $\mathcal{L}^2=\mathcal{L}^2(\mu)$ of square-integrable real-valued functions, defined by the functionals $f\mapsto\int f\times g$ for $g\in\mathcal{L}^2$. The essential fact we need is that norm-bounded sets are relatively weakly compact. In L^2, this is because Hilbert spaces are reflexive (4A4Ka). In $\mathcal{L}^2$, given an ultrafilter $\mathcal{F}$ containing a $\|\,\|_2$-bounded set $B\subseteq\mathcal{L}^2$, $v=\lim_{f\to\mathcal{F}}f^\bullet$ must be defined in L^2 for $\mathfrak{T}_s(L^2,L^2)$, and now there is a $g\in\mathcal{L}^2$ such that $g^\bullet=v$; in which case

$$\lim_{f\to\mathcal{F}}\int f\times h=\lim_{f\to\mathcal{F}}(f^\bullet|h^\bullet)=(g^\bullet|h^\bullet)=\int g\times h$$

for every $h\in\mathcal{L}^2$. Note that we are free to take g to be a Σ-measurable function with domain X (241Bk).

465F Lemma Let (X,Σ,μ) be a measure space, and $B\subseteq\mathcal{L}^2=\mathcal{L}^2(\mu)$ a $\|\,\|_2$-bounded set. Suppose that $h\in\mathcal{L}^2$ belongs to the closure of B for $\mathfrak{T}_s(\mathcal{L}^2,\mathcal{L}^2)$. Then for any $\delta>0$, $k\geq 1$ the set

$$W=\bigcup_{f\in B}\{w:w\in X^k,w(i)\in\operatorname{dom}f\cap\operatorname{dom}h \text{ and } f(w(i))\geq h(w(i))-\delta \text{ for every } i<k\}$$

is μ^k-conegligible in X^k.

proof (a) Since completing the measure μ does not change the space $\mathcal{L}^2$ (244Xa) nor the product measure μ^k (251Wn), we may suppose that μ is complete.

(b) The first substantive fact to note is that there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in B converging to h for $\mathfrak{T}_s=\mathfrak{T}_s(\mathcal{L}^2,\mathcal{L}^2)$. **P** Setting $C=\{f^\bullet:f\in B\}$, C is a bounded set in $L^2=L^2(\mu)$ and $h^\bullet$ belongs to the $\mathfrak{T}_s(L^2,L^2)$-closure of C. But L^2, being a normed space, is angelic in its weak topology (462D), and C is relatively compact in L^2, so there is a sequence in C converging to $h^\bullet$. We can represent this sequence as $\langle f_n^\bullet\rangle_{n\in\mathbb{N}}$ where $f_n\in B$ for every n, and now $\langle f_n\rangle_{n\in\mathbb{N}}\to h$ for $\mathfrak{T}_s$. **Q**

(c) The second component of the proof is the following simple idea. Suppose that $\langle E_n\rangle_{n\in\mathbb{N}}$ is a sequence in Σ such that $\bigcap_{n\in I}E_n$ is negligible for every infinite set $I\subseteq\mathbb{N}$. For $m\geq 1$, $I\subseteq\mathbb{N}$ set

$$V_m(I)=\textstyle\bigcap_{n\in I}\{w:w\in X^m,\,\exists\,i<m,\,w(i)\in E_n\}.$$

Then $\mu^mV_m(I)=0$ for every infinite $I\subseteq\mathbb{N}$. **P** Induce on m. For $m=1$ this is just the original hypothesis on $\langle E_n\rangle_{n\in\mathbb{N}}$. For the inductive step to $m+1$, identify μ^{m+1} with the product of μ^m and μ, and observe that

$$V_{m+1}(I)^{-1}[\{x\}] = \{w : (w,x) \in V_{m+1}(I)\} = V_m(\{n : n \in I,\ x \notin E_n\})$$

for every $x \in X$. Now, setting $I_x = \{n : n \in I,\ x \notin E_n\}$, $F = \{x : I_x \text{ is finite}\}$, we have

$$F = \bigcup_{r\in\mathbb{N}} \bigcap_{n\in I\setminus r} E_n,$$

so F is negligible, while if $x \notin F$ then $V_m(I_x)$ is negligible, by the inductive hypothesis. But this means that almost every horizontal section of $V_{m+1}(I)$ is negligible, and $V_{m+1}(I)$, being measurable, must be negligible, by Fubini's theorem (252F). Thus the induction proceeds. **Q**

(d) Now let us return to the main line of the argument from (b). For each $n \in \mathbb{N}$, set $E_n = \{x : x \in \operatorname{dom} f_n \cap \operatorname{dom} h,\ f_n(x) < h(x) - \delta\}$. If $I \subseteq \mathbb{N}$ is infinite, then $\bigcap_{n\in I} E_n$ is negligible. **P** Setting $G = \bigcap_{n\in I} E_n$, μG is finite (because $\delta\chi E_n \leq |h - f_n|$ a.e., so $\mu E_n < \infty$ for every n) and $\int_G f_n \leq \int_G h - \delta\mu G$ for every $n \in I$. But $\lim_{n\to\infty} \int_G f_n = \int_G h$ and I is infinite, so

$$\delta\mu G \leq \inf_{n\in I} |\textstyle\int_G h - \int_G f_n| = 0. \ \mathbf{Q}$$

By (c), it follows that

$$V = \bigcap_{n\in\mathbb{N}} \{w : w \in X^k,\ \exists\, i < k,\ w(i) \in E_n\}$$

is negligible. But if we set $Y = \bigcap_{n\in\mathbb{N}} \operatorname{dom} f_n \cap \operatorname{dom} h$, Y is a conegligible subset of X, Y^k is a conegligible subset of X^k, and $Y^k \setminus V \subseteq W$, so W is conegligible, as required.

465G Theorem Let (X, Σ, μ) be a semi-finite measure space, and $A \subseteq \mathcal{L}^0 = \mathcal{L}^0(\Sigma)$ a stable set of measurable functions. Let $\mathfrak{T}_p$ and $\mathfrak{T}_m$ be the topologies of pointwise convergence and convergence in measure, as in §463. Then the identity map from A to itself is $(\mathfrak{T}_p, \mathfrak{T}_m)$-continuous.

proof ? Suppose, if possible, otherwise.

(a) We must have an $f_0 \in A$, a set $F \in \Sigma$ of finite measure, and an $\epsilon > 0$ such that for every $\mathfrak{T}_p$-neighbourhood U of f_0 there is an $f \in A \cap U$ such that $\int_F \min(1, |f - f_0|)d\mu \geq \epsilon$. Set

$$B = \{\min(\chi F, |f - f_0|) : f \in A\}.$$

Then

$$B = \{\chi F - (\chi F - |f - f_0|)^+ : f \in A\}$$

is stable, by 465Cf, 465Ch, 465Cm and 465Ce, used repeatedly. Setting $B' = \{f : f \in B,\ \int f \geq \epsilon\}$, B' is again stable (465Ca). Our hypothesis is that f_0 is in the $\mathfrak{T}_p$-closure of $A' = \{f : f \in A,\ \int \min(\chi F, |f - f_0|) \geq \epsilon\}$; since $f \mapsto \min(\chi F, |f - f_0|)$ is $\mathfrak{T}_p$-continuous, 0 belongs to the $\mathfrak{T}_p$-closure of the image of A' under this map, which is B'.

(b) Let $\mathcal{F}$ be an ultrafilter on B' which $\mathfrak{T}_p$-converges to 0. Because B' is $\|\ \|_2$-bounded (since $0 \leq f \leq \chi F$ for every $f \in B'$), $\mathcal{F}$ also has a $\mathfrak{T}_s(\mathcal{L}^2, \mathcal{L}^2)$-limit h say, as noted in 465E; and we can suppose that h is measurable and defined everywhere. We must have

$$\textstyle\int_F h = \lim_{f\to\mathcal{F}} \int_F f \geq \epsilon > 0.$$

So there is a $\delta > 0$ such that $E = \{x : x \in F,\ h(x) \geq 3\delta\}$ has measure greater than 0.

(c) Because B' is stable, there must be some $k \geq 1$ such that $(\mu^{2k})^* D_k(B', E, \delta, 2\delta) < (\mu E)^{2k}$. Let $W \subseteq E^{2k} \setminus D_k(B', E, \delta, 2\delta)$ be a measurable set of positive measure. By Fubini's theorem, there must be $x_0, \ldots, x_{k-1}$ such that $\mu^k V$ is defined and greater than 0, where

$$V = \{v : v \in X^k,\ (x_0, v(0), x_1, v(1), \ldots, x_{k-1}, v(k-1)) \in W\}.$$

Set $C = \{f : f \in B',\ f(x_i) \leq \delta \text{ for every } i < k\}$; then $C \in \mathcal{F}$, because $\mathcal{F} \to 0$ for $\mathfrak{T}_p$. Accordingly h belongs to the $\mathfrak{T}_s(\mathcal{L}^2, \mathcal{L}^2)$-closure of C. But now 465F tells us that there must be a $v \in V$ and an $f \in C$ such that $f(v(i)) \geq h(v(i)) - \delta$ for every $i < k$.

Consider $w = (x_0, v(0), \ldots, x_{k-1}, v(k-1))$. We know that $w \in W$ (because $v \in V$), so, in particular, $w \in E^{2k}$ and $h(w(i)) \geq 3\delta$ for every $i < 2k$; accordingly

$$f(w(2i+1)) = f(v(i)) \geq h(v(i)) - \delta \geq 2\delta$$

for every $i < k$. On the other hand, $f(w(2i)) = f(x_i) \leq \delta$ for every $i < k$, because $f \in C$. But this means that f witnesses that $w \in D_k(B', E, \delta, 2\delta)$, which is supposed to be disjoint from W. **X**

This contradiction shows that the theorem is true.

465H We shall need some interesting and important general facts concerning powers of measures. I start with an important elaboration of the strong law of large numbers.

Theorem Let (X, Σ, μ) be any probability space. For $n \in \mathbb{N}$, write Λ_n for the domain of the product measure μ^n. For $w \in X^{\mathbb{N}}$, $k \geq 1$, $n \geq 1$ write ν_{wk} for the probability measure with domain $\mathcal{P}X$ defined by writing

$$\nu_{wk}(E) = \frac{1}{k}\#(\{i : i < k,\, w(i) \in E\})$$

for $E \subseteq X$, and ν_{wk}^n for the corresponding product measure on X^n.

Then whenever $n \geq 1$ and $f : X^n \to \mathbb{R}$ is bounded and Λ_n-measurable, $\lim_{k\to\infty} \int f d\nu_{wk}^n$ exists, and is equal to $\int f d\mu^n$, for $\mu^{\mathbb{N}}$-almost every $w \in X^{\mathbb{N}}$.

proof (a) For $k \geq n$ and $w \in X^{\mathbb{N}}$, set

$$h_k(w) = \tfrac{(k-n)!}{k!} \textstyle\sum_{\pi : n \to k \text{ is injective}} f(w\pi).$$

Setting $M = \sup_{v \in X^n} |f(v)|$, we have

$$\begin{aligned} |h_k(w) - \int f d\nu_{wk}^n| &= |\tfrac{(k-n)!}{k!} \sum_{\pi : n \to k \text{ is injective}} f(w\pi) - \tfrac{1}{k^n} \sum_{\pi : n \to k} f(w\pi)| \\ &\leq M\big(\tfrac{(k-n)!}{k!} - \tfrac{1}{k^n}\big) + \tfrac{M}{k^n}\big(k^n - \tfrac{k!}{(k-n)!}\big) \to 0 \end{aligned}$$

as $k \to \infty$, for every $w \in X^{\mathbb{N}}$.

(b) Write Λ for the domain of $\mu^{\mathbb{N}}$, and for $k \geq n$ set

$$\mathrm{T}_k = \{W : W \in \Lambda,\, w\pi \in W \text{ whenever } w \in W \text{ and } \pi \in S_k\},$$

where S_k is the set of bijections $\psi : \mathbb{N} \to \mathbb{N}$ such that $\psi(i) = i$ for $i \geq k$. Then $\langle \mathrm{T}_k \rangle_{k \geq n}$ is a non-increasing sequence of σ-subalgebras of Λ. For any injective function $\pi : n \to k$ there are just $(k-n)!$ extensions of π to a member of S_k. So

$$h_k(w) = \tfrac{1}{k!} \textstyle\sum_{\pi \in S_k} f(w\pi\restriction n)$$

for every w. Observe that $h_k(w\psi) = h_k(w)$ for every $\psi \in S_k$, $w \in X^{\mathbb{N}}$, so h_k is T_k-measurable. (Of course we need to look back at the definition of h_k to confirm that it is Λ-measurable.)

(c) For any $k \geq n$, h_k is a conditional expectation of h_n on T_k. **P** If $W \in \mathrm{T}_k$, then

$$\textstyle\int_W h_k(w)dw = \frac{1}{k!}\sum_{\pi \in S_k} \int_W f(w\pi\restriction n)dw = \frac{1}{k!}\sum_{\pi \in S_k} \int_W g(w\pi)dw$$

where $g(w) = f(w\restriction n)$ for $w \in X^{\mathbb{N}}$. Now observe that for every $\pi \in S_k$ the map $w \mapsto w\pi$ is a measure space automorphism of $X^{\mathbb{N}}$ which leaves W unchanged, because $W \in \mathrm{T}_k$; so that $\int_W g(w\pi)dw = \int_W g(w)dw$, by 235Ic. So $\int_W h_k = \int_W g$. But (since $W \in \mathrm{T}_k \subseteq \mathrm{T}_n$) $\int_W h_n$ is also equal to $\int_W g$, and $\int_W h_k = \int_W h_n$. As W is arbitrary, h_k is a conditional expectation of h_n on T_k. **Q**

(d) By the reverse martingale theorem (275K), $h_\infty(w) = \lim_{k\to\infty} h_k(w)$ is defined for almost every $w \in X^{\mathbb{N}}$. Accordingly $\lim_{k\to\infty} \int f d\nu_{wk}^n$ is defined for almost every w.

(e) To see that the limit is $\int f d\mu^n$, observe that if $W \in \mathrm{T}_\infty = \bigcap_{k\in\mathbb{N}} \mathrm{T}_k$ then $\mu^{\mathbb{N}}W$ must be either 0 or 1. **P** Set $\gamma = \mu^{\mathbb{N}}W$. Let $\epsilon > 0$. Then there is a $V \in \bigotimes_{\mathbb{N}} \Sigma$ (notation: 465Ad) such that $\mu^{\mathbb{N}}(W \triangle V) \leq \epsilon$ (254Fe). There is some k such that V is determined by coordinates in k. If we set $\pi(i) = 2k - i$ for $i < 2k$, i for $i \geq 2k$, then $V' = \{w\pi : w \in V\}$ is determined by coordinates in $2k \setminus k$, so $\mu^{\mathbb{N}}(V \cap V') = (\mu^{\mathbb{N}}V)^2$.

On the other hand, because $W \in \mathrm{T}_{2k}$, the measure space automorphism $w \mapsto w\pi$ does not move W, and $\mu^{\mathbb{N}}(W \setminus V') = \mu^{\mathbb{N}}(W \setminus V)$. Accordingly

$$\gamma = \mu^{\mathbb{N}}(W \cap W) \leq \mu^{\mathbb{N}}(V \cap V') + 2\mu^{\mathbb{N}}(W \setminus V) \leq (\mu^{\mathbb{N}}V)^2 + 2\epsilon \leq (\gamma + \epsilon)^2 + 2\epsilon.$$

As ϵ is arbitrary, $\gamma \leq \gamma^2$ and $\gamma \in \{0, 1\}$. **Q**

(e) Now 275K tells us that h_∞ is T_∞-measurable, therefore essentially constant, and must be equal to its expectation almost everywhere. But, setting $W = X^{\mathbb{N}}$ in (c), we see that $\int h_k = \int f(w\restriction n)dw = \int f d\mu^n$ for every k, so

$$\lim_{k\to\infty} \int f d\nu^n_{wk} = h_\infty(w) = \int f d\mu^n$$

for almost every w, as claimed.

465I Now for a string of lemmas, working towards the portmanteau Theorem 465M. The first is elementary.

Lemma Let X be a set, and Σ a σ-algebra of subsets of X. For $w \in X^{\mathbb{N}}$, $k \geq 1$, $n \geq 1$ write ν_{wk} for the probability measure with domain $\mathcal{P}X$ defined by writing

$$\nu_{wk}(E) = \frac{1}{k}\#(\{i : i < k,\, w(i) \in E\})$$

for $E \subseteq X$. Then for any $k \in \mathbb{N}$ and any set I, $w \mapsto \nu^I_{wk}(W)$ is $\widehat{\bigotimes}_{\mathbb{N}}\Sigma$-measurable (notation: 465Ad) for every $W \in \widehat{\bigotimes}_I \Sigma$.

proof Write $\mathcal{W}$ for the set of subsets W of X^I such that $w \mapsto \nu^I_{wk}(W)$ is $\widehat{\bigotimes}_{\mathbb{N}}\Sigma$-measurable. Then $X^I \in \mathcal{W}$, $W' \setminus W \in \mathcal{W}$ whenever $W, W' \in \mathcal{W}$ and $W \subseteq W'$, and $\bigcup_{n\in\mathbb{N}} W_n \in \mathcal{W}$ whenever $\langle W_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in $\mathcal{W}$. Write $\mathcal{V}$ for the set of Σ-cylinders in X^I, that is, sets expressible in the form $\{v : v(i) \in E_i \text{ for every } i \in J\}$, where $J \subseteq I$ is finite and $E_i \in \Sigma$ for $i \in J$. Then $\mathcal{V} \subseteq \mathcal{W}$. **P** If $J \subseteq I$ is finite and $E_i \in \Sigma$ for $i \in J$, then

$$w \mapsto \nu_{wk}E_i = \frac{1}{k}\sum_{j=0}^{k-1} \chi E_i(w(j))$$

is $\widehat{\bigotimes}_{\mathbb{N}}$-measurable for every $i \in J$. So

$$w \mapsto \nu^I_{wk}\{v : v(i) \in E_i \;\forall\, i \in J\} = \prod_{i\in J} \nu_{wk}E_i$$

is also measurable. **Q**

Because $V \cap V' \in \mathcal{V}$ for all V, $V' \in \mathcal{V}$, the Monotone Class Theorem (136B) tells us that $\mathcal{W}$ must include the σ-algebra generated by $\mathcal{V}$, which is $\widehat{\bigotimes}_I \Sigma$.

465J The next three lemmas are specifically adapted to the study of stable sets of functions.

Lemma Let (X, Σ, μ) be a probability space. For any $n \in \mathbb{N}$, $W \subseteq X^n$ I say that W is **symmetric** if $w\pi \in W$ whenever $w \in W$ and $\pi : n \to n$ is a bijection. Give each power X^n its product measure μ^n.

(a) Suppose that for each $n \geq 1$ we have a measurable set $W_n \subseteq X^n$, and that $W_{m+n} \subseteq W_m \times W_n$ for all m, $n \geq 1$, identifying X^{m+n} with $X^m \times X^n$. Then $\lim_{n\to\infty}(\mu^n W_n)^{1/n}$ is defined and equal to $\delta = \inf_{n\geq 1}(\mu^n W_n)^{1/n}$.

(b) Now suppose that each W_n is symmetric. Then there is an $E \in \Sigma$ such that $\mu E = \delta$ and $E^n \setminus W_n$ is negligible for every $n \in \mathbb{N}$.

(c) Next, let $\langle D_n\rangle_{n\geq 1}$ be a sequence of sets such that

$D_n \subseteq X^n$ is symmetric for every $n \geq 1$,

whenever $1 \leq m \leq n$, $v \in D_n$ then $v\restriction m \in D_m$.

Then $\delta = \lim_{n\to\infty}((\mu^n)^* D_n)^{1/n}$ is defined and there is an $E \in \Sigma$ such that $\mu E = \delta$ and $(\mu^n)^*(D_n \cap E^n) = (\mu E)^n$ for every $n \in \mathbb{N}$.

proof (a) For any $\eta > 0$, there is an $m \geq 1$ such that $\mu^m W_m \leq (\delta + \eta)^m$. If $n = mk + i$, where $k \geq 1$ and $i < m$, then (identifying X^n with $(X^m)^k \times X^i$) $W_n \subseteq (W_m)^k \times X^i$, so

$$\mu^n W_n \leq (\delta+\eta)^{mk} \leq \gamma(\delta+\eta)^{mk+i},$$

where

$$\gamma = \max_{i<m}(\tfrac{1}{\delta+\eta})^i.$$

So

$$\limsup_{n\to\infty}(\mu^n W_n)^{1/n} \leq (\delta+\eta)\limsup_{n\to\infty}\gamma^{1/n} = \delta+\eta.$$

As η is arbitrary,

$$\delta \leq \liminf_{n\to\infty}(\mu^n W_n)^{1/n} \leq \limsup_{n\to\infty}(\mu^n W_n)^{1/n} \leq \delta,$$

and $\lim_{n\to\infty}(\mu^n W_n)^{1/n} = \delta$.

(b) It is enough to consider the case $\delta > 0$.

(i) Consider the family $\mathbb{V}$ of sequences $\langle V_n\rangle_{n\geq 1}$ such that
for each $n \geq 1$, V_n is a symmetric measurable subset of X^n and $\mu^n V_n \geq \delta^n$,
if $1 \leq m \leq n$ then $v\restriction m \in V_m$ for every $v \in V_n$.

Observe that $\boldsymbol{W} = \langle W_n\rangle_{n\geq 1} \in \mathbb{V}$. Order $\mathbb{V}$ by saying that $\langle V_n\rangle_{n\geq 1} \leq \langle V'_n\rangle_{n\geq 1}$ if $V_n \subseteq V'_n$ for every n. For $\boldsymbol{V} = \langle V_n\rangle_{n\geq 1}$ in $\mathbb{V}$, set $\theta(\boldsymbol{V}) = \sum_{n=1}^{\infty} 2^{-n}\mu^n V_n$. Any non-increasing sequence $\langle\langle V_{kn}\rangle_{n\geq 1}\rangle_{k\in\mathbb{N}}$ in $\mathbb{V}$ has a lower bound $\langle\bigcap_{k\in\mathbb{N}} V_{kn}\rangle_{n\geq 1}$ in $\mathbb{V}$, so there must be a $\boldsymbol{W}' = \langle W'_n\rangle_{n\geq 1} \in \mathbb{V}$ such that $\boldsymbol{W}' \leq \boldsymbol{W}$ and $\theta(\boldsymbol{V}) = \theta(\boldsymbol{W}')$ whenever $\boldsymbol{V} \in \mathbb{V}$ and $\boldsymbol{V} \leq \boldsymbol{W}'$; that is, whenever $\langle V_n\rangle_{n\geq 1} \in \mathbb{V}$ and $V_n \subseteq W'_n$ for every $n \geq 1$, then $\mu^n V_n = \mu^n W'_n$ for every n.

(ii) For $x \in X$, $n \geq 1$ set $V_n^{(x)} = \{w : (x,w) \in W'_{n+1}\}$. Then $V_n^{(x)}$ is measurable for almost every x; let $X_1 \subseteq X$ be a conegligible set such that $V_n^{(x)}$ is measurable for every $x \in X_1$ and every $n \geq 1$. Every $V_n^{(x)}$ is symmetric, and if $1 \leq m \leq n$ and $v \in V_n^{(x)}$ then $v\restriction m \in V_m^{(x)}$. It follows that if m, $n \geq 1$ then $V_{m+n}^{(x)}$ becomes identified with a subset of $V_m^{(x)} \times V_n^{(x)}$.

From (a) we see that $\delta_x = \lim_{n\to\infty}(\mu^n V_n^{(x)})^{1/n}$ is defined for every $x \in X_1$. The map $x \mapsto \mu^n V_n^{(x)} : X_1 \to [0,\infty[$ is measurable for each n, by Fubini's theorem (252D), so $x \mapsto \delta_x$ is also measurable. Since $V_n^{(x)} \subseteq W'_n \subseteq W_n$ for every x and n, $\delta_x \leq \delta$ for every $x \in X_1$.

(iii) Set $E = \{x : x \in X_1, \delta_x = \delta\}$. Then $\mu E \geq \delta$. **P?** Otherwise, there is some $\beta < \delta$ such that $\mu F < \beta$, where $F = \{x : x \in X_1, \delta_x \geq \beta\}$. Now

$$\begin{aligned} X_1 \setminus F &= \{x : x \in X_1, \lim_{n\to\infty}(\mu^n V_n^{(x)})^{1/n} < \beta\} \\ &\subseteq \bigcup_{m\in\mathbb{N}}\bigcap_{n\geq m}\{x : x \in X_1, \mu^n V_n^{(x)} \leq \beta^n\}, \end{aligned}$$

so there is some $m \in \mathbb{N}$ such that $\mu H \geq 1-\beta$, where

$$H = \textstyle\bigcap_{n\geq m}\{x : x \in X_1, \mu^n V_n^{(x)} \leq \beta^n\}.$$

Set $\gamma_n = \mu^n W'_n \geq \delta^n$ for each n. Then, for any $n \geq m$,

$$\begin{aligned} \gamma_{n+1} = \mu^{n+1} W'_{n+1} &= \int_{X_1} \mu^n V_n^{(x)}\mu(dx) \\ &= \int_H \mu^n V_n^{(x)}\mu(dx) + \int_{X_1\setminus H} \mu^n V_n^{(x)}\mu(dx) \leq \beta^n + \beta\gamma_n \end{aligned}$$

because $V_n^{(x)} \subseteq W'_n$ for every x, and $\mu(X_1 \setminus H) \leq \beta$. An easy induction shows that $\gamma_{m+k} \leq k\beta^{m+k-1} + \beta^k\gamma_m$ for every $k \in \mathbb{N}$. But this means that

$$\delta^k = \delta^{-m}\delta^{m+k} \leq \delta^{-m}\gamma_{m+k} \leq \beta^k\delta^{-m}(k\beta^{m-1} + \gamma_m)$$

for every k; setting $\eta = (\delta-\beta)/\beta > 0$,

$$\tfrac{k(k-1)}{2}\eta \leq (1+\eta)^k = \big(\tfrac{\delta}{\beta}\big)^k \leq \delta^{-m}(k\beta^{m-1} + \gamma_m)$$

for every k, which is impossible. **XQ**

(iv) Next, for any $x \in E$, $\boldsymbol{V}^{(x)} = \langle V_n^{(x)} \rangle_{n\geq 1} \in \mathbb{V}$ and $\boldsymbol{V}^{(x)} \leq \boldsymbol{W}'$, so $\mu^n(W_n' \setminus V_n^{(x)}) = 0$ for every $n \geq 1$. This means that, given $n \geq 1$, every vertical section of $(E \times W_n') \setminus W_{n+1}'$ (regarded as a subset of $X \times X^n$) is negligible; so $(E \times W_n') \setminus W_{n+1}'$ is negligible. We are assuming that $\delta > 0$, so

$$E \subseteq \{x : x \in X_1, V_1^{(x)} \neq \emptyset\} \subseteq \{x : \{w : (x,w) \in W_2'\} \neq \emptyset\} \subseteq W_1'.$$

Now a simple induction shows that $E^n \setminus W_n'$ is negligible for every $n \geq 1$, so that $E^n \setminus W_n$ is negligible for every n, and we have an appropriate E. (Of course $\mu E = \delta$ exactly, because $(\mu E)^n \leq \mu W_n$ for every n.)

(c) For each $n \in \mathbb{N}$ let V_n be a measurable envelope of D_n in X^n. Define $\langle W_n \rangle_{n\geq 1}$ inductively by saying

$$W_1 = V_1,$$

$$W_{n+1} = \{w : w \in X^{n+1}, w\pi\restriction n \in W_n, \\ w\pi \in V_{n+1} \text{ for every bijection } \pi : n+1 \to n+1\}$$

for each $n \geq 1$. Then an easy induction on n shows that W_n is measurable and symmetric and that $D_n \subseteq W_n \subseteq V_n$, so that W_n is a measurable envelope of D_n and $\mu^n W_n = (\mu^n)^* D_n$.

Now $\langle W_n \rangle_{n\geq 1}$ satisfies the hypotheses of (b), so

$$\delta = \lim_{n\to\infty} (\mu^n W_n)^{1/n} = \lim_{n\to\infty} ((\mu^n)^* D_n)^{1/n}$$

is defined and there is a set $E \in \Sigma$, of measure δ, such that $E^n \setminus W_n$ is negligible for every n; but this means that

$$(\mu^n)^*(E^n \cap D_n) = \mu^n(E^n \cap W_n) = \delta^n$$

for every n, as required.

465K Lemma Let (X, Σ, μ) be a complete probability space, and $A \subseteq [0,1]^X$ a stable set. Suppose that $\epsilon > 0$ is such that $\int f d\mu \leq \epsilon^2$ for every $f \in A$. Then there are an $n \geq 1$ and a $W \in \widehat{\bigotimes}_n \Sigma$ (notation: 465Ad) and a $\gamma > \mu^n W$ such that $\int f d\nu \leq 3\epsilon$ whenever $f \in A$ and ν is a probability measure on X with domain including Σ such that $\nu^n W \leq \gamma$.

proof (a) For $n \in \mathbb{N}$ write $\tilde{C}_n = \bigcup_{f\in A} \{x : f(x) \geq \epsilon\}^n$. Then $\langle \tilde{C}_n \rangle_{n\in\mathbb{N}}$ satisfies the conditions of 465Jc, so $\delta = \lim_{n\to\infty} ((\mu^n)^* \tilde{C}_n)^{1/n}$ is defined, and there is an $E \in \Sigma$ such that $\mu E = \delta$ and $(\mu^n)^*(E^n \cap \tilde{C}_n) = \delta^n$ for every $n \in \mathbb{N}$. Now, for $B \subseteq [0,1]^X$ and $n \in \mathbb{N}$, write $C_n(B) = \bigcup_{f\in B} \{x : x \in E, f(x) \geq \epsilon\}^n$, so that $C_n(A) = E^n \cap \tilde{C}_n$, and $(\mu^n)^* C_n(A) = \delta^n$ for every n. For any $B \subseteq [0,1]^X$, $\langle C_n(B) \rangle_{n\in\mathbb{N}}$ also satisfies the conditions of 465Jc, and $\delta_B = \lim_{n\to\infty} ((\mu^n)^* C_n(B))^{1/n}$ is defined; we have $\delta_A = \delta$.

(b) If B, $B' \subseteq [0,1]^X$, then

$$(\mu^n)^* C_n(B) \leq (\mu^n)^* C_n(B \cup B') \leq (\mu^n)^* C_n(B) + (\mu^n)^* C_n(B')$$

for every n, so $\delta_B \leq \delta_{B\cup B'} \leq \max(\delta_B, \delta_{B'})$. It follows that if

$$\mathcal{G} = \{G : G \subseteq [0,1]^X \text{ is } \mathfrak{T}_p\text{-open}, \delta_{G\cap A} < \delta\},$$

where $\mathfrak{T}_p$ is the usual topology of $[0,1]^X$, no finite subfamily of $\mathcal{G}$ can cover A. Accordingly, since the $\mathfrak{T}_p$-closure $\overline{A}$ of A is $\mathfrak{T}_p$-compact, there is an $h \in \overline{A}$ such that $\delta_{G\cap A} = \delta$ for every $\mathfrak{T}_p$-open set G containing h.

(c) At this point recall that every function in $\overline{A}$ is measurable (465Cb, 465Da) and that $\int : \overline{A} \to [0,1]$ is $\mathfrak{T}_p$-continuous (465G). So $\int h \leq \epsilon^2$ and $\mu\{x : h(x) \geq \epsilon\} \leq \epsilon$.

? Suppose, if possible, that $\delta > \epsilon$. Then there is some $\eta > 0$ such that $\mu F > 0$, where $F = \{x : x \in E, h(x) < \epsilon - \eta\}$. For $k \in \mathbb{N}$, $u \in F^k$ set $G_u = \{f : f \in [0,1]^X, f(u(i)) < \epsilon - \eta \text{ for every } i < k\}$. Then G_u is an open neighbourhood of h, so $\delta_{G_u\cap A} = \delta$ and $(\mu^k)^*(C_k(G_u \cap A)) \geq \delta^k$. But because $F \subseteq E$ and $C_k(G_u \cap A) \subseteq E^k$, this means that $(\mu^k)^*(F^k \cap C_k(G_u \cap A)) = (\mu F)^k$.

For u, $v \in X^k$ write $u\#v$ for $(u(0), v(0), u(1), v(1), \dots, u(k-1), v(k-1)) \in X^{2k}$. Then

$$C_k(G_u \cap A) \cap F^k \subseteq \{v : u\#v \in D_k(A, F, \epsilon - \eta, \epsilon)\}$$

for any $u \in F^k$. So $(\mu^k)^*\{v : u\#v \in D_k(A, F, \epsilon - \eta, \epsilon)\} = (\mu F)^k$ for every $u \in F^k$. But this means that $(\mu^{2k})^* D_k(A, F, \epsilon - \eta, \epsilon) = (\mu F)^{2k}$. Since this is so for every $k \geq 1$, A is not stable. **X**

(d) Thus $\delta \leq \epsilon$. There is therefore some $n \geq 1$ such that $(\mu^n)^*\tilde{C}_n < (2\epsilon)^n$. Let $W \in \widehat{\bigotimes}_n \Sigma$ be a measurable envelope of $\tilde{C}_n$, and try $\gamma = (2\epsilon)^n$. If ν is any probability measure on X with domain including Σ such that $\nu^n W \leq \gamma$, then for any $f \in A$ we have

$$\{x : f(x) \geq \epsilon\}^n \subseteq \tilde{C}_n \subseteq W, \quad (\nu\{x : f(x) \geq \epsilon\})^n \leq \nu^n W \leq (2\epsilon)^n,$$

so that $\nu\{x : f(x) \geq \epsilon\} \leq 2\epsilon$. As $0 \leq f(x) \leq 1$ for every $x \in X$, $\int f d\nu \leq 3\epsilon$, as required.

465L Lemma (Talagrand 87) Let (X, Σ, μ) be a complete probability space, and $A \subseteq [0,1]^X$ a set which is not stable. Then there are measurable functions h_0, $h_1 : X \to [0,1]$ such that $\int h_0 \, d\mu < \int h_1 \, d\mu$ and $(\mu^{2k})^* \tilde{D}_k = 1$ for every $k \geq 1$, where

$$\tilde{D}_k = \bigcup_{f \in A} \{w : w \in X^{2k}, f(w(2i)) \leq h_0(w(2i)), \\ f(w(2i+1)) \geq h_1(w(2i+1)) \text{ for every } i < k\}. \qquad (*)$$

proof The proof divides into two cases.

case 1 Suppose that there is an ultrafilter $\mathcal{F}$ on A such that the $\mathfrak{T}_p$-limit g_0 of $\mathcal{F}$ is not measurable. Let h_0', h_1' be measurable functions such that $h_0' \leq g_0 \leq h_1'$ and $\int h_0' = \underline{\int} g_0$, $\int h_1' = \overline{\int} g_0$ (133Ja). Then $\delta = \frac{1}{5}\int h_1' - h_0' > 0$ (133Jd). Set $h_0 = h_0' + 2\delta\chi X$, $h_1 = h_1' - 2\delta\chi X$, so that $\int h_0 < \int h_1$.

Set $Q_0 = \{x : x \in X, g_0(x) \leq h_0(x) - \delta\}$. Then $\mu^* Q_0 = 1$. **P?** Otherwise, there is a non-negligible measurable set $F \subseteq X \setminus Q_0$. But in this case $h_0' + \delta\chi F \leq g_0$ and $\underline{\int} g_0 > \int h_0'$, which is impossible. **XQ** Similarly, $\mu^* Q_1 = 1$, where $Q_1 = \{x : g_0(x) \geq h_1(x) + \delta\}$.

If $k \geq 1$, $x_0, \ldots, x_{k-1} \in Q_0$ and $y_0, \ldots, y_{k-1} \in Q_1$, then there is an $f \in A$ such that $|f(x_i) - g_0(x_i)| \leq \delta$, $|f(y_i) - g_0(y_i)| \leq \delta$ for every $i < k$. But this means that $f(x_i) \leq h_0(x_i)$ and $f(y_i) \geq h_1(y_i)$ for every $i < k$, and $(x_0, y_0, \ldots, x_{k-1}, y_{k-1}) \in \tilde{D}_k$. Thus

$$\tilde{D}_k \supseteq Q_0 \times Q_1 \times Q_0 \times Q_1 \times \ldots \times Q_0 \times Q_1,$$

which has full outer measure, by 254L. As k is arbitrary, we have found appropriate h_0, h_1 in this case.

case 2 Now suppose that for every ultrafilter $\mathcal{F}$ on A, the $\mathfrak{T}_p$-limit of $\mathcal{F}$ is measurable.

(i) We are supposing that A is not stable, so there are $E \in \Sigma$ and $\alpha < \beta$ such that $\mu E > 0$ and $(\mu^{2k})^* D_k(A, E, \alpha, \beta) = (\mu E)^{2k}$ for every $k \geq 1$. The first thing to note is that if $\mathcal{I}$ is the set of those $B \subseteq A$ for which there is some $k \in \mathbb{N}$ such that $(\mu^{2k})^* D_k(B, E, \alpha, \beta) < (\mu E)^{2k}$, then $\mathcal{I}$ is an ideal of subsets of A. **P** Of course $\emptyset \in \mathcal{I}$ and $B \in \mathcal{I}$ whenever $B \subseteq B' \in \mathcal{I}$. Also 465Cc tells us that, if $B \in \mathcal{I}$, then $\lim_{k\to\infty} \frac{1}{(\mu E)^{2k}} (\mu^{2k})^* D_k(B, E, \alpha, \beta) = 0$. It follows easily (as in the proof of 465Cd) that $B \cup B' \in \mathcal{I}$ for all B, $B' \in \mathcal{I}$. **Q**

(ii) $\mathcal{I}$ is a proper ideal of subsets of A (by the choice of E, k, α and β), so there is an ultrafilter $\mathcal{F}$ on A such that $\mathcal{F} \cap \mathcal{I} = \emptyset$. Let g_0 be the $\mathfrak{T}_p$-limit of $\mathcal{F}$. Then g_0 is measurable.

Set $\delta = \frac{1}{3}(\beta - \alpha)\mu E > 0$, and define h_0, h_1 by setting

$$h_0(x) = \alpha \text{ if } x \in E, \\ = g_0(x) + \delta \text{ if } x \in X \setminus E, \\ h_1(x) = \beta \text{ if } x \in E, \\ = g_0(x) - \delta \text{ if } x \in X \setminus E.$$

Then

$$\int h_1 - \int h_0 \geq (\beta - \alpha)\mu E - 2\delta\mu(X \setminus E) > 0.$$

(iii) We shall need to know a little more about sets of the form $D_k(B, E, \alpha, \beta)$ for $B \in \mathcal{F}$. In fact, if $B \subseteq A$ and $B \notin \mathcal{I}$, then for any finite sets I and J

$$(\mu^I \times \mu^J)^* D_{I,J}(B, E, \alpha, \beta) = (\mu E)^{\#(I)+\#(J)},$$

where $D_{I,J}(B, E, \alpha, \beta)$ is

$$\bigcup_{f \in B} \{(u, v) : u \in E^I,\ v \in E^J,\ f(u(i)) \leq \alpha \text{ for } i \in I,\ f(v(j)) \geq \beta \text{ for } j \in J\}.$$

P We may suppose that $I = k$ and $J = l$ where $k, l \in \mathbb{N}$. Take $m = \max(1, k, l)$. Then we have an inverse-measure-preserving map $\phi : X^{2m} \to X^I \times X^J$ defined by saying that $\phi(w) = (u, v)$ where $u(i) = w(2i)$ for $i < k$ and $v(i) = w(2i + 1)$ for $i < l$. **?** If $(\mu^I \times \mu^J)^* D_{I,J}(B, E, \alpha, \beta) < (\mu E)^{k+l}$, there is a non-negligible measurable set $V \subseteq (E^I \times E^J) \setminus D_{I,J}(B, E, \alpha, \beta)$. Now $\phi^{-1}[V]$ is non-negligible and depends only on coordinates in $\{2i : i < k\} \cup \{2i + 1 : i < l\}$, so

$$\mu^{2m}(E^{2m} \cap \phi^{-1}[V]) = \mu^{2m}(\phi^{-1}[V]) \cdot (\mu E)^{2m-k-l} > 0.$$

But $\phi^{-1}[V] \cap D_m(B, E, \alpha, \beta) = \emptyset$, so $(\mu^{2m})^* D_m(B, E, \alpha, \beta) < (\mu E)^{2m}$, and $B \in \mathcal{I}$. **X** So $(\mu^I \times \mu^J)^* D_{I,J}(B, E, \alpha, \beta) = (\mu E)^{k+l}$, as required. **Q**

(iv) **?** Suppose, if possible, that $k \geq 1$ is such that $(\mu^{2k})^* \tilde{D}_k < 1$, where $\tilde{D}_k$ is defined from h_0 and h_1 by the formula (*) in the statement of the lemma. Let $W \subseteq X^{2k}$ be a measurable set of positive measure disjoint from $\tilde{D}_k$. For I, $J \subseteq k$ write W_{IJ} for

$$\{w : w \in W, w(2i) \in E \text{ for } i \in I,\ w(2i) \notin E \text{ for } i \in k \setminus I,$$
$$w(2i + 1) \in E \text{ for } i \in J,\ w(2i + 1) \notin E \text{ for } i \in k \setminus J\}.$$

Then there are I, $J \subseteq K$ such that $\mu^{2k} W_{IJ} > 0$.

We can identify X^{2k} with $X^I \times X^J \times X^{k \setminus I} \times X^{k \setminus J}$, matching any $w \in X^{2k}$ with (w_0, w_1, w_2, w_3) where

$$w_0(i) = w(2i) \text{ for } i \in I,$$
$$w_1(i) = w(2i + 1) \text{ for } i \in J,$$
$$w_2(i) = w(2i) \text{ for } i \in k \setminus I,$$
$$w_3(i) = w(2i + 1) \text{ for } i \in k \setminus J.$$

Write $\tilde{W}$ for the image of W_{IJ} under this matching. The condition $W_{IJ} \cap \tilde{D}_k = \emptyset$ translates into

(†) whenever $(w_0, w_1, w_2, w_3) \in \tilde{W}$, $f \in A$,
either there is an $i \in I$ such that $f(w_0(i)) > \alpha$
or there is an $i \in J$ such that $f(w_1(i)) < \beta$
or there is an $i \in k \setminus I$ such that $f(w_2(i)) > g_0(w_2(i)) + \delta$
or there is an $i \in k \setminus J$ such that $f(w_3(i)) < g_0(w_3(i)) - \delta$.

(v) By Fubini's theorem, applied to $(X^I \times X^J) \times (X^{k \setminus I} \times X^{k \setminus J})$, we can find $w_2 \in X^{k \setminus I}$, $w_3 \in X^{k \setminus J}$ such that $(\mu^I \times \mu^J)(V)$ is defined and greater than 0, where $V = \{(w_0, w_1) : (w_0, w_1, w_2, w_3) \in \tilde{W}\}$. Set

$$B = \{f : f \in A, |f(w_2(i)) - g_0(w_2(i))| \leq \delta \text{ for } i \in k \setminus I,$$
$$|f(w_3(i)) - g_0(w_3(i))| \leq \delta \text{ for } i \in k \setminus J\}.$$

Then $B \in \mathcal{F}$, because $\mathcal{F} \to g_0$ for $\mathfrak{T}_p$. So $(\mu^I \times \mu^J)^* D_{I,J}(B, E, \alpha, \beta) = (\mu E)^{\#(I)+\#(J)}$, by (iii) above. Since $W_{IJ} \subseteq W \subseteq E^{2k}$, V is included in $E^I \times E^J$ and meets $D_{I,J}(B, E, \alpha, \beta)$; that is, there are $f \in B$ and $(w_0, w_1) \in V$ such that $f(w_0(i)) \leq \alpha$ for $i \in I$ and $f(w_1(i)) \geq \beta$ for $i \in J$. But because $f \in B$ we also have $f(w_2(i)) \leq g_0(w_2(i)) + \delta$ for $i \in k \setminus I$ and $f(w_3(i)) \geq g_0(w_3(i)) - \delta$ for $i \in k \setminus J$; which contradicts the list in (†) above. **X**

(vi) Thus $(\mu^{2k})^* \tilde{D}_k = 1$ for every k, and in this case also we have an appropriate pair h_0, h_1.

465M Theorem (TALAGRAND 82, TALAGRAND 87) Let (X, Σ, μ) be a complete probability space, and A a non-empty uniformly bounded set of real-valued functions defined on X. Then the following are equiveridical:

(i) A is stable.

(ii) Every function in A is measurable, and $\lim_{k\to\infty} \sup_{f\in A} |\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \int f| = 0$ for almost every $w \in X^{\mathbb{N}}$.

(iii) Every function in A is measurable, and for every $\epsilon > 0$ there are a finite subalgebra T of Σ in which every atom of T is non-negligible and a sequence $\langle h_k\rangle_{k\geq 1}$ of measurable functions on $X^{\mathbb{N}}$ such that

$$h_k(w) \geq \sup_{f\in A} \frac{1}{k}\sum_{i=0}^{k-1} \big|f(w(i)) - \mathbb{E}(f|\mathrm{T})(w(i))\big|$$

for every $w \in X^{\mathbb{N}}$, $k \geq 1$, and

$$\limsup_{k\to\infty} h_k(w) \leq \epsilon$$

for almost every $w \in X^{\mathbb{N}}$. (Here $\mathbb{E}(f|\mathrm{T})$ is the (unique) conditional expectation of f on T.)

(iv) $\lim_{k,l\to\infty} \sup_{f\in A} |\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l}\sum_{i=0}^{l-1} f(w(i))| = 0$ for almost every $w \in X^{\mathbb{N}}$.

(v) $\lim_{k,l\to\infty} \overline{\int} \sup_{f\in A} |\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l}\sum_{i=0}^{l-1} f(w(i))| \mu^{\mathbb{N}}(dw) = 0$.

proof All the statements (i)-(v) are unaffected by translations (by constant functions) and scalar multiplications of the set A, so it will be enough to consider the case in which $A \subseteq [0,1]^X$.

As in 465I-465H, I write $\nu_{wk}(E) = \frac{1}{k}\#(\{i : i < k, w(i) \in E\})$ for $w \in X^{\mathbb{N}}$, $k \geq 1$ and $E \subseteq X$; so for any function $f : X \to \mathbb{R}$, $\int f d\nu_{wk} = \frac{1}{k}\sum_{i=0}^{r-1} f(w(i))$.

(a)(i)⇒(iii)(α) If A is stable, then every function in A is measurable, by 465Da. Let $\epsilon > 0$. Set $\eta = \frac{1}{27}\epsilon^2 > 0$. By 465Db, the $\mathfrak{T}_p$-closure $\overline{A}$ of A in $\mathbb{R}^X$ is a $\mathfrak{T}_p$-compact set of measurable functions, and by 465G it is $\mathfrak{T}_m$-compact; because A is uniformly bounded, it must be totally bounded for the norm $\|\ \|_1$. So there are $f_0, \dots, f_m \in A$ such that for every $f \in A$ there is an $i \leq m$ such that $\|f - f_i\|_1 \leq \eta$. Let T_0 be the finite subalgebra of Σ generated by the sets $\{x : j\eta \leq f_i(x) < (j+1)\eta\}$ for $i \leq m$ and $j \leq \frac{1}{\eta}$. Then T_0 may have negligible atoms, but if we absorb these into non-negligible atoms we get a finite subalgebra T of Σ such that $|f_i(x) - \mathbb{E}(f_i|\mathrm{T})(x)| \leq \eta$ for almost every $x \in X$, every $i \leq m$. (Because T is a finite algebra with non-negligible atoms, two T-measurable functions which are equal almost everywhere must be identical, and we have unique conditional expectations with respect to T.) Since $\|\mathbb{E}(f|\mathrm{T}) - \mathbb{E}(g|\mathrm{T})\|_1 \leq \|f - g\|_1$ for all integrable functions f and g (242Je), $\|f - \mathbb{E}(f|\mathrm{T})\|_1 \leq 3\eta$ for every $f \in A$.

(β) Set $A' = \{f - \mathbb{E}(f|\mathrm{T}) : f \in A\}$. Then A' is stable. **P** Suppose that $\mu E > 0$ and $\alpha < \beta$. The set $B = \{\mathbb{E}(f|\mathrm{T}) : f \in A\}$ is a uniformly bounded subset of a finite-dimensional space of functions, so is $\|\ \|_\infty$-compact. So there are $g_0, \dots, g_r \in B$ such that $B \subseteq \bigcup_{i\leq r}\{g : \|g - g_i\|_\infty \leq \frac{1}{3}(\beta - \alpha)\}$. By 465Cf and 465Cd, $C = \bigcup_{i\leq r} A - g_i$ is stable. So there is a $k \geq 1$ such that

$$(\mu^{2k})^* D_k(C, E, \tfrac{2}{3}\alpha + \tfrac{1}{3}\beta, \tfrac{1}{3}\alpha + \tfrac{2}{3}\beta) < (\mu E)^{2k}.$$

But for every $g \in A'$ there is an $h \in C$ such that $\|g - h\|_\infty \leq \frac{1}{3}(\beta - \alpha)$, so

$$D_k(A', E, \alpha, \beta) \subseteq D_k(C, E, \tfrac{2}{3}\alpha + \tfrac{1}{3}\beta, \tfrac{1}{3}\alpha + \tfrac{2}{3}\beta),$$

$$(\mu^{2k})^* D_k(A', E, \alpha, \beta) < (\mu E)^{2k}.$$

As E, α and β are arbitrary, A' is stable. **Q**

(γ) By 465Ch, $\{|g| : g \in A'\}$ is stable. By 465K there are an $n \geq 1$, a $W \in \widehat{\bigotimes}_n \Sigma$ and a $\gamma > \mu^n W$ such that $\int |g| d\nu \leq 3\sqrt{3\eta} = \epsilon$ whenever $g \in A'$ and ν is a probability measure on X, with domain Σ, such that $\nu^n W \leq \gamma$. Now ν_{wk} is a probability measure on X; set $q_k(w) = \nu_{wk}^n(W)$. Applying 465H to the characteristic function of W, we see that $\lim_{k\to\infty} q_k(w) = \mu^n W$ for almost every $w \in X^{\mathbb{N}}$. Also, because $W \in \widehat{\bigotimes}_n \Sigma$, every q_k is measurable, by 465I.

Set $h_k(w) = 1$ if $q_k(w) > \gamma$, ϵ if $q_k(w) \leq \gamma$. Then every h_k is measurable and $\lim_{k\to\infty} h_k(w) = \epsilon$ for almost every w.

For any $w \in X^{\mathbb{N}}$ and any $f \in A$, $g = f - \mathbb{E}(f|\mathrm{T}) \in A'$ and $\|g\|_\infty \leq 1$. So either $h_k(w) = 1$ and certainly $\int |g| d\nu_{wk} \leq h_k(w)$, or $h_k(w) = \epsilon$, $\nu^n_{wk}(W) \leq \gamma$ and $\int |g| d\nu_{wk} \leq \epsilon$. Thus we have

$$\tfrac{1}{k}\textstyle\sum_{i=0}^{k-1} \big|f(w(i)) - \mathbb{E}(f|\mathrm{T})(w(i))\big| = \int \big|f - \mathbb{E}(f|\mathrm{T})\big| d\nu_{wk} \leq h_k(w)$$

for every $w \in X^{\mathbb{N}}$ and every $f \in A$, as required by (iii).

(b)(iii)⇒(ii) & (v) Set

$$g_k(w) = \sup_{f\in A} |\tfrac{1}{k}\textstyle\sum_{i=0}^{k-1} f(w(i)) - \int f d\mu|,$$

$$g'_{kl}(w) = \sup_{f\in A} |\tfrac{1}{k}\textstyle\sum_{i=0}^{k-1} f(w(i)) - \tfrac{1}{l}\sum_{i=0}^{l-1} f(w(i))|,$$

for $w \in X^{\mathbb{N}}$ and k, $l \geq 1$. Let $\epsilon > 0$. Let T, $\langle h_k\rangle_{k\geq 1}$ be as in (iii), and let $E_0, \dots, E_r$ be the atoms of T. For $w \in X^{\mathbb{N}}$, $k \geq 1$, $j \leq r$ set $q_{kj}(w) = |\mu E_j - \nu_{wk} E_j|$. Then for any $f \in A$, $\mathbb{E}(f|\mathrm{T})$ is expressible as $\sum_{j=0}^r \alpha_j \chi E_j$ where $\alpha_j \in [0,1]$ for every j (remember that $A \subseteq [0,1]^X$), so

$$\begin{aligned}
|\tfrac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \int f d\mu| &= |\int f d\nu_{wk} - \int f d\mu| \\
&\leq |\int f d\nu_{wk} - \int \mathbb{E}(f|\mathrm{T}) d\nu_{wk}| + |\int \mathbb{E}(f|\mathrm{T}) d\nu_{wk} - \int \mathbb{E}(f|\mathrm{T}) d\mu| \\
&\leq \tfrac{1}{k}\sum_{i=0}^{k-1} \big|f(w(i)) - \mathbb{E}(f|\mathrm{T})(w(i))\big| + \sum_{j=0}^r \alpha_j \big|\mu E_j - \nu_{wk} E_j\big| \\
&\leq h_k(w) + \sum_{j=0}^r q_{kj}(w), \\
|\tfrac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \tfrac{1}{l}\sum_{i=0}^{l-1} f(w(i))\big| & \\
&\leq h_k(w) + \sum_{j=0}^r q_{kj}(w) + h_l(w) + \sum_{j=0}^r q_{lj}(w).
\end{aligned}$$

Taking the supremum over f, we have

$$g_k(w) \leq h_k(w) + \textstyle\sum_{j=0}^r q_{kj}(w),$$

$$g'_{kl}(w) \leq h_k(w) + \textstyle\sum_{j=0}^r q_{kj}(w) + \sum_{j=0}^r q_{lj}(w) + h_l(w).$$

But, for each $j \leq r$, $\lim_{k\to\infty} q_{kj}(w) = 0$ for almost every w, by 465H (or 273J) applied to the characteristic function of E_j. So

$$\limsup_{k\to\infty} g_k(w) \leq \limsup_{k\to\infty} h_k(w) \leq \epsilon$$

for almost every w. At the same time,

$$\limsup_{k,l\to\infty} \overline{\textstyle\int} g'_{kl} \leq \limsup_{k,l\to\infty} \textstyle\int h_k + \sum_{j=0}^r \int q_{kj} + \sum_{j=0}^r \int q_{lj} + \int h_l \leq 2\epsilon.$$

As ϵ is arbitrary, $\{w : \limsup_{k\to\infty} g_k(w) \geq 2^{-i}\}$ is negligible for every $i \in \mathbb{N}$, and $\lim_{k\to\infty} g_k = 0$ almost everywhere, as required, while equally $\lim_{k,l\to\infty} \overline{\int} g'_{kl} = 0$. Thus (ii) and (v) are true.

(c)(ii)⇒(iv) is trivial, since

$$|\tfrac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \tfrac{1}{l}\sum_{i=0}^{l-1} f(w(i))| \leq |\tfrac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \int f| + |\tfrac{1}{l}\sum_{i=0}^{l-1} f(w(i)) - \int f|.$$

(d) not-(i)⇒not-(iv) & not-(v) Now suppose that A is not stable.

(α) In this case, by 465L, there are measurable functions h_0, $h_1 : X \to [0,1]$ such that $\int h_0 d\mu < \int h_1 d\mu$ and $(\mu^{2k})^* \tilde{D}_k = 1$ for every $k \in \mathbb{N}$, where

$$\tilde{D}_k = \bigcup_{f \in A} \{w : w \in X^{2k}, f(w(2i)) \le h_0(w(2i)), \\ f(w(2i+1)) \ge h_1(w(2i+1)) \text{ for every } i < k\}.$$

(β) Set $\delta = \frac{1}{4}\int h_1 - h_0 > 0$. Let $k_0 \ge 1$ be so large that

$$\mu^k\{w : w \in X^k, |\int h_j - \tfrac{1}{k}\textstyle\sum_{i=0}^{k} h_j(w(i))| \le \delta\} \ge \frac{1}{2}$$

for both j and for every $k \ge k_0$ (273J or 465H. The point is that

$$\begin{aligned}\mu^k\{w : w \in X^k, |\int h_j - \frac{1}{k}\sum_{i=0}^{k} h_j(w(i))| \le \delta\} \\ &= \mu^{\mathbb{N}}\{w : w \in X^{\mathbb{N}}, |\int h_j - \frac{1}{k}\sum_{i=0}^{k} h_j(w(i))| \le \delta\} \\ &\to 1 \text{ as } k \to \infty.)\end{aligned}$$

Let $\langle k_n\rangle_{n\ge 1}$ be such that $k_n \ge \frac{2}{\delta}\sum_{i=0}^{n-1} k_i$ for every $n \ge 1$. Since $\delta \le \frac{1}{4}$, every k_n is at least as large as k_0.

(γ) Set $m_n = 2\sum_{i<n} k_i$ for each $n \in \mathbb{N}$. Then we have a measure space isomorphism $\phi : \prod_{n\in\mathbb{N}} X^{2k_n} \to X^{\mathbb{N}}$ defined by setting

$$\phi(w)(m_n + i) = w(n)(2i), \quad \phi(w)(m_n + k_n + i) = w(n)(2i+1)$$

for $n \in \mathbb{N}$ and $i < k_n$. For each $n \in \mathbb{N}$, $\tilde{D}_{k_n}$ has outer measure 1 in X^{2k_n}, so $\tilde{D} = \prod_{n\in\mathbb{N}} \tilde{D}_{k_n}$ has outer measure 1 in $\prod_{n\in\mathbb{N}} X^{2k_n}$, and $\phi[\tilde{D}]$ has outer measure 1 in $X^{\mathbb{N}}$. Note that $\phi[\tilde{D}]$ is just the set of $w \in X^{\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, there is an $f \in A$ such that

$$f(w(i)) \le h_0(w(i)) \text{ for } m_n \le i < m_n + k_n,$$
$$f(w(i)) \ge h_1(w(i)) \text{ for } m_n + k_n \le i < m_n + 2k_n = m_{n+1}.$$

If we set

$$V_{n0} = \{w : w \in X^{\mathbb{N}}, |\tfrac{1}{k_n}\textstyle\sum_{i=m_n}^{m_n+k_n-1} h_0(w(i)) - \int h_0| \le \delta\},$$

$$V_{n1} = \{w : w \in X^{\mathbb{N}}, |\tfrac{1}{k_n}\textstyle\sum_{i=m_n+k_n}^{m_n+2k_n-1} h_1(w(i)) - \int h_1| \le \delta\},$$

every V_{n0} and V_{n1} has measure at least $\frac{1}{2}$, because $k_n \ge k_0$.

(δ) Now suppose that $n \in \mathbb{N}$ and $w \in V_{n0} \cap V_{n1} \cap \phi[\tilde{D}]$. Then

$$\sup_{f\in A} \Big|\frac{1}{m_n+2k_n}\sum_{i=0}^{m_n+2k_n-1} f(w(i)) - \frac{1}{m_n+k_n}\sum_{i=0}^{m_n+k_n-1} f(w(i))\Big| \ge \frac{\delta}{(2+\delta)(1+\delta)}.$$

P Since $w \in \phi[\tilde{D}]$, there must be an $f \in A$ such that

$$f(w(i)) \le h_0(w(i)) \text{ for } m_n \le i < m_n + k_n,$$
$$f(w(i)) \ge h_1(w(i)) \text{ for } m_n + k_n \le i < m_n + 2k_n = m_{n+1}.$$

Set $s = \sum_{i=0}^{m_n-1} f(w(i))$, $t = \sum_{i=m_n}^{m_n+k_n} f(w(i))$ and $t' = \sum_{i=m_n+k_n}^{m_n+2k_n-1} f(w(i))$. Then

$$\begin{aligned}\frac{1}{m_n+2k_n}\sum_{i=0}^{m_n+2k_n-1} f(w(i)) - \frac{1}{m_n+k_n}\sum_{i=0}^{m_n+k_n-1} f(w(i))\\ &= \frac{s+t+t'}{m_n+2k_n} - \frac{s+t}{m_n+k_n} = \frac{(t'-t-s)k_n + t'm_n}{(m_n+2k_n)(m_n+k_n)}\\ &\geq \frac{(t'-t-m_n)k_n}{(m_n+2k_n)(m_n+k_n)} \geq \frac{t'-t-\delta k_n}{(2+\delta)(1+\delta)k_n}\end{aligned}$$

because $m_n \leq \delta k_n$, by the choice of k_n.

To estimate t and t', we have

$$t = \textstyle\sum_{i=m_n}^{m_n+k_n-1} f(w(i)) \leq \sum_{i=m_n}^{m_n+k_n-1} h_0(w(i)) \leq k_n(\int h_0 + \delta)$$

because $w \in V_{n0}$. On the other hand,

$$t' = \textstyle\sum_{i=m_n+k_n}^{m_n+2k_n-1} f(w(i)) \geq \sum_{i=m_n+k_n}^{m_n+2k_n-1} h_1(w(i)) \geq k_n(\int h_1 - \delta)$$

because $w \in V_{n1}$. So

$$t' - t \geq k_n(\textstyle\int h_1 - h_0 - 2\delta) = 2k_n\delta,$$

and

$$\begin{aligned}\frac{1}{m_n+2k_n}\sum_{i=0}^{m_n+2k_n-1} f(w(i)) - \frac{1}{m_n+k_n}\sum_{i=0}^{m_n+k_n-1} f(w(i))\\ &\geq \frac{2\delta k_n - \delta k_n}{(2+\delta)(1+\delta)k_n} = \frac{\delta}{(2+\delta)(1+\delta)}. \blacksquare\end{aligned}$$

(ϵ) The V_{n0} and V_{n1} are all independent. So $\mu^{\mathbb{N}}(V_{n0} \cap V_{n1}) \geq \frac{1}{4}$ for every n, and

$$W = \{w : w \in X^{\mathbb{N}}, \, w \in V_{n0} \cap V_{n1} \text{ for infinitely many } n\}$$

has measure 1 (by the Borel-Cantelli lemma, 273K, or otherwise). Accordingly $W \cap \phi[\tilde{D}]$ has outer measure 1 in $X^{\mathbb{N}}$. But if $w \in W \cap \phi[\tilde{D}]$, then (δ) tells us that

$$\limsup_{k,l\to\infty} \sup_{f\in A} \big|\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l}\sum_{i=0}^{l-1} f(w(i))\big| \geq \frac{\delta}{(2+\delta)(1+\delta)},$$

because there are infinitely many n such that $w \in V_{n0} \cap V_{n1} \cap \phi[\tilde{D}]$. So (iv) must be false.

(ζ) We see also that, for any $n \in \mathbb{N}$,

$$\begin{aligned}\overline{\int} g'_{m_n+k_n, m_n+2k_n} &\geq \frac{\delta}{(2+\delta)(1+\delta)}\mu^{\mathbb{N}}(V_{n0} \cap V_{n1})\\ &\geq \frac{\delta}{4(2+\delta)(1+\delta)},\end{aligned}$$

writing

$$g'_{kl}(w) = \sup_{f\in A} |\tfrac{1}{k}\textstyle\sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l}\sum_{i=0}^{l-1} f(w(i))|$$

for k, $l \in \mathbb{N}$. So $\limsup_{k,l\to\infty} \overline{\int} g'_{kl} > 0$, and (v) is false.

465N Corollary (Talagrand 84) Let (X, Σ, μ) be a measure space.

(a) Let $A \subseteq \mathbb{R}^X$ be a stable set. Suppose that there is a measurable function $g : X \to [0, \infty[$ such that $|f(x)| \leq g(x)$ for every $x \in X$, $f \in A$. Then the convex hull $\Gamma(A)$ of A in $\mathbb{R}^X$ is stable.

(b) Let A, $B \subseteq \mathbb{R}^X$ be two stable sets. Suppose that that there is a measurable function $g : X \to [0, \infty[$ such that $|f(x)| \leq g(x)$ for every $x \in X$, $f \in A \cup B$. Then $A + B = \{f_1 + f_2 : f_1 \in A, \, f_2 \in B\}$ is stable.

proof (a)(i) Consider first the case in which $\mu X = 1$ and $A \subseteq [-1,1]^X$. In this case,

$$\sup_{f\in\Gamma(A)} |\tfrac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \int f| = \sup_{f\in A} |\tfrac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \int f|$$

for every $k \geq 1$, $w \in X^{\mathbb{N}}$. So $\Gamma(A)$ satisfies condition (ii) of 465M whenever A does, and we have the result.

(ii) Now suppose just that $\mu X = 1$. Set $A' = \{\frac{f}{1+g} : f \in A\}$. Then A' is stable (465Ci), so $\Gamma(A')$ is stable, by (i), and $\Gamma(A) = \{f \times (1+g) : f \in A'\}$ is stable.

(iii) For the general case, take $E \in \Sigma$ and $\alpha < \beta$ such that $0 < \mu E < \infty$. Let ν be the indefinite-integral measure defined by μ and the function $(\mu E)^{-1}\chi E$. Then A is stable with respect to ν (465Ck), so $\Gamma(A)$ is stable with respect to ν, by (ii). There is therefore some $k \geq 1$ such that $(\nu^{2k})^* D_k(\Gamma(A), E, \alpha, \beta) < 1$. But (because ν^{2k} is the indefinite integral of $(\mu E)^{-2k}\chi E^{2k}$) $(\mu^{2k})^* D_k(\Gamma(A), E, \alpha, \beta) < (\mu E)^{2k}$. As E, α and β are arbitrary, $\Gamma(A)$ is stable with respect to μ.

(b) $A + B \subseteq 2\Gamma(A \cup B)$, so the result follows from 465Cd, (a) and 465Ce.

465O Stable sets in L^0 The notion of 'stability' as defined in 465B is applicable only to true functions; in such examples as 465Xj, the irregularity of the set A is erased entirely if we look at its image in the space L^0 of equivalence classes of measurable functions. We do, however, have a corresponding concept for subsets of function spaces, which can be expressed in the language of §325. If $(\mathfrak{A}, \bar\mu)$ is a semi-finite measure algebra, and $k \geq 1$, I write $(\widehat{\bigotimes}_k \mathfrak{A}, \bar\mu^k)$ for the localizable measure algebra free product of k copies of $(\mathfrak{A}, \bar\mu)$, as described in 325H. If $Q \subseteq L^0(\mathfrak{A})$, $k \geq 1$, $a \in \mathfrak{A}$ has finite measure and $\alpha < \beta$ in $\mathbb{R}$, set

$$d_k(Q, a, \alpha, \beta) = \sup_{v\in Q}(a \cap [\![v \leq \alpha]\!]) \otimes (a \cap [\![v \geq \beta]\!]) \otimes \ldots \otimes (a \cap [\![v \leq \alpha]\!]) \otimes (a \cap [\![v \geq \beta]\!])$$

in $\widehat{\bigotimes}_{2k}\mathfrak{A}$, taking k repetitions of the formula $(a \cap [\![v \leq \alpha]\!]) \otimes (a \cap [\![v \geq \beta]\!])$ to match the corresponding formula

$$D_k(A, E, \alpha, \beta) = \bigcup_{f\in A}((E \cap \{x : f(x) \leq \alpha\}) \times (E \cap \{x : f(x) \geq \beta))^k.$$

(Note that the supremum $\sup_{v\in Q} \ldots$ is defined because $a^{\otimes 2k} = a \otimes \ldots \otimes a$ has finite measure in the measure algebra $(\widehat{\bigotimes}_{2k}\mathfrak{A}, \bar\mu^{2k})$. Of course I mean to take $d_k(Q, a, \alpha, \beta) = 0$ if $Q = \emptyset$.) Now we can say that Q is **stable** if whenever $0 < \bar\mu a < \infty$ and $\alpha < \beta$ there is a $k \geq 1$ such that $\bar\mu^{2k} d_k(Q, a, \alpha, \beta) < (\bar\mu a)^{2k}$; that is, $d_k(Q, a, \alpha, \beta) \neq a \otimes \ldots \otimes a$.

We have the following relationships between the two concepts of stability.

465P Theorem Let (X, Σ, μ) be a semi-finite measure space, with measure algebra $(\mathfrak{A}, \bar\mu)$.

(a) Suppose that $A \subseteq \mathcal{L}^0(\Sigma)$ and that $Q = \{f^\bullet : f \in A\} \subseteq L^0(\mu)$, identified with $L^0 = L^0(\mathfrak{A})$ (364Jc). Then Q is stable in the sense of 465O iff every countable subset of A is stable in the sense of 465B.

(b) If μ is strictly localizable and $Q \subseteq L^0(\mu)$ is stable, then there is a stable set $B \subseteq \mathcal{L}^0$ such that $Q = \{f^\bullet : f \in B\}$.

proof (a)(i) Suppose that all countable subsets of A are stable, and take $a \in \mathfrak{A}$ such that $0 < \bar\mu a < \infty$ and $\alpha < \beta$ in $\mathbb{R}$. For each $k \in \mathbb{N}$ there is a countable set $Q_k \subseteq Q$ such that $d_k(Q_k, a, \alpha, \beta) = d_k(Q, a, \alpha, \beta)$, because $a^{\otimes 2k}$ is of finite measure in $\widehat{\bigotimes}_{2k}\mathfrak{A}$. Now there is a countable set $A' \subseteq A$ such that $\{f^\bullet : f \in A'\} = \bigcup_{k\in\mathbb{N}} Q_k$. Let $E \in \Sigma$ be such that $E^\bullet = a$. As $\mu E = \bar\mu a \in]0, \infty[$ and A' is stable, there is a $k \geq 1$ such that $(\mu^{2k})^* D_k(A', E, \alpha, \beta) < (\mu E)^{2k}$. Because A' is countable, $D_k(A', E, \alpha, \beta)$ is measurable. But 325He tells us that we have an order-continuous measure-preserving Boolean homomorphism π from the measure algebra of μ^{2k} to $\widehat{\bigotimes}_{2k}\mathfrak{A}$, such that $\pi(\prod_{i<2k} F_i)^\bullet = F_0^\bullet \otimes \ldots \otimes F_{2k-1}^\bullet$ for all $F_0, \ldots, F_{2k-1} \in \Sigma$; accordingly

$$\begin{aligned}\bar\mu^{2k} d_k(Q, a, \alpha, \beta) &= \bar\mu^{2k} d_k(Q_k, a, \alpha, \beta) \leq \bar\mu^{2k} d_k(\bigcup_{i\in\mathbb{N}} Q_i, a, \alpha, \beta)\\ &= \bar\mu^{2k}(\pi D_k(A', E, \alpha, \beta)^\bullet) = \mu^{2k} D_k(A', E, \alpha, \beta)\\ &< (\mu E)^{2k} = \bar\mu^{2k} a^{\otimes 2k}.\end{aligned}$$

As a, α and β are arbitrary, Q is stable.

(ii) Now suppose that Q is stable and that A' is any countable subset of A. Take $E \in \Sigma$ such that $0 < \mu E < \infty$, and $\alpha < \beta$ in $\mathbb{R}$. Set $a = E^\bullet \in \mathfrak{A}$. This time, writing Q' for $\{f^\bullet : f \in A'\}$, we have

$$\pi D_k(A', E, \alpha, \beta)^\bullet = d_k(Q', a, \alpha, \beta) \subseteq d_k(Q, a, \alpha, \beta)$$

for every $k \geq 1$. There is some k such that $d_k(Q, a, \alpha, \beta) \neq a^{\otimes 2k}$, and in this case $\mu^{2k} D_k(A', E, \alpha, \beta) < (\mu E)^{2k}$; as E, α and β are arbitrary, A' is stable.

(b)(i) If $\mu X = 0$ this is trivial; suppose that $\mu X > 0$. Replacing μ by its completion does not change either $L^0(\mu)$ or the stable subsets of $\mathbb{R}^X$ (241Xb, 465Cj), and leaves μ strictly localizable (212Ga), so we may suppose that μ is complete. Let $\langle E_i \rangle_{i \in I}$ be a decomposition of X into sets of finite measure. Amalgamating any negligible E_i into other non-negligible ones, we may suppose that $\mu E_i > 0$ for each i. Writing μ_i for the subspace measure on E_i, we have a consistent lifting ϕ_i for μ_i (346J). Set $\phi E = \bigcup_{i \in I} \phi_i(E \cap X_i)$ for $E \in \Sigma$; then ϕ is a lifting of μ. Let θ be the corresponding lifting from $\mathfrak{A}$ to Σ (341B) and $T : L^\infty(\mathfrak{A}) \to \mathfrak{L}^\infty(\Sigma)$ the associated linear operator, defined by saying that $T(\chi a) = \chi(\theta a)$ for every $a \in \mathfrak{A}$ (363F). Since $\theta(a)^\bullet = a$ for every $a \in \mathfrak{A}$, $(Tv)^\bullet = v$ for every $v \in L^\infty$.

(ii) We need to know that if $v \in L^\infty$ and $\alpha < \alpha'$, then $\{x : (Tv)(x) \leq \alpha\} \subseteq \phi(\{x : (Tv)(x) \leq \alpha'\})$. **P** Let $v' \in S(\mathfrak{A})$ be such that $\|v - v'\|_\infty \leq \frac{1}{2}(\alpha' - \alpha)$ (363C), and set $\gamma = \frac{1}{2}(\alpha + \alpha')$. Express v' as $\sum_{i=0}^n \alpha_i \chi a_i$ where $a_0, \ldots, a_n \in \mathfrak{A}$ are disjoint. Then $Tv' = \sum_{i=0}^n \alpha_i \chi(\theta a_i)$. Now

$$\|Tv - Tv'\|_\infty \leq \|v - v'\|_\infty \leq \gamma - \alpha,$$

so

$$\begin{aligned}\{x : (Tv)(x) \leq \alpha\} &\subseteq \{x : (Tv')(x) \leq \gamma\} \\ &= \bigcup \{\theta a_i : i \leq n,\ \alpha_i \leq \gamma\} = \phi(\bigcup \{\theta a_i : i \leq n,\ \alpha_i \leq \gamma\}) \\ &\text{(because } \phi(\theta a) = \theta a \text{ for every } a \in \mathfrak{A}) \\ &= \phi(\{x : (Tv')(x) \leq \gamma\}) \subseteq \phi(\{x : (Tv)(x) \leq \alpha'\}),\end{aligned}$$

as claimed. **Q**

Similarly, if $\beta' < \beta$ then $\{x : (Tv)(x) \geq \beta\} \subseteq \phi(\{x : (Tv)(x) \geq \beta'\})$.

(iii) For the moment, suppose that $Q \subseteq L^\infty(\mathfrak{A})$, which we may identify with $L^\infty(\mu)$ (363I). Set $B = T[Q]$, so that $Q = \{f^\bullet : f \in A\}$. Then B is stable. **P** Let $E \in \Sigma$ be such that $0 < \mu E < \infty$, and $\alpha < \beta$. Let $i \in I$ be such that $\mu(E \cap E_i) > 0$, and α', $\beta' \in \mathbb{R}$ such that $\alpha < \alpha' < \beta' < \beta$. Setting $a = (E \cap E_i)^\bullet$, we have $0 < \bar{\mu} a < \infty$, so there is some $k \in \mathbb{N}$ such that $d_k(Q, a, \alpha', \beta') \neq a^{\otimes 2k}$. Let π be the measure-preserving Boolean homomorphism from the measure algebra of μ^{2k} to $\widehat{\bigotimes}_{2k} \mathfrak{A}$ described in part (a) of this proof; as noted in 325He, the present context is enough to ensure that π is an isomorphism. So there is a $W \in \operatorname{dom} \mu^{2k}$ such that $\pi W^\bullet = d_k(Q, a, \alpha', \beta')$; since

$$d_k(Q, a, \alpha', \beta') \subseteq a^{\otimes 2k} \subseteq (E_i^{2k})^\bullet,$$

we may suppose that $W \subseteq E_i^{2k}$. If $f \in B$, then

$$\pi D_k(\{f\}, E, \alpha', \beta') = d_k(\{f^\bullet\}, a, \alpha', \beta') \subseteq d_k(Q, a, \alpha', \beta'),$$

so $D_k(\{f\}, E, \alpha', \beta') \setminus W$ is negligible.

At this point, recall that ϕ_i was supposed to be a consistent lifting of μ_i. So we have a lifting ϕ' of μ_i^{2k} such that $\phi'(\prod_{j<2k} F_j) = \prod_{j<2k} \phi_i F_j$ for all $F_0, \ldots, F_{2k-1} \in \Sigma_i$. In particular, if $f \in B$ and we set

$$F_{2j} = \{x : x \in E \cap E_i,\ f(x) \leq \alpha\}, \quad F_{2j+1} = \{x : x \in E \cap E_i,\ f(x) \geq \beta\},$$

$$F'_{2j} = \{x : x \in E \cap E_i,\ f(x) \leq \alpha'\}, \quad F'_{2j+1} = \{x : x \in E \cap E_i,\ f(x) \geq \beta'\},$$

for $j < k$, we shall have

$$\prod_{j<2k} F_j \subseteq \prod_{j<2k} \phi F'_j$$

(by (ii) above, because $f = Tv$ for some v)

$$= \prod_{j<2k} \phi_i F'_j = \phi'(\prod_{j<2k} F'_j) \subseteq \phi' W$$

because $\prod_{j<2k} F'_j = D_k(\{f\}, E, \alpha', \beta')$. As f is arbitrary, $D_k(B, E \cap E_i, \alpha, \beta) \subseteq \phi' W$. But now

$$(\mu^{2k})^* D_k(B, E \cap E_i, \alpha, \beta) \leq \mu^{2k}(\phi' W) = \mu_i^{2k}(\phi' W)$$

(251Wl)

$$= \mu_i^{2k} W = \mu^{2k} W$$
$$= \bar{\mu}^{2k} d_k(Q, a, \alpha', \beta') < \bar{\mu}^{2k} a^{\otimes 2k} = \mu(E \cap E_i)^{2k}.$$

As usual, it follows that $(\mu^{2k})^* D_k(B, E, \alpha, \beta) < (\mu E)^{2k}$; as E, α and β are arbitrary, B is stable, as claimed. **Q**

(iv) Thus the result is true if Q is included in the unit ball of L^∞. In general, set $h(\alpha) = \tanh \alpha$ for $\alpha \in \mathbb{R}$, and consider $A' = \{hf : f \in A\}$, $Q' = \{(hf)^\bullet : f \in A\}$. Since every countable subset of A is stable, so is every countable subset of A' (465Cl) and Q' is stable (by (a) of this theorem). But Q' is included in the unit ball of L^∞, so there is a stable set $B' \subseteq \mathcal{L}^0$ such that $Q' = \{g^\bullet : g \in B'\}$. Setting $B = \{h^{-1}g : g \in B'\}$, B is stable and $\{f^\bullet : f \in B\} = Q$. So we have the general theorem.

465Q Remarks Using 465Pa, we can work through the first part of this section to get a list of properties of stable subsets of L^0. For instance, the convex hull of an order-bounded stable set in L^0 is stable, as in 465Na. It is harder to relate such results as 465M to the idea of stability in L^0, but the argument of 465Pb gives a line to follow: if (X, Σ, μ) is complete and strictly localizable, there is a linear operator $T : L^\infty(\mu) \to \mathcal{L}^\infty(\Sigma)$, defined from a lifting, such that, for $Q \subseteq L^\infty$, $T[Q]$ is stable iff Q is stable. So when μ is a complete probability measure, we can look at the averages $\psi_{wk}(v) = \frac{1}{k}\sum_{i=0}^{k-1}(Tv)(w(i))$ for $v \in Q$, $w \in X^{\mathbb{N}}$ to devise criteria for stability of Q in terms of the linear functionals ψ_{wk}.

Working in L^1, however, we can look for results of a different type, as follows.

465R Theorem (Talagrand 84) Let $(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ be measure algebras, and $T : L^1(\mathfrak{A}, \bar{\mu}) \to L^1(\mathfrak{B}, \bar{\nu})$ a bounded linear operator. If Q is stable and order-bounded in $L^1(\mathfrak{A}, \bar{\mu})$, then $T[Q] \subseteq L^1(\mathfrak{B}, \bar{\nu})$ is stable.

proof (a) To begin with (down to (d) below) let us suppose that

$(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are the measure algebras of measure spaces (X, Σ, μ) and (Y, T, ν),

so that we can identify $L^1_{\bar{\mu}}$, $L^1_{\bar{\nu}}$ with $L^1(\mu)$ and $L^1(\nu)$ (365B),

Q is countable,

so that Q can be expressed as $\{f^\bullet : f \in A\}$, where $A \subseteq \mathcal{L}^0(\Sigma)$ is countable and stable (465P),

T is positive,

μ and ν are totally finite,

$T(\chi X^\bullet) = \chi Y^\bullet$,

$Q \subseteq L^\infty(\mathfrak{A})$ is $\| \|_\infty$-bounded,

so that we may take $A \subseteq \mathcal{L}^\infty$ to be $\| \|_\infty$-bounded,

$\nu Y = 1$,

$\mu Y = 1$,

and that $\|T\| \leq 1$.

(b) The idea of the argument is that for any $n \geq 1$ we have a positive linear operator $U_n : L^1(\mu^n) \to L^1(\nu^n)$ defined as follows.

If $f_0, \dots, f_{n-1} \in \mathcal{L}^1(\mu)$, set $(f_0 \otimes \dots \otimes f_{n-1})(w) = \prod_{i=0}^n f_i(w(i))$ whenever $w \in \prod_{i<n} \operatorname{dom} f_i$. Now we can define $u_0 \otimes u_1 \otimes \dots \otimes u_{n-1} \in L^1(\mu^n)$, for $u_0, \dots, u_{n-1} \in L^1(\mu)$, by saying that $f_0^\bullet \otimes \dots \otimes f_{n-1}^\bullet = (f_0 \otimes \dots \otimes f_{n-1})^\bullet$ for all $f_0, \dots, f_{n-1} \in \mathcal{L}^1(\mu)$, as in 253E.

Define the operators U_n inductively. $U_1 = T$. Given that $U_n : L^1(\mu^n) \to L^1(\nu^n)$ is a positive linear operator, then we have a bilinear operator $\psi : L^1(\mu^n) \times L^1(\mu) \to L^1(\nu^{n+1})$ defined by saying that $\psi(q,u) = U_n q \otimes Tu$ for $q \in L^1(\mu^n)$, $u \in L^1(\mu)$, where $\otimes : L^1(\nu^n) \times L^1(\nu) \to L^1(\nu^n \times \nu) \cong L^1(\nu^{n+1})$ is the operator of 253E. By 253F, there is a (unique) bounded linear operator $U_{n+1} : L^1(\mu^{n+1}) \to L^1(\nu^{n+1})$ such that $U_{n+1}(q \otimes u) = \psi(q,u)$ for all $q \in L^1(\mu^n)$, $u \in L^1(\mu)$. To see that U_{n+1} is positive, use 253G. (Remember that we are supposing that T is positive.) Continue.

Now it is easy to check that

$$U_n(u_0 \otimes \dots \otimes u_{n-1}) = Tu_0 \otimes \dots \otimes Tu_{n-1}$$

for all $u_0, \dots, u_{n-1} \in L^1(\mu)$. Moreover, $\|U_{n+1}\| \le \|U_n\|\|T\|$ for every n (see 253F), so $\|U_n\| \le 1$ for every n.

For $i < n \in \mathbb{N}$, we have a natural operator $R_{ni} : L^1(\mu) \to L^1(\mu^n)$, defined by saying that $R_{ni} f^\bullet = (f\pi_{ni})^\bullet$ for every $f \in \mathcal{L}^1(\mu)$, where $\pi_{ni}(w) = w(i)$ for $w \in X^n$. Similarly, we have an operator $S_{ni} : L^1(\nu) \to L^1(\nu^n)$. Observe that

$$R_{ni} u = e \otimes \dots \otimes e \otimes u \otimes e \otimes \dots \otimes e$$

where $e = \chi X^\bullet$ and the u is put in the position corresponding to the coordinate i. Since $Te = (\chi Y)^\bullet = e'$ say,

$$U_n R_{ni} u = e' \otimes \dots \otimes Tu \otimes \dots \otimes e' = S_{ni} Tu,$$

for every $u \in L^1(\mu)$.

(c) Let $B \subseteq \mathcal{L}^\infty(\mathrm{T})$ be a countable $\|\,\|_\infty$-bounded set such that $T[Q] = \{g^\bullet : g \in B\}$. ($T[Q]$ is $\|\,\|_\infty$-bounded because T is positive and $T(\chi X)^\bullet = (\chi Y)^\bullet$.) I seek to show that B is stable by using the criterion 465M(v). Let $\epsilon > 0$. Then there is an $m \ge 1$ such that $\int f_{kl}(w)\mu^{\mathbb{N}}(dw) \le \epsilon$ for any $k, l \ge m$, writing

$$f_{kl}(w) = \sup_{f \in A} |\tfrac{1}{k}\textstyle\sum_{i=0}^{k-1} f(w(i)) - \tfrac{1}{l}\sum_{i=0}^{l-1} f(w(i))|$$

for $w \in X^{\mathbb{N}}$; note that f_{kl} is measurable because A is countable.

Take any $k, l \ge m$ and consider

$$g_{kl}(z) = \sup_{g \in B} |\tfrac{1}{k}\textstyle\sum_{i=0}^{k-1} g(z(i)) - \tfrac{1}{l}\sum_{i=0}^{l-1} g(z(i))|$$

for $z \in Y^{\mathbb{N}}$. I claim that $\int g_{kl} d\nu^{\mathbb{N}} \le \epsilon$. **P** Set $n = \max(k,l)$. Then $\int g_{kl} d\nu^{\mathbb{N}} = \int \tilde{g}\, d\nu^n$, where

$$\tilde{g}(z) = \sup_{g \in B} |\tfrac{1}{k}\textstyle\sum_{i=0}^{k-1} g(z(i)) - \tfrac{1}{l}\sum_{i=0}^{l-1} g(z(i))|$$

for $z \in Y^n$. If we look at $\tilde{g}^\bullet$ in $L^1(\nu^n)$, we see that it is

$$\sup_{g \in B} |\tfrac{1}{k}\textstyle\sum_{i=0}^{k-1} S_{ni} g^\bullet - \tfrac{1}{l}\sum_{i=0}^{l-1} S_{ni} g^\bullet|$$

where $S_{ni} : L^1(\nu) \to L^1(\nu^n)$ is defined in (b) above. Thus

$$\tilde{g}^\bullet = \sup_{v \in T[Q]} |\tfrac{1}{k}\textstyle\sum_{i=0}^{k-1} S_{ni} v - \tfrac{1}{l}\sum_{i=0}^{l-1} S_{ni} v|.$$

Similarly, setting

$$\tilde{f}(w) = \sup_{f \in A} |\tfrac{1}{k}\textstyle\sum_{i=0}^{k-1} f(w(i)) - \tfrac{1}{l}\sum_{i=0}^{l-1} f(w(i))|$$

for $w \in X^n$,

$$\tilde{f}^\bullet = \sup_{u \in Q} |\tfrac{1}{k}\textstyle\sum_{i=0}^{k-1} R_{ni} u - \tfrac{1}{l}\sum_{i=0}^{l-1} R_{ni} u|.$$

Now consider $U_n \tilde{f}^\bullet$. For any $v \in T[Q]$, we can express v as Tu where $u \in Q$, so

$$\begin{aligned}|\frac{1}{k}\sum_{i=0}^{k-1} S_{ni}v - \frac{1}{l}\sum_{i=0}^{l-1} S_{ni}v| &= |\frac{1}{k}\sum_{i=0}^{k-1} S_{ni}Tu - \frac{1}{l}\sum_{i=0}^{l-1} S_{ni}Tu| \\ &= |\frac{1}{k}\sum_{i=0}^{k-1} U_nR_{ni}u - \frac{1}{l}\sum_{i=0}^{l-1} U_nR_{ni}u|\end{aligned}$$

(because $U_nR_{ni} = S_{ni}T$, as noted in (b) above)

$$\begin{aligned}&= |U_n(\frac{1}{k}\sum_{i=0}^{k-1} R_{ni}u - \frac{1}{l}\sum_{i=0}^{l-1} R_{ni}u)| \\ &\leq U_n(|\frac{1}{k}\sum_{i=0}^{k-1} R_{ni}u - \frac{1}{l}\sum_{i=0}^{l-1} R_{ni}u|)\end{aligned}$$

(because U_n is positive)

$$\leq U_n\tilde{f}^\bullet.$$

As v is arbitrary, $\tilde{g}^\bullet \leq U_n\tilde{f}^\bullet$, and

$$\int g_{kl}d\nu^{\mathbb{N}} = \int \tilde{g}\,d\nu^n = \|\tilde{g}^\bullet\|_1 \leq \|U_n\tilde{f}^\bullet\|_1 \leq \|\tilde{f}^\bullet\|_1$$

(because $\|U_n\| \leq 1$)

$$\leq \epsilon$$

because k, $l \geq m$. **Q**

(d) As ϵ is arbitrary, B satisfies the criterion 465M(v), and is stable. So $T[Q]$ is stable, by 465Pa in the other direction.

(e) Now let us seek to unwind the list of special assumptions used in the argument above. Suppose we drop the last two, and assume only that

$(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are the measure algebras of measure spaces (X, Σ, μ) and (Y, T, ν),
Q is countable,
T is positive,
μ and ν are totally finite,
$T(\chi X^\bullet) = \chi Y^\bullet$,
$Q \subseteq L^\infty(\mathfrak{A})$ is $\|\,\|_\infty$-bounded,
$\nu Y = 1$.

Then $T[Q]$ is stable. **P** Define $\mu_1 : \Sigma \to [0, \infty[$ by setting $\mu_1 E = \int T(\chi E^\bullet)$ for every $E \in \Sigma$. Then μ_1 is countably additive because T is (sequentially) order-continuous (355Ka). If $\mu E = 0$ then $\chi E^\bullet = 0$ in $L^1(\mu)$ and $\mu_1 E = 0$, so μ_1 is truly continuous with respect to μ (232Bb) and has a Radon-Nikodým derivative (232E). By 465Ck, A is stable with respect to μ_1, while $\mu_1 X = \nu Y = 1$, because $T(\chi X)^\bullet = \chi Y^\bullet$.

Let $(\mathfrak{A}_1, \bar{\mu}_1)$ be the measure algebra of μ_1. If $E \in \Sigma$ and $\mu_1 E = 0$, then $T(\chi E^\bullet) = 0$. Accordingly we can define an additive function $\theta : \mathfrak{A}_1 \to L^1(\nu)$ by setting $\theta E^\bullet = T(\chi E^\bullet)$ for every $E \in \Sigma$. (Note that the two $^\bullet$s here must be interpreted differently. In the formula $\theta E^\bullet$, the equivalence class $E^\bullet$ is to be taken in $\mathfrak{A}_1$. In the formula $\chi E^\bullet = (\chi E)^\bullet$, the equivalence class is to be taken in $L^0(\mu)$. In the rest of this proof I will pass over such points without comment; I hope the context will always make it clear how each $^\bullet$ is to be read.) Because T is positive, θ is non-negative, and by the definition of μ_1 we have $\|\theta a\|_1 = \bar{\mu}_1 a$ for every $a \in \mathfrak{A}_1$. So we have a positive linear operator $T_1 : L^1(\mathfrak{A}_1, \bar{\mu}_1) \to L^1(\nu)$ defined by setting

$$T_1(\chi E^\bullet) = \theta E^\bullet = T(\chi E^\bullet)$$

for every $E \in \Sigma$ (365I).

If $f : X \to \mathbb{R}$ is simple (that is, a linear combination of characteristic functions of sets in Σ), then $T_1 f^\bullet = Tf^\bullet$. So this is also true for every $f \in \mathcal{L}^\infty(\Sigma)$; in particular, it is true for every $f \in A$, so that

$T[Q] = T_1[Q_1]$, where $Q_1 = \{f^\bullet : f \in A\} \subseteq L^1(\mu_1)$. But μ_1, Q_1 and T_1 satisfy all the conditions of (a), so (b)-(d) tell us that $T_1[Q_1]$ is stable, and $T[Q]$ is stable, as required. **Q**

(f) The next step is to drop the condition '$\nu Y = 1$'. But this is elementary, since we are still assuming that ν is totally finite, and multiplying ν by a non-zero scalar doesn't change $L^0(\nu)$ or the stability of any of its subsets, while the case $\nu Y = 0$ is trivial. So we conclude that if

$(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are the measure algebras of measure spaces (X, Σ, μ) and (Y, T, ν),
Q is countable,
T is positive,
μ and ν are totally finite,
$T(\chi X^\bullet) = \chi Y^\bullet$,
$Q \subseteq L^\infty(\mathfrak{A})$ is $\| \|_\infty$-bounded,

then $T[Q]$ is stable.

(g) We can now attack what remains. We find that if

$(\mathfrak{A}, \bar{\mu})$ and $(\mathfrak{B}, \bar{\nu})$ are the measure algebras of measure spaces (X, Σ, μ) and (Y, T, ν),
Q is countable,
T is positive,

then $T[Q]$ is stable. **P** At this point recall that we are supposing that Q is order-bounded. Let $u_0 \in L^1(\mu)$ be such that $|u| \leq u_0$ for every $u \in Q$; let $f_0 \in \mathcal{L}^0(\Sigma)^+$ be such that $f_0^\bullet = u_0$. Setting $A' = \{(f \wedge f_0) \vee (-f_0) : f \in A\}$, A' is still stable, because its image in $L^1(\mu)$ is still Q, or otherwise. Set $v_0 = Tu_0$, and let $g_0 \in \mathcal{L}^0(\mathrm{T})^+$ be such that $g_0^\bullet = v_0$. Because T is positive, $|Tu| \leq T|u| \leq v_0$ for every $u \in Q$. So we can represent $T[Q]$ as $\{g^\bullet : g \in B\}$, where $B \subseteq \mathcal{L}^0(\mathrm{T})$ is a countable set and $|g| \leq g_0$ for every $g \in B$. Set $F_0 = \{y : g_0(y) \neq 0\}$.

Define measures μ_1, ν_1 by setting $\mu_1 E = \int_E f_0 d\mu$ for $E \in \Sigma$, $\nu_1 F = \int_F g_0 d\nu$ for $F \in \mathrm{T}$. Then both μ_1 and ν_1 are totally finite. By 465Ck, A' is stable with respect to μ_1. Set $A_1 = \{\frac{f}{f_0} : f \in A'\}$, interpreting $\frac{f}{f_0}(x)$ as 0 if $f_0(x) = 0$; then A_1 is stable with respect to μ_1, by 465Ci, and $\|f\|_\infty \leq 1$ for every $f \in A_1$. Take $Q_1 = \{f^\bullet : f \in A_1\} \subseteq L^1(\mu_1)$, so that Q_1 is stable.

We have a norm-preserving positive linear operator $R : L^1(\mu_1) \to L^1(\mu)$ defined by setting $Rf^\bullet = (f \times f_0)^\bullet$ for every $f \in \mathcal{L}^1(\mu_1)$ (use 235A). Observe that $R[Q_1] = Q$ and $R(\chi X)^\bullet = u_0$. Similarly, we have a norm-preserving positive linear operator $S : L^1(\nu_1) \to L^1(\nu)$ defined by setting $Sg^\bullet = (g \times g_0)^\bullet$ for $g \in \mathcal{L}^1(\nu_1)$. The set of values of S is just

$$\{g^\bullet : g \in \mathcal{L}^1(\nu),\ g(y) = 0 \text{ whenever } g_0(y) = 0\},$$

which is the band of $L^1(\nu)$ generated by v_0. So

$$\{u : u \in L^1(\mu_1),\ TR|u| \in S[L^1(\nu_1)]\}$$

is a band in $L^1(\nu_1)$ containing $\chi X^\bullet$, and must be the whole of $L^1(\mu_1)$. Thus we have a positive linear operator $T_1 = S^{-1}TR : L^1(\mu_1) \to L^1(\nu_1)$, and $T_1(\chi X)^\bullet = \chi Y^\bullet$ in $L^1(\nu_1)$.

By (f), $T_1[Q_1]$ is stable in $L^1(\nu_1)$. Observe that $T_1[Q_1] = \{S^{-1}g^\bullet : g \in B\} = \{g^\bullet : g \in B_1\}$, where $B_1 = \{\frac{g}{g_0} : g \in B\}$, interpreting $\frac{g}{g_0}(y)$ as 0 if $y \in Y \setminus F_0$. Consequently B_1 and $B = \{g \times g_0 : g \in B_1\}$ are stable with respect to ν_1. By 465Ck, once more, B_1 is stable in $L^1(\nu_0)$, where

$$\nu_0 F = \int_F \frac{1}{g_0} d\nu_1 = \nu(F \cap F_0)$$

for any $F \in \mathrm{T}$. But because $g(y) = 0$ whenever $g \in B$ and $y \in Y \setminus F_0$,

$$(\nu^{2k})^* D_k(B, F, \alpha, \beta) = (\nu^{2k})^* D_k(B, F \cap F_0, \alpha, \beta) = (\nu_0^{2k})^* D_k(B, F, \alpha, \beta)$$

whenever $F \in \mathrm{T}$, $\alpha < \beta$ and $k \geq 1$; so B is also stable with respect to ν, and $Q = \{g^\bullet : g \in B\}$ is stable in $L^1(\nu)$. **Q**

(h) The worst is over. If we are not told that T is positive, we know that it is expressible as the difference of positive linear operators T_1 and T_2 (371D); now $T_1[Q]$ and $T_2[Q]$ will be stable, by the work above, so $T[Q] \subseteq T_1[Q] - T_2[Q]$ is stable, by 465Nb. If we are not told that Q is countable, we refer to 465P to see that we need only check that countable subsets of $T[Q]$ are stable, and these are images of countable subsets of

Q. Finally, the identification of the abstract measure algebras $(\mathfrak{A}, \bar{\mu}_1)$ and $(\mathfrak{B}, \bar{\nu}_1)$ with the measure algebras of measure spaces is Theorem 321J.

***465S R-stable sets** The theory above has been developed in the context of general measure (or probability) spaces and the 'ordinary' product measure of measure spaces. For τ-additive measures – in particular, for Radon measures – we have an alternative product measure, as described in §417. If $(X, \mathfrak{T}, \Sigma, \mu)$ is a semi-finite τ-additive topological measure space such that μ is inner regular with respect to the Borel sets, write $\tilde{\mu}^I$ for the τ-additive product measure on X^I, as described in 417D (for the product of two spaces) and 417F (for the product of any family of probability spaces); just as in 251W, we can extend the construction of 417D to arbitrary finite products. Now say that $A \subseteq \mathbb{R}^X$ is **R-stable** if whenever $0 < \mu E < \infty$ and $\alpha < \beta$ there is a $k \geq 1$ such that $(\tilde{\mu}^{2k})^* D_k(A, E, \alpha, \beta) < (\mu E)^{2k}$. Because we have a version of Fubini's theorem for the products of τ-additive topological measures (417H), all the arguments of this section can be applied to R-stable sets, yielding criteria for R-stability exactly like those in 465M.

Because the τ-additive product measure extends the c.l.d. product measure, stable sets are always R-stable. (We must have

$$(\tilde{\mu}^{2k})^* D_k(A, E, \alpha, \beta) \leq (\mu^{2k})^* D_k(A, E, \alpha, \beta)$$

for all k, A, E, α and β.) For an example of an R-stable set which is not stable, see 465U.

The concept of 'R-stability' is used in Talagrand 84 in applications to the integration of vector-valued functions. I give one result, however, to show how it is relevant to a question with a natural expression in the language of this chapter.

***465T Proposition** (Talagrand 84) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a semi-finite τ-additive topological measure space such that μ is inner regular with respect to the Borel sets. If $A \subseteq C(X)$ is such that every countable subset of A is R-stable, then A is R-stable.

proof For any $\alpha < \beta$ and $k \geq 1$,

$$G_k(A, X, \alpha, \beta) = \bigcup_{f \in A} \{w : w \in X^{2k}, f(w(2i)) < \alpha, \\ f(w(2i+1)) > \beta \text{ for } i < k\}$$

is open. Suppose that $0 < \mu E < \infty$. Because all the product measures $\tilde{\mu}^{2k}$ are τ-additive, we can find a countable set $A' \subseteq A$ such that

$$\tilde{\mu}^{2k} G_k(A, E, \alpha, \beta) = \tilde{\mu}^{2k} G_k(A', E, \alpha, \beta)$$

for every $k \geq 1$ and all rational α, β. Now, if $\alpha < \beta$, there are rational α', β' such that $\alpha < \alpha' < \beta' < \beta$, and a $k \geq 1$ such that $\tilde{\mu}^{2k} D_k(A', E, \alpha', \beta') < (\mu E)^{2k}$; in which case

$$\begin{aligned}(\tilde{\mu}^{2k})^* D_k(A, E, \alpha, \beta) &\leq \tilde{\mu}^{2k} G_k(A, E, \alpha', \beta') = \tilde{\mu}^{2k} G_k(A', E, \alpha', \beta') \\ &\leq \tilde{\mu}^{2k} D_k(A', E, \alpha', \beta') < (\mu E)^{2k}.\end{aligned}$$

As E, α and β are arbitrary, A is R-stable.

***465U** I come now to the promised example of an R-stable set which is not stable. I follow the construction in Talagrand 88, which displays an interesting characteristic related to 465O-465P above.

Example There is a Radon probability space with an R-stable set of continuous functions which is not stable.

proof (a) Let (X, Σ, μ) be an atomless probability space (e.g., the unit interval with Lebesgue measure). Define $\langle r_n \rangle_{n \in \mathbb{N}}$ by setting $r_0 = 1$, $r_1 = 2$, $r_{n+1} = 2^{nr_n}$ for $n \geq 1$; then $2^n \leq r_n < r_{n+1}$ for every n. Let $\langle \Sigma_n \rangle_{n \in \mathbb{N}}$ be an increasing sequence of finite subalgebras of Σ, each Σ_n having r_n atoms of the same size; this is possible because r_{n+1} is always a multiple of r_n. Write $\mathcal{H}_n$ for the set of atoms of Σ_n. Next, for each $n \in \mathbb{N}$, let $\langle G_H \rangle_{H \in \mathcal{H}_n}$ be an independent family in Σ_{n+1} of sets of measure 2^{-n}; such a family exists because r_{n+1} is a multiple of 2^{nr_n}.

Let $\mathcal{E}$ be the family of all sets expressible in the form $E = \bigcup_{i\in\mathbb{N}} H_i$ where, for some strictly increasing sequence $\langle n_i\rangle_{i\in\mathbb{N}}$ in $\mathbb{N}$, $H_i \in \mathcal{H}_{n_i}$ and $H_j \subseteq G_{H_i}$ whenever $i < j$ in $\mathbb{N}$. Set $A = \{\chi E : E \in \mathcal{E}\} \subseteq \mathfrak{L}^0(\Sigma)$.

(b) A is stable. **P** Suppose that $F \in \Sigma$ and that $\mu F > 0$. Take $n \in \mathbb{N}$ so large that $3\cdot 2^{-n} < (\mu F)^2$. Set $\mathcal{H} = \{H : H \in \mathcal{H}_n, \mu(H\cap F) > 0\}$; enumerate $\mathcal{H}$ as $\langle H_i\rangle_{i<m}$; set $F_1 = F\cap\bigcup_{i<m} H_i$, $V = \prod_{j<m}(H_j\cap F_1) \subseteq F_1^m$. Because $\mu F_1 = \mu F$, $m \geq r_n\mu F \geq 3$.

Consider

$$U_k = \bigcup_{H\in\mathcal{H}_k}(H\times H)\cup(H\times G_H)\cup(G_H\times H) \subseteq X^2$$

for $k \in \mathbb{N}$. Then

$$\mu^2 U_k \leq r_k(\tfrac{1}{r_k^2} + \tfrac{1}{2^k r_k} + \tfrac{1}{2^k r_k}) \leq 3\cdot 2^{-k},$$

so

$$\mu^2(F_1^2 \setminus \bigcup_{k>n} U_k) \geq (\mu F)^2 - 3\textstyle\sum_{k=n+1}^{\infty} 2^{-k} > 0.$$

Set

$$V = \{w : w \in F_1^{2m},\ w(2i) \in H_i \text{ for every } i < m,\ (w(1), w(3)) \notin \textstyle\bigcup_{k>n} U_k\}.$$

Then

$$\mu^{2m}(V) \geq \textstyle\prod_{i=0}^{m-1} \mu(F_1\cap H_i)\cdot\mu^2(F_1^2\setminus\bigcup_{k>n}U_k)\cdot(\mu F)^{m-2} > 0.$$

? Suppose, if possible, that there is a point $w \in V \cap D_m(A, F, 0, 1)$. Then there is an $E \in \mathcal{E}$ such that $w(2i) \notin E$, $w(2i+1) \in E$ for $i < m$. Express E as $\bigcup_{j\in\mathbb{N}} E_j$ where $E_j \in \mathcal{H}_{n_j}$ for every $j \in \mathbb{N}$, where $\langle n_j\rangle_{j\in\mathbb{N}}$ is strictly increasing, and $E_k \subseteq G_{E_j}$ whenever $j < k$. Because $w(2i) \in F_1\cap H_i \setminus E$, E_j cannot include H_i for any $i < m$, $j \in \mathbb{N}$; so $E_j \cap H_i = \emptyset$ whenever $i < m$ and $n_j \leq n$. Because $w(1)$, $w(3) \in E$, there are j_0, $j_1 \in \mathbb{N}$ such that $w(1) \in E_{j_0}$ and $w(3) \in E_{j_1}$. Since both $w(1)$ and $w(3)$ belong to $F_1 \subseteq \bigcup_{i<m} H_i$, n_{j_0} and n_{j_1} are both greater than n. But now

— if $j_0 = j_1$, then $(w(1), w(3)) \in E_{j_0}^2$,

— if $j_0 < j_1$, then $(w(1), w(3)) \in E_{j_0}\times E_{j_1} \subseteq E_{j_0}\times G_{E_{j_0}}$,

— if $j_1 < j_0$, then $(w(1), w(3)) \in E_{j_0}\times E_{j_1} \subseteq G_{E_{j_1}}\times E_{j_1}$,

so in any case $(w(1), w(3)) \in U_k$ where $k = \min(j_0, j_1) > n$, which is impossible. **X**

Thus $D_m(A, F, 0, 1)$ does not meet V, and $(\mu^{2m})^* D_m(A, F, 0, 1) < (\mu F)^{2m}$.

Now if $\alpha < \beta$, then $D_m(A, F, \alpha, \beta) = D_m(A, F, 0, 1)$ if $0 \leq \alpha < \beta \leq 1$, $\emptyset$ otherwise; so in all cases $(\mu^{2m})^* D_m(A, F, \alpha, \beta) < (\mu F)^{2m}$. As F also is arbitrary, A is stable, as claimed. **Q**

(c) Now let $(Z, \mathfrak{S}, \mathrm{T}, \nu)$ be the Stone space of the measure algebra of (X, Σ, μ) (321K), so that ν is a Radon measure (411Pe). For $E \in \Sigma$, write E^* for the corresponding open-and-closed set in Z, so that $E \mapsto E^* : \Sigma \to \mathrm{T}$ is a measure-preserving Boolean homomorphism. Set $A^* = \{\chi E^* : E \in \mathcal{E}\} \subseteq C(Z)$. Write ν^m for the c.l.d. product measure on Z^m for $m \geq 1$. We already know that ν^2 is not a topological measure (419Xc).

(d) The point is that $(\nu^{2m})^* D_m(A^*, Z, 0, 1) = 1$ for every $m \geq 1$. **P?** Suppose, if possible, otherwise. Then there is a set $\tilde{W} \subseteq Z^{2m}$ such that $\nu^{2m}(\tilde{W}) > 0$ and $\tilde{W} \cap D_m(A^*, Z, 0, 1) > 0$. For u, $v \in Z^m$, write $u\#v = (u(0), v(0), u(1), v(1), \dots, u(m-1), v(m-1)) \in Z^{2m}$. Then there is an $\epsilon > 0$ such that

$$\tilde{V} = \{v : v \in Z^m,\ \nu^m\{u : u \in Z^m,\ u\#v \in \tilde{W}\} \text{ is defined and greater than } m\epsilon\}$$

has non-zero inner measure for ν^m. Now there are sets $\tilde{F}_{ij} \in \mathrm{T}$, for $i \in \mathbb{N}$ and $j < m$, such that

$$Z^m \setminus \tilde{V} \subseteq \textstyle\bigcup_{i\in\mathbb{N}} \prod_{j<m} \tilde{F}_{ij}, \quad \sum_{i=0}^{\infty}\prod_{j=0}^{m} \nu\tilde{F}_{ij} < 1.$$

Enlarging the $\tilde{F}_{ij}$ slightly if need be, we may suppose that they are all open-and-closed (322Qc), therefore expressible as F_{ij}^* where $F_{ij} \in \Sigma$ for $i \in \mathbb{N}$, $j < m$. Set $V = X^m \setminus \bigcup_{i\in\mathbb{N}} \prod_{j<m} F_{ij}$, so that

$$\mu^m V = \nu^m(Z^m \setminus \textstyle\bigcup_{i\in\mathbb{N}} \prod_{j<m} F_{ij}^*) > 0.$$

I seek to choose $\langle n_k\rangle_{k\in\mathbb{N}}$ in $\mathbb{N}$, $\langle H_k\rangle_{k\in\mathbb{N}}$ and $\langle E_{kj}\rangle_{k\in\mathbb{N}, j<m}$ in Σ inductively, in such a way that

$$2^{-n_0} \leq \tfrac{1}{2}\epsilon,$$

$$E_{kj} \subseteq E_{ij} \cap G_{H_i} \text{ for } i < k,\ j < m,$$

$$\prod_{j<m} E_{kj} \cap \prod_{j<m} F_{kj} = \emptyset,$$

$$\mu^m(V \cap G^m_{H_k} \cap \prod_{j<m} E_{kj}) > 0,$$

$$n_i < n_k \text{ for every } i < k,$$

$$H_k \in \mathcal{H}_{n_k},$$

$$\text{if } k = i_k m + j_k, \text{ where } i_k \in \mathbb{N},\ j_k < m, \text{ then } \mu(H_k \cap E_{kj_k}) > 0,$$

for every $k \in \mathbb{N}$. The induction proceeds as follows. Set $E'_{kj} = X$ if $k = 0$, $G_{H_{k-1}} \cap E_{k-1,j}$ otherwise, so that $\mu(V \cap \prod_{j<m} E'_{kj}) > 0$. Because $V \cap \prod_{j<m} F_{kj}$ is empty, we can find $E_{kj} \subseteq E'_{kj}$, for $j < m$, such that $\mu^m(V \cap \prod_{j<m} E_{kj}) > 0$ and $\prod_{j<m} E_{kj} \cap \prod_{j<m} F_{kj} = \emptyset$. Set $\eta = \mu E_{kj_k}$, $\delta = \mu^m(V \cap \prod_{j<m} E_{kj})$, so that η and δ are both strictly positive.

Now take n_k so large that

$$(\text{if } k = 0)\ 2^{-n_k} \leq \tfrac{1}{2}\epsilon, \quad n_k > n_i \text{ for } i < k, \quad (1 - 2^{-mn_k})^{\eta r_{n_k}} < \delta.$$

(This is possible because $\lim_{n\to\infty} 2^{-mn} r_n = \infty$.) Set $\mathcal{H} = \{H : H \in \mathcal{H}_{n_k}, \mu(H \cap E_{kj_k}) > 0\}$; then $\#(\mathcal{H}) \geq \eta r_{n_k}$. Consider the family $\langle G^m_H \rangle_{H\in\mathcal{H}}$. These are stochastically independent sets of measure 2^{-mn_k}, so their union has measure $1 - (1 - 2^{-mn_k})^{\#(\mathcal{H})} > 1 - \delta$, and there is an $H_k \in \mathcal{H}$ such that $\mu^m(V \cap G^m_{H_k} \cap \prod_{j<m} E_{kj}) > 0$. Thus the induction continues.

Look at the sequence $\langle H_k \rangle_{k\in\mathbb{N}}$ and its union E. We have $H_k \in \mathcal{H}_{n_k}$ for every k; moreover, if $i < k$, then $\mu(H_k \cap E_{kj_k}) > 0$, while $E_{kj_k} \subseteq G_{H_i}$; since H_k is an atom of Σ_{n_k}, while $G_{H_i} \in \Sigma_{n_k}$, $H_k \subseteq G_{H_i}$. Thus $E \in \mathcal{E}$. Next, whenever $i \leq k \in \mathbb{N}$,

$$\prod_{j<m} E_{kj} \cap \prod_{j<m} F_{ij} \subseteq \prod_{j<m} E_{ij} \cap \prod_{j<m} F_{ij} = \emptyset,$$

so $\prod_{j<m} E_{kj} \cap \bigcup_{i\leq k} \prod_{j<m} F_{ij} = \emptyset$. At the same time, we know that

$$E^m \cap \prod_{j<m} E_{kj} \supseteq \prod_{j<m} H_{km+j} \cap E_{kj} \supseteq \prod_{j<m} H_{km+j} \cap E_{km+j,j}$$

has non-zero measure. So $\mu^m(E^m \setminus \bigcup_{i\leq k} \prod_{j<m} F_{ij}) > 0$.

Moving back to Z, this translates into

$$\nu^m((E^*)^m \setminus \bigcup_{i\leq k} \prod_{j<m} F^*_{ij}) > 0.$$

But this means that $(E^*)^m$ is not included in $\bigcup_{i\leq k} \prod_{j<m} F^*_{ij}$, for any $k \in \mathbb{N}$. Because E^* is compact and every F^*_{ij} is open, $(E^*)^m$ is not included in $\bigcup_{i\in\mathbb{N}} \prod_{j<m} F^*_{ij}$, and there is some $v \in (E^*)^m \cap \tilde{V}$.

By the definition of $\tilde{V}$,

$$\begin{aligned}\nu^m\{u : u\#v \in \tilde{W}\} > m\epsilon &\geq m\sum_{k=0}^{\infty} 2^{-n_0-k} \geq m\sum_{k=0}^{\infty} 2^{-n_k} \\ &= m\sum_{k=0}^{\infty} \mu H_k \geq m\mu E = m\nu E^*.\end{aligned}$$

So there must be some u such that $u\#v \in \tilde{W}$ and $u(j) \notin E^*$ for every $j < m$. But now, setting $w = u\#v$, we have $w(2j) \notin E^*$, $w(2j+1) \in E^*$ for $j < m$, and $w \in D_m(A^*, Z, 0, 1) \cap \tilde{W}$; which is supposed to be impossible. **XQ**

(e) This shows that A^* is not stable. It is, however, R-stable. **P** We have a measure algebra isomorphism between the measure algebras of μ and ν defined by the map $E \mapsto E^* : \Sigma \to \mathrm{T}$. The corresponding isomorphism between $L^0(\mu)$ and $L^0(\nu)$ takes $\{f^\bullet : f \in A\}$ to $\{h^\bullet : h \in A^*\}$. By 465Pa and (b) above, $\{f^\bullet : f \in A\}$ is stable in $L^0(\mu)$, so $\{h^\bullet : h \in A^*\}$ is stable in $L^0(\nu)$, and every countable subset of A^* is stable. Since $A^* \subseteq C(X)$, it follows that A^* is stable (465T). **Q**

***465V Remark** This example is clearly related to 419E. The argument here is significantly deeper, but it does have an idea in common with that in 419E, besides the obvious point that both involve the Stone spaces of atomless probability spaces. Suppose that, in the context of 465U, we take $\mathcal{E}_0$ to be the family of sets E expressible as the union of a *finite* chain $H_0, \dots, H_k$ where $H_i \in \mathcal{H}_{n_i}$ for $i \leq k$ and $H_j \subseteq G_{H_i}$ for $i < j \leq k$. Then we find, on repeating the argument of (b) in the proof of 465U, that the countable set $A_0^* = \{\chi E^* : E \in \mathcal{E}_0\}$ is stable, so that, setting $W_m = D_m(A_0^*, Z, 0, 1)$, $\nu^{2m}W_m$ is small for large m. On the other hand, setting

$$\tilde{W}_m = \bigcup\{V : V \subseteq Z^m \text{ is open, } V \setminus W_m \text{ is negligible}\},$$

we see that $D_m(A^*, Z, 0, 1) \subseteq \tilde{W}_m$, so that $(\nu^{2m})^*\tilde{W}_m = 1$ for all $m \geq 1$. Of course, writing $\tilde{\nu}^{2m}$ for the Radon product measure on Z^{2m}, we have $\tilde{\nu}^{2m}(\tilde{W}_m) = \nu^{2m}W_m < 1$ for large m, just as in 419E.

Both $A \subseteq \mathcal{L}^0(\Sigma)$ and $A^* \subseteq \mathcal{L}^0(\mathrm{T})$ are relatively pointwise compact. Note that while I took (X, Σ, μ) and (Z, T, ν) to be quite separate, it is entirely possible for them to be actually the same space. In this case it is natural to take every Σ_n to consist of open-and-closed sets, so that every member of $\mathcal{E}$ is open, and E^* becomes identified with the closure of E for $E \in \mathcal{E}$.

465X Basic exercises (a) Let (X, Σ, μ) be a measure space, and $\langle f_n \rangle_{n\in\mathbb{N}}$ a sequence of measurable real-valued functions on X which converges a.e. Show that $\{f_n : n \in \mathbb{N}\}$ is stable.

(b) Let $\mathcal{C}$ be the family of convex sets in $\mathbb{R}^r$. Show that $\{\chi C : C \in \mathcal{C}\}$ is stable with respect to Lebesgue measure on $\mathbb{R}^r$, but that if $r \geq 2$ there is a Radon probability measure ν on $\mathbb{R}^r$ such that $\{\chi C : C \in \mathcal{C}, C \text{ is closed}\}$ is not stable with respect to ν.

(c) Show that, for any $M \geq 0$, the set of functions $f : \mathbb{R} \to \mathbb{R}$ of variation at most M is stable with respect to any Radon measure on $\mathbb{R}$. (*Hint*: show that if μ is a Radon measure on $\mathbb{R}$ and $E \in \operatorname{dom}\mu$ has non-zero finite measure, and $(2k-1)(\beta-\alpha) > M$, then $(\mu^{2k})^* D_k(A, E, \alpha, \beta) < (\mu E)^{2k}$.)

(d) Let (X, Σ, μ) be a measure space, and $A \subseteq \mathbb{R}^X$. Show that A is stable iff $\{f^+ : f \in A\}$ and $\{f^- : f \in A\}$ are both stable.

(e) Let (X, Σ, μ) and (Y, T, ν) be measure spaces, and $\phi : X \to Y$ an inverse-measure-preserving function. Show that if $B \subseteq \mathbb{R}^Y$ is stable with respect to ν, then $\{g\phi : g \in B\}$ is stable with respect to μ.

(f) Let (X, Σ, μ) be a semi-finite measure space and $A \subseteq \mathbb{R}^X$. (i) Suppose that μ is inner regular with respect to the family $\{F : F \in \Sigma, \{f \times \chi F : f \in A\} \text{ is stable}\}$. Show that A is stable. (ii) Use this together with 465Ce, 465Cf and 465Cl to prove 465Ci.

(g) Let (X, Σ, μ) be a totally finite measure space and T a σ-subalgebra of Σ. Let $A \subseteq \mathcal{L}^0(\mathrm{T})$ be any set. (i) Show that if A is $\mu{\restriction}\mathrm{T}$-stable then it is μ-stable. (ii) Give an example to show that A can be μ-stable and pointwise compact without being $\mu{\restriction}\mathrm{T}$-stable. (*Hint*: take μ to be Lebesgue measure on $[0,1]$ and T the countable-cocountable algebra.)

(h) Let (X, Σ, μ) be a measure space, and $A \subseteq \mathbb{R}^X$. For $g : X \to [0, \infty[$ set $A_g = \{(f \wedge g) \vee (-g) : f \in A\}$. Show that A is stable iff A_g is stable for every integrable $g : X \to [0, \infty[$.

(i) Let (X, Σ, μ) be a locally determined measure space and $A \subseteq \mathbb{R}^X$ a stable set. Let $\overline{A}$ be the closure of A for the topology of pointwise convergence. Show that $\{f^\bullet : f \in \overline{A}\}$ is just the closure of $\{f^\bullet : f \in A\} \subseteq L^0(\mu)$ for the topology of convergence in measure.

(j) Show that there is a disjoint family $\mathcal{I}$ of finite subsets of $[0,1]$ such that $A = \{\chi I : I \in \mathcal{I}\}$ is not stable, though A is pointwise compact and the identity map on A is continuous for the topology of pointwise convergence and the topology of convergence in measure.

(k) Let (X, Σ, μ) be a probability space. Show that $A \subseteq \mathbb{R}^X$ is stable iff

$$\inf_{m\in\mathbb{N}}\big((\mu^{2m})^* D_m(A, X, \alpha, \beta)\big)^{1/m} = 0$$

whenever $\alpha < \beta$ in $\mathbb{R}$.

(l) Let (X,Σ,μ) be a measure space. Show that a countable set $A \subseteq \mathcal{L}^0(\Sigma)$ is not stable iff there are $E \in \Sigma$ and $\alpha < \beta$ such that $\mu E > 0$ and $\mu^m \tilde{D}_m(A,E,\alpha,\beta) = (\mu E)^m$ for every $m \geq 1$, where $\tilde{D}_m(A,E,\alpha,\beta)$ is the set of those $w \in E^m$ such that for every $I \subseteq m$ there is an $f \in A$ such that $f(w(i)) \leq \alpha$ for $i \in I$, $f(w(i)) \geq \beta$ for $i \in m \setminus I$. (*Hint*: see part (iii) of case 2 of the proof of 465L.)

(m) Let (X,Σ,μ) be a semi-finite measure space, and $A \subseteq \mathcal{L}^0(\Sigma)$ a set which is compact and metrizable for the topology of pointwise convergence. Show that A is stable. (*Hint*: apply the ideas of case 2 in the proof of 465L to a countable dense subset of A.)

(n) Let (X,Σ,μ) be a probability space and $A \subseteq \mathcal{L}^0(\Sigma)$ a uniformly bounded set of functions. Show that A is stable iff

$$\lim_{n\to\infty} \overline{\int} \sup_{f\in A,k,l\geq n} \left|\frac{1}{k}\sum_{i=0}^{k-1} f(w(i)) - \frac{1}{l}\sum_{i=0}^{l-1} f(w(i))\right| \mu^{\mathbb{N}}(dw) = 0.$$

(o) Show that there is a set $A \subseteq \mathbb{R}^{[0,1]}$ such that $g(x) = \sup_{f\in A} |f(x)|$ is finite for every x, and A is stable with respect to Lebesgue measure, but its convex hull $\Gamma(A)$ is not. (*Hint*: take $A = \{g(x)\chi\{x\} : x \in [0,1]\}$, where g is such that for every $m \geq 1$, every non-negligible compact set $K \subseteq [0,1]^m$ there is a $w \in K$ such that $g(w(i)) = 2^{i+1}$ for every $i < m$.)

(p) Let (X,Σ,μ) be a strictly localizable measure space and $Q \subseteq L^0(\mu)$ a set which is stable in the sense of 465R. Show that the closure of Q (for the topology of convergence in measure) is stable. (*Hint*: 465Xi.)

(q) Let (X,Σ,μ) be a probability space, T a σ-subalgebra of Σ, and $A \subseteq \mathbb{R}^X$ a countable stable set of Σ-measurable functions such that $\int \sup_{f\in A} |f(x)|\mu(dx) < \infty$. Show that if for each $f \in A$ we choose a conditional expectation g_f of f on T (requiring each g_f to be T-measurable and defined everywhere on X), then $\{g_f : f \in A\}$ is stable.

(r) Give an example of a probability algebra $(\mathfrak{A},\bar{\mu})$, a conditional expectation operator $P : L^1_{\bar{\mu}} \to L^1_{\bar{\mu}}$ (365R), and a uniformly integrable stable set $A \subseteq L^1_{\bar{\mu}}$ such that $P[A]$ is not stable. (*Hint*: start by picking a sequence $\langle v_n\rangle_{n\in\mathbb{N}}$ in $P[L^1_{\bar{\mu}}]$ which is norm-convergent to 0 but not stable, and express this as $\langle Pu_n\rangle_{n\in\mathbb{N}}$ where $\bar{\mu}[\![u_n > 0]\!] \leq 2^{-n}$ for every n.)

465Y Further exercises (a) Find an integrable continuous function $f : [0,1[^2 \to [0,\infty[$ such that, in the notation of 465H, $\limsup_{k\to\infty} \int f d\nu^2_{wk} = \infty$ for almost every $w \in [0,1]^{\mathbb{N}}$. (*Hint*: arrange for $f(x,x)$ to grow very fast as $x \uparrow 1$.)

(b) Show that in 465M we may replace the condition 'A is uniformly bounded' by the condition '$|f| \leq f_0$ for every $f \in A$, where f_0 is integrable'. (*Hint*: Talagrand 87.)

(c) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a τ-additive topological measure space such that μ is inner regular with respect to the Borel sets. Show that a countable R-stable subset of $\mathbb{R}^X$ is stable.

(d) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a complete τ-additive topological probability space such that μ is inner regular with respect to the Borel sets, and $A \subseteq [0,1]^X$ an R-stable set. Suppose that $\epsilon > 0$ is such that $\int f d\mu \leq \epsilon^2$ for every $f \in A$. Show that there are an $n \geq 1$ and a Borel set $W \subseteq X^n$ and a $\gamma > \tilde{\mu}^n W$ (writing $\tilde{\mu}^n$ for the τ-additive product measure on X^n) such that $\int f d\nu \leq 3\epsilon$ whenever $f \in A$, $\nu : \mathcal{P}X \to [0,1]$ is a point-supported probability measure and $\nu^n W \leq \gamma$.

465 Notes and comments The definition in 465B arose naturally when M.Talagrand and I were studying pointwise compact sets of measurable functions; we found that in many cases a set of functions was relatively pointwise compact because it was stable (465Db). Only later did it appear that the concept was connected with 'Glivenko-Cantelli' classes in the theory of empirical measures, as explained in Talagrand 87.

It is not the case that all pointwise compact sets of measurable functions are stable. In fact I have already offered examples in 463Xi and 464E above. In both cases it is easy to check from the definition in 465B that they are not stable, as can also be deduced from 465G. Another example is in 465Xj. You will observe

however that all these examples are 'pathological' in the sense that either the measure space is irregular (from some points of view, indeed, any measure space not isomorphic to Lebesgue measure on the unit interval can be dismissed as peripheral), or the set of functions is uninteresting. It is clear from 465R and 465T, for instance, that we should start with countable sets. So it is natural to ask: if we have a *separable* pointwise compact set of real-valued measurable functions on the unit interval, must it be stable? It turns out that this is undecidable in Zermelo-Fraenkel set theory (Shelah & Fremlin 93); I hope to return to the question in Volume 5. (If we ask for 'metrizable', instead of 'separable', we get a positive answer; see 465Xm.)

The curious phrasing of the statement of 465M(iii), with the auxiliary functions h_k, turns on the fact that all the expressions '$\sup_{f\in A}\dots$' here give rise to functions which need not be measurable. Thus the simple pointwise convergence described in (ii) and (iv) is not at all the same thing as the convergence in (v), which may be thought of as a kind of $\|\ \|_1$-convergence if we write $\|g\|_1 = \overline{\int}|g|$ for arbitrary real-valued functions g. (Since the sets A here are uniformly bounded, it may equally be thought of as convergence in measure.) Similarly, 465M(iii) is saying much more than just

$$\lim_{k\to\infty}\sup_{f\in A}\frac{1}{k}\sum_{i=0}^{k-1}\big|f(w(i)) - \mathbb{E}(f|\mathrm{T})(w(i))\big| = 0 \text{ a.e.},$$

though of course for countable sets A these distinctions disappear. On the other hand, since the convergence is certainly not monotonic in k, $\|\ \|_1$-convergence does not imply pointwise convergence. So we can look for something stronger than either (iv) or (v) of 465M, as in 465Xn. But this is still nowhere near the strength of 465M(iii), in which $|\sum\dots|$ is replaced by $\sum|\dots|$. For further variations, see Talagrand 87 and Talagrand 96.

If we wish to adapt the ideas here to spaces of equivalence classes of functions rather than spaces of true functions, we find that problems of measurability evaporate, and that (because the definition of stability looks only at sets of finite measure) all the relevant suprema can be interpreted as suprema of countable sets. Consequently a subset of $L^0(\mu)$ or $L^0(\mathfrak{A})$ is stable if all its countable subsets are stable (465Pa). It is remarkable that, for strictly localizable measures μ, we can lift any stable set in L^0 to a stable set in $\mathcal{L}^0$ (465Pb, 465Q). By moving to function spaces we get a language in which to express a new kind of permanence property of stable sets (465R, 465Xq). See also Talagrand 89.

The definition of 'stable set' of functions seems to be utterly dependent on the underlying measure space. But 465R tells us that in fact the property of being an order-bounded stable subset of $L^1(\bar{\mu})$ is invariant under normed space automorphisms. ('Order-boundedness' is a normed space invariant in L^1 spaces by the Chacon-Krengel theorem, 371D.) Since stability can be defined in terms of order-bounded sets (465Xh), we could, for instance, develop a theory of stable sets in abstract L-spaces.

The theory of stable sets is of course bound intimately to the theory of product measures, and such results as 465J have independent interest as theorems about sets in product spaces. So any new theory of product measures will give rise to a new theory of stable sets. In particular, the τ-additive product measures in §417 lead to R-stable sets (465S). It is instructive to work through the details, observing how the properties of the product are employed. Primarily, of course, we need 'associative' and 'commutative' laws, and Fubini's theorem; but some questions of measurability need to be re-examined, as in 465Yd.

I have starred 465U because it involves the notion of R-stability. In fact this appears only in the final stage, and the construction, as set out in part (a) of the proof, is an instructive challenge to any intuitive concept of what stable sets are like.

466 Measures on linear topological spaces

In this section I collect a number of results on the special properties of topological measures on linear topological spaces. The most important is surely Phillips' theorem (466A-466B): on any Banach space, the weak and norm topologies give rise to the same totally finite Radon measures. This is not because the weak and norm topologies have the same Borel σ-algebras, though this does happen in interesting cases (466C-466E, §467). When the Borel σ-algebras are different, we can still ask whether the Borel measures are 'essentially' the same, that is, whether every (totally finite) Borel measure for the weak topology extends to a Borel measure for the norm topology. A construction due to M.Talagrand (466H) gives a negative answer.

Just as in $\mathbb{R}^r$, a totally finite quasi-Radon measure on a locally convex linear topological space is determined by its characteristic function (466K). I end the section with a note on measurability conditions sufficient to ensure that a linear operator between Banach spaces is continuous (466L-466M).

466A Theorem Let $(X, \mathfrak{T})$ be a metrizable locally convex linear topological space and μ a σ-finite measure on X which is quasi-Radon for the weak topology $\mathfrak{T}_s(X, X^*)$. Then the support of μ is separable, so μ is quasi-Radon for the original topology $\mathfrak{T}$. If X is complete and μ is locally finite with respect to $\mathfrak{T}$, then μ is Radon for $\mathfrak{T}$.

proof (a) Let $\langle V_n\rangle_{n\in\mathbb{N}}$ be a sequence running over a base of neighbourhoods of 0 in X, and for $n \in \mathbb{N}$ set $V_n^\circ = \{f : f \in X^*, |f(x)| \leq 1 \text{ for every } x \in V_n\}$. Then each V_n° is a convex $\mathfrak{T}_s$-compact subset of X^* (4A4Bh), and $X^* = \bigcup_{n\in\mathbb{N}} V_n^\circ$. Let $Z \subseteq X$ be the support of μ, and for $n \in \mathbb{N}$ set $K_n = \{f\restriction Z : f \in V_n^\circ\}$. Since the map $f \mapsto f\restriction Z : X^* \to C(Z)$ is linear and continuous for $\mathfrak{T}_s$ and the topology $\mathfrak{T}_p$ of pointwise convergence on Z, each K_n is convex and $\mathfrak{T}_p$-compact. Next, the subspace measure μ_Z on Z is σ-finite and strictly positive, so by 463E each K_n is $\mathfrak{T}_p$-metrizable.

Let $\mathcal{U}$ be the base for $\mathfrak{T}_p$ consisting of sets of the form

$$U(I, h, \epsilon) = \{f : f \in C(Z), |f(x) - h(x)| < \epsilon \text{ for every } x \in I\},$$

where $I \subseteq Z$ is finite, $h \in \mathbb{R}^I$ and $\epsilon > 0$. For each $n \in \mathbb{N}$, write $\mathcal{V}_n = \{K_n \cap U : U \in \mathcal{U}\}$, so that $\mathcal{V}_n$ is a base for the subspace topology of K_n (4A2B(a-iv)). Since this is compact and metrizable, therefore second-countable, there is a countable base $\mathcal{V}_n' \subseteq \mathcal{V}_n$ (4A2Ob). Now there is a countable set $\mathcal{U}' \subseteq \mathcal{U}$ such that $\mathcal{V}_n' \subseteq \{K_n \cap U : U \in \mathcal{U}'\}$ for every $n \in \mathbb{N}$, and a countable set $D \subseteq Z$ such that

$$\mathcal{U}' \subseteq \{U(I, h, \epsilon) : I \subseteq D \text{ is finite}, h \in \mathbb{R}^I, \epsilon > 0\}.$$

Let $D' \subseteq X$ be the linear span of D over the rationals. Then D' is again countable, and its $\mathfrak{T}$-closure Y is a linear subspace of X (4A4Bi). **?** If $Z \not\subseteq Y$, then $\mu(X \setminus Y) > 0$. But, writing

$$Y^\circ = \{f : f \in X^*, f(x) \leq 0 \text{ for every } x \in Y\} = \{f : f \in X^*, f(x) = 0 \text{ for every } x \in Y\},$$

$X \setminus Y$ must be $\bigcup_{f\in Y^\circ}\{x : |f(x)| > 0\}$, by 4A4Eb. Because μ is τ-additive, there must be an $f \in Y^\circ$ such that $\mu\{x : |f(x)| > 0\} > 0$. Let $n \in \mathbb{N}$ be such that $f \in V_n^\circ$. Since $f(x) = 0$ for every $x \in D$, $f\restriction Z$ belongs to the same members of $\mathcal{V}_n'$ as the zero function; since $\mathcal{V}_n'$ is a base for the Hausdorff subspace topology of K_n, $f\restriction Z$ actually is the zero function, and $\{x : |f(x)| > 0\} \subseteq X \setminus Z$, so $\mu Z < \mu X$, contrary to the choice of Z. **X**

Thus $Z \subseteq Y$. Because $Y = \overline{D'}$ is $\mathfrak{T}$-separable, and $\mathfrak{T}$ is metrizable, Z is also $\mathfrak{T}$-separable (4A2P(a-iv)).

(b) The subspace topology $\mathfrak{T}_Y$ induced on Y by $\mathfrak{T}$ is a separable metrizable locally convex linear space topology, so the Borel σ-algebras of $\mathfrak{T}_Y$ and the associated weak topology $\mathfrak{T}_s(Y, Y^*)$ are equal (4A3V). But $\mathfrak{T}_s(Y, Y^*)$ is just the subspace topology induced on Y by $\mathfrak{T}_s$ (4A4Ea), Accordingly the subspace measure μ_Y on Y is $\mathfrak{T}_s(Y, Y^*)$-quasi-Radon (415B). Since every $\mathfrak{T}_Y$-open set is $\mathfrak{T}_s(Y, Y^*)$-Borel, μ_Y is a topological measure for $\mathfrak{T}_Y$. Since $\mathfrak{T}_Y$ is finer than $\mathfrak{T}_s(Y, Y^*)$, μ_Y is effectively locally finite for $\mathfrak{T}_Y$ and inner regular with respect to the $\mathfrak{T}_Y$-closed sets, and therefore is a quasi-Radon measure for $\mathfrak{T}_Y$ (415D(i)).

By 415J, there is a measure $\tilde{\mu}$ on X, quasi-Radon for $\mathfrak{T}$, such that $\tilde{\mu}E = \mu_Y(E \cap Y)$ whenever $\tilde{\mu}$ measures E. But as Y is $\mathfrak{T}$-closed and μ-conegligible, and μ is complete, we have $\mu = \tilde{\mu}$, and μ is quasi-Radon for $\mathfrak{T}$.

(c) If X is complete and μ is locally finite with respect to $\mathfrak{T}$, then $(X, \mathfrak{T})$ is a pre-Radon space (434Jg), so μ is a Radon measure for $\mathfrak{T}$ (434Jb).

466B Corollary (Compare 462G.) If X is a Banach space and μ is a totally finite measure on X which is quasi-Radon for the weak topology of X, it is a Radon measure for both the norm topology and the weak topology.

proof By 466A, μ is a Radon measure for the norm topology; by 418I, or otherwise, it is a Radon measure for the weak topology.

Remark Thus Banach spaces, with their weak topologies, are pre-Radon.

466C Definition A normed space X has a **Kadec norm** (also called **Kadec-Klee norm**) if the norm and weak topologies coincide on the sphere $\{x : \|x\| = 1\}$.

Example For any set I, any $p \in [1, \infty[$, the Banach space $\ell^p(I)$ has a Kadec norm. **P** If $x \in \ell^p(I)$ and $\|x\|_p = 1$ and $\epsilon > 0$, let $J \subseteq I$ be a finite set such that $\sum_{i \in I \setminus J} |x(i)|^p \leq \epsilon$. Set $H = \{y : y \in \ell^p(I), \sum_{i \in J} |y(i) - x(i)|^p < \epsilon\}$; then H is open for the weak topology of $\ell^p(I)$. **Q**

For further examples, see §467.

466D Proposition (Hansell 01) Let X be a normed space with a Kadec norm. Then there is a network for the norm topology on X expressible in the form $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$, where for each $n \in \mathbb{N}$ $\mathcal{V}_n$ is an isolated family for the weak topology and $\bigcup \mathcal{V}_n$ is the difference of two closed sets for the weak topology.

proof Let $\mathcal{U}$ be a σ-disjoint base for the norm topology of X (4A2L(h-ii)); express it as $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ where every $\mathcal{U}_n$ is disjoint. For rational numbers q, q' with $0 < q < q'$, set $S_{qq'} = \{x : q < \|x\| \leq q'\}$, and for $A \subseteq X$ write $W(A, q, q')$ for the interior of $A \cap S_{qq'}$ taken in the subspace weak topology of $S_{qq'}$. Set $\mathcal{V}_{nqq'} = \{W(U, q, q') : U \in \mathcal{U}_n\}$, so that $\mathcal{V}_{nqq'}$ is a disjoint family of relatively-weakly-open subsets of $S_{qq'}$, and is an isolated family for the weak topology. Now $\bigcup_{n \in \mathbb{N}, q, q' \in \mathbb{Q}, 0 < q < q'} \mathcal{V}_{nqq'}$ is a network for the norm topology on $X \setminus \{0\}$. **P** If $x \in X \setminus \{0\}$ and $\epsilon > 0$, then take $n \in \mathbb{N}$, $U \in \mathcal{U}_n$ such that $x \in U \subseteq \{y : \|y - x\| \leq \epsilon\}$. Let $\delta > 0$ be such that $\{y : \|y - x\| \leq \delta\} \subseteq U$. Next, because $\| \, \|$ is a Kadec norm, there is a weak neighbourhood V of 0 such that $\|y - x\| \leq \frac{1}{2}\delta$ whenever $y \in x - V$ and $\|y\| = \|x\|$. Let V' be an open weak neighbourhood of 0 such that $V' + V' \subseteq V$. Let $\eta > 0$ be such that $\eta < 1$, $\eta\|x\| \leq \frac{1}{2}\delta$ and $y \in V'$ whenever $\|y\| \leq \eta\|x\|$. If $y \in x - V'$ and $(1 - \eta)\|x\| \leq \|y\| \leq (1 + \eta)\|x\|$, then $\frac{\|x\|}{\|y\|} y \in x + V$. Since

$$\|y - \frac{\|x\|}{\|y\|} y\| = |1 - \frac{\|y\|}{\|x\|}| \|x\| \leq \eta\|x\|,$$

$$x - \frac{\|x\|}{\|y\|} y = (x - y) + (y - \frac{\|x\|}{\|y\|} y) \in V' + V' \subseteq V,$$

so $\|x - \frac{\|x\|}{\|y\|} y\| \leq \frac{1}{2}\delta$ and $\|x - y\| \leq \delta$ and $y \in U$. This means that if we take q, $q' \in \mathbb{Q}$ such that $(1 - \eta)\|x\| \leq q \leq \|x\| \leq q' \leq (1 + \eta)\|x\|$, $(x - V') \cap S_{qq'} \subseteq U$ and $x \in W(U, q, q') \in \mathcal{V}_{nqq'}$. Since of course $W(U, q, q') \subseteq U \subseteq \{y : \|y - x\| \leq \epsilon\}$, and x and ϵ are arbitrary, we have the result. **Q**

To get a σ-isolated family for the weak topology which is a network for the norm topology on the whole of X, we just have to add the singleton set $\{0\}$. To see that the union of each of our isolated families is the difference of two weakly open sets, observe that $\bigcup \mathcal{V}_{nqq'}$ is a relatively weakly open subset of $S_{qq'}$, which is the difference of the weakly open sets $\{x : \|x\| > q\}$ and $\{x : \|x\| > q'\}$.

466E Corollary Let X be a normed space with a Kadec norm.
(a) The norm and weak topologies give rise to the same Borel σ-algebras.
(b) The weak topology has a σ-isolated network, so is hereditarily weakly θ-refinable.

proof (a) Write $\mathcal{B}_{\| \, \|}$, $\mathcal{B}_{\mathfrak{T}_s}$ for the Borel σ-algebras for the weak and norm topologies. Of course $\mathcal{B}_{\mathfrak{T}_s} \subseteq \mathcal{B}_{\| \, \|}$. Let $\langle \mathcal{V}_n \rangle_{n \in \mathbb{N}}$ be a sequence covering a network for the norm topology as in 466D. Because $\mathcal{V}_n$ is (for the weak topology) isolated and its union belongs to $\mathcal{B}_{\mathfrak{T}_s}$, $\bigcup \mathcal{W} \in \mathcal{B}_{\mathfrak{T}_s}$ for every $\mathcal{W} \subseteq \mathcal{V}_n$, $n \in \mathbb{N}$. But this means that $\bigcup \mathcal{W} \in \mathcal{B}_{\mathfrak{T}_s}$ for every $\mathcal{W} \subseteq \mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{V}_n$; and as $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is a network for the norm topology, every norm-open set belongs to $\mathcal{B}_{\mathfrak{T}_s}$, and $\mathcal{B}_{\| \, \|} \subseteq \mathcal{B}_{\mathfrak{T}_s}$. Thus the two Borel σ-algebras are equal.

(b) Of course $\mathcal{V}$ is also a network for the weak topology, so the weak topology has a σ-isolated network; by 438Ld, it is hereditarily weakly θ-refinable.

466F Proposition Let X be a Banach space with a Kadec norm. Then the following are equiveridical:
(i) X is a Radon space in its norm topology;
(ii) X is a Radon space in its weak topology;
(iii) the weight of X (for the norm topology) is measure-free in the sense of §438.

proof (a)(i) $\iff$ (ii) By 466Ea, the norm and weak topologies give rise to the same algebra $\mathcal{B}$ of Borel sets. If X is a Radon space in its norm topology, then any totally finite measure with domain $\mathcal{B}$ is inner regular with respect to the norm-compact sets, therefore inner regular with respect to the weakly compact sets, and X is Radon in its weak topology. If X is a Radon space in its weak topology, then any totally finite measure μ with domain $\mathcal{B}$ has a completion $\hat{\mu}$ which is a Radon measure for the weak topology, therefore also for the norm topology, by 466B; as μ is arbitrary, X is a Radon space for the norm topology.

(b)(i) $\iff$ (iii) is 438H.

466G Definition A partially ordered set X has the **σ-interpolation property** or **countable separation property** if whenever A, B are non-empty countable subsets of X and $x \leq y$ for every $x \in A$, $y \in B$, then there is a $z \in X$ such that $x \leq z \leq y$ for every $x \in A$, $y \in B$. A Dedekind σ-complete partially ordered set (314Ab) always has the σ-interpolation property.

466H Proposition (Jayne & Rogers 95) Let X be a Riesz space with a Riesz norm, given its weak topology $\mathfrak{T}_s = \mathfrak{T}_s(X, X^*)$. Suppose that ($\alpha$) X has the σ-interpolation property (β) there is a strictly increasing family $\langle p_\xi \rangle_{\xi<\omega_1}$ in X. Then there is a $\mathfrak{T}_s$-Borel probability measure μ on X such that

(i) μ is not inner regular with respect to the $\mathfrak{T}_s$-closed sets;

(ii) μ is not τ-additive for the topology $\mathfrak{T}_s$;

(iii) μ has no extension to a norm-Borel measure on X.

Accordingly $(X, \mathfrak{T}_s)$ is not a Radon space (indeed, is not Borel-measure-complete).

proof (a) Let K be the set

$$\{f : f \in X^*, f \geq 0, \|f\| \leq 1\} = \{f : \|f\| \leq 1\} \cap \textstyle\bigcap_{x \in X^+} \{f : f(x) \geq 0\},$$

so that K is a weak*-closed subset of the unit ball of X^* and is weak*-compact. Because X^* is a solid linear subspace of the order-bounded dual $X^\sim$ of X (356Da), every member of X^* is the difference of two non-negative members of X^*, and K spans X^*. We shall need to know that if $x < y$ in X, there is an $f \in K$ such that $f(x) < f(y)$; set $f = \frac{1}{\|g\|}|g|$ where $g \in X^*$ is such that $g(x) \neq g(y)$. (Recall that the norm of X^* is a Riesz norm, as also noted in 356Da.)

Set

$$A = \textstyle\bigcup_{\xi<\omega_1} \{x : x \in X, x \leq p_\xi\};$$

then every sequence in A has an upper bound in A, but A has no greatest member. It follows that if $B \subseteq K$ is countable there is an $x \in A$ such that $f(x) = \sup_{y \in A} f(y)$ for every $f \in B$, so that $f(x) = f(y)$ whenever $x \leq y \in A$ and $f \in B$.

Let $\mathcal{E}$ be the family of those sets $E \subseteq X$ such that $A \cap E$ is cofinal with A. Observe that if $\langle E_n \rangle_{n \in \mathbb{N}}$ is any sequence of subsets of X, and $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{E}$, then some E_n belongs to $\mathcal{E}$. **P?** Otherwise, there is for each $n \in \mathbb{N}$ an $x_n \in A$ such that $x_n \not\leq y$ for every $y \in A \cap E_n$. Now there is an $x \in A$ such that $x_n \leq x$ for every $n \in \mathbb{N}$, and $x \not\leq y$ for every $y \in \bigcup_{n \in \mathbb{N}} E_n$. **XQ**

(b) If $G \in \mathcal{E}$ is a $\mathfrak{T}_s$-open set, there is a $k \in \mathbb{N}$ such that

for every $z \in A$ there are $x \in A$ and $f_0, \ldots, f_k \in K$ such that $x \geq z$ and $\{y : y \in X, |f_i(y-x)| \leq 2^{-k}$ for every $i \leq k\}$ is included in G.

P For each $k \in \mathbb{N}$ let H_k be the set of those $x \in X$ such that $\{y : |f_i(y-x)| \leq 2^{-k}$ for every $i \leq k\}$ is included in G for some $f_0, \ldots, f_k \in K$. Because the linear span of K is X^*, and G is $\mathfrak{T}_s$-open, $G = \bigcup_{k \in \mathbb{N}} H_k$. So, as remarked in (a), there is some $k \in \mathbb{N}$ such that $H_k \in \mathcal{E}$, and this k will serve. **Q**

(c) (The key.) If $\langle G_n \rangle_{n \in \mathbb{N}}$ is any sequence of $\mathfrak{T}_s$-open sets belonging to $\mathcal{E}$, then $\bigcap_{n \in \mathbb{N}} G_n \in \mathcal{E}$. **P** Start from any $z^* \in A$. For each $n \in \mathbb{N}$, choose $k_n \in \mathbb{N}$ as in (b), so that there is a family $\langle f_{nzi} \rangle_{z \in A, i \leq k_n}$ in K such that

for every $z \in A$ there is an $x \in A$ such that $x \geq z$ and

$$\{y : |f_{nzi}(y-x)| \leq 2^{-k_n} \ \forall\ i \leq k_n\} \subseteq G_n.$$

Let $\mathcal{F}$ be any ultrafilter on A containing $\{x : x \in A, x \geq z\}$ for every $z \in A$, and write $f_{ni} = \lim_{z \to \mathcal{F}} f_{nzi}$ for $n \in \mathbb{N}$ and $i \leq k_n$, the limit being taken for the weak* topology on K. Let $z_1^* > z^*$ be such that $z_1^* \in A$ and $f_{ni}(x - z_1^*) = 0$ whenever $x \in A$, $x \geq z_1^*$, $n \in \mathbb{N}$ and $i \leq k_n$.

Choose sequences $\langle x_n \rangle_{n \in \mathbb{N}}$, $\langle y_n \rangle_{n \in \mathbb{N}}$ and $\langle z_n \rangle_{n \in \mathbb{N}}$ in A inductively, as follows. Set $y_0 = z_1^*$. Given y_n, the set

$$C_n = \{z : z \in A, z \geq y_n, |(f_{nzi} - f_{ni})(y_n - z_1^*)| \leq 2^{-k_n} \text{ for every } i \leq k_n\}$$

belongs to $\mathcal{F}$, so is not empty; take $z_n \in C_n$. Because $y_n \geq z_1^*$, we have $f_{ni}(y_n - z_1^*) = 0$ and $|f_{nz_ni}(y_n - z_1^*)| \leq 2^{-k_n}$, for every $i \leq k_n$. Let $x_n \in A$ be such that $x_n \geq z_n$ and $\{y : |f_{nz_ni}(y - x_n)| \leq 2^{-k_n} \text{ for every } i \leq k_n\} \subseteq G_n$. Now let $y_{n+1} \in A$ be such that $y_{n+1} \geq x_n$ and $f_{nz_ni}(y - y_{n+1}) = 0$ whenever $y \in A$ and $y \geq y_{n+1}$. Continue.

At the end of the induction, let z_2^* be an upper bound for $\{y_n : n \in \mathbb{N}\}$ in A. For $n \in \mathbb{N}$, set

$$u_n = z_1^* + \textstyle\sum_{j=0}^{n} x_j - y_j, \quad v_n = u_n + z_2^* - y_{n+1}.$$

Then $\langle u_n \rangle_{n \in \mathbb{N}}$ is non-decreasing and $u_n \leq v_n \leq z_2^*$ for every $n \in \mathbb{N}$; moreover,

$$v_n - v_{n+1} = y_{n+1} - x_{n+1} - y_{n+1} + y_{n+2} \geq 0$$

for every n, so $\langle v_n \rangle_{n \in \mathbb{N}}$ is non-increasing, and $u_m \leq v_n$ for all m, $n \in \mathbb{N}$. Because X has the σ-interpolation property, there is an $x \in X$ such that $u_n \leq x \leq v_n$ for every $n \in \mathbb{N}$. Since $z_1^* \leq x \leq z_2^*$, $x \in A$ and $x > z^*$.

Take any $n \in \mathbb{N}$. Then

$$f_{nz_ni}(x - u_n) \leq f_{nz_ni}(v_n - u_n) = f_{nz_ni}(z_2^* - y_{n+1}) = 0$$

for every $i \leq k_n$. On the other hand,

$$x_n - u_n = (y_n - z_1^*) - \textstyle\sum_{j=0}^{n-1}(x_j - y_j)$$

lies between 0 and $y_n - z_1^*$, so

$$0 \leq f_{nz_ni}(x_n - u_n) \leq 2^{-k_n}$$

for every $i \leq k_n$, and $|f_{nz_ni}(x - x_n)| \leq 2^{-k_n}$ for every i. Thus $x \in G_n$. As n is arbitrary, $x \in \bigcap_{n \in \mathbb{N}} G_n$.

As $x > z^*$, this shows that z^* is not an upper bound of $\bigcap_{n \in \mathbb{N}} G_n \cap A$. As z^* is arbitrary, $\bigcap_{n \in \mathbb{N}} G_n \in \mathcal{E}$. **Q**

(d) Set

$$\mathcal{K}_0 = \{G \setminus H : G, H \in \mathfrak{T}_s, G \in \mathcal{E}, H \notin \mathcal{E}\}.$$

Then $\bigcap_{n \in \mathbb{N}} E_n \neq \emptyset$ for any sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{K}_0$. **P** Express each E_n as $G_n \setminus H_n$ where G_n, H_n are $\mathfrak{T}_s$-open, $G_n \in \mathcal{E}$ and $H_n \notin \mathcal{E}$. Then $\bigcup_{n \in \mathbb{N}} H_n \notin \mathcal{E}$, as noted in (a), while $\bigcap_{n \in \mathbb{N}} G_n \in \mathcal{E}$, by (c); so $\bigcap_{n \in \mathbb{N}} E_n = \bigcap_{n \in \mathbb{N}} G_n \setminus \bigcup_{n \in \mathbb{N}} H_n$ is non-empty. **Q**

It follows that $\mathcal{K} = \mathcal{K}_0 \cup \{\emptyset\}$ is a countably compact class in the sense of 413L. Moreover, $E \cap E' \in \mathcal{K}$ for all E, $E' \in \mathcal{K}$ (using (b) and (c) again), so if we define $\phi_0 : \mathcal{K} \to \{0, 1\}$ by writing $\phi_0(E) = 1$ for $E \in \mathcal{K}_0$, $\phi_0(\emptyset) = 0$, then $\mathcal{K}$ and ϕ_0 will satisfy all the conditions of 413M. There is therefore a measure $\hat{\mu}$ on X extending ϕ_0 and inner regular with respect to $\mathcal{K}_\delta$, the family of sets expressible as intersections of sequences in $\mathcal{K}$. The domain of $\hat{\mu}$ must include every member of $\mathcal{K}$; but if $G \in \mathfrak{T}_s$ then either G or $X \setminus G$ belongs to $\mathcal{K}_0$, so is measured by $\hat{\mu}$, and $\hat{\mu}$ is a topological measure.

We need to observe that, because ϕ_0 takes only the values 0 and 1, $\hat{\mu}E \leq 1$ for every $E \in \mathcal{K}_\delta$, and $\hat{\mu}X \leq 1$; since $\phi_0 X = 1$, $\hat{\mu}X = 1$ and $\hat{\mu}$ is a probability measure.

(e) We may therefore take μ to be the restriction of $\hat{\mu}$ to the algebra $\mathcal{B}$ of Borel sets, and μ is a $\mathfrak{T}_s$-Borel probability measure. Now μ is not inner regular with respect to the $\mathfrak{T}_s$-closed sets. **P** For each $\xi < \omega_1$, $p_\xi < p_{\xi+1} < p_{\xi+2}$, so there are g_ξ, $h_\xi \in K$ such that $g_\xi(p_\xi) < g_\xi(p_{\xi+1})$ and $h_\xi(p_{\xi+1}) < h_\xi(p_{\xi+2})$. Let $D \subseteq \omega_1$ be any set such that D and $\omega_1 \setminus D$ are both uncountable, and set

$$G = \textstyle\bigcup_{\xi \in D}\{x : g_\xi(p_\xi) < g_\xi(x), h_\xi(x) < h_\xi(p_{\xi+2})\}.$$

Then $G \in \mathfrak{T}_s$, and $p_{\xi+1} \in G$ for every $\xi \in D$, so $G \in \mathcal{E}$ and $\mu G = \phi_0 G = 1$. On the other hand, if $\eta \in \omega_1 \setminus D$, then for every $\xi \in D$ either $\xi < \eta$ and $h_\xi(p_{\xi+2}) \leq h_\xi(p_{\eta+1})$, or $\xi > \eta$ and $g_\xi(p_{\eta+1}) \leq g_\xi(p_\xi)$; thus $p_{\eta+1} \notin G$

for any $\eta \in \omega_1 \setminus D$, and $X \setminus G \in \mathcal{E}$. But this means that if $F \subseteq G$ is closed then $X \setminus F \in \mathcal{K}_0$ and $\mu F = 0$. Thus $\mu G > \sup_{F \subseteq G \text{ is closed}} \mu F$. **Q**

(f) Because $\mathfrak{T}_s$ is regular, μ cannot be τ-additive, by 414Mb. It follows at once that $(X, \mathfrak{T}_s)$ is not Borel-measure-complete, and in particular is not a Radon space. To see that μ has no extension to a norm-Borel measure, we need to look again at the set A. For each $\xi < \omega_1$, $F_\xi = \{x : x \leq p_\xi\}$. Then every F_ξ is norm-closed (354Bc) and $\langle F_\xi \rangle_{\xi < \omega_1}$ is an increasing family with union A. Consequently, A is norm-closed (4A2Ld, 4A2Ka). At the same time, every F_ξ is convex (cf. 351Ce), so A is also convex. It follows that A, like every F_ξ, is $\mathfrak{T}_s$-closed (3A5Ee). So μ measures A and every F_ξ. Because $X \setminus F_\xi$ is a $\mathfrak{T}_s$-open set belonging to $\mathcal{E}$, $\mu F_\xi = 0$ for every $\xi < \omega_1$; because $X \setminus A$ is a $\mathfrak{T}_s$-open set not belonging to $\mathcal{E}$, $\mu A = 1$.

But ω_1 is a measure-free cardinal (438Cd), so 438I tells us that $\lambda A = \sup_{\xi < \omega_1} \lambda F_\xi$ for any semi-finite norm-Borel measure λ on X. Thus μ has no extension to a norm-Borel measure, and the proof is complete.

466I Examples The following spaces satisfy the hypotheses of 466H.

(a) (Talagrand 78a, or Talagrand 84, 16-1-2) $X = \ell^\infty(I)$ or $\{x : x \in \ell^\infty(I), \{i : x(i) \neq 0\}$ is countable$\}$, where I is uncountable. **P** X has the σ-interpolation property because it is Dedekind complete, and if $\langle i_\xi \rangle_{\xi < \omega_1}$ is any family of distinct elements of I, we can set $p_\xi(i_\eta) = 1$ for $\eta \leq \xi$, $p_\xi(i) = 0$ for all other $i \in I$ to obtain a strictly increasing family $\langle p_\xi \rangle_{\xi < \omega_1}$ in X. **Q**

(b) (de Maria & Rodriguez-Salinas 91) $X = \ell^\infty / \boldsymbol{c}_0$.

P (i) To see that X has the σ-interpolation property, let A, $B \subseteq X$ be non-empty countable sets such that $u \leq v$ for all $u \in A$, $v \in B$. Let $\langle x_n \rangle_{n \in \mathbb{N}}$, $\langle y_n \rangle_{n \in \mathbb{N}}$ be sequences in ℓ^∞ such that $A = \{x_n^\bullet : n \in \mathbb{N}\}$ and $B = \{y_n^\bullet : n \in \mathbb{N}\}$. Set $\tilde{x}_n = \sup_{i \leq n} x_i$, $\tilde{y}_n = \inf_{i \leq n} y_i$ for $n \in \mathbb{N}$; then $\tilde{x}_n^\bullet \leq \tilde{y}_n^\bullet$, so $(\tilde{x}_n - \tilde{y}_n)^+ \in \boldsymbol{c}_0$. Set

$$k_n = \max(\{n\} \cup \{i : \tilde{x}_n(i) \geq \tilde{y}_n(i) + 2^{-n}\})$$

for $n \in \mathbb{N}$, and define $x \in \ell^\infty$ by writing

$$\begin{aligned} x(i) &= 0 \text{ if } i \leq k_0, \\ &= \tilde{x}_n(i) \text{ if } k_n < i \leq k_{n+1}. \end{aligned}$$

Then it is easy to check that $u \leq x^\bullet \leq v$ for every $u \in A$, $v \in B$; as A and B are arbitrary, X has the σ-interpolation property.

(ii) To see that X has the other property, recall that there is a family $\langle I_\xi \rangle_{\xi < \omega_1}$ of infinite subsets of $\mathbb{N}$ such that $I_\xi \setminus I_\eta$ is finite if $\eta \leq \xi$, infinite if $\xi < \eta$ (4A1Fa). Setting $p_\xi = \chi(\mathbb{N} \setminus I_\xi)^\bullet$, we have a strictly increasing family $\langle p_\xi \rangle_{\xi < \omega_1}$ in X. **Q**

466J Proposition Let X be a linear topological space and Σ its cylindrical σ-algebra. If μ and ν are probability measures with domain Σ such that $\int e^{if(x)} \mu(dx) = \int e^{if(x)} \nu(dx)$ for every $f \in X^*$, then $\mu = \nu$.

proof Define $T : X \to \mathbb{R}^{X^*}$ by setting $(Tx)(f) = f(x)$ for $f \in X^*$, $x \in X$. Then T is linear and continuous for the weak topology of X. So if $F \subseteq \mathbb{R}^{X^*}$ is a Baire set for the product topology of $\mathbb{R}^{X^*}$, $T^{-1}[F]$ is a Baire set for the weak topology of X (4A3Kc) and belongs to Σ (4A3U). We therefore have Baire measures μ', ν' on $\mathbb{R}^{X^*}$ defined by saying that $\mu' F = \mu T^{-1}[F]$ and $\nu' F = \nu T^{-1}[F]$ for every Baire set $F \subseteq \mathbb{R}^{X^*}$.

If $h : \mathbb{R}^{X^*} \to \mathbb{R}$ is a continuous linear functional, it can be expressed in the form $h(z) = \sum_{i=0}^n \alpha_i z(f_i)$ where $f_0, \ldots, f_n \in X^*$ and $\alpha_0, \ldots, \alpha_n \in \mathbb{R}$. So

$$h(Tx) = \sum_{i=0}^n \alpha_i (Tx)(f_i) = \sum_{i=0}^n \alpha_i f_i(x) = f(x)$$

for every $x \in X$, where $f = \sum_{i=0}^n \alpha_i f_i$. This means that

$$\int e^{ih(z)} \mu'(dz) = \int e^{ih(Tx)} \mu(dx) = \int e^{if(x)} \mu(dx) = \int e^{if(x)} \nu(dx) = \int e^{ih(z)} \nu'(dz).$$

As h is arbitrary, $\mu' = \nu'$ (454Ma).

Now let Σ' be the family of subsets of X of the form $T^{-1}[F]$ where $F \subseteq \mathbb{R}^{X^*}$ is a Baire set. This is a σ-algebra and contains all sets of the form $\{x : f(x) \geq \alpha\}$ where $f \in X^*$ and $\alpha \in \mathbb{R}$. So every member of X^* is Σ'-measurable and Σ' must include the cylindrical σ-algebra of X. Since μ and ν agree on Σ' they must be identical.

466K Proposition If X is a locally convex linear topological space and μ, ν are quasi-Radon probability measures on X such that $\int e^{if(x)}\mu(dx) = \int e^{if(x)}\nu(dx)$ for every $f \in X^*$, then $\mu = \nu$.

proof Write $\mathfrak{T}$ for the given topology on X and $\mathfrak{T}_s = \mathfrak{T}_s(X, X^*)$ for the weak topology. By 466J, μ and ν must agree on the cylindrical σ-algebra Σ of X. Since Σ includes a base for $\mathfrak{T}_s$, every weakly open set G is the union of an upwards-directed family of open sets belonging to Σ; as μ and ν are τ-additive, $\mu G = \nu G$. Consequently μ and ν agree on $\mathfrak{T}_s$-closed sets, and therefore on $\mathfrak{T}$-closed convex sets (4A4Ed). Write $\mathcal{H}$ for the family of $\mathfrak{T}$-open sets which are expressible as the union of a non-decreasing sequence of $\mathfrak{T}$-closed convex sets; then μ and ν agree on $\mathcal{H}$. If τ is a $\mathfrak{T}$-continuous seminorm on X, $x_0 \in X$ and $\alpha > 0$, then $\{x : \tau(x - x_0) < \alpha\} \in \mathcal{H}$; and sets of this kind constitute a base for $\mathfrak{T}$ (4A4Cb). Also the intersection of two members of $\mathcal{H}$ belongs to $\mathcal{H}$. By 415H(v), $\mu = \nu$.

Remark This generalizes 285M and 454Xk, which are the special cases $X = \mathbb{R}^r$ (for finite r) and $X = \mathbb{R}^I$.

466L Proposition Suppose that X and Y are Banach spaces and that $T : X \to Y$ is a linear operator such that $gT : X \to \mathbb{R}$ is universally Radon-measurable, in the sense of 434Ec, for every $g \in Y^*$. Then T is continuous.

proof ? Suppose, if possible, otherwise. Then there is a $g \in Y^*$ such that gT is not continuous (4A4Ib). For each $n \in \mathbb{N}$, take $x_n \in X$ such that $\|x_n\| = 2^{-n}$ and $g(Tx_n) > 2n$. Define $h : \{0,1\}^{\mathbb{N}} \to X$ by setting $h(t) = \sum_{n=0}^{\infty} t(n)x_n$ (4A4Ie). Then h is continuous, because $\|h(t) - h(t')\| \leq \sum_{n=0}^{\infty} 2^{-n}|t(n) - t'(n)|$ for all t, $t' \in \{0,1\}^{\mathbb{N}}$. Let ν be the usual measure on $\{0,1\}^{\mathbb{N}}$; then the image measure $\mu = \nu h^{-1}$ is a Radon measure on X (418I), so gT must be $\operatorname{dom}\mu$-measurable, and $\phi = gTh$ is $\operatorname{dom}\nu$-measurable. In this case there is an $m \in \mathbb{N}$ such that $E = \{t : |\phi(t)| \leq m\}$ has measure greater than $\frac{1}{2}$. But as $g(Tx_m) > 2m$, we see that if $t \in E$ then $t' \notin E$, where t' differs from t at the mth coordinate only, so that $|\phi(t) - \phi(t')| = g(Tx_m)$. Since the map $t \mapsto t'$ is an automorphism of the measure space $(\{0,1\}^{\mathbb{N}}, \nu)$, $\nu E \leq \frac{1}{2}$, which is impossible. **X**

466M Corollary If X is a Banach space, Y is a separable Banach space, and $T : X \to Y$ is a linear operator such that the graph of T is a Souslin-F set in $X \times Y$, then T is continuous.

proof It is enough to show that $T\restriction Z$ is continuous for every separable closed linear subspace Z of X (because then it must be sequentially continuous, and we can use 4A2Ld). Write $\Gamma \subseteq X \times Y$ for the graph of T. If $H \subseteq Y$ is open, then $\Gamma \cap (Z \times H)$ is a Souslin-F set in the Polish space $Z \times Y$, so is analytic (423E), and its projection $(T\restriction Z)^{-1}[H]$ is also analytic (423Ba), therefore universally measurable (434Dc). Thus $T\restriction Z : Z \to Y$ is a universally measurable function, and $gT\restriction Z$ must be universally measurable for any $g \in Y^*$ (434Df). By 466L, $T\restriction Z$ is continuous; as Z is arbitrary, T is continuous.

466X Basic exercises (a) Let $(X, \mathfrak{T})$ be a metrizable locally convex linear topological space and μ a totally finite measure on X which is quasi-Radon for the topology $\mathfrak{T}$. Show that μ is quasi-Radon for the weak topology $\mathfrak{T}_s(X, X^*)$.

>(b) Let $\langle e_n \rangle_{n\in\mathbb{N}}$ be the usual orthonormal basis of ℓ^2. Give ℓ^2 the Radon probability measure ν such that $\nu\{e_n\} = 2^{-n-1}$ for every n. Let I be an uncountable set, and set $X = (\ell^2)^I$ with the product linear structure and the product topology $\mathfrak{T}$, each copy of ℓ^2 being given its norm topology; write $\mathfrak{T}_s$ for the weak topology of X. (i) Let λ be the τ-additive product of copies of ν (417G). Show that λ is quasi-Radon for $\mathfrak{T}$ but is not inner regular with respect to the $\mathfrak{T}_s$-closed sets. (ii) Let μ_s be the τ-additive product measure of copies of μ when each copy of ℓ^2 is given its weak topology instead of its norm topology. Show that μ_s is quasi-Radon for $\mathfrak{T}_s$ but does not measure every $\mathfrak{T}$-Borel set. (*Hint*: setting $E = \{e_n : n \in \mathbb{N}\}$, $\mu(E^I) = 1$ and E^I is relatively $\mathfrak{T}_s$-compact.)

>(c) Let X be a metrizable locally convex linear topological space and μ a τ-additive totally finite measure on X with domain the cylindrical σ-algebra of X. Show that μ has an extension to a quasi-Radon measure on X. (*Hint*: 4A3U, 415N.)

(d) Let X be a metrizable locally convex linear topological space which is Lindelöf in its weak topology, and Σ the cylindrical σ-algebra of X. Show that any totally finite measure with domain Σ has an extension to a quasi-Radon measure on X.

>(e) Let X be a separable Banach space. Show that it is a Radon space when given its weak topology. (*Hint*: 4A3V, 467E.)

(f) Let K be a compact metric space, and $C(K)$ the Banach space of continuous real-valued functions on K. Show that the σ-algebra of subsets of $C(K)$ generated by the functionals $x \mapsto x(t) : C(K) \to \mathbb{R}$, for $t \in K$, is just the cylindrical σ-algebra of $C(K)$. (*Hint*: 4A2Pe.) Show that $C(K)$ is a pre-Radon space when given its weak topology. (See 462Z.)

(g) Re-write part (d) of the proof of 466H to avoid any appeal to results from §413.

(h) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(1) = 1$ and $f(x + y) = f(x) + f(y)$ for all x, $y \in \mathbb{R}$. Show that the following are equiveridical: (i) $f(x) = x$ for every $x \in \mathbb{R}$; (ii) f is continuous at some point; (iii) f is bounded on some non-empty open set; (iv) f is bounded on some Lebesgue measurable set of non-zero measure; (v) f is Lebesgue measurable; (vi) f is Borel measurable; *(vii) f is bounded on some non-meager G_δ set; *(viii) f is $\widehat{\mathcal{B}}$-measurable, where $\widehat{\mathcal{B}}$ is the Baire property algebra of $\mathbb{R}$. (*Hint*: 443Db.)

(i) Set $X = \{x : x \in \mathbb{R}^{\mathbb{N}}, \{n : x(n) \neq 0\} \text{ is finite}\}$, and give X any norm. Show that any linear operator from X to any normed space is universally measurable.

(j) Let X be a locally convex linear topological space and μ, ν two totally finite quasi-Radon measures on X. Show that if μ and ν give the same measure to every half-space $\{x : f(x) \geq \alpha\}$, where $f \in X^*$ and $\alpha \in \mathbb{R}$, then $\mu = \nu$.

(k) Let X be a Hilbert space and μ, ν two totally finite Radon measures on X. Show that if μ and ν give the same measure to every ball $B(x, \delta)$, where $x \in X$ and $\delta \geq 0$, then $\mu = \nu$. (*Hint*: every open half-space is the union of a non-decreasing sequence of balls.)

(l) Let μ be a Radon measure on $\ell^\infty(I)$, where I is any set. Set $f_i(x) = x(i)$ for $x \in \ell^\infty(I)$, $i \in I$. Show that $\{f_i^\bullet : i \in I\}$ is relatively compact in $L^0(\mu)$. (*Hint*: 463Xc.)

>(m) Show that Talagrand's measure, intepreted as a measure on $\ell^\infty(I)$ (464R), is not τ-additive for the weak topology.

466Y Further exercises (a) Give an example of a Hausdorff locally convex linear topological space $(X, \mathfrak{T})$ with a probability measure μ on X which is a Radon measure for the weak topology $\mathfrak{T}_s(X, X^*)$ but not for the topology $\mathfrak{T}$. (*Hint*: take $C = C([0, 1])$ and $X = C^*$ with the Mackey topology for the dual pair (X, C), that is, the topology of uniform convergence on weakly compact subsets of C.)

(b) In 466H, show that there is a Borel set for the norm topology on X which is not a Borel set for the weak topology. (*Hint*: we can adjust the family $\langle p_\xi \rangle_{\xi < \omega_1}$ to ensure that, for some $\epsilon > 0$, $\|p_\xi - p_\eta\| \geq \epsilon$ whenever $\eta < \xi$. For $\xi < \omega_1$ set $F_\xi = \{x : x \leq p_\xi, x \not\leq p_\eta \text{ for any } \eta < \xi\}$. Show that $\bigcup_{\xi \in D} F_\xi$ is norm-Borel for every $D \subseteq \omega_1$, while $\mu^*(\bigcup_{\xi < \omega_1} F_\xi) = 1$.)

(c) Find Banach spaces X and Y and a linear operator from X to Y which is not continuous but whose graph is an F_σ set in $X \times Y$.

466Z Problems (a) Does every probability measure defined on the $\mathfrak{T}_s(\ell^\infty, (\ell^\infty)^*)$-Borel sets of $\ell^\infty = \ell^\infty(\mathbb{N})$ extend to a measure defined on the $\| \|_\infty$-Borel sets?

It is by no means obvious that the Borel sets of ℓ^∞ are different for the weak and norm topologies; for a proof see Talagrand 78B.

(b) Assume that $\mathfrak{c}$ is measure-free. Does it follow that ℓ^∞, with its weak topology, is a Radon space?

Note that a positive answer to 464Z (with $I = \mathbb{N}$) would settle this, since Talagrand's measure, when interpreted as a measure on ℓ^∞, cannot agree on the weakly Borel sets with any Radon measure on ℓ^∞ (466Xl, 466Xm).

466 Notes and comments I have given a proof of 466A using the machinery of §463; when the measure μ is known to be a Radon measure for the weak topology, rather than just a quasi-Radon measure or a τ-additive measure on the cylindrical algebra (466Xc), the theorem is older than this, and for an instructive alternative approach see TALAGRAND 84, 12-1-4. Another proof is in JAYNE & ROGERS 95.

On any Banach space we have at least three important σ-algebras: the norm-Borel σ-algebra (generated by the norm-open sets), the weak-Borel algebra (generated by the weakly open sets) and the cylindrical algebra (generated by the continuous linear functionals). (Note that the Baire σ-algebras corresponding to the norm and weak topologies are the norm-Borel algebra (4A3Kb) and the cylindrical algebra (4A3U).) If our Banach space is naturally represented as a subspace of some $\mathbb{R}^I$ (e.g., because it is a space of continuous functions), then we have in addition the σ-algebra generated by the functionals $x \mapsto x(i)$ for $i \in I$, and the Borel algebra for the topology of pointwise convergence. We correspondingly have natural questions concerning when these algebras coincide, as in 466E and 4A3V and 466Xf, and when a measure on one of the algebras leads to a measure on another, as in 466A-466B.

The question of which Banach spaces are Radon spaces in their norm topologies is, if not exactly 'solved', at least reducible to a classical problem in set theory by the results in §438. It seems much harder to decide which non-separable Banach spaces are Radon spaces in their weak topologies. We have a simple positive result for spaces with Kadec norms (466F), and after some labour a negative result for a couple of standard examples (466I). The case of ℓ^∞ seems still to be open in 'ordinary' set theories (466Zb). ℓ^∞ is of particular importance in this context because the dual of any separable Banach space is isometrically isomorphic to a linear subspace of ℓ^∞ (4A4Id). So a positive answer to either question in 466Z would have very interesting consequences – and would be correspondingly surprising.

466L and 466M belong to a large family of results of the general form: if, between spaces with both topological and algebraic structures, we have a homomorphism (for the algebraic structures) which is not continuous, then it is wildly irregular. I hope to return to some of these ideas in Volume 5. For the moment I just give a version of the classical result that an additive function $f : \mathbb{R} \to \mathbb{R}$ which is Lebesgue measurable must be continuous (466Xh). The definition of 'universally measurable' function which I gave in §434 has a number of paradoxical aspects. I have already remarked that in some contexts we might prefer to use the notion of 'universally Radon-measurable' function; this is also appropriate for 466L. But when our space X, for any reason, has few Borel measures, as in 466Xi, there are correspondingly many universally measurable functions defined on X. Of course 466M can also be thought of as a generalization of the closed graph theorem; but note that, unlike the closed graph theorem, it needs a separable codomain (466Yc).

A curious geometric question concerning measures on metric spaces is the following. If two totally finite Radon measures on a metric space agree on balls, must they be identical? It is known that (even for compact spaces) the answer, in general, is 'no' (DAVIES 71); in Hilbert spaces the answer is 'yes', for non-trivial reasons (466Xk); in any finite-dimensional normed space, the answer is again 'yes' (JACKSON & MAULDIN 99).

*467 Locally uniformly rotund norms

In the last section I mentioned Kadec norms. These are interesting in themselves, but the reason for including them in this book is that in a normed space with a Kadec norm the weak topology has the same Borel sets as the norm topology (466Ea). The same will evidently be true of any space which has an equivalent Kadec norm. Now Kadec norms themselves are not uncommon, but equivalent Kadec norms appear in a striking variety of cases. Here I describe the principal class of spaces (the 'weakly K-countably determined' Banach spaces, 467H) which have equivalent Kadec norms. In fact they have 'locally uniformly rotund' norms, which are much easier to do calculations with.

Almost everything here is pure functional analysis, mostly taken from DEVILLE GODEFROY & ZIZLER 93, which is why I have starred the section. The word 'measure' does not appear until 467P. At that point, however, we find ourselves with a striking result (Schachermayer's theorem) which appears to need the structure theory of weakly compactly generated Banach spaces developed in 467C-467M.

467A Definition Let X be a linear space with a norm $\| \|$. $\| \|$ is **locally uniformly rotund** or **locally uniformly convex** if whenever $\|x\| = 1$ and $\epsilon > 0$, there is a $\delta > 0$ such that $\|x - y\| \leq \epsilon$ whenever $\|y\| = 1$ and $\|x + y\| \geq 2 - \delta$.

If X has a locally uniformly rotund norm, then every subspace of X has a locally uniformly rotund norm.

467B Proposition A locally uniformly rotund norm is a Kadec norm.

proof Let X be a linear space with a locally uniformly rotund norm $\|\,\|$. Set $S_X = \{x : \|x\| = 1\}$. Suppose that G is open for the norm topology and that $x \in G \cap S_X$. Then there is an $\epsilon > 0$ such that $G \supseteq B(x,\epsilon) = \{y : \|y-x\| \leq \epsilon\}$. Let $\delta > 0$ be such that $\|x-y\| \leq \epsilon$ whenever $\|y\| = 1$ and $\|x+y\| \geq 2-\delta$. Now there is an $f \in X^*$ such that $f(x) = \|f\| = 1$ (3A5Ac). So $V = \{y : f(y) > 1-\delta\}$ is open for the weak topology. But if $y \in V \cap S_X$, then $\|x+y\| \geq f(x+y) \geq 2-\delta$, so $\|x-y\| \leq \epsilon$ and $y \in G$. As x is arbitrary, $G \cap S_X$ is open for the weak topology on S_X; as G is arbitrary, the norm and weak topologies agree on S_X.

467C A technical device (a) I will use the following notation for the rest of the section. Let X be a linear space and $p : X \to [0,\infty[$ a seminorm. Define $q_p : X \times X \to [0,\infty[$ by setting

$$q_p(x,y) = 2p(x)^2 + 2p(y)^2 - p(x+y)^2 = (p(x)-p(y))^2 + (p(x)+p(y))^2 - p(x+y)^2$$

for $x \in X$.

(b) A norm $\|\,\|$ on X is locally uniformly rotund iff whenever $x \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that $\|x-y\| \leq \epsilon$ whenever $q_{\|\,\|}(x,y) \leq \delta$.

P(i) Suppose that $\|\,\|$ is locally uniformly rotund, $x \in X$ and $\epsilon > 0$. (α) If $x = 0$ then $q_{\|\,\|}(x,y) = \|y\|^2 = \|x-y\|^2$ for every y so we can take $\delta = \epsilon^2$. (β) If $x \neq 0$ set $x' = \frac{1}{\|x\|}x$. Let $\eta > 0$ be such that $\|x'-y'\| \leq \frac{1}{2}\epsilon\|x\|$ whenever $\|y'\| = 1$ and $\|x'+y'\| \geq 2-\eta$. Let $\delta > 0$ be such that

$$\delta + 2\sqrt{\delta}\|x\| \leq \eta\|x\|^2, \quad \sqrt{\delta} < \|x\|, \quad \sqrt{\delta} \leq \tfrac{1}{2}\epsilon\|x\|^2.$$

Now if $q_{\|\,\|}(x,y) \leq \delta$, we must have

$$(\|x\| - \|y\|)^2 \leq \delta < \|x\|^2, \quad (\|x\| + \|y\|)^2 - \|x+y\|^2 \leq \delta,$$

so that $y \neq 0$ and

$$\big|\tfrac{1}{\|x\|} - \tfrac{1}{\|y\|}\big|\|y\| = \tfrac{|\|y\|-\|x\||}{\|x\|} \leq \tfrac{\sqrt{\delta}}{\|x\|},$$

$$\|x\| + \|y\| - \|x+y\| \leq \tfrac{\delta}{\|x\|+\|y\|+\|x+y\|} \leq \tfrac{\delta}{\|x\|}.$$

Set $y' = \frac{1}{\|y\|}y$, $y'' = \frac{1}{\|x\|}y$. Then $\|y'\| = 1$, and

$$\|y'-y''\| = \big|\tfrac{1}{\|y\|} - \tfrac{1}{\|x\|}\big|\|y\| \leq \tfrac{\sqrt{\delta}}{\|x\|} \leq \tfrac{1}{2}\epsilon\|x\|.$$

Accordingly

$$\begin{aligned}\|x'+y'\| &\geq \|x'+y''\| - \|y'-y''\| \geq \tfrac{1}{\|x\|}\|x+y\| - \tfrac{\sqrt{\delta}}{\|x\|}\\ &\geq \tfrac{1}{\|x\|}(\|x\| + \|y\| - \tfrac{\delta}{\|x\|}) - \tfrac{\sqrt{\delta}}{\|x\|} = 1 + \tfrac{\|y\|}{\|x\|} - \tfrac{\delta}{\|x\|^2} - \tfrac{\sqrt{\delta}}{\|x\|}\\ &\geq 1 + \tfrac{\|x\|-\sqrt{\delta}}{\|x\|} - \tfrac{\delta}{\|x\|^2} - \tfrac{\sqrt{\delta}}{\|x\|} = 2 - \tfrac{\delta}{\|x\|^2} - \tfrac{2\sqrt{\delta}}{\|x\|} \geq 2-\eta.\end{aligned}$$

But this means that $\|x'-y'\| \leq \frac{1}{2}\epsilon\|x\|$, so that $\|x'-y''\| \leq \epsilon\|x\|$ and $\|x-y\| \leq \epsilon$. As x and ϵ are arbitrary, the condition is satisfied.

(ii) Suppose the condition is satisfied. If $\|x\| = 1$ and $\epsilon > 0$, take $\delta \in]0,2]$ such that $\|x-y\| \leq \epsilon$ whenever $q(x,y) \leq 4\delta$; then if $\|y\| = 1$ and $\|x+y\| \geq 2-\delta$, $q(x,y) = 4 - \|x+y\|^2 \leq 4\delta$ and $\|x-y\| \leq \epsilon$. As x and ϵ are arbitrary, $\|\,\|$ is locally uniformly rotund. **Q**

(c) We have the following elementary facts. Let X be a linear space.

(i) For any seminorm p on X, $q_p(x,y) \geq 0$ for all x, $y \in X$. **P**

$$q_p(x,y) \geq (p(x)+p(y))^2 - p(x+y)^2 \geq 0$$

because $p(x+y) \leq p(x)+p(y)$. **Q**

(ii) Suppose that $\langle p_i \rangle_{i\in I}$ is a family of seminorms on X such that $\sum_{i\in I} p_i(x)^2$ is finite for every $x \in X$. Set $p(x) = \sqrt{\sum_{i\in I} p_i(x)^2}$ for $x \in X$; then p is a seminorm on X and $q_p = \sum_{i\in I} q_{p_i}$. **P** Of course $p(\alpha x) = |\alpha| p(x)$ for $\alpha \in \mathbb{R}$ and $x \in X$. If $x \in X$, then $p(x) = \|\phi(x)\|_2$, where $\phi(x) = \langle p_i(x)\rangle_{i\in I} \in \ell^2(I)$. Now for x, $y \in X$,

$$0 \leq \phi(x+y) \leq \phi(x) + \phi(y)$$

in $\ell^2(I)$, so

$$p(x+y) = \|\phi(x+y)\|_2 \leq \|\phi(x)+\phi(y)\|_2 \leq \|\phi(x)\|_2 + \|\phi(y)\|_2 = p(x)+p(y).$$

Thus p is a seminorm. Now the calculation of $q_p = \sum_{i\in I} q_{p_i}$ is elementary. **Q** In particular, $q_p \geq q_{p_i}$ for every $i \in I$.

(iii) If $\| \|$ is an inner product norm on X, then $q_{\| \|}(x,y) = \|x-y\|^2$ for all x, $y \in X$. **P**

$$\begin{aligned} 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 &= 2(x|x) + 2(y|y) - (x+y|x+y) \\ &= (x|x) + (y|y) - (x|y) - (y|x) = (x-y|x-y). \end{aligned}$$ **Q**

467D Lemma Let X be a normed space. Suppose that there are a normed space Y with a locally uniformly rotund norm and a bounded linear operator $T : Y \to X$ such that $T[Y]$ is dense in X and, for every $x \in X$ and $\gamma > 0$, there is a $z \in Y$ such that $\|x-Tz\|^2 + \gamma\|z\|^2 = \inf_{y\in Y} \|x-Ty\|^2 + \gamma\|y\|^2$. Then X has an equivalent locally uniformly rotund norm.

proof (a) For each $n \in \mathbb{N}$, $x \in X$ set

$$p_n(x) = \sqrt{\inf_{y\in Y} \|x-Ty\|^2 + 2^{-n}\|y\|^2}.$$

Then $p_n : X \to [0,\infty[$ is a norm on X, equivalent to $\| \|$. **P** (i) The functionals $(x,y) \mapsto \|x-Ty\|$, $(x,y) \mapsto 2^{-n/2}\|y\|$ from $X \times Y$ to $[0,\infty[$ are both seminorms, so the functional $(x,y) \mapsto \phi(x,y) = \sqrt{\|x-Ty\|^2 + 2^{-n}\|y\|^2}$ also is, by 467C(c-ii). (ii) If $x \in X$ and $\alpha \in \mathbb{R}$, take $z \in Y$ such that $p_n(x) = \phi(x,z)$; then

$$p_n(\alpha x) \leq \phi(\alpha x, \alpha z) = |\alpha|\phi(x,z) = |\alpha| p_n(x).$$

If $\alpha \neq 0$, apply the same argument to see that $p_n(x) \leq |\alpha|^{-1} p_n(\alpha x)$, so that $p_n(\alpha x) = |\alpha| p_n(x)$. (iii) Now take any x_1, $x_2 \in X$. Let z_1, $z_2 \in Y$ be such that $p_n(x_i) = \phi(x_i, z_i)$ for both i. Then

$$\begin{aligned} p_n(x_1+x_2) &= 2p_n(\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2) \leq 2\phi(\tfrac{1}{2}x_1 + \tfrac{1}{2}x_2, \tfrac{1}{2}z_1 + \tfrac{1}{2}z_2) \\ &\leq 2(\tfrac{1}{2}\phi(x_1,z_1) + \tfrac{1}{2}\phi(x_2,z_2)) = p_n(x_1) + p_n(x_2). \end{aligned}$$

Thus p_n is a seminorm. (iv) $p_n(x) \leq \phi(x,0) = \|x\|$ for every $x \in X$. (vi) For any $x \in X$ and $y \in Y$, either $\|Ty\| \geq \frac{1}{2}\|x\|$ and

$$\phi(x,y) \geq 2^{-n/2}\|y\|_2 \geq \frac{1}{2\cdot 2^{n/2}\|T\|}\|x\|,$$

or $\|Ty\| \leq \frac{1}{2}\|x\|$ and

$$\phi(x,y) \geq \|x-Ty\| \geq \frac{1}{2}\|x\|;$$

this shows that $p_n(x) \geq \min(\frac{1}{2}, \frac{1}{2}2^{-n/2}\|T\|^{-1})\|x\|$. (I am passing over the trivial case $X = \{0\}$, $\|T\| = 0$.) In particular, $p_n(x) = 0$ only when $x = 0$. Thus p_n is a norm equivalent to p. **Q**

(c) For any $x \in X$, $\lim_{n\to\infty} p_n(x) = 0$. **P** Let $\epsilon > 0$. Let $y \in Y$ be such that $\|x - Ty\| \leq \epsilon$; then

$$\limsup_{n\to\infty} p_n(x)^2 \leq \limsup_{n\to\infty} \|x - Ty\|^2 + 2^{-n}\|y\|_2^2 \leq \epsilon^2.$$

As ϵ is arbitrary, $\lim_{n\to\infty} p_n(x) = 0$. **Q**

(d) Set $\|x\|' = \sqrt{\sum_{n=0}^{\infty} 2^{-n} p_n(x)^2}$ for $x \in X$. The sum is always finite because $p_n(x) \leq \|x\|$ for every n, so $\|\,\|'$ is a seminorm; and it is a norm equivalent to $\|\,\|$ because p_0 is. Now $\|\,\|'$ is locally uniformly rotund. **P** Take $x \in X$ and $\epsilon > 0$. Let $n \in \mathbb{N}$ be such that $p_n(x) \leq \frac{1}{4}\epsilon$. Choose $y \in Y$ such that $p_n(x)^2 = \|x - Ty\|^2 + 2^{-n}\|y\|^2$. (This is where we really use the hypothesis that the infimum in the definition of p_n is attained.) Let $\delta > 0$ be such that $2^n\delta \leq (\frac{1}{4}\epsilon)^2$ and $\|T\|\|y' - y\| \leq \frac{1}{4}\epsilon$ whenever $q_{\|\,\|}(y', y) \leq 2^{2n}\delta$.

If $q_{\|\,\|'}(x, x') \leq \delta$, then $q_{2^{-n/2}p_n}(x, x') \leq \delta$, by 467C(c-ii), that is, $q_{p_n}(x, x') \leq 2^n\delta$. Let $y' \in Y$ be such that $p_n(x')^2 = \|x' - Ty'\|^2 + 2^{-n}\|y'\|_2^2$. Then

$$\begin{aligned} p_n(x + x')^2 &\leq \|x + x' - Ty - Ty'\|^2 + 2^{-n}\|y + y'\|^2 \\ &\leq (\|x - Ty\| + \|x' - Ty'\|)^2 + 2^{-n}\|y + y'\|^2, \end{aligned}$$

so

$$\begin{aligned} q_{p_n}(x, x') &= 2p_n(x)^2 + 2p_n(x')^2 - p_n(x + x')^2 \\ &\geq 2(\|x - Ty\|^2 + 2^{-n}\|y\|^2) + 2(\|x' - Ty'\|^2 + 2^{-n}\|y'\|^2) \\ &\qquad - (\|x - Ty\| + \|x' - Ty'\|)^2 - 2^{-n}\|y + y'\|^2 \\ &= (\|x - Ty\| - \|x' - Ty'\|)^2 + 2^{-n}(2\|y\|^2 + 2\|y'\|_2^2 - \|y + y'\|^2). \end{aligned}$$

This means that

$$q_{\|\,\|}(y, y') \leq 2^n q_{p_n(x,x')} \leq 2^{2n}\delta,$$

so $\|T\|\|y - y'\| \leq \frac{1}{4}\epsilon$, while

$$\|x - Ty\| + \|x' - Ty'\| \leq 2\|x - Ty\| + \sqrt{q_{p_n}(x, x')} \leq 2p_n(x) + 2^{n/2}\sqrt{\delta} \leq \frac{3}{4}\epsilon.$$

Finally

$$\|x - x'\| \leq \|T\|\|y - y'\| + \|x - Ty\| + \|x' - Ty'\| \leq \tfrac{1}{4}\epsilon + \tfrac{3}{4}\epsilon = \epsilon.$$

As x and ϵ are arbitrary, this shows that $\|\,\|'$ is locally uniformly rotund. **Q**

This completes the proof.

467E Theorem Let X be a separable normed space. Then it has an equivalent locally uniformly rotund norm.

proof (a) It is enough to show that the completion of X has an equivalent locally uniformly rotund norm; since the completion of X is separable, we may suppose that X itself is complete. Let $\langle x_i \rangle_{i\in\mathbb{N}}$ be a sequence in X running over a dense subset of X. Define $T : \ell^2 \to X$ by setting

$$Ty = \sum_{i=0}^{\infty} \frac{y(i)}{2^i(1 + \|x_i\|)} x_i$$

for $y \in \ell^2 = \ell^2(\mathbb{N})$; then Ty is always defined (4A4Ie); T is a linear operator and

$$\|Ty\| \leq \sum_{i=0}^{\infty} 2^{-i}|y(i)| \leq \sqrt{\sum_{i=0}^{\infty} 2^{-2i}}\|y\|_2$$

for every $y \in \ell^2$, by Cauchy's inequality (244E). So T is a bounded linear operator.

(b) T satisfies the conditions of 467D. **P** $T[\ell^2]$ is dense because it contains every x_i. Given $x \in X$ and $\gamma > 0$, $\alpha \geq 0$, the function $y \mapsto \sqrt{\|x - Ty\|^2 + \gamma\|y\|_2^2}$ is a seminorm and norm-continuous, so the set

$$C_\alpha(x) = \{y : y \in \ell^2,\ \|x - Ty\|^2 + \gamma\|y\|_2^2 \leq \alpha^2\}$$

is convex and norm-closed. Consequently, $C_\alpha(x)$ is weakly closed (4A4Ed); since $\|y\|_2 \leq \gamma^{-1/2}\alpha$ for every $y \in C_\alpha(x)$, $C_\alpha(x)$ is weakly compact (4A4Ka). Set $\beta = \inf_{y\in\ell^2} \|x - Ty\|^2 + \gamma\|y\|_2^2$. Then $\{C_\alpha(x) : \alpha > \beta\}$

is a downwards-directed set of non-empty weakly compact sets, so has non-empty intersection; taking any $z \in \bigcap_{\alpha>\beta} C_\alpha(x)$, $\beta = \|x - Tz\|^2 + \gamma\|z\|_2^2$. **Q**

(c) So 467D gives the result.

467F Lemma Let $(X, \|\,\|)$ be a Banach space, and $\langle T_i\rangle_{i\in I}$ a family of bounded linear operators from X to itself such that

(i) for each $i \in I$, the subspace $T_i[X]$ has an equivalent locally uniformly rotund norm,

(ii) for each $x \in X$, $\epsilon > 0$ there is a finite set $J \subseteq I$ such that $\|x - \sum_{i\in J} T_i x\| \leq \epsilon$,

(iii) for each $x \in X$, $\epsilon > 0$ the set $\{i : i \in I, \|T_i x\| \geq \epsilon\}$ is finite.

Then X has an equivalent locally uniformly rotund norm.

proof (a) Let $\|\,\|_i$ be a locally uniformly rotund norm on $X_i = T_i[X]$ equivalent to $\|\,\|$ on X_i. Reducing $\|\,\|_i$ by a scalar multiple if necessary, we may suppose that $\|T_i x\|_i \leq \|x\|$ for every $x \in X$, $i \in I$. By (iii), $\sup_{i\in I} \|T_i x\|$ is finite for every $x \in X$; by the Uniform Boundedness Theorem (3A5Ha), $M = \sup_{i\in I} \sup_{\|x\|\leq 1} \|T_i x\|$ is finite. For finite sets $J \subseteq I$ and $k \geq 1$, set

$$p_{Jk}(x) = \sqrt{\textstyle\sum_{i\in J} \|T_i x\|_i^2 + \frac{1}{k}\sum_{K\subseteq J} \|x - \sum_{i\in K} T_i x\|^2};$$

for $n \in \mathbb{N}$ and $k \geq 1$ set

$$p_k^{(n)}(x) = \sup\{p_{Jk}(x) : J \subseteq I, \#(J) \leq n\}.$$

By 467C(c-ii), as usual, all the p_{Jk} are seminorms, and it follows at once that the $p_k^{(n)}$ are seminorms. Observe that if $K \subseteq I$ is finite, then $\|x - \sum_{i\in K} T_i x\| \leq (1 + M\#(K))\|x\|$ for every x, so if $J \subseteq I$ is finite then

$$p_{Jk}(x) \leq \sqrt{\#(J) + 2^{\#(J)}(1 + M\#(J))}\|x\|,$$

and $p_k^{(n)}(x) \leq \sqrt{n + 2^n(1 + Mn)}\|x\|$ for every $n \in \mathbb{N}$, $k \geq 1$. Setting $\beta_{nk} = 2^{2n+k}$ for n, $k \in \mathbb{N}$,

$$\|x\|' = \sqrt{\textstyle\sum_{n=0}^\infty \sum_{k=1}^\infty \beta_{nk}^{-1} p_k^{(n)}(x)^2}$$

is finite for every $x \in X$, so that $\|\,\|'$ is a seminorm on X; moreover, $\|x\|' \leq \beta\|x\|$ for every $x \in X$, where

$$\beta = \sqrt{\textstyle\sum_{n=0}^\infty \sum_{k=1}^\infty \beta_{nk}^{-1}(n + 2^n(1 + Mn))}$$

is finite. Since we also have

$$\|x\|' \geq \tfrac{1}{\sqrt{2}} p_1^{(0)}(x) = \tfrac{1}{\sqrt{2}} p_{\emptyset 1}(x) = \tfrac{1}{\sqrt{2}}\|x\|$$

for every x, $\|\,\|'$ is a norm on X equivalent to $\|\,\|$.

(b) Now $\|\,\|'$ is locally uniformly rotund. **P** Take $x \in X$ and $\epsilon > 0$. Let $K \subseteq I$ be a finite set such that $\|x - \sum_{i\in K} T_i x\| \leq \frac{1}{4}\epsilon$; we may suppose that $T_i x \neq 0$ for every $i \in K$. Set $\alpha_1 = \min_{i\in K} \|T_i x\|_i$, $J = \{i : i \in I, \|T_i x\|_i \geq \alpha_1\}$ and $\alpha_0 = \sup_{i\in I\setminus J} \|Tx_i\|_i$. (For completeness, if $K = \emptyset$, take $J = \emptyset$, $\alpha_0 = \sup_{i\in I} \|T_i x\|_i$ and $\alpha_1 = \alpha_0 + 1$.) Then J is finite and $\alpha_0 < \alpha_1$, by hypothesis (iii) of the lemma. Set $n = \#(J)$. Let k be so large that $\frac{2^n(Mn+1)}{k}\|x\|^2 < \frac{1}{2}(\alpha_1^2 - \alpha_0^2)$. Let $\eta > 0$ be such that

$$(\beta_{nk} + 1)\eta \leq \min(\tfrac{\epsilon^2}{16k}, \alpha_1^2 - \alpha_0^2),$$

$$\|T_i x - z\| \leq \tfrac{\epsilon}{1+4n} \text{ whenever } i \in K,\ z \in X_i \text{ and } q_{\|\,\|_i}(T_i x, z) \leq (\beta_{nk} + 1)\eta;$$

this is where we use the hypothesis that every $\|\,\|_i$ is locally uniformly rotund (and equivalent to $\|\,\|$ on X_i).

Now suppose that $y \in X$ and $q_{\|\,\|'}(x, y) \leq \eta$. Then $q_{p_k^{(n)}}(x, y) \leq \beta_{nk}\eta$, by 467C(c-ii). Let $L \in [I]^n$ be such that $p_k^{(n)}(x + y)^2 \leq p_{Lk}(x + y)^2 + \eta$. Then

$$\begin{aligned} q_{p_{Lk}}(x, y) &= 2p_{Lk}(x)^2 + 2p_{Lk}(y)^2 - p_{Lk}(x + y)^2 \\ &\leq 2p_k^{(n)}(x)^2 + 2p_k^{(n)}(y)^2 - p_k^{(n)}(x + y)^2 + \eta \leq (\beta_{nk} + 1)\eta. \end{aligned}$$

We also have

$$\begin{aligned}2p_{Lk}(x)^2 &\geq 2p_{Lk}(x)^2 + 2p_{Lk}(y)^2 - 2p_k^{(n)}(y)^2 \geq p_{Lk}(x+y)^2 - 2p_k^{(n)}(y)^2\\ &\geq p_k^{(n)}(x+y)^2 - \eta - 2p_k^{(n)}(y)^2 \geq 2p_k^{(n)}(x)^2 - (\beta_{nk}+1)\eta,\end{aligned}$$

so

$$\begin{aligned}\sum_{i\in J}\|Tx_i\|_i^2 \leq p_{Jk}(x)^2 \leq p_k^{(n)}(x)^2 &\leq p_{Lk}(x)^2 + \tfrac{1}{2}(\beta_{nk}+1)\eta\\ &\leq \sum_{i\in L}\|Tx_i\|_i^2 + \tfrac{2^n(Mn+1)}{k}\|x\|^2 + \tfrac{1}{2}(\beta_{nk}+1)\eta\\ &< \sum_{i\in L}\|Tx_i\|_i^2 + \alpha_1^2 - \alpha_0^2.\end{aligned}$$

Since $\#(L) = \#(J)$ and

$$\|Tx_i\|_i^2 \leq \alpha_0^2 < \alpha_1^2 \leq \|Tx_j\|_j^2$$

whenever $j \in J$ and $i \in I \setminus J$, we must actually have $L = J$. In particular, $K \subseteq L$. But this means that (by 467C(c-ii) again)

$$q_{\|\,\|_i}(T_ix, T_iy) \leq q_{p_{Lk}}(x,y) \leq (\beta_{nk}+1)\eta$$

and (by the choice of η) $\|T_ix - T_iy\| \leq \frac{\epsilon}{1+4n}$ for every $i \in K$, so that $\|\sum_{i\in K} T_ix - \sum_{i\in K} T_iy\| \leq \frac{1}{4}\epsilon$.

The last element we need is that, setting $\tilde{p}(z) = \frac{1}{\sqrt{k}}\|z - \sum_{i\in K} T_iz\|$, $\tilde{p}$ is a seminorm on X and is one of the constituents of p_{Lk}; so that

$$\begin{aligned}\tfrac{1}{k}(\|x - \sum_{i\in K} T_ix\| - \|y - \sum_{i\in K} T_iy\|)^2 \leq q_{\tilde{p}}(x,y) &\leq q_{p_{Lk}}(x,y)\\ &\leq (\beta_{nk}+1)\eta \leq \frac{\epsilon^2}{16k},\end{aligned}$$

and $\big|\|x - \sum_{i\in K} T_ix\| - \|y - \sum_{i\in K} T_iy\|\big| \leq \frac{1}{4}\epsilon$. This means that

$$\|y - \textstyle\sum_{i\in K} T_iy\| \leq \frac{1}{4}\epsilon + \|x - \sum_{i\in K} T_ix\| \leq \frac{1}{2}\epsilon.$$

Putting these together,

$$\|x - y\| \leq \|x - \textstyle\sum_{i\in K} T_ix\| + \sum_{i\in K}\|T_ix - T_iy\| + \|y - \sum_{i\in K} T_iy\| \leq \epsilon.$$

And this is true whenever $q_{\|\,\|'}(x,y) \leq \eta$. As x and ϵ are arbitrary, $\|\,\|'$ is locally uniformly rotund. **Q**

467G Theorem Let X be a Banach space. Suppose that there are an ordinal ζ and a family $\langle P_\xi\rangle_{\xi\leq\zeta}$ of bounded linear operators from X to itself such that

(i) if $\xi \leq \eta \leq \zeta$ then $P_\xi P_\eta = P_\eta P_\xi = P_\xi$;

(ii) $P_0(x) = 0$ and $P_\zeta(x) = x$ for every $x \in X$;

(iii) if $\xi \leq \zeta$ is a non-zero limit ordinal, then $\lim_{\eta\uparrow\xi} P_\eta(x) = P_\xi(x)$ for every $x \in X$;

(iv) if $\xi < \zeta$ then $X_\xi = \{(P_{\xi+1} - P_\xi)(x) : x \in X\}$ has an equivalent locally uniformly rotund norm.

Then X has an equivalent locally uniformly rotund norm.

Remark A family $\langle P_\xi\rangle_{\xi\leq\zeta}$ satisfying (i), (ii) and (iii) here is called a **projectional resolution of the identity**.

proof For $\xi < \zeta$ set $T_\xi = P_{\xi+1} - P_\xi$. From condition (i) we see easily that $T_\xi T_\eta = T_\xi$ if $\xi = \eta$, 0 otherwise; and that $T_\xi P_\eta = T_\xi$ if $\xi < \eta$, 0 otherwise.

I seek to show that the conditions of 467F are satisfied by $\langle T_\xi\rangle_{\xi<\zeta}$. Condition (i) of 467F is just condition (iv) here. Let Z be the set of those $x \in X$ for which conditions (ii) and (iii) of 467F are satisfied; that is,

for each $\epsilon > 0$ there is a finite set $J \subseteq \zeta$ such that $\|x - \sum_{\xi\in J} T_\xi x\| \leq \epsilon$, and $\{\xi : \|T_\xi x\| \geq \epsilon\}$ is finite.

Then Z is a linear subspace of X. For $\xi \leq \zeta$, set $Y_\xi = P_\xi[X]$. Then $Y_\xi \subseteq Z$. **P** Induce on ξ. Since $Y_0 = \{0\}$, the induction starts. For the inductive step to a successor ordinal $\xi + 1 \leq \zeta$, $Y_{\xi+1} = Y_\xi + X_\xi \subseteq Z$. For the inductive step to a non-zero limit ordinal $\xi \leq \zeta$, given $x \in Y_\xi$ and $\epsilon > 0$, we know that there is a $\xi' < \xi$ such that $\|P_\eta x - P_\xi x\| \leq \frac{1}{3}\epsilon$ whenever $\xi' \leq \eta \leq \xi$. So $\|T_\eta x\| \leq \frac{2}{3}\epsilon$ whenever $\xi' \leq \eta < \xi$, and

$$\{\eta : \|T_\eta x\| \geq \epsilon\} = \{\eta : \eta < \xi',\ \|T_\eta x\| \geq \epsilon\} = \{\eta : \|T_\eta P_{\xi'} x\| \geq \epsilon\}$$

is finite, by the inductive hypothesis. Moreover, there is a finite set $J \subseteq \xi'$ such that $\|P_{\xi'}x - \sum_{\eta\in J} T_\eta P_{\xi'} x\| \leq \frac{2}{3}\epsilon$, and now $\|x - \sum_{\eta\in J} T_\eta x\| \leq \epsilon$. As x and ϵ are arbitrary, $Y_\xi \subseteq Z$. **Q**

In particular, $X = Y_\zeta \subseteq Z$ and conditions (ii) and (iii) of 467F are satisfied. So 467F gives the result.

467H Definitions (a) A topological space X is **K-countably determined** or a **Lindelöf-Σ** space if there are a subset A of $\mathbb{N}^{\mathbb{N}}$ and an usco-compact relation $R \subseteq A \times X$ such that $R[A] = X$. Observe that all K-analytic Hausdorff spaces (§422) are K-countably determined.

(b) A normed space X is **weakly K-countably determined** if it is K-countably determined in its weak topology.

(c) Let X be a normed space and Y, W closed linear subspaces of X, X^* respectively. I will say that (Y, W) is a **projection pair** if $X = Y \oplus W^\circ$ and $\|y + z\| \geq \|y\|$ for every $y \in Y$, $z \in W^\circ$, where

$$W^\circ = \{z : z \in X,\ f(z) \leq 1 \text{ for every } f \in W\} = \{z : z \in X,\ f(z) = 0 \text{ for every } f \in W\}.$$

467I Lemma (a) If X is a weakly K-countably determined normed space, then any closed linear subspace of X is weakly K-countably determined.

(b) If X is a weakly K-countably determined normed space, Y is a normed space, and $T : X \to Y$ is a continuous linear surjection, then Y is weakly K-countably determined.

(c) If X is a Banach space and $Y \subseteq X$ is a dense linear subspace which is weakly K-countably determined, then X is weakly K-countably determined.

proof (a) Let $A \subseteq \mathbb{N}^{\mathbb{N}}$, $R \subseteq A \times X$ be such that R is usco-compact (for the weak topology on X) and $R[A] = X$. Let Y be a (norm-)closed linear subspace of X; then Y is closed for the weak topology (3A5Ee). Also the weak topology on Y is just the subspace topology induced by the weak topology of X (4A4Ea). Set $R' = R \cap (A \times Y)$. Then R' is usco-compact whether regarded as a subset of $A \times X$ or as a subset of $A \times Y$ (422Da, 422Db, 422Dg). Since $Y = R'[A]$, Y is weakly K-countably determined.

(b) Let $A \subseteq \mathbb{N}^{\mathbb{N}}$, $R \subseteq A \times X$ be such that R is usco-compact for the weak topology on X and $R[A] = X$. Because T is continuous for the weak topologies on X and Y (3A5Ec),

$$R_1 = \{(\phi, y)\text{: there is some } x \in X \text{ such that } (\phi, x) \in R \text{ and } Tx = y\}$$

is usco-compact in $A \times Y$ (422Db, 422Df). Also $R_1[A] = T[R[A]] = Y$. So Y is weakly K-countably determined.

(c) Let $A \subseteq \mathbb{N}^{\mathbb{N}}$, $R \subseteq A \times Y$ be such that R is usco-compact (for the weak topology on Y) and $R[A] = Y$. Then, as in (a), R is usco-compact when regarded as a subset of $A \times X$. By 422Dd, the set

$$R_1 = \{(\langle\phi_n\rangle_{n\in\mathbb{N}}, \langle y_n\rangle_{n\in\mathbb{N}}) : (\phi_n, y_n) \in R \text{ for every } n \in \mathbb{N}\}$$

is usco-compact in $A^{\mathbb{N}} \times Y^{\mathbb{N}}$. Now examine

$$S = \{(\langle y_n\rangle_{n\in\mathbb{N}}, x) : x \in X,\ y_n \in Y \text{ and } \|y_n - x\| \leq 2^{-n} \text{ for every } n \in \mathbb{N}\}.$$

Then S is usco-compact in $Y^{\mathbb{N}} \times X$. **P** (Remember that we are using weak topologies on X and Y throughout.) If $\boldsymbol{y} = \langle y_n\rangle_{n\in\mathbb{N}}$ is a sequence in Y and $(\boldsymbol{y}, x) \in S$, then $x = \lim_{n\to\infty} y_n$; so $S[\{\boldsymbol{y}\}]$ has at most one member and is certainly compact. Let $F \subseteq X$ be a weakly closed set and $\boldsymbol{y} \in Y^{\mathbb{N}} \setminus S^{-1}[F]$.

case 1 If there are m, $n \in \mathbb{N}$ such that $\|y_m - y_n\| > 2^{-m} + 2^{-n}$, let $f \in Y^*$ be such that $\|f\| \leq 1$ and $f(y_m - y_n) > 2^{-m} + 2^{-n}$. Then $G = \{\boldsymbol{z} : f(z_m - z_n) > 2^{-m} + 2^{-n}\}$ is an open set in $Y^{\mathbb{N}}$ containing $\boldsymbol{y}$ and disjoint from $S^{-1}[F]$, because $S[G]$ is empty.

case 2 Otherwise, $\boldsymbol{y}$ is a Cauchy sequence and (because X is a Banach space) has a limit $x \in X$, which does not belong to F. Let $\delta > 0$ and $f_0, \dots, f_r \in X^*$ be such that $\{w : |f_i(w) - f_i(x)| \leq \delta \text{ for every } i \leq r\}$ does not meet F. Let $n \in \mathbb{N}$ be such that $2^{-n}\|f_i\| \leq \frac{1}{3}\delta$ for every $i \leq r$. Then $G = \{\boldsymbol{z} : |f_i(z_n) - f_i(y_n)| < \frac{1}{3}\delta$ for every $i \leq r\}$ is an open set in $Y^{\mathbb{N}}$ containing Y. If $\boldsymbol{z} \in G$ and $(\boldsymbol{z}, w) \in S$, then $\|z_n - w\| \leq 2^{-n}$ so

$$\begin{aligned}|f_i(w) - f_i(x)| &\leq |f_i(w) - f_i(z_n)| + |f_i(z_n) - f_i(y_n)| + |f_i(y_n) - f_i(x)| \\ &\leq 2^{-n}\|f_i\| + \frac{1}{3}\delta + 2^{-n}\|f_i\| \leq \delta\end{aligned}$$

for every $i \leq r$, and $w \notin F$. Thus again $G \cap S^{-1}[F]$ is empty.

This shows that there is always an open set containing y and disjoint from $S^{-1}[F]$. As y is arbitrary, $S^{-1}[F]$ is closed. As F is arbitrary, S is usco-compact. **Q**

It follows that $SR_1 \subseteq A^{\mathbb{N}} \times X$ is usco-compact (422Df), while

$$(SR_1)[A^{\mathbb{N}}] = S[R_1[A^{\mathbb{N}}]] = S[Y^{\mathbb{N}}] = X$$

because Y is dense in X. Finally, $A^{\mathbb{N}}$ is homeomorphic to a subset of $\mathbb{N}^{\mathbb{N}}$ because it is a subspace of $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \cong \mathbb{N}^{\mathbb{N}}$. So X is weakly K-countably determined.

467J Lemma Let X be a weakly K-countably determined Banach space. Then there is a family $\mathcal{M}$ of subsets of $X \cup X^*$ such that

(i) whenever $B \subseteq X \cup X^*$ there is an $M \in \mathcal{M}$ such that $B \subseteq M$ and $\#(M) \leq \max(\omega, \#(B))$;

(ii) whenever $\mathcal{M}' \subseteq \mathcal{M}$ is upwards-directed, then $\bigcup \mathcal{M}' \in \mathcal{M}$;

(iii) whenever $M \in \mathcal{M}$ then $(\overline{M \cap X}, \overline{M \cap X^*})$ (where the closures are taken for the norm topologies) is a projection pair of subspaces of X and X^*.

proof (a) Let $A \subseteq \mathbb{N}^{\mathbb{N}}$, $R \subseteq A \times X$ be such that R is usco-compact in $A \times X$ and $R[A] = X$. Set $S = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ and for $\sigma \in S$ set $F_\sigma = R[I_\sigma]$, where $I_\sigma = \{\phi : \sigma \subseteq \phi \in \mathbb{N}^{\mathbb{N}}\}$; set $S_0 = \{\sigma : \sigma \in S, F_\sigma \neq \emptyset\}$.

Let $\mathcal{M}$ be the family of those sets $M \subseteq X \cup X^*$ such that (α) whenever $x, y \in M \cap X$ and $q \in \mathbb{Q}$ then $x + y$ and qx belong to M (β) whenever $f, g \in M \cap X^*$ and $q \in \mathbb{Q}$ then $f + g$ and qf belong to M (γ) $\|x\| = \max\{f(x) : f \in M \cap X^*, \|f\| \leq 1\}$ for every $x \in M \cap X$ (δ) $\sup_{x \in F_\sigma} f(x) = \sup_{x \in F_\sigma \cap M} f(x)$ for every $f \in M \cap X^*$, $\sigma \in S_0$.

(b) For each $x \in X$, choose $h_x \in X^*$ such that $\|h_x\| \leq 1$ and $h_x(x) = \|x\|$; for each $f \in X^*$ and $\sigma \in S_0$ choose a countable set $C_{f\sigma} \subseteq F_\sigma$ such that $\sup\{f(x) : x \in C_{f\sigma}\} = \sup\{f(x) : x \in F_\sigma\}$. Given $B \subseteq X$, define $\langle B_n \rangle_{n \in \mathbb{N}}$ by setting

$$\begin{aligned}B_{n+1} = B_n &\cup \{x + y : x, y \in B_n \cap X\} \cup \{qx : q \in \mathbb{Q}, x \in B_n \cap X\} \\ &\cup \{f + g : f, g \in B_n \cap X\} \cup \{qf : q \in \mathbb{Q}, f \in B_n \cap X\} \\ &\cup \{h_x : x \in B_n \cap X\} \cup \bigcup \{C_{f\sigma} : f \in B_n \cap X^*, \sigma \in S_0\},\end{aligned}$$

for each $n \in \mathbb{N}$. Then $M = \bigcup_{n \in \mathbb{N}} B_n$ belongs to $\mathcal{M}$ and has cardinal at most $\max(\omega, \#(B))$.

(c) The definition of $\mathcal{M}$ makes it plain that if $\mathcal{M}' \subseteq \mathcal{M}$ is upwards-directed then $\bigcup \mathcal{M}'$ belongs to $\mathcal{M}$.

(d) Now take $M \in \mathcal{M}$ and set $Y = \overline{M \cap X}$, $W = \overline{M \cap X^*}$. These are linear subspaces (2A5Ec). If $y \in M \cap X$ and $z \in W^\circ$, then there is an $f \in M \cap X^*$ such that $\|f\| \leq 1$ and $f(y) = \|y\|$, so that

$$\|y + z\| \geq f(y + z) = f(y) = \|y\|.$$

Because the function $y \mapsto \|y+z\| - \|y\|$ is continuous, $\|y+z\| \geq \|y\|$ for every $y \in Y$ and $z \in W^\circ$. In particular, if $y \in Y \cap W^\circ$, $\|y\| \leq \|y - y\| = 0$ and $y = 0$, so $Y + W^\circ = Y \oplus W^\circ$. If $x \in \overline{Y + W^\circ}$, then there are sequences $\langle y_n \rangle_{n \in \mathbb{N}}$ in Y and $\langle z_n \rangle_{n \in \mathbb{N}}$ in W° such that $x = \lim_{n \to \infty} y_n + z_n$; now $\|y_m - y_n\| \leq \|(y_m + z_m) - (y_n + z_n)\| \to 0$ as $n \to \infty$, so (because X is a Banach space) $\langle y_n \rangle_{n \in \mathbb{N}}$ is convergent to y say; in this case, $y \in Y$ and $x - y = \lim_{n \to \infty} z_n$ belongs to W°, so $x \in Y + W^\circ$. This shows that $Y \oplus W^\circ$ is a closed linear subspace of X.

(e) **?** Suppose, if possible, that $Y \oplus W^\circ \neq X$. Then there is an $x_0 \in X \setminus (Y \oplus W^\circ)$. By 4A4Eb, there is an $f \in X^*$ such that $f(x_0) > 0$ and $f(y) = f(z) = 0$ whenever $y \in Y$ and $z \in W^\circ$; multiplying f by a scalar if necessary, we can arrange that $f(x_0) = 1$. By 4A4Eg, f belongs to the weak* closure of W in X^*.

Let $\phi \in A$ be such that $(\phi, x_0) \in R$. Then $K = R[\{\phi\}]$ is weakly compact. Now the weak* closure of W is also its closure for the Mackey topology of uniform convergence on weakly compact subsets of X (4A4F). So there is a $g \in W$ such that $|g(x) - f(x)| \leq \frac{1}{7}$ for every $x \in K$. Next, because K is bounded, and g belongs to the norm closure of $M \cap X^*$, there is an $h \in M \cap X^*$ such that $|h(x) - g(x)| \leq \frac{1}{7}$ for every $x \in K$. This means that $|h(x) - f(x)| \leq \frac{2}{7}$ for every $x \in K$, and K is included in the weakly open set $G = \{x : h(x) - f(x) < \frac{3}{7}\}$, that is, ϕ does not belong to $R^{-1}[X \setminus G]$, which is relatively closed in A, because R is usco-compact regarded as a relation between A and X with the weak topology. There is therefore a $\sigma \in S$ such that $\phi \in I_\sigma$ and $I_\sigma \cap R^{-1}[X \setminus G] = \emptyset$, that is, $F_\sigma \subseteq G$. In this case, $x_0 \in F_\sigma$, so $\sigma \in S_0$, while $h(x) - f(x) < \frac{3}{7}$ for every $x \in F_\sigma$. But, because $M \in \mathcal{M}$, there is a $y \in M \cap X \cap F_\sigma$ such that $h(y) \geq h(x_0) - \frac{1}{7}$, and as $y \in Y$ we must now have

$$\begin{aligned} 0 = f(y) &= h(y) - (h(y) - f(y)) > h(x_0) - \frac{1}{7} - \frac{3}{7} \\ &= f(x_0) - (f(x_0) - h(x_0)) - \frac{4}{7} \geq 1 - \frac{3}{7} - \frac{4}{7} = 0, \end{aligned}$$

which is absurd. **X**

Thus $X = Y \oplus W^\circ$ and (Y, W) is a projection pair. This completes the proof.

467K Theorem Let X be a weakly K-countably determined Banach space. Then it has an equivalent locally uniformly rotund norm.

proof Since the completion of X is weakly K-countably determined (467Ic), we may suppose that X is complete. The proof proceeds by induction on the weight of X.

(a) The induction starts by observing that if $w(X) \leq \omega$ then X is separable (4A2Li) so has an equivalent locally uniformly rotund norm by 467E.

(b) So let us suppose that $w(X) = \kappa > \omega$ and that any weakly K-countably determined Banach space of weight less than κ has an equivalent locally uniformly rotund norm.

Let $\mathcal{M}$ be a family of subsets of $X \cup X^*$ as in 467J. Then there is a non-decreasing family $\langle M_\xi \rangle_{\xi \leq \kappa}$ in $\mathcal{M}$ such that $\#(M_\xi) \leq \max(\omega, \#(\xi))$ for every $\xi \leq \kappa$, M_κ is dense in X, and $M_\xi = \bigcup_{\eta < \xi} M_\eta$ for every limit ordinal $\xi \leq \kappa$. **P** By 4A2Li, there is a dense subset of X of cardinal κ; enumerate it as $\langle x_\xi \rangle_{\xi < \kappa}$. Choose M_ξ inductively, as follows. $M_0 = \emptyset$. GIven M_ξ with $\#(M_\xi) \leq \max(\omega, \#(\xi))$, then by 467J(i) there is an $M_{\xi+1} \in \mathcal{M}$ such that $M_{\xi+1} \supseteq M_\xi \cup \{x_\xi\}$ and

$$\#(M_{\xi+1}) \leq \max(\omega, \#(M_\xi \cup \{x_\xi\}) \leq \max(\omega, \#(\xi + 1)).$$

Given that $\langle M_\eta \rangle_{\eta < \xi}$ is a non-decreasing family in $\mathcal{M}$ with $\#(M_\eta) \leq \max(\omega, \#(\eta))$ for every $\eta < \xi$, set $M_\xi = \bigcup_{\eta < \xi} M_\eta$; then 467J(ii) tells us that $M_\xi \in \mathcal{M}$, while $\#(M_\xi) \leq \max(\omega, \#(\xi))$, as required by the inductive hypothesis. **Q**

At the end of the induction, $M_\kappa \supseteq \{x_\xi : \xi < \kappa\}$ will be dense in X.

(c) For each $\xi < \kappa$, set $Y_\xi = \overline{M_\xi \cap X}$, $W_\xi = \overline{M_\xi \cap X^*}$. Then (Y_ξ, W_ξ) is a projection pair, by 467J(iii). Since $X = Y_\xi \oplus W_\xi^\circ$, we have a projection $P_\xi : X \to Y_\xi$ defined by saying that $P_\xi(y + z) = y$ whenever $y \in Y_\xi$ and $z \in W_\xi^\circ$. Now $P_\xi P_\eta = P_\eta P_\xi = P_\xi$ whenever $\xi \leq \eta$. **P** The point is that $Y_\xi \subseteq Y_\eta$ and $W_\xi \subseteq W_\eta$, so $W_\eta^\circ \subseteq W_\xi^\circ$. If $x \in X$, express it as $y + z_1$ where $y \in Y_\xi$ and $z_1 \in W_\xi^\circ$, and express z_1 as $y' + z$ where $y' \in Y_\eta$ and $z' \in W_\eta^\circ$. Then

$$P_\xi x = y \in Y_\xi \subseteq Y_\eta$$

so $P_\eta P_\xi x = P_\xi x$. On the other hand, $x = y + y' + z$, $y + y' \in Y_\eta$ and $z \in W_\eta^\circ$, so $P_\eta x = y + y'$; and as $y' = z_1 - z$ belongs to W_ξ°, $P_\xi(y + y') = y$, so $P_\xi P_\eta x = P_\xi x$. **Q**

Note that the condition

$$\|y + z\| \geq \|y\| \text{ whenever } y \in Y_\xi,\ z \in W_\xi^\circ$$

ensures that $\|P_\xi\| \leq 1$ for every ξ.

Next, if $\xi \leq \kappa$ is a non-zero limit ordinal, $P_\xi x = \lim_{\eta \uparrow \xi} P_\eta x$. **P** We know that

$$P_\xi x \in Y_\xi = \overline{M_\xi \cap X} = \overline{\bigcup_{\eta<\xi} M_\eta \cap X}.$$

So, given $\epsilon > 0$, there is an $x' \in \bigcup_{\eta<\xi} M_\eta$ such that $\|P_\xi x - x'\| \leq \frac{1}{2}\epsilon$. Let $\eta < \xi$ be such that $x' \in M_\eta$. If $\eta \leq \eta' \leq \xi$, then

$$\|P_\xi x - P_{\eta'} x\| = \|P_\xi(P_\xi x - x') - P_{\eta'}(P_\xi x - x')\|$$

(because $x' \in Y_\eta$, so $P_\xi x' = P_{\eta'} x' = x'$)

$$\leq 2\|P_\xi x - x'\| \leq \epsilon.$$

As ϵ is arbitrary, $P_\xi x = \lim_{\eta\uparrow\xi} P_\eta x$. **Q**

(d) Now observe that every Y_ξ is weakly K-countably determined (467Ia), while $w(Y_\xi) \leq \max(\omega, \#(\xi)) < \kappa$ for every $\xi < \kappa$ (using 4A2Li, as usual). So the inductive hypothesis tells us that $Y_\xi = P_\xi[X]$ has an equivalent locally uniformly rotund norm for every $\xi < \kappa$. By 467G, $X = P_\kappa[X]$ has an equivalent locally uniformly rotund norm. Thus the induction proceeds.

467L Weakly compactly generated Banach spaces The most important class of weakly K-countably determined spaces is the following. A normed space X is **weakly compactly generated** if there is a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of weakly compact subsets of X such that $\bigcup_{n\in\mathbb{N}} K_n$ is dense in X.

467M Proposition (Talagrand 75) A weakly compactly generated Banach space is weakly K-countably determined.

proof Let X be a Banach space with a sequence $\langle K_n\rangle_{n\in\mathbb{N}}$ of weakly compact subsets of X such that $\bigcup_{n\in\mathbb{N}} K_n$ is dense in X. Set

$$L_n = \{\textstyle\sum_{i=0}^n \alpha_i x_i : |\alpha_i| \leq n,\ x_i \in \bigcup_{j\leq n} K_j \text{ for every } i \leq n\}$$

for $n \in \mathbb{N}$. Then every L_n is weakly compact, and $Y = \bigcup_{n\in\mathbb{N}} L_n$ is a linear subspace of X including $\bigcup_{n\in\mathbb{N}} K_n$, therefore dense. Now Y is a countable union of weakly compact sets, therefore K-analytic for its weak topology (422Gc, 422Hc); in particular, it is weakly K-countably determined. By 467Ic, X is weakly K-countably determined.

467N Theorem Let X be a Banach lattice with an order-continuous norm (§354). Then it has an equivalent locally uniformly rotund norm.

proof (a) Consider first the case in which X has a weak order unit e. Then the interval $[0, e]$ is weakly compact. **P** We have $X^* = X^\times$ and $X^{**} = X^{\times\sim}$ (356D). The canonical identification of X with its image in X^{**} is an order-continuous Riesz homomorphism from X onto a solid order-dense Riesz subspace of $X^{\times\times}$ (356I) and therefore of $X^{\times\sim} = X^{**}$. In particular, $[0, e] \subseteq X$ is matched with an interval $[0, \hat{e}] \subseteq X^{**}$. But $[0, \hat{e}] = \{\theta : \theta \in X^{**},\ 0 \leq \theta(f) \leq f(e) \text{ for every } f \in (X^*)^+\}$ is weak*-closed and norm-bounded in X^{**}, therefore weak*-compact; as the weak* topology on X^{**} corresponds to the weak topology of X, $[0, e]$ is weakly compact. **Q**

Now, for each $n \in \mathbb{N}$, $K_n = [-ne, ne] = n[0, e] - n[0, e]$ is weakly compact. If $x \in X^+$, then $\langle x \wedge ne\rangle_{n\in\mathbb{N}}$ converges to X, because the norm of X is order-continuous; so for any $x \in X$, $\langle x^+ \wedge ne - x^- \wedge ne\rangle_{n\in\mathbb{N}}$ converges to x. Thus $\bigcup_{n\in\mathbb{N}} K_n$ is dense in X and X is weakly compactly generated. By 467M and 467K, X has an equivalent locally uniformly rotund norm.

(b) For the general case, let $\langle x_i\rangle_{i\in I}$ be a maximal disjoint family in X^+. For each $i \in I$ let X_i be the band in X generated by x_i, and $T_i : X \to X_i$ the band projection onto X_i (354Ee, 353Hb). Then $\langle T_i\rangle_{i\in I}$ satisfies the conditions of 467F. **P** (i) Each T_i is a continuous linear operator because the given norm $\|\ \|$ of X is a Riesz norm. Next, each X_i has a weak order unit x_i, and the norm on X_i is order-continuous, so (a) tells us that there is an equivalent locally uniformly rotund norm on $X_i = T_i[X]$. (ii) If $x \in X$, set $x' = \sup_{i\in I} T_i|x|$; then $(|x| - x') \wedge x_i = 0$ for every i, so, by the maximality of $\langle x_i\rangle_{i\in I}$, $x' = |x|$. If $\epsilon > 0$ then, because the norm of X is order-continuous, there is a finite $J \subseteq I$ such that

$$\|x - \textstyle\sum_{j\in J} T_i x\| = \|x' - \sup_{j\in J} T_i|x|\| \leq \epsilon.$$

Moreover, if $i \in I \setminus J$, then

$$\|T_i x\| \leq \|x - \sum_{j \in J} T_j x\| \leq \epsilon.$$

Thus conditions (ii) and (iii) of 467F are satisfied. **Q**

Accordingly 467F tells us that X has an equivalent locally uniformly rotund norm.

467O Eberlein compacta: Definition A topological space K is an **Eberlein compactum** if it is homeomorphic to a weakly compact subset of a Banach space.

467P Proposition Let K be a compact Hausdorff space.

(a) The following are equiveridical:

(i) K is an Eberlein compactum;

(ii) there is a set $L \subseteq C(K)$, separating the points of K, which is compact for the topology of pointwise convergence.

(b) Suppose that K is an Eberlein compactum.

(i) K has a σ-isolated network, so is hereditarily weakly θ-refinable.

(ii) (SCHACHERMAYER 77) If $w(K)$ is measure-free, K is a Radon space.

proof (a)(i)⇒(ii) If K is an Eberlein compactum, we may suppose that it is a weakly compact subset of a Banach space X. Set $L = \{f \restriction K : f \in X^*, \|f\| \leq 1\}$; since the map $f \mapsto f \restriction K : X^* \to C(K)\}$ is continuous for the weak* topology of X^* and the topology $\mathfrak{T}_p$ of pointwise convergence on $C(K)$, L is $\mathfrak{T}_p$-compact; and L separates the points of K because X^* separates the points of X.

(ii)⇒(i) If $L \subseteq C(K)$ is $\mathfrak{T}_p$-compact and separates the points of K, set $L_n = \{f : f \in L, \|f\|_\infty \leq n\}$ for each $n \in \mathbb{N}$. Then L_n is $\mathfrak{T}_p$-compact for each n. Set $L' = \{0\} \cup \bigcup_{n \in \mathbb{N}} 2^{-n} L_n$; then $L' \subseteq C(K)$ is norm-bounded and $\mathfrak{T}_p$-compact and separates the points of K. Now define $x \mapsto \hat{x} : K \to \mathbb{R}^{L'}$ by setting $\hat{x}(f) = f(x)$ for $x \in K$, $f \in L'$. Then $\hat{x} \in C(L')$ for every x and $x \mapsto \hat{x}$ is continuous for the given topology of K and the topology of pointwise convergence on $C(L')$; so the image $\hat{K} = \{\hat{x} : x \in K\}$ is $\mathfrak{T}_p$-compact. Since it is also bounded, it is weakly compact (462E). But $x \mapsto \hat{x}$ is injective, because L' separates the points of K; so K is homeomorphic to $\hat{K}$, and is an Eberlein compactum.

(b) Again suppose K is actually a weakly compact subset of a Banach space X. As in 467M, set $L_n = \{\sum_{i=0}^n \alpha_i x_i : |\alpha_i| \leq n, x_i \in K \text{ for every } i \leq n\}$ for each $n \in \mathbb{N}$. Then $Y = \overline{\bigcup_{n \in \mathbb{N}} L_n}$ is a weakly compactly generated Banach space. (I am passing over the trivial case $K = \emptyset$.) So Y has an equivalent locally uniformly rotund norm (467M, 467K), which is a Kadec norm (467B), and Y, with the weak topology, has a σ-isolated network (466Eb). It follows at once that K has a σ-isolated network (4A2B(a-ix)), so is hereditarily weakly θ-refinable (438Ld); and if $w(K)$ is measure-free, K is Borel-measure-complete (438M), therefore Radon (434Jf, 434Ka).

467X Basic exercises (a) (i) Show that a continuous image of a K-countably determined space is K-countably determined. (ii) Show that the product of a sequence of K-countably determined spaces is K-countably determined. (iii) Show that any K-countably determined topological space is Lindelöf. (*Hint*: 422De.) (iv) Show that any Souslin-F subset of a K-countably determined topological space is K-countably determined. (*Hint*: 422Hc.)

(b) Let X be a σ-compact Hausdorff space. Show that a subspace Y of X is K-countably determined iff there is a countable family $\mathcal{K}$ of compact subsets of X such that $\bigcap\{K : x \in K \in \mathcal{K}\} \subseteq Y$ for every $x \in Y$.

(c) Show that if X and Y are weakly K-countably determined normed spaces, then $X \times Y$, with an appropriate norm, is weakly K-countably determined.

(d) Show that a normed space X is weakly compactly generated iff there is a weakly compact set $K \subseteq X$ such that the linear subspace of X generated by K is dense in X.

(e) Show that any separable normed space is weakly compactly generated.

(f) Show that any reflexive Banach space is weakly compactly generated.

>(g) Show that if X is a weakly compactly generated Banach space, then it is K-analytic in its weak topology. (*Hint*: in 467M, use the proof of 467Ic.)

(h) Show that if X is a Banach space and there is a set $A \subseteq X$ such that A is K-countably determined for the weak topology and the linear subspace generated by A is dense, then X is weakly K-countably determined.

(i) Show that the one-point compactification of any discrete space is an Eberlein compactum.

467Y Further exercises (a) Let $\langle X_n \rangle_{n \in \mathbb{N}}$ be a sequence of weakly K-countably determined normed spaces. Investigate normed subspaces of $\prod_{n \in \mathbb{N}} X_n$ which will be weakly K-countably determined.

(b) (i) Show that if (Ω, Σ, μ) is a probability space, then $L^1(\mu)$ has a locally uniformly rotund Riesz norm. (*Hint*: apply the construction of 467D with $Y = L^2(\mu)$ and T the identity operator; show that all the norms p_n are Riesz norms.) (ii) Show that if X is any L-space then it has a locally uniformly rotund Riesz norm. (*Hint*: apply the construction of 467F/467N, noting that if the T_i in 467F are band projections and $\| \|$ and all the $\| \|_i$ are Riesz norms, then all the norms in the proof of 467F are Riesz norms.)

467 Notes and comments The purpose of this section has been to give an idea of the scope of Proposition 466F. 'Local uniform rotundity' has an important place in the geometrical theory of Banach spaces, but for the many associated ideas I refer you to Deville Godefroy & Zizler 93. From our point of view, Theorem 467E is therefore purely accessory, since we know by different arguments that on separable Banach spaces the weak and norm topologies have the same Borel σ-algebras (4A3V). We need it to provide the first step in the inductive proof of 467K.

Since the concept of 'K-analytic' space is one of the fundamental ideas of Chapter 43, it is natural here to look at 'K-countably determined' spaces, especially as many of the ideas of §422 are directly applicable (467Xa). But the goal of this part of the argument is Schachermayer's theorem 467P(b-ii), which uses 'weakly compactly generated' spaces (467L). 'Eberlein compacta' are of great interest in other ways; they are studied at length in Arkhangel'skii 92.

I mention order-continuous norms here (467N) because they are prominent in the theory of Banach lattices in Volume 3. Note that the methods here do *not* suffice in general to arrange that the locally uniformly rotund norm found on X is a Riesz norm; though see 467Yb. It is in fact the case that every Banach lattice with an order-continuous norm has an equivalent locally uniformly rotund Riesz norm, but this requires further ideas (see Davis Ghoussoub & Lindenstrauss 81).

Chapter 47

Geometric measure theory

I offer a chapter on geometric measure theory, continuing from Chapter 26. The greater part of it is directed specifically at a version of the Divergence Theorem (475N), and I do not attempt to provide a balanced view of the subject, for which I must refer you to Mattila 95, Evans & Gariepy 92 and Federer 69. However §472, at least, deals with something which must be central to any approach, Besicovitch's Density Theorem for Radon measures on $\mathbb{R}^r$. In §473 I examine Lipschitz functions, and give crude forms of some fundamental ineqalities relating integrals $\int \|\operatorname{grad} f\| d\mu$ with other measures of the variation of a function f. In §474 I introduce perimeter measures λ_E^∂ and outward-normal functions ψ_E as those for which the Divergence Theorem, in the form $\int_E \operatorname{div} \phi \, d\mu = \int \phi \cdot \psi_E \, d\lambda_E^\partial$, will be valid, and give the geometric description of $\psi_E(x)$ as the Federer exterior normal to E at x. In §475 I show that λ_E^∂ can be identified with normalized Hausdorff $r-1$-dimensional measure on the essential boundary of E.

§471 is devoted to Hausdorff measures on general metric spaces, extending the ideas introduced in §264 for Euclidean space, up to basic results on densities (471P) and Howroyd's theorem (471S). In §476 I turn to a different topic, the problem of finding the subsets of $\mathbb{R}^r$ on which Lebesgue measure is most 'concentrated' in some sense. I present a number of classical results, the deepest being the Isoperimetric Theorem (476L): among sets with a given measure, those with the smallest perimeters are the balls.

471 Hausdorff measures

I begin the chapter by returning to a class of measures which we have not examined since Chapter 26. The primary importance of these measures is in studying the geometry of Euclidean space; in §265 I looked briefly at their use in describing surface measures, which will reappear in §475. Hausdorff measures are also one of the basic tools in the study of fractals, but for such applications I must refer you to Falconer 90 and Mattila 95. All I shall attempt to do here is to indicate some of the principal ideas which are applicable to general metric spaces, and to look at some special properties of Hausdorff measures related to the concerns of this chapter and of §261.

471A Definition Let (X, ρ) be a metric space and $r \in]0, \infty[$. For $\delta > 0$ and $A \subseteq X$, set

$$\theta_{r\delta} A = \inf\{\sum_{n=0}^{\infty} (\operatorname{diam} D_n)^r : \langle D_n \rangle_{n\in\mathbb{N}} \text{ is a sequence of subsets of } X \text{ covering } A, \operatorname{diam} D_n \leq \delta \text{ for every } n \in \mathbb{N}\}.$$

(As in §264, take $\operatorname{diam} \emptyset = 0$ and $\inf \emptyset = \infty$.) It will be useful to note that every $\theta_{r\delta}$ is an outer measure. Now set

$$\theta_r A = \sup_{\delta>0} \theta_{r\delta} A$$

for $A \subseteq X$; θ_r also is an outer measure on X, as in 264B; this is **r-dimensional Hausdorff outer measure** on X. Let μ_{Hr} be the measure defined by Carathéodory's method from θ_r; μ_{Hr} is **r-dimensional Hausdorff measure** on X.

Notation It may help if I list some notation already used elsewhere. Suppose that (X, ρ) is a metric space. I write

$$B(x, \delta) = \{y : \rho(y, x) \leq \delta\}, \quad U(x, \delta) = \{y : \rho(y, x) < \delta\}$$

for $x \in X$, $\delta \geq 0$; recall that $U(x, \delta)$ is open (2A3G). For $x \in X$ and A, $A' \subseteq X$ I write

$$\rho(x, A) = \inf_{y\in A} \rho(x, y), \quad \rho(A, A') = \inf_{y\in A, z\in A'} \rho(y, z);$$

for definiteness, take $\inf \emptyset$ to be ∞, as before.

471B Definition Let (X, ρ) be a metric space. An outer measure θ on X is a **metric outer measure** if $\theta(A_1 \cup A_2) = \theta A_1 + \theta A_2$ whenever A_1, $A_2 \subseteq X$ and $\rho(A_1, A_2) > 0$.

471C Proposition Let (X,ρ) be a metric space and θ a metric outer measure on X. Let μ be the measure on X defined from θ by Carathéodory's method. Then μ is a topological measure.

proof (Compare 264E, part (b) of the proof.) Let $G \subseteq X$ be open, and A any subset of X such that $\theta A < \infty$. Set

$$A_n = \{x : x \in A,\ \rho(x, A \setminus G) \geq 2^{-n}\},$$

$$B_0 = A_0, \quad B_n = A_n \setminus A_{n-1} \text{ for } n > 1.$$

Observe that $A_n \subseteq A_{n+1}$ for every n and $\bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} B_n = A \cap G$. The point is that if m, $n \in \mathbb{N}$ and $n \geq m+2$, and if $x \in B_m$ and $y \in B_n$, then there is a $z \in A \setminus G$ such that $\rho(y,z) < 2^{-n+1} \leq 2^{-m-1}$, while $\rho(x,z)$ must be at least 2^{-m}, so $\rho(x,y) \geq \rho(x,z) - \rho(y,z) \geq 2^{-m-1}$. Thus $\rho(B_m, B_n) > 0$ whenever $n \geq m+2$. It follows that for any $k \geq 0$

$$\textstyle\sum_{m=0}^{k} \theta B_{2m} = \theta(\bigcup_{m\leq k} B_{2m}) \leq \theta(A \cap G) < \infty,$$

$$\textstyle\sum_{m=0}^{k} \theta B_{2m+1} = \theta(\bigcup_{m\leq k} B_{2m+1}) \leq \theta(A \cap G) < \infty.$$

Consequently $\sum_{n=0}^{\infty} \theta B_n < \infty$.

But now, given $\epsilon > 0$, there is an m such that $\sum_{n=m}^{\infty} \theta B_m \leq \epsilon$, so that

$$\begin{aligned}\theta(A \cap G) + \theta(A \setminus G) &\leq \theta A_m + \sum_{n=m}^{\infty} \theta B_n + \theta(A \setminus G)\\ &\leq \epsilon + \theta A_m + \theta(A \setminus G) = \epsilon + \theta(A_m \cup (A \setminus G))\end{aligned}$$

(since $\rho(A_m, A \setminus G) \geq 2^{-m}$)

$$\leq \epsilon + \theta A.$$

As ϵ is arbitrary, $\theta(A \cap G) + \theta(A \setminus G) \leq \theta A$. As A is arbitrary, G is measured by μ; as G is arbitrary, μ is a topological measure.

471D Theorem Let (X,ρ) be a metric space and $r > 0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X, and Σ its domain; write θ_r for r-dimensional Hausdorff outer measure on X, as defined in 471A.

(a) μ_{Hr} is a topological measure.

(b) For every $A \subseteq X$ there is a G_δ set $H \supseteq A$ such that $\mu_{Hr} H = \theta_r A$.

(c) θ_r is the outer measure defined from μ_{Hr} (that is, θ_r is a regular outer measure).

(d) Σ is closed under Souslin's operation.

(e) $\mu_{Hr} E = \sup\{\mu_{Hr} F : F \subseteq E \text{ is closed}\}$ whenever $E \in \Sigma$ and $\mu_{Hr} E < \infty$.

(f) If $A \subseteq X$ and $\theta_r A < \infty$ then A is separable and the set of isolated points of A is μ_{Hr}-negligible.

(g) μ_{Hr} is atomless.

(h) If μ_{Hr} is totally finite it is a quasi-Radon measure.

proof (a) The point is that θ_r, as defined in 471A, is a metric outer measure. **P** (Compare 264E, part (a) of the proof.) Let A_1, A_2 be subsets of X such that $\rho(A_1, A_2) > 0$. Of course $\theta(A_1 \cup A_2) \leq \theta_r A_1 + \theta_r A_2$, because θ_r is an outer measure. For the reverse inequality, we may suppose that $\theta_r(A_1 \cup A_2) < \infty$, so that $\theta_r A_1$ and $\theta_r A_2$ are both finite. Let $\epsilon > 0$ and let δ_1, $\delta_2 > 0$ be such that

$$\theta_r A_1 + \theta_r A_2 \leq \theta_{r\delta_1} A_1 + \theta_{r\delta_2} A_2 + \epsilon,$$

defining the $\theta_{r\delta_i}$ as in 471A. Set $\delta = \min(\delta_1, \delta_2, \frac{1}{2}\rho(A_1, A_2)) > 0$ and let $\langle D_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets of diameter at most δ, covering $A_1 \cup A_2$, and such that $\sum_{n=0}^{\infty} (\operatorname{diam} D_n)^r \leq \theta_{r\delta}(A_1 \cup A_2) + \epsilon$. Set

$$K = \{n : D_n \cap A_1 \neq \emptyset\}, \quad L = \{n : D_n \cap A_2 \neq \emptyset\}.$$

Because $\rho(x,y) > \operatorname{diam} D_n$ whenever $x \in A_1$, $y \in A_2$ and $n \in \mathbb{N}$, $K \cap L = \emptyset$; and of course $A_1 \subseteq \bigcup_{n\in K} D_k$, $A_2 \subseteq \bigcup_{n\in L} D_n$. Consequently

$$\theta_r A_1 + \theta_r A_2 \leq \epsilon + \theta_{r\delta_1} A_1 + \theta_{r\delta_2} A_2 \leq \epsilon + \sum_{n\in K} (\operatorname{diam} D_n)^r + \sum_{n\in L} (\operatorname{diam} D_n)^r$$
$$\leq \epsilon + \sum_{n=0}^{\infty} (\operatorname{diam} D_n)^r \leq 2\epsilon + \theta_{r\delta}(A_1 \cup A_2) \leq 2\epsilon + \theta_r(A_1 \cup A_2).$$

As ϵ is arbitrary, $\theta_r(A_1 \cup A_2) \geq \theta_r A_1 + \theta_r A_2$. The reverse inequality is true just because θ_r is an outer measure, so $\theta_r(A_1 \cup A_2) = \theta_r A_1 + \theta_r A_2$. As A_1 and A_2 are arbitrary, θ_r is a metric outer measure. **Q**

Now 471C tells us that μ_{Hr} must be a topological measure.

(b) (Compare 264Fa.) If $\theta_r A = \infty$ this is trivial. Otherwise, for each $n \in \mathbb{N}$, let $\langle D_{ni}\rangle_{i\in\mathbb{N}}$ be a sequence of sets of diameter at most 2^{-n} such that $A \subseteq \bigcup_{i\in\mathbb{N}} D_i$ and $\sum_{i=0}^{\infty} (\operatorname{diam} D_{ni})^r \leq \theta_{r,2^{-n}}(A) + 2^{-n}$, defining $\theta_{r,2^{-n}}$ as in 471A. Let $\eta_{ni} \in]0, 2^{-n}]$ be such that $(2\eta_{ni} + \operatorname{diam} D_{ni})^r \leq 2^{-n-i} + (\operatorname{diam} D_{ni})^r$, and set $G_{ni} = \{x : \rho(x, D_{ni}) < \eta_{ni}\}$ for all n, $i \in \mathbb{N}$; then $G_{ni} = \bigcup_{x\in D_{ni}} U(x, \eta_{ni})$ is an open set including D_{ni} and $(\operatorname{diam} G_{ni})^r \leq 2^{-n-i} + (\operatorname{diam} D_{ni})^r$. Set

$$H = \bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} G_{ni},$$

so that H is a G_δ set including A.

For any $\delta > 0$, there is an $n \in \mathbb{N}$ such that $3 \cdot 2^{-n} \leq \delta$, so that $\operatorname{diam} G_{mi} \leq \operatorname{diam} D_{mi} + 2\eta_{mi} \leq \delta$ for every $i \in \mathbb{N}$ and $m \geq n$, and

$$\theta_{r\delta}(H) \leq \sum_{i=0}^{\infty} (\operatorname{diam} G_{mi})^r \leq \sum_{i=0}^{\infty} 2^{-m-i} + (\operatorname{diam} D_{mi})^r$$
$$\leq 2^{-m+1} + \theta_{r,2^{-m}}(A) + 2^{-m} \leq 2^{-m+2} + \theta_r(A)$$

for every $m \geq n$. Accordingly $\theta_{r\delta} H \leq \theta_r A$ for every $\delta > 0$, so $\theta_r H \leq \theta_r A$. Of course this means that $\theta_r H = \theta_r A$; and since, by (a), μ_{Hr} measures H, we have $\mu_{Hr} H = \theta_r A$, as required.

(c) (Compare 264Fb.) If $A \subseteq X$,

$$\theta_r A \geq \mu^*_{Hr} A$$

(by (b))

$$= \inf\{\theta_r E : A \subseteq E \in \Sigma\} \geq \theta_r A.$$

(d) Use 431C.

(e) By (b), there is a Borel set $H \supseteq E$ such that $\mu_{Hr} H = \mu_{Hr} E$, and now there is a Borel set $H' \supseteq H \setminus E$ such that $\mu_{Hr} H' = \mu_{Hr}(H \setminus E) = 0$, so that $G = H \setminus H'$ is a Borel set included in E and $\mu_{Hr} G = \mu_{Hr} E$. Now G is a Baire set (4A3Kb), so is Souslin-F (421L), and $\mu_{Hr} G = \sup_{F\subseteq G \text{ is closed}} \mu_{Hr} F$, by 431E.

(f) For every $n \in \mathbb{N}$, there must be a sequence $\langle D_{ni}\rangle_{i\in\mathbb{N}}$ of sets of diameter at most 2^{-n} covering A; now if $D \subseteq A$ is a countable set which meets D_{ni} whenever i, $n \in \mathbb{N}$ and $A \cap D_{ni} \neq \emptyset$, D will be dense in A. Now if A_0 is the set of isolated points in A, it is still separable (4A2P(a-iv)); but as the only dense subset of A_0 is itself, it is countable. Since $\theta_{r\delta}\{x\} = (\operatorname{diam}\{x\})^r = 0$ for every $\delta > 0$, $\mu_{Hr}\{x\} = 0$ for every $x \in X$, and A_0 is negligible.

(g) In fact, if $A \subseteq X$ and $\theta_r A > 0$, there are disjoint A_0, $A_1 \subseteq A$ such that $\theta_r A_i > 0$ for both i. **P** (i) Suppose first that A is not separable. For each $n \in \mathbb{N}$, let $D_n \subseteq A$ be a maximal set such that $\rho(x, y) \geq 2^{-n}$ for all distinct x, $y \in D_n$; then $\bigcup_{n\in\mathbb{N}} D_n$ is dense in A, so there is some $n \in \mathbb{N}$ such that D_n is uncountable; if we take A_1, A_2 to be disjoint uncountable subsets of D_n, then $\theta_r A_1 = \theta_r A_2 = \infty$. (ii) If A is separable, then set $\mathcal{G} = \{G : G \subseteq X \text{ is open}, \theta_r(A \cap G) = 0\}$. Because A is hereditarily Lindelöf (4A2P(a-iii)), there is a countable subset $\mathcal{G}_0$ of $\mathcal{G}$ such that $A \cap \bigcup \mathcal{G} = A \cap \bigcup \mathcal{G}_0$ (4A2H(c-i)), so $A \cap \bigcup \mathcal{G}$ is negligible and $A \setminus \bigcup \mathcal{G}$ has at least two points x_0, x_1. If we set $A_i = U(x_i, \frac{1}{2}\rho(x_i, x_{1-i}))$ for each i, these are disjoint subsets of A of non-zero outer measure. **Q**

(h) If μ_{Hr} is totally finite, then it is inner regular with respect to the closed sets, by (e). Also, because X must be separable, by (f), therefore hereditarily Lindelöf, μ_{Hr} must be τ-additive (414O). Finally, μ_{Hr} is complete just because it is defined by Carathéodory's method. So μ_{Hr} is a quasi-Radon measure.

471E Corollary If (X,ρ) is a metric space, $r>0$ and $Y\subseteq X$ then r-dimensional Hausdorff measure $\mu_{Hr}^{(Y)}$ on Y extends the subspace measure $(\mu_{Hr}^{(X)})_Y$ on Y induced by r-dimensional Hausdorff measure $\mu_{Hr}^{(X)}$ on X; and if either Y is measured by $\mu_{Hr}^{(X)}$ or Y has finite r-dimensional Hausdorff outer measure in X, then $\mu_{Hr}^{(Y)}=(\mu_{Hr}^{(X)})_Y$.

proof Write $\theta_r^{(X)}$ and $\theta_r^{(Y)}$ for the two r-dimensional Hausdorff outer measures.

If $A\subseteq Y$ and $\langle D_n\rangle_{n\in\mathbb{N}}$ is any sequence of subsets of X covering A, then $\langle D_n\cap Y\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of Y covering A, and $\sum_{n=0}^\infty(\operatorname{diam}(D_n\cap Y))^r\le\sum_{n=0}^\infty(\operatorname{diam}D_n)^r$; moreover, when calculating $\operatorname{diam}(D_n\cap Y)$, it doesn't matter whether we use the metric ρ on X or the subspace metric $\rho\restriction Y\times Y$ on Y. What this means is that, for any $\delta>0$, $\theta_{r\delta}A$ is the same whether calculated in Y or in X, so that $\theta_r^{(Y)}A=\sup_{\delta>0}\theta_{r\delta}A=\theta_r^{(X)}A$.

Thus $\theta_r^{(Y)}=\theta_r^{(X)}\restriction\mathcal{P}Y$. Also, by 471Db, $\theta_r^{(X)}$ is a regular outer measure. So 214Hb gives the results.

471F Corollary Let (X,ρ) be an analytic metric space (that is, a metric space in which the topology is analytic in the sense of §423), and write μ_{Hr} for r-dimensional Hausdorff measure on X. Suppose that ν is a locally finite indefinite-integral measure over μ_{Hr}. Then ν is a Radon measure.

proof Since $\operatorname{dom}\nu\supseteq\operatorname{dom}\mu$, ν is a topological measure. Because X is separable, therefore hereditarily Lindelöf, ν is σ-finite and τ-additive, therefore locally determined and effectively locally finite. Next, it is inner regular with respect to the closed sets. **P** Let f be a Radon-Nikodým derivative of ν. If $\nu E>0$, there is an $E'\subseteq E$ such that

$$0<\nu E'=\int f\times\chi E'\,d\mu_{Hr}<\infty.$$

There is a μ_{Hr}-simple function g such that $g\le f\times\chi E'$ μ_{Hr}-a.e. and $\int g\,d\mu_{Hr}>0$; setting $H=E'\cap\{x:g(x)>0\}$, $\nu_{Hr}H<\infty$. Now there is a closed set $F\subseteq H$ such that $\mu_{Hr}F>0$, by 471De, and in this case $\nu F\ge\int_F g\,d\mu_{Hr}>0$. By 412B, this is enough to show that ν is inner regular with respect to the closed sets. **Q**

Since ν is complete (234A), it is a quasi-Radon measure, therefore a Radon measure (434Jf, 434Jb).

471G Increasing Sets Lemma(Davies 70) Let (X,ρ) be a metric space and $r>0$.

(a) Suppose that $\delta>0$ and that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of X with union A. Then $\theta_{r,6\delta}(A)\le(5^r+2)\sup_{n\in\mathbb{N}}\theta_{r\delta}(A_n)$.

(b) Suppose that $\delta>0$ and that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of X with union A. Then $\theta_{r\delta}A\le\sup_{n\in\mathbb{N}}\theta_{r\delta}A_n$.

proof (a) If $\sup_{n\in\mathbb{N}}\theta_{r\delta}(A_n)=\infty$ this is trivial; suppose otherwise.

(i) Take any $\gamma>\gamma'>\sup_{n\in\mathbb{N}}\theta_{r\delta}(A_n)$. For each $i\in\mathbb{N}$, let $\zeta_i\in\left]0,\frac{1}{6}\delta\right]$ be such that $(\alpha+2\zeta_i)^r\le\alpha^r+2^{-i-1}(\gamma-\gamma')$ whenever $0\le\alpha\le\delta$. For each $n\in\mathbb{N}$, there is a sequence $\langle C_{ni}\rangle_{i\in\mathbb{N}}$ of sets covering A_n such that $\operatorname{diam}C_{ni}\le\delta$ for every i and $\sum_{i=0}^\infty(\operatorname{diam}C_{ni})^r\le\gamma'$. Since $\sum_{i=0}^\infty(\operatorname{diam}C_{ni})^r$ is finite, $\lim_{i\to\infty}\operatorname{diam}C_{ni}=0$; rearranging the sequence, we may suppose that $\operatorname{diam}C_{ni}+\zeta_i\ge\operatorname{diam}C_{n,i+1}$ for every i.

In this case, $\lim_{i\to\infty}\sup_{n\in\mathbb{N}}\operatorname{diam}C_{ni}=0$. **P** Given $\epsilon>0$, take k such that $\sum_{j=0}^k\max(0,\epsilon-\zeta_j)^r>\gamma$. For any $i\ge k$ and $n\in\mathbb{N}$, $\operatorname{diam}C_{nj}\ge\max(0,\operatorname{diam}C_{ni}-\zeta_j)$ for every $j\le k$, and $\sum_{j=0}^k(\operatorname{diam}C_{nj})^r\le\gamma$, so $\operatorname{diam}C_{ni}\le\epsilon$. **Q**

(ii) By Ramsey's theorem (4A1G, with $n=2$), there is an infinite set $I\subseteq\mathbb{N}$ such that

for all i, $j\in\mathbb{N}$ there is an $s\in\mathbb{N}$ such that either $C_{mi}\cap C_{nj}=\emptyset$ whenever m, $n\in I$ and $s\le m<n$ or $C_{mi}\cap C_{nj}\ne\emptyset$ whenever m, $n\in I$ and $s\le m<n$,

for each $i\in\mathbb{N}$, $\alpha_i=\lim_{n\in I,n\to\infty}\operatorname{diam}C_{ni}$ is defined in $\mathbb{R}$.

(Apply 4A1Fb with the families

$$\begin{aligned}\mathcal{J}_{ij}=\{J:J\in[\mathbb{N}]^\omega,&\text{ either }C_{mi}\cap C_{nj}=\emptyset\text{ whenever }m,n\in J\text{ and }m<n\\&\text{or }C_{mi}\cap C_{nj}\ne\emptyset\text{ whenever }m,n\in J\text{ and }m<n\}\\\mathcal{J}'_{iq}=\{J:J\in[\mathbb{N}]^\omega,&\text{ either }\operatorname{diam}C_{ni}\le q\text{ for every }n\in J\\&\text{or }\operatorname{diam}C_{ni}\ge q\text{ for every }n\in J\}\end{aligned}$$

for i, $j \in \mathbb{N}$ and $q \in \mathbb{Q}$.)

Of course $\alpha_j - \zeta_i \leq \alpha_i \leq \delta$ whenever $i \leq j$, because $\operatorname{diam} C_{nj} - \zeta_i \leq \operatorname{diam} C_{ni} \leq \delta$ for every n. Set $D_{ni} = \{x : \rho(x, C_{ni}) \leq 2\alpha_i + 3\zeta_i\}$ for all n, $i \in \mathbb{N}$, and $D_i = \bigcup_{s\in\mathbb{N}} \bigcap_{n\in I\setminus s} D_{ni}$ for $i \in \mathbb{N}$. (I am identifying each $s \in \mathbb{N}$ with the set of its predecessors.)

(iii) Set

$$L = \{(i,j) : i, j \in \mathbb{N}, \forall\, s \in \mathbb{N}\ \exists\, m, n \in I, s \leq m < n \text{ and } C_{mi} \cap C_{nj} \neq \emptyset\}.$$

If $(i,j) \in L$ then there is an $s \in \mathbb{N}$ such that $C_{mi} \subseteq D_{\min(i,j)}$ whenever $m \in I$ and $m \geq s$. **P** By the choice of I, we know that there is an $s_0 \in \mathbb{N}$ such that $C_{mi} \cap C_{nj} \neq \emptyset$ whenever m, $n \in I$ and $s_0 \leq m < n$. Let $s_1 \geq s_0$ be such that

$$\operatorname{diam} C_{mi} \leq \alpha_i + \min(\zeta_i, \zeta_j), \quad \operatorname{diam} C_{mj} \leq \alpha_j + \min(\zeta_i, \zeta_j)$$

whenever $m \in I$ and $m \geq s_1$. Take $m_0 \in I$ such that $m_0 \geq s_1$, and set $s = m_0 + 1$. Let $m \in I$ be such that $m \geq s$.

($\boldsymbol{\alpha}$) Suppose that $i \leq j$ and $x \in C_{mi}$. Take any $n \in I$ such that $m \leq n$. Then there is an $n' \in I$ such that $n < n'$. We know that $C_{mi} \cap C_{n'j}$ and $C_{ni} \cap C_{n'j}$ are both non-empty. So

$$\rho(x, C_{ni}) \leq \operatorname{diam} C_{mi} + \operatorname{diam} C_{n'j} \leq \alpha_i + \zeta_i + \alpha_j + \zeta_i \leq 2\alpha_i + 3\zeta_i$$

and $x \in D_{ni}$. This is true for all $n \in I$ such that $n \geq m$, so $x \in D_i$. As x is arbitrary, $C_{mi} \subseteq D_i$.

($\boldsymbol{\beta}$) Suppose that $j \leq i$ and $x \in C_{mi}$. Take any $n \in I$ such that $n > m$. Then $C_{mi} \cap C_{nj}$ is not empty, so

$$\rho(x, C_{nj}) \leq \operatorname{diam} C_{mi} \leq \alpha_i + \zeta_j \leq \alpha_j + 2\zeta_j$$

and $x \in D_{nj}$. As x and n are arbitrary, $C_{mi} \subseteq D_j$.

Thus $C_{mi} \subseteq D_{\min(i,j)}$ in both cases. **Q**

(iv) Set

$$D = \textstyle\bigcup_{i\in\mathbb{N}} D_i, \quad J = \{i : i \in \mathbb{N}, \exists\, s \in \mathbb{N}, C_{ni} \subseteq D \text{ whenever } n \in I \text{ and } n \geq s\}.$$

If $i \in \mathbb{N} \setminus J$ and $j \in \mathbb{N}$, then (iii) tells us that $(i,j) \notin L$, so there is some $s \in \mathbb{N}$ such that $C_{mi} \cap C_{nj} = \emptyset$ whenever m, $n \in I$ and $s \leq m < n$.

(v) For $l \in \mathbb{N}$, $\mu^*_{Hr}(A_l \setminus D) \leq 2\gamma$. **P** Let $\epsilon > 0$. Then there is a $k \in \mathbb{N}$ such that $\operatorname{diam} C_{ni} \leq \epsilon$ whenever $n \in \mathbb{N}$ and $i > k$. Next, there is an $s \in \mathbb{N}$ such that

$$C_{ni} \subseteq D \text{ whenever } i \leq k,\ i \in J,\ n \in I \text{ and } s \leq n,$$

$$C_{mi} \cap C_{nj} = \emptyset \text{ whenever } i, j \leq k,\ i \notin J,\ m, n \in I \text{ and } s \leq m < n.$$

Take m, $n \in I$ such that $\max(l, s) \leq m < n$. Then

$$\begin{aligned}
A_l \setminus D &= \bigcup_{i\in\mathbb{N}} A_l \cap C_{mi} \setminus D \\
&\subseteq \bigcup_{i\leq k} (A_n \cap C_{mi} \setminus D) \cup \bigcup_{i>k} C_{mi} \\
&\subseteq \bigcup_{i\leq k, i\notin J} (A_n \cap C_{mi}) \cup \bigcup_{i>k} C_{mi} \\
&\subseteq \bigcup_{i\leq k, i\notin J, j\leq k} (C_{mi} \cap C_{nj}) \cup \bigcup_{j>k} C_{nj} \cup \bigcup_{i>k} C_{mi} \\
&= \bigcup_{j>k} C_{nj} \cup \bigcup_{i>k} C_{mi}.
\end{aligned}$$

Since $\operatorname{diam} C_{nj} \leq \epsilon$ and $\operatorname{diam} C_{mi} \leq \epsilon$ for all i, $j > k$,

$$\theta_{r\epsilon}(A_l \setminus D) \leq \textstyle\sum_{i=k+1}^{\infty} (\operatorname{diam} C_{ni})^r + \sum_{i=k+1}^{\infty} (\operatorname{diam} C_{mi})^r \leq 2\gamma.$$

This is true for every $\epsilon > 0$, so $\mu^*_{Hr}(A_l \setminus D) \leq 2\gamma$, as claimed. **Q**

(vi) This is true for each $l \in \mathbb{N}$. But this means that $\mu^*_{Hr}(A \setminus D) \leq 2\gamma$ (132Ae). Now $\theta_{r,6\delta}D \leq 5^r\gamma$. **P** For each $i \in \mathbb{N}$,

$$\begin{aligned}\operatorname{diam} D_i &\leq \lim_{n\in I, n\to\infty} \operatorname{diam} D_{ni}\\ &\leq \lim_{n\in I, n\to\infty} \operatorname{diam} C_{ni} + 4\alpha_i + 6\zeta_i = 5\alpha_i + 6\zeta_i \leq 6\delta.\end{aligned}$$

Next, for any $k \in \mathbb{N}$,

$$\textstyle\sum_{i=0}^k \alpha_i^r = \lim_{n\in I, n\to\infty} \sum_{i=0}^k (\operatorname{diam} C_{ni})^r \leq \gamma',$$

so

$$\sum_{i=0}^k (\operatorname{diam} D_i)^r \leq 5^r \sum_{i=0}^k (\alpha_i + 2\zeta_i)^r \leq 5^r (\sum_{i=0}^k \alpha_i^r + 2^{-i-1}(\gamma - \gamma'))$$

(by the choice of the ζ_i)

$$\leq 5^r\gamma.$$

Letting $k \to \infty$,

$$\theta_{r,6\delta}D \leq \textstyle\sum_{i=0}^\infty (\operatorname{diam} D_i)^r \leq 5^r\gamma.$$ **Q**

Putting these together,

$$\theta_{r,6\delta}A \leq \theta_{r,6\delta}D + \theta_{r,6\delta}(A \setminus D) \leq \theta_{r,6\delta}D + \mu^*_{Hr}(A \setminus D) \leq (5^r + 2)\gamma.$$

As γ is arbitrary, we have the preliminary result (a).

(b) Now let us turn to the sharp form (b). Once again, we may suppose that $\sup_{n\in\mathbb{N}} \theta_{r\delta}A_n$ is finite.

(i) Take γ, γ' such that $\sup_{n\in\mathbb{N}} \theta_{r\delta}A_n < \gamma' < \gamma$. As in (a)(i) above, we can find a family $\langle C_{ni}\rangle_{n,i\in\mathbb{N}}$ such that

$$A_n \subseteq \textstyle\bigcup_{i\in\mathbb{N}} C_{ni},$$

$$\operatorname{diam} C_{ni} \leq \delta \text{ for every } i \in \mathbb{N},$$

$$\textstyle\sum_{i=0}^\infty (\operatorname{diam} C_{ni})^r \leq \gamma'$$

for each n, and

$$\lim_{i\to\infty} \sup_{n\in\mathbb{N}} \operatorname{diam} C_{ni} = 0.$$

Replacing each C_{ni} by its closure if necessary, we may suppose that every C_{ni} is a Borel set.

Let $Q \subseteq X$ be a countable set which meets C_{ni} whenever n, $i \in \mathbb{N}$ and C_{ni} is not empty. This time, let $I \subseteq \mathbb{N}$ be an infinite set such that

$$\alpha_i = \lim_{n\in I, n\to\infty} \operatorname{diam} C_{ni} \text{ is defined in } [0, \delta] \text{ for every } i \in \mathbb{N},$$

$$\lim_{n\in I, n\to\infty} \rho(z, C_{ni}) \text{ is defined in } [0, \infty] \text{ for every } i \in \mathbb{N} \text{ and every } z \in Q.$$

(Take $\rho(z, \emptyset) = \infty$ if any of the C_{ni} are empty.) It will be helpful to note straight away that the limit $\lim_{n\in I, n\to\infty} \rho(x, C_{ni})$ is defined in $[0, \infty]$ for every $i \in \mathbb{N}$ and $x \in \overline{Q}$. **P** If $\lim_{n\in I, n\to\infty} \rho(y, C_{ni}) = \infty$ for any $z \in Q$, then $\lim_{n\in I, n\to\infty} \rho(x, C_{ni}) = \infty$, and we can stop. Otherwise, for any $\epsilon > 0$, there are a $z \in Q$ such that $\rho(x, z) \leq \epsilon$ and an $s \in \mathbb{N}$ such that C_{mi} is not empty and $|\rho(z, C_{mi}) - \rho(z, C_{ni})| \leq \epsilon$ whenever m, $n \in I \setminus s$; in which case $|\rho(x, C_{mi}) - \rho(x, C_{ni})| \leq 3\epsilon$ whenever m, $n \in I \setminus s$. As ϵ is arbitrary, $\lim_{n\in I, n\to\infty} \rho(x, C_{ni})$ is defined in $\mathbb{R}$. **Q**

Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbb{N}$ containing I, and for $i \in \mathbb{N}$ set

$$D_i = \{x : \lim_{n\to\mathcal{F}} \rho(x, C_{ni}) = 0\}.$$

Set $D = \bigcup_{i\in\mathbb{N}} \overline{D}_i$. (Actually it is easy to check that every D_i is closed.)

(ii) Set

$$A^* = \bigcup_{m\in\mathbb{N}} \bigcap_{n\in I\setminus m} \bigcup_{i\in\mathbb{N}} C_{ni} \setminus D;$$

note that A^* is a Borel set. For k, $m \in \mathbb{N}$, set

$$A^*_{km} = \bigcap_{n\in I\setminus m} \bigcup_{i\geq k} C_{ni}.$$

For fixed k, $\langle A^*_{km}\rangle_{m\in\mathbb{N}}$ is a non-decreasing sequence of sets. Also its union includes A^*. **P** Take $x \in A^*$.

(α)? If $x \notin \overline{Q}$, let $\epsilon > 0$ be such that $Q \cap B(x,\epsilon) = \emptyset$. Let $k \in \mathbb{N}$ be such that $\operatorname{diam} C_{ni} \leq \epsilon$ whenever $n \in \mathbb{N}$ and $i \geq k$; then $x \notin C_{ni}$ whenever $n \in \mathbb{N}$ and $i \geq k$. Let $m \in \mathbb{N}$ be such that $x \in \bigcup_{i\in\mathbb{N}} C_{ni}$ whenever $n \in I$ and $n \geq m$. Then $x \in \bigcup_{i<k} C_{ni}$ whenever $n \in I$ and $n \geq m$. But this means that there must be some $i < k$ such that $\{n : x \in C_{ni}\} \in \mathcal{F}$ and $x \in D_i \subseteq D$; which is impossible. **X**

(β) Thus $x \in \overline{Q}$, so $\lim_{n\in I, n\to\infty} \rho(x, C_{ni})$ is defined for each i (see (i) above), and must be greater than 0, since $x \notin D_i$. In particular, there is an $s \in \mathbb{N}$ such that $x \notin C_{ni}$ whenever $i < k$ and $n \in I \setminus s$; there is also an $m \in \mathbb{N}$ such that $x \in \bigcap_{n\in I\setminus m} \bigcup_{i\in\mathbb{N}} C_{ni}$; so that $x \in A^*_{k,\max(s,m)}$. As x is arbitrary, $A^* \subseteq \bigcup_{m\in\mathbb{N}} A^*_{km}$. **Q**

(iii) $\mu_{Hr}A^*$ is finite. **P** Take any $\epsilon > 0$. Let $k \in \mathbb{N}$ be such that $\operatorname{diam} C_{ni} \leq \epsilon$ whenever $i \geq k$ and $n \in \mathbb{N}$. For any $m \in I$, $\theta_{r\epsilon}A^*_{km} \leq \sum_{i=k}^{\infty} (\operatorname{diam} C_{mi})^r \leq \gamma$. By (a),

$$\begin{aligned}\theta_{r,6\epsilon}A^* &\leq (5^r+2)\sup_{m\in\mathbb{N}} \theta_{r\epsilon}A^*_{km}\\ &= (5^r+2)\sup_{m\in I} \theta_{r\epsilon}A^*_{km} \leq (5^r+2)\gamma.\end{aligned}$$

As ϵ is arbitrary, $\mu_{Hr}A^* \leq (5^r+2)\gamma < \infty$. **Q**

(iv) Actually, $\mu_{Hr}A^* \leq \gamma' - \sum_{i=0}^{\infty} \alpha_i^r$. **P?** Suppose, if possible, otherwise. Take β such that $\gamma' - \sum_{i=0}^{\infty} \alpha_i^r < \beta < \mu_{Hr}A^*$. For $x \in A^*$ and $k \in \mathbb{N}$, set $f_k(x) = \min\{n : n \in I,\ x \in A^*_{kn}\}$; then $\langle f_k\rangle_{k\in\mathbb{N}}$ is a non-decreasing sequence of Borel measurable functions from A^* to $\mathbb{N}$. Choose $\langle s_k\rangle_{k\in\mathbb{N}}$ inductively so that

$$\mu_{Hr}\{x : x \in A^*,\ f_j(x) \leq s_j \text{ for every } j \leq k\} \geq \beta$$

for every $k \in \mathbb{N}$. Set $\tilde{A} = \{x : x \in A^*,\ f_j(x) \leq s_j \text{ for every } j \in \mathbb{N}\}$; because $\mu_{Hr}A^*$ is finite, $\mu_{Hr}\tilde{A} \geq \beta$. Take $\epsilon > 0$ such that $\theta_{r\epsilon}\tilde{A} > \gamma' - \sum_{i=0}^{\infty} \alpha_i^r$. Let $k \in \mathbb{N}$ be such that $\theta_{r\epsilon}\tilde{A} + \sum_{i=0}^{k-1} \alpha_i^r > \gamma'$ and $\operatorname{diam} C_{ni} \leq \epsilon$ whenever $n \in \mathbb{N}$ and $i \geq k$. Take $n \in I$ such that $n \geq s_j$ for every $j < k$ and $\theta_{r\epsilon}\tilde{A} + \sum_{i=0}^{k-1} (\operatorname{diam} C_{ni})^r > \gamma'$. If $x \in \tilde{A}$, then $x \in A^*_{kn} \subseteq \bigcup_{i\geq k} C_{ni}$, so $\theta_{r\epsilon}\tilde{A} \leq \sum_{i=k}^{\infty} (\operatorname{diam} C_{ni})^r$; but this means that $\sum_{i=0}^{\infty} (\operatorname{diam} C_{ni})^r > \gamma'$, contrary to the choice of the C_{ni}. **XQ**

(v) Now observe that

$$A \subseteq \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} \bigcup_{i\in\mathbb{N}} C_{ni} \subseteq A^* \cup D.$$

Moreover, for any $i \in \mathbb{N}$, $\operatorname{diam} \overline{D}_i \leq \alpha_i \leq \delta$. **P** If x, $y \in D_i$ then for every $\epsilon > 0$

$$\rho(x, C_{ni}) \leq \epsilon, \quad \rho(y, C_{ni}) \leq \epsilon, \quad \operatorname{diam} C_{ni} \leq \alpha_i + \epsilon$$

for all but finitely many $n \in I$. So $\rho(x,y) \leq \alpha_i + 3\epsilon$. As x, y and ϵ are arbitrary, $\operatorname{diam} \overline{D}_i = \operatorname{diam} D_i \leq \alpha_i$. Of course $\alpha_i \leq \delta$ because $\operatorname{diam} C_{ni} \leq \delta$ for every n. **Q**

Now

$$\theta_{r\delta}D \leq \sum_{i=0}^{\infty} (\operatorname{diam} \overline{D}_i)^r \leq \sum_{i=0}^{\infty} \alpha_i^r.$$

Putting this together with (iv),

$$\theta_{r\delta}A \leq \theta_{r\delta}D + \theta_{r\delta}A^* \leq \theta_{r\delta}D + \mu_{Hr}A^* \leq \gamma.$$

As γ and γ' are arbitrary,

$$\theta_{r\delta}A \leq \sup_{n\in\mathbb{N}} \theta_{r\delta}A_n,$$

as required.

471H Corollary Let (X,ρ) be a metric space, and $r>0$. For $A\subseteq X$, set

$$\theta_{r\infty}A=\inf\{\textstyle\sum_{n=0}^{\infty}(\operatorname{diam}D_n)^r : \langle D_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of subsets of } X \text{ covering } A\}.$$

Then $\theta_{r\infty}$ is a capacity on X.

proof (a) Of course $0\le\theta_{r\infty}A\le\theta_{r\infty}B$ whenever $A\subseteq B\subseteq X$.

(b) Suppose that $\langle A_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of subsets of A with union A. By (a), $\gamma=\lim_{n\to\infty}\theta_{r\infty}A_n$ is defined and less than or equal to $\theta_{r\infty}A$. If $\gamma=\infty$, of course it is equal to $\theta_{r\infty}A$. Otherwise, take $\delta=(\gamma+1)^{1/r}$. For n, $k\in\mathbb{N}$ there is a sequence $\langle D_{nki}\rangle_{i\in\mathbb{N}}$ of sets, covering A_n, such that $\sum_{i=0}^{\infty}(\operatorname{diam}D_{nki})^r\le\gamma+2^{-k}$. But in this case $\operatorname{diam}D_{nki}\le\delta$ for all n, k and i, so the D_{nki} witness that $\theta_{r\delta}A_n\le\gamma$. By 471Gb, $\gamma\ge\theta_{r\delta}A\ge\theta_{r\infty}A$ and again we have $\gamma=\theta_{r\infty}A$.

(c) Let $K\subseteq X$ be any set, and suppose that $\gamma>\theta_{r\infty}K$. Let $\langle D_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets, covering K, such that $\sum_{n=0}^{\infty}(\operatorname{diam}D_n)^r<\gamma$. Let $\langle\epsilon_n\rangle_{n\in\mathbb{N}}$ be a sequence of strictly positive real numbers such that $\sum_{n=0}^{\infty}(\operatorname{diam}D_n+2\epsilon_n)^r\le\gamma$. Set $G_n=\{x:\rho(x,D_n)<\epsilon_n\}$ for each n; then G_n is open and $\operatorname{diam}G_n\le\operatorname{diam}D_n+2\epsilon_n$. So $G=\bigcup_{n\in\mathbb{N}}G_n$ is an open set including K, and $\langle G_n\rangle_{n\in\mathbb{N}}$ witnesses that $\theta_{r\infty}G\le\gamma$. As K and γ are arbitrary, condition (iii) of 432J is satisfied and $\theta_{r\infty}$ is a capacity.

Remark $\theta_{r\infty}$ is **r-dimensional Hausdorff capacity** on X.

471I Theorem Let (X,ρ) be a metric space, and $r>0$. Write μ_{Hr} for r-dimensional Hausdorff measure on X. If $A\subseteq X$ is analytic, then $\mu_{Hr}A$ is defined and equal to $\sup\{\mu_{Hr}K:K\subseteq A \text{ is compact}\}$.

proof (a) Before embarking on the main line of the proof, it will be convenient to set out a preliminary result. For $\delta>0$, $n\in\mathbb{N}$, $B\subseteq X$ set

$$\theta_{r\delta}^{(n)}(B)=\inf\{\textstyle\sum_{i=0}^{n}(\operatorname{diam}D_i)^r : B\subseteq\bigcup_{i\le n}D_i,\ \operatorname{diam}D_i\le\delta \text{ for every } i\le n\},$$

taking $\inf\emptyset=\infty$ as usual. Then $\theta_{r\delta}(B)\le\theta_{r\delta}^{(n)}(B)$ for every n. Now the point is that $\theta_{r\delta}^{(n)}(B)=\sup\{\theta_{r\delta}^{(n)}(I):I\subseteq B \text{ is finite}\}$. **P** Set $\gamma=\sup_{I\in[B]^{<\omega}}\theta_{r\delta}^{(n)}(I)$. Of course $\gamma\le\theta_{r\delta}^{(n)}(B)$. If $\gamma=\infty$ there is nothing more to say. Otherwise, take any $\gamma'>\gamma$. For each $I\in[B]^{<\omega}$, we have a function $f_I:I\to\{0,\dots,n\}$ such that $\sum_{i\in J}\rho(x_i,y_i)^r\le\gamma'$ whenever $J\subseteq\{0,\dots,n\}$ and x_i, $y_i\in I$ and $f_I(x_i)=f_I(y_i)=i$ for every $i\in J$, while $\rho(x,y)\le\delta$ whenever x, $y\in I$ and $f_I(x)=f_I(y)$. Let $\mathcal{F}$ be an ultrafilter on $[B]^{<\omega}$ such that $\{I:x\in I\in[B]^{<\omega}\}\in\mathcal{F}$ for every $x\in B$ (4A1Ia). Then for every $x\in B$ there is an $f(x)\in\{0,\dots,n\}$ such that $\{I:x\in I\in[B]^{<\omega},\ f_I(x)=f(x)\}\in\mathcal{F}$. Set $D_i=f^{-1}[\{i\}]$ for $i\le n$. If x, $y\in B$ and $f(x)=f(y)$, there is an $I\in[B]^{<\omega}$ containing both x and y such that $f_I(x)=f(x)=f(y)=f_I(y)$, so that $\rho(x,y)\le\delta$; thus $\operatorname{diam}D_i\le\delta$ for each i. If $J\subseteq\{0,\dots,n\}$ and for each $i\in J$ we take x_i, $y_i\in D_i$, then there is an $I\in[B]^{<\omega}$ such that $f_I(x_i)=f_I(y_i)=i$ for every $i\in J$, so $\sum_{i\in J}\rho(x_i,y_i)^r\le\gamma'$. This means that $\sum_{i\le n}(\operatorname{diam}D_i)^r\le\gamma'$, so that $\theta_{r\delta}^{(n)}(B)\le\gamma'$. As γ' is arbitrary, $\theta_{r\delta}^{(n)}(B)\le\gamma$, as claimed. **Q**

(b) Now let us turn to the set A. Because A is Souslin-F (422Ha), μ_{Hr} measures A (471Da, 471Dd). Set $\gamma=\sup\{\mu_{Hr}(K):K\subseteq A \text{ is compact}\}$.

? Suppose, if possible, that $\mu_{Hr}(A)>\gamma$. Take $\gamma'\in\,]\gamma,\mu_{Hr}(A)[$. Let $\delta>0$ be such that $\gamma'<\theta_{r\delta}(A)$. Let $f:\mathbb{N}^{\mathbb{N}}\to A$ be a continuous surjection. For $\sigma\in S^*=\bigcup_{n\in\mathbb{N}}\mathbb{N}^n$, set

$$F_\sigma=\{\phi:\phi\in\mathbb{N}^{\mathbb{N}},\ \phi(i)\le\sigma(i) \text{ for every } i<\#(\sigma)\},$$

so that $f[F_\emptyset]=A$. Now choose $\psi\in\mathbb{N}^{\mathbb{N}}$ and a sequence $\langle I_n\rangle_{n\in\mathbb{N}}$ of finite subsets of $\mathbb{N}^{\mathbb{N}}$ inductively, as follows. Given that $I_j\subseteq F_{\psi\restriction n}$ for every $j<n$ and that $\theta_{r\delta}(f[F_{\psi\restriction n}])>\gamma'$, then $\theta_{r\delta}^{(n)}(f[F_{\psi\restriction n}])>\gamma'$, so by (a) above there is a finite subset I_n of $F_{\psi\restriction n}$ such that $\theta_{r\delta}^{(n)}(f[I_n])\ge\gamma'$. Next,

$$\lim_{i\to\infty}\theta_{r\delta}f[F_{(\psi\restriction n)^\frown i}]=\theta_{r\delta}\bigcup_{i\in\mathbb{N}}f[F_{(\psi\restriction n)^\frown i}]$$

(by 471G)

$$=\theta_{r\delta}f[F_{\psi\restriction n}]>\gamma',$$

so we can take $\psi(n)$ such that $\bigcup_{j\le n} I_j \subseteq F_{\psi\restriction n+1}$ and $\theta_{r\delta} f[F_{\psi\restriction n+1}] > \gamma'$, and continue.

At the end of the induction, set $K = \{\phi : \phi \le \psi\}$. Then $f[K]$ is a compact subset of A, and $I_n \subseteq K$ for every $n \in \mathbb{N}$, so

$$\theta^{(n)}_{r\delta}(f[K]) \ge \theta^{(n)}_{r\delta}(f[I_n]) \ge \gamma'$$

for every $n \in \mathbb{N}$. On the other hand, $\mu_{Hr}(f[K]) \le \gamma$, so there is a sequence $\langle D_i\rangle_{i\in\mathbb{N}}$ of sets, covering $f[K]$, all of diameter less than δ, such that $\sum_{i=0}^{\infty}(\operatorname{diam} D_i)^r < \gamma'$. Enlarging the D_i slightly if need be, we may suppose that they are all open. But in this case there is some finite n such that $K \subseteq \bigcup_{i\le n} D_i$, and $\theta^{(n)}_{r\delta}(K) \le \sum_{i=0}^{n}(\operatorname{diam} D_i)^r < \gamma'$; which is impossible. **X**

This contradiction shows that $\mu_{Hr}(A) = \gamma$, as required.

471J Proposition Let (X,ρ) and (Y,σ) be metric spaces, and $f : X \to Y$ a function which is **γ-Lipschitz**, that is, such that $\sigma(f(x),f(y)) \le \gamma\rho(x,y)$ for all x, $y \in X$, where $\gamma \ge 0$. If $r > 0$ and $\theta^{(X)}_r$, $\theta^{(Y)}_r$ are the r-dimensional Hausdorff outer measures on X and Y respectively, then $\theta^{(Y)}_r f[A] \le \gamma^r \theta^{(X)}_r A$ for every $A \subseteq X$.

proof (Compare 264G.) Let $\delta > 0$. Set $\eta = \delta/(1+\gamma)$ and consider $\theta^{(X)}_{r\eta} : \mathcal{P}X \to [0,\infty]$, defined as in 471A. We know that $\theta^{(X)}_r A \ge \theta^{(X)}_{r\eta} A$, so there is a sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of sets, all of diameter at most η, covering A, with $\sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r \le \theta^{(X)}_r A + \delta$. Now $\phi[A] \subseteq \bigcup_{n\in\mathbb{N}} \phi[D_n]$ and

$$\operatorname{diam}\phi[D_n] \le \gamma \operatorname{diam} D_n \le \gamma\eta \le \delta$$

for every n. Consequently

$$\theta^{(Y)}_{r\delta}(\phi[A]) \le \textstyle\sum_{n=0}^{\infty}(\operatorname{diam}\phi[D_n])^r \le \sum_{n=0}^{\infty}\gamma^r(\operatorname{diam} D_n)^r \le \gamma^r(\theta^{(X)}_r + \delta),$$

and

$$\theta^{(Y)}_r(\phi[A]) = \lim_{\delta\downarrow 0}\theta^{(Y)}_{r\delta}(\phi[A]) \le \gamma^r\theta^{(X)}_r A,$$

as claimed.

471K Lemma Let (X,ρ) be a metric space, and $r > 0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X. If $A \subseteq X$, then $\mu_{Hr}A = 0$ iff for every $\epsilon > 0$ there is a countable family $\mathcal{D}$ of sets, covering A, such that $\sum_{D\in\mathcal{D}}(\operatorname{diam} D)^r \le \epsilon$.

proof If $\mu_{Hr}A = 0$ and $\epsilon > 0$, then, in the language of 471A, $\theta_{r1}A \le \epsilon$, so there is a sequence $\langle D_n\rangle_{n\in\mathbb{N}}$ of sets covering A such that $\sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r \le \epsilon$.

If the condition is satisfied, then for any ϵ, $\delta > 0$ there is a countable family $\mathcal{D}$ of sets, covering A, such that $\sum_{D\in\mathcal{D}}(\operatorname{diam} D)^r \le \min(\epsilon,\delta^r)$. If $\mathcal{D}$ is infinite, enumerate it as $\langle D_n\rangle_{n\in\mathbb{N}}$; if it is finite, enumerate it as $\langle D_n\rangle_{n<m}$ and set $D_n = \emptyset$ for $n \ge m$. Now $A \subseteq \bigcup_{n\in\mathbb{N}} D_n$ and $\operatorname{diam} D_n \le \delta$ for every $n \in \mathbb{N}$, so $\theta_{r\delta}A \le \sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r \le \epsilon$. As ϵ is arbitrary, $\theta_{r\delta}A = 0$; as δ is arbitrary, $\theta_r A = 0$; it follows at once that $\mu_{Hr}A$ is defined and is zero (113Xa).

471L Proposition Let (X,ρ) be a metric space and $0 < r < s$. If $A \subseteq X$ is such that $\mu^*_{Hr}A$ is finite, then $\mu^*_{Hs}A = 0$.

proof Let $\epsilon > 0$. Let $\delta > 0$ be such that $\delta^{s-r}(1 + \mu^*_{Hr}A) \le \epsilon$. Then there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of sets of diameter at most δ such that $A \subseteq \bigcup_{n\in\mathbb{N}} A_n$ and $\sum_{n=0}^{\infty}(\operatorname{diam} A_n)^r \le 1 + \mu^*_{Hr}A$. But now, by the choice of δ, $\sum_{n=0}^{\infty}(\operatorname{diam} A_n)^s \le \epsilon$. As ϵ is arbitrary, $\mu_{Hs}A = 0$, by 471K.

471M There is a generalization of the density theorems of §§223 and 261 for general Hausdorff measures, which (as one expects) depends on a kind of Vitali theorem. I will use the following notation for the next few paragraphs.

Definition If (X,ρ) is a metric space and $A \subseteq X$, write $A^{\sim}$ for $\{x : x \in X, \rho(x,A) \le 2\operatorname{diam} A\}$, where $\rho(x,A) = \inf_{y\in A}\rho(x,y)$. (Following the conventions of 471A, $\emptyset^{\sim} = \emptyset$.)

471N Lemma Let (X,ρ) be a metric space. Let $\mathcal{F}$ be a family of subsets of X such that $\{\operatorname{diam} F : F \in \mathcal{F}\}$ is bounded. Set

$$Y = \bigcap_{\delta>0} \bigcup \{F : F \in \mathcal{F}, \operatorname{diam} F \leq \delta\}.$$

Then there is a disjoint family $\mathcal{I} \subseteq \mathcal{F}$ such that

(i) $\bigcup \mathcal{F} \subseteq \bigcup_{F\in\mathcal{I}} F^\sim$;

(ii) $Y \subseteq \overline{\bigcup \mathcal{J}} \cup \bigcup_{F\in\mathcal{I}\setminus\mathcal{J}} F^\sim$ for every finite $\mathcal{J} \subseteq \mathcal{I}$.

proof (a) Let γ be an upper bound for $\{\operatorname{diam} F : F \in \mathcal{F}\}$. Choose $\langle \mathcal{I}_n\rangle_{n\in\mathbb{N}}$, $\langle \mathcal{J}_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. $\mathcal{I}_0 = \emptyset$. Given $\mathcal{I}_n$, set $\mathcal{F}'_n = \{F : F \in \mathcal{F}, \operatorname{diam} F \geq 2^{-n}\gamma, F \cap \bigcup \mathcal{I}_n = \emptyset\}$, and let $\mathcal{J}_n \subseteq \mathcal{F}'_n$ be a maximal disjoint set; now set $\mathcal{I}_{n+1} = \mathcal{I}_n \cup \mathcal{J}_n$, and continue.

At the end of the induction, set

$$\mathcal{I}' = \bigcup_{n\in\mathbb{N}} \mathcal{I}_n, \quad \mathcal{I} = \mathcal{I}' \cup \{\{x\} : x \in F \setminus \bigcup \mathcal{I}', \{x\} \in \mathcal{F}\}.$$

The construction ensures that every $\mathcal{I}_n$ is a disjoint subset of $\mathcal{F}$, so $\mathcal{I}'$ and $\mathcal{I}$ are also disjoint subfamilies of $\mathcal{F}$.

(b) ? Suppose, if possible, that there is a point x in $\bigcup \mathcal{F} \setminus \bigcup_{F\in\mathcal{I}} F^\sim$. Let $F \in \mathcal{F}$ be such that $x \in F$. Since $x \notin \bigcup \mathcal{I}'$ and $\{x\} \notin \mathcal{I}$, $\{x\} \notin \mathcal{F}$, and $\operatorname{diam} F > 0$; let $n \in \mathbb{N}$ be such that $2^{-n}\gamma \leq \operatorname{diam} F \leq 2^{-n+1}\gamma$. If $F \notin \mathcal{F}'_n$, there is a $D \in \mathcal{I}_n$ such that $F \cap D \neq \emptyset$; otherwise, since $\mathcal{J}_n$ is maximal and $F \notin \mathcal{J}_n$, there is a $D \in \mathcal{J}_n$ such that $F \cap D \neq \emptyset$. In either case, we have a $D \in \mathcal{I}$ such that $F \cap D \neq \emptyset$ and $\operatorname{diam} F \leq 2 \operatorname{diam} D$. But in this case $\rho(x, D) \leq \operatorname{diam} F \leq 2 \operatorname{diam} D$ and $x \in D^\sim$, which is impossible. **X**

(c) ? Suppose, if possible, that there are a point $x \in Y$ and a finite set $\mathcal{J} \subseteq \mathcal{I}$ such that $x \notin \overline{\bigcup \mathcal{J}} \cup \bigcup_{F\in\mathcal{I}\setminus\mathcal{J}} F^\sim$. Then there is an $F \in \mathcal{F}$ such that $x \in F$ and $\operatorname{diam} F < \rho(x, \bigcup \mathcal{J})$, so that $F \cap \bigcup \mathcal{J} = \emptyset$. As in (b), F cannot be $\{x\}$, and there must be an $n \in \mathbb{N}$ such that $2^{-n}\gamma < \operatorname{diam} F \leq 2^{-n+1}\gamma$. As in (b), there must be a $D \in \mathcal{I}_{n+1}$ such that $F \cap D \neq \emptyset$, so that $x \in D^\sim$; and as D cannot belong to $\mathcal{J}$, we again have a contradiction. **X**

471O Lemma Let (X,ρ) be a metric space, and $r > 0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X. Suppose that A, $\mathcal{F}$ are such that

(i) $\mathcal{F}$ is a family of closed subsets of X such that $\sum_{n=0}^\infty (\operatorname{diam} F_n)^r$ is finite for every disjoint sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$,

(ii) for every $x \in A$, $\delta > 0$ there is an $F \in \mathcal{F}$ such that $x \in F$ and $0 < \operatorname{diam} F \leq \delta$.

Then there is a countable disjoint family $\mathcal{I} \subseteq \mathcal{F}$ such that $A \setminus \bigcup \mathcal{I}$ has zero r-dimensional Hausdorff measure.

proof Replacing $\mathcal{F}$ by $\{F : F \in \mathcal{F}, 0 < \operatorname{diam} F \leq 1\}$ if necessary, we may suppose that $\sup_{F\in\mathcal{F}} \operatorname{diam} F$ is finite and that $\operatorname{diam} F > 0$ for every $F \in \mathcal{F}$. Take a disjoint family $\mathcal{I} \subseteq \mathcal{F}$ as in 471N. If $\mathcal{I}$ is finite, then $A \subseteq Y \subseteq \bigcup \mathcal{I}$, where Y is defined as in 471N, so we can stop. Otherwise, hypothesis (i) tells us that $\{F : F \in \mathcal{I}, \operatorname{diam} F \geq \delta\}$ is finite for every $\delta > 0$, so $\mathcal{I}$ is countable; enumerate it as $\langle F_n\rangle_{n\in\mathbb{N}}$; we must have $\sum_{n=0}^\infty (\operatorname{diam} F_n)^r < \infty$. Since $\operatorname{diam} F_n^\sim \leq 5 \operatorname{diam} F_n$ for every n, $\sum_{n=0}^\infty (\operatorname{diam} F_n^\sim)^r$ is finite, and $\inf_{n\in\mathbb{N}} \sum_{i=n}^\infty (\operatorname{diam} F_i^\sim)^r = 0$. But now observe that the construction ensures that $A \setminus \bigcup \mathcal{I} \subseteq \bigcup_{i\geq n} F_i^\sim$ for every $n \in \mathbb{N}$. By 471K, $\mu_{Hr}(A \setminus \bigcup \mathcal{I}) = 0$, as required.

471P Theorem Let (X,ρ) be a metric space, and $r > 0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X. Suppose that $A \subseteq X$ and $\mu^*_{Hr} A < \infty$.

(a) $\lim_{\delta\downarrow 0} \sup\{\frac{\mu^*_{Hr}(A\cap D)}{(\operatorname{diam} D)^r} : x \in D, 0 < \operatorname{diam} D \leq \delta\} = 1$ for μ_{Hr}-almost every $x \in A$.

(b) $\limsup_{\delta\downarrow 0} \frac{\mu^*_{Hr}(A\cap B(x,\delta))}{\delta^r} \geq 1$ for μ_{Hr}-almost every $x \in A$. So

$$2^{-r} \leq \limsup_{\delta\downarrow 0} \frac{\mu^*_{Hr}(A\cap B(x,\delta))}{(\operatorname{diam} B(x,\delta))^r} \leq 1$$

for μ_{Hr}-almost every $x \in A$.

(c) If A is measured by μ_{Hr}, then

$$\lim_{\delta\downarrow 0}\sup\{\tfrac{\mu^*_{Hr}(A\cap D)}{(\operatorname{diam} D)^r} : x\in D,\ 0<\operatorname{diam} D\le\delta\}=0$$

for μ_{Hr}-almost every $x\in X\setminus A$.

proof (a)(i) Note first that as the quantities

$$\sup\{\tfrac{\mu^*_{Hr}(A\cap D)}{(\operatorname{diam} D)^r} : x\in D,\ 0<\operatorname{diam} D\le\delta\}$$

decrease with δ, the limit is defined in $[0,\infty]$ for every $x\in X$. Moreover, since $\operatorname{diam} D=\operatorname{diam}\overline{D}$ and $\mu^*_{Hr}(A\cap\overline{D})\ge\mu^*_{Hr}(A\cap D)$ for every D,

$$\begin{aligned}\sup\{\tfrac{\mu^*_{Hr}(A\cap D)}{(\operatorname{diam} D)^r} &: x\in D\subseteq X,\ 0<\operatorname{diam} D\le\delta\}\\ &=\sup\{\tfrac{\mu^*_{Hr}(A\cap F)}{(\operatorname{diam} F)^r} : F\subseteq X \text{ is closed},\ x\in F,\ 0<\operatorname{diam} F\le\delta\}\end{aligned}$$

for every x and δ.

(ii) Fix ϵ for the moment, and set

$$A_\epsilon=\{x : x\in A,\ \lim_{\delta\downarrow 0}\sup\{\tfrac{\mu^*_{Hr}(A\cap D)}{(\operatorname{diam} D)^r} : x\in D,\ 0<\operatorname{diam} D\le\delta\}>1+\epsilon\}.$$

Then $\theta_{r\eta}(A)\le\mu^*_{Hr}A-\frac{\epsilon}{1+\epsilon}\mu^*_{Hr}A_\epsilon$ for every $\eta>0$, where $\theta_{r\eta}$ is defined in 471A. **P** Let $\mathcal{F}$ be the family

$$\{F : F\subseteq X \text{ is closed},\ 0<\operatorname{diam} F\le\eta,\ (1+\epsilon)(\operatorname{diam} F)^r\le\mu^*_{Hr}(A\cap F)\}.$$

Then every member of A_ϵ belongs to sets in $\mathcal{F}$ of arbitrarily small diameter. Also, if $\langle F_n\rangle_{n\in\mathbb{N}}$ is any disjoint sequence in $\mathcal{F}$,

$$\textstyle\sum_{n=0}^\infty(\operatorname{diam} F_n)^r\le\sum_{n=0}^\infty\mu^*_{Hr}(A\cap F_n)\le\mu^*_{Hr}A<\infty$$

because every F_n, being closed, is measured by μ_{Hr}. (If you like, $F_n\cap A$ is measured by the subspace measure on A for every n.) So 471O tells us that there is a countable disjoint family $\mathcal{I}\subseteq\mathcal{F}$ such that $A_\epsilon\setminus\bigcup\mathcal{I}$ is negligible, and $\mu^*_{Hr}A_\epsilon=\mu^*_{Hr}(A_\epsilon\cap\bigcup\mathcal{I})$.

Because $\theta_{r\eta}$ is an outer measure and $\theta_{r\eta}\le\mu^*_{Hr}$,

$$\theta_{r\eta}A\le\theta_{r\eta}(A\cap\bigcup\mathcal{I})+\theta_{r\eta}(A\setminus\bigcup\mathcal{I})\le\sum_{F\in\mathcal{I}}(\operatorname{diam} F)^r+\mu^*_{Hr}(A\setminus\bigcup\mathcal{I})$$

(because $\mathcal{I}$ is countable)

$$\begin{aligned}&\le\tfrac{1}{1+\epsilon}\mu^*_{Hr}(A\cap\bigcup\mathcal{I})+\mu^*_{Hr}(A\setminus\bigcup\mathcal{I})=\mu^*_{Hr}A-\tfrac{\epsilon}{1+\epsilon}\mu^*_{Hr}(A\cap\bigcup\mathcal{I})\\ &\le\mu^*_{Hr}A-\tfrac{\epsilon}{1+\epsilon}\mu^*_{Hr}(A_\epsilon\cap\bigcup\mathcal{I})=\mu^*_{Hr}A-\tfrac{\epsilon}{1+\epsilon}\mu^*_{Hr}A_\epsilon,\end{aligned}$$

as claimed. **Q**

(iii) Taking the supremum as $\eta\downarrow 0$, $\mu^*_{Hr}A\le\mu^*_{Hr}A-\frac{\epsilon}{1+\epsilon}\mu^*_{Hr}A_\epsilon$ and $\mu_{Hr}A_\epsilon=0$.

This is true for any $\epsilon>0$. But

$$\{x : x\in A,\ \lim_{\delta\downarrow 0}\sup\{\tfrac{\mu^*_{Hr}(A\cap D)}{(\operatorname{diam} D)^r} : x\in D,\ 0<\operatorname{diam} D\le\delta\}>1\}$$

is just $\bigcup_{n\in\mathbb{N}}A_{2^{-n}}$, so is negligible.

(iv) Next, for $0<\epsilon\le 1$, set

$$\begin{aligned}A'_\epsilon=\{x : x\in A,\ &\mu^*_{Hr}(A\cap D)\le(1-\epsilon)(\operatorname{diam} D)^r\\ &\text{whenever } x\in D \text{ and } 0<\operatorname{diam} D\le\epsilon\}.\end{aligned}$$

Then A'_ϵ is negligible. **P** Let $\langle D_n\rangle_{n\in\mathbb{N}}$ be any sequence of sets of diameter at most ϵ covering A'_ϵ. Set $K = \{n : D_n \cap A'_\epsilon \neq \emptyset\}$. Then

$$\begin{aligned}\mu^*_{Hr}A'_\epsilon &\leq \sum_{n\in K} \mu^*_{Hr}(A\cap D_n)\\ &\leq (1-\epsilon)\sum_{n\in K}(\operatorname{diam} D_n)^r \leq (1-\epsilon)\sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r.\end{aligned}$$

As $\langle D_n\rangle_{n\in\mathbb{N}}$ is arbitrary,

$$\mu^*_{Hr}A'_\epsilon \leq (1-\epsilon)\theta_{r\epsilon}A'_\epsilon \leq (1-\epsilon)\mu^*_{Hr}A'_\epsilon,$$

and $\mu^*_{Hr}A'_\epsilon$ (being finite) must be zero. **Q**

This means that

$$\{x : x\in A,\ \lim_{\delta\downarrow 0}\sup\{\tfrac{\mu^*_{Hr}(A\cap D)}{(\operatorname{diam} D)^r} : x\in D,\ 0<\operatorname{diam} D\leq\delta\} < 1\} \subseteq \textstyle\bigcup_{n\in\mathbb{N}} A'_{2^{-n}}$$

is also negligible, and we have the result.

(b) We need a slight modification of the argument in (a)(iv). This time, for $0<\epsilon\leq 1$, set

$$\tilde{A}'_\epsilon = \{x : x\in A,\ \mu^*_{Hr}(A\cap B(x,\delta)) \leq (1-\epsilon)\delta^r \text{ whenever } 0<\delta\leq\epsilon\}.$$

Then $\mu^*_{Hr}\tilde{A}'_\epsilon \leq \epsilon$. **P** Note first that, as $\mu_{Hr}\{x\} = 0$ for every x, $\mu^*_{Hr}(A\cap B(x,\delta)) \leq (1-\epsilon)\delta^r$ whenever $x\in\tilde{A}'_\epsilon$ and $0\leq\delta\leq\epsilon$. Let $\langle D_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets of diameter at most ϵ covering $\tilde{A}'_\epsilon$. Set $K = \{n : D_n\cap\tilde{A}'_\epsilon\neq\emptyset\}$, and for $n\in K$ choose $x_n\in D_n\cap\tilde{A}'_\epsilon$ and set $\delta_n = \operatorname{diam} D_n$. Then $D_n\subseteq B(x_n,\delta_n)$ and $\delta_n\leq\epsilon$ for each n, so $\tilde{A}'_\epsilon\subseteq\bigcup_{n\in K}B(x_n,\delta_n)$ and

$$\begin{aligned}\mu^*_{Hr}\tilde{A}'_\epsilon &\leq \sum_{n\in K}\mu^*_{Hr}(\tilde{A}'_\epsilon\cap B(x_n,\delta_n))\\ &\leq \sum_{n\in K}(1-\epsilon)\delta_n^r \leq (1-\epsilon)\sum_{n=0}^{\infty}(\operatorname{diam} D_n)^r.\end{aligned}$$

As $\langle D_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mu^*_{Hr}\tilde{A}'_\epsilon\leq(1-\epsilon)\mu^*_{Hr}\tilde{A}'_\epsilon$ and $\tilde{A}'_\epsilon$ must be negligible. **Q**

Now

$$\{x : x\in A,\ \limsup_{\delta\downarrow 0}\tfrac{\mu^*_{Hr}(A\cap B(x,\delta))}{\delta^r} < 1\} = \textstyle\bigcup_{n\in\mathbb{N}}\tilde{A}'_{2^{-n}}$$

is negligible. As for the second formula, we need note only that $\operatorname{diam} B(x,\delta)\leq 2\delta$ for every $x\in X$, $\delta>0$ to obtain the first inequality, and apply (a) to obtain the second.

(c) Let $\epsilon>0$. This time, write $\tilde{A}_\epsilon$ for

$$\{x : x\in X,\ \lim_{\delta\downarrow 0}\sup\{\tfrac{\mu^*_{Hr}(A\cap D)}{(\operatorname{diam} D)^r} : x\in D,\ 0<\operatorname{diam} D\leq\delta\} > \epsilon\}.$$

Let $E\subseteq A$ be a closed set such that $\mu(A\setminus E)\leq\epsilon^2$ (471De). For $\eta>0$, let $\mathcal{F}_\eta$ be the family

$$\{F : F\subseteq X\setminus E \text{ is closed},\ 0<\operatorname{diam} F\leq\eta,\ \mu_{Hr}(A\cap F)\geq\epsilon(\operatorname{diam} F)^r\}.$$

Just as in (a) above, every point in $\tilde{A}_\epsilon\setminus E$ belongs to members of $\mathcal{F}_\eta$ of arbitrarily small diameter. If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence in $\mathcal{F}_\eta$,

$$\textstyle\sum_{n=0}^{\infty}(\operatorname{diam} F)^r \leq \frac{1}{\epsilon}\mu_{Hr}(A\setminus E)\leq\epsilon$$

is finite. There is therefore a countable disjoint family $\mathcal{I}_\eta\subseteq\mathcal{F}_\eta$ such that $\mu_{Hr}((\tilde{A}_\epsilon\setminus E)\setminus\bigcup\mathcal{I}_\eta) = 0$. If $\theta_{r\eta}$ is the outer measure defined in 471A, we have

$$\begin{aligned}\theta_{r\eta}(\tilde{A}_\epsilon\setminus A) &\leq \theta_{r\eta}(\textstyle\bigcup\mathcal{I}_\eta) + \theta_{r\eta}(\tilde{A}_\epsilon\setminus(E\cup\bigcup\mathcal{I}_\eta))\\ &\leq \sum_{F\in\mathcal{I}_\eta}(\operatorname{diam} F)^r + \mu^*_{Hr}(\tilde{A}_\epsilon\setminus(E\cup\textstyle\bigcup\mathcal{I}_\eta)) \leq \epsilon.\end{aligned}$$

As η is arbitrary, $\mu^*_{Hr}(\tilde{A}_\epsilon \setminus A) \leq \epsilon$. But now

$$\{x : x \in X \setminus A,\ \lim_{\delta \downarrow 0} \sup\{\tfrac{\mu^*_{Hr}(A \cap D)}{(\operatorname{diam} D)^r} : x \in D,\ 0 < \operatorname{diam} D \leq \delta\} > 0\}$$

is $\bigcup_{n \in \mathbb{N}} \tilde{A}_{2^{-n}} \setminus A$, and is negligible.

471Q I now come to a remarkable fact about Hausdorff measures on analytic spaces: their Borel versions are semi-finite (471S). We need some new machinery.

Lemma Let (X, ρ) be a metric space, and $r > 0$, $\delta > 0$. Suppose that $\theta_{r\delta}(X)$, as defined in 471A, is finite.

(a) There is a non-negative additive functional ν on $\mathcal{P}X$ such that $\nu X = 5^{-r}\theta_{r\delta}(X)$ and $\nu A \leq (\operatorname{diam} A)^r$ whenever $A \subseteq X$ and $\operatorname{diam} A \leq \frac{1}{5}\delta$.

(b) If X is compact, there is a Radon measure μ on X such that $\mu X = 5^{-r}\theta_{r\delta}(X)$ and $\mu G \leq (\operatorname{diam} G)^r$ whenever $G \subseteq X$ is open and $\operatorname{diam} G \leq \frac{1}{5}\delta$.

proof (a) I use 391E. If $\theta_{r\delta}(X) = 0$ the result is trivial. Otherwise, set $\gamma = 5^r/\theta_{r\delta}(X)$ and define $\phi : \mathcal{P}X \to [0,1]$ by setting $\phi A = \min(1, \gamma(\operatorname{diam} A)^r)$ if $\operatorname{diam} A \leq \frac{1}{5}\delta$, 1 for other $A \subseteq X$. Now

whenever $\langle A_i \rangle_{i \in I}$ is a finite family of subsets of X, $m \in \mathbb{N}$ and $\sum_{i \in I} \chi A_i \geq m\chi X$, then $\sum_{i \in I} \phi A_i \geq m$.

P Discarding any A_i for which $\phi A_i = 1$, if necessary, we may suppose that $\operatorname{diam} A_i \leq \frac{1}{5}\delta$ and $\phi A_i = \gamma(\operatorname{diam} A_i)^r$ for every i. Choose $\langle I_j \rangle_{j \leq m}$, $\langle J_j \rangle_{j < m}$ inductively, as follows. $I_0 = I$. Given that $j < m$ and that $I_j \subseteq I$ is such that $\sum_{i \in I_j} \chi A_i \geq (m-j)\chi X$, apply 471N to $\{A_i : i \in I_j\}$ to find $J_j \subseteq I_j$ such that $\langle A_i \rangle_{i \in J_j}$ is disjoint and $\bigcup_{i \in I_j} A_i \subseteq \bigcup_{i \in J_j} A_j^\sim$. Set $I_{j+1} = I_j \setminus J_j$. Observe that $\sum_{i \in J_j} \chi A_j \leq \chi X$, so $\sum_{i \in I_{j+1}} \chi A_i \geq (m-j-1)\chi X$ and the induction proceeds.

Now note that, for each $j < m$, $\langle A_i^\sim \rangle_{i \in J_j}$ is a cover of $\bigcup_{i \in I_j} A_i = X$ by sets of diameter at most δ. So $\sum_{i \in J_j} (\operatorname{diam} A_i^\sim)^r \geq \theta_{r\delta}(X)$ for each j, and $\sum_{i \in I} (\operatorname{diam} A_i^\sim)^r \geq m\theta_{r\delta}(X)$. Accordingly

$$\begin{aligned}\sum_{i \in I} \phi A_i &= \gamma \sum_{i \in I} (\operatorname{diam} A_i)^r \geq 5^{-r}\gamma \sum_{i \in I} (\operatorname{diam} A_i^\sim)^r \\ &\geq 5^{-r} m \gamma \theta_{r\delta}(X) = m. \ \blacksquare\end{aligned}$$

By 391E, there is an additive functional $\nu_0 : \mathcal{P}X \to [0,1]$ such that $\nu_0 X = 1$ and $\nu_0 A \leq \phi A$ for every $A \subseteq X$. Setting $\nu = 5^{-r}\theta_{r\delta}(X)\nu_0$, we have the result.

(b) Now suppose that X is compact. By 416N, there is a Radon measure μ on X such that $\mu X \leq \nu X$ and $\mu K \geq \nu K$ for every compact $K \subseteq X$. Because X itself is compact, $\mu X = \nu X = 5^{-r}\theta_{r\delta}(X)$. If G is open and $\operatorname{diam} G \leq \frac{1}{5}\delta$,

$$\begin{aligned}\mu G &= \mu X - \mu(X \setminus G) \leq \nu X - \nu(X \setminus G) \\ &\text{(because } X \setminus G \text{ is compact)} \\ &= \nu G \leq (\operatorname{diam} G)^r,\end{aligned}$$

as required.

471R Lemma (Howroyd 95) Let (X, ρ) be a compact metric space and $r > 0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X. If $\mu_{Hr}(X) > 0$, there is a Borel set $H \subseteq X$ such that $0 < \mu_{Hr} H < \infty$.

proof (a) Let $\delta > 0$ be such that $\theta_{r,5\delta}(X) > 0$, where $\theta_{r,5\delta}$ is defined as in 471A. Then there is a family $\mathcal{V}$ of open subsets of X such that (i) $\operatorname{diam} V \leq \delta$ for every $V \in \mathcal{V}$ (ii) $\{V : V \in \mathcal{V}, \operatorname{diam} V \geq \epsilon\}$ is finite for every $\epsilon > 0$ (iii) whenever $A \subseteq X$ and $0 < \operatorname{diam} A < \frac{1}{4}\delta$ there is a $V \in \mathcal{V}$ such that $A \subseteq V$ and $\operatorname{diam} V \leq 8 \operatorname{diam} A$. **P** For each $k \in \mathbb{N}$, let I_k be a finite subset of X such that $X = \bigcup_{x \in I_k} B(x, 2^{-k-2}\delta)$; now set $\mathcal{V} = \{U(x, 2^{-k-1}\delta) : k \in \mathbb{N}, x \in I_k\}$. Then $\mathcal{V}$ is a family of open sets and (i) and (ii) are satisfied. If $A \subseteq X$ and $0 < \operatorname{diam} A < \frac{1}{4}\delta$, let $k \in \mathbb{N}$ be such that $2^{-k-3}\delta \leq \operatorname{diam} A < 2^{-k-2}\delta$. Take $x \in I_k$ such that $B(x, 2^{-k-2}\delta) \cap A \neq \emptyset$; then $A \subseteq U(x, 2^{-k-1}\delta) \in \mathcal{V}$ and $\operatorname{diam} U(x, 2^{-k-1}\delta) \leq 2^{-k}\delta \leq 8 \operatorname{diam} A$. $\blacksquare$

In particular, $\{V : V \in \mathcal{V}, \operatorname{diam} V \leq \epsilon\}$ covers X for every $\epsilon > 0$.

(b) Set

$$P = \{\mu : \mu \text{ is a Radon measure on } X, \mu V \leq (\operatorname{diam} V)^r \text{ for every } V \in \mathcal{V}\}.$$

P is non-empty (it contains the zero measure, for instance). Now if $G \subseteq X$ is open, $\mu \mapsto \mu G$ is lower semi-continuous for the narrow topology (437Jd), so P is a closed set in the narrow topology on the set of Radon measures on X, which may be identified with a compact subset of $C(X)^*$ with its weak* topology (437Kc). Moreover, since there is a finite subfamily of $\mathcal{V}$ covering X, $\gamma = \sup\{\mu X : \mu \in P\}$ is finite, and P is compact (437O). Because $\mu \mapsto \mu X$ is continuous, $P_0 = \{\mu : \mu \in P, \mu X = \gamma\}$ is non-empty. Of course P and P_0 are both convex, and P_0, like P, is compact. By the Kreĭn-Mil'man theorem (4A4Gb), applied in $C(X)^*$, P has an extreme point ν say.

Note next that $\theta_{r,5\delta}(X)$ is certainly finite, again because X is compact. By 471Qb, $\gamma > 0$, and ν is non-trivial. For any $\epsilon > 0$, there is a finite cover of X by sets in $\mathcal{V}$ of diameter at most ϵ, which have measure at most ϵ^r (for ν); so ν is atomless. In particular, $\nu\{x\} = 0$ for every $x \in X$.

(c) For $\epsilon > 0$, set

$$G_\epsilon = \bigcup\{V : V \in \mathcal{V}, 0 < \operatorname{diam} V \leq \epsilon \text{ and } \nu V \geq \tfrac{1}{2}(\operatorname{diam} V)^r\}.$$

Then G_ϵ is ν-conegligible. **P?** Otherwise, $\nu(X \setminus G_\epsilon) > 0$. Because $\mathcal{V}'_\epsilon = \{V : V \in \mathcal{V}, \operatorname{diam} V > \epsilon\}$ is finite, there is a Borel set $E \subseteq X \setminus G_\epsilon$ such that $\nu E > 0$ and, for every $V \in \mathcal{V}'_\epsilon$, either $E \subseteq V$ or $E \cap V = \emptyset$. Because ν is atomless, there is a measurable set $E_0 \subseteq E$ such that $\nu E_0 = \frac{1}{2}\nu E$ (215D); set $E_1 = E \setminus E_0$.

Define Radon measures ν_1, ν_2 on X by setting

$$\nu_i(F) = 2\nu(F \cap E_i) + \nu(F \setminus E)$$

whenever ν measures $F \setminus E_{1-i}$, for each i (use 416S if you feel the need to check that this defines a Radon measure on the definitions of this book). If $V \in \mathcal{V}$, then, by the choice of E,

either $E \subseteq V$ and $\nu_i V = \nu V \leq (\operatorname{diam} V)^r$

or $E \cap V = \emptyset$ and $\nu_i V = \nu V \leq (\operatorname{diam} V)^r$

or $0 < \operatorname{diam} V \leq \epsilon$ and $\nu V < \frac{1}{2}(\operatorname{diam} V)^r$, in which case $\nu_i V \leq 2\nu V \leq (\operatorname{diam} V)^r$

or $\operatorname{diam} V = 0$ and $\nu_i V = \nu V = 0 = (\operatorname{diam} V)^r$.

So both ν_i belong to P and therefore to P_0, since $\nu_i X = \nu X = \gamma$. But $\nu = \frac{1}{2}(\nu_1 + \nu_2)$ and $\nu_1 \neq \nu_2$, so this is impossible, because ν is supposed to be an extreme point of P_0. **XQ**

(d) Accordingly, setting $H = \bigcap_{n\in\mathbb{N}} G_{2^{-n}}$, $\nu H = \nu X = \gamma$. Now examine $\mu_{Hr} H$.

(i) $\mu_{Hr} H \geq 8^{-r}\gamma$. **P** Let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a sequence of sets covering H with $\operatorname{diam} A_n \leq \frac{1}{8}\delta$ for every n. Set $K = \{n : \operatorname{diam} A_n > 0\}$, $H' = H \cap \bigcup_{n\in K} A_n$; then $H \setminus H'$ is countable, so $\nu H' = \nu H$. For each $n \in K$, let $V_n \in \mathcal{V}$ be such that $A_n \subseteq V_n$ and $\operatorname{diam} V_n \leq 8 \operatorname{diam} A_n$ ((a) above). Then

$$\begin{aligned}\sum_{n=0}^{\infty} (\operatorname{diam} A_n)^r &= \sum_{n\in K} (\operatorname{diam} A_n)^r \geq 8^{-r} \sum_{n\in K} (\operatorname{diam} V_n)^r \\ &\geq 8^{-r} \sum_{n\in K} \nu V_n \geq 8^{-r} \nu H' = 8^{-r}\gamma.\end{aligned}$$

As $\langle A_n\rangle_{n\in\mathbb{N}}$ is arbitrary,

$$8^{-r}\gamma \leq \theta_{r,\delta/8}(H) \leq \mu^*_{Hr} H = \mu_{Hr} H. \ \textbf{Q}$$

(ii) $\mu_{Hr} H \leq 2\gamma$. **P** Let $\eta > 0$. Set $\mathcal{F} = \{\overline{V} : V \in \mathcal{V}, 0 < \operatorname{diam} V \leq \eta, \nu V \geq \frac{1}{2}(\operatorname{diam} V)^r\}$. Then $\mathcal{F}$ is a family of closed subsets of X, and (by the definition of G_ϵ) every member of H belongs to members of $\mathcal{F}$ of arbitrarily small diameter. Also $\nu F \geq \frac{1}{2}(\operatorname{diam} F)^r$ for every $F \in \mathcal{F}$, so

$$\textstyle\sum_{n=0}^{\infty} (\operatorname{diam} F_n)^r \leq 2 \sum_{n=0}^{\infty} \nu F_n < \infty$$

for any disjoint sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$. By 471O, there is a countable disjoint family $\mathcal{I} \subseteq \mathcal{F}$ such that $\mu_{Hr}(H \setminus \bigcup \mathcal{I}) = 0$. Accordingly

$$\theta_{r\eta}(H) \leq \sum_{F\in\mathcal{I}}(\operatorname{diam} F)^r + \theta_{r\eta}(H \setminus \bigcup\mathcal{I}) \leq \sum_{F\in\mathcal{I}} 2\nu F \leq 2\gamma.$$

As η is arbitrary, $\mu_{Hr}H = \mu^*_{Hr}H \leq 2\gamma$. **Q**

(e) But this means that we have found a Borel set H with $0 < \mu_{Hr}H < \infty$, as required.

471S Theorem (HOWROYD 95) Let (X, ρ) be an analytic metric space, and $r > 0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X, and $\mathcal{B}$ the Borel σ-algebra of X. Then the Borel measure $\mu_{Hr}\restriction\mathcal{B}$ is semi-finite and tight (that is, inner regular with respect to the compact sets).

proof Suppose that $E \in \mathcal{B}$ and $\mu_{Hr}E > 0$. Since E is analytic (423E), 471I above tells us that there is a compact set $K \subseteq E$ such that $\mu_{Hr}K > 0$. Next, by 471R, there is a Borel set $H \subseteq K$ such that $0 < \mu_{Hr}H < \infty$. (Strictly speaking, $\mu_{Hr}H$ here should be calculated as the r-dimensional Hausdorff measure of H defined by the subspace metric $\rho\restriction K \times K$ on K. By 471E we do not need to distinguish between this and the r-dimensional measure calculated from ρ itself.) By 471I again (applied to the subspace metric on H), there is a compact set $L \subseteq H$ such that $\mu_{Hr}L > 0$.

Thus E includes a non-negligible compact set of finite measure. As E is arbitrary, this is enough to show both that $\mu_{Hr}\restriction\mathcal{B}$ is semi-finite and that it is tight.

471X Basic exercises (a) Define a metric ρ on $X = \{0,1\}^{\mathbb{N}}$ by setting $\rho(x,y) = 2^{-n}$ if $x\restriction n = y\restriction n$ and $x(n) \neq y(n)$. Show that the usual measure μ on X is one-dimensional Hausdorff measure. (*Hint*: $\operatorname{diam} F \geq \mu F$ for every closed set $F \subseteq X$.)

(b) Let (X, ρ) be a metric space and $r > 0$; let μ_{Hr}, $\theta_{r\infty}$ be r-dimensional Hausdorff measure and capacity on X. (i) Show that, for $A \subseteq X$, $\mu_{Hr}A = 0$ iff $\theta_{r\infty}A = 0$. (ii) Suppose that $E \subseteq X$ and $\delta > 0$ are such that $\mu_{Hr}E < \infty$ and $\delta < \theta_{r\infty}E$. Show that there is a closed set $F \subseteq E$ such that $\mu_{Hr}F > 0$ and $\delta\mu_{Hr}(F \cap G) \leq (\operatorname{diam} G)^r$ whenever μ_{Hr} measures G. (*Hint*: show that $\{G^\bullet : \theta_{r\infty}G \leq \delta\mu_{Hr}(E \cap G)\}$ cannot be order-dense in the measure algebra of μ_{Hr}. This is a version of 'Frostman's Lemma'.)

(c) Let (X, ρ) be an analytic metric space, (Y, σ) a metric space, and $f : X \to Y$ a Lipschitz function, that is, a function which is γ-Lipschitz for some $\gamma \geq 0$. Show that if $r > 0$ and $A \subseteq X$ is measured by Hausdorff r-dimensional measure on X, with finite measure, then $f[A]$ is measured by Hausdorff r-dimensional measure on Y.

(d) Let (X, ρ) be a metric space and $r > 0$. Show that a set $A \subseteq X$ is negligible for Hausdorff r-dimensional measure on X iff there is a sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of subsets of X such that $\sum_{n=0}^{\infty}(\operatorname{diam} A_n)^r$ is finite and $A \subseteq \bigcap_{n\in\mathbb{N}}\bigcup_{m\geq n} A_m$.

(e) Let (X, ρ) be a metric space. (i) Show that there is a unique $\dim_H(X) \in [0,\infty]$ such that the r-dimensional Hausdorff measure of X is infinite if $0 < r < \dim_H(X)$, zero if $r > \dim_H(X)$. ($\dim_H(X)$ is the **Hausdorff dimension** of X.) (ii) Show that if $\langle A_n\rangle_{n\in\mathbb{N}}$ is any sequence of subsets of X, then $\dim_H(\bigcup_{n\in\mathbb{N}} A_n) = \sup_{n\in\mathbb{N}} \dim_H(A_n)$.

(f) Let (X, ρ) be a metric space, and μ any topological measure on X. Suppose that $E \subseteq X$ and that μE is defined and finite. (i) Show that $(x,\delta) \mapsto \mu(E \cap B(x,\delta)) : X \times [0,\infty[\to \mathbb{R}$ is upper semi-continuous. (ii) Show that $x \mapsto \limsup_{\delta\downarrow 0} \frac{1}{\delta^r}\mu(E \cap B(x,\delta)) : X \to [0,\infty]$ is Borel measurable. (iii) Show that if X is separable, then $\mu B(x,\delta) > 0$ for every $\delta > 0$, for almost every $x \in X$.

(g) Give $\mathbb{R}$ its usual metric. Let $C \subseteq \mathbb{R}$ be the Cantor set, and $r = \ln 2/\ln 3$. Show that

$$\liminf_{\delta\downarrow 0} \frac{\mu_{Hr}(C\cap B(x,\delta))}{(\operatorname{diam} B(x,\delta))^r} \leq 2^{-r}$$

for every $x \in \mathbb{R}$.

(h) Let (X, ρ) be a metric space and $r > 0$. Let μ_{Hr} be r-dimensional Hausdorff measure on X and $\tilde{\mu}_{Hr}$ its c.l.d. version (213D-213E). Show that $\tilde{\mu}_{Hr}$ is inner regular with respect to the closed sets, and that $\tilde{\mu}_{Hr}A = \mu_{Hr}A$ for every analytic set $A \subseteq X$.

471Y Further exercises (a) The next few exercises (down to 471Yd) will all be based on the following context. Let (X,ρ) be a metric space and $\psi : \mathcal{P}X \to [0,\infty]$ a function such that $\psi\emptyset = 0$ and $\psi A \leq \psi A'$ whenever $A \subseteq A' \subseteq X$. Set

$$\theta_{\psi\delta}A = \inf\{\sum_{n=0}^{\infty} \psi D_n : \langle D_n\rangle_{n\in\mathbb{N}} \text{ is a sequence of subsets of } X \text{ covering } A,$$
$$\operatorname{diam} D_n \leq \delta \text{ for every } n \in \mathbb{N}\}$$

for $\delta > 0$, and $\theta_\psi A = \sup_{\delta>0} \theta_{\psi\delta}A$ for $A \subseteq X$ Show that θ_ψ is a metric outer measure. Let μ_ψ be the measure defined from θ_ψ by Carathéodory's method.

(b) Suppose that $\psi A = \inf\{\psi E : E \text{ is a Borel set including } A\}$ for every $A \subseteq X$. Show that $\theta_\psi = \mu_\psi^*$ and that $\mu_\psi E = \sup\{\mu_\psi F : F \subseteq E \text{ is closed}\}$ whenever $\mu_\psi E < \infty$.

(c) Suppose that X is separable and that there is a $\beta \geq 0$ such that $\psi A^\sim \leq \beta\psi A$ for every $A \subseteq X$, where $A^\sim$ is defined in 471M. (i) Suppose that $A \subseteq X$ and $\mathcal{F}$ is a family of closed subsets of X such that $\sum_{n=0}^{\infty} \psi F_n$ is finite for every disjoint sequence $\langle F_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{F}$ and for every $x \in A$, $\delta > 0$ there is an $F \in \mathcal{F}$ such that $x \in F$ and $0 < \operatorname{diam} F \leq \delta$. Show that there is a disjoint family $\mathcal{I} \subseteq \mathcal{F}$ such that $\mu_\psi(A \setminus \bigcup\mathcal{I}) = 0$. (ii) Suppose that $\delta > 0$ and that $\theta_{\psi\delta}(X) < \infty$. Show that there is a non-negative additive functional ν on $\mathcal{P}X$ such that $\nu X = \frac{1}{\beta}\theta_{\psi\delta}(X)$ and $\nu A \leq \psi A$ whenever $A \subseteq X$ and $\operatorname{diam} A \leq \frac{1}{5}\delta$. (iii) Now suppose that for every $x \in X$, $\epsilon > 0$ there is a $\delta > 0$ such that $\psi B(x,\delta) \leq \epsilon$. Show that if X is compact and $\mu_\psi X > 0$ there is a compact set $K \subseteq X$ such that $0 < \mu_\psi K < \infty$.

(d) State and prove a version of 471P appropriate to this context.

(e) Give an example of a set $A \subseteq \mathbb{R}^2$ which is measured by Hausdorff 1-dimensional measure on $\mathbb{R}^2$ but is such that its projection onto the first coordinate is not measured by Hausdorff 1-dimensional measure on $\mathbb{R}$.

(f) Show that the space (X,ρ) of 471Xa can be isometrically embedded as a subset of a metric space (Y,σ) in such a way that (i) $\operatorname{diam} B(y,\delta) = 2\delta$ for every $y \in Y$ and $\delta \geq 0$ (ii) $Y \setminus X$ is countable. Show that if μ_{H1} is one-dimensional Hausdorff measure on Y, then $\mu_{H1}B(y,\delta) \leq \delta$ for every $y \in Y$, $\delta \geq 0$, so that

$$\limsup_{\delta\downarrow 0} \frac{\mu_{H1}B(y,\delta)}{\operatorname{diam} B(x,\delta)} = \frac{1}{2}$$

for every $y \in Y$.

(g) Let $\langle k_n\rangle_{n\in\mathbb{N}}$ be a sequence in $\mathbb{N} \setminus \{0,1,2,3\}$ such that $\sum_{n=0}^{\infty} \frac{1}{k_n} < \infty$. Set $X = \prod_{n\in\mathbb{N}} k_n$. Set $m_0 = 1$, $m_{n+1} = k_0k_1\dots k_n$ for $n \in \mathbb{N}$. Define a metric ρ on X by saying that

$$\rho(x,y) = 1/2m_n \text{ if } n = \min\{i : x(i) \neq y(i)\} \text{ and } \min(x(n),y(n)) = 0,$$
$$= 1/m_n \text{ if } n = \min\{i : x(i) \neq y(i)\} \text{ and } \min(x(n),y(n)) > 0.$$

Let ν be the product measure on X obtained by giving each factor k_n the uniform probability measure in which each singleton set has measure $1/k_n$. (i) Show that if $A \subseteq X$ then $\nu^* A \leq \operatorname{diam} A$. (ii) Show that ν is one-dimensional Hausdorff measure on X. (iii) Set $E = \bigcup_{n\in\mathbb{N}}\{x : x \in X, x(n) = 0\}$. Show that $\nu E < 1$. (iv) Show that

$$\limsup_{\delta\downarrow 0} \frac{\nu(E\cap B(x,\delta))}{\nu B(x,\delta)} \geq \frac{1}{2}$$

for every $x \in X$. (v) Show that there is a family $\mathcal{F}$ of closed balls in X such that every point of X is the centre of arbitrarily small members of $\mathcal{F}$, but $\nu(\bigcup\mathcal{I}) < 1$ for any disjoint subfamily $\mathcal{I}$ of $\mathcal{F}$.

(h) Let ρ be the metric on $\{0,1\}^{\mathbb{N}}$ defined in 471Xa. Show that for any integer $s \geq 1$ there is a bijection $f : [0,1]^s \to \{0,1\}^{\mathbb{N}}$ such that whenever $0 < r \leq s$, μ_{Hr}^* is Hausdorff r-dimensional outer measure on $[0,1]^s$ (for its usual metric) and $\tilde{\mu}_{H,r/s}^*$ is Hausdorff $\frac{r}{s}$-dimensional measure on $\{0,1\}^\omega$, then there is an $\alpha > 0$ such that $\mu_{Hr}^* f^{-1}[A] \leq \mu_{H,r/s}^* A \leq \alpha\mu_{Hr}^* f^{-1}[A]$ for every $A \subseteq \{0,1\}^{\mathbb{N}}$.

(i) Let (X, ρ) be a metric space, and $r > 0$; let $s \geq 1$ be an integer. Write $\mu_{Hr}^{(X)}$ and $\mu_{H,r+s}^{(X\times\mathbb{R}^s)}$ for Hausdorff r-dimensional measure on X and Hausdorff $r+s$-dimensional measure on $X \times \mathbb{R}^s$ respectively, and μ_{Ls} for Lebesgue measure on $\mathbb{R}^s$. (i) Show that there are non-zero constants c, c' such that $c\mu_{H,r+s}^{(X\times\mathbb{R}^s)}(E \times F) \leq \mu_{Hr}^{(X)}(E) \cdot \mu_{Ls}(F) \leq c'\mu_{H,r+s}^{(X\times\mathbb{R}^s)}(E \times F)$ for all Borel sets $E \subseteq X$, $F \subseteq \mathbb{R}^s$. (*Hint*: Federer 69, 2.10.45.) (ii) Write λ for the c.l.d. product of $\mu_{Hr}^{(X)}$ and μ_{Ls}, and $\tilde{\mu}_{H,r+s}^{(X\times\mathbb{R}^s)}$ for the c.l.d. version of $\mu_{H,r+s}^{(X\times\mathbb{R}^s)}$. Show that these have the same domain Λ and that $c\tilde{\mu}_{H,r+s}^{(X\times\mathbb{R}^s)}(W) \leq \lambda W \leq c'\tilde{\mu}_{H,r+s}^{(X\times\mathbb{R}^s)}(W)$ for every $W \in \Lambda$.

471 Notes and comments In the exposition above, I have worked throughout with simple r-dimensional measures for $r > 0$. As noted in 264Db, there are formulae in which it is helpful to interpret μ_{H0} as counting measure. More interestingly, when we come to use Hausdorff measures to give us information about the geometric structure of an object (e.g., in the observation that the Cantor set has $\ln 2 / \ln 3$-dimensional Hausdorff measure 1, in 264J), it is sometimes useful to refine the technique by using other functionals than $A \mapsto (\operatorname{diam} A)^r$ in the basic formulae of 264A or 471A. The most natural generalization is to functionals of the form $\psi A = h(\operatorname{diam} A)$ where $h : [0, \infty] \to [0, \infty]$ is a non-decreasing function (264Yo). But it is easy to see that many of the arguments are valid in much greater generality, as in 471Ya-471Yc. For more in these directions see Rogers 70 and Federer 69.

In the context of this book, the most conspicuous peculiarity of Hausdorff measures is that they are often very far from being semi-finite. (This is trivial for non-separable spaces, by 471Df. That Hausdorff one-dimensional measure on a subset of $\mathbb{R}^2$ can be purely infinite is not I think obvious; I gave an example in 439H.) The response I ordinarily recommend in such cases is to take the c.l.d. version. But then of course we need to know just what effect this will have. In geometric applications, one usually begins by checking that the sets one is interested in have σ-finite measure, and that therefore no problems arise; but it is a striking fact that Hausdorff measures behave relatively well on analytic sets, even when not σ-finite, provided we ask exactly the right questions (471I, 471S, 471Xh).

The geometric applications of Hausdorff measures, naturally, tend to rely heavily on density theorems; it is therefore useful to know that we have effective versions of Vitali's theorem available in this context (471N-471O), leading to a general density theorem (471P) similar to that in 261D; see also 472D below. I note that 471P is useful only after we have been able to concentrate our attention on a set of finite measure. And traps remain. For instance, the formulae of 261C-261D cannot be transferred to the present context without re-evaluation (471Yg).

472 Besicovitch's Density Theorem

The first step in the program of the next few sections is to set out some very remarkable properties of Euclidean space. We find that in $\mathbb{R}^r$, for geometric reasons (472A), we have versions of Vitali's theorem (472B-472C) and Lebesgue's Density Theorem (472D) for arbitrary Radon measures. I add a version of the Hardy-Littlewood Maximal Theorem (472F).

Throughout the section, $r \geq 1$ will be a fixed integer. As usual, I write $B(x, \delta)$ for the closed ball with centre x and radius δ. $\| \|$ will represent the Euclidean norm, and $x \cdot y$ the scalar product of x and y, so that $x \cdot y = \sum_{i=1}^r \xi_i \eta_i$ if $x = (\xi_1, \dots, \xi_r)$ and $y = (\eta_1, \dots, \eta_r)$.

472A Besicovitch's Covering Lemma Suppose that $\epsilon > 0$ is such that $(5^r + 1)(1 - \epsilon - \epsilon^2)^r > (5 + \epsilon)^r$. Let $x_0, \dots, x_n \in \mathbb{R}^r$, $\delta_0, \dots, \delta_n > 0$ be such that

$$\|x_i - x_j\| > \delta_i, \quad \delta_j \leq (1 + \epsilon)\delta_i$$

whenever $i < j \leq n$. Then

$$\#(\{i : i \leq n, \|x_i - x_n\| \leq \delta_i + \delta_n\}) \leq 5^r.$$

proof Set $I = \{i : i \leq n, \|x_i - x_n\| \leq \delta_n + \delta_i\}$.

(a) It will simplify the formulae of the main argument if we suppose for the time being that $\delta_n = 1$; in this case $1 \leq (1+\epsilon)\delta_i$, so that $\delta_i \geq \frac{1}{1+\epsilon}$ for every $i \leq n$, while we still have $\delta_i < \|x_i - x_n\|$ for every $i < n$, and $\|x_i - x_n\| \leq 1 + \delta_i$ for every $i \in I$.

For $i \in I$, define x_i' by saying that

– if $\|x_i - x_n\| \leq 2 + \epsilon$, $x_i' = x_i$;

– if $\|x_i - x_n\| > 2 + \epsilon$, x_i' is to be that point of the closed line segment from x_n to x_i which is at distance $2 + \epsilon$ from x_n.

(b) The point is that $\|x_i' - x_j'\| > 1 - \epsilon - \epsilon^2$ whenever i, j are distinct members of I. **P** We may suppose that $i < j$.

case 1 Suppose that $\|x_i - x_n\| \leq 2 + \epsilon$ and $\|x_j - x_n\| \leq 2 + \epsilon$. In this case

$$\|x_i' - x_j'\| = \|x_i - x_j\| \geq \delta_i \geq \frac{1}{1+\epsilon} \geq 1 - \epsilon.$$

case 2 Suppose that $\|x_i - x_n\| \geq 2 + \epsilon \geq \|x_j - x_n\|$. In this case

$$\begin{aligned}\|x_i' - x_j'\| = \|x_i' - x_j\| &\geq \|x_i - x_j\| - \|x_i - x_i'\| \\ &\geq \delta_i - \|x_i - x_n\| + 2 + \epsilon \geq \delta_i - \delta_i - 1 + 2 + \epsilon = 1 + \epsilon.\end{aligned}$$

case 3 Suppose that $\|x_i - x_n\| \leq 2 + \epsilon \leq \|x_j - x_n\|$. Then

$$\begin{aligned}\|x_i' - x_j'\| = \|x_i - x_j'\| &\geq \|x_i - x_j\| - \|x_j - x_j'\| > \delta_i - \|x_j - x_n\| + 2 + \epsilon \\ &\geq \delta_i - \delta_j - 1 + 2 + \epsilon \geq \delta_i - \delta_i(1+\epsilon) + 1 + \epsilon \geq 1 + \epsilon - \epsilon(2+\epsilon) \\ &\text{(because } \delta_i < \|x_i - x_n\| \leq 2 + \epsilon) \\ &= 1 - \epsilon - \epsilon^2.\end{aligned}$$

case 4 Suppose that $2 + \epsilon \leq \|x_j - x_n\| \leq \|x_i - x_n\|$. Let y be the point on the line segment between x_i and x_n which is the same distance from x_n as x_j. In this case

$$\|y - x_j\| \geq \|x_i - x_j\| - \|x_i - y\| \geq \delta_i - \|x_i - x_n\| + \|x_j - x_n\| \geq \|x_j - x_n\| - 1.$$

Because the triangles (x_n, y, x_j) and (x_n, x_i', x_j') are similar,

$$\|x_i' - x_j'\| = \tfrac{2+\epsilon}{\|x_j - x_n\|}\|y - x_j\| \geq (2+\epsilon)\tfrac{\|x_j - x_n\| - 1}{\|x_j - x_n\|} \geq 1 + \epsilon$$

because $\|x_j - x_n\| \geq 2 + \epsilon$.

case 5 Suppose that $2 + \epsilon \leq \|x_i - x_n\| \leq \|x_j - x_n\|$. This time, let y be the point on the line segment from x_n to x_j which is the same distance from x_n as x_i is. We now have

$$\begin{aligned}\|y - x_i\| &\geq \|x_i - x_j\| - \|x_j - y\| > \delta_i - \|x_j - x_n\| + \|x_i - x_n\| \\ &\geq \delta_j - \epsilon\delta_i - (\delta_j + 1) + \|x_i - x_n\| \\ &= \|x_i - x_n\| - 1 - \epsilon\delta_i \geq \|x_i - x_n\|(1 - \epsilon) - 1,\end{aligned}$$

so that

$$\begin{aligned}\|x_i' - x_j'\| = \tfrac{2+\epsilon}{\|x_i - x_n\|}\|y - x_i\| &> (2+\epsilon)\tfrac{\|x_i - x_n\|(1-\epsilon) - 1}{\|x_i - x_n\|} \\ &\geq (2+\epsilon)\tfrac{(2+\epsilon)(1-\epsilon) - 1}{2+\epsilon} = 1 - \epsilon - \epsilon^2.\end{aligned}$$

So we have the required inequality in all cases. **Q**

(c) Now consider the balls $B(x_i', \frac{1-\epsilon-\epsilon^2}{2})$ for $i \in I$. These are disjoint, all have Lebesgue measure $2^{-r}\beta_r(1-\epsilon-\epsilon^2)^r$ where β_r is the measure of the unit ball $B(\mathbf{0},1)$, and are all included in the ball $B(x_n, 2+\epsilon+\frac{1-\epsilon}{2})$, which has measure $2^{-r}\beta_r(5+\epsilon)^r$. So we must have

$$2^{-r}\beta_r(1-\epsilon-\epsilon^2)^r\#(I) \leq 2^{-r}\beta_r(5+\epsilon)^r.$$

But ϵ was declared to be so small that this implies that $\#(I) \leq 5^r$, as claimed.

(e) This proves the lemma in the case $\delta_n = 1$. For the general case, replace each x_i by $\delta_n^{-1}x_i$ and each δ_i by δ_i/δ_n; the change of scale does not affect the hypotheses or the set I.

472B Theorem Let $A \subseteq \mathbb{R}^r$ be a bounded set, and $\mathcal{I}$ a family of non-trivial closed balls in $\mathbb{R}^r$ such that every point of A is the centre of a member of $\mathcal{I}$. Then there is a family $\langle \mathcal{I}_k\rangle_{k<5^r}$ of countable subsets of $\mathcal{I}$ such that each $\mathcal{I}_k$ is disjoint and $\bigcup_{k<5^r}\mathcal{I}_k$ covers A.

proof (a) For each $x \in A$ let $\delta_x > 0$ be such that $B(x,\delta_x) \in \mathcal{I}$. If either A is empty or $\sup_{x\in A}\delta_x = \infty$, the result is trivial. (In the latter case, take $x \in A$ such that $\delta_x \geq \operatorname{diam} A$ and set $\mathcal{I}_0 = \{B(x,\delta_x)\}$, $\mathcal{I}_k = \emptyset$ for $k > 0$.) So let us suppose henceforth that $\{\delta_x : x \in A\}$ is bounded in $\mathbb{R}$. In this case, $C = \bigcup_{x\in A} B(x,\delta_x)$ is bounded in $\mathbb{R}^r$.

Fix $\epsilon > 0$ such that $(5^r+1)(1-\epsilon-\epsilon^2)^r > (5+\epsilon)^r$.

(b) Choose inductively a sequence $\langle B_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{I}\cup\{\emptyset\}$ as follows. Given $\langle B_i\rangle_{i<n}$, then if $A \subseteq \bigcup_{i<n} B_i$ set $B_n = \emptyset$. Otherwise, set $\alpha_n = \sup\{\delta_x : x \in A\setminus\bigcup_{i<n}B_i\}$, choose $x_n \in A\setminus\bigcup_{i<n}B_i$ such that $(1+\epsilon)\delta_{x_n} \geq \alpha_n$, set $B_n = B(x_n,\delta_{x_n})$ and continue.

Now whenever $n \in \mathbb{N}$, $I_n = \{i : i < n,\ B_i \cap B_n \neq \emptyset\}$ has fewer than 5^r members. **P** We may suppose that $B_n \neq \emptyset$, in which case $B_i = B(x_i,\delta_{x_i})$ for every $i \leq n$, and the x_i, δ_{x_i} are such that, whenever $i < j \leq n$,

$$x_j \notin B_i, \text{ i.e., } \|x_i - x_j\| > \delta_{x_i},$$

$$\delta_{x_j} \leq \alpha_i \leq (1+\epsilon)\delta_{x_i}.$$

But now 472A gives the result at once. **Q**

(c) We may therefore define a function $f : \mathbb{N} \to \{0,1,\dots,5^r-1\}$ by setting

$$f(n) = \min\{k : 0 \leq k < 5^r,\ f(i) \neq k \text{ for every } i \in I_n\}$$

for every $n \in \mathbb{N}$. Set $\mathcal{I}_k = \{B_i : i \in \mathbb{N},\ f(i) = k,\ B_i \neq \emptyset\}$ for each $k < 5^r$. By the choice of f, $i \notin I_j$, so that $B_i \cap B_j = \emptyset$, whenever $i < j$ and $f(i) = f(j)$; thus every $\mathcal{I}_k$ is disjoint. Since $B_i \subseteq C$ for every i, $\sum\{\mu B_i : f(i) = k\} \leq \mu^* C$ for every $k < 5^r$, and $\sum_{i=0}^{\infty}\mu B_i \leq 5^r\mu^* C$ is finite.

(d) **?** Suppose, if possible, that

$$A \not\subseteq \bigcup_{k<5^r}\bigcup\mathcal{I}_k = \bigcup_{n\in\mathbb{N}} B_n.$$

Take $x \in A \setminus \bigcup_{n\in\mathbb{N}} B_n$. Then, first, $A \not\subseteq \bigcup_{i<n} B_i$ for every n, so that α_n is defined; next, $\alpha_n \geq \delta_x$, so that $(1+\epsilon)\delta_{x_n} \geq \delta_x$ for every n. But this means that $\mu B_n \geq \beta_r\big(\frac{\delta_x}{1+\epsilon}\big)^r$ for every n, and $\sum_{n=0}^{\infty}\mu B_n = \infty$; which is impossible. **X**

(e) Thus $A \subseteq \bigcup_{k<5^r}\bigcup\mathcal{I}_k$, as required.

472C Theorem Let λ be a Radon measure on $\mathbb{R}^r$, A a subset of $\mathbb{R}^r$ and $\mathcal{I}$ a family of non-trivial closed balls in $\mathbb{R}^r$ such that every point of A is the centre of arbitrarily small members of $\mathcal{I}$. Then

(a) there is a countable disjoint $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $\lambda(A\setminus\bigcup\mathcal{I}_0) = 0$;

(b) for every $\epsilon > 0$ there is a countable $\mathcal{I}_1 \subseteq \mathcal{I}$ such that $A \subseteq \bigcup\mathcal{I}_1$ and $\sum_{B\in\mathcal{I}_1}\lambda B \leq \lambda^* A + \epsilon$.

proof (a)(i) The first step is to show that if $A' \subseteq A$ is bounded then there is a finite disjoint set $\mathcal{J} \subseteq \mathcal{I}$ such that $\lambda^*(A' \cap \bigcup\mathcal{J}) \geq 6^{-r}\lambda^* A'$. **P** If $\lambda^* A' = 0$ take $\mathcal{J} = \emptyset$. Otherwise, by 472B, there is a family $\langle\mathcal{J}_k\rangle_{k<5^r}$ of disjoint countable subsets of $\mathcal{I}$ such that $\bigcup_{k<5^r}\mathcal{J}_k$ covers A'. Accordingly

$$\lambda^* A' \leq \sum_{k=0}^{5^r-1}\lambda^*(A' \cap \bigcup\mathcal{J}_k)$$

and there is some $k < 5^r$ such that $\lambda^*(A' \cap \bigcup \mathcal{J}_k) \geq 5^{-r}\lambda^* A'$. Let $\langle B_i \rangle_{i\in\mathbb{N}}$ be a sequence running over $\mathcal{J}_k$; then

$$\lim_{n\to\infty} \lambda^*(A' \cap \bigcup_{i\leq n} B_i) = \lambda^*(A' \cap \bigcup \mathcal{J}_k) \geq 5^{-r}\lambda^* A',$$

so there is some $n \in \mathbb{N}$ such that $\lambda^*(A' \cap \bigcup_{i\leq n} B_i) \geq 6^{-r}\lambda^* A'$, and we can take $\mathcal{J} = \{B_i : i \leq n\}$. **Q**

(ii) Now choose $\langle \mathcal{K}_n \rangle_{n\in\mathbb{N}}$ inductively, as follows. Start by fixing on a sequence $\langle m_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that every member of $\mathbb{N}$ appears infinitely often as an m_n. Take $\mathcal{K}_0 = \emptyset$. Given that $\mathcal{K}_n$ is a finite disjoint subset of $\mathcal{I}$, set $\mathcal{I}' = \{B : B \in \mathcal{I},\ B \cap \bigcup \mathcal{K}_n = \emptyset\}$, $A_n = A \cap B(\mathbf{0}, m_n) \setminus \bigcup \mathcal{K}_n$. Because every point of A is the centre of arbitrarily small members of $\mathcal{I}$, and $\bigcup \mathcal{K}_n$ is closed, every member of A_n is the centre of (arbitrarily small) members of $\mathcal{I}'$, and (i) tells us that there is a finite disjoint set $\mathcal{J}_n \subseteq \mathcal{I}'$ such that $\lambda^*(A_n \cap \bigcup \mathcal{J}_n) \geq 6^{-r}\lambda^* A_n$. Set $\mathcal{K}_{n+1} = \mathcal{K} \cup \mathcal{J}_n$, and continue. At the end of the induction, set $\mathcal{I}_0 = \bigcup_{n\in\mathbb{N}} \mathcal{K}_n$; because $\langle \mathcal{K}_n \rangle_{n\in\mathbb{N}}$ is non-decreasing and every $\mathcal{K}_n$ is disjoint, $\mathcal{I}_0$ is disjoint, and of course $\mathcal{I}_0 \subseteq \mathcal{I}$.

The effect of this construction is to ensure that

$$\lambda^*(A \cap B(\mathbf{0}, m_n) \setminus \bigcup \mathcal{K}_{n+1}) = \lambda^*(A_n \setminus \bigcup \mathcal{J}_n) = \lambda^* A_n - \lambda^*(A_n \cap \bigcup \mathcal{J}_n)$$

(because $\bigcup \mathcal{J}_n$ is a closed set, therefore measured by λ)

$$\begin{aligned} &\leq (1 - 6^{-r})\lambda^* A_n \\ &= (1 - 6^{-r})\lambda^*(A \cap B(\mathbf{0}, m_n) \setminus \bigcup \mathcal{K}_n) \end{aligned}$$

for every n. So, for any $m \in \mathbb{N}$,

$$\lambda^*(A \cap B(\mathbf{0}, m) \setminus \bigcup \mathcal{K}_n) \leq \lambda^*(A \cap B(\mathbf{0}, m))(1 - 6^{-r})^{\#(\{j : j<n, m_j = m\})} \to 0$$

as $n \to \infty$, and $\lambda^*(A \cap B(\mathbf{0}, m) \setminus \bigcup \mathcal{I}_0) = 0$. As m is arbitrary, $\lambda^*(A \setminus \bigcup \mathcal{I}_0) = 0$, as required.

(b)(i) Let $E \supseteq A$ be such that $\lambda E = \lambda^* A$, and $H \supseteq E$ an open set such that $\lambda H \leq \lambda E + \frac{1}{2}\epsilon$ (256Bb). Set $\mathcal{I}' = \{B : B \in \mathcal{I},\ B \subseteq H\}$. Then every point of A is the centre of arbitrarily small members of $\mathcal{I}'$, so by (a) there is a disjoint family $\mathcal{I}_0 \subseteq \mathcal{I}'$ such that $\lambda(A \setminus \bigcup \mathcal{I}_0) = 0$. Of course

$$\sum_{B\in\mathcal{I}_0} \lambda B = \lambda(\bigcup \mathcal{I}_0) \leq \lambda H \leq \lambda^* A + \tfrac{1}{2}\epsilon.$$

(ii) For $m \in \mathbb{N}$ set $A_m = A \cap B(\mathbf{0}, m) \setminus \bigcup \mathcal{I}_0$. Then there is a $\mathcal{J}_m \subseteq \mathcal{I}$, covering A_m, such that $\sum_{B\in\mathcal{J}_m} \lambda B \leq 2^{-m-2}\epsilon$. **P** There is an open set $G \supseteq A_m$ such that $\lambda G \leq 5^{-r}2^{-m-2}\epsilon$. Now $\mathcal{I}'' = \{B : B \in \mathcal{I},\ B \subseteq G\}$ covers A_m, so there is a family $\langle \mathcal{J}_{mk} \rangle_{k<5^r}$ of disjoint countable subfamilies of $\mathcal{I}''$ such that $\mathcal{J}_m = \bigcup_{k<5^r} \mathcal{J}_{mk}$ covers A_m. For each k,

$$\sum_{B\in\mathcal{J}_{mk}} \lambda B = \lambda(\bigcup \mathcal{J}_{mk}) \leq \lambda G,$$

so

$$\sum_{B\in\mathcal{J}_m} \lambda B \leq 5^r \lambda G \leq 2^{-m-2}\epsilon.$$ **Q**

(iii) Setting $\mathcal{I}_1 = \mathcal{I}_0 \cup \bigcup_{m\in\mathbb{N}} \mathcal{J}_m$ we have a cover of A by members of $\mathcal{I}$, and

$$\begin{aligned} \sum_{B\in\mathcal{I}_1} \lambda B &\leq \sum_{B\in\mathcal{I}_0} \lambda B + \sum_{m=0}^{\infty} \sum_{B\in\mathcal{J}_m} \lambda B \\ &\leq \lambda^* A + \frac{1}{2}\epsilon + \sum_{m=0}^{\infty} 2^{-m-2}\epsilon = \lambda^* A + \epsilon. \end{aligned}$$

472D Besicovitch's Density Theorem Let λ be any Radon measure on $\mathbb{R}^r$. Then, for any locally λ-integrable real-valued function f,

(a) $f(y) = \lim_{\delta\downarrow 0} \frac{1}{\lambda B(y,\delta)} \int_{B(y,\delta)} f d\lambda$,

(b) $\lim_{\delta\downarrow 0} \frac{1}{\lambda B(y,\delta)} \int_{B(y,\delta)} |f(x) - f(y)| \lambda(dx) = 0$

for λ-almost every $y \in \mathbb{R}^r$.

Remark The theorem asserts that, for λ-almost every y, limits of the form $\lim_{\delta\downarrow 0} \frac{1}{\lambda B(y,\delta)} \dots$ are defined; in my usage, this includes the assertion that $\lambda B(y,\delta) > 0$ for all sufficiently small $\delta > 0$.

proof (Compare 261C and 261E.)

(a) Let Z be the support of λ (411Nd); then Z is λ-conegligible and $\lambda B(y,\delta) > 0$ whenever $y \in Z$ and $\delta > 0$. For $q < q'$ in $\mathbb{Q}$, $n \in \mathbb{N}$ set

$$A_{nqq'} = \{y : y \in Z \cap \operatorname{dom} f,\ \|y\| \le n,\ f(y) \le q,\ \limsup_{\delta\downarrow 0} \frac{1}{\lambda B(y,\delta)} \int_{B(y,\delta)} f d\lambda > q'\}.$$

Then $\lambda A_{nqq'} = 0$. **P** Let $\epsilon > 0$. Then there is an $\eta \in]0,\epsilon]$ such that $\int_F |f| d\lambda \le \epsilon$ whenever $F \subseteq B(\mathbf{0},n)$ and $\lambda F \le \eta$ (225A). Let E be a measurable envelope of $A_{nqq'}$ included in $\{y : y \in Z \cap \operatorname{dom} f,\ f(y) \le q\}$, and take an open set $G \supseteq E$ such that $\lambda(G \setminus E) \le \eta$ (256Bb). Let $\mathcal{I}$ be the family of non-singleton closed balls $B \subseteq G$ such that $\int_B f \ge q'\lambda B$. Then every point of $A_{nqq'}$ belongs to arbitrarily small members of $\mathcal{I}$, so there is a disjoint family $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $\lambda(A_{nqq'} \setminus \bigcup \mathcal{I}_0) = 0$ (472C). Now $\lambda(E \setminus \bigcup \mathcal{I}_0) = 0$ and $\lambda((\bigcup \mathcal{I}_0) \setminus E) \le \eta \le \epsilon$, so

$$\begin{aligned} q'\lambda E &\le q'\lambda(\textstyle\bigcup \mathcal{I}_0) + \epsilon|q'| = \sum_{B\in\mathcal{I}_0} q'\lambda B + \epsilon|q'| \\ &\le \sum_{B\in\mathcal{I}_0} \int_B f d\lambda + \epsilon|q'| = \int_{\bigcup\mathcal{I}_0} f d\lambda + \epsilon|q'| \\ &\le \int_E f d\lambda + \epsilon(1+|q'|) \le q\lambda E + \epsilon(1+|q'|), \end{aligned}$$

and

$$(q'-q)\lambda^* A_{nqq'} = (q'-q)\lambda E \le (1+|q'|)\epsilon.$$

As ϵ is arbitrary, $\lambda^* A_{nqq'} = 0$. **Q**

As n, q and q' are arbitrary,

$$\limsup_{\delta\downarrow 0} \frac{1}{\lambda B(y,\delta)} \int_{B(y,\delta)} f \le f(y)$$

for λ-almost every $y \in Z$, therefore for λ-almost every $y \in \mathbb{R}^r$. Similarly, or applying the same argument to $-f$,

$$\liminf_{\delta\downarrow 0} \frac{1}{\lambda B(y,\delta)} \int_{B(y,\delta)} f \ge f(y)$$

for λ-almost every y, and

$$\lim_{\delta\downarrow 0} \frac{1}{\lambda B(y,\delta)} \int_{B(y,\delta)} f \text{ exists} = f(y)$$

for λ-almost every y.

(b) Now, for each $q \in \mathbb{Q}$ set $g_q(x) = |f(x) - q|$ for $x \in \operatorname{dom} f$. By (a), we have a λ-conegligible set D such that

$$\lim_{\delta\downarrow 0} \frac{1}{\lambda B(y,\delta)} \int_{B(y,\delta)} g_q d\lambda = g_q(y)$$

for every $y \in D$ and $q \in \mathbb{Q}$. Now, if $y \in D$ and $\epsilon > 0$, there is a $q \in \mathbb{Q}$ such that $|f(y) - q| \le \epsilon$, and a $\delta_0 > 0$ such that

$$|\frac{1}{\lambda B(y,\delta)} \int_{B(y,\delta)} g_q d\lambda - g_q(y)| \le \epsilon$$

whenever $0 < \delta \le \delta_0$. But in this case

$$\begin{aligned}\frac{1}{\lambda B(y,\delta)}&\int_{B(y,\delta)}|f(x)-f(y)|\lambda(dx)\\ &\leq \frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}|f(x)-q|\lambda(dx)+\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}|q-f(y)|\lambda(dx)\\ &\leq 2\epsilon.\end{aligned}$$

As ϵ is arbitrary,

$$\lim_{\delta\downarrow 0}\frac{1}{\lambda B(y,\delta)}\int_{B(y,\delta)}|f(x)-f(y)|\lambda(dx)=0;$$

as this is true for every $y\in D$, the theorem is proved.

***472E Proposition** Let λ, λ' be Radon measures on $\mathbb{R}^r$, and $G\subseteq\mathbb{R}^r$ an open set. Let Z be the support of λ, and for $x\in Z\cap G$ set

$$M(x)=\sup\{\frac{\lambda' B}{\lambda B}: B\subseteq G \text{ is a non-trivial ball with centre } x\}.$$

Then

$$\lambda\{x: x\in Z,\, M(x)\geq t\}\leq\frac{5^r}{t}\lambda' G$$

for every $t>0$.

proof The function $M: Z\to[0,\infty]$ is lower semi-continuous. **P** If $M(x)>t\geq 0$, there is a $\delta>0$ such that $B(x,\delta)\subseteq G$ and $\lambda' B(x,\delta)>t\lambda B(x,\delta)$. Because λ is a Radon measure, there is an open set $V\supseteq B(x,\delta)$ such that $V\subseteq G$ and $\lambda' B(x,\delta)>t\lambda V$; because $B(x,\delta)$ is compact, there is an $\eta>0$ such that $B(x,\delta+2\eta)\subseteq V$. Now if $y\in Z$ and $\|y-x\|\leq\eta$,

$$B(x,\delta)\subseteq B(y,\delta+\eta)\subseteq V,$$

so $\lambda' B(y,\delta+\eta)>t\lambda B(y,\delta+\eta)$ and $M(y)>t$. **Q**

In particular, $H_t=\{x: x\in Z\cap G,\, M(x)>t\}$ is always measured by λ. Now, given $t>0$, let $\mathcal{I}$ be the set of non-trivial closed balls $B\subseteq G$ such that $\lambda' B>t\lambda B$. By 472B, there is a family $\langle\mathcal{I}_k\rangle_{k<5^r}$ of countable disjoint subsets of $\mathcal{I}$ such that $\bigcup_{k<5^r}\mathcal{I}_k$ covers H_t. So

$$\lambda H_t\leq\sum_{k=0}^{5^r-1}\sum_{B\in\mathcal{I}_k}\lambda B\leq\frac{1}{t}\sum_{k=0}^{5^r-1}\sum_{B\in\mathcal{I}_k}\lambda' B\leq\frac{5^r}{t}\lambda' G,$$

as claimed.

***472F Theorem** Let λ be a Radon measure on $\mathbb{R}^r$, and $f\in\mathcal{L}^p(\lambda)$ any function, where $1<p<\infty$. Let Z be the support of λ, and for $x\in Z$ set $f^*(x)=\sup_{\delta>0}\frac{1}{\lambda B(x,\delta)}\int_{B(x,\delta)}|f|d\lambda$. Then f^* is lower semi-continuous, and $\|f^*\|_p\leq 2\big(\frac{5^r p}{p-1}\big)^{1/p}\|f\|_p$.

proof (a) Replacing f by $|f|$ if necessary, we may suppose that $f\geq 0$. Z is λ-conegligible, so that f^* is defined λ-almost everywhere. Next, f^* is lower semi-continuous. **P** I repeat an idea from the proof of 472E. If $f^*(x)>t\geq 0$, there is a $\delta>0$ such that $\int_{B(x,\delta)}|f|d\lambda>t\lambda B(x,\delta)$. Because λ is a Radon measure, there is an open set $V\supseteq B(x,\delta)$ such that $\int_{B(x,\delta)}|f|d\lambda>t\lambda V$; because $B(x,\delta)$ is compact, there is an $\eta>0$ such that $B(x,\delta+2\eta)\subseteq V$; and now $f^*(y)>t$ for every $y\in Z\cap B(x,\eta)$. **Q**

(b) For $t>0$, set $H_t=\{x: x\in Z,\, f^*(x)>t\}$, $F_t=\{x: x\in\operatorname{dom} f,\, f(x)\geq t\}$. Then

$$\lambda H_t\leq\frac{2\cdot 5^r}{t}\int_{F_{t/2}}f d\lambda$$

for every $t>0$. **P** Set $g=f\times\chi F_{t/2}$. Because $(\frac{t}{2})^p\lambda F_{t/2}\leq\|f\|_p^p$ is finite, $\lambda F_{t/2}$ is finite, $\chi F_{t/2}\in\mathcal{L}^q(\lambda)$ (where $\frac{1}{p}+\frac{1}{q}=1$) and $g\in\mathcal{L}^1(\lambda)$ (244E). Let λ' be the indefinite-integral measure defined by g over λ (§234); then λ' is totally finite, and is a Radon measure (416S). Set

$$M(x) = \sup\{\frac{\lambda' B}{\lambda B} : B \subseteq \mathbb{R}^r \text{ is a non-trivial ball with centre } x\}$$

for $x \in Z$. Then $f^*(x) \leq M(x) + \frac{t}{2}$ for every $x \in Z$, just because

$$\int_B f d\lambda \leq \frac{t}{2}\lambda B + \int_B g\, d\lambda = \frac{t}{2}\lambda B + \lambda' B$$

for every closed ball B. Accordingly

$$\lambda H_t \leq \lambda\{x : M(x) > \frac{t}{2}\} \leq \frac{2\cdot 5^r}{t}\lambda' \mathbb{R}^r = \frac{2\cdot 5^r}{t}\int_{F_{t/2}} f d\lambda. \; \blacksquare$$

(c) As in part (c) of the proof of 286A, we now have

$$\begin{aligned}\int (f^*)^p d\lambda &= \int_0^\infty \lambda\{x : f^*(x)^p > t\}dt = p\int_0^\infty t^{p-1}\lambda\{x : f^*(x) > t\}dt \\ &\leq 2\cdot 5^r p \int_0^\infty t^{p-2}\int_{F_{t/2}} f d\lambda dt = 2\cdot 5^r p \int_{\mathbb{R}^r} f(x)\int_0^{2f(x)} t^{p-2} dt\lambda(dx) \\ &= 2\cdot 5^r p \int_{\mathbb{R}^r} \frac{2^{p-1}}{p-1} f(x)^p \lambda(dx) = \frac{2^p 5^r p}{p-1}\int f^p d\lambda.\end{aligned}$$

Taking pth roots, we have the result.

472X Basic exercises (a) Show that if λ, λ' are Radon measures on $\mathbb{R}^r$ which agree on closed balls, they are equal.

(b) Let λ be a Radon measure on $\mathbb{R}^r$. Let $A \subseteq \mathbb{R}^r$ be a non-empty set, and $\epsilon > 0$. Show that there is a sequence $\langle B_n\rangle_{n\in\mathbb{N}}$ of closed balls in $\mathbb{R}^r$, all of radius at most ϵ and with centres in A, such that $A \subseteq \bigcup_{n\in\mathbb{N}} B_n$ and $\sum_{n=0}^\infty \lambda B_n \leq \lambda^* A + \epsilon$.

(c) Let λ be a non-zero Radon measure on $\mathbb{R}^r$ and F its support. Show that we have a lower density ϕ (definition: 341C) for the subspace measure λ_F defined by setting $\phi E = \{x : x \in F, \lim_{\delta\downarrow 0}\frac{\lambda(E\cap B(x,\delta))}{\lambda B(x,\delta)} = 1\}$ whenever λ_F measures E.

472Y Further exercises (a) (i) Let $\mathcal{I}$ be a finite family of intervals (open, closed or half-open) in $\mathbb{R}$. Show that there are subfamilies $\mathcal{I}_0$, $\mathcal{I}_1 \subseteq \mathcal{I}$, both disjoint, such that $\mathcal{I}_0 \cup \mathcal{I}_1$ covers $\bigcup\mathcal{I}$. (*Hint*: induce on $\#(\mathcal{I})$.) Show that this remains true if any totally ordered set is put in place of $\mathbb{R}$. (ii) Show that if $\mathcal{I}$ is any family of non-trivial intervals in $\mathbb{R}$ such that none contains the centre of any other, then $\mathcal{I}$ is expressible as $\mathcal{I}_0 \cup \mathcal{I}_1$ where both $\mathcal{I}_0$ and $\mathcal{I}_1$ are disjoint.

(b) Let $m = m(r)$ be the largest number such that there are $u_1, \ldots, u_m \in \mathbb{R}^r$ such that $\|u_i\| = 1$ for every i, $\|u_i - u_j\| \geq 1$ for all $i \neq j$. Let $A \subseteq \mathbb{R}^r$ be a bounded set and $x \mapsto \delta_x : A \to]0,\infty[$ a bounded function; set $B_x = B(x,\delta_x)$ for $x \in A$. (i) Show that $m < 3^r$. (ii) Show that there is an $\epsilon \in \left]0, \frac{1}{10}\right]$ such that whenever $\|u_0\| = \ldots = \|u_m\| = 1$ there are distinct i, $j \leq m$ such that $u_i \cdot u_j > \frac{1}{2}(1+\epsilon)$. (iii) Suppose that u, $v \in \mathbb{R}^r$ are such that $\frac{1}{3} \leq \|u\| \leq 1$, $\|v\| \leq 1+\epsilon$ and $\|u - v\| > 1$. Show that the angle $u\widehat{\mathbf{0}}v$ has cosine at most $\frac{1}{2}(1+\epsilon)$. (*Hint*: maximise $\frac{a^2+b^2-c^2}{2ab}$ subject to $\frac{1}{3} \leq a \leq 1$, $b \leq 1+\epsilon$ and $c \geq 1$.) (iv) Suppose that $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in A such that $x_n \notin B_{x_i}$ for $i < n$ and $(1+\epsilon)\delta_{x_n} \geq \sup\{\delta_x : x \in A \setminus \bigcup_{i<n} B_{x_i}\}$ for every n. Show that $A \subseteq \bigcup_{n\in\mathbb{N}} B_{x_n}$. (v) Take $y \in \mathbb{R}^r$. Show that there is at most one n such that $\|y - x_n\| \leq \frac{1}{3}\delta_{x_n}$. (vi) Show that if $i < j$ and $\frac{1}{3}\delta_{x_i} \leq \|y - x_i\| \leq \delta_{x_i}$, $\|y - x_j\| \leq \delta_j$ then the cosine of the angle $x_i\widehat{y}x_j$ is at most $\frac{1}{2}(1+\epsilon)$. (vii) Show that $\#(\{i : y \in B_{x_i}\}) \leq m+1$.

Hence show that if $\mathcal{I}$ is any family of non-trivial closed balls such that every point of A is the centre of some member of $\mathcal{I}$, then there is a countable $\mathcal{I}_0 \subseteq \mathcal{I}$, covering A, such that no point of $\mathbb{R}^r$ belongs to more than 3^r members of $\mathcal{I}_0$.

(c) Use 472Yb to prove an alternative version of 472B, but with the constant 9^r in place of 5^r.

(d) Let λ be a Radon measure on $\mathbb{R}^r$, and f a λ-integrable real-valued function. Show that $f^*(x) = \sup_{\delta>0} \frac{1}{\lambda B(x,\delta)} \int_{B(x,\delta)} |f| d\lambda$ is defined and finite for λ-almost every $x \in \mathbb{R}^r$.

(e) Let λ, λ' be Radon measures on $\mathbb{R}^r$. (i) Show that $g(x) = \lim_{\delta \downarrow 0} \frac{\lambda' B_r(x)}{\lambda B_r(x)}$ is defined in $\mathbb{R}$ for λ-almost every x. (ii) Setting $\lambda_0 = \sup_{n\in\mathbb{N}} \lambda' \wedge n\lambda$ in the cone of Radon measures on $\mathbb{R}^r$ (437Yj), show that g is a Radon-Nikodým derivative of λ_0 with respect to λ. (*Hint*: show that if λ and λ' are mutually singular then $g = 0$ λ-a.e.)

472 Notes and comments I gave primacy to the 'weak' Vitali's theorem in 261B because I think it is easier than the 'strong' form in 472C, it uses the same ideas as the original one-dimensional theorem in 221A, and it is adequate for the needs of Volume 2. Any proper study of general measures on $\mathbb{R}^r$, however, will depend on the ideas in 472A-472C. You will see that in 472B, as in other forms of Vitali's theorem, there is a key step in which a sequence is chosen greedily. This time we must look much more carefully at the geometry of $\mathbb{R}^r$ because we can no longer rely on a measure to tell us what is happening. (Though you will observe that I still use the elementary properties of Euclidean volume in the argument of 472A.) Once we have reached 472C, however, we are in a position to repeat all the arguments of 261C-261E in much greater generality (472D), and, as a bonus, can refine 261F (472Xb). For more in this direction see Mattila 95 and Federer 69, §2.8.

It is natural to ask whether the constant '5^r' in 472A is best possible. The argument of 472A is derived from Sullivan 94, where a more thorough analysis is given. It seems that even for $r = 2$ the best constant is unknown. (For $r = 1$, the best constant is 2; see 472Ya.) Note that even for finite families $\mathcal{I}$ we should have to find the colouring number of a graph (counting two balls as linked if they intersect), so it may well be a truly difficult problem. The method in 472B amounts to using the greedy colouring algorithm after ordering the balls by size, and one does not expect such approaches to give exact colouring numbers. Of course the questions addressed here depend only on the existence of *some* function of r to do the job.

An alternative argument runs through a kind of pointwise version of 472A (472Yb-472Yc). It gives a worse constant but is attractive in other ways. For many of the applications of 472C, the result of 472Yb is already sufficient.

The constant $2\big(\frac{5^r p}{p-1}\big)^{1/p}$ in 472F makes no pretence to be 'best', or even 'good'. The only reason for giving a formula at all is to emphasize the remarkable fact that it does not depend on the measure λ. The theorems of this section are based on the metric geometry of Euclidean space, not on any special properties of Lebesgue measure. The constants *do* depend on the dimension, so that even in Hilbert space (for instance) we cannot expect any corresponding results.

473 Poincaré's inequality

In this section I embark on the main work of the chapter, leading up to the Divergence Theorem in §475. I follow the method in Evans & Gariepy 92. The first step is to add some minor results on differentiable and Lipschitz functions to those already set out in §262 (473B-473C). Then we need to know something about convolution products (473D), extending ideas in §§256 and 444; in particular, it will be convenient to have a fixed sequence $\langle \tilde{h}_n \rangle_{n\in\mathbb{N}}$ of smoothing functions with some useful special properties (473E).

The new ideas of the section begin with the Gagliardo-Nirenberg-Sobolev inequality, relating $\|f\|_{r/(r-1)}$ to $\|\operatorname{grad} f\|_1$. In its simplest form (473H) it applies only to functions with compact support; we need to work much harder to get results which we can use to estimate $\int_B |f|^{r/(r-1)}$ in terms of $\int_B \|\operatorname{grad} f\|$ and $\int_B f$ for balls B (473I, 473K).

473A Notation For the next three sections, $r \geq 2$ will be a fixed integer. For $x \in \mathbb{R}^r$ and $\delta \geq 0$, $B(x,\delta) = \{y : \|y - x\| \leq \delta\}$ will be the closed ball with centre x and radius δ. I will write $\partial B(x,\delta)$ for the

boundary of $B(x,\delta)$, the sphere $\{y : \|y-x\| = \delta\}$. $S_{r-1} = \partial B(\mathbf{0},1)$ will be the unit sphere. As in Chapter 26, I will use Greek letters to represent coordinates of vectors, so that $x = (\xi_1, \dots, \xi_r)$, etc.

μ will always be Lebesgue measure on $\mathbb{R}^r$. $\beta_r = \mu B(\mathbf{0},1)$ will be the r-dimensional volume of the unit ball, that is,

$$\begin{aligned}\beta_r &= \frac{\pi^k}{k!} \text{ if } r = 2k \text{ is even,}\\ &= \frac{2^{2k+1}k!\pi^k}{(2k+1)!} \text{ if } r = 2k+1 \text{ is odd}\end{aligned}$$

(252Q). ν will be normalized Hausdorff $r-1$-dimensional measure on $\mathbb{R}^r$, that is, $\nu = 2^{-r+1}\beta_{r-1}\mu_{H,r-1}$, where $\mu_{H,r-1}$ is $r-1$-dimensional Hausdorff measure on $\mathbb{R}^r$ as described in §264, and

$$\begin{aligned}\beta_{r-1} &= \frac{2^{2k}k!\pi^{k-1}}{(2k)!} \text{ if } r = 2k \text{ is even,}\\ &= \frac{\pi^k}{k!} \text{ if } r = 2k+1 \text{ is odd}\end{aligned}$$

is the Lebesgue measure of a ball of diameter 1 in $\mathbb{R}^{r-1}$ (264I). Recall from 265F and 265H that $\nu S_{r-1} = 2\pi\beta_{r-2} = r\beta_r$ (counting β_0 as 1).

473B Differentiable functions (a) Recall from §262 that a function ϕ from a subset of $\mathbb{R}^r$ to $\mathbb{R}^s$ (where $s \geq 1$) is differentiable at $x \in \mathbb{R}^r$, with derivative an $s \times r$ matrix T, if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|\phi(y) - \phi(x) - T(y-x)\| \leq \epsilon\|y-x\|$ whenever $\|y-x\| \leq \delta$; this includes the assertion that $B(x,\delta) \subseteq \operatorname{dom}\phi$. In this case, the coefficients of T are the partial derivatives $\frac{\partial\phi_j}{\partial\xi_i}(x)$ at x, where $\phi_1, \dots, \phi_s$ are the coordinate functions of ϕ, and $\frac{\partial}{\partial\xi_i}$ represents partial differentiation with respect to the ith coordinate (262Ic).

(b) When $s = 1$, so that we have a real-valued function f defined on a subset of $\mathbb{R}^r$, I will write $(\operatorname{grad} f)(x)$ for the derivative of f at x, the **gradient** of f. If we strictly adhere to the language of (a), $\operatorname{grad} f$ is a $1 \times r$ matrix $\begin{pmatrix}\frac{\partial f}{\partial\xi_1} & \dots & \frac{\partial f}{\partial\xi_r}\end{pmatrix}$; but it is convenient to treat it as a vector, so that $\operatorname{grad} f(x)$ (when defined) belongs to $\mathbb{R}^r$, and we can speak of $y \cdot \operatorname{grad} f(x)$ rather than $(\operatorname{grad} f(x))(y)$, etc.

(c) Chain rule for functions of many variables I find that I have not written out the following basic fact. Let $\phi : A \to \mathbb{R}^s$ and $\psi : B \to \mathbb{R}^p$ be functions, where $A \subseteq \mathbb{R}^r$ and $B \subseteq \mathbb{R}^s$. If $x \in A$ is such that ϕ is differentiable at x, with derivative S, and ψ is differentiable at $\phi(x)$, with derivative T, then the composition $\psi\phi$ is differentiable at x, with derivative TS.

P Recall that if we regard S and T as linear operators, they have finite norms (262H). Given $\epsilon > 0$, let $\eta > 0$ be such that $\eta\|T\| + \eta(\|S\| + \eta) \leq \epsilon$. Let $\delta_1, \delta_2 > 0$ be such that $\phi(y)$ is defined and $\|\phi(y) - \phi(x) - S(y-x)\| \leq \eta\|y-x\|$ whenever $\|y-x\| \leq \delta_1$, and $\psi(z)$ is defined and $\|\psi(z) - \psi\phi(x) - T(z-\phi(x))\| \leq \eta\|z-\phi(x)\|$ whenever $\|z-\phi(x)\| \leq \delta_2$. Set $\delta = \min(\delta_1, \frac{\delta_2}{\eta+\|S\|}) > 0$. If $\|y-x\| \leq \delta$, then $\phi(y)$ is defined and

$$\|\phi(y) - \phi(x)\| \leq \|S(y-x)\| + \|\phi(y) - \phi(x) - S(y-x)\| \leq (\|S\| + \eta)\|y-x\| \leq \delta_2,$$

so $\psi\phi(y)$ is defined and

$$\begin{aligned}&\|\psi\phi(y) - \psi\phi(x) - TS(y-x)\|\\ &\quad\leq \|\psi\phi(y) - \psi\phi(x) - T(\phi(y) - \phi(x))\| + \|T\|\|\phi(y) - \phi(x) - S(y-x)\|\\ &\quad\leq \eta\|\phi(y) - \phi(x) - S(y-x)\| + \|T\|\eta\|y-x\|\\ &\quad\leq \eta(\|S\| + \eta)\|y-x\| + \|T\|\eta\|y-x\| \leq \epsilon\|y-x\|;\end{aligned}$$

as ϵ is arbitrary, $\psi\phi$ is differentiable at x with derivative TS. **Q**

(d) It follows that if f and g are real-valued functions defined on a subset of $\mathbb{R}^r$, and $x \in \operatorname{dom} f \cap \operatorname{dom} g$ is such that $(\operatorname{grad} f)(x)$ and $(\operatorname{grad} g)(x)$ are both defined, then $\operatorname{grad}(f \times g)(x)$ is defined and equal to $f(x)\operatorname{grad} g(x) + g(x)\operatorname{grad} f(x)$. **P** Set $\phi(y) = \begin{pmatrix} f(y) \\ g(y) \end{pmatrix}$ for $y \in \operatorname{dom} f \cap \operatorname{dom} g$; then ϕ is differentiable at x with derivative the $2 \times r$ matrix $\begin{pmatrix} \operatorname{grad} f(x) \\ \operatorname{grad} g(x) \end{pmatrix}$ (262Ib). Set $\psi(z) = \zeta_1\zeta_2$ for $z = (\zeta_1, \zeta_2) \in \mathbb{R}^2$; then ψ is differentiable everywhere, with derivative the 1×2 matrix $(\,\zeta_2 \quad \zeta_1\,)$. So $f \times g = \psi\phi$ is differentiable at x with derivative

$$(\,g(x) \quad f(x)\,)\begin{pmatrix} \operatorname{grad} f(x) \\ \operatorname{grad} g(x) \end{pmatrix} = g(x)\operatorname{grad} f(x) + f(x)\operatorname{grad} g(x). \quad \textbf{Q}$$

(e) Let D be a subset of $\mathbb{R}^r$ and $\phi : D \to \mathbb{R}^s$ any function. Set $D_0 = \{x : x \in D, \phi \text{ is differentiable at } x\}$. Then the derivative of ϕ, regarded as a function from D_0 to $\mathbb{R}^{rs}$, is (Lebesgue) measurable. **P** Use 262P; the point is that, writing $T(x)$ for the derivative of ϕ at x, $T(x)$ is surely a derivative of $\phi\restriction D_0$, relative to D_0, at every point of D_0. **Q** (See also 473Ya.)

(f) A function $\phi : \mathbb{R}^r \to \mathbb{R}^s$ is **smooth** if it is differentiable arbitrarily often; that is, if all its repeated partial derivatives

$$\frac{\partial^m \phi_j}{\partial\xi_{i_1}\ldots\partial\xi_{i_m}}$$

are defined and continuous everywhere on $\mathbb{R}^r$. I will write $\mathcal{D}$ for the family of real-valued functions on $\mathbb{R}^r$ which are smooth and have compact support.

473C Lipschitz functions (a) If f and g are bounded real-valued Lipschitz functions, defined on any subsets of $\mathbb{R}^r$, then $f \times g$, defined on $\operatorname{dom} f \cap \operatorname{dom} g$, is Lipschitz. **P** Let γ_f, M_f, γ_g and M_g be such that $|f(x)| \le M_f$ and $|f(x)-f(y)| \le \gamma_f\|x-y\|$ for all $x, y \in \operatorname{dom} f$, while $|g(x)| \le M_g$ and $|g(x)-g(y)| \le \gamma_g\|x-y\|$ for all $x, y \in \operatorname{dom} g$. Then for any $x, y \in \operatorname{dom} f \cap \operatorname{dom} g$,

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &\le |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)| \\ &\le (M_f\gamma_g + M_g\gamma_f)\|x - y\|. \end{aligned}$$

So $M_f\gamma_g + M_g\gamma_f$ is a Lipschitz constant for $f \times g$. **Q**

(b) Suppose that F_1, $F_2 \subseteq \mathbb{R}^r$ are closed sets with convex union C. Let $f : C \to \mathbb{R}$ be a function such that $f\restriction F_1$ and $f\restriction F_2$ are both Lipschitz. Then f is Lipschitz. **P** For each j, let γ_j be a Lipschitz constant for $f\restriction F_j$, and set $\gamma = \max(\gamma_1, \gamma_2)$, so that γ is a Lipschitz constant for both $f\restriction F_1$ and $f\restriction F_2$. Take any x, $y \in C$. If both belong to the same F_j, then $|f(x) - f(y)| \le \gamma\|x - y\|$. If $x \in F_j$ and $y \notin F_j$, then y must belong to F_{3-j}, and $(1-t)x + ty \in F_1 \cup F_2$ for every $t \in [0,1]$, because C is convex. Set $t_0 = \sup\{t : t \in [0,1], (1-t)x + ty \in F_j\}$, $z = (1-t_0)x + t_0y$; then $z \in F_1 \cap F_2$, because both are closed, so

$$|f(x) - f(y)| \le |f(x) - f(z)| + |f(z) - f(y)| \le \gamma\|x - z\| + \gamma\|z - y\| = \gamma\|x - y\|.$$

As x and y are arbitrary, γ is a Lipschitz constant for f. **Q**

(c) Suppose that $f : \mathbb{R}^r \to \mathbb{R}$ is Lipschitz. Recall that by Rademacher's theorem (262Q), $\operatorname{grad} f$ is defined almost everywhere. All the partial derivatives of f are (Lebesgue) measurable, by 473Be, so $\operatorname{grad} f$ is (Lebesgue) measurable on its domain. If γ is a Lipschitz constant for f, $\|\operatorname{grad} f(x)\| \le \gamma$ whenever $\operatorname{grad} f(x)$ is defined. **P** If $z \in \mathbb{R}^r$, then

$$\lim_{t\downarrow 0} \frac{1}{t}|f(x + tz) - f(x) - tz\,.\operatorname{grad} f(x)| = 0,$$

so

$$|z\,.\operatorname{grad} f(x)| = \lim_{t\downarrow 0} \frac{1}{t}|f(x + tz) - f(x)| \le \gamma\|z\|;$$

as z is arbitrary, $\|\operatorname{grad} f(x)\| \le \gamma$. **Q**

(d) Conversely, if $f : \mathbb{R}^r \to \mathbb{R}$ is differentiable and $\|\operatorname{grad} f(x)\| \leq \gamma$ for every x, then γ is a Lipschitz constant for f. **P** Take $x, y \in \mathbb{R}^r$. Set $g(t) = f((1-t)x+ty)$ for $t \in \mathbb{R}$. The function $t \mapsto (1-t)x+ty : \mathbb{R} \to \mathbb{R}^r$ is everywhere differentiable, with constant derivative $y - x$, so by 473Bc g is differentiable, with derivative $g'(t) = y - x \cdot gradf(g(t))$ for every t; in particular, $|g'(t)| \leq \gamma\|y - x\|$ for every t. Now, by the Mean Value Theorem, there is a $t \in [0, 1]$ such that $g(1) - g(0) = g'(t)$, so that $|f(y) - f(x)| = |g'(t)| \leq \gamma\|y - x\|$. As x and y are arbitrary, f is γ-Lipschitz. **Q**

(e) Note that if $f \in \mathcal{D}$ then all its partial derivatives are continuous functions with compact support, so are bounded (436Ia), and f is Lipschitz as well as bounded, by (d) here.

(f)(i) If $D \subseteq \mathbb{R}^r$ is bounded and $f : D \to \mathbb{R}$ is Lipschitz, then there is a Lipschitz function $g : \mathbb{R}^r \to \mathbb{R}$, with compact support, extending f. **P** By 262Bb there is a Lipschitz function $f_1 : \mathbb{R}^r \to \mathbb{R}$ which extends f. Let $\gamma > 0$ be such that $D \subseteq B(0, \gamma)$ and γ is a Lipschitz constant for f_1; set $M = |f_1(0)| + \gamma^2$; then $|f_1(x)| \leq M$ for every $x \in D$, so if we set $f_2(x) = \max(-M, \min(f(x), M))$ for $x \in \mathbb{R}^r$, f_2 is a bounded Lipschitz function extending f. Let $f_3(x) = \max(0, 1 - \frac{\|x\|}{\gamma})$ for $x \in \mathbb{R}^r$; then f_3 is a bounded Lipschitz function with compact support. By (a), $g = f_3 \times f_2$ is Lipschitz, and $g : \mathbb{R}^r \to \mathbb{R}$ is a function with compact support extending f. **Q**

(ii) It follows that if $D \subseteq \mathbb{R}^r$ is bounded and $f : D \to \mathbb{R}^s$ is Lipschitz, then there is a Lipschitz function $g : \mathbb{R}^r \to \mathbb{R}^s$, with compact support, extending f. **P** By 262Ba, we need only apply (i) to each coordinate of f. **Q**

473D Smoothing by convolution We shall need a miscellany of facts, many of them special cases of results in §§255 and 444, concerning convolutions on $\mathbb{R}^r$. Recall that I write $(f * g)(x) = \int f(y)g(x-y)\mu(dy)$ whenever f and g are real-valued functions defined almost everywhere in $\mathbb{R}^r$ and the integral is defined. Recall that $f * g = g * f$ (255Fb, 444Og). Now we have the following.

Lemma Suppose that f and g are Lebesgue measurable real-valued functions defined μ-almost everywhere in $\mathbb{R}^r$.

(a) If f is integrable and g is essentially bounded, then their convolution $f * g$ is defined everywhere in $\mathbb{R}^r$ and uniformly continuous, and $\sup_{x \in \mathbb{R}^r} |(f * g)(x)| \leq \|f\|_1 \|g\|_\infty$.

(b) If f is locally integrable and g is bounded and has compact support, then $f * g$ is defined everywhere in $\mathbb{R}^r$ and is continuous.

(c) If f and g are defined everywhere in $\mathbb{R}^r$ and $x \in \mathbb{R}^r \setminus (\{y : f(y) \neq 0\} + \{z : g(z) \neq 0\})$, then $(f * g)(x)$ is defined and equal to 0.

(d) If f is integrable and g is bounded, Lipschitz and defined everywhere, then $f * \operatorname{grad} g$ and $\operatorname{grad}(f * g)$ are defined everywhere and equal, where $f * \operatorname{grad} g = (f * \frac{\partial g}{\partial \xi_1}, \dots, f * \frac{\partial g}{\partial \xi_r})$. Moreover, $f * \operatorname{grad} g = \operatorname{grad}(f * g)$ is Lipschitz.

(e) If f is locally integrable, and $g \in \mathcal{D}$, then $f * g$ is defined everywhere and is smooth.

(f) If f is essentially bounded and $g \in \mathcal{D}$, then $f * g$ is Lipschitz as well as smooth.

(g) If f is integrable and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a bounded measurable function with components $\phi_1, \dots, \phi_r$, and we write $(f * \phi)(x) = ((f * \phi_1)(x), \dots, (f * \phi_r)(x))$, then $\|(f * \phi)(x)\| \leq \|f\|_1 \sup_{y \in \mathbb{R}^r} \|\phi(y)\|$ for every $x \in \mathbb{R}^r$.

proof (a) See 255K.

(b) Suppose that $g(y) = 0$ when $\|y\| \geq n$. Given $x \in \mathbb{R}^r$, set $\tilde{f} = f \times \chi B(x, n + 1)$. Then $\tilde{f} * g$ is defined everywhere and continuous, by (a), while $(f * g)(z) = (\tilde{f} * g)(z)$ whenever $z \in B(x, 1)$; so $f * g$ is defined everywhere in $B(x, 1)$ and is continuous at x.

(c) We have only to note that $f(y)g(x - y) = 0$ for every y.

(d) Let γ be a Lipschitz constant for g. We know that $\operatorname{grad} g$ is defined almost everywhere, is measurable, and that $\|\operatorname{grad} g(x)\| \leq \gamma$ whenever it is defined (473Cc); so $(f * \operatorname{grad} g)(x)$ is defined for every x, by (a) here. Fix $x \in \mathbb{R}^r$. If $y, z \in \mathbb{R}^r$ set

$$\theta(y,z) = \frac{1}{\|z\|}\big(g(x-y+z) - g(x-y) - z\,.\,\operatorname{grad} g(x-y)\big)$$

whenever this is defined. Then $|\theta(y,z)| \leq 2\gamma$ whenever it is defined. Now suppose that $\langle z_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^r \setminus \{\mathbf{0}\}$ converging to $\mathbf{0}$. Then $\lim_{n\to\infty}\theta(y,z_n) = 0$ whenever $\operatorname{grad} g(x-y)$ is defined, which almost everywhere. So $\lim_{n\to\infty}\int f(y)\theta(y,z_n)\mu(dy) = 0$, by Lebesgue's Dominated Convergence Theorem. But this means that

$$\frac{1}{\|z_n\|}\big((f*g)(x+z_n) - (f*g)(x) - ((f*\operatorname{grad} g)(x))\,.\,z_n\big) \to 0$$

as $n\to\infty$. As $\langle z_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\operatorname{grad}(f*g)(x)$ is defined and is equal to $(f*\operatorname{grad} g)(x)$.

Now $\operatorname{grad} g$ is bounded, because g is Lipschitz, so $\operatorname{grad}(f*g) = f*\operatorname{grad} g$ is also bounded, by (a), and $f*g$ must be Lipschitz (473Cd).

(e) By (b), $f*g$ is defined everywhere and is continuous. Now, for any $i\leq r$, $\frac{\partial}{\partial\xi_i}(f*g) = f*\frac{\partial g}{\partial\xi_i}$ everywhere. **P** Let $n\in\mathbb{N}$ be such that $g(y)=0$ if $\|y\|\geq n$. Given $x\in\mathbb{R}^r$, set $\tilde f = f\times\chi B(x,n+1)$. Then $(f*g)(z) = (\tilde f*g)(z)$ for every $z\in B(x,1)$, so that

$$\frac{\partial f*g}{\partial\xi_i}(x) = \frac{\partial\tilde f*g}{\partial\xi_i}(x) = (\tilde f*\frac{\partial g}{\partial\xi_i})(x)$$

(by (d))

$$= (f*\frac{\partial g}{\partial\xi_i})(x)$$

(because of course $\frac{\partial g}{\partial\xi_i}$ is also zero outside $B(\mathbf{0},n)$). **Q** Inducing on k,

$$\frac{\partial^k}{\partial\xi_{i_1}\dots\partial\xi_{i_k}}(f*g)(x) = (f*\frac{\partial^k g}{\partial\xi_{i_1}\dots\partial\xi_{i_k}})(x)$$

for every $x\in\mathbb{R}^r$ and every $i_1,\dots,i_k$; so we have the result.

(f) The point is just that all the partial derivatives of g, being smooth functions with compact support, are integrable, and that

$$|\frac{\partial}{\partial\xi_i}(f*g)(x)| = |(f*\frac{\partial g}{\partial\xi})(x)| \leq \|f\|_\infty\|\frac{\partial g}{\partial\xi_i}\|_1$$

for every x and every $i\leq r$. This means that $\operatorname{grad}(f*g)$ is bounded, so that $f*g$ is Lipschitz, by 473Cd.

(g) If x, $z\in\mathbb{R}^r$, then

$$\begin{aligned} z\,.\,(f*\phi)(x) &= \sum_{i=1}^r \zeta_i(f*\phi_i)(x) = \int f(y)\sum_{i=1}^r\zeta_i\phi_i(x-y)\mu(dy)\\ &\leq \int |f(y)|\,|\sum_{i=1}^r\zeta_i\phi_i(x-y)|\mu(dy)\\ &\leq \int |f(y)|\|z\|\|\phi(x-y)\|\mu(dy) \leq \|z\|\|f\|_1\sup_{y\in\mathbb{R}^r}\|\phi(y)\|. \end{aligned}$$

As z is arbitrary, $\|(f*\phi)(x)\|\leq\|f\|_1\sup_{y\in\mathbb{R}^r}\|\phi(y)\|$.

473E Lemma (a) Define $h:\mathbb{R}\to[0,1]$ by setting $h(t) = \exp(\frac{1}{t^2-1})$ for $|t|<1$, 0 for $|t|\geq 1$. Then h is smooth, and $h'(t)\leq 0$ for $t\geq 0$.

(b) For $n\geq 1$, define $\tilde h_n:\mathbb{R}^r\to\mathbb{R}$ by setting

$$\alpha_n = \int h((n+1)^2\|x\|^2)\mu(dx), \quad \tilde h_n(x) = \frac{1}{\alpha_n}h((n+1)^2\|x\|^2)$$

for every $x \in \mathbb{R}^r$. Then $\tilde{h}_n \in \mathcal{D}$, $\tilde{h}_n(x) \geq 0$ for every x, $\tilde{h}_n(x) = 0$ if $\|x\| \geq \frac{1}{n+1}$, and $\int \tilde{h}_n d\mu = 1$.

(c) If $f \in \mathcal{L}^0(\mu)$, then $f(x) = \lim_{n\to\infty}(f * \tilde{h}_n)(x) = f(x)$ for every $x \in \operatorname{dom} f$ at which f is continuous.

(d) If $f : \mathbb{R}^r \to \mathbb{R}$ is uniformly continuous (in particular, if it is either Lipschitz or a continuous function with compact support), then $\lim_{n\to\infty} \|f - f * \tilde{h}_n\|_\infty = 0$.

(e) If $f \in \mathcal{L}^0(\mu)$ is locally integrable, then $f(x) = \lim_{n\to\infty}(f * \tilde{h}_n)(x)$ for μ-almost every $x \in \mathbb{R}^r$.

(f) If $f \in \mathcal{L}^p(\mu)$, where $1 \leq p < \infty$, then $\lim_{n\to\infty} \|f - f * \tilde{h}_n\|_p = 0$.

proof (a) Set $h_0(t) = \exp(-\frac{1}{t})$ for $t > 0$, 0 for $t \leq 0$. A simple induction on n shows that the nth derivative $h_0^{(n)}$ of h_0 is of the form

$$h_0^{(n)}(t) = q_n(\tfrac{1}{t}) \exp(-\tfrac{1}{t}) \text{ for } t > 0,$$
$$= 0 \text{ for } t \leq 0,$$

where each q_n is a polynomial of degree $2n$; the inductive hypothesis depends on the fact that $\lim_{s\to\infty} q(s)e^{-s} = 0$ for every polynomial q. So h_0 is smooth. Now $h(t) = h_0(1 - t^2)$ so h also is smooth. If $0 \leq t < 1$ then

$$h'(t) = -\exp(\tfrac{1}{t^2-1}) \cdot \tfrac{2t}{(t^2-1)^2} < 0;$$

if $t > 1$ then $h'(t) = 0$; since h' is continuous, $h'(t) \geq 0$ for every $t \geq 0$.

(b) We need only observe that

$$x \mapsto (n+1)^2\|x\|^2 = (n+1)^2 \textstyle\sum_{i=1}^r \xi_i^2$$

is smooth and that the composition of smooth functions is smooth (by 473Bc).

(c) If f is continuous at x and $\epsilon > 0$, let $n_0 \in \mathbb{N}$ be such that $|f(y) - f(x)| \leq \epsilon$ whenever $y \in \operatorname{dom} f$ and $\|y - x\| \leq \frac{1}{n_0+1}$. Then for any $n \geq n_0$,

$$\begin{aligned}|(f * \tilde{h}_n)(x) - f(x)| &= |\int f(x-y)\tilde{h}_n(y)\mu(dy) - \int f(x)\tilde{h}_n(y)\mu(dy)| \\ &\leq \int |f(x-y) - f(x)|\tilde{h}_n(y)\mu(dy) \leq \int \epsilon\tilde{h}_n(y)\mu(dy) = \epsilon.\end{aligned}$$

As ϵ is arbitrary, we have the result.

(d) Repeat the argument of (c), but 'uniformly in x'; that is, given $\epsilon > 0$, take n_0 such that $|f(y) - f(x)| \leq \epsilon$ whenever x, $y \in \mathbb{R}^r$ and $\|y - x\| \leq \frac{1}{n_0+1}$, and see that $|(f * \tilde{h}_n)(x) - f(x)| \leq \epsilon$ for every $n \geq n_0$ and every x.

(e) We know from 472Db or 261E that, for almost every $x \in \mathbb{R}^r$,

$$\lim_{\delta\downarrow 0} \tfrac{1}{\mu B(x,\delta)} \textstyle\int_{B(x,\delta)} |f(y) - f(x)|\mu(dy) = 0.$$

Take any such x. Set $\gamma = f(x)$, Set $g(y) = |f(y) - \gamma|$ for every $y \in \operatorname{dom} f$. Let $\epsilon > 0$. Then there is some $\delta > 0$ such that $\frac{q(t)}{\beta_r t^r} \leq \epsilon$ whenever $0 < t \leq \delta$, where

$$q(t) = \textstyle\int_{B(x,t)} g\, d\mu = \int_0^t \int_{\partial B(x,s)} g(y)\nu(dy)dt$$

by 265G, so $q'(t) = \int_{\partial B(y,t)} g\, d\nu$ for almost every $t \in [0, \delta]$, by 222E. If $n + 1 \geq \frac{1}{\delta}$, then

$$\begin{aligned}(g * \tilde{h}_n)(x) &= \int g(y)\tilde{h}_n(x-y)\mu(dy) = \int_{B(x,\delta)} g(y)\tilde{h}_n(x-y)\mu(dy) \\ &= \frac{1}{\alpha_n}\int_0^\delta \int_{\partial B(x,t)} g(y)h((n+1)^2t^2)\nu(dy)dt\end{aligned}$$

(265G again)

$$= \frac{1}{\alpha_n}\int_0^\delta h((n+1)^2t^2)q'(t)dt$$
$$= -\frac{1}{\alpha_n}\int_0^\delta 2(n+1)^2h'((n+1)^2t^2)q(t)dt$$

(integrating by parts (225F), because $q(0) = h((n+1)^2\delta^2) = 0$ and both q and h are absolutely continuous)

$$\leq -\frac{\epsilon}{\alpha_n}\int_0^\delta 2(n+1)^2h'((n+1)^2t^2)\beta_r t^r dt$$

(because $0 \leq q(t) \leq \epsilon\beta_r t^r$ and $h'((n+1)^2t^2) \leq 0$ for $0 \leq t \leq \delta$)

$$= \epsilon$$

(applying the same calculations with $\chi\mathbb{R}^r$ in place of g). But now, since $(\gamma\chi\mathbb{R}^r * \tilde{h}_n)(x) = \gamma$ for every n,

$$|(f * \tilde{h}_n)(x) - \gamma| = |\int (f(y)-\gamma)\tilde{h}_n(x-y)\mu(dy)| \leq \int |f(y)-\gamma|\tilde{h}_n(x-y)\mu(dy) \leq \epsilon$$

whenever $n+1 \geq \frac{1}{\delta}$. As ϵ is arbitrary, $f(x) = \gamma = \lim_{n\to\infty}(f * \tilde{h}_n)(x)$; and this is true for μ-almost every x.

(f) Apply 444T with ν the indefinite-integral measure over μ defined by $\tilde{h}_n$; use 444P for the identification of $\nu * f$ with $\tilde{h}_n * f = f * \tilde{h}_n$.

473F Lemma For any measure space (X, Σ, λ) and any non-negative $f_1, \dots, f_k \in \mathcal{L}^0(\lambda)$,

$$\int \prod_{i=1}^k f_i^{1/k} d\lambda \leq \prod_{i=1}^k \big(\int f_i d\lambda\big)^{1/k}.$$

proof Induce on k. Note that we can suppose that every f_i is integrable; for if any $\int f_i$ is zero, then $f_i = 0$ a.e. and the result is trivial; and if all the $\int f_i$ are greater than zero and any of them is infinite, the result is again trivial.

The induction starts with the trivial case $k = 1$. For the inductive step to $k \geq 2$, we have

$$\int \prod_{i=1}^k f_i^{1/k} d\lambda \leq \| \prod_{i=1}^{k-1} f_i^{1/k}\|_{k/(k-1)} \|f_k^{1/k}\|_k$$

(by Hölder's inequality, 244E)

$$= \big(\int \prod_{i=1}^{k-1} f_i^{1/(k-1)} d\lambda\big)^{(k-1)/k} \big(\int f_k d\lambda\big)^{1/k}$$
$$\leq \big(\prod_{i=1}^{k-1} (\int f_i d\lambda)^{1/(k-1)}\big)^{(k-1)/k} \big(\int f_k d\lambda\big)^{1/k}$$

(by the inductive hypothesis)

$$= \prod_{i=1}^k (\int f_i d\lambda)^{1/k},$$

as required.

473F Lemma Let (X, Σ, λ) be a σ-finite measure space and $k \geq 2$ an integer. Write λ_k for the product measure on X^k. For $x = (\xi_1, \dots, \xi_k) \in X^k$, $t \in X$, $1 \leq i \leq k$ set $S_i(x,t) = (\xi_1', \dots, \xi_k')$ where $\xi_i' = t$ and $\xi_j' = \xi_j$ for $j \neq i$. Then if $h \in \mathcal{L}^1(\lambda_k)$ is non-negative, and we set $h_i(x) = \int S_i(x,t)\lambda(dt)$ whenever this is defined in $\mathbb{R}$, we have

$$\int (\prod_{i=1}^k h_i)^{1/(k-1)} d\lambda_k \leq (\int h\, d\lambda_k)^{k/(k-1)}.$$

proof Induce on k.

(a) If $k = 2$, we have

$$\begin{aligned}\int h_1 \times h_2 d\lambda_k &= \iint \big(\int h(\tau_1, \xi_2) d\tau_1\big)\big(\int h(\xi_1, \tau_2) d\tau_2\big) d\xi_1 d\xi_2 \\ &= \iiint\int h(\tau_1, \xi_2) h(\xi_1, \tau_2) d\tau_1 d\tau_2 d\xi_1 d\xi_2 \\ &= \iint h(\tau_1, \xi_2) d\tau_1 d\xi_2 \cdot \iint h(\xi_1, \tau_2) d\tau_2 d\xi_1 = \big(\int h \, d\lambda_2\big)^2\end{aligned}$$

by Fubini's theorem (252B) used repeatedly, because (by 253D) $(\tau_1, \tau_2, \xi_1, \xi_2) \mapsto h(\xi_1, \tau_2) h(\tau_1, \xi_2)$ is λ_4-integrable. (See 251W for a sketch of the manipulations needed to apply 252B, as stated, to the integrals above.)

(b) For the inductive step to $k \geq 3$, argue as follows. For $y \in X^{k-1}$, set $g(y) = \int h(y,t) dt$ whenever this is defined in $\mathbb{R}$, identifying X^k with $X^{k-1} \times X$, so that $g(y) = h_k(y,t)$ whenever either is defined. If $1 \leq i < k$, we can consider $S_i(y,t)$ for $y \in X^{k-1}$ and $t \in X$, and we have

$$\int g(S_i(y,t)) dt = \iint h(S_i(y,t), u) du dt = \int h_i(y,t) dt$$

for almost every $y \in X^{k-1}$. So

$$\begin{aligned}\int (\prod_{i=1}^{k} h_i)^{1/(k-1)} d\lambda_k &= \iint (\prod_{i=1}^{k-1} h_i(y,t))^{1/(k-1)} g(y)^{1/(k-1)} dt \, \lambda_{k-1}(dy) \\ &= \int g(y)^{1/(k-1)} \int (\prod_{i=1}^{k-1} h_i(y,t))^{1/(k-1)} dt \, \lambda_{k-1}(dy) \\ &\leq \int g(y)^{1/(k-1)} \prod_{i=1}^{k-1} \big(\int h_i(y,t) dt\big)^{1/(k-1)} \lambda_{k-1}(dy)\end{aligned}$$

(473F)

$$= \int g(y)^{1/(k-1)} \prod_{i=1}^{k-1} g_i(y)^{1/(k-1)} \lambda_{k-1}(dy)$$

(where g_i is defined from g in the same way as h_i is defined from h)

$$\leq \big(\int g(y) \lambda_{k-1}(dy)\big)^{1/(k-1)} \big(\int \prod_{i=1}^{k-1} g_i(y)^{1/(k-2)} \lambda_{k-1}(dy)\big)^{(k-2)/(k-1)}$$

(by Hölder's inequality again, this time with $\frac{1}{k-1} + \frac{k-2}{k-1} = 1$)

$$\leq \big(\int g(y) \lambda_{k-1}(dy)\big)^{1/(k-1)} \cdot \int g(y) \lambda_{k-1}(dy)$$

(by the inductive hypothesis)

$$= \big(\int g(y) \lambda_{k-1}(dy)\big)^{k/(k-1)} = \big(\int h(x) \lambda_k(dx)\big)^{k/(k-1)},$$

and the induction proceeds.

473H Gagliardo-Nirenberg-Sobolev inequality Suppose that $f : \mathbb{R}^r \to \mathbb{R}$ is a Lipschitz function with compact support. Then $\|f\|_{r/(r-1)} \leq \int \|\operatorname{grad} f\| d\mu$.

proof By 473Cc, $\operatorname{grad} f$ is measurable and bounded, so $\|\operatorname{grad} f\|$ also is; since it must have compact support, it is integrable.

For $1 \leq i \leq r$, $x = (\xi_1, \ldots, \xi_r) \in \mathbb{R}^r$ and $t \in \mathbb{R}$ write $S_i(x,t) = (\xi_1', \ldots, \xi_r')$ where $\xi_i' = t$ and $\xi_j' = \xi_j$ for $j \neq i$. Set $h_i(x) = \int_{-\infty}^{\infty} \|\operatorname{grad} f(S_i(x,t))\| dt$ when this is defined, which will be the case for almost every x. Now, whenever $h_i(x)$ is defined,

$$|f(x)| = |f(S_i(x,\xi_i))| = |\int_{-\infty}^{\xi_i} \tfrac{\partial}{\partial t} f(S_i(x,t))dt| \leq h_i(x).$$

(Use 225E and the fact that a Lipschitz function on any bounded interval in $\mathbb{R}$ is absolutely continuous.) So $|f| \leq_{\text{a.e.}} h_i$ for every $i \leq r$. Accordingly

$$\int |f(x)|^{r/(r-1)}\mu(dx) \leq \int \prod_{i=1}^r h_i(x)^{1/(r-1)}\mu(dx) \leq \big(\int \|\operatorname{grad} f(x)\|\mu(dx)\big)^{r/(r-1)}$$

by 473F. Raising both sides to the power $(r-1)/r$ we have the result.

473I Lemma For any Lipschitz function $f : B(\mathbf{0},1) \to \mathbb{R}$,

$$\int_{B(\mathbf{0},1)} |f|^{r/(r-1)}d\mu \leq \big(2^{r+2}(1+\sqrt{r})\int_{B(\mathbf{0},1)} \|\operatorname{grad} f\| + |f|d\mu\big)^{r/(r-1)}.$$

proof (a) Set $g(x) = \max(0, 2\|x\|^2 - 1)$ for $x \in B(\mathbf{0},1)$. Then $\operatorname{grad} g$ is defined at every point x such that $\|x\| < 1$ and $\|x\| \neq \sqrt{2}$, and at all such points $\frac{\partial g}{\partial \xi_i}$ is either 0 or $4\xi_i$ for each i, so that $\|\operatorname{grad} g(x)\| \leq 4\|x\| \leq 4$. Hence (or otherwise) g is Lipschitz. So $f_1 = f \times g$ is Lipschitz (473Ca).

By Rademacher's theorem again, $\operatorname{grad} f_1$ is defined almost everywhere in $B(\mathbf{0},1)$. Now

$$\begin{aligned}\int_{B(\mathbf{0},1)} \|\operatorname{grad} f_1\| &= \int_{B(\mathbf{0},1)} \|f(x)\operatorname{grad} g(x) + g(x)\operatorname{grad} f(x)\|\mu(dx)\\ &\leq \int_{B(\mathbf{0},1)} 4|f| + \|\operatorname{grad} f\|d\mu.\end{aligned} \tag{473Bd}$$

(b) It will be convenient to have an elementary fact out in the open. Set $\phi(x) = \frac{x}{\|x\|^2}$ for $x \in \mathbb{R}^r \setminus \{\mathbf{0}\}$; note that $\phi^2(x) = x$. Then $\phi\restriction \mathbb{R}^r \setminus B(\mathbf{0},\delta)$ is Lipschitz, for any $\delta > 0$. **P** If $\|x\| = \alpha \geq \delta$, $\|y\| = \beta \geq \delta$, then we have

$$\begin{aligned}\|\phi(x) - \phi(y)\|^2 &= \tfrac{1}{\alpha^4}\|x\|^2 - \tfrac{2}{\alpha^2\beta^2}x\boldsymbol{.}y + \tfrac{1}{\beta^4}\|y\|^2\\ &= \tfrac{1}{\alpha^2\beta^2}\big(\|y\|^2 - 2x\boldsymbol{.}y + \|x\|^2\big) \leq \tfrac{1}{\delta^4}\|x-y\|^2,\end{aligned}$$

so $\frac{1}{\delta^2}$ is a Lipschitz constant for $\phi\restriction \mathbb{R}^r \setminus B(\mathbf{0},\delta)$. **Q**

(c) Set $f_2(x) = f(x)$ if $\|x\| \leq 1$, $f_1\phi(x)$ if $\|x\| \geq 1$. Then f_2 is well-defined (because $f_1(x) = f(x)$ if $\|x\| = 1$), is zero outside $B(\mathbf{0},\sqrt{2})$ (because $g(x) = 0$ if $\|x\| \leq \frac{1}{\sqrt{2}}$), and is Lipschitz. **P** By 473Cb, it will be enough to show that $f_2\restriction F$ is Lipschitz, where $F = \{x : \|x\| \geq 1\}$. But (b) shows that $\phi\restriction F$ is 1-Lipschitz on F, so any Lipschitz constant for f_1 is also a Lipschitz constant for $f_2\restriction F$. **Q**

If $\|x\| > 1$, then, for any $i \leq r$,

$$\begin{aligned}\tfrac{\partial f_2}{\partial \xi_i}(x) &= \sum_{j=1}^r \tfrac{\partial f_1}{\partial \xi_j}(\phi(x)) \cdot \tfrac{\partial}{\partial \xi_i}(\tfrac{\xi_j}{\|x\|^2})\\ &= \tfrac{\partial f_1}{\partial \xi_i}(\phi(x)) \cdot \tfrac{1}{\|x\|^2} - \sum_{j=1}^r \tfrac{\partial f}{\partial \xi_j}(\phi(x)) \cdot \tfrac{\xi_i\xi_j}{\|x\|^4}\end{aligned}$$

wherever the right-hand side is defined, that is, wherever all the partial derivatives $\frac{\partial f_1}{\partial \xi_i}(\phi(x))$ are defined. But $H = B(\mathbf{0},1) \setminus \operatorname{dom}(\operatorname{grad} f_1)$ is negligible, and does not meet $\{x : \|x\| < \sqrt{2}\}$, so $\phi\restriction H$ is Lipschitz and $\phi[H] = \phi^{-1}[H]$ is negligible (262D); while $\operatorname{grad} f_1(\phi(x))$ is defined whenever $\|x\| > 1$ and $x \notin \phi^{-1}[H]$. So the formula here is valid for almost every $x \in F$, and

$$|\frac{\partial f_2}{\partial \xi_i}(x)| \leq \|\operatorname{grad} f_1(\phi(x))\| \cdot \frac{1}{\|x\|^2} + \|\operatorname{grad} f_1(\phi(x))\| \cdot \frac{|\xi_i|}{\|x\|^4} \sqrt{\sum_{j=1}^{r} \xi_j^2}$$

$$= \|\operatorname{grad} f_1(\phi(x))\| \frac{\|x\|+|\xi_i|}{\|x\|^3}$$

for almost every $x \in F$. But (since we know that $\operatorname{grad} f_2$ is defined almost everywhere, by Rademacher's theorem, as usual) we have

$$\|\operatorname{grad} f_2(x)\| \leq \frac{1+\sqrt{r}}{\|x\|^2} \|\operatorname{grad} f_1(\phi(x))\| \leq (1+\sqrt{r}) \|\operatorname{grad} f_1(\phi(x))\|$$

for almost every $x \in F$.

(c) We are now in a position to estimate

$$\int_F \|\operatorname{grad} f_2\| d\mu = \int_{B(\mathbf{0},\sqrt{2})} \|\operatorname{grad} f_2\| d\mu - \int_{B(\mathbf{0},1)} \|\operatorname{grad} f_2\| d\mu$$

(because $\operatorname{grad} f_1(x) = \mathbf{0}$ if $\|x\| < \sqrt{2}$)

$$= \int_1^{\sqrt{2}} \int_{\partial B(\mathbf{0},t)} \|\operatorname{grad} f_2(x)\| \nu(dx) dt$$

(265G, as usual)

$$\leq (1+\sqrt{r}) \int_1^{\sqrt{2}} \int_{\partial B(\mathbf{0},t)} \|\operatorname{grad} f_1(\tfrac{1}{t^2}x)\| \nu(dx) dt$$

(by (b) above)

$$\leq (1+\sqrt{r}) \int_1^{\sqrt{2}} \int_{\partial B(\mathbf{0},1/t)} t^{2r-2} \|\operatorname{grad} f_1(y)\| \nu(dy) dt$$

substituting $x = t^2 y$ in the inner integral; the point being that as the function $y \mapsto t^2 y$ changes all distances by a scalar multiple t^2, it must transform Hausdorff $r-1$-dimensional measure by a multiple t^{2r-2}. But now, substituting $s = \frac{1}{t}$ in the outer integral, we have

$$\begin{aligned}
\int_F \|\operatorname{grad} f_2\| d\mu &\leq (1+\sqrt{r}) \int_{1/\sqrt{2}}^1 \frac{1}{s^{2r}} \int_{\partial B(\mathbf{0},s)} \|\operatorname{grad} f_1(y)\| \nu(dy) ds \\
&\leq 2^r (1+\sqrt{r}) \int_{1/\sqrt{2}}^1 \int_{\partial B(\mathbf{0},s)} \|\operatorname{grad} f_1(y)\| \nu(dy) ds \\
&= 2^r (1+\sqrt{r}) \int_{B(\mathbf{0},1)} \|\operatorname{grad} f_1\| d\mu \\
&\leq 2^r (1+\sqrt{r}) \int_{B(\mathbf{0},1)} 4|f| + \|\operatorname{grad} f\| d\mu.
\end{aligned}$$

by (a) above.

(d) Accordingly

$$\begin{aligned}
\int_{\mathbb{R}^r} \|\operatorname{grad} f_2\| d\mu &\leq \int_{B(\mathbf{0},1)} \|\operatorname{grad} f\| d\mu + \int_F \|\operatorname{grad} f_2\| d\mu \\
&\leq 2^{r+2} (1+\sqrt{r}) \int_{B(\mathbf{0},1)} |f| + \|\operatorname{grad} f\| d\mu.
\end{aligned}$$

But now we can apply 473H to see that

$$\int_{B(\mathbf{0},1)} |f|^{r/(r-1)}d\mu \leq \int |f_2|^{r/r-1}d\mu \leq (\int \|\operatorname{grad} f_2\|d\mu)^{r/(r-1)}$$
$$\leq \big(2^{r+2}(1+\sqrt{r})\int_{B(\mathbf{0},1)} |f| + \|\operatorname{grad} f\|d\mu\big)^{r/(r-1)},$$

as claimed.

473J Lemma Let $f : \mathbb{R}^r \to \mathbb{R}$ be a Lipschitz function. Then

$$\int_{B(y,\delta)} |f(x) - f(z)|\mu(dx) \leq \frac{2^r}{r}\delta^r \int_{B(y,\delta)} \|\operatorname{grad} f(x)\|\|x-z\|^{1-r}\mu(dx)$$

whenever $y \in \mathbb{R}^r$, $\delta > 0$ and $z \in B(y,\delta)$.

proof (a) To begin with, suppose that f is smooth. In this case, for any x, $z \in B(y,\delta)$,

$$|f(x) - f(z)| = \big|\int_0^1 \frac{d}{dt} f(z + t(x-z))dt\big|$$
$$= \big|\int_0^1 (x-z)\,.\operatorname{grad} f(z+t(x-z))dt\big|$$
$$\leq \|x-z\| \int_0^1 \|\operatorname{grad} f(z+t(x-z))\|dt.$$

So, for $\eta > 0$,

$$\int_{B(y,\delta)\cap\partial B(z,\eta)} |f(x) - f(z)|\nu(dx)$$
$$\leq \eta \int_0^1 \int_{B(y,\delta)\cap\partial B(z,\eta)} \|\operatorname{grad} f(z+t(x-z))\|\nu(dx)dt$$

(grad f is continuous and bounded, and the subspace measure induced by ν on $\partial B(z,\eta)$ is a (quasi-)Radon measure (471E, 471Dh), so its product with Lebesgue measure also is (417T), and there is no difficulty with the change in order of integration)

$$= \eta \int_0^1 \frac{1}{t^{r-1}} \int_{B(y,\delta)\cap\partial B(z,t\eta)} \|\operatorname{grad} f(w)\|\nu(dw)dt$$

(because if $\phi(x) = z + t(x-z)$, then $\nu\phi^{-1}[E] = \frac{1}{t^{r-1}}\nu E$ whenever ν measures E and $t > 0$)

$$= \eta^r \int_0^1 \int_{B(y,\delta)\cap\partial B(z,t\eta)} \|\operatorname{grad} f(w)\|\|w-z\|^{1-r}\nu(dw)dt$$
$$= \eta^{r-1} \int_0^\eta \int_{B(y,\delta)\cap\partial B(z,s)} \|\operatorname{grad} f(w)\|\|w-z\|^{1-r}\nu(dw)ds$$

(substituting $s = t\eta$)

$$= \eta^{r-1} \int_{B(y,\delta)\cap B(z,\eta)} \|\operatorname{grad} f(w)\|\|w-z\|^{1-r}\mu(dw)$$

by 265G. By 265G again,

$$\begin{aligned}\int_{B(y,\delta)} |f(x)-f(z)|\mu(dx) &= \int_0^{2\delta}\int_{B(y,\delta)\cap\partial B(z,\eta)} |f(x)-f(z)|\nu(dx)d\eta\\ &\le \int_0^{2\delta}\eta^{r-1}\int_{B(y,\delta)\cap B(z,\eta)} \|\operatorname{grad} f(w)\|\|w-z\|^{1-r}\mu(dw)d\eta\\ &\le \int_0^{2\delta}\eta^{r-1}\int_{B(y,\delta)} \|\operatorname{grad} f(w)\|\|w-z\|^{1-r}\mu(dw)d\eta\\ &= \frac{2^r}{r}\delta^r\int_{B(y,\delta)} \|\operatorname{grad} f(w)\|\|w-z\|^{1-r}\mu(dw).\end{aligned}$$

(b) Now turn to the general case in which f is not necessarily differentiable everywhere, but is known to be Lipschitz. We need to know that $\int_{B(y,\delta)}\|x-z\|^{1-r}\mu(dx)$ is finite; this is because

$$\int_{B(y,\delta)}\|x-z\|^{1-r}\mu(dx) \le \int_{B(z,2\delta)}\|x-z\|^{1-r}\mu(dx) = \int_0^{2\delta} t^{1-r}\nu(\partial B(z,t))dt$$

(265G once more)

$$= \int_0^{2\delta} t^{1-r}r\beta_r t^{r-1}dt = 2\delta r\beta_r.$$

Now take the sequence $\langle\tilde{h}_n\rangle_{n\in\mathbb{N}}$ from 473E. Then $\langle f*\tilde{h}_n\rangle_{n\in\mathbb{N}}$ converges uniformly to f (473Ed), while $\langle\operatorname{grad}(f*\tilde{h}_n)\rangle_{n\in\mathbb{N}} = \langle\tilde{h}_n*\operatorname{grad} f\rangle_{n\in\mathbb{N}}$ (473Dd) is uniformly bounded (473Cc, 473Dg) and converges almost everywhere to $\operatorname{grad} f$ (473Ee). But this means that, setting $f_n = f*\tilde{h}_n$,

$$\begin{aligned}\int_{B(y,\delta)} |f(x)-f(z)|\mu(dx) &= \lim_{n\to\infty}\int_{B(y,\delta)} |f_n(x)-f_n(z)|\mu(dx)\\ &\le \lim_{n\to\infty}\frac{2^r}{r}\delta^r\int_{B(y,\delta)} \|\operatorname{grad} f_n(x)\|\|x-z\|^{1-r}\mu(dx)\end{aligned}$$

(because every f_n is differentiable, by 473De)

$$= \frac{2^r}{r}\delta^r\int_{B(y,\delta)} \|\operatorname{grad} f(x)\|\|x-z\|^{1-r}\mu(dx)$$

by Lebesgue's Dominated Convergence Theorem.

473K Poincaré's inequality for balls Let $B\subseteq\mathbb{R}^r$ be a non-trivial closed ball, and $f : B\to\mathbb{R}$ a Lipschitz function. Set $\gamma = \frac{1}{\mu B}\int_B f d\mu$. Then

$$\big(\int_B |f-\gamma|^{r/(r-1)}d\mu\big)^{(r-1)/r} \le c\int_B \|\operatorname{grad} f\|d\mu,$$

where $c = 2^{r+2}(1+\sqrt{r})(1+2^{r+1})$.

proof (a) To begin with (down to the end of (b)) suppose that B is the unit ball $B(\mathbf{0},1)$. Then, for any $x\in B$,

$$\begin{aligned}|f(x)-\gamma| &= \frac{1}{\mu B}\big|\int_B f(x)-f(z)\mu(dz)\big|\\ &\le \frac{1}{\mu B}\int_B |f(x)-f(z)|\mu(dz)\\ &\le \frac{2^r}{r}\cdot\frac{1}{\mu B}\int_B \|\operatorname{grad} f(z)\|\|z-x\|^{1-r}\mu(dz),\end{aligned}$$

by 473J. Also

$$\int_{B(z,2)} \|x-z\|^{1-r}\mu(dx) = \int_0^2 \int_{\partial B(z,t)} \|x-z\|^{1-r}\nu(dx)dt$$

(265G)

$$= \int_0^2 t^{1-r}\nu(\partial B(z,t))dt = \int_0^1 t^{1-r}t^{r-1}\nu(\partial B(\mathbf{0},1))dt$$
$$= 2\nu(\partial B(\mathbf{0},1)) = 2r\beta_r$$

by 265H. So

$$\begin{aligned}\int_B |f(x)-\gamma|\mu(dx) &\le \frac{2^r}{r}\cdot\frac{1}{\mu B}\int_B\int_B \|\operatorname{grad} f(z)\|\|z-x\|^{1-r}\mu(dz)\mu(dx)\\ &= \frac{2^r}{r\beta_r}\int_B\int_B \|\operatorname{grad} f(z)\|\|z-x\|^{1-r}\mu(dx)\mu(dz)\\ &\le \frac{2^r}{r\beta_r}\int_B \|\operatorname{grad} f(z)\|\int_{B(z,2)}\|z-x\|^{1-r}\mu(dx)\mu(dz)\\ &\le 2^{r+1}\int_B \|\operatorname{grad} f(z)\|\mu(dz).\end{aligned}$$

(b) Now apply 473I to $g = f(x)-\gamma$. We have

$$\begin{aligned}\int_B |f-\gamma|^{r/(r-1)}d\mu &\le \big(2^{r+2}(1+\sqrt{r})\int_B \|\operatorname{grad} f\| + |g|d\mu\big)^{r/(r-1)}\\ &\le \big(2^{r+2}(1+\sqrt{r})(1+2^{r+1})\int_B \|\operatorname{grad} f\|d\mu\big)^{r/(r-1)}\end{aligned}$$

(by (a))

$$= \big(c\int_B \|\operatorname{grad} f\|d\mu\big)^{r/(r-1)}.$$

(c) For the general case, express B as $B(y,\delta)$, and set $h(x) = f(y+\delta x)$ for $x \in B(\mathbf{0},1)$. Then $\operatorname{grad} h(x) = \delta \operatorname{grad} f(y+\delta x)$ for almost every $x \in B(\mathbf{0},1)$. Now

$$\textstyle\int_{B(\mathbf{0},1)} h\,d\mu = \frac{1}{\delta^r}\int_{B(y,\delta)} f d\mu,$$

so

$$\textstyle\frac{1}{\mu B(\mathbf{0},1)}\int_{B(\mathbf{0},1)} h d\mu = \frac{1}{\mu B(y,\delta)}\int_{B(y,\delta)} f d\mu = \gamma.$$

We therefore have

$$\begin{aligned}\int_{B(y,\delta)} |f-\gamma|^{r/(r-1)}d\mu &= \delta^r\int_{B(\mathbf{0},1)} |h-\gamma|^{r/(r-1)}d\mu\\ &\le \delta^r\big(c\int_{B(\mathbf{0},1)}\|\operatorname{grad} h\|d\mu\big)^{r/(r-1)}\end{aligned}$$

(by (a)-(b) above)

$$\begin{aligned}&= \delta^r\big(\tfrac{\delta c}{\delta^r}\int_{B(y,\delta)}\|\operatorname{grad} f\|d\mu\big)^{r/(r-1)}\\ &= \big(c\int_{B(y,\delta)}\|\operatorname{grad} f\|d\mu\big)^{r/(r-1)}.\end{aligned}$$

Raising both sides to the power $(r-1)/r$ we have the result as stated.

Remark As will be plain from the way in which the proof here is constructed, there is no suggestion that the formula offered for c gives anything near the best possible value.

473L Corollary Let $B \subseteq \mathbb{R}^r$ be a non-trivial closed ball, and $f : B \to [0,1]$ a Lipschitz function. Set

$$F_0 = \{x : x \in B,\, f(x) \leq \tfrac{1}{4}\}, \quad F_1 = \{x : x \in B,\, f(x) \geq \tfrac{3}{4}\}.$$

Then

$$\big(\min(\mu F_0, \mu F_1)\big)^{(r-1)/r} \leq 4c \int_B \|\operatorname{grad} f\| d\mu,$$

where $c = 2^{r+2}(1+\sqrt{r})(1+2^{r+1})$.

proof Setting $\gamma = \frac{1}{\mu B}\int_B f d\mu$,

$$\int_B |f-\gamma|^{r/(r-1)} d\mu \geq \tfrac{1}{4^{r/(r-1)}}\mu F_0 \text{ if } \gamma \geq \tfrac{1}{2},$$
$$\geq \tfrac{1}{4^{r/(r-1)}}\mu F_1 \text{ if } \gamma \leq \tfrac{1}{2}.$$

So 473K tells us that

$$\tfrac{1}{4}\big(\min(\mu F_0, \mu F_1)\big)^{(r-1)/r} \leq c \int_B \|\operatorname{grad} f\| d\mu,$$

as required.

473M The case $r = 1$ The general rubric for this section declares that r is taken to be at least 2, which is clearly necessary for the formula in 475J to be appropriate. For the sake of an application in the next section, however, I mention the elementary corresponding result when $r = 1$. In this case, B is just a closed interval, and $\operatorname{grad} f$ is the ordinary deriivative of f; interpreting $(\int_B |f-\gamma|^{r/(r-1)})^{(r-1)/r}$ as $\|f \times \chi B - \gamma \chi B\|_{r/(r-1)}$, it is natural to look at

$$\|f \times \chi B - \gamma \chi B\|_\infty = \sup_{x \in B} |f(x) - \gamma| \leq \sup_{x,y \in B} |f(x) - f(y)| \leq \int_B |f'| d\mu,$$

giving a version of 473K for $r = 1$. We see that the formula for c remains valid in the case $r = 1$, with a good deal to spare. As for 473L, if $\int_B |f'| < \frac{1}{2}$ then at least one of F_0, F_1 must be empty.

473X Basic exercises (a) Set $f(x) = \max(0, -\ln\|x\|)$, $f_k(x) = \min(k, f(x))$ for $x \in \mathbb{R}^2 \setminus \{0\}$, $k \in \mathbb{N}$. Show that $\lim_{k\to\infty} \|f - f_k\|_2 = \lim_{k\to\infty} \|\operatorname{grad} f - \operatorname{grad} f_k\|_1 = 0$, so that all the inequalities 473H-473L are valid for f.

(b) Let $k \in [1, r]$ be an integer, and set $m = \frac{(r-1)!}{(k-1)!(r-k)!}$. Let $e_1, \ldots, e_r$ be the standard orthonormal basis of $\mathbb{R}^r$ and $\mathcal{J}$ the family of subsets of $\{1, \ldots, r\}$ with k members. For $J \in \mathcal{J}$ let V_J be the linear span of $\{e_i : i \in J\}$, $\pi_J : \mathbb{R}^r \to V_J$ the orthogonal projection and ν_J the normalized k-dimensional Hausdorff measure on V_J. Show that if $A \subseteq \mathbb{R}^r$ then $(\mu^* A)^m \leq \prod_{J\in\mathcal{J}} \nu_J^* \pi_J[A]$. (*Hint*: start with $A \subseteq [0,1]^r$ and note that $([0,1]^r)^m$ can be identified with $\prod_{J\in\mathcal{J}} [0,1]^J$.)

473Y Further exercises (a) Let $D \subseteq \mathbb{R}^r$ be any set and $\phi : D \to \mathbb{R}^s$ any function. Show that $D_0 = \{x : x \in D, \phi \text{ is differentiable at } x\}$ is a Borel subset of $\mathbb{R}^r$, and that the derivative of ϕ is a Borel measurable function. (Compare 225J.)

473 Notes and comments The point of all the inequalities 473H-473L is that they bound some measure of variance of a function f by the integral of $\|\operatorname{grad} f\|$. If $r = 2$, indeed, we are looking at $\|f\|_2$ (473H) or $\int_B |f|^2$ (473I) or something essentially equal to the variance of probability theory (473K). In higher dimensions we need to look at $\|\,\|_{r/(r-1)}$ in place of $\|\,\|_2$, and when $r = 1$ we can interpret the inequalities in terms of the supremum norm $\|\,\|_\infty$ (473M). In all cases we want to develop inequalities which will enable us to discuss a function in terms of its first derivative. In one dimension, this is the familiar Fundamental Theorem of Calculus (Chapter 22). We find there a straightforward criterion ('absolute continuity') to determine whether a given function of one variable is an indefinite integral, and that if so it is the indefinite integral of its own derivative. Even in two dimensions, this simplicity disappears. The essential problem is that a

function can be the indefinite integral of an integrable gradient function without being bounded (473Xa). The principal results of this section are stated for Lipschitz functions, but in fact they apply much more widely. The argument suggested in 473Xa involves approximating the unbounded function f by Lipschitz functions f_k in a sharp enough sense to make it possible to read off all the inequalities for f from the corresponding inequalities for the f_k. This idea leads naturally to the concept of 'Sobolev space', which I leave on one side for the moment; see EVANS & GARIEPY 92, chap. 4, for details.

474 The distributional perimeter

The next step is a dramatic excursion, defining (for appropriate sets E) a perimeter *measure* for which a version of the Divergence Theorem is true (474E). I begin the section with elementary notes on the divergence of a vector field (474B-474C). I then define 'locally finite perimeter' (474D), 'perimeter measure' and 'outward normal' (474F) and 'reduced boundary' (474H). The definitions rely on the Riesz representation theorem, and we have to work very hard to relate them to any geometrically natural idea of 'boundary'. Even half-spaces (474I) demand some attention. From Poincaré's inequality (473K) we can prove isoperimetric inequalities for perimeter measures (474L). With some effort we can locate the reduced boundary as a subset of the topological boundary (474Xc), and obtain asymptotic inequalities on the perimeter measures of small balls (474N). With much more effort we can find a geometric description of outward normal functions in terms of 'Federer exterior normals' (474Q), and get a tight asymptotic description of the perimeter measures of small balls (474R). I end with the Compactness Theorem for sets of bounded perimeter (474S).

474A Notation I had better repeat some of the notation from §473. $r \geq 2$ is a fixed integer. μ is Lebesgue measure on $\mathbb{R}^r$, and $\beta_r = \mu B(\mathbf{0}, 1)$ is the volume of the unit ball. $S_{r-1} = \partial B(\mathbf{0}, 1)$ is the unit sphere. ν is normalized $r-1$-dimensional Hausdorff measure on $\mathbb{R}^r$. We shall sometimes need to look at Lebesgue measure on $\mathbb{R}^{r-1}$, which I will denote μ_{r-1}.

$\mathcal{D}$ is the set of smooth functions $f : \mathbb{R}^r \to \mathbb{R}$ with compact support; $\mathcal{D}_r$ the set of smooth functions $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support, that is, the set of functions $\phi = (\phi_1, \ldots, \phi_r) : \mathbb{R}^r \to \mathbb{R}^r$ such that $\phi_i \in \mathcal{D}$ for every i.

I continue to use the sequence $\langle \tilde{h}_n \rangle_{n \in \mathbb{N}}$ from 473E; these functions all belong to $\mathcal{D}$, are non-negative everywhere and zero outside $B(\mathbf{0}, \frac{1}{n+1})$, are even, and have integral 1.

474B The divergence of a vector field (a) For a function ϕ from a subset of $\mathbb{R}^r$ to $\mathbb{R}^r$, write $\operatorname{div} \phi = \sum_{i=1}^r \frac{\partial \phi_i}{\partial \xi_i}$, where $\phi = (\phi_1, \ldots, \phi_r)$; for definiteness, let us take the domain of $\operatorname{div} \phi$ to be the set of points at which ϕ is differentiable (in the strict sense of 262Fa). Note that $\operatorname{div} \phi \in \mathcal{D}$ for every $\phi \in \mathcal{D}_r$. We need the following elementary facts.

(b) If $f : \mathbb{R}^r \to \mathbb{R}$ and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ are functions, then $\operatorname{div}(f \times \phi) = \phi \boldsymbol{.} \operatorname{grad} f + f \times \operatorname{div} \phi$ at any point at which f and ϕ are both differentiable. (Use 473Bc-473Bd.)

(c) If $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a Lipschitz function with compact support, then $\int \operatorname{div} \phi \, d\mu = 0$. **P** $\operatorname{div} \phi$ is defined almost everywhere (by Rademacher's theorem, 262Q), measurable (473Be), bounded (473Cc) and with compact support, so

$$\int \operatorname{div} \phi \, d\mu = \sum_{i=1}^r \int \frac{\partial \phi_i}{\partial \xi_i} d\mu$$

is defined in $\mathbb{R}$. For each $i \leq r$, Fubini's theorem tells us that we can replace integration with respect to μ by a repeated integral, in which the inner integral is

$$\int_{-\infty}^{\infty} \frac{\partial \phi_i}{\partial \xi_i}(\xi_1, \ldots, \xi_r) d\xi_i = 0$$

because $\phi_i(\xi_1, \ldots, \xi_r) = 0$ whenever $|\xi_i|$ is large enough. So $\int \frac{\partial \phi_i}{\partial \xi_i} d\mu$ is also zero. Summing over i, we have the result. **Q**

(d) If $\phi : \mathbb{R}^r \to \mathbb{R}^r$ and $f : \mathbb{R}^r \to \mathbb{R}$ are Lipschitz functions, one of which has compact support, then

$$\int \phi \boldsymbol{.} \operatorname{grad} f \, d\mu + \int f \times \operatorname{div} \phi \, d\mu = 0.$$

P f and ϕ are both differentiable almost everywhere. So

$$\int \phi \boldsymbol{.} \operatorname{grad} f \, d\mu + \int f \times \operatorname{div} \phi \, d\mu = \int \operatorname{div}(f \times \phi) d\mu = 0$$

by (c) above. **Q**

(e) If $f \in \mathcal{L}^\infty(\mu)$, $g \in \mathcal{L}^1(\mu)$ is even (that is, $g(-x)$ is defined and equal to $g(x)$ for every $x \in \operatorname{dom} g$), and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a Lipschitz function with compact support, then $\int (f * g) \times \operatorname{div} \phi = \int f \times \operatorname{div}(g * \phi)$, where $g * \phi = (g * \phi_1, \dots, g * \phi_r)$. **P** For each i,

$$\int (f * g) \times \frac{\partial \phi_i}{\partial \xi_i} d\mu = \iint f(x) g(y) \frac{\partial \phi_i}{\partial \xi_i}(x + y) \mu(dy) \mu(dx)$$

(255G/444Od)

$$= \iint f(x) g(-y) \frac{\partial \phi_i}{\partial \xi_i}(x + y) \mu(dy) \mu(dx)$$

(because g is even)

$$= \int f \times (g * \frac{\partial \phi_i}{\partial \xi_i}) d\mu = \int f \times \frac{\partial}{\partial \xi_i}(g * \phi_i) d\mu$$

by 473Dd. Now take the sum over i of both sides. **Q**

(f) If $g \in \mathcal{L}^1(\mu)$ and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a bounded Lebesgue measurable function, then

$$\|(g * \phi)(x)\| \leq \|g\|_1 \sup_{y \in \mathbb{R}^r} \|\phi(y)\|$$

for every $x \in \mathbb{R}^r$. **P** We can find $z = (\zeta_1, \dots, \zeta_r) \in S_{r-1}$ such that

$$\|(g * \phi)(x)\| = (g * \phi)(x) \boldsymbol{.} z = \sum_{i=1}^r \int \zeta_i g(x - y) \phi_i(y) \mu(dy)$$

$$\leq \int |g(x - y)| \|\phi(y)\| \mu(dy) \leq \|g\|_1 \sup_{y \in \mathbb{R}^r} \|\phi(y)\|. \; \textbf{Q}$$

474C Invariance under isometries: Proposition Suppose that $T : \mathbb{R}^r \to \mathbb{R}^r$ is any isometry, and that ϕ is a function from a subset of $\mathbb{R}^r$ to $\mathbb{R}^r$. Then

$$\operatorname{div}(T^{-1} \phi T) = (\operatorname{div} \phi) T.$$

proof Set $z = T(\mathbf{0})$. By 4A4Jb, the isometry $x \mapsto Tx - z$ is linear and preserves inner products, so there is an orthogonal matrix S such that $T(x) = z + Sx$ for every $x \in \mathbb{R}^r$. Now suppose that $x \in \mathbb{R}^r$ is such that $(\operatorname{div} \phi)(Tx)$ is defined. Then $T(y) - T(x) - S(y - x) = 0$ for every y, so T is differentiable at x, with derivative S, and ϕT is differentiable at x, with derivative DS, where D is the derivative of ϕ at $T(x)$, by 473Bc. Also $T^{-1}(y) = S^{-1}(y - z)$ for every y, so T^{-1} is differentiable at $\phi(T(x))$ with derivative S^{-1}, and $T^{-1} \phi T$ is differentiable at x, with derivative $S^{-1} DS$. Now if D is $\langle \delta_{ij} \rangle_{1 \leq i,j \leq r}$ and S is $\langle \sigma_{ij} \rangle_{1 \leq i,j \leq r}$ and $S^{-1} DS$ is $\langle \tau_{ij} \rangle_{1 \leq i,j \leq r}$, then S^{-1} is the transpose $\langle \sigma_{ji} \rangle_{1 \leq i,j \leq r}$ of S, because S is orthogonal, so

$$\operatorname{div}(T^{-1} \phi T)(x) = \sum_{i=1}^r \tau_{ii} = \sum_{i=1}^r \sum_{j=1}^r \sigma_{ji} \sum_{k=1}^r \delta_{jk} \sigma_{ki}$$

$$= \sum_{j=1}^r \sum_{k=1}^r \delta_{jk} \sum_{i=1}^r \sigma_{ji} \sigma_{ki} = \sum_{j=1}^r \delta_{jj} = \operatorname{div} \phi(x)$$

because $\sum_{i=1}^r \sigma_{ji} \sigma_{jk} = 1$ if $j = k$ and 0 otherwise. If $\operatorname{div}(T^{-1} \phi T)(x)$ is defined, then (because T^{-1} is also an isometry)

$$(\operatorname{div}\phi)(Tx) = \operatorname{div}(TT^{-1}\phi TT^{-1})(Tx) = \operatorname{div}(T^{-1}\phi T)(T^{-1}Tx) = \operatorname{div}(T^{-1}\phi T)(x).$$

So the functions $\operatorname{div}(T^{-1}\phi T)$ and $(\operatorname{div}\phi)T$ are identical.

474D Locally finite perimeter: Definition Let $E \subseteq \mathbb{R}^r$ be a Lebesgue measurable set. Its **perimeter** per E is

$$\sup\{|\textstyle\int_E \operatorname{div}\phi\, d\mu| : \phi : \mathbb{R}^r \to B(\mathbf{0},1) \text{ is a Lipschitz function with compact support}\}.$$

E has **locally finite perimeter** if

$$\sup\{|\textstyle\int_E \operatorname{div}\phi\, d\mu| : \phi : \mathbb{R}^r \to \mathbb{R}^r \text{ is a Lipschitz function, } \|\phi\| \leq \chi B(\mathbf{0},n)\}$$

is finite for every $n \in \mathbb{N}$.

474E Theorem Suppose that $E \subseteq \mathbb{R}^r$ has locally finite perimeter.

(i) There are a Radon measure λ_E^∂ on $\mathbb{R}^r$ and a function $\psi : \mathbb{R}^r \to S_{r-1}$ such that

$$\textstyle\int_E \operatorname{div}\phi\, d\mu = \int \phi \bullet \psi\, d\lambda_E^\partial$$

for every Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support.

(ii) This formula uniquely determines λ_E^∂, which can also be defined by saying that

$$\lambda_E^\partial(G) = \sup\{|\textstyle\int_E \operatorname{div}\phi\, d\mu| : \phi : \mathbb{R}^r \to \mathbb{R}^r \text{ is Lipschitz, } \|\phi\| \leq \chi G\}$$

whenever $G \subseteq \mathbb{R}^r$ is open.

(iii) If $\hat\psi$ is another function satisfying the formula in (i), then $\hat\psi = \psi$ λ_E^∂-almost everywhere.

proof (a) For each $l \in \mathbb{N}$, set

$$\gamma_l = \sup\{|\textstyle\int_E \operatorname{div}\phi\, d\mu| : \phi : \mathbb{R}^r \to \mathbb{R}^r \text{ is Lipschitz, } \|\phi\| \leq \chi B(\mathbf{0},l)\}.$$

Write C_k for the space of continuous functions with compact support from $\mathbb{R}^r$ to $\mathbb{R}$. By 473Dc and 473De, $f * \tilde h_n \in \mathcal{D}$ for every $f \in C_k$, $n \in \mathbb{N}$. Now the point is that

$$L_i(f) = -\lim_{n\to\infty} \textstyle\int_E \frac{\partial}{\partial\xi_i}(f * \tilde h_n)d\mu$$

is defined for every $f \in C_k$, $i \leq r$. **P** Applying 473Ed, we see that $\|f - f * \tilde h_n\|_\infty \to 0$ as $n \to \infty$. But this means that $\|(f * \tilde h_m) - (f * \tilde h_n)\|_\infty \to 0$ as $m, n \to \infty$. Set $\phi^{(n)}(x) = (f * \tilde h_n)(x)e_i$, where e_i is the unit vector $(0,\dots,1,\dots,0)$ with a 1 in the ith place. We have some $l \in \mathbb{N}$ such that $|f| \leq l\chi B(\mathbf{0},l)$; in which case $\phi^{(n)}(x) = \mathbf{0}$ whenever $\|x\| > l+1$ and $n \in \mathbb{N}$. This means that

$$\begin{aligned}|\textstyle\int_E \frac{\partial}{\partial\xi_i}(f * \tilde h_m)d\mu - \int_E \frac{\partial}{\partial\xi_i}(f * \tilde h_n)d\mu| &= |\textstyle\int \operatorname{div}(\phi^{(m)} - \phi^{(n)})d\mu| \\ &\leq \gamma_{l+1} \sup_{x\in\mathbb{R}^r} \|\phi^{(m)}(x) - \phi^{(n)}(x)\| \\ &\text{(applying the definition of } \gamma_{l+1} \text{ to a suitable multiple of } \phi^{(m)} - \phi^{(n)}) \\ &= \gamma_{l+1}\|(f * \tilde h_m) - (f * \tilde h_n)\|_\infty \to 0\end{aligned}$$

as $m, n \to \infty$. Thus $\langle \int_E \frac{\partial}{\partial\xi_i}(f * \tilde h_n)d\mu\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence and must have a limit. **Q**

If f is Lipschitz, then $\lim_{n\to\infty} \|f - f * \tilde h_n\|_\infty = 0$ and $|f - f * \tilde h_n| \leq \|f - f * \tilde h_n\|_\infty \chi B(\mathbf{0}, l+1)$ for every n, so $\lim_{n\to\infty} \int_E \frac{\partial}{\partial\xi_i}(f - f * \tilde h_n)d\mu = 0$, and $L_i(f) = -\int_E \frac{\partial f}{\partial\xi_i}d\mu$.

Note that the argument shows also that $|L_i(f)| \leq \gamma_{l+1}\|f\|_\infty$ if $f(x) = 0$ for $x \in \mathbb{R}^r \setminus B(\mathbf{0},l)$.

(b) Because all the functionals $f \mapsto \int_E \frac{\partial}{\partial\xi_i}(f * \tilde h_n)d\mu$ are linear, L_i is linear. Moroever, by the last remark in (a), it is order-bounded when regarded as a linear functional on the Riesz space C_k, so is expressible as a difference $L_i^+ - L_i^-$ of positive linear functionals (356B).

By the Riesz Representation Theorem (436J), we have Radon measures λ_i^+, λ_i^- on $\mathbb{R}^r$ such that $L_i^+(f) = \int f d\lambda_i^+$, $L_i^-(f) = \int f d\lambda_i^-$ for every $f \in C_k$. If we set $L = \sum_{i=1}^r L_i^+ + L_i^-$, there is another Radon measure $\tilde{\lambda}$ on $\mathbb{R}^r$ such that $L(f) = \int f d\tilde{\lambda}$ for every $f \in C_k$. Now each λ_i^+, λ_i^- is an indefinite-integral measure with respect to $\tilde{\lambda}$. **P** Because all these measures are σ-finite and the completions of their restrictions to the Borel σ-algebra, 234G tells us that it is enough to check that $\lambda_i^+(F) = \lambda_i^-(F) = 0$ whenever $\tilde{\lambda}F = 0$. But this is immediate from 416E(a-v). **Q**

For each $i \leq r$, let g_i^+, g_i^- be Radon-Nikodým derivatives of λ_i^+, λ_i^- with respect to $\tilde{\lambda}$. Adjusting them on a $\tilde{\lambda}$-negligible set if necessary, we may suppose that they are all bounded non-negative Borel measurable functions from $\mathbb{R}^r$ to $\mathbb{R}$. (Recall from 256C that $\tilde{\lambda}$ must be the completion of its restriction to the Borel σ-algebra.) Set $g_i = g_i^+ - g_i^-$ for each i. Then

$$\begin{aligned} -\int_E \frac{\partial f}{\partial \xi_i} d\mu &= L_i^+(f) - L_i^-(f) = \int f d\lambda_i^+ - \int f d\lambda_i^- \\ &= \int f \times g_i^+ d\tilde{\lambda} - \int f \times g_i^- d\tilde{\lambda} = \int f \times g_i d\tilde{\lambda} \end{aligned}$$

for every Lipschitz function f with compact support (235M). Set $g = \sqrt{\sum_{i=1}^r g_i^2}$. For $i \leq r$, set $\psi_i(x) = -\frac{g_i(x)}{g(x)}$ when $g(x) \neq 0$, $\frac{1}{\sqrt{r}}$ when $g(x) = 0$, so that $\psi = (\psi_1, \dots, \psi_r) : \mathbb{R}^r \to S_{r-1}$ is Borel measurable. Let λ_E^∂ be the indefinite-integral measure over $\tilde{\lambda}$ defined by g; then λ_E^∂ is a Radon measure on $\mathbb{R}^r$ (256E/416S).

(c) Now take any Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support. Express it as $(\phi_1, \dots, \phi_r)$ where $\phi_i : \mathbb{R}^r \to \mathbb{R}$ is a Lipschitz function with compact support for each i. Then

$$\begin{aligned} \int_E \operatorname{div} \phi \, d\mu &= \sum_{i=1}^r \int_E \frac{\partial \phi_i}{\partial \xi_i} d\mu = -\sum_{i=1}^r L_i(\phi_i) \\ &= -\sum_{i=1}^r L_i^+(\phi_i) + \sum_{i=1}^r L_i^-(\phi_i) = -\sum_{i=1}^r \int \phi_i d\tilde{\lambda}_i^+ + \sum_{i=1}^r \int \phi_i d\tilde{\lambda}_i^- \\ &= -\sum_{i=1}^r \int \phi_i \times g_i^+ d\tilde{\lambda} + \sum_{i=1}^r \int \phi_i \times g_i^- d\tilde{\lambda} \end{aligned}$$

(by 235M)

$$= -\sum_{i=1}^r \int \phi_i \times g_i d\tilde{\lambda} = \sum_{i=1}^r \int \phi_i \times \psi_i d\lambda_E^\partial$$

(by 235M again, because $\psi_i \times g = -g_i$)

$$= \int \phi \boldsymbol{.} \psi \, d\lambda_E^\partial.$$

So we have λ_E^∂ and ψ satisfying (i).

(d) Now suppose that $G \subseteq \mathbb{R}^r$ is open. If $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a Lipschitz function with compact support and $\|\phi\| \leq \chi G$, then

$$|\textstyle\int_E \operatorname{div} \phi \, d\mu| = |\int \phi \boldsymbol{.} \psi \, d\lambda_E^\partial| \leq \int \|\phi\| d\lambda_E^\partial \leq \lambda_E^\partial(G).$$

On the other hand, if $\gamma < \lambda_E^\partial(G)$, let $G_0 \subseteq G$ be a bounded open set such that $\gamma < \lambda_E^\partial(G_0)$, and set $\epsilon = \frac{1}{2}(\lambda_E^\partial(G_0) - \gamma)$. Let $K \subseteq G_0$ be a compact set such that $\lambda_E^\partial(G_0 \setminus K) \leq \epsilon$. Let $\delta > 0$ be such that $\|x - y\| \geq 2\delta$ whenever $y \in K$ and $x \in \mathbb{R}^r \setminus G_0$, and set $H = \{x : \inf_{y \in K} \|x - y\| < \delta\}$. Now there are $f_1, \dots, f_r \in C_k$ such that

$$\textstyle\sum_{i=1}^r f_i^2 \leq \chi H, \quad \int \sum_{i=1}^r f_i \times \psi_i d\lambda_E^\partial \geq \gamma.$$

P For each $i \leq r$, we can find a sequence $\langle g_{mi} \rangle_{m \in \mathbb{N}}$ in C_k such that $\int |g_{mi} - (\psi_i \times \chi K)| d\lambda_E^\partial \leq 2^{-m}$ for every $m \in \mathbb{N}$ (416I); multiplying the g_{mi} by a function which takes the value 1 on K and 0 outside H if necessary, we can suppose that $g_{mi}(x) = 0$ for $x \notin H$. Set

$$f_{mi} = \frac{g_{mi}}{\max(1, \sqrt{\sum_{j=1}^r g_{mj}^2})} \in C_k$$

for each m and i. Now $\lim_{m\to\infty} f_{mi}(x) = \psi_i(x)$ for every $i \le r$ whenever $\lim_{m\to\infty} g_{mi}(x) = \psi_i(x)$ for every $i \le r$, which is the case for λ_E^∂-almost every $x \in K$. Also $\sum_{i=1}^r f_{mi}^2 \le \chi H$ for every m, so $|\sum_{i=1}^r f_{mi} \times \psi_i| \le \chi H$ for every m, while

$$\lim_{m\to\infty} \sum_{i=1}^r \int_K f_{mi} \times \psi_i d\lambda_E^\partial = \sum_{i=1}^r \int_K \psi_i^2 d\lambda_E^\partial = \lambda_E^\partial(K).$$

At the same time,

$$|\sum_{i=1}^r \int_{\mathbb{R}^r \setminus K} f_{mi} \times \psi_i d\lambda_E^\partial| \le \lambda_E^\partial(H \setminus K) \le \epsilon$$

for every m, so

$$\sum_{i=1}^r \int f_{mi} \times \psi_i d\lambda_E^\partial \ge \lambda_E^\partial(G_0) - 2\epsilon = \gamma$$

for all m large enough, and we may take $f_i = f_{mi}$ for such an m. **Q**

Now, for $n \in \mathbb{N}$, set

$$\phi_n = (f_1 * \tilde{h}_n, \dots, f_r * \tilde{h}_n) \in \mathcal{D}_r.$$

For all n large enough, we shall have $\|x - y\| \ge \frac{1}{n+1}$ for every $x \in \mathbb{R}^r \setminus G_0$ and $y \in H$, so that $\phi_n(x) = 0$ if $x \notin G_0$. By 473Dg,

$$\|\phi_n(x)\| \le \sup_{y\in\mathbb{R}} \sqrt{\sum_{i=1}^r f_i(y)^2} \le 1$$

for every x and n, so that $\|\phi_n\| \le \chi G_0$ for all n large enough. Next, $\phi_n(x) \to (f_1(x), \dots, f_r(x))$ for every $x \in \mathbb{R}^r$ (473Ed), so

$$\int_E \operatorname{div} \phi_n d\mu = \int \phi_n \bullet \psi \, d\lambda_E^\partial \to \int \sum_{i=1}^r f_i \times \psi_i d\lambda_E^\partial \ge \gamma$$

as $n \to \infty$, by Lebesgue's Dominated Convergence Theorem. As γ is arbitrary,

$$\sup\{\int_E \operatorname{div} \phi \, d\mu : \phi \in \mathcal{D}_r, \, \|\phi\| \le \chi G\} \ge \lambda_E^\partial(G),$$

and the two must be equal.

(e) Thus λ_E^∂ must satisfy (ii). By 416Eb, it is uniquely defined. Now suppose that $\hat{\psi}$ is another function from a λ_E^∂-conegligible set to $\mathbb{R}^r$ and satisfies (i). Then

$$\int \phi \bullet \psi \, d\lambda_E^\partial = \int \phi \bullet \hat{\psi} \, d\lambda_E^\partial$$

for every Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support. Take any $i \le r$ and any compact set $K \subseteq \mathbb{R}^r$. For $m \in \mathbb{N}$, set $f_m(x) = \max(0, 1 - 2^m \inf_{y\in K} \|y - x\|)$ for $x \in \mathbb{R}^r$, so that $\langle f_m \rangle_{m\in\mathbb{N}}$ is a sequence of Lipschitz functions with compact support and $\lim_{m\to\infty} f_m = \chi K$. Set

$$\phi_m = (0, \dots, f_m, \dots, 0),$$

where the non-zero term is in the ith position. Then

$$\begin{aligned}\int_K \psi_i d\lambda_E^\partial &= \lim_{m\to\infty} \int f_m \times \psi_i d\lambda_E^\partial = \lim_{m\to\infty} \int \phi_m \bullet \psi \, d\lambda_E^\partial \\ &= \lim_{m\to\infty} \int \phi_m \bullet \hat{\psi} \, d\lambda_E^\partial = \int_K \hat{\psi}_i d\lambda_E^\partial.\end{aligned}$$

By the Monotone Class Theorem (136B), or otherwise, $\int_F \hat{\psi}_i d\lambda_E^\partial = \int_F \psi_i d\lambda_E^\partial$ for every bounded Borel set F, so that $\hat{\psi}_i = \psi_i$ λ_E^∂-a.e.; as i is arbitrary, $\psi = \hat{\psi}$ λ_E^∂-a.e. This completes the proof.

474F Definitions In the context of 474E, I will call λ_E^∂ the **perimeter measure** of E, and if ψ is a function from a λ_E^∂-conegligible subset of $\mathbb{R}^r$ to S_{r-1} which has the property in (i) of the theorem, I will call it an **outward-normal** function for E.

The words 'perimeter' and 'outward normal' are intended to suggest geometric interpretations; much of this section and the next will be devoted to validating this suggestion.

Observe that if E has locally finite perimeter, then $\operatorname{per} E = \lambda_E^\partial(\mathbb{R}^r)$.

474G The reduced boundary Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter; let λ_E^∂ be its perimeter measure and ψ an outward-normal function for E. The **reduced boundary** $\partial^\$ E$ is the set of those $y \in \mathbb{R}^r$ such that, for some $z \in S_{r-1}$,

$$\lim_{\delta \downarrow 0} \frac{1}{\lambda_E^\partial B(y,\delta)} \int_{B(y,\delta)} \|\psi(x) - z\| \, d\lambda_E^\partial = 0.$$

(When requiring that the limit be defined, I mean to insist that $\lambda_E^\partial B(y,\delta)$ should be non-zero for every $\delta > 0$, that is, that y belongs to the support of $\partial\lambda_E^\$$. **Warning!** Some authors use the phrase 'reduced boundary' for a slightly larger set.) Note that, writing $\psi = (\psi_1, \dots, \psi_r)$ and $z = (\zeta_1, \dots, \zeta_r)$, we must have

$$\zeta_i = \lim_{\delta \downarrow 0} \frac{1}{\lambda_E^\partial B(y,\delta)} \int_{B(y,\delta)} \psi_i d\lambda_E^\partial,$$

so that z is uniquely defined; call it $\psi_E(y)$. Of course $\partial^\$ E$ and ψ_E are determined entirely by the set E, because λ_E^∂ is uniquely determined and ψ is determined up to a λ_E^∂-negligible set (474E).

By Besicovitch's Density Theorem (472D),

$$\lim_{\delta \downarrow 0} \frac{1}{\lambda_E^\partial B(x,\delta)} \int_{B(x,\delta)} |\psi_i(x) - \psi_i(y)| \lambda_E^\partial(dx) = 0$$

for λ_E^∂-almost every $y \in \mathbb{R}^r$, for every $i \le r$; and for any such y, $\psi_E(y)$ is defined and equal to $\psi(y)$. Thus $\partial^\$ E$ is λ_E^∂-conegligible and ψ_E is an outward-normal function for E. I will call $\psi_E : \partial^\$ E \to S_{r-1}$ the **canonical outward-normal function** of E.

474H Invariance under isometries: Proposition Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter. Let λ_E^∂ be its perimeter measure, and ψ an outward normal function for E. If $T : \mathbb{R}^r \to \mathbb{R}^r$ is any isometry, then $T[E]$ has locally finite perimeter, $\lambda_{T[E]}^\partial$ is the image measure $\lambda_E^\partial T^{-1}$, and $S\psi T^{-1}$ is an outward normal function for $T[E]$, where S is the derivative of T.

proof (a) As noted in 474C, the derivative of T is constant, and is an orthogonal matrix. Suppose that $n \in \mathbb{N}$. Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a Lipschitz function such that $\|\phi\| \le \chi B(\mathbf{0}, n)$. Then

$$\left| \int_{T[E]} \operatorname{div} \phi \, d\mu \right| = \left| \int_E (\operatorname{div} \phi) T \, d\mu \right|$$

(263D, because $|\det S| = 1$)

$$= \left| \int_E \operatorname{div}(T^{-1}\phi T) d\mu \right|$$

(474C)

$$= \left| \int_E \operatorname{div}(S^{-1}\phi T) d\mu \right|$$

(because $S^{-1}\phi T$ and $T^{-1}\phi T$ differ by a constant, and must have the same derivative)

$$\le \lambda_E^\partial(T^{-1}[B(\mathbf{0}, n)])$$

because $S^{-1}\phi T$ is a Lipschitz function and

$$\|S^{-1}\phi T\| = \|\phi T\| \le \chi T^{-1}[B(\mathbf{0}, n)].$$

Since $T^{-1}[B(\mathbf{0}, n)]$ is bounded, $\lambda_E^\partial(T^{-1}[B(\mathbf{0}, n)])$ is finite for every n, and $T[E]$ has locally finite perimeter.

(b) We can therefore speak of its perimeter measure $\lambda_{T[E]}^\partial$. Let $G \subseteq \mathbb{R}^r$ be an open set. If $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a Lipschitz function and $\|\phi\| \le \chi T[G]$, then

$$\left| \int_{T[E]} \operatorname{div} \phi \, d\mu \right| = \left| \int_E \operatorname{div}(S^{-1}\phi T) d\mu \right| \le \lambda_E^\partial(G)$$

because $S^{-1}\phi T$ is a Lipschitz function dominated by χG. As ϕ is arbitrary, $\lambda_{T[E]}^\partial(T[G]) \le \lambda_E^\partial(G)$. Applying the same argument in reverse, with T^{-1} in the place of T, we see that $\lambda_E^\partial(G) \le \lambda_{T[E]}^\partial(T[G])$, so the two are

equal. Now this means that the Radon measures $\lambda^\partial_{T[E]}$ and $\lambda^\partial_E T^{-1}$ (418I) agree on open sets, and must be identical (416Eb).

(c) Now consider $S\psi T^{-1}$. Since ψ is defined λ^∂_E-almost everywhere and takes values in S_{r-1}, ψT and $S\psi T^{-1}$ are defined $\lambda^\partial_{T[E]}$-almost everywhere and take values in S_{r-1}. If $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a Lipschitz function with compact support,

$$\begin{aligned}\int_{T[E]} \operatorname{div}\phi\, d\mu &= \int_E (\operatorname{div}\phi)T\, d\mu = \int_E \operatorname{div}(T^{-1}\phi T)d\mu \\ &= \int_E \operatorname{div}(S^{-1}\phi T)d\mu = \int (S^{-1}\phi T)\boldsymbol{.}\psi\, d\lambda^\partial_E \\ &= \int (\phi T)\boldsymbol{.}(S\psi)d\lambda^\partial_E\end{aligned}$$

(because S is orthogonal)

$$= \int \phi\boldsymbol{.}(S\psi T^{-1})d(\lambda^\partial_E T^{-1})$$

(235I)

$$= \int \phi\boldsymbol{.}(S\psi T^{-1})d\lambda^\partial_{T[E]}$$

Accordingly $S\psi T^{-1}$ is an outward-normal function for $T[E]$.

474I Half-spaces It will be useful, and perhaps instructive, to check the most elementary special case.

Proposition Let $H \subseteq \mathbb{R}^r$ be a half-space $\{x : x\boldsymbol{.}v \leq \alpha\}$, where $v \in S^{r-1}$. Then H has locally finite perimeter; its perimeter measure λ^∂_H is defined by saying

$$\lambda^\partial_H(F) = \nu(F \cap \partial H)$$

whenever $F \subseteq \mathbb{R}^r$ is such that ν measures $F \cap \partial H$, and the constant function with value v is an outward-normal function for H.

proof (a) Suppose, to begin with, that v is the unit vector $(0, \dots, 1)$ and that $\alpha = 0$, so that $H = \{x : \xi_r \leq 0\}$. Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a Lipschitz function with compact support. Then for any $i < r$

$$\int_H \frac{\partial \phi_i}{\partial \xi_i}\mu(dx) = 0$$

because we can regard this as a multiple integral in which the inner integral is with respect to ξ_i and is therefore always zero. On the other hand, integrating with respect to the rth coordinate first,

$$\begin{aligned}\int_H \frac{\partial \phi_r}{\partial \xi_r}\mu(dx) &= \int_{\mathbb{R}^{r-1}} \int_{-\infty}^0 \frac{\partial \phi_r}{\partial \xi_r}(z,t)dt\, \mu_{r-1}(dz) \\ &= \int_{\mathbb{R}^{r-1}} \phi_r(z,0)\mu_{r-1}(dz) = \int_{\partial H} \phi_r(x)\nu(dx)\end{aligned}$$

(identifying ν on $\mathbb{R}^{r-1} \times \{0\}$ with μ_{r-1} on $\mathbb{R}^{r-1}$)

$$= \int_{\partial H} \phi\boldsymbol{.}v\, d\nu = \int \phi\boldsymbol{.}v\, d\lambda$$

where λ is the indefinite-integral measure over ν defined by the function $\chi(\partial H)$. Note that (by 234Da) λ can also be regarded as $\nu_{\partial H}\iota^{-1}$, where $\nu_{\partial H}$ is the subspace measure on ∂H and $\iota : \partial H \to \mathbb{R}^r$ is the identity map. Now $\nu_{\partial H}$ can be identified with Lebesgue measure on $\mathbb{R}^{r-1}$, by 265B or otherwise, so in particular is a Radon measure, and λ also is a Radon measure, by 418I or otherwise.

This means that λ and the constant function with value v satisfy the conditions of 474E, and must be the perimeter measure of H and an outward-normal function.

(b) For the general case, let S be an orthogonal matrix such that $S(0,\dots,1) = v$, and set $T(x) = Sx + \alpha v$ for every x, so that $H = T[\{x : \xi_r \leq 0\}]$. By 474H, the perimeter measure of H is λT^{-1} and the constant function with value v is an outward-normal function for H. Now the Radon measure $\lambda_H^\partial = \lambda T^{-1}$ is defined by saying that

$$\lambda_H^\partial F = \lambda T^{-1}[F] = \nu(T^{-1}[F] \cap \{x : \xi_r = 0\}) = \nu(F \cap T[\{x : \xi_r = 0\}]) = \nu(F \cap \partial H)$$

whenever $\nu(F \cap \partial H)$ is defined, because ν (being a scalar multiple of a Hausdorff measure) must be invariant under the isometry T.

474J Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter. Let λ_E^∂ be the perimeter measure of E, and ψ_E its canonical outward-normal function. Then $\mathbb{R}^r \setminus E$ also has locally finite perimeter; its perimeter measure is λ_E^∂, its reduced boundary is $\partial^\$ E$, and its canonical outward-normal function is $-\psi_E$.

proof By 474Bc,

$$\int_{\mathbb{R}^r \setminus E} \operatorname{div} \phi \, d\mu = -\int_E \operatorname{div} \phi \, d\mu = \int \phi \boldsymbol{.} (-\psi_E) \, d\lambda_E^\partial$$

for every Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support. The uniqueness assertions in 474E tell us that $\mathbb{R}^r \setminus E$ has locally finite perimeter, that its perimeter measure is λ_E^∂, and that $-\psi_E$ is an outward-normal function for $\mathbb{R}^r \setminus E$. Referring to the definition of 'reduced boundary' in 474G, we see at once that $\partial^\$(\mathbb{R}^r \setminus E) = \partial^\$ E$ and that $\psi_{\mathbb{R}^r \setminus E} = -\psi_E$.

474K Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter; let λ_E^∂ be its perimeter measure, and ψ an outward-normal function for E. Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a Lipschitz function with compact support, and $g \in \mathcal{D}$ an even function. Then

$$\int \phi \boldsymbol{.} \operatorname{grad}(g * \chi E) d\mu + \int (g * \phi) \boldsymbol{.} \psi \, d\lambda_E^\partial = 0.$$

proof

$$\int \phi \boldsymbol{.} \operatorname{grad}(g * \chi E) d\mu = -\int (g * \chi E) \times \operatorname{div} \phi \, d\mu$$

(474Bd, using 473Df to see that $g * \chi E$ is Lipschitz)

$$= -\int \chi E \times \operatorname{div}(g * \phi) d\mu$$

(474Be)

$$= -\int (g * \phi) \boldsymbol{.} \psi \, d\lambda_E^\partial$$

(because $g * \phi$ is smooth and has compact support, so is Lipschitz), as required.

474L Two isoperimetric inequalities: Theorem Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and λ_E^∂ its perimeter measure.

(a) If E is bounded, then $(\mu E)^{(r-1)/r} \leq \operatorname{per} E$.

(b) If $B \subseteq \mathbb{R}^r$ is a closed ball, then

$$\min(\mu(B \cap E), \mu(B \setminus E))^{(r-1)/r} \leq 2c\lambda_E^\partial(\operatorname{int} B),$$

where $c = 2^{r+2}(1 + \sqrt{r})(1 + 2^{r+1})$.

proof (a) Let $\epsilon > 0$. By 473Ef, there is an $n \in \mathbb{N}$ such that $\|f - \chi E\|_{r/(r-1)} \leq \epsilon$, where $f = \chi E * \tilde{h}_n$. Note that f is smooth (473De) and has compact support, because E is bounded. Let $\eta > 0$ be such that

$$\int \|\operatorname{grad} f\| d\mu \leq \int \frac{\|\operatorname{grad} f\|^2}{\sqrt{\eta + \|\operatorname{grad} f\|^2}} \, d\mu + \epsilon,$$

and set $\phi = \frac{\operatorname{grad} f}{\sqrt{\eta + \|\operatorname{grad} f\|^2}}$. Then $\phi \in \mathcal{D}_r$ and $\|\phi(x)\| \leq 1$ for every $x \in \mathbb{R}^r$. Now we can estimate

$$\begin{aligned}\int \|\operatorname{grad} f\| d\mu &\leq \int \phi \boldsymbol{.} \operatorname{grad} f \, d\mu + \epsilon \\ &= -\int (\tilde{h}_n * \phi) \boldsymbol{.} \psi \, d\lambda_E^\partial + \epsilon\end{aligned}$$

(where ψ is an outward-normal function for E, by 474K)

$$\leq \int \|\tilde{h}_n * \phi\| d\lambda_E^\partial + \epsilon \leq \operatorname{per} E + \epsilon$$

because $\|(\tilde{h}_n * \phi)(x)\| \leq 1$ for every $x \in \mathbb{R}^r$, by 473Dg. Accordingly

$$(\mu E)^{(r-1)/r} = \|\chi E\|_{r/(r-1)} \leq \|f\|_{r/(r-1)} + \epsilon \leq \int \|\operatorname{grad} f\| d\mu + \epsilon$$

(473H)

$$\leq \operatorname{per} E + 2\epsilon.$$

As ϵ is arbitrary, we have the result.

(b)(i) Set $\alpha = \min(\mu(B \cap E), \mu(B \setminus E))$. If $\alpha = 0$, the result is trivial; so suppose that $\alpha > 0$. Take any $\epsilon \in]0, \alpha]$. Let B_1 be a closed ball, with the same centre as B and strictly smaller non-zero radius, such that $\mu(B \setminus B_1) \leq \epsilon$; then $\alpha - \epsilon \leq \min(\mu(B_1 \cap E), \mu(B_1 \setminus E))$. For $f \in \mathcal{L}^{r/(r-1)}(\mu)$ set

$$\gamma_0(f) = \frac{1}{\mu B_1} \int_{B_1} f d\mu, \quad \gamma_1(f) = \|(f \times \chi B_1) - \gamma_0(f) \chi B_1\|_{r/(r-1)};$$

then both γ_0 and γ_1 are continuous functions on $\mathcal{L}^{r/(r-1)}(\mu)$ if we give it its usual pseudometric $(f, g) \mapsto \|f - g\|_{r/(r-1)}$. Now $\gamma_1(\chi(E \cap B)) \geq \frac{1}{2}(\alpha - \epsilon)^{(r-1)/r}$. **P** We have

$$\gamma_0(\chi(E \cap B)) = \tfrac{\mu(B_1 \cap E)}{\mu B_1},$$

$$\begin{aligned}\gamma_1(\chi(E \cap B))^{r/(r-1)} &= \int_{B_1} |\chi(E \cap B) - \gamma_0(\chi(E \cap B))|^{r/(r-1)} \\ &= \mu(B_1 \cap E)\big(1 - \tfrac{\mu(B_1 \cap E)}{\mu B_1}\big)^{r/(r-1)} + \mu(B_1 \setminus E)\big(\tfrac{\mu(B_1 \cap E)}{\mu B_1}\big)^{r/(r-1)} \\ &= \mu(B_1 \cap E)\big(\tfrac{\mu(B_1 \setminus E)}{\mu B_1}\big)^{r/(r-1)} + \mu(B_1 \setminus E)\big(\tfrac{\mu(B_1 \cap E)}{\mu B_1}\big)^{r/(r-1)}.\end{aligned}$$

Either $\mu(B_1 \cap E) \geq \frac{1}{2}\mu B_1$ or $\mu(B_1 \setminus E) \geq \frac{1}{2}\mu B_1$; suppose the former. Then

$$\gamma_1(\chi(E \cap B))^{r/(r-1)} \geq \frac{1}{2^{r/(r-1)}} \mu(B_1 \setminus E) \geq \frac{1}{2^{r/(r-1)}}(\alpha - \epsilon)$$

and $\gamma_1(\chi(E \cap B)) \geq \frac{1}{2}(\alpha - \epsilon)^{r/(r-1)}$. Exchanging $B_1 \cap E$ and $B_1 \setminus E$ we have the same result if $\mu(B_1 \cap E) \geq \frac{1}{2}\mu B_1$. **Q**

(ii) Express B as $B(y, \delta)$ and B_1 as $B(y, \delta_1)$. Take $n_0 \geq \frac{2}{\delta - \delta_1}$. Because γ_1 is $\| \|_{r/(r-1)}$-continuous, there is an $n \geq n_0$ such that $\gamma_1(f) \geq \frac{1}{2}(\alpha - \epsilon)^{(r-1)/r} - \epsilon$, where $f = \tilde{h}_n * \chi(E \cap B)$ (473Ef); as in part (a) of the proof, $f \in \mathcal{D}$. Let $\eta > 0$ be such that

$$\int_{B_1} \frac{\|\operatorname{grad} f\|^2}{\sqrt{\eta + \|\operatorname{grad} f\|^2}} d\mu \geq \int_{B_1} \|\operatorname{grad} f\| d\mu - \epsilon.$$

Let $m \geq n_0$ be such that $\int \phi \boldsymbol{.} \operatorname{grad} f \, d\mu \geq \int_{B_1} \|\operatorname{grad} f\| d\mu - 2\epsilon$, where

$$\phi = \tilde{h}_m * \big(\frac{\operatorname{grad} f}{\sqrt{\eta + \|\operatorname{grad} f\|^2}} \times \chi B_1\big).$$

Note that $\phi(x) = 0$ if $\|x - y\| \geq \frac{1}{2}(\delta + \delta_1)$, so that $(\tilde{h}_n * \phi)(x) = 0$ if $x \notin \operatorname{int} B$. By 473Dg, $\|\phi(x)\| \leq 1$ for every x and $\|(\tilde{h}_n * \phi)(x)\| \leq 1$ for every x, so $\|\tilde{h}_n * \phi\| \leq \chi(\operatorname{int} B)$.

Now we have

$$\begin{aligned}\int \phi \boldsymbol{.} \operatorname{grad} f \, d\mu &= \int \phi \boldsymbol{.} \operatorname{grad}(\tilde{h}_n * \chi(E \cap B)) \, d\mu \\ &= -\int (\tilde{h}_n * \chi(E \cap B)) \times \operatorname{div} \phi \, d\mu \end{aligned}$$

(474Bd)

$$= -\int_{E \cap B} \operatorname{div}(\tilde{h}_n * \phi) d\mu$$

(474Be)

$$= -\int_E \operatorname{div}(\tilde{h}_n * \phi) d\mu \leq \lambda_E^{\partial}(\operatorname{int} B)$$

(474E).

(iii) Accordingly

$$\frac{1}{2}(\alpha - \epsilon)^{(r-1)/r} - \epsilon \leq \gamma_1(f) \leq c \int_{B_1} \| \operatorname{grad} f \| d\mu$$

(473K)

$$\leq c\Big(\int \phi \boldsymbol{.} \operatorname{grad} f \, d\mu + 2\epsilon \Big) \leq c(\lambda_E^{\partial}(\operatorname{int} B) + 2\epsilon).$$

As ϵ is arbitrary, $\alpha^{(r-1)/r} \leq 2c\lambda_E^{\partial}(\operatorname{int} B)$, as claimed.

474M Lemma Suppose that $E \subseteq \mathbb{R}^r$ has locally finite perimeter, with perimeter measure λ_E^{∂} and an outward-normal function ψ. Then for any $y \in \mathbb{R}^r$ and any Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$,

$$\int_{E \cap B(y,\delta)} \operatorname{div} \phi \, d\mu = \int_{B(y,\delta)} \phi \boldsymbol{.} \psi \, d\lambda_E^{\partial} + \int_{E \cap \partial B(y,\delta)} \phi(x) \boldsymbol{.} \frac{1}{\delta}(x - y) \, \nu(dx)$$

for almost every $\delta > 0$.

proof (a) For $t > 0$, set

$$w(t) = \int_{E \cap \partial B(y,t)} \phi(x) \boldsymbol{.} \frac{1}{t}(x - y) \, \nu(dx)$$

when this is defined. By 265G, applied to functions of the form

$$x \mapsto \begin{cases} \phi(x) \boldsymbol{.} \frac{x-y}{\|x-y\|} & \text{if } x \in E \text{ and } 0 < \|x - y\| \leq \alpha, \\ 0 & \text{otherwise,} \end{cases}$$

w is defined almost everywhere in $]0, \infty[$ and is measurable (for Lebesgue measure on $\mathbb{R}$).

Let $\delta > 0$ be any point in the Lebesgue set of w (223D). Then

$$\lim_{t \downarrow 0} \frac{1}{t} \int_{\delta}^{\delta+t} |w(s) - w(\delta)| ds \leq 2 \lim_{t \downarrow 0} \frac{1}{2t} \int_{\delta-t}^{\delta+t} |w(s) - w(\delta)| = 0.$$

Let $\epsilon > 0$. Then there is an $\eta > 0$ such that

$$\frac{1}{\eta} \int_{\delta}^{\delta+\eta} |w(s) - w(\delta)| ds \leq \epsilon, \quad \int_{B(y,\delta+\eta) \setminus B(y,\delta)} \|\phi\| d\lambda_E^{\partial} \leq \epsilon,$$

$$\int_{B(y,\delta+\eta) \setminus B(y,\delta)} \| \operatorname{div} \phi \| d\mu \leq \epsilon.$$

(b) Set

$$\begin{aligned} g(x) &= 1 \text{ if } \|x-y\| \leq \delta, \\ &= 1 - \frac{1}{\eta}(\|x-y\| - \delta) \text{ if } \delta \leq \|x-y\| \leq \delta+\eta, \\ &= 0 \text{ if } \|x-y\| \geq \delta+\eta. \end{aligned}$$

Then g is continuous, and $\operatorname{grad} g(x) = \mathbf{0}$ if $\|x-y\| < \delta$ or $\|x-y\| > \delta+\eta$; while if $\delta < \|x-y\| < \delta+\eta$, $\operatorname{grad} g(x) = -\frac{x-y}{\eta\|x-y\|}$. This means that

$$\begin{aligned} \int_E \phi \boldsymbol{.} \operatorname{grad} g \, d\mu &= -\frac{1}{\eta}\int_\delta^{\delta+\eta} \int_{E\cap\partial B(y,t)} \frac{1}{t}(x-y) \boldsymbol{.} \phi(x)\, \nu(dx)dt \\ &= -\frac{1}{\eta}\int_\delta^{\delta+\eta} w(t)dt. \end{aligned}$$

By the choice of η,

$$|\textstyle\int_E \phi \boldsymbol{.} \operatorname{grad} g \, d\mu + w(\delta)| \leq \epsilon.$$

(c) By 474E and 474Bb we have

$$\int (g\times\phi)\boldsymbol{.}\psi \, d\lambda_E^\partial = \int_E \operatorname{div}(g\times\phi)d\mu$$

(of course $g\times\phi$ is Lipschitz, by 473Ca and 262Ba)

$$= \int_E \phi \boldsymbol{.} \operatorname{grad} g \, d\mu + \int_E g \times \operatorname{div}\phi \, d\mu.$$

Next, by the choice of η,

$$|\textstyle\int((g\times\phi)-\phi)\boldsymbol{.}\psi d\lambda_E^\partial| \leq \int_{B(y,\delta+\eta)\setminus B(y,\delta)} \|\phi\| d\lambda_E^\partial \leq \epsilon$$

and

$$\begin{aligned} \Big|\int_E \phi \boldsymbol{.} \operatorname{grad} g \, d\mu + \int_{E\cap\partial B(y,\delta)} \phi(x) \boldsymbol{.} \tfrac{1}{\delta}(x-y)\nu(dx)\Big| \\ = \Big|-\int_\delta^{\delta+\eta}\int_{\partial B(y,t)} \tfrac{x-y}{\eta\|x-y\|} \boldsymbol{.} \phi(x)\nu(dx)dt + w(\delta)\Big| \end{aligned}$$

(265G)

$$= \Big|-\frac{1}{\eta}\int_\delta^{\delta+\eta} w(t)dt + w(\delta)\Big| \leq \epsilon$$

and

$$\begin{aligned} \Big|\int_E g\times\operatorname{div}\phi\, d\mu - \int_{E\cap B(y,\delta)} \operatorname{div}\phi\, d\mu\Big| \\ \leq \int_{B(y,\delta+\eta)\setminus B(y,\delta)} \|\operatorname{div}\phi\| d\mu \leq \epsilon. \end{aligned}$$

Putting these together, we have

$$|\textstyle\int_{E\cap B(y,\delta)} \operatorname{div}\phi\, d\mu - \int_{B(y,\delta)} \phi\boldsymbol{.}\psi \, d\lambda_E^\partial - \int_{E\cap\partial B(y,\delta)} \phi(x)\boldsymbol{.}\frac{1}{\delta}(x-y)\nu(dx)| \leq 3\epsilon.$$

As ϵ is arbitrary, this gives the result.

474N Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and λ_E^∂ its perimeter measure. Then, for any $y \in \partial^\$ E$,

(i) $\liminf_{\delta\downarrow 0} \frac{\mu(B(y,\delta)\cap E)}{\mu B(y,\delta)} \geq \frac{1}{(3r)^r\beta_r}$;

(ii) $\liminf_{\delta\downarrow 0} \frac{\mu(B(y,\delta)\setminus E)}{\mu B(y,\delta)} \geq \frac{1}{(3r)^r\beta_r}$;

(iii) $\liminf_{\delta\downarrow 0} \frac{\lambda_E^\partial B(y,\delta)}{\delta^{r-1}} \geq \frac{1}{2c(3r)^r}$,

where $c = 2^{r+2}(1+\sqrt{r})(1+2^{r+1})$;

(iv) $\limsup_{\delta\downarrow 0} \frac{\lambda_E^\partial B(y,\delta)}{\delta^{r-1}} \leq 4\pi\beta_{r-2}$.

proof (a) Let ψ_E be the canonical outward-normal function of E (474G). Take $y \in \partial^\$ E$. Set

$$\Phi = \{\phi : \phi \text{ is a Lipschitz function from } \mathbb{R}^r \text{ to } B(\mathbf{0},1)\}.$$

Because the space $L^1(\mu)$ is separable in its usual (norm) topology (244I), so is $\{(\operatorname{div}\phi \times \chi B_1)^\bullet : \phi \in \Phi\}$, (4A2P(a-iv)), and there must be a countable set $\Phi_0 \subseteq \Phi$ such that

whenever $\phi \in \Phi$ and $m \in \mathbb{N}$ there is a $\hat\phi \in \Phi_0$ such that $\int_{B(y,1)} |\operatorname{div}\phi - \operatorname{div}\hat\phi| d\mu \leq 2^{-m}$.

Now, for each $\phi \in \Phi_0$,

$$\begin{aligned}|\int_{E\cap B(y,\delta)} \operatorname{div}\phi\, d\mu| &= |\int_{B(y,\delta)} \phi\cdot\psi_E\, d\lambda_E^\partial + \int_{E\cap\partial B(y,\delta)} \phi(x)\cdot\tfrac{1}{\delta}(x-y)\nu(dx)| \\ &\leq \lambda_E^\partial B(y,\delta) + \nu(E\cap\partial B(y,\delta))\end{aligned}$$

for almost every $\delta > 0$, by 474M. But this means that, for almost every $\delta \in]0,1]$,

$$\begin{aligned}\operatorname{per}(E\cap B(y,\delta)) &= \sup_{\phi\in\Phi} |\int_{E\cap B(y,\delta)} \operatorname{div}\phi\, d\mu| \\ &= \sup_{\phi\in\Phi_0} |\int_{E\cap B(y,\delta)} \operatorname{div}\phi\, d\mu| \leq \lambda_E^\partial B(y,\delta) + \nu(E\cap\partial B(y,\delta)).\end{aligned}$$

(b) It follows that, for some $\delta_0 > 0$,

$$\operatorname{per}(E\cap B(y,\delta)) \leq 3\nu(E\cap\partial B(y,\delta))$$

for almost every $\delta \in]0,\delta_0]$. **P** Applying 474M with $\phi(x) = \psi_E(y)$ for every x, we have

$$0 = \int_{B(y,\delta)} \psi_E(y)\cdot\psi_E(x)\lambda_E^\partial(dx) + \int_{E\cap\partial B(y,\delta)} \psi_E(y)\cdot\tfrac{1}{\delta}(x-y)\nu(dx)$$

for almost every $\delta \in [0,1]$. But by the definition of $\psi_E(y)$,

$$\lim_{\delta\downarrow 0} \frac{1}{\lambda_E^\partial B(y,\delta)} \int_{B(y,\delta)} \psi_E(y)\cdot\psi_E\, d\lambda_E^\partial = \lim_{\delta\downarrow 0} \frac{1}{\lambda_E^\partial B(y,\delta)} \int_{B(y,\delta)} \psi_E(y)\cdot\psi_E(y)\, d\lambda_E^\partial = 1.$$

So there is some $\delta_0 > 0$ such that, for almost every $\delta \in]0,\delta_0]$,

$$\begin{aligned}\lambda_E^\partial B(y,\delta) &\leq 2\int_{B(y,\delta)} \psi_E(y)\cdot\psi_E\, d\lambda_E^\partial \\ &= -2\int_{E\cap\partial B(y,\delta)} \psi_E(y)\cdot\tfrac{1}{\delta}(x-y)\nu(dx) \leq 2\nu(E\cap\partial B(y,\delta)).\end{aligned} \tag{1}$$

But this means that, for almost every such δ,

$$\operatorname{per}(E\cap B(y,\delta)) \leq \lambda_E^\partial B(y,\delta) + \nu(E\cap\partial B(y,\delta)) \leq 3\nu(E\cap\partial B(y,\delta)). \text{ } \mathbf{Q}$$

(c) Set $g(t) = \mu(E\cap B(y,t))$ for $t \geq 0$. By 265G, $g(t) = \int_0^t \nu(E\cap\partial B(y,s))ds$ for every t, so g is absolutely continuous on $[0,1]$ and $g'(t) = \nu(E\cap\partial B(y,t))$ for almost every t. Now we can estimate

$$g(t)^{(r-1)/r} = \mu(E \cap B(y,t))^{(r-1)/r} \leq \operatorname{per}(E \cap B(y,t))$$

(474La)

$$\leq 3\nu(E \cap \partial B(y,t)) = 3g'(t)$$

for almost every $t \in [0, \delta_0]$. So

$$\frac{d}{dt}\big(g(t)^{1/r}\big) = \frac{1}{r} g(t)^{(1-r)/r} g'(t) \geq \frac{1}{3r}$$

for almost every $t \in [0, \delta_0]$; since $t \mapsto g(t)^{1/r}$ is non-decreasing, $g(t)^{1/r} \geq \frac{t}{3r}$ (222C) and $g(t) \geq (3r)^{-r} t^r$ for every $t \in [0, \delta_0]$.

(d) Accordingly

$$\liminf_{\delta \downarrow 0} \frac{\mu(B(y,\delta) \cap E)}{\mu B(y,\delta)} \geq \inf_{0<\delta\leq\delta_0} \frac{\mu(B(y,\delta)\cap E)}{\mu B(y,\delta)} = \inf_{0<\delta\leq\delta_0} \frac{g(\delta)}{\beta_r \delta^r} \geq \frac{1}{(3r)^r \beta_r}.$$

This proves (i).

(e) Because $\lambda^{\partial}_{\mathbb{R}^r \setminus E} = \lambda^{\partial}_E$ and $-\psi_E$ is the canonical outward-normal function of $\mathbb{R}^r \setminus E$ (474J), y also belongs to $\partial^{\$}(\mathbb{R}^r \setminus E)$, so the second formula of this lemma follows from the first.

(f) By 474Lb,

$$\lambda^{\partial}_E B(y,\delta) \geq \frac{1}{2c} \min\big(\mu(B(y,\delta) \cap E), \mu(B(y,\delta) \setminus E)\big)^{(r-1)/r}$$

for every $\delta \geq 0$. So

$$\begin{aligned}\liminf_{\delta \downarrow 0} \frac{\lambda^{\partial}_E B(y,\delta)}{\delta^{r-1}} &\geq \frac{1}{2c} \liminf_{\delta \downarrow 0} \min\big(\frac{\mu(B(y,\delta) \cap E)}{\delta^r}, \frac{\mu(B(y,\delta) \setminus E)}{\delta^r}\big)^{(r-1)/r} \\ &\geq \frac{1}{2c(3r)^r}.\end{aligned}$$

Thus (iii) is true.

(g) Returning to the inequality (1) in the proof of (b) above, we have a $\delta_0 > 0$ such that

$$\lambda^{\partial}_E B(y,\delta) \leq 2\nu(E \cap \partial B(y,\delta)) \leq 2\nu(\partial B(y,\delta)) = 4\pi\beta_{r-2}\delta^{r-1}$$

for almost every $\delta \in]0, \delta_0]$. But this means that, for any $\delta \in [0, \delta_0[$,

$$\lambda^{\partial}_E B(y,\delta) \leq \inf_{t>\delta} \lambda^{\partial}_E B(y,t) \leq \inf_{t>\delta} 4\pi\beta_{r-2}t^{r-1} = 4\pi\beta_{r-2}\delta^{r-1},$$

and (iv) is true.

474O Definition Let $A \subseteq \mathbb{R}^r$ be any set, and $y \in \mathbb{R}^r$. A **Federer exterior normal to** A **at** y is a $v \in S_{r-1}$ such that,

$$\lim_{\delta \downarrow 0} \frac{\mu^*((H \triangle A) \cap B(y,\delta))}{\mu B(y,\delta)} = 0,$$

where H is the half-space $\{x : (x-y) \boldsymbol{.} v \leq 0\}$.

474P Lemma If $A \subseteq \mathbb{R}^r$ and $y \in \mathbb{R}^r$, there can be at most one Federer exterior normal to A at y.

proof Suppose that v, $v' \in S_{r-1}$ are two Federer exterior normals to E at y. Set

$$H = \{x : (x-y) \boldsymbol{.} v \leq 0\}, \quad H' = \{x : (x-y) \boldsymbol{.} v' \leq 0\}.$$

Then

$$\lim_{\delta \downarrow 0} \frac{\mu((H \triangle H') \cap B(y,\delta))}{\mu B(y,\delta)} \leq \lim_{\delta \downarrow 0} \frac{\mu^*((H \triangle A) \cap B(y,\delta))}{\mu B(y,\delta)} + \lim_{\delta \downarrow 0} \frac{\mu^*((H' \triangle A) \cap B(y,\delta))}{\mu B(y,\delta)} = 0.$$

But for any $\delta > 0$,

$$(H \triangle H') \cap B(y,\delta) = y + \delta((H_0 \triangle H_0') \cap B(\mathbf{0},1)),$$

where

$$H_0 = \{x : x \boldsymbol{.} v \leq 0\}, \quad H_0' = \{x : x \boldsymbol{.} v' \leq 0\}.$$

So

$$0 = \lim_{\delta \downarrow 0} \frac{\mu((H \triangle H') \cap B(y,\delta))}{\mu B(y,\delta)} = \frac{\mu((H_0 \triangle H_0') \cap B(\mathbf{0},1))}{\mu B(\mathbf{0},1)} = \frac{\mu((H_0 \triangle H_0') \cap B(\mathbf{0},n))}{\mu B(\mathbf{0},n)}$$

for every $n \geq 1$, and $\mu(H_0 \triangle H_0') = 0$. Since μ is strictly positive, and H_0 and H_0' are both the closures of their interiors, they must be identical; and it follows that $v = v'$.

474Q Theorem Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, λ_E^∂ its perimeter measure, ψ_E its canonical outward-normal function, and y any point of its reduced boundary $\partial^\$ E$. Then $\psi_E(y)$ is the Federer exterior normal to E at y.

proof (a) To begin with (down to the end of (i) below) suppose that $y = \mathbf{0}$ and that $\psi_E(y) = (0, \ldots, 0, 1) = v$ say. Set

$$c_0 = \frac{2^{r+2}\pi\beta_{r-1}}{r\beta_r}, \quad c' = 2^{r+1}(1 + \sqrt{r-1})(1 + 2^r),$$

(counting β_0 as 1, if $r = 2$),

$$c_1 = 1 + \max\big(4\pi\beta_{r-2}, (3r)^r 2^{r+3}(1 + \sqrt{r})(1 + 2^{r+1})\big),$$

$$c_2 = 2^{(r+1)/2} c_1, \quad c_3 = \frac{c_2}{2} + \frac{8c'}{\beta_{r-1}}.$$

For $\delta > 0$ let C_δ be the solid cylinder

$$\{x : \textstyle\sum_{i=1}^{r-1} \xi_i^2 \leq \delta^2, \, |\xi_r| \leq \delta\}.$$

For $\gamma \in \mathbb{R}$ set

$$\tilde{H}(\gamma) = \{x : \xi_r \leq \gamma\}.$$

For each $\delta > 0$, choose $\gamma_\delta \in \mathbb{R}$ such that

$$\mu((E \triangle \tilde{H}(\gamma_\delta)) \cap C_\delta) \leq \delta \mu C_\delta + \inf_{\gamma \in \mathbb{R}} \mu((E \triangle \tilde{H}(\gamma)) \cap C_\delta).$$

(b) Take any $\epsilon > 0$ such that

$$\epsilon < \frac{1}{\sqrt{2}}, \quad 4c'\epsilon < \frac{1}{8}\beta_{r-1}, \quad c_2\epsilon^3 < \frac{1}{4}\beta_{r-1}.$$

Then there is a $\delta_0 \in]0,1]$ such that

$$\frac{1}{\lambda_E^\partial B(\mathbf{0},\delta)} \int_{B(\mathbf{0},\delta)} \|\psi_E(x) - v\| \lambda_E^\partial(dx) \leq \epsilon^3,$$

$$\frac{1}{c_1}\delta^{r-1} \leq \lambda_E^\partial B(\mathbf{0},\delta) \leq c_1 \delta^{r-1}$$

for every $\delta \in \left]0, \delta_0\sqrt{2}\right]$ (using 474N(iii) and 474N(iv) for the inequalities bounding $\lambda_E^\partial B(\mathbf{0},\delta)$). Then

$$B(\mathbf{0},\delta) \subseteq C_\delta \subseteq B(\mathbf{0},\delta\sqrt{2}),$$

so

$$\begin{aligned}\int_{C_\delta} \|\psi_E(x) - v\| \lambda_E^\partial(dx) &\leq \int_{B(\mathbf{0},\delta\sqrt{2})} \|\psi_E(x) - v\| \lambda_E^\partial(dx) \\ &\leq \epsilon^3 \lambda_E^\partial B(\mathbf{0},\delta\sqrt{2}) \leq 2^{(r-1)/2} c_1 \epsilon^3 \delta^{r-1} = \frac{1}{2} c_2 \epsilon^3 \delta^{r-1}\end{aligned}$$

whenever $0 < \delta \leq \delta_0$.

For each $n \in \mathbb{N}$, set $f_n = \tilde{h}_n * \chi E$, where $\tilde{h}_n$ is the smoothing function of 473E, so that f_n is smooth and Lipschitz (473Df) and $\operatorname{grad} f_n$ is bounded (473Cc). For $n \in \mathbb{N}$, $t \in \mathbb{R}$ set

$$F_n = \{x : x \in \mathbb{R}^r,\ f_n(x) \geq \tfrac{3}{4}\}, \quad F_n' = \{x : x \in \mathbb{R}^r,\ f_n(x) \leq \tfrac{1}{4}\},$$

$$f_{nt}(z) = f_n(z,t) \text{ for } z \in \mathbb{R}^{r-1},$$

identifying $\mathbb{R}^r$ with $\mathbb{R}^{r-1} \times \mathbb{R}$,

$$F_{nt} = \{z : z \in \mathbb{R}^{r-1},\ f_{nt}(z) \geq \tfrac{3}{4}\}, \quad F_{nt}' = \{z : z \in \mathbb{R}^{r-1},\ f_{nt}(z) \leq \tfrac{1}{4}\}.$$

Write $V_\delta \subseteq \mathbb{R}^{r-1}$ for the ball $\{z : \|z\| \leq \delta\}$, so that C_δ is $V_\delta \times [-\delta, \delta]$. Set

$$\gamma_{\delta n} = \sup\big(\{-\delta\} \cup \{t : t \in [-\delta,\delta],\ \mu_{r-1}(F_{nt} \cap V_\delta) \geq \tfrac{3}{4}\mu V_\delta\}\big),$$

$$G_{\delta n} = \{t : t \in [-\delta,\delta],\ \textstyle\int_{V_\delta} \|\operatorname{grad} f_{nt}\| d\mu_{r-1} > \epsilon^2(\delta + \frac{1}{n+1})^{r-2}\}.$$

The aim of the main part of the proof is to show that, for small δ and large n, $C_\delta \cap F_n$ is well approximated by $C_\delta \cap \tilde{H}(\gamma_{\delta n})$.

(c) For the time being (down to the end of (f) below) fix $\delta > 0$ and $n \in \mathbb{N}$ such that

$$\delta + \tfrac{1}{n+1} \leq \delta_0, \quad 4c'\epsilon(\delta + \tfrac{1}{n+1})^{r-1} \leq \tfrac{1}{8}\beta_{r-1}\delta^{r-1},$$

$$c_2\epsilon^3(\delta + \tfrac{1}{n+1})^{r-1} < \tfrac{1}{4}\beta_{r-1}\delta^{r-1}.$$

For $f : \mathbb{R}^r \to \mathbb{R}$ set

$$\operatorname{grad}_{r-1} f = \operatorname{grad} f - (v \boldsymbol{.} \operatorname{grad} f)v = (\tfrac{\partial f}{\partial \xi_1}, \dots, \tfrac{\partial f}{\partial \xi_{r-1}}, 0);$$

for definiteness, take the domain of $\operatorname{grad}_{r-1} f$ to be the set of points at which f is differentiable. Then

$$\textstyle\int_{C_\delta} \|\operatorname{grad}_{r-1} f_n\| + \max(\frac{\partial f_n}{\partial \xi_r}, 0) d\mu \leq c_2\epsilon^3(\delta + \frac{1}{n+1})^{r-1}.$$

P? Suppose, if possible, otherwise. Note that because $\mu C_\delta = 2\beta_{r-1}\delta^r$ (by Fubini's theorem), $\lim_{\delta' \uparrow \delta} \mu C_{\delta'} = \mu C_\delta$, so there is some $\delta' < \delta$ such that

$$\textstyle\int_{C_{\delta'}} \|\operatorname{grad}_{r-1} f_n\| + \max(\frac{\partial f_n}{\partial \xi_r}, 0) d\mu > c_2\epsilon^3(\delta + \frac{1}{n+1})^{r-1}.$$

For $1 \leq i < r$, set

$$\begin{aligned} \theta_i(x) &= \frac{\frac{\partial f_n}{\partial \xi_i}(x)}{\|\operatorname{grad}_{r-1} f_n(x)\|} \text{ when } x \in C_{\delta'} \text{ and } \operatorname{grad}_{r-1} f_n(x) \neq 0, \\ &= 0 \text{ otherwise;} \end{aligned}$$

and set

$$\begin{aligned} \theta_r(x) &= \frac{\max(\frac{\partial f_n}{\partial \xi_r}(x), 0)}{|\frac{\partial f_n}{\partial \xi_r}(x)|} \text{ when } x \in C_{\delta'} \text{ and } \tfrac{\partial f_n}{\partial \xi_r}(x) \neq 0, \\ &= 0 \text{ otherwise.} \end{aligned}$$

Then $\theta : \mathbb{R}^r \to \mathbb{R}^r$ is Lebesgue measurable and $\|\theta\| \leq \sqrt{2}\chi C_{\delta'}$, so all the θ_i are μ-integrable, while

$$\textstyle\int \theta \boldsymbol{.} \operatorname{grad} f_n d\mu = \int_{C_{\delta'}} \|\operatorname{grad}_{r-1} f_n\| + \max(\frac{\partial f_n}{\partial \xi_r}, 0) d\mu > c_2\epsilon^3(\delta + \frac{1}{n+1})^{r-1}.$$

By 473Ef, $\langle \|\theta_i - \theta_i * \tilde{h}_k\|_1 \rangle_{k \in \mathbb{N}} \to 0$ for each i; since $\operatorname{grad} f_n$ is bounded,

$$\int (\tilde{h}_k * \theta) \cdot \operatorname{grad} f_n \, d\mu > c_2 \epsilon^3 (\delta + \tfrac{1}{n+1})^{r-1}$$

for any k large enough; if we ensure also that $\frac{1}{k+1} \leq \delta - \delta'$, and set $\phi = \tilde{h}_k * \theta$, we shall get a smooth function ϕ, with $\|\phi\| \leq \sqrt{2}\chi C_\delta$ (by 473Dc and 473Dg), such that

$$\int \phi \cdot \operatorname{grad} f_n d\mu > c_2 \epsilon^3 (\delta + \tfrac{1}{n+1})^{r-1}.$$

Moreover,

$$(\tilde{h}_n * \phi)(x) \cdot v = (\tilde{h}_n * (\tilde{h}_k * \theta_r))(x) \geq 0$$

for every x, because $\tilde{h}_n$, $\tilde{h}_k$ ant θ_r are all non-negative.

Now ϕ is smooth (473De) and has compact support, so

$$\begin{aligned} c_2 \epsilon^3 (\delta + \tfrac{1}{n+1})^{r-1} &< \int \phi \cdot \operatorname{grad} f_n d\mu = \int \phi \cdot \operatorname{grad}(\tilde{h}_n * \chi E) d\mu \\ &= -\int (\tilde{h}_n * \phi) \cdot \psi_E \, d\lambda_E^\partial \end{aligned}$$

(474K)

$$\leq \int (\tilde{h}_n * \phi) \cdot (v - \psi_E) \, d\lambda_E^\partial \leq 2 \int_{C_{\delta + (1/n+1)}} \|v - \psi_E\| \, d\lambda_E^\partial$$

(because $\|(\tilde{h}_n * \phi)(x)\| \leq \sqrt{2}$ for every x, by 473Dg again, and $(\tilde{h}_n * \phi)(x) = 0$ if $x \notin C_{\delta+(1/n+1)}$)

$$\leq c_2 \epsilon^3 (\delta + \tfrac{1}{n+1})^{r-1};$$

which is absurd. **XQ**

(d) Observe that

$$\operatorname{grad}_{r-1} f_n(z,t) = (\operatorname{grad} f_{nt}(z), 0)$$

for all z and t. So (c) tells us that

$$\begin{aligned} \int_{-\delta}^{\delta} \int_{V_\delta} \|\operatorname{grad} f_{nt}(z)\| \mu_{r-1}(dz) dt &= \int_{C_\delta} \|\operatorname{grad}_{r-1} f_n(x)\| \mu(dx) \\ &\leq c_2 \epsilon^3 (\delta + \tfrac{1}{n+1})^{r-1}. \end{aligned}$$

So, writing μ_1 for Lebesgue measure on $\mathbb{R}$, $\mu_1 G_{\delta n} \leq c_2 \epsilon (\delta + \frac{1}{n+1})$.

If $t \in [-\delta, \delta] \setminus G_{\delta n}$, then

$$\min(\mu_{r-1}(F'_{nt} \cap V_\delta), \mu_{r-1}(F_{nt} \cap V_\delta)) \leq 4c' \epsilon (\delta + \tfrac{1}{n+1})^{r-1}.$$

P If $r > 2$,

$$\min\big(\mu_{r-1}(F'_{nt} \cap V_\delta), \mu_{r-1}(F_{nt} \cap V_\delta)\big)^{(r-2)/(r-1)} \leq 4c' \int_{V_\delta} \|\operatorname{grad} f_{nt}\| d\mu_{r-1}$$

(473L)

$$\leq 4c' \epsilon^2 (\delta + \tfrac{1}{n+1})^{r-2},$$

so that

$$\min(\mu_{r-1}(F'_{nt}\cap V_\delta),\mu_{r-1}(F_{nt}\cap V_\delta)) \le (4c'\epsilon^2)^{(r-1)/(r-2)}(\delta+\tfrac{1}{n+1})^{r-1}$$
$$\le 4c'\epsilon(\delta+\tfrac{1}{n+1})^{r-1}$$

because $4c'\ge 1$ and $\frac{2(r-1)}{r-2}\ge 1$ and $\epsilon\le 1$. If $r=2$, then

$$\int_{V_\delta}\|\operatorname{grad} f_{nt}\|d\mu_{r-1}\le\epsilon^2<\tfrac{1}{2},$$

so at least one of $F'_{nt}\cap V_\delta$, $F_{nt}\cap V_\delta$ is empty, as noted in 473M, and $\min(\mu_{r-1}(F'_{nt}\cap V_\delta),\mu_{r-1}(F_{nt}\cap V_\delta))=0$. **Q**

(e) The next thing to note is that $\mu_{r-1}(F'_{ns}\cap V_\delta)<\frac{3}{4}\mu_{r-1}V_\delta$ whenever $-\delta\le s<\gamma_{\delta n}$. **P?** Otherwise, we have $-\delta\le s\le t\le\delta$ such that

$$\mu_{r-1}(F'_{ns}\cap V_\delta)\ge\tfrac{3}{4}\mu_{r-1}V_\delta,\quad \mu_{r-1}(F_{nt}\cap V_\delta)\ge\tfrac{3}{4}\mu_{r-1}V_\delta,$$

so that

$$\mu_{r-1}(F'_{ns}\cap F_{nt}\cap V_\delta)\ge\tfrac{1}{2}\mu_{r-1}V_\delta.$$

But for any $z\in F'_{ns}\cap F_{nt}\cap V_\delta$, $f_n(z,s)\le\frac14$ and $f_n(z,t)\ge\frac34$, so

$$\int_{-\delta}^{\delta}\max(\tfrac{\partial f_n}{\partial\xi_r}(z,\xi),0)d\xi\ge\int_s^t\tfrac{\partial f_n}{\partial\xi_r}(z,\xi)d\xi=f_n(z,t)-f_n(z,s)\ge\tfrac12.$$

Accordingly

$$\int_{C_\delta}\max(\tfrac{\partial f_n}{\partial\xi_r}(x),0)\mu(dx)=\int_{V_\delta}\int_{-\delta}^{\delta}\max(\tfrac{\partial f_n}{\partial\xi_r}(z,\xi),0)d\xi\,\mu_{r-1}(dz)$$
$$\ge\tfrac12\mu_{r-1}(F'_{ns}\cap F_{nt}\cap V_\delta)\ge\tfrac14\mu_{r-1}V_\delta$$
$$=\tfrac14\beta_{r-1}\delta^{r-1}>c_2\epsilon^3(\delta+\tfrac{1}{n+1})^{r-1}$$

by the choice of δ and n; but this contradicts (c). **XQ**

(f) Now

$$\mu((F_n\triangle\tilde H(\gamma_{\delta n}))\cap C_\delta)\le 9\mu(C_\delta\setminus(F_n\cup F'_n))+c_2\epsilon(\delta+\tfrac{1}{n+1})\mu_{r-1}V_\delta$$
$$+16c'\epsilon\delta(\delta+\tfrac{1}{n+1})^{r-1}.$$

P Set

$$\tilde G=\{t:-\delta\le t\le\delta,\ \mu_{r-1}(V_\delta\setminus(F_{nt}\cup F'_{nt}))\ge\tfrac18\mu_{r-1}V_\delta\},$$

$$\hat G=\{t:-\delta\le t\le\delta,\ \mu_{r-1}(F_{nt}\cap V_\delta)\le 4c'\epsilon(\delta+\tfrac{1}{n+1})^{r-1}\},$$

$$\hat G'=\{t:-\delta\le t\le\delta,\ \mu_{r-1}(F'_{nt}\cap V_\delta)\le 4c'\epsilon(\delta+\tfrac{1}{n+1})^{r-1}\};$$

so that $G_{\delta n}\cup\hat G\cup\hat G'=[-\delta,\delta]$, by (d). Then

$$\tfrac18\mu_{r-1}V_\delta\cdot\mu_1\tilde G\le\mu(C_\delta\setminus(F_n\cup F'_n)),$$

$$\mu(F_n \cap (V_\delta \times \hat{G})) \leq 8c'\epsilon\delta(\delta + \tfrac{1}{n+1})^{r-1},$$

$$\mu(F'_n \cap (V_\delta \times \hat{G}')) \leq 8c'\epsilon\delta(\delta + \tfrac{1}{n+1})^{r-1}.$$

So if we set

$$W = (C_\delta \setminus (F_n \cup F'_n)) \cup (V_\delta \times (\tilde{G} \cup G_{\delta n} \cup \{\gamma_{\delta n}\})) \cup (F_n \cap (V_\delta \times \hat{G})) \cup (F'_n \cap (V_\delta \times \hat{G}')),$$

we shall have

$$\begin{aligned}\mu W &\leq \mu(C_\delta \setminus (F_n \cup F'_n)) + \mu_1\tilde{G} \cdot \mu_{r-1}V_\delta + \mu_1 G_{\delta n} \cdot \mu_{r-1}V_\delta + 16c'\epsilon\delta(\delta + \tfrac{1}{n+1})^{r-1} \\ &\leq 9\mu(C_\delta \setminus (F_n \cup F'_n)) + c_2\epsilon(\delta + \tfrac{1}{n+1})\mu_{r-1}V_\delta + 16c'\epsilon\delta(\delta + \tfrac{1}{n+1})^{r-1}\end{aligned}$$

(using the first part of (d) to estimate $\mu_1 G_{\delta n}$).

? Suppose, if possible, that there is a point $(z,t) \in (F_n \triangle \tilde{H}(\gamma_{\delta n})) \cap C_\delta \setminus W$. Since $t \notin G_{\delta n}$, (d) tells us that

$$\min(\mu_{r-1}(F'_{nt} \cap V_\delta), \mu_{r-1}(F_{nt} \cap V_\delta)) \leq 4c'\epsilon(\delta + \tfrac{1}{n+1})^{r-1} \leq \tfrac{1}{8}\mu_{r-1}V_\delta$$

by the choice of δ and n. So $t \in \hat{G} \cup \hat{G}'$. Also, since $t \notin \tilde{G}$, $\mu_{r-1}F_{nt} + \mu_{r-1}F'_{nt} \geq \frac{7}{8}\mu_{r-1}V_\delta$; so (since $t \neq \gamma_{\delta n}$) either $\mu_{r-1}F_{nt} \geq \frac{3}{4}\mu_{r-1}V_\delta$ and $t < \gamma_{\delta n}$, or $\mu_{r-1}F'_{nt} \geq \frac{3}{4}\mu_{r-1}V_\delta$ and $t > \gamma_{\delta n}$ (by (e)). Now

$$\begin{aligned}t < \gamma_{\delta n} &\implies \mu_{r-1}(F_{nt} \cap V_\delta) \geq \tfrac{3}{4}\mu_{r-1}V_\delta \\ &\implies \mu_{r-1}(F'_{nt} \cap V_\delta) \leq 4c'\epsilon(\delta + \tfrac{1}{n+1})^{r-1} \\ &\implies t \in \hat{G}' \\ &\implies (z,t) \notin F'_n \\ &\text{(because } (z,t) \notin F'_n \cap (V_\delta \times \hat{G}')) \\ &\implies (z,t) \in F_n \\ &\text{(because } (z,t) \notin C_\delta \setminus (F_n \cup F'_n)) \\ &\implies (z,t) \in F_n \cap \tilde{H}(\gamma_{\delta n}) \cap C_\delta,\end{aligned}$$

which is impossible. And similarly

$$\begin{aligned}t > \gamma_{\delta n} &\implies \mu_{r-1}(F'_{nt} \cap V_\delta) \geq \tfrac{3}{4}\mu_{r-1}V_\delta \\ &\implies \mu_{r-1}(F_{nt} \cap V_\delta) \leq 4c'\epsilon(\delta + \tfrac{1}{n+1})^{r-1} \\ &\implies t \in \hat{G} \\ &\implies (z,t) \notin F_n \\ &\implies (z,t) \notin F_n \cup \tilde{H}(\gamma_{\delta n}),\end{aligned}$$

which is equally impossible. **X**

Thus $(F_n \triangle \tilde{H}(\gamma_{\delta n})) \cap C_\delta \subseteq W$ has measure at most

$$9\mu(C_\delta \setminus (F_n \cup F'_n)) + c_2\epsilon(\delta + \tfrac{1}{n+1})\mu_{r-1}V_\delta + 16c'\epsilon\delta(\delta + \tfrac{1}{n+1})^{r-1},$$

as claimed. **Q**

(g) All this has been with fixed δ and n. Now take $\delta \in]0, \delta_0[$ and let $n \to \infty$. In this case, $f_n \to \chi E$ a.e. (473Ee), so

$$\lim_{n\to\infty} \mu((E \triangle F_n) \cap C_\delta = \lim_{n\to\infty} \mu(C_\delta \setminus (F_n \cup F_n')) = 0,$$

and

$$\begin{aligned}\limsup_{n\to\infty} \mu((E \triangle \tilde{H}(\gamma_{\delta n})) \cap C_\delta) &= \limsup_{n\to\infty} \mu((F_n \triangle \tilde{H}(\gamma_{\delta n})) \cap C_\delta) \\ &\leq c_2 \epsilon \delta \mu_{r-1} V_\delta + 16 c' \epsilon \delta^r = c_3 \epsilon \mu C_\delta.\end{aligned}$$

Now

$$\begin{aligned}\mu((E \triangle \tilde{H}(\gamma_\delta)) \cap C_\delta) &\leq \delta \mu C_\delta + \inf_{n\in\mathbb{N}} \mu((E \triangle \tilde{H}(\gamma_{\delta n})) \cap C_\delta) \\ &\leq (\delta + c_3 \epsilon) \mu C_\delta\end{aligned}$$

by the choice of γ_δ ((a) above). Since ϵ is arbitrary, this means that

$$\limsup_{\delta\downarrow 0} \frac{1}{\mu C_\delta} \mu((E \triangle \tilde{H}(\gamma_\delta)) \cap C_\delta) = 0.$$

(h) Now $\lim_{\delta\downarrow 0} \frac{\gamma_\delta}{\delta} = 0$. **P** Let $\epsilon \in]0, 1]$. Then by (g) above and 474N there is a $\delta_0 > 0$ such that

$$\mu((E \triangle \tilde{H}(\gamma_\delta)) \cap C_\delta) < \frac{\epsilon^r}{4\beta_{r-1}(3r)^r} \mu C_\delta,$$

$$\mu(B(\mathbf{0}, \delta) \cap E) \geq \frac{1}{2(3r)^r \beta_r} \mu B(\mathbf{0}, \delta),$$

$$\mu B(\mathbf{0}, \delta) \setminus E) \geq \frac{1}{2(3r)^r \beta_r} \mu B(\mathbf{0}, \delta)$$

whenever $0 < \delta \leq \delta_0$. **?** Suppose, if possible, that $0 < \delta \leq \delta_0$ and $\gamma_\delta \leq -\epsilon\delta$. Then

$$\begin{aligned}\mu(E \cap B(\mathbf{0}, \epsilon\delta)) &\leq \mu(E \cap C_\delta \setminus \tilde{H}(\gamma_\delta)) \\ &< \frac{\epsilon^r}{4\beta_{r-1}(3r)^r} \mu C_\delta = \frac{1}{2\beta_r (3r)^r} \mu B(\mathbf{0}, \epsilon\delta)\end{aligned}$$

which is impossible. **X** So $-\epsilon\delta \leq \gamma_\delta$ for all $\delta \in]0, \delta_0]$. In the same way, $\gamma_\delta \leq \epsilon\delta$ for all $\delta \in]0, \delta_0]$. As ϵ is arbitrary, we have the result. **Q**

(i) This means that

$$\lim_{\delta\downarrow 0} \frac{\mu((E \triangle \tilde{H}(0)) \cap C_\delta)}{\mu C_\delta} \leq \lim_{\delta\downarrow 0} \frac{\mu((E \triangle \tilde{H}(\gamma_\delta)) \cap C_\delta)}{\mu C_\delta} + \lim_{\delta\downarrow 0} \frac{\gamma_\delta}{\delta} = 0,$$

and

$$\lim_{\delta\downarrow 0} \frac{\mu((E \triangle \tilde{H}(0)) \cap B_\delta)}{\mu B_\delta} \leq \frac{2\beta_{r-1}}{\beta_r} \lim_{\delta\downarrow 0} \frac{\mu((E \triangle \tilde{H}(0)) \cap C_\delta)}{\mu C_\delta} = 0.$$

(j) Thus v is a Federer exterior normal to E at $\mathbf{0}$ if $\psi_E(\mathbf{0}) = v$. For the general case, let S be an orthogonal matrix such that $S\psi_E(y) = v$, and set $T(x) = S(x - y)$ for every x, so that $T(y) = \mathbf{0}$. The point is of course that $\mathbf{0} \in \partial^{\$} T[E]$ and $\psi_{T[E]}(\mathbf{0}) = v$. **P** Because $\lambda^\partial_{T[E]} = \lambda^\partial_E T^{-1}$ and $S\psi_E T^{-1}$ is an outward-normal function for $T[E]$ (474H),

$$\frac{1}{\lambda^\partial_{T[E]}B(\mathbf{0},\delta)}\int_{B(\mathbf{0},\delta)}\|S\psi_E T^{-1}(x)-v\|\lambda^\partial_{T[E]}(x)$$
$$=\frac{1}{\lambda^\partial_E T^{-1}[B(\mathbf{0},\delta)]}\int_{T[B(y,\delta)]}\|S\psi_E T^{-1}(x)-v\|(\lambda^\partial_E T^{-1})(dx)$$
$$=\frac{1}{\lambda^\partial_E B(y,\delta)}\int_{B(y,\delta)}\|S\psi_E(x)-v\|\lambda^\partial_E(dx)$$

(by 235I, because T^{-1} is inverse-measure-preserving for $\lambda^\partial_E T^{-1}$ and λ^∂_E)

$$=\frac{1}{\lambda^\partial_E B(y,\delta)}\int_{B(y,\delta)}\|\psi_E(x)-S^{-1}v\|\lambda^\partial_E(dx)$$
$$=\frac{1}{\lambda^\partial_E B(y,\delta)}\int_{B(y,\delta)}\|\psi_E(x)-\psi_E(y)\|\lambda^\partial_E(dx)\to 0$$

as $\delta\downarrow 0$. **Q**

Now consider

$$H=\{x:(x-y)\boldsymbol{.}\psi_E(y)\le 0\}=\{x:T(x)\boldsymbol{.}v\le 0\}=T^{-1}[\tilde H(0)].$$

Then

$$\frac{\mu((H\triangle E)\cap B(y,\delta))}{\mu B(y,\delta)}=\frac{\mu((\tilde H(0)\triangle T[E])\cap B(\mathbf{0},\delta))}{\mu B(\mathbf{0},\delta)}\to 0$$

as $\delta\downarrow 0$, so $\psi_E(y)$ is a Federer exterior normal to E at y, as required.

474R Corollary Let $E\subseteq\mathbb{R}^r$ be a set with locally finite perimeter, and λ^∂_E its perimeter measure. Let y be any point of the reduced boundary of E. Then

$$\lim_{\delta\downarrow 0}\frac{\lambda^\partial_E B(y,\delta)}{\beta_{r-1}\delta^{r-1}}=1.$$

proof (a) Set $v=\psi_E(y)$ and $H=\{x:(x-y)\boldsymbol{.}v\le 0\}$, as in 474Q. Now

$$\int_{H\cap\partial B(y,\delta)}v\boldsymbol{.}\tfrac{1}{\delta}(x-y)\nu(dx)=-\beta_{r-1}\delta^{r-1}$$

for almost every $\delta>0$. **P** Set $\phi(x)=v$ for every $x\in\mathbb{R}^r$. By 474I, ϕ is an outward-normal function for H, so 474M tells us that, for almost every $\delta>0$,

$$\int_{H\cap\partial B(y,\delta)}v\boldsymbol{.}\tfrac{1}{\delta}(x-y)\,\nu(dx)=\int_{H\cap B(y,\delta)}\operatorname{div}\phi\,d\mu-\int_{B(y,\delta)}v\boldsymbol{.}v\,d\lambda^\partial_H$$
$$=-\lambda^\partial_H B(y,\delta)=-\nu(B(y,\delta)\cap\partial H)$$

(using the identification of λ^∂_H in 474I)

$$=-\beta_{r-1}\delta^{r-1}$$

(identifying ν on the hyperplane ∂H with Lebesgue measure on $\mathbb{R}^{r-1}$, as usual). **Q**

(b) Now, given $\epsilon>0$, there is a $\delta_0>0$ such that whenever $0<\delta\le\delta_0$ there is an η such that $\delta\le\eta\le\delta(1+\epsilon)$ and $|\lambda^\partial_E B(y,\eta)-\beta_{r-1}\eta^{r-1}|\le\epsilon\delta^{r-1}$. **P** Let $\zeta>0$ be such that

$$\zeta(1+\frac{5\pi}{r}\beta_{r-2})(1+\epsilon)^r\le\epsilon^2.$$

By 474N(iv) and 474Q and the definition of ψ_E, there is a $\delta_0>0$ such that

$$\lambda^\partial_E B(y,\delta)\le 5\pi\beta_{r-2}\delta^{r-1},$$

$$\mu((E\triangle H)\cap B(y,\delta))\le\zeta\delta^r,$$

$$\int_{B(y,\delta)} \|\psi_E(x) - v\| \lambda_E^\partial(dx) \leq \zeta \lambda_E^\partial B(y,\delta)$$

whenever $0 < \delta \leq (1+\epsilon)\delta_0$. Take $0 < \delta \leq \delta_0$. Then, for almost every $\eta > 0$, we have

$$\int_{B(y,\eta)} v \bullet \psi_E(x)\, \lambda_E^\partial(dx) + \int_{E \cap \partial B(y,\eta)} v \bullet \frac{1}{\eta}(x-y)\, \nu(dx) = 0$$

by 474M, applied with $\phi \equiv v$. Putting this together with (a), we see that, for almost every $\eta \in]0, \delta_0]$,

$$\begin{aligned}
&|\lambda_E^\partial B(y,\eta) - \beta_{r-1}\eta^{r-1}| \\
&\quad \leq \int_{B(y,\eta)} \|\psi_E(x) - v\| \lambda_E^\partial(dx) \\
&\qquad + |\int_{B(y,\eta)} v \bullet \psi_E(x)\, \lambda_E^\partial(dx) + \int_{H \cap \partial B(y,\eta)} v \bullet \frac{1}{\eta}(x-y)\, \nu(dx)| \\
&\quad \leq \zeta \lambda_E^\partial B(y,\eta) \\
&\qquad + |\int_{H \cap \partial B(y,\eta)} v \bullet \frac{1}{\eta}(x-y)\, \nu(dx) - \int_{E \cap \partial B(y,\eta)} v \bullet \frac{1}{\eta}(x-y)\, \nu(dx)| \\
&\quad \leq 5\pi\beta_{r-2}\zeta\eta^{r-1} + \nu((E \triangle H) \cap \partial B(y,\eta)).
\end{aligned}$$

Integrating with respect to η, we have

$$\int_0^{\delta(1+\epsilon)} |\lambda_E^\partial B(y,\eta) - \beta_{r-1}\eta^{r-1}| d\eta \leq \frac{5\pi}{r}\beta_{r-2}\zeta\delta^r(1+\epsilon)^r + \mu((E \triangle H) \cap B(y, \delta(1+\epsilon)))$$

(using 265G, as usual)

$$\leq \frac{5\pi}{r}\beta_{r-2}\zeta\delta^r(1+\epsilon)^r + \zeta\delta^r(1+\epsilon)r \leq \epsilon^2\delta^r$$

by the choice of ζ. But this means that there must be some $\eta \in [\delta, \delta(1+\epsilon)]$ such that

$$|\lambda_E^\partial B(y,\eta) - \beta_{r-1}\eta^{r-1}| \leq \epsilon\delta^{r-1} \leq \epsilon\eta^{r-1}. \quad \blacksquare$$

(c) Now we see that

$$\lambda_E^\partial B(y,\delta) \leq \lambda_E^\partial B(y,\eta) \leq (\beta_{r-1} + \epsilon)\eta^{r-1} \leq (\beta_{r-1} + \epsilon)(1+\epsilon)^{r-1}\delta^{r-1}.$$

But by the same argument we have an $\hat\eta \in [\frac{\delta}{1+\epsilon}, \delta]$ such that $|\lambda_E^\partial B(y,\hat\eta) - \beta_{r-1}\hat\eta^{r-1}| \leq \epsilon\hat\eta^{r-1}$, so that

$$\lambda_E^\partial B(y,\delta) \geq \lambda_E^\partial B(y,\hat\eta) \geq (\beta_{r-1} - \epsilon)\hat\eta^{r-1} \geq (\beta_{r-1} - \epsilon)(1+\epsilon)^{1-r}\delta^{r-1}.$$

Thus, for every $\delta \in]0, \delta_0]$,

$$(\beta_{r-1} - \epsilon)(1+\epsilon)^{1-r}\delta^{r-1} \leq \lambda_E^\partial B(y,\delta) \leq (\beta_{r-1} + \epsilon)(1+\epsilon)^{r-1}\delta^{r-1}.$$

As ϵ is arbitrary,

$$\lim_{\delta \downarrow 0} \frac{\lambda_E^\partial B(y,\delta)}{\delta^{r-1}} = \beta_{r-1},$$

as claimed.

474S The Compactness Theorem Let Σ be the algebra of Lebesgue measurable subsets of $\mathbb{R}^r$, and give it the topology $\mathfrak{T}_m$ of convergence in measure defined by the pseudometrics $\rho_H(E,F) = \mu((E \triangle F) \cap H)$ for measurable sets H of finite measure (cf. §§245 and 323). Then

(a) $\mathrm{per} : \Sigma \to [0,\infty]$ is lower semi-continuous.

(b) For any $\gamma < \infty$, $\{E : E \in \Sigma, \mathrm{per}\, E \leq \gamma\}$ is compact.

proof (a) Let $\langle E_n \rangle_{n \in \mathbb{N}}$ be any $\mathfrak{T}_m$-convergent sequence in Σ with limit $E \in \Sigma$. If $\phi : \mathbb{R}^r \to B(\mathbf{0}, 1)$ is a Lipschitz function with compact support, then $\mathrm{div}\, \phi$ is integrable, so $F \mapsto \int_F \mathrm{div}\, \phi\, d\mu$ is truly continuous (225A), and

$$|\int_E \operatorname{div}\phi\, d\mu| = \lim_{n\to\infty} |\int_{E_n} \operatorname{div}\phi\, d\mu| \le \sup_{n\in\mathbb{N}} \operatorname{per} E_n.$$

As ϕ is arbitrary, $\operatorname{per} E \le \sup_{n\in\mathbb{N}} \operatorname{per} E_n$. This means that $\{E : \operatorname{per} E \le \gamma\}$ is sequentially closed, therefore closed (4A2Ld), for any γ, and per is lower semi-continuous.

(b) Set $\mathcal{A} = \{E : \operatorname{per} E \le \gamma\}$. Let us say that a 'dyadic cube' is a set expressible in the form $D = \prod_{i<r} [2^{-n}k_i, 2^{-n}(k_i+1)[$ where n, $k_0, \dots, k_{r-1} \in \mathbb{Z}$.

(i) For $E \in \mathcal{A}$, $n \in \mathbb{N}$ and $\epsilon \in]0,1]$ let $G(E,n,\epsilon)$ be the union of all the dyadic cubes D with side length 2^{-n} such that $\epsilon\mu D \le \mu(E\cap D) \le (1-\epsilon)\mu D$. Then $\mu G(E,n,\epsilon) \le \frac{c_1}{2^n\epsilon}\gamma$, where $c_1 = 2^{r+3}(1+2^{r+1})(1+\sqrt{r})^{r+1}$.

P Express $G(E,n,\epsilon)$ as a disjoint union $\bigcup_{i<k} D_i$ where each D_i is a dyadic cube of side length 2^{-n} and $\min(\mu(D_i\cap E), \mu(D_i\setminus E)) \ge \epsilon\mu D_i$. Let x_i be the centre of D_i and B_i the ball $B(x_i, 2^{-n-1}\sqrt{r})$, so that $D_i \subseteq B_i$ and $\mu B_i = \beta_r(\frac{1}{2}\sqrt{r})^r \mu D_i$. For any $x \in \mathbb{R}^r$, the ball $B(x, 2^{-n-1}\sqrt{r})$ is included in a closed cube with side length $2^{-n}\sqrt{r}$, so can contain at the very most $(1+\sqrt{r})^r$ different x_i, because different x_i differ by at least 2^{-n} in some coordinate. Turning this round, $\sum_{i=0}^{k-1} \chi B_i \le (1+\sqrt{r})^r \chi(\mathbb{R}^r)$.

Set $c = 2^{r+2}(1+\sqrt{r})(1+2^{r+1})$. Then, for each $i<k$,

$$\begin{aligned} 2c\lambda_E^\partial B_i &\ge \min(\mu(B_i\cap E), \mu(B_i\setminus E))^{(r-1)/r} \\ &\ge \min(\mu(D_i\cap E), \mu(D_i\setminus E))^{(r-1)/r} \ge (\epsilon\mu D_i)^{(r-1)/r} \ge 2^{-n(r-1)}\epsilon. \end{aligned}$$

((474Lb))

So

$$\begin{aligned} \mu G(E,n,\epsilon) = 2^{-nr}k &\le \frac{2c}{2^n\epsilon}\sum_{i=0}^{k} \lambda_E^\partial B_i \\ &\le \frac{2c(1+\sqrt{r})^r}{2^n\epsilon}\lambda_E^\partial(\mathbb{R}^r) = \frac{c_1}{2^n\epsilon}\gamma. \ \mathbf{Q} \end{aligned}$$

(ii) Now let $\langle E_n\rangle_{n\in\mathbb{N}}$ be any sequence in $\mathcal{A}$. Then we can find a subsequence $\langle E_n'\rangle_{n\in\mathbb{N}}$ such that whenever $n \in \mathbb{N}$, D is a dyadic cube of side length 2^{-n} meeting $B(\mathbf{0},n)$, and i, $j \ge n$, then $|\mu(D\cap E_i') - \mu(D\cap E_j')| \le \frac{1}{(n+1)^{r+2}}\mu D$. Now

$$\mu((E_i'\triangle E_j')\cap B(\mathbf{0},n)) \le \frac{3\beta_r(n+\sqrt{r})^r}{(n+1)^{r+2}} + 2^{-n}(n+1)^{r+1}c_1\gamma$$

whenever $n \ge 1$ and i, $j \ge n$. **P** Let $\mathcal{E}$ be the set of dyadic cubes of side length 2^{-n} meeting $B(\mathbf{0},n)$; then every member of $\mathcal{E}$ is included in $B(\mathbf{0}, n+2^{-n}\sqrt{r})$, so $\mu(\bigcup\mathcal{E}) \le \beta_r(n+\sqrt{r})^r$. Let $\mathcal{E}_1$ be the collection of those dyadic cubes of side length 2^{-n} included in $G(E_i', n, \frac{1}{(n+1)^{r+2}})$. If $D \in \mathcal{E}\setminus\mathcal{E}_1$, either $\mu(E_i'\cap D) \le \frac{1}{(n+1)^{r+2}}\mu D$ and $\mu(E_j'\cap D) \le \frac{2}{(n+1)^{r+2}}\mu D$ and $\mu((E_i'\triangle E_j')\cap D) \le \frac{3}{(n+1)^{r+2}}\mu D$, or $\mu(D\setminus E_i') \le \frac{1}{(n+1)^{r+2}}\mu D$ and $\mu(D\setminus E_j') \le \frac{2}{(n+1)^{r+2}}\mu D$ and $\mu((E_i'\triangle E_j')\cap D) \le \frac{3}{(n+1)^{r+2}}\mu D$. So

$$\begin{aligned} \mu((E_i'\triangle E_j')\cap B(\mathbf{0},n)) &\le \sum_{D\in\mathcal{E}} \mu((E_i'\triangle E_j')\cap D) \\ &\le \sum_{D\in\mathcal{E}\setminus\mathcal{E}_1} \mu((E_i'\triangle E_j')\cap D) + \mu(\bigcup\mathcal{E}_1) \\ &\le \frac{3}{(n+1)^{r+2}}\mu(\bigcup\mathcal{E}) + \mu G(E_i', n, \frac{1}{(n+1)^{r+2}}) \\ &\le \frac{3\beta_r(n+\sqrt{r})^r}{(n+2)^{r+1}} + 2^{-n}(n+1)^{r+2}c_1\gamma, \end{aligned}$$

as claimed. **Q**

(iii) This means that $\sum_{i=0}^{\infty} \mu((E_i' \triangle E_{i+1}') \cap B(\mathbf{0},n))$ is finite for each $n \in \mathbb{N}$, so that if we set $E = \bigcup_{i\in\mathbb{N}} \bigcap_{j\geq i} E_j'$, then

$$\mu((E\triangle E_i') \cap B(\mathbf{0},n)) \leq \sum_{j=i}^{\infty} \mu((E_j' \triangle E_{j+1}') \cap B(\mathbf{0},n)) \to 0$$

as $i \to \infty$ for every $n \in \mathbb{N}$. It follows that $\lim_{i\to\infty} \rho_H(E, E_i') = 0$ whenever $\mu H < \infty$ (see the proofs of 245Eb and 323Gb). Thus we have a subsequence $\langle E_i' \rangle_{i\in\mathbb{N}}$ of the original sequence $\langle E_i \rangle_{i\in\mathbb{N}}$ which is convergent for the topology $\mathfrak{T}_m$ of convergence in measure. By (a), its limit belongs to $\mathcal{A}$. But since $\mathfrak{T}_m$ is pseudometrizable (245Eb/323Gb), this is enough to show that A is compact for $\mathfrak{T}_m$ (4A2Lf).

474X Basic exercises (a) Show directly from 474E that if $E \subseteq \mathbb{R}^r$ has locally finite perimeter and $F \subseteq \mathbb{R}^r$ is such that $\mu(E\triangle F) = 0$, then F has locally finite perimeter, $\lambda_F^\partial = \lambda_E^\partial$, and any outward-normal function for E is an outward-normal function for F.

(b) Show that for any $E \subseteq \mathbb{R}^r$ with locally finite perimeter, its reduced boundary is a Borel set.

(c) Show that if $E \subseteq \mathbb{R}^r$ has locally finite perimeter, then $\partial^{\$} E \subseteq \partial E$. (*Hint*: 474N(i)-(ii).)

(d) Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and $y \in \partial^{\$} E$. Show that $\lim_{\delta\downarrow 0} \frac{\mu(E\cap B(y,\delta))}{\mu B(y,\delta)} = \frac{1}{2}$.

(e) In the proof of 474R, use 265E to show that

$$\int_{H\cap\partial B(y,\delta)} v \bullet \tfrac{1}{\delta}(x-y)\nu(dx) = -\beta_{r-1}\delta^{r-1}$$

for *every* $\delta > 0$.

474Y Further exercises (a) In 474E, explain how to interpret the pair $(\psi, \lambda_E^\partial)$ as a vector measure (definition: 393O) $\theta_E : \mathcal{B} \to \mathbb{R}^r$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{R}^r$, in such a way that we have $\int_E \operatorname{div}\phi\, d\mu = \int \phi\, d\theta_E$ for Lipschitz functions ϕ with compact support.

474 Notes and comments When we come to the Divergence Theorem itself in the next section, it will be nothing but a repetition of Theorem 474E with the perimeter measure and the outward-normal function properly identified. The idea of the indirect approach here is to start by defining the pair $(\psi_E, \lambda_E^\partial)$ as a kind of 'distributional derivative' of the set E. I take the space to match the details with the language of the rest of this treatise, but really 474E amounts to nothing more than the Riesz representation theorem; since the functional $\phi \mapsto \int_E \operatorname{div}\phi\, d\mu$ is linear, and we restrict attention to sets E for which it is continuous in an appropriate sense (and can therefore be extended to arbitrary continuous functions ϕ with compact support), it must be representable by a (vector) measure, as in 474Ya. For the process to be interesting, we have to be able to identify at least some of the appropriate sets E with their perimeter measures and outward-normal functions. Half-spaces are straightforward enough (474I), and 474Q tells us what the outward-normal functions have to be; but for a proper description of the family of sets with locally finite perimeter we must wait until the next section. I see no quick way to show from the results here that (for instance) the union of two sets with finite perimeter again has finite perimeter.

I have tried to find the shortest path to the Divergence Theorem itself, and have not attempted to give 'best' results in the intermediate material. In particular, in the isoperimetric inequality 474La, I show only that the measure of a set E is controlled by the magnitude of its perimeter measure. Simple scaling arguments show that if there is any such control, then it must be of the form $\gamma(\mu E)^{(r-1)/r} \leq \operatorname{per} E$; the identification of the best constant γ as $r\beta_r^{1/r}$, giving equality for balls, is the real prize, to which I shall come in 476L.

I leave the Compactness Theorem (474S) to the end of the section because it is off the line I have chosen to the Divergence Theorem (though it can be used to make the proof of 474Q more transparent; see Evans & Gariepy 92, 5.7.2). I have expressed 474S in terms of the topology of convergence in measure on the algebra of Lebesgue measurable sets. But since the perimeter of a measurable set E is not altered if we

change E by a negligible set, 'perimeter' can equally well be regarded as a function defined on the measure algebra, in which case 474S becomes a theorem about the usual topology of the measure algebra of Lebesgue measure, as described in §323.

475 The essential boundary

The principal aim of this section is to translate Theorem 474E into geometric terms. We have already identified the Federer exterior normal as an outward-normal function (474Q), so we need to find a description of perimeter measures. Most remarkably, it turns out that these are nothing but simple indefinite-integral measures over normalized Hausdorff measures (475G). This description needs the concept of 'essential boundary' (475B). In order to complete the programme, we need to be able to determine which sets have 'locally finite perimeter'; there is a natural criterion in the same language (475L). We now have all the machinery for a direct statement of the Divergence Theorem (for Lipschitz functions) which depends on nothing more advanced than the definition of Hausdorff measure (475N). (The definitions, at least, of 'Federer exterior normal' and 'essential boundary' are elementary.)

This concludes the main work of the middle part of this chapter. But since we are now within reach of a reasonably direct proof of a fundamental fact about the $r-1$-dimensional Hausdorff measure of the boundaries of subsets of $\mathbb{R}^r$ (475Q), I continue up to Cauchy's Perimeter Theorem and the Isoperimetric Theorem for convex sets (475S, 475T).

475A Notation As in the last two sections, r will be an integer (greater than or equal to 2, unless explicitly permitted to take the value 1). μ will be Lebesgue measure on $\mathbb{R}^r$; I will sometimes write μ_{r-1} for Lebesgue measure on $\mathbb{R}^{r-1}$ and μ_1 for Lebesgue measure on $\mathbb{R}$. $\beta_r = \mu B(\mathbf{0},1)$ will be the measure of the unit ball in $\mathbb{R}^r$, and $S_{r-1} = \partial B(\mathbf{0},1)$ will be the unit sphere. ν will be normalized $r-1$-dimensional Hausdorff measure on $\mathbb{R}^r$ (265A), that is, $\nu = 2^{-r+1}\beta_{r-1}\mu_{H,r-1}$, where $\mu_{H,r-1}$ is $r-1$-dimensional Hausdorff measure on $\mathbb{R}^r$. I will take it for granted that $x \in \mathbb{R}^r$ has coordinates $(\xi_1, \dots, \xi_r)$.

If $E \subseteq \mathbb{R}^r$ has locally finite perimeter (474D), λ_E^∂ will be its perimeter measure (474F), $\partial^\$ E$ its reduced boundary (474G) and ψ_E its canonical outward normal (474G).

475B The essential boundary (In this paragraph I allow $r = 1$.) Let $A \subseteq \mathbb{R}^r$ be any set. The **essential closure** of A is the set

$$\mathrm{cl}^*A = \{x : \limsup_{\delta \downarrow 0} \tfrac{\mu^*(B(x,\delta)\cap A)}{\mu B(x,\delta)} > 0\}.$$

Similarly, the **essential interior** of A is the set

$$\mathrm{int}^*A = \{x : \liminf_{\delta \downarrow 0} \tfrac{\mu_*(B(x,\delta)\cap A)}{\mu B(x,\delta)} = 1\}.$$

(If A is Lebesgue measurable, this is the lower Lebesgue density of A, as defined in 341E.) Finally, the **essential boundary** ∂^*A of A is the difference $\mathrm{cl}^*A \setminus \mathrm{int}^*A$.

475C Lemma (In this lemma I allow $r = 1$.) Let A, $A' \subseteq \mathbb{R}^r$.

(a)

$$\mathrm{int}\, A \subseteq \mathrm{int}^*A \subseteq \mathrm{cl}^*A \subseteq \overline{A}, \quad \partial^*A \subseteq \partial A,$$

$$\mathrm{cl}^*A = \mathbb{R}^r \setminus \mathrm{int}^*(\mathbb{R}^r \setminus A), \quad \partial^*(\mathbb{R}^r \setminus A) = \partial^*A.$$

(b) If $A \setminus A'$ is negligible, then $\mathrm{cl}^*A \subseteq \mathrm{cl}^*A'$ and $\mathrm{int}^*A \subseteq \mathrm{int}^*A'$; in particular, if A itself is negligible, cl^*A, int^*A and ∂^*A are all empty.

(c) int^*A, cl^*A and ∂^*A are Borel sets.

(d) $\mathrm{cl}^*(A\cup A') = \mathrm{cl}^*A \cup \mathrm{cl}^*A'$ and $\mathrm{int}^*(A\cap A') = \mathrm{int}^*A \cap \mathrm{int}^*A'$, so $\partial^*(A\cup A')$, $\partial^*(A\cap A')$ and $\partial^*(A\triangle A')$ are all included in $\partial^*A \cup \partial^*A'$.

(e) $\mathrm{cl}^*A \cap \mathrm{int}^*A' \subseteq \mathrm{cl}^*(A\cap A')$, $\partial^*A \cap \mathrm{int}^*A' \subseteq \partial^*(A\cap A')$ and $\partial^*A \setminus \mathrm{cl}^*A' \subseteq \partial^*(A\cup A')$.

(f) $\partial^*(A\cap A') \subseteq (\mathrm{cl}^*A' \cap \partial A) \cup (A \cap \partial^*A')$.

(g) If $E \subseteq \mathbb{R}^r$ is Lebesgue measurable, then $E\triangle\mathrm{int}^*E$, $E\triangle\mathrm{cl}^*E$ and ∂^*E are Lebesgue negligible.

(h) A is Lebesgue measurable iff ∂^*A is Lebesgue negligible.

proof (a) It is obvious that

$$\operatorname{int} A \subseteq \operatorname{int}^*A \subseteq \operatorname{cl}^*A \subseteq \overline{A},$$

so that $\partial^*A \subseteq \partial A$. Since

$$\frac{\mu^*(B(x,\delta)\cap A)}{\mu B(x,\delta)} + \frac{\mu_*(B(x,\delta)\setminus A)}{\mu B(x,\delta)} = 1$$

for every $x \in \mathbb{R}^r$ and every $\delta > 0$ (413Ec), $\mathbb{R}^r \setminus \operatorname{int}^*A = \operatorname{cl}^*(\mathbb{R}^r \setminus A)$. It follows that

$$\begin{aligned}\partial^*(\mathbb{R}^r \setminus A) &= \operatorname{cl}^*(\mathbb{R}^r \setminus A) \triangle \operatorname{int}^*(\mathbb{R}^r \setminus A)\\ &= (\mathbb{R}^r \setminus \operatorname{int}^*A) \triangle (\mathbb{R}^r \setminus \operatorname{cl}^*A) = \operatorname{int}^*A \triangle \operatorname{cl}^*A = \partial^*A.\end{aligned}$$

(b) If $A \setminus A'$ is negligible, then

$$\mu_*(B(x,\delta)\cap A) \le \mu_*(B(x,\delta)\cap A'), \quad \mu^*(B(x,\delta)\cap A) \le \mu^*(B(x,\delta)\cap A')$$

for all x and δ, so $\operatorname{int}^*A \subseteq \operatorname{int}^*A'$ and $\operatorname{cl}^*A \subseteq \operatorname{cl}^*A'$.

(c) The point is just that $(x,\delta) \mapsto \mu^*(A \cap B(x,\delta))$ is continuous. **P** For any x, $y \in \mathbb{R}^r$ and δ, $\eta \ge 0$ we have

$$\begin{aligned}|\mu^*(A\cap B(y,\eta)) - \mu^*(A\cap B(x,\delta))| &\le \mu(B(y,\eta)\triangle B(x,\delta))\\ &= 2\mu(B(x,\delta)\cup B(y,\eta)) - \mu B(x,\delta) - \mu B(y,\eta)\\ &\le \beta_r\big(2(\max(\delta,\eta) + \|x-y\|)^r - \delta^r - \eta^r\big) \to 0\end{aligned}$$

as $(y,\eta) \to (x,\delta)$. **Q** So

$$x \mapsto \limsup_{\delta\downarrow 0} \frac{\mu^*(A\cap B(x,\delta))}{\mu B(x,\delta)} = \inf_{\alpha\in\mathbb{Q},\alpha>0} \sup_{\beta\in\mathbb{Q},0<\beta\le\alpha} \frac{1}{\beta_r\beta^r}\mu^*(A\cap B(x,\beta))$$

is Borel measurable, and

$$\operatorname{cl}^*A = \{x : \limsup_{\delta\downarrow 0} \frac{\mu^*(A\cap B(x,\delta))}{\mu B(x,\delta)} = 1\}$$

is a Borel set.

Accordingly $\operatorname{int}^*A = \mathbb{R}^r \setminus \operatorname{cl}^*(\mathbb{R}^r \setminus A)$ and ∂^*A are also Borel sets.

(d) For any $x \in \mathbb{R}^r$,

$$\begin{aligned}\limsup_{\delta\downarrow 0} \frac{\mu^*((A\cup A')\cap B(x,\delta))}{\mu B(x,\delta)} &\le \limsup_{\delta\downarrow 0} \frac{\mu^*(A\cap B(x,\delta))}{\mu B(x,\delta)} + \frac{\mu^*(A'\cap B(x,\delta))}{\mu B(x,\delta)}\\ &\le \limsup_{\delta\downarrow 0} \frac{\mu(A\cap B(x,\delta))}{\mu B(x,\delta)} + \limsup_{\delta\downarrow 0} \frac{\mu(A'\cap B(x,\delta))}{\mu B(x,\delta)},\end{aligned}$$

so $\operatorname{cl}^*(A\cup A') \subseteq \operatorname{cl}^*A \cup \operatorname{cl}^*A'$. Of course $\operatorname{cl}^*A \cup \operatorname{cl}^*A' \subseteq \operatorname{cl}^*(A\cup A')$, so we have equality. Accordingly

$$\operatorname{int}^*(A\cap A') = \mathbb{R}^r \setminus \operatorname{cl}^*((\mathbb{R}^r\setminus A)\cup(\mathbb{R}^r\setminus A')) = \operatorname{int}^*A \cap \operatorname{int}^*A'.$$

Since of course $\operatorname{int}^*(A\cup A') \supseteq \operatorname{int}^*A \cup \operatorname{int}^*A'$, $\partial^*(A\cup A') \subseteq \partial^*A \cup \partial^*A'$. Now

$$\partial^*(A\cap A') = \partial^*(\mathbb{R}^r\setminus(A\cap A')) \subseteq \partial^*(\mathbb{R}^r\setminus A)\cup\partial^*(\mathbb{R}^r\setminus A')) = \partial^*A\cup\partial^*A'$$

and

$$\partial^*(A\triangle A') \subseteq \partial^*(A\cap(\mathbb{R}^r\setminus A'))\cup\partial^*(A'\cap(\mathbb{R}^r\setminus A)) \subseteq \partial^*A\cup\partial^*A'.$$

(e) If $x \in \operatorname{cl}^*A \cap \operatorname{int}^*A'$, then $x \notin \operatorname{cl}^*(\mathbb{R}^r\setminus A')$ so $x \notin \operatorname{cl}^*(A\setminus A')$. But from (d) we know that $x \in \operatorname{cl}^*(A\cap A')\cup\operatorname{cl}^*(A\setminus A')$, so $x \in \operatorname{cl}^*(A\cap A')$.

Now

$$\begin{aligned}\partial^*A \cap \operatorname{int}^*A' &= (\operatorname{cl}^*A \cap \operatorname{int}^*A') \setminus \operatorname{int}^*A \\ &\subseteq \operatorname{cl}^*(A \cap A') \setminus \operatorname{int}^*(A \cap A') = \partial^*(A \cap A'),\end{aligned}$$

$$\begin{aligned}\partial^*A \setminus \operatorname{cl}^*A' &= \partial^*(\mathbb{R}^r \setminus A) \cap \operatorname{int}^*(\mathbb{R}^r \setminus A') \\ &\subseteq \partial^*((\mathbb{R}^r \setminus A) \cap (\mathbb{R}^r \setminus A')) \\ &= \partial^*(\mathbb{R}^r \setminus (A \cup A')) = \partial^*(A \cup A').\end{aligned}$$

(f) If $x \in \partial^*(A \cap A') \cap \partial A$, then of course $x \in \operatorname{cl}^*(A \cap A') \subseteq \operatorname{cl}^*A$, so $x \in \operatorname{cl}^*A \cap \partial A$. If $x \in \partial^*(A \cap A') \setminus \partial A$, then surely $x \in \overline{A}$, so $x \in \operatorname{int} A$. But this means that

$$\mu_*(B(x,\delta) \cap A') = \mu_*(B(x,\delta) \cap A \cap A'), \quad \mu^*(B(x,\delta) \cap A') = \mu^*(B(x,\delta) \cap A \cap A')$$

for all δ small enough, so $x \in \partial^*A'$ and $x \in A \cap \partial^*A'$.

(g) Applying 472D to μ and χE, we see that

$$\begin{aligned}\partial^*E &\subseteq (E \triangle \operatorname{cl}^*E) \cup (E \triangle \operatorname{int}^*E) \\ &\subseteq \{x : \chi E(x) \neq \lim_{\delta \downarrow 0} \tfrac{\mu(E \cap B(x,\delta))}{\mu B(x,\delta)}\}\end{aligned}$$

are all μ-negligible.

(h) If A is Lebesgue measurable then (g) tells us that ∂^*A is negligible. If ∂^*A is negligible, let E be a measurable envelope of A. Then $\mu(E \cap B(x,\delta)) = \mu^*(A \cap B(x,\delta))$ for all x and δ, so $\operatorname{cl}^*E = \operatorname{cl}^*A$. Similarly, if F is a measurable envelope of $\mathbb{R}^r \setminus A$, then $\operatorname{cl}^*F = \operatorname{cl}^*(\mathbb{R}^r \setminus A) = \mathbb{R}^r \setminus \operatorname{int}^*A$ (using (a)). Now (g) tells us that

$$\mu(E \cap F) = \mu(\operatorname{cl}^*E \cap \operatorname{cl}^*F) = \mu(\operatorname{cl}^*A \setminus \operatorname{int}^*A) = 0.$$

But now $A \setminus E$ and $E \setminus A \subseteq (E \cap F) \cup ((\mathbb{R}^r \setminus A) \setminus F)$ are Lebesgue negligible, so A is Lebesgue measurable.

475D Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and $\partial^{\$}E$ its reduced boundary (474G). Then $\partial^{\$}E \subseteq \partial^*E$ and $\nu(\partial^*E \setminus \partial^{\$}E) = 0$.

proof (a) By 474N(i), $\partial^{\$}E \subseteq \operatorname{cl}^*E$; by 474N(ii), $\partial^{\$}E \cap \operatorname{int}^*E = \emptyset$; so $\partial^{\$}E \subseteq \partial^*E$.

(b) For any $y \in \partial^*E$,

$$\limsup_{\delta \downarrow 0} \tfrac{1}{\delta^{r-1}} \lambda^{\partial}_E B(y,\delta) > 0.$$

P We have an $\epsilon \in \left]0, \frac{1}{2}\right]$ such that

$$\liminf_{\delta \downarrow 0} \tfrac{\mu(E \cap B(y,\delta))}{\mu B(y,\delta)} < 1 - \epsilon, \quad \limsup_{\delta \downarrow 0} \tfrac{\mu(E \cap B(y,\delta))}{\mu B(y,\delta)} > \epsilon.$$

Since the function $\delta \mapsto \frac{\mu(E \cap B(y,\delta))}{\mu B(y,\delta)}$ is continuous, there is a sequence $\langle \delta_n \rangle_{n \in \mathbb{N}}$ in $]0, \infty[$ such that $\lim_{n \to \infty} \delta_n = 0$ and

$$\epsilon \mu B(y,\delta_n) \leq \mu(E \cap B(y,\delta_n)) \leq (1-\epsilon)\mu B(y,\delta_n)$$

for every n. Now from 474Lb we have

$$\begin{aligned}(\epsilon\beta_r)^{(r-1)/r}\delta_n^{r-1} &= (\epsilon \mu B(y,\delta_n))^{(r-1)/r} \\ &\leq \min(\mu(B(y,\delta_n) \cap E), \mu(B(y,\delta_n) \setminus E))^{(r-1)/r} \leq 2c\lambda^{\partial}_E B(y,\delta_n)\end{aligned}$$

for every n, where c is the constant there. But this means that

$$\limsup_{\delta \downarrow 0} \tfrac{1}{\delta^{r-1}} \lambda^{\partial}_E B(y,\delta) \geq \limsup_{n \to \infty} \tfrac{1}{\delta_n^{r-1}} \lambda^{\partial}_E B(y,\delta_n) \geq \tfrac{1}{2c}(\epsilon\beta_r)^{(r-1)/r} > 0. \quad \textbf{Q}$$

(c) Let $\epsilon > 0$. Set

$$F_\epsilon = \{y : y \in \mathbb{R}^r \setminus \partial^{\$} E, \limsup_{\delta \downarrow 0} \frac{1}{\delta^{r-1}} \lambda_E^\partial B(y,\delta) > \epsilon\}.$$

Because $\partial^{\$} E$ is λ_E^∂-conegligible (474G), $\lambda_E^\partial F_\epsilon = 0$. So there is an open set $G \supseteq F_\epsilon$ such that $\lambda_E^\partial G \leq \epsilon^2$ (256Bb). Let $\delta > 0$. Let $\mathcal{I}$ be the family of all those non-singleton closed balls $B \subseteq G$ such that $\operatorname{diam} B \leq \delta$ and $\lambda_E^\partial B \geq 2^{-r+1}\epsilon(\operatorname{diam} B)^{r-1}$. Then every point of F_ϵ is the centre of arbitrarily small members of $\mathcal{I}$. By Besicovitch's Covering Lemma (472B), there is a family $\langle \mathcal{I}_k \rangle_{k<5^r}$ of disjoint countable subsets of $\mathcal{I}$ such that $\mathcal{I}^* = \bigcup_{k<5^r} \bigcup \mathcal{I}_k$ covers F_ϵ. Now

$$\sum_{B\in\mathcal{I}^*} (\operatorname{diam} B)^{r-1} \leq \sum_{k<5^r} \sum_{B\in\mathcal{I}_k} \frac{2^{r-1}}{\epsilon} \lambda_E^\partial B \leq \frac{5^r 2^{r-1}}{\epsilon} \lambda_E^\partial G \leq 5^r 2^{r-1}\epsilon.$$

As δ is arbitrary, $\mu^*_{H,r-1} F_\epsilon$ is at most $5^r 2^{r-1}\epsilon$ (264Fb/471Dc) and $\nu^* F_\epsilon \leq 5^r \beta_{r-1}\epsilon$. As ϵ is arbitrary,

$$\partial^* E \setminus \partial^{\$} E \subseteq \{y : y \in \mathbb{R}^r \setminus \partial^{\$} E, \limsup_{\delta\downarrow 0} \frac{1}{\delta^{r-1}} \lambda_E^\partial B(y,\delta) > 0\}$$

is ν-negligible, as claimed.

475E Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter.
(a) If $A \subseteq \partial^{\$} E$, then $\nu^* A \leq (\lambda_E^\partial)^* A$.
(b) If $A \subseteq \mathbb{R}^r$ and $\nu A = 0$, then $\lambda_E^\partial A = 0$.

proof (a) Given $\epsilon, \delta > 0$ let $\mathcal{I}$ be the family of non-trivial closed balls $B \subseteq \mathbb{R}^r$ of diameter at most δ such that $\beta_{r-1}(\frac{1}{2}\operatorname{diam} B)^{r-1} \leq (1+\epsilon)\lambda_E^\partial B$. By 474R, every point of A is the centre of arbitrarily small members of $\mathcal{I}$. By 472C, there is a countable family $\mathcal{I}_1 \subseteq I$ such that $A \subseteq \bigcup \mathcal{I}_1$ and $\sum_{B\in\mathcal{I}_1} \lambda_E^\partial B \leq (\lambda_E^\partial)^* A + \epsilon$. But this means that

$$\sum_{B\in\mathcal{I}_1} (\operatorname{diam} B)^{r-1} \leq (1+\epsilon)\frac{2^{r-1}}{\beta_{r-1}}((\lambda_E^\partial)^* A + \epsilon).$$

As δ is arbitrary,

$$\nu^* A = \frac{\beta_{r-1}}{2^{r-1}} \mu^*_{H,r-1} A \leq (1+\epsilon)((\lambda_E^\partial)^* A + \epsilon).$$

As ϵ is arbitrary, we have the result.

(b) For $n \in \mathbb{N}$, set

$$A_n = \{x : x \in A, \lambda_E^\partial B(x,\delta) \leq 2\beta_{r-1}\delta^{r-1} \text{ whenever } 0 < \delta \leq 2^{-n}\}.$$

Now, given $\epsilon > 0$, there is a sequence $\langle D_i \rangle_{i\in\mathbb{N}}$ of sets covering A_n such that $\operatorname{diam} D_i \leq 2^{-n}$ for every i and $\sum_{i=0}^\infty (\operatorname{diam} D_i)^{r-1} \leq \epsilon$. Passing over the trivial case $A_n = \emptyset$, we may suppose that for each $i \in \mathbb{N}$ there is an $x_i \in A_n \cap D_i$, so that $D_i \subseteq B(x_i, \operatorname{diam} D_i)$ and

$$\begin{aligned}(\lambda_E^\partial)^* A_n &\leq \sum_{i=0}^\infty (\lambda_E^\partial)^* D_i \leq \sum_{i=0}^\infty \lambda_E^\partial B(x_i, \operatorname{diam} D_i) \\ &\leq \sum_{i=0}^\infty 2\beta_{r-1}(\operatorname{diam} D_i)^{r-1} \leq 2\beta_{r-1}\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\lambda_E^\partial A_n = 0$. And this is true for every n. As $A \setminus \bigcup_{n\in\mathbb{N}} A_n$ is λ_E^∂-negligible, by 474G and 474R, so is A.

475F Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and $\epsilon > 0$. Then λ_E^∂ is inner regular with respect to the family $\mathcal{E} = \{F : F \subseteq \mathbb{R}^r \text{ is Borel}, \lambda_E^\partial F \leq (1+\epsilon)\nu F\}$.

proof (a) We need some elementary bits of geometry.

(i) If $x \in \mathbb{R}^r$, $\delta > 0$, $\alpha \geq 0$ and $v \in S_{r-1}$, then

$$\mu\{z : z \in B(x,\delta),\ |(z-x)\boldsymbol{.}v| \leq \alpha\} \leq 2\alpha\beta_{r-1}\delta^{r-1}.$$

P Translating and rotating, if necessary, we can reduce to the case $x = \mathbf{0}$, $v = (0,\dots,1)$. In this case we are looking at

$$\{z : \|z\| \leq \delta,\ |\zeta_r| \leq \alpha\} \subseteq \{u : u \in \mathbb{R}^{r-1},\ \|u\| \leq \delta\} \times [-\alpha,\alpha]$$

which has measure $2\alpha\beta_{r-1}\delta^{r-1}$. **Q**

(ii) If $x \in \mathbb{R}^r$, $\delta > 0$, $0 < \eta \leq \frac{1}{2}$, $v \in S_{r-1}$, $H = \{z : z\boldsymbol{.}v \leq \alpha\}$ and

$$|\mu(H \cap B(x,\delta)) - \tfrac{1}{2}\mu B(x,\delta)| < 2^{-r+1}\beta_{r-1}\eta\delta^r,$$

then $|x\boldsymbol{.}v - \alpha| \leq \eta\delta$. **P** Again translating and rotating if necessary, we may suppose that $x = \mathbf{0}$ and $v = (0,\dots,1)$. Set $H_0 = \{z : \zeta_r \leq 0\}$. **?** If $\alpha > \eta\delta$, then $H \cap B(\mathbf{0},\delta)$ includes

$$H_0 \cap B(\mathbf{0},\delta) \cup \{u : u \in \mathbb{R}^{r-1},\ \|u\| \leq \tfrac{1}{2}\delta\} \times [0,\alpha],$$

so

$$\begin{aligned}\mu(H \cap B(\mathbf{0},\delta)) - \tfrac{1}{2}\mu B(\mathbf{0},\delta) &= \mu(H \cap B(\mathbf{0},\delta)) - \mu(H_0 \cap B(\mathbf{0},\delta))\\ &\geq 2^{-r+1}\beta_{r-1}\delta^{r-1}\alpha > 2^{-r+1}\beta_{r-1}\delta^r\eta,\end{aligned}$$

contrary to hypothesis. **X** Similarly, if $\alpha < -\eta\delta$, then $H \cap B(\mathbf{0},\delta)$ is included in

$$H_0 \cap B(\mathbf{0},\delta) \setminus \{u : u \in \mathbb{R}^{r-1},\ \|u\| \leq \tfrac{1}{2}\delta\} \times]\alpha,0],$$

so

$$\begin{aligned}\tfrac{1}{2}\mu B(\mathbf{0},\delta) - \mu(H \cap B(\mathbf{0},\delta)) &= \mu(H_0 \cap B(\mathbf{0},\delta)) - \mu(H \cap B(\mathbf{0},\delta))\\ &\geq 2^{-r+1}\beta_{r-1}\delta^{r-1}|\alpha| > 2^{-r+1}\beta_{r-1}\delta^r\eta,\end{aligned}$$

which is equally impossible. So $|\alpha| \leq \eta\delta$. **Q**

(b) Let F be such that $\lambda_E^\partial F > 0$. Let η, $\zeta > 0$ be such that

$$\eta < 1, \quad \tfrac{(1+\eta)^2}{(1-\eta)^{r-1}} \leq 1+\epsilon, \quad 2(1+2^r)\beta_r\zeta < 2^{-r}\beta_{r-1}\eta.$$

Because $\partial^\$E$ is λ_E^∂-conegligible (474G), $\lambda_E^\partial(F \cap \partial^\$E) > 0$. Because λ_E^∂ is a Radon measure (474E) and $\psi_E : \partial^\$E \to S_{r-1}$ is $\mathrm{dom}(\lambda_E^\partial)$-measurable (474G), there is a compact set $K_1 \subseteq F \cap \partial^\$E$ such that $\lambda_E^\partial K_1 > 0$ and $\psi_E\restriction K_1$ is continuous, by Lusin's theorem (418J). For $x \in \partial^\$E$, set $H_x = \{z : (z-x)\boldsymbol{.}\psi_E(x) \leq 0\}$. The function

$$(x,\delta) \mapsto \mu((E \triangle H_x) \cap B(x,\delta)) : K_1 \times [0,\infty[\to \mathbb{R}$$

is Borel measurable. **P** Take a Borel set E' such that $\mu(E \triangle E') = 0$. Then

$$\{(x,\delta,z) : x \in K_1,\ z \in (E' \triangle H_x) \cap B(x,\delta)\}$$

is a Borel set in $\mathbb{R}^r \times [0,\infty[\times \mathbb{R}^r$, so its sectional measure is a Borel measurable function, by 252P. **Q**

For each $x \in K_1$,

$$\lim_{n\to\infty} \tfrac{\mu((E\triangle H_x)\cap B(x,2^{-n}))}{\mu B(x,2^{-n})} = 0$$

(474P). So there is an $n_0 \in \mathbb{N}$ such that $\lambda_E^\partial F_1 > 0$, where F_1 is the Borel set

$$\{x : x \in K_1,\ \mu((E \triangle H_x) \cap B(x,2^{-n})) \leq \zeta\mu B(x,2^{-n}) \text{ for every } n \geq n_0\}.$$

Let $K_2 \subseteq F_1$ be a compact set such that $\lambda_E^\partial K_2 > 0$.

For each $n \in \mathbb{N}$, the function

$$x \mapsto \lambda_E^\partial B(x,2^{-n})) = \lambda_E^\partial(x + B(\mathbf{0},2^{-n})$$

is Borel measurable (444Fe). Let $y \in K_2$ be such that $\lambda_E^\partial(K_2 \cap B(y,\delta)) > 0$ for every $\delta > 0$ (cf. 411Nd). Set $v = \psi_E(y)$. Let $n \geq n_0$ be so large that $2\beta_{r-1}\|\psi_E(x) - v\| \leq \beta_r\zeta$ whenever $x \in K_1$ and $\|x - y\| \leq 2^{-n}$.

(c) We have $|(x-z)\cdot v| \leq \eta\|x-z\|$ whenever $x,\ z \in K_2 \cap B(y, 2^{-n-1})$. **P** If $x = z$ this is trivial. Otherwise, let $k \geq n$ be such that $2^{-k-1} \leq \|x-z\| \leq 2^{-k}$, and set $\delta = 2^{-k}$. Set

$$H'_x = \{w : (w-x)\cdot v \leq 0\}, \quad H'_z = \{w : (w-z)\cdot v \leq 0\}.$$

Since $|(w-x)\cdot v - (w-x)\cdot\psi_E(x)| \leq 2\delta\|\psi_E(x) - v\|$ whenever $w \in B(x, 2\delta)$,

$$(H_x \triangle H'_x) \cap B(x,2\delta) \subseteq \{w : w \in B(x,2\delta),\ |(w-x)\cdot v| \leq 2\delta\|\psi_E(x) - v\|\}$$

has measure at most

$$4\delta\|\psi_E(x) - v\|\beta_{r-1}(2\delta)^{r-1} \leq 2\delta\beta_r\zeta(2\delta)^{r-1} = \zeta\mu B(x,2\delta),$$

by (a-i). So

$$\begin{aligned}\mu((E\triangle H'_x)\cap B(x,2\delta)) &\leq \mu((E\triangle H_x)\cap B(x,2\delta)) + \mu((H_x\triangle H'_x)\cap B(x,2\delta))\\ &\leq 2\zeta\mu B(x,2\delta)\end{aligned}$$

because $k \geq n_0$ and $x \in F_1$. Similarly, $\mu((E\triangle H'_z)\cap B(z,\delta)) \leq 2\zeta\mu B(z,\delta)$. Now observe that because $\|x-z\| \leq \delta$, $B(z,\delta) \subseteq B(x,2\delta)$ and

$$\mu((E\triangle H'_x)\cap B(z,\delta)) \leq 2\zeta\mu B(x,2\delta) = 2^{r+1}\zeta\mu B(z,\delta),$$

and

$$\begin{aligned}\mu((H'_x\triangle H'_z)\cap B(z,\delta)) &\leq \mu((E\triangle H'_x)\cap B(z,\delta)) + \mu((E\triangle H'_z)\cap B(z,\delta))\\ &\leq (2+2^{r+1})\zeta\mu B(z,\delta).\end{aligned}$$

Since $\mu(H'_z \cap B(z,\delta)) = \frac{1}{2}\mu B(z,\delta)$,

$$|\mu(H'_x\cap B(z,\delta)) - \tfrac{1}{2}\mu B(z,\delta)| \leq 2(1+2^r)\zeta\mu B(z,\delta) < 2^{-r}\beta_{r-1}\eta\delta^r,$$

and (using (a-ii) above)

$$|(x-z)\cdot v| \leq \tfrac{1}{2}\eta\delta \leq \eta\|x-v\|. \ \textbf{Q}$$

(d) Let V be the hyperplane $\{w : w\cdot v = 0\}$, and let $T : K_2 \to V$ be the orthogonal projection, that is, $T(x) = x - (x\cdot v)v$ for every $x \in K_2$. Then (c) tells us that if $x,\ z \in F_1$,

$$\|T(x) - T(z)\| \geq \|x-z\| - |(x-z)\cdot v| \geq (1-\eta)\|x-z\|.$$

Because $\eta < 1$, T is injective. Consider the compact set $T[K_2]$. The inverse T^{-1} of T is $\frac{1}{1-\eta}$-Lipschitz, and $\nu K_2 > 0$ (by 475Eb), so $\nu T[K_2] \geq \frac{1}{(1-\eta)^{r-1}}\nu F_1 > 0$ (264G/471J). Let $G \supseteq T[K_2]$ be an open set such that $\nu(G\cap V) \leq (1+\eta)\nu T[K_2]$. (I am using the fact that the subspace measure ν_V induced by ν on V is a copy of Lebesgue measure on $\mathbb{R}^{r-1}$, so is a Radon measure.) Let $\mathcal{I}$ be the family of non-trivial closed balls $B \subseteq G$ such that $(1-\eta)^{r-1}\lambda_E^\partial T^{-1}[B] \leq (1+\eta)\nu(B\cap V)$. Then every point of $T[K_2]$ is the centre of arbitrarily small members of $\mathcal{I}$. **P** If $x \in K_2$ and $\delta_0 > 0$, there is a $\delta \in]0,\delta_0]$ such that $B(T(x),\delta) \subseteq G$ and $\lambda_E^\partial B(x,\delta) \leq (1+\eta)\beta_{r-1}\delta^{r-1}$ (474R). Now consider $B = B(T(x),(1-\eta)\delta)$. Then $T^{-1}[B] \subseteq B(x,\delta)$, so

$$\lambda_E^\partial T^{-1}[B] \leq \lambda_E^\partial B(x,\delta) \leq (1+\eta)\beta_{r-1}\delta^{r-1} = \frac{1+\eta}{(1-\eta)^{r-1}}\nu(B\cap V). \ \textbf{Q}$$

By 261B/472C, applied in $V \cong \mathbb{R}^{r-1}$, there is a countable disjoint family $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $\nu(T[K_2] \setminus \bigcup\mathcal{I}_0) = 0$.

Now $\nu(K_2 \setminus \bigcup_{B\in\mathcal{I}_0} T^{-1}[B]) = 0$, because T^{-1} is Lipschitz, so $\lambda_E^\partial(K_2 \setminus \bigcup_{B\in\mathcal{I}_0} T^{-1}[B]) = 0$ (475Eb again), and

$$\lambda_E^\partial K_2 \le \sum_{B\in\mathcal{I}_0} \lambda_E^\partial T^{-1}[B] \le \frac{1+\eta}{(1-\eta)^{r-1}} \sum_{B\in\mathcal{I}_0} \nu(B\cap V)$$
$$\le \frac{1+\eta}{(1-\eta)^{r-1}}\nu(G\cap V) \le \frac{(1+\eta)^2}{(1-\eta)^{r-1}}\nu T[K_2] \le \frac{(1+\eta)^2}{(1-\eta)^{r-1}}\nu K_2$$

(because T is 1-Lipschitz)

$$\le (1+\epsilon)\nu K_2$$

by the choice of η. Thus $K_2 \in \mathcal{E}$.

(e) This shows that every non-negligible set measured by λ_E^∂ includes a non-negligible member of $\mathcal{E}$. As $\mathcal{E}$ is closed under disjoint unions, λ_E^∂ is inner regular with respect to $\mathcal{E}$ (412Aa).

475G Theorem Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter. Then $\lambda_E^\partial = \nu\llcorner\partial^*E$, that is, for $F \subseteq \mathbb{R}^r$, $\lambda_E^\partial F = \nu(F\cap\partial^*E)$ whenever either is defined.

proof (a) Suppose first that F is a Borel set included in the reduced boundary $\partial^\$E$ of E. Then $\nu F \le \lambda_E^\partial F$, by 475Ea. On the other hand, for any $\epsilon > 0$ and $\gamma < \lambda_E^\partial F$, there is an $F_1 \subseteq F$ such that

$$\gamma \le \lambda_E^\partial F_1 \le (1+\epsilon)\nu F_1 \le (1+\epsilon)\nu F,$$

by 475F; so we must have $\lambda_E^\partial F = \nu F$.

(b) Now suppose that F is measured by $\lambda_E^\partial F$. Because $\partial^\$E$ is λ_E^∂-conegligible, and λ_E^∂ is a σ-finite Radon measure, there is a Borel set $F' \subseteq F\cap\partial^\$E$ such that $\lambda_E^\partial(F\setminus F') = 0$. Now $\nu F' = \lambda_E^\partial F'$, by (a), and $\nu(F\cap\partial^\$E\setminus F') = 0$, by 475Ea, and $\nu(\partial^*E\setminus\partial^\$E) = 0$, by 475D; so $\nu(F\cap\partial^*E)$ is defined and equal to $\lambda_E^\partial F$.

(c) Let $\langle K_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of compact subsets of $\partial^\$E$ such that $\bigcup_{n\in\mathbb{N}} K_n$ is λ_E^∂-conegligible. By (b), $\nu(\partial^*E\setminus\bigcup_{n\in\mathbb{N}}K_n) = 0$, while $\nu K_n = \lambda_E^\partial K_n$ is finite for each n. For each n, the subspace measure ν_{K_n} on K_n is a multiple of Hausdorff $r-1$-dimensional measure on K_n (471E), so is a Radon measure (471Dh/471F), and so is $(\lambda_E^\partial)_{K_n}$; since, by (b), ν_{K_n} and $(\lambda_E^\partial)_{K_n}$ agree on the Borel subsets of K_n, they are actually identical. So if $F\subseteq\mathbb{R}^r$ is such that ν measures $F\cap\partial^*E$, λ_E^∂ will measure $F\cap K_n$ for every n, and therefore will measure F; so that in this case also $\lambda_E^\partial F = \nu(F\cap\partial^*E)$.

475H Proposition Let $V\subseteq\mathbb{R}^r$ be a hyperplane, and $T:\mathbb{R}^r\to V$ the orthogonal projection. Suppose that $A\subseteq\mathbb{R}^r$ is such that νA is defined and finite, and for $u\in V$ set

$$f(u) = \#(A\cap T^{-1}[\{u\}]) \text{ if this is finite,}$$
$$= \infty \text{ otherwise.}$$

Then $\int_V f(u)\nu(du)$ is defined and at most νA.

proof (a) Because ν is invariant under isometries, we can suppose that $V = \{x : \xi_r = 0\}$, so that $T(x) = (\xi_1,\dots,\xi_{r-1},0)$ for $x = (\xi_1,\dots,\xi_r)$. For m, $n\in\mathbb{N}$ and $u\in V$ set

$$f_{mn}(u) = \#(\{k : k\in\mathbb{Z},\ |k|\le 4^m,\ A\cap(\{u\}\times[2^{-m}k, 2^{-m}(k+1-2^{-n})[) \ne\emptyset\});$$

so that $f(u) = \lim_{m\to\infty}\lim_{n\to\infty} f_{mn}(u)$ for every $u\in V$.

(b) Suppose for the moment that A is actually a Borel set. Then $T[A\cap(\mathbb{R}^r\times[\alpha,\beta[)]$ is always analytic (423E, 423Bb), therefore measured by ν (432A), and every f_{mn} is measurable. Next, given m, $n\in\mathbb{N}$ and $\epsilon>0$, there is a sequence $\langle F_i\rangle_{i\in\mathbb{N}}$ of closed sets of diameter at most 2^{-m-n}, covering A, such that $\sum_{i=0}^\infty 2^{-r+1}\beta_{r-1}(\operatorname{diam}F_i)^{r-1} \le \nu A+\epsilon$. Now each $T[F_i]$ is a compact set of diameter at most $\operatorname{diam}F_i$, so $\nu T[F_i] \le 2^{-r+1}\beta_{r-1}(\operatorname{diam}F_i)^{r-1}$ (264H); and if we set $g = \sum_{i=0}^\infty \chi T[F_i]$, $f_{mn}\le g$ everywhere on V, so $\int f_{mn}d\nu \le \int g\,d\nu \le \nu A+\epsilon$. As ϵ is arbitary, $\int f_{mn}d\nu\le\nu A$. Accordingly

$$\int f d\nu = \lim_{m\to\infty}\lim_{n\to\infty}\int f_{mn}d\nu \le \nu A.$$

(c) In general, there are Borel sets E, F such that $E \setminus F \subseteq A \subseteq E$ and $\nu F = 0$, by 264Fc or 471Db. By (b),

$$\int \#(E \cap T^{-1}[\{u\}])\nu(du) \leq \nu E, \quad \int \#(F \cap T^{-1}[\{u\}])\nu(du) \leq \nu F = 0,$$

so $\int \#(A \cap T^{-1}[\{u\}])\nu(du)$ is defined and equal to $\int \#(E \cap T^{-1}[\{u\}])\nu(du) \leq \nu A$.

475I Lemma (In this lemma I allow $r = 1$.) Let $E \subseteq \mathbb{R}^r$ be a Lebesgue measurable set, and $\gamma < \mu E$. Then there is a compact set $K \subseteq E$ such that $\mu K \geq \gamma$ and $K = \text{cl}^*K$.

proof **(a)** Write $\mathcal{C}^*$ for the set of dyadic (half-open) cubes in $\mathbb{R}^r$, that is, sets expressible in the form $\prod_{i<r} [2^{-m}k_i, 2^{-m}(k_i+1)[$ where m, $k_0, \ldots, k_{r-1} \in \mathbb{Z}$. For $m \in \mathbb{N}$, $x \in \mathbb{R}^r$ write $C(x,m)$ for the dyadic cube with side of length 2^{-m} which contains x. Then, for any $A \subseteq \mathbb{R}^r$,

$$\lim_{m\to\infty} 2^{mr}\mu_*(A \cap C(x,m)) = 1$$

for every $x \in \text{int}^*A$. **P** $C(x,m) \subseteq B(x, 2^{-m}\sqrt{r})$ for each m, so

$$2^{mr}\mu^*(C(x,m) \setminus A) \leq \beta_r r^{r/2} \frac{\mu(B(x,2^{-m}\sqrt{r}) \setminus A)}{\mu B(x,2^{-m}\sqrt{r})} \to 0$$

as $m \to \infty$. **Q**

(b) Now, given $\gamma < \mu E$, choose $\langle E_n\rangle_{n\in\mathbb{N}}$, $\langle \gamma_n\rangle_{n\in\mathbb{N}}$, $\langle m_n\rangle_{n\in\mathbb{N}}$ and $\langle K_n\rangle_{n\in\mathbb{N}}$ inductively, as follows. Start with $E_0 = E$ and $\gamma_0 = \gamma$. Given that $\gamma_n < \mu E_n$, let $K_n \subseteq E_n$ be a compact set of measure greater than γ_n. Now, by (a), there is an $m_n \geq n$ such that $\mu^* E_{n+1} > \gamma_n$, where

$$E_{n+1} = \{x : x \in K_n,\ \mu(K_n \cap C(x,m_n)) \geq \tfrac{2}{3}\mu C(x,m_n)\};$$

note that E_{n+1} is of the form $K_n \cap \bigcup \mathcal{D}$ for some set $\mathcal{D}$ of half-open cubes of side 2^{-m_n}, so that E_{n+1} is measurable and

$$\mu(E_{n+1} \cap C(x,m_n)) = \mu(K_n \cap C(x,m_n)) \geq \tfrac{2}{3}\mu C(x,m_n)$$

for every $x \in E_{n+1}$. Now set

$$\gamma_{n+1} = \max(\gamma_n, \mu E_{n+1} - \tfrac{1}{3}\cdot 2^{-m_n r}),$$

and continue.

At the end of the induction, set $K = \bigcap_{n\in\mathbb{N}} K_n = \bigcap_{n\in\mathbb{N}} E_n$. Then K is compact, $K \subseteq E$ and

$$\mu K = \lim_{n\to\infty} \mu K_n = \lim_{n\to\infty} \gamma_n \geq \gamma.$$

If $x \in K$ and $n \in \mathbb{N}$, then

$$\begin{aligned}\mu(K \cap B(x,2^{-m_n}\sqrt{r})) &\geq \mu(K \cap C(x,m_n)) \geq \mu(E_{n+1} \cap C(x,m_n)) - \mu E_{n+1} + \mu K\\ &\geq \tfrac{2}{3}\mu C(x,m_n) - \mu E_{n+1} + \gamma_{n+1}\\ &\geq \tfrac{1}{3}\cdot 2^{-m_n r} \geq \frac{1}{3\beta_r r^{r/2}}\mu B(x,2^{-m_n}\sqrt{r}),\end{aligned}$$

so

$$\limsup_{\delta\downarrow 0} \frac{\mu(K\cap B(x,\delta))}{\mu B(x,\delta)} \geq \frac{1}{3\beta_r r^{r/2}} > 0$$

and $x \in \text{cl}^*K$.

(c) Thus $K \subseteq \text{cl}^*K$. Since certainly $\text{cl}^*K \subseteq \overline{K} = K$, we have $K = \text{cl}^*K$.

475J Lemma Let E be a Lebesgue measurable subset of $\mathbb{R}^r$, identified with $\mathbb{R}^{r-1} \times \mathbb{R}$. For $u \in \mathbb{R}^{r-1}$, set $G_u = \{t : (u,t) \in \text{int}^*E\}$, $H_u = \{t : (u,t) \in \text{int}^*(\mathbb{R}^r \setminus E)\}$ and $D_u = \{t : (u,t) \in \partial^*E\}$, so that G_u, H_u and D_u are disjoint and cover $\mathbb{R}$.

(a) There is a μ_{r-1}-conegligible set $Z \subseteq \mathbb{R}^{r-1}$ such that whenever $u \in Z$, $t < t'$ in $\mathbb{R}$, $t \in G_u$ and $t' \in H_u$, there is an $s \in D_u \cap]t,t'[$.

(b) If E has locally finite perimeter, we can arrange further that, for every $u \in Z$, $D_u \cap [-n,n]$ is finite for every $n \in \mathbb{N}$, G_u and H_u are open, and $D_u = \partial G_u = \partial H_u$, so that the constituent intervals of $\mathbb{R} \setminus D_u$ lie alternately in G_u and H_u.

proof (a)(i) For $q \in \mathbb{Q}$, set $f_q(u) = \sup(G_u \cap]-\infty, q[)$, taking $-\infty$ for $\sup \emptyset$. Then $f : \mathbb{R}^{r-1} \to [-\infty, q]$ is Lebesgue measurable. **P** For $\alpha < q$,

$$\{u : f_q(u) > \alpha\} = \{u : \text{there is some } t \in]\alpha, q[\text{ such that } (u,t) \in \text{int}^*E\}.$$

Because int^*E is a Borel set (475Cc), $\{u : f_q(u) > \alpha\}$ is analytic (423E, 423Bc), therefore measurable (432A). **Q**

(ii) For any $q \in \mathbb{Q}$, $W_q = \{u : f_q(u) < q,\ f_q(u) \in G_u\}$ is negligible. **P?** Suppose, if possible, otherwise. Let $n \in \mathbb{N}$ be such that $\{u : u \in W_q,\ f_q(u) > -n\}$ is not negligible, and set $g(u) = \max(-n, f_q(u))$ for $u \in \mathbb{R}^{r-1}$. Then g is measurable, so by Lusin's theorem (418J) there is a Borel set $F \subseteq \mathbb{R}^{r-1}$ such that $g \restriction F$ is continuous and $F_1 = \{u : u \in F \cap W_q,\ f_q(u) \in G_u,\ f_q(u) > -n\}$ is not negligible. Note that F_1 is measurable, being the projection of the Borel set $\{(u, g(u)) : u \in F,\ -n < g(u) < q\} \cap \text{int}^*E$. By 475I, there is a non-negligible compact set $K \subseteq F_1$ such that $K \subseteq \text{cl}^*K$, interpreting cl^*K in $\mathbb{R}^{r-1}$. Because $g \restriction K$ is continuous, it attains its maximum at $u \in K$ say.

Set $x = (u, f_q(u))$. Then, whenever $0 < \delta \leq \min(n + f_q(u), q - f_q(u))$,

$$B(x, 2\delta) \setminus \text{int}^*E \supseteq \{(w,t) : w \in K \cap V(u,\delta),\ |t - f_q(u)| \leq \delta,\ f_q(w) < t < q\}$$

(writing $V(u,\delta)$ for $\{w : w \in \mathbb{R}^{r-1},\ \|w - u\| \leq \delta\}$)

$$\supseteq \{(w,t) : w \in K \cap V(u,\delta),\ f_q(u) < t \leq f_q(u) + \delta\}$$

because $f_q(w) = g(w) \leq g(u) = f_q(u)$ for every $w \in K$. So, for such δ,

$$\begin{aligned}\mu(B(x,2\delta) \setminus \text{int}^*E) &\geq \delta \mu_{r-1}(K \cap V(u,\delta)) \\ &= \frac{\beta_{r-1}}{2^r \beta_r} \cdot \frac{\mu_{r-1}(K \cap V(u,\delta))}{\mu_{r-1} V(u,\delta)} \cdot \mu B(x, 2\delta),\end{aligned}$$

and

$$\begin{aligned}\limsup_{\delta \downarrow 0} \frac{\mu(B(x,\delta) \setminus E)}{\mu B(x,\delta)} &= \limsup_{\delta \downarrow 0} \frac{\mu(B(x,2\delta) \setminus \text{int}^*E)}{\mu B(x,2\delta)} \\ &\geq \frac{\beta_{r-1}}{2^r \beta_r} \limsup_{\delta \downarrow 0} \frac{\mu_{r-1}(K \cap V(u,\delta))}{\mu_{r-1} V(u,\delta)} > 0\end{aligned}$$

because $u \in \text{cl}^*K$; but $x = (u, f_q(u))$ is supposed to belong to int^*E. **XQ**

(iii) Similarly, setting

$$f'_q(u) = \inf(H_u \cap]u, \infty[), \quad W'_q = \{u : q < f'_q(u) < \infty,\ f'_q(u) \in H_u\},$$

every W'_q is negligible. So $Z = \mathbb{R}^{r-1} \setminus \bigcup_{q \in \mathbb{Q}} (W_q \cup W'_q)$ is μ_{r-1}-conegligible.

Now if $u \in Z$, $t \in G_u$ and $t' \in H_u$, where $t < t'$, there is some $s \in]t,t'[\cap D_u$. **P?** Suppose, if possible, otherwise. Since, by hypothesis, neither t nor t' belongs to D_u, $D_u \cap [t,t'] = \emptyset$. Note that, because $u \in Z$,

$$s = \inf(G_u \cap]s, \infty[) \text{ for every } s \in G_u, \quad s = \sup(H_u \cap]-\infty, s[) \text{ for every } s \in H_u.$$

Choose $\langle s_n \rangle_{n \in \mathbb{N}}$, $\langle s'_n \rangle_{n \in \mathbb{N}}$ and $\langle \delta_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. Set $s_0 = t$, $s'_0 = t'$. Given that $t \leq s_n < s'_n \leq t'$ and $s_n \in G_u$ and $s'_n \in H_u$, where n is even, set $s'_{n+1} = \sup(G_u \cap [s_n, s'_n])$. Then either $s'_{n+1} = s'_n$, so $s'_{n+1} \in H_u$, or $s'_{n+1} < s'_n$ and $\left]s'_{n+1}, s'_n\right] \cap G_u = \emptyset$, so $s'_{n+1} \notin G_u$ and again $s'_{n+1} \in H_u$. Let $\delta_n > 0$ be

such that $\delta_n \leq 2^{-n}$ and $\mu(E \cap B((u, s'_{n+1}), \delta_n)) < \frac{1}{2}\beta_r \delta_n^r$. Because the function $s \mapsto \mu(E \cap B((u, s), \delta_n))$ is continuous (443C, or otherwise), there is an $s_{n+1} \in G_u \cap \left[s_n, s'_{n+1}\right[$ such that $\mu(E \cap B((u, s), \delta_n)) \leq \frac{1}{2}\beta_r \delta_n^r$ whenever $s \in [s_{n+1}, s'_{n+1}]$.

This is the inductive step from even n. If n is odd and $s_n \in G_u$, $s'_n \in H_u$ and $s_n < s'_n$, set $s_{n+1} = \inf(H_u \cap [s_n, s'_n])$. This time we find that s_{n+1} must belong to G_u. Let $\delta_n \in]0, 2^{-n}]$ be such that $\mu(E \cap B((u, s_{n+1}), \delta_n)) > \frac{1}{2}\beta_r \delta_n^r$, and let $s'_{n+1} \in H_u \cap]s_{n+1}, s'_n]$ be such that $\mu(E \cap B((u, s), \delta_n)) \geq \frac{1}{2}\beta_r \delta_n^r$ whenever $s \in [s_{n+1}, s'_{n+1}]$.

The construction provides us with a non-increasing sequence $\langle [s_n, s'_n] \rangle_{n \in \mathbb{N}}$ of closed intervals, so there must be some s in their intersection. In this case

$$\mu(E \cap B((u, s), \delta_n) \leq \tfrac{1}{2}\beta_r \delta_n^r \text{ if } n \text{ is even},$$

$$\mu(E \cap B((u, s), \delta_n) \geq \tfrac{1}{2}\beta_r \delta_n^r \text{ if } n \text{ is odd}.$$

Since $\lim_{n \to \infty} \delta_n = 0$,

$$\liminf_{\delta \downarrow 0} \frac{\mu(E \cap B((u,s),\delta))}{\mu B((u,s),\delta)} \leq \frac{1}{2}, \quad \limsup_{\delta \downarrow 0} \frac{\mu(E \cap B((u,s),\delta))}{\mu B((u,s),\delta)} \geq \frac{1}{2},$$

and $s \in D_u$, while of course $t \leq s \leq t'$. **XQ**

(iv) Thus the conegligible set Z has the property required by (a).

(b) Now suppose that E has locally finite perimeter. Applying (a) to $\mathbb{R}^r \setminus E$, there is a conegligible set $Z_1 \subseteq \mathbb{R}^{r-1}$ such that if $u \in Z_1$, $t \in H_u$, $t' \in G_u$ and $t < t'$, then D_u meets $]t, t'[$. Next, we know that $\nu(\partial^* E \cap B(\mathbf{0}, n)) = \lambda_E^{\partial} B(\mathbf{0}, n)$ is finite for every $n \in \mathbb{N}$ (475G). By 475H,

$$\int_{\|u\| \leq n} \#(D_u \cap [-n, n]) \mu_{r-1}(du) \leq \nu(\partial^* E \cap B(\mathbf{0}, 2n)) < \infty$$

for every $n \in \mathbb{N}$; but this means that $D_u \cap [-n, n]$ must be finite for almost every u such that $\|u\| \leq n$, for every n, and therefore that

$$Z_2 = \{u : u \in Z \cap Z_1,\ D_u \cap [-n, n] \text{ is finite for every } n\}$$

is conegligible. For $u \in Z_2$, $\mathbb{R} \setminus D_u$ is an open set, so is made up of a disjoint sequence of intervals with endpoints in $D_u \cup \{-\infty, \infty\}$ (see 2A2I); and because $u \in Z \cap Z_1$, each of these intervals is included in either G_u or H_u. Now

$$A = \{u : \text{there are successive constituent intervals of } \mathbb{R} \setminus D_u \text{ both included in } G_u\}$$

is negligible. **P?** Otherwise, there are rationals $q < q'$ such that

$$F = \{u :]q, q'[\setminus G_u \text{ contains exactly one point}\}$$

is not negligible. Note that, writing $T : \mathbb{R}^r \to \mathbb{R}^{r-1}$ for the orthogonal projection,

$$F = T[(\mathbb{R}^{r-1} \times]q, q'[) \setminus \text{int}^* E] \setminus \bigcup_{q'' \in \mathbb{Q}, q < q'' < q'} T[(\mathbb{R}^{r-1} \times]q, q''[) \setminus \text{int}^* E] \cap T[(\mathbb{R}^{r-1} \times]q'', q'[) \setminus \text{int}^* E]$$

is measurable. Take any $u \in F \cap \text{int}^* F$, and let t be the unique point in $]q, q'[\setminus G_u$. Then whenever $0 < \delta \leq \min(t - q, q' - t)$ we shall have

$$\mu(B((u, t), \delta) \setminus E) = \mu(B((u, t), \delta) \setminus \text{int}^* E) \leq 2\delta \mu_{r-1}(V(u, \delta) \setminus F),$$

because if $w \in F$ then $(w, s) \in \text{int}^* E$ for almost every $s \in [t - \delta, t + \delta]$. So

$$\limsup_{\delta \downarrow 0} \frac{\mu(B((u,t),\delta) \setminus E)}{\mu B((u,t),\delta)} \leq \frac{2\beta_{r-1}}{\beta_r} \limsup_{\delta \downarrow 0} \frac{\mu_{r-1}(V(u,\delta) \setminus F)}{\mu_{r-1} V(u,\delta)} = 0,$$

and $(u, t) \in \text{int}^* E$. **XQ**

Similarly,

$$A' = \{u : \text{there are successive constituent intervals of } \mathbb{R} \setminus D_u \text{ both included in } H_u\}$$

is negligible. So $Z_2 \setminus (A \cup A')$ is a conegligible set of the kind we need.

475K Lemma Suppose that $h : \mathbb{R}^r \to [-1,1]$ is a Lipschitz function with compact support; let $n \in \mathbb{N}$ be such that $h(x) = 0$ for $\|x\| \geq n$. Suppose that $E \subseteq \mathbb{R}^r$ is a Lebesgue measurable set. Then

$$|\int_E \frac{\partial h}{\partial \xi_r} d\mu| \leq 2\big(\beta_{r-1} n^{r-1} + \nu(\partial^* E \cap B(\mathbf{0}, n))\big).$$

proof (a) Since (as noted in 473Cc) $\frac{\partial h}{\partial \xi_r}$ is defined almost everywhere, is measurable and bounded, and is zero outside $B(\mathbf{0}, n)$, the integral is well-defined. If $\nu(\partial^* E \cap B(\mathbf{0}, n))$ is infinite, the result is trivial; so henceforth let us suppose that $\nu(\partial^* E \cap B(\mathbf{0}, n)) < \infty$. Identify $\mathbb{R}^r$ with $\mathbb{R}^{r-1} \times \mathbb{R}$. For $u \in \mathbb{R}^{r-1}$, set

$$\begin{aligned} f(u) &= \#(\{t : (u,t) \in \partial^* E \cap B(\mathbf{0}, n)\}) \text{ if this is finite,} \\ &= \infty \text{ otherwise.} \end{aligned}$$

By 475H, $\int f d\mu_{r-1} \leq \nu(\partial^* E \cap B(\mathbf{0}, n))$. By 475J, there is a μ_{r-1}-conegligible set $Z \subseteq \mathbb{R}^{r-1}$ such that

whenever $u \in Z$ and $(u,t) \in \operatorname{int}^* E$ and $(u,t') \in \mathbb{R}^r \setminus \operatorname{cl}^* E$, there is an s lying between t and t' such that $(u,s) \in \partial^* E$;

we may suppose also that $f(u)$ is finite for every $u \in Z$. For $u \in Z$, set $D'_u = \{t : (u,t) \in B(\mathbf{0}, n) \setminus \partial^* E\}$, and define $g_u : D'_u \to \{0,1\}$ by setting

$$\begin{aligned} g_u(t) &= 1 \text{ if } u \in Z \text{ and } (u,t) \in B(\mathbf{0}, n) \cap \operatorname{int}^* E, \\ &= 0 \text{ if } u \in Z \text{ and } (u,t) \in B(\mathbf{0}, n) \setminus \operatorname{cl}^* E. \end{aligned}$$

Now if $t, t' \in D'_u$ and $g(t) \neq g(t')$, there is a point s between t and t' such that $(u,s) \in \partial^* E$; so the variation $\operatorname{Var}_{D_u} g_u$ of g_u (224A) is at most $f(u)$. Setting $h_u(t) = h(u,t)$ for $u \in \mathbb{R}^{r-1}$ and $t \in \mathbb{R}$, we now have

$$\int_{-\infty}^{\infty} \frac{\partial h}{\partial \xi_r}(u,t) \chi(B(\mathbf{0}, n) \cap \operatorname{int}^* E)(u,t) dt = \int_{D_u} h'_u(t) g_u(t) dt$$

(because $h'_u(t) = 0$ if $(u,t) \notin B(\mathbf{0}, n)$)

$$\leq \big(1 + \operatorname*{Var}_{D'_u} g_u\big) \sup_{a \leq b} |\int_a^b h'_u(t)| dt$$

(by 224J, recalling that D'_u is either empty or a bounded interval with finitely many points deleted)

$$\leq (1 + f(u)) \sup_{a \leq b} |h_u(b) - h_u(a)|$$

(because h_u is Lipschitz, therefore absolutely continuous on any bounded interval)

$$\leq 2(1 + f(u)).$$

Integrating over u, we now have

$$|\int_E \frac{\partial h}{\partial \xi_r} d\mu| = |\int_{B(\mathbf{0}, n) \cap \operatorname{int}^* E} \frac{\partial h}{\partial \xi_r} d\mu|$$

(475Cg)

$$= \int_{V(\mathbf{0}, n)} \int_{-\infty}^{\infty} \frac{\partial h}{\partial \xi_r}(u,t) \chi(B(\mathbf{0}, n) \cap \operatorname{int}^* E)(u,t) dt \, \mu_{r-1}(du)$$

(where $V(\mathbf{0}, n) = \{u : u \in \mathbb{R}^{r-1}, \|u\| \leq n\}$)

$$\leq \int_{V(\mathbf{0}, n)} 2(1 + f(u)) \mu_{r-1}(du)$$

$$\leq 2\big(\beta_{r-1} n^{r-1} + \nu(\partial^* E \cap B(\mathbf{0}, n))\big).$$

475L Theorem Suppose that $E \subseteq \mathbb{R}^r$. Then E has locally finite perimeter iff $\nu(\partial^*E \cap B(\mathbf{0},n))$ is finite for every $n \in \mathbb{N}$.

proof If E has locally finite perimeter, then $\nu(\partial^*E \cap B(\mathbf{0},n)) = \lambda_E^\partial B(\mathbf{0},n)$ is finite for every n, by 475G. So let us suppose that $\nu(\partial^*E \cap B(\mathbf{0},n))$ is finite for every $n \in \mathbb{N}$.

If $n \in \mathbb{N}$, then

$$\sup\{|\int_E \operatorname{div}\phi\, d\mu| : \phi : \mathbb{R}^r \to \mathbb{R}^r \text{ is Lipschitz, } \|\phi\| \leq \chi B(\mathbf{0},n)\} \leq 2r\big((\beta_{r-1}n^{r-1} + \nu(\partial^*E \cap B(\mathbf{0},n))\big)$$

is finite. **P** If $\phi = (\phi_1,\dots,\phi_r) : \mathbb{R}^r \to \mathbb{R}^r$ is a Lipschitz function and $\|\phi\| \leq \chi B(\mathbf{0},n)$, then 475K tells us that

$$|\int_E \frac{\partial\phi_r}{\partial\xi_r} d\mu| \leq 2\big(\beta_{r-1}n^{r-1} + \nu(\partial^*E \cap B(\mathbf{0},n))\big).$$

But of course it is equally true that

$$|\int_E \frac{\partial\phi_j}{\partial\xi_j} d\mu| \leq 2\big(\beta_{r-1}n^{r-1} + \nu(\partial^*E \cap B(\mathbf{0},n))\big)$$

for every $j \leq r$; adding, we have the result. **Q**

Since n is arbitrary, E has locally finite perimeter.

475M Corollary (a) The family of sets with locally finite perimeter is a subalgebra of the algebra of Lebesgue measurable subsets of $\mathbb{R}^r$.

(b) A set $E \subseteq \mathbb{R}^r$ has finite perimeter iff $\nu(\partial^*E) < \infty$, and in this case $\nu(\partial^*E)$ is the perimeter of E.

(c) If $E \subseteq \mathbb{R}^r$ has finite measure, then $\operatorname{per} E = \liminf_{\alpha\to\infty} \operatorname{per}(E \cap B(\mathbf{0},\alpha))$.

proof (a) Recall that the definition in 474D insists that sets with locally finite perimeter should be Lebesgue measurable. If $E \subseteq \mathbb{R}^r$ has locally finite perimeter, then so does $\mathbb{R}^r \setminus E$, by 474J. If E, $F \subseteq \mathbb{R}^r$ have locally finite perimeter, then

$$\nu(\partial^*(E \cup F) \cap B(\mathbf{0},n)) \leq \nu(\partial^*E \cap B(\mathbf{0},n)) + \nu(\partial^*F \cap B(\mathbf{0},n))$$

is finite for every $n \in \mathbb{N}$, by 475L and 475Cd; by 475L in the other direction, $E \cup F$ has locally finite perimeter.

(b) If E has finite perimeter (on the definition in 474D), then it is measurable and $\nu(\partial^*E) = \lambda_E^\partial\mathbb{R}^r$ is the perimeter of E, by 475G. If $\nu(\partial^*E) < \infty$, then $\mu(\partial^*E) = 0$ (471L) so E is measurable (475Ch); now E has locally finite perimeter, by 475L, and 475G again tells us that $\nu(\partial^*E) = \lambda_E^\partial\mathbb{R}^r$ is the perimeter of E.

(c) Now suppose that μE is finite. For any $\alpha \geq 0$,

$$\partial^*(E \cap B(\mathbf{0},\alpha)) \subseteq (\partial^*E \cap B(\mathbf{0},\alpha)) \cup \partial B(\mathbf{0},\alpha)$$

by 475Cf; since certainly

$$\partial^*(E \cap B(\mathbf{0},\alpha)) \subseteq \operatorname{cl}^*(E \cap B(\mathbf{0},\alpha)) \subseteq \operatorname{cl}^*E,$$

$$\partial^*(E \cap B(\mathbf{0},\alpha)) \subseteq \partial^*E \cup (\operatorname{cl}^*E \cap \partial B(\mathbf{0},\alpha)).$$

Now we know also that

$$\int_0^\infty \nu(\operatorname{cl}^*E \cap \partial B(\mathbf{0},\alpha))d\alpha = \mu E < \infty$$

(265G), so $\liminf_{\alpha\to\infty} \nu(\operatorname{cl}^*E \cap \partial B(\mathbf{0},\alpha)) = 0$. This means that

$$\liminf_{\alpha\to\infty} \operatorname{per}(E \cap B(\mathbf{0},\alpha)) = \liminf_{\alpha\to\infty} \nu(\partial^*(E \cap B(\mathbf{0},\alpha))) \leq \liminf_{\alpha\to\infty} \nu(\partial^*E) + \nu(\operatorname{cl}^*E \cap \partial B(\mathbf{0},\alpha)) = \nu(\partial^*E).$$

In the other direction,

$$\partial^*E \cap U(\mathbf{0},\alpha) = \partial^*E \cap \operatorname{int}^*B(\mathbf{0},\alpha) \subseteq \partial^*(E \cap B(\mathbf{0},\alpha))$$

for every α, by 475Cb, so

$$\begin{aligned}\operatorname{per} E = \nu(\partial^* E) &= \lim_{\alpha\to\infty} \nu(\partial^* E \cap U(\mathbf{0},\alpha)) \\ &\leq \liminf_{\alpha\to\infty} \nu(\partial^*(E\cap B(\mathbf{0},\alpha))) = \liminf_{\alpha\to\infty} \operatorname{per}(E\cap B(\mathbf{0},\alpha))\end{aligned}$$

and we must have equality.

Remark See 475Xj, 475Xk.

475N The Divergence Theorem Let $E \subseteq \mathbb{R}^r$ be such that $\nu(\partial^* E \cap B(\mathbf{0},n))$ is finite for every $n \in \mathbb{N}$.
(a) For ν-almost every $x \in \partial^* E$, there is a Federer exterior normal v_x of E at x.
(b) For every Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$ with compact support,

$$\int_E \operatorname{div}\phi \, d\mu = \int_{\partial^* E} \phi(x) \boldsymbol{.} v_x \, \nu(dx).$$

proof By 475L, E has locally finite perimeter. By 474Q, there is a Federer exterior normal $v_x = \psi_E(x)$ of E at x for every $x \in \partial^{\$} E$; by 475D, ν-almost every point in $\partial^* E$ is of this kind. By 474E-474F,

$$\int_E \operatorname{div}\phi \, d\mu = \int \phi \boldsymbol{.} \psi_E \, d\lambda_E^{\partial} = \int_{\partial^{\$} E} \phi \boldsymbol{.} \psi_E \, d\lambda_E^{\partial},$$

and this is also equal to $\int_{\partial^* E} \phi \boldsymbol{.} \psi_E \, d\lambda_E^{\partial}$, because $\partial^{\$} E \subseteq \partial^* E$. But λ_E^{∂} and ν induce the same subspace measures on $\partial^* E$, by 475G, so

$$\int_E \operatorname{div}\phi \, d\mu = \int_{\partial^* E} \phi \boldsymbol{.} \psi_E \, d\nu = \int_{\partial^* E} \phi(x) \boldsymbol{.} v_x \, \nu(dx),$$

as claimed.

475O Lemma Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and ψ_E its canonical outward normal. Let v be the unit vector $(0,\dots,0,1)$. Identify $\mathbb{R}^r$ with $\mathbb{R}^{r-1}\times\mathbb{R}$. Then we have sequences $\langle F_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ and $\langle g_n'\rangle_{n\in\mathbb{N}}$ such that
(i) for each $n \in \mathbb{N}$, F_n is a Lebesgue measurable subset of $\mathbb{R}^{r-1}$, and g_n, $g_n' : F_n \to [-\infty,\infty]$ are Lebesgue measurable functions such that $g_n(u) < g_n'(u)$ for every $u \in F_n$;
(ii) if m, $n \in \mathbb{N}$ then $g_m(u) \neq g_n'(u)$ for every $u \in F_m \cap F_n$;
(iii) $\sum_{n=0}^{\infty} \int_{F_n} g_n' - g_n d\mu_{r-1} = \mu E$;
(iv) for any continuous function $h : \mathbb{R}^r \to \mathbb{R}$ with compact support,

$$\int_{\partial^* E} h(x) v \boldsymbol{.} \psi_E(x) \, \nu(dx) = \sum_{n=0}^{\infty} \int_{F_n} h(u, g_n'(u)) - h(u, g_n(u)) \mu_{r-1}(du),$$

where we interpret $h(u,\infty)$ and $h(u,-\infty)$ as 0 if necessary;
(v) for μ_{r-1}-almost every $u \in \mathbb{R}^{r-1}$,

$$\begin{aligned}\{t : (u,t) \in \partial^* E\} = \{g_n(u) : n \in \mathbb{N},\, u \in F_n,\, g_n(u) \neq -\infty\} \\ \cup \{g_n'(u) : n \in \mathbb{N},\, u \in F_n,\, g_n'(u) \neq \infty\}.\end{aligned}$$

proof (a) Take a conegligible set $Z \subseteq \mathbb{R}^{r-1}$ as in 475Jb. It will be convenient to shrink Z down further, if necessary, to a conegligible Borel set. For $u \in Z$ set $D_u = \{t : (u,t) \in \partial^* E\}$.

(b) For each $q \in \mathbb{Q}$, set $F_q' = \{u : u \in Z,\, (u,q) \in \operatorname{int}^* E\}$, so that F_q' is a Borel set, and for $u \in F_q'$ set

$$f_q(u) = \sup(D_u \cap]-\infty, q[), \quad f_q'(u) = \inf(D_u \cap]-\infty, q[),$$

allowing $-\infty$ as $\sup\emptyset$ and ∞ as $\inf\emptyset$. Observe that (because $D_u \cap [q-1, q+1]$ is finite) $f_q(u) < q < f_q'(u)$. Now f_q and f_q' are measurable, by the argument of part (a-i) of the proof of 475J. Enumerate $\mathbb{Q}$ as $\langle q_n\rangle_{n\in\mathbb{N}}$, and set

$$F_n = F_{q_n} \setminus \bigcup_{m<n} \{u : u \in F_{q_m},\, f_{q_m}(u) < q_n < f_{q_m}'(u)\}$$

for $n \in \mathbb{N}$. Set $g_n(u) = f_{q_n}(u)$, $g_n'(u) = f_{q_n}'(u)$ for $u \in F_n$.

The effect of this construction is to ensure that, for any $u \in Z$, $u \in F_{q_n}$ iff q_n is the first rational lying in one of those constituent intervals I of $\mathbb{R} \setminus D_u$ such that $\{u\} \times I \subseteq \operatorname{int}^* E$, and that now $g_n(u)$ and $g_n'(u)$ are the endpoints of that interval, allowing $\pm\infty$ as endpoints.

(c) Now let us look at the items (i)-(v) of the statement of this lemma. We have already achieved (i). If $u \in F_m \cap F_n$, then $g_m(u)$ is the infimum of one of the constituent intervals of G_u, and $g'_n(u)$ is either ∞ or the infimum of one of the constituent intervals of H_u; so $g_m(u) \neq g'_n(u)$. Thus (ii) is true. For any $u \in Z$,

$$\sum_{n\in\mathbb{N},u\in F_n} g'_n(u) - g_n(u) = \mu_1\{t : (u,t) \in \text{int}^*E\},$$

so

$$\sum_{n=0}^{\infty}\int_{F_n} g'_n - g_n d\mu_{r-1} = \int_Z \mu_1\{t : (u,t) \in \text{int}^*E\}\mu_{r-1}(du) = \mu(\text{int}^*E) = \mu E.$$

So (iii) is true. Also, for $u \in Z$,

$$\begin{aligned}\{t : (u,t) \in \partial^*E\} &= D_u \\ &= \{g_n(u) : n \in \mathbb{N},\, u \in F_n,\, g_n(u) \neq -\infty\} \\ &\qquad \cup \{g'_n(u) : n \in \mathbb{N},\, u \in F_n,\, g'_n(u) \neq \infty\},\end{aligned}$$

so (v) is true.

(d) As for (iv), suppose first that $h : \mathbb{R}^r \to \mathbb{R}$ is a Lipschitz function with compact support. Set $\phi(x) = (0, \dots, h(x))$ for $x \in \mathbb{R}^r$. Then

$$\begin{aligned}\int_{\partial^*E} h(x)v.\psi_E(x)\,\nu(dx) &= \int h(x)v.\psi_E(x)\,\lambda_E^{\partial}(dx) \\ &\text{(475G)} \\ &= \int \phi.\psi_E\,\lambda_E^{\partial}(dx) = \int_E \operatorname{div}\phi\,d\mu \\ &\text{(474E)} \\ &= \int_E \frac{\partial h}{\partial\xi_r}\,d\mu = \int_{\text{int}^*E} \frac{\partial h}{\partial\xi_r}\,d\mu \\ &= \int \sum_{n\in\mathbb{N},u\in F_n} \int_{g_n(u)}^{g'_n(u)} \frac{\partial h}{\partial\xi_r}(u,t)dt\,\mu_{r-1}(du) \\ &= \int \sum_{n\in\mathbb{N},u\in F_n} h(u,g'_n(u)) - h(u,g_n(u))\mu_{r-1}(du) \\ &\text{(with the convention that } h(u,\pm\infty) = \lim_{t\to\pm\infty} h(u,t) = 0) \\ &= \sum_{n=0}^{\infty}\int_{F_n} h(u,g'_n(u)) - h(u,g_n(u))\mu_{r-1}(du)\end{aligned}$$

because if $|h| \leq m\chi B(\mathbf{0},m)$ then

$$\begin{aligned}\int \sum_{n\in\mathbb{N},u\in F_n} &|h(u,g'_n(u)) - h(u,g_n(u))|\mu_{r-1}(du) \\ &\leq 2m\int \#(\{t : (u,t) \in B(\mathbf{0},m) \cap \partial^*E\})\mu_{r-1}(du) \\ &\leq 2m\nu(B(\mathbf{0},m) \cap \partial^*E)\end{aligned}$$

(475H) is finite.

For a general continuous function h of compact support, consider the convolutions $h_k = h * \tilde{h}_k$ for large k, where $\tilde{h}_k$ is defined in 473E. If $|h| \leq m\chi B(\mathbf{0},m)$ then $|h_k| \leq m\chi B(\mathbf{0},m+1)$ for every k, so that

$$\sum_{n\in\mathbb{N},u\in F_n} |h_k(g'_n(u)) - h_k(g_n(u))| \leq 2m\#(\{t : (u,t) \in B(\mathbf{0},m+1) \cap \partial^*E\})$$

for every $u \in \mathbb{R}^r$, $k \in \mathbb{N}$. Since $h_k \to h$ uniformly (473Ed),

$$\begin{aligned}\int_{\partial^*E} h(x)v\boldsymbol{.}\psi_E(x)\,\nu(dx) &= \lim_{k\to\infty}\int_{\partial^*E} h_k(x)v\boldsymbol{.}\psi_E(x)\,\nu(dx)\\ &= \lim_{k\to\infty}\int \sum_{n\in\mathbb{N},u\in F_n} h_k(u,g_n'(u)) - h_k(u,g_n(u))\mu_{r-1}(du)\\ &= \int \sum_{n\in\mathbb{N},u\in F_n} h(u,g_n'(u)) - h(u,g_n(u))\mu_{r-1}(du)\\ &= \sum_{n=0}^{\infty}\int_{F_n} h(u,g_n'(u)) - h(u,g_n(u))\mu_{r-1}(du),\end{aligned}$$

as required.

475P Lemma Let $v \in S_{r-1}$ be any unit vector, and $V \subseteq \mathbb{R}^r$ the hyperplane $\{x : x\boldsymbol{.}v = 0\}$. Let $T : \mathbb{R}^r \to V$ be the orthogonal projection. If $E \subseteq \mathbb{R}^r$ is any set with locally finite perimeter and canonical outward normal ψ_E, then

$$\int_{\partial^*E} |v\boldsymbol{.}\psi_E|d\nu = \int_V \#(\partial^*E \cap T^{-1}[\{u\}])\nu(du).$$

proof As usual, we may suppose that the structure (E, v) is rotated until $v = (0, \dots, 1)$, so that we can identify $T(\xi_1, \dots, \xi_r)$ with $(\xi_1, \dots, \xi_{r-1}) \in \mathbb{R}^{r-1}$, and $\int_V \#(\partial^*E \cap T^{-1}[\{u\}])\nu(du)$ with $\int_{\mathbb{R}^{r-1}} \#(D_u)\mu_{r-1}(du)$, where $D_u = \{t : (u,t) \in \partial^*E\}$.

(a) Write Φ for the set of continuous functions $h : \mathbb{R}^r \to [-1,1]$ with compact support. Then

$$\int_{\partial^*E} |v\boldsymbol{.}\psi_E|d\nu = \sup_{h\in\Phi}\int_{\partial^*E} h(x)v\boldsymbol{.}\psi_E(x)\,\nu(dx).$$

P Of course

$$\int_{\partial^*E} |v\boldsymbol{.}\psi_E|d\nu \geq \int_{\partial^*E} h(x)v\boldsymbol{.}\psi_E(x)\,\nu(dx)$$

for any $h \in \Phi$. On the other hand, if

$$\gamma < \int_{\partial^*E} |v\boldsymbol{.}\psi_E|d\nu = \int |v\boldsymbol{.}\psi_E|d\lambda_E^{\partial}$$

(475G), then, because λ_E^{∂} is a Radon measure, there is an $n \in \mathbb{N}$ such that

$$\gamma < \int_{B(\mathbf{0},n)} |v\boldsymbol{.}\psi_E|d\lambda_E^{\partial} = \int h_0(x)v\boldsymbol{.}\psi_E(x)\,\lambda_E^{\partial}(dx)$$

where

$$\begin{aligned}h_0(x) &= \frac{v\boldsymbol{.}\psi_E(x)}{|v\boldsymbol{.}\psi_E(x)|} \text{ if } x \in B(\mathbf{0},n) \text{ and } v\boldsymbol{.}\psi_E(x) \neq 0\\ &= 0 \text{ otherwise.}\end{aligned}$$

Because λ_E^{∂} is a Radon measure, there is a continuous function $h_1 : \mathbb{R}^r \to \mathbb{R}$ with compact support such that

$$\int |h_1 - h_0|d\lambda_E^{\partial} \leq \int_{B(\mathbf{0},n)} |v\boldsymbol{.}\psi_E|d\lambda_E^{\partial} - \gamma$$

(416I); of course we may suppose that h_1, like h, takes values in $[-1,1]$, so that $h_1 \in \Phi$. Now

$$\int_{\partial^*E} h_1(x)v\boldsymbol{.}\psi_E(x)\nu(dx) = \int h_1(x)v\boldsymbol{.}\psi_E(x)\,\lambda_E^{\partial}(dx) \geq \gamma.$$

As γ is arbitrary,

$$\sup_{h\in\Phi}\int h(x)v\boldsymbol{.}\psi_E(x)\,\nu(dx) = \int |v\boldsymbol{.}\psi_E|d\nu$$

as required. **Q**

(b) Now take $\langle F_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ and $\langle g_n'\rangle_{n\in\mathbb{N}}$ as in 475O. Then

$$\int \#(D_u)\mu_{r-1}(du) = \sup_{h\in\Phi}\textstyle\sum_{n=0}^{\infty}\int_{F_n} h(u,g_n'(u)) - h(u,g_n(u))\mu_{r-1}(du).$$

P By 475O(ii), $g'_m(u) \neq g_n(u)$ whenever $m, n \in \mathbb{N}$ and $u \in F_m \cap F_n$, while for almost all $u \in \mathbb{R}^{r-1}$

$$D_u = (\{g_n(u) : n \in \mathbb{N},\, u \in F_n\} \setminus \{-\infty\}) \cup (\{g'_n(u) : n \in \mathbb{N},\, u \in F_n\} \setminus \{\infty\}).$$

So if $h \in \Phi$,

$$\begin{aligned}\sum_{n\in\mathbb{N},u\in F_n} h(u,g'_n(u)) - h(u,g_n(u)) &\leq \#(\{g_n(u) : u \in F_n,\, g_n(u) \neq -\infty\}) \\ &\quad + \#(\{g'_n(u) : u \in F_n,\, g'_n(u) \neq \infty\}) \\ &= \#(D_u)\end{aligned}$$

for almost every $u \in \mathbb{R}^{r-1}$, and

$$\textstyle\sum_{n=0}^{\infty} \int_{F_n} h(u,g'_n(u)) - h(u,g_n(u))\mu_{r-1}(du) \leq \int \#(D_u)\mu_{r-1}(du).$$

In the other direction, given $\gamma < \int \#(D_u)\mu_{r-1}(du)$, there is an $m \in \mathbb{N}$ such that $\gamma < \int \#(D_u^{(m)})\mu_{r-1}(du)$, where $D_u^{(m)} = \{t : (u,t) \in \partial^* E,\, \|(u,t)\| < m\}$. Note that $\int \#(D_u^{(m)})\mu_{r-1}(du) \leq \nu(\partial^* E \cap B(\mathbf{0},m))$ is finite. Setting

$$H_n = \{u : u \in F_n,\, g_n(u) \in D_u^{(m)}\}, \quad H'_n = \{u : u \in F_n,\, g'_n(u) \in D_u^{(m)}\}$$

for $n \in \mathbb{N}$, we have

$$\gamma < \textstyle\sum_{n=0}^{\infty} \mu_{r-1}H_n + \sum_{n=0}^{\infty} \mu_{r-1}H'_n < \infty.$$

By Lusin's theorem (418J), we can find $n \in \mathbb{N}$ and compact sets $K_i \subseteq H_i$, $K'_i \subseteq H'_i$ such that $g_i{\restriction}K_i$ and $g'_i{\restriction}K'_i$ are continuous for every i and

$$\begin{aligned}\gamma &\leq \sum_{i=0}^{n} (\mu_{r-1}K_i + \mu_{r-1}K'_i) - \sum_{i=n+1}^{\infty} (\mu_{r-1}H_i + \mu_{r-1}H'_i) \\ &\quad - \sum_{i=0}^{n} (\mu_{r-1}(H_i \setminus K_i) + \mu_{r-1}(H'_i \setminus K'_i)).\end{aligned}$$

Set

$$K = \textstyle\bigcup_{i\leq n} \{(u, g_i(u)) : u \in K_i\}, \quad K' = \bigcup_{i\leq n} \{(u, g'_i(u)) : u \in K'_i\},$$

so that K and K' are disjoint compact subsets of $\operatorname{int} B(\mathbf{0},m)$. Let $h : \mathbb{R}^r \to \mathbb{R}$ be a continuous function such that

$$\begin{aligned}h(x) &= 1 \text{ for } x \in K', \\ &= -1 \text{ for } x \in K, \\ &= 0 \text{ for } x \in \mathbb{R}^r \setminus \operatorname{int} B(\mathbf{0},m)\end{aligned}$$

(4A2F(d-viii)); we can suppose that $-1 \leq h(x) \leq 1$ for every x. Then $h \in \Phi$, and

$$\begin{aligned}&\sum_{i=0}^{\infty} \int_{F_i} h(u,g'_i(u)) - h(u,g_i(u))\mu_{r-1}(du) \\ &\qquad = \sum_{i=0}^{\infty} \int_{H'_i} h(u,g'_i(u))\mu_{r-1}(du) - \sum_{i=0}^{\infty} \int_{H_i} h(u,g_i(u))\mu_{r-1}(du) \\ &\text{(because } h \text{ is zero outside } \operatorname{int} B(\mathbf{0},m)) \\ &\qquad \geq \sum_{i=0}^{n} \int_{K'_i} h(u,g'_i(u))\mu_{r-1}(du) - \sum_{i=0}^{n} \int_{K_i} h(u,g_i(u))\mu_{r-1}(du) \\ &\qquad\quad - \sum_{i=0}^{n} (\mu(H'_i \setminus K'_i) + \mu(H_i \setminus K_i)) - \sum_{i=n+1}^{\infty} (\mu H'_i + \mu H_i) \\ &\qquad \geq \gamma.\end{aligned}$$

This shows that

$$\int \#(D_u)\mu_{r-1}(du) \leq \sup_{h\in\Phi}\sum_{n=0}^{\infty}\int_{F_n} h(u,g_n'(u)) - h(u,g_n(u))\mu_{r-1}(du),$$

and we have equality. **Q**

(c) Putting (a) and (b) together with 475O(iv), we have the result.

475Q Theorem Let $E \subseteq \mathbb{R}^r$ be a set with finite perimeter. For $v \in S^{r-1}$ write V_v for $\{x : x.v = 0\}$, and let $T_v : \mathbb{R}^r \to V_v$ be the orthogonal projection. Then

$$\nu(\partial^*E) = \frac{1}{2\beta_{r-1}}\int_{S_{r-1}}\int_{V_v} \#(\partial^*E \cap T_v^{-1}[\{u\}])\nu(du)\nu(dv).$$

proof (a) I start with an elementary fact: there is a constant c such that $\int |w.v|\lambda^{\partial}_{S_{r-1}}(dv) = c$ for every $w \in S_{r-1}$; this is because whenever w, $w' \in S_{r-1}$ there is an orthogonal linear transformation taking w to w', and this transformation is an automorphism of the structure $(\mathbb{R}^r, \nu, S_{r-1}, \lambda^{\partial}_{S_{r-1}})$ (474H). (Later in the proof I will come to the calculation of c.)

(b) Now, writing ψ_E for the canonical outward normal of E, we have

$$\int_{S_{r-1}}\int_{V_v} \#(\partial^*E \cap T_v^{-1}[\{u\}])\nu(du)\nu(dv) = \int_{S_{r-1}}\int_{\partial^*E} |\psi_E(x).v|\nu(dx)\nu(dv)$$

(475P)

$$= \iint |\psi_E(x).v|\lambda^{\partial}_E(dx)\lambda^{\partial}_{S_{r-1}}(dv)$$

(by 235M, recalling that $\lambda^{\partial}_{S_{r-1}}$ and λ^{∂}_E are just indefinite-integral measures over ν, while S_{r-1} and ∂^*E are Borel sets)

$$= \iint |\psi_E(x).v|\lambda^{\partial}_{S_{r-1}}(dv)\lambda^{\partial}_E(dx)$$

(because $.$ is continuous, so $(x,v) \mapsto \psi_E(x).v$ is measurable, while $\lambda^{\partial}_{S_{r-1}}$ and λ^{∂}_E are totally finite, so we can use Fubini's theorem)

$$= c\operatorname{per} E$$

(by (a) above)

$$= c\nu(\partial^*E).$$

(c) We have still to identify the constant c. But observe that the argument above applies whenever $\nu(\partial^*E)$ is finite, and in particular applies to $E = B(\mathbf{0},1)$. In this case, $\partial^*B(\mathbf{0},1) = S_{r-1}$, and for any $v \in S_{r-1}$, $u \in V_v$ we have

$$\begin{aligned}\#(\partial^*B(\mathbf{0},1) \cap T^{-1}[\{u\}]) &= 2 \text{ if } \|u\| < 1,\\ &= 1 \text{ if } \|u\| = 1,\\ &= 0 \text{ if } \|u\| > 1.\end{aligned}$$

Since we can identify ν on V_v as a copy of Lebesgue measure μ_{r-1},

$$\int_{V_v} \#(\partial^*B(\mathbf{0},1) \cap T^{-1}[\{u\}])\nu(du) = 2\nu\{u : u \in V_v, \|u\| < 1\} = 2\beta_{r-1}.$$

This is true for every $v \in V_u$, so from (b) we get

$$2\beta_{r-1}\nu S_{r-1} = \int_{S_{r-1}}\int_{V_v} \#(\partial^*E \cap T_v^{-1}[\{u\}])\nu(du)\nu(dv) = c\nu S_{r-1},$$

and $c = 2\beta_{r-1}$. Substituting this into (b), we have the result.

475R Convex sets in $\mathbb{R}^r$ For the next theorem it will help to have some elementary facts about convex sets in finite-dimensional spaces out in the open.

Lemma (In this lemma I allow $r = 1$.) Let $C \subseteq \mathbb{R}^r$ be a convex set.
(a) If $x \in C$ and $y \in \operatorname{int} C$, then $ty + (1-t)x \in \operatorname{int} C$ for every $t \in]0,1]$.
(b) $\overline{C}$ and $\operatorname{int} C$ are convex.
(c) If $\operatorname{int} C \neq \emptyset$ then $\overline{C} = \overline{\operatorname{int} C}$.
(d) If $\operatorname{int} C = \emptyset$ then C lies within some hyperplane.
(e) $\operatorname{int} \overline{C} = \operatorname{int} C$.

proof (a) Setting $\phi(z) = x + t(z-x)$ for $z \in \mathbb{R}^r$, $\phi : \mathbb{R}^r \to \mathbb{R}^r$ is a homeomorphism and $\phi[C] \subseteq C$, so

$$\phi(y) \in \phi[\operatorname{int} C] = \operatorname{int} \phi[C] \subseteq \operatorname{int} C.$$

(b) It follows at once from (a) that $\operatorname{int} C$ is convex; $\overline{C}$ is convex because $(x,y) \mapsto tx + (1-t)y$ is continuous for every $t \in [0,1]$.

(c) From (a) we see also that if $\operatorname{int} C \neq \emptyset$ then $C \subseteq \overline{\operatorname{int} C}$, so that $\overline{C} \subseteq \overline{\operatorname{int} C}$ and $\overline{C} = \overline{\operatorname{int} C}$.

(d) It is enough to consider the case in which $\mathbf{0} \in C$, since if $C = \emptyset$ the result is trivial. **?** If $x_1, \dots, x_r$ are linearly independent elements of C, set $x = \frac{1}{r+1}\sum_{i=1}^r x_i$; then

$$x + \textstyle\sum_{i=1}^r \alpha_i x_i = \sum_{i=1}^r (\alpha_i + \frac{1}{r+1}) x_i \in C$$

whenever $\sum_{i=1}^r |\alpha_i| \leq \frac{1}{r+1}$. Also, writing $e_1, \dots, e_r$ for the standard orthonormal basis of $\mathbb{R}^r$, we can express e_i as $\sum_{j=1}^r \alpha_{ij} x_j$ for each j; setting $M = (r+1)\max_{i \leq r} \sum_{j=1}^r |\alpha_{ij}|$, we have $x \pm \frac{1}{M} e_i \in C$ for every $i \leq r$, so that $x + y \in C$ whenever $\|y\| \leq \frac{\sqrt{r}}{M}$, and $x \in \operatorname{int} C$. **X**

So the linear subspace of $\mathbb{R}^r$ spanned by C has dimension at most $r-1$.

(e) If $\operatorname{int} C$ is empty, then (d) shows that C is included in a hyperplane, so that $\operatorname{int} \overline{C}$ is empty. Otherwise, if $x \in \mathbb{R}^r \setminus \operatorname{int} C$, there is an $e \in \mathbb{R}^r$ such that $e \cdot y \leq e \cdot x$ for every $y \in \operatorname{int} C$ (4A4Db, or otherwise). Now, by (c), $e \cdot y \leq e \cdot x$ for every $y \in \overline{C}$, so $x \notin \operatorname{int} \overline{C}$. This shows that $\operatorname{int} \overline{C} \subseteq \operatorname{int} C$, so that the two are equal.
It follows at once that $\partial C = \partial \overline{C}$.

475S Corollary: Cauchy's Perimeter Theorem Let $C \subseteq \mathbb{R}^r$ be a bounded convex set with non-empty interior. For $v \in S^{r-1}$ write V_v for $\{x : x \cdot v = 0\}$, and let $T_v : \mathbb{R}^r \to V_v$ be the orthogonal projection. Then

$$\nu(\partial C) = \frac{1}{\beta_{r-1}} \int_{S_{r-1}} \nu(T_v[C]) \nu(dv).$$

proof (a) The first thing to note is that $\partial^* C = \partial C$. **P** Of course $\partial^* C \subseteq \partial C$ (475Ca). If $x \in \partial C$, there is a half-space V containing x and disjoint from $\operatorname{int} C$ (4A4Db), so that

$$\limsup_{\delta \downarrow 0} \frac{\mu(B(x,\delta) \setminus C)}{\mu B(x,\delta)} \geq \limsup_{\delta \downarrow 0} \frac{\mu(B(x,\delta) \setminus \operatorname{int} V)}{\mu B(x,\delta)} = \frac{1}{2},$$

and $x \notin \operatorname{int}^* C$. On the other hand, if $\eta > 0$ is such that $B(x_0, \eta) \subseteq \operatorname{int} C$, then for any $\delta > 0$ we can set $t = \frac{\delta}{\eta + \|x_0 - x\|}$, and then

$$B(x + t(x_0 - x), t\eta) \subseteq B(x,\delta) \cap \operatorname{int} \overline{C} \subseteq B(x,\delta) \cap C,$$

so that

$$\mu(B(x,\delta) \cap C) \geq \beta_r t^r \eta^r = \big(\frac{\eta}{\|x_0 - x\| + \eta}\big)^r \mu B(x,\delta);$$

as δ is arbitrary, $x \in \operatorname{cl}^* C$. This shows that $\partial C \subseteq \partial^* C$ so that $\partial^* C = \partial C$. **Q**

(b) We have a function $\phi : \mathbb{R}^r \to \overline{C}$ defined by taking $\phi(x)$ to be the unique point of C closest to x, for every $x \in \mathbb{R}^r$ (3A5Ld). This function is 1-Lipschitz. **P** Take any x, $y \in \mathbb{R}^r$ and set $e = \phi(x) - \phi(y)$. We know that $\phi(x) - \epsilon e \in \overline{C}$, so that $\|x - \phi(x) - \epsilon e\| \geq \|x - \phi(x)\|$, for $0 \leq \epsilon \leq 1$; it follows that $(x - \phi(x)) \cdot e \geq 0$. Similarly, $(y - \phi(y)) \cdot (-e) \geq 0$. Accordingly $(x - y) \cdot e \geq e \cdot e$ and $\|x - y\| \geq \|e\|$. As x and y are arbitrary, ϕ is 1-Lipschitz. **Q**

Now suppose that that $C' \supseteq C$ is a closed bounded convex set. Then $\nu(\partial C') \geq \nu(\partial C)$. **P** Let ϕ be the function defined just above. By 264G/471J, $\nu^*(\phi[\partial C']) \leq \nu(\partial C')$. But if $x \in \partial C$, there is an $e \in \mathbb{R}^r \setminus \{0\}$ such that $x \cdot e \geq y \cdot e$ for every $y \in C$. Then $\phi(x + \alpha e) = x$ for every $\alpha \geq 0$. Because C' is closed and bounded, and $x \in C \subseteq C'$, there is a greatest $\alpha \geq 0$ such that $x + \alpha e \in C'$, and in this case $x + \alpha e \in \partial C'$; since $\phi(x + \alpha e) = x$, $x \in \phi[\partial C']$. As x is arbitrary, $\partial C \subseteq \phi[\partial C']$, and

$$\nu(\partial C) \leq \nu^*(\phi[\partial C']) \leq \nu(\partial C'). \quad \textbf{Q}$$

Since we can certainly find a closed convex set $C' \supseteq C$ such that $\nu(\partial C')$ is finite (e.g., any sufficiently large ball or cube), $\nu(\partial C) < \infty$. It follows at once that $\mu(\partial C) = 0$ (471L).

(c) The argument so far applies, of course, to every $r \geq 1$ and every bounded convex set with non-empty interior in $\mathbb{R}^r$. Moving to the intended case $r \geq 2$, and fixing $v \in S_{r-1}$ for the moment, we see that, because T_v is an open map (if we give V_v its subspace topology), $T_v[C]$ is again a bounded convex set with non-empty (relative) interior. Since the subspace measure induced by ν on V is just a copy of Lebesgue measure, (b) tells us that $\nu T_v[C] = \nu(\operatorname{int} T_v[C])$, where here I write $\operatorname{int} T_v[C]$ for the interior of $T_v[C]$ in the subspace topology of V_v. Now the point is that $\operatorname{int} T_v[C] \subseteq T_v[\operatorname{int} C]$. **P** $\operatorname{int} C$ is dense in C (475Rc), so $W = T_v[\operatorname{int} C]$ is an open convex set which is dense in $T_v[C]$; now $W = \operatorname{int} \overline{W}$, taking the interior in V (475Re), so $W \supseteq \operatorname{int} T_v[C]$. **Q**

It follows that $\#(\partial C \cap T_v^{-1}[\{u\}]) = 2$ for every $u \in \operatorname{int} T_v[C]$. **P** $T_v^{-1}[\{u\}]$ is a straight line meeting $\operatorname{int} C$ in y_0 say. Because $\overline{C}$ is a bounded convex set, it meets $T_v^{-1}[\{u\}]$ in a bounded convex set, which must be a non-trivial closed line segment with endpoints y_1, y_2 say. Now certainly neither y_1 nor y_2 can be in the interior of C. Moreover, the open line segments between y_1 and y_0, and between y_2 and y_0, are covered by $\operatorname{int} C$, by 475Ra; so $T_v^{-1}[\{u\}] \cap \partial C = \{y_1, y_2\}$ has just two members. **Q**

(d) This is true for every $v \in S_{r-1}$. But this means that we can apply 475Q to see that

$$\begin{aligned}
\nu(\partial C) = \nu(\partial^* C) &= \frac{1}{2\beta_{r-1}} \int_{S_{r-1}} \int_{V_v} \#(\partial^* E \cap T_v^{-1}[\{u\}]) \nu(du) \nu(dv) \\
&= \frac{1}{2\beta_{r-1}} \int_{S_{r-1}} \int_{T_v[C]} \#(\partial^* E \cap T_v^{-1}[\{u\}]) \nu(du) \nu(dv) \\
&= \frac{1}{2\beta_{r-1}} \int_{S_{r-1}} \int_{\operatorname{int} T_v[C]} \#(\partial^* E \cap T_v^{-1}[\{u\}]) \nu(du) \nu(dv) \\
&= \frac{1}{\beta_{r-1}} \int_{S_{r-1}} \nu(\operatorname{int} T_v[C]) \nu(dv) = \frac{1}{\beta_{r-1}} \int_{S_{r-1}} \nu T_v[C] \nu(dv),
\end{aligned}$$

as required.

475T Corollary: the Convex Isoperimetric Theorem If $C \subseteq \mathbb{R}^r$ is a bounded convex set, then $\nu(\partial C) \leq r\beta_r (\frac{1}{2} \operatorname{diam} C)^{r-1}$.

proof (a) If C is included in some $r-1$-dimensional affine subspace, then

$$\nu(\partial C) = \nu \overline{C} \leq \beta_{r-1} (\tfrac{1}{2} \operatorname{diam} C)^{r-1}$$

by 264H. For completeness, I should check that $\beta_{r-1} \leq r\beta_r$. **P** Comparing 265F with 265H, or working from the formulae in 252Q, we have $r\beta_r = 2\pi\beta_{r-2}$. On the other hand, by the argument of 252Q,

$$\beta_{r-1} = \beta_{r-2} \int_{-\pi/2}^{\pi/2} \cos^{r-1} t \, dt \leq \pi \beta_{r-2},$$

so (not coincidentally) we have a factor of two to spare. **Q**

(b) Otherwise, C has non-empty interior (475Rd), and for any orthogonal projection T of $\mathbb{R}^r$ onto an $r-1$-dimensional linear subspace, $\operatorname{diam} T[C] \leq \operatorname{diam} C$, so $\nu(T[C]) \leq \beta_{r-1}(\frac{1}{2}\operatorname{diam} C)^{r-1}$. Now 475S tells us that

$$\nu(\partial C) \leq (\tfrac{1}{2}\operatorname{diam} C)^{r-1}\nu S_{r-1} = r\beta_r(\tfrac{1}{2}\operatorname{diam} C)^{r-1}.$$

475X Basic exercises (a) Show that if $C \subseteq \mathbb{R}^r$ is convex, then either $\mu C = 0$ and $\partial^* C = \emptyset$, or $\partial^* C = \partial C$.

(b) Let A, $A' \subseteq \mathbb{R}^r$ be any sets. Show that

$$(\partial^* A \cap \operatorname{int}^* A') \cup (\partial^* A' \cap \operatorname{int}^* A) \subseteq \partial^*(A \cap A') \subseteq (\partial^* A \cap \operatorname{cl}^* A') \cup (\partial^* A' \cap \operatorname{cl}^* A).$$

(c) Let $A \subseteq \mathbb{R}^r$ be any set, and B a non-trivial closed ball. Show that

$$\partial^*(A \cap B) \triangle ((B \cap \partial^* A) \cup (A \cap \partial B)) \subseteq ((\operatorname{cl}^* A \setminus A) \cup (A \setminus \operatorname{int}^* A)) \cap \partial B.$$

>(d) Let E, $F \subseteq \mathbb{R}^r$ be measurable sets, and v a Federer exterior normal to E at $x \in \operatorname{int}^* F$. Show that v is a Federer exterior normal to $E \cap F$ at x.

(e) Let $\mathfrak{T}$ be the density topology on $\mathbb{R}^r$ (414P) defined from lower Lebesgue density (341E). Show that, for any $A \subseteq \mathbb{R}^r$, $A \cup \operatorname{cl}^* A$ is the $\mathfrak{T}$-closure of A and $\operatorname{int}^* A$ is the $\mathfrak{T}$-interior of the $\mathfrak{T}$-closure of A.

(f) Let $A \subseteq \mathbb{R}^r$ be any set. Show that $A \setminus \operatorname{cl}^* A$ and $\operatorname{int}^* A \setminus A$ are Lebesgue negligible.

>(g) Let $E \subseteq \mathbb{R}^r$ be such that $\nu(\partial^* E)$ and μE are both finite. Show that, taking v_x to be a Federer exterior normal to E at any point x where this is defined,

$$\int_E \operatorname{div} \phi \, d\mu = \int_{\partial^* E} \phi(x) \boldsymbol{.} v_x \, \nu(dx).$$

for every bounded Lipschitz function $\phi : \mathbb{R}^r \to \mathbb{R}^r$.

>(h) Let $\langle E_n \rangle_{n \in \mathbb{N}}$ be a sequence of measurable subsets of $\mathbb{R}^r$ such that (i) there is a measurable set E such that $\lim_{n\to\infty} \mu((E_n \triangle E) \cap B(\mathbf{0}, m)) = 0$ for every $m \in \mathbb{N}$ (ii) $\sup_{n\in\mathbb{N}} \nu(\partial^* E_n \cap B(\mathbf{0}, m))$ is finite for every $m \in \mathbb{N}$. Show that E has locally finite perimeter. (*Hint*: $\int_E \operatorname{div} \phi \, d\mu = \lim_{n\to\infty} \int_{E_n} \operatorname{div} \phi \, d\mu$ for every Lipschitz function ϕ with compact support.)

(i) Give an example of bounded convex sets E and F such that $\partial^{\$}(E \cup F) \not\subseteq \partial^{\$} E \cup \partial^{\$} F$.

(j) (i) Show that if A, $B \subseteq \mathbb{R}^r$ then $\partial^*(A \cap B) \cap \partial^*(A \cup B) \subseteq \partial^* A \cap \partial^* B$. (ii) Show that if E, $F \subseteq \mathbb{R}$ have finite perimeter then $\operatorname{per}(E \cap F) + \operatorname{per}(E \cup F) \leq \operatorname{per} E + \operatorname{per} F$.

(k) Let $E \subseteq \mathbb{R}^r$ be a set with finite Lebesgue measure and finite perimeter. (i) Show that if $H \subseteq \mathbb{R}^r$ is a half-space, then $\operatorname{per}(E \cap H) \leq \operatorname{per} E$. (*Hint*: 475Ja.) (ii) Show that if $C \subseteq \mathbb{R}^r$ is convex, then $\operatorname{per}(E \cap C) \leq \operatorname{per} E$. (*Hint*: by the Hahn-Banach theorem, C is a limit of polytopes; use 474Sa.) (iii) Show that in 475Mc we have $\operatorname{per} E = \lim_{\alpha\to\infty} \operatorname{per}(E \cap B(\mathbf{0}, \alpha))$.

475Y Further exercises (a) Let B be a non-trivial ball in $\mathbb{R}^r$ with centre y, and v, v' two unit vectors in $\mathbb{R}^r$. Set

$$H = \{x : x \in \mathbb{R}^r, (x-y)\boldsymbol{.}v \leq 0\}, \quad H' = \{x : x \in \mathbb{R}^r, (x-y)\boldsymbol{.}v' \leq 0\}.$$

Show that $\mu((H \triangle H') \cap B) = \frac{1}{\pi} \arccos(v \boldsymbol{.} v') \mu B$.

(b) Show that μ is inner regular with respect to the family of compact sets $K \subseteq \mathbb{R}^r$ such that $\limsup_{\delta\downarrow 0} \frac{\mu(B(x,\delta)\cap K)}{\mu B(x,\delta)} \geq \frac{1}{2}$ for every $x \in K$.

(c) Show that if $E \subseteq \mathbb{R}^r$ has finite perimeter then one of E, $\mathbb{R}^r \setminus E$ has finite measure.

475 Notes and comments The successful identification of the distributionally-defined notion of 'perimeter', as described in §474, with the geometrically accessible concept of Hausdorff measure of an appropriate boundary, is of course the key to any proper understanding of the results of the last section as well as this one. The very word 'perimeter' would be unfair if the perimeter of $E \cup F$ were unrelated to the perimeters of E and F; and from this point of view the reduced boundary is less suitable than the essential boundary (475Cd, 475Xh). If we re-examine 474M, we see that it is saying, in effect, that for many balls B the boundary $\partial^*(E \cap B)$ is nearly $(B \cap \partial^* E) \cup (E \cap \partial B)$, and that an outward-normal function for $E \cap B$ can be assembled from outward-normal functions for E and B. But looking at 475Xc-475Xd we see that this is entirely natural; we need only ensure that $\nu(F \cap \partial B) = 0$ for a μ-negligible set F defined from E; and the 'almost every δ' in the statement of 474M is fully enough to arrange this. On the other hand, 475Xg seems to be very hard to prove without using the identification between $\nu(\partial^* E)$ and $\operatorname{per} E$.

Concerning 475Q, I ought to emphasize that it is *not* generally true that

$$\nu F = \frac{1}{2\beta_{r-1}} \int_{S_{r-1}} \int_{V_v} \#(F \cap T_v^{-1}[\{u\}])\nu(du)\nu(dv)$$

even for $r = 2$ and compact sets F with $\nu F < \infty$. We are here approaching one of the many fundamental concepts of geometric measure theory which I am ignoring. The key word is 'rectifiability'; for 'rectifiable' sets a wide variety of concepts of k-dimensional measure coincide, including the integral-geometric form above, and $\partial^* E$ is rectifiable whenever E has locally finite perimeter (Evans & Gariepy 92, 5.7.3). For the general theory of rectifiable sets, see the last quarter of Mattila 95, or Chapter 3 of Federer 69.

I have already noted that the largest volumes for sets of given diameter or perimeter are provided by balls (see 264H and the notes to §474). The isoperimetric theorem for convex sets (475T) is of the same form: once again, the best constant (here, the largest perimeter for a convex set of given diameter, or the smallest diameter for a convex set of given perimeter) is provided by balls.

476 Concentration of measure

Among the myriad special properties of Lebesgue measure, a particularly interesting one is 'concentration of measure'. For a set of given measure in the plane, it is natural to feel that it is most 'concentrated' if it is a disk. There are many ways of defining 'concentration', and I examine three of them in this section (476J, 476K and 476L); all lead us to Euclidean balls as the 'most concentrated' shapes. On the sphere the same criteria lead us to caps (476O, 476Xj).

All the main theorems of this section will be based on the fact that semi-continuous functions on compact spaces attain their bounds. The compact spaces in question will be spaces of subsets, and I begin with some very general remarks on topologies on such spaces (476A-476F). The facts to be used here will mostly relate to topologies induced by Hausdorff metrics (476A), but I include a mention of Vietoris topologies (476C) as what might be an appropriate alternative if we had to start from a non-metrizable space. The particular geometric properties of Euclidean space which make all these results possible are described in 476H-476I, where I describe concentrating operators based on reflections. The actual theorems 476J-476L and 476O can now almost be mass-produced.

476A Hausdorff metrics Let (X, ρ) be a metric space, and $\mathcal{C}$ the family of closed subsets of X.

(a) For a non-empty subset A of X, write $\rho(x, A) = \inf_{y \in A} \rho(x, y)$ for every $x \in X$. Note that $\rho(x, A) \leq \rho(x, y) + \rho(y, A)$ for all x, $y \in X$, so that $x \mapsto \rho(x, A) : X \to \mathbb{R}$ is 1-Lipschitz.

For E, $F \in \mathcal{C} \setminus \{\emptyset\}$, set

$$\tilde{\rho}(E, F) = \max(\sup_{x \in E} \rho(x, F), \sup_{y \in F} \rho(y, E)).$$

If E, $F \in \mathcal{C} \setminus \{\emptyset\}$ and $z \in X$, then $\rho(z, F) \leq \rho(z, E) + \sup_{x \in E} \rho(x, F)$; from this it is easy to see that $\tilde{\rho}$ is a metric on $\mathcal{C} \setminus \{\emptyset\}$, the **Hausdorff metric**. Observe that $x \mapsto \{x\} : X \to \mathcal{C} \setminus \{\emptyset\}$ is an isometry.

(b) If (X,ρ) is complete so is $(\mathcal{C}\setminus\{\emptyset\},\tilde{\rho})$. **P** Suppose that $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathcal{C}\setminus\{\emptyset\}$ such that $\tilde{\rho}(F_m,F_n)\le 2^{-m}$ whenever $m\le n$. Set

$$F=\limsup_{n\to\infty}F_n=\bigcap_{n\in\mathbb{N}}\overline{\bigcup_{i\ge n}F_i}.$$

Take any $n\in\mathbb{N}$ and $x\in F_n$. Then we can choose $\langle x_i\rangle_{i\in\mathbb{N}}$ inductively by setting $x_0=x$ and, given $x_i\in F_{n+i}$, taking $x_{i+1}\in F_{n+i+1}$ such that $\rho(x_{i+1},x_i)\le 2^{-n-i+1}$; this is possible because $\rho(x_i,F_{n+i+1})\le\tilde{\rho}(F_{n+i},F_{n+i+1})\le 2^{-n-i}$. Now $\langle x_i\rangle_{i\in\mathbb{N}}$ is Cauchy, so has a limit z in X. For every $m\in\mathbb{N}$,

$$z\in\overline{\{x_i:i\ge m\}}\subseteq\overline{\bigcup_{i\ge m}F_i},$$

so $z\in F$; accordingly

$$\rho(x,F)\le\rho(x,z)\le\textstyle\sum_{i=0}^{\infty}\rho(x_i,x_{i+1})\le 2^{-n+2}.$$

Since F_n is not empty, this shows incidentally that F is not empty and belongs to $\mathcal{C}\setminus\{\emptyset\}$.

Next, $\{y:\rho(y,F_n)\le 2^{-n}\}$ is a closed set including F_i for every $i\ge n$, so includes F. This shows that $\tilde{\rho}(F_n,F)\le\max(2^{-n+2},2^{-n})=2^{-n+2}$. As n is arbitrary, $F=\lim_{n\to\infty}F_n$ in $\mathcal{C}\setminus\{\emptyset\}$. As $\langle F_n\rangle_{n\in\mathbb{N}}$ is arbitrary, $\mathcal{C}\setminus\{\emptyset\}$ is complete under $\tilde{\rho}$. **Q**

(c) If (X,ρ) is totally bounded, so is $(\mathcal{C}\setminus\{\emptyset\},\tilde{\rho})$. **P** Let $\epsilon>0$. Then there is a finite set $I\subseteq X$ such that $\rho(x,I)\le\epsilon$ for every $x\in X$. Set $\mathcal{E}=\mathcal{P}I\setminus\{\emptyset\}$. Then $\mathcal{E}$ is a finite subset of $\mathcal{C}\setminus\{\emptyset\}$. If $F\in\mathcal{C}\setminus\{\emptyset\}$, set $E=\{x:x\in I,\ \rho(x,F)\le\epsilon\}$. Then $\tilde{\rho}(E,F)\le\epsilon$. Thus $\mathcal{C}\setminus\{\emptyset\}$ can be covered by finitely many closed balls of radius ϵ. As ϵ is arbitrary, $(\mathcal{C}\setminus\{\emptyset\},\tilde{\rho})$ is totally bounded. **Q**

(d) It follows that if (X,ρ) is compact, so is $(\mathcal{C}\setminus\{\emptyset\},\tilde{\rho})$ (4A2Jd).

476B Lemma Let (X,ρ) be a metric space, $\mathcal{C}_1$ the set of closed subsets of X, and $\tilde{\rho}$ the Hausdorff metric on $\mathcal{C}\setminus\{0\}$. Let σ be the metric on $X\times X$ defined by setting $\sigma((x_1,x_2),(y_1,y_2))=\max(\rho(x_1,y_1),\rho(x_2,y_2))$, $\mathcal{C}_2$ the set of closed subsets of $X\times X$ and $\tilde{\sigma}$ the Hausdorff metric on $\mathcal{C}_2\setminus\{\emptyset\}$. Then the map $F\mapsto F\times F:\mathcal{C}\setminus\{\emptyset\}\to\mathcal{C}_2\setminus\{\emptyset\}$ is an isometry.

proof If E, $F\in\mathcal{C}\setminus\{\emptyset\}$, then

$$\rho(x,F)=\inf_{y\in F}\rho(x,y)=\inf_{y\in F}\sigma((x,x),(y,y))\le\tilde{\sigma}(E\times E,F\times F)$$

for any $x\in E$, and similarly $\rho(y,E)\le\tilde{\sigma}(E\times E,F\times F)$ for every $y\in F$; so $\tilde{\rho}(E,F)\le\sigma(E\times E,F\times F)$. In the other direction,

$$\begin{aligned}\sigma((x_1,x_2),F\times F)&=\inf_{y_1,y_2\in F}\sigma((x_1,y_1),(x_2,y_2))=\inf_{y_1,y_2\in F}\max(\rho(x_1,y_1),\rho(x_2,y_2))\\&=\max\big(\inf_{y_1\in F}\rho(x_1,y_1),\inf_{y_2\in F}\rho(x_2,y_2)\big)\le\tilde{\rho}(E,F)\end{aligned}$$

for all x_1, $x_2\in E$, and similarly $\sigma((y_1,y_2),E\times E)\le\tilde{\rho}(E,F)$ for all y_1, $y_2\in F$, so $\tilde{\sigma}(E\times E,F\times F)\le\tilde{\rho}(E,F)$.

476C Vietoris topologies Let X be a topological space, and write $\mathcal{C}$ for the family of closed subsets of X.

(a) The **Vietoris topology** on $\mathcal{C}$ is the topology generated by sets of the form

$$\{F:F\in\mathcal{C},\ F\subseteq G\},\quad\{F:F\in\mathcal{C},\ F\cap G\ne\emptyset\}$$

where $G\subseteq X$ is open.

(b) If X is regular, then $\mathcal{C}$ is Hausdorff in its Vietoris topology. **P** Let E_1, $E_2\in\mathcal{C}$ be distinct. Then at least one of $E_1\setminus E_2$, $E_2\setminus E_1$ is non-empty; suppose the former. Take $x\in E_1\setminus E_2$ and an open set G containing x such that $\overline{G}\cap E_2=\emptyset$. Then $\{F:F\in\mathcal{C},\ F\cap G\ne\emptyset\}$ and $\{F:F\in\mathcal{C},\ F\subseteq X\setminus\overline{G}\}$ are disjoint open subsets of $\mathcal{C}$ containing E_1, E_2 respectively. **Q**

(c) If X is regular and normal, then $\mathcal{C}$ is regular in its Vietoris topology. **P** Suppose that $F\in\mathcal{C}$ and that $\mathcal{E}$ is an open subset of $\mathcal{C}$ containing F. Then there must be a finite family $\mathcal{G}$ of open subsets of X and an open subset H_0 of X such that

$$F \subseteq \{E : E \in \mathcal{C},\, E \cap G \neq \emptyset \;\forall\, G \in \mathcal{G},\, E \subseteq H_0\} \subseteq \mathcal{E}.$$

Let H_1 be an open set such that $F \subseteq H_1 \subseteq \overline{H}_1 \subseteq H_0$. For each $G \in \mathcal{G}$ take a point $x_G \in F \cap G$ and an open set G' containing x_G such that $\overline{G'} \subseteq G$. Now consider

$$\mathcal{E}_1 = \{E : E \in \mathcal{C},\, E \cap G' \neq \emptyset \;\forall\, G \in \mathcal{G},\, E \subseteq H_0\},$$

$$\mathcal{E}_2 = \textstyle\bigcup_{G \in \mathcal{G}} \{E : E \in \mathcal{C},\, E \subseteq X \setminus \overline{G'}\} \cup \{E : E \in \mathcal{C},\, E \cap (X \setminus \overline{H}_1) \neq \emptyset\}.$$

Then $\mathcal{E}_1$ and $\mathcal{E}_2$ are disjoint open sets in $\mathcal{C}$, $F \in \mathcal{E}_1$ and $\mathcal{E}_2 \supseteq \mathcal{C} \setminus \mathcal{E}$. As F and $\mathcal{E}$ are arbitrary, $\mathcal{C}$ is regular. **Q**

(d) A variant of the Vietoris topology will be useful in §495. I will say that the **k-Vietoris topology** on $\mathcal{C}$ is the topology generated by sets of the form

$$\{F : F \in \mathcal{C},\, F \cap G \neq \emptyset\}, \quad \{F : F \in \mathcal{C},\, F \cap K = \emptyset\}$$

where $G \subseteq X$ is open and $K \subseteq X$ is compact. Now we have the following easy results.

(i) $\mathcal{C}$ is compact in its k-Vietoris topology. **P** Let $\mathfrak{F}$ be an ultrafilter on $\mathcal{C}$. Set $C_0 = \bigcap_{\mathcal{E} \in \mathfrak{F}} \overline{\bigcup \mathcal{E}}$. ($\alpha$) If $G \subseteq X$ is open and $G \cap C_0 \neq \emptyset$, set $\mathcal{E} = \{C : C \in \mathcal{C},\, C \cap G = \emptyset\}$. Then $\overline{\bigcup \mathcal{E}}$ does not meet G, so does not include C_0, and $\mathcal{E} \notin \mathfrak{F}$. Accordingly $\{C : C \cap G \neq \emptyset\} = \mathcal{C} \setminus \mathcal{E}$ belongs to $\mathcal{F}$. (β) If $K \subseteq X$ is compact and $C_0 \cap K = \emptyset$, then $\{K \cap \overline{\bigcup \mathcal{E}} : \mathcal{E} \in \mathfrak{F}\}$ is a downwards-directed family of relatively closed subsets of K with empty intersection so must contain the empty set, and there is an $\mathcal{E} \in \mathfrak{F}$ such that $K \cap C = \emptyset$ for every $C \in \mathcal{E}$, that is, $\{C : C \cap K = \emptyset\}$ belongs to $\mathcal{F}$. (γ) By 4A2B(a-v), $\mathfrak{F} \to C_0$. As $\mathfrak{F}$ is arbitrary, $\mathcal{C}$ is compact. **Q**

(ii) If X is Hausdorff, the k-Vietoris topology is coarser than the Vietoris topology. If X is compact and Hausdorff the two topologies agree; in particular, the Vietoris topology on $\mathcal{C}$ is compact.

(iii) If X is locally compact and Hausdorff, the k-Vietoris topology on X is Hausdorff. **P** Let C_1, $C_2 \in \mathcal{C}$ be distinct. Then there is a point belonging to one and not the other; suppose that $x \in C_1 \setminus C_2$. Because X is locally compact and Hausdorff, there is a relatively compact open set G such that $x \in G$ and $\overline{G} \cap C_2 = \emptyset$. Now $\{C : C \cap G \neq \emptyset\}$ and $\{C : C \cap \overline{G} = \emptyset\}$ are disjoint open sets in $\mathcal{C}$ containing C_1, C_2 respectively. **Q**

476D Proposition Let (X, ρ) be a metric space, and $\tilde{\rho}$ the Hausdorff metric on $\mathcal{C} \setminus \{\emptyset\}$, where $\mathcal{C}$ is the family of closed subsets of X. Write $\mathcal{K} \subseteq \mathcal{C}$ for the family of compact subsets of X.

(a) If μ is a totally finite topological measure on X, then $\mu\restriction\mathcal{C} \setminus \{\emptyset\}$ is upper semi-continuous.

(b) If μ is a locally finite topological measure on X, then $\mu\restriction\mathcal{K} \setminus \{\emptyset\}$ is upper semi-continuous for the subspace topology on $\mathcal{K} \setminus \{\emptyset\}$.

(c) If μ is a topological measure on X and f is a non-negative μ-integrable real-valued function, then $F \mapsto \int_F f d\mu : \mathcal{C} \setminus \{\emptyset\} \to \mathbb{R}$ is upper semi-continuous.

proof (a) If $F \in \mathcal{C} \setminus \{\emptyset\}$ and $\mu F < \alpha$, then for each $n \in \mathbb{N}$ set $F_n = \{x : \rho(x, F) \leq 2^{-n}\}$. Since $\langle F_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence of closed sets with intersection F, and μ is totally finite, there is an n such that $\mu F_n < \alpha$. If now we take $E \in \mathcal{C} \setminus \{\emptyset\}$ such that $\tilde{\rho}(E, F) \leq 2^{-n}$, then $E \subseteq F_n$ so $\mu E < \alpha$. As F and α are arbitrary, $\mu\restriction\mathcal{C} \setminus \{\emptyset\}$ is upper semi-continuous.

(b) The argument is the same as in (a), with one refinement. If $K \in \mathcal{K} \setminus \{\emptyset\}$ and we set $F_n = \{x : \rho(x, K) \leq 2^{-n}\}$, then we do not know that every F_n has finite measure. However, since μ is locally finite, every point of K has an open neighbourhood of finite measure; as K is compact, we can cover it by finitely many open sets of finite measure, and $K \subseteq G$ where $G \subseteq X$ is open and $\mu G < \infty$. Next, the function $x \mapsto \rho(x, X \setminus G)$ is continuous and strictly positive on K, so has a non-zero lower bound on K, and there is some $m \in \mathbb{N}$ such that $\rho(x, X \setminus G) > 2^{-m}$ for every $x \in K$, in which case $F_m \subseteq G$. It follows now that there is some $n \geq m$ such that $\mu F_n < \alpha$, and we can continue as before.

(c) Apply (a) to the indefinite-integral measure over μ defined by f.

476E Proposition Let X be a topological space, and $\mathcal{C}$ the family of closed subsets of X, with the Vietoris topology. Write $\mathcal{K} \subseteq \mathcal{C}$ for the family of closed compact subsets of X.

(a) If μ is a totally finite topological measure on X which is inner regular with respect to the closed sets, then $\mu\restriction\mathcal{C}$ is upper semi-continuous.

(b) If μ is a locally finite topological measure on X which is inner regular with respect to the closed sets, then $\mu\restriction\mathcal{K}$ is upper semi-continuous.

proof (a) Suppose that $F \in \mathcal{C}$ and that $\mu F < \alpha$. Because μX is finite, there is a closed set $F_1 \subseteq X \setminus F$ such that $\mu F_1 > \mu X - \alpha$, so that $\mu(X \setminus F_1) < \alpha$. Now $\mathcal{E} = \{E : E \in \mathcal{C}, E \subseteq X \setminus F_1\}$ is an open set containing F, and $\mu E < \alpha$ for every $E \in \mathcal{E}$. As F and α are arbitrary, $\mu\restriction\mathcal{C}$ is upper semi-continuous.

(b) If $K \in \mathcal{K}$ and $\mu K < \alpha$ then, because μ is locally finite, there is an open set $G \supseteq K$ such that $\mu G < \infty$. Now there is a closed set $F_1 \subseteq G$ such that $\mu F_1 > \mu G - \alpha$, so that $\mathcal{L} = \{L : L \in \mathcal{K}, L \subseteq G \setminus F_1\}$ is an open subset of $\mathcal{K}$ and $\mu L < \alpha$ for every $L \in \mathcal{L}$. As K and α are arbitrary, $\mu\restriction\mathcal{K}$ is upper semi-continuous.

476F Proposition Let (X, ρ) be a metric space, and $\tilde{\rho}$ the Hausdorff metric on $\mathcal{C} \setminus \{\emptyset\}$, where $\mathcal{C}$ is the family of closed subsets of X. For non-empty sets $A \subseteq X$ and $\epsilon > 0$, set $U(A, \epsilon) = \{x : x \in X, \rho(x, A) < \epsilon\}$. Then for any τ-additive topological measure μ on X, the function

$$(F, \epsilon) \mapsto \mu U(F, \epsilon) : (\mathcal{C} \setminus \{\emptyset\}) \times]0, \infty[\to [0, \infty]$$

is lower semi-continuous.

proof Suppose that $\alpha \in]0, \infty[$ and that $F_0 \in \mathcal{C} \setminus \{\emptyset\}$, $\epsilon_0 \in]0, \infty[$ are such that $\mu U(F_0, \epsilon_0) > \alpha$. Then $\{U(I, \delta) : I \in [F_0]^{<\omega}, \delta < \epsilon_0\}$ is an upwards-directed family of open sets with union $U(F_0, \epsilon_0)$. Because μ is τ-additive, there are $\delta < \epsilon_0$, $I \in [F_0]^{<\omega}$ such that $\mu U(I, \delta) > \alpha$. Now suppose that $F \in \mathcal{C} \setminus \{\emptyset\}$ and ϵ are such that $\tilde{\rho}(F, F_0) < \frac{1}{2}(\epsilon_0 - \delta)$ and $\epsilon > \frac{1}{2}(\delta + \epsilon_0)$. Then $\rho(x, F) < \frac{1}{2}(\epsilon_0 - \delta) < \epsilon - \delta$ for every $x \in I$, so $U(I, \delta) \subseteq U(F, \epsilon)$ and $\mu U(F, \epsilon) > \alpha$. As α, F_0 and ϵ_0 are arbitrary, $(F, \epsilon) \mapsto \mu U(F, \epsilon)$ is lower semi-continuous.

Remark Recall that all 'ordinary' topological measures on metric spaces are τ-additive; see 438J.

476G Proposition Let (X, ρ) be a non-empty compact metric space, and suppose that its isometry group G acts transitively on X. Then X has a unique G-invariant Radon probability measure μ, which is strictly positive.

proof By 441G, G, with its topology of pointwise convergence, is a compact topological group, and the action of G on X is continuous. So 443Td gives the result.

476H Concentration by partial reflection The following construction will be used repeatedly in the rest of the section. Let X be an inner product space. (In this section, X will be usually be $\mathbb{R}^r$, but in 493J it will be helpful to be able to speak of abstract Hilbert spaces.) For any unit vector $e \in X$ and any $\alpha \in \mathbb{R}$, write $R_{e\alpha} : X \to X$ for the reflection in the hyperplane $V_{e\alpha} = \{x : x \in X, (x|e) = \alpha\}$, so that $R_{e\alpha}(x) = x + 2(\alpha - (x|e))e$ for every $x \in X$. Next, for any $A \subseteq X$, we can define a set $\psi_{e\alpha}(A)$ by setting

$$\begin{aligned}\psi_{e\alpha}(A) &= \{x : x \in A, (x|e) \geq \alpha\} \cup \{x : x \in A, (x|e) < \alpha, R_{e\alpha}(x) \in A\} \\ &\qquad \cup \{x : x \in \mathbb{R}^r \setminus A, (x|e) \geq \alpha, R_{e\alpha}(x) \in A\} \\ &= (W \cap (A \cup R_{e\alpha}[A])) \cup (A \cap R_{e\alpha}[A] \setminus W),\end{aligned}$$

where W is the half-space $\{x : (x|e) \geq \alpha\}$. Geometrically, we construct $\psi_{e\alpha}(A)$ by moving those points of A on the 'wrong' side of the hyperplane $V_{e\alpha}$ to their reflections, provided those points are not already occupied. We have the following facts.

(a) For non-empty $A \subseteq X$ and $\epsilon > 0$, set $U(A, \epsilon) = \{x : \rho(x, A) < \epsilon\}$, where ρ is the standard metric on X. Now $U(\psi_{e\alpha}(A), \epsilon) \subseteq \psi_{e\alpha}(U(A, \epsilon))$. **P** Take $x \in U(\psi_{e\alpha}(A), \epsilon)$. Then there is a $y \in \psi_{e\alpha}(A)$ such that $\|x - y\| < \epsilon$.

case 1 Suppose $(x|e) \geq \alpha$. If $x \in U(A,\epsilon)$ then certainly $x \in \psi_{e\alpha}(U(A,\epsilon))$. Otherwise, because $\|x-y\| < \epsilon$, $y \notin A$, so $R_{e\alpha}(y) \in A$. But $\|R_{e\alpha}(x) - R_{e\alpha}(y)\| = \|x-y\| < \epsilon$, so $R_{e\alpha}(x) \in U(A,\epsilon)$ and $x \in \psi_{e\alpha}(U(A,\epsilon))$.

case 2a Suppose $(x|e) < \alpha$, $(y|e) \geq \alpha$. Then

$$\|R_{e\alpha}(x) - y\| = \|x - R_{e\alpha}(y)\| \leq \|R_{e\alpha}(x) - R_{e\alpha}(y)\| = \|x-y\| < \epsilon.$$

At least one of y, $R_{e\alpha}(y)$ belongs to A, so both x and $R_{e\alpha}(x)$ belong to $U(A,\epsilon)$ and $x \in \psi_{e\alpha}(U(A,\epsilon))$.

case 2b Suppose $(x|e) < \alpha$ and $(y|e) < \alpha$. In this case, both y and $R_{e\alpha}(y)$ belong to A, so both x and $R_{e\alpha}(x)$ belong to $U(A,\epsilon)$ and again $x \in \psi_{e\alpha}(U(A,\epsilon))$.

Thus $x \in \psi_{e\alpha}(U(A,\epsilon))$ in all cases; as x is arbitrary, we have the result. **Q**

(b) If $F \subseteq X$ is closed, then $\psi_{e\alpha}(F)$ is closed. **P** Use the second formula for $\psi_{e\alpha}(F)$. **Q**

476I Lemma Let X be an inner product space, $e \in X$ a unit vector and $\alpha \in \mathbb{R}$. Let $R = R_{e\alpha} : X \to X$ be the reflection operator, and $\psi = \psi_{e\alpha} : \mathcal{P}X \to \mathcal{P}X$ the associated transformation, as described in 476H. For $x \in A \subseteq X$, define

$$\begin{aligned}\phi_A(x) &= x \text{ if } (x|e) \geq \alpha,\\ &= x \text{ if } (x|e) < \alpha \text{ and } R(x) \in A,\\ &= R(x) \text{ if } (x|e) < \alpha \text{ and } R(x) \notin A.\end{aligned}$$

Let ν be a topological measure on X which is R-invariant, that is, ν is the image measure νR^{-1}.

(a) For any $A \subseteq X$, $\phi_A : A \to \psi(A)$ is a bijection. If $\alpha < 0$, then $\|\phi_A(x)\| \leq \|x\|$ for every $x \in A$, with $\|\phi_A(x)\| < \|x\|$ iff $(x|e) < \alpha$ and $R(x) \notin A$.

(b) If $E \subseteq X$ is measured by ν, then $\psi(E)$ is measurable and $\nu\psi(E) = \nu E$; moreover, ϕ_E is a measure space isomorphism for the subspace measures on E and $\psi(E)$.

(c) If $\alpha < 0$ and $E \subseteq X$ is measurable, then $\int_E \|x\|\nu(dx) \geq \int_{\psi(E)} \|x\|\nu(dx)$, with equality iff $\{x : x \in E,\ (x|e) < \alpha,\ R(x) \notin E\}$ is negligible.

(d) Suppose that X is separable. Let λ be the c.l.d. product measure of ν with itself on $X \times X$. If $\alpha \leq 0$ and $E \subseteq X$ is measurable, then

$$\int_{E\times E} \|x-y\|\lambda(d(x,y)) \geq \int_{\psi(E)\times\psi(E)} \|x-y\|\lambda(d(x,y)).$$

(e) Now suppose that $X = \mathbb{R}^r$. Then $\nu(\partial^*\psi(A)) \leq \nu(\partial^* A)$ for every $A \subseteq \mathbb{R}^r$, where $\partial^* A$ is the essential boundary of A (definition: 475B).

proof (a) That $\phi_A : A \to \psi(A)$ is a bijection is immediate from the definitions of ψ and ϕ_A. If $\alpha < 0$, then for any $x \in A$ either $\phi_A(x) = x$ or $(x|e) < \alpha$ and $R(x) \notin A$. In the latter case

$$\begin{aligned}\|\phi_A(x)\|^2 = \|R(x)\|^2 &= \|x + 2\gamma e\|^2\\ \text{(where } \gamma = \alpha - (x|e) > 0)\\ &= \|x\|^2 + 4\gamma(x|e) + 4\gamma^2 = \|x\|^2 + 4\gamma\alpha < \|x\|^2,\end{aligned}$$

so $\|\phi_A(x)\| < \|x\|$.

(b) If we set

$$E_1 = \{x : x \in E,\ (x|e) \geq \alpha\},$$

$$E_2 = \{x : x \in E, (x|e) < \alpha,\ R_{e\alpha}(x) \in E\},$$

$$E_3 = \{x : x \in E,\ (x|e) < \alpha,\ R_{e\alpha}(x) \notin E\},$$

$$E_4 = \{x : x \in \mathbb{R}^r \setminus E,\ (x|e) > \alpha,\ R_{e\alpha}(x) \in E\},$$

then E_1, E_2, E_3 and E_4 are disjoint measurable sets, $E = E_1 \cup E_2 \cup E_3$, $\psi(E) = E_1 \cup E_2 \cup E_4$ and $\phi_E{\restriction}E_3 = R{\restriction}E_3$ is a measure space isomorphism for the subspace measures on E_3 and E_4.

(c) By (a),

$$\int_E \|x\|\nu(dx) \geq \int_E \|\phi_E(x)\|\nu(dx) = \int_{\psi(E)} \|x\|\nu(dx)$$

by 235Ic, because $\phi_E : E \to \psi(E)$ is inverse-measure-preserving, with equality only when

$$\{x : \|x\| > \|\phi_E(x)\|\} = \{x : x \in E, (x|e) < \alpha, R(x) \notin E\}$$

is negligible.

(d) Note first that if Λ is the domain of λ then Λ includes the Borel algebra of $X \times X$ (because X is second-countable, so this is just the σ-algebra generated by products of Borel sets, by 4A3G); so that $(x, y) \mapsto \|x - y\|$ is Λ-measurable, and the integral is defined in $[0, \infty]$. Now consider the sets

$$W_1 = \{(x, y) : x \in E, y \in E, R(x) \notin E, R(y) \in E, (x|e) < \alpha, (y|e) < \alpha\},$$

$$W_1' = \{(x, y) : x \in E, y \in E, R(x) \notin E, R(y) \in E, (x|e) < \alpha, (y|e) > \alpha\},$$

$$W_2 = \{(x, y) : x \in E, y \in E, R(x) \in E, R(y) \notin E, (x|e) < \alpha, (y|e) < \alpha\},$$

$$W_2' = \{(x, y) : x \in E, y \in E, R(x) \in E, R(y) \notin E, (x|e) > \alpha, (y|e) < \alpha\}.$$

Then $(x, y) \mapsto (x, R(y)) : W_1' \to W_1$ is a measure space isomorphism for the subspace measures induced on W_1 and W_1' by λ, so

$$\begin{aligned}\int_{W_1} \|\phi_E(x) - \phi_E(y)\|\lambda(d(x, y)) &= \int_{W_1} \|R(x) - y\|\lambda(d(x, y)) \\ &= \int_{W_1'} \|R(x) - R(y)\|\lambda(d(x, y)) \\ &= \int_{W_1'} \|x - y\|\lambda(d(x, y)).\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{W_1'} \|\phi_E(x) - \phi_E(y)\|\lambda(d(x, y)) &= \int_{W_1'} \|R(x) - y\|\lambda(d(x, y)) \\ &= \int_{W_1} \|R(x) - R(y)\|\lambda(d(x, y)) \\ &= \int_{W_1} \|x - y\|\lambda(d(x, y)).\end{aligned}$$

So we get

$$\int_{W_1 \cup W_1'} \|\phi_E(x) - \phi_E(y)\|\lambda(d(x, y)) = \int_{W_1 \cup W_1'} \|x - y\|\lambda(d(x, y)).$$

In the same way, $(x, y) \mapsto (R(x), y)$ is an isomorphism of the subspace measures on W_2 and W_2', and we have

$$\int_{W_2 \cup W_2'} \|\phi_E(x) - \phi_E(y)\|\lambda(d(x, y)) = \int_{W_2 \cup W_2'} \|x - y\|\lambda(d(x, y)).$$

On the other hand, for all $(x, y) \in (E \times E) \setminus (W_1 \cup W_1' \cup W_2 \cup W_2')$, we have $\|\phi_E(x) - \phi_E(y)\| \leq \|x - y\|$. (Either x and y are both left fixed by ϕ, or both are moved, or one is on the reflecting hyperplane, or one is moved to the same side of the reflecting hyperplane as the other.) So we get

$$\begin{aligned}\int_{E \times E} \|x - y\|\lambda(d(x, y)) &\geq \int_{E \times E} \|\phi_E(x) - \phi_E(y)\|\lambda(d(x, y)) \\ &= \int_{\psi(E) \times \psi(E)} \|x - y\|\lambda(d(x, y))\end{aligned}$$

because $(x, y) \mapsto (\phi_E(x), \phi_E(y))$ is an inverse-measure-preserving transformation for the subspace measures on $E \times E$ and $\psi(E) \times \psi(E)$.

(e)(i) Because R is both an isometry and a measure space automorphism, $\mathrm{cl}^*R[A] = R[\mathrm{cl}^*A]$ and $\mathrm{int}^*R[A] = R[\mathrm{int}^*A]$, where cl^*A and int^*A are the essential closure and the essential interior of A, as in 475B. Recall that cl^*A, int^*A and ∂^*A are all Borel sets (475Cc), so that ∂^*A and $\partial^*\psi(A)$ are measurable.

(ii) Suppose that $x.e = \alpha$. Then $x \in \partial^*\psi(A)$ iff $x \in \partial^*A$. **P** It is easy to check that $B(x,\delta) \cap \psi(A) = \psi(B(x,\delta) \cap A)$ for any $\delta > 0$, so that $\nu^*(B(x,\delta) \cap \psi(A)) = \nu^*(\psi(B(x,\delta) \cap A))$ for every $\delta > 0$ and $x \in \mathrm{cl}^*A$ iff $x \in \mathrm{cl}^*\psi(A)$. If $x = R(x) \in \mathrm{int}^*A$, then $x \in \mathrm{int}^*R[A]$ so $x \in \mathrm{int}^*(A \cap R[A])$ (475Cd) and $x \in \mathrm{int}^*\psi(A)$. If $x \in \mathrm{int}^*\psi(A)$ then $x \in \mathrm{int}^*(R[\psi(A)] \cap \psi(A)) \subseteq \mathrm{int}^*A$. **Q**

(iii) If $x \in \partial^*\psi(A) \setminus \partial^*A$ then $R(x) \in \partial^*A \setminus \partial^*\psi(A)$. **P** By (ii), $x.e \neq \alpha$.

case 1 Suppose that $x.e > \alpha$. Setting $\delta = x.e - \alpha$, we see that $U(x,\delta) \cap \psi(A) = U(x,\delta) \cap (A \cup R[A])$, while $U(R(x),\delta) \cap \psi(A) = U(R(x),\delta) \cap A \cap R[A]$. Since $x \notin \mathrm{int}^*\psi(A)$, $x \notin \mathrm{int}^*(A \cup R[A])$ and $x \notin \mathrm{int}^*A$; since x also does not belong to ∂^*A, $x \notin \mathrm{cl}^*A$. However, $x \in \mathrm{cl}^*(A \cup R[A]) = \mathrm{cl}^*A \cup \mathrm{cl}^*R[A]$ (475Cd), so $x \in \mathrm{cl}^*R[A]$ and $R(x) \in \mathrm{cl}^*A$. Next, $x \notin \mathrm{int}^*R[A]$, so $R(x) \notin \mathrm{int}^*A$ and $R(x) \in \partial^*A$. Since $x \notin \mathrm{cl}^*A$, $R(x) \notin \mathrm{cl}^*R[A]$ and $R(x) \notin \mathrm{cl}^*\psi(A)$; so $R(x) \in \partial^*A \setminus \partial^*\psi(A)$.

case 2 Suppose that $x.e < \alpha$. This time, set $\delta = \alpha - x.e$, so that $U(x,\delta) \cap \psi(A) = U(x,\delta) \cap A \cap R[A]$ and $U(R(x),\delta) \cap \psi(A) = U(R(x),\delta) \cap (A \cup R[A])$. As $x \in \mathrm{cl}^*\psi(A)$, $x \in \mathrm{cl}^*(A \cap R[A])$ and $R(x) \in \mathrm{cl}^*A$. Also $x \in \mathrm{cl}^*A$; as $x \notin \partial^*A$, $x \in \mathrm{int}^*A$, $R(x) \in \mathrm{int}^*R[A]$ and $R(x) \in \mathrm{int}^*\psi(A)$, so that $R(x) \notin \partial^*\psi(A)$. Finally, we know that $x \in \mathrm{int}^*A$ but $x \notin \mathrm{int}^*(A \cap R[A])$ (because $x \notin \mathrm{int}^*\psi(A)$; it follows that $x \notin \mathrm{int}^*R[A]$ so $R(x) \notin \mathrm{int}^*A$ and $R(x) \in \partial^*A \setminus \partial^*\psi(A)$.

Thus all possibilities are covered and we have the result. **Q**

(iv) What this means is that if we set $E = \partial^*\psi(A) \setminus \partial^*A$ then $R[E] \subseteq \partial^*A \setminus \partial^*\psi(A)$. So

$$\nu\partial^*\psi(A) = \nu E + \nu(\partial^*\psi(A) \cap \partial^*A) = \nu R[E] + \nu(\partial^*\psi(A) \cap \partial^*A) \leq \nu\partial^*A,$$

as required in (e). This ends the proof of the lemma.

476J Theorem Let $r \geq 1$ be an integer, and let μ be Lebesgue measure on $\mathbb{R}^r$. For non-empty $A \subseteq \mathbb{R}^r$ and $\epsilon > 0$, write $U(A,\epsilon)$ for $\{x : \rho(x,A) < \epsilon\}$, where ρ is the Euclidean metric on $\mathbb{R}^r$. If μ^*A is finite, then $\mu U(A,\epsilon) \geq \mu U(B_A,\epsilon)$, where B_A is the closed ball with centre $\mathbf{0}$ and measure μ^*A.

proof (a) To begin with, suppose that A is bounded. Set $\gamma = \mu^*A$, $\beta = \mu U(A,\epsilon)$. If $\gamma = 0$ then (because $A \neq \emptyset$)

$$\mu U(A,\epsilon) \geq \mu U(\{\mathbf{0}\},\epsilon) = \mu U(B_A,\epsilon),$$

and we can stop. So let us suppose henceforth that $\gamma > 0$. Let $M \geq 0$ be such that $A \subseteq B(\mathbf{0},M)$, and consider the family

$$\mathcal{F} = \{F : F \in \mathcal{C} \setminus \{\emptyset\},\ \mu F \geq \gamma,\ \mu U(F,\epsilon) \leq \beta\},$$

where $\mathcal{C}$ is the family of closed subsets of $B(\mathbf{0},M)$. Because $U(\overline{A},\epsilon) = U(A,\epsilon)$, $\overline{A} \in \mathcal{F}$ and $\mathcal{F}$ is non-empty. By 476D and 476F, $\mathcal{F}$ is closed for the Hausdorff metric on $\mathcal{C} \setminus \{\emptyset\}$, therefore compact, by 476Ad, because $B(\mathbf{0},M)$ is compact. Next, the function

$$F \mapsto \int_F (M - \|x\|)\mu(dx) : \mathcal{C} \setminus \{\emptyset\} \to [0,\infty[$$

is upper semi-continuous, by 476Dc. It therefore attains its supremum on $\mathcal{F}$ at some $F_0 \in \mathcal{F}$ (4A2Gk). Let $F_1 \subseteq F_0$ be a closed self-supporting set of the same measure as F_0; then $U(F_1,\epsilon) \subseteq U(F_0,\epsilon)$ and $\mu F_1 = \mu F_0$, so $F_1 \in \mathcal{F}$; also

$$\int_{F_1} M - \|x\|\mu(dx) = \int_{F_0} M - \|x\|\mu(dx) \geq \int_F M - \|x\|\mu(dx)$$

for every $F \in \mathcal{F}$.

(b) Now F_1 is a ball with centre $\mathbf{0}$. **P?** Suppose, if possible, otherwise. Then there are $x_1 \in F_1$ and $x_0 \in \mathbb{R}^r \setminus F_1$ such that $\|x_0\| < \|x_1\|$. Set $e = \frac{1}{\|x_0 - x_1\|}(x_0 - x_1)$, so that e is a unit vector, and $\alpha = \frac{1}{2}e.(x_0 + x_1)$; then

$$\alpha = \frac{1}{2\|x_0 - x_1\|}(x_0 - x_1).(x_0 + x_1) = \frac{1}{2\|x_0 - x_1\|}(\|x_0\|^2 - \|x_1\|^2) < 0.$$

Define $R = R_{e\alpha} : \mathbb{R}^r \to \mathbb{R}^r$ and $\psi = \psi_{e\alpha}$ as in 476H. Set $F = \psi(F_1)$. Then F is closed (476Hb) and $\mu F = \mu F_1 \geq \mu^* A$ (476Ib). Also $U(F,\epsilon) \subseteq \psi(U(F_1,\epsilon))$ (476Ha), so

$$\mu U(F,\epsilon) \leq \mu(\psi(U(F_1,\epsilon)) = \mu U(F_1,\epsilon) \leq \beta$$

and $F \in \mathcal{F}$. It follows that $\int_F M - \|x\|\mu(dx) \leq \int_{F_1} M - \|x\|\mu(dx)$; as $\mu F = \mu F_1$, $\int_F \|x\|\mu(dx) \geq \int_{F_1} \|x\|\mu(dx)$. By 476Ic, $G = \{x : x \in F_1, x.e < \alpha, R(x) \notin F_1\}$ is negligible. But G contains x_1 and is relatively open in F_1, and F_1 is supposed to be self-supporting; so this is impossible. **XQ**

(c) Since $\mu F_1 \geq \gamma$, $F_1 \supseteq B_A$, and

$$\mu U(B_A,\epsilon) \leq \mu U(F_1,\epsilon) \leq \beta = \mu U(A,\epsilon).$$

So we have the required result for bounded A. In general, given an unbounded set A of finite measure, let δ be the radius of B_A; then

$$\begin{aligned}\mu U(B_A,\epsilon) &= \mu B(\mathbf{0},\delta+\epsilon) = \sup_{\alpha<\delta} \mu B(\mathbf{0},\alpha+\epsilon)\\ &\leq \sup_{A'\subseteq A \text{ is bounded}} \mu U(B_{A'},\epsilon) \leq \sup_{A'\subseteq A \text{ is bounded}} \mu U(A,\epsilon) = \mu U(A,\epsilon)\end{aligned}$$

because $\{U(A',\epsilon) : A' \subseteq A \text{ is bounded}\}$ is an upwards-directed family of open sets with union $U(A,\epsilon)$, and μ is τ-additive. So the theorem is true for unbounded A as well.

476K Theorem Let $r \geq 1$ be an integer, and let μ be Lebesgue measure on $\mathbb{R}^r$; write λ for the product measure on $\mathbb{R}^r \times \mathbb{R}^r$. For any measurable set $E \subseteq \mathbb{R}^r$ of finite measure, write B_E for the closed ball with centre $\mathbf{0}$ and the same measure as E. Then

$$\int_{E\times E} \|x-y\|\lambda(d(x,y)) \geq \int_{B_E\times B_E} \|x-y\|\lambda(d(x,y)).$$

proof (a) Suppose for the moment that E is compact and not empty, and that $\epsilon > 0$. Let $M \geq 0$ be such that $\|x\| \leq M$ for every $x \in E$. For non-empty sets $A \subseteq \mathbb{R}^r$ set $U(A,\epsilon) = \{x : \rho(x,A) < \epsilon\}$, where ρ is Euclidean distance on $\mathbb{R}^r$. Set $\beta = \int_{U(E,\epsilon)\times U(E,\epsilon)} \|x-y\|\lambda(d(x,y))$. Let $\mathcal{F}$ be the family of non-empty closed subsets F of the ball $B(\mathbf{0},M) = \{x : \|x\| \leq M\}$ such that $\mu F \geq \mu E$ and $\int_{U(F,\epsilon)\times U(F,\epsilon)} \|x-y\|\lambda(d(x,y)) \leq \beta$. Then $\mathcal{F}$ is compact for the Hausdorff metric on the family $\mathcal{C}\setminus\{\emptyset\}$ of non-empty closed subsets of $B(\mathbf{0},M)$. **P** We know from 476Ad that $\mathcal{C}\setminus\{\emptyset\}$ is compact, and from 476Da that $\{F : \mu F \geq \mu E\}$ is closed. To see that $\mathcal{F}$ is closed, consider the metric σ on $\mathbb{R}^r \times \mathbb{R}^r$ defined by setting $\sigma((x,y),(x',y')) = \max(\|x-x'\|, \|y-y'\|)$, and the measure ν which is the indefinite-integral measure over λ defined by the function $(x,y) \mapsto \|x-y\|$. Writing $\mathcal{C}_2$ for the family of non-empty closed subsets of $\mathbb{R}^r$, the map $F \mapsto F\times F : \mathcal{C}\setminus\{\emptyset\} \to \mathcal{C}_2\setminus\{\emptyset\}$ is continuous (476B) and the map $W \mapsto \nu U(W,\epsilon) : \mathcal{C}_2\setminus\{\emptyset\} \to \leftarrow$ is lower semi-continuous, where $U(W,\epsilon) = \{(x,y) : \sigma((x,y),W) < \epsilon\}$ (476F). But this means that $F \mapsto \int_{U(F,\epsilon)\times U(F,\epsilon)} \|x-y\|\lambda(d(x,y))$ is lower semi-continuous, and $\{F : \int_{U(F,\epsilon)\times U(F,\epsilon)} \|x-y\|\lambda(d(x,y)) \leq \beta\}$ is closed. Thus $\mathcal{F}$ is a closed subset of $\mathcal{C}\setminus\{\emptyset\}$ and is compact. **Q**

(b) Since $E \in \mathcal{F}$, $\mathcal{F}$ is not empty. By 476Dc, there is an $F_0 \in \mathcal{F}$ such that $\int_{F_0} M - \|x\|\mu(dx) \geq \int_F M - \|x\|\mu(dx)$ for every $F \in \mathcal{F}$. Let $F_1 \subseteq F_0$ be a closed self-supporting set of the same measure; then $U(F_1,\epsilon) \subseteq U(F_0,\epsilon)$, so $\int_{U(F_1,\epsilon)\times U(F_1,\epsilon)} \|x-y\|\lambda(d(x,y)) \leq \beta$ and $F_1 \in \mathcal{F}$; also

$$\int_{F_1} M - \|x\|\mu(dx) = \int_{F_0} M - \|x\|\mu(dx) \geq \int_F M - \|x\|\mu(dx)$$

for every $F \in \mathcal{F}$.

Now F_1 is a ball with centre $\mathbf{0}$. **P?** Suppose, if possible, otherwise. Then (just as in the proof of 476K) there are $x_1 \in F_1$ and $x_0 \in \mathbb{R}^r\setminus F_1$ such that $\|x_1\| > \|x_0\|$. Set $e = \frac{1}{\|x_0-x_1\|}(x_0-x_1)$ and $\alpha = \frac{1}{2}e.(x_0+x_1) < 0$. Define $R = R_{e\alpha} : \mathbb{R}^r \to \mathbb{R}^r$ and $\psi = \psi_{e\alpha}$ as in 476H. Set $F = \psi(F_1)$. Then F is closed and $\mu F = \mu F_1 \geq \mu E$ and $U(F,\epsilon) \subseteq \psi(U(F_1,\epsilon))$. So

$$\begin{aligned}\int_{U(F,\epsilon)\times U(F,\epsilon)} \|x-y\|\lambda(d(x,y)) &\leq \int_{\psi(U(F_1,\epsilon))\times\psi(U(F_1,\epsilon))} \|x-y\|\lambda(d(x,y)) \\ &\leq \int_{U(F_1,\epsilon))\times U(F_1,\epsilon)} \|x-y\|\lambda(d(x,y)) \\ &\leq \beta. \end{aligned} \tag{476Id}$$

This means that $F \in \mathcal{F}$. Accordingly $\int_{F_1} M-\|x\|\mu(dx) \geq \int_F M-\|x\|\mu(dx)$; since $\mu F = \mu F_1$, $\int_{F_1}\|x\|\mu(dx) \leq \int_F \|x\|\mu(dx)$. By 476Ic, $G = \{x : x \in F_1, x\boldsymbol{.}e < \alpha, R(x) \notin F_1\}$ must be negligible. But G contains x_1 and is relatively open in F_1, and F_1 is supposed to be self-supporting; so this is impossible. **XQ**

(c) Since $\mu F_1 \geq \mu E$, $F_1 \supseteq B_E$, and

$$\begin{aligned}\int_{B_E\times B_E} \|x-y\|\lambda(d(x,y)) &\leq \int_{U(F_1,\epsilon)\times U(F_1,\epsilon)} \|x-y\|\lambda(d(x,y)) \\ &\leq \beta = \int_{U(E,\epsilon)\times U(E,\epsilon)} \|x-y\|\lambda(d(x,y)).\end{aligned}$$

At this point, recall that ϵ was arbitrary. Since E is compact,

$$\begin{aligned}\int_{E\times E} \|x-y\|\lambda(d(x,y)) &= \inf_{\epsilon>0} \int_{U(E,\epsilon)\times U(E,\epsilon)} \|x-y\|\lambda(d(x,y)) \\ &\geq \int_{B_E\times B_E} \|x-y\|\lambda(d(x,y)).\end{aligned}$$

(d) Thus the result is proved for non-empty compact sets E. In general, given a measurable set E of finite measure, then if E is negligible the result is trivial; and otherwise, writing δ for the radius of B_E,

$$\begin{aligned}\int_{B_E\times B_E} \|x-y\|\lambda(d(x,y)) &= \sup_{\alpha<\delta} \int_{B(\mathbf{0},\alpha)\times B(\mathbf{0},\alpha)} \|x-y\|\lambda(d(x,y)) \\ &\leq \sup_{K\subseteq E \text{ is compact}} \int_{B_K\times B_K} \|x-y\|\lambda(d(x,y)) \\ &\leq \sup_{K\subseteq E \text{ is compact}} \int_{K\times K} \|x-y\|\lambda(d(x,y)) \\ &= \int_{E\times E} \|x-y\|\lambda(d(x,y)),\end{aligned}$$

so the proof is complete.

476L The Isoperimetric Theorem Let $r \geq 1$ be an integer, and let μ be Lebesgue measure on $\mathbb{R}^r$. If $E \subseteq \mathbb{R}^r$ is a measurable set of finite measure, then $\operatorname{per} E \geq \operatorname{per} B_E$, where B_E is the closed ball with centre $\mathbf{0}$ and the same measure as E, while $\operatorname{per} E$ is the perimeter of E as defined in 474D.

proof (a) Suppose to begin with that $E \subseteq B(\mathbf{0}, M)$, where $M \geq 0$, and that $\operatorname{per} E < \infty$. Let $\mathcal{F}$ be the family of measurable sets $F \subseteq \mathbb{R}^r$ such that $F \setminus B(\mathbf{0}, M)$ is negligible, $\mu F \geq \mu E$ and $\operatorname{per} F \leq \operatorname{per} E$, with the topology of convergence in measure (474S). Then

$$\begin{aligned}\mathcal{F} = \{F : \operatorname{per} F \leq \operatorname{per} E,\ &\mu(F \cap B(\mathbf{0}, M)) \geq \mu E, \\ &\mu(F \cap B(\mathbf{0},\alpha)) \leq \mu(F \cap B(\mathbf{0}, M)) \text{ for every } \alpha \geq 0\}\end{aligned}$$

is a closed subset of $\{F : \operatorname{per} F \leq \operatorname{per} E\}$, which is compact (474Sb), so $\mathcal{F}$ is compact. For $F \in \mathcal{F}$ set $h(F) = \int_F \|x\|\mu(dx)$; then $|h(F) - h(F')| \leq M\mu((F \triangle F') \cap B(\mathbf{0}, M))$ for all F, $F' \in \mathcal{F}$, so h is continuous.

There is therefore an $F_0 \in \mathcal{F}$ such that $h(F_0) \leq h(F)$ for every $F \in \mathcal{F}$. Set $F_1 = \text{cl}^*F_0$, so that $F_1 \triangle F_0$ is negligible (475Cg), $F_1 \in \mathcal{F}$, $F_1 \subseteq B(\mathbf{0}, M)$ and $h(F_1) = h(F_0)$.

(b) Writing $\delta = \sup_{x \in F_1} \|x\|$, we have $U(\mathbf{0}, \delta) \subseteq F_1$. **P?** Otherwise, there are $x_0 \in \mathbb{R}^r \setminus F_1$ and $x_1 \in F_1$ such that $\|x_0\| < \|x_1\|$. Set $e = \frac{1}{\|x_0 - x_1\|}(x_0 - x_1)$, $\alpha = \frac{1}{2} e \cdot (x_0 + x_1) < 0$, $R = R_{e\alpha} : \mathbb{R}^r \to \mathbb{R}^r$ and $\psi = \psi_{e\alpha}$. Set $F = \psi(F_1)$ and let $\phi = \phi_{F_1} : F_1 \to F$ be the function described in 476I. Then $\|\phi(x)\| \leq \|x\|$ for every $x \in F_1$ (476Ia). In particular, $F = \phi[F_1] \subseteq B(\mathbf{0}, M)$. Now F is measurable and $\mu F = \mu F_1 \geq \mu E$, by 476Ib. Also, writing ν for normalized $r-1$-dimensional Hausdorff measure on $\mathbb{R}^r$,

$$\text{per}\, F = \nu(\partial^* F) \leq \nu(\partial^* F_1) = \text{per}\, F_1 \leq \text{per}\, E,$$

by 475Mb and 476Ie. So $F \in \mathcal{F}$, and

$$\int_F \|x\| \mu(dx) \geq \int_{F_0} \|x\| \mu(dx) = \int_{F_1} \|x\| \mu(dx).$$

By 476Ic, $G = \{x : x \in F_1, x \cdot e < \alpha, R(x) \notin F_1\}$ is negligible. But now consider $G \cap U(x_1, \eta)$ for small $\eta > 0$. Since x_1 belongs to $F_1 = \text{cl}^*F_0 = \text{cl}^*F_1$, but x_0 does not,

$$\limsup_{\eta \downarrow 0} \frac{\mu(F_1 \cap B(x_1, \eta))}{\mu B(x_1, \eta)} > 0 = \limsup_{\eta \downarrow 0} \frac{\mu(F_1 \cap B(x_0, \eta))}{\mu B(x_0, \eta)}.$$

There must therefore be some $\eta > 0$ such that $\eta < \frac{1}{2}\|x_1 - x_0\|$ and $\mu(F_1 \cap B(x_0, \eta)) < \mu(F_1 \cap B(x_1, \eta))$. In this case, however, $G \supseteq F_1 \cap B(x_1, \eta) \setminus R[F_1 \cap B(x_0, \eta)]$ has measure at least $\mu(F_1 \cap B(x_1, \eta)) - \mu(F_1 \cap B(x_0, \eta)) > 0$, which is impossible. **XQ**

(c) Thus $U(\mathbf{0}, \delta) \subseteq F_1 \subseteq B(\mathbf{0}, \delta)$ and $\text{per}\, F_1 = \text{per}\, B(\mathbf{0}, \delta)$. Since $\mu F_1 \geq \mu E$, the radius of B_E is at most δ and

$$\text{per}\, B_E \leq \text{per}\, B(\mathbf{0}, \delta) = \text{per}\, F_1 \leq \text{per}\, E.$$

(d) Thus the result is proved when E is bounded and has finite perimeter. Of course it is trivial when E has infinite perimeter. Now suppose that E is any measurable set with finite measure and finite perimeter. Set $E_\alpha = E \cap B(\mathbf{0}, \alpha)$ for $\alpha \geq 0$; then $\text{per}\, E = \liminf_{\alpha \to \infty} \text{per}\, E_\alpha$ (475Mc). By (a)-(c), $\text{per}\, E_\alpha \geq \text{per}\, B_{E_\alpha}$; since $\text{per}\, B_{E_\alpha} \to \text{per}\, B_E$ as $\alpha \to \infty$, $\text{per}\, E \geq \text{per}\, B_E$ in this case also.

476M Spheres in inner product spaces For the rest of the section I will use the following notation. Let X be a (real) inner product space. Then S_X will be the unit sphere $\{x : x \in X, \|x\| = 1\}$. Let H_X be the isometry group of S_X with its topology of pointwise convergence (441G).

A **cap** in S_X will be a set of the form $\{x : x \in S_X, (x|e) \geq \alpha\}$ where $e \in S_X$ and $-1 \leq \alpha \leq 1$.

When X is finite-dimensional, it is isomorphic, as inner product space, to $\mathbb{R}^r$, where $r = \dim X$ (4A4Jd). In this case S_X is compact, so has a unique H_X-invariant Radon probability measure ν_X, which is strictly positive (476G). If $r \geq 1$ is an integer, we know that the $r-1$-dimensional Hausdorff measure of the sphere $S_{\mathbb{R}^r}$ is finite and non-zero (265F). Since Hausdorff measures are invariant under isometries (264G, 471J), and are quasi-Radon measures when totally finite (471Dh), $r-1$-dimensional Hausdorff measure on $S_{\mathbb{R}^r}$ is a multiple of the normalized invariant measure $\nu_{\mathbb{R}^r}$, by 476G. The same is therefore true in any r-dimensional inner product space.

476N Lemma Let X be a real inner product space and $f \in H_X$. Then $(f(x)|f(y)) = (x|y)$ for all x, $y \in S_X$. Consequently $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ whenever x, $y \in S_X$ and α, $\beta \in \mathbb{R}$ are such that $\alpha x + \beta y \in S_X$.

proof (a) We have

$$\rho(x, y)^2 = (x - y | x - y) = (x|x) - 2(x|y) + (y|y) = 2 - 2(x|y),$$

so

$$(x|y) = 1 - \tfrac{1}{2}\rho(x, y)^2 = 1 - \tfrac{1}{2}\rho(f(x), f(y))^2 = (f(x)|f(y)).$$

(b)

$$\begin{aligned}\|f(\alpha x+\beta y)-\alpha f(x)-\beta f(y)\|^2&\\ &=(f(\alpha x+\beta y)-\alpha f(x)-\beta f(y)|f(\alpha x+\beta y)-\alpha f(x)-\beta f(y))\\ &=1+\alpha^2+\beta^2-2\alpha(f(\alpha x+\beta y)|f(x))-2\beta(f(\alpha x+\beta y)|f(y))+2\alpha\beta(f(x)|f(y))\\ &=1+\alpha^2+\beta^2-2\alpha(\alpha x+\beta y|x)-2\beta(\alpha x+\beta y|y)+2\alpha\beta(x|y)\\ &=\|(\alpha x+\beta y)-\alpha x-\beta y\|^2=0.\end{aligned}$$

476O I give a theorem on concentration of measure on the sphere corresponding to 476J.

Theorem Let X be a finite-dimensional inner product space of dimension at least 2, S_X its unit sphere and ν_X the invariant Radon probability measure on S_X. For a non-empty set $A\subseteq S_X$ and $\epsilon>0$, write $U(A,\epsilon)=\{x:\rho(x,A)<\epsilon\}$, where ρ is the usual metric of X. Then there is a cap $C\subseteq S_X$ such that $\nu_X C=\nu_X^*A$, and $\nu_X(S_X\cap U(A,\epsilon))\geq\nu_X(S_X\cap U(C,\epsilon))$ for any such C.

proof In order to apply the results of 476H-476I directly, and simplify some of the formulae slightly, it will be helpful to write ν for the Radon measure on X defined by setting $\nu E=\nu_X(E\cap S_X)$ whenever this is defined. By 214Cd, ν^* agrees with ν_X^* on $\mathcal{P}S_X$.

(a) The first step is to check that there is a cap C of S_X such that $\nu C=\nu^*A$. **P** Take any $e_0\in S_X$, and set $C_\alpha=\{x:x\in S_X,\,(x|e_0)\geq\alpha\}$ for $\alpha\in[-1,1]$. νC_α is defined for every $\alpha\in\mathbb{R}$ because every C_α is closed and ν is a topological measure. Now examine the formulae of 265F. We can identify X with $\mathbb{R}^{r+1}$ where $r+1=\dim X$; do this in such a way that e_0 corresponds to the unit vector $(0,\dots,0,1)$. We have a parametrization $\phi_r:D_r\to S_X$, where D_r is a Borel subset of $\mathbb{R}^r$ with interior $]-\pi,\pi[\times]0,\pi[^{r-1}$ and ϕ_r is differentiable with continuous derivative. Moreover, if $x=(\xi_1,\dots,\xi_r)\in D_r$, then $\phi_r(x)\cdot e_0=\cos\xi_r$, and the Jacobian J_r of ϕ_r is bounded by 1 and never zero on $\operatorname{int}D_r$. Finally, the boundary ∂D_r is negligible. What this means is that $\nu_r C_\alpha=\int_{E_\alpha}J_r d\mu_r$, where μ_r is Lebesgue measure on $\mathbb{R}^r$, ν_r is normalized Hausdorff r-dimensional measure on $\mathbb{R}^{r+1}$, and $E_\alpha=\{x:x\in D_r,\cos\xi_r\geq\alpha\}$. So if $-1\leq\alpha\leq\beta\leq1$ then

$$\nu_r C_\alpha-\nu_r C_\beta\leq\mu_r(E_\alpha\setminus E_\beta)\leq 2\pi^{r-1}(\arccos\alpha-\arccos\beta);$$

because arccos is continuous, so is $\alpha\mapsto\nu_r C_\alpha$. Also, if $\alpha<\beta$, then $E_\alpha\setminus E_\beta$ is non-negligible, so $\int_{E_\alpha\setminus E_\beta}J_r d\mu_r\neq0$ and $\nu_r C_\alpha>\nu_r C_\beta$.

This shows that $\alpha\mapsto\nu_r C_\alpha$ is continuous and strictly decreasing; since ν_r is just a multiple of ν on S_X, the same is true of $\alpha\mapsto\nu C_\alpha$.

Since $\nu C_{-1}=\nu S_X=1$ and $\nu C_1=\nu\{e_0\}=0$, the Intermediate Value Theorem tells us that there is a unique α such that $\nu C_\alpha=\nu^*A$, and we can set $C=C_\alpha$. **Q**

(b) Now take any non-empty set $A\subseteq S_X$ and any $\epsilon>0$, and set $\gamma=\nu^*A$, $\beta=\nu U(A,\epsilon)$. Let C be a cap such that $\nu^*A=\nu C$; let e_0 be the centre of C. Consider the family

$$\mathcal{F}=\{F:F\in\mathcal{C}\setminus\{\emptyset\},\,F\subseteq S_X,\,\nu F\geq\gamma,\,\nu U(F,\epsilon)\leq\beta\},$$

where $\mathcal{C}$ is the family of closed subsets of X. Because $U(\overline{A},\epsilon)=U(A,\epsilon)$, $\overline{A}\in\mathcal{F}$ and $\mathcal{F}$ is non-empty. By 476D and 476F, $\mathcal{F}$ is closed for the Hausdorff metric on $\mathcal{C}\setminus\{\emptyset\}$, therefore compact, by 475Ad, because S_X is compact. Next, the function

$$F\mapsto\int_F\max(0,1+(x|e_0))\nu(dx):\mathcal{C}\setminus\{\emptyset\}\to[0,\infty[$$

is upper semi-continuous, by 476Dc. It therefore attains its supremum on $\mathcal{F}$ at some $F_0\in\mathcal{F}$. Let $F_1\subseteq F_0$ be a self-supporting closed set with the same measure as F_0; then $F_1\in\mathcal{F}$ and $\int_F(1+(x|e_0))\nu(dx)\geq\int_{F_1}(1+(x|e_0))\nu(dx)$ for every $F\in\mathcal{F}$.

(c) F_1 is a cap with centre e_0. **P?** Otherwise, there are $x_0\in S_X\setminus F_1$ and $x_1\in F_1$ such that $(x_0|e_0)>(x_1|e_0)$. Set $e=\frac{1}{\|x_0-x_1\|}(x_0-x_1)$. Then $e\in S_X$ and $(e|e_0)>0$. Set $R=R_{e0}$ and $\psi=\psi_{e0}$ as defined in 476H; write F for $\psi(F_1)$. Note that $(x_0+x_1|x_0-x_1)=\|x_0\|^2-\|x_1\|^2=0$, so $(x_0+x_1|e)=0$ and

$R(x_0) = x_1$. Also $R[S_X] = S_X$, so ν is R-invariant, because ν is a multiple of Hausdorff $r-1$-dimensional measure on S_X and must be invariant under isometries of S_X.

We have $\nu F = \nu F_1 \geq \gamma$, by 476Ib, and

$$\nu U(F,\epsilon) \leq \nu\psi(U(F_1,\epsilon)) = \nu U(F_1,\epsilon) \leq \nu U(F_0,\epsilon) \leq \beta$$

by 476Ha. So $F \in \mathcal{F}$. But consider the standard bijection $\phi = \phi_{F_1} : F_1 \to F$ as defined in 476I. We have

$$\int_{F_1}(1+(\phi(x)|e_0))\nu(dx) = \int_F (1+(x|e_0))\nu(dx) \leq \int_{F_1}(1+(x|e_0))\nu(dx).$$

But if we examine the definition of ϕ, we see that $\phi(x) \neq x$ only when $(x|e) < 0$ and $\phi(x) = R(x)$, so that in this case $\phi(x) - x$ is a positive multiple of e and $(\phi(x)|e_0) > (x|e_0)$. So $G = \{x : x \in F_1,\ (x|e) < 0,\ R(x) \notin F_1\}$ must be ν-negligible. But G includes a relative neighbourhood of x_1 in F_1 and F_1 is supposed to be self-supporting for ν, so this is impossible. **XQ**

(e) Now $\nu^* A = \gamma \leq \nu F_1$, so $C \subseteq F_1$ and

$$\nu U(A,\epsilon) = \beta \geq \nu U(F_1,\epsilon) \geq \nu U(C,\epsilon),$$

as claimed.

476P Corollary For any $\epsilon > 0$, there is an $r_0 \geq 2$ such that whenever X is a finite-dimensional inner product space of dimension at least r_0, A_1, $A_2 \subseteq S_X$ and $\min(\nu_X^* A_1, \nu_X^* A_2) \geq \epsilon$, then there are $x \in A_1$, $y \in A_2$ such that $\|x - y\| \leq \epsilon$.

proof Take $r_0 \geq 2$ such that $r_0\epsilon^3 > 2$. Suppose that $\dim X = r \geq r_0$. Fix $e_0 \in S_X$. We need an estimate of $\nu_X C_{\epsilon/2}$, where $C_{\epsilon/2} = \{x : x \in S_X,\ (x|e_0) \geq \epsilon/2\}$ as in 476O. To get this, let $e_1, \ldots, e_{r-1}$ be such that $e_0, \ldots, e_{r-1}$ is an orthonormal basis of X (4A4Kc). For each $i < r$, there is an $f \in H_X$ such that $f(e_i) = e_0$, so that $(x|e_i) = (f(x)|e_0)$ for every x (476N), and

$$\int (x|e_i)^2 \nu_X(dx) = \int (f(x)|e_0)^2 \nu_X(dx) = \int (x|e_0)^2 \nu_X(dx),$$

because $f : S_X \to S_X$ is inverse-measure-preserving for ν_X.

Accordingly

$$\begin{aligned}
\nu_X C_{\epsilon/2} &= \tfrac{1}{2}\nu_X\{x : x \in S_X,\ |(x|e_0)| \geq \epsilon/2\} \\
&\leq \frac{2}{\epsilon^2}\int_{S_X}(x|e_0)^2\nu_X(dx) < r\epsilon\int_{S_X}(x|e_0)^2\nu_X(dx) \\
&= \epsilon\sum_{i=0}^{r-1}\int_{S_X}(x|e_i)^2\nu_X(dx) = \epsilon\int_{S_X}\sum_{i=0}^{r-1}(x|e_i)^2\nu_X(dx) = \epsilon \leq \nu_X^* A_1.
\end{aligned}$$

So, taking C to be the cap of S_X with centre e_0 and measure $\nu^* A_1$, $C = C_\alpha$ where $\alpha < \frac{1}{2}\epsilon$, and

$$\nu_X(S_X \cap U(A_1, \tfrac{1}{2}\epsilon)) \geq \nu_X(S_X \cap U(C_\alpha, \tfrac{1}{2}\epsilon)) \geq \nu_X C_{\alpha - \epsilon/2} > \frac{1}{2}.$$

Similarly, $\nu_X(S_X \cap U(A_2, \frac{1}{2}\epsilon)) > \frac{1}{2}$ and there must be some $z \in S_X \cap U(A_1, \frac{1}{2}\epsilon) \cap U(A_2, \frac{1}{2}\epsilon)$. Take $x \in A_1$ and $y \in A_2$ such that $\|x - z\| < \frac{1}{2}\epsilon$ and $\|y - z\| < \frac{1}{2}\epsilon$; then $\|x - y\| \leq \epsilon$, as required.

476X Basic exercises (a) Let (X,ρ) be a metric space and give $\mathcal{C} \setminus \{\emptyset\}$ its Hausdorff metric, where $\mathcal{C}$ is the family of closed subsets of X. (i) Show that $x \mapsto \{x(i) : i < n\} : X^n \to \mathcal{C} \setminus \{\emptyset\}$ is continuous for every $n \geq 1$. (ii) Show that $(E,F) \mapsto E \cup F : (\mathcal{C} \setminus \{\emptyset\})^2 \to \mathcal{C} \setminus \{\emptyset\}$ is continuous. (iii) Show that $\operatorname{diam} : \mathcal{C} \setminus \{\emptyset\} \to [0,\infty]$ is continuous.

(b) Let (X,ρ) be a metric space and $\mathcal{K}$ the family of compact subsets of X. (i) Show that the Hausdorff metric topology and the Vietoris topology coincide on $\mathcal{K} \setminus \{\emptyset\}$. (ii) Show that if G is the isometry group of X with its topology of pointwise convergence (441G), then $(g,K) \mapsto g[K]$ is a continuous action of G on $\mathcal{K}$ if $\mathcal{K}$ is given the topology induced by the Vietoris topology.

(c) Let (X,ρ) be a metric space, and $\mathcal{C}$ the set of closed subsets of X. Show that the topology induced by the Hausdorff metric on $\mathcal{C}\setminus\{0\}$ is finer than the subspace topology induced by the k-Vietoris topology on $\mathcal{C}$.

(d) Let (X,ρ) be a locally compact metric space, and $\mathcal{C}$ the set of closed subsets of X with its k-Vietoris topology. (i) Show that if G is the isometry group of X with its topology of pointwise convergence, then $(g,C)\mapsto g[C]$ is a continuous action of G on $\mathcal{C}$. (ii) Show that if X is separable then $\mathcal{C}$ is metrizable.

(e) Let (X,ρ) be a metric space and give $\mathcal{C}\setminus\{\emptyset\}$ its Hausdorff metric, where $\mathcal{C}$ is the family of closed subsets of X. Show that whenever μ is a topological measure on X and f is a μ-integrable real-valued function, then $F\mapsto\int_F f\,d\mu:\mathcal{C}\setminus\{0\}\to\mathbb{R}$ is Borel measurable.

(f) Let X be a topological space and give $\mathcal{C}$ its Vietoris topology, where $\mathcal{C}$ is the family of closed subsets of X. (i) Show that if X is T_0 then $x\mapsto\{x(i):i<n\}:X^n\to\mathcal{C}\}$ is continuous for every $n\geq 1$. (ii) Show that $(E,F)\mapsto E\cup F:\mathcal{C}^2\to\mathcal{C}$ is continuous.

(g) In the context of 476H, show that $\operatorname{diam}\psi_{e\alpha}(A)\leq\operatorname{diam}A$ for all A, e and α.

>(h) Find an argument along the lines of those in 476J and 476K to prove 264H. (*Hint*: 476Xg.)

>(i) Let X be an inner product space and S_X its unit sphere. Show that every isometry $f:S_X\to S_X$ extends uniquely to an isometry $T_f:X\to X$ which is a linear operator. (*Hint*: first check the cases in which $\dim X\leq 2$.) Show that f is surjective iff T_f is, so that we have a natural isomorphism between the isometry group of S_X and the group of invertible isometric linear operators. Show that this isomorphism is a homeomorphism for the topologies of pointwise convergence.

(j) Let X be a finite-dimensional inner product space, ν_X the invariant Radon probability measure on the sphere S_X, and $E\in\operatorname{dom}\nu_X$; let $C\subseteq S_X$ be a cap with the same measure as E, and let λ be the product measure of ν_X with itself on $S_X\times S_X$. Show that $\int_{C\times C}\|x-y\|\lambda(d(x,y))\leq\int_{E\times E}\|x-y\|\lambda(d(x,y))$.

(k) Let X be a finite-dimensional inner product space and ν_X the invariant Radon probability measure on the sphere S_X. (i) Without appealing to the formulae in §265, show that $\nu_X(S_X\cap H)=0$ for whenever $H\subseteq X$ is a proper affine subspace. (*Hint*: induce on $\dim H$.) (ii) Use this to prove that if $e\in S_X$ then $\alpha\mapsto\nu_X\{x:(x|e)\geq\alpha\}$ is continuous.

476Y Further exercises (a) Let (X,ρ) and (Y,σ) be metric spaces, and $f:X\to Y$ a function. Write $\mathcal{C}_X$, $\mathcal{C}_Y$ for the spaces of closed subsets of X, Y respectively, and give $\mathcal{C}_X\setminus\{\emptyset\}$, $\mathcal{C}_Y\setminus\{\emptyset\}$ their Hausdorff metrics. For $F\in\mathcal{C}_X\setminus\{\emptyset\}$ set $\tilde{f}(F)=\overline{f[F]}\in\mathcal{C}_Y\setminus\{\emptyset\}$. Show that the following are equiveridical: (i) f is uniformly continuous (ii); $\tilde{f}$ is continuous; (iii) $\tilde{f}$ is uniformly continuous.

(b) Let X and Y be topological spaces, and $\mathcal{C}_X$, $\mathcal{C}_Y$ the spaces of closed subsets of X, Y respectively, with their Vietoris topologies. Suppose that $f:X\to Y$ is a continuous function and that Y is normal. Show that $F\mapsto\overline{f[F]}:\mathcal{C}_X\to\mathcal{C}_Y$ is continuous.

(c) Let X be a compact metric space and G its isometry group. Suppose that $H\subseteq G$ is a subgroup such that the action of H on X is transitive. Show that X has a unique H-invariant Radon probability measure which is also G-invariant.

476 Notes and comments The main theorems here (476J-476K, 476O), like 264H, are all 'classical'; they go back to the roots of geometric measure theory, and the contribution of the twentieth century has been to extend the classes of sets for which balls or caps provide the bounding examples. It is very striking that they can all be proved with the same tools (see 476Xh). Of course I should remark that the Compactness Theorem (474S) lies at a much deeper level than the rest of the ideas here. (The proof of 474S relies on the distributional definition of 'perimeter' in 474D, while the arguments of 476Ie and 476L work with the Hausdorff measures of essential boundaries; so that we can join these ideas together only after proving all the

principal theorems of §§472-475.) So while 'Steiner symmetrization' (264H) and 'concentration by partial reflection' (476H) are natural companions, 476L is essentially harder than the other results.

In all the theorems here, as in 264H, I have been content to show that a ball or a cap is an optimum for whatever inequality is being considered. I have not examined the question of whether, and in what sense, the optimum is unique. It seems that this requires deeper analysis.

Chapter 48

Gauge integrals

For the penultimate chapter of this volume I turn to a completely different approach to integration which has been developed in the last forty years, following KURZWEIL 57 and HENSTOCK 63. This depends for its inspiration on a particular formulation of the Riemann integral (see 481Xe), and leads in particular to some remarkable extensions of the Lebesgue integral (§§483-484). While (in my view) it remains peripheral to the most important parts of measure theory, it has deservedly attracted a good deal of interest in recent years, and is entitled to a place here.

From the very beginning, in the definitions of §122, I have presented the Lebesgue integral in terms of almost-everywhere approximations by simple functions. Because the integral $\int \lim_{n\to\infty} f_n$ of a limit is *not* always the limit $\lim_{n\to\infty} \int f_n$ of the integrals, we are forced, from the start, to constrain ourselves by the ordering, and to work with monotone or dominated sequences. This almost automatically leads us to an 'absolute' integral, in which $|f|$ is integrable whenever f is, whether we start from measures (as in Chapter 11) or from linear functionals (as in §436). For four volumes now I have been happily developing the concepts and intuitions appropriate to such integrals. But if we return to one of the foundation stones of Lebesgue's theory, the Fundamental Theorem of Calculus, we find that it is easy to construct a differentiable function f such that the absolute value $|f'|$ of its derivative is not integrable (483Xd). It was observed very early (PERRON 14) that the Lebesgue integral can be extended to integrate the derivative of any function which is differentiable everywhere. The achievement of HENSTOCK 63 was to find a formulation of this extension which was conceptually transparent enough to lend itself to a general theory, fragments of which I will present here.

The first step is to set out the essential structures on which the theory depends (§481), with a first attempt at a classification scheme. (One of the most interesting features of the Kurzweil-Henstock approach is that we have an extraordinary degree of freedom in describing our integrals, and apart from the Henstock integral itself it is not clear that we have yet found the right canonical forms to use.) In §482 I give a handful of general theorems showing what kind of results can be expected and what difficulties arise. In §483, I work through the principal properties of the Henstock integral on the real line, showing, in particular, that it coincides with the Perron and special Denjoy integrals. Finally, in §484, I look at a very striking integral on $\mathbb{R}^r$, due to W.F.Pfeffer.

481 Tagged partitions

I devote this section to establishing some terminology (481A-481B, 481E-481G) and describing a variety of examples (481I-481Q), some of which will be elaborated later. The clearest, simplest and most important example is surely Henstock's integral on a closed bounded interval (481J), so I recommend turning immediately to that paragraph and keeping it in mind while studying the notation here. It may also help you to make sense of the definitions here if you glance at the statements of some of the results in §482; in this section I give only the formula defining gauge integrals (481C), with some elementary examples of its use (481Xb-481Xh).

481A Tagged partitions and Riemann sums The common idea underlying all the constructions of this chapter is the following. We have a set X and a functional ν defined on some family $\mathcal{C}$ of subsets of X. We seek to define an integral $\int f d\nu$, for functions f with domain X, as a limit of *finite* **Riemann sums** $\sum_{i=0}^{n} f(x_i)\nu C_i$, where $x_i \in X$ and $C_i \in \mathcal{C}$ for $i \leq n$. There is no strict reason, at this stage, to forbid repetitions in the string $(x_0, C_0), \dots, (x_n, C_n)$, but also little to be gained from allowing them, and it will simplify some of the formulae below if I say from the outset that a **tagged partition** on X will be a finite subset $\boldsymbol{t}$ of $X \times \mathcal{P}X$.

So one necessary element of the definition will be a declaration of which tagged partitions $\{(x_0, C_0), \dots, (x_n, C_n)\}$ will be employed, in terms, for instance, of which sets C_i are permitted, whether they are allowed to overlap at their boundaries, whether they are required to cover the space, and whether each **tag** x_i is required to belong to the corresponding C_i. The next element of the definition will be a description of a filter $\mathcal{F}$ on the set T of tagged partitions, so that the integral will be the limit (when it exists) of the sums along the filter, as in 481C below.

In the formulations studied in this chapter, the C_i will generally be disjoint, but this is not absolutely essential, and it is occasionally convenient to allow them to overlap in 'small' sets, as in 481Ya. In some cases, we can restrict attention to families for which the C_i are non-empty and have union X, so that $\{C_0, \dots, C_n\}$ is a partition of X in the strict sense.

481B Notation Let me immediately introduce notations which will be in general use throughout the chapter.

(a) First, a shorthand to describe a particular class of sets of tagged partitions. If X is a set, a **straightforward set of tagged partitions** on X is a set of the form

$$T = \{\boldsymbol{t} : \boldsymbol{t} \in [Q]^{<\omega},\ C \cap C' = \emptyset \text{ whenever } (x, C), (x', C') \text{ are distinct members of } \boldsymbol{t}\}$$

where $Q \subseteq X \times \mathcal{P}X$; I will say that T is **generated** by Q. In this case, of course, Q can be recovered from T, since $Q = \bigcup T$. Note that no control is imposed on the tags at this point. It remains theoretically possible that a pair $(x, \emptyset)$ should belong to Q, though in essentially all applications this will be excluded in one way or another.

(b) If X is a set and $\boldsymbol{t} \subseteq X \times \mathcal{P}X$ is a tagged partition, I write

$$W_{\boldsymbol{t}} = \bigcup\{C : (x, C) \in \boldsymbol{t}\}.$$

(c) If X is a set, $\mathcal{C}$ is a family of subsets of X, $f : X \to \mathbb{R}$ and $\nu : \mathcal{C} \to \mathbb{R}$ are functions and $\boldsymbol{t} \in [X \times \mathcal{C}]^{<\omega}$ is a tagged partition, then

$$S_{\boldsymbol{t}}(f, \nu) = \textstyle\sum_{(x,C)\in\boldsymbol{t}} f(x)\nu C.$$

481C Proposition Let X be a set, $\mathcal{C}$ a family of subsets of X, $T \subseteq [X \times \mathcal{C}]^{<\omega}$ a non-empty set of tagged partitions and $\mathcal{F}$ a filter on T. For any functions $f : X \to \mathbb{R}$ and $\nu : \mathcal{C} \to \mathbb{R}$, set

$$I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}} S_{\boldsymbol{t}}(f, \nu)$$

if this is defined in $\mathbb{R}$.

(a) I_ν is a linear functional defined on a linear subspace of $\mathbb{R}^X$.

(b) Now suppose that $\nu C \geq 0$ for every $C \in \mathcal{C}$. Then

(i) I_ν is a positive linear functional (definition: 351F);

(ii) if f, $g : X \to \mathbb{R}$ are such that $|f| \leq g$ and $I_\nu(g)$ is defined and equal to 0, then $I_\nu(f)$ is defined and equal to 0.

proof (a) We have only to observe that if f, $g \in \mathbb{R}^X$ and $\alpha \in \mathbb{R}$, then

$$S_{\boldsymbol{t}}(f + g, \nu) = S_{\boldsymbol{t}}(f, \nu) + S_{\boldsymbol{t}}(g, \nu), \quad S_{\boldsymbol{t}}(\alpha f, \nu) = \alpha S_{\boldsymbol{t}}(f, \nu)$$

for every $\boldsymbol{t} \in T$, and apply 2A3Sf.

(b) If $g \geq 0$ in $\mathbb{R}^X$, that is, $g(x) \geq 0$ for every $x \in X$, then $S_{\boldsymbol{t}}(g, \nu) \geq 0$ for every $\boldsymbol{t} \in T$, so the limit $I_\nu(g)$, if defined, will also be non-negative. Next, if $|f| \leq g$, then $|S_{\boldsymbol{t}}(f, \nu)| \leq S_{\boldsymbol{t}}(g, \nu)$ for every $\boldsymbol{t}$, so if $I_\nu(g) = 0$ then $I_\nu(f)$ is also zero.

481D Remarks (a) Functionals $I_\nu = \lim_{\boldsymbol{t}\to\mathcal{F}} S_{\boldsymbol{t}}(., \nu)$, as described above, are called **gauge integrals**.

(b) In fact even greater generality is possible at this point. There is no reason why f and ν should take real values. All we actually need is an interpretation of sums of products $f(x) \times \nu C$ in a space in which we can define limits. So for any linear spaces U, V and W with a bilinear functional $\phi : U \times V \to W$ (253A) and a Hausdorff linear space topology on W, we can set out to construct an integral of a function $f : X \to U$ with respect to a functional $\nu : \mathcal{C} \to V$ as a limit of sums $S_{\boldsymbol{t}}(f, \nu) = \sum_{(x,C)\in\boldsymbol{t}} \phi(f(x), \nu C)$ in W. I will not go farther along this path here. But it is worth noting that the constructions of this chapter lead the way to interesting vector integrals of many types.

(c) An extension which is, however, sometimes useful is to allow ν to be undefined (or take values outside $\mathbb{R}$, such as $\pm\infty$) on part of $\mathcal{C}$. In this case, set $\mathcal{C}_0 = \nu^{-1}[\mathbb{R}]$. Provided that $T \cap [X \times \mathcal{C}_0]^{<\omega}$ belongs to $\mathcal{F}$, we can still define I_ν, and 481C will still be true.

481E Gauges The most useful method (so far) of defining filters on sets of tagged partitions is the following. If X is a set, a **gauge** on X is a subset δ of $X \times \mathcal{P}X$. For a gauge δ, a tagged partition $\boldsymbol{t}$ is **δ-fine** if $\boldsymbol{t} \subseteq \delta$. Now, for a set Δ of gauges and a set T of tagged partitions, we can seek to define a filter $\mathcal{F}$ on T as the filter generated by sets of the form $T_\delta = \{\boldsymbol{t} : \boldsymbol{t} \in T \text{ is } \delta\text{-fine}\}$ as δ runs over Δ. Of course we shall need to establish that T and Δ are compatible in the sense that $\{T_\delta : \delta \in \Delta\}$ has the finite intersection property; this will ensure that there is indeed a filter containing every T_δ (4A1Ia).

In nearly all cases, Δ will be non-empty and downwards-directed (that is, for any δ_1, $\delta_2 \in \Delta$ there will be a $\delta \in \Delta$ such that $\delta \subseteq \delta_1 \cap \delta_2$); in this case, we shall need only to establish that T_δ is non-empty for every $\delta \in \Delta$. Note that the filter on T generated by $\{T_\delta : \delta \in \Delta\}$ depends only on T and the filter on $X \times \mathcal{P}X$ generated by Δ.

The most important gauges (so far) are 'neighbourhood gauges'. If $(X, \mathfrak{T})$ is a topological space, a **neighbourhood gauge** on X is a set expressible in the form $\delta = \{(x, C) : x \in X,\ C \subseteq G_x\}$ where $\langle G_x \rangle_{x \in X}$ is a family of open sets such that $x \in G_x$ for every $x \in X$. It is useful to note (i) that the family $\langle G_x \rangle_{x \in X}$ can be recovered from δ, since $G_x = \bigcup\{A : (x, A) \in \delta\}$ (ii) that $\delta_1 \cap \delta_2$ is a neighbourhood gauge whenever δ_1 and δ_2 are. When (X, ρ) is a metric space, we can define a neighbourhood gauge δ_h from any function $h : X \to]0, \infty[$, setting

$$\delta_h = \{(x, C) : x \in X,\ C \subseteq X,\ \rho(y, x) < h(x) \text{ for every } y \in C\}.$$

The set of gauges expressible in this form is coinitial with the set of all neighbourhood gauges and therefore defines the same filter on any compatible set T of tagged partitions. Specializing yet further, we can restrict attention to constant functions h, obtaining the **uniform metric gauges**

$$\delta_\eta = \{(x, C) : x \in X,\ C \subseteq X,\ \rho(x, y) < \eta \text{ for every } y \in C\}$$

for $\eta > 0$, used in the Riemann integral (481I). (The use of the symbol 'δ' to represent a gauge has descended from its traditional appearance in the definition of the Riemann integral.)

481F Residual sets The versatility and power of the methods being introduced here derives from the insistence on taking *finite* sums $\sum_{(x,C) \in \boldsymbol{t}} f(x)\nu C$, so that all questions about convergence are concentrated in the final limit $\lim_{\boldsymbol{t} \to \mathcal{F}} S_{\boldsymbol{t}}(f, \nu)$. Since in a given Riemann sum we can look at only finitely many sets C of finite measure, we cannot insist, even when ν is Lebesgue measure on $\mathbb{R}$, that $W_{\boldsymbol{t}}$ should always be X. There are many other cases in which it is impossible or inappropriate to insist that $W_{\boldsymbol{t}} = X$ for every tagged partition in T. We shall therefore need to add something to the definition of the filter $\mathcal{F}$ on T beyond what is possible in the language of 481E. In the examples below, the extra condition will always be of the following form. There will be a collection $\mathfrak{R}$ of **residual families** $\mathcal{R} \subseteq \mathcal{P}X$. It will help to have a phrase corresponding to the phrase 'δ-fine': if $\mathcal{R} \subseteq \mathcal{P}X$, and $\boldsymbol{t}$ is a tagged partition on X, I will say that $\boldsymbol{t}$ is **$\mathcal{R}$-filling** if $X \setminus W_{\boldsymbol{t}} \in \mathcal{R}$. Now, given a family $\mathfrak{R}$ of residual sets, and a family Δ of gauges on X, we can seek to define a filter $\mathcal{F}(T, \Delta, \mathfrak{R})$ on T as that generated by sets of the form

$$T_{\delta\mathcal{R}} = \{\boldsymbol{t} : \boldsymbol{t} \in T \text{ is } \delta\text{-fine and } \mathcal{R}\text{-filling}\}$$

where δ runs over Δ and $\mathcal{R}$ runs over $\mathfrak{R}$. When there is such a filter, that is, the family $\{T_{\delta R} : \delta \in \Delta,\ \mathcal{R} \in \mathfrak{R}\}$ has the finite intersection property, I will say that T is **compatible** with Δ and $\mathfrak{R}$.

It is important here to note that we shall *not* suppose that, for a typical residual family $\mathcal{R} \in \mathfrak{R}$, subsets of members of $\mathcal{R}$ again belong to $\mathcal{R}$; there will frequently be a restriction on the 'shape' of members of $\mathcal{R}$ as well as on their size. On the other hand, it will usually be helpful to arrange that $\mathfrak{R}$ is a filter base, so that (if Δ is also downwards-directed) we need only show that every $T_{\delta\mathcal{R}}$ is non-empty, and $\{T_{\delta\mathcal{R}} : \delta \in \Delta,\ \mathcal{R} \in \mathfrak{R}\}$ will be a base for $\mathcal{F}(T, \Delta, \mathfrak{R})$.

If the filter $\mathcal{F}$ is defined as in 481E, with no mention of a family $\mathfrak{R}$, we can still bring the construction into the framework considered here by setting $\mathfrak{R} = \{\mathcal{P}X\}$. If it is convenient to define T in terms which do not impose any requirement on the sets $W_{\boldsymbol{t}}$, but nevertheless we wish to restrict attention to sums $S_{\boldsymbol{t}}(f, \nu)$ for which the tagged partition covers the whole space X, we can do so by setting $\mathfrak{R} = \{\{\emptyset\}\}$.

481G Subdivisions When we come to analyse the properties of integrals constructed by the method of 481C, there is an important approach which depends on the following combination of features. I will say that $(X, T, \Delta, \mathfrak{R})$ is a **tagged-partition structure allowing subdivisions**, **witnessed by** $\mathcal{C}$, if

(i) X is a set.

(ii) Δ is a downwards-directed family of gauges on X.

(iii)(α) $\mathfrak{R}$ is a non-empty downwards-directed collection of families of subsets of X, all containing $\emptyset$;

(β) for every $\mathcal{R} \in \mathfrak{R}$ there is an $\mathcal{R}' \in \mathfrak{R}$ such that $A \cup B \in \mathcal{R}$ whenever A, $B \in \mathcal{R}'$ are disjoint.

(iv) $\mathcal{C}$ is a family of subsets of X such that whenever C, $C' \in \mathcal{C}$ then $C \cap C' \in \mathcal{C}$ and $C \setminus C'$ is expressible as the union of a disjoint finite subset of $\mathcal{C}$.

(v) Whenever $\mathcal{C}_0 \subseteq \mathcal{C}$ is finite and $\mathcal{R} \in \mathfrak{R}$, there is a finite set $\mathcal{C}_1 \subseteq \mathcal{C}$, including $\mathcal{C}_0$, such that $X \setminus \bigcup \mathcal{C}_1 \in \mathcal{R}$.

(vi) $T \subseteq [X \times \mathcal{C}]^{<\omega}$ is (in the language of 481Ba) a straightforward set of tagged partitions on X.

(vii) Whenever $C \in \mathcal{C}$, $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ there is a δ-fine tagged partition $\boldsymbol{t} \in T$ such that $W_{\boldsymbol{t}} \subseteq C$ and $C \setminus W_{\boldsymbol{t}} \in \mathcal{R}$.

481H Remarks(a) Conditions (ii) and (iii-α) of 481G are included primarily for convenience, since starting from any Δ and $\mathfrak{R}$ we can find directed sets leading to the same filter $\mathcal{F}(T, \Delta, \mathfrak{R})$. (iii-$\beta$), on the other hand, is saying something new.

(b) It is important to note, in (vii) of 481G, that the tags of $\boldsymbol{t}$ there are *not* required to belong to the set C.

(c) All the applications below will fall into one of two classes. In one type, the residual families $\mathcal{R} \in \mathfrak{R}$ will be families of 'small' sets, in some recognisably measure-theoretic sense, and, in particular, we shall have subsets of members of any $\mathcal{R}$ belonging to $\mathcal{R}$. In the other type, (vii) of 481G will be true because we can always find $\boldsymbol{t} \in T$ such that $W_{\boldsymbol{t}} = C$.

(d) The following elementary fact got left out of §136 and Chapter 31. Let $\mathfrak{A}$ be a Boolean algebra and $C \subseteq \mathfrak{A}$. Set

$$E = \{\sup C_0 : C_0 \subseteq C \text{ is finite and disjoint}\}.$$

If $c \cap c'$ and $c \setminus c'$ belong to E for all c, $c' \in C$, then E is a subring of $\mathfrak{A}$. **P** Write $\mathcal{D}$ for the family of finite disjoint subsets of C. (i) If C_0, $C_1 \in \mathcal{D}$, then for $c \in C_0$, $c' \in C_1$ there is a $D_{cc'} \in \mathcal{D}$ with supremum $c \cap c'$. Now $D = \sup_{c \in C_0, c' \in C_1} D_{cc'}$ belongs to $\mathcal{D}$ and has supremum $(\sup C_0) \cap (\sup C_1)$. Thus $e \cap e' \in E$ for all e, $e' \in E$. (ii) Of course $e \cup e' \in E$ whenever e, $e' \in E$ and $e \cap e' = 0$. (iii) Again suppose that C_0, $C_1 \in \mathcal{D}$. Then $c \setminus c' \in E$ for all $c \in C_0$, $c' \in C_1$. By (i), $c \setminus \sup C_1 \in E$ for every $c \in C$; by (ii), $(\sup C_0) \setminus (\sup C_1) \in E$. Thus $e \setminus e' \in E$ for all e, $e' \in E$. (iv) Putting (ii) and (iii) together, $e \triangle e' \in E$ for all e, $e' \in E$. (v) As $0 = \sup \emptyset$ belongs to E, E is a subring of $\mathfrak{A}$. **Q**

In particular, if $\mathcal{C} \subseteq \mathcal{P}X$ has the properties in (iv) of 481G, then

$$\mathcal{E} = \{\bigcup \mathcal{C}_0 : \mathcal{C}_0 \subseteq \mathcal{C} \text{ is finite and disjoint}\}$$

is a ring of subsets of X.

(e) Suppose that X is a set and that $\mathfrak{R} \subseteq \mathcal{P}\mathcal{P}X$ satisfies (iii) of 481G. Then for every $\mathcal{R} \in \mathfrak{R}$ there is a non-increasing sequence $\langle \mathcal{R}_n \rangle_{n \in \mathbb{N}}$ in $\mathfrak{R}$ such that $\bigcup_{i \le n} A_i \in \mathcal{R}$ whenever $A_i \in \mathcal{R}_i$ for $i \le n$ and $\langle A_i \rangle_{i \le n}$ is disjoint. **P** Take $\mathcal{R}_0 \in \mathfrak{R}$ such that $\mathcal{R}_0 \subseteq \mathcal{R}$ and $A \cup B \in \mathcal{R}$ for all disjoint A, $B \in \mathcal{R}_0$; similarly, for $n \in \mathbb{N}$, choose $\mathcal{R}_{n+1} \in \mathfrak{R}$ such that $\mathcal{R}_{n+1} \subseteq \mathcal{R}_n$ and $A \cup B \in \mathcal{R}_n$ for all disjoint A, $B \in \mathcal{R}_{n+1}$. Now, given that $A_i \in \mathcal{R}_i$ for $i \le n$ and $\langle A_i \rangle_{i \le n}$ is disjoint, we see by downwards induction on m that $\bigcup_{m < i \le n} A_i \in \mathcal{R}_m$ for each $m \le n$, so that $\bigcup_{i \le n} A_i \in \mathcal{R}$. **Q**

(f) If $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, then T is compatible with Δ and $\mathfrak{R}$ in the sense of 481F. **P** $\emptyset \in T$ so T is not empty. Take $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$. Let $\langle \mathcal{R}_i \rangle_{i \in \mathbb{N}}$ be a sequence in $\mathfrak{R}$ such that $\bigcup_{i \le n} A_i \in \mathcal{R}$ whenever $A_i \in \mathcal{R}_i$ for $i \le n$ and $\langle A_i \rangle_{i \le n}$ is disjoint ((e) above). There is a

finite set $\mathcal{C}_1 \subseteq \mathcal{C}$ such that $X \setminus \bigcup \mathcal{C}_1 \in \mathcal{R}_0$, by 481G(iv). By (d), there is a disjoint family $\mathcal{C}_0 \subseteq \mathcal{C}$ such that $\bigcup \mathcal{C}_0 = \bigcup \mathcal{C}_1$; enumerate $\mathcal{C}_0$ as $\langle C_i \rangle_{i<n}$. For each $i < n$, there is a δ-fine $\boldsymbol{t}_i \in T$ such that $W_{\boldsymbol{t}_i} \subseteq C_i$ and $C_i \setminus W_{\boldsymbol{t}_i} \in \mathcal{R}_{i+1}$, by 481G(vii). Set $\boldsymbol{t} = \bigcup_{i<n} \boldsymbol{t}_i$; then $\boldsymbol{t} \in T$ is δ-fine, and

$$X \setminus W_{\boldsymbol{t}} = (X \setminus \bigcup \mathcal{C}_1) \cup \bigcup_{i<n} (C_i \setminus W_{\boldsymbol{t}_i}) \in \mathcal{R}$$

by the choice of $\langle \mathcal{R}_i \rangle_{i \in \mathbb{N}}$. Thus we have a δ-fine $\mathcal{R}$-filling member of T; as δ and $\mathcal{R}$ are arbitrary, T is compatible with Δ and $\mathfrak{R}$. **Q**

(g) For basic results which depend on 'subdivisions' as described in 481G(vii), see 482A-482B below. A different hypothesis asserting the existence of a sort of subdivision appears in 482G(iv).

481I I now run through some simple examples of these constructions, limiting myself for the moment to the definitions, the proofs that T is compatible with Δ and $\mathfrak{R}$, and (when appropriate) the proofs that the structures allow subdivisions.

The proper Riemann integral Fix a non-empty closed interval $X = [a, b] \subseteq \mathbb{R}$. Write $\mathcal{C}$ for the set of all non-empty intervals (open, closed or half-open) included in $[a, b]$, and set $Q = \{(x, C) : C \in C,\ x \in \overline{C}\}$; let T be the straightforward set of tagged partitions generated by Q. Let Δ be the set of uniform metric gauges on X, and $\mathfrak{R} = \{\{\emptyset\}\}$. Then $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof (i), (iii) and (vi) of 481G are trivial and (ii), (iv) and (v) are elementary. As for (vii), given $\eta > 0$ and $C \in \mathcal{C}$, take a disjoint family $\langle C_i \rangle_{i \in I}$ of intervals of length less than 2η covering C, and x_i to be the midpoint of C_i for $i \in I$; then $\boldsymbol{t} = \{(x_i, C_i) : i \in I\}$ belongs to T and is δ_η-fine, in the language of 481E, and $W_{\boldsymbol{t}} = C$.

481J The Henstock integral on a bounded interval (HENSTOCK 63) Take X, $\mathcal{C}$, T and $\mathfrak{R}$ as in 481I. This time, let Δ be the set of *all* neighbourhood gauges on $[a, b]$. Then $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof Again, only 481G(vii) needs more than a moment's consideration. Take any $C \in \mathcal{C}$. Set $a_0 = \inf C$, $b_0 = \sup C$ and let T_0 be the family of δ-fine partitions $\boldsymbol{t} \in T$ such that $W_{\boldsymbol{t}}$ is a relatively closed initial subinterval of C, that is, is of the form $C \cap [a_0, y_{\boldsymbol{t}}]$ for some $y_{\boldsymbol{t}} \in [a_0, b_0]$. Set $A = \{y_{\boldsymbol{t}} : \boldsymbol{t} \in T_0\}$. I have to show that there is a $\boldsymbol{t} \in T_0$ such that $W_{\boldsymbol{t}} = C$, that is, that $b_0 \in A$.

Observe that there is an $\eta_0 > 0$ such that $(a_0, A) \in \delta$ whenever $A \subseteq [a, b] \cap [a_0 - \eta_0, a_0 + \eta_0]$, and now $\{(a_0, [a_0, a_0 + \eta_0] \cap C)\}$ belongs to T_0, so $\min(a_0 + \eta_0, b_0) \in A$ and A is a non-empty subset of $[a_0, b_0]$. It follows that $c = \sup A$ is defined in $[a_0, b_0]$. Let $\eta > 0$ be such that $(c, A) \in \delta$ whenever $A \subseteq [a, b] \cap [c - \eta, c + \eta]$. There is some $\boldsymbol{t} \in T_0$ such that $y_{\boldsymbol{t}} \geq c - \eta$. If $y_{\boldsymbol{t}} = b_0$, we can stop. Otherwise, set $C' = C \cap]y_{\boldsymbol{t}}, c + \eta]$. Then $(c, C') \in \delta$ and $c \in \overline{C'}$ and $C' \cap W_{\boldsymbol{t}}$ is empty, so $\boldsymbol{t}' = \boldsymbol{t} \cup \{(c, C')\}$ belongs to T_0 and $y_{\boldsymbol{t}'} = \min(c + \eta, b_0)$. Since $y_{\boldsymbol{t}'} \leq c$, this shows that $y_{\boldsymbol{t}'} = c = b_0$ and again $b_0 \in A$, as required.

481K The Henstock integral on $\mathbb{R}$ This time, set $X = \mathbb{R}$ and let $\mathcal{C}$ be the family of all non-empty intervals in $\mathbb{R}$. Let T be the straightforward set of tagged partitions generated by $\{(x, C) : C \in \mathcal{C},\ x \in \overline{C}\}$. Following 481J, let Δ be the set of all neighbourhood gauges on $\mathbb{R}$. This time, set $\mathfrak{R} = \{\mathcal{R}_{ab} : a \leq b \in \mathbb{R}\}$, where $\mathcal{R}_{ab} = \{\mathbb{R} \setminus [c, d] : c \leq a,\ d \geq b\} \cup \{\emptyset\}$. Then $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof This time we should perhaps take a moment to look at (iii) of 481G. But all we need to note is that $\mathcal{R}_{ab} \cap \mathcal{R}_{a'b'} = \mathcal{R}_{\min(a,a'),\max(b,b')}$, and that any two members of $\mathcal{R}_{ab}$ have non-empty intersection. Conditions (i), (ii), (iv), (v) and (vi) of 481G are again elementary, so once more we are left with (vii). But this can be dealt with by exactly the same argument as in 481J.

481L The symmetric Riemann-complete integral (cf. CARRINGTON 72, chap. 3) Again take $X = \mathbb{R}$, and $\mathcal{C}$ the set of all non-empty bounded intervals of X. This time, take T to be the straightforward set of tagged partitions generated by the set of pairs (x, C) where $C \in \mathcal{C}$ and x is the *midpoint* of C. As in

481K, take Δ to be the set of all neighbourhood gauges on $\mathbb{R}$; but this time take $\mathfrak{R} = \{\mathcal{R}'_a : a \geq 0\}$, where $\mathcal{R}'_a = \{\mathbb{R} \setminus [-c, c] : c \geq a\} \cup \{\emptyset\}$. Then T is compatible with Δ and $\mathfrak{R}$.

proof Take $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$. For $x \geq 0$, let $\theta(x) > 0$ be such that $(x, D) \in \delta$ whenever $D \subseteq [x - \theta(x), x + \theta(x)]$ and $(-x, D) \in \delta$ whenever $D \subseteq [-x - \theta(x), -x + \theta(x)]$. Write A for the set of those $a > 0$ such that there is a finite sequence $a_0, \ldots, a_n$ such that $0 < a_0 < a_1 < \ldots < a_n = a$, $a_0 \leq \theta(0)$ and $a_{i+1} - a_i \leq 2\theta(\frac{1}{2}(a_i + a_{i+1}))$ for $i < n$.

? Suppose, if possible, that A is bounded above. Then $c = \inf([0, \infty[\setminus \overline{A})$ is defined in $[0, \infty[$. Observe that if $0 < a \leq \theta(0)$, then the one-term sequence a witnesses that $a \in A$. So $c \geq \theta(0) > 0$. Now there must be u, v such that $c < u < v < \min(c + \theta(c), 2c)$ and $]u, v[\cap A = \emptyset$; on the other hand, the interval $]2c - v, 2c - u[$ must contain a point x of A. Set $y = 2c - x$. Then we can find $a_0 < \ldots < a_n = x$ such that $0 < a_0 \leq \theta(0)$ and $a_{i+1} - a_i \leq 2\theta(\frac{1}{2}(a_i + a_{i+1}))$ for $i < n$; setting $a_{n+1} = y$, we see that $\langle a_i \rangle_{i \leq n+1}$ witnesses that $y \in A$, though $y \in]u, v[$. **X**

This contradiction shows that A is unbounded above. So now suppose that $\mathcal{R} = \mathcal{R}_a$ where $a \geq 0$. Take $a_0, \ldots, a_n$ such that $0 < a_0 < \ldots < a_n$, $0 < a_0 \leq \theta(0)$ and $a_{i+1} - a_i \leq 2\theta(\frac{1}{2}(a_i + a_{i+1}))$ for $i < n$, and $a_n \geq a$. For $i < n$, set $x_i = \frac{1}{2}(a_i + a_{i+1})$, $C_i =]a_i, a_{i+1}]$, $x'_i = -x_i$, $C'_i = [-a_{i+1}, a_i[$. Then x_i, x'_i are the midpoints of C_i, C'_i and (by the choice of the function θ) $(x_i, C_i) \in \delta$, $(x'_i, C'_i) \in \delta$ for $i < n$. So if we set

$$\boldsymbol{t} = \{(x_i, C_i) : i < n\} \cup \{(x'_i, C'_i) : i < n\} \cup \{(0, [-a_0, a_0])\}$$

we shall obtain a δ-fine $\mathcal{R}$-filling member of T.

As Δ and $\mathfrak{R}$ are both downwards-directed, this is enough to show that T is compatible with Δ and $\mathfrak{R}$.

481M The McShane integral on an interval (McShane 73) As in 481J, take $X = [a, b]$ and let $\mathcal{C}$ be the family of non-empty subintervals of $[a, b]$. This time, take T to be the straightforward set of tagged partitions generated by $Q = X \times \mathcal{C}$, so that *no* condition is imposed relating the tags to their associated intervals. As in 481J, let Δ be the set of all neighbourhood gauges on X, and $\mathfrak{R} = \{\{\emptyset\}\}$. Proceed as before. Since the only change is that Q and T have been enlarged, $(X, T, \Delta, \mathfrak{R})$ is still a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

481N The McShane integral on a topological space (Fremlin 95) Now let $(X, \mathfrak{T}, \Sigma, \mu)$ be any effectively locally finite τ-additive topological measure space, and take $\mathcal{C} = \{E : E \in \Sigma, \mu E < \infty\}$, $Q = X \times \mathcal{C}$; let T be the straightforward set of tagged partitions generated by Q. Again let Δ be the set of all neighbourhood gauges on X. This time, define $\mathfrak{R}$ as follows. For any set $E \in \Sigma$ of finite measure and $\eta > 0$, let $\mathcal{R}_{E\eta}$ be the set $\{F : F \in \Sigma, \mu(F \cap E) \leq \eta\}$, and set $\mathfrak{R} = \{\mathcal{R}_{E\eta} : \mu E < \infty, \eta > 0\}$. Then $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof As usual, everything is elementary except perhaps 481G(vii). But if $C \in \mathcal{C}$, $\delta \in \Delta$, $E \in \Sigma$, $\mu E < \infty$ and $\eta > 0$, take for each $x \in X$ an open set G_x containing x such that $(x, A) \in \delta$ whenever $A \subseteq G_x$. $\{G_x : x \in X\}$ is an open cover of X, so by 414Ea there is a finite family $\langle x_i \rangle_{i<n}$ in X such that $\mu(E \cap C \setminus \bigcup_{i<n} G_{x_i}) \leq \eta$; setting $C_i = C \cap G_{x_i} \setminus \bigcup_{j<i} G_{x_j}$ for $i < n$, we get a δ-fine tagged partition $\boldsymbol{t} = \{(x_i, C_i) : i < n\}$ such that $C \setminus W_{\boldsymbol{t}} \in \mathcal{R}_{E\eta}$.

481O Convex partitions in $\mathbb{R}^r$ Fix $r \geq 1$. Let us say that a **convex polytope** in $\mathbb{R}^r$ is a non-empty bounded set expressible as the intersection of finitely many open or closed half-spaces; let $\mathcal{C}$ be the family of convex polytopes in $X = \mathbb{R}^r$, and T the straightforward set of tagged partitions generated by $\{(x, C) : x \in \overline{C}\}$. Let Δ be the set of neighbourhood gauges on $\mathbb{R}^r$. For $a \geq 0$, let $\mathcal{C}_a$ be the set of closed convex polytopes $C \subseteq \mathbb{R}^r$ such that, for some $b \geq a$, $B(0, b) \subseteq C \subseteq B(0, 2b)$, where $B(0, b)$ is the ordinary Euclidean ball with centre 0 and radius b; set $\mathcal{R}_a = \{\mathbb{R}^r \setminus C : C \in \mathcal{C}_a\} \cup \{\emptyset\}$, and $\mathfrak{R} = \{\mathcal{R}_a : a \geq 0\}$. Then $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof As usual, only 481G(vii) requires thought.

(a) We need a geometrical fact: if $C \in \mathcal{C}$, $x \in \overline{C}$ and $y \in C$, then $\alpha y + (1 - \alpha)x \in C$ for every $\alpha \in]0, 1]$. **P** The family of sets $C \subseteq \mathbb{R}^r$ with this property is closed under finite intersections and contains all half-spaces. **Q** It follows that if C_1, $C_2 \in \mathcal{C}$ are not disjoint, then $\overline{C_1 \cap C_2} = \overline{C}_1 \cap \overline{C}_2$.

(b) Write $\mathcal{D} \subseteq \mathcal{C}$ for the family of products of non-empty bounded intervals in $\mathbb{R}$. The next step is to show that if $D \in \mathcal{D}$ and $\delta \in \Delta$, then there is a δ-fine tagged partition $\boldsymbol{t} \in T$ such that $W_{\boldsymbol{t}} = D$ and $\boldsymbol{t} \subseteq \mathbb{R}^r \times \mathcal{D}$. **P** Induce on r. For $r = 1$ this is just 481J again. For the inductive step to $r+1$, suppose that $D \subseteq \mathbb{R}^{r+1}$ is a product of bounded intervals and that δ is a neighbourhood gauge on $\mathbb{R}^{r+1}$. Identifying $\mathbb{R}^{r+1}$ with $\mathbb{R}^r \times \mathbb{R}$, express D as $D' \times L$, where $D' \subseteq \mathbb{R}^r$ is a product of bounded intervals and $L \subseteq \mathbb{R}$ is a bounded interval. For $y \in \mathbb{R}^r$, $\alpha \in \mathbb{R}$ let $G(y,\alpha)$, $H(y,\alpha)$ be open sets containing y, α respectively such that $((y,\alpha), A) \in \delta$ whenever $A \subseteq G(y,\alpha) \times H(y,\alpha)$. For $y \in \mathbb{R}^r$, set $\delta_y = \{(\alpha, A) : \alpha \in \mathbb{R}, A \subseteq H(y,\alpha)\}$; then δ_y is a neighbourhood gauge on $\mathbb{R}$.

By the one-dimensional case there is a δ_y-fine tagged partition $\boldsymbol{s}_y \in T_1$ such that $W_{\boldsymbol{s}_y} = L$, where I write T_1 for the set of tagged partitions used in 481K. Set

$$\delta' = \{(y, A) : y \in \mathbb{R}^r, A \subseteq G(y,\alpha) \text{ for every } (\alpha, F) \in \boldsymbol{s}_y\}.$$

δ' is a neighbourhood gauge on $\mathbb{R}^r$. By the inductive hypothesis, there is a δ'-fine tagged partition $\boldsymbol{u} \in T_r$ such that $W_{\boldsymbol{u}} = D'$, where here T_r is the set of tagged partitions on $\mathbb{R}^r$ corresponding to the r-dimensional version of this result. Consider the family

$$\boldsymbol{t} = \{((y,\alpha), E \times F) : (y, E) \in \boldsymbol{u}, (\alpha, F) \in \boldsymbol{s}_y\}.$$

For $(y, E) \in \boldsymbol{u}$, $(\alpha, F) \in \boldsymbol{s}_y$, we have

$$y \in \overline{E}, \quad E \subseteq G(y,\alpha), \quad \alpha \in \overline{F}, \quad F \subseteq H(y,\alpha),$$

so

$$(y,\alpha) \in \overline{E \times F}, \quad E \times F \subseteq G(y,\alpha) \times H(y,\alpha),$$

and $((y,\alpha), E \times F) \in \delta$. If $((y,\alpha), E \times F)$, $((y',\alpha'), E' \times F')$ are distinct members of $\boldsymbol{t}$, then either $(y, E) \neq (y', E')$ so $E \cap E' = \emptyset$ and $(E \times F) \cap (E' \times F')$ is empty, or $y = y'$ and (α, F), (α', F') are distinct members of $\boldsymbol{s}_y$, so that $F \cap F' = \emptyset$ and again $E \times F$, $E' \times F'$ are disjoint. Thus $\boldsymbol{t}$ is a δ-fine member of T_{r+1}. Finally,

$$W_{\boldsymbol{t}} = \bigcup_{(y,E) \in \boldsymbol{u}} \bigcup_{(\alpha,F) \in \boldsymbol{s}_y} E \times F = \bigcup_{(y,E) \in \boldsymbol{u}} E \times L = D' \times L = D.$$

So the induction proceeds. **Q**

(c) Now suppose that C_0 is an arbitrary member of $\mathcal{C}$ and that δ is a neighbourhood gauge on $\mathbb{R}^r$. Set

$$\delta' = \delta \cap \{(x, A) : \text{either } x \in \overline{C}_0 \text{ or } A \cap \overline{C}_0 = \emptyset\}.$$

Then δ' is a neighbourhood gauge on $\mathbb{R}^r$, being the intersection of δ with the neighbourhood gauge associated with the family $\langle U_x \rangle_{x \in \mathbb{R}^r}$, where $U_x = \mathbb{R}^r$ if $x \in \overline{C}_0$, $\mathbb{R}^r \setminus \overline{C}_0$ otherwise. Let $D \in \mathcal{D}$ be such that $C_0 \subseteq D$. By (b), there is a δ'-fine tagged partition $\boldsymbol{t} \in T$ such that $W_{\boldsymbol{t}} = D$. Set $\boldsymbol{s} = \{(x, C \cap C_0) : (x, C) \in \boldsymbol{t}, C \cap C_0 \neq \emptyset\}$. Since $\boldsymbol{t} \subseteq \delta'$, $x \in \overline{C}_0$ whenever $(x, C) \in \boldsymbol{t}$ and $C \cap C_0 \neq \emptyset$. By (a), $x \in \overline{C \cap C_0}$ for all such pairs (x, C); and of course $(x, C \cap C_0) \in \delta$ for every $(x, C) \in \boldsymbol{t}$. So $\boldsymbol{s}$ belongs to T, and $W_{\boldsymbol{s}} = W_{\boldsymbol{t}} \cap C_0 = C_0$. As C_0 and δ are arbitrary, 481G(vii) is satisfied.

481P Box products (cf. Muldowney 87, Prop. 1) Let $\langle (X_i, \mathfrak{T}_i) \rangle_{i \in I}$ be a non-empty family of non-empty compact metrizable spaces with product $(X, \mathfrak{T})$. Set $\pi_i(x) = x(i)$ for $x \in X$, $i \in I$. For each $i \in I$, let $\mathcal{C}_i \subseteq \mathcal{P}X_i$ be such that (α) whenever E, $E' \in \mathcal{C}_i$ then $E \cap E' \in \mathcal{C}_i$ and $E \setminus E'$ is expressible as the union of a disjoint finite subset of $\mathcal{C}_i$ (β) $\mathcal{C}_i$ includes a base for $\mathfrak{T}_i$.

Let $\mathcal{C}$ be the set of subsets of X of the form

$$C = \{X \cap \textstyle\bigcap_{i \in J} \pi_i^{-1}[E_i] : J \in [I]^{<\omega}, E_i \in \mathcal{C}_i \text{ for every } i \in J\},$$

and let T be the straightforward set of tagged partitions generated by $\{(x, C) : C \in \mathcal{C}, x \in \overline{C}\}$. Let Δ be the set of those neighbourhood gauges δ on X defined by families $\langle G_x \rangle_{x \in X}$ of open sets such that, for some countable $J \subseteq I$, every G_x is determined by coordinates in J (definition: 254M). Then $(X, T, \Delta, \{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof Conditions (i), (iii) and (vi) of 481G are trivial, and (ii), (iv) and (v) are elementary; so we are left with (vii), as usual. **?** Suppose, if possible, that $C \in \mathcal{C}$ and $\delta \in \Delta$ are such that there is no δ-fine $\boldsymbol{t} \in T$ with $W_{\boldsymbol{t}} = C$. Let $\langle G_x \rangle_{x \in X}$ be the family of open sets determining δ, and $J \subseteq I$ a non-empty countable set

such that G_x is determined by coordinates in J for every $x \in X$. For $i \in J$, let $\mathcal{C}'_i \subseteq \mathcal{C}_i$ be a countable base for $\mathfrak{T}_i$ (4A2P(a-iii)), and take a sequence $\langle (i_n, E_n)\rangle_{n\in\mathbb{N}}$ running over $\{(i,E) : i \in J,\ E \in \mathcal{C}'_i\}$.

Write $\mathcal{D} = \{W_{\boldsymbol{t}} : \boldsymbol{t} \in T \text{ is } \delta\text{-fine}\}$. Note that if D_1, $D_2 \in \mathcal{D}$ are disjoint then $D_1 \cup D_2 \in \mathcal{D}$. So if $D \in \mathcal{C}\setminus\mathcal{D}$ and $C \in \mathcal{C}$, there must be some $D' \in \mathcal{C}\setminus\mathcal{D}$ such that either $D' \subseteq D \cap C$ or $D' \subseteq D \setminus C$, just because $\mathcal{C}$ satisfies 481G(iv). Now choose $\langle C_n\rangle_{n\in\mathbb{N}}$ inductively so that $C_0 = C$ and

$$C_n \in \mathcal{C}\setminus\mathcal{D},$$

$$\text{either } C_{n+1} \subseteq C_n \cap \pi_{i_n}^{-1}[E_n] \text{ or } C_{n+1} \subseteq C_n \setminus \pi_{i_n}^{-1}[E_n]$$

for every $n \in \mathbb{N}$. Because X, being a product of compact spaces, is compact, there is an $x \in \bigcap_{n\in\mathbb{N}} \overline{C_n}$. We know that G_x is determined by coordinates in J, so $G_x = \tilde{\pi}^{-1}[\tilde{\pi}[G_x]]$, where $\tilde{\pi}$ is the canonical map from X onto $Y = \prod_{i\in J} X_i$. $V = \tilde{\pi}[G_x]$ is open, so there must be a finite set $K \subseteq J$ and a family $\langle V_i\rangle_{i\in K}$ such that $x(i) \in V_i \in \mathfrak{T}_i$ for every $i \in K$ and $\{y : y \in Y,\ y(i) \in V_i \text{ for every } i \in K\}$ is included in V. This means that $\{z : z \in X,\ z(i) \subseteq V_i \text{ for every } i \in K\}$ is included in G_x. Now, for each $i \in K$, there is some $m \in \mathbb{N}$ such that $i = i_m$ and $x(i) \in E_m \subseteq V_i$. Because $x \in \overline{C}_{m+1}$, $\pi_{i_m}^{-1}[E_m]$ cannot be disjoint from C_{m+1}, and $C_{m+1} \subseteq \pi_{i_m}^{-1}[E_m] \subseteq \pi_i^{-1}[V_i]$.

But this means that, for any n large enough, $C_n \subseteq G_x$ and $\boldsymbol{t} = \{(x, C_n)\}$ is a δ-fine member of T with $W_{\boldsymbol{t}} = C_n$; contradicting the requirement that $C_n \notin \mathcal{D}$. **X**

This contradiction shows that 481G(vii) is also satisfied.

481Q The approximately continuous Henstock integral (Gordon 94, chap. 16) Let μ be Lebesgue measure on $\mathbb{R}$. As in 481K, let $\mathcal{C}$ be the family of non-empty bounded intervals in $\mathbb{R}$, and Q the set of pairs (x, C) where $C \in \mathcal{C}$ and $x \in \overline{C}$; let T be the straightforward set of tagged partitions defined from Q; and set $\mathfrak{R} = \{\mathcal{R}_{ab} : a,\ b \in \mathbb{R}\}$, where $\mathcal{R}_{ab} = \{\mathbb{R}\setminus[c,d] : c \le a,\ d \ge b\} \cup \{\emptyset\}$ for a, $b \in \mathbb{R}$.

This time, define gauges as follows. Let E be the set of families $\boldsymbol{e} = \langle E_x\rangle_{x\in\mathbb{R}}$ where every E_x is a measurable set containing x such that x is a density point of E_x (definition: 223B). For $\boldsymbol{e} = \langle E_x\rangle_{x\in\mathbb{R}} \in \mathrm{E}$, set

$$\delta_{\boldsymbol{e}} = \{(x,C) : C \in \mathcal{C},\ x \in \overline{C},\ \inf C \in E_x \text{ and } \sup C \in E_x\}.$$

Set $\Delta = \{\delta_{\boldsymbol{e}} : \boldsymbol{e} \in \mathrm{E}\}$. Then $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof (a) Turning to 481G, we find, as usual, that most of the conditions are satisfied for elementary reasons. Since we have here a new kind of gauge, we had better check 481G(ii); but if $\boldsymbol{e} = \langle E_x\rangle_{x\in\mathbb{R}}$ and $\boldsymbol{e}' = \langle E'_x\rangle_{x\in\mathbb{R}}$ both belong to E, so does $\boldsymbol{e}\wedge\boldsymbol{e}' = \langle E_x \cap E'_x\rangle_{x\in\mathbb{R}}$, because

$$\begin{aligned}\liminf_{\eta\downarrow 0} \frac{1}{2\eta}\mu([x-\eta, x+\eta] \cap E_x \cap E'_x)&\\ \ge \lim_{\eta\downarrow 0} \frac{1}{2\eta}\big(\mu([x-\eta,x+\eta]\cap E_x) &+ \mu([x-\eta,x+\eta]\cap E'_x) - 2\eta\big) = 1\end{aligned}$$

for every x; and now $\delta_{\boldsymbol{e}} \cap \delta_{\boldsymbol{e}'} = \delta_{\boldsymbol{e}\wedge\boldsymbol{e}'}$ belongs to Δ. Everything else we have done before, except, of course, (vii).

(b) So take any $\delta \in \Delta$; express δ as $\delta_{\boldsymbol{e}}$, where $\boldsymbol{e} = \langle E_x\rangle_{x\in\mathbb{R}} \in \mathrm{E}$. For x, $y \in \mathbb{R}$, write $x \frown y$ if $x \le y$ and $E_x \cap E_y \cap [x,y] \ne \emptyset$; note that we always have $x \frown x$. For $x \in \mathbb{R}$, let $\eta_x > 0$ be such that $\mu(E_x \cap [x-\eta, x+\eta]) \ge \frac{5}{3}\eta$ whenever $0 \le \eta \le \eta_x$.

Fix $a < b$ in $\mathbb{R}$ for the moment. Say that a finite string $(x_0, \dots, x_n)$ is 'acceptable' if $a \le x_0 \frown \dots \frown x_n \le b$ and $\mu([x_0, x_n] \cap \bigcup_{i<n} E^+_{x_i}) \ge \frac{1}{2}(x_n - x_0)$, where $E^+_x = E_x \cap [x, \infty[$ for $x \in \mathbb{R}$. Observe that if $(x_0, \dots, x_m)$ and $(x_m, x_{m+1}, \dots, x_n)$ are both acceptable, so is $(x_0, \dots, x_n)$. For $x \in [a,b]$, set

$$h(x) = \sup\{x_n : (x, x_1, \dots, x_n) \text{ is acceptable}\};$$

this is defined in $[x,b]$ because the string (x) is acceptable. If $a \le x < b$, then (x,y) is acceptable whenever $y \in E_x$ and $0 \le y \le \min(b, x+\eta_x)$, so $h(x) > x$. Now choose sequences $\langle x_i\rangle_{i\in\mathbb{N}}$, $\langle n_k\rangle_{k\in\mathbb{N}}$ inductively, as follows. $n_0 = 0$ and $x_0 = a$. Given that $x_i \frown x_{i+1}$ for $i < n_k$ and that $(x_{n_j}, \dots, x_{n_k})$ is acceptable for

every $j \leq k$, let $n_{k+1} > n_k$, $(x_{n_k+1}, \ldots, x_{n_{k+1}})$ be such that $(x_{n_k}, \ldots, x_{n_{k+1}})$ is acceptable and $x_{n_{k+1}} \geq \frac{1}{2}(x_{n_k} + h(x_{n_k}))$; then $(x_{n_j}, \ldots, x_{n_{k+1}})$ is acceptable for any $j \leq k+1$; continue.

At the end of the induction, set

$$c = \sup_{i \in \mathbb{N}} x_i = \sup_{k \in \mathbb{N}} x_{n_k}.$$

Then there are infinitely many i such that $x_i \frown c$. **P** For any $k \in \mathbb{N}$,

$$\begin{aligned}\mu([x_{n_k}, c] \cap \bigcup_{i \geq n_k} E^+_{x_i}) &= \lim_{l \to \infty} \mu\big([x_{n_k}, x_{n_l}] \cap \bigcup_{n_k \leq i < n_l} E^+_{x_i}\big) \\ &\geq \lim_{l \to \infty} \frac{1}{2}(x_{n_l} - x_{n_k})\end{aligned}$$

(because $(x_{n_k}, \ldots, x_{n_l})$ is always an acceptable string)

$$= \frac{1}{2}(c - x_{n_k}).$$

But this means that if we take k so large that $x_{n_k} \geq c - \eta_c$, so that $\mu(E_c \cap [x_{n_k}, c]) \geq \frac{2}{3}(c - x_{n_k})$, there must be some $z \in E_c \cap \bigcup_{i \geq n_k} E^+_i \cap [x_{n_k}, c]$; and if $i \geq n_k$ is such that $z \in E^+_i$, then z witnesses that $x_i \frown c$. As k is arbitrarily large, we have the result. **Q**

? If $c < b$, take any $y \in E_c$ such that $c < y \leq \min(c + \eta_c, b)$. Take $k \in \mathbb{N}$ such that $x_{n_k} \geq c - \frac{1}{3}(y - c)$, and $j \geq n_k$ such that $x_j \frown c$. In this case, $(x_{n_k}, x_{n_k+1}, \ldots, x_j, c, y)$ is an acceptable string, because

$$\mu E^+_c \cap [x_{n_k}, y] \geq \mu E_c \cap [c, y] \geq \tfrac{2}{3}(y - c) \geq \tfrac{1}{2}(y - x_{n_k}).$$

But this means that $h(x_{n_k}) \geq y$, so that

$$x_{n_{k+1}} \leq c < \frac{1}{2}(x_{n_k} + h(x_{n_k})),$$

contrary to the choice of $x_{n_k+1}, \ldots, x_{n_{k+1}}$. **X**

Thus $c = b$. We therefore have a $j \in \mathbb{N}$ such that $x_j \frown b$, and $a = x_0 \frown \ldots \frown x_j \frown b$.

(c) Now suppose that $C \in \mathcal{C}$. Set $a = \inf C$ and $b = \sup C$. If $a = b$, then $\boldsymbol{t} = \{(a, C)\} \in T$ and $W_{\boldsymbol{t}} = C$. Otherwise, (b) tells us that we have $x_0, \ldots, x_n$ such that $a = x_0 \frown \ldots \frown x_n = b$. Choose $a_i \in [x_{i-1}, x_i] \cap E_{x_{i-1}} \cap E_{x_i}$ for $1 \leq i \leq n$. Set $C_i = [a_i, a_{i+1}[$ for $1 \leq i < n$, $C_0 = [a, a_1[$, $C_n = [a_n, b]$; set $I = \{i : i \leq n, C \cap C_i \neq \emptyset\}$; and check that $(x_i, C \cap C_i) \in \delta$ for $i \in I$, so that $\boldsymbol{t} = \{(x_i, C \cap C_i) : i \in I\}$ is a δ-fine member of T with $W_{\boldsymbol{t}} = C$. As C and δ are arbitrary, 481G(vii) is satisfied.

481X Basic exercises (a) Let X, $\mathcal{C}$, T and $\mathcal{F}$ be as in 481C. Show that if $f : X \to \mathbb{R}$, $\mu : \mathcal{C} \to \mathbb{R}$ and $\nu : \mathcal{C} \to \mathbb{R}$ are functions, then $I_{\mu+\nu}(f) = I_\mu(f) + I_\nu(f)$ whenever the right-hand side is defined.

>(b) Let I be any set. Set $T = \{\{(i, \{i\}) : i \in J\} : J \in [I]^{<\omega}\}$, $\delta = \{(i, \{i\}) : i \in I\}$, $\Delta = \{\delta\}$. For $J \in [I]^{<\omega}$ set $\mathcal{R}_J = \{I \setminus K : J \subseteq K \in [I]^{<\omega}\} \cup \{\emptyset\}$; set $\mathfrak{R} = \{\mathcal{R}_J : J \in [I]^{<\omega}\}$. Show that $(I, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $[I]^{<\omega}$. Let $\nu : [I]^{<\omega} \to \mathbb{R}$ be any additive functional. Show that, for a function $f : I \to \mathbb{R}$, $\lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \nu) = \sum_{i \in I} f(i)\nu(\{i\})$ if either exists in $\mathbb{R}$.

>(c) Set $T = \{\{(n, \{n\}) : n \in I\} : I \in [\mathbb{Z}]^{<\omega}\}$, $\delta = \{(n, \{n\}) : n \in \mathbb{Z}\}$, $\Delta = \{\delta\}$. For $I \in [\mathbb{Z}]^{<\omega}$ and $m, n \in \mathbb{N}$ set $R_{mn} = \mathbb{Z} \setminus \{i : -m \leq i \leq n\}$, $\mathcal{R}'_n = \{R_{kl} : k, l \geq n\} \cup \{\emptyset\}$, $\mathcal{R}''_n = \{R_{kk} : k \geq n\} \cup \{\emptyset\}$, $\mathfrak{R}' = \{\mathcal{R}'_n : n \in \mathbb{N}\}$, $\mathfrak{R}'' = \{\mathcal{R}''_n : n \in \mathbb{N}\}$. Show that $(\mathbb{Z}, T, \Delta, \mathfrak{R}')$ and $(\mathbb{Z}, T, \Delta, \mathfrak{R}'')$ are tagged-partition structures allowing subdivisions, witnessed by $[\mathbb{Z}]^{<\omega}$. Let μ be counting measure on $\mathbb{Z}$. Show that, for a function $f : \mathbb{Z} \to \mathbb{R}$, (i) $\lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\mathfrak{R}')} S_{\boldsymbol{t}}(f, \mu) = \lim_{m,n \to \infty} \sum_{i=-m}^{n} f(i)$ if either is defined in $\mathbb{R}$ (ii) $\lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\mathfrak{R}'')} S_{\boldsymbol{t}}(f, \mu) = \lim_{n \to \infty} \sum_{i=-n}^{n} f(i)$ if either is defined in $\mathbb{R}$.

(d) Set $X = \mathbb{N} \cup \{\infty\}$, and let T be the straightforward set of tagged partitions generated by $\{(n, \{n\}) : n \in \mathbb{N}\} \cup \{(\infty, X \setminus n) : n \in \mathbb{N}\}$ (interpreting a member of $\mathbb{N}$ as the set of its predecessors). For $n \in \mathbb{N}$ set $\delta_n = \{(i, \{i\}) : i \in \mathbb{N}\} \cup \{(\infty, A) : A \subseteq X \setminus n\}$; set $\Delta = \{\delta_n : n \in \mathbb{N}\}$. Show that $(X, T, \Delta, \{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C} = [\mathbb{N}]^{<\omega} \cup \{X \setminus I : I \in [\mathbb{N}]^{<\omega}\}$. Let $h : \mathbb{N} \to \mathbb{R}$ be

any function, and define $\nu : \mathcal{C} \to \mathbb{R}$ by setting $\nu I = \sum_{i \in I} h(i)$, $\nu(X \setminus I) = -\nu I$ for $I \in [\mathbb{N}]^{<\omega}$. Let $f : X \to \mathbb{R}$ be any function such that $f(\infty) = 0$. Show that $\lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\{\{\emptyset\}\})} S_{\boldsymbol{t}}(f,\mu) = \lim_{n \to \infty} \sum_{i=0}^{n} f(i)h(i)$ if either is defined in $\mathbb{R}$.

>(e) Take X, T, Δ and $\mathfrak{R}$ as in 481I. Show that if μ is Lebesgue measure on $[a,b]$ then the gauge integral $I_\mu = \lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(.,\mu)$ is the ordinary Riemann integral $\int_a^b$ as described in 134K. (*Hint*: show first that they agree on step-functions.)

>(f) Let (X, Σ, μ) be a semi-finite measure space, and let Σ^f be the family of measurable sets of finite measure. Let T be the straightforward set of tagged partitions generated by $\{(x,E) : x \in E \in \Sigma^f\}$. For $E \in \Sigma^f$ and $\epsilon > 0$ set $\mathcal{R}_{E\epsilon} = \{F : F \in \Sigma, \mu(E \setminus F) \leq \epsilon\}$; set $\mathfrak{R} = \{\mathcal{R}_{E\epsilon} : E \in \Sigma^f, \epsilon > 0\}$. Let $\mathfrak{E}$ be the family of countable partitions of X into measurable sets, and set $\delta_{\mathcal{E}} = \bigcup_{E \in \mathcal{E}} \{(x,A) : x \in E, A \subseteq E\}$ for $\mathcal{E} \in \mathfrak{E}$, $\Delta = \{\delta_{\mathcal{E}} : \mathcal{E} \in \mathfrak{E}\}$. Show that $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by Σ^f. Show that, for a function $f : X \to \mathbb{R}$, $\int f d\mu = \lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\mu)$ if either is defined in $\mathbb{R}$. (*Hint*: when showing that if $I_\mu(f)$ is defined then f is μ-virtually measurable, you will need 413G or something similar; compare 482E.)

(g) Let (X, Σ, μ) be a totally finite measure space, and T the straightforward set of tagged partitions generated by $\{(x,E) : x \in E \in \Sigma\}$. Let $\mathfrak{E}$ be the family of finite partitions of X into measurable sets, and set $\delta_{\mathcal{E}} = \bigcup_{E \in \mathcal{E}} \{(x,A) : x \in E, A \subseteq E\}$ for $\mathcal{E} \in \mathfrak{E}$, $\Delta = \{\delta_{\mathcal{E}} : \mathcal{E} \in \mathfrak{E}\}$. Show that $(X, T, \Delta, \{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by Σ. Show that, for a function $f : X \to \mathbb{R}$, $I_\mu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\mu)$ is defined iff $f \in \mathcal{L}^\infty(\mu)$ (definition: 243A), and that then $I_\mu(f) = \int f d\mu$.

(h) Let X be a zero-dimensional compact Hausdorff space and $\mathcal{E}$ the algebra of open-and-closed subsets of X. Let T be the straightforward set of tagged partitions generated by $\{(x,E) : x \in E \in \mathcal{E}\}$. Let Δ be the set of all neighbourhood gauges on X. Show that $(X, T, \Delta, \{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{E}$. Now let $\nu : \mathcal{E} \to \mathbb{R}$ be an additive functional, and set $I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\{\{\emptyset\}\})} S_{\boldsymbol{t}}(f,\nu)$ when $f : X \to \mathbb{R}$ is such that the limit is defined. (i) Show that $I_\nu(\chi E) = \nu E$ for every $E \in \mathcal{E}$. (ii) Show that if ν is bounded then $I_\nu(f)$ is defined for every $f \in C(X)$.

(i) Let X be a set, Δ a set of gauges on X, $\mathfrak{R}$ a collection of families of subsets of X, and T a set of tagged partitions on X which is compatible with Δ and $\mathfrak{R}$. Let $H \subseteq X$ be such that there is a $\tilde{\delta} \in \Delta$ such that $H \cap A = \emptyset$ whenever $x \in X \setminus H$ and $(x,A) \in \tilde{\delta}$, and set $\delta_H = \{(x, A \cap H) : x \in H, (x,A) \in \delta\}$ for $\delta \in \Delta$, $\Delta_H = \{\delta_H : \delta \in \Delta\}$, $\mathfrak{R}_H = \{\{R \cap H : R \in \mathcal{R}\} : \mathcal{R} \in \mathfrak{R}\}$, $T_H = \{\{(x, C \cap H) : (x,C) \in \boldsymbol{t}, x \in H\} : \boldsymbol{t} \in T\}$. Show that T_H is compatible with Δ_H and $\mathfrak{R}_H$.

481Y Further exercises (a) Suppose that $[a,b]$, $\mathcal{C}$, T and Δ are as in 481J. Let $T' \subseteq [[0,1] \times \mathcal{C}]^{<\omega}$ be the set of tagged partitions $\boldsymbol{t} = \{(x_i, [a_i, a_{i+1}]) : i < n\}$ where $a = a_0 \leq x_0 \leq a_1 \leq x_2 \leq a_2 \leq \ldots \leq x_{n-1} \leq a_n = b$. Show that T', as well as T, is compatible with Δ in the sense of 481E; let $\mathcal{F}'$, $\mathcal{F}$ be the corresponding filters on T' and T. Show that if $\nu : \mathcal{C} \to \mathbb{R}$ is a functional which is additive in the sense that $\nu(C \cup C') = \nu C + \nu C'$ whenever C, C' are disjoint members of $\mathcal{C}$ with union in $\mathcal{C}$, and if $\nu\{x\} = 0$ for every $x \in [a,b]$ and $f : [a,b] \to \mathbb{R}$ is any function, then $I'_\nu(f) = I_\nu(f)$ if either is defined, where

$$I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}} S_{\boldsymbol{t}}(f,\nu), \quad I'_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}'} S_{\boldsymbol{t}}(f,\nu)$$

are the gauge integrals associated with $(T, \mathcal{F})$ and $(T', \mathcal{F}')$.

(b) Let us say that a family $\mathfrak{R}$ of residual families is 'the simple residual structure complementary to $\mathcal{H} \subseteq \mathcal{P}X$' if $\mathfrak{R} = \{\mathcal{R}_H : H \in \mathcal{H}\}$, where $\mathcal{R}_H = \{X \setminus H' : H \subseteq H' \in \mathcal{H}\} \cup \{\emptyset\}$ for each $H \in \mathcal{H}$. Suppose that, for each member i of a non-empty finite set I, $(X_i, T_i, \Delta_i, \mathfrak{R}_i)$ is a tagged-partition structure allowing subdivisions, witnessed by an upwards-directed family $\mathcal{C}_i \subseteq \mathcal{P}X_i$, where X_i is a topological space, Δ_i is the set of all neighbourhood gauges on X_i, and $\mathfrak{R}_i$ is the simple residual structure complementary to $\mathcal{C}_i$. Set $X = \prod_{i \in I} X_i$ and let Δ be the set of neighbourhood gauges on X; let $\mathcal{C}$ be $\{\prod_{i \in I} C_i : C_i \in \mathcal{C}_i \text{ for each } i \in I\}$, and $\mathfrak{R}$ the simple residual structure based on $\mathcal{C}$; and let T be the straightforward set of tagged partitions generated by $\{(\langle x_i \rangle_{i \in I}, \prod_{i \in I} C_i) : \{(x_i, C_i)\} \in T_i \text{ for every } i \in I\}$. Show that $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

(c) Give an example to show that, in 481Xi, $(X, T, \Delta, \Re)$ can be a tagged-partition structure allowing subdivisions, while $(H, T_H, \Delta_H, \Re_H)$ is not.

481 Notes and comments In the examples above I have tried to give an idea of the potential versatility of the ideas here. Further examples may be found in HENSTOCK 91. The goal of 481A-481F is the formula $I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\Re)} S_{\boldsymbol{t}}(f, \nu)$; the elaborate notation reflects the variety of the applications. One of these is a one-step definition of the ordinary integral (481Xf). In §483 I will show that the Henstock integral (481J) properly extends the Lebesgue integral on $\mathbb{R}$. In 481Xc I show how adjusting $\Re$ can change the class of integrable functions; in 481Xd I show how a similar effect can sometimes be achieved by adding a point at infinity and adjusting T and Δ. As will become apparent in later sections, one of the great strengths of gauge integrals is their ability to incorporate special limiting processes. Another is the fact that we don't need to assume that the functionals ν are countably additive; see 481Xd. In the formulae of this section, I don't even ask for finite additivity; but of course the functional I_ν is likely to have a rather small domain if ν behaves too erratically.

'Gauges', as I describe them here, have moved rather briskly forward from the metric gauges δ_h (481E), which have sufficed for most of the gauge integrals so far described. But the generalization affects only the notation, and makes it clear why so much of the theory of the ordinary Henstock integral applies equally well to the 'approximately continuous Henstock integral' (481Q), for instance. You will observe that the sets Q of 481Ba are just 'gauges' in the wide sense used here. But (as the examples of this section show clearly) we generally use them in a different way.

In ordinary measure theory, we have a fairly straightforward theory of subspaces (§214) and a rather deeper theory of product spaces (chap. 25). For gauge integrals, there are significant difficulties in the theory of subspaces, some of which will appear in the next section (see 482G-482H). For closed subspaces, something can be done, as in 481Xi; but the procedure suggested there may lose some essential element of the original tagged-partition structure (481Yc). For products of gauge integrals, we do have a reasonably satisfying version of Fubini's theorem (482M); I offer 481O and 481P as alternative approaches.

You will note the concentration on 'neighbourhood gauges' in the work above. The set Δ of all neighbourhood gauges on a topological space X has the following special property: if $x \mapsto \delta_x : X \to \Delta$ is any function, then $\{(x, A) : x \in X, (x, A) \in \delta_x\}$ belongs to Δ. By analogy with the notion of 'full group of automorphisms' (381Be), we might say that a set Δ of gauges is 'full' if it has this property, and 'countably full' if whenever $\langle \delta_n \rangle_{n \in \mathbb{N}}$ is a sequence in Δ, and $\phi : X \to \mathbb{N}$ is a function, then $\{(x, A) : x \in X, (x, A) \in \delta_{\phi(x)}\}$ belongs to Δ. As will appear repeatedly in the next section, this kind of flexibility in constructing gauges is just what one needs when proving that functions are gauge-integrable.

While I have used such phrases as 'Henstock integral', 'symmetric Riemann-complete integral' above, I have not in fact discussed integrals here, except in the exercises; in most of the examples in 481I-481Q there is no mention of any functional ν from which a gauge integral I_ν can be defined. The essence of the method is that we can set up a tagged-partition structure quite independently of any set function, and it turns out that the properties of a gauge integral depend more on this structure than on the measure involved.

482 General theory

I turn now to results which can be applied to a wide variety of tagged-partition structures. The first step is a 'Saks-Henstock' lemma (482B), a fundamental property of tagged-partition structures allowing subdivisions. In order to relate gauge integrals to the ordinary integrals treated elsewhere in this treatise, we need to know when gauge-integrable functions are measurable (482E) and when integrable functions are gauge-integrable (482F). There are significant difficulties when we come to interpret gauge integrals over subspaces, but I give partial results in 482G and 482L. 482J, 482K and 482M are gauge-integral versions of the fundamental theorem of calculus, B.Levi's theorem and Fubini's theorem, while 482H is a new theorem on integrability of a limit function.

Henstock's integral (481J-481K) remains the most important example and the natural test case for the ideas here; I will give the details in the next section, and you may wish to take the two sections in parallel.

482A Lemma Suppose that $(X, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions (481G), witnessed by $\mathcal{C} \subseteq \mathcal{P}X$.

(a) Whenever $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ and E belongs to the subalgebra of $\mathcal{P}X$ generated by $\mathcal{C}$, there is a δ-fine $\boldsymbol{s} \in T$ such that $W_{\boldsymbol{s}} \subseteq E$ and $E \setminus W_{\boldsymbol{s}} \in \mathcal{R}$.

(b) Whenever $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ and $\boldsymbol{t} \in T$ is δ-fine, there is a δ-fine $\mathcal{R}$-filling $\boldsymbol{t}' \in T$ including $\boldsymbol{t}$.

(c) Suppose that $f : X \to \mathbb{R}$, $\nu : \mathcal{C} \to \mathbb{R}$, $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ and $\epsilon \geq 0$ are such that $|S_{\boldsymbol{t}}(f, \nu) - S_{\boldsymbol{t}'}(f, \nu)| \leq \epsilon$ whenever $\boldsymbol{t}$, $\boldsymbol{t}' \in T$ are δ-fine and $\mathcal{R}$-filling. Then

(i) $|S_{\boldsymbol{t}}(f, \nu) - S_{\boldsymbol{t}'}(f, \nu)| \leq \epsilon$ whenever $\boldsymbol{t}$, $\boldsymbol{t}' \in T$ are δ-fine and $W_{\boldsymbol{t}} = W_{\boldsymbol{t}'}$;

(ii) whenever $\boldsymbol{t} \in T$ is δ-fine, and $\delta' \in \Delta$, there is a δ'-fine $\boldsymbol{s} \in T$ such that $W_{\boldsymbol{s}} \subseteq W_{\boldsymbol{t}}$ and $|S_{\boldsymbol{s}}(f, \nu) - S_{\boldsymbol{t}}(f, \nu)| \leq \epsilon$.

(d) Suppose that $f : X \to [0, \infty[$ and $\nu : \mathcal{C} \to [0, \infty[$ are such that $I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(f, \nu)$ is defined. Then for any $\epsilon > 0$ there is a $\delta \in \Delta$ such that $S_{\boldsymbol{t}}(f, \nu) \leq I_\nu(f) + \epsilon$ for every δ-fine $\boldsymbol{t} \in T$.

proof (a) By 481He, there is a non-increasing sequence $\langle \mathcal{R}_k \rangle_{k \in \mathbb{N}}$ in $\mathfrak{R}$ such that $\bigcup_{i \leq k} A_i \in \mathcal{R}$ whenever $A_i \in \mathcal{R}_i$ for every $i \leq k$ and $\langle A_i \rangle_{i \leq k}$ is disjoint. Let $\mathcal{C}_0$ be a finite subset of $\mathcal{C}$ such that E belongs to the subalgebra of $\mathcal{P}X$ generated by $\mathcal{C}_0$, and let $\mathcal{C}_1 \supseteq \mathcal{C}_0$ be a finite subset of $\mathcal{C}$ such that $X \setminus W \in \mathcal{R}_0$, where $W = \bigcup \mathcal{C}_1$ (481G(v)). Then either $E \subseteq W$ or $E \supseteq X \setminus W$. In either case, $E \cap W$ belongs to the ring generated by $\mathcal{C}$, so is expressible as $\bigcup_{i<n} C_i$ where $\langle C_i \rangle_{i<n}$ is a disjoint family in $\mathcal{C}$ (481Hd).

For each $i < n$, let $\boldsymbol{s}_i$ be a δ-fine member of T such that $W_{\boldsymbol{s}_i} \subseteq C_i$ and $C_i \setminus W_{\boldsymbol{s}_i} \in \mathcal{R}_{i+1}$ (481G(vii)). Set $\boldsymbol{s} = \bigcup_{i<n} \boldsymbol{s}_i$. Because $\langle W_{\boldsymbol{s}_i} \rangle_{i<n}$ is disjoint and T is a straightforward set of tagged partitions, $\boldsymbol{s} \in T$; $\boldsymbol{s}$ is δ-fine because every $\boldsymbol{s}_i$ is; $W_{\boldsymbol{s}} = \bigcup_{i<n} W_{\boldsymbol{s}_i}$ is included in E; and $E \setminus W_{\boldsymbol{s}}$ is either $\bigcup_{i<n} C_i \setminus W_{\boldsymbol{s}_i}$ or $(X \setminus W) \cup \bigcup_{i<n} (C_i \setminus W_{\boldsymbol{s}_i})$, and in either case belongs to $\mathcal{R}$, by the choice of $\langle \mathcal{R}_n \rangle_{n \in \mathbb{N}}$.

(b) Set $E = X \setminus W_{\boldsymbol{t}}$. By (a), there is a δ-fine $\boldsymbol{s} \in T$ such that $W_{\boldsymbol{s}} \subseteq E$ and $E \setminus W_{\boldsymbol{s}} \in \mathcal{R}$. Set $\boldsymbol{t}' = \boldsymbol{t} \cup \boldsymbol{s}$; this works.

(c)(i) As in (b), there is a δ-fine $\boldsymbol{s} \in T$ such that $W_{\boldsymbol{s}} \cap W_{\boldsymbol{t}} = \emptyset$ and $\boldsymbol{t} \cup \boldsymbol{s}$ is $\mathcal{R}$-filling. Now $W_{\boldsymbol{t} \cup \boldsymbol{s}} = W_{\boldsymbol{t}' \cup \boldsymbol{s}}$, so $\boldsymbol{t}' \cup \boldsymbol{s}$ is also $\mathcal{R}$-filling, and

$$|S_{\boldsymbol{t}}(f, \nu) - S_{\boldsymbol{t}'}(f, \nu)| = |S_{\boldsymbol{t} \cup \boldsymbol{s}}(f, \nu) - S_{\boldsymbol{t}' \cup \boldsymbol{s}}(f, \nu)| \leq \epsilon.$$

(ii) Replacing δ' by a lower bound of $\{\delta, \delta'\}$ in Δ if necessary, we may suppose that $\delta' \subseteq \delta$. Enumerate $\boldsymbol{t}$ as $\langle (x_i, C_i) \rangle_{i<n}$. Let $\langle \mathcal{R}_k \rangle_{k \in \mathbb{N}}$ be a sequence in $\mathfrak{R}$ such that $\bigcup_{i \leq k} A_i \in \mathcal{R}$ whenever $\langle A_i \rangle_{i \leq k}$ is disjoint and $A_i \in \mathcal{R}_i$ for every $i \leq k$. For each $i < n$, let $\boldsymbol{s}_i$ be a δ'-fine member of T such that $W_{\boldsymbol{s}_i} \subseteq C_i$ and $C_i \setminus W_{\boldsymbol{s}_i} \in \mathcal{R}_{i+1}$, and set $\boldsymbol{s} = \bigcup_{i<n} \boldsymbol{s}_i$, so that $\boldsymbol{s} \in T$ is δ'-fine. By (a), there is a δ-fine $\boldsymbol{u} \in T$ such that $W_{\boldsymbol{u}} \cap W_{\boldsymbol{t}} = \emptyset$ and $X \setminus (W_{\boldsymbol{t}} \cup W_{\boldsymbol{u}}) \in \mathcal{R}_0$. Set $\boldsymbol{t}' = \boldsymbol{t} \cup \boldsymbol{u}$, $\boldsymbol{s}' = \boldsymbol{s} \cup \boldsymbol{u}$; then $\boldsymbol{t}'$ and $\boldsymbol{s}'$ are δ-fine and $\mathcal{R}$-filling, because

$$X \setminus W_{\boldsymbol{s}'} = (X \setminus (W_{\boldsymbol{t}} \cup W_{\boldsymbol{u}})) \cup \bigcup_{i<n} (C_i \setminus W_{\boldsymbol{s}_i}) \in \mathcal{R},$$

by the choice of $\langle \mathcal{R}_k \rangle_{k \in \mathbb{N}}$. So

$$|S_{\boldsymbol{t}}(f, \nu) - S_{\boldsymbol{s}}(f, \nu)| = |S_{\boldsymbol{t}'}(f, \nu) - S_{\boldsymbol{s}'}(f, \nu)| \leq \epsilon,$$

as required.

(d) There are $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ such that $|S_{\boldsymbol{t}}(f, \nu) - I_\nu(f)| \leq \epsilon$ whenever $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}$-filling. If $\boldsymbol{t} \in T$ is an arbitrary δ-fine tagged partition, there is a δ-fine $\mathcal{R}$-filling $\boldsymbol{t}' \supseteq \boldsymbol{t}$, by (b), so

$$S_{\boldsymbol{t}}(f, \nu) \leq S_{\boldsymbol{t}'}(f, \nu) \leq I_\nu(f) + \epsilon,$$

as claimed.

482B Saks-Henstock Lemma Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $f : X \to \mathbb{R}$, $\nu : \mathcal{C} \to \mathbb{R}$ functions such that $I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(f, \nu)$ is defined. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$. Then there is a unique additive functional $F : \mathcal{E} \to \mathbb{R}$ such that for every $\epsilon > 0$ there are $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ such that

(α) $\sum_{(x,C) \in \boldsymbol{t}} |F(C) - f(x) \nu C| \leq \epsilon$ for every δ-fine $\boldsymbol{t} \in T$,

(β) $|F(E)| \leq \epsilon$ whenever $E \in \mathcal{E} \cap \mathcal{R}$.

Moreover, $F(X) = I_\nu(f)$.

proof (a) For $E \in \mathcal{E}$, write T_E for the set of those $\boldsymbol{t} \in T$ such that, for every $(x, C) \in \boldsymbol{t}$, either $C \subseteq E$ or $C \cap E = \emptyset$. For any $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ and finite $\mathcal{D} \subseteq \mathcal{E}$ there is a δ-fine $\boldsymbol{t} \in \bigcap_{E \in \mathcal{D}} T_E$ such that $E \setminus W_{\boldsymbol{t}} \in \mathcal{R}$ for every $E \in \mathcal{D}$. **P** Let $\langle \mathcal{R}_n \rangle_{n \in \mathbb{N}}$ be a sequence in $\mathfrak{R}$ such that whenever $A_i \in \mathcal{R}_i$ for $i \leq n$ and $\langle A_i \rangle_{i \leq n}$ is disjoint then $\bigcup_{i \leq n} A_i \in \mathcal{R}$ (481He). Let $\mathcal{E}_0$ be the subalgebra of $\mathcal{E}$ generated by $\mathcal{D}$, and enumerate the atoms of $\mathcal{E}_0$ as $\langle E_i \rangle_{i<n}$. By 482Aa, there is for each $i < n$ a δ-fine $\boldsymbol{s}_i \in T$ such that $W_{\boldsymbol{s}_i} \subseteq E_i$ and $E_i \setminus W_{\boldsymbol{s}_i} \in \mathcal{R}_i$. Set $\boldsymbol{t} = \bigcup_{i<n} \boldsymbol{s}_i$. If $E \in \mathcal{D}$ then $E = \bigcup_{i \in J} E_i$ for some $J \subseteq n$. For any $(x, C) \in \boldsymbol{t}$, there is some $i < n$ such that $C \subseteq E_i$, so that $C \subseteq E$ if $i \in J$, $C \cap E = \emptyset$ otherwise; thus $\boldsymbol{t} \in T_E$. Moreover, $E \setminus W_{\boldsymbol{t}} = \bigcup_{i \in J} (E_i \setminus W_{\boldsymbol{s}_i})$ belongs to $\mathcal{R}$. **Q**

(b) We therefore have a filter $\mathcal{F}^*$ on T generated by sets of the form

$$T_{E\delta\mathcal{R}} = \{\boldsymbol{t} : \boldsymbol{t} \in T_E \text{ is } \delta\text{-fine}, E \setminus W_{\boldsymbol{t}} \in \mathcal{R}\}$$

as δ runs over Δ, $\mathcal{R}$ runs over $\mathfrak{R}$ and E runs over $\mathcal{E}$. For $\boldsymbol{t} \in T$, $E \subseteq X$ set $\boldsymbol{t}_E = \{(x, C) : (x, C) \in \boldsymbol{t}, C \subseteq E\}$. Now $F(E) = \lim_{\boldsymbol{t} \to \mathcal{F}^*} S_{\boldsymbol{t}_E}(f, \nu)$ is defined for every $E \in \mathcal{E}$. **P** For any $\epsilon > 0$, there are $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ such that $|I_\nu(f) - S_{\boldsymbol{t}}(f, \nu)| \leq \epsilon$ for every δ-fine $\mathcal{R}$-filling $\boldsymbol{t} \in T$. Let $\mathcal{R}' \in \mathfrak{R}$ be such that $A \cup B \in \mathcal{R}$ for all disjoint A, $B \in \mathcal{R}'$. If $\boldsymbol{t}$, $\boldsymbol{t}'$ belong to $T_{E,\delta,\mathcal{R}'} = T_{X \setminus E,\delta,\mathcal{R}'}$, then set

$$\boldsymbol{s} = \{(x, C) : (x, C) \in \boldsymbol{t}', C \subseteq E\} \cup \{(x, C) : (x, C) \in \boldsymbol{t}, C \cap E = \emptyset\}.$$

Then $\boldsymbol{s} \in T_E$ is δ-fine, and also $E \setminus W_{\boldsymbol{s}} = E \setminus W_{\boldsymbol{t}'}$, $(X \setminus E) \setminus W_{\boldsymbol{s}} = (X \setminus E) \setminus W_{\boldsymbol{t}}$ both belong to $\mathcal{R}'$; so their union $X \setminus W_{\boldsymbol{s}}$ belongs to $\mathcal{R}$, and $\boldsymbol{s}$ is $\mathcal{R}$-filling. Accordingly

$$\begin{aligned} |S_{\boldsymbol{t}_E}(f, \nu) - S_{\boldsymbol{t}'_E}(f, \nu)| &= |S_{\boldsymbol{t}}(f, \nu) - S_{\boldsymbol{s}}(f, \nu)| \\ &\leq |S_{\boldsymbol{t}}(f, \nu) - I_\nu(f)| + |I_\nu(f) - S_{\boldsymbol{s}}(f, \nu)| \leq 2\epsilon. \end{aligned}$$

As ϵ is arbitrary, this is enough to show that $\liminf_{\boldsymbol{t} \to \mathcal{F}^*} S_{\boldsymbol{t}_E}(f, \nu) = \limsup_{\boldsymbol{t} \to \mathcal{F}^*} S_{\boldsymbol{t}_E}(f, \nu)$, so that the limit $\lim_{\boldsymbol{t} \to \mathcal{F}^*} S_{\boldsymbol{t}_E}(f, \nu)$ is defined (2A3S). **Q**

(c) If E, $E' \in \mathcal{E}$ are disjoint, then

$$S_{\boldsymbol{t}_{E \cup E'}}(f, \nu) = S_{\boldsymbol{t}_E}(f, \nu) + S_{\boldsymbol{t}_{E'}}(f, \nu)$$

for any $\boldsymbol{t} \in T_E \cap T_{E'}$; since both T_E and $T_{E'}$ belong to $\mathcal{F}^*$, $F(E \cup E') = F(E) + F(E')$. Thus F is additive.

(d) Now suppose that $\epsilon > 0$. Let $\delta \in \Delta$, $\mathcal{R}^* \in \mathfrak{R}$ be such that $|I_\nu(f) - S_{\boldsymbol{t}}(f, \nu)| \leq \frac{1}{4}\epsilon$ for every δ-fine, $\mathcal{R}^*$-filling $\boldsymbol{t} \in T$. Let $\mathcal{R} \in \mathfrak{R}$ be such that $A \cup B \in \mathcal{R}^*$ for all disjoint A, $B \in \mathcal{R}$.

(i) If $\boldsymbol{t} \in T$ is δ-fine, then $|F(W_{\boldsymbol{t}}) - S_{\boldsymbol{t}}(f, \nu)| \leq \frac{1}{2}\epsilon$. **P** For any $\eta > 0$, there is a δ-fine $\boldsymbol{s} \in T$ such that
$|I_\nu(f) - S_{\boldsymbol{s}}(f, \nu)| \leq \eta$,
for every $(x, C) \in \boldsymbol{s}$, either $C \subseteq W_{\boldsymbol{t}}$ or $C \cap W_{\boldsymbol{t}} = \emptyset$,
$(X \setminus W_{\boldsymbol{t}}) \setminus W_{\boldsymbol{s}} \in \mathcal{R}$, $W_{\boldsymbol{t}} \setminus W_{\boldsymbol{s}} \in \mathcal{R}$,
$|F(W_{\boldsymbol{t}}) - \sum_{(x,C) \in \boldsymbol{s}, C \subseteq W_{\boldsymbol{t}}} f(x)\nu C| \leq \eta$
because the set of $\boldsymbol{s}$ with these properties belongs to $\mathcal{F}^*$. Now, setting $\boldsymbol{s}_1 = \{(x, C) : (x, C) \in \boldsymbol{s}, C \subseteq W_{\boldsymbol{t}}\}$ and $\boldsymbol{t}' = \boldsymbol{t} \cup \boldsymbol{s} \setminus \boldsymbol{s}_1$, $\boldsymbol{t}'$ is δ-fine and $\mathcal{R}^*$-filling, like $\boldsymbol{s}$, so

$$\begin{aligned} |F(W_{\boldsymbol{t}}) - S_{\boldsymbol{t}}(f, \nu)| &\leq |F(W_{\boldsymbol{t}}) - S_{\boldsymbol{s}_1}(f, \nu)| + |S_{\boldsymbol{s}_1}(f, \nu) - S_{\boldsymbol{t}}(f, \nu)| \\ &\leq \eta + |S_{\boldsymbol{s}}(f, \nu) - S_{\boldsymbol{t}'}(f, \nu)| \\ &\leq \eta + |S_{\boldsymbol{s}}(f, \nu) - I_\nu(f)| + |I_\nu(f) - S_{\boldsymbol{t}'}(f, \nu)| \leq \eta + \frac{1}{2}\epsilon. \end{aligned}$$

As η is arbitrary we have the result. **Q**

(ii) So if $\boldsymbol{t} \in T$ is δ-fine, $\sum_{(x,C) \in \boldsymbol{t}} |F(C) - f(x)\nu C| \leq \epsilon$. **P** Set $\boldsymbol{t}' = \{(x, C) : (x, C) \in \boldsymbol{t}, F(C) \leq f(x)\nu C\}$, $\boldsymbol{t}'' = \boldsymbol{t} \setminus \boldsymbol{t}''$. Then both $\boldsymbol{t}'$ and $\boldsymbol{t}''$ are δ-fine, so

$$\begin{aligned} &\sum_{(x,C) \in \boldsymbol{t}} |F(C) - f(x)\nu C| \\ &\quad = |\sum_{(x,C) \in \boldsymbol{t}'} F(C) - f(x)\nu C| + |\sum_{(x,C) \in \boldsymbol{t}''} F(C) - f(x)\nu C| \leq \epsilon. \end{aligned}$$ **Q**

(iii) Now suppose that $E \in \mathcal{E} \cap \mathcal{R}$. Then $|F(E)| \leq \epsilon$. **P** Let $\mathcal{R}' \in \mathfrak{R}$ be such that $A \cup B \in \mathcal{R}$ whenever A, $B \in \mathcal{R}'$ are disjoint. Let $\boldsymbol{t}$ be such that

$\boldsymbol{t} \in T_E$ is δ-fine,

$E \setminus W_{\boldsymbol{t}}$ and $(X \setminus E) \setminus W_{\boldsymbol{t}}$ both belong to $\mathcal{R}'$,

$|F(E) - S_{\boldsymbol{t}_E}(f, \nu)| \leq \frac{1}{2}\epsilon$;

once again, the set of candidates belongs to $\mathcal{F}^*$, so is not empty. Then $\boldsymbol{t}$ and $\boldsymbol{t}_{X \setminus E}$ are both $\mathcal{R}^*$-filling and δ-fine, so

$$\begin{aligned} |F(E)| &\leq \tfrac{1}{2}\epsilon + |S_{\boldsymbol{t}_E}(f, \nu)| \\ &= \tfrac{1}{2}\epsilon + |S_{\boldsymbol{t}}(f, \nu) - S_{\boldsymbol{t}_{X \setminus E}}(f, \nu)| \leq \epsilon. \end{aligned}$$ **Q**

As ϵ is arbitrary, this shows that F has all the required properties.

(d) I have still to show that F is unique. Suppose that $F' : \mathcal{E} \to \mathbb{R}$ is another functional with the same properties, and take $E \in \mathcal{E}$ and $\epsilon > 0$. Then there are δ, $\delta' \in \Delta$ and $\mathcal{R}$, $\mathcal{R}' \in \mathfrak{R}$ such that

$\sum_{(x,C) \in \boldsymbol{t}} |F(C) - f(x)\nu C| \leq \epsilon$ for every δ-fine $\boldsymbol{t} \in T$,

$\sum_{(x,C) \in \boldsymbol{t}} |F'(C) - f(x)\nu C| \leq \epsilon$ for every δ'-fine $\boldsymbol{t} \in T$,

$|F(R)| \leq \epsilon$ whenever $R \in \mathcal{E} \cap \mathcal{R}$,

$|F'(R)| \leq \epsilon$ whenever $R \in \mathcal{E} \cap \mathcal{R}'$.

Now taking $\delta'' \in \Delta$ such that $\delta'' \subseteq \delta \cap \delta'$, and $\mathcal{R}'' \in \mathfrak{R}$ such that $\mathcal{R}'' \subseteq \mathcal{R} \cap \mathcal{R}'$, there is a δ''-fine $\boldsymbol{t} \in T$ such that $E' = W_{\boldsymbol{t}}$ is included in E and $E \setminus E' \in \mathcal{R}''$. In this case

$$|F(E) - F'(E)| \leq |F(E \setminus E')| + \sum_{(x,C) \in \boldsymbol{t}} |F(C) - F'(C)| + |F'(E \setminus E')|$$

(because F and F' are both additive)

$$\leq 2\epsilon + \sum_{(x,C) \in \boldsymbol{t}} |F(C) - f(x)\nu C| + \sum_{(x,C) \in \boldsymbol{t}} |F'(C) - f(x)\nu C| \leq 4\epsilon.$$

As ϵ and E are arbitrary, $F = F'$, as required.

(e) Finally, to calculate $F(X)$, take any $\epsilon > 0$. Let $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ be such that $\sum_{(x,C) \in \boldsymbol{t}} |F(C) - f(x)\nu C| \leq \epsilon$ for every δ-fine $\boldsymbol{t} \in T$ and $|F(E)| \leq \epsilon$ whenever $E \in \mathcal{R}$. Let $\boldsymbol{t}$ be any δ-fine $\mathcal{R}$-filling member of T such that $S_{\boldsymbol{t}}(f, \nu) - I_\nu(f)| \leq \epsilon$. Then, because F is additive,

$$|F(X) - I_\nu(f)| \leq |F(X) - F(W_{\boldsymbol{t}})| + |\textstyle\sum_{(x,C) \in \boldsymbol{t}} F(C) - f(x)\nu C| + |S_{\boldsymbol{t}} - I_\nu(f)| \leq 3\epsilon.$$

As ϵ is arbitrary, $F(X) = I_\nu(f)$.

482C Definition In the context of 482B, I will call the function F the **Saks-Henstock indefinite integral** of f; of course it depends on the whole structure $(X, T, \Delta, \mathfrak{R}, \mathcal{C}, f, \nu)$ and not just on (X, f, ν). You should *not* take it for granted that $F(E) = I_\nu(f \times \chi E)$; but see 482G below.

482D The Saks-Henstock lemma characterizes the gauge integral, as follows.

Theorem Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $\nu : \mathcal{C} \to \mathbb{R}$ any function. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$. If $f : X \to \mathbb{R}$ is any function, then the following are equiveridical:

(i) $I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f, \nu)$ is defined in $\mathbb{R}$;

(ii) there is an additive functional $F : \mathcal{E} \to \mathbb{R}$ such that

(α) for every $\epsilon > 0$ there is a $\delta \in \Delta$ such that $\sum_{(x,C) \in \boldsymbol{t}} |F(C) - f(x)\mu C| \leq \epsilon$ for every δ-fine $\boldsymbol{t} \in T$,

(β) for every $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(E)| \leq \epsilon$ for every $E \in \mathcal{E} \cap \mathcal{R}$.

proof (i)⇒(ii) is just 482B above; so let us assume (ii) and seek to prove (i). Given $\epsilon > 0$, take $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ such that (α) and (β) are satisfied. Let $\boldsymbol{t} \in T_\alpha$ be δ-fine and $\mathcal{R}$-filling. Then

$$|F(\mathbb{R}^r) - S_{\boldsymbol{t}}(f,\mu)| \leq |F(\mathbb{R}^r \setminus W_{\boldsymbol{t}})| + \sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\mu C|$$
$$\leq 2\epsilon.$$

As ϵ is arbitrary, $I_\nu(f)$ is defined and equal to $F(\mathbb{R}^r)$.

482E Theorem Let (X,ρ) be a metric space and μ a complete locally determined measure on X with domain Σ. Let $\mathcal{C}$, Q, T, Δ and $\mathfrak{R}$ be such that

(i) $\mathcal{C} \subseteq \Sigma$ and μC is finite for every $C \in \mathcal{C}$;

(ii) $Q \subseteq X \times \mathcal{C}$, and for $C \in \mathcal{C}$, $(x,C) \in Q$ for almost every $x \in C$;

(iii) T is the straightforward set of tagged partitions generated by Q;

(iv) Δ is a downwards-directed family of neighbourhood gauges on X containing all the uniform metric gauges;

(v) $\mathfrak{R}$ is a downwards-directed collection of families of subsets of X such that whenever $E \in \Sigma$, $\mu E < \infty$ and $\epsilon > 0$, there is an $\mathcal{R} \in \mathfrak{R}$ such that $\mu^*(E \cap R) \leq \epsilon$ for every $R \in \mathcal{R}$;

(vi) T is compatible with Δ and $\mathfrak{R}$.

Let $f : X \to \mathbb{R}$ be any function such that $I_\mu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\mu)$ is defined. Then f is Σ-measurable.

proof ? Suppose, if possible, otherwise.

Because μ is complete and locally determined, there are a measurable set E of non-zero finite measure and $\alpha < \beta$ in $\mathbb{R}$ such that

$$\mu^*\{x : x \in E,\ f(x) \leq \alpha\} = \mu^*\{x : x \in E,\ f(x) \geq \beta\} = \mu E$$

(413G). Let $\epsilon > 0$ be such that $(\beta-\alpha)(\mu E - 3\epsilon) > 2\epsilon$. Let $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ be such that $|S_{\boldsymbol{t}}(f,\mu) - I_\nu(f)| \leq \epsilon$ whenever $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}$-filling. Let $\langle G_x\rangle_{x\in X}$ be a family of open sets such that $x \in G_x$ for every $x \in X$ and $\delta = \{(x,C) : x \in X,\ C \in \mathcal{C},\ C \subseteq G_x\}$. For $m \geq 1$, set

$$A_m = \{x : x \in E,\ f(x) \leq \alpha,\ U_{1/m}(x) \subseteq G_x\},$$

writing $U_{1/m}(x)$ for $\{y : \rho(y,x) < \frac{1}{m}\}$,

$$B_m = \{x : x \in E,\ f(x) \geq \beta,\ U_{1/m}(x) \subseteq G_x\}.$$

Then there is some $m \geq 1$ such that $\mu^* A_m \geq \mu E - \epsilon$, $\mu^* B_m \geq \mu E - \epsilon$. Let $\delta' \in \Delta$ be such that

$$\delta' \subseteq \delta \cap \{(x,C) : x \in C \subseteq U_{1/3m}(x)\}.$$

Let $\mathcal{R}' \in \mathfrak{R}$ be such that $\mathcal{R}' \subseteq \mathcal{R}$ and $\mu^*(R \cap E) \leq \epsilon$ for every $R \in \mathcal{R}'$.

Let $\boldsymbol{t}$ be any δ'-fine $\mathcal{R}'$-filling member of T. Enumerate $\boldsymbol{t}$ as $\langle (x_i, C_i)\rangle_{i<n}$. Set

$$J = \{i : i < n,\ C_i \cap A_m \text{ is negligible}\}, \quad J' = \{i : i < n,\ C_i \cap B_m \text{ is negligible}\}.$$

Then

$$\mu(E \cap \textstyle\bigcup_{i\in J} C_i) \leq \mu_*(E \setminus A_m) = \mu E - \mu^*(E \cap A_m) \leq \epsilon,$$

and similarly $\mu(E \cap \bigcup_{i\in J'} C_i) \leq \epsilon$. Also, because $\bigcup_{i<m} C_i = W_{\boldsymbol{t}}$ belongs to $\mathcal{R}'$, $\mu(E \setminus \bigcup_{i<m} C_i) \leq \epsilon$. So, setting $K = n \setminus (J \cup J')$, $\sum_{i\in K} \mu C_i \geq \mu E - 3\epsilon$.

For $i \in K$, $\mu^*(C_i \cap A_m) > 0$, while $\{x : x \in C_i,\ (x,C_i) \notin Q\}$ is negligible, so we can find $x_i' \in C_i \cap A_m$ such that $(x_i', C_i) \in Q$; similarly, there is an $x_i'' \in C_i \cap B_m$ such that $(x_i'', C_i) \in Q$. For other $i < n$, set $x_i' = x_i'' = x_i$. By conditions (ii)-(iii) of this proposition, $\boldsymbol{s} = \{(x_i', C_i) : i < n\}$ and $\boldsymbol{s}' = \{(x_i'', C_i) : i < n\}$ belong to T. Of course they are $\mathcal{R}'$-filling, therefore $\mathcal{R}$-filling, because $\boldsymbol{t}$ is. We also see that, because $(x_i, C_i) \in \delta'$, the diameter of C_i is at most $\frac{2}{3m}$ for each $i < n$, so that $C_i \subseteq G_{x_i'}$ and $(x_i', C_i) \in \delta$ for each $i \in K$. But since surely $(x_i, C_i) \in \delta' \subseteq \delta$ for $i \in n \setminus K$, this means that $\boldsymbol{s}$ is δ-fine. Similarly, $\boldsymbol{s}'$ is δ-fine.

We must therefore have

$$|S_{\boldsymbol{s}'}(f,\mu) - S_{\boldsymbol{s}}(f,\mu)| \leq |S_{\boldsymbol{s}'}(f,\mu) - I_\mu(f)| + |S_{\boldsymbol{s}}(f,\mu) - I_\mu(f)| \leq 2\epsilon.$$

But

$$S_{\boldsymbol{s}'}(f,\mu) - S_{\boldsymbol{s}}(f,\mu) = \sum_{i\in K}(f(x_i'') - f(x_i'))\mu C_i$$
$$\geq (\beta-\alpha)\sum_{i\in K}\mu C_i \geq (\beta-\alpha)(\mu E - 3\epsilon) > 2\epsilon$$

by the choice of ϵ. **X**

So we have the result.

482F Proposition Let X, Σ, μ, $\mathfrak{T}$, T, Δ and $\mathfrak{R}$ be such that

(i) (X,Σ,μ) is a measure space;

(ii) $\mathfrak{T}$ is a topology on X such that μ is inner regular with respect to the closed sets and outer regular with respect to the open sets;

(iii) $T \subseteq [X\times\Sigma]^{<\omega}$ is a set of tagged partitions such that $C\cap C'$ is empty whenever (x,C), (x',C') are distinct members of any $\boldsymbol{t}\in T$;

(iv) Δ is a set of gauges on X containing every neighbourhood gauge on X;

(v) $\mathfrak{R}$ is a collection of families of subsets of X such that whenever $\mu E<\infty$ and $\epsilon>0$ there is an $\mathcal{R}\in\mathfrak{R}$ such that $\mu^*(E\cap R)\leq\epsilon$ for every $R\in\mathcal{R}$;

(vi) T is compatible with Δ and $\mathfrak{R}$.

Then $I_\mu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\mu)$ is defined and equal to $\int f d\mu$ for every μ-integrable function $f: X\to\mathbb{R}$.

proof (a) It is worth noting straight away that we can replace (X,Σ,μ) by its completion $(X,\hat\Sigma,\hat\mu)$. **P** We need to check that $\hat\mu$ is inner and outer regular. But inner regularity is 412Ha, and outer regularity is equally elementary: if $\hat\mu E<\gamma$, there is an $E'\in\Sigma$ such that $E\subseteq E'$ and $\mu E' = \hat\mu E$ (212C), and now there is an open set $G\in\Sigma$ such that $E'\subseteq G$ and $\mu G\leq\gamma$, so that $E\subseteq G$ and $\hat\mu G\leq\gamma$. Since we are not changing T or Δ or $\mathfrak{R}$, $I_{\hat\mu}(f) = I_\mu(f)$ if either is defined; while also $\int f d\mu = \int f d\hat\mu$ if either is defined, by 212Fb. **Q**

So let us suppose that μ is actually complete.

(b) In this case, f is measurable. Suppose to begin with that it is non-negative. Let $\epsilon>0$. For $m\in\mathbb{Z}$, set $E_m = \{x : x\in X,\ (1+\epsilon)^m\leq f(x)<(1+\epsilon)^{m+1}\}$. Then E_m is measurable and has finite measure, so there is a measurable open set $G_m\supseteq E_m$ such that $(1+\epsilon)^{m+1}\mu(G_m\setminus E_m)\leq 2^{-|m|}\epsilon$.

Take a set H_0 of finite measure and $\eta_0>0$ such that $\int_E f d\mu\leq\epsilon$ whenever $E\in\Sigma$ and $\mu(E\cap H_0)\leq 2\eta_0$ (225A); replacing H_0 by $\{x : x\in H_0,\ f(x)>0\}$ if necessary, we may suppose that $H_0\subseteq\bigcup_{m\in\mathbb{Z}} E_m$. Let $F\subseteq H_0$ be a closed set such that $\mu(H_0\setminus F)\leq\eta_0$.

Define $\langle V_x\rangle_{x\in X}$ by setting $V_x = G_m$ if $m\in\mathbb{Z}$ and $x\in E_m$, $V_x = X\setminus F$ if $f(x)=0$. Let $\delta\in\Delta$ be the corresponding neighbourhood gauge $\{(x,C) : x\in X,\ C\subseteq V_x\}$. Let $\mathcal{R}\in\mathfrak{R}$ be such that $\mu^*(R\cap H_0)\leq\eta_0$ for every $R\in\mathcal{R}$.

Suppose that $\boldsymbol{t}$ is any δ-fine $\mathcal{R}$-filling member of T. Enumerate $\boldsymbol{t}$ as $\langle(x_i,C_i)\rangle_{i<n}$. For each $m\in\mathbb{Z}$, set $J_m = \{i : i<n,\ x_i\in E_m\}$. Then $C_i\subseteq V_{x_i}\subseteq G_m$ for every $i\in J_m$, so

$$S_{\boldsymbol{t}}(f,\mu) = \sum_{i<n} f(x_i)\mu C_i = \sum_{m\in\mathbb{Z}}\sum_{i\in J_m} f(x_i)\mu C_i \leq \sum_{m\in\mathbb{Z}}(1+\epsilon)^{m+1}\mu G_m$$
$$\leq (1+\epsilon)\sum_{m\in\mathbb{Z}}(1+\epsilon)^m\mu E_m + \sum_{m\in\mathbb{Z}}(1+\epsilon)^{m+1}\mu(G_m\setminus E_m)$$
$$\leq (1+\epsilon)\int f d\mu + \sum_{m\in\mathbb{Z}} 2^{-|m|}\epsilon = (1+\epsilon)\int f d\mu + 3\epsilon.$$

On the other hand, set $F' = F\cap\bigcup_{i<n} C_i$. Because $\bigcup_{i<n} C_i\in\mathcal{R}$, $\mu(H_0\setminus F')\leq 2\eta_0$, and

$$S_{\boldsymbol{t}}(f,\mu) \geq \sum_{m\in\mathbb{Z}}\sum_{i\in J_m} f(x_i)\mu(C_i\cap F)$$
$$\geq (1+\epsilon)^{-1}\sum_{m\in\mathbb{Z}}\sum_{i\in J_m}(1+\epsilon)^{m+1}\mu(C_i\cap F)$$
$$\geq (1+\epsilon)^{-1}\int_{F'} f d\mu \geq (1+\epsilon)^{-1}(\int f d\mu - \epsilon).$$

What this means is that

$$\{\boldsymbol{t} : (1+\epsilon)^{-1}(\int f d\mu - \epsilon) \leq S_{\boldsymbol{t}}(f,\nu) \leq (1+\epsilon)\int f d\mu + 3\epsilon\}$$

belongs to $\mathcal{F}(T,\Delta,\Re)$, for any $\epsilon > 0$. So $\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\Re)} S_{\boldsymbol{t}}(f,\mu)$ is defined and equal to $\int f d\mu$.

(c) In general, f is expressible as $f^+ - f^-$ where f^+ and f^- are non-negative integrable functions, so

$$I_\nu(f) = I_\nu(f^+) - I_\nu(f^-) = \int f d\mu$$

by 481Ca.

482G Proposition Let X, $\mathfrak{T}$, $\mathcal{C}$, ν, T, Δ and $\Re$ be such that

(i) $(X,T,\Delta,\Re)$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$;

(ii) $\mathfrak{T}$ is a topology on X, and Δ is the set of neighbourhood gauges on X;

(iii) $\nu : \mathcal{C} \to \mathbb{R}$ is a functional such that

(α) ν is additive in the sense that $\nu C = \sum_{i=0}^n \nu C_i$ whenever $C_0,\dots,C_n \in \mathcal{C}$ are disjoint and $\bigcup_{i\leq n} C_i = C$ belongs to $\mathcal{C}$,

(β) whenever $C \in \mathcal{C}$ and $\epsilon > 0$, there are closed sets $F \subseteq C$, $F' \subseteq X \setminus C$ such that

$$\sup\{\textstyle\sum_{i=0}^n |\nu C_i| : C_0,\dots,C_n \in \mathcal{C} \text{ are disjoint, } \bigcup_{i\leq n} C_i \cap (F \cup F') = \emptyset\} \leq \epsilon;$$

(iv) whenever $E \in \mathcal{C}$ there is a $\delta_0 \in \Delta$ such that whenever $\{(x,C)\} \in T \cap \delta_0$ and $x \in \partial E$ there is a partition $C_0,\dots,C_n$ of C such that, for each $i \leq n$, $\{(x,C_i)\} \in T$ and either $C_i \subseteq E$ or $C_i \cap E = \emptyset$;

(v) whenever $C \in \mathcal{C}$ and $\mathcal{R} \in \Re$, there is an $\mathcal{R}' \in \Re$ such that $R \cap C \in \mathcal{R}$ for every $R \in \mathcal{R}'$.

Let $f : X \to \mathbb{R}$ be a function such that $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\Re)} S_{\boldsymbol{t}}(f,\nu)$ is defined. Let $\mathcal{E}$ be the algebra of subsets of X generated by $\mathcal{C}$, and $F : \mathcal{E} \to \mathbb{R}$ the Saks-Henstock indefinite integral of f. Then $I_\nu(f \times \chi H)$ is defined and equal to $F(H)$ for every $H \in \mathcal{E}$.

proof (a) Because both F and I_ν are additive, and $F(X) = I_\nu(f)$ (482B), and either H or its complement is a finite disjoint union of members of $\mathcal{C}$ (481Hd), it is enough to consider the case in which $H \in \mathcal{C}$. Let $\epsilon > 0$. Let $\delta_1 \in \Delta$ and $\mathcal{R}_1 \in \Re$ be such that $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\nu C| \leq \epsilon$ for every δ_1-fine $\boldsymbol{t} \in T$, and $|F(E)| \leq \epsilon$ for every $E \in \mathcal{E} \cap \mathcal{R}_1$. Let $\delta_0 \in \Delta$ be such that whenever $\{(x,C)\} \in T \cap \delta_0$ and $x \in \partial H$ there is a partition $C_0,\dots,C_n$ of C such that, for each $i \leq n$, $\{(x,C_i)\} \in T$ and either $C_i \subseteq H$ or $C_i \cap H = \emptyset$.

For each $n \in \mathbb{N}$ let $F_n \subseteq H$, $F_n' \subseteq X \setminus H$ be such that

$$\sup\{\textstyle\sum_{i=0}^n |\nu C_i| : C_0,\dots,C_n \in \mathcal{C} \text{ are disjoint, } \bigcup_{i\leq n} C_i \cap (F \cup F') = \emptyset\} \leq \frac{2^{-n}\epsilon}{n+1}.$$

Define $\langle V_x\rangle_{x\in X}$ by setting

$$\begin{aligned} V_x &= \operatorname{int} H \text{ if } x \in \operatorname{int} H,\\ &= X \setminus \overline{H} \text{ if } x \in X \setminus \overline{H},\\ &= X \setminus F_n \text{ if } x \in \overline{H} \setminus H \text{ and } n \leq |f(x)| < n+1,\\ &= X \setminus F_n' \text{ if } x \in H \setminus \operatorname{int} H \text{ and } n \leq |f(x)| < n+1, \end{aligned}$$

so that each V_x is an open set containing x. Let δ_2 be the neighbourhood gauge corresponding to the family $\langle V_x\rangle_{x\in X}$, and set $\delta = \delta_0 \cap \delta_1 \cap \delta_2 \in \Delta$. Let $\mathcal{R} \in \Re$ be such that $R \cap H \in \mathcal{R}_1$ whenever $R \in \mathcal{R}$.

(b) Let T' be the set of those $\boldsymbol{t} \in T$ such that, for each $(x,C) \in \boldsymbol{t}$, either $C \subseteq H$ or $C \cap H = \emptyset$. The key to the proof is the following fact: if $\boldsymbol{t} \in T$ is δ-fine, then there is a δ-fine $\boldsymbol{s} \in T'$ such that $W_{\boldsymbol{s}} = W_{\boldsymbol{t}}$ and $S_{\boldsymbol{s}}(g,\nu) = S_{\boldsymbol{t}}(g,\nu)$ for every $g : X \to \mathbb{R}$. **P** Enumerate $\boldsymbol{t}$ as $\langle (x_i,C_i)\rangle_{i<n}$. For each $i < n$ define $\boldsymbol{s}_i \in T$ as follows. If $x_i \notin \partial H$, then take $\boldsymbol{s}_i = \{(x_i,C_i)\}$. Observe that because $C_i \subseteq V_{x_i}$, we either have $x_i \in \operatorname{int} H$ and $C_i \subseteq H$ or $x_i \notin \overline{H}$ and $C_i \cap H = \emptyset$. If $x_i \in \partial H$, then, because $(x_i,C_i) \in \delta_0$, we can find a finite partition $\mathcal{D}_i$ of C_i into sets such that, for each $D \in \mathcal{D}_i$, $\{(x_i,D)\} \in T$ and D is either included in H or disjoint from it; set $\boldsymbol{s}_i = \{(x_i,D) : D \in \mathcal{D}_i\}$. Because δ is a neighbourhood gauge, $(x_i,D) \in \delta$ for every $D \in \mathcal{D}_i$. Now take $\boldsymbol{s} = \bigcup_{i<n} \boldsymbol{s}_i$; it is easy to see that this works, because ν is additive, so $\nu C_i = \sum_{D\in\mathcal{D}_i} \nu D$ when $x_i \in \partial H$. **Q**

(c) Now $|S_{\boldsymbol{t}}(f\times\chi H,\nu)-F(H)|\le 4\epsilon$ for every δ-fine $\mathcal{R}$-filling $\boldsymbol{t}\in T$. **P** By (b), there is a δ-fine $\mathcal{R}$-filling $\boldsymbol{s}\in T'$ such that $S_{\boldsymbol{t}}(f\times\chi H,\nu)=S_{\boldsymbol{s}}(f\times\chi H,\nu)$. Discarding any terms of the form $(x,\emptyset)$, we may suppose that $C\neq\emptyset$ whenever $(x,C)\in\boldsymbol{s}$. Consider

$$\boldsymbol{u}=\{(x,C):(x,C)\in\boldsymbol{s},\ C\subseteq H\},\quad \boldsymbol{v}=\{(x,C):(x,C)\in\boldsymbol{s},\ x\in H\}.$$

Then $W_{\boldsymbol{u}}=H\cap W_{\boldsymbol{s}}$, so we have

$$\begin{aligned}S_{\boldsymbol{s}}(f\times\chi H,\nu)&=S_{\boldsymbol{v}}(f,\nu),\\ |F(H)-S_{\boldsymbol{u}}(f,\nu)|&\le|F(H)-F(W_{\boldsymbol{u}})|+\sum_{(x,C)\in\boldsymbol{u}}|F(C)-f(x)\nu C|\\ &\le|F(H\setminus W_{\boldsymbol{s}})|+\epsilon\end{aligned}$$

(because $\boldsymbol{u}$ is δ-fine)

$$\le 2\epsilon$$

because $X\setminus W_{\boldsymbol{s}}\in\mathcal{R}$, so $H\setminus W_{\boldsymbol{s}}\in\mathcal{R}_1$.

Enumerate $\boldsymbol{s}$ as $\langle(x_i,C_i)\rangle_{i<n}$, so that

$$S_{\boldsymbol{u}}(f,\nu)=\textstyle\sum_{i\in J}f(x_i)\nu C_i,\quad S_{\boldsymbol{v}}(f\times\chi H,\nu)=\sum_{i\in K}f(x_i)\nu C_i,$$

where $J=\{i:i<n,\ C_i\subseteq H\}$, $K=\{i:i<n,\ x_i\in H\}$. For $m\in\mathbb{N}$, set $J_m=\{i:i\in J\setminus K,\ m\le|f(x)|<m+1\}$, $K_m=\{i:i\in K\setminus J,\ m\le|f(x)|<m+1\}$. If $i\in J_m$, then $x_i\notin H$ but $\emptyset\neq C_i\subseteq H$, so $V_{x_i}\cap H\neq\emptyset$ and $x_i\in\overline{H}\setminus H$; which implies that $V_{x_i}=X\setminus F_m$ and $C_i\subseteq H\setminus F_m$. Similarly, if $i\in K_m$, then $x_i\in H$ and $C_i\cap H=\emptyset$, so $x_i\in H\setminus\operatorname{int}H$ and $C_i\subseteq(X\setminus H)\setminus F'_m$. Now, for each i, $\langle C_i\rangle_{i\in J_m\cup K_m}$ is a disjoint family in $\mathcal{C}$ with union included in $X\subseteq(F_m\cup F'_m)$, so

$$\textstyle\sum_{i\in J_m\cup K_m}|\nu C_i|\le\frac{2^{-m}\epsilon}{m+1}.$$

So we have

$$\begin{aligned}|S_{\boldsymbol{u}}(f,\nu)-S_{\boldsymbol{v}}(f,\nu)|&=|\sum_{i\in J}f(x_i)\nu C_i-\sum_{i\in K}f(x_i)\nu C_i|\le\sum_{i\in J\triangle K}|f(x_i)\nu C_i|\\ &\le\sum_{m=0}^{\infty}\big(\sum_{i\in J_m}|f(x_i)\nu C_i|+\sum_{i\in K_m}|f(x_i)\nu C_i|\big)\\ &\le\sum_{m=0}^{\infty}2^{-m}\epsilon=2\epsilon.\end{aligned}$$

Putting these together,

$$\begin{aligned}|S_{\boldsymbol{t}}(f\times\chi H,\nu)-F(H)|&\le|S_{\boldsymbol{v}}(f,\nu)-S_{\boldsymbol{u}}(f,\nu)|+|S_{\boldsymbol{u}}(f,\nu)-F(H)|\\ &\le 2\epsilon+2\epsilon=4\epsilon.\ \textbf{Q}\end{aligned}$$

(d) As ϵ is arbitrary, $I_\nu(f\times\chi H)$ is defined and equal to $F(H)$, as claimed.

482H Proposition Suppose that X, $\mathfrak{T}$, $\mathcal{C}$, ν, T, Δ and $\mathfrak{R}$ satisfy the conditions (i)-(v) of 482G, and that $f:X\to\mathbb{R}$, $\langle H_n\rangle_{n\in\mathbb{N}}$, H and γ are such that

(vi) $\langle H_n\rangle_{n\in\mathbb{N}}$ is a sequence of open subsets of X with union H,

(vii) $I_\nu(f\times\chi H_n)$ is defined for every $n\in\mathbb{N}$,

(viii) $\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})}I_\nu(f\times\chi W_{\boldsymbol{t}\restriction H})$ is defined and equal to γ,

where $\boldsymbol{t}\restriction H=\{(x,C):(x,C)\in\boldsymbol{t},\ x\in H\}$ for $\boldsymbol{t}\in T$. Then $I_\nu(f\times\chi H)$ is defined and equal to γ.

proof Let $\epsilon>0$. For each $n\in\mathbb{N}$, let F_n be the Saks-Henstock indefinite integral of $f\times\chi H_n$. Let $\delta_n\in\Delta$ be such that

$$\begin{aligned}2^{-n}\epsilon &\geq \sum_{(x,C)\in\boldsymbol{s}} |F_n(C) - (f\times\chi H_n)(x)\nu C|\\ &\geq |F_n(W_{\boldsymbol{s}}) - S_{\boldsymbol{s}}(f\times\chi H_n,\nu)|\end{aligned}$$

whenever $\boldsymbol{s}\in T$ is δ_n-fine. Set

$$\begin{aligned}\tilde\delta = \{(x,A) &: x\in X\setminus H,\ A\subseteq X\}\\ &\cup \bigcup_{n\in\mathbb{N}} \{(x,A) : x\in H_n\setminus\bigcup_{i<n} H_i,\ A\subseteq H_n,\ (x,A)\in\delta_n\},\end{aligned}$$

so that $\tilde\delta\in\Delta$. Note that if $x\in H$ and $C\in\mathcal{C}$ and $(x,C)\in\tilde\delta$, then there is some $n\in\mathbb{N}$ such that $x\in H_n$ and $C\subseteq H_n$, so that

$$I_\nu(f\times\chi C) = I_\nu((f\times\chi H_n)\times\chi C) = F_n(C)$$

is defined, by 482G; this means that $I_\nu(f\times\chi W_{\boldsymbol{t}\restriction H})$ will be defined for every $\tilde\delta$-fine $\boldsymbol{t}\in T$. Let $\delta\in\Delta$, $\mathcal{R}\in\mathfrak{R}$ be such that $|\gamma - I_\nu(f\times\chi W_{\boldsymbol{t}\restriction H})|\leq\epsilon$ whenever $\boldsymbol{t}\in T$ is δ-fine and $\mathcal{R}$-filling.

Let $\boldsymbol{t}\in T$ be $(\delta\cap\tilde\delta)$-fine and $\mathcal{R}$-filling. For $n\in\mathbb{N}$, set $\boldsymbol{t}_n = \{(x,C) : (x,C)\in\boldsymbol{t},\ x\in H_n\setminus\bigcup_{i<n}H_i\}$. Then $\boldsymbol{t}\restriction H = \bigcup_{n\in\mathbb{N}}\boldsymbol{t}_n$, and $\boldsymbol{t}_n$ is δ_n-fine and $W_{\boldsymbol{t}_n}\subseteq H_n$ for every n. So

$$\begin{aligned}|\gamma - S_{\boldsymbol{t}}(f\times\chi H,\nu)| &= |\gamma - \sum_{n=0}^\infty S_{\boldsymbol{t}_n}(f\times\chi H_n,\nu)|\\ &\leq |\gamma - I_\nu(f\times\chi W_{\boldsymbol{t}\restriction H})| + \sum_{n=0}^\infty |I_\nu(f\times\chi W_{\boldsymbol{t}_n}) - S_{\boldsymbol{t}_n}(f\times\chi H_n,\nu)|\end{aligned}$$

(note that $\boldsymbol{t}_n=\emptyset$ for all but finitely many n, so that $I_\nu(f\times\chi W_{\boldsymbol{t}\restriction H}) = \sum_{n=0}^\infty I_\nu(f\times\chi W_{\boldsymbol{t}_n})$)

$$\leq \epsilon + \sum_{n=0}^\infty |I_\nu(f\times\chi H_n\times\chi W_{\boldsymbol{t}_n}) - S_{\boldsymbol{t}_n}(f\times\chi H_n,\nu)|$$

(because $\boldsymbol{t}$ is δ-fine and $\mathcal{R}$-filling, while $W_{\boldsymbol{t}_n}\subseteq H_n$ for each n)

$$= \epsilon + \sum_{n=0}^\infty |F_n(W_{\boldsymbol{t}_n}) - S_{\boldsymbol{t}_n}(f\times\chi H_n,\nu)|$$

(by 482G)

$$\leq \epsilon + \sum_{n=0}^\infty 2^{-n}\epsilon$$

(because every $\boldsymbol{t}_n$ is δ_n-fine)

$$= 3\epsilon.$$

As ϵ is arbitrary, $\gamma = I_\nu(f\times\chi H)$, as claimed.

Remark For applications of this result see 483Bd and 483N.

482I Definitions (a) Let X be a set and $\Delta\subseteq\mathcal{P}(X\times\mathcal{P}X)$ a family of gauges on X. I will say that Δ is **countably full** if whenever $\langle\delta_n\rangle_{n\in\mathbb{N}}$ is a sequence in Δ, and $\phi : X\to\mathbb{N}$ is any function, then there is a $\delta\in\Delta$ such that $(x,C)\in\delta_{\phi(x)}$ whenever $(x,C)\in\delta$. I will say that Δ is **full** if whenever $\langle\delta_x\rangle_{x\in X}$ is a family in Δ, then there is a $\delta\in\Delta$ such that $(x,C)\in\delta_x$ whenever $(x,C)\in\delta$.

Of course a full set of gauges is countably full. Observe that if $(X,\mathfrak{T})$ is any topological space, the set of all neighbourhood gauges on X is full.

(b) Let X be a set, $\mathcal{C}$ a family of subsets of X, $T\subseteq[X\times\mathcal{C}]^{<\omega}$ a family of tagged partitions, $\nu:\mathcal{C}\to[0,\infty[$ a function, and Δ a family of gauges on X. I will say that ν is **moderated** (with respect to T and Δ) if there are a $\delta\in\Delta$ and a function $h:X\to]0,\infty[$ such that $S_{\boldsymbol{t}}(h,\nu)\leq 1$ for every δ-fine $\boldsymbol{t}\in T$.

482J Integrating a derivative As will appear in the next two sections, the real strength of gauge integrals is in their power to integrate derivatives. I give an elementary general expression of this fact. In the formulae below, we can think of f as a 'derivative' of F if $\nu = \theta$ is strictly positive and additive and we rephrase condition (iii) as

$$\text{`}\lim_{C \to \mathcal{G}_x} \tfrac{F(C)}{\nu C} = f(x)\text{'},$$

where $\mathcal{G}_x$ is the filter on $\mathcal{C}$ generated by the sets $\{C : (x, C) \in \delta\}$ as δ runs over Δ.

Theorem Let X, $\mathcal{C} \subseteq \mathcal{P}X$, $\Delta \subseteq \mathcal{P}(X \times \mathcal{P}X)$, $\mathfrak{R} \subseteq \mathcal{P}\mathcal{P}X$, $T \subseteq [X \times \mathcal{C}]^{<\omega}$, $f : X \to \mathbb{R}$, $\nu : \mathcal{C} \to \mathbb{R}$, $F : \mathcal{C} \to \mathbb{R}$, $\theta : \mathcal{C} \to [0, 1]$ and $\gamma \in \mathbb{R}$ be such that

(i) T is a straightforward set of tagged partitions which is compatible with Δ and $\mathfrak{R}$,

(ii) Δ is a full set of gauges on X,

(iii) for every $x \in X$ and $\epsilon > 0$ there is a $\delta \in \Delta$ such that $|f(x)\nu C - F(C)| \leq \epsilon\theta C$ whenever $(x, C) \in \delta$,

(iv) $\sum_{i=0}^{n} \theta C_i \leq 1$ whenever $C_0, \ldots, C_n \in \mathcal{C}$ are disjoint,

(v) for every $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|\gamma - \sum_{C \in \mathcal{C}_0} F(C)| \leq \epsilon$ whenever $\mathcal{C}_0 \subseteq \mathcal{C}$ is a finite disjoint set and $X \setminus \bigcup \mathcal{C}_0 \in \mathcal{R}$.

Then $I_\nu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T, \Delta, \mathfrak{R})} S_{\boldsymbol{t}}(f, \nu)$ is defined and equal to γ.

proof Let $\epsilon > 0$. For each $x \in X$ let $\delta_x \in \Delta$ be such that $|f(x)\nu C - F(C)| \leq \epsilon\theta C$ whenever $(x, C) \in \delta_x$. Because Δ is full, there is a $\delta \in \Delta$ such that $(x, C) \in \delta_x$ whenever $(x, C) \in \delta$. Let $\mathcal{R} \in \mathfrak{R}$ be as in (v). If $\boldsymbol{t} \in T$ is δ-fine and $\mathcal{R}$-filling, then

$$\begin{aligned} |S_{\boldsymbol{t}}(f, \nu) - \gamma| &\leq \sum_{(x,C) \in \boldsymbol{t}} |f(x)\nu C - F(C)| + |\gamma - \sum_{(x,C) \in \boldsymbol{t}} F(C)| \\ &\leq \sum_{(x,C) \in \boldsymbol{t}} \epsilon\theta C + \epsilon \end{aligned}$$

(because $(x, C) \in \delta_x$ whenever $(x, C) \in \boldsymbol{t}$, while $X \setminus \bigcup_{(x,C) \in \boldsymbol{t}} C \in \mathcal{R}$)

$$\leq 2\epsilon$$

by condition (ii). As ϵ is arbitrary, we have the result.

482K B.Levi's Theorem Let $(X, T, \Delta, \mathfrak{R})$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, such that Δ is countably full. Let $\nu : \mathcal{E} \to \mathbb{R}$ be a non-negative finitely additive functional, where $\mathcal{E}$ is a ring of subsets of X including $\mathcal{C}$, which is moderated with respect to T and Δ.

Let $\langle f_n \rangle_{n \in \mathbb{N}}$ be a non-decreasing sequence of functions from X to $\mathbb{R}$ with supremum $f : X \to \mathbb{R}$. If $\gamma = \lim_{n \to \infty} I_\nu(f_n)$ is defined in $\mathbb{R}$, then $I_\nu(f)$ is defined and equal to γ.

proof As in the proof of 123A, we may, replacing $\langle f_n \rangle_{n \in \mathbb{N}}$ by $\langle f_n - f_0 \rangle_{n \in \mathbb{N}}$ if necessary, suppose that $f_n(x) \geq 0$ for every $n \in \mathbb{N}$, $x \in X$.

(a) Take $\epsilon > 0$. Then there is a $\delta \in \Delta$ such that $S_{\boldsymbol{t}}(f, \nu) \leq \gamma + 5\epsilon$ for every δ-fine $\boldsymbol{t} \in T$.

P Fix a strictly positive function $h : X \to]0, \infty[$ and a $\tilde{\delta} \in \Delta$ such that $S_{\boldsymbol{t}}(h, \nu) \leq 1$ for every $\tilde{\delta}$-fine $\boldsymbol{t} \in T$. Let $\epsilon > 0$. For each $n \in \mathbb{N}$ choose $\delta_n \in \Delta$, $\mathcal{R}_n \in \mathfrak{R}$ such that $|I_\nu(f_n) - S_{\boldsymbol{t}}(f, \nu)| \leq 2^{-n}\epsilon$ for every δ_n-fine $\mathcal{R}_n$-filling $\boldsymbol{t} \in T$. For each $x \in X$, take $r_x \in \mathbb{N}$ such that $f(x) \leq f_{r_x}(x) + \epsilon h(x)$. Let $\delta \in \Delta$ be such that $(x, C) \in \tilde{\delta} \cap \delta_{r_x}$ for every $(x, C) \in \delta$.

Suppose that $\boldsymbol{t} \in T$ is δ-fine. Enumerate $\boldsymbol{t}$ as $\langle (x_i, C_i) \rangle_{i < n}$. Let $k \in \mathbb{N}$ be so large that $S_{\boldsymbol{t}}(f, \nu) \leq S_{\boldsymbol{t}}(f_k, \nu) + \epsilon$ and $r_{x_i} \leq k$ for every $i < n$. For $m \leq k$, set $J_m = \{i : i < n, r_{x_i} = m\}$. For each $i \in J_m$, $(x_i, C_i) \in \delta_m$. By 482A(c-ii), there is a δ_k-fine $\boldsymbol{s}_m \in T$ such that $W_{\boldsymbol{s}_m} \subseteq \bigcup_{i \in J_m} C_i$ and $|S_{\boldsymbol{s}_m}(f_m, \nu) - \sum_{i \in J_m} f_m(x_i)\nu C_i| \leq 2^{-m}\epsilon$.

Set $\boldsymbol{s} = \bigcup_{m \leq k} \boldsymbol{s}_m$, so that $\boldsymbol{s}$ is a δ_k-fine member of T and

$$\begin{aligned} \sum_{m=0}^{k} \sum_{i \in J_m} f_m(x_i)\nu C_i &\leq \sum_{m=0}^{k} S_{\boldsymbol{s}_m}(f_m, \nu) + 2^{-m}\epsilon \leq \sum_{m=0}^{k} S_{\boldsymbol{s}_m}(f_k, \nu) + 2^{-m}\epsilon \\ &\leq S_{\boldsymbol{s}}(f_k, \nu) + 2\epsilon \leq I_\nu(f_k) + 3\epsilon \end{aligned}$$

(because $\boldsymbol{s}$ extends to a δ_k-fine $\mathcal{R}_k$-filling member of T, by 482Ab)

$$\leq \gamma + 3\epsilon.$$

Now $\boldsymbol{t}$ is $\tilde{\delta}$-fine, so $S_{\boldsymbol{t}}(h,\nu) \leq 1$. Accordingly

$$\begin{aligned} S_{\boldsymbol{t}}(f) &= \sum_{i<n} f(x_i)\nu C_i = \sum_{m=0}^{k} \sum_{i\in J_m} f(x_i)\nu C_i \\ &\leq \sum_{m=0}^{k} \sum_{i\in J_m} (f_m(x_i) + \epsilon h(x_i))\nu C_i \leq \gamma + 3\epsilon + \epsilon S_{\boldsymbol{t}}(h,\nu) \leq \gamma + 4\epsilon, \end{aligned}$$

as required. **Q**

As ϵ is arbitrary, $\limsup_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\Re)} S_{\boldsymbol{t}}(f,\nu)$ is at most γ.

(b) On the other hand, given $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $I_\nu(f) \geq \gamma - \epsilon$. So taking $\delta_n \in \Delta$, $\mathcal{R}_n \in \Re$ as in (a) above,

$$S_{\boldsymbol{t}}(f,\nu) \geq S_{\boldsymbol{t}}(f_n,\nu) \geq I_\nu(f_n) - \epsilon \geq \gamma - 2\epsilon$$

for every δ_n-fine $\mathcal{R}_n$-filling $\boldsymbol{t} \in T$. So $\liminf_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\Re)} S_{\boldsymbol{t}}(f,\nu)$ is at least γ, and $I_\nu(f) = \gamma$.

482L Lemma Let X be a set, $\mathcal{C}$ a family of subsets of X, Δ a countably full downwards-directed set of gauges on X, $\Re \subseteq \mathcal{PP}X$ a downwards-directed collection of residual families, and $T \subseteq [X \times \mathcal{C}]^{<\omega}$ a straightforward set of tagged partitions of X compatible with Δ and $\Re$. Suppose further that whenever $\delta \in \Delta$, $\mathcal{R} \in \Re$ and $\boldsymbol{t} \in T$ is δ-fine, there is a δ-fine $\mathcal{R}$-filling $\boldsymbol{t}' \in T$ including $\boldsymbol{t}$. If $\nu : \mathcal{C} \to [0,\infty[$ and $f : X \to [0,\infty[$ are such that $I_\nu(f) = 0$, and $g : X \to \mathbb{R}$ is such that $g(x) = 0$ whenever $f(x) = 0$, then $I_\nu(g) = 0$.

proof Let $\epsilon > 0$. For each $n \in \mathbb{N}$, let $\delta_n \in \Delta$, $\mathcal{R}_n \in \Re$ be such that $S_{\boldsymbol{t}}(f,\nu) \leq 2^{-n}\epsilon$ for every δ_n-fine $\mathcal{R}_n$-filling $\boldsymbol{t} \in T$. For $x \in X$, set $\phi(x) = \min\{n : |g(x)| \leq nf(x)\}$; let $\delta \in \Delta$ be such that $(x,C) \in \delta_{\phi(x)}$ whenever $(x,C) \in \delta$.

Let $\boldsymbol{t}$ be any δ-fine member of T. Then $|S_{\boldsymbol{t}}(g,\nu)| \leq 2\epsilon$. **P** Enumerate $\boldsymbol{t}$ as $\langle (x_i,C_i)\rangle_{i<n}$. For each $m \in \mathbb{N}$, set $K_m = \{i : i < n,\ \phi(x_i) = m\}$; then $\{(x_i,C_i) : i \in K_m\}$ is a δ_m-fine member of T, so extends to a δ_m-fine $\mathcal{R}_m$-filling member $\boldsymbol{t}_m$ of T, and

$$\textstyle\sum_{i\in K_m} |g(x_i)\nu C_i| \leq m \sum_{i\in K_m} f(x_i)\nu C_i \leq m S_{\boldsymbol{t}_m}(f,\nu) \leq 2^{-m} m\epsilon.$$

Summing over m,

$$|S_{\boldsymbol{t}}(g,\nu)| \leq \textstyle\sum_{m=0}^{\infty} 2^{-m} m\epsilon = 2\epsilon. \ \textbf{Q}$$

Because T is compatible with Δ and $\mathcal{R}$, this is enough to show that $I_\nu(g)$ is defined and equal to 0.

482M Fubini's theorem Suppose that, for $i = 1$ and $i = 2$, we have X_i, $\mathfrak{T}_i$, T_i, Δ_i, $\mathcal{C}_i$ and ν_i such that

(i) $(X_i,\mathfrak{T}_i)$ is a topological space;

(ii) Δ_i is the set of neighbourhood gauges on X_i;

(iii) $T_i \subseteq [X_i \times \mathcal{C}_i]^{<\omega}$ is a straightforward set of tagged partitions, compatible with Δ_i and $\{\{\emptyset\}\}$;

(iv) $\nu_i : \mathcal{C}_i \to [0,\infty[$ is a function;

(v) ν_1 is moderated with respect to T_1 and Δ_1.

Write X for $X_1 \times X_2$; Δ for the set of neighbourhood gauges on X; $\mathcal{C}$ for $\{C \times D : C \in \mathcal{C}_1,\ D \in \mathcal{C}_2\}$; Q for $\{((x,y), C \times D) : \{(x,C)\} \in T_1,\ \{(y,D)\} \in T_2\}$; T for the straightforward set of tagged partitions generated by Q; and set $\nu(C \times D) = \nu_1 C \cdot \nu_2 D$ for $C \in \mathcal{C}_1$, $D \in \mathcal{C}_2$.

(a) T is compatible with Δ and $\{\{\emptyset\}\}$.

(b) Let I_{ν_1}, I_{ν_2} and I_ν be the gauge integrals defined by these structures as in 481C-481F. Suppose that $f : X \to \mathbb{R}$ is such that $I_\nu(f)$ is defined. Set $f_x(y) = f(x,y)$ for $x \in X_1$, $y \in X_2$. Let $g : X_1 \to \mathbb{R}$ be any function such that $g(x) = I_{\nu_2}(f_x)$ whenever this is defined. Then $I_{\nu_1}(g)$ is defined and equal to $I_\nu(f)$.

proof (a) Let $\delta \in \Delta$; we seek a tagged partition $\boldsymbol{u} \in T$ such that $W_{\boldsymbol{u}} = X$. Let $\langle V_{xy}\rangle_{(x,y)\in X}$ be the family of open sets in X defining δ; choose open sets $G_{xy} \subseteq X_1$, $H_{xy} \subseteq X_2$ such that $(x,y) \in G_{xy} \times H_{xy} \subseteq V_{xy}$

for all $x \in X_1$, $y \in X_2$. For each $x \in X_1$, let δ_x be the neighbourhood gauge on X_2 defined from the family $\langle H_{xy}\rangle_{y\in X_2}$. Then there is a δ_x-fine tagged partition $\boldsymbol{t}_x \in T_2$ such that $W_{\boldsymbol{t}_x} = X_2$. Set $G_x = X_1 \cap \bigcap_{(y,D)\in\boldsymbol{t}_x} G_{xy}$.

The family $\langle G_x\rangle_{x\in X_1}$ defines a neighbourhood gauge δ^* on X_1, so there is a δ^*-fine $\boldsymbol{s} \in T_1$ such that $W_{\boldsymbol{s}} = X_1$. Now consider

$$\boldsymbol{u} = \{((x,y), C\times D) : (x,C) \in \boldsymbol{s},\ (y,D) \in \boldsymbol{t}_x\}.$$

Then it is easy to check (just as in part (b) of the proof of 481O) that $\boldsymbol{u}$ is a δ-fine member of T with $W_{\boldsymbol{u}} = X$.

(b)(i) Let A be $\{x : x \in X_1,\ I_{\nu_2}(f_x) \text{ is defined}\}$. Let $h : X_1 \to \mathbb{R}$ is any function such that $h(x) = 0$ for every $x \in A$. For $x \in X_1$, set

$$h_0(x) = \inf(\{1\} \cup \{\sup_{\boldsymbol{t},\boldsymbol{t}'\in F} S_{\boldsymbol{t}}(f_x,\nu_2) - S_{\boldsymbol{t}'}(f_x,\nu_2) : F \in \mathcal{F}(T_2, \Delta_2, \{\{\emptyset\}\})\}).$$

(Thus $I_{\nu_2}(f_x)$ is defined iff $h_0(x) = 0$.) Then $I_{\nu_1}(h_0) = 0$. **P** Let $\epsilon > 0$. Then there is a $\delta \in \Delta$ such that $S_{\boldsymbol{u}}(f,\nu) - S_{\boldsymbol{u}'}(f,\nu) \le \epsilon$ whenever $\boldsymbol{u}, \boldsymbol{u}' \in T$ are δ-fine and $W_{\boldsymbol{u}} = W_{\boldsymbol{u}'} = X$. Define $\langle V_{xy}\rangle_{(x,y)\in X}$, $\langle G_{xy}\rangle_{(x,y)\in X}$, $\langle H_{xy}\rangle_{(x,y)\in X}$ and $\langle\delta_x\rangle_{x\in X_1}$ from δ as in (a) above. For each $x \in X_1$, we can find δ_x-fine partitions $\boldsymbol{t}_x, \boldsymbol{t}'_x \in T_2$ such that $W_{\boldsymbol{t}_x} = W_{\boldsymbol{t}'_x} = X_2$ and $S_{\boldsymbol{t}_x}(f_x,\nu_2) - S_{\boldsymbol{t}'_x(f_x,\nu_1} \ge \frac{1}{2}h_0(x)$. Set $G_x = X_1 \cap \bigcap_{(y,D)\in\boldsymbol{t}_x\cup\boldsymbol{t}'_x} G_{xy}$.

Let δ^* be the neighbourhood gauge on X_1 defined from $\langle G_x\rangle_{x\in X_1}$. Let $\boldsymbol{s}$ be any δ^*-fine member of T_1 with $W_{\boldsymbol{s}} = X_1$. Set

$$\boldsymbol{u} = \{((x,y), C\times D) : (x,C) \in \boldsymbol{s},\ (y,D) \in \boldsymbol{t}_x\},$$

$$\boldsymbol{u}' = \{((x,y), C\times D) : (x,C) \in \boldsymbol{s},\ (y,D) \in \boldsymbol{t}'_x\},$$

Then $\boldsymbol{u}$ and $\boldsymbol{u}'$ are δ-fine members of T with $W_{\boldsymbol{u}} = W_{\boldsymbol{u}'} = X$, so

$$\begin{aligned}
S_{\boldsymbol{t}}(h_0,\nu_1) &= \sum_{(x,C)\in\boldsymbol{t}} h_0(x)\nu_1(C) \le 2\sum_{(x,C)\in\boldsymbol{t}} (S_{\boldsymbol{t}_x}(f_x,\nu_2) - S_{\boldsymbol{t}'_x}(f_x,\nu_2))\nu_1(C)\\
&= 2\Big(\sum_{\substack{(x,C)\in\boldsymbol{t}\\(y,D)\in\boldsymbol{t}_x}} f(x,y)\nu_1(C)\nu_2(D) - \sum_{\substack{(x,C)\in\boldsymbol{t}\\(y,D)\in\boldsymbol{t}'_x}} f(x,y)\nu_1(C)\nu_2(D)\Big)\\
&= 2\big(S_{\boldsymbol{u}}(f,\nu) - S_{\boldsymbol{u}'}(f,\nu)\big) \le 2\epsilon.
\end{aligned}$$

As ϵ is arbitrary, $I_{\nu_1}(h_0) = 0$. **Q**

By 482L, $I_{\nu_1}(h)$ is also zero.

(ii) Again take any $\epsilon > 0$, and let $\delta \in \Delta$ be such that $|S_{\boldsymbol{u}}(f,\nu) - I_\nu(f)| \le \epsilon$ for every δ-fine $\boldsymbol{u} \in T$ such that $W_{\boldsymbol{u}} = X$. Define $\langle V_{xy}\rangle_{(x,y)\in X}$, $\langle G_{xy}\rangle_{(x,y)\in X}$, $\langle H_{xy}\rangle_{(x,y)\in X}$ and $\langle\delta_x\rangle_{x\in X_1}$ from δ as in (a) and (i) above. Let $\tilde\delta \in \Delta_1$, $\tilde h : X_1 \to]0,\infty[$ be such that $S_{\boldsymbol{s}}(\tilde h,\nu_1) \le 1$ for every $\tilde\delta$-fine $\boldsymbol{s} \in T_1$.

For $x \in A$, let $\boldsymbol{t}_x \in T_2$ be δ_x-fine and such that $W_{\boldsymbol{t}_x} = X_2$ and $|S_{\boldsymbol{t}_x}(f_x,\nu_2) - I_{\nu_2}(f_x)| \le \epsilon\tilde h(x)$; for $x \in X_1 \setminus A$, let $\boldsymbol{t}_x$ be any δ_x-fine member of T_2 such that $W_{\boldsymbol{t}_x} = X_2$. Set $G_x = X_1 \cap \bigcap_{(y,D)\in\boldsymbol{t}_x} G_{xy}$ for every $x \in X_1$. Let δ^* be the neighbourhood gauge on X_1 defined by $\langle G_x\rangle_{x\in X_1}$.

Set $h_1(x) = g(x) - S_{\boldsymbol{t}_x}(f_x,\nu_2)$ for $x \in X_1 \setminus A$, 0 for $x \in A$. Then $I_{\nu_1}(|h_1|) = 0$, by (a). Let $\delta' \in \Delta_1$ be such that $\delta' \subseteq \delta^* \cap \tilde\delta$ and $S_{\boldsymbol{s}}(|h_1|,\nu_1) \le \epsilon$ for every δ'-fine $\boldsymbol{s} \in T_1$ such that $W_{\boldsymbol{s}} = X_1$.

Now suppose that $\boldsymbol{s} \in T_1$ is δ'-fine and that $W_{\boldsymbol{s}} = X_1$. Set

$$\boldsymbol{u} = \{((x,y), C\times D) : (x,C) \in \boldsymbol{s},\ (y,D) \in \boldsymbol{t}_x\},$$

so that $\boldsymbol{u} \in T$ is δ-fine and $W_{\boldsymbol{u}} = X$. Then

$$\begin{aligned}|S_{\boldsymbol{s}}(g,\nu_1) - I_\nu(f)| &\le |S_{\boldsymbol{u}}(f,\nu) - I_\nu(f)| + |S_{\boldsymbol{s}}(g,\nu_1) - S_{\boldsymbol{u}}(f,\nu)| \\ &\le \epsilon + \big|\sum_{(x,C)\in\boldsymbol{s}} \big(g(x) - \sum_{(y,D)\in\boldsymbol{t}_x} f(x,y)\nu_2 D\big)\nu_1 C\big| \\ &= \epsilon + \big|\sum_{(x,C)\in\boldsymbol{s}} \big(g(x) - S_{\boldsymbol{t}_x}(f_x,\nu_2)\big)\nu_1 C\big| \\ &\le \epsilon + \sum_{(x,C)\in\boldsymbol{s}} |h_1(x)|\nu_1 C \\ &\qquad + \sum_{(x,C)\in\boldsymbol{s},x\in A} |g(x) - \sum_{(y,D)\in\boldsymbol{t}_x} f(x,y)\nu_2 D|\nu_1 C \\ &\le 2\epsilon + \sum_{(x,C)\in\boldsymbol{s},x\in A} \epsilon\tilde{h}(x)\nu_1(C) \le 2\epsilon + \epsilon S_{\boldsymbol{s}}(\tilde{h},\nu_1) \le 3\epsilon.\end{aligned}$$

As ϵ is arbitrary, $I_\nu(g)$ is defined and equal to $I_\nu(f)$, as claimed.

482X Basic exercises (a) Let $(X,T,\Delta,\Re)$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $\mathcal{E}$ the algebra of subsets of X generated by $\mathcal{C}$. Write $\mathfrak{I}$ for the set of pairs (f,ν) such that $f : X \to \mathbb{R}$ and $\nu : \mathcal{C} \to \mathbb{R}$ are functions and $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\Re)} S_{\boldsymbol{t}}(f,\nu)$ is defined; for $(f,\nu) \in \mathfrak{I}$, let $F_{f\nu} : \mathcal{E} \to \mathbb{R}$ be the corresponding Saks-Henstock indefinite integral. Show that $(f,\nu) \mapsto F_{f\nu}$ is bilinear in the sense that

$$F_{f+g,\nu} = F_{f\nu} + F_{g\nu}, \quad F_{\alpha f,\nu} = F_{f,\alpha\nu} = \alpha F_{f\nu}, \quad F_{f,\mu+\nu} = F_{f\mu} + F_{f\nu}$$

whenever (f,ν), (g,ν) and (f,μ) belong to $\mathfrak{I}$ and $\alpha \in \mathbb{R}$.

(b) Let $(X,T,\Delta,\Re)$ be a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, and $\mathcal{E}$ the algebra of subsets of X generated by $\mathcal{C}$. Let $f : X \to \mathbb{R}$ and $\nu : \mathcal{C} \to [0,\infty[$ be functions such that $I_\nu(f) = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\Re)} S_{\boldsymbol{t}}(f,\nu)$ is defined, and let $F : \mathcal{E} \to \mathbb{R}$ be the corresponding Saks-Henstock indefinite integral. Show that F is non-negative iff $I_\nu(f^-) = 0$, where $f^-(x) = \max(0, f(x))$ for every $x \in X$.

>(c) Let X be a zero-dimensional compact Hausdorff space, $\mathcal{E}$ the algebra of open-and-closed subsets of X, $Q = \{(x,E) : x \in E \in \mathcal{E}\}$, T the straightforward set of tagged partitions generated by Q, Δ the set of neighbourhood gauges on X and $\nu : \mathcal{E} \to \mathbb{R}$ a non-negative additive functional. Let μ be the corresponding Radon measure on X (416Q) and I_ν the gauge integral defined by $(X,T,\Delta,\{\{\emptyset\}\})$ (cf. 481Xh). Show that, for $f : X \to \mathbb{R}$, $I_\nu(f) = \int f d\mu$ if either is defined in $\mathbb{R}$. (*Hint*: if f is measurable but not μ-integrable, take x_0 such that f is not integrable on any neighbourhood of x_0. Given $\delta \in \Delta$, fix a δ-fine partition containing (x_0,V_0) for some V_0; now replace (x_0,V_0) by refinements $\{(x_0,V_0'),(x_1,V_1),\dots,(x_n,V_n)\}$, where $\sum_{i=1}^n f(x_i)\nu V_i$ is large, to show that $S_{\boldsymbol{t}}(f,\nu)$ cannot be controlled by δ.)

>(d) Let $(X,\mathfrak{T},\Sigma,\mu)$ be an effectively locally finite τ-additive topological measure space in which μ is inner regular with respect to the closed sets and outer regular with respect to the open sets (see 412W). Let T be the straightforward set of tagged partitions generated by $X \times \{E : \mu E < \infty\}$, Δ the set of neighbourhood gauges on X, and $\Re = \{\mathcal{R}_{E\eta} : \mu E < \infty, \eta > 0\}$, where $\mathcal{R}_{E\eta} = \{F : \mu(F \cap E) \le \eta\}$, as in 481N. Show that if I_μ is the associated gauge integral, and $f : X \to \mathbb{R}$ is a function, then $I_\mu(f) = \int f d\mu$ if either is defined in $\mathbb{R}$.

(e) Let $\mathcal{C}$ be the set of non-empty subintervals of $X = [0,1]$, T the straightforward tagged-partition structure generated by $[0,1] \times \mathcal{C}$, and Δ the set of neighbourhood gauges on $[0,1]$, as in 481M. Let μ be any Radon measure on $[0,1]$, and I_μ the gauge integral defined from μ and the tagged-partition structure $([0,1],T,\Delta,\{\{\emptyset\}\})$. Show that, for any $f : [0,1] \to \mathbb{R}$, $I_\mu(f) = \int f d\mu$ if either is defined in $\mathbb{R}$.

(f) Let $\mathcal{C}$ be the set of non-empty subintervals of $[0,1]$, T the straightforward tagged-partition structure generated by $\{(x,C) : C \in \mathcal{C}, x \in \overline{C}\}$, and Δ the set of neighbourhood gauges on X, as in 481J. Let $\nu : \mathcal{C} \to \mathbb{R}$ be a bounded additive functional, and I_ν the gauge integral defined from ν and $(X,T,\Delta,\{\{\emptyset\}\})$. Show that $I_\nu(\chi[a,b[) = \lim_{x\uparrow a, y\uparrow b} \nu([x,y])$ whenever $0 < a < b \le 1$.

(g) Let $\mathcal{C}$ be the set of non-empty subintervals of $X = [0,1]$, T the straightforward tagged-partition structure generated by $\{(x,C) : C \in \mathcal{C},\ x \in \overline{C}\}$, and Δ the set of uniform metric gauges on $[0,1]$, as in 481I. Let μ be the Radon probability measure on $[0,1]$ such that $\mu\{\frac{1}{2}\} = 1$, and let $I_\mu = \lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\Re)} S_{\boldsymbol{t}}(\ ,\mu)$ be the corresponding gauge integral. Show that $I_\mu(\chi[0,1])$ is defined but that $I_\mu(\chi[0,\frac{1}{2}])$ is not.

>(h) (i) Show that the McShane integral on an interval $[a,b]$ as described in 481M coincides with the Lebesgue integral on $[a,b]$. (ii) Show that if $(X,\mathfrak{T},\Sigma,\mu)$ is a quasi-Radon measure space and μ is outer regular with respect to the open sets then the McShane integral as described in 481N coincides with the usual integral.

(i) Explain how the results in 481Xb-481Xc can be regarded as special cases of 482H.

(j) Let $(X,\mathfrak{T})$ be a topological space, $\mathcal{C}$ a ring of subsets of X, $T \subseteq [X \times \mathcal{C}]^{<\omega}$ a straightforward set of tagged partitions, $\nu : \mathcal{C} \to [0,\infty[$ an additive function, and Δ the family of neighbourhood gauges on X. Suppose that there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of open sets, covering X, such that $\sup\{\nu C : C \in \mathcal{C},\ C \subseteq G_n\}$ is finite for every $n \in \mathbb{N}$. Show that ν is moderated with respect to T and Δ.

(k) Let $(X,\mathfrak{T},\Sigma,\mu)$ be a quasi-Radon measure space. Let T be the straightforward tagged-partition structure generated by $\{(x,E) : \mu E < \infty,\ x \in E\}$ and Δ the set of all neighbourhood gauges on X. Show that μ is moderated with respect to T and Δ iff there is a sequence of open sets of finite measure covering X.

482Y Further exercises (a) Set $X = \mathbb{R}$ and let $\mathcal{C}$ be the family of non-empty bounded intervals in X; set $Q = \{(x,C) : C \in \mathcal{C},\ x = \inf C\}$, and let T be the straightforward set of tagged partitions generated by Q. Let $\mathfrak{T}$ be the Sorgenfrey right-facing topology on X, and Δ the set of neighbourhood gauges for $\mathfrak{T}$. Set $\mathcal{R}_n = \{E : E \in \Sigma,\ \mu([-n,n] \cap E) \leq 2^{-n}\}$ for $n \in \mathbb{N}$, where μ is Lebesgue measure on $\mathbb{R}$ and Σ its domain, and write $\mathfrak{R} = \{\mathcal{R}_n : n \in \mathbb{N}\}$. Show that $(X,T,\Delta,\mathfrak{R})$ is a tagged-partition structure allowing subdivisions. Show that if $f : X \to \mathbb{R}$ is such that $I_\mu(f)$ is defined, then f is Lebesgue measurable.

(b) Give an example of X, $\mathfrak{T}$, Σ, μ, T, Δ, $\mathcal{C}$, f and C such that $(X,\mathfrak{T},\Sigma,\mu)$ is a compact metrizable Radon probability space, Δ is the set of all neighbourhood gauges on X, $(X,T,\Delta,\{\{\emptyset\}\})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$, $f : X \to \mathbb{R}$ is a function such that $I_\mu(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T,\Delta,\{\{\emptyset\}\})} S_{\boldsymbol{t}}(f,\mu)$ is defined, $C \in \mathcal{C}$ is a closed set with negligible boundary, and $I_\mu(f \times \chi C)$ is not defined.

(c) Suppose that, for $i = 1$ and $i = 2$, we have a tagged-partition structure $(X_i,T_i,\Delta_i,\mathfrak{R}_i)$ allowing subdivisions, witnessed by a ring $\mathcal{C}_i \subseteq \mathcal{P}X_i$, where X_i is a topological space, Δ_i is the set of all neighbourhood gauges on X_i, and $\mathfrak{R}_i$ is the simple residual structure complementary to $\mathcal{C}_i$, as in 481Yb. Set $X = X_1 \times X_2$ and let Δ be the set of neighbourhood gauges on X; set $\mathcal{C} = \{C \times D : C \in \mathcal{C}_1,\ D \in \mathcal{C}_2\}$; let $\mathfrak{R}$ be the simple residual structure on X complementary to $\mathcal{C}$; and let T be the straightforward tagged-partition structure generated by $\{((x,y),C \times D) : \{(x,C)\} \in T_1,\ \{(y,D)\} \in T_2\}$. For each i, let $\nu_i : \mathcal{C}_i \to [0,\infty[$ be a function moderated with respect to T_i and Δ_i, and define $\nu : \mathcal{C} \to [0,\infty[$ by setting $\nu(C \times D) = \nu_1 C \cdot \nu_2 D$ for $C \in \mathcal{C}_1$, $D \in \mathcal{C}_2$. Show that T is compatible with Δ and $\mathfrak{R}$. Let I_{ν_1}, I_{ν_2}, I_ν be the gauge integrals defined by these structures. Suppose that $f : X \to \mathbb{R}$ is such that $I_\nu(f)$ is defined. Set $f_x(y) = f(x,y)$ for $x \in X_1$, $y \in X_2$. Let $g : X_1 \to \mathbb{R}$ be any function such that $g(x) = I_{\nu_2}(f_x)$ whenever this is defined. Show that $I_{\nu_1}(g)$ is defined and equal to $I_\nu(f)$.

482 Notes and comments In 482E, 482F and 482G the long lists of conditions reflect the variety of possible applications of these arguments. The price to be paid for the versatility of the constructions here is a theory which is rather weak in the absence of special hypotheses. As everywhere in this book, I try to set ideas out in maximal convenient generality; you may feel that in this section the generality is becoming inconvenient; but the theory of gauge integrals has not, to my eye, matured to the point that we can classify the systems here even as provisionally as I set out to classify topological measure spaces in Chapters 41 and 43.

Enthusiasts for gauge integrals offer two substantial arguments for taking them seriously, apart from the universal argument in pure mathematics, that these structures offer new patterns for our delight and new challenges to our ingenuity. First, they say, gauge integrals integrate more functions than Lebesgue-type integrals, and it is the business of a theory of integration to integrate as many functions as possible; and secondly, gauge integrals offer an easier path to the principal theorems. I have to say that I think the first argument is sounder than the second. It is quite true that the Henstock integral on $\mathbb{R}$ (481K) can be rigorously defined in fewer words, and with fewer concepts, than the Lebesgue integral. The style of Chapters 11 and 12 is supposed to be better adapted to the novice than the style of this chapter, but you will have no difficulty in putting the ideas of 481A, 481C, 481J and 481K together into an elementary definition of an integral for real functions in which the only non-trivial argument is that establishing the existence of enough tagged partitions (481J), corresponding I suppose to Proposition 114D. But the path I took in defining the integral in §122, though arduous at that point, made (I hope) the convergence theorems of §123 reasonably natural; the proof of 482K, on the other hand, makes significant demands on our technique. Furthermore, the particular clarity of the one-dimensional Henstock integral is not repeated in higher dimensions. Fubini's theorem, with exact statement and full proof, even for products of Lebesgue measures on Euclidean spaces, is a lot to expect of an undergraduate; but Lebesgue measure on $\mathbb{R}^r$ makes sense in a way that it is quite hard to repeat with gauge integrals. (For instance, Lebesgue measure is invariant under isometries; this is not particularly easy to prove – see 263A – but at least it is true; if we want a gauge integral which is invariant under isometries, then we have to use a construction such as 481O, which does not directly match any natural general definition of 'product gauge integral' along the lines of 481P, 482M or 482Yc.)

In my view, a stronger argument for taking gauge integrals seriously is their 'power', that is, their ability to provide us with integrals of many functions in consistent ways. 482E, 482F and 482J give us an idea of what to expect. If we start from a measure space (X, Σ, μ) and build a gauge integral I_μ from a set $T \subseteq [X \times \Sigma]^{<\omega}$ of tagged partitions, then we can hope that integrable functions will be gauge-integrable, with the right integrals (482F); while gauge-integrable functions will be measurable (482E). What this means is that for *non-negative* functions, the integrals will coincide. Any 'new' gauge-integrable functions f will be such that $\int f^+ = \int f^- = \infty$; the gauge integral will offer a process for cancelling the divergent parts of these integrals. On the other hand, we can hope for a large class of gauge-integrable derivatives. In the next two sections, I will explain how this works in the Henstock and Pfeffer integrals. For simple examples calling for such procedures, see the formulae of §§282 and 283; for radical applications of the idea, see Muldowney 87.

Against this, gauge integrals are not effective in 'general' measure spaces, and cannot be, because there is no structure in an abstract measure space (X, Σ, μ) which allows us to cancel an infinite integral $\int f^+ = \int_F f$ against $\int f^- = \int_{X \setminus F} f$. Put another way, if a tagged-partition structure is invariant under all automorphisms of the structure (X, Σ, μ), as in 481Xf-481Xg, we cannot expect anything better than the standard integral. In order to get something new, the most important step seems to be the specification of a family $\mathcal{C}$ of 'regular' sets, preliminary to describing a set T of tagged partitions. To get a 'powerful' gauge integral, we want a fine filter on T, corresponding to a small set $\mathcal{C}$ and a large set of gauges. The residual families of 481F are generally introduced just to ensure 'compatibility' in the sense described there; as a rule, we try to keep them simple. But even if we take the set of all neighbourhood gauges, as in the Henstock integral, this is not enough unless we also sharply restrict both the family $\mathcal{C}$ and the permissible tags (482Xc-482Xe). The most successful restrictions, so far, have been 'geometric', as in 481J and 481O, and 484F below. Further limitations on admissible pairs (x, C), as in 481L and 481Q, in which $\mathcal{C}$ remains the set of intervals, but fewer tags are permitted, also lead to very interesting results.

Another limitation in the scope of gauge integrals is the difficulty they have in dealing with spaces of infinite measure. Of course we expect to have to specify a limiting procedure if we are to calculate $I_\mu(f)$ from sums $S_{\boldsymbol{t}}(f, \mu)$ which necessarily consider only sets of finite measure, and this is one of the functions of the collections $\mathfrak{R}$ of residual families. But this is not yet enough. In B.Levi's theorem (482K) we already need to suppose that our set-function ν is 'moderated' in order to determine how closely $f_n(x)$ needs to approximate each $f(x)$. The condition

$$S_{\boldsymbol{t}}(h, \nu) \leq 1 \text{ for every } \delta\text{-fine } \boldsymbol{t}$$

of 482Ib is very close to saying that $I_\mu(h) \leq 1$. But it is *not* the same as saying that μ is σ-finite; it suggests rather that X should be covered by a sequence of open sets of finite measure (482Xk).

Because gauge integrals are not absolute – that is, we can have $I_\nu(f)$ defined and finite while $I_\nu(|f|)$ is not – we are bound to have difficulties with integrals $\int_H f$, even if we interpret these in the simplest way, as $I_\nu(f \times \chi H)$, so that we do not need a theory of subspaces as developed in §214. 482G(iii)-(v) are an attempt to find a reasonably versatile sufficient set of conditions. The 'multiplier problem', for a given gauge integral I_ν, is the problem of characterizing the functions g such that $I_\nu(f \times g)$ is defined whenever $I_\nu(f)$ is defined, and even for some intensively studied integrals remains challenging. In 484L I will give an important case which is not covered by 482G.

One of the striking features of gauge integrals is that there is no need to assume that the set-function ν is countably additive. We can achieve countable additivity of the integral – in the form of B.Levi's theorem, for instance – by requiring only that the set of gauges should be 'countably full' (482K, 482L; contrast 482Xg). If we watch our definitions carefully, we can make this match the rules for Stieltjes integrals (114Xa, 482Xf). In 481Db I have already remarked on the potential use of gauge integrals in vector integration.

It is important to recognise that a value $F(E)$ of a Saks-Henstock indefinite integral (482B-482C) cannot be identified with either $I_\nu(f \times \chi E)$ or with $I_{\nu\restriction\mathcal{P}E}(f\restriction E)$, because in the formula $F(E) = \lim_{\boldsymbol{t}\to\mathcal{F}^*} S_{\boldsymbol{t}_E}(f,\nu)$ the tags of the partitions $\boldsymbol{t}_E$ need not lie in E. The idea of 482G is to impose sufficient conditions to ensure that the contributions of 'straddling' elements (x, C), where either $x \in H$ and $C \not\subseteq H$ or $x \notin H$ and $C \cap H \neq \emptyset$, are insignificant. To achieve this we seem to need both a regularity condition on the functional ν (condition (iii-β) of 482G) and a geometric condition on the set $\mathcal{C}$ underlying T (482G(iv)). As usual, the regularity condition required is closer to *outer* than to *inner* regularity, in contexts in which there is a distinction.

I am not sure that I have the 'right' version of Proposition 482E. The hypothesis there is that we have a metric space. But in the principal non-metrizable cases the result is still valid (482Xc-482Xe). Proposition 482H is a 'new' limit theorem; it has no counterpart in the ordinary theory of integration. The hypotheses seem, from where we are standing at the moment, to be exceedingly restrictive. In the leading examples in §483, however, the central requirement 482H(viii) is satisfied for straightforward reasons.

Gauge integrals insist on finite functions defined everywhere. But since we have an effective theory of negligible sets (482L), we can easily get a consistent theory of integration for functions which are defined and real-valued almost everywhere if we say that

$$I_\nu(f) = I_\nu(g) \text{ whenever } g : X \to \mathbb{R} \text{ extends } f\restriction f^{-1}[\mathbb{R}]$$

whenever $I_\nu(g) = I_\nu(g')$ for all such extensions.

483 The Henstock integral

I come now to the original gauge integral, the 'Henstock integral' for real functions. The first step is to check that the results of §482 can be applied to show that this is an extension of both the Lebesgue integral and the improper Riemann integral (483B), coinciding with the Lebesgue integral for non-negative functions (483C). It turns out that any Henstock integrable function can be approximated in a strong sense by a sequence of Lebesgue integrable functions (483G). The Henstock integral can be identified with the Perron and special Denjoy integrals (483J, 483N). Much of the rest of the section is concerned with indefinite Henstock integrals. Some of the results of §482 on tagged-partition structures allowing subdivisions condense into a particularly strong Saks-Henstock lemma (483F). If f is Henstock integrable, it is equal almost everywhere to the derivative of its Henstock integral (483I). Finally, indefinite Henstock integrals can be characterized as continuous ACG_* functions (483R).

483A Definition The following notation will apply throughout the section. Let $\mathcal{C}$ be the family of non-empty bounded intervals in $\mathbb{R}$, and let $T \subseteq [\mathbb{R}\times\mathcal{C}]^{<\omega}$ be the straightforward set of tagged partitions generated by $\{(x, C) : C \in \mathcal{C}, x \in \overline{C}\}$. Let Δ be the set of all neighbourhood gauges on $\mathbb{R}$. Set $\mathfrak{R} = \{\mathcal{R}_{ab} : a \leq b \in \mathbb{R}\}$, where $\mathcal{R}_{ab} = \{\mathbb{R} \setminus [c, d] : c \leq a,\ d \geq b\} \cup \{\emptyset\}$. Then $(\mathbb{R}, T, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions (481K), so T is compatible with Δ and $\mathfrak{R}$ (481Hf). The **Henstock integral** is the gauge integral defined by the process of 481E-481F from $(\mathbb{R}, T, \Delta, \mathfrak{R})$ and Lebesgue measure μ. For a function $f : \mathbb{R} \to \mathbb{R}$ I will say that f is **Henstock integrable**, and that $\int f = \gamma$, if $\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\mathfrak{R})} S_{\boldsymbol{t}}(f,\mu)$ is defined and equal to $\gamma \in \mathbb{R}$. For α, $\beta \in [-\infty, \infty]$ I will write $\int_\alpha^\beta f$ for $\int f \times \chi\,]\alpha, \beta[$ if this is defined in $\mathbb{R}$. I will use the symbol $\int$ for the ordinary integral, so that $\int f d\mu$ is the Lebesgue integral of f.

483B Tracing through the theorems of §482, we have the following.

Theorem (a) Every Henstock integrable function on $\mathbb{R}$ is Lebesgue measurable.

(b) Every Lebesgue integrable function $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, with the same integral.

(c) If f is Henstock integrable so is $f \times \chi C$ for every interval $C \subseteq \mathbb{R}$.

(d) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is any function. Then

(α) if $\beta \in]-\infty, \infty]$, $\int_{-\infty}^{\beta} f = \lim_{b \uparrow \beta} \int_{-\infty}^{b} f$ if either is defined in $\mathbb{R}$,

(β) if $\alpha \in [-\infty, \infty[$, $\int_{\alpha}^{\infty} f = \lim_{a \downarrow \alpha} \int_{a}^{\infty} f$ if either is defined in $\mathbb{R}$.

proof (a) Apply 482E.

(b) Apply 482F, referring to 134F to confirm that condition 482F(ii) is satisfied.

(c)-(d)(i) The first thing to check is that the conditions of 482G are satisfied by $\mathbb{R}$, T, Δ, $\mathfrak{R}$, $\mathcal{C}$ and μ. **P** 482G(i) is just 481K, and 482G(ii) is trivial. 482G(iii-β) and 482G(v) are elementary, and so is 482G(iii-β); if you like, it is a special case of 412W(b-iii). As for 482G(iv), if $E \in \mathcal{C}$ is a singleton $\{x\}$, then whenever $(x, C) \in \mathcal{C}$ we can express C as a union of one or more of the sets $C \cap]-\infty, x[$, $C \cap \{x\}$ and $C \cap]x, \infty[$, and for any non-empty C' of these we have $\{(x, C')\} \in T$ and either $C' \subseteq E$ or $C' \cap E = \emptyset$. Otherwise, let $\eta > 0$ be half the length of E, and let δ be the uniform metric gauge $\{(x, A) : A \subseteq]x - \eta, x + \eta[\}$. Then if $x \in \partial E$ and $(x, C) \in T \cap \tilde{\delta}$, we can again express C as a union of one or more of the sets $C \cap]-\infty, x[$, $C \cap \{x\}$ and $C \cap]x, \infty[$, and these will witness that 482G(iv) is satisfied. **Q**

(ii) Now suppose that f is Henstock integrable. Then $\lim_{a \to -\infty, b \to \infty} \int_a^b f$ is defined and equal to $\int f$. **P** Let F be the Saks-Henstock indefinite integral of f (482B-482C). Then 482G tells us that $\int_a^b f$ is defined and equal to $F(]a, b[)$ whenever $a < b$ in $\mathbb{R}$. Now, given $\epsilon > 0$, there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(\mathbb{R} \setminus C)| \leq \epsilon$ whenever $C \in \mathcal{C}$ and $\mathbb{R} \setminus C \in \mathcal{R}$; that is, there are $a_0 \leq b_0$ such that $|\int f - \int_a^b f| \leq \epsilon$ whenever $a \leq a_0 \leq b_0 \leq b$. As ϵ is arbitrary, $\lim_{a \to -\infty, b \to \infty} \int_a^b f$ is defined and equal to $\int f$. **Q**

(iii) It follows that if f is Henstock integrable and $c \in \mathbb{R}$, $\lim_{b \to \infty} \int_c^b f$ is defined. **P** Let $\epsilon > 0$. Then there are $a_0 \leq c$, $b_0 \geq c$ such that $|\int_a^b f - \int f| \leq \epsilon$ whenever $a \leq a_0$ and $b \geq b_0$. But this means that if b, $b' \geq b_0$ then

$$|\int_c^b f - \int_c^{b'} f| = |\int_{a_0}^b f - \int_{a_0}^{b'} f| \leq 2\epsilon$$

whenever b, $b' \geq b_0$. As ϵ is arbitrary, $\lim_{b \to \infty} \int_c^b f$ is defined. **Q**

Similarly, $\lim_{a \to -\infty} \int_a^c f$ is defined.

(iv) Moreover, if f is Henstock integrable and $a < b$ in $\mathbb{R}$, then $\lim_{c \uparrow b} \int_a^c f$ is defined and equal to $\int_a^b f$. **P** Again let F be the Saks-Henstock indefinite integral of f. Let $\epsilon > 0$. Then there is a $\delta \in \Delta$ such that $\sum_{(x,C) \in \boldsymbol{t}} |F(C) - f(x)\mu C| \leq \epsilon$ whenever $\boldsymbol{t} \in T$ is δ-fine. Let $\eta > 0$ be such that $\eta |f(b)| \leq \epsilon$ and $(b, [c, b[) \in \delta$ whenever $b - \eta \leq c < b$. Then whenever $\max(a, b - \eta) \leq c < b$,

$$\int_a^b f - \int_a^c f = F([c, b[) \leq |F([c, b[) - f(b)\mu[c, b[| + |f(b)|\mu[c, b[\leq 2\epsilon.$$

As ϵ is arbitrary, $\lim_{c \uparrow b} \int_a^c f = \int_a^b f$. **Q**

Similarly, $\lim_{c \downarrow a} \int_c^b f$ is defined and equal to $\int_a^b f$.

(v) Next, if $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$ are such that $\gamma = \lim_{a \downarrow c, b \to \infty} \int_a^b f$ is defined in $\mathbb{R}$, then $\int_c^{\infty} f$ is defined and equal to γ. **P** I seek to apply 482H, with $H =]c, \infty[$. We have already seen that the conditions of 482G are satisfied, and 482H(vi)-(vii) are elementary, if we set $H_n =]c + 2^{-n}, c + 2^n[$. So we are left with 482H(viii). Given $\epsilon > 0$, let $\eta \in]0, 1]$, $b_0 \geq c + 1$ be such that $|\int_a^b f - \gamma| \leq \epsilon$ whenever $c < a \leq c + \eta$ and $b \geq b_0$. If we set

$$\begin{aligned}\delta_0 = &\{(x, A) : x < c,\ A \subseteq \left]-\infty, \tfrac{1}{2}(x + c)\right[\} \\ &\cup \{(c, A) : A \subseteq]c - \eta, c + \eta[\} \\ &\cup \{(x, A) : x > c,\ A \subseteq \left]\tfrac{1}{2}(x + c), \infty\right[\},\end{aligned}$$

$\delta_0 \in \Delta$. If $\boldsymbol{t} \in T$ is δ_0-fine and $\mathcal{R}_{c,b_0}$-filling, $W_{\boldsymbol{t}}$ is an interval including $[c, b_0]$ and $W_{\boldsymbol{t}\restriction H}$ must be an interval with infimum in $]c, c+\eta[$ and supremum at least b_0. So $|\gamma - \int f \times \chi W_{\boldsymbol{t}\restriction H}| \leq \epsilon$. As ϵ is arbitrary, condition 482H(viii) is satisfied, and $\int_c^\infty f$ is defined and equal to γ. **Q**

Similarly, $\int_\infty^c f = \lim_{a\to-\infty, b\uparrow c} \int_a^b f$ if the latter is defined.

(vi) We are now in a position to show that if f is Henstock integrable then $\int_c^\infty f = \lim_{b\to\infty} \int_c^b f$ is defined for every $c \in \mathbb{R}$. **P** By (iii) and (iv), $\lim_{a\downarrow c} \int_a^{c+1} f = \int_c^{c+1} f$ and $\lim_{b\to\infty} \int_{c+1}^b$ are both defined. Because $\int f \times \chi\{c\} = 0$, by (b),

$$\lim_{a\downarrow c, b\to\infty} \int_a^b = \int_c^{c+1} f + \lim_{b\to\infty} \int_{c+1}^b = \lim_{b\to\infty} \int_c^b f$$

is defined, and is equal to $\int_c^\infty f$, by (v). **Q**

Similarly, $\int_{-\infty}^c f = \lim_{a\to-\infty} \int_a^c f$ is defined. Putting these together, we see that if f is Henstock integrable and $a < b$ in $\mathbb{R}$,

$$\int_a^b f = \int (f \times \chi]a, \infty[) \times \chi]-\infty, b[$$

is defined. Once again, because $\int f \times \chi\{c\}$ is always defined, $\int f \times \chi C$ is defined for every interval C, and the proof of (c) is complete.

(vii) Now consider (d). If $\int_{-\infty}^\beta f$ is defined, then $g = f \times \chi]-\infty, \beta[$ is Henstock integrable. Taking any real $c < \beta$, we have

$$\lim_{b\uparrow\beta} \int_{-\infty}^b f = \lim_{b\uparrow\beta} \int_{-\infty}^b g = \int_{-\infty}^c g + \lim_{b\uparrow\beta} \int_c^b g = \int_{-\infty}^c g + \int_c^\beta g$$

(by (iii) or (iv))

$$= \int_\infty^\beta f.$$

In the other direction, suppose that $\gamma = \lim_{b\uparrow\beta} \int_{-\infty}^b f$ is defined, where $\beta \in \mathbb{R}$. Then

$$\lim_{b\uparrow\beta} \int_{\beta-1}^b f = \gamma - \int_{-\infty}^{\beta-1} f$$

is defined, and also $\lim_{a\to-\infty} \int_a^{\beta-1} f = \int_{-\infty}^{\beta-1} f$, by (iii). So $\lim_{a\to-\infty, b\uparrow b} \int_a^b f = \gamma$, and must be equal to $\int_\infty^\beta f$, by (v).

For the case $\beta = \infty$, suppose that $\gamma = \lim_{b\to\infty} \int_{-\infty}^b f$ is defined. Then

$$\lim_{a\downarrow 0, b\to\infty} \int_a^b f = \lim_{a\downarrow 0} \int_a^1 f + \lim_{b\to\infty} \int_1^b f$$
$$= \int_0^1 f + \gamma - \int_{-\infty}^1 f = \gamma - \int_{-\infty}^0 f$$

so, by (v), $\int_0^\infty f = \gamma - \int_{-\infty}^0 f$ and $\int f = \gamma$ are defined.

(viii) Similarly, if $\alpha \in [-\infty, \infty[$, $\int_\alpha^\infty f = \lim_{a\downarrow\alpha} \int_a^\infty f$ if either is defined in $\mathbb{R}$. This completes the proof.

483C Corollary The Henstock and Lebesgue integrals agree on non-negative functions, in the sense that if $f : \mathbb{R} \to [0, \infty[$ then $\int f = \int f d\mu$ if either is defined in $\mathbb{R}$.

proof If f is Lebesgue integrable, it is Henstock integrable, with the same integral, by 483Bb. If it is Henstock integrable, then it is measurable, by 483Ba, so that $\int f d\mu$ is defined in $[0, \infty]$; but

$$\int f d\mu = \sup\{\int g\, d\mu : g \leq f \text{ is a non-negative simple function}\}$$

(213B)

$$= \sup\{ \text{⨍} g : g \leq f \text{ is a non-negative simple function}\} \leq \text{⨍} f$$

(481Cb) is finite, so f is Lebesgue integrable.

483D Corollary If $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, then $\alpha \mapsto \text{⨍}_{-\infty}^{\alpha} f : [\infty, \infty] \to \mathbb{R}$ and $(\alpha, \beta) \mapsto \text{⨍}_{\alpha}^{\beta} f : [-\infty, \infty]^2 \to \mathbb{R}$ are continuous and bounded.

proof Set $F(x) = \text{⨍}_{-\infty}^{x} f$ for $x \in \mathbb{R}$. Take any $x_0 \in \mathbb{R}$ and $\epsilon > 0$. By 483B(d-i), there is an $\eta_1 > 0$ such that $|\text{⨍}_{-\infty}^{x} f - \text{⨍}_{-\infty}^{x_0} f| \leq \epsilon$ whenever $x_0 - \eta_1 \leq x \leq x_0$. By 483B(d-ii), there is an $\eta_2 > 0$ such that $|\text{⨍}_{x}^{\infty} f - \text{⨍}_{x_0}^{\infty} f| \leq \epsilon$ whenever $x_0 \leq x \leq x_0 + \eta_2$. But this means that $|F(x) - F(x_0)| \leq \epsilon$ whenever $x_0 - \eta_1 \leq x \leq x_0 + \eta_2$. As ϵ is arbitrary, F is continuous at x_0.

We know also that $\lim_{x\to\infty} F(x) = \text{⨍} f$ is defined in $\mathbb{R}$; while

$$\lim_{x\to-\infty} F(x) = \text{⨍} f - \lim_{x\to-\infty} \text{⨍}_{x}^{\infty} f = 0$$

is also defined, by 483B(d-ii), as usual. So F is continuous at $\pm\infty$.

Now writing $G(\alpha, \beta) = \text{⨍}_{\alpha}^{\beta} f$, we have $G(\alpha, \beta) = F(\beta) - F(\alpha)$ if $\alpha \leq \beta$ and zero if $\beta \leq \alpha$. So G also is continuous. F and G are bounded because $[-\infty, \infty]$ is compact.

483E Definition If $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, then its **indefinite Henstock integral** is the function $F : \mathbb{R} \to \mathbb{R}$ defined by saying that $F(x) = \text{⨍}_{-\infty}^{x} f$ for every $x \in \mathbb{R}$.

483F In the present context, the Saks-Henstock lemma can be sharpened, as follows.

Theorem Let $f : \mathbb{R} \to \mathbb{R}$ and $F : \mathbb{R} \to \mathbb{R}$ be functions. Then the following are equiveridical:

(i) f is Henstock integrable and F is its indefinite Henstock integral;

(ii)(α) F is continuous,

(β) $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x)$ is defined in $\mathbb{R}$,

(γ) for every $\epsilon > 0$ there are a gauge $\delta \in \Delta$ and a non-decreasing function $\phi : \mathbb{R} \to [0, \epsilon]$ such that $|f(x)(b-a) - F(b) + F(a)| \leq \phi(b) - \phi(a)$ whenever $a \leq x \leq b$ in $\mathbb{R}$ and $(x, [a, b]) \in \delta$.

proof (i)⇒(ii) (α)-(β) are covered by 483D. As for (γ), 482G tells us that we can identify the Saks-Henstock indefinite integral of f with $E \mapsto \text{⨍} f \times \chi E : \mathcal{E} \to \mathbb{R}$, where $\mathcal{E}$ is the algebra generated by $\mathcal{C}$. Let $\epsilon > 0$. Then there is a $\delta \in \Delta$ such that $\sum_{(x,C)\in\boldsymbol{t}} |f(x)\mu C - \text{⨍} f \times \chi C| \leq \epsilon$ for every δ-fine $\boldsymbol{t} \in T$. Set

$$\phi(a) = \sup_{\boldsymbol{t}\in T\cap\delta} \textstyle\sum_{(x,C)\in\boldsymbol{t}, C\subseteq]-\infty,a]} |f(x)\mu C - \text{⨍} f \times \chi C|,$$

so that $\phi : \mathbb{R} \to [0, \epsilon]$ is a non-decreasing function. Now suppose that $a \leq y \leq b$ and that $(y, [a, b]) \in \delta$. In this case, whenever $\boldsymbol{t} \in T \cap \delta$, $\boldsymbol{s} = \{(x, C) : (x, C) \in \boldsymbol{t},\ C \subseteq]-\infty, a]\} \cup \{(y,]a, b[)\}$ also belongs to $T \cap \delta$. Now $F(b) - F(a) = \text{⨍}_{a}^{b} f$, so

$$\begin{aligned}\phi(b) &\geq \sum_{(x,C)\in\boldsymbol{s}} |f(x)\mu C - \text{⨍} f \times \chi C| \\ &= \sum_{(x,C)\in\boldsymbol{t}, C\subseteq]-\infty,a]} |f(x)\mu C - \text{⨍} f \times \chi C| + |f(y)(b-a) - F(b) + F(a)|.\end{aligned}$$

As $\boldsymbol{t}$ is arbitrary,

$$\phi(b) \geq \phi(a) + |f(y)(b-a) - F(b) + F(a)|,$$

that is, $|f(y)(b-a) - F(b) + F(a)| \leq \phi(b) - \phi(a)$, as called for by ($\gamma$).

(ii)⇒(i) Assume (ii). Set $\gamma = \lim_{x\to\infty} F(x)$. Let $\epsilon > 0$. Let $a \le b$ be such that $|F(x)| \le \epsilon$ for every $x \le a$ and $|F(x) - \gamma| \le \epsilon$ for every $x \ge b$. Let $\delta \in \Delta$, $\phi : \mathbb{R} \to [0, \epsilon]$ be such that ϕ is non-decreasing and $|f(x)(\beta - \alpha) - F(\beta) + F(\alpha)| \le \phi(\beta) - \phi(\alpha)$ whenever $\alpha \le x \le \beta$ and $(x, [\alpha, \beta]) \in \delta$. Let $\delta' \in \Delta$ be such that $(x, \overline{A}) \in \delta$ whenever $(x, A) \in \delta'$. For $C \in \mathcal{C}$ set $\lambda C = F(\sup C) - F(\inf C)$, $\nu C = \phi(\sup C) - \phi(\inf C)$; then if $(x, C) \in \delta'$, $(x, [\inf C, \sup C]) \in \delta$, so $|f(x)\mu C - \lambda C| \le \nu C$. Note that λ and ν are both additive in the sense that $\lambda(C \cup C') = \lambda C + \lambda C'$, $\nu(C \cup C') = \nu C + \nu C'$ whenever C, C' are disjoint members of $\mathcal{C}$ such that $C \cup C' \in \mathcal{C}$ (cf. 482G(iii-α)).

Let $\boldsymbol{t} \in T$ be δ'-fine and $\mathcal{R}_{ab}$-filling. Then $W_{\boldsymbol{t}}$ is of the form $[c, d]$ where $c \le a$ and $b \le d$. So

$$\begin{aligned}|S_{\boldsymbol{t}}(f,\mu) - \gamma| &\le 2\epsilon + |S_{\boldsymbol{t}}(f,\mu) - F(d) + F(c)| = 2\epsilon + |S_{\boldsymbol{t}}(f,\mu) - \lambda[c,d]| \\ &= 2\epsilon + |\sum_{(x,C)\in\boldsymbol{t}} f(x)\mu C - \lambda C| \le 2\epsilon + \sum_{(x,C)\in\boldsymbol{t}} \nu C \\ &= 2\epsilon + \nu[c,d] \le 3\epsilon.\end{aligned}$$

As ϵ is arbitrary, f is Henstock integrable, with integral γ.

I still have to check that F is the indefinite integral of f. Set $F_1(x) = \int_{-\infty}^{x} f$ for $x \in \mathbb{R}$, and $G = F - F_1$. Then (ii) applies equally to the pair (f, F_1), because (i)⇒(ii). So, given $\epsilon > 0$, we have δ, $\delta_1 \in \Delta$ and non-decreasing functions ϕ, $\phi_1 : \mathbb{R} \to [0, \epsilon]$ such that

$|f(x)(b-a) - F(b) + F(a)| \le \phi(b) - \phi(a)$ whenever $a \le x \le b$ in $\mathbb{R}$ and $(x, [a, b]) \in \delta$,

$|f(x)(b-a) - F_1(b) + F_1(a)| \le \phi_1(b) - \phi_1(a)$ whenever $a \le x \le b$ in $\mathbb{R}$ and $(x, [a, b]) \in \delta_1$.

Putting these together,

$|G(b) - G(a)| \le \psi(b) - \psi(a)$ whenever $a \le x \le b$ in $\mathbb{R}$ and $(x, [a, b]) \in \delta \cap \delta_1$,

where $\psi = \phi + \phi_1$. But if $a \le b$ in $\mathbb{R}$, there are $a_0 \le x_0 \le a_1 \le x_1 \le \ldots \le x_{n-1} \le a_n$ such that $a = a_0$, $a_n = b$ and $(x_i, [a_i, a_{i+1}]) \in \delta$ for $i < n$ (481J), so that

$$\begin{aligned}|G(b) - G(a)| &\le \sum_{i=0}^{n-1} |G(a_{i+1}) - G(a_i)| \\ &\le \sum_{i=0}^{n-1} \psi(a_{i+1}) - \psi(a_i) = \psi(b) - \psi(a) \le 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, G is constant. As $\lim_{x\to-\infty} F(x) = \lim_{x\to-\infty} F_1(x) = 0$, $F = F_1$, as required.

483G Theorem (see Gordon 94, 9.18) Let $f : \mathbb{R} \to \mathbb{R}$ be a Henstock integrable function. Then there is a countable cover $\mathcal{K}$ of $\mathbb{R}$ by compact sets such that $f \times \chi K$ is Lebesgue integrable for every $K \in \mathcal{K}$.

proof (a) For $n \in \mathbb{N}$ set $E_n = \{x : |x| \le n, |f(x)| \le n\}$. By 483Ba, f is Lebesgue measurable, so for each $n \in \mathbb{N}$ we can find a compact set $K_n \subseteq E_n$ such that $\mu(E_n \setminus K_n) \le 2^{-n}$; f is Lebesgue integrable over K_n, and $Y = \mathbb{R} \setminus \bigcup_{n\in\mathbb{N}} K_n$ is Lebesgue negligible.

Let F be the indefinite Henstock integral of f, and take a gauge $\delta_0 \in \Delta$ and a non-decreasing function $\phi : \mathbb{R} \to [0, 1]$ such that $|f(x)(b-a) - F(b) + F(a)| \le \phi(b) - \phi(a)$ whenever $a \le x \le b$ and $(x, [a, b]) \in \delta_0$ (483F). Because $\int |f| \times \chi Y = \int_Y |f| d\mu = 0$ (483Bb), there is a $\delta_1 \in \Delta$ such that $S_{\boldsymbol{t}}(|f| \times \chi Y, \mu) \le 1$ whenever $\boldsymbol{t} \in T$ is δ_1-fine (482Ad). For $C \in \mathcal{C}$, set $\lambda C = F(\sup C) - F(\inf C)$, $\nu C = \phi(\sup C) - \phi(\inf C)$. Set

$$D_n = \{x : x \in E_n \cap Y, (x, [a, b]) \in \delta_0 \cap \delta_1 \text{ whenever } x - 2^{-n} \le a \le x \le b \le x + 2^{-n}\},$$

so that $\bigcup_{n\in\mathbb{N}} D_n = Y$; set $K_n' = \overline{D}_n$, so that K_n' is compact and $\bigcup_{n\in\mathbb{N}} K_n' \supseteq Y$.

(b) The point is that $f \times \chi K_n'$ is Lebesgue integrable for each n. **P** For $k \in \mathbb{N}$, let A_k be the set of points $x \in K_n'$ such that $\left]x, x + 2^{-k}\right] \cap K_n' = \emptyset$; then A_k is finite, because $K_n' \subseteq [-n, n]$ is bounded. Similarly, if $A_k' = \{x : x \in K_n', \left[x - 2^{-k}, x\right[\cap K_n' = \emptyset\}$, A_k' is finite. Set $B = K_n' \setminus \bigcup_{k\in\mathbb{N}}(A_k \cup A_k')$, so that $K_n' \setminus B$ is countable.

Set

$$\delta = \delta_0 \cap \delta_1 \cap \{(x, A) : x \in \mathbb{R}, A \subseteq \left]x - 2^{-n-1}, x + 2^{-n-1}\right[\},$$

so that $\delta \in \Delta$. Note that if $C \in \mathcal{C}$, $x \in B \cap \overline{C}$ and $(x, C) \in \delta$ and $\mu C > 0$, then $\operatorname{int} C$ meets K'_n (because there are points of K'_n arbitrarily close to x on both sides) so $\operatorname{int} C$ meets D_n; and if $y \in D_n \cap \operatorname{int} C$ then $(y, C) \in \delta_0 \cap \delta_1$, because $\operatorname{diam} C \leq 2^{-n}$. This means that if $\boldsymbol{t} \in T$ is δ-fine and $\boldsymbol{t} \subseteq B \times \mathcal{C}$, then there is a $\delta_0 \cap \delta_1$-fine $\boldsymbol{s} \in T$ such that $\boldsymbol{s} \subseteq D_n \times \mathcal{C}$, $W_{\boldsymbol{s}} \subseteq W_{\boldsymbol{t}}$ and whenever $(x, C) \in \boldsymbol{t}$ and C is not a singleton, there is a y such that $(y, C) \in \boldsymbol{s}$. Accordingly

$$\begin{aligned}\sum_{(x,C)\in\boldsymbol{t}} |\lambda C| &\leq \sum_{(y,C)\in\boldsymbol{s}} |\lambda C| \leq \sum_{(y,C)\in\boldsymbol{s}} |f(y)\mu C - \lambda C| + \sum_{(y,C)\in\boldsymbol{s}} |f(y)|\mu C \\ &\leq \sum_{(y,C)\in\boldsymbol{s}} \nu C + S_{\boldsymbol{s}}(|f| \times \chi Y, \mu) \leq 2.\end{aligned}$$

But this means that if $\boldsymbol{t} \in T$ is δ-fine,

$$S_{\boldsymbol{t}}(|f \times \chi B|, \mu) = \sum_{(x,C)\in\boldsymbol{t}\restriction B} |f(x)\mu C|$$

(where $\boldsymbol{t}\restriction B = \boldsymbol{t} \cap (B \times \mathcal{C})$)

$$\begin{aligned}&\leq \sum_{(x,C)\in\boldsymbol{t}\restriction B} |f(x)\mu C - \lambda C| + \sum_{(x,C)\in\boldsymbol{t}\restriction B} |\lambda C| \\ &\leq \sum_{(x,C)\in\boldsymbol{t}\restriction B} \nu C + 2 \leq 3.\end{aligned}$$

It follows that if g is a μ-simple function and $0 \leq g \leq |f \times \chi B|$,

$$\begin{aligned}\int g\,d\mu = \fint g &\leq \sup_{\boldsymbol{t}\in T \text{ is } \delta\text{-fine}} S_{\boldsymbol{t}}(g, \mu) \\ &\leq \sup_{\boldsymbol{t}\in T \text{ is } \delta\text{-fine}} S_{\boldsymbol{t}}(|f \times \chi B|, \mu) \leq 3,\end{aligned}$$

and $|f \times \chi B|$ is μ-integrable, by 213B, so $f \times \chi B$ is μ-integrable, by 122P. As $K'_n \setminus B$ is countable, therefore negligible, $f \times \chi K'_n$ is μ-integrable. **Q**

(c) So if we set $\mathcal{K} = \{K_n : n \in \mathbb{N}\} \cup \{K'_n : n \in \mathbb{N}\}$, we have a suitable family.

483H Upper and lower derivates: Definition Let $F : \mathbb{R} \to \mathbb{R}$ be any function. For $x \in \mathbb{R}$, set

$$D^*F(x) = \limsup_{y\to x} \frac{F(y)-F(x)}{y-x}, \quad D_*F(x) = \liminf_{y\to x} \frac{F(y)-F(x)}{y-x}$$

in $[-\infty, \infty]$.

483I Theorem Suppose that $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, and F is its indefinite Henstock integral. Then $F'(x)$ is defined and equal to $f(x)$ for almost every $x \in \mathbb{R}$.

proof For $n \in \mathbb{N}$, set $A_n = \{x : |x| \leq n, D^*F(x) > f(x) + 2^{-n}\}$. Then $\mu^* A_n \leq 2^{-n+1}$. **P** Let $\delta \in \Delta$ and $\phi : \mathbb{R} \to [0, 4^{-n}]$ be such that ϕ is non-decreasing and $|f(x)(b-a) - F(b) + F(a)| \leq \phi(b) - \phi(a)$ whenever $a \leq x \leq b$ and $(x, [a, b]) \in \delta$ (483F). Let $\mathcal{I}$ be the set of non-trivial closed intervals $[a, b] \subseteq \mathbb{R}$ such that, for some $x \in [a, b] \cap A_n$, $(x, [a, b]) \in \delta$ and $\frac{F(b)-F(a)}{b-a} \geq f(x) + 2^{-n}$. By Vitali's theorem (221A) we can find a countable disjoint family $\mathcal{I}_0 \subseteq \mathcal{I}$ such that $A_n \setminus \bigcup \mathcal{I}_0$ is negligible; so we have a finite family $\mathcal{I}_1 \subseteq \mathcal{I}_0$ such that $\mu^*(A_n \setminus \bigcup \mathcal{I}_1) \leq 2^{-n}$. Enumerate $\mathcal{I}_1$ as $\langle [a_i, b_i] \rangle_{i<m}$, and for each $i < m$ take $x_i \in [a_i, b_i] \cap A_n$ such that $(x_i, [a_i, b_i]) \in \delta$ and $F(b_i) - F(a_i) \geq (b_i - a_i)(f(x_i) + 2^{-n})$. Then

$$\phi(b_i) - \phi(a_i) \geq |f(x_i)(b_i - a_i) - F(b_i) + F(a_i)| \geq 2^{-n}(b_i - a_i)$$

for each $i < m$, so

$$\mu(\bigcup \mathcal{I}_1) = \sum_{i<m} b_i - a_i \leq 2^n \sum_{i<m} \phi(b_i) - \phi(a_i) \leq 2^{-n},$$

and $\mu^* A_n \leq 2^{-n+1}$. **Q**

Accordingly $\{x : D^*F(x) > f(x)\} = \bigcup_{m\in\mathbb{N}} \bigcap_{n\geq m} A_n$ is negligible. Similarly, or applying the argument to $-f$, $\{x : D_*F(x) < f(x)\}$ is negligible. So $D^*F \leq_{\text{a.e.}} f \leq_{\text{a.e.}} D_*F$. Since $D_*F \leq D^*F$ everywhere, $D^*F =_{\text{a.e.}} D_*F =_{\text{a.e.}} f$. But $F'(x) = f(x)$ whenever $D^*F(x) = D_*F(x) = f(x)$, so we have the result.

483J Theorem Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then the following are equiveridical:

(i) f is Henstock integrable;

(ii) for every $\epsilon > 0$ there are functions F_1, $F_2 : \mathbb{R} \to \mathbb{R}$, with finite limits at both $-\infty$ and ∞, such that $D^*F_1(x) \leq f(x) \leq D_*F_2(x)$ and $0 \leq F_2(x) - F_1(x) \leq \epsilon$ for every $x \in \mathbb{R}$.

proof (i)⇒(ii) Suppose that f is Henstock integrable and that $\epsilon > 0$. Set $F(x) = \int_{-\infty}^{x} f$ for $x \in \mathbb{R}$. Let $\delta \in \Delta$, $\phi : \mathbb{R} \to [0, \frac{1}{2}\epsilon]$ be such that ϕ is non-decreasing and $|f(x)(b-a) - F(b) + F(a)| \leq \phi(b) - \phi(a)$ whenever $a \leq x \leq b$ and $(x, [a, b]) \in \delta$ (483F). Set $F_1 = F - \phi$, $F_2 = F + \phi$; then $F_1(x) \leq F_2(x) \leq F_1(x) + \epsilon$ for every $x \in \mathbb{R}$, and the limits at $\pm\infty$ are defined because F and ϕ both have limits at both ends. If $x \in \mathbb{R}$, there is an $\eta > 0$ such that $(x, A) \in \delta$ whenever $A \subseteq [x - \eta, x + \eta]$. So if $x - \eta \leq a \leq x \leq b \leq x + \eta$ and $a < b$,

$$|\tfrac{F(b)-F(a)}{b-a} - f(x)| \leq \tfrac{\phi(b)-\phi(a)}{b-a},$$

and

$$\tfrac{F_1(b)-F_1(a)}{b-a} \leq f(x) \leq \tfrac{F_2(b)-F_2(a)}{b-a}.$$

In particular, this is true whenever $x - \eta \leq a < x = b$ or $x = a < b \leq x + \eta$. So $D^*F_1(x) \leq f(x) \leq D_*F_2(x)$. As x is arbitrary, we have a suitable pair F_1, F_2.

(ii)⇒(i) Suppose that (ii) is true. Take any $\epsilon > 0$. Let F_1, $F_2 : \mathbb{R} \to \mathbb{R}$ be as in the statement of (ii).

(α) We need to know that $F_2 - F_1$ is non-decreasing. **P** Set $G = F_2 - F_1$. Then

$$\begin{aligned}\liminf_{y\to x} \tfrac{G(y)-G(x)}{y-x} &= \liminf_{y\to x} \tfrac{F_2(y)-F_2(x)}{y-x} - \tfrac{F_1(y)-F_1(x)}{y-x} \\ &\geq \liminf_{y\to x} \tfrac{F_2(y)-F_2(x)}{y-x} - \limsup_{y\to x} \tfrac{F_1(y)-F_1(x)}{y-x} \\ &\text{(2A3Sf)} \\ &= D_*F_2(x) - D^*F_1(x) \geq 0.\end{aligned}$$

? If $a < b$ and $G(a) > G(b)$, set $\gamma = \frac{G(a)-G(b)}{2(b-a)}$, and choose $\langle a_n\rangle_{n\in\mathbb{N}}$, $\langle b_n\rangle_{n\in\mathbb{N}}$ inductively as follows. $a_0 = a$ and $b_0 = b$. Given that $a_n < b_n$ and $G(a_n) - G(b_n) > \gamma(b_n - a_n)$, set $c = \frac{1}{2}(a_n + b_n)$; then either $G(a_n) - G(c) > \gamma(c - a_n)$ or $G(c) - G(b_n) \geq \gamma(b_n - c)$; in the former case, take $a_{n+1} = a_n$ and $b_{n+1} = c$; in the latter, take $a_{n+1} = c$ and $b_{n+1} = b_n$. Set $x = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$. Then for each n, either $G(a_n) - G(x) > \gamma(x - a_n)$ or $G(x) - G(b_n) > \gamma(b_n - x)$. In either case, we have a y such that $0 < |y - x| \leq 2^{-n}(b - a)$ and $\frac{G(y)-G(x)}{y-x} < -\gamma$. So $D_*G(x) \leq -\gamma < 0$, which is impossible. **X**

Thus G is non-decreasing, as required. **Q**

(β) Let $a \leq b$ be such that $|F_1(x) - F_1(a)| \leq \epsilon$ whenever $x \leq a$ and $|F_1(x) - F_1(b)| \leq \epsilon$ whenever $x \geq b$. Let $h : \mathbb{R} \to]0, \infty[$ be a strictly positive integrable function such that $\int h\, d\mu \leq \epsilon$. Then $\int h \leq \epsilon$, by 483Bb, so there is a $\delta_0 \in \Delta$ such that $S_{\boldsymbol{t}}(h, \mu) \leq 2\epsilon$ for every δ_0-fine $\boldsymbol{t} \in T$ (482Ad). For $x \in \mathbb{R}$ let $\eta_x > 0$ be such that

$$\tfrac{F_1(y)-F_1(x)}{y-x} \leq f(x) + h(x), \quad \tfrac{F_2(y)-F_2(x)}{y-x} \geq f(x) - h(x)$$

whenever $0 < |y - x| \leq 2\eta_x$; set $\delta = \{(x, A) : (x, A) \in \delta_0, A \subseteq]x - \eta_x, x + \eta_x[\}$, so that $\delta \in \Delta$. Note that if $x \in \mathbb{R}$ and $x - \eta_x \leq \alpha \leq x \leq \beta \leq x + \eta_x$, then

$$F_1(\beta)-F_1(x)\le(\beta-x)(f(x)+h(x)),\quad F_1(x)-F_1(\alpha)\le(x-\alpha)(f(x)+h(x)),$$

so that $F_1(\beta)-F_1(\alpha)\le(\beta-\alpha)(f(x)+h(x))$; and similarly $F_2(\beta)-F_2(\alpha)\ge(\beta-\alpha)(f(x)-h(x))$.

For $C\in\mathcal{C}$, set

$$\lambda_1C=F_1(\sup C)-F_2(\sup C),\quad \lambda_2C=F_2(\sup C)-F_2(\sup C).$$

Then if $C\in\mathcal{C}$, $x\in\overline{C}$ and $(x,C)\in\delta$,

$$\lambda_1C\le(f(x)+h(x))\mu C,\quad \lambda_2C\ge(f(x)-h(x))\mu C.$$

Suppose that $\boldsymbol{t}\in T$ is δ-fine and $\mathcal{R}_{ab}$-filling. Then $W_{\boldsymbol{t}}=[\alpha,\beta]$ for some $\alpha\le a$, $\beta\ge b$, so that

$$\begin{aligned}S_{\boldsymbol{t}}(f,\mu)&=\sum_{(x,C)\in\boldsymbol{t}}f(x)\mu C\le\sum_{(x,C)\in\boldsymbol{t}}\lambda_2C+h(x)\mu C=\lambda_2[\alpha,\beta]+S_{\boldsymbol{t}}(h,\mu)\\&\le F_2(\beta)-F_2(\alpha)+2\epsilon\le F_1(\beta)-F_1(\alpha)+3\epsilon\le F_1(b)-F_1(a)+5\epsilon.\end{aligned}$$

Similarly,

$$\begin{aligned}S_{\boldsymbol{t}}(f,\mu)&=\sum_{(x,C)\in\boldsymbol{t}}f(x)\mu C\ge\sum_{(x,C)\in\boldsymbol{t}}\lambda_1C-h(x)\mu C\\&=\lambda_1[\alpha,\beta]-S_{\boldsymbol{t}}(h,\mu)\ge F_1(\beta)-F_1(\alpha)-2\epsilon\ge F_1(b)-F_1(a)-4\epsilon.\end{aligned}$$

But this means that if $\boldsymbol{t}$, $\boldsymbol{t}'$ are two δ-fine $\mathcal{R}_{ab}$-filling members of T, $|S_{\boldsymbol{t}}(f,\mu)-S_{\boldsymbol{t}'}(f,\mu)|\le9\epsilon$. As ϵ is arbitrary,

$$\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\Re)}S_{\boldsymbol{t}}(f,\mu)=\text{⨍}f$$

is defined.

Remark The formulation (ii) above is a version of the method of integration described by Perron 14.

483K Proposition Let $f:\mathbb{R}\to\mathbb{R}$ be a Henstock integrable function, and F its indefinite Henstock integral. Then $F[E]$ is Lebesgue negligible for every Lebesgue negligible set $E\subseteq\mathbb{R}$.

proof Let $\epsilon>0$. By 483C and 482Ad, as usual, together with 483F, there are a $\delta\in\Delta$ and a non-decreasing $\phi:\mathbb{R}\to[0,\epsilon]$ such that

$$S_{\boldsymbol{t}}(|f|\times\chi E,\mu)\le\epsilon,\quad |f(x)(b-a)-F(b)+F(a)|\le\phi(b)-\phi(a)$$

whenever $\boldsymbol{t}\in T$ is δ-fine, $a\le x\le b$ and $(x,[a,b])\in\delta$. For $n\in\mathbb{N}$ and $i\in\mathbb{Z}$, set

$$E_{ni}=\{x:x\in E\cap[2^{-n}i,2^{-n}(i+1)[\,,\,(x,A)\in\delta\text{ whenever }A\subseteq[x-2^{-n},x+2^{-n}]\}.$$

Set $J_n=\{i:i\in\mathbb{Z},\,-4^n<i\le4^n,\,E_{ni}\ne\emptyset\}$. Observe that

$$E=\textstyle\bigcup_{n\in\mathbb{N}}\bigcap_{m\ge n}\bigcup_{i\in J_m}E_{mi}.$$

For $i\in J_n$, take x_{ni}, $y_{ni}\in E_{ni}$ such that $x_{ni}\le y_{ni}$ and

$$\min(F(x_{ni}),F(y_{ni}))\le\inf F[E_{ni}]+4^{-n}\epsilon,$$

$$\max(F(x_{ni}),F(y_{ni}))\ge\sup F[E_{ni}]-4^{-n}\epsilon,$$

so that $\mu^*F[E_{ni}]\le|F(y_{ni})-F(x_{ni})|+2^{-2n+1}\epsilon$. Now, for each $i\in J_n$, $(x_{ni},[x_{ni},y_{ni}])\in\mathcal{C}\cap\delta$, while $[x_{ni},y_{ni}]\subseteq[2^{-n}i,2^{-n}(i+1)[$, so $\boldsymbol{t}=\{(x_{ni},[x_{ni},y_{ni}]):i\in J_n\}$ is a δ-fine member of T, and

$$\begin{aligned}\mu^*F[\bigcup_{i\in J_n}E_{ni}]&\le\sum_{i\in J_n}\mu^*F[E_{ni}]\le\sum_{i\in J_n}2^{-2n+1}\epsilon+|F(y_{ni})-F(x_{ni})|\\&\le4\epsilon+\sum_{i\in J_n}|f(x_{ni})(y_{ni}-x_{ni})|+\sum_{i\in J_n}\phi(y_{ni})-\phi(x_{ni})\\&\le4\epsilon+S_{\boldsymbol{t}}(|f|\times\chi E,\mu)+\epsilon\le6\epsilon.\end{aligned}$$

Since this is true for every $n\in\mathbb{N}$,

$$\begin{aligned}\mu^*F[E] &= \mu^*F[\bigcup_{n\in\mathbb{N}}\bigcap_{m\geq n}\bigcup_{i\in J_m}E_i]\\ &= \mu^*(\bigcup_{n\in\mathbb{N}}F[\bigcap_{m\geq n}\bigcup_{i\in J_m}E_i]) = \sup_{n\in\mathbb{N}}\mu^*F[\bigcap_{m\geq n}\bigcup_{i\in J_m}E_i]\\ &\leq 6\epsilon.\end{aligned} \tag{132Ae}$$

As ϵ is arbitrary, $F[E]$ is negligible, as claimed.

483L Definition If $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, I write $\|f\|_H$ for $\sup_{a\leq b} |\int_a^b f|$. It is elementary to check that this is a seminorm on the linear space of all Henstock integrable functions. (It is finite-valued by 483D.)

483M Proposition (a) If $f : \mathbb{R} \to \mathbb{R}$ is Henstock integrable, then $|\int f| \leq \|f\|_H$, and $\|f\|_H = 0$ iff $f = 0$ a.e.

(b) Write $\mathcal{HL}^1$ for the linear space of all Henstock integrable real-valued functions on $\mathbb{R}$, and HL^1 for $\{f^\bullet : f \in \mathcal{HL}^1\} \subseteq L^0(\mu)$ (§241). If we write $\|f^\bullet\|_H = \|f\|_H$ for every $f \in \mathcal{HL}^1$, then HL^1 is a normed space. The ordinary space $L^1(\mu)$ of equivalence classes of Lebesgue integrable functions is a linear subspace of HL^1, and $\|u\|_H \leq \|u\|_1$ for every $u \in L^1(\mu)$.

(c) We have a linear operator $T : HL^1 \to C_b(\mathbb{R})$ defined by saying that $T(f^\bullet)$ is the indefinite Henstock integral of f for every $f \in \mathcal{HL}^1$, and $\|T\| = 1$.

proof (a) Of course

$$|\int f| = \lim_{a\to-\infty,b\to\infty} |\int_a^b f| \leq \|f\|_H$$

(using 438Bd). Set $F(x) = \int_{-\infty}^x f$, so that $F(b) - F(a) = \int_a^b f$ whenever $a \leq b$. If $f = 0$ a.e., then $F(x) = \int_{-\infty}^x f d\mu = 0$ for every x, by 483B, so $\|f\|_H = 0$. If $\|f\|_H = 0$, then F is constant, so $f = F' = 0$ a.e., by 483I.

(b) This follows immediately from (a). (Compare the definitions of the norms $\|\ \|_p$ on L^p, for $1 \leq p \leq \infty$, in §§242-244.) By 483Bb, $L^1(\mu) \subseteq HL^1$, and

$$\|u\|_H \leq \|u^+\|_H + \|u^-\|_H = \|u^+\|_1 + \|u^-\|_1 = \|u\|_1$$

for every $u \in L^1(\mu)$, writing u^+ and u^- for the positive and negative parts of u, as in Chapter 24.

(c) If $f, g \in \mathcal{HL}^1$ and $f^\bullet = g^\bullet$, then f and g have the same indefinite integral, by 483Bb or otherwise; so T is defined as a function from HL^1 to $\mathbb{R}^{\mathbb{R}}$. By 483F, Tu is continuous and bounded for every $u \in HL^1$, and by 481C T is linear. If $f \in \mathcal{HL}^1$ and $Tf^\bullet = F$, then $\|f\|_H = \sup_{x,y\in\mathbb{R}} |F(y) - F(x)|$; since $\lim_{x\to-\infty} F(x) = 0$, $\|f\|_H \geq \|F\|_\infty$; as f is arbitrary, $\|T\| \leq 1$. On the other hand, for any non-negative Lebesgue integrable function f, $\|Tf^\bullet\|_\infty = \|f\|_1 = \|f\|_H$, so $\|T\| = 1$.

483N Proposition Suppose that $\langle I_k\rangle_{k\in K}$ is a disjoint family of open intervals in $\mathbb{R}$ with union G, and that $f : \mathbb{R} \to \mathbb{R}$ is a function such that $f_k = f \times \chi I_k$ is Henstock integrable for every $k \in K$. If $\sum_{k\in K} \|f_k\|_H < \infty$, then $f \times \chi G$ is Henstock integrable, and $\int f \times \chi G = \sum_{k\in K} \int f_k$.

proof I seek to apply 482H. We have already seen, in the proof of 483B, that the conditions of 482G are satisfied by $\mathbb{R}$, T, Δ, $\mathfrak{R}$, $\mathcal{C}$, $\mathfrak{T}$ and μ. Of course $G = \bigcup_{k\in K} I_k$ is the union of a sequence of open sets over which f is Henstock integrable. So we have only to check 482H(viii).

Set

$$\delta_0 = \bigcup_{k\in K}\{(x, A) : x \in I_k,\ A \subseteq I_k\} \cup \{(x, A) : x \in \mathbb{R} \setminus G,\ A \subseteq \mathbb{R}\},$$

so that $\delta_0 \in \Delta$. For each $k \in K$ let F_k be the Saks-Henstock indefinite integral of f_k. Let $\epsilon > 0$. Then there is a finite set $K_0 \subseteq K$ such that $\sum_{k\in K\setminus K_0} \|f_k\|_H \leq \epsilon$. Next, there must be $\delta_1 \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$ such that

$$\sum_{k\in K_0} |\fint f_k - S_{\boldsymbol{t}}(f_k,\mu)| \le \epsilon$$

for every δ-fine $\mathcal{R}$-filling $\boldsymbol{t}\in T$, and $\delta_2\in\Delta$ such that $\sum_{k\in K_0}|S_{\boldsymbol{t}}(f_k,\mu)-F_k(W_{\boldsymbol{t}})|\le\epsilon$ for every δ_2-fine $\boldsymbol{t}\in T$.

Now let $\boldsymbol{t}\in T$ be $(\delta_0\cap\delta_1\cap\delta_2)$-fine and $\mathcal{R}$-filling. For each $k\in K$ set $\boldsymbol{t}_k=\boldsymbol{t}\restriction I_k$, so that $\boldsymbol{t}\restriction G=\bigcup_{k\in K}\boldsymbol{t}_k$. Because $W_{\boldsymbol{t}}$ is an interval, each $W_{\boldsymbol{t}_k}$ must be an interval, and it is a subinterval of I_k because $\boldsymbol{t}$ is δ_0-fine. So (using 482G)

$$|F_k(W_{\boldsymbol{t}_k})| = |\fint f_k\times\chi W_{\boldsymbol{t}_k}| \le \|f_k\|_H.$$

Also

$$\begin{aligned}\sum_{k\in K_0}|\fint f_k - \fint f\times\chi W_{\boldsymbol{t}_k}| &\le \sum_{k\in K_0}|\fint f_k - S_{\boldsymbol{t}}(f_k,\mu)| + \sum_{k\in K_0}|S_{\boldsymbol{t}}(f_k,\mu)-F_k(W_{\boldsymbol{t}_k}|\\ &\le 2\epsilon.\end{aligned}$$

On the other hand,

$$\sum_{k\in K\setminus K_0}|\fint f_k - \fint f\times\chi W_{\boldsymbol{t}_k}| \le 2\sum_{k\in K\setminus K_0}\|f_k\|_H \le 2\epsilon.$$

Putting these together,

$$|\fint f\times\chi W_{\boldsymbol{t}\restriction G} - \sum_{k\in K}\fint f_k| = |\sum_{k\in K}\fint f\times\chi W_{\boldsymbol{t}_k} - \sum_{k\in K}\fint f_k|$$

(because $\boldsymbol{t}$ is finite, so all but finitely many terms in the sum $\sum_{k\in K} f\times\chi W_{\boldsymbol{t}_k}$ are zero)

$$\le \sum_{k\in K}|\fint f\times\chi W_{\boldsymbol{t}_k} - \fint f_k| \le 4\epsilon.$$

As ϵ is arbitrary, condition 482H(viii) is satisfied, with

$$\lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\Re)}\fint f\times\chi W_{\boldsymbol{t}\restriction G} = \sum_{k\in K}\fint f_k,$$

and 482H gives the result we seek.

483O Definitions (a) For any real-valued function F, write $\omega(F)$ for $\sup_{x,y\in\operatorname{dom}F}|F(x)-F(y)|$, the **oscillation** of F. (Interpret $\sup\emptyset$ as 0, so that $\omega(\emptyset)=0$.)

(b) Let $F:\mathbb{R}\to\mathbb{R}$ be a function. For $A\subseteq\mathbb{R}$, we say that F is $\mathbf{AC}_*$ **on** A if for every $\epsilon>0$ there is an $\eta>0$ such that $\sum_{I\in\mathcal{I}}\omega(F\restriction\overline{I})\le\epsilon$ whenever $\mathcal{I}$ is a disjoint family of open intervals with endpoints in A and $\sum_{I\in\mathcal{I}}\mu I\le\eta$. Note that whether F is AC_* on A is *not* determined by $F\restriction A$, since it depends on the behaviour of F on intervals with endpoints in A.

(c) Finally, F is $\mathbf{ACG}_*$ if it is continuous and there is a countable family $\mathcal{A}$ of sets, covering $\mathbb{R}$, such that F is AC_* on every member of $\mathcal{A}$.

483P Elementary results (a)(i) If $F,G:\mathbb{R}\to\mathbb{R}$ are functions and $A\subseteq B\subseteq\mathbb{R}$, then $\omega(F+G\restriction A)\le\omega(F\restriction A)+\omega(G\restriction A)$ and $\omega(F\restriction A)\le\omega(F\restriction B)$.

(ii) If F is the indefinite Henstock integral of $f:\mathbb{R}\to\mathbb{R}$ and $C\subseteq\mathbb{R}$ is an interval, then $\|f\times\chi C\|_H=\omega(F\restriction C)$.

(iii) If $F:\mathbb{R}\to\mathbb{R}$ is continuous, then $(a,b)\mapsto\omega(F\restriction[a,b]):\mathbb{R}^2\to\mathbb{R}$ is continuous, and $\omega(F\restriction\overline{A})=\omega(F\restriction A)$ for every set $A\subseteq\mathbb{R}$.

(b)(i) If $F:\mathbb{R}\to\mathbb{R}$ is AC_* on $A\subseteq\mathbb{R}$, it is AC_* on every subset of A.

(ii) If $F:\mathbb{R}\to\mathbb{R}$ is continuous and is AC_* on $A\subseteq\mathbb{R}$, it is AC_* on $\overline{A}$. **P** Let $\epsilon>0$. Let $\eta>0$ be such that $\sum_{I\in\mathcal{I}}\omega(F\restriction\overline{I})\le\epsilon$ whenever $\mathcal{I}$ is a disjoint family of open intervals with endpoints in A and $\sum_{I\in\mathcal{I}}\mu I\le\eta$. Let $\mathcal{I}$ be a disjoint family of open intervals with endpoints in $\overline{A}$ and $\sum_{I\in\mathcal{I}}\mu I\le\frac12\eta$. Let

$\mathcal{I}_0 \subseteq \mathcal{I}$ be a non-empty finite set; then we can enumerate $\mathcal{I}_0$ as $\langle]a_i, b_i[\rangle_{i \le n}$ where $a_0, b_0, \ldots, a_n, b_n \in \overline{A}$ and $a_0 \le b_0 \le a_1 \le b_1 \le \ldots \le a_n \le b_n$. Because $(a,b) \mapsto \omega(F\restriction[a,b])$ is continuous, as noted in (a-ii) above, we can find $a_0', \ldots, b_n' \in A$ such that $a_0' \le b_0' \le a_1' \le \ldots \le a_n' \le b_n'$, $\sum_{i=0}^n b_i' - a_i' \le \eta$, and $\sum_{i=0}^n |\omega(F\restriction[a_i', b_i']) - \omega(F\restriction[a_i, b_i])| \le \epsilon$; so that

$$\sum_{I \in \mathcal{I}_0} \omega(F\restriction\overline{I}) \le \sum_{i=0}^n \omega(F\restriction[a_i', b_i']) \le 2\epsilon.$$

As $\mathcal{I}_0$ is arbitrary, $\sum_{I \in \mathcal{I}} \omega(F\restriction\overline{I}) \le 2\epsilon$; as ϵ is arbitrary, F is AC_* on $\overline{A}$. **Q**

483Q Lemma Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function, and $K \subseteq \mathbb{R}$ a non-empty compact set such that F is AC_* on K. Write $\mathcal{I}$ for the family of non-empty bounded open intervals, disjoint from K, with endpoints in K.

(a) $\sum_{I \in \mathcal{I}} \omega(F\restriction I)$ is finite.

(b) Write a^* for $\inf K = \min K$. Then there is a Lebesgue integrable function $g : \mathbb{R} \to \mathbb{R}$ such that

$$F(x) - F(a^*) = \int_{a^*}^x g + \sum_{J \in \mathcal{I}, J \subseteq [a^*, x]} F(\sup J) - F(\inf J)$$

for every $x \in K$.

proof (a) Let $\eta > 0$ be such that $\sum_{I \in \mathcal{J}} \omega(F\restriction\overline{I}) \le 1$ whenever $\mathcal{J}$ is a disjoint family of open intervals with endpoints in K and $\sum_{I \in \mathcal{J}} \mu I \le \eta$. Let m_0, $m_1 \in \mathbb{Z}$ be such that $K \subseteq [m_0\eta, m_1\eta]$. For integers m between m_0 and m_1, let $\mathcal{I}_m$ be the set of intervals in $\mathcal{I}$ included in $]j\eta, (j+1)\eta[$. Then $\sum_{I \in \mathcal{I}_m} \mu I \le \eta$, so $\sum_{I \in \mathcal{I}_m} \omega(F\restriction I) \le 1$ for each j. Also every member of $\mathcal{J} = \mathcal{I} \setminus \bigcup_{m_0 \le m < m_1} \mathcal{I}_m$ contains $m\eta$ for some m between m_0 and m_1, so $\#(\mathcal{J}) \le m_1 - m_0$. Accordingly

$$\begin{aligned}\sum_{I \in \mathcal{I}} \omega(F\restriction\overline{I}) &\le \sum_{I \in \mathcal{J}} \omega(F\restriction\overline{I}) + \sum_{m=m_0}^{m_1-1} \sum_{I \in \mathcal{I}_m} \omega(F\restriction\overline{I}) \\ &\le \sum_{I \in \mathcal{J}} \omega(F\restriction\overline{I}) + m_1 - m_0 < \infty\end{aligned}$$

because F is continuous, therefore bounded on every bounded interval.

(b)(i) Set $b^* = \sup K = \max K$. Define $G : [a^*, b^*] \to \mathbb{R}$ by setting

$$\begin{aligned}G(x) &= F(x) \text{ if } x \in K, \\ &= \tfrac{F(b)(x-a)+F(a)(b-x)}{b-a} \text{ if } x \in]a,b[\in \mathcal{I}.\end{aligned}$$

Then G is absolutely continuous. **P** G is continuous because F is. Let $\epsilon > 0$. Let $\eta_1 > 0$ be such that $\sum_{I \in \mathcal{J}} \omega(F\restriction\overline{I}) \le \epsilon$ whenever $\mathcal{J}$ is a disjoint family of open intervals with endpoints in K and $\sum_{I \in \mathcal{J}} \mu I \le \eta_1$. Let $\mathcal{I}_0 \subseteq \mathcal{I}$ be a finite set such that $\sum_{I \in \mathcal{I} \setminus \mathcal{I}_0} \omega(F\restriction\overline{I}) \le \epsilon$, and take $M > 0$ such that $|F(b) - F(a)| \le M(b-a)$ whenever $]a,b[\in \mathcal{I}_0$; set $\eta = \min(\eta_1, \frac{\epsilon}{M}) > 0$.

Let $\mathcal{J}^*$ be the set of non-empty open subintervals J of $[a^*, b^*]$ such that either $J \cap K = \emptyset$ or both endpoints of J belong to K. Let $\mathcal{J} \subseteq \mathcal{J}^*$ be a disjoint family such that $\sum_{I \in \mathcal{J}} \mu I \le \eta$. Set $\mathcal{J}' = \{J : J \in \mathcal{J},\ J \cap K = \emptyset\}$. Then

$$\begin{aligned}\sum_{J \in \mathcal{J} \setminus \mathcal{J}'} |G(\sup J) - G(\inf J)| &= \sum_{J \in \mathcal{J} \setminus \mathcal{J}'} |F(\sup J) - F(\inf J)| \\ &\le \sum_{J \in \mathcal{J} \setminus \mathcal{J}'} \omega(F\restriction\overline{J}) \le \epsilon.\end{aligned}$$

On the other hand,

$$\sum_{J\in\mathcal{J}'} |G(\sup J) - G(\inf J)| = \sum_{I\in\mathcal{I}_0} \sum_{J\in\mathcal{J},J\subseteq I} |G(\sup J) - G(\inf J)| + \sum_{I\in\mathcal{I}\setminus\mathcal{I}_0} \sum_{J\in\mathcal{J},J\subseteq I} |G(\sup J) - G(\inf J)|$$
$$\leq M \sum_{I\in\mathcal{I}_0} \sum_{J\in\mathcal{J},J\subseteq I} \mu J + \sum_{I\in\mathcal{I}\setminus\mathcal{I}_0} |F(\sup I) - F(\inf I)|$$

(because G is monotonic on $\overline{I}$ for each $I \in \mathcal{I}$)

$$\leq M\eta + \epsilon \leq 2\epsilon,$$

so $\sum_{J\in\mathcal{J}} |G(\sup J) - G(\inf J)| \leq 3\epsilon$.

Generally, if J is any non-empty open subinterval of $[a^*, b^*]$, we can split it into at most three intervals belonging to $\mathcal{J}^*$. So if $\mathcal{J}$ is any disjoint family of non-empty open subintervals of $[a^*, b^*]$ with $\sum_{J\in\mathcal{J}} \mu J \leq \eta$, we can find a family $\tilde{\mathcal{J}} \subseteq \mathcal{J}^*$ with $\sum_{J\in\tilde{\mathcal{J}}} \mu J = \sum_{J\in\mathcal{J}} \mu J$ and $\sum_{J\in\tilde{\mathcal{J}}} |G(\sup J) - G(\inf J)| \geq \sum_{J\in\mathcal{J}} |G(\sup J) - G(\inf J)|$. But this means that $\sum_{J\in\mathcal{J}} |G(\sup J) - G(\inf J)| \leq 3\epsilon$. As ϵ is arbitrary, G is absolutely continuous. **Q**

(ii) By 225E, G' is Lebesgue integrable and $G(x) = G(a^*) + \int_{a^*}^x G'$ for every $x \in [a^*, b^*]$. Set $g(x) = G'(x)$ when $x \in K$ and $G'(x)$ is defined, 0 for other $x \in \mathbb{R}$, so that $g : \mathbb{R} \to \mathbb{R}$ is Lebesgue integrable. Now take any $x \in K$. Then

$$F(x) = G(x) = G(a^*) + \int_{a^*}^x G'$$
$$= F(a^*) + \int_{a^*}^x g + \int_{[a^*,x]\setminus K} G'$$
$$= F(a^*) + \int_{a^*}^x g + \sum_{I\in\mathcal{I},I\subseteq[a^*,x]} \int_I G'$$

(because $\mathcal{I}$ is a disjoint countable family of measurable sets, and $\bigcup_{I\in\mathcal{I},I\subseteq[a^*,x]} I = [a^*, x] \setminus K$)

$$= F(a^*) + \int_{a^*}^x g + \sum_{I\in\mathcal{I},I\subseteq[a^*,x]} G(\sup I) - G(\inf I)$$

(note that this sum is absolutely summable)

$$= F(a^*) + \int_{a^*}^x g + \sum_{I\in\mathcal{I},I\subseteq[a^*,x]} F(\sup I) - F(\inf I)$$

as required.

483R Theorem (see GORDON 94, 11.14) Let $F : \mathbb{R} \to \mathbb{R}$ be a function. Then F is an indefinite Henstock integral iff it is ACG_*, $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x)$ is defined in $\mathbb{R}$.

proof (a) Suppose that F is the indefinite Henstock integral of $f : \mathbb{R} \to \mathbb{R}$.

(i) By 483F, F is continuous, with limit zero at $-\infty$ and finite at ∞. So I have just to show that there is a sequence of sets, covering $\mathbb{R}$, on each of which F is AC_*. Recall that there is a sequence $\langle K_m\rangle_{m\in\mathbb{N}}$ of compact sets, covering $\mathbb{R}$, such that $f \times K_m$ is Lebesgue integrable for every $m \in \mathbb{N}$ (483G). By the arguments of (i)⇒(ii) in the proof of 483J, there is a function $F_2 \geq F$ such that $D_*F_2 \geq f$ and $F_2 - F$ is non-decreasing and takes values between 0 and 1. For $n \in \mathbb{N}$, $j \in \mathbb{Z}$ set

$$I_{nj} = [2^{-n}j, 2^{-n}(j+1)],$$

$$B_{nj} = \{x : x \in I_{nj}, \tfrac{F_2(y)-F_2(x)}{y-x} \geq -n \text{ whenever } y \in I_{nj} \setminus \{x\}\}.$$

Observe that $\bigcup_{n\in\mathbb{N},j\in\mathbb{Z}} B_{nj} = \mathbb{R}$, so that $\{B_{nj}\cap K_m : m, n\in\mathbb{N}, j\in\mathbb{Z}\}$ is a countable cover of $\mathbb{R}$. It will therefore be enough to show that F is AC_* on every $B_{nj}\cap K_m$.

(ii) Fix $m, n\in\mathbb{N}$ and $j\in\mathbb{Z}$, and set $A = B_{nj}\cap K_m$. If $A=\emptyset$, then of course F is AC_* on A; suppose that A is not empty. Set $G(x) = F_2(x) + nx$ for $x\in I_{nj}$, so that $F = G - (F_2 - F) - H$ on I_{nj}, where $H(x) = nx$. Whenever a, $b\in B_{nj}$ and $a\le x\le b$, then $G(a)\le G(x)\le G(b)$, because $x\in I_{nj}$. So if $a_0, b_0, a_1, b_1, \dots, a_k, b_k\in A$ and $a_0\le b_0\le a_1\le b_1\le\ldots\le a_k\le b_k$,

$$\textstyle\sum_{i=0}^k \omega(G\restriction[a_i,b_i]) \le \sum_{i=0}^k G(b_i) - G(a_i) = G(b_k) - G(a_0) \le \omega(G\restriction I_{nj}),$$

and

$$\begin{aligned}\sum_{i=0}^k \omega(F\restriction[a_i,b_i]) &\le \sum_{i=0}^k \omega(G\restriction[a_i,b_i]) + \sum_{i=0}^k \omega(F-F_2\restriction[a_i,b_i]) + \sum_{i=0}^k \omega(H\restriction[a_i,b_i])\\ &\le \omega(G\restriction I_{nj}) + \omega(F-F_2\restriction I_{nj}) + \omega(H\restriction I_{nj})\end{aligned}$$

(because $F-F_2$ and H are monotonic)

$$\begin{aligned}&\le \omega(F\restriction I_{nj}) + 2\omega(F-F_2\restriction I_{nj}) + 2\omega(H\restriction I_{nj})\\ &\le \omega(F\restriction I_{nj}) + 2(1+n\mu I_{nj}) = M\end{aligned}$$

say, which is finite, because F is bounded.

(iii) By 2A2I (or 4A2Rj), the open set $\mathbb{R}\setminus\overline{A}$ is expressible as a countable union of disjoint non-empty open intervals. Two of these are unbounded; let $\mathcal{I}$ be the set consisting of the rest, so that $\overline{A}\cup\bigcup\mathcal{I} = [a^*, b^*]$ is a closed interval included in I_{nj}. If I, I' are distinct members of $\mathcal{I}$ and $\inf I\le\inf I'$, then $\sup I\le\inf I'$, because $I\cap I' = \emptyset$, and there must be a point of $\overline{A}$ in the interval $[\sup I, \inf I']$; so in fact there must be a point of A in this interval. It follows that $\sum_{I\in\mathcal{I}}\omega(F\restriction I)\le M$. **P** If $\mathcal{I}_0\subseteq\mathcal{I}$ is finite and non-empty, we can enumerate it as $\langle I_i\rangle_{i\le k}$ where $\sup I_i\le\inf I_{i'}$ whenever $i<i'\le k$. We can find $a_0,\dots,a_{k+1}\in A$ such that $a_0\le\inf I_0$, $\sup I_i\le a_{i+1}\le\inf I_{i+1}$ for every $i<k$, and $\sup I_k\le a_{k+1}$; so that

$$\textstyle\sum_{I\in\mathcal{I}_0}\omega(F\restriction I) = \sum_{i=0}^k \omega(F\restriction I_i) \le \sum_{i=0}^k \omega(F\restriction[a_i,a_{i+1}]) \le M.$$

As $\mathcal{I}_0$ is arbitrary, this gives the result. **Q**

(iv) Whenever $a^*\le x\le y\le b^*$,

$$|F(y)-F(x)| \le \textstyle\int_{\overline{A}\cap[x,y]} |f|d\mu + \sum_{I\in\mathcal{I}}\omega(F\restriction I\cap\,]x,y[).$$

P For each $I\in\mathcal{I}$,

$$\|f\times\chi(I\cap\,]x,y[)\|_H = \omega(F\restriction I\cap\,]x,y[) \le \omega(F\restriction I)$$

(483P(a-i)). So

$$\textstyle\sum_{I\in\mathcal{I}}\|f\times\chi(I\cap\,]x,y[)\|_H \le \sum_{I\in\mathcal{I}}\omega(F\restriction I)$$

is finite. Writing $H = \,]x,y[\,\cap\bigcup\mathcal{I}$, $\int f\times\chi H$ is defined and equal to $\sum_{I\in\mathcal{I}}\int f\times\chi(I\cap\,]x,y[)$, by 483N. On the other hand,

$$]x,y[\setminus H\subseteq\overline{A}\subseteq K_m,$$

so $f\times\chi(]x,y[\setminus H)$ is Lebesgue integrable. Accordingly

$$\begin{aligned}|F(y)-F(x)| &= |\textstyle\int f\times\chi\,]x,y[\,|\\ &\le |\textstyle\int f\times\chi H| + |\int f\times\chi(]x,y[\setminus H)d\mu|\\ &\le \sum_{I\in\mathcal{I}} |\textstyle\int f\times\chi(I\cap\,]x,y[)| + \int_{\overline{A}\cap[x,y]}|f|d\mu\\ &\le \sum_{I\in\mathcal{I}}\omega(F\restriction I\cap\,]x,y[) + \int_{\overline{A}\cap[x,y]}|f|d\mu,\end{aligned}$$

as claimed. **Q**

(v) So if $a^* \le a \le b \le b^*$,

$$\omega(F\restriction[a,b]) \le \sum_{I\in\mathcal{I}} \omega(F\restriction I\cap[a,b]) + \int_{\overline{A}\cap[a,b]} |f|d\mu.$$

P We have only to observe that if $a \le x \le y \le b$, then $\omega(F\restriction I\cap]x,y[) \le \omega(F\restriction I\cap[a,b])$ for every $I\in\mathcal{I}$, and $\int_{\overline{A}\cap]x,y[}|f|d\mu \le \int_{\overline{A}\cap[a,b]}|f|d\mu$. **Q**

(vi) Now let $\epsilon>0$. Setting $\tilde{F}(x) = \int_{a^*}^x f\times\chi\overline{A}d\mu$ for $x\in[a^*,b^*]$, $\tilde{F}$ is absolutely continuous (225E), and there is an $\eta_0>0$ such that $\sum_{i=0}^k |\tilde{F}(b_i)-\tilde{F}(a_i)|\le\epsilon$ whenever $a^*\le a_0\le b_0\le a_1\le b_1\le\ldots\le a_k\le b_k\le b^*$ and $\sum_{i=0}^k b_i-a_i\le\eta_0$. Take $\mathcal{I}_0\subseteq\mathcal{I}$ to be a finite set such that $\sum_{I\in\mathcal{I}\setminus\mathcal{I}_0}\omega(F\restriction I)\le\epsilon$, and let $\eta>0$ be such that $\eta\le\eta_0$ and $\eta<\operatorname{diam} I$ for every $I\in\mathcal{I}_0$.

Suppose that $a_0,b_0,\ldots,a_k,b_k\in A$ are such that $a_0\le b_0\le\ldots\le a_k\le b_k$ and $\sum_{i=0}^k b_i-a_i\le\eta$. Then no member of $\mathcal{I}_0$ can be included in any interval $[a_i,b_i]$, and therefore, because no a_i or b_i can belong to $\bigcup\mathcal{I}$, no member of $\mathcal{I}_0$ meets any $[a_i,b_i]$. Also, of course, $a^*\le a_0$ and $b_k\le b^*$. We therefore have

$$\begin{aligned}\sum_{i=0}^k \omega(F\restriction[a_i,b_i]) &\le \sum_{i=0}^k\sum_{I\in\mathcal{I}}\omega(F\restriction I\cap[a_i,b_i]) + \int_{\overline{A}\cap[a_i,b_i]}|f|d\mu\\ &= \sum_{i=0}^k\sum_{I\in\mathcal{I}\setminus\mathcal{I}_0}\omega(F\restriction I\cap[a_i,b_i]) + \sum_{i=0}^k \tilde{F}(b_i)-\tilde{F}(a_i)\\ &\le \sum_{i=0}^k \sum_{I\in\mathcal{I}\setminus\mathcal{I}_0, I\subseteq[a_i,b_i]}\omega(F\restriction I)+\epsilon\end{aligned}$$

(because if $I\in\mathcal{I}$ meets $[a_i,b_i]$, it is included in it)

$$\le \sum_{I\in\mathcal{I}\setminus\mathcal{I}_0}\omega(F\restriction I)+\epsilon\le 2\epsilon.$$

As ϵ is arbitrary, F is AC_* on A. This completes the proof that F is ACG_* and therefore satisfies the conditions (α) and (β).

(b) Now suppose that F satisfies the conditions. Set $F(-\infty)=0$ and $F(\infty)=\lim_{x\to\infty}F(x)$, so that $F:[-\infty,\infty]\to\mathbb{R}$ is continuous. For $x\in\mathbb{R}$, set $f(x)=F'(x)$ if this is defined, 0 otherwise. Let $\mathcal{J}$ be the family of all non-empty intervals $C\subseteq\mathbb{R}$ such that $\int f\times\chi C$ is defined and equal to $F(\sup C)-F(\inf C)$, and let $\mathcal{I}$ be the set of non-empty open intervals I such that every non-empty subinterval of I belongs to $\mathcal{J}$. I seek to show that $\mathbb{R}$ belongs to $\mathcal{I}$.

(i) Of course singleton intervals belong to $\mathcal{J}$. If $C_1,C_2\in\mathcal{J}$ are disjoint and $C=C_1\cup C_2$ is an interval, then

$$\begin{aligned}\int f\times\chi C &= \int f\times\chi C_1 + \int f\times\chi C_2\\ &= F(\sup C_1)-F(\inf C_1)+F(\sup C_2)-F(\inf C_2)\\ &= F(\sup C)-F(\inf C)\end{aligned}$$

and $C\in\mathcal{J}$. If $I_1,I_2\in\mathcal{I}$ and $I_1\cap I_2$ is non-empty, then $I_1\cup I_2$ is an interval; also any subinterval C of $I_1\cup I_2$ is either included in one of the I_j or is expressible as a disjoint union $C_1\cup C_2$ where C_j is a subinterval of I_j for each j; so $C\in\mathcal{J}$ and $I\in\mathcal{I}$. If $I_1,I_2\in\mathcal{I}$ and $\sup I_1=\inf I_2$, then $I=I_1\cup I_2\cup\{\sup I_1\}\in\mathcal{I}$, because any subinterval of I is expressible as the disjoint union of at most three intervals in $\mathcal{J}$.

(ii) If $\mathcal{I}_0\subseteq\mathcal{I}$ is non-empty and upwards-directed, then $\bigcup\mathcal{I}_0\in\mathcal{I}$. **P** This is a consequence of 483Bd. If we take a non-empty open subinterval J of $\bigcup\mathcal{I}_0$ and express it as $]\alpha,\beta[$, where $-\infty\le\alpha<\beta\le\infty$, and take any $x\in J$, then for any $b\in[x,\beta[$ there is some member I of $\mathcal{I}$ including $[x,b]$, so $\int f\times\chi\,]x,b[= F(b)-F(x)$; now $\lim_{b\uparrow\beta}F(b)=F(\beta)$, so 483Bd tells us that $\int f\times\chi\,]x,b[$ is defined and equal to $F(b)-F(x)$. Similarly,

using 483B(d-β), $\int f\times\chi\,]a,x[= F(x)-F(a)$, so $\int f\times\chi\,]a,b[= F(b)-F(a)$. Thus $J\in\mathcal{J}$. I wrote this out for open intervals, for convenience; but any non-empty subinterval of $\bigcup\mathcal{I}_0$ is either a singleton or expressible as an open interval with at most two points added, so belongs to $\mathcal{J}$. Accordingly $\bigcup\mathcal{I}_0\in\mathcal{I}$. **Q**

(iii) It follows that every member of $\mathcal{I}$ is included in a maximal member of $\mathcal{I}$. Let $\mathcal{I}^*$ be the set of maximal members of $\mathcal{I}$. By (i), these are all disjoint, so no endpoint of any member of $\mathcal{I}^*$ can belong to $\bigcup\mathcal{I}^*$.

? Suppose, if possible, that $\mathbb{R}\notin\mathcal{I}$. Then $\bigcup\mathcal{I}=\bigcup\mathcal{I}^*$ cannot be $\mathbb{R}$, and $V=\mathbb{R}\setminus\bigcup\mathcal{I}$ is a non-empty closed set. By (i), no two distinct members of $\mathcal{I}^*$ can share a boundary point, so V has no isolated points.

We are supposing that F is ACG_*, so there is a countable family $\mathcal{A}$ of sets, covering $\mathbb{R}$, such that F is AC_* on A for every $A\in\mathcal{A}$. By Baire's theorem (3A3G or 4A2Ma), applied to the locally compact Polish space V, $V\setminus\overline{A}$ cannot be dense in V for every $A\in\mathcal{A}$, so there are an $A\in\mathcal{A}$ and a bounded open interval $\tilde{J}$ such that $\emptyset\neq V\cap\tilde{J}\subseteq\overline{A}$. Set $K=\overline{\tilde{J}\cap A}$; by 483Pc, F is AC_* on K, and $V\cap\tilde{J}\subseteq K$. Because V has no isolated points, $V\cap\tilde{J}$ is infinite, so, setting $a^*=\min K$ and $b^*=\max K$, $V\cap\,]a^*,b^*[$ is non-empty.

By 483Q, there is a Lebesgue integrable function $g:\mathbb{R}\to\mathbb{R}$ such that

$$F(x)=F(a^*)+\int_{a^*}^x g+\sum_{I\in\mathcal{I}_0,I\subseteq[a^*,x]}F(\sup I)-F(\inf I)$$

for every $x\in K$, where $\mathcal{I}_0$ is the family of non-empty bounded open intervals, disjoint from K, with endpoints in K, and $\sum_{I\in\mathcal{I}_0}\omega(F\restriction I)$ is finite. Since every member of $\mathcal{I}_0$ is disjoint from V, it is included in some member of $\mathcal{I}^*$ and belongs to $\mathcal{J}$, so $F(\sup I)-F(\inf I)=\int f\times\chi I$ for every $I\in\mathcal{I}_0$. If $I\in\mathcal{I}_0$, then

$$\begin{aligned}\|f\times\chi I\|_H&=\sup_{C\in\mathcal{C}}|\int f\times\chi I\times\chi C|=\sup_{C\in\mathcal{C},C\subseteq I}|\int f\times\chi C|\\&=\sup_{C\in\mathcal{C},C\subseteq I}F(\sup C)-F(\inf C)=\omega(F\restriction I)\end{aligned}$$

because every non-empty subinterval of I belongs to $\mathcal{J}$. So $\sum_{I\in\mathcal{I}_0}\|f\times\chi I\|_H$ is finite, and $f\times\chi H$ is Henstock integrable, where $H=\bigcup\mathcal{I}_0$, by 483N. Moreover, if $x\in K$, then

$$\begin{aligned}\int_{a^*}^x f\times\chi H&=\int f\times\chi(\bigcup\{I:I\in\mathcal{I}_0,\,I\subseteq[a^*,x]\})\\&=\sum_{I\in\mathcal{I}_0,I\subseteq[a^*,x]}\int f\times\chi I=\sum_{I\in\mathcal{I}_0,I\subseteq[a^*,x]}F(\sup I)-F(\inf I).\end{aligned}$$

But this means that if $y\in[a^*,b^*]$, and $x=\max(K\cap[a^*,y])$, so that $]x,y[\subseteq H$, then

$$\begin{aligned}F(y)&=F(x)+F(y)-F(x)\\&=F(a^*)+\int_{a^*}^x g+\sum_{I\in\mathcal{I}_0,I\subseteq[a^*,x]}(F(\sup I)-F(\inf I))+\int f\times\chi\,]x,y[\\&=F(a^*)+\int g\times\chi(K\cap[a^*,y[)+\int_{a^*}^x f\times\chi H+\int f\times\chi\,]x,y[\\&=F(a^*)+\int_{-\infty}^y h\end{aligned}$$

where $h=f\times\chi H+g\times\chi K$ is Henstock integrable because $f\times\chi H$ is Henstock integrable and $g\times\chi K$ is Lebesgue integrable.

Accordingly, if C is any non-empty subinterval of $[a^*,b^*]$, $F(\sup C)-F(\inf C)=\int h\times\chi C$. But we also know that $\frac{d}{dy}\int_{-\infty}^y h=h(y)$ for almost every y, by 483I. So $F'(y)$ is defined and equal to $h(y)$ for almost every $y\in[a^*,b^*]$, and $h=f$ a.e. on $[a^*,b^*]$. This means that $F(\sup C)-F(\inf C)=\int f\times\chi C$ for any non-empty subinterval C of $[a^*,b^*]$, and $]a^*,b^*[\,\in\mathcal{I}$. But $]a^*,b^*[$ meets V, so this is impossible. **X**

(iv) This contradiction shows that $\mathbb{R}\in\mathcal{I}$ and that F is an indefinite Henstock integral, as required.

483X Basic exercises >(a) Let $[a,b]\subseteq\mathbb{R}$ be a non-empty closed interval, and let I_μ be the gauge integral on $[a,b]$ defined from Lebesgue measure and the tagged-partition structure defined in 481J. Show that, for $f:\mathbb{R}\to\mathbb{R}$, $I_\mu(f\restriction[a,b])=\int f\times\chi[a,b]$ if either is defined.

>(b) Extract ideas from the proofs of 482G and 482H to give a direct proof of 483B(c)-(d).

(c) Set $f(0) = 0$ and $f(x) = \frac{\sin x}{x}$ for other real x. Show that f is Henstock integrable, and that $\int_0^\infty f = \frac{\pi}{2}$. (*Hint*: 283Da.)

(d) Set $f(x) = \frac{1}{x}\cos(\frac{1}{x^2})$ for $0 < x \leq 1$, 0 for other real x. Show that f is Henstock integrable but not Lebesgue integrable. (*Hint*: by considering $\frac{d}{dx}(x^2 \sin\frac{1}{x^2})$, show that $\lim_{a\downarrow 0}\int_a^1 f$ is defined.)

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be a Henstock integrable function. Show that there is a finitely additive functional $\lambda : \mathcal{P}\mathbb{R} \to \mathbb{R}$ such that for every $\epsilon > 0$ there are a gauge $\delta \in \Delta$ and a Radon measure ν on $\mathbb{R}$ such that $\nu\mathbb{R} \leq \epsilon$ and $|S_{\boldsymbol{t}}(f,\mu) - \lambda W_{\boldsymbol{t}}| \leq \nu W_{\boldsymbol{t}}$ for every δ-fine $\boldsymbol{t} \in T$.

(f) Let $f : \mathbb{R} \to \mathbb{R}$ be a Henstock integrable function. Show that $\bigcup\{G : G \subseteq \mathbb{R} \text{ is open, } f \text{ is Lebesgue integrable over } G\}$ is dense. (*Hint*: 483G.)

>(g) Let $f : \mathbb{R} \to \mathbb{R}$ be a Henstock integrable function and F its indefinite Henstock integral. Show that f is Lebesgue integrable iff F is of bounded variation on $\mathbb{R}$. (*Hint*: 224I.)

>(h) Let $F : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x)$ is defined in $\mathbb{R}$, and $F'(x)$ is defined for all but countably many $x \in \mathbb{R}$. Show that F is the indefinite Henstock integral of any function $f : \mathbb{R} \to \mathbb{R}$ extending F'. (*Hint*: in 483J, take F_1 and F_2 differing from F by saltus functions.)

(i) Let $f : \mathbb{R} \to \mathbb{R}$ be a Henstock integrable function, and $\langle I_n\rangle_{n\in\mathbb{N}}$ a disjoint sequence of intervals in $\mathbb{R}$. Show that $\lim_{n\to\infty} \|f \times \chi I_n\|_H = 0$.

(j) Show that HL^1 is not a Banach space. (*Hint*: there is a continuous function which is nowhere differentiable.)

>(k) Use 483N to replace half of the proof of 483Bd.

(l) Let U be a linear subspace of $\mathbb{R}^{\mathbb{R}}$ and $\phi : U \to \mathbb{R}$ a linear functional such that (i) $f \in U$ and $\phi f = \int f d\mu$ for every Lebesgue integrable function $f : \mathbb{R} \to \mathbb{R}$ (ii) $f \times \chi C \in U$ whenever $f \in U$ and $C \in \mathcal{C}$ (iii) whenever $f \in U$ and $\mathcal{I}$ is a disjoint family of non-empty open intervals such that $\sum_{I\in\mathcal{I}} \sup_{C\subseteq I, C\in\mathcal{C}} |\phi(f \times \chi C)| < \infty$, then $f \times \chi(\bigcup\mathcal{I}) \in U$ and $\phi(f \times \chi(\bigcup\mathcal{I})) = \sum_{I\in\mathcal{I}} \phi(f \times \chi I)$. Show that if $f : \mathbb{R} \to \mathbb{R}$ is any Henstock integrable function, then $f \in U$ and $\phi(f) = \int f$. (*Hint*: use the argument of part (b) of the proof of 483R.)

(m) Let $F : \mathbb{R} \to \mathbb{R}$ be such that D^*F and D_*F are both finite everywhere. Show that $F(b) - F(a) = \int D^*F \times \chi[a,b]$ whenever $a \leq b$ in $\mathbb{R}$.

>(n) Let ν be any Radon measure on $\mathbb{R}$, and let I_ν be the gauge integral defined from ν and the tagged-partition structure of 481K and this section.

(i) Show that if $I_\nu(f)$ is defined, then f is $\operatorname{dom}\nu$-measurable.

(ii) Show that if $f : \mathbb{R} \to \mathbb{R}$ is such that $\int f d\nu$ is defined in $\mathbb{R}$, then $I_\nu(f)$ is defined and equal to $\int f d\nu$.

(iii) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function, and $\alpha \in]-\infty,\infty]$. Show that $I_\nu(f \times \chi]-\infty,\alpha[) = \lim_{\beta\uparrow\alpha} I_\nu(f \times \chi]-\infty,\beta[)$ if either is defined in $\mathbb{R}$.

(iv) Show that if $I_\nu(f)$ is defined, then there is a countable cover of $\mathbb{R}$ by compact sets K such that $\int_K |f| d\nu < \infty$.

(o) (Bongiorno di Piazza & Preiss 00) Let $\mathcal{C}$ be the set of non-empty subintervals of a closed interval $[a,b] \subseteq \mathbb{R}$, and T the straightforward set of tagged partitions generated by $[a,b] \times \mathcal{C}$. Let Δ be the set of neighbourhood gauges on $[a,b]$. For $\alpha \geq 0$ set

$$T_\alpha = \{\boldsymbol{t} : \boldsymbol{t} \in T, \textstyle\sum_{(x,C)\in\boldsymbol{t}} \rho(x,C) \leq \alpha\},$$

writing $\rho(x,C) = \inf_{y\in C} |x - y|$ as usual. Show that T_α is compatible with Δ and $\mathfrak{R} = \{\{\emptyset\}\}$ in the sense of 481F. Show that if I_α is the gauge integral defined from $[a,b]$, T_α, Δ, $\mathfrak{R}$ and Lebesgue measure, then I_α extends the ordinary Lebesgue integral and $I_\alpha(F') = F(b) - F(a)$ whenever $F : [a,b] \to \mathbb{R}$ is differentiable.

483Y Further exercises (a) Let us say that a **Lebesgue measurable neighbourhood gauge** on $\mathbb{R}$ is a neighbourhood gauge of the form $\{(x,A) : x \in \mathbb{R},\ A \subseteq]x-\eta_x, x+\eta_x[\}$ where $x \mapsto \eta_x$ is a Lebesgue measurable function from $\mathbb{R}$ to $[0,\infty[$. Let $\tilde{\Delta}$ be the set of Lebesgue measurable neighbourhood gauges. Show that the gauge integral defined by the tagged-partition structure $(\mathbb{R}, T, \tilde{\Delta}, \Re)$ and μ is the Henstock integral.

(b) Show that if $\Delta_0 \subseteq \Delta$ is any set of cardinal at most $\mathfrak{c}$, then the gauge integral defined by $(\mathbb{R}, T, \Delta_0, \Re)$ and μ does not extend the Lebesgue integral, so is not the Henstock integral.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a Henstock integrable function with indefinite Henstock integral F, and ν a totally finite Radon measure on $\mathbb{R}$. Set $G(x) = \nu\,]-\infty, x]$ for $x \in \mathbb{R}$. Show that $f \times G$ is Henstock integrable, with indefinite Henstock integral H, where $H(x) = F(x)G(x) - \int_{]-\infty,x]} F\,d\nu$ for $x \in \mathbb{R}$.

(d) Let $f : \mathbb{R} \to \mathbb{R}$ be a Henstock integrable function, and $G : \mathbb{R} \to \mathbb{R}$ a function of bounded variation. Show that $f \times G$ is Henstock integrable, and

$$\left|\int f \times G\right| \leq (\lim_{x\to\infty} |G(x)| + \operatorname{Var}_{\mathbb{R}} G) \sup_{x\in\mathbb{R}} \left|\int_{-\infty}^{x} f\right|.$$

(Compare 224J.)

(e) Let $f : \mathbb{R} \to \mathbb{R}$ be a Henstock integrable function and $g : \mathbb{R} \to \mathbb{R}$ a Lebesgue integrable function; let F and G be their indefinite (Henstock) integrals. Show that $\int f \times G + \int g \times F\,d\mu$ is defined and equal to $\lim_{x\to\infty} F(x)G(x)$.

(f) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Show that f is Lebesgue integrable iff $f \times g$ is Henstock integrable for every bounded continuous function $g : \mathbb{R} \to \mathbb{R}$.

(g) Let V be a Banach space and $f : \mathbb{R} \to V$ a function. For $\boldsymbol{t} \in T$, set $S_{\boldsymbol{t}}(f,\mu) = \sum_{(x,C)\in\boldsymbol{t}} f(x)\mu C$. We say that f is **Henstock integrable**, with **Henstock integral** $u = \int f \in V$, if $u = \lim_{\boldsymbol{t}\to\mathcal{F}(T,\Delta,\Re)} S_{\boldsymbol{t}}(f,\mu)$.

(i) Show that the set $\mathcal{HL}^1_V$ of Henstock integrable functions from $\mathbb{R}$ to V is a linear subspace of $V^{\mathbb{R}}$ including the space $\mathcal{L}^1_V$ of Bochner integrable functions (253Yf), and that $\int : \mathcal{HL}^1_V \to V$ is a linear operator extending the Bochner integral.

(ii) Show that if $f : \mathbb{R} \to V$ is Henstock integrable, so is $f \times \chi C$ for every interval $C \subseteq \mathbb{R}$, and that $(a,b) \mapsto \int f \times \chi[a,b]$ is continuous. Set $\|f\|_H = \sup_{C\in\mathcal{C}} \|\int f \times \chi C\|$.

(iii) Show that if $\mathcal{I}$ is a disjoint family of open intervals in $\mathbb{R}$, and $f : \mathbb{R} \to V$ is such that $f \times \chi I \in \mathcal{HL}^1_V$ for every $I \in \mathcal{I}$ and $\sum_{I\in\mathcal{I}} \|f \times \chi I\|_H$ is finite, then $\int f \times \chi(\bigcup \mathcal{I})$ is defined and equal to $\sum_{I\in\mathcal{I}} \int f \times \chi I$.

(iv) Define $f : \mathbb{R} \to \ell^\infty([0,1])$ by setting $f(x) = \chi([0,1] \cap]-\infty, x])$ for $x \in \mathbb{R}$. Show that f is Henstock integrable, but that if $F(x) = \int f \times \chi\,]-\infty, x[$ for $x \in \mathbb{R}$, then $\lim_{y\to x} \frac{1}{y-x}(F(y) - F(x))$ is not defined in $\ell^\infty([0,1])$ for any $x \in [0,1]$.

483 Notes and comments I hope that the brief account here (largely taken from Gordon 94) will give an idea of the extraordinary power of gauge integrals. While what I am calling the 'Henstock integral', regarded as a linear functional on a space of real functions, was constructed long ago by Perron and Denjoy, the gauge integral approach makes it far more accessible, and gives clear pathways to corresponding Stieltjes and vector integrals (483Xn, 483Yg).

Starting from our position in the fourth volume of a book on measure theory, it is natural to try to describe the Henstock integral in terms of the Lebesgue integral, as in 483C (they agree on non-negative functions) and 483Xl (offering an extension process to generate the Henstock integral from the Lebesgue integral); on the way, we see that Henstock integrable functions are necessarily Lebesgue integrable over many intervals (483Xf). Alternatively, we can set out to understand indefinite Henstock integrals and their derivatives, just as Lebesgue integrable functions can be characterized as almost everywhere equal to derivatives of absolutely continuous functions (222E, 225E), because if f is Henstock integrable then it is equal almost everywhere to the derivative of its indefinite integral (483I). Any differentiable function (indeed, any continuous function differentiable except on a countable set) is an indefinite Henstock integral (483Xh). Recall that the Cantor

function (134H) is continuous and differentiable almost everywhere but is not an indefinite integral, so we have to look for a characterization which can exclude such cases. For this we have to work quite hard, but we find that 'ACG$_*$ functions' are the appropriate class (483R).

Gauge integrals are good at integrating derivatives (see 483Xh), but bad at integrating over subspaces. Even to show that $f \times \chi[0,\infty[$ is Henstock integrable whenever f is (482Bc) involves us in some unexpected manoeuvres. I give an argument which is designed to show off the general theory of §482, and I recommend you to look for short cuts (483Xb), but any method must depend on careful examination of the exact classes $\mathcal{C}$ and $\mathfrak{R}$ chosen for the definition of the integral. We do however have a new kind of convergence theorem in 482H and 483N.

One of the incidental strengths of the Henstock integral is that it includes the improper Riemann integral (483Bd, 483Xc); so that, for instance, Carleson's theorem (286U) can be written in the form

$$\hat{f}(y) = \tfrac{1}{\sqrt{2\pi}} \text{⨍} e^{-ixy} f(x)dx \text{ for almost every } y$$

if $f : \mathbb{R} \to \mathbb{C}$ is square-integrable. But to represent the many formulae of the type $\lim_{a\to\infty} \int_{-a}^{a} f$ in §283 (e.g., 283F, 283I and 283L) directly in the form $I_\mu(f)$ we need to change $\mathfrak{R}$, as in 481L or 481Xc.

484 The Pfeffer integral

I give brief notes on what seems at present to be the most interesting of the multi-dimensional versions of the Henstock integral, leading to Pfeffer's Divergence Theorem (484N).

484A Notation This section will depend heavily on Chapter 47, and will use much of the same notation. $r \geq 2$ will be a fixed integer, and μ will be Lebesgue measure on $\mathbb{R}^r$, while μ_{r-1} is Lebesgue measure on $\mathbb{R}^{r-1}$. As in §§473-475, let ν be 'normalized' $(r-1)$-dimensional Hausdorff outer measure on $\mathbb{R}^r$, as described in §265; that is, $\nu = 2^{-r+1}\beta_{r-1}\mu_{H,r-1}$, where $\mu_{H,r-1}$ is $r-1$-dimensional Hausdorff measure on $\mathbb{R}^r$ as described in §264, and

$$\begin{aligned}\beta_{r-1} &= \frac{2^{2k}k!\pi^{k-1}}{(2k)!} \text{ if } r = 2k \text{ is even},\\ &= \frac{\pi^k}{k!} \text{ if } r = 2k+1 \text{ is odd}\end{aligned}$$

is the Lebesgue measure of a ball of diameter 1 in $\mathbb{R}^{r-1}$ (264I). For this section only, let us say that a subset of $\mathbb{R}^r$ is **thin** if it is of the form $\bigcup_{n\in\mathbb{N}} A_n$ where $\nu^* A_n$ is finite for every n. Note that by 471L, $\mu A = 0$ for every thin set A. For $A \subseteq \mathbb{R}^r$, write ∂A for its ordinary topological boundary. If $x \in \mathbb{R}^r$ and $\epsilon > 0$, $B(x,\epsilon)$ will be the closed ball $\{y : \|y - x\| \leq \epsilon\}$.

I will use the term **dyadic cube** for sets of the form $\prod_{i<r} [2^{-m}k_i, 2^{-m}(k_i+1)[$ where $m, k_0, \ldots, k_{r-1} \in \mathbb{Z}$; write $\mathcal{D}$ for the set of dyadic cubes in $\mathbb{R}^r$. Note that if $D, D' \in \mathcal{D}$, either $D \subseteq D'$ or $D' \subseteq D$ or $D \cap D' = \emptyset$; so if $\mathcal{D}_0 \subseteq \mathcal{D}$, the maximal members of $\mathcal{D}_0$ are disjoint.

It will be helpful to have an abbreviation for the following expression: set

$$\alpha^* = \min(\frac{1}{r^{r/2}}, \frac{2^{r-2}}{r\beta_r^{(r-1)/r}}).$$

(As will become apparent, the actual value of this constant is of no importance; but the strict logic of the arguments below depends on α^* being small enough.)

As in §475, I write int$^*A$, cl$^*A$ and ∂^*A for the essential interior, essential closure and essential boundary of a set $A \subseteq \mathbb{R}^r$ (475B). Recall that a set $A \subseteq \mathbb{R}^r$ has finite perimeter in the sense of 474D iff $\nu(\partial^* A)$ is finite, and then

$$\nu(\partial^* A) = \lambda_A^\partial(\mathbb{R}^r) = \operatorname{per} A$$

is the perimeter of A (475M); we shall also need to remember that A is necessarily Lebesgue measurable.

$\mathcal{C}$ will be the family of subsets of $\mathbb{R}^r$ with locally finite perimeter, and $\mathcal{V}$ the family of bounded sets in $\mathcal{C}$, that is, the family of bounded sets with bounded perimeter. Note that $\mathcal{C}$ is an algebra of subsets of $\mathbb{R}^r$ (475Ma), and that $\mathcal{V}$ is an ideal in $\mathcal{C}$.

484B Theorem (Tamanini & Giacomelli 89) Let $E \subseteq \mathbb{R}^r$ be a Lebesgue measurable set of finite measure and perimeter, and $\epsilon > 0$. Then there is a Lebesgue measurable set $G \subseteq E$ such that $\operatorname{per} G \leq \operatorname{per} E$, $\mu(E \setminus G) \leq \epsilon$ and $\operatorname{cl}^*G = \overline{G}$.

proof (Pfeffer 91b) **(a)** Set $\alpha = \frac{1}{\epsilon} \operatorname{per} E$. For measurable sets $G \subseteq E$ set $q(G) = \operatorname{per} G - \alpha\mu G$. Then there is a self-supporting measurable set $G \subseteq E$ such that $q(G) \leq q(G')$ whenever $G' \subseteq E$ is measurable.

P Write Σ for the family of Lebesgue measurable subsets of $\mathbb{R}^r$; give Σ the topology of convergence in measure defined by the pseudometrics $\rho_H(G, G') = \mu((G \triangle G') \cap H)$ for measurable sets H of finite measure (cf. 474S). Extend q to Σ by setting $q(G) = \operatorname{per}(E \cap G) - \alpha\mu(E \cap G)$ for every $G \in \Sigma$. Because $\operatorname{per} : \Sigma \to [0, \infty]$ is lower semi-continuous for the topology of convergence in measure (474Sa), and $G \mapsto E \cap G$, $G \mapsto \mu(E \cap G)$ are continuous, $q : \Sigma \to [0, \infty[$ is lower semi-continuous (4A2B(d-ii)). Next, $\mathcal{K} = \{G : G \in \Sigma, \operatorname{per} G \leq \operatorname{per} E\}$ is compact (474Sb), while $\mathcal{L} = \{G : \mu(G \setminus E) = 0\}$ is closed, so there is an $G_0 \in \mathcal{L} \cap \mathcal{K}$ such that $q(G_0) = \inf_{G \in \mathcal{L} \cap \mathcal{K}} q(G)$ (4A2Gk). Since $G_0 \in \mathcal{L}$, $\operatorname{per}(G_0 \cap E) = \operatorname{per}(G_0)$ and $\mu(G_0 \cap E) = \mu G_0$, so we may suppose that $G_0 \subseteq E$. Moreover, there is a self-supporting set $G \subseteq G_0$ such that $G_0 \setminus G$ is negligible (414F), and we still have $q(G) = q(G_0)$. Of course $q(G) \leq q(E)$, just because $E \in \mathcal{L} \cap \mathcal{K}$.

? If there is a measurable set $G' \subseteq E$ such that $q(G') < q(G)$, then

$$\operatorname{per} G' = q(G') + \alpha\mu G' \leq q(E) + \alpha\mu E = \operatorname{per} E,$$

so $G' \in \mathcal{K}$; but this means that $G' \in \mathcal{L} \cap \mathcal{K}$ and $q(G) = q(G_0) \leq q(G')$. **X** So G has the required properties. **Q**

(b) Since $q(G) \leq q(E)$, we must have

$$\operatorname{per} G = q(G) + \alpha\mu G \leq q(E) + \alpha\mu E = \operatorname{per} E$$

(just as in the last part of the proof of (a)). Also $\operatorname{per} G + \alpha\mu(E \setminus G) \leq \operatorname{per} E$, so $\mu(E \setminus G) \leq \epsilon$.

(c) Next, $\overline{G} \subseteq \operatorname{cl}^*G$. **P** Let $x \in \overline{G}$. For every $t > 0$, set $U_t = \{y : \|y - x\| < t\}$; then $q(G \setminus U_t) \leq q(G)$. Now

$$\begin{aligned}
\operatorname{per}(G \cap U_t) + \operatorname{per}(G \setminus U_t) &= \nu(\partial^*(G \cap U_t)) + \nu(\partial^*(G \setminus U_t)) \\
&\leq \nu(\partial^*G \cap U_t) + \nu(\operatorname{cl}^*G \cap \partial U_t) \\
&\qquad + \nu(\partial^*G \setminus U_t) + \nu(\operatorname{cl}^*G \cap \partial U_t)
\end{aligned}$$

(475Cf, because $\partial(\mathbb{R}^r \setminus U_t) = \partial U_t$))

$$= \nu(\partial^*G) + 2\nu(\operatorname{cl}^*G \cap \partial U_t) = \operatorname{per} G + 2\nu(G \cap \partial U_t)$$

for almost every $t > 0$, because

$$\int_0^\infty \nu((G \triangle \operatorname{cl}^*G) \cap \partial U_t)dt = \mu(G \triangle \operatorname{cl}^*G) = 0$$

(265G). So, for almost every t,

$$\mu(G \cap U_t)^{(r-1)/r} \leq \operatorname{per}(G \cap U_t)$$

(474La)

$$\begin{aligned}
&\leq \operatorname{per} G + 2\nu(G \cap \partial U_t) - \operatorname{per}(G \setminus U_t) \\
&= q(G) + \alpha\mu(G \cap U_t) + 2\nu(G \cap \partial U_t) - q(G \setminus U_t) \\
&\leq \alpha\mu(G \cap U_t) + 2\nu(G \cap \partial U_t)
\end{aligned}$$

because $q(G)$ is minimal.

? Suppose, if possible, that $x \notin \operatorname{cl}^*G$. Take $\beta > 0$ such that $2\beta < \frac{1}{r}\beta^{(r-1)/r}$. Then there is some $\gamma > 0$ such that $\mu(G \cap U_t) \leq \beta t^r$ for every $t \in]0, \gamma]$, and we may suppose that $2\beta < \frac{1}{r}\beta^{(r-1)/r} - \frac{1}{r+1}\alpha\gamma$. In this case, for almost every $t \in]0, \gamma]$,

$$\nu(G\cap\partial U_t)\geq\tfrac{1}{2}(\beta^{(r-1)/r}t^{r-1}-\alpha t^r),$$

so

$$\beta\gamma^r\geq\mu(G\cap U_\gamma)=\textstyle\int_0^\gamma\nu(G\cap\partial U_t)dt\geq\frac{1}{2}\big(\frac{\beta^{(r-1)/r}}{r}\gamma^r-\frac{\alpha}{r+1}\gamma^{r+1}\big)$$

and

$$2\beta\geq\tfrac{1}{r}\beta^{(r-1)/r}-\tfrac{1}{r+1}\alpha\gamma>2\beta,$$

which is absurd. **X**

Thus $x\in\text{cl}^*G$. As x is arbitrary, $\overline{G}\subseteq\text{cl}^*G$. **Q**

Since we certainly have $\text{cl}^*G\subseteq\overline{G}$, this G serves.

484C Lemma Let $E\in\mathcal{V}$ and $l\in\mathbb{N}$ be such that $\max(\text{per}\,E,\text{diam}\,E)\leq l$. Then E is expressible as $\bigcup_{i<n}E_i$ where $\langle E_i\rangle_{i<n}$ is disjoint, $\text{per}\,E_i\leq 1$ for each $i<n$ and n is at most $2^r(l+2)^r(4r(4l^2+1))^{r/(r-1)}+2^{r+2}l^2$.

proof For $D\in\mathcal{D}$, write $\mathcal{D}_D$ for $\{D':D'\in\mathcal{D},\,D'\subseteq D,\,\text{diam}\,D'=\frac{1}{2}\text{diam}\,D\}$, the family of the 2^r dyadic subcubes of D at the next level down.

(a) If $l\leq 1$ the result is trivial, so let us suppose that $l\geq 2$. Let $m\in\mathbb{N}$ be minimal subject to $4r(4l^2+1)\leq 2^{m(r-1)}$, so that $2^m\leq 2(4r(4l^2+1))^{1/(r-1)}$. Then we can cover E by a family $\mathcal{L}_0$ of dyadic cubes of side 2^{-m} with

$$\#(\mathcal{L}_0)\leq(2^ml+2)^r\leq 2^r(l+2)^r(4r(4l^2+1))^{r/(r-1)}.$$

(b) Let $\mathcal{L}_1$ be the set of those $D\in\mathcal{D}$ such that D is included in some member of $\mathcal{L}_0$ and $\lfloor 2\nu(D'\cap\partial^*E)\rfloor<\lfloor 2\nu(D\cap\partial^*E)\rfloor$ for every $D'\in\mathcal{D}_D$. Then $\#(\mathcal{L}_1)\leq(2l)^2$. **P** I show by induction on k that if $D_0\in\mathcal{D}$ and $\lfloor 2\nu(D\cap\partial^*E)\rfloor=k$ then $\#(\{D:D\in\mathcal{L}_1,\,D\subseteq D_0\}\leq k^2$. For $k=0$ this is trivial. For the inductive step to $k>0$, given that $\lfloor 2\nu(D_0\cap\partial^*E)\rfloor=k$, let $\mathcal{L}_1'$ be the set of maximal elements in

$$\{D:D\in\mathcal{L}_1,\,D\subseteq D_0,\,\lfloor 2\nu(D\cap\partial^*E)\rfloor<k\}.$$

Now there can be at most one member D of $\mathcal{L}_1$ included in D_0 and with $\lfloor 2\nu(D\cap\partial^*E)\rfloor\geq k$. So

$$\begin{aligned}\#(\{D:D\in\mathcal{L}_1,\,D\subseteq D_0\})&\leq 1+\sum_{D_1\in\mathcal{L}_1'}\#(\{D:D\in\mathcal{L}_1,\,D\subseteq D_1\})\\&\leq 1+\sum_{D_1\in\mathcal{L}_1'}(\lfloor 2\nu(D_1\cap\partial^*E)\rfloor)^2\end{aligned}$$

(by the inductive hypothesis)

$$\leq(\lfloor 2\nu(D_0\cap\partial^*E)\rfloor)^2$$

(because $(\lfloor\alpha\rfloor)^2+(\lfloor\beta\rfloor)^2<(\lfloor\alpha+\beta\rfloor)^2$ whenever $\alpha,\,\beta\geq 0$ and $\max(\lfloor\alpha\rfloor,\lfloor\beta\rfloor)<\lfloor\alpha+\beta\rfloor$))

$$\leq k^2.$$

So the induction proceeds. Taking $k=2l$, we have the result. **Q**

(c) Set $\mathcal{L}_2=\mathcal{L}_0\cup\bigcup\{\mathcal{D}_D:D\in\mathcal{L}_1\}$. For $D\in\mathcal{L}_2$, set $\tilde{D}=D\setminus\bigcup\{D':D'\in\mathcal{L}_1,\,D'\subseteq D\}$, and consider $\mathcal{K}=\{\tilde{D}:D\in\mathcal{L}_2\}$, so that

$$\begin{aligned}\#(\mathcal{K})\leq\#(\mathcal{L}_2)&\leq\#(\mathcal{L}_0)+2^r\#(\mathcal{L}_1)\\&\leq 2^r(l+2)^r(4r(4l^2+1))^{r/(r-1)}+2^{r+2}l^2.\end{aligned}$$

(d)(i) $\bigcup\mathcal{K}\supseteq E$. **P** If $x\in E$, there is a smallest member D of $\mathcal{L}_2$ containing it, because certainly $x\in\bigcup\mathcal{L}_0$. But now x cannot belong to any member of $\mathcal{L}_1$ included in D, so $x\in\tilde{D}$. **Q**

(ii) $\mathcal{K}$ is disjoint. **P** If D_1, $D_2 \in \mathcal{L}_2$ are disjoint, then of course $\tilde{D}_1 \cap \tilde{D}_2 = \emptyset$. If $D_1 \subset D_2$, then $D_1 \notin \mathcal{L}_0$, so there is a $D \in \mathcal{L}_1$ such that $D_1 \in \mathcal{D}_D$; in this case $D \subseteq D_2$ so $\tilde{D}_2 \subseteq D_2 \setminus D$ is disjoint from D_1. **Q**

(iii) $\nu(D \cap \partial^* E) \leq \frac{1}{2}$ for every $D \in \mathcal{K}$. **P?** If $D_0 \in \mathcal{L}_2$ and $\nu(\tilde{D}_0 \cap \partial^* E) > \frac{1}{2}$, set $j = \lfloor 2\nu(D_0 \cap \partial^* E) \rfloor \geq 1$. Given that $D_i \in \mathcal{D}$ is included in D_0 and $\lfloor 2\nu(D_i \cap \partial^* E) \rfloor = j$, then $\nu((D_0 \setminus D_i) \cap \partial^* E) < \frac{1}{2}$, so $\tilde{D}_0 \cap D_i$ is non-empty and $D_i \notin \mathcal{L}_1$. There must therefore be a $D_{i+1} \in \mathcal{D}_{D_i}$ such that $\lfloor 2\nu(D_{i+1} \cap \partial^* E) \rfloor = j$. Thus we have a strictly decreasing sequence $\langle D_i \rangle_{i \in \mathbb{N}}$ in $\mathcal{D}$ such that $\nu(D_i \cap \partial^* E) \geq j$ for every i. But (because $\operatorname{per} E$ is finite) this means that, writing x for the unique member of $\bigcap_{i \in \mathbb{N}} \overline{D}_i$, $\nu\{x\} \geq j$, which is absurd. **X** **Q**

(iv) $\operatorname{per}(D \cap E) \leq 1$ for every $D \in \mathcal{K}$. **P** If $D \in \mathcal{L}_2$,

$$\nu(\partial \tilde{D}) \leq \nu(\partial D) + \textstyle\sum_{D' \in \mathcal{L}_1} \nu(\partial D') \leq 2r(4l^2+1)2^{-m(r-1)} \leq \frac{1}{2}$$

by the choice of m. So

$$\operatorname{per}(D \cap E) \leq \nu(\partial D) + \nu(D \cap \partial^* E) \leq \tfrac{1}{2} + \tfrac{1}{2} = 1$$

by 475Cf. **Q**

(e) So if we take $\langle E_i \rangle_{i<n}$ to be an enumeration of $\{E \cap D : D \in \mathcal{K}\}$, we shall have the required result.

484D Definitions The gauge integrals of this section will be based on the following residual families. Let H be the family of strictly positive sequences $\eta = \langle \eta(i) \rangle_{i \in \mathbb{N}}$ in $\mathbb{R}$. For $\eta \in \mathrm{H}$, write $\mathcal{M}_\eta$ for the set of disjoint sequences $\langle E_i \rangle_{i \in \mathbb{N}}$ of measurable subsets of $\mathbb{R}^r$ such that $\mu E_i \leq \eta(i)$ and $\operatorname{per} E_i \leq 1$ for every $i \in \mathbb{N}$, and E_i is empty for all but finitely many i. Let $\mathcal{R}_\eta$ be the set of unions of members of $\mathcal{M}_\eta$. Next, for $V \in \mathcal{V}$ and $\eta \in \mathrm{H}$, set $\mathcal{R}_\eta^{(V)} = \{R : R \subseteq \mathbb{R}^r,\ R \cap V \in \mathcal{R}_\eta\}$; finally, set $\mathfrak{R} = \{\mathcal{R}_\eta^{(V)} : V \in \mathcal{V},\ \eta \in \mathrm{H}\}$.

484E Lemma (a)(i) For every $\mathcal{R} \in \mathfrak{R}$, there is an $\eta \in \mathrm{H}$ such that $\mathcal{R}_\eta \subseteq \mathcal{R}$.

(ii) If $\mathcal{R} \in \mathfrak{R}$ and $C \in \mathcal{C}$, there is an $\mathcal{R}' \in \mathfrak{R}$ such that $C \cap R \in \mathcal{R}$ whenever $R \in \mathcal{R}'$.

(b)(i) If $\eta \in \mathrm{H}$ and $\gamma \geq 0$, there is an $\epsilon > 0$ such that $R \in \mathcal{R}_\eta$ whenever $\mu R \leq \epsilon$, $\operatorname{diam} R \leq \gamma$ and $\operatorname{per} R \leq \gamma$.

(ii) If $\mathcal{R} \in \mathfrak{R}$ and $\gamma \geq 0$, there is an $\epsilon > 0$ such that $R \in \mathcal{R}$ whenever $\mu R \leq \epsilon$ and $\operatorname{per} R \leq \gamma$.

(c) If $\mathcal{R} \in \mathfrak{R}$ there is an $\mathcal{R}' \in \mathfrak{R}$ such that $R \cup R' \in \mathcal{R}$ whenever R, $R' \in \mathcal{R}'$ and $R \cap R' = \emptyset$.

(d)(i) If $\eta \in \mathrm{H}$ and $A \subseteq \mathbb{R}^r$ is a thin set, then there is a set $\mathcal{D}_0 \subseteq \mathcal{D}$ such that every point of A belongs to the interior of $\bigcup \mathcal{D}_1$ for some finite $\mathcal{D}_1 \subseteq \mathcal{D}_0$, and $\bigcup \mathcal{D}_1 \in \mathcal{R}_\eta$ for every finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$.

(ii) If $\mathcal{R} \in \mathfrak{R}$ and $A \subseteq \mathbb{R}^r$ is a thin set, then there is a set $\mathcal{D}_0 \subseteq \mathcal{D}$ such that every point of A belongs to the interior of $\bigcup \mathcal{D}_1$ for some finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$, and $\bigcup \mathcal{D}_1 \in \mathcal{R}$ for every finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$.

proof (a)(i) Express $\mathcal{R}$ as $\mathcal{R}_{\eta'}^{(V)}$ where $\eta' \in \mathrm{H}$ and $V \in \mathcal{V}$. Let $l \in \mathbb{N}$ be such that $\max(\operatorname{diam} V, 1 + \operatorname{per} V) \leq l$, and take $n \geq 2^r(l+2)^r(4r(4l^2+1))^{r/(r-1)} + 2^{r+2}l^2$. Set $\eta(i) = \min\{\eta'(j) : ni \leq j < n(i+1)\}$ for every $i \in \mathbb{N}$, so that $\eta \in \mathrm{H}$. If $R \in \mathcal{R}_\eta$, express it as $\bigcup_{i \in \mathbb{N}} E_i$ where $\langle E_i \rangle_{i \in \mathbb{N}} \in \mathcal{M}_\eta$. For each i, $\max(\operatorname{diam}(E_i \cap V), \operatorname{per}(E_i \cap V)) \leq l$, so by 484C we can express $E_i \cap V$ as $\bigcup_{ni \leq j < n(i+1)} E'_j$, where $\langle E'_j \rangle_{ni \leq j < n(i+1)}$ is disjoint and $\operatorname{per} E'_j \leq 1$ for each j. Now

$$\mu E'_j \leq \mu E_i \leq \eta(i) \leq \eta'(j)$$

for $ni \leq j < n(i+1)$. Also $\{j : E'_j \neq \emptyset\}$ is finite because $\{j : E_j = \emptyset\}$ is finite. So $\langle E'_j \rangle_{j \in \mathbb{N}} \in \mathcal{M}_{\eta'}$ and $R \cap V = \bigcup_{j \in \mathbb{N}} E'_j$ belongs to $\mathcal{R}_{\eta'}$, that is, $R \in \mathcal{R}$. As R is arbitrary, $\mathcal{R}_\eta \subseteq \mathcal{R}$.

(ii) Express $\mathcal{R}$ as $\mathcal{R}_\eta^{(V)}$, where $V \in \mathcal{V}$ and $\eta \in \mathrm{H}$. By (i), there is an $\eta' \in \mathrm{H}$ such that $\mathcal{R}_{\eta'} \subseteq \mathcal{R}_\eta^{(C \cap V)}$. Set $\mathcal{R}' = \mathcal{R}_{\eta'}^{(V)} \in \mathfrak{R}$. If $R \in \mathcal{R}'$, then $R \cap V \in \mathcal{R}_{\eta'}$, so $C \cap R \cap V \in \mathcal{R}_\eta$ and $C \cap R \in \mathcal{R}$.

(b)(i) Take $l \geq \gamma$ and $n \geq 2^r(l+2)^r(4r(4l^2+1))^{r/(r-1)} + 2^{r+2}l^2$, and set $\epsilon = \min_{i<n} \eta(i)$. If $\mu R \leq \epsilon$, $\operatorname{diam} R \leq l$ and $\operatorname{per} R \leq l$, then R is expressible as $\bigcup_{i<n} E_i$ where $\langle E_i \rangle_{i<n}$ is disjoint and $\operatorname{per} E_i \leq 1$ for each $i < n$. Since $\mu E_i \leq \mu R \leq \eta(i)$ for each i, $R \in \mathcal{R}_\eta$.

(ii) Express $\mathcal{R}$ as $\mathcal{R}_\eta^{(V)}$. By (i) just above, there is an $\epsilon > 0$ such that $R \in \mathcal{R}_\eta$ whenever $\operatorname{diam} R \leq \operatorname{diam} V$, $\operatorname{per} R \leq \gamma + \operatorname{per} V$ and $\mu R \leq \epsilon$; and this ϵ serves.

(c) Express $\mathcal{R}$ as $\mathcal{R}_\eta^{(V)}$. Set $\eta'(i) = \min(\eta(2i), \eta(2i+1))$ for every i; then $\eta' \in \mathrm{H}$ and if $\langle E_i \rangle_{i\in\mathbb{N}}$, $\langle E_i' \rangle_{i\in\mathbb{N}}$ belong to $\mathcal{M}_{\eta'}$ and have disjoint unions, then $(E_0, E_0', E_1, E_1', \dots) \in \mathcal{M}_\eta$; this is enough to show that $R \cup R' \in \mathcal{R}_\eta$ whenever R, $R' \in \mathcal{R}_{\eta'}$ are disjoint, so that $R \cup R' \in \mathcal{R}$ whenever R, $R' \in \mathcal{R}_{\eta'}^{(V)}$ are disjoint.

(d)(i)($\boldsymbol{\alpha}$) We can express A as $\bigcup_{i\in\mathbb{N}} A_i$ where $\mu^*_{H,r-1} A_i < 1/2^{r+1} r$ for every i. **P** Because A is thin, it is the union of a sequence of sets of finite outer measure for ν, and therefore for $\mu_{H,r-1}$. On each of these the subspace measure is atomless (471E, 471Dg), so that the set can be dissected into finitely many sets of measure less than $1/2^{r+1}r$ (215D). **Q**

($\boldsymbol{\beta}$) For each $i \in \mathbb{N}$, we can cover A_i by a sequence $\langle A_{ij} \rangle_{j\in\mathbb{N}}$ of sets such that $\operatorname{diam} A_{ij} < \eta(i)$ for every j and $\sum_{j=0}^\infty (\operatorname{diam} A_{ij})^{r-1} < 1/2^{r+1}r$; enlarging the A_{ij} slightly if need be, we can suppose that they are all open. Now we can cover each A_{ij} by 2^r cubes $D_{ijk} \in \mathcal{D}$ in such a way that the side length of each D_{ijk} is at most the diameter of A_{ij}.

Setting $\mathcal{D}_0 = \{D_{ijk} : i,\, j \in \mathbb{N},\, k < 2^r\}$, we see that every point of A belongs to an open set A_{ij} which is covered by a finite subset of $\mathcal{D}_0$. If $\mathcal{D}_1 \subseteq \mathcal{D}_0$ is finite, let $\mathcal{D}_1'$ be the family of maximal elements of $\mathcal{D}_1$, so that $\mathcal{D}_1'$ is disjoint and $\bigcup \mathcal{D}_1' = \bigcup \mathcal{D}_1$. Express $\mathcal{D}_1'$ as $\{D_{ijk} : (i,j,k) \in I\}$ where $I \subseteq \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ and $\langle D_{ijk} \rangle_{(i,j,k)\in I}$ is disjoint. Set $I_i = \{(j,k) : (i,j,k) \in I\}$ and $E_i = \bigcup_{(j,k)\in I_i} D_{ijk}$ for $i \in \mathbb{N}$. Then $\langle E_i \rangle_{i\in\mathbb{N}}$ is disjoint, $E_i = \emptyset$ for all but finitely many i, and for each $i \in \mathbb{N}$

$$\begin{aligned}
\mu E_i &\leq \sum_{j=0}^\infty \sum_{k=0}^{2^r-1} \mu D_{ijk} \leq \sum_{j=0}^\infty \sum_{k=0}^{2^r-1} (\operatorname{diam} A_{ij})^r \\
&\leq 2^r \sum_{j=0}^\infty \eta(i) (\operatorname{diam} A_{ij})^{r-1} \leq \eta(i), \\
\nu(\partial E_i) &\leq \sum_{(j,k)\in I_i} \nu(\partial D_{ijk}) \leq \sum_{j=0}^\infty \sum_{k=0}^{2^r-1} \nu(\partial D_{ijk}) \\
&\leq \sum_{j=0}^\infty \sum_{k=0}^{2^r-1} 2r (\operatorname{diam} A_{ij})^{r-1} \leq \sum_{j=0}^\infty 2^{r+1} r (\operatorname{diam} A_{ij})^{r-1} \leq 1.
\end{aligned}$$

So $\bigcup \mathcal{D}_1 = \bigcup_{i\in\mathbb{N}} E_i$ belongs to $\mathcal{R}_\eta$.

(ii) By (a-i), there is an $\eta \in \mathrm{H}$ such that $\mathcal{R}_\eta \subseteq \mathcal{R}$. By (i) here, there is a $\mathcal{D}_0 \subseteq \mathcal{D}$ such that every point of A belongs to the interior of $\bigcup \mathcal{D}_1$ for some finite $\mathcal{D}_1 \subseteq \mathcal{D}_0$, and $\bigcup \mathcal{D}_1 \in \mathcal{R}_\eta \subseteq \mathcal{R}$ for every finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$.

484F A family of tagged-partition structures For $\alpha > 0$, let $\mathcal{C}_\alpha$ be the family of those $C \in \mathcal{V}$ such that $\mu C \geq \alpha (\operatorname{diam} C)^r$ and $\alpha \operatorname{per} C \leq (\operatorname{diam} C)^{r-1}$, and let T_α be the straightforward set of tagged partitions generated by the set

$$\{(x, C) : C \in \mathcal{C}_\alpha,\, x \in \mathrm{cl}^* C\}.$$

Note that if $0 < \alpha' \leq \alpha$ then $\mathcal{C}_\alpha \subseteq \mathcal{C}_{\alpha'}$ and $T_\alpha \subseteq T_{\alpha'}$. Let Θ be the set of functions $\theta : \mathbb{R}^r \to [0, \infty[$ such that $\{x : \theta(x) = 0\}$ is thin (definition: 484A), and set $\Delta = \{\delta_\theta : \theta \in \Theta\}$, where $\delta_\theta = \{(x, A) : x \in \mathbb{R}^r,\, \theta(x) > 0,\, A \subseteq B(x, \theta(x))\}$.

Then whenever $0 < \alpha < \alpha^*$, $(\mathbb{R}^r, T_\alpha, \Delta, \mathfrak{R})$ is a tagged-partition structure allowing subdivisions, witnessed by $\mathcal{C}$.

proof (a) We had better look again at all the conditions in 481G.

(i) and (vi) really are trivial. (iv) is true because $\mathcal{C}$ is actually an algebra of sets, and (v) is true because $\mathcal{V} \subseteq \mathcal{C}$ and $\emptyset \in \mathcal{R}_\eta$ for every $\eta \in \mathrm{H}$.

For (ii), we have to observe that the union of two thin sets is thin, so that $\theta \wedge \theta' \in \Theta$ for all θ, $\theta' \in \Theta$; since $\delta_{\theta_1} \cap \delta_{\theta_2} = \delta_{\theta_1 \wedge \theta_2}$, this is all we need.

(iii)(α) follows from 484Ea: given V, $V' \in \mathcal{V}$ and η, $\eta' \in \mathrm{H}$, take $\tilde\eta$, $\tilde\eta' \in \mathrm{H}$ such that $\mathcal{R}_{\tilde\eta} \subseteq \mathcal{R}_\eta^{(V)}$ and $\mathcal{R}_{\tilde\eta'} \subseteq \mathcal{R}_{\eta'}^{(V')}$. Then $V \cup V' \in \mathcal{V}$ (475Cd) and $\mathcal{R}_\eta^{(V)} \cap \mathcal{R}_{\eta'}^{(V')} \supseteq \mathcal{R}_{\tilde\eta\wedge\tilde\eta'}^{(V\cup V'))}$. ($\beta$) is just 484Ec.

(b) Now let us turn to 481G(vii).

(i) Fix $C \in \mathcal{C}$, $\delta \in \Delta$ and $\mathcal{R} \in \mathfrak{R}$. Express δ as δ_θ where $\theta \in \Theta$. Let $\mathcal{R}' \in \mathfrak{R}$ be such that $A \cup A' \in \mathcal{R}$ whenever A, $A' \in \mathcal{R}'$ are disjoint. Express $\mathcal{R}'$ as $\mathcal{R}_\eta^{(V)}$ where $V \in \mathcal{V}$ and $\eta \in \mathrm{H}$. By 475E(b-ii), there is an $\epsilon > 0$ such that $R \in \mathcal{R}'$ whenever $\operatorname{diam} R \le \operatorname{diam} V$, $\operatorname{per} R \le 2\operatorname{per}(C \cap V)$ and $\mu R \le \epsilon$.

By 484B, there is an $E \subseteq C \cap V$ such that $\operatorname{per} E \le \operatorname{per}(C \cap V)$, $\mu((C \cap V) \setminus E) \le \epsilon$ and $\mathrm{cl}^* E = \overline{E}$. In this case, $\operatorname{diam}(C \cap V \setminus E) \le \operatorname{diam} V$ and

$$\operatorname{per}(C \cap V \setminus E) \le \operatorname{per}(C \cap V) + \operatorname{per} E \le 2\operatorname{per}(C \cap V),$$

so $C \cap V \setminus E \in \mathcal{R}'$, by the choice of ϵ. By 475E(a-ii), there is an $\mathcal{R}'' \in \mathfrak{R}$ such that $E \cap R \in \mathcal{R}'$ whenever $R \in \mathcal{R}''$.

Now consider

$$A = \{x : \theta(x) = 0\} \cup \partial^* E \cup \{x : \limsup_{\zeta \downarrow 0} \sup_{x \in G, 0 < \operatorname{diam} G \le \zeta} \tfrac{\nu^*(G \cap \partial^* E)}{(\operatorname{diam} G)^{r-1}} > 0\}.$$

Because $\operatorname{per} E < \infty$ and

$$\{x : x \in \mathbb{R}^r \setminus \partial^* E, \limsup_{\zeta \downarrow 0} \sup_{x \in G, 0 < \operatorname{diam} G \le \zeta} \tfrac{\nu^*(G \cap \partial^* E)}{(\operatorname{diam} G)^{r-1}} > 0\}$$

is ν-negligible (471Pc), A is thin. By 484E(d-ii), there is a set $\mathcal{D}_0 \subseteq \mathcal{D}$ such that

$$A \subseteq \bigcup\{\operatorname{int}(\bigcup \mathcal{D}_1) : \mathcal{D}_1 \in [\mathcal{D}_0]^{<\omega}\}, \quad \bigcup \mathcal{D}_1 \in \mathcal{R}'' \text{ for every } \mathcal{D}_1 \in [\mathcal{D}_0]^\omega.$$

(ii) Write T' for the set of those δ-fine $\boldsymbol{t} \in T_\alpha$ such that every member of $\boldsymbol{t}$ is of the form $(x, D \cap E)$ for some $D \in \mathcal{D}$. If $x \in \operatorname{int}^* E \setminus A$, there is an $h(x) > 0$ such that $h(x) \le \theta(x)$ and $\{(x, D \cap E)\} \in T'$ whenever $D \in \mathcal{D}$, $x \in \overline{D}$ and $\operatorname{diam} D \le h(x)$. **P** Let $\epsilon_1 > 0$ be such that $\beta_r r^{r/2} \epsilon_1 \le \frac{1}{2}$ and

$$\alpha \le \tfrac{1 - r^{r/2}\epsilon_1}{r^{r/2}}, \quad \frac{1}{\alpha} \ge (2r + r^{r/2}\epsilon_1) \cdot \frac{1}{2^{r-1}} \big(\tfrac{\beta_r}{1 - r^{r/2}\epsilon_1}\big)^{(r-1)/r};$$

this is where we need to know that $\alpha < \alpha^*$. Because $x \notin A$, $\theta(x) > 0$; let $h(x) \le \theta(x)$ be such that (α) $\nu^*(D \cap \partial^* E) \le \epsilon_1 (\operatorname{diam} D)^{r-1}$ whenever $D \in \mathcal{D}$ and $x \in \overline{D}$ and $\operatorname{diam} D \le h(x)$ (β) $\mu(B(x,t) \setminus E) \le \epsilon_1 \mu B(x,t)$ whenever $0 \le t \le h(x)$. Now suppose that $D \in \mathcal{D}$ and $x \in \overline{D}$ and $\operatorname{diam} D \le h(x)$. Then, writing γ for the side length of D,

$$\begin{aligned}
\mu(D \cap E) &\ge \mu D - \mu(B(x, \operatorname{diam} D) \setminus E) \ge \gamma^r - \epsilon_1 \beta_r (\operatorname{diam} D)^r \\
&= \gamma^r (1 - r^{r/2}\epsilon_1) \ge \alpha \gamma^r r^{r/2} \\
&= \alpha (\operatorname{diam} D)^r \ge \alpha \operatorname{diam}(D \cap E)^r.
\end{aligned}$$

Since also $\mu(D \cap E) \le \frac{1}{2^r} \beta_r \operatorname{diam}(D \cap E)^r$ (264H), we see that

$$\operatorname{diam}(D \cap E) \ge 2\gamma \big(\tfrac{1 - r^{r/2}\epsilon_1}{\beta_r}\big)^{1/r}.$$

Next,

$$\operatorname{per}(D \cap E) = \nu(\partial^*(D \cap E)) \le \nu(\partial D) + \nu(D \cap \partial^* E)$$

(475Cf)

$$\begin{aligned}
&\le 2r\gamma^{r-1} + \epsilon_1 (\operatorname{diam} D)^{r-1} = \gamma^{r-1}(2r + r^{r/2}\epsilon_1) \\
&\le (2r + r^{r/2}\epsilon_1) \cdot \frac{1}{2^{r-1}} \big(\tfrac{\beta_r}{1 - \epsilon_r r^{r/2}}\big)^{(r-1)/r} \operatorname{diam}(D \cap E)^{r-1} \\
&\le \frac{1}{\alpha} \operatorname{diam}(D \cap E)^{r-1}.
\end{aligned}$$

So $D \cap E \in \mathcal{C}_\alpha$. Also, for every $s > 0$, there is a $D' \in \mathcal{D}$ such that $D' \subseteq D$ and $x \in \overline{D'}$ and $\operatorname{diam} D' \leq s$. In this case

$$\mu(B(x, \operatorname{diam} D') \setminus E) \leq \epsilon_1 \mu B(x, \operatorname{diam} D') = \epsilon_1 \beta_r (\operatorname{diam} D')^r = \epsilon_1 \beta_r r^{r/2} \mu D' \leq \tfrac{1}{2} \mu D',$$

so

$$\begin{aligned} \mu(D \cap E \cap B(x, \operatorname{diam} D')) &\geq \mu D' - \mu(B(x, \operatorname{diam} D') \setminus E) \\ &\geq \tfrac{1}{2} \mu D' = \frac{1}{2\beta_r r^{r/2}} \mu B(x, \operatorname{diam} D'). \end{aligned}$$

As $\operatorname{diam} D'$ is arbitrarily small, $x \in \operatorname{cl}^*(D \cap E)$ and $\boldsymbol{t} = \{(x, D \cap E)\} \in T_\alpha$. Finally, since $\operatorname{diam} D \leq \theta(x)$, $(x, D \cap E) \in \delta$ and $\boldsymbol{t} \in T'$. **Q**

(iii) Let $\mathcal{H}$ be the set of those $H \subseteq \mathbb{R}$ such that $W_{\boldsymbol{t}} \subseteq E \cap H \subseteq W_{\boldsymbol{t}} \cup \bigcup \mathcal{D}_1$ for some δ-fine $\boldsymbol{t} \in T'$ and finite $\mathcal{D}_1 \subseteq \mathcal{D}_0$. Then $H \cup H' \in \mathcal{H}$ whenever H, $H' \in \mathcal{H}$ are disjoint. If $\langle D_n \rangle_{n \in \mathbb{N}}$ is any strictly decreasing sequence in $\mathcal{D}$, then some D_n belongs to $\mathcal{H}$. **P** Let x be the unique point of $\bigcap_{n \in \mathbb{N}} \overline{D_n}$.

case 1 If $x \in A$, then there is a finite subset $\mathcal{D}_1$ of $\mathcal{D}_0$ whose union is a neighbourhood of x, and therefore includes D_n for some n; so $\boldsymbol{t} = \emptyset$ and $\mathcal{D}_1$ witness that $D_n \in \mathcal{H}$.

case 2 If $x \in E \setminus A$, then $x \in \operatorname{int}^* E$, so $h(x) > 0$ and there is some $n \in \mathbb{N}$ such that $\operatorname{diam} D_n \leq h(x)$. In this case $\boldsymbol{t} = \{(x, D_n \cap E)\}$ belongs to T', by the choice of $h(x)$, so that $\boldsymbol{t}$ and $\emptyset$ witness that $D_n \in \mathcal{H}$.

case 3 Finally, if $x \notin E \cup A$, then $x \notin \operatorname{cl}^* E$ so $x \notin \overline{E}$ and there is some n such that $D_n \cap E = \emptyset$, in which case $\boldsymbol{t} = \mathcal{D}_1 = \emptyset$ witness that $D_n \in \mathcal{H}$. **Q**

(iv) In fact $\mathbb{R}^r \in \mathcal{H}$. **P?** Otherwise, because $E \subseteq C$ is bounded, it can be covered by a finite disjoint family in $\mathcal{D}$, and there must be some $D_0 \in \mathcal{D} \setminus \mathcal{H}$. Now we can find $\langle D_n \rangle_{n \geq 1}$ in $\mathcal{D} \setminus \mathcal{H}$ such that $D_n \subseteq D_{n-1}$ and $\operatorname{diam} D_n = \frac{1}{2} \operatorname{diam} D_{n-1}$ for every n. But this contradicts (iii). **XQ**

(v) We therefore have a δ-fine $\boldsymbol{t} \in T'$ and a finite set $\mathcal{D}_1 \subseteq \mathcal{D}_0$ such that $W_{\boldsymbol{t}} \subseteq E \subseteq W_{\boldsymbol{t}} \cup \bigcup \mathcal{D}_1$. Now we can find $\boldsymbol{t}' \subseteq \boldsymbol{t}$ and $\mathcal{D}_1' \subseteq \mathcal{D}_1$ such that $W_{\boldsymbol{t}'} \cap \bigcup \mathcal{D}_1' = \emptyset$ and $W_{\boldsymbol{t}'} \cup \bigcup \mathcal{D}_1' = W_{\boldsymbol{t}'} \cup \bigcup \mathcal{D}_1$.

By the choice of $\mathcal{D}_0$, $\bigcup \mathcal{D}_1 \in \mathcal{R}''$; by the choice of $\mathcal{R}''$, $E \setminus W_{\boldsymbol{t}} = E \cap \bigcup \mathcal{D}_1$ belongs to $\mathcal{R}'$. But we know also that $C \cap V \setminus E \in \mathcal{R}'$, that is, $C \setminus E \in \mathcal{R}'$, because $\mathcal{R}' = \mathcal{R}_\eta^{(V)}$. By the choice of $\mathcal{R}'$, $C \setminus W_{\boldsymbol{t}} \in \mathcal{R}$. And $\boldsymbol{t} \in T'$ is a δ-fine member of T_α. As C, δ and $\mathcal{R}$ are arbitrary, 481G(vii) is satisfied, and the proof is complete.

484G The Pfeffer integral (a) For $\alpha \in]0, \alpha^*[$, write I_α for the linear functional defined by setting

$$I_\alpha(f) = \lim_{\boldsymbol{t} \to \mathcal{F}(T_\alpha, \Delta, \Re)} S_{\boldsymbol{t}}(f, \mu)$$

whenever $f : \mathbb{R}^r \to \mathbb{R}$ is such that the limit is defined. Then if $0 < \beta \leq \alpha$ and $I_\beta(f)$ is defined, so is $I_\alpha(f)$, and the two are equal. **P** All we have to observe is that $\mathcal{C}_\alpha \subseteq \mathcal{C}_\beta$ so that $T_\alpha \subseteq T_\beta$, while $\mathcal{F}(T_\alpha, \Delta, \Re)$ is just $\{A \cap T_\alpha : A \in \mathcal{F}(T_\beta, \Delta, \Re)\}$. **Q**

(b) Let $f : \mathbb{R}^r \to \mathbb{R}$ be a function. I will say that it is **Pfeffer integrable**, with **Pfeffer integral** $\text{⨍}_{\text{f}} f$, if

$$\text{⨍}_{\text{f}} f = \lim_{\alpha \downarrow 0} I_\alpha(f)$$

is defined; that is to say, if $I_\alpha(f)$ is defined whenever $0 < \alpha < \alpha^*$.

484H The first step is to work through the results of §482 to see which ideas apply directly to the limit integral ⨍_{f}.

Proposition (a) The domain of ⨍_{f} is a linear space of functions, and ⨍_{f} is a positive linear functional.

(b) If f, $g : \mathbb{R}^r \to \mathbb{R}$ are such that $|f| \leq g$ and $\text{⨍}_{\text{f}} g = 0$, then $\text{⨍}_{\text{f}} f$ is defined and equal to 0.

(c) If $f : \mathbb{R}^r \to \mathbb{R}$ is Pfeffer integrable, then there is a unique additive functional $F : \mathcal{C} \to \mathbb{R}$ such that whenever $\epsilon > 0$ and $0 < \alpha < \alpha^*$ there are $\delta \in \Delta$ and $\mathcal{R} \in \Re$ such that

$$\textstyle\sum_{(x,C) \in \boldsymbol{t}} |F(C) - f(x)\mu C| \leq \epsilon \text{ for every } \delta\text{-fine } \boldsymbol{t} \in T_\alpha,$$

$$|F(E)| \leq \epsilon \text{ whenever } E \in \mathcal{R}.$$

Moreover, $F(\mathbb{R}^r) = \text{Pff}\, f$.

(d) Every Pfeffer integrable function is Lebesgue measurable.

(e) Every Lebesgue integrable function is Pfeffer integrable, with the same integral.

(f) A non-negative function is Pfeffer integrable iff it is Lebesgue integrable.

proof (a)-(b) Immediate from 481C.

(c) For each $\alpha \in]0, \alpha^*[$ let F_α be the Saks-Henstock indefinite integral corresponding to the the structure $(\mathbb{R}^r, T_\alpha, \Delta, \mathfrak{R}, \mu)$. Then all the F_α coincide. **P** Suppose that $0 < \beta \leq \alpha < \alpha^*$. Then, for any $\epsilon > 0$, there are $\delta \in \Delta$, $\mathcal{R} \in \mathfrak{R}$ such that

$$\textstyle\sum_{(x,C)\in \boldsymbol{t}} |F_\beta(C) - f(x)\mu C| \leq \epsilon \text{ for every } \delta\text{-fine } \boldsymbol{t} \in T_\beta,$$

$$|F_\beta(E)| \leq \epsilon \text{ whenever } E \in \mathcal{R}.$$

Since $T_\alpha \subseteq T_\beta$, this means that

$$\textstyle\sum_{(x,C)\in \boldsymbol{t}} |F_\beta(C) - f(x)\mu C| \leq \epsilon \text{ for every } \delta\text{-fine } \boldsymbol{t} \in T_\alpha.$$

And this works for any $\epsilon > 0$. By the uniqueness assertion in 482B, F_β must be exactly the same as F_α. **Q**

So we have a single functional F; and 482B also tells us that

$$F(\mathbb{R}^r) = I_\alpha(f) = \text{Pff}\, f$$

for every α.

(d) In fact if there is any α such that $I_\alpha(f)$ is defined, f must be Lebesgue measurable. **P** We have only to check that the conditions of 482E are satisfied by μ, $\mathcal{C}$, T_α, Δ and $\mathfrak{R}$. The only items not covered by the checks above are (ii) and (v) of the list there. 482E(ii) is true because $C \setminus \text{cl}^*C$ is negligible for every $C \in \mathcal{C}$ (475C(f)). 482E(v) is true because if $\mu E < \infty$ and $\epsilon > 0$, there are $n \in \mathbb{N}$ and $\eta \in \text{H}$ such that $\mu(E \setminus B(\mathbf{0}, n) \leq \frac{1}{2}\epsilon$ and $\sum_{i=0}^{\infty} \eta(i) \leq \frac{1}{2}\epsilon$, so that

$$\mu(E \cap R) \leq \mu(E \setminus B(\mathbf{0}, n)) + \mu(R \cap B(\mathbf{0}, n)) \leq \epsilon$$

for every $R \in \mathcal{R}_\eta^{(B(\mathbf{0},n))}$. **Q**

(e) This time, we have to check that the conditions of 482F are satisfied by T_α, Δ and $\mathfrak{R}$ whenever $0 < \alpha < \alpha^*$. **P** Of course μ is inner regular with respect to the closed sets and outer regular with respect to the open sets (134F). Condition 482F(v) just repeats 482E(v), verified in (d) above. **Q**

(f) If $f \geq 0$ is integrable in the ordinary sense, then it is Pfeffer integrable, by (e). If it is Pfeffer integrable, then it is measurable; but also $\int g\, d\mu = \text{Pff}\, g \leq \text{Pff}\, f$ for every simple function $g \leq f$, so f is integrable (213B).

484I Definition If $f : \mathbb{R}^r \to \mathbb{R}$ is Pfeffer integrable, I will call the function $F : \mathcal{C} \to \mathbb{R}$ defined in 484Hc the **Saks-Henstock indefinite integral** of f.

484J In fact 484Hc characterizes the Pfeffer integral, just as the Saks-Henstock lemma can be used to define general gauge integrals based on tagged-partition structures allowing subdivisions.

Proposition Suppose that $f : \mathbb{R}^r \to \mathbb{R}$ and $F : \mathcal{C} \to \mathbb{R}$ are such that

(i) F is additive,

(ii) whenever $0 < \alpha < \alpha^*$ and $\epsilon > 0$ there is a $\delta \in \Delta$ such that $\sum_{(x,C)\in \boldsymbol{t}} |F(C) - f(x)\mu C| \leq \epsilon$ for every δ-fine $\boldsymbol{t} \in T_\alpha$,

(iii) for every $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(E)| \leq \epsilon$ for every $E \in \mathcal{C} \cap \mathcal{R}$.

Then f is Pfeffer integrable and F is the Saks-Henstock indefinite integral of f.

proof By 482D, the gauge integral $I_\alpha(f)$ is defined and equal to $F(\mathbb{R}^r)$ for every $\alpha \in]0, \alpha^*[$. So f is Pfeffer integrable. Now 484Hc tells us that F must be its Saks-Henstock indefinite integral.

484K Lemma Suppose that $\alpha > 0$ and $0 < \alpha' < \alpha \min(\frac{1}{2}, 2^{r-1}(\frac{\alpha}{2\beta_r})^{(r-1)/r})$. If $E \in \mathcal{C}$ is such that $E \subseteq \text{cl}^*E$, then there is a $\delta \in \Delta$ such that $\{(x, C \cap E)\} \in T_{\alpha'}$ whenever $(x, C) \in \delta$, $x \in E$ and $\{(x, C)\} \in T_\alpha$.

proof Take $\epsilon > 0$ such that $\beta_r\epsilon \leq \frac{1}{2}\alpha$ and

$$\frac{1}{\alpha} + 2^{r-1}\epsilon \leq \frac{2^{r-1}}{\alpha'}\big(\frac{\alpha}{2\beta_r}\big)^{(r-1)/r}.$$

Set

$$A = \partial^*E \cup \{x : \lim_{\zeta\downarrow 0} \sup_{x\in G, 0<\operatorname{diam} G\leq\zeta} \frac{\nu^*(G\cap\partial^*E)}{(\operatorname{diam} G)^{r-1}} > 0\},$$

so that A is a thin set, as in the proof of 484F. (Of course ∂^*E is thin because $\nu(\partial^*E \cap B(\mathbf{0}, n))$ is finite for every $n \in \mathbb{N}$.)

For $x \in E \setminus A$, we have $x \in \operatorname{int}^*E$ (because $E \subseteq \operatorname{cl}^*E$), so there is a $\theta(x) > 0$ such that

$$\mu(B(x,\gamma) \setminus E) \leq \epsilon\mu B(x,\gamma), \quad \nu(\partial^*E \cap B(x,\gamma)) \leq \epsilon(2\gamma)^{r-1}$$

whenever $0 < \gamma \leq 2\theta(x)$. If we set $\theta(x) = 0$ for $x \in E \cap A$ and $\theta(x) = 1$ for $x \in \mathbb{R}^r \setminus E$, then $\theta \in \Theta$ and $\delta_\theta \in \Delta$.

Now suppose that $x \in E$, $(x, C) \in \delta_\theta$ and $\{(x, C)\} \in T_\alpha$, that is, that $C \in \mathcal{C}_\alpha$ and $x \in (E \cap \operatorname{cl}^*C) \setminus A$ and $C \subseteq B(x, \theta(x))$. Set $\gamma = \operatorname{diam} C \leq 2\theta(x)$. Then

$$\mu(C \setminus E) \leq \mu(B(x,\gamma) \setminus E) \leq \epsilon\mu B(x,\gamma) = \beta_r\epsilon\gamma^r,$$

so

$$\begin{aligned}\mu(C \cap E) &\geq \mu C - \beta_r\epsilon\gamma^r \geq (\alpha - \beta_r\epsilon)\gamma^r \\ &\geq \tfrac{1}{2}\alpha\gamma^r \geq \alpha'\gamma^r \geq \alpha' \operatorname{diam}(C \cap E)^r.\end{aligned}$$

Next,

$$\begin{aligned}\operatorname{per}(C \cap E) &= \nu(\partial^*(C \cap E)) \leq \nu(\partial^*C) + \nu(\operatorname{cl}^*C \cap \partial^*E) \\ &\leq \operatorname{per} C + \nu(B(x,\gamma) \cap \partial^*E) \leq \frac{1}{\alpha}\gamma^{r-1} + \epsilon(2\gamma)^{r-1} = (\tfrac{1}{\alpha} + 2^{r-1}\epsilon)\gamma^{r-1}.\end{aligned}$$

Moreover,

$$2^{-r}\beta_r \operatorname{diam}(C \cap E)^r \geq \mu(C \cap E) \geq \tfrac{1}{2}\alpha\gamma^r$$

(264H), so $\operatorname{diam}(C \cap E) \geq 2(\frac{\alpha}{2\beta_r})^{1/r}\gamma$ and

$$\operatorname{per}(C \cap E) \leq (\tfrac{1}{\alpha} + 2^{r-1}\epsilon) \cdot \frac{1}{2^{r-1}}\big(\frac{2\beta_r}{\alpha}\big)^{(r-1)/r} \operatorname{diam}(C \cap E)^{r-1} \leq \frac{1}{\alpha'} \operatorname{diam}(C \cap E)^{r-1}.$$

Putting these together, we see that $C \in \mathcal{C}_{\alpha'}$.

Finally, because $x \in \operatorname{cl}^*C \cap \operatorname{int}^*E \subseteq \operatorname{cl}^*(C \cap E)$, $\{(x, C \cap E)\} \in T_{\alpha'}$.

484L Proposition Suppose that $f : \mathbb{R}^r \to \mathbb{R}$ is Pfeffer integrable, and that $F : \mathcal{C} \to \mathbb{R}$ is its Saks-Henstock indefinite integral. Then $\int f \times \chi E$ is defined and equal to $F(E)$ for every $E \in \mathcal{C}$.

proof (a) To begin with (down to the end of (d) below), suppose that $E \in \mathcal{C}$ is such that $\operatorname{int}^*E \subseteq E \subseteq \operatorname{cl}^*E$. For $G \in \mathcal{C}$ set $F_1(G) = F(G \cap E)$. I seek to show that F_1 satisfies the conditions of 484J.

Of course $F_1 : \mathcal{C} \to \mathbb{R}$ is additive. If $\epsilon > 0$, there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(G)| \leq \epsilon$ whenever $G \in \mathcal{R}$, by 484H; now there is an $\mathcal{R}' \in \mathfrak{R}$ such that $G \cap E \in \mathcal{R}$ for every $G \in \mathcal{R}'$ (484E(a-ii)), so that $|F_1(G)| \leq \epsilon$ for every $G \in \mathcal{C} \cap \mathcal{R}'$. Thus F_1 satisfies (iii) of 484J.

(b) Take $\alpha \in]0, \alpha^*[$ and $\epsilon > 0$. Take α' such that $0 < \alpha' < \alpha \min(\frac{1}{2}, 2^{r-1}(\frac{\alpha}{2\beta_r})^{(r-1)/r})$. Applying 484K to E and its complement, and appealing to the definition of F, we see that there is a $\delta \in \Delta$ such that

(α) $\{(x, C \cap E)\} \in T_{\alpha'}$ whenever $(x, C) \in \delta$, $x \in E$ and $\{(x, C)\} \in T_\alpha$,

(β) $\{(x, C \setminus E)\} \in T_{\alpha'}$ whenever $(x, C) \in \delta$, $x \in \mathbb{R}^r \setminus E$ and $\{(x, C)\} \in T_\alpha$,

(γ) $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\mu C| \leq \epsilon$ for every δ-fine $\boldsymbol{t} \in T_{\alpha'}$.

(For (α), we need to know that $E \subseteq \text{cl}^*E$ and for (β) we need $\text{int}^*E \subseteq E$.) Next, choose for each $n \in \mathbb{N}$ closed sets $H_n \subseteq E$, $H_n' \subseteq \mathbb{R}^r \setminus E$ such that $\mu(E \setminus H_n) \leq 2^{-n}\epsilon$ and $\mu((\mathbb{R}^r \setminus E) \setminus H_n') \leq 2^{-n}\epsilon$. Define $\theta : \mathbb{R}^r \to]0, \infty[$ by setting

$$\theta(x) = \min(1, \tfrac{1}{2}\rho(x, H_n')) \text{ if } x \in E \text{ and } n \leq |f(x)| < n+1,$$

$$= \min(1, \tfrac{1}{2}\rho(x, H_n)) \text{ if } x \in \mathbb{R}^r \setminus E \text{ and } n \leq |f(x)| < n+1,$$

writing $\rho(x, A) = \inf_{y \in A} \|x - y\|$ if $A \subseteq \mathbb{R}^r$ is non-empty, ∞ if $A = \emptyset$. Then $\delta_\theta \in \Delta$. Let $\mathcal{R}_1 \in \mathfrak{R}$ be such that $A \cap E \in \mathcal{R}$ whenever $A \in \mathcal{R}_1$ (484E(a-ii)).

(c) Write f_E for $f \times \chi E$. Then $\sum_{(x,C) \in \boldsymbol{t}} |F_1(C) - f_E(x)\mu(C)| \leq 11\epsilon$ whenever $\boldsymbol{t} \in T_\alpha$ is $\delta \cap \delta_\theta$-fine. **P** Set

$$\boldsymbol{t}' = \{(x, C \cap E) : (x, C) \in \boldsymbol{t},\, x \in E\}.$$

By clause (α) of the choice of δ, $\boldsymbol{t}' \in T_{\alpha'}$, and of course it is δ-fine. So

$$\textstyle\sum_{(x,C) \in \boldsymbol{t}, x \in E} |F(C \cap E) - f(x)\mu(C \cap E)| \leq \epsilon$$

by clause (γ) of the choice of δ. Next,

$$\sum_{(x,C) \in \boldsymbol{t}, x \in E} |f(x)\mu(C \setminus E)| = \sum_{n=0}^{\infty} \sum_{\substack{(x,C) \in \boldsymbol{t}, x \in E \\ n \leq |f(x)| < n+1}} |f(x)|\mu(C \setminus E)$$

$$\leq \sum_{n=0}^{\infty} (n+1)\mu((\mathbb{R}^r \setminus E) \setminus H_n')$$

(because $\text{diam}\, C \leq \theta(x)$, so $C \cap H_n' = \emptyset$ whenever $(x, C) \in \boldsymbol{t}$, $x \in E$ and $n \leq |f(x)| < n+1$)

$$\leq \sum_{n=0}^{\infty} 2^{-n}(n+1)\epsilon = 4\epsilon,$$

and

$$\textstyle\sum_{(x,C) \in \boldsymbol{t}, x \in E} |F(C \cap E) - f(x)\mu C| \leq 5\epsilon.$$

Similarly,

$$\textstyle\sum_{(x,C) \in \boldsymbol{t}, x \notin E} |F(C \setminus E) - f(x)\mu C)| \leq 5\epsilon.$$

But as

$$\textstyle\sum_{(x,C) \in \boldsymbol{t}, x \notin E} |F(C) - f(x)\mu C| \leq \epsilon$$

(because surely $\boldsymbol{t}$ itself belongs to $T_{\alpha'}$), we have

$$\textstyle\sum_{(x,C) \in \boldsymbol{t}, x \notin E} |F(C \cap E)| \leq 6\epsilon.$$

Putting these together,

$$\sum_{(x,C) \in \boldsymbol{t}} |F_1(C) - f_E(x)\mu C| = \sum_{\substack{(x,C) \in \boldsymbol{t} \\ x \in E}} |F(C \cap E) - f(x)\mu C| + \sum_{\substack{(x,C) \in \boldsymbol{t} \\ x \notin E}} |F(C \cap E)|$$

$$\leq 5\epsilon + 6\epsilon = 11\epsilon. \textbf{ Q}$$

(d) As α and ϵ are arbitrary, condition (iii) of 484J is satisfied by F_1 and f_E, so $\int_{\text{ff}} f_E = F_1(\mathbb{R}^r) = F(E)$.

(e) This completes the proof when $\text{int}^*E \subseteq E \subseteq \text{cl}^*E$. For a general set $E \in \mathcal{C}$, set $E_1 = (E \cup \text{int}^*E) \cap \text{cl}^*E$. Then $E \triangle E_1$ is negligible, so

$$\text{int}^*E_1 = \text{int}^*E \subseteq E_1 \subseteq \text{cl}^*E = \text{cl}^*E_1.$$

Also $\int\!\!\!\!\!\!\int f \times \chi(E \setminus E_1) = \int f \times \chi(E \setminus E_1)d\mu$, $\int\!\!\!\!\!\!\int f \times \chi(E_1 \setminus E) = \int f \times \chi(E \setminus E_1)d\mu$ are both zero, and

$$\int\!\!\!\!\!\!\int f \times \chi E = \int\!\!\!\!\!\!\int f \times \chi E_1 = F(E_1) = F(E).$$

(To see that $F(E_1) = F(E)$, note that $E \setminus E_1$ and $E_1 \setminus E$, being negligible sets, have empty essential boundary and zero perimeter, so belong to every member of $\mathfrak{R}$, by 484E(b-ii), or otherwise.)

484M Lemma Let G, $H \in \mathcal{C}$ be disjoint and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ a continuous function. If *either* $G \cup H$ is bounded *or* ϕ has compact support,

$$\int_{\partial^*(G\cup H)} \phi.\psi_{G\cup H}\, d\nu = \int_{\partial^* G} \phi.\psi_G\, d\nu + \int_{\partial^* H} \phi.\psi_G\, d\nu,$$

where ψ_G, ψ_H and $\psi_{G\cup H}$ are the canonical outward-normal functions (474G).

proof (a) Suppose first that ϕ is a Lipschitz function with compact support. Then 475N tells us that

$$\begin{aligned}\int_{\partial^*(G\cup H)} \phi.\psi_{G\cup H}\, d\nu &= \int_{G\cup H} \operatorname{div}\phi\, d\mu = \int_G \operatorname{div}\phi\, d\mu + \int_H \operatorname{div}\phi\, d\mu \\ &= \int_{\partial^* G} \phi.\psi_G\, d\nu + \int_{\partial^* H} \phi.\psi_G\, d\nu.\end{aligned}$$

(Recall from 474Q that we can identify outward-normal functions with Federer exterior normals, as in the statment of 475N.)

(b) Now suppose that ϕ is a continuous function with compact support. Let $\langle \tilde{h}_n\rangle_{n\in\mathbb{N}}$ be the smoothing sequence of 473E. Then all the functions $\phi * \tilde{h}_n$ are Lipschitz (473Ca), and $\langle \phi * \tilde{h}_n\rangle_{n\in\mathbb{N}}$ converges uniformly to ϕ (473Df, 473Ed). So

$$\begin{aligned}\int_{\partial^*(G\cup H)} \phi.\psi_{G\cup H}\, d\nu &= \lim_{n\to\infty} \int_{\partial^*(G\cup H)} (\phi * \tilde{h}_n).\psi_{G\cup H}\, d\nu \\ &= \lim_{n\to\infty} \int_{\partial^* G} (\phi * \tilde{h}_n).\psi_G\, d\nu + \lim_{n\to\infty} \int_{\partial^* H} (\phi * \tilde{h}_n).\psi_H\, d\nu \\ &= \int_{\partial^* G} \phi.\psi_G\, d\nu + \int_{\partial^* H} \phi.\psi_G\, d\nu.\end{aligned}$$

(c) If, on the other hand, ϕ is continuous and G and H are bounded, then we can find a continuous function $\tilde{\phi}$ with compact support agreeing with ϕ on $\overline{G \cup H}$ (4A2G(e-i), or otherwise); applying (b) to $\tilde{\phi}$, we get the required result for ϕ.

484N Pfeffer's Divergence Theorem Let $E \subseteq \mathbb{R}^r$ be a set with locally finite perimeter, and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ a continuous function with compact support such that $\{x : x \in \mathbb{R}^r, \phi \text{ is not differentiable at } x\}$ is thin. Let v_x be the Federer exterior normal to E at any point x where the normal exists. Then $\int\!\!\!\!\!\!\int \operatorname{div}\phi \times \chi E$ is defined and equal to $\int_{\partial^* E} \phi(x).v_x\nu(dx)$.

proof (a) Let n be such that $\phi(x) = \mathbf{0}$ for $\|x\| \geq n$. For $C \in \mathcal{C}$, set $F(C) = \int_{\partial^* C} \phi.\psi_C\nu(dx)$. By 484M, F is additive.

(b) If $0 < \alpha < \alpha^*$ and $\epsilon > 0$ and $x \in \mathbb{R}^r$ is such that ϕ is differentiable at x, there is a $\gamma > 0$ such that $|F(C) - \operatorname{div}\phi(x)\mu C| \leq \epsilon\mu C$ whenever $C \in \mathcal{C}_\alpha$, $x \in \overline{C}$ and $\operatorname{diam} C \leq \gamma$. **P** Let T be the derivative of ϕ at x. Let $\gamma > 0$ be such that $\|\phi(y) - \phi(x) - T(y-x)\| \leq \alpha^2\epsilon\|y - x\|$ whenever $\|y - x\| \leq \gamma$. Let $\tilde{\phi} : \mathbb{R}^r \to \mathbb{R}^r$ be a Lipschitz function with compact support such that $\tilde{\phi}(y) = \phi(x) + (y - x)$ whenever $\|y - x\| \leq \gamma$ (473Cf). If $C \in \mathcal{C}_\alpha$ has diameter at most γ and $x \in \overline{C}$, then

$$|F(C) - \operatorname{div}\phi(x)\mu C| = |\int_{\partial^* C} \phi.\psi_C\nu(dx) - \int_C \operatorname{div}\tilde{\phi}\, d\mu|$$

(because T is the derivative of $\tilde{\phi}$ everywhere on $B(x,\gamma)$, so $\operatorname{div}\tilde{\phi}(y) = \operatorname{div}\phi(x)$ for every $y \in C$)

$$= |\int_{\partial^* C} \phi - \tilde{\phi}.\psi_C \nu(dx)|$$

(applying the Divergence Theorem to $\tilde{\phi}$)

$$\leq \nu(\partial^* C) \sup_{y \in C} \|\phi(y) - \tilde{\phi}(y)\| \leq \alpha^2 \epsilon \operatorname{diam} C \operatorname{per} C$$
$$\leq \alpha\epsilon(\operatorname{diam} C)^r \leq \epsilon\mu C$$

because $C \in \mathcal{C}_\alpha$. **Q**

(c) If $\epsilon > 0$ and $\alpha \in]0, \alpha^*[$, there is a $\delta \in \Delta$ such that $\sum_{(x,C)\in \boldsymbol{t}} |F(C) - \operatorname{div}\phi(x)\mu C| \leq \epsilon$ whenever $\boldsymbol{t} \in T_\alpha$ is δ-fine. **P** Let $\zeta > 0$ be such that $\zeta\mu B(\mathbf{0}, n+2) \leq \epsilon$. Set $A = \{x : x \in \mathbb{R}^r, \phi \text{ is not differentiable at } x\}$, and for $x \in \mathbb{R}^r \setminus A$, let $\theta(x) \in]0, \frac{1}{2}]$ be such that $|F(C) - \operatorname{div}\phi(x)\mu C| \leq \zeta\mu C$ whenever $C \in \mathcal{C}_\alpha$, $x \in \overline{C}$ and $\operatorname{diam} C \leq \theta(x)$; for $x \in A$ set $\theta(x) = 0$. Now suppose that $\boldsymbol{t} \in T_\alpha$ is δ_θ-fine. Then

$$\sum_{(x,C)\in\boldsymbol{t}} |F(C) - \operatorname{div}\phi(x)\mu C| = \sum_{(x,C)\in\boldsymbol{t}, x\in B(\mathbf{0},n+1)} |F(C) - \operatorname{div}\phi(x)\mu C|$$

(because $\operatorname{diam} C \leq 1$ whenever $(x, C) \in \boldsymbol{t}$, so if $\|x\| > n+1$ then $F(C) = \operatorname{div}\phi(x) = 0$)

$$\leq \sum_{(x,C)\in\boldsymbol{t}, x\in B(\mathbf{0},n+1)} \zeta\mu C \leq \zeta\mu B(\mathbf{0}, n+2) \leq \epsilon. \ \mathbf{Q}$$

(d) Because ϕ is a continuous function with compact support, it is uniformly continuous (apply 4A2Je to each of the coordinates of ϕ). For $\zeta > 0$, let $\gamma(\zeta) > 0$ be such that $\|\phi(x) - \phi(y)\| \leq \zeta$ whenever $\|x - y\| \leq \gamma(\zeta)$.

If $C \in \mathcal{C}$ and $\operatorname{per} C \leq 1$ and $\mu C \leq \zeta\gamma(\zeta)$, where $\zeta > 0$, then $|F(C)| \leq r\zeta(2\|\phi\|_\infty + \frac{1}{2})$, writing $\|\phi\|_\infty$ for $\sup_{x\in\mathbb{R}^r} \|\phi(x)\|$. **P** For $1 \leq i \leq r$, let $\phi_i : \mathbb{R}^r \to \mathbb{R}$ be the ith component of ϕ, and v_i the ith unit vector $(0, \dots, 1, \dots, 0)$; write

$$\beta_i = \textstyle\int_{\partial^* C} \phi_i(x)(v_i | \psi_E(x))\nu(dx),$$

so that $F(C) = \sum_{i=1}^r \beta_i$. I start by examining β_r. By 475O, we have sequences $\langle H_n\rangle_{n\in\mathbb{N}}$, $\langle g_n\rangle_{n\in\mathbb{N}}$ and $\langle g'_n\rangle_{n\in\mathbb{N}}$ such that

(i) for each $n \in \mathbb{N}$, H_n is a Lebesgue measurable subset of $\mathbb{R}^{r-1}$, and g_n, $g'_n : H_n \to [-\infty, \infty]$ are Lebesgue measurable functions such that $g_n(u) < g'_n(u)$ for every $u \in F_n$;

(ii) if m, $n \in \mathbb{N}$ then $g_m(u) \neq g'_n(u)$ for every $u \in H_m \cap H_n$;

(iii) $\sum_{n=0}^\infty \int_{H_n} g'_n - g_n d\mu_{r-1} = \mu C$;

(iv)

$$\beta_r = \textstyle\sum_{n=0}^\infty \int_{H_n} \phi_r(u, g'_n(u)) - \phi_r(u, g_n(u))\mu_{r-1}(du),$$

where we interpret $\phi_r(u, \infty)$ and $\phi_r(u, -\infty)$ as 0 if necessary;

(v) for μ_{r-1}-almost every $u \in \mathbb{R}^{r-1}$,

$$\{t : (u,t) \in \partial^* C\} = \{g_n(u) : n \in \mathbb{N}, u \in F_n, g_n(u) \neq -\infty\} \cup \{g'_n(u) : n \in \mathbb{N}, u \in F_n, g'_n(u) \neq \infty\}.$$

From (iii) we see that g'_n and g_n are both finite almost everywhere on H_n, for every n. Consequently, by (v),

$$2\textstyle\sum_{n=0}^\infty \mu_{r-1}H_n = \int \#(\{t : (u,t) \in \partial^* C\})\mu_{r-1}(du) \leq \nu(\partial^* C) \leq 1$$

by 475H. For each n, set

$$H'_n = \{u : u \in H_n, g'_n(u) - g_n(u) > \gamma(\zeta)\}.$$

Then $\gamma(\zeta)\sum_{n=0}^\infty \mu H'_n \leq \mu C$ so $\sum_{n=0}^\infty \mu H'_n \leq \zeta$ and

$$|\sum_{n=0}^\infty \int_{H'_n} \phi_r(u, g'_n(u)) - \phi_r(u, g_n(u))\mu_{r-1}(du)| \leq 2\|\phi\|_\infty \sum_{n=0}^\infty \mu H'_n \leq 2\zeta\|\phi\|_\infty.$$

On the other hand, for $n \in \mathbb{N}$ and $u \in H_n \setminus H_n'$, $|\phi_r(u, g_n'(u)) - \phi_r(u, g_n(u))| \leq \zeta$, so

$$|\sum_{n=0}^{\infty} \int_{H_n \setminus H_n'} \phi_r(u, g_n'(u)) - \phi_r(u, g_n(u))\mu_{r-1}(du)| \leq \sum_{n=0}^{\infty} \zeta\mu(H_n \setminus H_n') \leq \tfrac{1}{2}\zeta.$$

Putting these together,

$$\beta_r \leq 2\zeta\|\phi\|_\infty + \tfrac{1}{2}\zeta.$$

But of course the same arguments apply to all the β_i, so

$$|F(C)| \leq \textstyle\sum_{i=1}^{r} |\beta_i| \leq r\zeta(2\|\phi\|_\infty + \frac{1}{2}),$$

as claimed. **Q**

(e) If $\epsilon > 0$ there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(C)| \leq \epsilon$ for every $C \in \mathcal{C} \cap \mathcal{R}$. **P** Let $\langle \epsilon_i \rangle_{i \in \mathbb{N}}$ be a sequence of strictly positive real numbers such that $r(2\|\phi\|_\infty + \frac{1}{2}) \sum_{i=0}^{\infty} \epsilon_i \leq \epsilon$. For each $i \in \mathbb{N}$, set $\eta(i) = \epsilon_i \gamma(\epsilon_i) > 0$. Set $V = B(\mathbf{0}, n+1)$, and take any $C \in \mathcal{C} \cap \mathcal{R}_\eta^{(V)}$. Then $F(C \setminus V) = 0$, so $F(C) = F(C \cap V)$, while $C \cap V \in \mathcal{R}_\eta$. Express $C \cap V$ as $\bigcup_{i \leq n} E_i$, where $\langle E_i \rangle_{i \leq n}$ is disjoint, $\operatorname{per} E_i \leq 1$ and $\mu E_i \leq \eta(i)$ for each i. Then $|F(E_i)| \leq r\epsilon_i(2\|\phi\|_\infty + \frac{1}{2})$ for each i, by (d), so

$$|F(C \cap V)| \leq \textstyle\sum_{i=0}^{n} |F(E_i)| \leq \epsilon,$$

as required. **Q**

(f) By 484J, $\operatorname{div} \phi$ is Pfeffer integrable. Moreover, by the uniqueness assertion in 484Hc, its Saks-Henstock indefinite integral is just the function F here. By 484L, $F(E) = \int \operatorname{div} \phi \times \chi E$, as required.

484O Differentiating the indefinite integral: Theorem Let $f : \mathbb{R}^r \to \mathbb{R}$ be a Pfeffer integrable function, and F its Saks-Henstock indefinite integral. Then whenever $0 < \alpha < \alpha^*$,

$$\begin{aligned} f(x) &= \lim_{\zeta \downarrow 0} \sup\{\frac{F(C)}{\mu C} : C \in \mathcal{C}_\alpha,\ x \in \overline{C},\ 0 < \operatorname{diam} C \leq \zeta\} \\ &= \lim_{\zeta \downarrow 0} \inf\{\frac{F(C)}{\mu C} : C \in \mathcal{C}_\alpha,\ x \in \overline{C},\ 0 < \operatorname{diam} C \leq \zeta\} \end{aligned}$$

for μ-almost every $x \in \mathbb{R}^r$.

proof (a) It will be useful to know the following: if $C \in \mathcal{C}_\alpha$, $\operatorname{diam} C > 0$, $x \in \overline{C}$ and $\epsilon > 0$, then for any sufficiently small $\zeta > 0$, $C \cup B(x, \zeta) \in C_{\alpha/2}$ and $|F(C \cup B(x, \zeta)) - F(C)| \leq \epsilon$. **P** Let $\mathcal{R} \in \mathfrak{R}$ be such that $|F(R)| \leq \epsilon$ whenever $R \in \mathcal{C} \cap \mathcal{R}$, let $\mathcal{R}' \in \mathfrak{R}$ be such that $(\mathbb{R}^r \setminus C) \cap R \in \mathcal{R}$ whenever $R' \in \mathcal{R}'$, and let $\eta \in \mathrm{H}$ be such that $\mathcal{R}_\eta \subseteq \mathcal{R}'$. Then for all sufficiently small $\zeta > 0$, we shall have $\operatorname{per} B(x, \zeta) \leq 1$ and $\mu B(x, \zeta) \leq \eta(0)$, so that $B(x, \zeta) \in \mathcal{R}_\eta$ and

$$|F(C \cup B(x, \zeta)) - F(C)| = |F(B(x, \zeta) \setminus C| \leq \epsilon.$$

Next, for all sufficiently small $\zeta > 0$,

$$\begin{aligned} \mu(C \cup B(x, \zeta)) &\geq \mu C \geq \alpha(\operatorname{diam} C)^r \\ &\geq \frac{\alpha}{2}(\zeta + \operatorname{diam} C)^r \geq \frac{\alpha}{2} \operatorname{diam}(C \cup B(x, \zeta))^r \end{aligned}$$

and

$$\begin{aligned} \operatorname{per}(C \cup B(x, \zeta)) &\leq \operatorname{per} C + \operatorname{per} B(x, \zeta) \leq \frac{1}{\alpha}(\operatorname{diam} C)^{r-1} + \operatorname{per} B(x, \zeta) \\ &\leq \frac{2}{\alpha}(\operatorname{diam} C)^{r-1} \leq \frac{2}{\alpha} \operatorname{diam}(C \cup B(x, \zeta))^{r-1}, \end{aligned}$$

so that $C \cup B(x, \zeta) \in \mathcal{C}_{\alpha/2}$. **Q**

(b) For $x \in \mathbb{R}$, set

$$g(x) = \lim_{\zeta \downarrow 0} \sup\{\frac{F(C)}{\mu C} : C \in \mathcal{C}_\alpha,\ x \in \overline{C},\ \operatorname{diam} C \leq \delta\}.$$

? Suppose, if possible, that there are rational numbers $q < q'$ and $n \in \mathbb{N}$ such that $A = \{x : \|x\| \leq n,\ f(x) \leq q < q' < g(x)\}$ is not μ-negligible. Set

$$\epsilon = \frac{(q-q')\alpha}{4 \cdot 5^r \beta_r} \mu^* A > 0.$$

Let $\theta \in \Theta$ be such that

$$\textstyle\sum_{(x,C) \in \boldsymbol{t}} |F(C) - f(x)\mu C| \leq \epsilon$$

for every δ_θ-fine $\boldsymbol{t} \in T_{\alpha/2}$. Let $\mathcal{I}$ be the family of all balls $B(x, \zeta)$ where $x \in A$, $0 < \zeta \leq \theta(x)$ and there is a $C \in \mathcal{C}_\alpha$ such that $x \in \overline{C}$, $\operatorname{diam} C = \zeta$ and $\frac{F(C)}{\mu C} > q'$. Then every member of $A_1 = A \setminus \theta^{-1}[\{0\}]$ is the centre of arbitrarily small members of $\mathcal{I}$, so by 472B there is a countable disjoint family $\mathcal{J}_0 \subseteq \mathcal{I}$ such that

$$\mu(\textstyle\bigcup \mathcal{J}_0) \geq 5^{-r} \mu^* A_1 = 5^{-r} \mu^* A.$$

There is therefore a finite family $\mathcal{J}_1 \subseteq \mathcal{J}_0$ such that $\mu(\bigcup \mathcal{J}_1) > \frac{1}{2 \cdot 5^r} \mu^* A$; enumerate $\mathcal{J}_1$ as $\langle B(x_i, \zeta_i) \rangle_{i \leq n}$ where, for each $i \leq n$, $x_i \in A$, $0 < \zeta_i \leq \theta_i(x)$ and there is a $C_i \in \mathcal{C}_\alpha$ such that $x_i \in \overline{C_i}$, $\operatorname{diam} C_i = \zeta_i$ and $F(C_i) > q' \mu C_i$. By (a), we can enlarge C_i by adding a sufficiently small ball around x_i to form a $C_i' \in \mathcal{C}_{\alpha/2}$ such that $x_i \in \operatorname{int} C_i'$, $C_i' \subseteq B(x_i, \zeta_i)$ and $F(C_i') \geq q' \mu C_i'$.

Consider $\boldsymbol{t} = \{(x_i, C_i') : i \leq n\}$. Then, because the balls $B(x_i, \zeta_i)$ are disjoint, and $x_i \in \operatorname{int} C_i' \subseteq \operatorname{cl}^* C_i'$ for every i, $\boldsymbol{t}$ is a δ_θ-fine member of $T_{\alpha/2}$. So $\sum_{i=0}^n F(C_i') \leq \epsilon + \sum_{i=0}^n f(x_i) \mu C_i'$. But as $F(C_i') \geq q' \mu C_i'$ and $f(x_i) \leq q$ for every i, this means that $(q' - q) \sum_{i=0}^n \mu C_i' \leq \epsilon$.

But now remember that $\operatorname{diam} C_i' \geq \operatorname{diam} C_i = \zeta_i$ and that $C_i' \in \mathcal{C}_{\alpha/2}$ for each i. This means that

$$\mu C_i' \geq \frac{\alpha}{2} \zeta_i^r \geq \frac{\alpha}{2\beta_r} \mu B(x_i, \zeta_i)$$

for each i, and

$$\begin{aligned}
\epsilon &\geq (q - q') \sum_{i=0}^n \mu C_i' \\
&\geq \frac{(q-q')\alpha}{2\beta_r} \sum_{i=0}^n \mu B(x_i, \zeta_i) \\
&> \frac{(q-q')\alpha}{4 \cdot 5^r \beta_r} \mu^* A = \epsilon
\end{aligned}$$

which is absurd. **X**

(c) Since q, q' and n are arbitrary, this means that $g \leq_{\text{a.e.}} f$. Similarly (or applying (b) to $-f$ and $-F$)

$$f(x) \leq \lim_{\zeta \downarrow 0} \inf\{\frac{F(C)}{\mu C} : C \in \mathcal{C}_\alpha,\ x \in \overline{C},\ 0 < \operatorname{diam} C \leq \zeta\}$$

for almost all x, as required.

484P Lemma Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be an injective Lipschitz function, and H the set of points at which it is differentiable; for $x \in H$, write $T(x)$ for the derivative of ϕ at x and $J(x)$ for $|\det T(x)|$. Then, for μ-almost every $x \in \mathbb{R}^r$,

$$\begin{aligned}
J(x) &= \lim_{\zeta \downarrow 0} \sup\{\frac{\mu\phi[C]}{\mu C} : C \in \mathcal{C}_\alpha,\ x \in \overline{C},\ 0 < \operatorname{diam} C \leq \zeta\} \\
&= \lim_{\zeta \downarrow 0} \inf\{\frac{\mu\phi[C]}{\mu C} : C \in \mathcal{C}_\alpha,\ x \in \overline{C},\ 0 < \operatorname{diam} C \leq \zeta\}
\end{aligned}$$

for every $\alpha > 0$.

proof By Rademacher's theorem (262Q), H is conegligible. Let H' be the Lebesgue set of J, so that H' is also conegligible (261E). Take any $x \in H'$ and $\epsilon > 0$. Then there is a $\zeta_0 > 0$ such that $\int_{B(x,\zeta)} |J(y) - J(x)|\mu(dy) \leq \epsilon\mu B(x,\zeta)$ for every $\zeta \in [0,\zeta_0]$. Now suppose that $C \in \mathcal{C}_\alpha$, $x \in \overline{C}$ and $0 < \operatorname{diam} C \leq \zeta_0$. Then $\mu(C \setminus H) = 0$ so $\mu\phi[C \setminus H] = 0$ (262D), and

$$\mu\phi[C] = \mu\phi[C \cap H] = \int_{C\cap H} J\,d\mu$$

(263D(iv)). So

$$\begin{aligned}|\mu\phi[C] - J(x)\mu C| &= |\int_{C\setminus H} J\,d\mu - J(x)\mu(C\cap H)| \leq \int_{C\setminus H} |J(y)-J(x)|d\mu\\ &\leq \int_{B(x,\operatorname{diam} C)} |J(y)-J(x)|d\mu \leq \epsilon\mu B(x,\operatorname{diam} C)\\ &= \beta_r\epsilon(\operatorname{diam} C)^r \leq \frac{\beta_r\epsilon}{\alpha}\mu C.\end{aligned}$$

Thus $|\frac{\mu\phi[C]}{\mu C} - J(x)| \leq \frac{\beta_r}{\alpha}\epsilon$ whenever $C \in \mathcal{C}_\alpha$, $x \in \overline{C}$ and $0 < \operatorname{diam} C \leq \zeta_0$; as ϵ is arbitrary,

$$\begin{aligned}J(x) &= \lim_{\zeta\downarrow 0} \sup\{\tfrac{\mu\phi[C]}{\mu C} : C \in \mathcal{C}_\alpha,\ x \in \overline{C},\ 0 < \operatorname{diam} C \leq \zeta\}\\ &= \lim_{\zeta\downarrow 0} \inf\{\tfrac{\mu\phi[C]}{\mu C} : C \in \mathcal{C}_\alpha,\ x \in \overline{C},\ 0 < \operatorname{diam} C \leq \zeta\}.\end{aligned}$$

And this is true for μ-almost every x.

484Q Definition If (X,ρ) and (Y,σ) are metric spaces, a function $\phi : X \to Y$ is a **lipeomorphism** if it is bijective and both ϕ and ϕ^{-1} are Lipschitz. Of course a lipeomorphism is a homeomorphism.

484R Lemma Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a lipeomorphism.

(a) For any set $A \subseteq \mathbb{R}^r$,

$$\text{cl}^*(\phi[A]) = \phi[\text{cl}^*A], \quad \text{int}^*(\phi[A]) = \phi[\text{int}^*A], \quad \partial^*(\phi[A]) = \phi[\partial^*A].$$

(b) $\phi[C] \in \mathcal{C}$ for every $C \in \mathcal{C}$, and $\phi[V] \in \mathcal{V}$ for every $V \in \mathcal{V}$.

(c) For any $\alpha > 0$ there is an $\alpha' \in]0,\alpha]$ such that $\phi[C] \in \mathcal{C}_{\alpha'}$ for every $C \in \mathcal{C}_\alpha$ and $\{(\phi(x),\phi[C]) : (x,C) \in \boldsymbol{t}\}$ belongs to $T_{\alpha'}$ for every $\boldsymbol{t} \in T_\alpha$.

(d) For any $\mathcal{R} \in \mathfrak{R}$ there is an $\mathcal{R}' \in \mathfrak{R}$ such that $\phi[R] \in \mathcal{R}$ for every $R \in \mathcal{R}'$.

(e) $\theta\phi : \mathbb{R}^r \to [0,\infty[$ belongs to Θ for every $\theta \in \Theta$.

proof Let γ be so large that it is a Lipschitz constant for both ϕ and ϕ^{-1}. Observe that in this case

$$\phi^{-1}[B(\phi(x),\tfrac{\zeta}{\gamma})] \subseteq B(x,\zeta), \quad \phi[B(x,\zeta)] \supseteq B(\phi(x),\tfrac{\zeta}{\gamma})$$

for every $x \in \mathbb{R}^r$, $\zeta \geq 0$.

(a) If $A \subseteq \mathbb{R}^r$ and $x \in \text{cl}^*A$, set

$$\epsilon = \frac{1}{2}\limsup_{\zeta\downarrow 0} \frac{\mu^*(B(x,\zeta)\cap A}{\mu B(x,\zeta)} > 0.$$

Take any $\zeta_0 > 0$. Then there is a ζ such that $0 < zeta \leq \zeta_0$ and $\mu^*(B(x,\frac{\zeta}{\gamma}) \cap A)] \geq \epsilon\mu B(x,\frac{\zeta}{\gamma})$, so that

$$\mu^*(B(\phi(x),\zeta) \cap \phi[A]) \geq \mu^*\phi[B(x,\tfrac{\zeta}{\gamma}) \cap A] \geq \epsilon\mu B(x,\tfrac{\zeta}{\gamma}) = \frac{\epsilon}{\gamma^r}\mu B(\phi(x),\zeta).$$

As ζ_0 is arbitrary,

$$\limsup_{\zeta\downarrow 0} \frac{\mu^*(B(\phi(x),\zeta)\cap\phi[A]}{\mu B(\phi(x),\zeta)} \geq \frac{\epsilon}{\gamma^r} > 0,$$

and $\phi(x) \in \text{cl}^*(\phi[A])$.

This shows that $\phi[\text{cl}^*A] \subseteq \text{cl}^*(\phi[A])$. The same argument applies to ϕ^{-1} and $\phi[A]$, so that $\phi[\text{cl}^*A]$ must be equal to $\text{cl}^*(\phi[A])$. Taking complements, $\phi[\text{int}^*A] = \text{int}^*(\phi[A])$, so that $\phi[\partial^*A] = \partial^*(\phi[A])$.

(b) Take $C \in \mathcal{C}$. Then, for any $n \in \mathbb{N}$, $\phi^{-1}[B(\mathbf{0}, n)]$ is bounded, so is included in $B(\mathbf{0}, m)$ for some m. Now

$$\begin{aligned}\nu(\partial^*\phi[C] \cap B(\mathbf{0}, n)) &= \nu(\phi[\partial^*C] \cap B(\mathbf{0}, n)) \\ &\leq \nu(\phi[\partial^*C \cap B(\mathbf{0}, m)]) \leq \gamma^{r-1}\nu(\partial^*C \cap B(\mathbf{0}, m))\end{aligned}$$

(264G/471J), so is finite. This shows that $\phi[C]$ has locally finite perimeter and belongs to $\mathcal{C}$. Since $\phi[V]$ is bounded whenever V is bounded, $\phi[V] \in \mathcal{V}$ whenever $V \in \mathcal{V}$.

(c) Set $\alpha' = \gamma^{-2r}\alpha$. Note that as γ^2 is a Lipschitz constant for the identity map, $\gamma \geq 1$, and $\gamma^{2-2r}\alpha \leq \alpha' \leq \alpha$. If $C \in \mathcal{C}_\alpha$, then

$$\begin{aligned}\mu\phi[C] &\geq \frac{1}{\gamma^r}\mu C \geq \frac{\alpha}{\gamma^r}(\operatorname{diam} C)^r \\ &\geq \frac{1}{\alpha}(\frac{1}{\gamma}\operatorname{diam}\phi[C])^r \geq \alpha'(\operatorname{diam}\phi[C])^r,\end{aligned}$$

$$\begin{aligned}\operatorname{per}\phi[C] &= \nu(\partial^*(\phi[C])) = \nu(\phi[\partial^*C]) \leq \gamma^{r-1}\nu(\partial^*C) \\ &\leq \frac{\gamma^{r-1}}{\alpha}(\operatorname{diam} C)^{r-1} \leq \frac{\gamma^{r-1}}{\alpha}(\gamma\operatorname{diam}\phi[C])^{r-1} \leq \frac{1}{\alpha'}(\operatorname{diam}\phi[C])^{r-1}.\end{aligned}$$

So $C \in \mathcal{C}_{\alpha'}$.

If now $\boldsymbol{t} \in T_\alpha$, then, for any $(x, C) \in \boldsymbol{t}$, $\phi[C] \in \mathcal{C}_{\alpha'}$ and $\phi(x) \in \text{cl}^*\phi[C]$; also, because ϕ is injective, $\langle\phi[C]\rangle_{(x,C)\in\boldsymbol{t}}$ is disjoint, so $\{(\phi(x), \phi[C]) : (x, C) \in \boldsymbol{t}\} \in T_{\alpha'}$.

(d) Express $\mathcal{R}$ as $\mathcal{R}_\eta^{(V)}$ where $V \in \mathcal{V}$ and $\eta \in \text{H}$, so that $R \in \mathcal{R}$ whenever $R \cap V \in \mathcal{R}$. By 484Ec and 481He, there is a sequence $\langle\mathcal{Q}_i\rangle_{i\in\mathbb{N}}$ in $\mathfrak{R}$ such that $\bigcup_{i\leq n} A_i \in \mathcal{R}$ whenever $n \in \mathbb{N}$, $\langle A_i\rangle_{i\leq n}$ is disjoint and $A_i \in \mathcal{Q}_i$ for every i. By 484E(b-ii), there is an $\eta' \in \text{H}$ such that $R \in \mathcal{Q}_i$ whenever $i \in \mathbb{N}$ and R is such that $\mu R \leq \gamma^r\eta'(i)$ and $\operatorname{per} R \leq \gamma^{r-1}$. Try $\mathcal{R}' = \mathcal{R}_{\eta'}^{(\phi^{-1}[V])} \in \mathfrak{R}$. If $R \in \mathcal{R}'$, we can express $R \cap \phi^{-1}[V]$ as $\bigcup_{i\leq n} E_i$ where $\operatorname{per} E_i \leq 1$ and $\mu E_i \leq \eta'_i$ for each $i \leq n$, and $\langle E_i\rangle_{i\leq n}$ is disjoint. So $\phi[R] \cap V = \bigcup_{i\leq n}\phi[E_i]$ and $\langle\phi[E_i]\rangle_{i\leq n}$ is disjoint. Now, for each i,

$$\mu\phi[E_i] \leq \gamma^r\mu E_i \leq \gamma^r\eta'(i), \quad \operatorname{per}\phi[E_i] \leq \gamma^{r-1}\operatorname{per} E_i \leq \gamma^{r-1},$$

so $\phi[E_i] \in \mathcal{Q}_i$. By the choice of $\langle\mathcal{Q}_i\rangle_{i\in\mathbb{N}}$, $\phi[R] \cap V \in \mathcal{R}$ and $\phi[R] \in \mathcal{R}$. So $\mathcal{R}'$ has the property we need.

(e) We have only to observe that if A is the thin set $\theta^{-1}[\{0\}]$, then $(\theta\phi)^{-1}[\{0\}] = \phi^{-1}[A]$ is also thin. **P** If $A = \bigcup_{n\in\mathbb{N}} A_n$ where ν^*A_n is finite for every n, then $\phi^{-1}[A] = \bigcup_{n\in\mathbb{N}}\phi^{-1}[A_n]$, while $\nu^*\phi^{-1}[A_n] \leq \gamma^{r-1}\nu^*A_n$ (264G/471J) is finite for every $n \in \mathbb{N}$. **Q**

484S Theorem Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be a lipeomorphism. Let H be the set of points at which ϕ is differentiable. For $x \in H$, write $T(x)$ for the derivative of ϕ at x; set $J(x) = |\det T(x)|$ for $x \in H$, 0 for $x \in \mathbb{R}^r \setminus H$. Then, for any function $f : \mathbb{R}^r \to \mathbb{R}^r$,

$$\int f = \int J \times f\phi$$

if either is defined in $\mathbb{R}$.

proof (a) Let $H' \subseteq H$ be a conegligible set such that

$$\begin{aligned}J(x) &= \lim_{\zeta\downarrow 0}\sup\{\frac{\mu\phi[C]}{\mu C} : C \in \mathcal{C}_\alpha,\ x \in \overline{C},\ 0 < \operatorname{diam} C \leq \zeta\} \\ &= \lim_{\zeta\downarrow 0}\inf\{\frac{\mu\phi[C]}{\mu C} : C \in \mathcal{C}_\alpha,\ x \in \overline{C},\ 0 < \operatorname{diam} C \leq \zeta\}\end{aligned}$$

for every $\alpha > 0$ and every $x \in H'$ (484P). To begin with (down to the end of (c)), suppose that f is Pfeffer integrable and that $f\phi(x) = 0$ for every $x \in \mathbb{R}^r \setminus H'$. Let F be the Saks-Henstock indefinite integral of f, and define $G : \mathcal{C} \to \mathbb{R}$ by setting $G(C) = F(\phi[C])$ for every $C \in \mathcal{C}$ (using 484Rb to see that this is well-defined).

(b) G and $J \times f\phi$ satisfy the conditions of 484J.

P (i) Of course G is additive, because F is.

(ii) Suppose that $0 < \alpha < \alpha^*$ and $\epsilon > 0$. Let $\alpha' \in]0, \alpha^*[$ be such that $\{(\phi(x), \phi[C]) : (x, C) \in \boldsymbol{t}\} \in T_{\alpha'}$ whenever $\boldsymbol{t} \in T_\alpha$ (484Rc). Let $\theta_1 \in \Theta$ be such that $\sum_{(x,C)\in\boldsymbol{t}} |F(C) - f(x)\mu C| \leq \frac{1}{2}\epsilon$ for every δ_{θ_1}-fine $\boldsymbol{t} \in T_{\alpha'}$. Let $\theta_2 : \mathbb{R}^r \to]0, 1]$ be such that whenever $x \in H'$ and $n \leq \|x\| + |f\phi(x)| < n + 1$ then

$$|\mu\phi[C] - J(x)\mu C| \leq \frac{\mu C}{2^{n+3}\beta_r(n+2)^r(n+1)}$$

whenever $C \in \mathcal{C}_{\alpha'}$, $x \in \overline{C}$ and $\operatorname{diam} C \leq 2\theta_2(x)$. Set $\theta(x) = \min(\frac{1}{\gamma}\theta_1\phi(x), \theta_2(x))$ for $x \in \mathbb{R}^r$, so that $\theta \in \Theta$ (484Re).

If $\boldsymbol{t} \in T_\alpha$ is δ_θ-fine, set $\boldsymbol{t}' = \{(\phi(x), \phi[C]) : (x, C) \in \boldsymbol{t}\}$. Then $\boldsymbol{t} \in T_{\alpha'}$, by the choice of α'. If $(x, C) \in \boldsymbol{t}$, then $\theta(x) > 0$ so $\theta_1\phi(x) > 0$; also $C \subseteq B(x, \theta(x))$ so

$$\phi[C] \subseteq B(\phi(x), \gamma\theta(x)) \subseteq B(\phi(x), \theta_1\phi(x)).$$

This shows that $\boldsymbol{t}'$ is δ_{θ_1}-fine. We therefore have

$$\begin{aligned}
\sum_{(x,C)\in\boldsymbol{t}} |G(C) - J(x)f(\phi(x))\mu C| &\leq \sum_{(x,C)\in\boldsymbol{t}} |F(\phi[C]) - f(\phi(x))\mu\phi[C]| \\
&\qquad + \sum_{(x,C)\in\boldsymbol{t}} |f(\phi(x))||\mu\phi[C] - J(x)\mu C| \\
&\leq \sum_{(x,C)\in\boldsymbol{t}'} |F(C) - f(x)\mu C| \\
&\qquad + \sum_{\substack{(x,C)\in\boldsymbol{t} \\ x\in H'}} |f(\phi(x))||\mu\phi[C] - J(x)\mu C|
\end{aligned}$$

(because if $x \notin H'$ then $f(\phi(x)) = 0$)

$$\begin{aligned}
&\leq \tfrac{1}{2}\epsilon + \sum_{n=0}^{\infty} \sum_{\substack{(x,C)\in\boldsymbol{t}, x\in H' \\ n\leq\|x\|+|f(\phi(x))|<n+1}} \frac{(n+1)\mu C}{2^{n+3}\beta_r(n+2)^r(n+1)} \\
&\leq \tfrac{1}{2}\epsilon + \sum_{n=0}^{\infty} \frac{(n+1)\mu B(\mathbf{0}, n+2)}{2^{n+3}\beta_r(n+2)^r(n+1)}
\end{aligned}$$

(remembering that $\theta_2(x) \leq 1$, so $C \subseteq B(\mathbf{0}, n+2)$ whenever $(x, C) \in \boldsymbol{t}$ and $\|x\| < n + 1$)

$$= \epsilon.$$

As $\boldsymbol{t}$ is arbitrary, this shows that G and $J \times f\phi$ satisfy (ii) of 484J.

(iii) Given $\epsilon > 0$, there is an $\mathcal{R} \in \mathfrak{R}$ such that $|F(C)| \leq \epsilon$ for every $C \in \mathcal{C} \cap \mathcal{R}$. Now by 484Rd there is an $\mathcal{R}' \in \mathfrak{R}$ such that $\phi[C] \in \mathcal{R}$ for every $R \in \mathcal{R}'$, so that $|G(C)| \leq \epsilon$ for every $C \in \mathcal{C} \cap \mathcal{R}'$. Thus G satisfies (iii) of 484J. **Q**

(c) This shows that $J \times f\phi$ is Pfeffer integrable, with Saks-Henstock indefinite integral G; so, in particular,

$$\text{Pf} \int f \times \phi = G(\mathbb{R}^r) = F(\mathbb{R}^r) = \text{Pf} \int f.$$

(d) Now suppose that f is an arbitrary Pfeffer integrable function. In this case set $f_1 = f \times \chi\phi[H']$. Because $\mathbb{R}^r \setminus H'$ is μ-negligible, so is $\phi[\mathbb{R}^r \setminus H']$, and $f_1 = f$ μ-a.e. Also, of course, $f\phi = f_1\phi$ μ-a.e. Because the Pfeffer integral extends the Lebesgue integral (484He),

$$\text{⨍}\!\!\!\!\!\!\int J \times f\phi = \text{⨍}\!\!\!\!\!\!\int J \times f_1\phi = \text{⨍}\!\!\!\!\!\!\int f_1 = \text{⨍}\!\!\!\!\!\!\int f.$$

(e) All this has been on the assumption that f is Pfeffer integrable. If $g = J \times f\phi$ is Pfeffer integrable, consider $\tilde{J} \times g\phi^{-1}$, where $\tilde{J}(x) = |\det \tilde{T}(x)|$ whenever the derivative $\tilde{T}(x)$ of ϕ^{-1} at x is defined, and otherwise is zero. Now

$$\tilde{J}(x)g(\phi^{-1}(x)) = \tilde{J}(x) \cdot J(\phi^{-1}(x))f(x)$$

for every x. But, for μ-almost every x,

$$\tilde{J}(x)J(\phi^{-1}(x)) = |\det \tilde{T}(x)||\det T(\phi^{-1}(x))| = |\det \tilde{T}(x)T(\phi^{-1}(x))| = 1$$

because $\tilde{T}(x)T(\phi^{-1}(x))$ is (whenever it is defined) the derivative at $\phi^{-1}(x)$ of the identity function $\phi^{-1}\phi$, by 473Bc. (I see that we need to know that $\{x : \phi \text{ is differentiable at } \phi^{-1}(x)\} = \phi[H]$ is conegligible.) So $\tilde{J} \times g\phi^{-1} = f$ μ-a.e., and f is Pfeffer integrable. This completes the proof.

484X Basic exercises (a) Show that for every $\mathcal{R} \in \mathfrak{R}$ there are $\eta \in \mathrm{H}$ and $n \in \mathbb{N}$ such that $\mathcal{R}_\eta^{(B(\mathbf{0},n))} \subseteq \mathcal{R}$.

>(b) (Pfeffer 91a) For $\alpha > 0$ let $\mathcal{C}'_\alpha$ be the family of bounded Lebesgue measurable sets C such that $\mu C \geq \alpha \operatorname{diam} C \operatorname{per} C$, and T'_α the straightforward set of tagged partitions generated by $\{(x, C) : C \in \mathcal{C}'_\alpha, x \in \mathrm{cl}^* C\}$. (i) Show that if $\alpha' = \min(\alpha^r, 2^{-r}\beta_r/\alpha)$ then $\mathcal{C}_{\alpha^2} \subseteq \mathcal{C}'_\alpha \subseteq \mathcal{C}_\alpha$. (*Hint*: 474La.) (ii) Show that T'_α is compatible with Δ and $\mathfrak{R}$ whenever $0 < \alpha < 1/2r\sqrt{r}$.

>(c) For $\alpha > 0$, let $\mathcal{C}''_\alpha$ be the family of convex sets C such that $\mu C \geq \alpha(\operatorname{diam} C)^r$. Show that $\mathcal{C}''_\alpha \subseteq \mathcal{C}_\alpha$. (*Hint*: 475T.) Show that if $0 < \alpha < \dfrac{2^{r-1}}{r\beta_r}$ and $T''_\alpha = T_\alpha \cap [\mathbb{R}^r \times \mathcal{C}'_\alpha]^{<\omega}$ then T''_α is compatible with Δ and $\mathfrak{R}$. (*Hint*: use the argument of 484F, but in part (b) take $C \in \mathcal{D}$.)

(d) Describe a suitable filter $\mathcal{F}$ to express the Pfeffer integral directly in the form considered in 481C.

(e) Let $f : \mathbb{R}^r \to \mathbb{R}$ be a Pfeffer integrable function. Show that there is some $n \in \mathbb{N}$ such that $\int_{\mathbb{R}^r \setminus B(\mathbf{0},n)} |f| d\mu$ is finite.

(f) Let $E \subseteq \mathbb{R}^r$ be a bounded set with finite perimeter, and $\phi : \mathbb{R}^r \to \mathbb{R}^r$ a differentiable function. Let v_x be the Federer exterior normal to E at any point x where the normal exists. Show that $\text{⨍}\!\!\!\!\!\!\int \operatorname{div} \phi \times \chi E$ is defined and equal to $\int_{\partial^* E} \phi(x) \cdot v_x \nu(dx)$.

484Y Further exercises (a) Let $E \subseteq \mathbb{R}^r$ be any Lebesgue measurable set, and $\epsilon > 0$. Show that there is a Lebesgue measurable set $G \subseteq E$ such that $\operatorname{per} G \leq \operatorname{per} E$, $\mu(E \setminus G) \leq \epsilon$ and $\mathrm{cl}^* G = \overline{G}$.

(b) Give an example of a compact set $K \subseteq \mathbb{R}^2$ with zero one-dimensional Hausdorff measure such that whenever $\theta : K \to]0, \infty[$ is a strictly positive function, and $\gamma \in \mathbb{R}$, there is a disjoint family $\langle B(x_i, \zeta_i)\rangle_{i \leq n}$ of balls such that $x_i \in K$ and $\zeta_i \leq \theta(x_i)$ for every i, while $\operatorname{per}(\bigcup_{i \leq n} B(x_i, \zeta_i)) \geq \gamma$.

484 Notes and comments Observe that 484B can be thought of as a refinement of 475I. As usual, the elaborate formula in the statement of 484C is there only to emphasize that we have a constant depending only on r.

The whole point of 'gauge integrals' is that we have an enormous amount of freedom within the framework of §§481-482. There is a corresponding difficulty in making exact choices. The essential ideology of the 'Pfeffer integral' is that we take an intersection of a family of gauge integrals, each determined by a family $\mathcal{C}_\alpha$ of sets which are 'Saks regular' in the sense that their measures, perimeters and diameters are linked. Shrinking $\mathcal{C}_\alpha$ and T_α, while leaving Δ and $\mathfrak{R}$ unchanged, of course leads to a more 'powerful' integral (supposing, at least, that we do not go so far that T_α is no longer compatible with Δ and $\mathfrak{R}$), so that Pfeffer's Divergence Theorem will remain true. One possibility is to turn to convex sets (484Xc).

Similar technical questions arise when choosing the family $\mathfrak{R}$ of residual classes. Here I use the construction $\mathcal{R}_\eta^{(V)} = \{R : R \cap V \in \mathcal{R}_\eta\}$ (484D) as a quick method of eliminating any difficulties at infinity (484Xe).

We do not of course need to look at arbitrary sets $V \in \mathcal{V}$ here (484Xa). One of the important simplifications in the one-dimensional theory is that we can work with a very much simpler family of residual classes; this is because 'thin' sets become countable (see 484Xf), and can be controlled by neighbourhood gauges. I give 484Yb as a pointer to the problems which arise if we do not exclude thin sets by the device of allowing the gauge functions θ to be zero on such sets, thereby removing some potential tags.

The many equivalent formulations of the Denjoy-Perron-Henstock integral challenge us to seek corresponding descriptions of integrals on $\mathbb{R}^r$. 484O shows that we can express a Pfeffer integrable function as the derivative almost everywhere (in a strong sense) of its Saks-Henstock indefinite integral.

I include 484S as a worked example on the techniques already developed. Note that it depends much more on the fact that the Pfeffer integral can be characterized in the language of 475J, than on the exact choices made in forming $\mathfrak{R}$ and the $\mathcal{C}_\alpha$. (Of course if we restricted the C_α to the point that they contained only convex sets, we could not expect invariance under lipeomorphisms.) For a discussion of integrals *defined* by Saks-Henstock lemmas, see Pfeffer 01.

Chapter 49

Further topics

I conclude the volume with notes on five almost unconnected special topics. In §491 I look at equidistributed sequences and the ideal $\mathcal{Z}$ of sets with asymptotic density zero. I give the principal theorems on the existence of equidistributed sequences in abstract topological measure spaces, and examine the way in which an equidistributed sequence induces an embedding of a measure algebra in the quotient algebra $\mathcal{P}\mathbb{N}/\mathcal{Z}$. In §492 I present some forms of 'concentration of measure' which echo ideas from §476 in combinatorial, rather than geometric, contexts, with theorems of Talagrand and Maurey on product measures and the Haar measure of a permutation group. In §493 I show how the ideas of §§449, 476 and 492 can be put together in the theory of 'extremely amenable' topological groups. In §494, I give a pair of simple, but surprising, results on subsets of sets of positive measure in product spaces. Finally, in §495, I introduce Poisson point processes, with notes on disintegrations and some special cases in which they can be represented by Radon measures.

491 Equidistributed sequences

In many of the most important topological measure spaces, starting with Lebesgue measure (491Xf), there are sequences which are equidistributed in the sense that, in the limit, they spend the right proportion of their time in each part of the space (491Ye). I give the basic results on existence of equidistributed sequences in 491E-491G and 491Q. Investigating such sequences, we are led to some interesting properties of the asymptotic density ideal $\mathcal{Z}$ and the quotient algebra $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ (491A, 491I-491K). For 'effectively regular' measures (491L-491M), equidistributed sequences lead to embeddings of measure algebras in $\mathfrak{Z}$ (491N).

491A The asymptotic density ideal (a) If $I \subseteq \mathbb{N}$, its **upper asymptotic density** is $d^*(I) = \limsup_{n\to\infty} \frac{1}{n}(I \cap n)$, and its **asymptotic density** is $d(I) = \lim_{n\to\infty} \frac{1}{n}\#(I \cap n)$ if this is defined. It is easy to check that d^* is a submeasure on $\mathcal{P}\mathbb{N}$ (definition: 392A), so that

$$\mathcal{Z} = \{I : I \subseteq \mathbb{N},\, d^*(I) = 0\} = \{I : I \subseteq \mathbb{N},\, d(I) = 0\}$$

is an ideal, the **asymptotic density ideal**.

(b) Note that

$$\mathcal{Z} = \{I : I \subseteq \mathbb{N},\, \lim_{n\to\infty} 2^{-n}\#(I \cap 2^{n+1} \setminus 2^n) = 0\}.$$

P If $I \subseteq \mathbb{N}$ and $d^*(I) = 0$, then

$$2^{-n}\#(I \cap 2^{n+1} \setminus 2^n) \leq 2 \cdot 2^{-n-1}\#(I \cap 2^{n+1}) \to 0$$

as $n \to \infty$. In the other direction, if $\lim_{n\to\infty} 2^{-n}\#(I \cap 2^{n+1} \setminus 2^n) = 0$, then for any $\epsilon > 0$ there is an $m \in \mathbb{N}$ such that $\#(I \cap 2^{k+1} \setminus 2^k) \leq 2^k\epsilon$ for every $k \geq m$. In this case, for $n \geq 2^m$, take k_n such that $2^{k_n} \leq n < 2^{k_n+1}$, and see that

$$\frac{1}{n}\#(I \cap n) \leq 2^{-k_n}(\#(I \cap 2^m) + \textstyle\sum_{k=m}^{k_n} 2^k\epsilon) \leq 2^{-k_n}\#(I \cap 2^m) + 2\epsilon \to 2\epsilon$$

as $n \to \infty$, and $d^*(I) \leq 2\epsilon$; as ϵ is arbitrary, $I \in \mathcal{Z}$. **Q**

(c) Writing $\mathcal{D}$ for the domain of d,

$$\begin{aligned}\mathcal{D} &= \{I : I \subseteq \mathbb{N},\, \limsup_{n\to\infty} \frac{1}{n}\#(I \cap n) = \liminf_{n\to\infty} \frac{1}{n}\#(I \cap n)\}\\ &= \{I : I \subseteq \mathbb{N},\, d^*(I) = 1 - d^*(\mathbb{N} \setminus I)\},\end{aligned}$$

$$\mathbb{N} \in \mathcal{D}, \quad \text{if } I,\, J \in \mathcal{D} \text{ and } I \subseteq J \text{ then } J \setminus I \in \mathcal{D},$$

$$\text{if } I,\, J \in \mathcal{D} \text{ and } I \cap J = \emptyset \text{ then } I \cup J \in \mathcal{D} \text{ and } d(I \cup J) = d(I) + d(J).$$

It follows that if $\mathcal{I} \subseteq \mathcal{D}$ and $I \cap J \in \mathcal{I}$ for all $I,\, J \in \mathcal{I}$, then the subalgebra of $\mathcal{P}\mathbb{N}$ generated by $\mathcal{I}$ is included in $\mathcal{D}$ (313Ga). But note that $\mathcal{D}$ itself is *not* a subalgebra of $\mathcal{P}\mathbb{N}$ (491Xa).

(d) The following elementary fact will be useful. If $\langle l_n\rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$ such that $\lim_{n\to\infty} l_{n+1}/l_n = 1$, and $I \subseteq \mathbb{R}$, then

$$d^*(I) \leq \limsup_{n\to\infty} \frac{1}{l_{n+1}-l_n}\#(I \cap l_{n+1} \setminus l_n).$$

P Set $\gamma = \limsup_{n\to\infty} \frac{1}{l_{n+1}-l_n}\#(I \cap l_{n+1} \setminus l_n)$, and take $\epsilon > 0$. Let n_0 be such that $\#(I \cap l_{n+1} \setminus l_n) \leq (\gamma+\epsilon)(l_{n+1}-l_n)$ and $l_{n+1} - l_n \leq \epsilon l_n$ for every $n \geq n_0$, and write M for $\#(I \cap l_{n_0})$. If $m > l_{n_0}$, take k such that $l_k \leq m < l_{k+1}$; then

$$\begin{aligned}\#(I \cap m) &\leq M + \sum_{n=n_0}^{k-1} \#(I \cap l_{n+1} \setminus l_n) + (m - l_k)\\ &\leq M + \sum_{n=n_0}^{k-1} (\gamma+\epsilon)(l_{n+1}-l_n) + l_{k+1} - l_k \leq M + m(\gamma+\epsilon) + \epsilon m,\end{aligned}$$

so

$$\frac{1}{m}\#(I \cap m) \leq \frac{M}{m} + \gamma + 2\epsilon.$$

Accordingly $d^*(I) \leq \gamma + 2\epsilon$; as ϵ is arbitrary, $d^*(I) \leq \gamma$. **Q**

491B Equidistributed sequences Let X be a topological space and μ a probability measure on X. I say that a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X is **(asymptotically) equidistributed** or **uniformly distributed** if $\mu F \geq d^*(\{i : x_i \in F\})$ for every measurable closed set $F \subseteq X$; equivalently, if $\mu G \leq \liminf_{n\to\infty} \frac{1}{n}\#(\{i : i < n,\ x_i \in G\})$ for every measurable open set $G \subseteq X$.

Remark Traditionally, equidistributed sequences have been defined in terms of their action on continuous functions, as in 491Cf. I have adopted the definition here in order to deal both with Radon measures on spaces which are not completely regular (so that we cannot identify the measure with an integral) and with Baire measures (so that there may be closed sets which are not measurable). Note that we cannot demand that the sets $\{i : x_i \in F\}$ should have well-defined densities (491Xh).

491C I work through a list of basic facts. The technical details (if we do not specialize immediately to metrizable or compact spaces) are not quite transparent, so I set them out carefully.

Proposition Let X be a topological space, μ a probability measure on X and $\langle x_n\rangle_{n\in\mathbb{N}}$ a sequence in X. Write $C_b(X)$ for the space of bounded continuous real-valued functions on X.

(a) $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed iff $\int f d\mu \leq \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(x_i)$ for every measurable bounded lower semi-continuous function $f : X \to \mathbb{R}$.

(b) If μ measures every zero set and $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed, then $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(x_i) = \int f d\mu$ for every $f \in C_b(X)$.

(c) Suppose that μ measures every zero set in X. If $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(x_i) = \int f d\mu$ for every $f \in C_b(X)$, then $d^*(\{n : x_n \in F\}) \leq \mu F$ for every zero set $F \subseteq X$.

(d) Suppose that X is normal and that μ measures every zero set and is inner regular with respect to the closed sets. If $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(x_i) = \int f d\mu$ for every $f \in C_b(X)$, then $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed.

(e) Suppose that μ is τ-additive and there is a base $\mathcal{G}$ for the topology of X, consisting of measurable sets and closed under finite unions, such that $\mu G \leq \liminf_{n\to\infty} \frac{1}{n+1}\#(\{i : i \leq n,\ x_i \in G\})$ for every $G \in \mathcal{G}$. Then $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed.

(f) Suppose that X is completely regular and that μ measures every zero set and is τ-additive. Then $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed iff $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(x_i) = \int f d\mu$ for every $f \in C_b(X)$.

(g) Suppose that X is metrizable and that μ is a topological measure. Then $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed iff $\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^n f(x_i) = \int f d\mu$ for every $f \in C_b(X)$.

(h) Suppose that X is compact, Hausdorff and zero-dimensional, and that μ is a Radon measure on X. Then $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed iff $d(\{n : x_n \in G\}) = \mu G$ for every open-and-closed subset G of X.

proof (a)(i) Suppose that $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed. Let $f : X \to [0,1]$ be a measurable lower semi-continuous function and $k \geq 1$. For each $j < k$ set $G_j = \{x : f(x) > \frac{j}{k}\}$. Then

$$\liminf_{n\to\infty} \frac{1}{n+1}\sum_{j=1}^{n} \chi G_j(x_i) = \liminf_{n\to\infty} \frac{1}{n+1}\#(\{i : i \leq n,\, x_i \in G_j\}) \geq \mu G_j$$

because $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed and G_j is a measurable open set. Also $f - \frac{1}{k}\chi X \leq \frac{1}{k}\sum_{j=1}^{k} \chi G_j \leq f$, so

$$\begin{aligned}\int f d\mu - \frac{1}{k} &\leq \frac{1}{k}\sum_{j=1}^{k} \mu G_j \leq \frac{1}{k}\sum_{j=1}^{k} \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} \chi G_j(x_i)\\ &\leq \frac{1}{k}\liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n}\sum_{j=1}^{k} \chi G_j(x_i) \leq \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i).\end{aligned}$$

As k is arbitrary, $\int f d\mu \leq \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i)$.

The argument above depended on f taking values in $[0,1]$. But multiplying by an appropriate positive scalar we see that $\int f d\mu \leq \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i)$ for every bounded measurable lower semi-continuous $f : X \to [0,\infty[$, and adding a multiple of χX we see that the same formula is valid for all bounded measurable lower semi-continuous $f : X \to \mathbb{R}$.

(ii) Conversely, if $\int f d\mu \leq \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i)$ for every bounded measurable lower semi-continuous $f : X \to \mathbb{R}$, and $G \subseteq X$ is a measurable open set, then χG is lower semi-continuous, so $\mu G \leq \liminf_{n\to\infty} \frac{1}{n+1}\#(\{i : i \leq n,\, x_i \in G\})$. As G is arbitrary, $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed.

(b) Apply (a) to the lower semi-continuous functions f and $-f$.

(c) Recall that if μ measures every zero set, then every bounded continuous real-valued function is integrable, by 4A3L. Let $F \subseteq X$ be a zero set, and $\epsilon > 0$. Then there is a continuous $f : X \to \mathbb{R}$ such that $F = f^{-1}[\{0\}]$. Let $\delta > 0$ be such that $\mu\{x : 0 < |f(x)| \leq \delta\} \leq \epsilon$, and set $g = (\chi X - \frac{1}{\delta}|f|)^+$. Then $g : X \to [0,1]$ is continuous and $\chi F \leq g$, so

$$\begin{aligned}d^*(\{n : x_n \in F\}) &\leq \limsup_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} g(x_i)\\ &= \int g\, d\mu \leq \mu\{x : |f(x)| \leq \delta\} \leq \mu F + \epsilon.\end{aligned}$$

As ϵ and F are arbitrary, we have the result.

(d) Let $F \subseteq X$ be a measurable closed set and $\epsilon > 0$. Because μ is inner regular with respect to the closed sets, there is a measurable closed set $F' \subseteq X \setminus F$ such that $\mu F' \geq \mu(X \setminus F) - \epsilon$. Because X is normal, there is a continuous function $f : X \to [0,1]$ such that $\chi F \leq f \leq \chi(X \setminus F')$. Now

$$d^*(\{n : x_n \in F\}) \leq \limsup_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i) = \int f d\mu \leq \mu(X \setminus F') \leq \mu F + \epsilon.$$

As F and ϵ are arbitrary, $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed.

(e) Let $F \subseteq X$ be a measurable closed set, and $\epsilon > 0$. Let $\mathcal{G}_1$ be the family of members of $\mathcal{G}$ disjoint from F. Then $\mathcal{G}_1$ is upwards-directed and $\bigcup \mathcal{G}_1 = X \setminus F$; because μ is τ-additive, there is a $G \in \mathcal{G}_1$ such that $\mu G > \mu(X \setminus F) - \epsilon$. Now

$$\begin{aligned}d^*(\{n : x_n \in F\}) &\leq \limsup_{n\to\infty} \frac{1}{n+1}\#(\{i : i \leq n,\, x_i \in X \setminus G\})\\ &= 1 - \liminf_{n\to\infty} \frac{1}{n+1}\#(\{i : i \leq n,\, x_i \in G\}) \leq 1 - \mu G \leq \mu F + \epsilon.\end{aligned}$$

As F and ϵ are arbitrary, $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed.

(f) (i) If $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed then (b) tells us that $\int f d\mu = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i)$ for every $f \in C_b(X)$. (ii) Suppose that $\int f d\mu = \lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} f(x_i)$ for every $f \in C_b(X)$. If $G \subseteq X$ is a cozero

set, we can apply (c) to its complement to see that $\mu G \leq \liminf_{n\to\infty} \frac{1}{n+1}\#(\{i : i \leq n, x_i \in G\})$. So applying (e) with $\mathcal{G}$ the family of cozero sets we see that $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed.

(g) Because every closed set is a zero set, this follows at once from (b) and (c).

(h) If $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed and $G \subseteq X$ is open-and-closed, then $d^*(\{n : x_n \in G\}) \leq \mu G$ because G is closed and $d^*(\{n : x_n \notin G\}) \leq 1 - \mu G$ because G is open; so $d(\{n : x_n \in G\}) = \mu G$. If the condition is satisfied, then (e) tells us that $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed.

491D The next lemma provides a useful general criterion for the existence of equidistributed sequences.

Lemma Let X be a topological space and μ a probability measure on X. Suppose that there is a sequence $\langle \nu_n\rangle_{n\in\mathbb{N}}$ of point-supported probability measures on X such that $\mu F \geq \limsup_{n\to\infty} \nu_n F$ for every measurable closed set $F \subseteq X$. Then μ has an equidistributed sequence.

proof For each $n \in \mathbb{N}$, let $q_n : X \to [0,1]$ be such that $\nu_n E = \sum_{x\in E} q_n(x)$ for every $E \subseteq X$. Let $q'_n : X \to [0,1]$ be such that $\sum_{x\in X} q'_n(x) = 1$, $K_n = \{x : q'_n(x) > 0\}$ is finite, $q'_n(x)$ is rational for every x, and $\sum_{x\in X} |q_n(x) - q'_n(x)| \leq 2^{-n}$; then $\mu F \geq \limsup_{n\to\infty} \nu'_n F$ for every measurable closed F, where ν'_n is defined from q'_n. For each n, let $s_n \geq 1$ be such that $r_n(x) = q'_n(x)s_n$ is an integer for every $x \in K_n$. Let $\langle x_{ni}\rangle_{i<s_n}$ be a family in K_n such that $\#(\{i : i < s_n,\ x_{ni} = x\}) = r_n(x)$ for each $x \in K_n$; then $\nu'_n E = \frac{1}{s_n}\#(\{i : i < s_n, x_{ni} \in E\})$ for every $E \subseteq X$.

Let $\langle m_k\rangle_{k\in\mathbb{N}}$ be such that $s_{k+1} \leq 2^{-k}\sum_{j=0}^{k} m_j s_j$ for each k. Set $l_0 = 0$. Given l_n, take the largest k such that $\sum_{j=0}^{k-1} m_j s_j \leq l_n$; set $l_{n+1} = l_n + s_k$ and $x_i = x_{k,i-l_n}$ for $l_n \leq i < l_{n+1}$; continue. By the choice of the m_k, $l_{n+1}/l_n \to 1$ as $n \to \infty$. For any $E \subseteq X$, $\#(\{i : l_n \leq i < l_{n+1}, x_i \in E\}) = \#(\{j : j < s_k, x_{kj} \in E\})$ whenever $\sum_{j=0}^{k-1} m_j s_j \leq l_n < \sum_{j=0}^{k} m_j s_j$. So for any measurable closed set $F \subseteq X$,

$$\begin{aligned} d^*(\{i : x_i \in F\}) &\leq \limsup_{k\to\infty} \frac{1}{s_k}\#(\{j : j < s_k, x_{kj} \in F\}) \\ &= \limsup_{k\to\infty} \nu'_k F \leq \mu F. \end{aligned} \tag{491Ad}$$

As F is arbitrary, $\langle x_n\rangle_{n\in\mathbb{N}}$ is an equidistributed sequence for μ.

491E Proposition (a)(i) Suppose that X and Y are topological spaces, μ a probability measure on X and $f : X \to Y$ a continuous function. If $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in X which is equidistributed with respect to μ, then $\langle f(x_n)\rangle_{n\in\mathbb{N}}$ is equidistributed with respect to the image measure μf^{-1}.

(ii) Suppose that (X,μ) and (Y,ν) are topological probability spaces and $f : X \to Y$ is a continuous inverse-measure-preserving function. If $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in X which is equidistributed with respect to μ, then $\langle f(x_n)\rangle_{n\in\mathbb{N}}$ is equidistributed with respect to ν.

(b) Let X be a topological space and μ a probability measure on X, and suppose that X has a countable network consisting of sets measured by μ. Let λ be the ordinary product measure on $X^{\mathbb{N}}$. Then λ-almost every sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in $X^{\mathbb{N}}$ is μ-equidistributed.

proof (a)(i) Let $F \subseteq Y$ be a closed set which is measured by μf^{-1}. Then $f^{-1}[F]$ is a closed set in X measured by μ. So

$$d^*(\{n : f(x_n) \in F\}) = d^*(\{n : x_n \in f^{-1}[F]\}) \leq \mu f^{-1}[F].$$

(ii) Replace 'μf^{-1}' above by 'ν'.

(b) Let $\mathcal{A}$ be a countable network for the given topology $\mathfrak{S}$ of X consisting of measurable sets, and let $\mathcal{E}$ be the countable subalgebra of $\mathcal{P}X$ generated by $\mathcal{A}$. Let $\mathfrak{T} \supseteq \mathfrak{S}$ be the second-countable topology generated by $\mathcal{E}$; then μ is a τ-additive topological measure with respect to $\mathfrak{T}$, and $\mathcal{E}$ is a base for $\mathfrak{T}$ closed under finite unions. If $E \in \mathcal{E}$, then $d(\{n : x_n \in E\}) = \mu E$ for λ-almost every sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X, by the strong law of large numbers (273J). So

$$d(\{n : x_n \in E\}) = \mu E \text{ for every } E \in \mathcal{E}$$

for λ-almost every $\langle x_n\rangle_{n\in\mathbb{N}}$. Now 491Ce tells us that any such sequence is equidistributed with respect to $\mathfrak{T}$ and therefore with respect to $\mathfrak{S}$.

491F Theorem Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of τ-additive topological probability spaces, each of which has an equidistributed sequence. If $\#(I) \leq \mathfrak{c}$, then the τ-additive product measure λ on $X = \prod_{i\in I} X_i$ (definition: 417G) has an equidistributed sequence.

proof (a) For the time being (down to the end of (f)), let us suppose that every μ_m is inner regular with respect to the Borel sets. (This will make it possible to use the theorems of §417, in particular 417H and 417J.)

The formulae of some of the arguments below will be simplified if we immediately re-index the family $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ as $\langle (X_A, \mathfrak{T}_A, \Sigma_A, \mu_A)\rangle_{A\in\mathcal{A}}$ where $\mathcal{A} \subseteq \mathcal{P}\mathbb{N}$. For each $A \in \mathcal{A}$, let $\langle t_{An}\rangle_{n\in\mathbb{N}}$ be an equidistributed sequence in X_A; for $n \in \mathbb{N}$, let ν_{An} be the point-supported measure on X_A defined by setting $\nu_{An}E = \frac{1}{n+1}\#(\{i : i \leq n,\ t_{Ai} \in E\})$ for $E \subseteq X_A$. For each finite set $\mathcal{I} \subseteq \mathcal{A}$, set $Y_{\mathcal{I}} = \prod_{A\in\mathcal{I}} X_A$; set $\pi_{\mathcal{I}}(x) = x{\restriction}\mathcal{I} \in Y_{\mathcal{I}}$ for $x \in X$. Let $\lambda_{\mathcal{I}}$ be the τ-additive product of $\langle \mu_I\rangle_{A\in\mathcal{I}}$ and, for each n, let $\check{\nu}_{\mathcal{I}n}$ be the product of the measures $\langle \nu_{An}\rangle_{A\in\mathcal{I}}$. (Because $\mathcal{I}$ is finite, this is a point-supported probability measure. I do not say 'τ-additive product' here because I do not wish to assume that all singleton sets are Borel, so the ν_{An} may not be inner regular with respect to Borel sets.)

(b) Suppose that $\mathcal{I} \subseteq \mathcal{A}$ is finite and that $W \subseteq Y_{\mathcal{I}}$ is an open set. Then $\lambda_{\mathcal{I}}W \leq \liminf_{n\to\infty} \check{\nu}_{\mathcal{I}n}W$. **P** Induce on $\#(\mathcal{I})$. If $\mathcal{I} = \emptyset$, $Y_{\mathcal{I}}$ is a singleton and the result is trivial. For the inductive step, if $\mathcal{I} \neq \emptyset$, take any $A \in \mathcal{I}$ and set $\mathcal{I}' = \mathcal{I} \setminus \{A\}$. Then we can identify $Y_{\mathcal{I}}$ with $Y_{\mathcal{I}'} \times X_A$, $\lambda_{\mathcal{I}}$ with the τ-additive product of $\lambda_{\mathcal{I}'}$ and μ_A (417J), and each $\check{\nu}_{\mathcal{I}n}$ with the product of $\check{\nu}_{\mathcal{I}'n}$ and ν_{An}.

Let $\mathcal{V}$ be the family of those subsets V of $Y_{\mathcal{I}}$ which are expressible as a finite union of sets of the form $U \times H$ where $U \subseteq Y_{\mathcal{I}'}$ and $H \subseteq X_A$ are open. Then $\mathcal{V}$ is a base for the topology of $Y_{\mathcal{I}}$ closed under finite unions. Let $\epsilon > 0$. Because $\lambda_{\mathcal{I}}$ is τ-additive, there is a $V \in \mathcal{V}$ such that $\lambda_{\mathcal{I}}V \geq \lambda W - \epsilon$. The function $t \mapsto \lambda_{\mathcal{I}'}V[\{t\}] : X_A \to [0,1]$ is lower semi-continuous (417Ba), so 491Ca tells us that

$$\begin{aligned}\lambda_{\mathcal{I}}V &= \int \lambda_{\mathcal{I}'}V[\{t\}]\mu_A(dt)\\ &\leq \liminf_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n} \lambda_{\mathcal{I}'}V[\{t_{Ai}\}] = \liminf_{n\to\infty} \int \lambda_{\mathcal{I}'}V[\{t\}]\nu_{An}(dt).\end{aligned}$$

At the same time, there are only finitely many sets of the form $V[\{t\}]$, and for each of these we have $\lambda_{\mathcal{I}'}V[\{t\}] \leq \liminf_{n\to\infty} \check{\nu}_{\mathcal{I}'n}V[\{t\}]$, by the inductive hypothesis. So there is an $n_0 \in \mathbb{N}$ such that $\lambda_{\mathcal{I}'}V[\{t\}] \leq \check{\nu}_{\mathcal{I}'n}V[\{t\}] + \epsilon$ for every $n \geq n_0$ and every $t \in X_A$. We must therefore have

$$\begin{aligned}\lambda_{\mathcal{I}}W \leq \lambda_{\mathcal{I}}V + \epsilon &\leq \liminf_{n\to\infty} \int \lambda_{\mathcal{I}'}V[\{t\}]\nu_{An}(dt) + \epsilon\\ &\leq \liminf_{n\to\infty} \int \check{\nu}_{\mathcal{I}'n}V[\{t\}]\nu_{An}(dt) + 2\epsilon\\ &= \liminf_{n\to\infty} \check{\nu}_{\mathcal{I}n}V + 2\epsilon \leq \liminf_{n\to\infty} \check{\nu}_{\mathcal{I}n}W + 2\epsilon.\end{aligned}$$

As ϵ and W are arbitrary, the induction proceeds. **Q**

(c) For $K \subseteq m \in \mathbb{N}$, set $\mathcal{A}_{mK} = \{A : A \in \mathcal{A},\ A \cap m = K\}$ and $Z_{mK} = \prod_{A\in\mathcal{A}_{mK}} X_A$. (If $\mathcal{A}_{mK} = \emptyset$ then $Z_{mK} = \{\emptyset\}$.) Then for each $m \in \mathbb{N}$ we can identify X with the finite product $\prod_{K\subseteq m} Z_{mK}$. For $K \subseteq m \in \mathbb{N}$ and $n \in \mathbb{N}$, define $z_{mKn} \in Z_{mK}$ by setting $z_{mKn}(A) = t_{An}$ for $A \in \mathcal{A}_{mK}$; let $\tilde{\nu}_{mKn}$ be the point-supported measure on Z_{mK} defined by setting $\tilde{\nu}_{mKn}W = \frac{1}{n+1}\#(\{i : i \leq n,\ z_{mKi} \in W\})$ for each $W \subseteq Z_{mK}$. For $n \in \mathbb{N}$ let $\tilde{\nu}_n$ be the measure on X which is the product of the measures $\tilde{\nu}_{nKn}$ for $K \subseteq n$; this too is point-supported.

(d) If $\mathcal{I} \subseteq \mathcal{A}$ is finite, there is an $m \in \mathbb{N}$ such that $\check{\nu}_{\mathcal{I}n} = \tilde{\nu}_n\pi_{\mathcal{I}}^{-1}$ for every $n \geq m$. **P** Let m be such that $A \cap m \neq A' \cap m$ for all distinct A, $A' \in \mathcal{I}$. If $n \geq m$, then $\tilde{\nu}_n$ is the product of the $\tilde{\nu}_{nKn}$ for $K \subseteq n$. Now $\pi_{\mathcal{I}}$, interpreted as a function from $\prod_{K\subseteq n} Z_{nK}$ onto $Y_{\mathcal{I}}$, is of the form $\pi_{\mathcal{I}}(\langle z_K\rangle_{K\subseteq n}) = \langle z_{A\cap n}(A)\rangle_{A\in\mathcal{I}}$, so the

image measure $\tilde{\nu}_n\pi_{\mathcal{I}}^{-1}$ is the product of the family $\langle\tilde{\nu}_{n,A\cap n,n}\hat{\pi}_A^{-1}\rangle_{A\in\mathcal{I}}$, writing $\hat{\pi}_A(z) = z(A)$ when $A\cap n = K$ and $z \in Z_{nK}$ (254H, or otherwise). But, looking back at the definitions,

$$\begin{aligned}\tilde{\nu}_{n,A\cap n,n}\hat{\pi}_A^{-1}[E] &= \frac{1}{n+1}\#(\{i : i \le n,\, z_{n,A\cap n,i} \in \hat{\pi}_A^{-1}[E]\})\\ &= \frac{1}{n+1}\#(\{i : i \le n,\, z_{n,A\cap n,i}(A) \in E\})\\ &= \frac{1}{n+1}\#(\{i : i \le n,\, t_{Ai} \in E\}) = \nu_{An}E\end{aligned}$$

for every $E \subseteq X_A$. So $\tilde{\nu}_n\pi_{\mathcal{I}}^{-1}$ is the product of $\langle\nu_{An}\rangle_{A\in\mathcal{I}}$, which is $\check{\nu}_{\mathcal{I}n}$. **Q**

(e) Let $\mathcal{W}$ be the family of those open sets $W \subseteq X$ expressible in the form $\pi_{\mathcal{I}}^{-1}[W']$ for some finite $\mathcal{I} \subseteq \mathcal{A}$ and some open $W' \subseteq Y_{\mathcal{I}}$. If $W \in \mathcal{W}$, then $\lambda W \le \liminf_{n\in\mathbb{N}} \tilde{\nu}_n W$. **P** Take $\mathcal{I} \in [\mathcal{A}]^{<\omega}$ and an open $W' \subseteq Y_{\mathcal{I}}$ such that $W = \pi_{\mathcal{I}}^{-1}[W']$. Then

$$\begin{aligned}\lambda W &= \lambda_{\mathcal{I}} W'\\ &\text{(417K)}\\ &\le \liminf_{n\to\infty} \check{\nu}_{\mathcal{I}n} W'\\ &\text{(by (b) above)}\\ &= \liminf_{n\to\infty} \tilde{\nu}_n \pi_{\mathcal{I}}^{-1}[W']\\ &\text{(by (d))}\\ &= \liminf_{n\to\infty} \tilde{\nu}_n W.\end{aligned}$$

Q

(f) If now $F \subseteq X$ is any closed set and $\epsilon > 0$, then (because $\mathcal{W}$ is a base for the topology of X closed under finite unions) there is a $W \in \mathcal{W}$ such that $W \subseteq X \setminus F$ and $\lambda W \ge 1 - \lambda F + \epsilon$. In this case

$$\limsup_{n\to\infty} \tilde{\nu}_n F \le 1 - \liminf_{n\to\infty} \tilde{\nu}_n W \le 1 - \lambda W \le \lambda F + \epsilon.$$

As ϵ is arbitrary, $\limsup_{n\to\infty} \tilde{\nu}_n F \le \lambda F$; as F is arbitrary, 491D tells us that there is an equidistributed sequence in X.

(g) All this was on the assumption that every μ_i is inner regular with respect to the Borel sets. For the superficially more general case enunciated, given only that each μ_i is a τ-additive topological measure with an equidistributed sequence, let μ_i' be the restriction of μ_i to the Borel σ-algebra of X_i for each $i \in I$. Each μ_i' is still τ-additive and equidistributed sequences for μ_i are of course equidistributed for μ_i'. If we take λ' to be the τ-additive product of $\langle\mu_i'\rangle_{i\in I}$, then it must agree with λ on the open sets of X and therefore on the closed sets, and an equidistributed sequence for λ' will be an equidistributed sequence for λ. This completes the proof.

491G Corollary The usual measure of $\{0,1\}^{\mathfrak{c}}$ has an equidistributed sequence.

proof The usual measure of $\{0,1\}$ of course has an equidistributed sequence (just set $x_i = 0$ for even i, $x_i = 1$ for odd i), so 491F gives the result at once.

491H Theorem (VEECH 71) Any separable compact Hausdorff topological group has an equidistributed sequence for its Haar probability measure.

proof Let X be a separable compact Hausdorff topological group. Recall that X has exactly one Haar probability measure μ (442Ic), which is both a left Haar measure and a right Haar measure.

(a) We need some elementary facts about convolutions.

(i) If ν_1 and ν_2 are point-supported probability measures on X, then $\nu_1 * \nu_2$ is point-supported. **P** If $\nu_1 E = \sum_{x\in E} q_1(x)$ and $\nu_2 E = \sum_{x\in E} q_2(x)$ for every $E \subseteq X$, then

$$(\nu_1 * \nu_2)(E) = (\nu_1 \times \nu_2)\{(x,y) : xy \in E\} = \sum_{xy\in E} q_1(x)q_2(y) = \sum_{z\in E} q(z)$$

where $q(z) = \sum_{x\in X} q_1(x)q_2(x^{-1}z)$ for $z \in X$. **Q**

(ii) Let ν, λ be Radon probability measures on X. Suppose that $f \in C(X)$, $\alpha \in \mathbb{R}$ and $\epsilon > 0$ are such that $|\int f(yxz)\nu(dx) - \alpha| \leq \epsilon$ for every y, $z \in X$. Then $|\int f(yxz)(\lambda * \nu)(dx) - \alpha| \leq \epsilon$ and $|\int f(yxz)(\lambda * \nu)(dx) - \alpha| \leq \epsilon$ for every y, $z \in X$. **P**

$$\begin{aligned}
&|\int f(yxz)(\lambda * \nu)(dx) - \alpha| \\
&\qquad = |\iint f(ywxz)\nu(dx)\lambda(dw) - \alpha| \\
&\qquad \leq \int |\int f(ywxz)\nu(dx) - \alpha|\lambda(dw) \leq \int \epsilon\lambda(dw) = \epsilon, \qquad \text{(444C)} \\
&|\int f(yxz)(\nu * \lambda)(dx) - \alpha| \\
&\qquad = |\iint f(yxwz)\nu(dx)\lambda(dw) - \alpha| \\
&\qquad \leq \int |\int f(yxwz)\nu(dx) - \alpha|\lambda(dw) \leq \int \epsilon\lambda(dw) = \epsilon. \ \mathbf{Q}
\end{aligned}$$

(b) Let $A \subseteq X$ be a countable dense set. Let N be the set of point-supported probability measures ν on X which are defined by functions q such that $\{x : q(x) > 0\}$ is a finite subset of A and $q(x)$ is rational for every x. Then N is countable. Now, for every $f \in C(X)$ and $\epsilon > 0$, there is a $\nu \in$ N such that $|\int f(yxz)\nu(dx) - \int f d\mu| \leq \epsilon$ for all y, $z \in X$. **P** Because X is compact, f is uniformly continuous for the right uniformity of X (4A2Je), so there is a neighbourhood U of the identity e such that $|f(x') - f(x)| \leq \frac{1}{2}\epsilon$ whenever $x'x^{-1} \in U$. Next, again because X is compact, there is a neighbourhood V of e such that $yxy^{-1} \in U$ whenever $x \in V$ and $y \in X$ (4A5Ej). Because A is dense, $xV^{-1} \cap A \neq \emptyset$ is not empty for every $x \in X$, that is, $AV = X$; once more because X is compact, there are $x_0, \ldots, x_n \in A$ such that $X = \bigcup_{i\leq n} x_iV$. Set $E_i = x_iV \setminus \bigcup_{j<i} x_jV$ for each $i \leq n$. Let $\alpha_0, \ldots, \alpha_n \in [0,1] \cap \mathbb{Q}$ be such that $\sum_{i=0}^n \alpha_i = 1$ and $\|f\|_\infty \sum_{i=0}^n |\alpha_i - \mu E_i| \leq \frac{1}{2}\epsilon$, and define $\nu \in$ N by setting $\nu E = \sum\{\alpha_i : i \leq n, x_i \in E\}$ for every $E \subseteq X$.

Let y, $z \in X$. If $i \leq n$ and $x \in E_i$, then $xx_i^{-1} \in V$ so $(yxz)(yx_iz)^{-1} = yxx_i^{-1}y^{-1} \in U$ and $|f(yxz) - f(yx_iz)| \leq \frac{1}{2}\epsilon$. Accordingly

$$\begin{aligned}
&|\int f(yxz)\nu(dx) - \int f(x)\mu(dx)| \\
&\qquad = |\sum_{i=0}^n \alpha_i f(yx_iz) - \int f(yxz)\mu(dx)| \\
&\qquad \text{(because } X \text{ is unimodular)} \\
&\qquad \leq \sum_{i=0}^n |\alpha_i f(yx_iz) - \int_{E_i} f(yxz)\mu(dx)| \\
&\qquad \leq \sum_{i=0}^n |\alpha_i - \mu E_i||f(yx_iz)| + \sum_{i=0}^n |f(yx_iz)\mu E_i - \int_{E_i} f(yxz)\mu(dx)| \\
&\qquad \leq \|f\|_\infty \sum_{i=0}^n |\alpha_i - \mu E_i| + \sum_{i=0}^n \int_{E_i} |f(yxz) - f(yx_iz)|\mu(dx) \\
&\qquad \leq \frac{\epsilon}{2} + \sum_{i=0}^n \frac{\epsilon}{2}\mu E_i = \epsilon. \ \mathbf{Q}
\end{aligned}$$

(c) Let $\langle \nu_n \rangle_{n\in\mathbb{N}}$ be a sequence running over N, and set $\lambda_n = \nu_0 * \nu_1 * \ldots * \nu_n$ for each n. Then each λ_n is a point-supported probability measure on X, by (a)(i). Also $\lim_{n\to\infty} \int f d\lambda_n = \int f d\mu$ for every $f \in C(X)$. **P** If $f \in C(X)$ and $\epsilon > 0$, then (b) tells us that there is an $m \in \mathbb{N}$ such that $|\int f(yxz)\nu_m(dx) - \int f d\mu| \leq \epsilon$ for all y, $z \in X$. For any $n \geq m$, λ_n is of the form $\lambda' * \nu_m * \lambda''$. By (a)(ii), used in both parts successively, $|\int f d\lambda_n - \int f d\mu| \leq \epsilon$. As ϵ is arbitrary, we have the result. **Q**

(d) If $F \subseteq X$ is closed, then

$$\begin{aligned}\mu F &= \inf\{\int f d\mu : \chi F \leq f \in C(X)\} \\ &= \inf_{\chi F \leq f} \limsup_{n\to\infty} \int f d\lambda_n \geq \limsup_{n\to\infty} \lambda_n F.\end{aligned}$$

By 491D, μ has an equidistributed sequence.

491I The quotient $\mathcal{P}\mathbb{N}/\mathcal{Z}$ I now return to the asymptotic density ideal $\mathcal{Z}$, moving towards a striking relationship between the corresponding quotient algebra and equidistributed sequences. Since $\mathcal{Z} \lhd \mathcal{P}\mathbb{N}$, we can form the quotient algebra $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$. The functionals d and d^* descend naturally to $\mathfrak{Z}$ if we set

$$\bar{d}^*(I^\bullet) = d^*(I), \quad \bar{d}(I^\bullet) = d(I) \text{ whenever } d(I) \text{ is defined.}$$

(a) $\bar{d}^*$ is a strictly positive submeasure on $\mathfrak{Z}$. **P** $\bar{d}^*$ is a submeasure on $\mathfrak{Z}$ because d^* is a submeasure on $\mathcal{P}\mathbb{N}$. $\bar{d}^*$ is strictly positive because $\mathcal{Z} \supseteq \{I : d^*(I) = 0\}$. **Q**

(b) Let $\bar{\rho}$ be the metric on $\mathfrak{Z}$ defined by saying that $\bar{\rho}(a, b) = \bar{d}^*(a \triangle b)$ for all a, $b \in \mathfrak{Z}$. Under ρ, the Boolean operations $\cup$, $\cap$, $\triangle$ and $\setminus$ and the function $\bar{d}^* : \mathfrak{Z} \to [0, 1]$ are uniformly continuous (393Bb), and $\mathfrak{Z}$ is complete. **P** Let $\langle c_n \rangle_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{Z}$ such that $\bar{\rho}(c_{n+1}, c_n) \leq 2^{-n}$ for every $n \in \mathbb{N}$; then $\bar{\rho}(c_r, c_i) \leq 2^{-i+1}$ for $i \leq r$. For each $n \in \mathbb{N}$ choose $C_n \subseteq \mathbb{N}$ such that $C_n^\bullet = c_n$; then $d^*(C_r \triangle C_i) \leq 2^{-i+1}$ for $i \leq r$. Choose a strictly increasing sequence $\langle k_n \rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $k_{n+1} \geq 2k_n$ for every n and, for each $n \in \mathbb{N}$,

$$\frac{1}{m}\#((C_n \triangle C_i) \cap m) \leq 2^{-i+2} \text{ whenever } i \leq n,\ m \geq k_n.$$

Set $C = \bigcup_{n\in\mathbb{N}} C_n \cap k_{n+1} \setminus k_n$, and $c = C^\bullet \in \mathfrak{Z}$. If $n \in \mathbb{N}$ and $m \geq k_{n+1}$, then take $r > n$ such that $k_r \leq m < k_{r+1}$; in this case $k_i \leq 2^{i-r}m$ for $i \leq r$, so

$$\begin{aligned}\#((C \triangle C_n) \cap m) &\leq k_n + \sum_{i=n}^{r-1} \#((C \triangle C_n) \cap k_{i+1} \setminus k_i) + \#((C \triangle C_n) \cap m \setminus k_r) \\ &= k_n + \sum_{i=n}^{r-1} \#((C_i \triangle C_n) \cap k_{i+1} \setminus k_i) + \#((C_r \triangle C_n) \cap m \setminus k_r) \\ &\leq k_n + \sum_{i=n+1}^{r-1} \#((C_i \triangle C_n) \cap k_{i+1}) + \#((C_r \triangle C_n) \cap m) \\ &\leq k_n + \sum_{i=n+1}^{r-1} 2^{-n+2}k_{i+1} + 2^{-n+2}m \\ &\leq k_n + \sum_{i=n+1}^{r-1} 2^{-n+2}2^{i+1-r}m + 2^{-n+2}m \\ &\leq k_n + 2^{-n+3}m + 2^{-n+2}m.\end{aligned}$$

But this means that

$$\bar{\rho}(c, c_n) = d^*(C \triangle C_n) \leq \lim_{m\to\infty} \tfrac{k_n}{m} + 2^{-n+3} + 2^{-n+2} \leq 2^{-n+4}$$

for every n, and $\langle c_n \rangle_{n\in\mathbb{N}}$ converges to c in $\mathfrak{Z}$. **Q**

***(c)** If $\langle a_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence in $\mathfrak{Z}$, there is an $a\in\mathfrak{Z}$ such that $a\subseteq a_n$ for every n and $\bar{d}^*(a)=\inf_{n\in\mathbb{N}}\bar{d}^*(a_n)$. **P** For each $n\in\mathbb{N}$, choose $I_n\subseteq\mathbb{N}$ such that $I_n^\bullet=a_n$; replacing I_n by $\bigcap_{j\le n}I_j$ if necessary, we can arrange that $I_{n+1}\subseteq I_n$ for every n. Set $\gamma=\inf_{n\in\mathbb{N}}\bar{d}^*(a_n)=\inf_{n\in\mathbb{N}}d^*(I_n)$. Let $\langle k_n\rangle_{n\in\mathbb{N}}$ be a strictly increasing sequence in $\mathbb{N}$ such that $k_{n+1}\ge 2^nk_n$ and $\#(I_n\cap k_{n+1})\ge(\gamma-2^{-n})k_{n+1}$ for every n. Set $K_n=I_n\cap k_{n+1}\setminus k_n$ for each n, and $I=\bigcup_{n\in\mathbb{N}}K_n$, $a=I^\bullet\in\mathfrak{Z}$. Then

$$\#(I\cap k_{n+1})\ge\#(K_n)\ge(\gamma-2^{-n+1})k_{n+1}$$

for each n, so $\bar{d}^*(a)=d^*(I)\ge\gamma$. Also $I\setminus I_n\subseteq k_n$ is finite, so $a\subseteq a_n$ for each n. Of course it follows at once that $\bar{d}^*(a)=\gamma$ exactly, as required. **Q**

***(d)** $\bar{d}^*$ is a Maharam submeasure on $\mathfrak{Z}$. (Immediate from (c).)

491J Lemma Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{Z}=\mathcal{P}\mathbb{N}/\mathcal{Z}$ such that $\lim_{n\to\infty}\bar{d}^*(a_n)+\bar{d}^*(1\setminus a_n)=1$. Then $\langle a_n\rangle_{n\in\mathbb{N}}$ is topologically convergent to $a\in\mathfrak{Z}$; $a=\sup_{n\in\mathbb{N}}a_n$ in $\mathfrak{Z}$ and $d^*(a)+d^*(1\setminus a)=1$.

proof (a) The point is that if $m\le n$ then $\bar{d}^*(a_n\setminus a_m)\le\bar{d}^*(a_n)+\bar{d}^*(1\setminus a_m)-1$. **P** Let I, $J\subseteq\mathbb{N}$ be such that $I^\bullet=a_m$ and $J^\bullet=a_n$. For any $k\ge1$,

$$\tfrac{1}{k}\#(k\cap J)+\tfrac{1}{k}\#(k\setminus I)=\tfrac{1}{k}\#(k\cap J\setminus I)+\tfrac{1}{k}(k\setminus(I\setminus J)),$$

so

$$\begin{aligned}d^*(J)+d^*(\mathbb{N}\setminus I)&=\limsup_{k\to\infty}\tfrac{1}{k}\#(k\cap J)+\limsup_{k\to\infty}\tfrac{1}{k}\#(k\setminus I)\\&\ge\limsup_{k\to\infty}\tfrac{1}{k}\#(k\cap J)+\tfrac{1}{k}\#(k\setminus I)\\&=\limsup_{k\to\infty}\tfrac{1}{k}\#(k\cap J\setminus I)+\tfrac{1}{k}(k\setminus(I\setminus J))\\&\ge\limsup_{k\to\infty}\tfrac{1}{k}\#(k\cap J\setminus I)+\liminf_{k\to\infty}\tfrac{1}{k}(k\setminus(I\setminus J))=d^*(J\setminus I)+1\end{aligned}$$

because $a_m\subseteq a_n$, so $I\setminus J\in\mathcal{Z}$. But this means that

$$\bar{d}^*(a_n\setminus a_m)=d^*(J\setminus I)\le d^*(J)+d^*(\mathbb{N}\setminus I)-1=\bar{d}^*(a_n)+\bar{d}^*(1\setminus a_m)-1.$$ **Q**

(b) Accordingly

$$\begin{aligned}\limsup_{m\to\infty}\sup_{n\ge m}\bar{\rho}(a_m,a_n)&=\limsup_{m\to\infty}\sup_{n\ge m}\bar{d}^*(a_n\setminus a_m)\\&\le\limsup_{m\to\infty}\sup_{n\ge m}\bar{d}^*(a_n)+\bar{d}^*(1\setminus a_m)-1\\&=\limsup_{m\to\infty}\sup_{n\ge m}\bar{d}^*(a_n)-\bar{d}^*(a_m)\end{aligned}$$

(because $\lim_{m\to\infty}\bar{d}^*(a_m)+\bar{d}^*(1\setminus a_m)=1$)

$$=0,$$

and $\langle a_n\rangle_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathfrak{Z}$.

(c) Because $\mathfrak{Z}$ is complete, $a=\lim_{n\to\infty}a_n$ is defined in $\mathfrak{Z}$. For each $m\in\mathbb{N}$, $a_m\setminus a=\lim_{n\to\infty}a_m\setminus a_n=0$ (because $\setminus$ is continuous), so $a_m\subseteq a$; thus a is an upper bound of $\{a_n:n\in\mathbb{N}\}$. If b is any upper bound of $\{a_n:n\in\mathbb{N}\}$, then $a\setminus b=\lim_{n\to\infty}a_n\setminus b=0$; so $a=\sup_{n\in\mathbb{N}}a_n$. Finally,

$$\bar{d}^*(a)+\bar{d}^*(1\setminus a)=\lim_{n\to\infty}\bar{d}^*(a_n)+\bar{d}^*(1\setminus a_n)=1.$$

491K Corollary Set $D = \{I^\bullet : I \subseteq \mathbb{N}, d(I) \text{ is defined}\} \subseteq \mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$.

(a) If $I \subseteq \mathbb{N}$ and $I^\bullet \in D$ then $d(I)$ is defined.

(b) $D = \{a : a \in \mathfrak{Z}, \bar{d}^*(a) + \bar{d}^*(1 \setminus a) = 1\}$; if $a \in D$ then $1 \setminus a \in D$; if a, $b \in D$ and $a \cap b = 0$, then $a \cup b \in D$ and $\bar{d}^*(a \cup b) = \bar{d}^*(a) + \bar{d}^*(b)$; if a, $b \in D$ and $a \subseteq b$ then $b \setminus a \in D$ and $\bar{d}^*(b \setminus a) = \bar{d}^*(b) - \bar{d}^*(a)$.

(c) D is a topologically closed subset of $\mathfrak{Z}$.

(d) If $A \subseteq D$ is upwards-directed, then $\sup A$ is defined in $\mathfrak{Z}$ and belongs to D; moreover there is a sequence in A with the same supremum as A, and $\sup A$ belongs to the topological closure of A.

(e) Let $\mathfrak{B} \subseteq D$ be a subalgebra of $\mathfrak{Z}$. Then the following are equiveridical:

(i) $\mathfrak{B}$ is topologically closed in $\mathfrak{Z}$;

(ii) $\mathfrak{B}$ is order-closed in $\mathfrak{Z}$;

(iii) setting $\bar{\nu} = \bar{d}^* \restriction \mathfrak{B}$, $(\mathfrak{B}, \bar{\nu})$ is a probability algebra.

In this case, $\mathfrak{B}$ is regularly embedded in $\mathfrak{Z}$.

(f) If $I \subseteq D$ is closed under either $\cap$ or $\cup$, then the topologically closed subalgebra of $\mathfrak{Z}$ generated by I, which is also the order-closed subalgebra of $\mathfrak{Z}$ generated by I, is included in D.

proof (a) There is a $J \subseteq \mathbb{N}$ such that $d(J)$ is defined and $I \triangle J \in \mathcal{Z}$. But in this case $d(I \triangle J) = 0$, so $d(I \cup J) = d(J) + d(I \setminus J)$ is defined; also $d(J \setminus I) = 0$, so $d(I) = d(I \cup J) - d(J \setminus I)$ is defined.

(b) These facts all follow directly from the corresponding results concerning $\mathcal{P}\mathbb{N}$ and d (491Ac).

(c) All we have to know is that $a \mapsto \bar{d}^*(a)$, $a \mapsto 1 \setminus a$ are continuous; so that $\{a : \bar{d}^*(a) + \bar{d}^*(1 \setminus a) = 1\}$ is closed.

(d) Because A is upwards-directed, and $\bar{d}^*$ is a non-decreasing functional on $\mathfrak{Z}$, there is a non-decreasing sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ in A such that $\lim_{n \to \infty} \bar{d}^*(a_n) = \sup_{a \in A} \bar{d}^*(a) = \gamma$ say. By 491J, $b = \lim_{n \to \infty} a_n = \sup_{n \in \mathbb{N}} a_n$ is defined in $\mathfrak{Z}$ and belongs to D. If $a \in A$ and $\epsilon > 0$, there is an $n \in \mathbb{N}$ such that $\bar{d}^*(a_n) \geq \gamma - \epsilon$. Let $a' \in A$ be a common upper bound of a and a_n. Then

$$\bar{d}^*(a \setminus b) \leq \bar{d}^*(a' \setminus a_n) = \bar{d}^*(a') - \bar{d}^*(a_n) \leq \gamma - \bar{d}^*(a_n) \leq \epsilon.$$

As ϵ is arbitrary, $a \subseteq b$; as a is arbitrary, b is an upper bound of A; as $b = \sup_{n \in \mathbb{N}} a_n$, b must be the supremum of A.

(e)(i)⇒(ii) Suppose that $\mathfrak{B}$ is topologically closed. If $A \subseteq \mathfrak{B}$ is a non-empty upwards-directed subset with supremum $b \in \mathfrak{Z}$, then (d) tells us that $b \in \overline{A} \subseteq \mathfrak{B}$. It follows that $\mathfrak{B}$ is order-closed in $\mathfrak{Z}$ (313E).

(ii)⇒(iii) Suppose that $\mathfrak{B}$ is order-closed in $\mathfrak{Z}$. If $A \subseteq \mathfrak{B}$ is non-empty, then $A' = \{a_0 \cup \ldots \cup a_n : a_0, \ldots, a_n \in A\}$ is non-empty and upwards-directed, so has a supremum in $\mathfrak{Z}$, which must belong to $\mathfrak{B}$, and must be the least upper bound of A in $\mathfrak{B}$. Thus $\mathfrak{B}$ is Dedekind (σ-)complete. Now let $\langle a_n \rangle_{n \in \mathbb{N}}$ be a disjoint sequence in $\mathfrak{B}$ and set $b_n = \sup_{i \leq n} a_i$ for each n. Then $\langle b_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in D so has a limit and supremum $b \in D$, and $b \in \mathfrak{B}$. Also $\bar{d}^*(b_n) = \sum_{i=0}^n \bar{d}^*(a_i)$ for each n (induce on n), so

$$\bar{\nu} b = \bar{d}^*(b) = \lim_{n \to \infty} \bar{d}^*(b_n) = \sum_{i=0}^\infty \bar{d}^*(a_i) = \sum_{i=0}^\infty \bar{\nu} a_i.$$

Since certainly $\bar{\nu} 0 = 0$, $\bar{\nu} 1 = 1$ and $\bar{\nu} b > 0$ whenever $b \in \mathfrak{B} \setminus \{0\}$, $(\mathfrak{B}, \bar{\nu})$ is a probability algebra.

(iii)⇒(i) Suppose that $(\mathfrak{B}, \bar{\nu})$ is a probability algebra. Then it is complete under its measure metric (323Gc), which agrees on $\mathfrak{B}$ with the metric $\bar{\rho}$ of $\mathfrak{Z}$; so $\mathfrak{B}$ must be topologically closed in $\mathfrak{Z}$.

We see also that $\mathfrak{B}$ is regularly embedded in $\mathfrak{Z}$. **P** (Compare 323H.) If $A \subseteq \mathfrak{B}$ is non-empty and downwards-directed and has infimum 0 in $\mathfrak{B}$, and $b \in \mathfrak{Z}$ is any lower bound of A in $\mathfrak{Z}$, then

$$\bar{d}^*(b) \leq \inf_{a \in A} \bar{d}^*(a) = \inf_{a \in A} \bar{\nu} a = 0$$

(321F), so $b = 0$. Thus $\inf A = 0$ in $\mathfrak{Z}$. As A is arbitrary, this is enough to show that the identity map from $\mathfrak{B}$ to $\mathfrak{Z}$ is order-continuous (313Lb), that is, that $\mathfrak{B}$ is regularly embedded in $\mathfrak{Z}$. **Q**

(f) Suppose that $I \subseteq D$ is closed under $\cap$. By 313Gc, D includes the order-closed subalgebra of $\mathfrak{Z}$ generated by I. If, on the other hand, I is closed under $\cup$, then $I' = \{1 \setminus a : a \in I\}$ is a subset of D closed under $\cap$, so the order-closed subalgebra generated by I, which is also the order-closed subalgebra generated by I', is again included in D.

491L Effectively regular measures The examples 491Xe and 491Yb show that the definition in 491B is drawn a little too wide for comfort, and allows some uninteresting pathologies. These do not arise in the measure spaces we care most about, and the following definitions provide a fire-break. Let (X, Σ, μ) be a measure space, and $\mathfrak{T}$ a topology on X.

(a) I will say that a measurable subset K of X of finite measure is **regularly enveloped** if for every $\epsilon > 0$ there are an open measurable set G and a closed measurable set F such that $K \subseteq G \subseteq F$ and $\mu(F \setminus K) \leq \epsilon$.

(b) Note that the family $\mathcal{K}$ of regularly enveloped measurable sets of finite measure is closed under finite unions and countable intersections. **P** (i) If K_1, $K_2 \in \mathcal{K}$ and $*$ is either $\cup$ or $\cap$, let $\epsilon > 0$. Take measurable open sets G_1, G_2 and measurable closed sets F_1, F_2 such that $K_i \subseteq G_i \subseteq F_i$ and $\mu(F_i \setminus K_i) \leq \frac{1}{2}\epsilon$ for both i. Then $G_1 * G_2$ is a measurable open set, $F_1 * F_2$ is a measurable closed set, $K_1 * K_2 \subseteq G_1 * G_2 \subseteq F_1 * F_2$ and $\mu((F_1 * F_2) \setminus (K_1 * K_2)) \leq \epsilon$. As ϵ is arbitrary, $K_1 * K_2 \in \mathcal{K}$. (ii) If $\langle K_n \rangle_{n \in \mathbb{N}}$ is a non-increasing sequence in $\mathcal{K}$ with intersection K and $\epsilon > 0$, let $n \in \mathbb{N}$ be such that $\mu K_n < \mu K + \epsilon$. Then we can find a measurable open set G and a measurable closed set F such that $K_n \subseteq G \subseteq F$ and $\mu F \leq \mu K + \epsilon$. As ϵ is arbitrary, $K \in \mathcal{K}$. Together with (i), this is enough to show that $\mathcal{K}$ is closed under countable intersections. **Q**

(c) Now I say that μ is **effectively regular** if it is inner regular with respect to the regularly enveloped sets of finite measure.

491M Examples (a) Any totally finite Radon measure is effectively regular. **P** Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a totally finite Radon measure space. If $K \subseteq X$ is compact and $\epsilon > 0$, let $L \subseteq X \setminus K$ be a compact set such that $\mu L \geq 1 - \mu K + \epsilon$. Let G, H be disjoint open sets including K, L respectively (4A2F(h-i)). Then $K \subseteq G \subseteq X \setminus H$, G is open, $X \setminus H$ is closed, both G and $X \setminus H$ are measurable, and $\mu((X \setminus H) \setminus K) \leq \epsilon$. This shows that every compact set is regularly enveloped, and μ is effectively regular. **Q**

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a quasi-Radon measure space such that $\mathfrak{T}$ is a regular topology. Then μ is effectively regular. **P** Let $E \in \Sigma$ and take $\gamma < \mu E$. Choose sequences $\langle E_n \rangle_{n \in \mathbb{N}}$ and $\langle G_n \rangle_{n \in \mathbb{N}}$ inductively, as follows. $E_0 \subseteq E$ is to be any measurable set such that $\gamma < \mu E_0 < \infty$. Given that $\mu E_n > \gamma$, let G be an open set of finite measure such that $\mu(E_n \cap G) > \gamma$ (414Ea), and $F \subseteq G \setminus E_n$ a closed set such that $\mu F \geq \mu(G \setminus E_n) - 2^{-n}$. Let $\mathcal{H}$ be the family of open sets H such that $\overline{H} \subseteq G \setminus F$. Then $\mathcal{H}$ is upwards-directed and covers E_n (because $\mathfrak{T}$ is regular), so there is a $G_n \in \mathcal{H}$ such that $\mu(E_n \cap G_n) > \gamma$ (414Ea again). Now $\mu(\overline{G}_n \setminus E_n) \leq 2^{-n}$. Set $E_{n+1} = E_n \cap G_n$, and continue.

At the end of the induction, set $K = \bigcap_{n \in \mathbb{N}} E_n$. For each n, $K \subseteq G_n \subseteq \overline{G}_n$ and

$$\lim_{n \to \infty} \mu(\overline{G}_n \setminus K) \leq \lim_{n \to \infty} 2^{-n} + \mu(E_n \setminus K) = 0,$$

so K is regularly enveloped. At the same time, $K \subseteq E$ and $\mu K \geq \gamma$. As E and γ are arbitrary, μ is effectively regular. **Q**

(c) Any totally finite Baire measure is effectively regular. **P** Let μ be a totally finite Baire measure on a topological space X. If $F \subseteq X$ is a zero set, let $f : X \to \mathbb{R}$ be a continuous function such that $F = f^{-1}[\{0\}]$. For each $n \in \mathbb{N}$, set $G_n = \{x : |f(x)| < 2^{-n}\}$, $F_n = \{x : |f(x)| \leq 2^{-n}\}$; then G_n is a measurable open set, F_n is a measurable closed set, $F \subseteq G_n \subseteq F_n$ for every n and $\lim_{n \to \infty} \mu F_n = \mu F$ (because μ is totally finite). This shows that every zero set is regularly enveloped; as μ is inner regular with respect to the zero sets (412D), μ is effectively regular. **Q**

(d) A totally finite completion regular topological measure is effectively regular. (As in (c), all zero sets are regularly enveloped.)

491N Theorem Let X be a topological space and μ an effectively regular probability measure on X, with measure algebra $(\mathfrak{A}, \bar{\mu})$. Suppose that $\langle x_n \rangle_{n \in \mathbb{N}}$ is an equidistributed sequence in X. Then we have a unique order-continuous Boolean homomorphism $\pi : \mathfrak{A} \to \mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ such that $\pi G^\bullet \subseteq \{n : x_n \in G\}^\bullet$ for every measurable open set $G \subseteq X$, and $\bar{d}^*(\pi a) = \bar{\mu} a$ for every $a \in \mathfrak{A}$.

proof (a) Define $\theta : \mathcal{P}X \to \mathfrak{Z}$ by setting $\theta A = \{n : x_n \in A\}^\bullet$ for $A \subseteq X$; then θ is a Boolean homomorphism. If $F \subseteq X$ is closed and measurable, then $\bar{d}^*(\theta F) \leq \mu F$, because $\langle x_n \rangle_{n \in \mathbb{N}}$ is equidistributed. Write $\mathcal{K}$ for the family of regularly enveloped measurable sets.

If $K \in \mathcal{K}$, then $\pi_0 K = \inf\{\theta G : K \subseteq G \in \Sigma \cap \mathfrak{T}\}$ is defined in $\mathfrak{Z}$, $\bar{d}^*(\pi_0 K) = \mu K$ and $\pi_0 K \in D$ as defined in 491K. **P** For each $n \in \mathbb{N}$, let G_n, $F_n \in \Sigma$ be such that $K \subseteq G_n \subseteq F_n$, G_n is open, F_n is closed and $\mu(F_n \setminus K) \leq 2^{-n}$. Set $H_n = X \setminus \bigcap_{i \leq n} G_i$. Then

$$\begin{aligned}\bar{d}^*(\theta H_n) + \bar{d}^*(1 \setminus \theta H_n) &\leq \bar{d}^*(\theta H_n) + \bar{d}^*(\theta(\bigcap_{i \leq n} F_i)) \leq \mu H_n + \mu(\bigcap_{i \leq n} F_i) \\ &\leq \mu(X \setminus K) + \mu F_n \leq 1 + 2^{-n}.\end{aligned}$$

Also $\langle \theta H_n \rangle_{n \in \mathbb{N}}$ is a non-decreasing sequence in $\mathfrak{Z}$. By 491J, $a = \lim_{n \to \infty} \theta H_n = \sup_{n \in \mathbb{N}} \theta H_n$ is defined in $\mathfrak{Z}$ and belongs to D. Set

$$b = 1 \setminus a = \lim_{n \to \infty} 1 \setminus \theta H_n = \lim_{n \to \infty} \theta(\textstyle\bigcap_{i \leq n} G_i),$$

so that b also belongs to D. If $K \subseteq G \in \Sigma \cap \mathfrak{T}$, then

$$b \setminus \theta G = \lim_{n \to \infty} \theta(\textstyle\bigcap_{i \leq n} G_i) \setminus \theta G = \lim_{n \to \infty} \theta(\textstyle\bigcap_{i \leq n} G_i \setminus G),$$

and

$$\begin{aligned}\bar{d}^*(b \setminus \theta G) &= \lim_{n \to \infty} \bar{d}^*(\theta(\bigcap_{i \leq n} G_i \setminus G)) \\ &\leq \lim_{n \to \infty} \bar{d}^*(\theta(\bigcap_{i \leq n} F_i \setminus G)) \leq \lim_{n \to \infty} \mu(\bigcap_{i \leq n} F_i \setminus G) = 0.\end{aligned}$$

This shows that $b \subseteq \theta G$ whenever $K \subseteq G \in \Sigma \cap \mathfrak{T}$. On the other hand, any lower bound of $\{\theta G : K \subseteq G \in \Sigma \cap \mathfrak{T}\}$ is also a lower bound of $\{\theta(\bigcap_{i \leq n} G_i) : n \in \mathbb{N}\}$, so is included in b. Thus $b = \inf\{\theta G : K \subseteq G \in \Sigma \cap \mathfrak{T}\}$.

To compute $\bar{d}^*(b)$, observe first that $b \subseteq 1 \setminus \theta H_n \subseteq \theta F_n$ for every n, so

$$\bar{d}^*(b) \leq \inf_{n \in \mathbb{N}} \bar{d}^*(\theta F_n) \leq \inf_{n \in \mathbb{N}} \mu F_n = \mu K.$$

On the other hand,

$$\bar{d}^*(\theta(\textstyle\bigcap_{i \leq n} G_i)) \geq 1 - \bar{d}^*(\theta H_n) \geq 1 - \mu H_n \geq \mu K$$

for every n, so

$$\bar{d}^*(b) = \lim_{n \to \infty} \bar{d}^*(\theta(\textstyle\bigcap_{i \leq n} G_i)) \geq \mu K.$$

Accordingly $\bar{d}^*(b) = \mu K$, and we can take b for $\pi_0 K$. **Q**

(b) If K, $L \in \mathcal{K}$, then $\pi_0(K \cap L) = \pi_0 K \cap \pi_0 L$. **P** We know that $K \cap L \in \mathcal{K}$ (491Lb). Now

$$\begin{aligned}\pi_0 K \cap \pi_0 L &= \inf\{\theta G : K \subseteq G \in \mathfrak{T}\} \cap \inf\{\theta H : L \subseteq H \in \mathfrak{T}\} \\ &= \inf\{\theta G \cap \theta H : K \subseteq G \in \mathfrak{T},\ L \subseteq H \in \mathfrak{T}\} \\ &= \inf\{\theta(G \cap H) : K \subseteq G \in \mathfrak{T},\ L \subseteq H \in \mathfrak{T}\} \supseteq \pi_0(K \cap L).\end{aligned}$$

Now suppose that $U \supseteq K \cap L$ is a measurable open set and $\epsilon > 0$. Let G, G' be measurable open sets and F, F' measurable closed sets such that $K \subseteq G \subseteq F$, $L \subseteq G' \subseteq F'$, $\mu(F \setminus K) \leq \epsilon$ and $\mu(F' \setminus L) \leq \epsilon$. Then

$$\begin{aligned}\bar{d}^*(\pi_0 K \cap \pi_0 L \setminus \theta U) &\leq \bar{d}^*(\theta G \cap \theta G' \setminus \theta U) = \bar{d}^*(\theta(G \cap G' \setminus U)) \\ &\leq \bar{d}^*(\theta(F \cap F' \setminus U)) \leq \mu(F \cap F' \setminus U) \leq 2\epsilon.\end{aligned}$$

As ϵ is arbitrary, $\pi_0 K \cap \pi_0 L \subseteq \theta U$; as U is arbitrary, $\pi_0 K \cap \pi_0 L \subseteq \pi_0(K \cap L)$. **Q**

This means that $\{\pi_0 K : K \subseteq X \text{ is a regularly embedded measurable set}\}$ is a subset of D closed under $\cap$. By 491Kf, the topologically closed subalgebra $\mathfrak{B}$ of $\mathfrak{Z}$ generated by this family is included in D; by 491Ke, $\mathfrak{B}$ is order-closed and regularly embedded in $\mathfrak{Z}$, and $(\mathfrak{B}, \bar{d}^* \restriction \mathfrak{B})$ is a probability algebra.

(c) Now observe that if we set $Q = \{K^\bullet : K \in \mathcal{K}\} \subseteq \mathfrak{A}$, we have a function $\pi : Q \to \mathfrak{B}$ defined by setting $\pi K^\bullet = \pi_0 K$ whenever $K \in \mathcal{K}$. **P** Suppose that K, $L \in \mathcal{K}$ are compact and $\mu(K \triangle L) = 0$. Then

$$\bar{d}^*(\pi_0 K \triangle \pi_0 L) = \bar{d}^*(\pi_0 K) + \bar{d}^*(\pi_0 L) - 2\bar{d}^*(\pi_0 K \cap \pi_0 L)$$

(because $\pi_0 K$ and $\pi_0 L$ belong to $\mathfrak{B} \subseteq D$)

$$= \bar{d}^*(\pi_0 K) + \bar{d}^*(\pi_0 L) - 2\bar{d}^*(\pi_0(K \cap L))$$
$$= \mu K + \mu L - 2\mu(K \cap L) = 0.$$

So $\pi_0 K = \pi_0 L$ and either can be used to define $\pi K^\bullet$. **Q** Next, the same formulae show that $\pi : Q \to \mathfrak{B}$ is an isometry when Q is given the measure metric of $\mathfrak{A}$, since if K, $L \subseteq X$ are compact

$$\bar{\rho}(\pi K^\bullet, \pi L^\bullet) = \bar{d}^*(\pi_0 K \triangle \pi_0 L) = \mu K + \mu L - 2\mu(K \cap L) = \mu(K \triangle L) = \bar{\mu}(K^\bullet \triangle L^\bullet).$$

As Q is dense in $\mathfrak{A}$ (412N), there is a unique extension of π to an isometry from $\mathfrak{A}$ to $\mathfrak{B}$.

(d) Because

$$\pi(K^\bullet \cap L^\bullet) = \pi(K \cap L)^\bullet = \pi_0(K \cap L) = \pi_0 K \cap \pi_0 L = \pi K^\bullet \cap \pi L^\bullet$$

for all K, $L \in \mathcal{K}$, $\pi(a \cap a') = \pi a \cap \pi a'$ for all a, $a' \in \mathfrak{A}$. It follows that π is a Boolean homomorphism. **P** The point is that $\bar{d}^*(\pi a) = \bar{\mu} a$ for every $a \in Q$, and therefore for every $a \in \mathfrak{A}$. Now if $a \in \mathfrak{A}$, $\pi(1 \setminus a)$ must be disjoint from πa (since certainly $\pi 0 = 0$), and has the same measure as $1 \setminus \pi a$ (remember that we know that $(\mathfrak{B}, \bar{d}^* \restriction \mathfrak{B})$ is a measure algebra), so must be equal to $1 \setminus \pi a$. By 312H, π is a Boolean homomorphism. **Q**

By 324G, π is order-continuous when regarded as a function from $\mathfrak{A}$ to $\mathfrak{B}$. Because $\mathfrak{B}$ is regularly embedded in $\mathfrak{Z}$, π is order-continuous when regarded as a function from $\mathfrak{A}$ to $\mathfrak{Z}$.

(e) Let $G \in \Sigma \cap \mathfrak{T}$. For any $\epsilon > 0$, there is a $K \in \mathcal{K}$ such that $K \subseteq G$ and $\mu(G \setminus K) \leq \epsilon$. In this case, $\pi K^\bullet = \pi_0 K \subseteq \theta G$. So

$$\bar{d}^*(\pi G^\bullet \setminus \theta G) \leq \bar{d}^*(\pi G^\bullet \setminus \pi K^\bullet) = \bar{\mu}(G^\bullet \setminus K^\bullet) = \mu(G \setminus K) \leq \epsilon.$$

As ϵ is arbitrary, $\pi G^\bullet \subseteq \theta G$.

(f) This shows that we have a homomorphism π with the required properties. To see that π is unique, suppose that $\pi' : \mathfrak{A} \to \mathfrak{Z}$ is any homomorphism of the same kind. In this case

$$\bar{d}^*(1 \setminus \pi' a) = \bar{d}^*(\pi'(1 \setminus a)) = \bar{\mu}(1 \setminus a) = 1 - \bar{\mu} a = 1 - \bar{d}^*(\pi' a),$$

so $\pi' a \in D$, for every $a \in \mathfrak{A}$. If $K \in \mathcal{K}$, then $\pi' K^\bullet \subseteq \theta G$ whenever $K \subseteq G \in \Sigma \cap \mathfrak{T}$, so $\pi' K^\bullet \subseteq \pi_0 K = \pi K^\bullet$. As both $\pi K^\bullet$ and $\pi' K^\bullet$ belong to D,

$$\bar{d}^*(\pi K^\bullet \setminus \pi' K^\bullet) = \bar{d}^*(\pi K^\bullet) - \bar{d}^*(\pi' K^\bullet) = \mu K - \mu K = 0,$$

and $\pi K^\bullet = \pi' K^\bullet$. As $\{K^\bullet : K \in \mathcal{K}\}$ is topologically dense in $\mathfrak{A}$, and both π and π' are continuous, they must be equal.

491O Proposition Let X be a topological space and μ an effectively regular probability measure on X which measures every zero set, and suppose that $\langle x_n \rangle_{n \in \mathbb{N}}$ is an equidistributed sequence in X. Let $\mathfrak{A}$ be the measure algebra of μ and $\pi : \mathfrak{A} \to \mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ the regular embedding described in 491N; let $T_\pi : L^\infty(\mathfrak{A}) \to L^\infty(\mathfrak{Z})$ be the corresponding order-continuous Banach algebra embedding (363F). Let $S : \ell^\infty(X) \to \ell^\infty$ be the Riesz homomorphism defined by setting $(Sf)(n) = f(x_n)$ for $f \in \ell^\infty(X)$ and $n \in \mathbb{N}$, and $R : \ell^\infty \to L^\infty(\mathfrak{Z})$ the Riesz homomorphism corresponding to the Boolean homomorphism $I \mapsto I^\bullet : \mathcal{P}\mathbb{N} \to \mathfrak{Z}$. For $f \in \mathfrak{L}^\infty(\mu)$ let $f^\bullet$ be the corresponding member of $L^\infty(\mu) \cong L^\infty(\mathfrak{A})$ (363I). Then $T_\pi(f^\bullet) = RSf$ for every $f \in C_b(X)$.

proof To begin with, suppose that $f : X \to [0,1]$ is continuous and $k \geq 1$. For each $i \leq k$ set $G_i = \{x : f(x) > \frac{i}{k}\}$, $F_i = \{x : f(x) \geq \frac{i}{k}\}$. Then $\frac{1}{k}\sum_{i=1}^k \chi F_i \leq f \leq \frac{1}{k}\sum_{i=0}^k \chi G_i$. So

$$\frac{1}{k}\sum_{i=1}^k \chi(\pi F_i^\bullet) \leq T_\pi f^\bullet \leq \frac{1}{k}\sum_{i=0}^k \chi(\pi G_i^\bullet),$$

$$\frac{1}{k}\sum_{i=1}^k \chi(\theta F_i) \leq RSf \leq \frac{1}{k}\sum_{i=0}^k \chi(\theta G_i)$$

where $\theta : \mathcal{P}X \to \mathfrak{Z}$ is the Boolean homomorphism described in 491N, because $RS : \ell^\infty \to L^\infty(\mathfrak{Z})$ is the Riesz homomorphism corresponding to θ (see 363Fa, 363Fg). Now 491N tells us that $\pi G^\bullet \subseteq \theta G$ for every open $G \subseteq X$, so

$$\begin{aligned} T_\pi f^\bullet &\leq \frac{1}{k}\sum_{i=0}^{k} \chi(\pi G_i^\bullet) \leq \frac{1}{k}\sum_{i=0}^{k} \chi(\theta G_i) \\ &\leq \frac{1}{k}\sum_{i=0}^{k} \chi(\theta F_i) = \frac{1}{k}e + \sum_{i=1}^{k} \chi(\theta F_i) \leq \frac{1}{k}e + RSf \end{aligned}$$

where e is the standard order unit of the M-space $L^\infty(\mathfrak{Z})$. But looking at complements we see that we must have $\pi F^\bullet \supseteq \theta F$ for every closed set $F \subseteq X$, so

$$\begin{aligned} RSf &\leq \frac{1}{k}\sum_{i=0}^{k} \chi(\theta G_i) \leq \frac{1}{k}\sum_{i=0}^{k} \chi(\theta F_i) \\ &\leq \frac{1}{k}\sum_{i=0}^{k} \chi(\pi F_i^\bullet) = \frac{1}{k}e + \sum_{i=1}^{k} \chi(\pi F_i^\bullet) \leq \frac{1}{k}e + T_\pi f^\bullet. \end{aligned}$$

This means that $|T_\pi f^\bullet - RSf| \leq \frac{1}{k}e$ for every $k \geq 1$, so that $T_\pi f^\bullet = RSf$. This is true whenever $f \in C_b(X)$ takes values in $[0,1]$; as all the operators here are linear, it is true for every $f \in C_b(X)$.

491P Proposition Any probability algebra $(\mathfrak{A}, \bar\mu)$ of cardinal at most $\mathfrak{c}$ can be regularly embedded as a subalgebra of $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ in such a way that $\bar\mu$ is identified with the restriction of the submeasure $\bar d^*$ to the image of $\mathfrak{A}$.

proof The usual measure of $\{0,1\}^{\mathfrak{c}}$ is a totally finite Radon measure, so is effectively regular (491Ma). It has an equidistributed sequence (491G), so its measure algebra $(\mathfrak{B}_{\mathfrak{c}}, \bar\nu_{\mathfrak{c}})$ can be regularly embedded in $\mathfrak{Z}$ in a way which matches $\bar\nu_{\mathfrak{c}}$ with $\bar d^*$ (491N). Now if $(\mathfrak{A}, \bar\mu)$ is any probability algebra of cardinal at most $\mathfrak{c}$, it can be regularly embedded (by a measure-preserving homomorphism) in $(\mathfrak{B}_{\mathfrak{c}}, \bar\nu_{\mathfrak{c}})$ (332N), and therefore in $(\mathfrak{Z}, \bar d^*)$. This completes the proof.

491Q Corollary Every Radon probability measure on $\{0,1\}^{\mathfrak{c}}$ has an equidistributed sequence.

proof Let μ be a Radon probability measure on $\{0,1\}^{\mathfrak{c}}$, and $(\mathfrak{A}, \bar\mu)$ its measure algebra. By 491P, there is a measure-preserving embedding $\pi : \mathfrak{A} \to \mathfrak{Z}$, and $\pi[\mathfrak{A}] \subseteq D$ as defined in 491K. For $\xi < \mathfrak{c}$ let $a_\xi \in \mathfrak{A}$ be the equivalence class of $\{x : x(\xi) = 1\}$, and let $I_\xi \subseteq \mathbb{N}$ be such that $I_\xi^\bullet = \pi a_\xi$ in $\mathfrak{Z}$. Define $x_n(\xi)$, for $n \in \mathbb{N}$ and $\xi < \mathfrak{c}$, by setting $x_n(\xi) = 1$ if $n \in I_\xi$, 0 otherwise. Now suppose that $E \subseteq \{0,1\}^{\mathfrak{c}}$ is a basic open set of the form $\{x : x(\xi) = 1 \text{ for } \xi \in K,\ 0 \text{ for } \xi \in L\}$, where K, $L \subseteq \mathfrak{c}$ are finite. Set $b = \pi E^\bullet$ in $\mathfrak{Z}$,

$$I = \{n : x_n \in E\} = \mathbb{N} \cap \textstyle\bigcap_{\xi\in K} I_\xi \setminus \bigcup_{\xi\in L} I_\xi.$$

Then

$$\begin{aligned} b = \pi E^\bullet &= \pi(\inf_{\xi\in K} a_\xi \setminus \sup_{\xi\in L} a_\xi) \\ &= \inf_{\xi\in K} \pi a_\xi \setminus \sup_{\xi\in L} \pi a_\xi = \inf_{\xi\in K} I_\xi^\bullet \setminus \sup_{\xi\in L} I_\xi^\bullet = I^\bullet. \end{aligned}$$

Since $b \in D$, $d(I)$ is defined and is equal to $\bar d^*(b) = \bar\mu E^\bullet = \mu E$.

If we now take E to be an open-and-closed subset of $\{0,1\}^{\mathfrak{c}}$, it can be expressed as a disjoint union of finitely many basic open sets of the type just considered; because d is additive on disjoint sets, $d(\{n : x_n \in E\})$ is defined and equal to μE. But this is enough to ensure that $\langle x_n\rangle_{n\in\mathbb{N}}$ is equidistributed, by 491Ch.

491X Basic exercises (a) (i) Show that if I, $J \in \mathcal{D} = \operatorname{dom} d$ as defined in 491A, then $I \cup J \in \mathcal{D}$ iff $I \cap J \in \mathcal{D}$ iff $I \setminus J \in \mathcal{D}$ iff $I \triangle J \in \mathcal{D}$. (ii) Show that if $\mathcal{E} \subseteq \mathcal{D}$ is an algebra of sets, then $d\restriction\mathcal{E}$ is additive. (iii) Find I, $J \in \mathcal{D}$ such that $I \cap J \notin \mathcal{D}$.

>(b) Suppose that $I \subseteq \mathbb{N}$ and that $f : \mathbb{N} \to \mathbb{N}$ is strictly increasing. Show that $d^*(f[I]) \leq d^*(I)d^*(f[\mathbb{N}])$, with equality if either I or $f[\mathbb{N}]$ has asymptotic density.

(c) Show that if $A \subseteq \mathbb{N}$ and $0 \leq \alpha \leq d^*(A)$ there is a $B \subseteq A$ such that $d^*(B) = \alpha$ and $d^*(A \setminus B) = d^*(A) - \alpha$.

(d) (i) Let $I \subseteq \mathbb{N}$ be such that $d(J) = d^*(J \cap I) + d^*(J \setminus I)$ for every $J \subseteq \mathbb{N}$ such that $d(J)$ is defined. Show that either $I \in \mathcal{Z}$ or $\mathbb{N} \setminus I \mathcal{Z}$. (ii) Show that for every $\epsilon > 0$ there is an $I \subseteq \mathbb{N}$ such that $d^*(I) = \epsilon$ but $d(J) = 1$ whenever $J \supseteq I$ and $d(J)$ is defined.

(e) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a compact Radon probability space. Take any point ∞ not belonging to X, and give $X \cup \{\infty\}$ the topology generated by $\{G \cup \{\infty\} : G \in \mathfrak{T}\}$. Show that $X \cup \{\infty\}$ is compact and that the image measure μ_∞ of μ under the identity map from X to $X \cup \{\infty\}$ is a quasi-Radon measure, inner regular with respect to the compact sets. Show that if we set $x_n = \infty$ for every n, then $\langle x_n \rangle_{n \in \mathbb{N}}$ is equidistributed for μ_∞.

>(f) (i) Show that a sequence $\langle t_n \rangle_{n \in \mathbb{N}}$ in $[0,1]$ is equidistributed with respect to Lebesgue measure iff $\lim_{n \to \infty} \frac{1}{n+1} \#(\{i : i \leq n, t_i \leq \beta\}) = \beta$ for every $\beta \in [0,1]$. (ii) Show that if $\alpha \in \mathbb{R}$ is irrational then the sequence $\langle <n\alpha> \rangle_{n \in \mathbb{N}}$ of fractional parts of multiples of α is equidistributed in $[0,1]$ with respect to Lebesgue measure. (*Hint*: 281Yi.)

(g) Show that the usual measure on the split interval (419L) has an equidistributed sequence.

>(h) Show that if X is a regular Hausdorff space and $f : \mathbb{N} \to X$ is injective, then there is an open set $G \subseteq X$ such that $f^{-1}[G]$ does not have asymptotic density.

>(i) Let Z, μ, $X = Z \times \{0,1\}$ and ν be as described in 439K, so that μ is a Radon probability measure on the compact metrizable space Z, X has a compact Hausdorff topology finer than the product topology, and ν is a measure on Z extending μ. (i) Show that if $f \in C(X)$, then $\{t : t \in Z, f(t,0) \neq f(t,1)\}$ is countable. (*Hint*: 4A2F(h-vii).) (ii) Show that $\int f(t,0)\mu(dt) = \int f(t,1)\nu(dt)$ for every $f \in C(X)$. (iii) Let λ be the measure νg^{-1}, where $g(t) = (t,1)$ for $t \in Z$. Show that there is a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in $Z \times \{0\}$ such that $\int f d\lambda = \lim_{n \to \infty} \frac{1}{n+1} \sum_{i=0}^{n} f(x_i)$ for every $f \in C(X)$, but that $\langle x_n \rangle_{n \in \mathbb{N}}$ is not λ-equidistributed.

(j) Show that a function $f : [0,1] \to \mathbb{R}$ is Riemann integrable iff $\lim_{n \to \infty} \frac{1}{n+1} \sum_{i=0}^{n} f(x_i)$ is defined in $\mathbb{R}$ for every sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in $[0,1]$ which is equidistributed for Lebesgue measure.

(k) Let X be a metrizable space, and μ a quasi-Radon probability measure on X. (i) Show that there is an equidistributed sequence for μ. (ii) Show that if the support of μ is not compact, and $\langle x_n \rangle_{n \in \mathbb{N}}$ is an equidistributed sequence for μ, then there is a continuous integrable function $f : X \to \mathbb{R}$ such that $\lim_{n \to \infty} \frac{1}{n+1} \sum_{i=0}^{n} f(x_i) = \infty$.

(l) Let $\phi : \mathfrak{c} \to \mathcal{P}\mathbb{N}$ be an injective function. For each $n \in \mathbb{N}$ let λ_n be the uniform measure on $\mathcal{P}(\mathcal{P}n)$, giving measure 2^{-2^n} to each point. Define $\psi_n : \mathcal{P}(\mathcal{P}n) \to \{0,1\}^{\mathfrak{c}}$ by setting $\psi_n(\mathcal{I})(\xi) = 1$ if $\phi(\xi) \cap n \in \mathcal{I}$, 0 otherwise, and let ν_n be the image measure $\lambda_n \psi_n^{-1}$. Show that $\nu_n E$ is the usual measure of E whenever $E \subseteq \{0,1\}^{\mathfrak{c}}$ is determined by coordinates in a finite set on which the map $\xi \mapsto \phi(\xi) \cap n$ is injective. Use this with 491D to prove 491G.

>(m) (i) Let Z be the Stone space of the measure algebra of Lebesgue measure on $[0,1]$, with its usual measure. Show that there is no equidistributed sequence in Z. (*Hint*: meager sets in Z have negligible closures.) (ii) Show that Dieudonné's measure on ω_1 (411Q) has no equidistributed sequence. (iii) Show that if $\#(I) > \mathfrak{c}$ then the usual measure on $\{0,1\}^I$ has no equidistributed sequence. (*Hint*: if $\langle x_n \rangle_{n \in \mathbb{N}}$ is any sequence in $\{0,1\}^I$, there is an infinite $J \subseteq I$ such that $\langle x_n(\eta) \rangle_{n \in \mathbb{N}} = \langle x_n(\xi) \rangle_{n \in \mathbb{N}}$ for all $\eta, \xi \in J$.) (iv) Show that if X is a topological group with a Haar probability measure μ, and X is not separable, then μ has no equidistributed sequence. (*Hint*: use 443D to show that every separable subset is negligible.)

(n) Let X be a compact Hausdorff abelian topological group and μ its Haar probability measure. Show that a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X is equidistributed for μ iff $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}\chi(x_i)=0$ for every non-trivial character $\chi: X\to S^1$. (*Hint*: 281G.)

(o) Show that $\mathfrak{Z}$ is not Dedekind σ-complete. (*Hint*: 392Hc.)

(p) Let $\langle a_n\rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence in $\mathfrak{Z}$. Show that there is an $a\in\mathfrak{Z}$ such that $a_n\subseteq a$ for every $n\in\mathbb{N}$ and $\bar{d}^*(a)=\sup_{n\in\mathbb{N}}\bar{d}^*(a_n)$.

(q) Let $\mathfrak{Z}$, $\bar{d}^*$ and D be as in 491K. Show that if $a\in D\setminus\{0\}$ and $\mathfrak{Z}_a$ is the principal ideal of $\mathfrak{Z}$ generated by a, then $(\mathfrak{Z}_a,\bar{d}^*\restriction\mathfrak{Z}_a)$ is isomorphic, up to a scalar multiple of the submeasure, to $(\mathfrak{Z},\bar{d}^*)$.

(r) Let (X,Σ,μ) be a semi-finite measure space and $\mathfrak{T}$ a topology on X. Show that μ is effectively regular iff whenever $E\in\Sigma$, $\mu E<\infty$ and $\epsilon>0$ there are a measurable open set G and a measurable closed set $F\supseteq G$ such that $\mu(F\setminus E)+\mu(E\setminus G)\leq\epsilon$.

(s) Let X be a normal topological space and μ a topological measure on X which is inner regular with respect to the closed sets and effectively locally finite. Show that μ is effectively regular.

(t) Let X be a topological space and μ an effectively regular measure on X. (i) Show that the completion and c.l.d. version of μ are also effectively regular. (ii) Show that if $Y\subseteq X$ then the subspace measure is again effectively regular. (iii) Show that any totally finite indefinite-integral measure over μ is effectively regular.

(u) (i) Let X_1, X_2 be topological spaces with effectively regular measures μ_1, μ_2. Show that the c.l.d. product measure on $X_1\times X_2$ is effectively regular with respect to the product topology. (*Hint*: 412R.) (ii) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces and μ_i an effectively regular probability measure on X_i for each i. Show that the product probability measure on $\prod_{i\in I}X_i$ is effectively regular.

(v) Give $[0,1]$ the topology $\mathfrak{T}$ generated by the usual topology and $\{[0,1]\setminus A: A\subseteq\mathbb{Q}\}$. Let μ_L be Lebesgue measure on $[0,1]$, and Σ its domain. For $E\in\Sigma$ set $\mu E=\mu_L E+\#(E\cap\mathbb{Q})$ if $E\cap\mathbb{Q}$ is finite, ∞ otherwise. Show that μ is a σ-finite quasi-Radon measure with respect to the topology $\mathfrak{T}$, but is not effectively regular.

(w) Let (X,ρ) be a separable metric space and μ a Borel probability measure on X. (i) Show that there is an equidistributed sequence in X. (ii) Show that if $\langle x_n\rangle_{n\in\mathbb{N}}$ is an equidistributed sequence in X, and $\langle y_n\rangle_{n\in\mathbb{N}}$ is a sequence in X such that $\lim_{n\to\infty}\rho(x_n,y_n)=0$, then $\langle y_n\rangle_{n\in\mathbb{N}}$ is equidistributed. (iii) Show that if $f: X\to\mathbb{R}$ is a bounded function, then $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}f(x_i)-f(y_i)=0$ for all equidistributed sequences $\langle x_n\rangle_{n\in\mathbb{N}}$, $\langle y_n\rangle_{n\in\mathbb{N}}$ in X iff $\{x: f$ is continuous at $x\}$ is conegligible, and in this case $\lim_{n\to\infty}\frac{1}{n+1}\sum_{i=0}^{n}f(x_i)=\int f d\mu$ for every equidistributed sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in X. (*Hint*: $\{x: f$ is continuous at $x\}=\bigcap_{m\in\mathbb{N}}G_m$, where $G_m=\bigcup\{G: G\subseteq X$ is open, $\sup_{x,y\in G}|f(x)-f(y)|\leq 2^{-m}\}$.) Compare 134L.

(x) Let $(X,\mathfrak{T})$ be a topological space, μ a probability measure on X, and $\phi: X\to X$ an inverse-measure-preserving function. (i) Suppose that $\mathfrak{T}$ has a countable network consisting of measurable sets, and that ϕ is ergodic. Show that $\langle\phi^n(x)\rangle_{n\in\mathbb{N}}$ is equidistributed for almost every $x\in X$. (*Hint*: 372Qb.) (ii) Suppose that μ is *either* inner regular with respect to the closed sets *or* effectively regular, and that $\{x:\langle\phi^n(x)\rangle_{n\in\mathbb{N}}$ is equidistributed$\}$ is not negligible. Show that ϕ is ergodic.

(y) Let $\mathfrak{A}$ be a countable Boolean algebra and ν a finitely additive functional on $\mathfrak{A}$ such that $\nu 1=1$. Show that there is a Boolean homomorphism $\pi:\mathfrak{A}\to\mathcal{P}\mathbb{N}$ such that $d(\pi a)$ is defined and equal to νa for every $a\in\mathfrak{A}$ (i) from 491Xc (ii) from 393B, 491P and 341Xf.

(z) Let X be a dyadic space. (i) Show that there is a Radon probability measure on X with support X. (ii) Show that the following are equiveridical: (α) $w(X)\leq\mathfrak{c}$; (β) every Radon probability measure on X has an equidistributed sequence; (γ) X is separable. (*Hint*: 4A2Dd, 418L.)

491Y Further exercises (a) Show that every subset A of $\mathbb{N}$ is expressible in the form $I_A \triangle J_A$ where $d(I_A) = d(J_A) = \frac{1}{2}$ (i) by a direct construction, with $A \mapsto I_A$ a continuous function (ii) using 443D.

(b) Find a topological space X with a τ-additive probability measure μ on X, a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in X and a base $\mathcal{G}$ for the topology of X, consisting of measurable sets and closed under finite intersections, such that $\mu G \leq \liminf_{n\to\infty} \frac{1}{n+1}\#(\{i : i \leq n, x_i \in G\})$ for every $G \in \mathcal{G}$ but $\langle x_n \rangle_{n\in\mathbb{N}}$ is not equidistributed. (*Hint*: take $\#(X) = 4$.)

(c) Let X be a compact Hausdorff space on which every Radon probability measure has an equidistributed sequence. Show that the cylindrical σ-algebra of $C(X)$ is the σ-algebra generated by sets of the form $\{f : f \in C(X), f(x) > \alpha\}$ where $x \in X$ and $\alpha \in \mathbb{R}$. (*Hint*: 436J, 491Cb.)

(d) Give ω_1+1 and $[0,1]$ their usual compact Hausdorff topologies. Let $\langle t_i \rangle_{i\in\mathbb{N}}$ be a sequence in $[0,1]$ which is equidistributed for Lebesgue measure μ_L, and set $Q = \{t_i : i \in \mathbb{N}\}$, $X = (\omega_1 \times ([0,1] \setminus Q)) \cup (\{\omega_1\} \times Q)$, with the subspace topology inherited from $(\omega_1 + 1) \times [0,1]$. (i) Set $F = \{\omega_1\} \times Q$. Show that F is a closed Baire set in the completely regular Hausdorff space X. (ii) Show that if $f \in C_b(X)$ then there are a $g_f \in C([0,1])$ and a $\zeta < \omega_1$ such that such that $f(\xi, t) = g_f(t)$ whenever $(\xi, t) \in X$ and $\zeta \leq \xi \leq \omega_1$. (iii) Show that there is a Baire measure μ on X such that $\int f d\mu = \int g_f d\mu_L$ for every $f \in C_b(X)$. (iv) Show that $\mu F = 0$. (v) Show that $\int f d\mu = \lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} f(\omega_1, t_i)$ for every $f \in C_b(X)$, but that $\langle (\omega_1, t_i) \rangle_{i\in\mathbb{N}}$ is not equidistributed with respect to μ.

(e) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a topological measure space. Let $\mathcal{E}$ be the Jordan algebra of X (411Yc). (i) Suppose that μ is a complete probability measure on X and $\langle x_n \rangle_{n\in\mathbb{N}}$ an equidistributed sequence in X. Show that the asymptotic density $d(\{n : x_n \in E\})$ is defined and equal to μE for every $E \in \mathcal{E}$. (ii) Suppose that μ is a probability measure on X and that $\langle x_n \rangle_{n\in\mathbb{N}}$ is a sequence in X such that $d(\{n : x_n \in E\})$ is defined and equal to μE for every $E \in \mathcal{E}$. Show that $\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} f(x_i) = \int f d\mu$ for every $f \in C_b(X)$.

(f) Show that a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in $[0,1]$ is equidistributed for Lebesgue measure iff there is some $r_0 \in \mathbb{N}$ such that $\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} x_i^r = \frac{1}{r+1}$ for every $r \geq r_0$.

(g) Show that a Radon measure on an extremally disconnected compact Hausdorff space has an equidistributed sequence iff it is point-supported. (*Hint*: see the hint for 326Yo.)

(h) Show that there is a countable dense set $D \subseteq [0,1]^{\mathfrak{c}}$ such that no sequence in D is equidistributed for the usual measure on $[0,1]^{\mathfrak{c}}$.

(i) Let X be a compact Hausdorff topological group. Show that a sequence $\langle x_n \rangle_{n\in\mathbb{N}}$ in X is equidistributed for the Haar probability measure on X iff $\lim_{n\to\infty} \frac{1}{n+1} \sum_{i=0}^{n} \phi(x_i) = 0$ for every non-trivial finite-dimensional representation ϕ of X. (*Hint*: 446Yb.)

(j) Show that $\mathfrak{Z}$ is weakly σ-distributed.

(k) Show that $\mathfrak{Z} \cong \mathfrak{Z}^{\mathbb{N}}$.

(l) Show that $\mathfrak{Z}$ has the σ-interpolation property (definition: 466G).

(m) Let $\langle X_\xi \rangle_{\xi<\mathfrak{c}}$ be a family of topological spaces with countable networks consisting of Borel sets, and μ a τ-additive topological probability measure on $X = \prod_{\xi<\mathfrak{c}} X_\xi$. Show that μ has an equidistributed sequence.

491Z Problem It is known that for almost every $x > 1$ the sequence $\langle < x^n > \rangle_{n\in\mathbb{N}}$ of fractional parts of powers of x is equidistributed for Lebesgue measure on $[0,1]$ (Kuipers & Niederreiter 74, p. 35). But is $\langle < (\frac{3}{2})^n > \rangle_{n\in\mathbb{N}}$ equidistributed?

491 Notes and comments The notations d^*, d (491A) are standard, and usefully suggestive. But coming from measure theory we have to remember that d^*, although a submeasure, is not an outer measure, the domain of d is not an algebra of sets (491Xa), and d and d^* are related by only one of the formulae we expect to connect a measure with an outer measure (491Ac, 491Xd). The limits $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n} f(x_i)$ are the **Cesaro means** of the sequences $\langle f(x_n)\rangle_{n\in\mathbb{N}}$. The delicacy of the arguments here arises from the fact that the family of (bounded) sequences with Cesaro means, although a norm-closed linear subspace of ℓ^∞, is neither a sublattice nor a subalgebra. When we turn to the quotient algebra $\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$, we find ourselves with a natural submeasure to which we can apply results from §393 to good effect (491I). What is striking is that equidistributed sequences induce regular embeddings of measure algebras in $\mathfrak{Z}$ which can be thought of as measure-preserving (491N).

Most authors have been content to define an 'equidistributed sequence' to be one such that the integrals of bounded continuous functions are correctly specified (491Cf, 491Cg); that is, that the point-supported measures $\frac{1}{n+1}\sum_{i=0}^{n} \delta_{x_i}$ converge to μ in the vague topology on an appropriate class of measures (437J). I am going outside this territory in order to cover some ideas I find interesting. 491Xi shows that it makes a difference; there are Borel measures on compact Hausdorff spaces which have sequences which give the correct Cesaro means for continuous functions, but lie within negligible closed sets; and the same can happen with Baire measures (491Yd). It seems to be difficult, in general, to determine whether a topological probability space – even a compact Radon probability space – has an equidistributed sequence. In the proofs of 491D-491G I have tried to collect the principal techniques for showing that spaces do have equidistributed sequences. In the other direction, it is obviously impossible for a space to have an equidistributed sequence if every separable subspace is negligible (491Xm). For an example of a separable compact Hausdorff space with a Radon measure which does not have an equidistributed sequence, we seem to have to go deeper (491Yg).

491Z is a famous problem. It is not clear that it is a problem in measure theory, and there is no reason to suppose that any of the ideas of this treatise beyond 491Xf(i) are relevant. I mention it because I think everyone should know that it is there.

492 Combinatorial concentration of measures

'Concentration of measure' takes its most dramatic form in the geometrically defined notions of concentration explored in §476. But the phenomenon is observable in many other contexts, if we can devise the right abstract geometries to capture it. In this section I present one of Talagrand's theorems on the concentration of measure in product spaces, using the Hamming metric (492D), and Maurey's theorem on concentration of measure in permutation groups (492H).

492A Lemma Let (X, Σ, μ) be a totally finite measure space, $\alpha < \beta$ in $\mathbb{R}$, $\phi : [\alpha, \beta] \to \mathbb{R}$ a convex function, and $f : X \to [\alpha, \beta]$ a Σ-measurable function. Then

$$\int \phi(f(x))\mu(dx) \leq \tfrac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha} \int f d\mu + \tfrac{\beta\phi(\alpha)-\alpha\phi(\beta)}{\beta-\alpha}\mu X.$$

proof If $t \in [\alpha, \beta]$ then $t = \frac{t-\alpha}{\beta-\alpha}\beta + \frac{\beta-t}{\beta-\alpha}\alpha$, so

$$\phi(t) \leq \tfrac{t-\alpha}{\beta-\alpha}\phi(\beta) + \tfrac{\beta-t}{\beta-\alpha}\phi(\alpha) = \tfrac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}t + \tfrac{\beta\phi(\alpha)-\alpha\phi(\beta)}{\beta-\alpha}.$$

Accordingly

$$\phi(f(x)) \leq \tfrac{\phi(\beta)-\phi(\alpha)}{\beta-\alpha}f(x) + \tfrac{\beta\phi(\alpha)-\alpha\phi(\beta)}{\beta-\alpha}$$

for every $x \in X$; integrating with respect to x, we have the result.

492B Corollary Let (X, Σ, μ) be a probability space and $f : X \to [\alpha, 1]$ a measurable function, where $0 < \alpha \leq 1$. Then $\int \frac{1}{f} d\mu \cdot \int f d\mu \leq \frac{(1+\alpha)^2}{4\alpha}$.

proof Set $\gamma = \int f d\mu$, so that $\alpha \leq \gamma \leq 1$. By 492A, with $\phi(t) = \frac{1}{t}$,

$$\int \frac{1}{f} d\mu \leq \frac{\gamma}{1-\alpha}(1 - \frac{1}{\alpha}) + \frac{1}{1-\alpha}(\frac{1}{\alpha} - \alpha) = \frac{1+\alpha-\gamma}{\alpha}.$$

Now $\frac{1+\alpha-\gamma}{\alpha} \cdot \gamma$ takes its maximum value $\frac{(1+\alpha)^2}{4\alpha}$ when $\gamma = \frac{1+\alpha}{2}$, so this is also the maximum possible value for $\int 1/f \cdot \int f$.

492C Lemma $\frac{1}{2}(1 + \cosh t) \leq e^{t^2/4}$ for every $t \in \mathbb{R}$.

proof For $k \geq 1$, $4^k k! \leq 2(2k)!$ (induce on k), so

$$1 + \cosh t = 2 + \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} \leq 2 + \sum_{k=1}^{\infty} \frac{2t^{2k}}{4^k k!} = 2e^{t^2/4}.$$

492D Theorem (TALAGRAND 95) Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i<n}$ be a non-empty finite family of probability spaces with product (X, Λ, λ). Let ρ be the **normalized Hamming metric** on X defined by setting $\rho(x, y) = \frac{1}{n}\#(\{i : i < n, x(i) \neq y(i)\})$ for x, $y \in X$. If $W \in \Lambda$ and $\lambda W > 0$, then

$$\overline{\int} e^{t\rho(x,W)} \lambda(dx) \leq \frac{1}{\lambda W} e^{t^2/4n}$$

for every $t \geq 0$.

proof The formulae below will go much more smoothly if we work with the simple Hamming metric $\sigma(x, y) = \#(\{i : x(i) \neq y(i)\}$ instead of ρ. In this case, we can make sense of the case $n = 0$, and this will be useful. In terms of σ, our target is to prove that if $W \in \Lambda$ and $\lambda W > 0$, then

$$\overline{\int} e^{t\sigma(x,W)} \lambda(dx) \leq \frac{1}{\lambda W} e^{nt^2/4}$$

for every $t \geq 0$.

(a) To begin with, suppose that every $X_i = Z = \{0, 1\}^{\mathbb{N}}$, every μ_i is a Borel measure, and W is compact. Note that in this case λ is a Radon measure (because the X_i are compact and metrizable), and

$$\{x : \sigma(x, W) \leq m\} = \bigcup_{I \subseteq n, \#(I) \leq m} \{x : \exists\, y \in W,\, x\restriction n \setminus I = y\restriction n \setminus I\}$$

is compact for every m, so the function $x \mapsto \sigma(x, W)$ is measurable.

Induce on n. If $n = 0$ we must have $W = X = \{\emptyset\}$ and $\sigma(x, W) = 0$ for every x, so the result is trivial. For the inductive step to $n \geq 1$, we have $W \subseteq X \times X_n$, where $X = \prod_{i<n} X_i$, and we are looking at $\iint e^{t\sigma((x,\xi),W)} \lambda(dx) \mu_n(d\xi)$. Now, setting $V_\xi = \{x : (x, \xi) \in W\}$ for $\xi \in X_n$, $V = \{x : \exists\, \xi \in X_n,\, (x, \xi) \in W\}$, we have

$$\sigma((x, \xi), W) \leq \min(\sigma(x, V_\xi), 1 + \sigma(x, V))$$

for all x and ξ, counting $\sigma(x, \emptyset)$ as ∞ if V_ξ is empty. So, for any $\xi \in X_n$,

$$\begin{aligned}\int e^{t\sigma((x,\xi),W)} \lambda(dx) &\leq \min(\int e^{t\sigma(x,V_\xi)} \lambda(dx), e^t \int e^{t\sigma(x,V)} \lambda(dx)) \\ &\leq e^{nt^2/4} \min(\frac{1}{\lambda V_\xi}, \frac{e^t}{\lambda V})\end{aligned}$$

by the inductive hypothesis, counting $\min(\frac{1}{0}, \frac{e^t}{\lambda V})$ as $\frac{e^t}{\lambda V}$.

It follows that if we set $f(\xi) = \max(e^{-t}, \frac{\lambda V_\xi}{\lambda V})$ for $\xi \in X_n$,

$$\begin{aligned}(\lambda\times\mu_n)(W)\int e^{t\sigma((x,\xi),W)}\lambda(dx)\mu_n(d\xi)&\\ &\le \int \lambda V_\xi \mu_n(d\xi)\cdot e^{nt^2/4}\int \min(\tfrac{1}{\lambda V_\xi},\tfrac{e^t}{\lambda V})\mu_n(d\xi)\\ &= e^{nt^2/4}\int \tfrac{\lambda V_\xi}{\lambda V}\mu_n(d\xi)\cdot\int \min(e^t,\tfrac{\lambda V}{\lambda V_\xi})\mu_n(d\xi)\\ &\le e^{nt^2/4}\int f(\xi)\mu_n(d\xi)\cdot\int \tfrac{1}{f(\xi)}\mu_n(d\xi)\\ &\le e^{nt^2/4}\cdot\tfrac{(1+e^{-t})^2}{4e^{-t}}\end{aligned}$$

(492B)

$$= \tfrac{1}{2}e^{nt^2/4}(1+\cosh t)\le e^{(n+1)t^2/4}$$

by 492C, and

$$\int e^{t\sigma((x,\xi),W)}\lambda(dx)\mu_n(d\xi)\le \frac{1}{(\lambda\times\mu_n)(W)}e^{(n+1)t^2/4},$$

so the induction continues.

(b) Now turn to the general case. If $W\in\Lambda$, there is a $W_1\subseteq W$ such that $W_1\in\widehat{\bigotimes}_{i<n}\Sigma_i$ and $\lambda W_1=\lambda W$ (254Ff). There must be countably-generated σ-subalgebras Σ_i' of Σ_i such that $W_1\in\widehat{\bigotimes}_{i<n}\Sigma_i'$. For each $i<n$, let $\langle E_{ik}\rangle_{k\in\mathbb{N}}$ be a sequence in Σ_i generating Σ_i', and let $h_i:X_i\to Z$ be the corresponding Marczewski functional, so that $h_i(\xi)=\langle\chi E_{ik}(\xi)\rangle_{k\in\mathbb{N}}$ for $\xi\in X_i$. Let μ_i' be the Borel measure on Z defined by setting $\mu_i'F=\mu_i h_i^{-1}[F]$ for every Borel set $F\subseteq Z$, and let ν be the product of the measures μ_i' on $Y=Z^n$. If we set $h(x)=\langle h_i(x(i))\rangle_{i<n}$ for $x\in X$, then $h:X\to Y$ is inverse-measure-preserving for λ and ν (254H). Moreover, by the choice of the E_{ik}, $W_1=h^{-1}[V]$ for some Borel set $V\subseteq Y$.

Because Y is a compact metrizable space, ν is the completion of a Borel measure and is a Radon measure (433Cb). For each $I\subseteq n$, write ν_I for the product measure on Z^I, and set

$$V_I=\{u:u\in Z^{n\setminus I},\ \nu_I\{v:v\in Z^I,\ (u,v)\in V\}>0\},$$

$$V_I'=\{y:y\in Y,\ y\restriction n\setminus I\in V_I\}.$$

Then $\nu(V\setminus V_I')=0$ for every $I\subseteq n$, so if we set $V'=\bigcap_{I\subseteq n}V_I'$ then $\nu V'=\nu V$. (Of course $V'\subseteq V_\emptyset'=V$.)

Take any $\gamma\in\,]0,\lambda W[\,=\,]0,\nu V'[$. Let $K\subseteq V'$ be a compact set such that $\nu K\ge\gamma$. Set $g(y)=e^{t\sigma(y,K)}$ for $y\in Y$, where I write σ for the Hamming metric on Y (regarded as a product of n factor spaces). Then $g:Y\to\mathbb{R}$ is Borel measurable and $gh:X\to\mathbb{R}$ is Λ-measurable. Also, for any $x\in X$, $\sigma(x,W)\le\sigma(h(x),K)$. **P** Take $y\in K$ such that $\sigma(h(x),y)=\sigma(h(x),K)$, and set

$$I=\{i:h(x)(i))\ne y(i)\},\quad u=h(x)\restriction n\setminus I=y\restriction n\setminus I.$$

Because $y\in V'$, $u\in V_I$ and $\nu_I H>0$, where $H=\{v:v\in Z^I,\ (u,v)\in V\}$. But if we write λ_I for the product measure on $\prod_{i\in I}X_i$, and $h_I(z)=\langle h_i(z(i))\rangle_{i\in I}$ for $z\in\prod_{i\in I}X_i$, then h_I is inverse-measure-preserving for λ_I and ν_I; in particular, $h_I^{-1}[H]$ is non-empty. This means that we can find an $x'\in X$ such that $x'\restriction n\setminus I=x\restriction n\setminus I$ and $x'\restriction I\in h_I^{-1}[H]$. In this case, $h(x')\in V$, so $x'\in W_1\subseteq W$, and

$$\sigma(x,W)\le\sigma(x,x')\le\#(I)=\sigma(h(x),K).\ \textbf{Q}$$

Accordingly

$$e^{t\sigma(x,W)}\le e^{t\sigma(h(x),K)}=g(h(x))$$

for every $x\in X$, and

$$\overline{\int}e^{t\sigma(x,W)}\lambda(dx)\le\int gh\,d\lambda=\int g\,d\nu$$

(because g is ν-integrable and h is inverse-measure-preserving, see 235I)

$$\leq \frac{1}{\nu K} e^{nt^2/4}$$

(by (a))

$$\leq \frac{1}{\gamma} e^{nt^2/4}.$$

As γ is arbitrary,

$$\overline{\int} e^{t\sigma(x,W)} \lambda(dx) \leq \frac{1}{\lambda W} e^{nt^2/4},$$

as claimed.

492E Corollary (TALAGRAND 95) Let $\langle (X_i, \Sigma_i, \mu_i) \rangle_{i<n}$ be a non-empty finite family of probability spaces with product (X, Λ, λ).

(a) Let ρ be the normalized Hamming metric on X. If $W \in \Lambda$ and $\lambda W > 0$, then

$$\lambda^*\{x : \rho(x, W) \geq \gamma\} \leq \frac{1}{\lambda W} e^{-n\gamma^2}$$

for every $\gamma \geq 0$.

(b) If W, $W' \in \Lambda$ and $\gamma > 0$ are such that $e^{-n\gamma^2} < \lambda W \cdot \lambda W'$ then there are $x \in W$, $x' \in W'$ such that $\#(\{i : i < n, x(i) \neq x'(i)\}) < n\gamma$.

proof (a) Set $t = 2n\gamma$. By 492D, there is a measurable function $f : X \to \mathbb{R}$ such that $f(x) \geq e^{t\rho(x,W)}$ for every $x \in X$ and $\int f d\lambda \leq \frac{1}{\lambda W} e^{t^2/4n}$. Then

$$\begin{aligned}\lambda^*\{x : \rho(x, W) \geq \gamma\} &\leq \lambda\{x : f(x) \geq e^{t\gamma}\} \leq e^{-t\gamma} \int f d\lambda \\ &\leq \frac{1}{\lambda W} e^{-t\gamma + t^2/4n} = \frac{1}{\lambda W} e^{-n\gamma^2}.\end{aligned}$$

(b) By (a), $\lambda^*\{x : \rho(x, W') \geq \gamma\} < \lambda W$, so there must be an $x \in W$ such that $\rho(x, W') < \gamma$.

492F The next theorem concerns concentration of measure in permutation groups. I approach this through a general result about slowly-varying martingales (492G).

Lemma $e^t \leq t + e^{t^2}$ for every $t \in \mathbb{R}$.

proof If $t \geq 1$ then $t \leq t^2$ so

$$e^t \leq e^{t^2} \leq t + e^{t^2}.$$

If $0 \leq t \leq 1$ then

$$\begin{aligned}e^t = 1 + t + \sum_{n=2}^{\infty} \frac{t^n}{n!} &\leq 1 + t + \sum_{k=1}^{\infty} (\frac{1}{(2k)!} + \frac{1}{(2k+1)!}) t^{2k} \\ &\leq 1 + t + \sum_{k=1}^{\infty} \frac{t^{2k}}{k!} = t + e^{t^2}.\end{aligned}$$

If $t \leq 0$ then

$$\begin{aligned}e^t = 1 + t + \sum_{n=2}^{\infty} \frac{t^n}{n!} &\leq 1 + t + \sum_{k=1}^{\infty} \frac{t^{2k}}{(2k)!} \\ &\leq 1 + t + \sum_{k=1}^{\infty} \frac{t^{2k}}{k!} = t + e^{t^2}.\end{aligned}$$

492G Lemma (MILMAN & SCHECHTMAN 86) Let (X, Σ, μ) be a probability space, and $\langle f_n \rangle_{n\in\mathbb{N}}$ a martingale on X. Suppose that $f_n \in \mathcal{L}^\infty(\mu)$ for every n, and set $\alpha_n = \text{ess sup}\,|f_n - f_{n-1}|$ for $n \geq 1$. Then for any $n \geq 1$ and $\gamma \geq 0$,

$$\Pr(f_n - f_0 \geq \gamma) \leq \exp(-\gamma^2/4\textstyle\sum_{i=1}^n \alpha_i^2),$$

interpreting $\exp(-\gamma^2/0)$ as 0 if $f_n =_{\text{a.e.}} f_0$.

proof Let $\langle \Sigma_n \rangle_{n\in\mathbb{N}}$ be a non-decreasing sequence of σ-subalgebras of Σ to which $\langle f_n \rangle_{n\in\mathbb{N}}$ is a adapted. Extending the functions f_n if necessary, we may suppose that they are all defined on the whole of X.

(a) I show first that

$$\mathbb{E}(\exp(\lambda(f_n - f_0))) \leq \exp(\lambda^2 \textstyle\sum_{i=1}^n \alpha_i^2)$$

for any $n \geq 0$ and any $\lambda > 0$. **P** Induce on n. For $n = 0$, interpreting $\sum_{i=1}^0$ as 0, this is trivial. For the inductive step to $n+1$, set $g = f_n - f_{n-1}$ and let g_1, g_2 be conditional expectations of $\exp(\lambda g)$ and $\exp(\lambda^2 g^2)$ on Σ_{n-1}. Because $|g| \leq \alpha_n$ a.e., $\exp(\lambda^2 g^2) \leq \exp(\lambda^2 \alpha_n^2)$ a.e. and $g_2 \leq \exp(\lambda^2 \alpha_n^2)$ a.e. Because $\exp(\lambda g) \leq \lambda g + \exp(\lambda^2 g^2)$ wherever g is defined (492F), and 0 is a conditional expectation of g on Σ_{n-1}, $g_1 \leq g_2 \leq \exp(\lambda^2 \alpha_n^2)$ a.e.

Now observe that $f_{n-1} - f_0$ is Σ_{n-1}-measurable, so that $\exp(\lambda(f_{n-1} - f_0)) \times g_1$ is a conditional expectation of $\exp(\lambda(f_{n-1} - f_0)) \times \exp(\lambda g) = \exp(\lambda(f_n - f_0))$ on Σ_{n-1} (233Eg). Accordingly

$$\begin{aligned}
\mathbb{E}(\exp(\lambda(f_n - f_0))) &= \mathbb{E}(\exp(\lambda(f_{n-1} - f_0)) \times g_1) \\
&\leq \text{ess sup}\,|g_1| \cdot \mathbb{E}(\exp(\lambda(f_{n-1} - f_0))) \\
&\leq \exp(\lambda^2 \alpha_n^2) \exp(\lambda^2 \sum_{i=1}^{n-1} \alpha_i^2)
\end{aligned}$$

(by the inductive hypothesis)

$$= \exp(\lambda^2 \sum_{i=1}^{n} \alpha_i^2)$$

and the induction continues. **Q**

(b) Now take $n \geq 1$ and $\gamma \geq 0$. If $\sum_{i=1}^n \alpha_i^2 = 0$ then $f_n =_{\text{a.e.}} f_0$ and the result is trivial. Otherwise, set $\lambda = \gamma/2\sum_{i=1}^n \alpha_i^2$. Then

$$\begin{aligned}
\Pr(f_n - f_0 \geq \gamma) &= \Pr(\exp(\lambda(f_n - f_0)) \geq e^{\lambda\gamma}) \\
&\leq e^{-\lambda\gamma}\mathbb{E}(\exp(\lambda(f_n - f_0))) \leq e^{-\lambda\gamma} \exp(\lambda^2 \sum_{i=1}^{n} \alpha_i^2)
\end{aligned}$$

(by (a) above)

$$= e^{-\lambda\gamma/2} = \exp(-\gamma^2/4 \sum_{i=1}^{n} \alpha_i^2)$$

as claimed.

492H Theorem (MAUREY 79) Let X be a non-empty finite set and G the group of all permutations of X with the discrete topology. For π, $\phi \in G$ set

$$\rho(\pi, \phi) = \frac{\#(\{x : x \in X, \pi(x) \neq \phi(x)\})}{\#(X)}.$$

Then ρ is a metric on G. Give G its Haar probability measure, and let $f : G \to \mathbb{R}$ be a 1-Lipschitz function. Then

$$\Pr(f - \mathbb{E}(f) \geq \gamma) \leq \exp(-\frac{\gamma^2 \#(X)}{16})$$

for any $\gamma \geq 0$.

proof We may suppose that $X = n = \{0, \dots, n-1\}$ where $n = \#(X)$. For $m \leq n$, $p : m \to n$ set $A_p = \{\pi : \pi \in G, \pi\restriction m = p\}$, and let Σ_m be the subalgebra of $\mathcal{P}G$ generated by $\{A_p : p \in n^m\}$. Then

$$\{\emptyset, G\} = \Sigma_0 \subseteq \Sigma_1 \subseteq \ldots \subseteq \Sigma_{n-1} = \Sigma_n = \mathcal{P}G;$$

for $m > n$ set $\Sigma_m = \mathcal{P}G$. For each m let f_m be the (unique) conditional expectation of f on Σ_m, so that

$$f_m(\pi) = \frac{1}{\#(A_p)} \sum_{\phi \in A_p} f(\phi)$$

whenever $\pi \in G$ and $p = \pi\restriction m$. Now we find that $|f_m(\pi) - f_{m-1}(\pi)| \leq \frac{2}{n}$ for every $m \geq 1$ and $\pi \in G$. **P** If $m > n$ this is trivial. Otherwise, set $p = \pi\restriction m - 1$ and $k = \pi(m-1)$. Set $J = p[m-1] = \{\pi(i) : i < m-1\}$, and for $j \in n \setminus J$ let $p_j = p^\frown j$ be that function from m to n which extends p and takes the value j at $m-1$; let α_j be the common value of $f_m(\phi)$ for $\phi \in A_{p_j}$, so that $f_m(\pi) = \alpha_k$. Now, for each $j \in n \setminus (J \cup \{k\})$, the function $\phi \mapsto (\overleftarrow{j\,k})\phi$ is a bijection from A_{p_k} to A_{p_j}, where $(\overleftarrow{j\,k}) \in G$ is the transposition which exchanges j and k. But this means that

$$\begin{aligned}
|\alpha_j - \alpha_k| &= \big|\tfrac{1}{(n-m)!} \sum_{\phi \in A_{p_j}} f(\phi) - \tfrac{1}{(n-m)!} \sum_{\phi \in A_{p_k}} f(\phi)\big| \\
&= \tfrac{1}{(n-m)!}\big| \sum_{\phi \in A_{p_j}} f(\phi) - f((\overleftarrow{j\,k})\phi)\big| \\
&\leq \sup_{\phi \in A_{p_j}} \big|f(\phi) - f((\overleftarrow{j\,k})\phi)\big| \leq \frac{2}{n}
\end{aligned}$$

because f is 1-Lipschitz and $\rho(\phi, (\overleftarrow{j\,k})\phi) = \frac{2}{n}$ for every ϕ. And this is true for every $j \in n \setminus (J \cup \{k\})$.

Accordingly

$$\begin{aligned}
|f_{m-1}(\pi) - f_m(\pi)| &= \big|\tfrac{1}{(n-m+1)!} \sum_{\phi \in A_p} f(\phi) - \alpha_k\big| = \big|\tfrac{1}{n-m+1} \sum_{j \in n \setminus J} \alpha_j - \alpha_k\big| \\
&\leq \tfrac{1}{n-m+1} \sum_{j \in n \setminus J} |\alpha_j - \alpha_k| \leq \tfrac{1}{n-m+1} \sum_{j \in n \setminus J} \frac{2}{n} = \frac{2}{n},
\end{aligned}$$

as claimed. **Q**

(b) Now observe that $f = f_{n-1}$ and that f_0 is the constant function with value $\mathbb{E}(f)$, so that

$$\Pr(f - \mathbb{E}(f) \geq \gamma) = \Pr(f_{n-1} - f_0 \geq \gamma) \leq \exp(-\gamma^2/4 \sum_{i=1}^{n-1} (\frac{2}{n})^2)$$

(492G)

$$\leq \exp(-\frac{n\gamma^2}{16}),$$

which is what we were seeking to prove.

492I Corollary Let X be a non-empty finite set and G the group of all permutations of X; write $n = \#(X)$. Let μ be the Haar probability measure of G when given its discrete topology. Suppose that $A \subseteq G$ and $\mu A \geq \frac{1}{2}$. Then

$$\mu\{\pi : \pi \in G, \exists\, \phi \in A, \#(\{x : x \in X, \pi(x) \neq \phi(x)\}) \leq k\} \geq 1 - \exp(-\frac{k^2}{64n})$$

for every $k \leq n$.

proof If $\exp(-\frac{k^2}{64n}) \geq \frac{1}{2}$, this is trivial, since the left-hand-side of the inequality is surely at least $\frac{1}{2}$. Otherwise, set $g(\pi) = \frac{1}{n}\min_{\phi\in A}\#(\{x : x \in X, \pi(x) \neq \phi(x)\})$ for $\pi \in G$, so that g is 1-Lipschitz for the metric ρ of 492H. Applying 492H to $f = -g$, we see that

$$\Pr(\mathbb{E}(g) - g \geq \frac{k}{2n}) \leq \exp(-\frac{k^2}{64n}) < \frac{1}{2},$$

and there must be some $\pi \in A$ such that $\mathbb{E}(g) - g(\pi) < \frac{k}{2n}$, so that $\mathbb{E}(g) < \frac{k}{2n}$. This means that

$$\begin{aligned}\mu\{\pi : \pi \in G, \exists\, \phi \in A, \#(\{x : x \in X, \pi(x) \neq \phi(x)\}) \leq k\}&\\ &= 1 - \mu\{\pi : \pi \in G, g(\pi) > \frac{k}{n}\}\\ &\geq 1 - \Pr(g - \mathbb{E}(g) \geq \frac{k}{2n}) \geq 1 - \exp(-\frac{k^2}{64n}),\end{aligned}$$

applying 492H to g itself.

492X Basic exercises (a) Let (X, Σ, μ) be a probability space, and $\langle f_n\rangle_{n\in\mathbb{N}}$ a martingale on X. Suppose that $f_n \in \mathcal{L}^\infty(\mu)$ for every n, and that $\sigma = \sqrt{\sum_{n=1}^\infty \alpha_n^2}$ is finite and not zero, where $\alpha_n = \operatorname{ess\,sup}|f_n - f_{n-1}|$ for $n \geq 1$. Show that $f = \lim_{n\to\infty} f_n$ is defined a.e., and that $\Pr(f - f_0 \geq \gamma) \leq \exp(-\gamma^2/4\sigma^2)$ for every $\gamma \geq 0$. (*Hint*: show first that $\|f_n\|_1 \leq \|f_n\|_2 \leq \sigma + \|f_0\|_2$ for every n, so that we can apply 275G.)

(b) Let (X, ρ) be a metric space and μ a topological probability measure on X. Suppose that $\gamma, \epsilon > 0$ are such that $\Pr(f - \mathbb{E}(f) \geq \gamma) \leq \epsilon$ whenever $f : X \to [-\gamma, \gamma]$ is 1-Lipschitz. Show that if $\mu F \geq \frac{1}{2}$ then $\mu\{x : \rho(x, F) \geq 2\gamma\} \leq \epsilon$.

(c) Let (X, ρ) be a metric space and μ a topological probability measure on X. Suppose that $\gamma, \epsilon > 0$ are such that $\mu\{x : \rho(x, F) > \gamma\} \leq \epsilon$ whenever $\mu F \geq \frac{1}{2}$. Show that if $f : X \to [-1, 1]$ is a 1-Lipschitz function then $\Pr(f - \mathbb{E}(f) > 2\gamma + 2\epsilon) \leq \epsilon$.

(d) Use 492G to show that if $\langle(X_i, \Sigma_i, \mu_i)\rangle_{i<n}$ is a non-empty finite family of probability spaces with product (X, Λ, λ), and X is given its normalized Hamming metric, and $f \in \mathcal{L}^\infty(\lambda)$ is 1-Lipschitz, then $\Pr(f - \mathbb{E}(f) \geq \gamma) \leq e^{-n\gamma^2/4}$ for every $\gamma \geq 0$. (*Hint*: if $\Sigma_k \subseteq \Lambda$ is the σ-algebra of subsets of X determined by coordinates in k, and f_k is a conditional expectation of f on Σ_k, then $\operatorname{ess\,sup}|f_{k+1} - f_k| \leq \frac{1}{n}$.)

492 Notes and comments In metric spaces, we can say that a probability measure is 'concentrated' if every Lipschitz function f is almost constant in the sense that, for some α, the sets $\{x : |f(x) - \alpha| \geq \gamma\}$ have small measure. What is astonishing is that this does not mean that the measure itself is concentrated on a small set. In 492H, the measure is the Haar probability measure, spread as evenly as it well could be. Of course, when I say that $\{x : |f(x) - \alpha| \geq \gamma\}$ has 'small' measure, I have to let some other parameter – in 492H, the size of X – vary, while γ itself is fixed. Also the shapes of the formulae depend on which normalizations we choose (observe the effect of moving from ρ to σ in the proof of 492D). But the value of 492H is that it gives a strong bound which is independent of the particular function f, provided that it is 1-Lipschitz. This kind of concentration of measure can be described either in terms of the variation of Lipschitz functions from their means or in terms of the measures of neighbourhoods of sets of measure $\frac{1}{2}$ (492Xb-492Xc). The latter, in a more abstract context, is what is described by the concentration functions of measures on uniform spaces (493D).

The martingale method can be used to prove a version of 492E (492Xd). The method of 492D gives a better exponent ($e^{-n\gamma^2}$ in place of $e^{-n\gamma^2/4}$) and also information of a slightly different kind, in that it can be applied directly to sets W of small measure, at least provided that $\gamma > \frac{1}{\sqrt{n}}$ in 492E. We also need a little more measure theory here, since sets which are measured by product measures can be geometrically highly irregular, and our Lipschitz functions $x \mapsto \rho(x, W)$ need not be measurable.

In the proof of 492G we have an interesting application of the idea of 'martingale'. The inequality here is quite different from the standard martingale inequalities like 275D or 275F or 275Yc. It gives a very

strong inequality concerning the difference $f_n - f_0$, at the cost of correspondingly strong hypotheses on the differences $f_i - f_{i-1}$; but since we need control of $\sum_i \text{ess sup} |f_i - f_{i-1}|^2$, not of $\sum_i \text{ess sup} |f_i - f_{i-1}|$, there is scope for applications like 492H. What the inequality tells us is that most of the time the differences $f_i - f_{i-1}$ cancel out, just as in the Central Limit Theorem, and that once again we have a vaguely Gaussian sum $f_n - f_0$.

Concentration of measure, in many forms, has been studied intensively in the context of the geometry of normed spaces, as in MILMAN & SCHECHTMAN 86, from which 492F-492I are taken.

493 Extremely amenable groups

A natural variation on the idea of 'amenable group' (§449) is the concept of 'extremely amenable' group (493A). Expectedly, most of the ideas of 449C-449E can be applied to extremely amenable groups (493B-493C); unexpectedly, we find not only that there are interesting extremely amenable groups, but that we need some of the central ideas of measure theory to study them. I give a criterion for extreme amenability of a group in terms of the existence of suitably concentrated measures (493D) before turning to four examples: measure algebras under symmetric difference (493E), L^0 spaces (493F), automorphism groups of measure algebras (493H) and isometry groups of spheres in infinite-dimensional Hilbert spaces (493J).

493A Definition Let G be a topological group. Then G is **extremely amenable** or has the **fixed point on compacta property** if every continuous action of G on a compact Hausdorff space has a fixed point.

493B Proposition (a) Let G and H be topological groups such that there is a continuous surjective homomorphism from G onto H. If G is extremely amenable, so is H.

(b) Let G be a topological group and suppose that there is a dense subset A of G such that every finite subset of A is included in an extremely amenable subgroup of G. Then G is extremely amenable.

(c) Let G be a topological group with an extremely amenable normal subgroup H such that G/H is extremely amenable. Then G is extremely amenable.

(d) The product of any family of extremely amenable topological groups is extremely amenable.

proof We can use the same arguments as in 449C, with some simplifications.

(a) As in 449Ca, let $\phi : G \to H$ be a continuous surjective homomorphism, X a non-empty compact Hausdorff space and $\bullet : H \times X \to X$ a continuous action. Let $\bullet_1$ be the continuous action of G on X defined by the formula $a\bullet_1 x = \phi(a)\bullet x$. Then any fixed point for $\bullet_1$ is a fixed point for $\bullet$.

(b) Let X be a non-empty compact Hausdorff space and $\bullet$ a continuous action of G on X. For $I \in [A]^{<\omega}$ let H_I be an extremely amenable subgroup of G including I. The restriction of the action to $H_I \times X$ is a continuous action of H_I on X, so

$$\{x : a\bullet x = x \text{ for every } a \in I\} \supseteq \{x : a\bullet x = x \text{ for every } a \in I\}$$

is closed and non-empty. Because X is compact, there is an $x \in X$ such that $a\bullet x = x$ for every $a \in A$. Now $\{a : a\bullet x = x\}$ includes the dense set A, so is the whole of G, and x is fixed under the action of G. As X and $\bullet$ are arbitrary, G is extremely amenable.

(c) Let X be a compact Hausdorff space and $\bullet$ a continuous action of G on X. Set $Q = \{x : x \in X, a\bullet x = x \text{ for every } a \in H\}$; then Q is a closed subset of X and, because H is extremely amenable, is non-empty. Next, $b\bullet x \in Q$ for every $x \in Q$ and $b \in G$. **P** If $a \in H$, then

$$a\bullet(b\bullet x) = (ab)\bullet x = (bb^{-1}ab)\bullet x = b\bullet((b^{-1}ab)\bullet x) = b\bullet x,$$

because H is normal, so $b^{-1}ab \in H$. As a is arbitrary, $b\bullet x \in Q$. **Q** Accordingly we have a continuous action of G on the compact Hausdorff space Q.

If $b \in G$, $a \in H$ and $x \in Q$, then $(ba)\bullet x = b\bullet x$. So we have an action of G/H on Q defined by saying that $b^\bullet\bullet x = b\bullet x$ for every $b \in G$ and $x \in Q$, and this is continuous for the quotient topology on G/H, as in the proof of 449Cc. Because G/H is amenable, there is a point x of Q which is fixed under the action of G/H.

So $b\bullet x = b^\bullet\bullet x = x$ for every $b \in G$, and x is fixed under the action of G. As X and $\bullet$ are arbitrary, G is extremely amenable.

(d) By (c), the product of two extremely amenable topological groups is extremely amenable, since each can be regarded as a normal subgroup of the product. It follows that the product of finitely many extremely amenable topological groups is extremely amenable. Now let $\langle G_i\rangle_{i\in I}$ be any family of extremely amenable topological groups with product G. For finite $J \subseteq I$ let H_J be the set of those $a \in G$ such that $a(i)$ is the identity in G_i for every $i \in I \setminus J$. Then H_J is isomorphic (as topological group) to $\prod_{i\in J} G_i$, so is extremely amenable. Since $\{H_J : J \in [I]^{<\omega}\}$ is an upwards-directed family of subgroups of G with dense union, (b) tells us that G is extremely amenable.

493C Proposition Let G be a topological group. Then G is extremely amenable iff there is a point in the greatest ambit Z of G (definition: 449D) which is fixed by the action of G on Z.

proof Repeat the arguments of 449E(i) $\iff$ (ii), noting that if $z_0 \in Z$ is a fixed point under the action of G on Z, then its images under the canonical maps ϕ of 449Dd will be fixed for other actions.

493D Theorem Let G be a topological group, and $\mathcal{B}$ its Borel σ-algebra. Suppose that for every open neighbourhood V of the identity of G, $\epsilon > 0$, finite set $I \subseteq G$ and finite family $\mathcal{E}$ of zero sets in G there is a finitely additive functional $\nu : \mathcal{B} \to [0,1]$ such that $\nu G = 1$ and

(i) $\nu(VF) \geq 1 - \epsilon$ whenever $F \in \mathcal{E}$ and $\nu F \geq \frac{1}{2}$,

(ii) $|\nu(aF) - \nu F| \leq \epsilon$ for every $F \in \mathcal{E}$ and $a \in I$.

Then G is extremely amenable.

proof (a) Write M for the set of finitely additive functionals $\nu : \mathcal{B} \to [0,1]$ such that $\nu G = 1$. If V is an open neighbourhood of the identity e of G, $\epsilon > 0$, $I \in [G]^{<\omega}$ and $\mathcal{E}$ is a finite family of zero sets in G, let $A(V,\epsilon,I,\mathcal{E})$ be the set of those $\nu \in M$ satisfying (i) and (ii) above. Our hypothesis is that none of these sets $A(V,\epsilon,I,\mathcal{E})$ are empty; since $A(V,\epsilon,I,\mathcal{E}) \subseteq A(V',\epsilon',I',\mathcal{E}')$ whenever $V \subseteq V'$, $\epsilon \leq \epsilon'$, $I \supseteq I'$ and $\mathcal{E} \supseteq \mathcal{E}'$, there is an ultrafilter $\mathcal{F}$ on M containing all these sets.

Let U be the space of bounded real-valued functionals on G which are uniformly continuous for the right uniformity on G. If we identify $L^\infty(\mathcal{B})$ with the space of bounded Borel measurable real-valued functions on G (363Ha), then U is a Riesz subspace of $L^\infty(\mathcal{B})$. For each $\nu \in M$ we have a positive linear functional $\int d\nu : L^\infty(\mathcal{B}) \to \mathbb{R}$ (363L). For $f \in U$ set $z(f) = \lim_{\nu\to\mathcal{F}} \int f d\nu$.

(b) $z : U \to \mathbb{R}$ is a Riesz homomorphism, and $z(\chi G) = 1$. **P** Of course z is a positive linear functional taking the value 1 at χG, just because all the integrals $\int d\nu$ are. Now suppose that f_0, $f_1 \in U$ and $f_0 \wedge f_1 = 0$. Take any $\epsilon > 0$. Then there is an open neighbourhood V of e such that $|f_i(x) - f_i(y)| \leq \epsilon$ whenever $xy^{-1} \in V$ and $i \in \{0,1\}$. Set $F_i = \{x : f_i(x) = 0\}$, $E_i = VF_i$ for each i. Then $F_0 \cup F_1 = X$, so $\nu F_0 + \nu F_1 \geq 1$ for every $\nu \in M$, and there is a $j \in \{0,1\}$ such that $A_0 = \{\nu : \nu F_j \geq \frac{1}{2}\} \in \mathcal{F}$. Next,

$$A_1 = \{\nu\text{: if } \nu F_j \geq \frac{1}{2} \text{ then } \nu(VF_j) \geq 1-\epsilon\}$$

belongs to $\mathcal{F}$. Accordingly $\lim_{\nu\to\mathcal{F}} \nu E_j \geq 1 - \epsilon$. As $f_j(x) \leq \epsilon$ for every $x \in E_j$,

$$z(f_j) = \lim_{\nu\to\mathcal{F}} \int f_j d\nu \leq \epsilon(1 + \|f_j\|_\infty).$$

This shows that $\min(z(f_0), z(f_1)) \leq \epsilon(1 + \|f_1\|_\infty + \|f_2\|_\infty)$. As ϵ is arbitrary, $\min(z(f_0), z(f_1)) = 0$; as f_0 and f_1 are arbitrary, z is a Riesz homomorphism (352G(iv)). **Q**

Thus z belongs to the greatest ambit Z of G.

(c) $a\bullet z = z$ for every $a \in G$. **P** Take any non-negative $f \in U$. For $n \in \mathbb{N}$ set $F_n = \{x : x \in G, f(x) \geq n\epsilon\}$. Set $m = \lfloor \frac{1}{\epsilon}\|f\|_\infty \rfloor$, so that $F_n = \emptyset$ for every $n > m$. Because $\{\nu : |\nu(a^{-1}F_n) - \nu F_n| \leq \delta\} \in \mathcal{F}$ for every $n \in \mathbb{N}$ and every $\delta > 0$, $\lim_{\nu\to\mathcal{F}} \nu F_n = \lim_{\nu\to\mathcal{F}} \nu(a^{-1}F_n)$ for every n. Next, setting $g = \sum_{n=1}^m \epsilon\chi F_n$, we have $g \in L^\infty(\mathcal{B})$ and $g \leq f \leq g + \epsilon\chi G$. Since $a^{-1}\bullet g$ (in the language of 441Ac) is just $\sum_{n=1}^m \epsilon\chi(a^{-1}F_n)$, we have $\lim_{\nu\to\mathcal{F}} \int g\, d\nu = \lim_{\nu\to\mathcal{F}} \int (a^{-1}\bullet g) d\nu$. But also $a^{-1}\bullet g \leq a^{-1}\bullet f \leq a^{-1}\bullet g + \epsilon\chi G$. So

$$|\int f d\nu - \int g\, d\nu| \leq \epsilon, \quad |\int (a^{-1}\bullet f) d\nu - \int (a^{-1}\bullet g) d\nu| \leq \epsilon$$

for every $\nu \in M$. This means that

$$|\lim_{\nu\to\mathcal{F}}\int f d\nu - \lim_{\nu\to\mathcal{F}}\int (a^{-1}\bullet f)d\nu| \leq 2\epsilon.$$

As ϵ is arbitrary,

$$(a\bullet z)(f) = z(a^{-1}\bullet f) = \lim_{\nu\to\mathcal{F}}\int (a^{-1}\bullet f)d\nu = \lim_{\nu\to\mathcal{F}}\int f d\nu = z(f).$$

This is true for every non-negative $f \in U$; as both z and $a\bullet z$ are linear, $(a\bullet z)(f) = z(f)$ for every $f \in U$, and $a\bullet z = z$. **Q**

(d) Thus $z \in Z$ is fixed under the action of G on Z; by 493C, this is enough to ensure that G is extremely amenable.

493E I turn now to examples of extremely amenable groups. The first three are groups which we have already studied for other reasons.

Theorem Let $(\mathfrak{A}, \bar{\mu})$ be an atomless measure algebra. Then $\mathfrak{A}$, with the group operation $\triangle$ and the measure-algebra topology (definition: 323A), is an extremely amenable group.

proof (a) To begin with let us suppose that $(\mathfrak{A}, \bar{\mu})$ is a probability algebra; write σ for the measure metric of $\mathfrak{A}$, so that $\sigma(a, a') = \bar{\mu}(a \triangle a')$ for a, $a' \in \mathfrak{A}$. I seek to apply 493D.

(i) Let V be an open neighbourhood of 0 in $\mathfrak{A}$, $\epsilon > 0$, $I \in [\mathfrak{A}]^{<\omega}$ and $\mathcal{E}$ a finite family of zero sets in $\mathfrak{A}$. Let $\gamma > 0$ be such that $V \supseteq \{a : \bar{\mu}a \leq 2\gamma\}$. Let $\mathfrak{B}_0$ be the finite subalgebra of $\mathfrak{A}$ generated by I and B_0 the set of atoms in $\mathfrak{B}_0$. Set

$$t = \frac{1}{\gamma}\ln\frac{3}{\epsilon}, \quad n = \lceil \max(t^2, \sup_{b\in B_0}\frac{1}{\bar{\mu}b})\rceil.$$

Because $\mathfrak{A}$ is atomless, we can split any member of $\mathfrak{A}\setminus\{0\}$ into two parts of equal measure (331C); if, starting from the disjoint set B_0, we successively split the largest elements until we have a disjoint set B with just n elements, then we shall have $\bar{\mu}b \leq \frac{2}{n}$ for every $b \in B$. We have a natural identification between $\{0,1\}^B$ and the subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ generated by B, matching $x \in \{0,1\}^B$ with $f(x) = \sup\{b : b \in B, x(b) = 1\}$. Writing ρ for the normalized Hamming metric on $\{0,1\}^B$ (492D), we have $\sigma(f(x), f(y)) \leq 2\rho(x, y)$ for all x, $y \in \{0,1\}^B$. **P** Set $J = \{b : b \in B, x(b) \neq y(b)\}$, so that

$$\sigma(f(x), f(y)) = \bar{\mu}(f(x) \triangle f(y)) = \bar{\mu}(\sup J) = \textstyle\sum_{b\in J}\bar{\mu}b \leq \frac{2}{n}\#(J) = 2\rho(x, y).$$ **Q**

(ii) Let λ be the usual measure on $\{0,1\}^B$ and set $\nu E = \lambda f^{-1}[E]$ for every Borel set $E \subseteq \mathfrak{A}$. Then ν is a probability measure. Note that $f : \{0,1\}^B \to \mathfrak{B}$ is a group isomorphism if we give $\{0,1\}^B$ the addition $+_2$ corresponding to its identification with $\mathbb{Z}_2^B$, and $\mathfrak{B}$ the operation $\triangle$. Because λ is translation-invariant for $+_2$, its copy, the subspace measure $\nu_{\mathfrak{B}}$ on the ν-conegligible finite set $\mathfrak{B}$, is translation-invariant for $\triangle$. But this means that $\nu\{b \triangle d : d \in F\} = \nu F$ whenever $b \in \mathfrak{B}$ and $F \subseteq \mathfrak{B}$, and therefore that $\nu\{b \triangle d : d \in F\} = \nu F$ whenever $b \in I$ and $F \in \mathcal{E}$. This shows that ν satisfies condition (ii) of 493D.

(iii) Now suppose that $F \in \mathcal{E}$ and that $\nu F \geq \frac{1}{2}$. Set $W = f^{-1}[F]$, so that $\lambda W \geq \frac{1}{2}$. By 492D,

$$\int e^{t\rho(x,W)}\lambda(dx) \leq 2e^{t^2/4n} \leq 2e^{1/4} \leq 3,$$

so

$$\lambda\{x : \rho(x, W) \geq \gamma\} = \lambda\{x : t\rho(x, W) \geq \ln\tfrac{3}{\epsilon}\} = \lambda\{x : e^{t\rho(x,W)} \geq \tfrac{3}{\epsilon}\} \leq \epsilon.$$

Accordingly

$$\begin{aligned}\nu\{a\triangle d : a \in V, d \in F\} &\geq \nu\{a : \sigma(a, F) \leq 2\gamma\} = \lambda\{x : \sigma(f(x), F) \leq 2\gamma\}\\ &\geq \lambda\{x : \sigma(f(x), f[W]) \leq 2\gamma\} \geq \lambda\{x : \rho(x, W) \leq \gamma\}\end{aligned}$$

(because f is 2-Lipschitz)

$$\geq 1 - \epsilon.$$

So ν also satisfies (i) of 493D.

(iv) Since V, ϵ, I and $\mathcal{E}$ are arbitrary, 493D tells us that $\mathfrak{A}$ is an extremely amenable group, at least when $(\mathfrak{A}, \bar{\mu})$ is an atomless probability algebra.

(b) For the general case, observe first that if $(\mathfrak{A}, \bar{\mu})$ is atomless and totally finite then $(\mathfrak{A}, \triangle)$ is an extremely amenable group; this is trivial if $\mathfrak{A} = \{0\}$, and otherwise there is a probability measure on $\mathfrak{A}$ which induces the same topology, so we can apply (a). For a general atomless measure algebra $(\mathfrak{A}, \bar{\mu})$, set $\mathfrak{A}^f = \{c : c \in \mathfrak{A}, \bar{\mu}c < \infty\}$ and for $c \in \mathfrak{A}^f$ let $\mathfrak{A}_c$ be the principal ideal generated by c. Then $\mathfrak{A}_c$ is a subgroup of $\mathfrak{A}$ and the measure-algebra topology of $\mathfrak{A}_c$, regarded as a measure algebra in itself, is the subspace topology induced by the measure-algebra topology of $\mathfrak{A}$. So $\{\mathfrak{A}_c : c \in \mathfrak{A}^f\}$ is an upwards-directed family of extremely amenable subgroups of $\mathfrak{A}$ with union which is dense in $\mathfrak{A}$, so $\mathfrak{A}$ itself is extremely amenable, by 493Bb. This completes the proof.

493F Theorem (PESTOV P01) Let (X, Σ, μ) be an atomless measure space. Then $L^0(\mu)$, with the group operation $+$ and the topology of convergence in measure, is an extremely amenable group.

proof It will simplify some of the formulae if we move at once to the space $L^0(\mathfrak{A})$, where $(\mathfrak{A}, \bar{\mu})$ is the measure algebra of (X, Σ, μ); for the identification of $L^0(\mathfrak{A})$ with $L^0(\mu)$ see 364Jc; for a note on convergence in measure in $L^0(\mathfrak{A})$, see 367M; of course $\mathfrak{A}$ is atomless if (X, Σ, μ) is (322Bg).

(a) I seek to prove that $S(\mathfrak{A})$, with the group operation of addition and the topology of convergence in measure, is extremely amenable. As in 493E, I start with the case in which $(\mathfrak{A}, \bar{\mu})$ is a probability algebra, and use 493D.

(i) Take an open neighbourhood V of 0 in $S(\mathfrak{A})$, an $\epsilon > 0$, a finite set $I \subseteq S(\mathfrak{A})$ and a finite family $\mathcal{E}$ of zero sets in $S(\mathfrak{A})$. Let $\gamma > 0$ be such that $u \in V$ whenever $u \in S(\mathfrak{A})$ and $\bar{\mu}[\![u \neq 0]\!] \leq 2\gamma$. Let $\mathfrak{B}_0$ be a finite subalgebra of $\mathfrak{A}$ such that I is included in the linear subspace of $S(\mathfrak{A})$ generated by $\{\chi b : b \in \mathfrak{B}_0\}$, and B_0 the set of atoms of $\mathfrak{B}_0$. As in the proof of 493E, set

$$t = \frac{1}{\gamma} \ln \frac{3}{\epsilon}, \quad n = \lceil \max(t^2, \sup_{b \in B_0} \frac{1}{\bar{\mu}b}) \rceil,$$

and let $B \subseteq \mathfrak{A} \setminus \{0\}$ be a partition of unity with n elements, refining B_0, such that $\bar{\mu}b \leq \frac{2}{n}$ for every $b \in B$. We have a natural identification between $\mathbb{R}^B$ and the linear subspace of $S(\mathfrak{A})$ generated by $\{\chi b : b \in B\}$, matching $x \in \mathbb{R}^B$ with $f(x) = \sum_{b \in B} x(b)\chi b$, which is continuous if $\mathbb{R}^B$ is given its product topology. Writing ρ for the normalized Hamming metric on $\mathbb{R}^B$, we have

$$\bar{\mu}[\![f(x) \neq f(y)]\!] = \sum_{x(b) \neq y(b)} \bar{\mu}b \leq \frac{2}{n} \#(\{b : x(b) \neq y(b)\}) = 2\rho(x, y)$$

for all x, $y \in \mathbb{R}^B$.

(ii) Set $\beta = \sup_{v \in I} \|v\|_\infty$ (if $I = \emptyset$, take $\beta = 0$). Let $M > 0$ be so large that $(M + \beta)^n \leq (1 + \frac{1}{2}\epsilon)M^n$. On $\mathbb{R}$, write μ_L for Lebesgue measure and μ'_L for the indefinite-integral measure over μ_L defined by the function $\frac{1}{2M}\chi[-M, M]$, so that $\mu'_L E = \frac{1}{2M}\mu_L(E \cap [-M, M])$ whenever $E \subseteq \mathbb{R}$ and $E \cap [-M, M]$ is Lebesgue measurable. Let λ, λ' be the product measures on $\mathbb{R}^B$ defined from μ_L and μ'_L. Let ν be the Borel probability measure on $S(\mathfrak{A})$ defined by setting $\nu F = \lambda' f^{-1}[F]$ for every Borel set $F \subseteq S(\mathfrak{A})$.

Now $|\nu(v + F) - \nu F| \leq \epsilon$ for every $v \in I$ and Borel set $F \subseteq S(\mathfrak{A})$. **P** Because B refines B_0, v is expressible as $f(y)$ for some $y \in \mathbb{R}^B$; because $\|v\|_\infty \leq \beta$, $|y(b)| \leq \beta$ for every $b \in B$. Because $f : \mathbb{R}^B \to S(\mathfrak{A})$ is linear, $f^{-1}[v + F] = y + f^{-1}[F]$. Now

$$\begin{aligned} |\nu(v + F) - \nu F| &= |\lambda' f^{-1}[v + F] - \lambda' f^{-1}[F]| \\ &= \frac{1}{(2M)^n} \big|\lambda(f^{-1}[v + F] \cap [-M, M]^n) - \lambda(f^{-1}[F] \cap [-M, M]^n)\big| \end{aligned}$$

(use 253I, or otherwise)

$$= \frac{1}{(2M)^n}\big|\lambda((y+f^{-1}[F])\cap[-M,M]^n) - \lambda(f^{-1}[F]\cap[-M,M]^n)\big|$$
$$= \frac{1}{(2M)^n}\big|\lambda(f^{-1}[F]\cap([-M,M]^n - y)) - \lambda(f^{-1}[F]\cap[-M,M]^n)\big|$$
$$\leq \frac{1}{(2M)^n}\lambda(([-M,M]^n - y)\triangle[-M,M]^n)$$
$$= \frac{2}{(2M)^n}\lambda(([-M,M]^n - y)\setminus[-M,M]^n)$$
$$\leq \frac{2}{(2M)^n}\lambda(([-M-\beta,M+\beta]^n\setminus[-M,M]^n)$$
$$= \frac{2}{M^n}((M+\beta)^n - M^n) \leq \epsilon. \blacksquare$$

So ν satisfies (ii) of 493D.

(iii) Now suppose that $F \in \mathcal{E}$ and $\nu F \geq \frac{1}{2}$. Set $W = f^{-1}[F]$, so that $\lambda' W \geq \frac{1}{2}$. Just as in the proof of 493E, $\int e^{t\rho(x,W)}\lambda'(dx) \leq 2e^{t^2/4n} \leq 3$, so

$$\lambda'\{x : \rho(x,W) \geq \gamma\} = \lambda'\{x : e^{t\rho(x,W)} \geq \tfrac{3}{\epsilon}\} \leq \epsilon,$$

and

$$\begin{aligned}\nu\{v+u : v \in V,\, u \in F\} &\geq \nu\{w : \exists\, u \in F,\, \bar\mu[\![u \neq w]\!] \leq 2\gamma\}\\ &\geq \lambda'\{x : \rho(x,W) \leq \gamma\} \geq 1-\epsilon.\end{aligned}$$

So ν also satisfies (i) of 493D.

(iv) Since V, ϵ, I and $\mathcal{E}$ are arbitrary, 493D tells us that $S(\mathfrak{A})$ is an extremely amenable group, at least when $(\mathfrak{A},\bar\mu)$ is an atomless probability algebra.

(b) The rest of the argument is straightforward, as in 493E. First, $S(\mathfrak{A})$ is extremely amenable whenever $(\mathfrak{A},\bar\mu)$ is an atomless totally finite measure algebra. For a general atomless measure algebra $(\mathfrak{A},\bar\mu)$, set $\mathfrak{A}^f = \{c : \bar\mu c < \infty\}$. For each $c \in \mathfrak{A}^f$, let $\mathfrak{A}_c$ be the corresponding principal ideal of $\mathfrak{A}$. Then we can identify $S(\mathfrak{A}_c)$, as topological group, with the linear subspace of $L^0(\mathfrak{A})$ generated by $\{\chi a : a \in \mathfrak{A}_c\}$, and it is extremely amenable. Since $\{S(\mathfrak{A}_c) : c \in \mathfrak{A}^f\}$ is an upwards-directed family of extremely amenable subgroups of $L^0(\mathfrak{A})$ with dense union in $L^0(\mathfrak{A})$, $L^0(\mathfrak{A})$ itself is extremely amenable, by 493Bb, as before.

493G I take the proof of the next theorem in two parts, the first being the leading special case.

Lemma Let κ be an infinite cardinal, λ the usual measure on $\{0,1\}^\kappa$ and $(\mathfrak{B},\bar\lambda)$ its measure algebra with its measure metric. Let $\mathrm{Aut}_{\bar\lambda}\mathfrak{B}$ be the group of measure-preserving automorphisms of $\mathfrak{B}$ with the topology $\mathfrak{T}$ of pointwise convergence inherited from the isometry group of $\mathfrak{B}$ (441G). Then $\mathrm{Aut}_{\bar\lambda}\mathfrak{B}$ is extremely amenable.

proof (a) For each $I \in [\kappa]^{<\omega}$, let H_I be the group of bijections from $\{0,1\}^I$ to itself. For $q \in H_I$ let $f_q : \{0,1\}^\kappa \to \{0,1\}^\kappa$ be the measure space automorphism defined by saying that

$$f_q(x)\restriction\kappa\setminus I = x\restriction\kappa\setminus I, \quad f_q(x)\restriction I = q(x\restriction I)$$

for every $x \in \{0,1\}^\kappa$, and ψ_q the member of $\mathrm{Aut}_{\bar\lambda}\mathfrak{B}$ defined by the formula

$$\psi_q(E^\bullet) = f_q[E]^\bullet \text{ for every measurable } E \subseteq \{0,1\}^\kappa.$$

Note that $f_{qq'} = f_q f_{q'}$ for all q, $q' \in H_I$, so that $q \mapsto \psi_q : H_I \to \mathrm{Aut}_{\bar\lambda}\mathfrak{B}$ is a group homomorphism. Set $G_I = \{\psi_q : q \in H_I\}$, so that G_I is a subgroup of $\mathrm{Aut}_{\bar\lambda}\mathfrak{B}$.

If $I \subseteq J \in [\kappa]^{<\omega}$ then $G_I \subseteq G_J$. **P** For any $q \in H_I$ we can define $q' \in H_J$ by setting

$$q'(z)\restriction J\setminus I = z\restriction J\setminus I, \quad q'(z)\restriction I = q(z\restriction I)$$

for every $z \in \{0,1\}^J$; then $f_{q'} = f_q$ so $\psi_{q'} = \psi_q$. As q is arbitrary, $G_I \subseteq G_J$. $\blacksquare$

Set $G = \bigcup\{G_I : I \in [\kappa]^{<\omega}\}$, so that G is a subgroup of $\mathrm{Aut}_{\bar\lambda}\mathfrak{B}$.

(b) Let V be a neighbourhood of the identity of G (with the subspace topology inherited from $\mathrm{Aut}_{\bar\lambda}\mathfrak{B}$), $\epsilon > 0$ and $K \subseteq G$ a finite set. Then there is a (purely atomic) Borel probability measure ν on G such that

(i) $\nu(\pi B) = \nu B$ for every $B \subseteq G$ and every $\pi \in K$,

(ii) $\nu(VB \cap G) \geq 1 - \epsilon$ whenever $B \subseteq G$ and $\nu B \geq \frac{1}{2}$.

P There are a finite set $A \subseteq \mathfrak{B}$ and a $\delta > 0$ such that $\pi \in V$ whenever $\pi \in G$ and $\bar\lambda(a \triangle \pi a) \leq \delta$ for every $a \in A$. Let $I \in [\kappa]^{<\omega}$ be such that $K \subseteq G_I$ and $2^n\delta - 1 \geq \sqrt{2^{n+6}\ln\frac{1}{\epsilon}}$, where $n = \#(I)$. (This is where we need to know that κ is infinite.) Let ν be the Borel probability measure on G such that $\nu\{\psi_q\} = 1/(2^n)!$ for every $q \in H_I$. Since ν acts on G_I as its Haar probability measure,

$$\nu(\pi B) = \nu(G_I \cap \pi B) = \nu(\pi(G_I \cap B)) = \nu(G_I \cap B) = \nu B$$

for every $B \subseteq G$ and every $\pi \in K \subseteq G_I$.

Now suppose that $B \subseteq G$ and $\nu B \geq \frac{1}{2}$. Set $C = \{q : q \in H_I, \psi_q \in B\}$, so that $\#(C) \geq \frac{1}{2}\#(H_I) = \frac{1}{2}(2^n)!$. Set $r = \lfloor 2^n\delta \rfloor$, so that $\exp(-r^2/2^{n+6}) \leq \epsilon$. Let $\tilde C$ be the set of those $\tilde q \in H_I$ such that there is some $q \in C$ with $\#(\{z : z \in \{0,1\}^I, \tilde q(z) \neq q(z)\}) \leq r$; then

$$\#(\tilde C) \geq (1 - \exp(-r^2/2^{n+6}))(2^n)! \geq (1-\epsilon)(2^n)!$$

by 492I. Writing $B' = \{\psi_q : q \in \tilde C\}$, $\nu B' \geq 1 - \epsilon$. Now if $\pi \in B'$, there are $\tilde q \in \tilde C$, $q \in C$ such that $\pi = \psi_{\tilde q}$ and $\#(\{z : z \in \{0,1\}^I, \tilde q(z) \neq q(z)\}) \leq r$. But this means that $\psi_q \in B$, while

$$\#(\{z : q^{-1}(z) \neq \tilde q^{-1}(z)\}) = \#(\{z : \tilde q q^{-1}(z) \neq q q^{-1}(z)\}) = \#(\{z : \tilde q(z) \neq q(z)\}) \leq r.$$

Setting

$$E_0 = \{x : x \in \{0,1\}^\kappa, \tilde q^{-1}(x\restriction I) \neq q^{-1}(x\restriction I)\},$$

we see that $\lambda E_0 \leq 2^{-n}r \leq \delta$ and $f_q^{-1}(x) = f_{\tilde q}^{-1}(x)$ for every $x \in \{0,1\}^\kappa \setminus E_0$, so that $f_q[E] \triangle f_{\tilde q}[E] \subseteq E_0$ for every $E \subseteq \{0,1\}^\kappa$. Accordingly $\pi a \triangle \psi_q a \subseteq E_0^\bullet$ for every $a \in \mathfrak{B}$, and

$$\bar\lambda(a \triangle \pi\psi_{q^{-1}}a) = \bar\lambda(\pi(\psi_{q^{-1}}a) \triangle \psi_q(\psi_{q^{-1}}a)) \leq \bar\lambda E_0^\bullet \leq \delta$$

for every $a \in A$. This shows that $\pi\psi_q^{-1} \in V$ so that $\pi \in VB$. As π is arbitrary, $B' \subseteq VB$ and $\nu(G \cap VB) \geq \nu B' \geq 1 - \epsilon$. As B is arbitrary, ν satisfies the condition (ii). **Q**

(c) By 493D, G is extremely amenable. Now G is dense in $\mathrm{Aut}_{\bar\lambda}\mathfrak{B}$. **P** Take any $\pi \in \mathrm{Aut}_{\bar\lambda}\mathfrak{B}$ and any neighbourhood V of π in $\mathrm{Aut}_{\bar\lambda}\mathfrak{B}$. Then there are $a_0, \ldots, a_k \in \mathfrak{B}$ and $\epsilon > 0$ such that

$$V' = \{\psi : \psi \in \mathrm{Aut}_{\bar\lambda}\mathfrak{B}, \bar\lambda(\psi a_i \triangle \pi a_i) \leq \epsilon \text{ for every } i \leq k\}$$

is included in V. Let $b_0, \ldots, b_m$ be the atoms of the finite subalgebra of $\mathfrak{B}$ generated by $\{a_i : i \leq k\} \cup \{\pi a_i : i \leq k\}$. Set $\delta = \epsilon/12(m+1)(2m+1) > 0$. For $j \leq m$, let $E_j, F_j \subseteq \{0,1\}^\kappa$ be such that $E_j^\bullet = b_j$ and $F_j^\bullet = \pi b_j$; let E_j', F_j' be open-and-closed sets such that $\lambda(E \triangle E_j') + \lambda(F_j \triangle F_j') \leq \delta$. Let $I \subseteq \kappa$ be a finite set such that E_j' and F_j' are determined by coordinates in I for every $j \leq m$. For each $j \leq m$, set $E_j'' = E_j' \setminus \bigcup_{i<j} E_i'$ and $F_j'' = F_j' \setminus \bigcup_{i<j} F_i'$, so that E_j'' and F_j'' are still determined by coordinates in I; finally, let $\tilde E_j \subseteq E_j''$ and $\tilde F_j \subseteq F_j''$ be sets determined by coordinates in I such that $\lambda\tilde E_j = \lambda\tilde F_j = \min(\lambda E_j'', \lambda F_j'')$.

For each $j \leq m$, we have $\lambda(E_j \triangle E_j') \leq \delta$; since $\lambda(E_i \cap E_j) = 0$ if $i < j$, $\lambda(E_i' \cap E_j') \leq 2\delta$ if $i < j$, and $\lambda(E_j' \setminus E_j'') \leq 2m\delta$; accordingly $\lambda(E_j \triangle E_j'') \leq (2m+1)\delta$, for every $j \leq m$. Similarly $\lambda(F_j \triangle F_j'') \leq (2m+1)\delta$; because $\lambda E_j = \lambda F_j$,

$$\lambda(E_j'' \setminus \tilde E_j) \leq |\lambda E_j'' - \lambda F_j''| \leq 4(m+1)\delta, \quad \lambda(E_j \triangle \tilde E_j) \leq 6(m+1)\delta$$

for each j. Similarly $\lambda(F_j \triangle \tilde F_j) \leq 6(m+1)\delta$ for each j.

Because $\tilde E_j$ is determined by coordinates in I, it is of the form $\{x : x\restriction I \in C_j\}$ for some $C_j \subseteq \{0,1\}^I$. Because $\tilde E_0, \ldots, \tilde E_m$ are disjoint, so are $C_0, \ldots, C_m$. Similarly, $\tilde F_j$ can be expressed as $\{x : x\restriction I \in D_j\}$ where $D_0, \ldots, D_m \subseteq \{0,1\}^I$ are disjoint. Because $\lambda\tilde E_j = \lambda\tilde F_j$, $\#(C_j) = \#(D_j)$ for each j. There is therefore a $q \in H_I$ such that $q[C_j] = D_j$ for each j. Now $\psi_q\tilde E_j^\bullet = \tilde F_j^\bullet$, so

$$\begin{aligned}\bar\lambda(\psi_q b_j \triangle \pi b_j) &\leq \bar\lambda(\psi_q b_j \triangle \tilde F_j^\bullet) + \bar\lambda(\tilde F_j^\bullet \triangle F_j^\bullet)\\ &= \bar\lambda(b_j \triangle \tilde E_j^\bullet) + \lambda(\tilde F_j \triangle F_j)\\ &\leq \lambda(E_j \triangle \tilde E_j) + 6(2m+1)\delta \leq 12(2m+1)\delta\end{aligned}$$

for each j. Since every a_i is a union of some of the b_j,

$$\bar\lambda(\psi_q a_i \triangle \pi a_i) \leq 12(2m+1)(m+1)\delta = \epsilon$$

for every $i \leq k$, and $\psi_q \in V' \subseteq V$. Thus $G \cap V \neq \emptyset$; as V is arbitrary, G is dense in $\mathrm{Aut}_{\bar\lambda}\mathfrak{B}$. **Q**

(d) By 493Bb, with $\mathcal{H} = \{G\}$, $\mathrm{Aut}_{\bar\lambda}\mathfrak{B}$ is extremely amenable.

493H Theorem (Giordano & Pestov 02) Let $(\mathfrak{A}, \bar\mu)$ be an atomless measure algebra, and set $\mathfrak{A}^f = \{a : \bar\mu a < \infty\}$. Let $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ be the group of measure-preserving automorphisms of $\mathfrak{A}$. For π, $\phi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ and $c \in \mathfrak{A}^f$, set $\sigma_c(\pi,\phi) = \bar\mu(\pi c \triangle \phi c)$. Then $\{\sigma_c : c \in \mathfrak{A}^f\}$ is a family of pseudometrics on $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. Under the topology it induces, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is an extremely amenable topological group.

proof (a) It is elementary to check that every σ_c is a pseudometric. To see that the topology $\mathfrak{T}$ generated by these pseudometrics is compatible with the group structure, recall that $(a,b) \mapsto \bar\mu(a \triangle b)$ is a metric on $\mathfrak{A}^f$ (323Ae). Now if we set $\theta(\pi) = \pi\restriction\mathfrak{A}^f$, then θ is a group homomorphism from $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ to the isometry group $\mathrm{Isom}(\mathfrak{A}^f)$ of $\mathfrak{A}^f$, and $\mathfrak{T}$ is just $\{\theta^{-1}[V] : V \in \mathfrak{S}\}$, where $\mathfrak{S}$ is the topology of pointwise convergence in $\mathrm{Isom}(\mathfrak{A}^f)$. Since $(\mathrm{Isom}(\mathfrak{A}^f), \mathfrak{S})$ is a topological group (441Ga), so is $(\mathrm{Aut}_{\bar\mu}\mathfrak{A}, \mathfrak{T})$.

(b) For $a \in \mathfrak{A}$ write $\mathfrak{A}_a$ for the principal ideal of $\mathfrak{A}$ generated by a and G_a for the subgroup of those $\pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$ which are supported by a (definition: 381Ba), so that G_a is isomorphic (as topological group) to the group of measure-preserving automorphisms of $\mathfrak{A}_a$.

(c) Let A be the set of those $a \in \mathfrak{A}^f$ such that $\mathfrak{A}_a$ is homogeneous. Then G_a is extremely amenable for every $a \in A$. **P** If $a = 0$ this is trivial, since G_a has just one element. Otherwise, $\mathfrak{A}_a$ is isomorphic, up to a scalar multiple of the measure, to the measure algebra $\mathfrak{B}$ of the usual measure λ on $\{0,1\}^\kappa$ where κ is infinite, by Maharam's theorem (331L). So G_a is isomorphic, as topological group, to $\mathrm{Aut}_{\bar\lambda}\mathfrak{B}$, which is extremely amenable, by 493G. **Q**

(d) Set $I = \{a_0 \cup \ldots \cup a_n : a_0, \ldots, a_n \in A\}$. Then G_b is extremely amenable for every $b \in I$. **P** Since $\mathfrak{A}_{a\cup b}$ is homogeneous whenever $\mathfrak{A}_a$, $\mathfrak{A}_b$ are homogeneous and of the same infinite Maharam type, we can express b as $a_0 \cup \ldots \cup a_n$ where all the $\mathfrak{A}_{a_i}$ are homogeneous and have different Maharam types. (This is really where I use the hypothesis that $\mathfrak{A}$ is atomless.) In this case, G_b will be isomorphic (as topological group) to $\prod_{i=0}^k G_{a_i}$, so is extremely amenable (493Bd). **Q**

Note that as $\pi a \in A$ whenever $a \in A$ and $\pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$, $\pi b \in I$ whenever $b \in I$ and $\pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$. Also $a' \in A$ whenever $a \in A$ and $a' \subseteq a$, so I is an ideal of $\mathfrak{A}$.

(e) Set $G = \bigcup_{b\in I} G_b$. Then G is dense in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. **P** Suppose that $\pi \in \mathrm{Aut}_{\bar\mu}\mathfrak{A}$, $a_0, \ldots, a_n \in \mathfrak{A}^f$ and $\epsilon > 0$. Set $c = \sup_{i\leq n} a_i$. By 332A, there is a $b_0 \in I$ such that $b_0 \subseteq c$ and $\bar\mu(c \setminus b_0) \leq \frac{1}{2}\epsilon$. Set $b = b_0 \cup \pi b_0 \in I$. Then there is a $\phi \in G_b$ such that $\phi d = \pi d$ for every $d \subseteq b_0$ (332L). Now

$$\phi a_i \triangle \pi a_i \subseteq \phi(a_i \setminus b_0) \cup \pi(a_i \setminus b_0)$$

has measure at most $2\bar\mu(c \setminus b_0) \leq \epsilon$ for every $i \leq n$. As π, $a_0, \ldots, a_n$ and ϵ are arbitrary, G is dense in $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$. **Q**

(f) Thus $\{G_b : b \in I\}$ is an upwards-directed family of extremely amenable subgroups of $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ with dense union. By 493Bb, $\mathrm{Aut}_{\bar\mu}\mathfrak{A}$ is extremely amenable.

493I Returning to the ideas of §476, we find another remarkable example of an extremely amenable topological group. I recall the notation of 476M. Let X be a (real) inner product space. S_X will be the unit sphere $\{x : x \in X, \|x\| = 1\}$. Let H_X be the isometry group of S_X with its topology of pointwise convergence. When X is finite-dimensional, it is isomorphic, as inner product space, to $\mathbb{R}^r$, where $r = \dim X$. In this case S_X is compact, so (if $r \geq 1$) has a unique H_X-invariant Radon probability measure ν_X, which is

strictly positive, and is a multiple of $(r-1)$-dimensional Hausdorff measure; also H_X is compact (441Gb), so has a unique Haar probability measure λ_X.

Lemma For any $m \in \mathbb{N}$ and any $\epsilon > 0$, there is an $r(m,\epsilon) \geq \max(2,m)$ such that whenever X is a finite-dimensional inner product space of dimension at least $r(m,\epsilon)$, $x_0, \ldots, x_{m-1} \in S_X$, Q_1, $Q_2 \subseteq H_X$ are closed sets and $\min(\lambda_X Q_1, \lambda_X Q_2) \geq \epsilon$, then there are $f_1 \in Q_1$, $f_2 \in Q_2$ such that $\|f_1(x_i) - f_2(x_i)\| \leq \epsilon$ for every $i < m$.

proof Induce on m. For $m = 0$, the result is trivial. For the inductive step to $m+1$, take $r(m+1,\epsilon) > r(m, \frac{1}{3}\epsilon)$ such that whenever $r(m+1,\epsilon) \leq \dim X < \omega$ and A_1, $A_2 \subseteq S_X$ and $\min(\nu_X^* A_1, \nu_X^* A_2) \geq \frac{1}{2}\epsilon$ then there are $x \in A_1$, $y \in A_2$ such that $\|x - y\| \leq \frac{1}{3}\epsilon$; this is possible by 476P.

Now take any inner product space X of finite dimension $r \geq r(m+1,\epsilon)$, closed sets Q_1, $Q_2 \subseteq H_X$ such that $\min(\lambda_X Q_1, \lambda_X Q_2) \geq \epsilon$, and orthonormal $x_0, \ldots, x_m \in S_X$. Let Y be the $(r-1)$-dimensional subspace $\{x : x \in X, (x|x_m) = 0\}$, so that $\dim Y \geq r(m, \frac{1}{3}\epsilon)$, and for $i < m$ let $y_i \in Y$ be a unit vector such that x_i is a linear combination of y_i and x_m. Set $H'_Y = \{f : f \in H_X, f(x_m) = x_m\}$; then $f \mapsto f{\restriction}S_Y$ is a topological group isomorphism from H'_Y to H_Y. **P** (i) If $f \in H'_Y$ and $x \in S_X$, then

$$x \in S_Y \iff (x|x_m) = 0 \iff (f(x)|f(x_m)) = 0 \iff f(x) \in S_Y,$$

so $f{\restriction}S_Y$ is a bijection from S_Y to itself and belongs to H_Y. (ii) If $g \in H_Y$, we can define $f \in H'_Y$ by setting $f(\alpha x + \beta x_m) = \alpha g(x) + \beta x_m$ whenever $x \in S_Y$ and $\alpha^2 + \beta^2 = 1$, and $f{\restriction}S_Y = g$. (iii) Note that H'_Y is a closed subgroup of H_X, so in itself is a compact Hausdorff topological group. Since the map $f \mapsto f{\restriction}S_Y : H'_Y \to H_Y$ is a bijective continuous group homomorphism between compact Hausdorff topological groups, it is a topological group isomorphism. **Q**

Let λ'_Y be the Haar probability measure of H'_Y. Then $\lambda_X Q_1 = \int \lambda'_Y(H'_Y \cap f^{-1}Q_1)\lambda_X(df)$ (443Te), so $\lambda'_X Q_1 \geq \frac{1}{2}\epsilon$, where $Q'_1 = \{f : \lambda'_Y(H'_Y \cap f^{-1}Q_1) \geq \frac{1}{2}\epsilon\}$. Similarly, setting $Q'_2 = \{f : \lambda'_Y(H'_Y \cap f^{-1}Q_2) \geq \frac{1}{2}\epsilon\}$, $\lambda_X Q'_2 \geq \frac{1}{2}\epsilon$. Next, setting $\theta(f) = f(x_m)$ for $f \in H_X$, $\lambda_X\theta^{-1}$ is an X-invariant Radon probability measure on S_X (443Ta), so must be equal to ν_X. Accordingly

$$\nu_X(\theta[Q'_j]) = \lambda_X(\theta^{-1}[\theta[Q'_j]]) \geq \lambda_X Q'_j \geq \frac{1}{2}\epsilon$$

for both j.

We chose $r(m+1,\epsilon)$ so large that we can be sure that there are $z_1 \in \theta[Q'_1]$, $z_2 \in \theta[Q'_2]$ such that $\|z_1 - z_2\| \leq \frac{1}{3}\epsilon$. Let $h_1 \in Q'_1$, $h_2 \in Q'_2$ be such that $h_1(x_m) = \theta(h_1) = z_1$ and $h_2(x_m) = z_2$. Let $h \in H_X$ be such that $h(z_1) = z_2$ and $\|h(x) - x\| \leq \frac{1}{3}\epsilon$ for every $x \in S_X$ (4A4Jf). Set $\tilde{h}_2 = hh_1$, so that $\tilde{h}_2(x_m) = z_2$ and $\|h_1(x) - \tilde{h}_2(x)\| \leq \frac{1}{3}\epsilon$ for every $x \in S_X$. Note that $\tilde{h}_2^{-1}h_2 \in H'_Y$, so that $\tilde{h}_2$ and h_2 belong to the same left coset of H'_Y, and

$$\lambda'_Y(H'_Y \cap \tilde{h}_2^{-1}Q_2) = \lambda'_Y(H'_Y \cap h_2^{-1}Q_2) \geq \frac{1}{2}\epsilon$$

by 443Pa.

At this point, recall that $\dim Y \geq r(m, \frac{1}{3}\epsilon)$, and that λ'_Y is a copy of λ_Y. So we have $g_1 \in H'_Y \cap h_1^{-1}Q_1$, $g_2 \in H'_Y \cap \tilde{h}_2^{-1}Q_2$ such that $\|g_1(y_i) - g_2(y_i)\| \leq \frac{1}{3}\epsilon$ for every $i < m$. We have $f_1 = h_1 g_1 \in Q_1$ and $f_2 = \tilde{h}_2 g_2 \in Q_2$. For any $i < m$,

$$\begin{aligned}\|f_1(y_i) - f_2(y_i)\| &\leq \|h_1 g_1(y_i) - h_1 g_2(y_i)\| + \|h_1 g_2(y_i) - \tilde{h}_2 g_2(y_i)\| \\ &\leq \|g_1(y_i) - g_2(y_i)\| + \tfrac{1}{3}\epsilon \leq \tfrac{2}{3}\epsilon.\end{aligned}$$

Also $g_1(x_m) = g_2(x_m) = x_m$, so

$$\|f_1(x_m) - f_2(x_m)\| = \|h_1(x_m) - \tilde{h}_2(x_m)\| \leq \tfrac{1}{3}\epsilon.$$

If $i < m$, then $x_i = (x_i|x_m)x_m + (x_i|y_i)y_i$, so $f_j(x_i) = (x_i|x_m)f_j(x_m) + (x_i|y_i)f_j(y_i)$ for both j (476N) and

$$\|f_1(x_i) - f_2(x_i)\| \leq \tfrac{1}{3}\epsilon|(x_i|x_m)| + \tfrac{2}{3}\epsilon|(x_i|y_i)|$$
$$\leq \sqrt{(\tfrac{1}{3}\epsilon)^2 + (\tfrac{2}{3}\epsilon)^2}\sqrt{(x_i|x_m)^2 + (x_i|y_i)^2} \leq \epsilon.$$

So f_1 and f_2 witness that the induction proceeds.

493J Theorem Let X be an infinite-dimensional inner product space. Then the isometry group H_X of its unit sphere S_X, with its topology of pointwise convergence, is extremely amenable.

proof I use the same approach as in 493G.

(a) Let $\mathcal{Y}$ be the family of finite-dimensional subspaces of X. For $Y \in \mathcal{Y}$, write $Y^\perp$ for the orthogonal complement of Y, so that $X = Y \oplus Y^\perp$ (4A4Je). For $q \in H_Y$ define $\theta_Y(q) : S_X \to S_X$ by saying that $\theta_Y(q)(\alpha y + \beta z) = \alpha q(y) + \beta z$ whenever $y \in S_Y$, $z \in S_{Y^\perp}$ and $\alpha^2 + \beta^2 = 1$. Then $\theta_Y : H_Y \to H_X$ is a injective group homomorphism. Also it is continuous, because $q \mapsto \alpha q(y) + \beta z$ is continuous for all relevant α, β, y and z.

If Y, $W \in \mathcal{Y}$ and $Y \subseteq W$ then $\theta_Y[H_Y] \subseteq \theta_W[H_W]$. **P** For any $q \in H_Y$ we can define $q' \in H_W$ by saying that $q'(\alpha y + \beta x) = \alpha q(y) + \beta x$ whenever $y \in S_Y$, $x \in S_{W \cap Y^\perp}$ and $\alpha^2 + \beta^2 = 1$. Now $\theta_Y(q) = \theta_W(q') \in \theta_W[H_W]$. **Q**

Set $G^* = \bigcup_{Y \in \mathcal{Y}} \theta_Y[H_Y]$, so that G^* is a subgroup of H_X.

(b) Let V be an open neighbourhood of the identity in G^* (with the subspace topology inherited from the topology of pointwise convergence on H_X), $\epsilon > 0$ and $I \subseteq G^*$ a finite set. Then there is a Borel probability measure λ on G^* such that

(i) $\lambda(fQ) = \lambda Q$ for every $f \in I$ and every closed set $Q \subseteq G^*$,

(ii) $\lambda(VQ) \geq 1 - \epsilon$ whenever $Q \subseteq G^*$ is closed and $\lambda Q \geq \frac{1}{2}$.

P We may suppose that $\epsilon \leq \frac{1}{2}$. Let $J \in [S_X]^{<\omega}$ and $\delta > 0$ be such that $f \in V$ whenever $f \in G^*$ and $\|f(x) - x\| \leq \delta$ for every $x \in J$. We may suppose that J is non-empty; set $m = \#(J)$. Let $Y \in \mathcal{Y}$ be such that $J \subseteq Y$ and $I \subseteq \theta_Y[H_Y]$ and $\dim Y = r \geq r(m, \epsilon)$, as chosen in 493I. (This is where we need to know that X is infinite-dimensional.) Set $\lambda F = \lambda_Y \theta_Y^{-1}[F]$ for every Borel set $F \subseteq G^*$.

If $f \in I$ and $F \subseteq G^*$ is closed, then

$$\lambda(fF) = \lambda_Y \theta_Y^{-1}[fF] = \lambda_Y(\theta_Y^{-1}(f)\theta_Y^{-1}[F]) = \lambda_Y \theta_Y^{-1}[F] = \lambda F.$$

So λ satisfies condition (i).

? Suppose, if possible, that $Q_1 \subseteq G^*$ is a closed set such that $\lambda Q_1 \geq \frac{1}{2}$ and $\lambda(VQ_1) < 1 - \epsilon$; set $Q_2 = G^* \setminus VQ_1$. Then $\theta_Y^{-1}[Q_1]$ and $\theta_Y^{-1}[Q_2]$ are subsets of H_Y both of measure at least ϵ. Set $R_j = \{q : q \in H_Y, q^{-1} \in \theta_Y^{-1}[Q_j]\}$ for each j; because H_Y is compact, therefore unimodular,

$$\lambda_Y R_j = \lambda_Y \theta_Y^{-1}[Q_j] = \lambda Q_j \geq \epsilon$$

for both j. Because $\dim Y \geq r(m, \epsilon)$, there are $q_1 \in R_1$, $q_2 \in R_2$ such that $\|q_1(x) - q_2(x)\| \leq \epsilon$ for $x \in J$. Set $f = \theta_Y(q_2^{-1}q_1)$. If $x \in J$, then

$$\|f(x) - x\| = \|q_2^{-1}q_1(x) - x\| = \|q_1(x) - q_2(x)\| \leq \epsilon.$$

As this is true for every $x \in J$, $f \in V$. On the other hand, $\theta_Y(q_1^{-1}) \in Q_1$ and $\theta_Y(q_2^{-1}) \in Q_2$ and $f\theta_Y(q_1^{-1}) = \theta_Y(q_2^{-1})$, so $\theta_Y(q_2^{-1}) \in VQ_1 \cap Q_2$, which is impossible. **X**

Thus λ satisfies (ii). **Q**

(c) By 493D, G^* is extremely amenable. But G^* is dense in H_X. **P** If $f \in H_X$ and $I \subseteq S_X$ is finite and not empty, let Y_1 be the linear subspace of X generated by I, and let $(y_1, \ldots, y_m)$ be an orthonormal basis of Y_1. Set $z_j = f(y_j)$ for each j, so that $(z_1, \ldots, z_m)$ is orthonormal (476N); let Y be the linear subspace of X generated by $y_1, \ldots, y_m, z_1, \ldots, z_m$. Set $r = \dim Y$ and extend the orthonormal sets $(y_1, \ldots, y_m)$ and $(z_1, \ldots, z_m)$ to orthonormal bases $(y_1, \ldots, y_r)$ and $(z_1, \ldots, z_r)$ of Y. Then we have an isometric linear operator $T : Y \to Y$ defined by saying that $Ty_i = z_i$ for each i; set $q = T \upharpoonright S_Y \in H_Y$. By 476N, $q(x) = f(x)$ for every $x \in I$, so $\theta_Y(q)$ agrees with f on I, while $\theta_Y(q) \in G^*$. As f and I are arbitrary, G^* is dense in G. **Q**

So 493Bb tells us that H_X is extremely amenable, and the proof is complete.

493K The following result shows why extremely amenable groups did not appear in Chapter 44.

Theorem (VEECH 77) If G is a locally compact Hausdorff topological group with more than one element, it is not extremely amenable.

proof If G is compact, this is trivial, since the left action of G on itself has no fixed point; so let us assume henceforth that G is not compact.

(a) Let Z be the greatest ambit of G, $a \mapsto \hat{a} : G \to Z$ the canonical map, and U the space of bounded right-uniformly continuous real-valued functions on G. (I aim to show that the action of G on Z has no fixed point.) Take any $z^* \in Z$. Let V_0 be a compact neighbourhood of the identity e in G, and let $B_0 \subseteq G$ be a maximal set such that $V_0 b \cap V_0 c = \emptyset$ for all distinct b, $c \in B_0$. Then for any $a \in G$ there is a $b \in B_0$ such that $V_0 a \cap V_0 b \neq \emptyset$, that is, $a \in V_0^{-1} V_0 B_0$. So if we set $Y_0 = \overline{\{\hat{b} : b \in B\}} \subseteq Z$, $\{a \bullet y : a \in V_0^{-1}V_0,\ y \in Y_0\}$ is a compact subset of Z including $\{\hat{a} : a \in G\}$, and is therefore the whole of Z (449Dc). Let $a_0 \in V_0^{-1}V_0$, $y_0 \in Y_0$ be such that $a_0 \bullet y_0 = z^*$, and set $B_1 = a_0 B_0$, $V_1 = a_0 V_0 a_0^{-1}$; then $z^* \in \overline{\{\hat{b} : b \in B_1\}}$ and $V_1 b \cap V_1 c = \emptyset$ for all distinct b, $c \in B_1$.

(b) Because V_1 is compact and G is not compact, there is an $a_1 \in G \setminus V_1$. Let $V_2 \subseteq V_1$ be a neighbourhood of e such that $a_1^{-1} V_2 V_2^{-1} a_1 \subseteq V_1$. Then we can express B_1 as $D_0 \cup D_1 \cup D_2$ where $a_1 D_i \cap V_2 D_i = \emptyset$ for all i. **P** Consider $\{(b, c) : b,\ c \in B_1,\ a_1 b \in V_2 c\}$. Because $V_2 c \cap V_2 c' \subseteq V_1 c \cap V_1 c' = \emptyset$ for all distinct c, $c' \in B_1$, this is the graph of a function $h : D \to B_1$ for some $D \subseteq B_1$. **?** If h is not injective, we have distinct b, $c \in B_1$ and $d \in B_1$ such that $a_1 b$ and $a_1 c$ both belong to $V_2 d$. But in this case b and c both belong to $a_1^{-1} V_2 d$ and $bc^{-1} \in a_1^{-1} V_2 d d^{-1} V_2^{-1} a_1 \subseteq V_1$ and $b \in V_1 c$, which is impossible. **X** At the same time, if $b \in B_1$, then $a_1 b \notin V_2 b$ because $a_1 \notin V_2$, so $h(b) \neq b$ for every $b \in D$.

Let $D_0 \subseteq D$ be a maximal set such that $h[D_0] \cap D_0 = \emptyset$, and set $D_1 = h[D_0]$, $D_2 = B_1 \setminus (D_0 \cup D_1)$. Then $h[D_0] \cap D_0 = \emptyset$ by the choice of D_0; $h[D \cap D_1] \cap D_1 = \emptyset$ because h is injective and $D_1 \subseteq h[D \setminus D_1]$; and $h[D \cap D_2] \subseteq D_0$ because if $b \in D \cap D_2$ there must have been some reason why we did not put b into D_0, and it wasn't because $b \in h[D_0]$ or because $h(b) = b$. So $h[D_i] \cap D_i = \emptyset$ for all i, which is what was required. **Q**

(c) Since $z^* \in \overline{\{\hat{b} : b \in B_1\}}$, there must be some $j \leq 2$ such that $z^* \in \overline{\{\hat{b} : b \in D_j\}}$. Now recall that the right uniformity on G, like any uniformity, can be defined by some family of pseudometrics (4A2Ja). There is therefore a pseudometric ρ on G such that $W_\epsilon = \{(a, b) : a,\ b \in G,\ \rho(a, b) \leq \epsilon\}$ is a member of the right uniformity on G for every $\epsilon > 0$ and $W_1 \subseteq \{(a, b) : ab^{-1} \subseteq V_2\}$. If now we set

$$f(a) = \min(1, \rho(a, D_j)) = \min(1, \inf\{\rho(a, b) : b \in D_j\})$$

for $a \in G$, $f : G \to \mathbb{R}$ is bounded and uniformly continuous for the right uniformity, so belongs to U. On the other hand, if b, $c \in D_j$, then $a_1 b \notin V_2 c$, that is, $a_1 b c^{-1} \notin V_2$ and $\rho(a_1 b, c) > 1$; as c is arbitrary, $f(a_1 b) = 1$.

(d) Now $\hat{b}(f) = f(b) = 0$ for every $b \in D_j$, so $z^*(f) = 0$. On the other hand, because $z \mapsto a_1 \bullet z$ is continuous,

$$a_1 \bullet z^* \in \overline{\{a_1 \bullet \hat{b} : b \in D_j\}} = \overline{\{\widehat{a_1 b} : b \in D_j\}},$$

so

$$(a_1 \bullet z^*)(f) \geq \inf_{b \in D_j} \widehat{a_1 b}(f) = \inf_{b \in D_j} f(a_1 b) = 1,$$

so $a_1 \bullet z^* \neq z^*$. As z^* is arbitrary, this shows that the action of G on Z has no fixed point, and G is not extremely amenable.

493X Basic exercises (a) Show that a dense subgroup of an extremely amenable topological group is extremely amenable.

(b) Let G be a Hausdorff topological group, and $\widehat{G}$ its completion with respect to its bilateral uniformity. Show that G is extremely amenable iff $\widehat{G}$ is.

>(c) Let X be a set with more than one member and ρ the metric on X such that $\rho(x, y) = 1$ for all distinct x, $y \in X$. Let G be the isometry group of X with the topology of pointwise convergence. Show that G is not extremely amenable. (*Hint*: give X a total ordering $\leq$, and let x, y be any two points of X.

For $a \in G$ set $f(a) = 1$ if $a^{-1}(x) < a^{-1}(y)$, -1 otherwise. Show that, in the language of 449D, $f \in U$. Show that if $(\overleftarrow{x\,y})$ is the transposition exchanging x and y then $(\overleftarrow{x\,y})\bullet f = -f$, while $|z(f)| = 1$ for every z in the greatest ambit of G.) (Compare 449Xc.)

(d) Show that under the conditions of 493D (i) there is a finitely additive functional $\nu : \mathcal{B} \to [0,1]$ such that $\nu(aF) = \nu F$ for every $a \in G$ and every zero set $F \subseteq G$, while $\nu(VF) = 1$ whenever V is a neighbourhood of the identity, F is a zero set and $\nu F > \frac{1}{2}$ (ii) there is a positive linear functional $p : C_b(G) \to \mathbb{R}$ such that $p(a\bullet f) = p(f)$ for every $f \in C_b(G)$ and $a \in G$.

(e) Let $(\mathfrak{A}, \bar{\mu})$ be a σ-finite measure algebra of countable Maharam type. Show that $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$, with the topology described in 493H, is a Polish group. (*Hint*: 441Xm.)

(f) Let X be any inner product space. Show that the isometry group of X, with its topology of pointwise convergence, is amenable. (*Hint*: 449Cd.)

(g) Let X be a separable Hilbert space. Show that the isometry group of its unit sphere, with its topology of pointwise convergence, is a Polish group.

(h) Let G be a topological group carrying Haar measures. Show that it is extremely amenable iff its topology is the indiscrete topology. (*Hint*: 443L.)

493Y Further exercises (a) For a Boolean algebra $\mathfrak{A}$ and a group G with identity e, write $S(\mathfrak{A}; G)$ for the set of partitions of unity $\langle a_g\rangle_{g\in G}$ in $\mathfrak{A}$ such that $\{g : a_g \neq e\}$ is finite. For $\langle a_g\rangle_{g\in G}$, $\langle b_g\rangle_{g\in G} \in S(A)$, write $\langle a_g\rangle_{g\in G} \cdot \langle b_g\rangle_{g\in G} = \langle c_g\rangle_{g\in G}$ where $c_g = \sup\{a_h \cap b_{h^{-1}g} : h \in G\}$ for $g \in G$. (i) Show that under this operation $S(\mathfrak{A}; G)$ is a group. (ii) Show that if we write $h\chi a$ for the member $\langle a_g\rangle_{g\in G}$ of $S(\mathfrak{A}; G)$ such that $a_h = a$ and $a_g = 0$ for other $g \in G$, then $g\chi a \cdot h\chi b = (gh)\chi(a \cap b)$, and $S(\mathfrak{A}; G)$ is generated by $\{g\chi a : g \in G, a \in \mathfrak{A}\}$. (iii) Show that if $\mathfrak{A} = \Sigma/\mathcal{I}$ where Σ is an algebra of subsets of a set X and $\mathcal{I}$ is an ideal of Σ, then $S(\mathfrak{A}; G)$ can be identified with a space of equivalence classes in a suitable subgroup of G^X. (iv) Devise a universal mapping theorem for the construction $S(\mathfrak{A}; G)$ to match 361F in the case $(G, \cdot) = (\mathbb{R}, +)$. (v) Now suppose that $(\mathfrak{A}, \bar{\mu})$ is a measure algebra and that G is a topological group. Show that we have a topology on $S(\mathfrak{A}, G)$, making it a topological group, for which basic neighbourhoods of the identity $e\chi 1$ are of the form $V(c, \epsilon, U) = \{\langle a_g\rangle_{g\in G} : \bar{\mu}(c \cap \sup_{g\in G\setminus U} a_g) \leq \epsilon\}$ with $\bar{\mu}c < \infty$, $\epsilon > 0$ and U a neighbourhood of the identity in G. (vi) Show that if G is an amenable locally compact Hausdorff group and $(\mathfrak{A}, \bar{\mu})$ is an atomless measure algebra, then $S(\mathfrak{A}; G)$ is extremely amenable. (*Hint*: Pestov p01.) *(vi) Explore possible constructions of spaces $L^0(\mathfrak{A}; G)$. (See Hartman & Mycielski 58.)

(b) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless totally finite measure algebra; give $\mathfrak{A}$ its measure metric. Show that the isometry group of $\mathfrak{A}$, with its topology of pointwise convergence, is extremely amenable. (*Hint*: every isometry of $\mathfrak{A}$ is of the form $a \mapsto c \triangle \pi a$, where $c \in \mathfrak{A}$ and $\pi \in \mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$; now use 493Bc.)

(c) Let $(\mathfrak{A}, \bar{\mu})$ be an atomless probability algebra and $\pi : \mathfrak{A} \to \mathfrak{A}$ an ergodic measure-preserving automorphism. (i) Show that for any $n \geq 1$ there is a periodic measure-preserving automorphism ϕ of period n and support 1 such that the support of $\phi^{-1}\pi$ has measure at most $\frac{2}{n}$. (*Hint*: 386C.) (ii) Show that $\{\phi\pi\phi^{-1} : \phi \in \mathrm{Aut}_{\bar{\mu}}\mathfrak{A}\}$ is dense in $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ with its pointwise topology as described in 493H. (*Hint*: any two automorphisms with period n and support 1 are conjugate.)

(d) Let $(\mathfrak{A}, \bar{\mu})$ be the measure algebra of Lebesgue measure on $[0,1]$, and $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ the Polish group of measure-preserving automorphisms of $\mathfrak{A}$ (493Xe). (i) Show that the entropy function h of 385M is Borel measurable. (*Hint*: 385Xr.) (ii) Show that the set of ergodic measure-preserving automorphisms is a G_δ set in $\mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$. (*Hint*: let $D \subseteq \mathfrak{A}$ be a countable dense set. Show that $\pi \in \mathrm{Aut}_{\bar{\mu}}\mathfrak{A}$ is ergodic iff $\lim_{n\to\infty} \|\frac{1}{n+1}\sum_{i=0}^{n} \chi(\pi^i d) - \bar{\mu}d \cdot \chi d\|_1 = 0$ for every $d \in D$.)

493 Notes and comments In writing this section I have relied heavily on PESTOV 99 and PESTOV P01, where you may find many further examples of extremely amenable groups. It is a striking fact that while the theories of locally compact groups and extremely amenable groups are necessarily almost entirely separate (493K), both are dependent on measure theory. Curiously, what seems to have been the first non-trivial extremely amenable group to be described was found in the course of investigating the Control Measure Problem (HERER & CHRISTENSEN 75).

The theory of locally compact groups has for seventy years now been a focal point for measure theory. Extremely amenable groups have not yet had such an influence. But they encourage us to look again at concentration-of-measure theorems, which are of the highest importance for quite separate reasons. In all four of the principal examples of this section we need concentration of measure in product spaces (493E-493F), permutation groups (493H) or on spheres in Euclidean space (493J). 493E and 493F are special cases of a general result in PESTOV P01 (493Ya(vi)) which itself extends an idea from GLASNER 98. I note that 493E needs only concentration of measure in $\{0,1\}^I$, while 493F demands something rather closer to the full strength of Talagrand's theorem 492D.

I have expressed 493J as a theorem about the isometry groups of spheres in infinite-dimensional inner product spaces; of course these are isomorphic to the orthogonal groups of the whole spaces with their strong operator topologies (476Xi). Adapting the basic concentration-of-measure theorem 476O to the required lemma 493I involves an instructive application of ideas from §443.

494 Cubes in product spaces

I offer a brief note on a special property of (Radon) product measures.

494A Proposition Let $(\mathfrak{A},\bar\mu)$ be a semi-finite measure algebra with its measure-algebra topology (323A). Suppose that $A\subseteq\mathfrak{A}$ is an uncountable analytic set. Then there is a compact set $L\subseteq A$, homeomorphic to $\{0,1\}^{\mathbb{N}}$, such that $\inf L\neq 0$ in $\mathfrak{A}$.

proof $A\setminus\{0\}$ is still an uncountable analytic subset of $\mathfrak{A}$. By 423J, it has a subset homeomorphic to $\{0,1\}^{\mathbb{N}}\cong\mathcal{P}\mathbb{N}$; let $f:\mathcal{P}\mathbb{N}\to A\setminus\{0\}$ be an injective continuous function. Because $(\mathfrak{A},\bar\mu)$ is semi-finite, there is an $a\subseteq f(\emptyset)$ such that $0<\bar\mu a<\infty$; set $\delta=\bar\mu a$. Note that $(I,J)\mapsto\bar\mu(a\cap f(I)\setminus f(J)):(\mathcal{P}\mathbb{N})^2\to\mathbb{R}$ is continuous. Choose a sequence $\langle k_n\rangle_{n\in\mathbb{N}}$ in $\mathbb{N}$ inductively, as follows. Given $\langle k_i\rangle_{i<n}$, set $K_n=\{k_i:i<n\}$. For each $J\subseteq K_n$ we have $\lim_{r\to\infty}\bar\mu(a\cap f(J)\setminus f(J\cup\{r\}))=0$, so there is a k_n, greater than k_i for every $i<n$, such that $\bar\mu(a\cap f(J)\setminus f(J\cup\{k_n\}))\leq 2^{-2n-2}\delta$ for every $J\subseteq K_n$; continue.

Now

$$\bar\mu(a\cap\inf_{J\subseteq K_n}f(J))\geq\delta(\tfrac12+2^{-n-1})$$

for every $n\in\mathbb{N}$. **P** Induce on n. If $n=0$, then $a\cap\inf_{J\subseteq K_n}f(J)=a\cap f(\emptyset)$ has measure $\delta=\delta(\frac12+2^{-1})$. For the inductive step to $n+1\geq 1$, observe that

$$\begin{aligned}\bar\mu(a\cap\inf_{J\subseteq K_{n+1}}f(J))&=\bar\mu(a\cap\inf_{J\subseteq K_n}f(J)\cap f(J\cup\{k_n\}))\\&\geq\bar\mu(a\cap\inf_{J\subseteq K_n}f(J))-\sum_{J\subseteq K_n}\bar\mu(a\cap f(J)\setminus f(J\cup\{k_n\}))\\&\geq\delta(\tfrac12+2^{-n-1})-\sum_{J\subseteq K_n}2^{-2n-2}\delta\end{aligned}$$

(by the inductive hypothesis and the choice of k_n)

$$=\delta(\tfrac12+2^{-n-2}).$$

So the induction proceeds. **Q**

Set $K=\{k_i:i\in\mathbb{N}\}$, $c=\inf\{f(J):J\subseteq K\text{ is finite}\}$. Then

$$\bar\nu(a\cap c)=\inf_{n\in\mathbb{N}}\bar\mu(a\cap\inf_{J\subseteq K_n}f(J))\geq\tfrac{1}{2}\delta,$$

and $c\neq 0$. But now observe that $L=f[\mathcal{P}K]$ is a subset of A homeomorphic to $\mathcal{P}K$ and therefore to $\{0,1\}^{\mathbb{N}}$. Also $\{b:b\supseteq c\}$ is closed (323D(d-i)), so $C=\{J:f(J)\supseteq c\}$ is closed in $\mathcal{P}K$; as it includes the dense set $[K]^{<\omega}$, $C=\mathcal{P}K$ and $\inf L\supseteq c$ is non-zero.

494B Proposition (see Brodskiĭ 49, Eggleston 54) Let $(X,\mathfrak{T},\Sigma,\mu)$ be an atomless Radon measure space, $(Y,\mathfrak{S},\mathrm{T},\nu)$ an effectively locally finite τ-additive topological measure space, and $\tilde\lambda$ the τ-additive product measure on $X\times Y$ (417G). Then if $W\subseteq X\times Y$ and $\tilde\lambda W>0$ there are a non-scattered compact set $K\subseteq X$ and a set $F\subseteq Y$ of positive measure such that $K\times F\subseteq W$.

proof (a) To begin with (down to the end of (c)), let us suppose that both μ and ν are totally finite. Let $(\mathfrak{B},\bar\nu)$ be the measure algebra of ν. Because $\tilde\lambda$ is inner regular with respect to the closed sets, we may suppose that W is closed. In this case, writing λ for the c.l.d. product measure on $X\times Y$, there is a $W'\supseteq W$ such that $\lambda W'$ is defined and equal to $\tilde\lambda W$ (417C(iv)). By 418Tb, there is a μ-conegligible set X_0 such that $W'[\{x\}]\in\mathrm{T}$ for every $x\in X_0$, $B=\{W'[\{x\}]^\bullet:x\in X_0\}$ is separable for the measure-algebra topology of $\mathfrak{B}$, and $x\mapsto W'[\{x\}]^\bullet:X_0\to\mathfrak{B}$ is measurable. Now Fubini's theorem, applied in the form of 252D to λ and in the form of 417Ha to $\tilde\lambda$, tells us that

$$\int\nu W'[\{x\}]\mu(dx)=\lambda W'=\tilde\lambda W=\int\nu W[\{x\}]\mu(dx).$$

So $X_1=\{x:x\in X_0,\ W'[\{x\}]^\bullet=W[\{x\}]^\bullet\}$ is μ-conegligible. Since the topology of $\mathfrak{B}$ is metrizable (323Ad), B is separable and metrizable, and $x\mapsto W[\{x\}]^\bullet:X_1\to B$ is almost continuous (418J, applied to the subspace measure on X_1). Let $K^*\subseteq X_1$ be a compact set of non-zero measure such that $x\mapsto W[\{x\}]^\bullet:K^*\to\mathfrak{B}$ is continuous.

(b) There is a non-zero $c\in\mathfrak{B}$ such that $K_c=\{x:x\in K^*,\ c\subseteq W[\{x\}]^\bullet\}$ is compact and not scattered. **P** Because $x\mapsto W[\{x\}]^\bullet$ is continuous on K^*, $B^*=\{W[\{x\}]^\bullet:x\in K^*\}$ is compact and every K_c is compact. (i) If B^* is countable, then $K^*=\bigcup_{b\in B^*}K_b$, so there is some $c\in B^*$ such that $\mu K_c>0$. Let E be a non-negligible self-supporting subset of K_c; then (because μ is atomless, therefore zero on singletons) E has no isolated points. So K_c is not scattered. (ii) If B^* is uncountable, then by 494A there is a set $D\subseteq B^*$, homeomorphic to $\{0,1\}^{\mathbb{N}}$, with a non-zero lower bound c in $\mathfrak{B}$. Now $\{W[\{x\}]^\bullet:x\in K_c\}$ includes D, so $\{0,1\}^{\mathbb{N}}$ and therefore $[0,1]$ are continuous images of closed subsets of K_c and K_c is not scattered (4A2G(i-iv)). **Q**

(c) Set $K=K_c$. Then 414Ac tells us that

$$(\textstyle\bigcap_{x\in K}W[\{x\}])^\bullet=\inf_{x\in K}W[\{x\}]^\bullet\supseteq c$$

is non-zero, so $F=\bigcap_{x\in K}W[\{x\}]$ is non-negligible; while $K\times F\subseteq W$. So we have found appropriate sets K and F, at least when μ and ν are totally finite.

(d) For the general case, we need observe only that by 417C(iii) there are $X'\in\Sigma$ and $Y'\in\mathrm{T}$, both of finite measure, such that $\tilde\lambda(W\cap(X'\times Y'))>0$. Now the subspace measure $\mu_{X'}$ on X' is atomless and Radon (214Ja, 416Rb), the subspace measure $\nu_{Y'}$ on Y' is τ-additive (414K), and the τ-additive product of $\mu_{X'}$ and $\nu_{Y'}$ is the subspace measure on $X'\times Y'$ induced by $\tilde\lambda$ (417I). So we can apply (a)-(c) to $\mu_{X'}$ and $\nu_{Y'}$ to see that there are a non-scattered compact set $K\subseteq X'$ and a non-negligible measurable set $F\subseteq Y'$ such that $K\times F\subseteq W$.

494C Proposition (see Ciesielski & Pawlikowski 03) Let $\langle(X_i,\mathfrak{T}_i,\Sigma_i,\mu_i)\rangle_{i\in I}$ be a countable family of atomless Radon probability spaces, and $\tilde\lambda$ the product Radon probability measure on $X=\prod_{i\in I}X_i$. If $W\subseteq X$ and $\tilde\lambda W>0$, there is a family $\langle K_i\rangle_{i\in I}$ such that $K_i\subseteq X_i$ is a non-scattered compact set for each $i\in I$ and $\prod_{i\in I}K_i\subseteq W$.

proof (a) To begin with, let us suppose that $I=\mathbb{N}$. For each $n\in\mathbb{N}$, set $Y_n=\prod_{i\geq n}X_i$ and let $\tilde\lambda_n$ be the product Radon probability measure on Y_n, so that $\tilde\lambda_0=\tilde\lambda$ and $\tilde\lambda_n$ can be identified with the product of μ_n and $\tilde\lambda_{n+1}$ (417J). Using 494B repeatedly, we can find non-scattered compact sets $K_n\subseteq X_n$ and closed non-negligible sets $W_n\subseteq Y_n$ such that $W_0\subseteq W$ and $K_n\times W_{n+1}\subseteq W_n$ for every n. In this case,

$\prod_{i<n} K_i \times W_n \subseteq W_0$ for every n. If $x \in \prod_{i\in\mathbb{N}} K_i$, then there is for each $n \in \mathbb{N}$ an $x_n \in \prod_{i<n} K_i \times W_n$ such that $x_n{\restriction}n = x{\restriction}n$, just because W_n is not empty. But now every $x_n \in W_0$ and

$$x = \lim_{n\to\infty} x_n \in W_0 \subseteq W.$$

As x is arbitrary, $\prod_{i\in\mathbb{N}} K_i \subseteq W$.

(b) For the general case, we may suppose that $I \subseteq \mathbb{N}$. For $i \in \mathbb{N} \setminus I$, take $(X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)$ to be $[0,1]$ with Lebesgue measure. Set $\tilde{W} = \{x : x \in \prod_{i\in\mathbb{N}} X_i,\ x{\restriction}I \in W\}$. By (a), there are non-scattered compact sets $K_i \subseteq X_i$ such that $\prod_{i\in\mathbb{N}} K_i \subseteq \tilde{W}$, in which case $\prod_{i\in I} K_i \subseteq W$, as required.

494X Basic exercises (a) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a Radon measure space, $(Y, \mathfrak{S}, \mathrm{T}, \nu)$ an effectively locally finite τ-additive topological measure space, and $\tilde{\lambda}$ the τ-additive product measure on $X \times Y$. Show that if $W \subseteq X \times Y$ and $\tilde{\lambda}W > 0$ there are a compact set $K \subseteq X$ and a set $F \subseteq Y$ of positive measure such that $K \times F \subseteq W$ and K is either non-scattered or non-negligible.

(b) Let $(X, \mathfrak{T}, \Sigma, \mu)$ be an atomless Radon measure space, (Y, T, ν) any measure space, and λ the c.l.d. product measure on $X \times Y$. Show that if $W \subseteq X \times Y$ and $\lambda W > 0$ there are a non-scattered compact set $K \subseteq X$ and a set $F \subseteq Y$ of positive measure such that $K \times F \subseteq W$. (*Hint*: reduce to the case in which ν is totally finite and T is countably generated, so that the completion of ν is a quasi-Radon measure for an appropriate second-countable topology.)

(c) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be any family of atomless Radon probability spaces, and λ the ordinary product measure on $X = \prod_{i\in I} X_i$. Show that if $W \subseteq X$ and $\lambda W > 0$ then there are non-scattered compact sets $K_i \subseteq X_i$ for $i \in I$ such that $\prod_{i\in I} K_i \subseteq W$.

(d) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a countable family of atomless Radon probability spaces, and $W \subseteq \prod_{i\in I} X_i$ a set with positive measure for the Radon product of $\langle \mu_i\rangle_{i\in I}$. Show that there are atomless Radon probability measures ν_i on X_i such that W is conegligible for the Radon product of $\langle \nu_i\rangle_{i\in I}$. (*Hint*: 439Xh(vii).)

494Y Further exercises (a) Let $\langle (X_i, \mathfrak{T}_i, \Sigma_i, \mu_i)\rangle_{i\in I}$ be a family of atomless perfect probability spaces, and λ the ordinary product measure on $X = \prod_{i\in I} X_i$. Show that if $W \subseteq X$ and $\lambda W > 0$ then there are sets $K_i \subseteq X_i$ for $i \in I$, all of cardinal $\mathfrak{c}$, such that $\prod_{i\in I} K_i \subseteq W$.

494 Notes and comments I have already noted (325Yc) that a set W of positive measure in a product space need not include the product of two sets of positive measure; this fact is also the basis of 419E. Here, however, we see that if one of the factors is a Radon measure space then W does include the product of a non-trivial compact set and a set of positive measure. There are many possible variations on the result, corresponding to different product measures (494B, 494Xb) and different notions of 'non-trivial' (494Xa, 494Ya). The most important of the latter seems to be the idea of a 'non-scattered' compact set K; this is a quick way of saying that $[0,1]$ is a continuous image of K, which is a little stronger than saying that $\#(K) \geq \mathfrak{c}$, and arises naturally from the proof of 494B.

495 Poisson point processes

A classical challenge in probability theory is to formulate a consistent notion of 'random set'. Simple geometric considerations lead us to a variety of measures which are both interesting and important. All these are manifestly special constructions. Even in the most concrete structures, we have to make choices which come to seem arbitrary as soon as we are conscious of the many alternatives. There is however one construction which has a claim to pre-eminence because it is both robust under the transformations of abstract measure theory and has striking properties when applied to familiar measures (to the point, indeed, that it is relevant to questions in physics and chemistry). This gives the 'Poisson point processes' of 495D-495E. In this section I give a brief introduction to the measure-theoretic aspects of this construction.

495A Poisson distributions We need a little of the elementary theory of Poisson distributions.

(a) The **Poisson distribution** with parameter $\gamma > 0$ is the point-supported Radon probability measure ν_γ on $\mathbb{R}$ such that $\nu_\gamma\{n\} = \frac{\gamma^n}{n!}e^{-\gamma}$ for every $n \in \mathbb{N}$. (See 285Q and 285Xo.) Its expectation is $\sum_{n=1}^\infty \frac{\gamma^n}{(n-1)!}e^{-\gamma} = \gamma$. Since $\nu_\gamma\mathbb{N} = 1$, ν_γ can be identified with the corresponding subspace measure on $\mathbb{N}$. It will be convenient to allow $\gamma = 0$, so that the probability measure on $\mathbb{R}$ or $\mathbb{N}$ with support $\{0\}$ is a 'Poisson distribution with expectation 0'.

(b) The convolution of two Poisson distributions is a Poisson distribution. **P** If α, $\beta > 0$ then

$$(\nu_\alpha * \nu_\beta)(\{n\}) = \int \nu_\beta(\{n\} - t)\nu_\alpha(dt)$$

(444A)

$$= \sum_{i=0}^n \frac{\beta^{n-i}}{(n-i)!}e^{-\beta} \cdot \frac{\alpha^i}{i!}e^{-\alpha}$$

$$= e^{-\alpha-\beta}\frac{1}{n!}\sum_{i=0}^n \frac{n!}{i!(n-i)!}\alpha^i\beta^{n-i} = \frac{(\alpha+\beta)^n}{n!}e^{-\alpha-\beta}$$

for every $n \in \mathbb{N}$, so $\nu_\alpha * \nu_\beta = \nu_{\alpha+\beta}$. **Q** So if f and g are independent random variables with Poisson distributions then $f + g$ has a Poisson distribution (272S).

(c) If $\langle f_i\rangle_{i\in I}$ is a countable independent family of random variables with Poisson distributions, and $\alpha = \sum_{i\in I}\mathbb{E}(f_i)$ is finite, then $f = \sum_{i\in I} f_i$ is defined a.e. and has a Poisson distribution with expectation α. **P** For finite I we can induce on $\#(I)$, using (b) (and 272L) for the inductive step. For the infinite case we can suppose that $I = \mathbb{N}$. In this case $f_i \geq 0$ a.e. for each i so $f = \sum_{i=0}^\infty f_i$ is defined a.e. and has expectation α, by B.Levi's theorem. Setting $g_n = \sum_{i=0}^n f_i$ for each n, so that g_n has a Poisson distribution with expectation $\beta_n = \sum_{i=0}^n \alpha_i$, we have

$$\Pr(f \leq \gamma) = \lim_{n\to\infty}\Pr(g_n \leq \gamma) = \lim_{n\to\infty}\sum_{i=0}^{\lfloor\gamma\rfloor}\frac{\beta_n^i}{i!}e^{-\beta_n} = \sum_{i=0}^{\lfloor\gamma\rfloor}\frac{\alpha^i}{i!}e^{-\alpha}$$

for every $\gamma \geq 0$, so f has a Poisson distribution with expectation α. **Q**

495B Theorem Let (X, Σ, μ) be a measure space. Set $\Sigma^f = \{E : E \in \Sigma, \mu E < \infty\}$. Then for any $\gamma > 0$ there are a probability space $(\Omega, \Lambda, \lambda)$ and a family $\langle g_E\rangle_{E\in\Sigma^f}$ of random variables on Ω such that

(i) for every $E \in \Sigma^f$, g_E has a Poisson distribution with expectation $\gamma\mu E$;

(ii) whenever $\langle E_i\rangle_{i\in I}$ is a disjoint family in Σ^f, then $\langle g_{E_i}\rangle_{i\in I}$ is stochastically independent;

(iii) whenever $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in Σ^f with union $E \in \Sigma^f$, then $g_E =_{\text{a.e.}} \sum_{i=0}^\infty g_{E_i}$.

proof (a) Let $\mathcal{H} \subseteq \{H : H \in \Sigma, 0 < \mu H < \infty\}$ be a maximal family such that $H \cap H'$ is negligible for all distinct H, $H' \in \mathcal{H}$. For $H \in \mathcal{H}$, let μ'_H be the normalized subspace measure defined by setting $\mu'_H E = \mu E/\mu H$ for $E \in \Sigma \cap \mathcal{P}H$, and λ_H the corresponding product probability measure on $H^{\mathbb{N}}$. Next, for $H \in \mathcal{H}$, let ν_H be the Poisson distribution with expectation $\gamma\mu H$, regarded as a probability measure on $\mathbb{N}$. Let λ be the product measure on $\Omega = \prod_{H\in\mathcal{H}}(\mathbb{N} \times H^{\mathbb{N}})$, giving each $\mathbb{N} \times H^{\mathbb{N}}$ the product measure $\nu_H \times \lambda_H$. For $\omega \in \Omega$, write $m_H(\omega)$, $x_{Hj}(\omega)$ for its coordinates, so that $\omega = \langle(m_H(\omega), \langle x_{Hj}(\omega)\rangle_{j\in\mathbb{N}})\rangle_{H\in\mathcal{H}}$.

(b) For $H \in \mathcal{H}$ and $E \in \Sigma$, set $g_{HE}(\omega) = \#(\{j : j < m_H(\omega), x_{Hj}(\omega) \in E\})$. Then g_{HE} is measurable and has a Poisson distribution with expectation $\gamma\mu(H \cap E)$; moreover, if $E_0, \ldots, E_r \in \Sigma$ are disjoint, then $g_{HE_0}, \ldots, g_{HE_r}$ are independent. **P** It is enough to examine the case in which the E_i cover X. Then for any $n_0, \ldots, n_r \in \mathbb{N}$ with sum n,

$$\begin{aligned}
&\lambda\{\omega : g_{HE_i}(\omega) = n_i \text{ for every } i \le r\}\\
&\quad = \lambda\{\omega : \#(\{j : j < m_H(\omega),\, x_{Hj} \in E_i\}) = n_i \text{ for every } i \le r\}\\
&\quad = \lambda\{\omega : m_H(\omega) = n,\, \#(\{j : j < n,\, x_{Hj} \in E_i\}) = n_i \text{ for every } i \le r\}\\
&\quad = \sum_{\substack{J_0,\dots,J_r \text{ partition } n\\ \#(J_i)=n_i \text{ for each } i\le r}} \lambda\{\omega : m_H(\omega) = n,\, x_{Hj} \in E_i \text{ whenever } i \le r,\, j \in I_i\}\\
&\quad = \sum_{\substack{J_0,\dots,J_r \text{ partition } n\\ \#(J_i)=n_i \text{ for each } i\le r}} \frac{(\gamma\mu H)^n}{n!} e^{-\gamma\mu H} \prod_{i=0}^{r} \big(\frac{\mu(H\cap E_i)}{\mu H}\big)^{n_i}\\
&\quad = \frac{n!}{n_0!\dots n_r!}\frac{1}{n!} e^{-\gamma\mu H} \prod_{i=0}^{r} (\gamma\mu(H\cap E_i))^{n_i}\\
&\quad = \prod_{i=0}^{r} \frac{(\gamma\mu(H\cap E_i))^{n_i}}{n_i!} e^{-\gamma\mu(H\cap E_i)},
\end{aligned}$$

which is just what we wanted to know. **Q**

Obviously $g_{HE} = \sum_{i=0}^{\infty} g_{HE_i}$ whenever $\langle E_i\rangle_{i\in\mathbb{N}}$ is a sequence in Σ with union E.

(c) Suppose that $H_0, \dots, H_n \in \mathcal{H}$ are distinct and $E_0, \dots, E_r \in \Sigma$ are disjoint. Then the random variables $g_{H_jE_i}$ are independent. **P** For each $j \le n$, $g_{H_jE_i}$ is Λ_{H_j}-measurable, where Λ_{H_j} is the σ-algebra of subsets of Ω which are measured by λ and determined by the single coordinate H_j in the product $\prod_{H\in\mathcal{H}}(\mathbb{N}\times H^{\mathbb{N}})$. Now the σ-algebras Λ_{H_j} are independent (272Ma). So if we have any family $\langle n_{ij}\rangle_{i\le r, j\le n}$ in $\mathbb{N}$,

$$\begin{aligned}
&\lambda\{\omega : g_{H_jE_i}(\omega) = n_{ij} \text{ for every } i \le r, j \le n\}\\
&\qquad = \prod_{j=0}^{n} \lambda\{\omega : g_{H_jE_i}(\omega) = n_{ij} \text{ for every } i \le r\}\\
&\qquad = \prod_{j=0}^{n}\prod_{i=0}^{r} \lambda\{\omega : g_{H_jE_i}(\omega) = n_{ij}\}
\end{aligned}$$

by (b); and this is what we need to know. **Q**

(d) For $E \in \Sigma^f$, set $\mathcal{H}_E = \{H : H \in \mathcal{H},\, \mu(E\cap H) > 0\}$; then $\mathcal{H}_E$ is countable, because $\mathcal{H}$ is almost disjoint, and $\mu E = \sum_{H\in\mathcal{H}_E} \mu(H\cap E)$, because $\mathcal{H}$ is maximal. Set $g_E(\omega) = \sum_{H\in\mathcal{H}_E} g_{HE}(\omega)$ when this is finite. Then g_E is defined a.e. and has a Poisson distribution with expectation $\gamma\mu E$ (495Ac). Also $\langle g_{E_i}\rangle_{i\in I}$ are independent whenever $\langle E_i\rangle_{i\in I}$ is a disjoint family in Σ. **P** It is enough to deal with the case of finite I (272Bb). Set $\mathcal{H}^* = \bigcup_{i\in I}\mathcal{H}_{E_i}$, so that $\mathcal{H}^*$ is countable, and for $i \in I$ set $g_i' = \sum_{H\in\mathcal{H}^*} g_{HE_i}$. Then each g_i' is equal almost everywhere to the corresponding g_{E_i}, and $\langle g_i'\rangle_{i\in I}$ is independent, by 272K. (The point is that each g_i' is Λ_i^*-measurable, where Λ_i^* is the σ-algebra generated by $\{g_{HE_i} : H \in \mathcal{H}\}$, and 272K assures us that the Λ_i^* are independent.) It follows at once that $\langle g_{E_i}\rangle_{i\in I}$ is independent (272H). **Q** This proves (i) and (ii).

(e) Similarly, if $\langle E_i\rangle_{i\in\mathbb{N}}$ is a disjoint sequence in Σ^f with union $E \in \Sigma^f$, set $\mathcal{H}^* = \mathcal{H}_E \cup \bigcup_{i\in\mathbb{N}}\mathcal{H}_{E_i}$. For each $i \in \mathbb{N}$, set $g_i' = \sum_{H\in\mathcal{H}^*} g_{HE_i}$; then $g_i' =_{\text{a.e.}} g_{E_i}$. Now

$$\sum_{i=0}^{\infty} g_{E_i} =_{\text{a.e.}} \sum_{i=0}^{\infty} g_i' = \sum_{H\in\mathcal{H}^*}\sum_{i=0}^{\infty} g_{HE_i} = \sum_{H\in\mathcal{H}^*} g_{HE} =_{\text{a.e.}} g_E,$$

as required by (iii).

495C Lemma Let X be a set and $\mathcal{E}$ a subring of the Boolean algebra $\mathcal{P}X$. Let $\mathcal{H}$ be the family of sets of the form

$$\{S : S \subseteq X, \#(S \cap E_i) = n_i \text{ for every } i \in I\}$$

where $\langle E_i\rangle_{i\in I}$ is a finite disjoint family in $\mathcal{E}$ and $n_i \in I$ for every $i \in I$. Then the Dynkin class $\mathrm{T} \subseteq \mathcal{P}(\mathcal{P}X)$ generated by $\mathcal{H}$ is the σ-algebra of subsets of $\mathcal{P}X$ generated by $\mathcal{H}$.

proof Let Q be the set of functions q from finite subsets of $\mathcal{E}$ to $\mathbb{N}$, and for $q \in Q$ set

$$H_q = \{S : S \subseteq X, \#(S \cap E) = q(E) \text{ for every } E \in \operatorname{dom} q\}.$$

Our family $\mathcal{H}$ is just $\{H_q : q \in Q, \operatorname{dom} q \text{ is disjoint}\}$.

If $q \in Q$ and $\operatorname{dom} q$ is a subring of $\mathcal{E}$, then $H_q \in \mathrm{T}$. **P** Being a finite Boolean ring, $\operatorname{dom} q$ is a Boolean algebra; let $\mathcal{A}$ be the set of its atoms. Then H_q is either empty or equal to $H_{q\restriction\mathcal{A}}$; in either case it belongs to T. **Q**

If q is any member of Q, then $H_q \in \mathrm{T}$. **P** Let $\mathcal{E}$ be the subring of $\mathcal{P}X$ generated by $\operatorname{dom} q$. Then $H_q = \bigcup_{q\subseteq q'\in Q, \operatorname{dom} q' = \mathcal{E}} H_{q'}$ is the union of a finite disjoint family in T, so belongs to T. **Q**

Now observe that $\mathcal{H}_1 = \{H_q : q \in Q\} \cup \{\emptyset\}$ is a subset of T closed under finite intersections, so by the Monotone Class Theorem (136B) T includes the σ-algebra generated by $\mathcal{H}_1$, and must be precisely the σ-algebra generated by $\mathcal{H}$.

495D Theorem Let (X, Σ, μ) be an atomless measure space. Set $\Sigma^f = \{E : E \in \Sigma, \mu E < \infty\}$; for $E \in \Sigma^f$, set $f_E(S) = \#(S \cap E)$ when $S \subseteq X$ meets E in a finite set. Let T be the σ-algebra of subsets of $\mathcal{P}X$ generated by sets of the form $\{S : f_E(S) = n\}$ where $E \in \Sigma^f$ and $n \in \mathbb{N}$. Then for any $\gamma > 0$ there is a unique probability measure ν with domain T such that

(i) for every $E \in \Sigma^f$, f_E is measurable and has a Poisson distribution with expectation $\gamma\mu E$;

(ii) whenever $\langle E_i\rangle_{i\in I}$ is a disjoint family in Σ^f, then $\langle f_{E_i}\rangle_{i\in I}$ is stochastically independent.

proof (a) Let $\mathcal{H}$, $\langle\nu_H\rangle_{H\in\mathcal{H}}$, $\langle\mu_H\rangle_{H\in\mathcal{H}}$, $\langle\mu'_H\rangle_{H\in\mathcal{H}}$, $\langle\lambda_H\rangle_{H\in\mathcal{H}}$, Ω, λ, $\langle\mathcal{H}_E\rangle_{E\in\Sigma^f}$ and $\langle g_E\rangle_{E\in\Sigma^f}$ be as in the proof of 495B. Note that all the μ'_H are atomless (234Ff). Define $\phi : \Omega \to \mathcal{P}X$ by setting

$$\phi(\omega) = \{x_{Hj}(\omega) : H \in \mathcal{H}, j < m_H(\omega)\}$$

for $\omega \in \Omega$.

(b) For $E \in \Sigma^f$, let A_E be the set of those $\omega \in \Omega$ such that

either there are $H \in \mathcal{H} \setminus \mathcal{H}_E$, $j \in \mathbb{N}$ such that $x_{Hj}(\omega) \in E$

or there are distinct H, $H' \in \mathcal{H}_E$ and $j \in \mathbb{N}$ such that $x_{Hj}(\omega) \in H'$

or there is an $H \in \mathcal{H}$ such that the $x_{Hj}(\omega)$, for $j \in \mathbb{N}$, are not all distinct.

Then for any sequence $\langle E_i\rangle_{i\in\mathbb{N}}$ in Σ^f, $\lambda_*(\bigcup_{i\in\mathbb{N}} A_{E_i}) = 0$. **P** Set $\mathcal{H}^* = \bigcup_{i\in\mathbb{N}} \mathcal{H}_{E_i}$, so that $\mathcal{H}^*$ is a countable subset of $\mathcal{H}$. For $H \in \mathcal{H}$, set

$$F_H = H \setminus (\bigcup\{E_i : i \in \mathbb{N}, H \cap E_i \text{ is negligible}\} \cup \bigcup\{H' : H' \in \mathcal{H}^*, H' \neq H\}),$$

$$W_H = \{\boldsymbol{x} : \boldsymbol{x} \in H^{\mathbb{N}} \text{ is injective}\},$$

so that F_H is μ'_H-conegligible and W_H is λ_H-conegligible (because μ'_H is atomless, see 254V). Now

$$\Omega \setminus \bigcup_{k\in\mathbb{N}} A_{E_k} \supseteq \prod_{H\in\mathcal{H}} (\mathbb{N} \times (W_H \cap F_H^{\mathbb{N}}))$$

has full outer measure in Ω, by 254Lb, and its complement has zero inner measure (413Ec). **Q**

It follows that there is a probability measure $\tilde\lambda$ on Ω, extending λ, such that $\tilde\lambda A_E = 0$ for every $E \in \Sigma^f$ (417A). Let ν_0 be the image measure $\tilde\lambda\phi^{-1}$.

(c) If $E \in \Sigma^f$ and $\omega \in \Omega \setminus A_E$, then $f_E(\phi(\omega)) = g_E(\omega)$ if either is defined. **P** If $H \in \mathcal{H}$, then all the $x_{Hj}(\omega)$ are distinct; if $H \in \mathcal{H} \setminus \mathcal{H}_E$, no $x_{Hj}(\omega)$ can belong to E; if H, $H' \in \mathcal{H}_E$ are distinct, then no $x_{Hj}(\omega)$ can belong to H'. So all the $x_{Hj}(\omega)$, $x_{H'k}(\omega)$ for H, $H' \in \mathcal{H}_E$ and j, $k \in \mathbb{N}$ must be distinct, and

$$\begin{aligned} f_E(\phi(\omega)) &= \#(\{x_{Hj}(\omega) : H \in \mathcal{H},\, j < m_H(\omega),\, x_{Hj}(\omega) \in E\}) \\ &= \#(\{(H,j) : H \in \mathcal{H}_E,\, j < m_H(\omega),\, x_{Hj}(\omega) \in E\}) \\ &= \sum_{H \in \mathcal{H}_E} g_{HE}(\omega) = g_E(\omega) \end{aligned}$$

if any of these is finite. **Q** It follows at once that if $E_0, \dots, E_r \in \Sigma^f$ are disjoint, then $\{\omega : f_{E_i}(\phi(\omega)) = g_{E_i}(\omega)$ for every $i \le r\}$ is $\tilde{\lambda}$-conegligible, so that if $n_0, \dots, n_r \in \mathbb{N}$ then

$$\begin{aligned} \nu_0\{S : f_{E_i}(S) = n_i \text{ for every } i \le r\} &= \tilde{\lambda}\{\omega : f_{E_i}(\phi(\omega)) = n_i \text{ for every } i \le r\} \\ &= \tilde{\lambda}\{\omega : g_{E_i}(\omega) = n_i \text{ for every } i \le r\} \\ &= \prod_{i=0}^{r} \frac{(\gamma\mu E_i)^{n_i}}{n_i!} e^{-\gamma\mu E_i}. \end{aligned}$$

Thus every f_{E_i} is finite ν_0-a.e., belongs to $\mathcal{L}^0(\nu_0)$ and has a Poisson distribution with the appropriate expectation, and they are independent.

(d) As T is defined to be the σ-algebra generated by the family $\{f_E : E \in \Sigma^f\}$, it is included in the domain of ν_0. Set $\nu = \nu_0 \restriction \mathrm{T}$; then ν has the properties (i) and (ii). To see that it is unique, observe that if ν' also has these properties, then $\{A : \nu A = \nu' A\}$ is a Dynkin class containing every set of the form

$$\{S : f_{E_i}(S) = n_i \text{ for } i \le r\}$$

where $E_0, \dots, E_r \in \Sigma^f$ are disjoint and $n_0, \dots, n_r \in \mathbb{N}$. By 495C it contains the σ-algebra generated by this family, which is T. So ν and ν' agree on T, and are equal.

495E Definition In the context of 495D, I will call the completion of ν the **Poisson point process** on X with density γ.

Note that the Poisson point process on (X, μ) with density $\gamma > 0$ is identical with the Poisson point process on $(X, \gamma\mu)$ with density 1. There would therefore be no real loss of generality in the main theorems of this section if I spoke only of point processes with density 1. I retain the extra parameter because applications frequently demand it, and the formulae will be more useful with the γs in their proper places; moreover, there are important ideas associated with variations in γ, as in 495Xd.

495F Proposition Let (X, Σ, μ) be a perfect atomless measure space, and $\gamma > 0$. Then the Poisson point process on X with density γ is a perfect probability measure.

proof I refer to the construction in 495B-495D. In (b) of the proof of 495D, use the construction set out in the proof of 417A, so that the domain $\tilde{\Lambda}$ of $\tilde{\lambda}$ is precisely the family of sets of the form $W \triangle A$ where W belongs to the domain Λ of the product measure λ and A belongs to the σ-ideal $\mathcal{A}^*$ generated by $\{A_E : E \in \Sigma^f\}$. Then $\tilde{\lambda}$ is perfect. **P** Let $h : \Omega \to \mathbb{R}$ be a $\tilde{\Lambda}$-measurable function and $W \in \tilde{\Lambda}$ a set of non-zero measure. Then there are a $W' \in \Lambda$ and an $A \in \mathcal{A}^*$ such that $W \triangle W' \subseteq A$ and $h \restriction \Omega \setminus A$ is Λ-measurable; let $\langle E_n \rangle_{n \in \mathbb{N}}$ be a sequence in Σ^f such that $A \subseteq \bigcup_{n \in \mathbb{N}} A_{E_n}$, and $h_1 : \Omega \to \mathbb{R}$ a Λ-measurable function agreeing with h on $\Omega \setminus A$. Set $\mathcal{H}^* = \bigcup_{n \in \mathbb{N}} \mathcal{H}_{E_n}$, so that $\mathcal{H}^*$ is countable. As in the proof of 495D, set

$$F_H = H \setminus (\bigcup\{E_n : n \in \mathbb{N},\, H \cap E_n \text{ is negligible}\} \cup \bigcup\{H' : H' \in \mathcal{H}^*,\, H' \ne H\}),$$

$$W_H = \{\boldsymbol{x} : \boldsymbol{x} \in H^{\mathbb{N}} \text{ is injective}\},$$

for $H \in \mathcal{H}$, so that $W'_H = W_H \cap F_H^{\mathbb{N}}$ is λ_H-conegligible. Set $\Omega' = \prod_{H \in \mathcal{H}} (\mathbb{N} \times W'_H)$. This is disjoint from every A_{E_n} (as in 495D) and therefore from A. The subspace measure $\lambda_{\Omega'}$ on Ω' induced by λ is just the product of the measures on $\mathbb{N} \times W'_H$ (254La). All of these are perfect (451Jc, 451Dc), so $\lambda_{\Omega'}$ is also perfect (451Jc again). Now

$$\lambda_{\Omega'}(W \cap \Omega') = \lambda_{\Omega'}(W' \cap \Omega') = \lambda W' = \tilde{\lambda} W > 0.$$

It follows that there is a compact set $K \subseteq h_1[W \cap \Omega']$ such that $\lambda_{\Omega'}(h_1^{-1}[K] \cap \Omega') > 0$. As h and h_1 agree on Ω', $K \subseteq h[W]$, while

$$\tilde{\lambda}h^{-1}[K] = \tilde{\lambda}h_1^{-1}[K]$$

(because $\tilde{\lambda}A = 0$)

$$= \lambda h_1^{-1}[K] = \lambda_{\Omega'}(h^{-1}[K] \cap \Omega') > 0.$$

As W and h are arbitrary, $\tilde{\lambda}$ is perfect. **Q**

It follows at once that the image measure $\tilde{\lambda}\phi^{-1}$ and its restriction to T are perfect (451Ea); finally, the completion is perfect, by 451Gc.

495G Proposition Let (X_1, Σ_1, μ_1) and (X_2, Σ_2, μ_2) be atomless measure spaces, and $f : X_1 \to X_2$ an inverse-measure-preserving function. Let $\gamma > 0$, and let ν_1, ν_2 be the Poisson point processes on X_1, X_2 respectively with density γ. Then $S \mapsto f[S] : \mathcal{P}X_1 \to \mathcal{P}X_2$ is inverse-measure-preserving for ν_1 and ν_2; in particular, $\mathcal{P}A$ has full outer measure for ν_2 whenever $A \subseteq X_2$ has full outer measure for μ_2.

proof Set $\psi(S) = f[S]$ for $S \subseteq X_1$.

(a) If $F \in \Sigma_2^f$, then $\{S : S \subseteq X_1,\ f\restriction f^{-1}[F] \cap S \text{ is not injective}\}$ is ν_1-negligible. **P** Let $n \in \mathbb{N}$. Set $\alpha = \frac{1}{n+1}\mu_2 F$. Because μ_2 is atomless, we can find a partition of F into sets $F_0, \dots, F_n$ of measure α. Now

$$\{S : f\restriction f^{-1}[F] \cap S \text{ is not injective}\} \subseteq \bigcup_{i \le n} \{S : \#(S \cap f^{-1}[F_i]) > 1\}$$

has ν_1-outer measure at most

$$(n+1)(1 - e^{-\gamma\alpha}(1 + \gamma\alpha)) \le (n+1)\alpha^2\gamma^2 = \frac{1}{n+1}(\gamma\mu_2 F)^2.$$

As n is arbitrary, $\{S : f\restriction f^{-1}[F] \cap S \text{ is not injective}\}$ is negligible. **Q**

(b) It follows that, for any $F \in \Sigma_2^f$ and $n \in \mathbb{N}$,

$$\{S : \#(f[S] \cap F) = n\} \triangle \{S : \#(S \cap f^{-1}[F]) = n\}$$

is ν_1-negligible and $\{S : \#(f[S] \cap F) = n\}$ is measured by ν_1. So if T_2 is the σ-algebra of subsets of $\mathcal{P}X_2$ generated by sets of the form $\{T : \#(F \cap T) = n\}$ for $F \in \Sigma_2^f$ and $n \in \mathbb{N}$, then ν_1 measures $\psi^{-1}[H]$ for every $H \in \mathrm{T}_2$. Next, if $\langle F_i \rangle_{i \in I}$ is a finite disjoint family in Σ_2^f and $n_i \in \mathbb{N}$ for $i \in I$,

$$\begin{aligned}\nu_1\{S : \#(f[S] \cap F_i) = n_i \text{ for every } i \in I\} &= \nu_1\{S : \#(S \cap f^{-1}[F_i]) = n_i \text{ for every } i \in I\} \\ &= \prod_{i \in I} \frac{(\gamma\mu_1 f^{-1}[F_i])^n}{n!} e^{-\gamma\mu_1 f^{-1}[F_i]}\end{aligned}$$

(because $\langle f^{-1}[F_i] \rangle_{i \in I}$ is a disjoint family in Σ_1^f)

$$= \prod_{i \in I} \frac{(\gamma\mu_2 F_i)^n}{n!} e^{-\gamma\mu_2 F_i}.$$

So the image measure $\nu_1\psi^{-1}$ satisfies (i) and (ii) of 495D, and must agree with ν_2 on T_2; that is, ψ is inverse-measure-preserving for ν_1 and $\nu_2\restriction \mathrm{T}_2$. As ν_1 is complete, ψ is inverse-measure-preserving for ν_1 and ν_2 (235Hc).

(c) If $A \subseteq X_2$ has full outer measure, then we can take μ_1 to be the subspace measure on $X_1 = A$ and $f(x) = x$ for $x \in A$. In this case, $\mathcal{P}A = \psi[\mathcal{P}A]$ must have full outer measure for ν_2.

495H Lemma Let $(\tilde{X}, \tilde{\Sigma}, \tilde{\mu})$ be an atomless σ-finite measure space, and $\gamma > 0$. Write μ_L for Lebesgue measure on $[0,1]$, μ' for the product measure on $X' = \tilde{X} \times [0,1]$, and λ' for the product measure on $\Omega' = [0,1]^{\tilde{X}}$. Let $\tilde{\nu}$, ν' be the Poisson point processes on $\tilde{X}$, X' respectively with density γ. For $T \subseteq \tilde{X}$

define $\psi_T : \Omega' \to \mathcal{P}X'$ by setting $\psi_T(z) = \{(t, z(t)) : t \in T\}$ for $z \in \Omega'$; let ν'_T be the image measure $\lambda'\psi_T^{-1}$ on $\mathcal{P}X'$. Then $\langle \nu'_T \rangle_{T \subseteq \tilde{X}}$ is a disintegration of ν' over $\tilde{\nu}$ (definition: 452E).

proof (a) Let $E \subseteq X'$ be a measurable set with finite measure, and write $H_E = \{S : S \cap E \neq \emptyset\}$. Then $\nu' H_E = 1 - e^{-\gamma\mu' E} \leq \gamma\mu' E$; but also $\overline{\int} \nu'_T(H_E)\tilde{\nu}(dT) \leq 2\gamma\mu' E$. **P** We know that $\int \mu_L E[\{t\}]\tilde{\mu}(dt) = \mu' E$ (252D). Let $Y \subseteq \tilde{X}$ be a conegligible set such that $E[\{t\}]$ is measurable for every $t \in Y$ and $t \mapsto \mu_L E[\{t\}] : Y \to [0,1]$ is measurable. Set $F_i = \{t : t \in Y,\ 2^{-i-1} < \mu_L E[\{t\}] \leq 2^{-i}\}$ for each $i \in \mathbb{N}$; let $\langle F'_i \rangle_{i \in \mathbb{N}}$ be a sequence of sets of finite measure with union $\tilde{X} \setminus \bigcup_{i \in \mathbb{N}} F_i$. Let W be the set of those $T \subseteq \tilde{X}$ such that $T \cap (\tilde{X} \setminus Y)$ is empty and $T \cap F_i$, $T \cap F'_i$ are finite for every $i \in \mathbb{N}$; then W is $\tilde{\nu}$-conegligible.

For any $T \in W$,

$$\begin{aligned}\psi_T^{-1}[H_E] &= \{z : \psi_T(z) \cap E \neq \emptyset\} \\ &= \bigcup_{i \in \mathbb{N}} \bigcup_{t \in T \cap F_i} \{z : z(t) \in E[\{t\}]\} \cup \bigcup_{i \in \mathbb{N}} \bigcup_{t \in T \cap F'_i} \{z : z(t) \in E[\{t\}]\}\end{aligned}$$

is measured by λ' and has measure at most $\sum_{i=0}^{\infty} 2^{-i} \#(T \cap F_i)$, because $\mu_L E[\{t\}]$ has measure at most 2^{-i} if $t \in T \cap F_i$, and is zero if $t \in T \cap F'_i$. So

$$\begin{aligned}\overline{\int} \nu'_T(H_E)\tilde{\nu}(dT) &= \overline{\int} \lambda'\psi_T^{-1}[H_E]\tilde{\nu}(dT) \leq \int \sum_{i=0}^{\infty} 2^{-i} \#(T \cap F_i)\tilde{\nu}(dT) \\ &= \sum_{i=0}^{\infty} 2^{-i} \int \#(T \cap F_i)\tilde{\nu}(dT) = \sum_{i=0}^{\infty} 2^{-i}\gamma\tilde{\mu}F_i\end{aligned}$$

(because $T \mapsto \#(T \cap F_i)$ has expectation $\gamma\tilde{\mu}F_i$)

$$\leq 2\gamma \int \mu_L E[\{t\}]\tilde{\nu}(dt) = 2\gamma\mu' E. \ \textbf{Q}$$

(b) Suppose that $\langle F_j \rangle_{j<s}$, $\langle C_{ij} \rangle_{i<r, j<s}$ and $\langle n_{ij} \rangle_{i<r, j<s}$ are such that
r, $s \in \mathbb{N}$,
$n_{ij} \in \mathbb{N}$ for $i < r$, $j < s$,
$\langle F_j \rangle_{j<s}$ is a disjoint family of subsets of $\tilde{X}$ with finite measure,
for each $j < s$, $\langle C_{ij} \rangle_{i<r}$ is a disjoint family of measurable subsets of $[0,1]$.
Set $E_{ij} = F_j \times C_{ij}$ for $i < r$ and $j < s$, and $H = \{S : S \subseteq X',\ \#(S \cap E_{ij}) = n_{ij}$ for every $i < r$, $j < s\}$. Then $\int \nu'_T(H)\tilde{\nu}(dT) = \nu' H$.

P (i) To begin with, suppose that $\bigcup_{i<r} C_{ij} = [0,1]$ for every j. Set $n_j = \sum_{i=0}^{r-1} n_{ij}$ for each j, and let W be the set of those $T \subseteq \tilde{X}$ such that $\#(T \cap F_j) = n_j$ for every j. Then $\tilde{\nu}W = \prod_{j=0}^{s-1} \frac{(\gamma\tilde{\mu}F_j)^{n_j}}{n_j!} e^{-\gamma\tilde{\mu}F_j}$. If $T \subseteq \tilde{X}$ and $z \in \psi_T^{-1}[H]$, then for each $j < s$ we must have

$$\begin{aligned}\#(T \cap F_j) &= \#(\{t : t \in T \cap F_j,\ z(t) \in \bigcup_{i<r} C_{ij}\}) \\ &= \sum_{i=0}^{r-1} \#(\{t : t \in T \cap F_j,\ z(t) \in C_{ij}\}) \\ &= \sum_{i=0}^{r-1} \#(\psi_T(z) \cap E_{ij}) = \sum_{i=0}^{r-1} n_{ij} = n_j.\end{aligned}$$

Turning this round, we see that if $T \notin W$ then $\psi_T^{-1}[H] = \emptyset$ and $\nu'_T H = 0$.

If $T \in W$, let Q be the set of all $q = \langle q(i,j) \rangle_{i<r, j<s}$ such that $\langle q(i,j) \rangle_{i<r}$ is a disjoint family of subsets of $T \cap F_j$ for each j and $\#(q(i,j)) = n_{ij}$ for all i and j. Then $\#(Q) = \prod_{j=0}^{s-1} \frac{n_j!}{\prod_{i=0}^{r-1} n_{ij}!}$. Accordingly

$$\begin{aligned}\nu'_T H &= \lambda'\{z : \psi_T(z) \in H\}\\ &= \lambda'\{z : \#(\{t : t \in T \cap F_j,\, z(t) \in C_{ij}\}) = n_{ij} \text{ for all } i,\, j\}\\ &= \sum_{q\in Q} \lambda\{z : z(t) \in C_{ij} \text{ whenever } i < r,\, j < s,\, t \in q(i,j)\}\\ &= \sum_{q\in Q} \prod_{\substack{i<r, j<s\\ t\in q(i,j)}} \mu_L C_{ij} = \sum_{q\in Q} \prod_{i<r, j<s} (\mu_L C_{ij})^{n_{ij}} = \prod_{j=0}^{s-1}(n_j! \prod_{i=0}^{r-1} \frac{(\mu_L C_{ij})^{n_{ij}}}{n_{ij}!}).\end{aligned}$$

It follows that

$$\begin{aligned}\int \nu'_T(H)\tilde\nu(dT) &= \tilde\nu W \cdot \prod_{j=0}^{s-1}(n_j! \prod_{i=0}^{r-1} \frac{(\mu_L C_{ij})^{n_{ij}}}{n_{ij}!})\\ &= \prod_{j=0}^{s-1} \frac{(\gamma\tilde\mu F_j)^{n_j}}{n_j!} e^{-\gamma\tilde\mu F_j} \cdot \prod_{j=0}^{s-1}(n_j! \prod_{i=0}^{r-1} \frac{(\mu_L C_{ij})^{n_{ij}}}{n_{ij}!})\\ &= \prod_{j=0}^{s-1} (\gamma\tilde\mu F_j)^{n_j} e^{-\gamma\tilde\mu F_j} \prod_{i=0}^{r-1} \frac{(\mu_L C_{ij})^{n_{ij}}}{n_{ij}!}\\ &= \prod_{j=0}^{s-1}\prod_{i=0}^{r-1} e^{-\gamma\mu' E_{ij}} \frac{(\gamma\mu' E_{ij})^{n_{ij}}}{n_{ij}!} = \nu' H,\end{aligned}$$

as required.

(ii) For the general case, set $C_{rj} = [0,1] \setminus \bigcup_{i<r} C_{ij}$, $E_{rj} = F_j \times C_{rj}$ for each $j < s$. For $\sigma \in \mathbb{N}^s$, set $H_\sigma = \{S : S \in H,\, \#(S \cap E_{rj}) = \sigma(j) \text{ for every } j < s\}$; then (i) tells us that $\nu' H_\sigma = \int \nu'_T(H_\sigma)\tilde\nu(dT)$ for every σ, so that

$$\nu' H = \textstyle\sum_{\sigma\in\mathbb{N}^s} \nu' H_\sigma = \sum_{\sigma\in\mathbb{N}^s} \int \nu'_T(H_\sigma)\tilde\nu(dT) = \int \nu'_T(H)\tilde\nu(dT),$$

as required. **Q**

(c) Suppose that $\langle E_i\rangle_{i<r}$ is a disjoint family in $\tilde\Sigma \otimes \Sigma_L$ such that all the projections of the E_i onto $\tilde X$ have finite measure, and $n_i \in \mathbb{N}$ for each $i < r$. Set $H = \{S : S \subseteq X',\, \#(S \cap E_i) = n_i \text{ for every } i < r\}$. Then $\int \nu'_T(H)\tilde\nu(dT) = \nu' H$.

P Let $\mathcal{E}$ be a finite subalgebra of $\tilde\Sigma$ such that every E_i belongs to $\mathcal{E} \otimes \Sigma_L$, and let $\langle F_j\rangle_{j<s}$ enumerate the atoms of $\mathcal{E}$ of finite measure; extend this to an enumeration $\langle F_j\rangle_{j<s'}$ of all the atoms of $\mathcal{E}$. Then we can express each E_i as $\bigcup_{j<s'} F_j \times C_{ij}$ where each $C_{ij} \in \Sigma_L$; but as the projection of E_i has finite measure, C_{ij} must be empty for every $j \geq s$, so $E_i = \bigcup_{j<s} F_j \times C_{ij}$. Let Q be the set of all $q \in \mathbb{N}^{r\times s}$ such that $\sum_{j=0}^{s-1} q(i,j) = n_i$ for every $i < r$. For $q \in Q$ set

$$H_q = \{S : S \subseteq X',\, \#(S \cap (F_j \times C_{ij})) = q(i,j) \text{ for every } i < r,\, j < s\}.$$

Then $\langle H_q\rangle_{q\in Q}$ is disjoint and has union H, so

$$\textstyle\int \nu'_T(H)\tilde\nu(dT) = \sum_{q\in Q} \int \nu'_T(H_q)\tilde\nu(dT) = \sum_{q\in Q} \nu' H_q = \nu' H,$$

using (b) for the middle equality. **Q**

(d) Now let $\langle E_i\rangle_{i<r}$ be a finite disjoint family of subsets of X' of finite measure, and $\langle n_i\rangle_{i<r}$ a family in $\mathbb{N}$. Set $H = \{S : S \subseteq X',\, \#(S \cap E_i) = n_i \text{ for every } i < r\}$. Then $\int \nu'_T(H)\tilde\nu(dT)$ is defined and equal to $\nu' H$.

P Let $\epsilon > 0$. For each $i < r$ we can find an $E'_i \in \tilde\Sigma \otimes \Sigma_L$ such that $\mu'(E_i \triangle E'_i) \leq \epsilon$ (251Ie). Discarding a negligible set from E'_i if necessary, we may suppose that the projection of E'_i on $\tilde X$ has finite measure. Set $\hat E_i = E'_i \setminus \bigcup_{k<i} E'_k$ for each i, so that $\langle \hat E_i\rangle_{i<r}$ is a disjoint family in $\tilde\Sigma \otimes \Sigma_L$, and the projections of the $\hat E_i$

are still of finite measure. Set $\hat{H} = \{S : S \subseteq X', \#(S \cap \hat{E}_i) = n_i \text{ for every } i < r\}$. Then (c) tells us that $\int \nu'_T(\hat{H})\tilde{\nu}(dT) = \nu'\hat{H}$.

Set $E = \bigcup_{i<r}(E_i \triangle E'_i)$. Then $\mu' E \leq r\epsilon$, while E includes $E_i \triangle \hat{E}_i$ for every i, so

$$\hat{H} \setminus H_E \subseteq H \subseteq \hat{H} \cup H_E,$$

where $H_E = \{S : S \cap E \neq \emptyset\}$ as in (a). Accordingly

$$\begin{aligned}
\nu'H - 3r\gamma\epsilon &\leq \nu'\hat{H} - 2r\gamma\epsilon \\
&\text{(by (a))} \\
&= \int \nu'_T(\hat{H})\tilde{\nu}(dT) - 2r\gamma\epsilon \\
&\text{(by (c))} \\
&\leq \int \nu'_T(\hat{H}) - \nu'_T(H_E)\tilde{\nu}(dT) \\
&\text{(by the other part of (a))} \\
&\leq \underline{\int}(\nu'_T)_*(H)\tilde{\nu}(dT) \leq \overline{\int}(\nu'_T)^*(H)\tilde{\nu}(dT) \\
&\leq \overline{\int} \nu'_T(\hat{H}) + \nu'_T(H_E)\tilde{\nu}(dT) \leq \nu'\hat{H} + 2r\gamma\epsilon \leq \nu'H + 3r\gamma\epsilon.
\end{aligned}$$

As ϵ is arbitrary,

$$\nu'H = \underline{\int}(\nu'_T)_*(H)\tilde{\nu}(dT) = \overline{\int}(\nu'_T)^*(H)\tilde{\nu}(dT).$$

In particular, $(\nu'_T)_*(H) = (\nu'_T)^*(H)$ for almost every T; since ν'_T is the image of the complete measure λ, it is complete (212Bd), and H is measured by almost every ν'_T; finally, $\int \nu'_T(H)\tilde{\nu}(dT)$ is defined and equal to $\nu'H$. **Q**

(e) So if we write $\mathcal{H}$ for the family of subsets H of $\mathcal{P}X'$ such that $\int \nu'_T(H)\tilde{\nu}(dT)$ is defined and equal to $\nu'H$, and $\mathcal{H}_0$ for the family of sets of the form $H = \{S : S \subseteq X', \#(S \cap \hat{E}_i) = n_i \text{ for every } i < r\}$ where $\langle E_i\rangle_{i<r}$ is a disjoint family of sets of finite measure and $n_i \in \mathbb{N}$ for $i < r$, we have $\mathcal{H} \supseteq \mathcal{H}_0$. But $\mathcal{H}$ is a Dynkin class, so includes the σ-algebra T generated by $\mathcal{H}_0$, by 495C. Since every ν'-negligible set is included in a ν'-negligible member of T′, $\mathcal{H}$ contains every ν'-negligible set, and therefore every set measured by ν'; which is what we need to know.

495I Theorem Let (X, Σ, μ) and $(\tilde{X}, \tilde{\Sigma}, \tilde{\mu})$ be atomless σ-finite measure spaces and $\gamma > 0$. Let ν, $\tilde{\nu}$ be the Poisson point processes on X, $\tilde{X}$ respectively with density γ. Suppose that $f : X \to \tilde{X}$ is inverse-measure-preserving and that $\langle\mu_t\rangle_{t\in\tilde{X}}$ is a disintegration of μ over $\tilde{\mu}$ consistent with f (definition: 452E) such that every μ_t is a probability measure. Write λ for the product measure $\prod_{t\in\tilde{X}} \mu_t$ on $\Omega = X^{\tilde{X}}$, and for $T \subseteq \tilde{X}$ define $\phi_T : \Omega \to \mathcal{P}X$ by setting $\phi_T(z) = z[T]$ for $z \in \Omega$; let ν_T be the image measure $\lambda\phi_T^{-1}$ on $\mathcal{P}X$. Then $\langle\nu_T\rangle_{T\subseteq\tilde{X}}$ is a disintegration of ν over $\tilde{\nu}$. Moreover

(i) setting $\tilde{f}(S) = f[S]$ for $S \subseteq X$, $\langle\nu_T\rangle_{T\subseteq\tilde{X}}$ is consistent with $\tilde{f} : \mathcal{P}X \to \mathcal{P}\tilde{X}$;

(ii) if μ is strongly consistent with f, then $\langle\nu_T\rangle_{T\subseteq\tilde{X}}$ is strongly consistent with $\tilde{f}$.

proof (a) For $T \subseteq \tilde{X}$ let V_T be the set of those $z \in \Omega$ such that $fz\restriction T$ is injective. We need to know that

$$W = \{T : T \subseteq \tilde{X}, T \text{ is countable}, V_T \text{ is } \lambda\text{-conegligible}\}$$

is $\tilde{\nu}$-conegligible. **P** Write $\tilde{\Sigma}^f = \{F : F \in \tilde{\Sigma}, \tilde{\mu}F < \infty\}$. Because $\tilde{\mu}$ is atomless and σ-finite, there is a countable subalgebra $\mathcal{E}$ of $\tilde{\Sigma}$ such that for every $\epsilon > 0$ there is a cover of $\tilde{X}$ by members of $\mathcal{E}$ of measure at most ϵ. Set

$$Y = \{t : t \in \tilde{X}, \mu_t f^{-1}[F] = (\chi F)(t) \text{ for every } F \in \mathcal{E}\},$$

so that Y is $\tilde{\mu}$-conegligible and $\mathcal{P}Y$ is $\tilde{\nu}$-conegligible. For $F \in \mathcal{E}$, let W_F be the set of those $T \subseteq Y$ such that for every $t \in T \cap F$ there is an $F' \in \mathcal{E}$ such that $T \cap F' = \{t\}$. Now, given $F \in \mathcal{E} \cap \tilde{\Sigma}^f$ and $\epsilon > 0$, there is a partition $\langle F_i \rangle_{i \in I}$ of F into members of $\mathcal{E}$ of measure at most ϵ. Then

$$\begin{aligned}\tilde{\nu}^*(\mathcal{P}Y \setminus W_F) &\leq \tilde{\nu}\{T : \#(T \cap F_i) > 1 \text{ for some } i \in I\} \\ &\leq \sum_{i \in I} 1 - e^{-\gamma\tilde{\mu}F_i}(1 + \gamma\tilde{\mu}F_i) \leq \sum_{i \in I} (\gamma\tilde{\mu}F_i)^2 \leq \epsilon\gamma^2 \sum_{i \in I} \tilde{\mu}F_i = \epsilon\gamma^2\tilde{\mu}F.\end{aligned}$$

As ϵ is arbitary, W_F is $\tilde{\nu}$-conegligible; accordingly $W' = \bigcap\{W_F : F \in \mathcal{E} \cap \tilde{\Sigma}^f\}$ is $\tilde{\nu}$-conegligible.

Now suppose that $T \in W'$. Because $\tilde{X}$ is covered by $\mathcal{E} \cap \tilde{\Sigma}^f$, we see that for every $t \in T$ there is an $F \in \mathcal{E} \cap \tilde{\Sigma}^f$ containing t, and now there is an $F' \in \mathcal{E}$ such that $T \cap F' = \{t\}$. In particular, T is countable, so

$$U_T = \{z : z(t) \in f^{-1}[F] \text{ whenever } F \in \mathcal{E} \text{ and } t \in T \cap F\}$$

is λ-conegligible. Take $z \in U_T$. If t, t' are distinct points of T, there is an $F \in \mathcal{E}$ containing t but not t', and now F contains $f(z(t))$ but not $f(z(t'))$. So $z \restriction T$ is injective. Thus $U_T \subseteq V_T$ and V_T is λ-conegligible. This is true for every $T \in W'$, so $W \supseteq W'$ is $\tilde{\nu}$-conegligible. **Q**

(b) Suppose that $\langle E_i \rangle_{i<r}$ is a disjoint family of subsets of X with finite measure, and $n_i \in \mathbb{N}$ for $i < r$. Set $H = \{S : S \subseteq X, \#(S \cap E_i) = n_i \text{ for every } i < r\}$. Then $\int \nu_T(H)\tilde{\nu}(dT)$ is defined and equal to νH. **P** As in 495H, set $X' = \tilde{X} \times [0,1]$ with the product measure μ', and write λ' for the product measure on $\Omega' = [0,1]^{\tilde{X}}$. Let ν' be the Poisson point process on X' with density γ. For $T \subseteq \tilde{X}$ define $\psi_T : \Omega' \to \mathcal{P}X'$ by setting $\psi_T(z) = \{(t, z(t)) : t \in T\}$ for $z \in \Omega'$ and let ν'_T be the image measure $\lambda\psi_T^{-1}$ on $\mathcal{P}X'$.

For each $i < r$, $\int \mu_t(E_i)\tilde{\mu}(dt) = \mu E_i$; set $Y_1 = \{t : \mu_t E_i \text{ is defined for every } i < r\}$, so that $Y_1 \subseteq \tilde{X}$ is $\tilde{\nu}$-conegligible. Set $f_i(t) = \sum_{j<i} \mu_t E_j$ for $t \in Y_1$ and $i < r$, and

$$E'_i = \{(t, \alpha) : t \in Y_1,\ f_i(t) \leq \alpha < f_{i+1}(t)\}$$

for $i < r$. Then $\mu' E'_i = \int f_{i+1} - f_i \, d\tilde{\nu} = \mu E_i$ for each i, by 252N.

Set

$$H' = \{S : S \subseteq X', \#(S' \cap E'_i) = n_i \text{ for every } i < r\},$$

$$W_1 = \{T : T \in W,\ T \subseteq Y_1,\ \nu'_T H' \text{ is defined}\},$$

so that W_1 is $\tilde{\nu}$-conegligible. Let T be any member of W_1. Let Q be the set of partitions $q = \langle q(i) \rangle_{i \leq r}$ of T such that $\#(q(i)) = n_i$ for every $i < r$; because T is countable, so is Q. Set $E_r = X \setminus \bigcup_{i<r} E_i$ and $E'_r = X' \setminus \bigcup_{i<r} E'_i$. Then $\mu_t E_r = 1 - \sum_{i=0}^{r-1} \mu_t E_i = \mu_L E'_r[\{t\}]$ for every $t \in Y_1$. Now

$$\begin{aligned}\nu'_T H' &= \lambda'\{z : z \in \Omega',\ \psi_T(z) \in H'\} \\ &= \lambda'\{z : z \in \Omega',\ \#(\{t : t \in T,\ z(t) \in E'_i[\{t\}]\}) = n_i \text{ for every } i < r\} \\ &= \sum_{q \in Q} \lambda'\{z : z \in \Omega',\ z(t) \in E'_i[\{t\}] \text{ whenever } i \leq r \text{ and } t \in q(i)\} \\ &= \sum_{q \in Q} \prod_{i=0}^{r} \prod_{t \in q(i)} \mu_L E'_i[\{t\}] \\ &= \sum_{q \in Q} \prod_{i=0}^{r} \prod_{t \in q(i)} \mu_t E_i \\ &= \sum_{q \in Q} \lambda\{z : z \in \Omega,\ z(t) \in E_i \text{ whenever } i \leq r \text{ and } t \in q(i)\} \\ &= \lambda\{z : z \in \Omega,\ \#(\{t : t \in T,\ z(t) \in E_i\}) = n_i \text{ for every } i < r\} \\ &= \lambda\{z : z \in V_T,\ \#(\{t : t \in T,\ z(t) \in E_i\}) = n_i \text{ for every } i < r\} \\ &= \lambda\{z : z \in V_T,\ \#(z[T] \cap E_i) = n_i \text{ for every } i < r\} \\ &= \lambda\{z : z \in V_T,\ \phi_T(z) \in H\} = \nu_T H.\end{aligned}$$

Since this is true for $\tilde{\nu}$-almost every T,

$$\int \nu_T(H)\tilde{\nu}(dT) = \int \nu'_T(H')\tilde{\nu}(dT) = \nu'H'$$

(495H)

$$= \prod_{i<r} \frac{(\gamma\mu' E'_i)^{n_i}}{n_i!} e^{-\gamma\mu' E'_i} = \prod_{i<r} \frac{(\gamma\mu E_i)^{n_i}}{n_i!} e^{-\gamma\mu E_i} = \nu H. \text{ Q}$$

Now, just as in part (e) of the proof of 495H, 495C tells us that $\int \nu_T(H)\tilde{\nu}(dT) = \nu H$ whenever ν measures H, so that $\langle \nu_T \rangle_{T\subseteq\tilde{X}}$ is a disintegration of ν over $\tilde{\nu}$.

(c) Let $\langle F_i \rangle_{i<r}$ be a disjoint family in $\tilde{\Sigma}^f$, and take $n_i \in \mathbb{N}$ for $i < r$. Set

$$F_r = \tilde{X} \setminus \bigcup_{i<r} F_i,$$

$$Y_2 = \{t : t \in \tilde{X},\ \mu_t f^{-1}[F_i] = (\chi F_i)(t) \text{ for every } i \leq r\},$$

$$W_2 = \{T : T \in W,\ T \subseteq Y_2\},$$

so that Y_2 is $\tilde{\mu}$-conegligible and W_2 is $\tilde{\nu}$-conegligible. Set

$$\tilde{H} = \{T : T \in W_2,\ \#(T \cap F_i) = n_i \text{ for every } i < r\}, \quad H = \tilde{f}^{-1}[\tilde{H}] = \{S : S \subseteq X,\ f[S] \in \tilde{H}\}.$$

Then $\nu_T H = \chi\tilde{H}(T)$ for every $T \in W_2 \cap \tilde{H}$. **P** For $i \leq r$ and $t \in T \cap F_i$, we have

$$\lambda\{z : z(t) \in f^{-1}[F_i]\} = \mu_t f^{-1}[F_i] = 1$$

because $T \subseteq Y_2$. So

$$V = \{z : z \in V_T,\ f(z(t)) \in F_i \text{ whenever } i \leq r \text{ and } t \in T \cap F_i\}$$

is λ-conegligible. But if $z \in V$ then $fz\restriction T$ is injective, so

$$\#(f[z[T]] \cap F_i) = \#(T \cap (fz)^{-1}[F_i]) = \#(T \cap F_i)$$

for every $i < r$, and $z[T] \in H$ iff $T \in \tilde{H}$. Thus

$$\nu_T H = \lambda\{z : z[T] \in H\} = \lambda V = 1 \text{ if } T \in \tilde{H},$$
$$= \lambda(X^{\tilde{X}} \setminus V) = 0 \text{ otherwise. } \text{Q}$$

Setting

$$\mathcal{H} = \{\tilde{H} : \tilde{H} \subseteq \tilde{X},\ \nu_T \tilde{f}^{-1}[\tilde{H}] = \chi\tilde{H}(T) \text{ for } \tilde{\nu}\text{-almost every } T \in \tilde{H}\},$$

it is easy to check that $\mathcal{H}$ is a Dynkin class containing all sets of the form $\{T : T \subseteq \tilde{X},\ \#(T \cap F_i) = n_i \text{ for every } i < r\}$ where $\langle F_i \rangle_{i<r}$ is a disjoint family in $\tilde{\Sigma}^f$, and therefore including the σ-algebra generated by such sets, by 495C. But as $\mathcal{H}$ also contains any subset of a negligible set belonging to $\mathcal{H}$ (remember that all the ν_T are complete probability measures, like λ), it includes the domain of $\tilde{\nu}$, and $\langle \nu_T \rangle_{T\subseteq\tilde{X}}$ is consistent with $\tilde{f}$.

(d) Now suppose that $\langle \mu_t \rangle_{t\in\tilde{X}}$ is strongly consistent with f. Set $Y_3 = \{t : \mu_t f^{-1}[\{t\}] = 1\}$ and $W_3 = \{T : T \in W,\ T \subseteq Y_3\}$. Then $\nu_T \tilde{f}^{-1}[\{T\}] = 1$ for every $T \in W_3$. **P** Set $V'_T = \{z : z \in X^{\tilde{X}},\ f(z(t)) = t \text{ for every } t \in T\}$. For each $t \in T$,

$$\lambda\{z : f(z(t)) = t\} = \mu_t f^{-1}[\{t\}] = 1,$$

because $T \subseteq W_3$. As T is countable, V'_T is λ-conegligible. But now

$$\nu_T \tilde{f}^{-1}[\{T\}] = \lambda\{z : f[z[T]] = T\} = \lambda V'_T = 1. \text{ Q}$$

As W_3 is $\tilde{\nu}$-conegligible, $\langle \nu_T \rangle_{T\subseteq\tilde{X}}$ is strongly consistent with $\tilde{f}$.

495J Proposition Let $(\mathfrak{A},\bar\mu)$ be a measure algebra, and $\gamma>0$. Then there are a probability algebra $(\mathfrak{B},\bar\lambda)$ and a function $\theta:\mathfrak{A}\to\mathfrak{B}$ such that

(i) $\theta(\sup A)=\sup\theta[A]$ for every non-empty $A\subseteq\mathfrak{A}$ such that $\sup A$ is defined in $\mathfrak{A}$;

(ii) $\bar\lambda\theta(a)=1-e^{-\gamma\bar\mu a}$ for every $a\in\mathfrak{A}$, interpreting $e^{-\infty}$ as 0;

(iii) whenever $\langle a_i\rangle_{i\in I}$ is a disjoint family in $\mathfrak{A}$ and $\mathfrak{C}_i$ is the closed subalgebra of $\mathfrak{B}$ generated by $\{\theta(a):a\subseteq a_i\}$ for each i, then $\langle\mathfrak{C}_i\rangle_{i\in I}$ is stochastically independent.

proof (a) We may suppose that $(\mathfrak{A},\bar\mu)$ is the measure algebra of a measure space (X,Σ,μ) (321J). Set $\Sigma^f=\{E:E\in\Sigma,\ \mu E<\infty\}$ and $\mathfrak{A}^f=\{a:a\in\mathfrak{A},\ \bar\mu a<\infty\}$. Let (Ω,Λ,λ) and $\langle g_E\rangle_{E\in\Sigma^f}$ be as in 495B, and take $(\mathfrak{B},\bar\lambda)$ to be the measure algebra of (Ω,Λ,λ). Note that if E, $F\in\Sigma^f$ and $\mu(E\triangle F)=0$, then $g_{E\setminus F}$ and $g_{F\setminus E}$ have Poisson distributions with expectation 0, so are zero almost everywhere, while $g_E=_{\text{a.e.}}g_{E\cap F}+g_{E\setminus F}$ and $g_F=_{\text{a.e.}}g_{E\cap F}+g_{F\setminus E}$; so that $g_E=_{\text{a.e.}}g_F$. This means that we can define $\theta:\mathfrak{A}^f\to\mathfrak{B}$ by setting $\theta(E^\bullet)=\{\omega:g_E(\omega)\neq0\}^\bullet$ whenever $E\in\Sigma^f$, and we shall have $\bar\lambda(\theta a)=1-e^{-\gamma\bar\mu a}$ because g_E has a Poisson distribution with expectation $\bar\mu a$ whenever $E\in\Sigma^f$ and $E^\bullet=a$. For $a\in\mathfrak{A}\setminus\mathfrak{A}^f$ set $\theta(a)=1_{\mathfrak{B}}$.

(b) If a, $b\in\mathfrak{A}^f$ are disjoint, they can be represented as $E^\bullet$, $F^\bullet$ where E, $F\in\Sigma^f$ are disjoint. In this case, $g_{E\cup F}=_{\text{a.e.}}g_E+g_F$, so $\theta(a\cup b)=\theta(a)\cup\theta(b)$. Of course the same is true if a, $b\in\mathfrak{A}$ are disjoint and either has infinite measure. It follows at once that for any a, $b\in\mathfrak{A}$,

$$\theta(a\cup b)=\theta(a\setminus b)\cup\theta(a\cap b)\cup\theta(b\setminus a)=\theta(a)\cup\theta(b).$$

Consequently $\theta(\sup A)=\sup\theta[A]$ for any finite set $A\subseteq\mathfrak{A}$. If $A\subseteq\mathfrak{A}$ is an infinite set with supremum a^*, then $A'=\{\sup B:B\in[A]^{<\omega}\}$ is an upwards-directed set with supremum a^*, so there is a non-decreasing sequence $\langle a_n\rangle_{n\in\mathbb{N}}$ in A' such that $\lim_{n\to\infty}\bar\mu a_n=\bar\mu a^*$ (321D). In this case, $b^*=\sup_{n\in\mathbb{N}}\theta(a_n)$ is defined in $\mathfrak{B}$ and

$$\bar\lambda b^*=\lim_{n\to\infty}\bar\lambda\theta(a_n)=\lim_{n\to\infty}1-e^{-\gamma\bar\mu a_n}=1-e^{-\gamma\bar\mu a^*}=\bar\lambda\theta(a^*).$$

So $b^*=\theta(a^*)$; since $\theta(a^*)$ is certainly an upper bound of $\theta[A']$, it must actually be the supremum of $\theta[A']$ and therefore (because θ preserves finite suprema) of $\theta[A]$.

(c) Thus θ satisfies (i) and (ii). As for (iii), note first that if $\langle a_i\rangle_{i\in I}$ is a finite disjoint family in $\mathfrak{A}$, then $\bar\lambda(\inf_{i\in I}\theta(a_i))=\prod_{i\in I}\bar\lambda\theta(a_i)$. **P** Set $J=\{i:i\in I,\ \bar\mu a_i<\infty\}$. For $i\in J$, represent a_i as $E_i^\bullet$ where $\langle E_i\rangle_{i\in J}$ is a disjoint family in Σ^f. Then $\langle g_{E_i}\rangle_{i\in J}$ is independent, so

$$\begin{aligned}\bar\lambda(\inf_{i\in I}\theta(a_i))&=\bar\lambda(\inf_{i\in J}\theta(a_i))=\lambda(\Omega\cap\bigcap_{i\in J}\{\omega:g_{E_i}(\omega)=0\})\\&=\prod_{i\in J}\lambda\{\omega:g_{E_i}(\omega)=0\}=\prod_{i\in J}\bar\lambda\theta(a_i)=\prod_{i\in I}\bar\lambda\theta(a_i).\ \textbf{Q}\end{aligned}$$

Now suppose that $\langle a_i\rangle_{i\in I}$ is a finite disjoint family in $\mathfrak{A}$ and that $\mathfrak{D}_i$ is the subalgebra of $\mathfrak{B}$ generated by $D_i=\{\theta(a):a\subseteq a_i\}$ for each i. We know that each D_i is closed under $\cup$ (by (i)) and that $\bar\lambda(\inf_{i\in J}d_i)=\prod_{i\in J}\bar\lambda d_i$ whenever $J\subseteq I$ and $d_i\in D_i$ for each $i\in J$, that is, that $\langle d_i\rangle_{i\in I}$ is stochastically independent whenever $d_i\in D_i$ for each i. Setting $D_i'=\{1\setminus d:d\in D_i\}\cup\{0\}$, we see that D_i' is closed under $\cap$ and that $\langle d_i\rangle_{i\in I}$ is stochastically independent whenever $d_i\in D_i'$ for each i (as in 272F). An induction on $\#(J)$, using 313Ga for the inductive step, shows that if $J\subseteq I$, $d_i\in\mathfrak{D}_i$ for $i\in J$, and $d_i\in D_i'$ for $i\in I\setminus J$, then $\bar\lambda(\inf_{i\in I}d_i)=\prod_{i\in I}\bar\lambda d_i$. At the end of the induction, we see that $\bar\lambda(\inf_{i\in I}d_i)=\prod_{i\in I}\bar\lambda d_i$ whenever $d_i\in\mathfrak{D}_i$ for each i, and therefore whenever d_i belongs to the topological closure of $\mathfrak{D}_i$ for each i, where $\mathfrak{B}$ is given its measure-algebra topology (§323).

Finally, suppose that $\langle a_i\rangle_{i\in I}$ is any disjoint family in $\mathfrak{A}$, and $\mathfrak{C}_i$ is the closed subalgebra of $\mathfrak{B}$ generated by $D_i=\{\theta(a):a\subseteq a_i\}$ for each i. Take a finite set $J\subseteq I$ and $c_i\in\mathfrak{C}_i$ for each $i\in J$. By 323J, $\mathfrak{C}_i$ is the topological closure of the subalgebra $\mathfrak{D}_i$ of $\mathfrak{B}$ generated by $\{\theta(a):a\subseteq a_i\}$; so $\bar\lambda(\inf_{i\in J}c_i)=\prod_{i\in J}\bar\lambda c_i$. As $\langle c_i\rangle_{i\in J}$ is arbitrary, $\langle\mathfrak{C}_i\rangle_{i\in I}$ is independent.

495K Proposition Let U be any L-space. Then there are a probability space (Ω,Λ,λ) and a positive linear operator $T:U\to L^1(\lambda)$ such that $\|Tu\|_1=\|u\|_1$ whenever $u\in L^1(\mu)^+$ and $\langle Tu_i\rangle_{i\in I}$ is stochastically independent in $L^0(\lambda)$ whenever $\langle u_i\rangle_{i\in I}$ is a disjoint family in $L^1(\mu)$.

Remarks Recall that a family $\langle u_i\rangle_{i\in I}$ in a Riesz space is 'disjoint' if $|u_i|\wedge|u_j|=0$ for all distinct i, $j\in I$ (352C). A family $\langle v_i\rangle_{i\in I}$ in $L^0(\lambda)$ is 'independent' if $\langle g_i\rangle_{i\in I}$ is an independent family of random variables whenever $g_i\in\mathcal{L}^0(\lambda)$ represents v_i for each i; compare 367W.

proof (a) By Kakutani's theorem, there is a measure algebra $(\mathfrak{A},\bar\mu)$ such that U is isomorphic, as Banach lattice, to $L^1(\mathfrak{A},\bar\mu)$; now $(\mathfrak{A},\bar\mu)$ can be represented as the measure algebra of a measure space (X,Σ,μ), and we can identify U and $L^1(\mathfrak{A},\bar\mu)$ with $L^1(\mu)$ (365B). Set $\Sigma^f=\{E:E\in\Sigma,\ \mu E<\infty\}$ and $\mathfrak{A}^f=\{a:a\in\mathfrak{A},\ \bar\mu a<\infty\}$ as usual. Take (Ω,Λ,λ) and $\langle g_E\rangle_{E\in\Sigma^f}$ from 495B, with $\gamma=1$. As in the proof of 495J, we have $g_E=_{\text{a.e.}}g_F$ whenever E, $F\in\Sigma^f$ and $\mu(E\triangle F)=0$; consequently we can define $\psi:\mathfrak{A}^f\to L^1(\lambda)$ by setting $\psi a=g_E^\bullet$ whenever $E\in\Sigma^f$ and $E^\bullet=a$. Again as in 495J, $g_{E\cup F}=_{\text{a.e.}}g_E+g_F$ whenever E, $F\in\Sigma^f$ are disjoint, so ψ is additive. Also

$$\|\psi a\|_1=\int g_E\,d\lambda=\mu E=\bar\mu a$$

whenever $E\in\Sigma^f$ represents $a\in\mathfrak{A}^f$. By 365I, there is a unique bounded linear operator $T:L^1(\mathfrak{A},\bar\mu)\to L^1(\lambda)$ such that $T(\chi a)=\psi a$ for every $a\in\mathfrak{A}^f$. By 365Ka, T is a positive operator. The set $\{u:u\in L^1(\mathfrak{A},\bar\mu)^+,\|Tu\|_1=\|u\|_1\}$ is closed under addition, norm-closed and contains $\alpha\chi a$ for every $a\in\mathfrak{A}^f$ and $\alpha\geq0$, so is the whole of $L^1(\mathfrak{A},\bar\mu)^+$, by 365F.

(b) Now let $\langle u_i\rangle_{i\in I}$ be a disjoint family in $L^1(\mathfrak{A},\bar\mu)$. Then $\langle Tu_i\rangle_{i\in I}$ is independent. **P?** Otherwise, there are a finite set $J\subseteq I$ and a family $\langle V_i\rangle_{i\in J}$ such that V_i is a neighbourhood of Tu_i in the topology of convergence in measure on $L^0(\mu)$ for each $i\in J$, and $\langle v_i\rangle_{i\in J}$ is not independent whenever $v_i\in V_i$ for each i (367W). Because the embedding $L^1(\lambda)\subseteq L^0(\lambda)$ is continuous for the norm topology on $L^1(\lambda)$ and the topology of convergence in measure (245G), there is a $\delta>0$ such that $Tu_i'\in V_i$ whenever $i\in J$, $u_i'\in L^1(\mathfrak{A},\bar\mu)$ and $\|u_i'-u_i\|_1\leq\delta$. Now we can find such $u_i'\in S(\mathfrak{A}^f)$ with $|u_i'|\leq|u_i|$ (365F).

Express each u_i' as $\sum_{k=0}^{n_i}\alpha_{ik}\chi a_{ik}$ where $\langle a_{ik}\rangle_{k\leq n_i}$ is a disjoint family in $\mathfrak{A}^f$ and no α_{ik} is zero (361Eb). In this case, all the a_{ik}, for $i\in J$ and $k\leq n_i$, are disjoint, so all the $\psi(a_{ik})$ are independent. But this means that $\langle Tu_i'\rangle_{i\in J}=\langle\sum_{k=0}^{n_i}\alpha_{ik}\psi(a_{ik})\rangle_{i\in J}$ is independent (272K); which is impossible, because $Tu_i'\in V_i$ for every $i\in J$. **XQ**

So T, regarded as a function from U to $L^1(\lambda)$, has the required properties.

495L The following is a more concrete expression of the same ideas.

Proposition Let (X,Σ,μ) be an atomless measure space, and ν the Poisson point process on X with density $\gamma>0$.

(a) If $h\in\mathcal{L}^1(\mu)$, $Q_h(S)=\sum_{x\in S\cap\operatorname{dom}h}h(x)$ is defined for ν-almost every $S\subseteq X$, and $\int Q_h\,d\nu=\gamma\int h\,d\mu$.

(b) We have a positive linear operator $T:L^1(\mu)\to L^1(\nu)$ defined by setting $T(h^\bullet)=Q_h^\bullet$ for every $h\in\mathcal{L}^1(\mu)$.

(c) $\|Tu\|_1=\gamma\|u\|_1$ whenever $u\in L^1(\mu)^+$ and $\langle Tu_i\rangle_{i\in I}$ is stochastically independent in $L^0(\lambda)$ whenever $\langle u_i\rangle_{i\in I}$ is a disjoint family in $L^1(\mu)$.

proof (a) In the language of 495D, $Q_{\chi E}=f_E$ for every $E\in\Sigma^f$. So $Q_{\chi E}\in\mathcal{L}^1(\nu)$ and $\int Q_{\chi E}\,d\nu=\gamma\mu E$ for every $E\in\Sigma^f$. If $h=\sum_{i=0}^r\alpha_i\chi E_i$ is a simple function on X, then $Q_h=_{\text{a.e.}}\sum_{i=0}^r\alpha_iQ_{\chi E_i}\in\mathcal{L}^1(\nu)$ and $\int Q_h\,d\nu=\int h\,d\mu$. If $h\in\mathcal{L}^1(\mu)$ is zero a.e., then $\{S:S\subseteq h^{-1}[\{0\}]\}$ is ν-conegligible, so $Q_h=0$ a.e. It follows that if $h\in\mathcal{L}^1(\mu)$ is non-negative, and $\langle h_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence of simple functions converging to h almost everywhere, then $Q_h=_{\text{a.e.}}\lim_{n\to\infty}Q_{h_n}$, while $Q_{h_n}\leq_{\text{a.e.}}Q_{h_{n+1}}$ for every n; so Q_h is ν-integrable and

$$\int Q_h\,d\nu=\lim_{n\to\infty}\int Q_{h_n}\,d\nu=\lim_{n\to\infty}\gamma\int h_n\,d\mu=\gamma\int h\,d\mu.$$

(b) Since $Q_h=_{\text{a.e.}}Q_{h'}$ whenever $h=_{\text{a.e.}}h'$ in $\mathcal{L}^1(\mu)$, we can define $T:L^1(\mu)\to L^1(\nu)$ by setting $T(h^\bullet)=(Q_h)^\bullet$ for every $h\in\mathcal{L}^1(\mu)$; because $Q_{\alpha h}=_{\text{a.e.}}\alpha Q_h$ and $Q_{h+h'}=_{\text{a.e.}}Q_h+Q_{h'}$ whenever h, $h'\in\mathcal{L}^1(\mu)$ and $\alpha\in\mathbb{R}$, T is linear; because $Q_h\geq0$ a.e. whenever $h\geq0$ a.e., T is positive.

(c) Because $\int Q_h\,d\nu=\gamma\int h\,d\mu$ for every $h\in\mathcal{L}^1(\mu)$, and T is positive, $\|Tu\|_1=\gamma\|u\|_1$ for every $u\in L^1(\mu)^+$. Finally, if $\langle u_i\rangle_{i\in I}$ is a finite disjoint family in $L^1(\mu)$, we can find a family $\langle h_i\rangle_{i\in I}$ of measurable functions from X to $\mathbb{R}$ such that $h_i^\bullet=u_i$ for each i and the sets $E_i=\{x:h_i(x)\neq0\}$ are disjoint. For each $i\in I$, let T_i be the σ-algebra of subsets of $\mathcal{P}X$ generated by sets of the form $\{S:\#(S\cap E)=n\}$ where

$E \subseteq E_i$ has finite measure and $n \in \mathbb{N}$. Then $\langle \mathrm{T}_i \rangle_{i\in I}$ is independent (as in part (c) of the proof of 495J), and each Q_{h_i} is T_i-measurable, so $\langle Q_{h_i}\rangle_{i\in I}$ is independent and $\langle Tu_i\rangle_{i\in I}$ is independent.

495M We can identify the characteristic functions of the random variables Q_f as defined above.

Proposition Let (X, Σ, μ) be an atomless measure space, and ν the Poisson point process on X with density $\gamma > 0$. For $f \in \mathcal{L}^1(\mu)$ set $Q_f(S) = \sum_{x\in S\cap \mathrm{dom}\, f} f(x)$ when $S \subseteq X$ and the sum is defined in $\mathbb{R}$. Then

$$\int_{\mathcal{P}X} e^{iyQ_f}\, d\nu = \exp\big(\gamma \int_X (e^{iyf} - 1) d\mu\big)$$

for any $y \in \mathbb{R}$.

proof Note that Q_f is defined ν-almost everywhere, by 495La.

(a) Consider first the case in which f is a simple function, expressed as $\sum_{j=0}^n \alpha_j \chi F_j$ where $\langle F_j\rangle_{j\le n}$ is a disjoint family of sets of finite measure and $\alpha_j \in \mathbb{R}$ for each j. Then $Q_f(S) = \sum_{j=0}^n \alpha_j \#(S \cap F_j)$ for ν-almost every S, so

$$\int e^{iyQ_f}\, d\nu = \int \prod_{j=0}^n e^{iy\alpha_j \#(S\cap F_j)} \nu(dS) = \prod_{j=0}^n \int e^{iy\alpha_j \#(S\cap F_j)} \nu(dS)$$

(because the functions $S \mapsto \#(S \cap F_j)$ are independent)

$$\begin{aligned} &= \prod_{j=0}^n \sum_{k=0}^\infty \frac{(\gamma\mu F_j)^k}{k!} e^{-\gamma\mu F_j} e^{iy\alpha_j k} = \prod_{j=0}^n e^{-\gamma\mu F_j} \sum_{k=0}^\infty \frac{(e^{iy\alpha_j}\gamma\mu F_j)^k}{k!} \\ &= \prod_{j=0}^n \exp\big((e^{iy\alpha_j} - 1)\gamma\mu F_j\big) = \exp\big(\gamma \sum_{j=0}^n (e^{iy\alpha_j} - 1)\mu F_j\big) \\ &= \exp\big(\gamma \int (e^{iyf} - 1) d\mu\big). \end{aligned}$$

(b) Now suppose that f is any integrable function. Then there is a sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ of simple functions such that $|f_n| \le_{\text{a.e.}} |f|$ for every n and $\lim_{n\to\infty} f_n =_{\text{a.e.}} f$. Write q_n, q for Q_{f_n}, Q_f. Set

$$D = \{x : x \in \mathrm{dom}\, f,\ |f_n(x)| \le |f(x)| \text{ for every } n \text{ and } \lim_{n\to\infty} f_n(x) = f(x)\},$$

so that D is μ-conegligible. If $S \subseteq D$ and $Q_{|f|}(S)$ is defined, then $q(S) = \lim_{n\to\infty} q_n(S)$, and this is true for ν-almost every S; so $\int e^{iyq} d\nu = \lim_{n\to\infty} e^{iyq_n} d\nu$, by Lebesgue's Dominated Convergence Theorem. On the other hand,

$$|e^{i\alpha} - 1| = |\textstyle\int_0^\alpha \frac{1}{i} e^{it} dt| \le \alpha, \quad |e^{-i\alpha} - 1| = |\int_0^\alpha \frac{1}{i} e^{-it} dt| \le \alpha$$

for every $\alpha \ge 0$. So if we set $g(x) = e^{iyf(x)} - 1$, $g_n(x) = e^{iyf_n(x)} - 1$ when these are defined, we have $|g_n| \le_{\text{a.e.}} |yf_n| \le_{\text{a.e.}} |yf|$ for every n. Accordingly

$$\textstyle\int (e^{iyf} - 1) d\mu = \int \lim_{n\to\infty} (e^{iyf_n} - 1) d\mu = \lim_{n\to\infty} \int (e^{iyf_n} - 1) d\mu$$

by Lebesgue's theorem again. It follows that

$$\int e^{iyQ_f}\, d\nu = \lim_{n\to\infty} \int e^{iyq_n} d\nu = \lim_{n\to\infty} \exp\big(\gamma \int (e^{iyf_n} - 1) d\mu\big)$$

(by (a))

$$= \exp\big(\gamma \lim_{n\to\infty} \int (e^{iyf_n} - 1) d\mu\big) = \exp\big(\gamma \int (e^{iyf} - 1) d\mu\big),$$

as claimed.

Remark Recall that a Poisson random variable with expectation γ has characteristic function $y \mapsto \exp(\gamma(e^{iy} - 1))$ (part (a) of the proof of 285Q), corresponding to the case $f = \chi F$ where $\mu F = 1$. The random variables Q_f have **compound Poisson** distributions.

495N If our underlying measure is a Radon measure, we can look for Radon measures on $\mathcal{P}X$ to represent the Poisson point processes on X. There seem to be difficulties in general, but I can offer the following. See also 495Yc.

Proposition Let $(X, \mathfrak{T}, \Sigma, \mu)$ be a σ-compact locally compact Radon measure space, and $\gamma > 0$. Give the space $\mathcal{C}$ of closed subsets of X its k-Vietoris topology (476Cd).

(i) There is a unique Radon measure $\tilde{\nu}$ on $\mathcal{C}$ such that

$$\tilde{\nu}\{C : \#(C \cap E_i) \neq \emptyset \text{ for every } i \leq r\} = \prod_{i=0}^{r} 1 - e^{-\gamma\mu E_i}$$

whenever $E_0, \ldots, E_r \subseteq X$ are disjoint sets of finite measure.

(ii) If $E_0, \ldots, E_r$ are disjoint sets of finite measure, none including any singleton set of non-zero measure, and $n_i \in \mathbb{N}$ for $i \leq r$, then

$$\tilde{\nu}\{C : \#(C \cap E_i) = n_i \text{ for every } i \leq r\} = \prod_{i=0}^{r} \frac{(\gamma\mu E_i)^{n_i}}{n_i!} e^{-\gamma\mu E_i}.$$

(iii) If μ is atomless and ν is the Poisson point process on X with density γ, then $\mathcal{C}$ is ν-conegligible and $\tilde{\nu}$ extends the subspace measure $\nu|\mathcal{C}$.

proof (a) Recall that $\mathcal{C}$ is Hausdorff (476Cd). Because X is locally compact and σ-compact, there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of relatively compact open sets covering X. For each n set $H'_n = H_n \setminus \bigcup_{i<n} H_i$; then $\langle H'_n\rangle_{n\in\mathbb{N}}$ is a disjoint sequence of Borel sets of finite measure covering X. Set $\mathcal{H} = \{H'_n : n \in \mathbb{N}, \mu H'_n > 0\}$. Then $\mathcal{H}$ is a countable maximal almost-disjoint family of measurable sets of non-zero finite measure. Build Ω, λ and $\phi : \Omega \to \mathcal{P}X$ from $\mathcal{H}$ as in the proof of 495B. This time, however, observe that all the μ'_H are Radon measures (416Rb), while surely the Poisson distributions ν_H on $\mathbb{N}$ also are, so that we have a Radon measure $\tilde{\lambda}$ on Ω extending λ (417Q).

(b) $\phi(\omega) \in \mathcal{C}$ for every $\omega \in \Omega$. **P** For each $H \in \mathcal{H}$, $H \cap \phi(\omega) = \{x_{Hn}(\omega) : n < m_H(\omega)\}$ is finite. Since each H_n meets only finitely many members of $\mathcal{H}$, $\phi(\omega) \cap H_n$ is finite for each n; because $\{H_n : n \in \mathbb{N}\}$ is an open cover of the Hausdorff space X, this is enough to ensure that $\phi(\omega)$ is closed. **Q**

(c) $\phi : \Omega \to \mathcal{C}$ is continuous. **P** (i) If $G \subseteq X$ is open, then

$$\{\omega : G \cap \phi(\omega) \neq \emptyset\} = \bigcup_{H\in\mathcal{H}, n\in\mathbb{N}} \{\omega : m_H(\omega) > n, x_{Hn}(\omega) \in G\}$$

is open. (ii) If $K \subseteq X$ is compact, there is some $r \in \mathbb{N}$ such that $K \subseteq \bigcup_{i\leq r} H_i$, so that $\mathcal{H}_0 = \{H : H \in \mathcal{H}, H \cap K \neq \emptyset\}$ is finite. Now

$$\{\omega : \phi(\omega) \cap K = \emptyset\} = \bigcap_{H\in\mathcal{H}_0} \bigcup_{n\in\mathbb{N}} \bigcap_{i<n} \{\omega : m_H(\omega) = n, x_{Hi}(\omega) \notin K\}$$

is open. (iii) By 4A2B(a-ii), ϕ is continuous. **Q**

(d) Accordingly we have a Radon measure $\tilde{\nu}$ on $\mathcal{C}$ defined by setting $\tilde{\nu}(\mathcal{S}) = \tilde{\lambda}\phi^{-1}[\mathcal{S}]$ whenever $\mathcal{S} \subseteq \mathcal{C}$ and $\phi^{-1}[\mathcal{S}]$ is measured by $\tilde{\lambda}$ (418I).

If $E_0, \ldots, E_r \in \Sigma$ are disjoint sets of finite measure, then, defining g_{E_i} as in 495B,

$$\begin{aligned}\tilde{\nu}\{C : C \cap E_i \neq \emptyset \text{ for every } i \leq r\} &= \tilde{\lambda}\{\omega : g_{E_i}(\omega) \geq 1 \text{ for every } i \leq r\} \\ &= \prod_{i=0}^{r} 1 - e^{-\gamma\mu E_i}\end{aligned}$$

as required by (i).

(e) To see that $\tilde{\nu}$ is uniquely defined, let $\tilde{\nu}'$ be another Radon probability measure on $\mathcal{C}$ with the same property. Suppose that we have a compact set $K \subseteq X$ and open sets $G_i \subseteq X$, for $i < r$, and set

$$V = \{C : C \in \mathcal{C}, C \cap K = \emptyset, C \cap G_i \neq \emptyset \text{ for every } i < r\}.$$

Then $\tilde{\nu}V = \tilde{\nu}'V$. **P** For each $n \in \mathbb{N}$, set $G_{ni} = G_i \cap \bigcup_{j\leq n} H_j$ for each i, and set

$$V_n = \{C : C \in \mathcal{C}, C \cap K = \emptyset, C \cap G_{ni} \neq \emptyset \text{ for every } i < r\}.$$

Let $\mathcal{E}_n$ be the finite subalgebra of $\mathcal{P}X$ generated by $\{G_{ni} : i < r\} \cup \{K\}$, $\mathcal{A}_n$ the set of atoms of $\mathcal{E}_n$ and $\mathcal{A}_n^f$ the set of members of $\mathcal{A}_n$ of finite measure. For $\mathcal{I} \subseteq \mathcal{A}_n^f$ set

$$V_{n\mathcal{I}} = \{C : C \in \mathcal{C}, \mathcal{I} = \{A : A \in \mathcal{A}_n^f, C \cap A \neq \emptyset\}\}.$$

Then $V_n = \bigcup_{\mathcal{I} \in \mathfrak{I}_n} V_{n\mathcal{I}}$, where

$$\mathfrak{I}_n = \{\mathcal{I} : \mathcal{I} \subseteq \mathcal{A}_n^f, A \cap K = \emptyset \text{ for every } A \in \mathcal{I}, \text{ for every } i < r \text{ there is an } A \in \mathcal{I} \text{ such that } A \subseteq G_{ni}\}.$$

Now the defining formula for $\tilde{\nu}$ shows that $\tilde{\nu}V_{n\mathcal{I}} = \tilde{\nu}'V_{n\mathcal{I}}$ for every $\mathcal{I} \subseteq \mathcal{A}_n^f$, so that $\tilde{\nu}V_n = \tilde{\nu}'V_n$; and now

$$\tilde{\nu}V = \lim_{n\to\infty} \tilde{\nu}V_n = \lim_{n\to\infty} \tilde{\nu}'V_n = \tilde{\nu}'V. \; \mathbf{Q}$$

Since sets V of the type described form a base for the k-Vietoris topology closed under finite intersections, $\tilde{\nu} = \tilde{\nu}'$ (415H(v)).

(f) If $E \in \Sigma$ has finite measure and does not include any non-negligible singleton, then $\#(E \cap \phi(\omega)) = g_E(\omega)$ for almost every $\omega \in \Omega$. **P** As in part (c) of the proof of 495D, $\#(E \cap \phi(\omega)) = g_E(\omega)$ for every $\omega \in \Omega \setminus A_E$, where

$$A_E = \{\omega : \text{ there are distinct } H, H' \in \mathcal{H} \text{ and } j \in \mathbb{N} \text{ such that } x_{Hj}(\omega) \in H'\} \cup \{\omega : \text{ there are } H \in \mathcal{H} \text{ and distinct } i, j \in \mathbb{N} \text{ such that } x_{Hi}(\omega) = x_{Hj}(\omega) \in E\},$$

and A_E is negligible, because the subspace measure on E is atomless (414G/416Xa). **Q** So if $E_0, \ldots, E_r$ are disjoint sets of finite measure, none including any non-negligible singleton, and $n_i \in \mathbb{N}$ for $i \leq r$,

$$\begin{aligned}\tilde{\nu}\{C : \#(C \cap E_i) = n_i \text{ for every } i \leq r\} &= \tilde{\lambda}\{\omega : g_{E_i}(\omega) = n_i \text{ for every } i \leq r\} \\ &= \prod_{i=0}^{r} \frac{(\gamma\mu E_i)^{n_i}}{n_i!} e^{-\gamma\mu E_i},\end{aligned}$$

as required by (ii).

(g) Now suppose that μ is atomless. Let ν be the Poisson point process on X with density γ. For each n, the set $\{S : S \cap H_n \text{ is finite}\}$ is ν-conegligible, so $\{S : S \cap H_n \text{ is finite for every } n\}$ is ν-conegligible; but as $\langle H_n \rangle_{n\in\mathbb{N}}$ is an open cover of X, this is included in $\mathcal{C}$. So $\mathcal{C}$ is ν-conegligible.

Let T be the σ-algebra of subsets of $\mathcal{P}X$ generated by sets of the form $\{S : \#(S \cap E) = n\}$ where $\mu E < \infty$ and $n \in \mathbb{N}$. Then (ii) here and 495C tell us that $\tilde{\nu}(\mathcal{C} \cap \mathcal{S}) = \nu\mathcal{S}$ whenever $\mathcal{S}$ belongs to T and therefore whenever ν measures $\mathcal{S}$; so that $\tilde{\nu}$ extends the subspace measure $\nu_{\mathcal{C}}$.

495O Proposition Let $(X, \mathfrak{T})$ be a σ-compact locally compact Hausdorff space and $M_{\mathrm{R}}^+(X)$ the set of Radon measures on X. Give $M_{\mathrm{R}}^+(X)$ the topology generated by sets of the form $\{\mu : \mu G > \alpha\}$ and $\{\mu : \mu K < \alpha\}$ where $G \subseteq X$ is open, $K \subseteq X$ is compact and $\alpha \in \mathbb{R}$. Let $\mathcal{C}$ be the space of closed subsets of X with the k-Vietoris topology, and $P_{\mathrm{R}}(\mathcal{C})$ the set of Radon probability measures on $\mathcal{C}$ with its narrow topology (definition: 437Jd). Fix $\gamma > 0$, and for $\mu \in M_{\mathrm{R}}^+(X)$ let $\tilde{\nu}_\mu$ be the Radon measure on $\mathcal{C}$ defined in 495N. Then the function $\mu \mapsto \tilde{\nu}_\mu : M_{\mathrm{R}}^+(X) \to P_{\mathrm{R}}(\mathcal{C})$ is continuous.

proof (a) Recall that because X is locally compact and Hausdorff, $\mathcal{C}$ is compact and Hausdorff (476Cd). Fix an open set $W_0 \subseteq \mathcal{C}$, $\alpha_0 > 0$ and $\mu_0 \in M_{\mathrm{R}}^+(X)$ such that $\tilde{\nu}_{\mu_0} W_0 > \alpha_0$. Let $\mathcal{E}$ be the family of relatively compact Borel subsets E of X such that $\mu_0(\partial E) = 0$. Then $\mathcal{E}$ is a subring of $\mathcal{P}X$ (4A2Bi). Also $\mu \mapsto \mu E : M_{\mathrm{R}}^+(X) \to [0, \infty[$ is continuous at μ_0 for every $E \in \mathcal{E}$. **P** If $E \in \mathcal{E}$ and $\epsilon > 0$, then

$$\begin{aligned}&\{\mu : \mu_0 E - \eta < \mu E < \mu_0 E + \eta\} \\ &\quad \supseteq \{\mu : \mu(\operatorname{int} E) > \mu_0(\operatorname{int} E) - \eta, \, \mu(X \setminus \overline{E}) > \mu_0(X \setminus \overline{E}) - \eta\}\end{aligned}$$

is a neighbourhood of μ_0. **Q**

(b) Next, $\mathcal{U} = \mathcal{E} \cap \mathfrak{T}$ is a base for $\mathfrak{T}$ (411Gi). It follows that the family $\mathcal{V}$ of sets of the form

$$\{C : C \in \mathcal{C}, C \cap U_i \neq \emptyset \text{ for } i \leq r, C \cap \overline{U} = \emptyset\},$$

where $U, U_0, \ldots, U_r \in \mathcal{U}$, is a base for the k-Vietoris topology on $\mathcal{C}$. **P** If $W \subseteq \mathcal{C}$ is open for the k-Vietoris topology and $C_0 \in W$, there are $r \in \mathbb{N}$, open sets $G_i \subseteq X$ for $i \leq r$ and a compact set $K \subseteq X$ such that

$$C_0 \in \{C : C \cap G_i \neq \emptyset \text{ for each } i \leq r,\ C \cap K = \emptyset\} \subseteq W.$$

For each $i \leq r$ choose $x_i \in C_0 \cap G_i$ and $U_i \in \mathcal{U}$ such that $x_i \in U_i \subseteq G_i$. Because X is locally compact and Hausdorff, it is regular, so every point of K belongs to a member of $\mathcal{U}$ with closure disjoint from C_0; because $\mathcal{U}$ is closed under finite unions, there is a $U \in \mathcal{U}$ such that $K \subseteq U$ and $C_0 \cap \overline{U} = \emptyset$. Now

$$\{C : C \in \mathcal{C},\ C \cap U_i \neq \emptyset \text{ for } i \leq r,\ C \cap \overline{U} = \emptyset\}$$

belongs to $\mathcal{V}$, contains C_0 and is included in W. As C_0 and W are arbitrary, $\mathcal{V}$ is a base for the k-Vietoris topology on $\mathcal{C}$. **Q**

(c) If $V \in \mathcal{V}$, then $\mu \mapsto \tilde{\nu}_\mu V : M^+_{\mathrm{R}}(X) \to [0,1]$ is continuous at μ_0. **P** Express V as $\{C : C \in \mathcal{C}, C \cap U_i \neq \emptyset \text{ for } i \leq r,\ C \cap \overline{U} = \emptyset\}$, where U_i, $U \in \mathcal{U}$. Let $\mathcal{A}$ be the set of atoms of the finite subring of $\mathcal{E}$ generated by $\{U_i : i \leq r\} \cup \{\overline{U}\}$. For $\mathcal{I} \subseteq \mathcal{A}$ set

$$V_{\mathcal{I}} = \{C : C \in \mathcal{C},\ \mathcal{I} = \{A : A \in \mathcal{A},\ C \cap A \neq \emptyset\}.$$

Let $\mathfrak{I}$ be the set of those $\mathcal{I} \subseteq \mathcal{A}$ such that $A \cap \overline{U} = \emptyset$ for every $A \in \mathcal{I}$ and for every $i \leq r$ there is an $A \in \mathcal{I}$ such that $A \subseteq U_i$. Then $\langle V_{\mathcal{I}} \rangle_{\mathcal{I} \in \mathfrak{I}}$ is a partition of V. Moreover, for any $\mu \in M^+_{\mathrm{R}}(X)$ and $\mathcal{I} \subseteq \mathcal{A}$,

$$\tilde{\nu}_\mu V_{\mathcal{I}} = \prod_{A \in \mathcal{I}} e^{-\gamma\mu A} \prod_{A \in \mathcal{A} \setminus \mathcal{I}} (1 - e^{-\gamma \mu A}).$$

Since each $\mu \mapsto \mu A$ is continuous at μ_0, by (a), so are the functionals $\mu \mapsto \tilde{\nu}_\mu V_{\mathcal{I}}$, for $\mathcal{I} \subseteq \mathcal{A}$, and $\mu \mapsto \tilde{\nu}_\mu V = \sum_{\mathcal{I} \in \mathfrak{I}} \tilde{\nu}_\mu V_{\mathcal{I}}$. **Q**

(d) Let $\mathcal{V}^*$ be the family of Borel subsets V of $\mathcal{C}$ such that $\mu \mapsto \tilde{\nu}_\mu V : M^+_{\mathrm{R}}(X) \to [0, \infty[$ is continuous at μ_0. Then $\mathcal{V} \subseteq \mathcal{V}^*$ (by (c)), $\mathcal{C} \in \mathcal{V}^*$ and $V \setminus V' \in \mathcal{V}^*$ whenever V, $V' \in \mathcal{V}^*$ and $V' \subseteq V$. Because $\mathcal{V}$ is closed under finite intersections, it follows that $\mathcal{V}^*$ includes the algebra of subsets of $\mathcal{C}$ generated by $\mathcal{V}$ (313Ga); in particular, any finite union of members of $\mathcal{V}$ belongs to $\mathcal{V}^*$.

(e) Let us return to the open set $W_0 \subseteq \mathcal{C}$ and the $\alpha_0 \in \mathbb{R}$ of part (a). Because $\tilde{\nu}_{\mu_0}$ is τ-additive and $\mathcal{V}$ is a base for the topology of $\mathcal{C}$ ((b) above), there is a finite family $\mathcal{V}_0 \subseteq \mathcal{V}$ such that $V_0 = \bigcup \mathcal{V}_0$ is included in W_0 and $\tilde{\nu}_{\mu_0} V_0 > \alpha_0$. But this means that

$$\{\mu : \mu \in M^+_{\mathrm{R}}(X),\ \tilde{\nu}_\mu W_0 > \alpha_0\} \supseteq \{\mu : \mu \in M^+_{\mathrm{R}}(X),\ \tilde{\nu}_\mu V_0 > \alpha_0\}$$

is a neighbourhood of μ_0. As μ_0 is arbitrary, $\{\mu : \tilde{\nu}_\mu W_0 > \alpha_0\}$ is open; as W_0 and α_0 are arbitrary, $\mu \mapsto \tilde{\nu}_\mu$ is continuous.

495P There are many constructions which, in particular cases, can be used as an alternative to the method of 495B-495D in setting up Poisson point processes. I give one which applies to the half-line $[0, \infty[$ with Lebesgue measure.

Theorem Let $\gamma > 0$, and let ν be the Poisson point process on $[0, \infty[$, with Lebesgue measure, with density γ. Let λ_0 be the exponential distribution with expectation $1/\gamma$, regarded as a Radon probability measure on $]0, \infty[$, and λ the corresponding product measure on $]0, \infty[^{\mathbb{N}}$. Define $\phi :]0, \infty[^{\mathbb{N}} \to \mathcal{P}([0, \infty[)$ by setting $\phi(x) = \{\sum_{i=0}^n x(i) : n \in \mathbb{N}\}$ for $x \in]0, \infty[^{\mathbb{N}}$. Then ϕ is a measure space isomorphism between $]0, \infty[^{\mathbb{N}}$ and a ν-conegligible subset of $\mathcal{P}([0, \infty[)$.

Remark As I seem not to have mentioned exponential distributions earlier in this treatise, I remark now that the **exponential distribution** with parameter γ has distribution function

$$F(t) = 0 \text{ if } t < 0,\ 1 - e^{-\gamma t} \text{ if } t \geq 0,$$

and probability density function

$$f(t) = 0 \text{ if } t \leq 0,\ \gamma e^{-\gamma t} \text{ if } t > 0;$$

its expectation is

$$\int_0^\infty \gamma t e^{-\gamma t} dt = -\int_0^\infty \tfrac{d}{dt} \tfrac{\gamma t + 1}{\gamma} e^{-\gamma t} dt = \frac{1}{\gamma}.$$

Because (when regarded as a Radon probability measure on $\mathbb{R}$, following my ordinary rule set out in §271) it gives measure zero to $]-\infty, 0]$, it can be identified with the subspace measure on $]0, \infty[$, as here.

proof (a) For each $n \in \mathbb{N}$, $\#(S \cap [0,n])$ is finite for ν-almost every S; so the set

$$Q_0 = \{S : S \subseteq [0,\infty[,\ \#(S \cap [0,n]) \text{ is finite for every } n\}$$

is ν-conegligible. Next, the sets $\{S : S \cap [n, n+1[\neq \emptyset\}$ are ν-independent and have measure $1 - e^{-\gamma} > 0$, so

$$\{S : S \cap [n, n+1[\neq \emptyset \text{ for infinitely many } n\}$$

is ν-conegligible (273K). Finally, $\nu\{S : 0 \in S\} = 0$, so $Q = \{S : S \in Q_0,\ 0 \notin S,\ S \text{ is infinite}\}$ is ν-conegligible. For $S \in Q$, let $\langle g_n(S)\rangle_{n\in\mathbb{N}}$ be the increasing enumeration of S. Let T be the σ-algebra of subsets of $\mathcal{P}([0,\infty[)$ generated by sets of the form $\{S : \#(S \cap E) = n\}$ where $E \subseteq [0,\infty[$ has finite measure and $n \in \mathbb{N}$. Then, for $n \in \mathbb{N}$ and $\alpha \geq 0$, $\{S : g_n(S) \leq \alpha\} = \{S : \#(S \cap [0,\alpha]) \geq n+1\}$ belongs to T, so g_n is T-measurable. Set $h_0(S) = g_0(S)$, $h_n(S) = g_n(S) - g_{n-1}(S)$ for $n \geq 1$, and $h(S) = \langle h_n(S)\rangle_{n\in\mathbb{N}}$; then $h : Q \to]0,\infty[^{\mathbb{N}}$ is a bijection, and its inverse is ϕ.

(b) For each $k \in \mathbb{N}$, $I_S = \{i : i \in \mathbb{N},\ S \cap \left[2^{-k}i, 2^{-k}(i+1)\right[\neq \emptyset\}$ is infinite for every $S \in Q$. So we can define $g_{kn} : Q \to]0,\infty[$, for each n, by taking $g_{kn}(S) = 2^{-k}(j+1)$ if $j \in I_S$ and $\#(I_S \cap j) = n$. Because all the sets $\{S : j \in I_S\}$ belong to T, each g_{kn} is T-measurable, and $\langle g_{kn}\rangle_{k\in\mathbb{N}}$ is a non-increasing sequence with limit g_n. Set $h_{k0}(S) = g_{k0}(S)$ and $h_{kn}(S) = g_{k,n+1}(S) - g_{kn}(S)$ for $n \geq 1$. Then $h_n = \lim_{k\to\infty} h_{kn}$.

(d) For any $n \in \mathbb{N}$, $j_0, \ldots, j_n \in \mathbb{N}$, $k \in \mathbb{N}$, set $j'_r = \sum_{i=0}^{r} j_i$ for $r \leq n$. Then

$$\begin{aligned}
&\nu\{S : S \in Q, h_{ki}(S) = 2^{-k}(j_i+1) \text{ for every } i \leq n\}\\
&\quad = \nu\{S : S \in Q,\ g_{kr}(S) = 2^{-k}(r+1+j'_r) \text{ for every } r \leq n\}\\
&\quad = \nu\{S : S \cap \left[2^{-k}(r+j'_r), 2^{-k}(r+1+j'_r)\right[\neq \emptyset \text{ for every } r \leq n,\\
&\qquad\qquad S \cap \left[0, 2^{-k}j_0\right[= \emptyset,\\
&\qquad\qquad S \cap \left[2^{-k}(r+1+j'_r), 2^{-k}(r+1+j'_{r+1})\right[= \emptyset \text{ for every } r < n\}\\
&\quad = (1-\exp(-2^{-k}\gamma))^{n+1}\exp(-2^{-k}\gamma j_0)\prod_{r<n}\exp(-2^{-k}\gamma(j'_{r+1}-j'_r))\\
&\quad = \prod_{i=0}^{n}(1-\exp(-2^{-k}\gamma))\exp(-2^{-k}\gamma j_i)
\end{aligned}$$

This means that the h_{ki}, for $i \in \mathbb{N}$ are independent, with

$$\Pr(h_{ki} = 2^{-k}(j+1)) = (1-e^{-2^{-k}\gamma})e^{-2^{-k}\gamma j}$$

for each j. Since $h_{ki} \to h_i$ ν-a.e. for each i, $\langle h_i\rangle_{i\in\mathbb{N}}$ is also independent (367W). Now, for any $\alpha > 0$,

$$\begin{aligned}
\Pr(h_{ki} \leq \alpha) &= \sum_{2^{-k}(j+1)\leq\alpha} (1-\exp(-2^{-k}\gamma))\exp(-2^{-k}\gamma j)\\
&= 1 - \exp(-2^{-k}\gamma\lfloor 2^k\alpha\rfloor) \to 1 - e^{-\gamma\alpha}
\end{aligned}$$

as $k \to \infty$. So

$$\begin{aligned}
\Pr(h_i \leq \alpha) &= \inf_{\beta>\alpha} \liminf_{i\to\infty} \Pr(h_{ki} \leq \beta)\\
&= \inf_{\beta>\alpha} 1 - e^{-\gamma\beta} = 1 - e^{-\gamma\alpha}
\end{aligned} \tag{271L}$$

for every $\alpha \geq 0$ and every $i \in \mathbb{N}$.

(e) Accordingly $\langle h_i\rangle_{i\in\mathbb{N}}$ is an independent sequence of random variables, each exponentially distributed with expectation $1/\gamma$. It follows that $h : Q \to]0,\infty[^{\mathbb{N}}$ is inverse-measure-preserving for the subspace measure ν_Q and λ (254G).

Observe next that if $E \subseteq [0,\infty[$ is Lebesgue measurable and $n \in \mathbb{N}$, then

$$\{x : x \in]0,\infty[^{\mathbb{N}}, \#(\phi(x) \cap E) = n\} = \bigcup_{I \in [\mathbb{N}]^n} \{x : \textstyle\sum_{i=0}^{j} x(i) \in E \iff j \in I\} \in \Lambda,$$

writing Λ for the domain of λ. So ϕ is (Λ, T)-measurable. Now, for any $W \in \mathrm{T}$,

$$\lambda\phi^{-1}[W] = \nu(h^{-1}[\phi^{-1}[W]]) = \nu(W \cap Q) = \nu W.$$

So ϕ is inverse-measure-preserving for λ and for $\nu\restriction\mathrm{T}$. Since λ is complete and ν is defined as the completion of its restriction to T, ϕ is inverse-measure-preserving for λ and ν. Thus ϕ and h are the two halves of an isomorphism between $(]0,\infty[^{\mathbb{N}}, \lambda)$ and the subspace (Q, ν_Q), as claimed.

495X Basic exercises (a) Let (X, Σ, μ) be an atomless measure space, and ν a Poisson point process on X. (i) Show that $[X]^{\leq\omega}$ has full outer measure for ν. (ii) Show that if μ is semi-finite then $[X]^{\leq\omega}$ is conegligible iff μ is σ-finite. (iii) Show that if μ is semi-finite, then $[X]^{<\omega}$ is non-negligible iff $[X]^{<\omega}$ is conegligible iff μ is totally finite.

(b) Let (X, Σ, μ) be an atomless measure space and $\mathcal{E}$ a countable partition of X into measurable sets; fix $\gamma > 0$. Let ν be the Poisson point process on X with density γ, and for each $E \in \mathcal{E}$ let ν_E be the Poisson point process on E with density γ corresponding to the subspace measure μ_E on E. Let λ be the product of the family $\langle \nu_E \rangle_{E\in\mathcal{E}}$. Show that the map $S \mapsto \langle S \cap E\rangle_{E\in\mathcal{E}} : \mathcal{P}X \to \prod_{E\in\mathcal{E}} \mathcal{P}E$ is a measure space isomorphism for ν and λ.

>(c) Let (X, Σ, μ) be an atomless measure space and $\mathfrak{T}$ a topology on X such that X is covered by a sequence of open sets of finite outer measure. Let ν be a Poisson point process on X. Show that ν-almost every set $S \subseteq X$ is locally finite in the sense that X is covered by the open sets meeting S in finite sets; in particular, if X is Hausdorff, then ν-almost every subset of X is closed.

>(d) Let (X, Σ, μ) be an atomless measure space, and for $\gamma > 0$ let ν_γ be the Poisson point process on X with density γ. Show that for any γ, $\delta > 0$ the map $(S, T) \mapsto S \cup T : \mathcal{P}X \times \mathcal{P}X \to \mathcal{P}X$ is inverse-measure-preserving for the product measure $\nu_\gamma \times \nu_\delta$ and $\nu_{\gamma+\delta}$.

(e) Let (X, Σ, μ) be an atomless semi-finite measure space, and ν the Poisson point process on X with density 1. Show that ν is perfect iff μ is.

(f) Let $(\mathfrak{A}, \bar\mu)$ be a measure algebra. Show that there is a probability measure λ on $\mathbb{R}^{\mathfrak{A}^f}$ such that (i) for every $a \in \mathfrak{A}^f$ the corresponding marginal measure on $\mathbb{R}$ is the Poisson distribution with expectation $\bar\mu a$ (ii) whenever $a_0, \dots, a_n \in \mathfrak{A}^f$ are disjoint, the functions $z \mapsto z(a_i) : \mathbb{R}^{\mathfrak{A}^f} \to \mathbb{R}$ are stochastically independent with respect to λ. (*Hint*: prove the result for finite $\mathfrak{A}$ and use 454D.) Use this to prove 495J.

(g) Let U be a Hilbert space. Show that there are a probability algebra $(\mathfrak{B}, \bar\lambda)$ and a linear operator $T : U \to L^0(\mathfrak{B})$ such that (i) for every $u \in U$, Tu has a normal distribution with expectation 0 and variance $\|u\|_2^2$ (ii) if $\langle u_i\rangle_{i\in I}$ is an orthogonal family in U then $\langle Tu_i\rangle_{i\in I}$ is $\bar\lambda$-independent. (*Hint*: see the proof of 456G.)

(h) Let U be an L-space. Show that there are a probability algebra $(\mathfrak{B}, \bar\lambda)$ and a linear operator $T : U \to L^1(\mathfrak{B}, \bar\lambda)$ such that (i) for every $u \in U$, Tu has a standard gamma distribution (definition: 455Xe) with expectation $\|u\|$ (ii) if $\langle u_i\rangle_{i\in I}$ is a disjoint family in U then $\langle Tu_i\rangle_{i\in I}$ is $\bar\lambda$-independent.

(i) Let ν be the Poisson point process with density 1 on $[0,\infty[$ with Lebesgue measure. Set $Q_0 = \{S : S \subseteq [0,\infty[, S\cap[0,n]$ is finite for every $n\}$ and for $S \in Q_0$ set $\psi(S)(t) = \#(S\cap[0,t])$ for $t \in [0,\infty[$. Show that ψ is inverse-measure-preserving for the subspace measure ν_{Q_0} and the Baire measure on $\mathbb{R}^{[0,\infty[}$ described in 455Xc.

(j) Let (Y, T, ν) be a probability space, and λ_0 the exponential distribution with expectation 1, regarded as a Radon measure on $]0,\infty[$. Let λ be the product measure $\lambda_0^{\mathbb{N}} \times \nu^{\mathbb{N}}$ on $]0,\infty[^{\mathbb{N}} \times Y^{\mathbb{N}}$. Set $\phi(x, y) = \{(\sum_{i=0}^{n} x(i), y(n)) : n \in \mathbb{N}\}$ for $x \in]0,\infty[^{\mathbb{N}}$ and $y \in Y^{\mathbb{N}}$. Show that $\phi :]0,\infty[^{\mathbb{N}} \times Y^{\mathbb{N}} \to \mathcal{P}(]0,\infty[\times Y)$ is a measure space isomorphism between $(]0,\infty[^{\mathbb{N}} \times Y^{\mathbb{N}}, \lambda)$ and a conegligible set for the Poisson point process on $]0,\infty[\times Y$ with density 1 for the c.l.d. product measure $\mu_L \times \nu$, where μ_L is Lebesgue measure.

(k) Let $\mathcal{C}$ be the family of closed subsets of $[0,\infty[$. Let ρ be the usual metric on $[0,\infty[$ and $\tilde{\rho}$ the corresponding Hausdorff metric on $\mathcal{C}\setminus\{\emptyset\}$. Let ν the Poisson point process on $[0,\infty[$ with density 1 over Lebesgue measure. Show that any $\tilde{\rho}$-compact subset of $\mathcal{C}\setminus\{\emptyset\}$ is ν-negligible.

(l) Show that the topology on $M_{\mathrm{R}}^{+}(X)$ described in 495O is just the topology induced by the natural embedding of $M_{\mathrm{R}}(X)$ into $C_k(X)^{\sim}$ (436J) and the weak topology $\mathfrak{T}_s(C_k(X)^{\sim},C_k(X))$, where $C_k(X)$ is the Riesz space of continuous real-valued functions on X with compact support.

(m) Let X be a σ-compact locally compact Hausdorff space, $\mathcal{C}$ the family of closed subsets of X with its k-Vietoris topology, and $P_{\mathrm{R}}(\mathcal{C})$ the set of Radon probability measures on $\mathcal{C}$ with its narrow topology. Let μ be a Radon measure on X and $\gamma>0$, and $\tilde{\nu}$ the Radon measure on $\mathcal{C}$ defined from μ and γ as in 495N. Let Q be the set of families $q=\langle(x_{qi},E_{qi})\rangle_{i\in I_q}$ such that I_q is a countable set, $\langle E_{qi}\rangle_{i\in I_q}$ is a disjoint family of measurable sets of finite measure, and $\langle x_{qi}\rangle_{i\in I_q}$ is a family of distinct points in X; for $q\in Q$ set $A_q=\{x_{qi}:i\in I_q\}$. (i) Show that for each $q\in Q$ we have a Radon measure $\tilde{\nu}_q$ on $\mathcal{C}$ defined by saying that $\tilde{\nu}_q(\mathcal{P}A_q)=1$ and

$$\tilde{\nu}_q\{C:x_{qi}\notin C\text{ for }i\in J\}=\exp(-\gamma\textstyle\sum_{i\in J}\mu E_{qi})$$

for every $J\subseteq I_q$. (ii) Show that for every neighbourhood V of $\tilde{\nu}$ for the narrow topology on $P_{\mathrm{R}}(\mathcal{C})$ there are an open cover $\mathcal{H}$ of X, a compact set $K\subseteq X$ and an $\epsilon>0$ such that $\tilde{\nu}_q\in V$ whenever $q\in Q$, $\mu(K\setminus\bigcup_{i\in I_q}E_{qi})\leq\epsilon$ and for every $i\in I_q$ there is an $H\in\mathcal{H}$ including $E_{qi}\cup\{x_{qi}\}$.

(n) Let $\mathcal{C}$ be the set of closed subsets of $[0,\infty[$ with its k-Vietoris topology. For $\delta\in]0,1]$ let λ_δ be the measure on $\{0,1\}^{\mathbb{N}}$ which is the product of copies of the measure on $\{0,1\}$ in which $\{1\}$ is given measure δ. Define $\phi_\delta:\{0,1\}^{\mathbb{N}}\to\mathcal{C}$ by setting $\phi_\delta(x)=\{n\delta:n\in\mathbb{N},x(n)=1\}$, and let $\tilde{\nu}_\delta$ be the Radon measure $\lambda_\delta\phi_\delta^{-1}$ on $\mathcal{C}$. Show that the Radon measure on $\mathcal{C}$ representing the Poisson point process on $[0,\infty[$ with density 1 over Lebesgue measure is the limit $\lim_{\delta\downarrow0}\tilde{\nu}_\delta$ for the narrow topology on the space of Radon probability measures on $\mathcal{C}$.

(o) Show that the standard gamma distribution with expectation 1 is the exponential distribution with expectation 1.

(p) Let $r\geq1$ be an integer; let μ be Lebesgue measure on $\mathbb{R}^r$ and β_r the volume of the unit ball in $\mathbb{R}^r$. Set $\psi(t)=(t/\beta_r)^{1/r}$ for $t\geq0$, so that the volume of a ball of radius $\psi(t)$ is t. Let S_{r-1} be the unit sphere in $\mathbb{R}^r$ and θ the invariant Radon probability measure on S_{r-1}, so that θ is a multiple of $r-1$-dimensional Hausdorff measure (476M). Let λ_0 be the exponential distribution with expectation 1, regarded as a Radon probability measure on $]0,\infty[$, and λ the product measure $\lambda_0^{\mathbb{N}}\times\theta^{\mathbb{N}}$ on $]0,\infty[^{\mathbb{N}}\times S_{r-1}^{\mathbb{N}}$. Set

$$\phi(x,z)=\{\psi(\textstyle\sum_{i=0}^{n}x(i))z(n):n\in\mathbb{N}\}$$

for $x\in]0,\infty[^{\mathbb{N}}$ and $z\in S_{r-1}^{\mathbb{N}}$. Show that $\phi:]0,\infty[^{\mathbb{N}}\times S_{r-1}^{\mathbb{N}}\to\mathcal{P}(\mathbb{R}^r)$ is a measure space isomorphism between $]0,\infty[^{\mathbb{N}}\times S_{r-1}^{\mathbb{N}}$ and a conegligible set for the Poisson point process on $\mathbb{R}^r$ with density 1.

495Y Further exercises (a) Let U be an L-space. Show that there are a probability algebra $(\mathfrak{B},\bar{\lambda})$ and a linear operator $T:U\to L^0(\mathfrak{B})$ such that (i) for every $u\in L^1(\mathfrak{A},\bar{\mu})$, Tu has a Cauchy distribution with centre 0 and scale parameter $\|u\|$ (ii) if $\langle u_i\rangle_{i\in I}$ is a disjoint family in U then $\langle Tu_i\rangle_{i\in I}$ is $\bar{\lambda}$-independent.

(b) Let $(\mathfrak{A},\bar{\mu})$ be a measure algebra. For $\alpha,\ y\in\mathbb{R}$ set $h_y(\alpha)=e^{iy\alpha}$, and let $\bar{h}_y:L^0(\mathfrak{A})\to L^0_{\mathbb{C}}(\mathfrak{B})$ (definition: 364Yn) be the corresponding operator (to be defined, following the ideas of 364I or otherwise). Show that there are a probability algebra $(\mathfrak{B},\bar{\lambda})$ and a positive linear operator $T:L^1(\mathfrak{A},\bar{\mu})\to L^1(\mathfrak{B},\bar{\lambda})$ such that (i) $\|Tu\|_1=\|u\|_1$ whenever $u\in L^1(\mathfrak{A},\bar{\mu})^+$ (ii) $\langle Tu_i\rangle_{i\in I}$ is $\bar{\lambda}$-independent in $L^0(\mathfrak{B})$ whenever $\langle u_i\rangle_{i\in I}$ is a disjoint family in $L^1(\mathfrak{A},\bar{\mu})$ (iii) $\int\bar{h}_y(Tu)d\bar{\lambda}=\exp(\int(\bar{h}_y(u)-\chi1)d\bar{\mu})$ for every $u\in L^1(\mathfrak{A},\bar{\mu})$ and $y\in\mathbb{R}$.

(c) Let (X,ρ) be a totally bounded metric space, μ a non-totally-finite Radon measure on X, and $\gamma>0$. Let $\mathcal{C}$ be the set of closed subsets of X, and $\tilde{\rho}$ the Hausdorff metric on $\mathcal{C}\setminus\{\emptyset\}$. (i) Show that there is a unique Radon measure $\tilde{\nu}$ on $\mathcal{C}\setminus\{\emptyset\}$ such that

$$\tilde{\nu}\{C:\#(C\cap E_i)\neq\emptyset\text{ for every }i\leq r\}=\textstyle\prod_{i=0}^{r}1-e^{-\gamma\mu E_i}$$

whenever $E_0, \dots, E_r \subseteq X$ are disjoint sets of finite measure. (ii) Show that if $E_0, \dots, E_r$ are disjoint sets of finite measure, none including any singleton set of non-zero measure, and $n_i \in \mathbb{N}$ for $i \leq r$, then

$$\tilde{\nu}\{C : \#(C \cap E_i) = n_i \text{ for every } i \leq r\} = \prod_{i=0}^{r} \frac{(\gamma\mu E_i)^{n_i}}{n_i!} e^{-\gamma\mu E_i}.$$

(iii) Show that if μ is atomless and ν is the Poisson point process on X with density γ, then $\mathcal{C} \setminus \{\emptyset\}$ is ν-conegligible and $\tilde{\nu}$ extends the subspace measure $\nu{\restriction}\mathcal{C} \setminus \{\emptyset\}$.

495 Notes and comments The underlying fact on which this section relies is that the Poisson distributions form a one-parameter semigroup of infinitely divisible distributions, with $\nu_\alpha * \nu_\beta = \nu_{\alpha+\beta}$ for all $\alpha, \beta > 0$. Other well-known families with this property are normal distributions, Cauchy distributions and gamma distributions; for each of these we have results corresponding to 495B and 495K (495Xg, 495Ya, 495Xh). The same distributions appeared, for the same reason, in the evolving processes of 455C and 455Xc-455Xe. Observe that the version for the normal distribution is related to the Gaussian processes of §456. The 'compound Poisson' distributions of 495M provide further examples, which approach the general form of infinitely divisible distributions (Loève 77, §23, or Fristedt & Gray 97, §16.3).

The special feature of the Poisson point process, in this context, is the fact that (for atomless measure spaces (X, μ)) it can be represented by a measure on $\mathcal{P}X$ rather than on some abstract auxiliary space (495D); so that we have a notion of 'random subset', and can discuss the expected topological properties of subsets of X (495Xa, 495Xc). In Euclidean spaces the geometric properties of these random subsets are also of great interest; see Meester & Roy 96. Here I look at the relations between this construction and others which have been prominent in this book, such as inverse-measure-preserving functions (495G) and disintegrations (495H-495I). In the latter we find ourselves in an interesting difficulty. If, as in 495H, we have a measure space $X = \tilde{X} \times [0,1]$, where $\tilde{X}$ is an atomless measure space, then it is natural to suppose that our Poisson process on X can be represented by picking a random subset T of $\tilde{X}$ and then, for each $t \in T$, a random $(t, \alpha) \in X$. The obvious model for this idea is the map $(T, z) \mapsto \{(t, z(t)) : t \in T\} : \mathcal{P}\tilde{X} \times [0,1]^{\tilde{X}} \to \mathcal{P}X$. The problem with this model is that the map is simply not measurable for the standard σ-algebras on $\mathcal{P}\tilde{X}$, $\mathcal{P}\tilde{X} \times [0,1]^{\tilde{X}}$ and $\mathcal{P}X$. When we have a canonical ordering in order type ω of almost every subset of $\tilde{X}$ ('almost every' with respect to the Poisson point process on $\tilde{X}$, of course), as in 495Xp, there can be a way around this, cutting $[0,1]^{\tilde{X}}$ down to a countable product and re-inventing the representation of pairs (T, z) as subsets of X. But in the general case it seems that we have to set up a disintegration of the Poisson point process on X over the Poisson point process on $\tilde{X}$ which does not correspond to any measure on a product $\mathcal{P}\tilde{X} \times \Omega$.

Following my usual custom, I have expressed the theorems of this section in terms of arbitrary (atomless) measure spaces. The results are not quite without interest when applied to totally finite measures, but their natural domain is the class of non-totally-finite σ-finite measures, as in 495N-495P. There is an unavoidable obstacle if we wish to extend the ideas to measure spaces which are not atomless. The functions $S \mapsto \#(S \cap E)$ can no longer have Poisson distributions, since if E is a singleton of positive measure then we shall have a non-trivial two-valued random variable. In 495N-495O I take one of the possible resolutions of this, with measures $\tilde{\nu}$ on spaces of subsets for which at least the functions $S \mapsto \#(S \cap E)$, for disjoint E, are independent (495Xm). An alternative which is sometimes appropriate is to work with functions $h : X \to \mathbb{N}$ and $\sum_{x \in E} h(x)$ in place of subsets S of X and $\#(S \cap E)$; see Fristedt & Gray 97, §29.

In 495J-495L we have a little cluster of results which are relevant to rather different questions, to which I expect to return in Chapter 52 of Volume 5. The objective here is to connect the structure of a measure algebra or Banach lattice of arbitrarily large cellularity with something which can be realized in a probability space. In each case, disjointness is transformed into stochastic independence. Once again, the special feature of the Poisson point process is that we have a concrete representation of a linear operator which can also be described in a more abstract way (495L).

The construction of 495B-495D seems to be the most straightforward way to generate Poisson point processes. It fails however to give a direct interpretation of one of the most important approaches to these processes, as limits of purely atomic processes in which sets are chosen by including or excluding individual points independently (495Xm-495Xn). In order to make sense of the limit here it seems that we need to put some further structure onto the underlying measure space, and 'σ-finite locally compact Radon measure space' is sufficient to give an interesting pair of positive results (495N-495O).

Appendix to Volume 4

Useful facts

As is to be expected, we are coming in this volume to depend on a wide variety of more or less recondite information, and only an exceptionally broad mathematical education will have covered it all. While all the principal ideas are fully expressed in standard textbooks, there are many minor points where I need to develop variations on the familiar formulations. A little under half the material, by word-count, is in general topology (§4A2), where I begin with some pages of definitions. I follow this with a section on Borel and Baire σ-algebras, Baire property algebras and cylindrical algebras (§4A3), worked out a little more thoroughly than the rest of the material. The other sections are on set theory (§4A1), linear analysis (§4A4), topological groups (§4A5) and Banach algebras (§4A6).

4A1 Set theory

For this volume, we need fragments from four topics in set theory and one in Boolean algebra. The most important are the theory of closed cofinal sets and stationary sets (4A1B-4A1C) and infinitary combinatorics (4A1D-4A1H). Rather more specialized, we have the theory of normal (ultra)filters (4A1J-4A1L) and a mention of Ostaszewski's ♣ (4A1M-4A1N), used for an example in §439. I conclude with a simple result on the cardinality of σ-algebras (4A1O).

4A1A Cardinals again (a) An infinite cardinal κ is **regular** if it is not the supremum of fewer than κ smaller cardinals, that is, if $\operatorname{cf}\kappa = \kappa$. Any infinite successor cardinal is regular. (Kunen 80, I.10.37; Jech 78, p. 40; Just & Weese 96, 11.18; Levy 79, IV.3.11.) In particular, $\omega_1 = \omega^+$ is regular.

(b) If ζ is an ordinal, X a set then I say that a family $\langle x_\xi\rangle_{\xi<\zeta}$ in X **runs over** X **with cofinal repetitions** if $\{\xi : \xi < \zeta,\ x_\xi = x\}$ is cofinal with ζ for every $x \in X$. Now if X is any non-empty set and κ is a cardinal greater than or equal to $\max(\omega, \#(X))$, there is a family $\langle x_\xi\rangle_{\xi<\kappa}$ running over X with cofinal repetitions. **P** By 3A1Ca, there is a surjection $\xi \mapsto (x_\xi, \alpha_\xi) : \kappa \to X \times \kappa$. **Q**

(c) The cardinal $\mathfrak{c}$ (i) Every non-trivial interval in $\mathbb{R}$ has cardinal $\mathfrak{c}$. (Enderton 77, p. 131.)

(ii) $\operatorname{cf}(2^\kappa) > \kappa$ for every infinite cardinal κ; in particular, $\operatorname{cf}\mathfrak{c} > \omega$. (Kunen 80, I.10.40; Jech 78, p. 46; Just & Weese 96, 11.24; Levy 79, V.5.2.)

(d) The Continuum Hypothesis This is the statement '$\mathfrak{c} = \omega_1$'; it is neither provable nor disprovable from the ordinary axioms of mathematics, including the Axiom of Choice. As such, it belongs to Volume 5 rather than to the present volume. But I do at one point refer to one of its immediate consequences. If the continuum hypothesis is true, then there is a well-ordering $\preccurlyeq$ of $[0,1]$ such that $([0,1], \preccurlyeq)$ has order type ω_1 (because there is a bijection $f : [0,1] \to \omega_1$, and we can set $s \preccurlyeq t$ if $f(s) \leq f(t)$).

4A1B Closed cofinal sets Let κ be a cardinal.

(a) Note that a subset F of κ is closed in the order topology iff $\sup A \in F$ whenever $A \subseteq F$ is non-empty and $\sup A < \kappa$. (4A2S(a-ii).)

(b) If κ has uncountable cofinality, and $A \subseteq \kappa$ has supremum κ, then $A' = \{\xi : 0 < \xi < \kappa, \xi = \sup(A \cap \xi)\}$ is a closed cofinal set in κ. **P** (i) For any $\eta < \kappa$ we can choose inductively a strictly increasing sequence $\langle \xi_n\rangle_{n\in\mathbb{N}}$ in A starting from $\xi_0 \geq \eta$; now $\xi = \sup_{n\in\mathbb{N}} \xi_n \in A'$ and $\xi \geq \eta$; this shows that A' is cofinal with κ. (ii) If $\emptyset \neq B \subseteq A'$ and $\sup B = \xi < \kappa$, then $A \cap \xi \supseteq A \cap \eta$ for every $\eta \in B$, so

$$\xi = \sup B = \sup_{\eta\in B} \sup(A \cap \eta) = \sup(\textstyle\bigcup_{\eta\in B} A \cap \eta) \leq \sup A \cap \xi \leq \xi$$

and $\xi \in A'$. **Q**

In particular, taking $A = \kappa = \omega_1$, the set of non-zero countable limit ordinals is a closed cofinal set in ω_1.

(c) If $\mathcal{F}$ is a non-empty family of closed cofinal sets in κ and $\#(\mathcal{F}) < \operatorname{cf}\kappa$, then $\bigcap\mathcal{F}$ is a closed cofinal set in κ. (Kunen 80, II.6.8; Just & Weese 97, Theorem 21.3; Levy 79, IV.4.15.) In particular, the intersection of any sequence of closed cofinal sets in ω_1 is again a closed cofinal set in ω_1.

(d) If $\langle F_\xi\rangle_{\xi<\kappa}$ is a family of subsets of κ, the **diagonal intersection** of $\langle F_\xi\rangle_{\xi<\kappa}$ is $\{\xi : \xi < \kappa, \xi \in F_\eta \text{ for every } \eta < \xi\}$. If κ is a regular uncountable cardinal and $\langle F_\xi\rangle_{\xi<\kappa}$ is any family of closed cofinal sets in κ, its diagonal intersection is again a closed cofinal set in κ. (Kunen 80, II.6.14; Jech 78, p. 58; Just & Weese 97, Theorem 21.7; Levy 79, IV.4.20.) In particular, if $f : \kappa \to \kappa$ is any function, then $\{\xi : \xi < \kappa, f(\eta) < \xi \text{ for every } \eta < \xi\}$ is a closed cofinal set in κ, being the diagonal intersection of $\langle \kappa \setminus (f(\xi)+1)\rangle_{\xi<\kappa}$.

4A1C Stationary sets (a) Let κ be a cardinal. A subset of κ is **stationary** in κ if it meets every closed cofinal set in κ; otherwise it is **non-stationary**.

(b) If κ is a cardinal of uncountable cofinality, the intersection of any stationary subset of κ with a closed cofinal set in κ is again a stationary set (because the intersection of two closed cofinal sets is a closed cofinal set); the family of non-stationary subsets of κ is a σ-ideal. (Kunen 80, II.6.9; Just & Weese 97, Lemma 21.11; Levy 79, IV.4.35.)

(c) Pressing-Down Lemma (Fodor's theorem) If κ is a regular uncountable cardinal, $A \subseteq \kappa$ is stationary and $f : A \to \kappa$ is such that $f(\xi) < \xi$ for every $\xi \in A$, then there is a stationary set $B \subseteq A$ such that f is constant on B. (Kunen 80, II.6.15; Jech 78, Theorem 22; Just & Weese 97, Theorem 21.2; Levy 79, IV.4.40.)

(d) There are disjoint stationary sets A, $B \subseteq \omega_1$. (This is easily deduced from 419G or 438Cd, and is also a special case of very much stronger results. See Kunen 80, II.6.12; Jech 78, p. 59; Just & Weese 97, Corollary 23.4; Levy 79, IV.4.48.)

4A1D Δ-systems (a) A family $\langle I_\xi\rangle_{\xi\in A}$ of sets is a **Δ-system** with **root** I if $I_\xi \cap I_\eta = I$ for all distinct ξ, $\eta \in A$.

(b) Δ-system Lemma If $\#(A)$ is a regular uncountable cardinal and $\langle I_\xi\rangle_{\xi\in A}$ is any family of finite sets, there is a set $D \subseteq A$ such that $\#(D) = \#(A)$ and $\langle I_\xi\rangle_{\xi\in D}$ is a Δ-system. (Kunen 80, II.1.6; Just & Weese 97, Theorem 16.3. For the present volume we need only the case $\#(A) = \omega_1$, which is treated in Jech 78, p. 225.)

4A1E Free sets (a) Let A be a set of cardinal at least ω_2, and $\langle J_\xi\rangle_{\xi\in A}$ a family of countable sets. Then there are distinct ξ, $\eta \in A$ such that $\xi \notin J_\eta$ and $\eta \notin J_\xi$. **P** Let $K \subseteq A$ be a set of cardinal ω_1, and set $L = K \cup \bigcup_{i\in K} J_i$. Then L has cardinal ω_1, so there is a $\xi \in A \setminus L$. Now there is an $\eta \in K \setminus J_\xi$, and this pair (ξ, η) serves. **Q**

(b) If $\langle K_\xi\rangle_{\xi\in A}$ is a disjoint family of sets indexed by an uncountable subset A of ω_1, and $\langle J_\eta\rangle_{\eta<\omega_1}$ is a family of countable sets, there is an uncountable $B \subseteq A$ such that $K_\xi \cap J_\eta = \emptyset$ whenever η, $\xi \in B$ and $\eta < \xi$. **P** Choose $\langle \xi_\zeta\rangle_{\zeta<\omega_1}$ inductively in such a way that $\xi_\zeta \in A$ and $K_{\xi_\zeta} \cap J_{\xi_\eta} = \emptyset$, $\xi_\zeta > \xi_\eta$ for every $\eta < \xi$. Set $B = \{\xi_\zeta : \zeta < \omega_1\}$. **Q**

4A1F Selecting subsequences (a) Let $\langle K_i\rangle_{i\in I}$ be a countable family of sets such that $\bigcap_{i\in J} K_i$ is infinite for every finite subset J of I. Then there is an infinite set K such that $K \setminus K_i$ is finite and $K_i \setminus K$ is infinite for every $i \in I$. **P** We can suppose that $I \subseteq \mathbb{N}$. Choose $\langle k_n\rangle_{n\in\mathbb{N}}$ inductively such that $k_n \in \bigcap_{i\in I, i\le n} K_i \setminus \{k_i : i < n\}$ for every $n \in \mathbb{N}$, and set $K = \{k_{2n} : n \in \mathbb{N}\}$. **Q**

Consequently there is a family $\langle K_\xi\rangle_{\xi<\omega_1}$ of infinite subsets of $\mathbb{N}$ such that $K_\xi \setminus K_\eta$ is finite if $\eta \le \xi$, infinite if $\xi < \eta$. (Choose the K_ξ inductively.)

(b) Let $\langle \mathcal{J}_i\rangle_{i\in I}$ be a countable family of subsets of $[\mathbb{N}]^\omega$ such that $\mathcal{J}_i \cap \mathcal{P}K \neq \emptyset$ for every $K \in [\mathbb{N}]^\omega$ and $i \in I$. Then there is an infinite $K \subseteq \mathbb{N}$ such that for every $i \in I$ there is a $J \in \mathcal{J}_i$ such that $K \setminus J$ is finite. **P** The case $I = \emptyset$ is trivial; suppose that $\langle i_n\rangle_{n\in\mathbb{N}}$ runs over I. Choose K_n, k_n inductively, for $n \in \mathbb{N}$, by taking

$$K_0 = \mathbb{N}, \quad k_n \in K_n,$$

$$K_{n+1} \subseteq K_n \setminus \{k_n\}, \quad K_{n+1} \in \mathcal{J}_{i_n}$$

for every n; set $K = \{k_n : n \in \mathbb{N}\}$. **Q**

4A1G Ramsey's Theorem If $n \in \mathbb{N}$, K is finite and $h : [\mathbb{N}]^n \to K$ is any function, there is an infinite $I \subseteq \mathbb{N}$ such that h is constant on $[I]^n$. (BOLLOBÁS 79, p. 105, Theorem 3; JECH 78, 29.1; JUST & WEESE 97, 15.3; LEVY 79, IX.3.7. For the present volume we need only the case $n = \#(K) = 2$.)

4A1H The Marriage Lemma again In 449J it will be useful to have an infinitary version of the Marriage Lemma available.

Proposition Let X and Y be sets, and $R \subseteq X \times Y$ a set such that $R[\{x\}]$ is finite for every $x \in X$ and $\#(R[I]) \geq \#(I)$ for every finite set $I \subseteq X$. Then there is an injective function $f : X \to Y$ such that $(x, f(x)) \in R$ for every $x \in X$.

proof For each finite $J \subseteq X$ there is an injective function $f_J : J \to Y$ such that $f_J \subseteq R$ (identifying f_J with its graph), by the ordinary Marriage Lemma (3A1K) applied to $R \cap (J \times R[J])$. Let $\mathcal{F}$ be any ultrafilter on $[X]^{<\omega}$ containing $\{J : I \subseteq J \in [X]^{<\omega}\}$ for every $I \in [X]^{<\omega}$; then for each $x \in X$ there must be an $f(x) \in R[\{x\}]$ such that $\{J : f_J(x) = f(x)\} \in \mathcal{F}$, because $R[\{x\}]$ is finite. Now $f \subseteq R$ is a function from X to Y and must be injective, because for any x, $x' \in X$ there is a $J \in [X]^{<\omega}$ such that f and f_J agree on $\{x, x'\}$.

4A1I Filters (a) Let X be a non-empty set. If $\mathcal{E} \subseteq \mathcal{P}X$ is non-empty and has the finite intersection property,

$$\mathcal{F} = \{A : A \subseteq X,\ A \supseteq \bigcap \mathcal{E}' \text{ for some non-empty finite } \mathcal{E}' \subseteq \mathcal{E}\}$$

is the smallest filter on X including $\mathcal{E}$, the filter **generated** by $\mathcal{E}$.

If $\mathcal{E} \subseteq \mathcal{P}X$ is non-empty and downwards-directed, then it has the finite intersection property iff it does not contain $\emptyset$; in this case we say that $\mathcal{E}$ is a **filter base**; $\mathcal{F} = \{A : A \subseteq X,\ A \supseteq E \text{ for some } E \in \mathcal{E}\}$, and $\mathcal{E}$ is a base for the filter $\mathcal{F}$.

In general, if $\mathcal{E}$ is a family of subsets of a non-empty set X, then there is a filter on X including $\mathcal{E}$ iff $\mathcal{E}$ has the finite intersection property; in this case, there is an ultrafilter on X including $\mathcal{E}$ (2A1O).

(b) If κ is a cardinal and $\mathcal{F}$ is a filter then $\mathcal{F}$ is **κ-complete** if $\bigcap \mathcal{E} \in \mathcal{F}$ whenever $\mathcal{E} \subseteq \mathcal{F}$ and $0 < \#(\mathcal{E}) < \kappa$. Every filter is ω-complete.

(c) A filter $\mathcal{F}$ on a regular uncountable cardinal κ is **normal** if (i) $\kappa \setminus \xi \in \mathcal{F}$ for every $\xi < \kappa$ (ii) whenever $\langle F_\xi \rangle_{\xi<\kappa}$ is a family in $\mathcal{F}$, its diagonal intersection belongs to $\mathcal{F}$.

4A1J Lemma A normal filter $\mathcal{F}$ on a regular uncountable cardinal κ is κ-complete.

proof If $\lambda < \kappa$ and $\langle F_\xi \rangle_{\xi<\lambda}$ is a family in $\mathcal{F}$, set $F_\xi = \kappa$ for $\lambda \leq \xi < \kappa$, and let F be the diagonal intersection of $\langle F_\xi \rangle_{\xi<\kappa}$; then $\bigcap_{\xi<\lambda} F_\xi \supseteq F \setminus \lambda$ belongs to $\mathcal{F}$.

4A1K Theorem Let X be a set and $\mathcal{F}$ a non-principal ω_1-complete ultrafilter on X. Let κ be the least cardinal of any non-empty set $\mathcal{E} \subseteq \mathcal{F}$ such that $\bigcap \mathcal{E} \notin \mathcal{F}$. Then κ is a regular uncountable cardinal, $\mathcal{F}$ is κ-complete, and there is a function $g : X \to \kappa$ such that $g[[\mathcal{F}]]$ is a normal ultrafilter on κ.

proof (a) By the definition of κ, $\mathcal{F}$ is κ-complete. Because $\mathcal{F}$ is ω_1-complete, $\kappa > \omega$. Let H be the set of all functions $h : X \to \kappa$ such that $h^{-1}[\kappa \setminus \xi] \in \mathcal{F}$ for every $\xi < \kappa$. Then H is not empty. **P** Let $\langle E_\xi \rangle_{\xi<\kappa}$ be a family in $\mathcal{F}$ such that $E = \bigcap_{\xi<\kappa} E_\xi \notin \mathcal{F}$. Because $\mathcal{F}$ is an ultrafilter, $X \setminus E \in \mathcal{F}$. Set $h(x) = 0$ if $x \in E$, $h(x) = \min\{\xi : x \notin E_\xi\}$ if $x \in X \setminus E$; then

$$h^{-1}[\kappa \setminus \xi] \supseteq (X \setminus E) \cap \bigcap_{\eta<\xi} E_\eta \in \mathcal{F}$$

for every $\xi < \kappa$, because $\mathcal{F}$ is κ-complete, so $h \in H$. **Q**

(b) For h, $h' \in H$, say that $h \prec h'$ if $\{x : h(x) < h'(x)\} \in \mathcal{F}$. Then there is a $g \in H$ such that $h \not\prec g$ for any $h \in H$. **P?** Otherwise, there is a sequence $\langle h_n \rangle_{n\in\mathbb{N}}$ in H such that $h_{n+1} \prec h_n$ for every $n \in \mathbb{N}$. Then $E_n = \{x : h_{n+1}(x) < h_n(x)\} \in \mathcal{F}$ for every n. Because $\mathcal{F}$ is ω_1-complete, there is an $x \in \bigcap_{n\in\mathbb{N}} E_n$; but now $\langle h_n(x) \rangle_{n\in\mathbb{N}}$ is a strictly decreasing sequence of ordinals, which is impossible. **XQ**

(c) I should check that κ is regular. **P** If $\langle\alpha_\xi\rangle_{\xi<\lambda}$ is any family in κ with $\lambda<\kappa$, then $g^{-1}[\kappa\setminus\alpha_\xi]\in\mathcal{F}$ for every ξ, so (because $\mathcal{F}$ is κ-complete)

$$g^{-1}[\kappa\setminus\sup_{\xi<\lambda}\alpha_\xi]=\bigcap_{\xi<\lambda}g^{-1}[\kappa\setminus\alpha_\xi]\in\mathcal{F},$$

and $\sup_{\xi<\lambda}\alpha_\xi\neq\kappa$. **Q**

(d) The image filter $g[[\mathcal{F}]]$ is an ultrafilter on κ, by 2A1N. Because $g\in H$, $g^{-1}[\kappa\setminus\xi]\in\mathcal{F}$ and $\kappa\setminus\xi\in g[[\mathcal{F}]]$ for any $\xi<\kappa$. **?** Suppose, if possible, that $g[[\mathcal{F}]]$ is not normal. Then there is a family $\langle A_\xi\rangle_{\xi<\kappa}$ in $g[[\mathcal{F}]]$ such that its diagonal intersection A does not belong to $g[[\mathcal{F}]]$, that is, $g^{-1}[A]\notin\mathcal{F}$ and $X\setminus g^{-1}[A]\in\mathcal{F}$. Define $h:X\to\kappa$ by setting

$$\begin{aligned}h(x)&=0 \text{ if } g(x)\in A\\ &=\min\{\eta:\eta<g(x),\ g(x)\notin A_\eta\} \text{ if } g(x)\notin A.\end{aligned}$$

Then

$$h^{-1}[\kappa\setminus\xi]\supseteq(X\setminus g^{-1}[A])\cap\textstyle\bigcap_{\eta<\xi}g^{-1}[A_\eta]\in\mathcal{F}$$

for every $\xi<\kappa$. Thus $h\in H$. But also $h(x)<g(x)$ for every $x\in X\setminus g^{-1}[A]$, so $h\prec g$, contrary to the choice of g. **X**

Thus $g[[\mathcal{F}]]$ is a normal filter, and the theorem is proved.

4A1L Theorem Let κ be a regular uncountable cardinal, and $\mathcal{F}$ a normal ultrafilter on κ. If $S\subseteq[\kappa]^{<\omega}$, there is a set $F\in\mathcal{F}$ such that, for each $n\in\mathbb{N}$, $[F]^n$ is either a subset of S or disjoint from S. (Jech 78, Theorem 70; Just & Weese 97, Corollary 23.18; Levy 79, IX.4.30. Compare 4A1G.)

4A1M Ostaszewski's ♣ This is the statement

> Let Ω be the family of non-zero countable limit ordinals. Then there is a family $\langle\theta_\xi(n)\rangle_{\xi\in\Omega,n\in\mathbb{N}}$ such that (i) for each $\xi\in\Omega$, $\langle\theta_\xi(n)\rangle_{n\in\mathbb{N}}$ is a strictly increasing sequence with supremum ξ (ii) for any uncountable $A\subseteq\omega_1$ there is a $\xi\in\Omega$ such that $\theta_\xi(n)\in A$ for every $n\in\mathbb{N}$.

This is an immediate consequence of Jensen's $\diamondsuit$ (Just & Weese 97, Exercise 22.9), which is itself a consequence of Gödel's Axiom of Constructibility (Kunen 80, §II.7; Jech 78, §22; Just & Weese 97, §22).

4A1N Lemma Assume ♣. Then there is a family $\langle C_\xi\rangle_{\xi<\omega_1}$ of sets such that (i) $C_\xi\subseteq\xi$ for every $\xi<\omega_1$ (ii) $C_\xi\cap\eta$ is finite whenever $\eta<\xi<\omega_1$ (iii) for any uncountable sets A, $B\subseteq\omega_1$ there is a $\xi<\omega_1$ such that $A\cap C_\xi$ and $B\cap C_\xi$ are both infinite.

proof (a) Let $\langle\theta_\xi(n)\rangle_{\xi\in\Omega,n\in\mathbb{N}}$ be a family as in 4A1M. Let $f:\omega_1\to[\omega_1]^2$ be a surjection (3A1Cd). For $\xi\in\Omega$, set

$$C_\xi=\textstyle\bigcup_{i\in\mathbb{N}}f(\theta_\xi(i+1))\cap\xi\setminus\theta_\xi(i).$$

Then $C_\xi\subseteq\xi$, and if $\eta<\xi$ there is some $n\in\mathbb{N}$ such that $\theta_\xi(n)\geq\eta$, so that

$$C_\xi\cap\eta\subseteq\textstyle\bigcup_{i\leq n}f(\theta_\xi(i))$$

is finite. For $\xi\in\omega_1\setminus\Omega$ set $C_\xi=\emptyset$. Then $\langle C_\xi\rangle_{\xi<\omega_1}$ satisfies (i) and (ii) above.

(b) Now suppose that A, $B\subseteq\omega_1$ are uncountable. Choose $\langle\alpha_\xi\rangle_{\xi<\omega_1}$, $\langle\beta_\xi\rangle_{\xi<\omega_1}$, $\langle I_\xi\rangle_{\xi<\omega_1}$ inductively, as follows. β_ξ is to be the smallest ordinal such that $\{\alpha_\eta:\eta<\xi\}\cup\bigcup_{\eta<\xi}I_\eta\subseteq\beta_\xi$; I_ξ is to be a doubleton subset of $\omega_1\setminus(\beta_\xi\cup\bigcup_{\eta\leq\beta_\xi}f(\eta))$ meeting both A and B; and $\alpha_\xi<\omega_1$ is to be such that $f(\alpha_\xi)=I_\xi$. Set $D=\{\alpha_\xi:\xi<\omega_1\}$. This construction ensures that $\langle\alpha_\xi\rangle_{\xi<\omega_1}$ and $\langle\beta_\xi\rangle_{\xi<\omega_1}$ are strictly increasing, with $\beta_\xi<\alpha_\xi<\beta_{\xi+1}$ for every ξ, so that $f(\delta)$ meets both A and B for every $\delta\in D$, while $f(\delta)\subseteq\delta'$ and $f(\delta')\cap f(\delta)=\emptyset$ whenever $\delta<\delta'$ in D.

By the choice of $\langle\theta_\xi(n)\rangle_{\xi\in\Omega,n\in\mathbb{N}}$, there is a $\xi\in\Omega$ such that $\theta_\xi(n)\in D$ for every $n\in\mathbb{N}$. But this means that

$$f(\theta_\xi(i))\subseteq\theta_\xi(i+1)\subseteq\xi,\quad f(\theta_\xi(i+1))\cap f(\theta_\xi(i))=\emptyset$$

for every $i\in\mathbb{N}$, so $C_\xi=\bigcup_{i\in\mathbb{N}}f(\theta_\xi(i))$ meets both A and B in infinite sets.

4A1O The size of σ-algebras: Proposition Let $\mathfrak{A}$ be a Boolean algebra, B a subset of $\mathfrak{A}$, and $\mathfrak{B}$ the σ-subalgebra of $\mathfrak{A}$ generated by B (331E). Then $\#(\mathfrak{B}) \leq \max(4, \#(B^{\mathbb{N}}))$. In particular, if $\#(B) \leq \mathfrak{c}$ then $\#(\mathfrak{B}) \leq \mathfrak{c}$.

proof (a) If $\#(B) \leq 1$, this is trivial, since then $\#(\mathfrak{B}) \leq 4$. So we need consider only the case $\#(B) \geq 2$.

(b) Set $\kappa = \#(B^{\mathbb{N}})$; then whenever $\#(A) \leq \kappa$, that is, there is an injection from A to $B^{\mathbb{N}}$, then

$$\#(A^{\mathbb{N}}) \leq \#((B^{\mathbb{N}})^{\mathbb{N}}) = \#(B^{\mathbb{N}\times\mathbb{N}}) = \#(B^{\mathbb{N}}) = \kappa.$$

As we are supposing that B has more than one element, $\kappa \geq \#(\{0,1\}^{\mathbb{N}}) = \mathfrak{c} \geq \omega_1$.

(c) Define $\langle B_\xi\rangle_{\xi<\omega_1}$ inductively, as follows. $B_0 = B \cup \{0\}$. Given $\langle B_\eta\rangle_{\eta<\xi}$, where $0 < \xi < \omega_1$, set $B'_\xi = \bigcup_{\eta<\xi} B_\eta$ and

$$\begin{aligned} B_\xi = &\{1 \setminus b : b \in B'_\xi\} \\ &\cup \{\sup_{n\in\mathbb{N}} b_n : \langle b_n\rangle_{n\in\mathbb{N}} \text{ is a sequence in } B'_\xi \text{ with a supremum in } \mathfrak{A}\}; \end{aligned}$$

continue.

An easy induction on ξ (relying on 3A1Cc and (b) above) shows that every B_ξ has cardinal at most κ. So $C = \bigcup_{\xi<\omega_1} B_\xi$ has cardinal at most κ.

(d) Now $\langle B_\xi\rangle_{\xi<\omega_1}$ is a non-decreasing family, so if $\langle c_n\rangle_{n\in\mathbb{N}}$ is any sequence in C there is some $\xi < \omega_1$ such that every c_n belongs to $B_\xi \subseteq B'_{\xi+1}$. But this means that if $\sup_{n\in\mathbb{N}} c_n$ is defined in $\mathfrak{A}$, it belongs to $B_{\xi+1} \subseteq C$. At the same time,

$$1 \setminus c_0 \in B_{\xi+1} \subseteq C.$$

This shows that C is closed under complementation and countable suprema; since it contains 0, it is a σ-subalgebra of $\mathfrak{A}$; since it includes B, it includes $\mathfrak{B}$, and $\#(\mathfrak{B}) \leq \#(C) \leq \kappa$, as claimed.

(d) Finally, if $\#(B) \leq \mathfrak{c} = \#(\{0,1\}^{\mathbb{N}})$, then

$$\#(\mathfrak{B}) \leq \kappa \leq \#(\{0,1\}^{\mathbb{N}\times\mathbb{N}}) = \mathfrak{c}.$$

4A1P An incidental fact If I is a countable set and $\epsilon > 0$, there is a family $\langle\epsilon_i\rangle_{i\in I}$ of strictly positive real numbers such that $\sum_{i\in I} \epsilon_i \leq \epsilon$. **P** Let $f : I \to \mathbb{N}$ be an injection and set $\epsilon_i = 2^{-f(i)-1}\epsilon$. **Q**

4A2 General topology

Even more than in previous volumes, naturally enough, the work of this volume depends on results from general topology. We have now reached the point where some of the facts I rely on are becoming hard to find as explicitly stated theorems in standard textbooks. I find myself therefore writing out rather a lot of proofs. You should not suppose that the results to which I attach proofs, rather than references, are particularly deep; on the contrary, in many cases I am merely spelling out solutions to classic exercises.

The style of 'general' topology, as it has evolved over the last ninety years, is to develop a language capable of squeezing the utmost from every step of argument. While this does sometimes lead to absurdly obscure formulations, it remains a natural, and often profitable, response to the remarkably dense network of related ideas in this area. I therefore follow the spirit of the subject in giving the results I need in the full generality achievable within the terminology I use. For the convenience of anyone coming to the theory for the first time, I repeat some of them in the forms in which they are actually applied. I should remark, however, that in some cases materially stronger results can be proved with little extra effort; as always, this appendix is to be thought of not as a substitute for a thorough study of the subject, but as a guide connecting standard approaches to the general theory with the special needs of this volume.

4A2A Definitions I begin the section with a glossary of terms not defined elsewhere.

Base of neighbourhoods If X is a topological space and $x \in X$, a **base of neighbourhoods** of x is a family $\mathcal{V}$ of neighbourhoods of x such that every neighbourhood of x includes some member of $\mathcal{V}$.

boundary If X is a topological space and $A \subseteq X$, the **boundary** of A is $\partial A = \overline{A} \setminus \operatorname{int} A$.

Čech-complete A completely regular Hausdorff topological space X is **Čech-complete** if it is homeomorphic to a G_δ subset of a compact Hausdorff space.

closed interval Let X be a totally ordered set. A **closed interval** in X is an interval of one of the forms $[x, y]$, $]-\infty, y]$, $[x, \infty[$ or $X =]-\infty, \infty[$ where x, $y \in X$ (see the definition of 'interval' below); for definiteness, let me say that I count $\emptyset$ as a 'closed interval'.

coarser topology If $\mathfrak{S}$ and $\mathfrak{T}$ are two topologies on a set X, we say that $\mathfrak{S}$ is **coarser** than $\mathfrak{T}$ if $\mathfrak{S} \subseteq \mathfrak{T}$.

compact support Let X be a topological space and $f : X \to \mathbb{R}$ a function. I say that f has **compact support** if $\overline{\{x : x \in X, f(x) \neq 0\}}$ is compact in X.

countably compact A topological space X is **countably compact** if every countable open cover of X has a finite subcover. (**Warning!** some authors reserve the term for Hausdorff spaces.) A subset of a topological space is countably compact if it is countably compact in its subspace topology.

countably paracompact A topological space X is **countably paracompact** if given any countable open cover $\mathcal{G}$ of X there is a locally finite family $\mathcal{H}$ of open sets which refines $\mathcal{G}$ and covers X.

countably tight A topological space X is **countably tight** (or has **countable tightness**) if whenever $A \subseteq X$ and $x \in \overline{A}$ there is a countable set $B \subseteq A$ such that $x \in \overline{B}$.

direct sum, disjoint union Let $\langle (X_i, \mathfrak{T}_i) \rangle_{i \in I}$ be a family of topological spaces, and set $X = \{(x, i) : i \in I, x \in X_i\}$. The **disjoint union topology** on X is $\mathfrak{T} = \{G : G \subseteq X, \{x : (x, i) \in G\} \in \mathfrak{T}_i \text{ for every } i \in I\}$; $(X, \mathfrak{T})$ is the **(direct) sum** of $\langle (X_i, \mathfrak{T}_i) \rangle_{i \in I}$.

If X is a set, $\langle X_i \rangle_{i \in I}$ a partition of X, and $\mathfrak{T}_i$ a topology on X_i for every $i \in I$, then the **disjoint union topology** or **sum topology** on X is $\{G : G \subseteq X, G \cap X_i \in \mathfrak{T}_i \text{ for every } i \in I\}$.

dyadic A Hausdorff space is **dyadic** if it is a continuous image of $\{0, 1\}^I$ for some set I.

finer topology If $\mathfrak{S}$ and $\mathfrak{T}$ are two topologies on a set X, we say that $\mathfrak{S}$ is **finer** than $\mathfrak{T}$ if $\mathfrak{S} \supseteq \mathfrak{T}$.

first-countable A topological space X is **first-countable** if every point has a countable base of neighbourhoods.

half-open Let X be a totally ordered set. A **half-open interval** in X is a set of one of the forms $[x, y[$, $]x, y]$ where x, $y \in X$ and $x < y$ (see the definition of 'interval' below).

hereditarily Lindelöf A topological space is **hereditarily Lindelöf** if every subspace is Lindelöf.

hereditarily metacompact A topological space is **hereditarily metacompact** if every subspace is metacompact.

indiscrete If X is any set, the **indiscrete** topology on X is the topology $\{\emptyset, X\}$.

interval Let $(X, \leq)$ be a totally ordered set. Adjoin points ∞, $-\infty$ to X and extend $\leq$ to a total order on $X \cup \{-\infty, \infty\}$ by declaring that $-\infty < x < \infty$ for every $x \in X$. For u, $v \in X \cup \{-\infty, \infty\}$, set $]u, v[= \{x : u < x < v\}$, $[u, v] = \{x : u \leq x \leq v\}$, $]u, v] = \{x : u < x \leq v\}$, $[u, v[= \{x : u \leq x < v\}$; call sets of these forms **intervals**. Note that if X is not Dedekind complete then not every set which is order-convex in the sense of 351Xb will be an 'interval' in this sense.

isolated If X is a topological space, a family $\mathcal{A}$ of subsets of X is **isolated** if $A \cap \overline{\bigcup(\mathcal{A} \setminus \{A\})}$ is empty for every $A \in \mathcal{A}$; that is, if $\mathcal{A}$ is disjoint and every member of $\mathcal{A}$ is a relatively open set in $\bigcup \mathcal{A}$.

Lindelöf A topological space is **Lindelöf** if every open cover has a countable subcover. (**Warning!** some authors reserve the term for regular spaces.)

locally finite If X is a topological space, a family $\mathcal{A}$ of subsets of X is **locally finite** if for every $x \in X$ there is an open set which contains x and meets only finitely many members of $\mathcal{A}$.

lower semi-continuous If X is a topological space and Y a totally ordered set, a function $f : X \to Y$ is **lower semi-continuous** if $\{x : f(x) > y\}$ is open for every $y \in Y$. (Cf. 225H, 3A3Cf.)

metacompact A topological space is **metacompact** if every open cover has a point-finite refinement which is an open cover. (**Warning!** some authors reserve the term for Hausdorff spaces.)

neighbourhood If X is a topological space and $x \in X$, a **neighbourhood** of x is any subset of X including an open set which contains x.

network Let $(X, \mathfrak{T})$ be a topological space. A **network** for $\mathfrak{T}$ is a family $\mathcal{E} \subseteq \mathcal{P}X$ such that whenever $x \in G \in \mathfrak{T}$ there is an $E \in \mathcal{E}$ such that $x \in E \subseteq G$.

normal A topological space X is **normal** if for any disjoint closed sets E, $F \subseteq X$ there are disjoint open sets G, H such that $E \subseteq G$ and $F \subseteq H$. (**Warning!** some authors reserve the term for Hausdorff spaces.)

open interval Let X be a totally ordered set. An **open interval** in X is a set of the form $]u, v[$ where u, $v \in X \cup \{-\infty, \infty\}$ (see the definition of 'interval' above).

open map If $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ are topological spaces, a function $f : X \to Y$ is **open** if $f[G] \in \mathfrak{S}$ for every $G \in \mathfrak{T}$.

order topology Let $(X, \leq)$ be a totally ordered set. Its **order topology** is that generated by sets of the form $]x, \infty[= \{y : y > x\}$, $]-\infty, x[= \{y : y < x\}$ as x runs over X.

paracompact A topological space is **paracompact** if every open cover has a locally finite refinement which is an open cover. (**Warning!** some authors reserve the term for Hausdorff spaces.)

perfectly normal A topological space is **perfectly normal** if it is normal and every closed set is a G_δ set. (**Warning!** remember that some authors reserve the term 'normal' for Hausdorff spaces.)

point-countable, point-finite A family $\mathcal{A}$ of sets is **point-countable** if no point belongs to more than countably many members of $\mathcal{A}$. Similarly, an indexed family $\langle A_i \rangle_{i \in I}$ of sets is **point-finite** if $\{i : x \in A_i\}$ is finite for every x.

Polish A topological space X is **Polish** if it is separable and its topology can be defined from a metric under which X is complete.

pseudometrizable A topological space $(X, \mathfrak{T})$ is **pseudometrizable** if $\mathfrak{T}$ is defined by a single pseudometric (2A3F).

refine(ment) If $\mathcal{A}$ is a family of sets, a **refinement** of $\mathcal{A}$ is a family $\mathcal{B}$ of sets such that every member of $\mathcal{B}$ is included in some member of $\mathcal{A}$; in this case I say that $\mathcal{B}$ **refines** $\mathcal{A}$. (**Warning!** I do not suppose that $\bigcup \mathcal{B} = \bigcup \mathcal{A}$.)

relatively countably compact If X is a topological space, a subset A of X is **relatively countably compact** if every sequence in A has a cluster point in X. (**Warning!** This is *not* the same as supposing that A is included in a countably compact subset of X.)

scattered A topological space X is **scattered** if every non-empty subset of X has an isolated point (in its subspace topology).

second-countable A topological space is **second-countable** if the topology has a countable base, that is, if its weight is at most ω.

semi-continuous *see* lower semi-continuous, upper semi-continuous.

separately continuous Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces with product X, and Y another topological space. Then a function $f : X \to Y$ is **separately continuous** if for every $j \in I$ and $z \in \prod_{i \in I \setminus \{j\}} X_i$ the function $t \mapsto f(z^\frown t) : X_j \to Y$ is continuous, where $z^\frown t$ is the member of X extending z and such that $(z^\frown t)(j) = t$.

sequential A topological space is **sequential** if every sequentially closed set in X is closed.

sequentially closed If X is a topological space, a subset A of X is **sequentially closed** if $x \in A$ whenever $\langle x_n \rangle_{n \in \mathbb{N}}$ is a sequence in A converging to $x \in X$.

sequentially compact A topological space is **sequentially compact** if every sequence has a convergent sequence. A subset of a topological space is sequentially compact if it is sequentially compact in its subspace topology. (**Warning!** some authors reserve the term for Hausdorff spaces.)

sequentially continuous If X and Y are topological spaces, a function $f : X \to Y$ is **sequentially continuous** if $\langle f(x_n) \rangle_{n \in \mathbb{N}} \to f(x)$ in Y whenever $\langle x_n \rangle_{n \in \mathbb{N}} \to x$ in X.

subbase If $(X, \mathfrak{T})$ is a topological space, a **subbase** for $\mathfrak{T}$ is a family $\mathcal{U} \subseteq \mathfrak{T}$ which generates $\mathfrak{T}$, in the sense that $\mathfrak{T}$ is the coarsest topology on X including $\mathcal{U}$. (**Warning!** most authors reserve the term for families $\mathcal{U}$ with union X.)

totally bounded If $(X, \mathcal{W})$ is a uniform space, a subset A of X is **totally bounded** if for every $W \in \mathcal{W}$ there is a finite set $I \subseteq X$ such that $A \subseteq W[I]$. If (X, ρ) is a metric space, a subset of X is totally bounded if it is totally bounded for the associated uniformity (3A4B).

uniform convergence If X is a set and $\mathcal{A}$ is a family of subsets of X then the **topology of uniform convergence** on members of $\mathcal{A}$ is the topology on $\mathbb{R}^X$ generated by the pseudometrics $(f, g) \mapsto \min(1, \sup_{x \in A} |f(x) - g(x)|)$ as A runs through $\mathcal{A} \setminus \{\emptyset\}$. (It is elementary to verify that the formula here defines a pseudometric.)

upper semi-continuous If X is a topological space, a function $f : X \to \mathbb{R}$ is **upper semi-continuous** if $\{x : f(x) < \alpha\}$ is open for every $\alpha \in \mathbb{R}$.

weakly α-favourable A topological space $(X,\mathfrak{T})$ is **weakly α-favourable** if there is a function $\sigma : \bigcup_{n\in\mathbb{N}}(\mathfrak{T}\setminus\{\emptyset\})^{n+1}\to\mathfrak{T}\setminus\{\emptyset\}$ such that (i) $\sigma(G_0,\dots,G_n)\subseteq G_n$ whenever $G_0,\dots,G_n$ are non-empty open sets (ii) whenever $\langle G_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{T}\setminus\{\emptyset\}$ such that $G_{n+1}\subseteq\sigma(G_0,\dots,G_n)$ for every n, then $\bigcap_{n\in\mathbb{N}}G_n$ is non-empty.

weight If X is a topological space, its **weight** $w(X)$ is the smallest cardinal of any base for the topology.

F_σ If X is a topological space, an $\mathbf{F}_\sigma$ set in X is one expressible as the union of a sequence of closed sets.

G_δ If X is a topological space, a $\mathbf{G}_\delta$ set in X is one expressible as the intersection of a sequence of open sets.

K_σ If X is a topological space, a $\mathbf{K}_\sigma$ set in X is one expressible as the union of a sequence of compact sets.

lim sup If $\langle A_i\rangle_{i\in I}$ is a non-empty family of sets in a topological space and $\mathcal{F}$ is a filter on I, I write $\limsup_{i\to\mathcal{F}}A_i$ for $\bigcap_{J\in\mathcal{F}}\overline{\bigcup_{i\in J}A_i}$.

$\mathcal{P}X$ If X is any set, the **usual topology** on $\mathcal{P}X$ is that generated by the sets $\{A : A\cap J = K\}$ where $J\subseteq X$ is finite and $K\subseteq J$.

T_0 If $(X,\mathfrak{T})$ is a topological space, we say that it is $\mathbf{T}_0$ if for any two distinct points of X there is an open set containing one but not the other.

T_1 If $(X,\mathfrak{T})$ is a topological space, we say that it is $\mathbf{T}_1$ if $\{x\}$ is a closed set for every $x\in X$.

π-base If $(X,\mathfrak{T})$ is a topological space, a **π-base** for $\mathfrak{T}$ is a set $\mathcal{U}\subseteq\mathfrak{T}$ such that every non-empty open set includes a non-empty member of $\mathcal{U}$.

σ-compact A topological space X is **σ-compact** if there is a sequence of compact sets covering X.

σ-disjoint A family $\mathcal{A}$ of sets is **σ-disjoint** if it is expressible as $\bigcup_{n\in\mathbb{N}}\mathcal{A}_n$ where every $\mathcal{A}_n$ is disjoint.

σ-isolated If X is a topological space, a family of subsets of X is **σ-isolated** if it is expressible as $\bigcup_{n\in\mathbb{N}}\mathcal{A}_n$ where every $\mathcal{A}_n$ is an isolated family.

σ-metrically-discrete If (X,ρ) is a metric space, a family $\mathcal{A}$ of subsets of X is **σ-metrically-discrete** if it is expressible as $\bigcup_{n\in\mathbb{N}}\mathcal{A}_n$ where $\rho(x,y)\geq 2^{-n}$ whenever $n\in\mathbb{N}$, A and B are distinct members of $\mathcal{A}_n$, $x\in A$ and $y\in B$.

4A2B Elementary facts about general topological spaces (a) Bases and networks (i) Let $(X,\mathfrak{T})$ be a topological space and $\mathcal{U}$ a subbase for $\mathfrak{T}$. Then $\{X\}\cup\{U_0\cap U_1\cap\ldots\cap U_n : U_0,\dots,U_n\in\mathcal{U}\}$ is a base for $\mathfrak{T}$. (For this is a base for a topology, by 3A3Mc.)

(ii) Let X and Y be topological spaces, and $\mathcal{V}$ a subbase for the topology of Y. Then a function $f : X\to Y$ is continuous iff $f^{-1}[V]$ is open for every $V\in\mathcal{V}$. (Engelking 89, 1.4.1(ii)).

(iii) If X and Y are topological spaces, $\mathcal{E}$ is a network for the topology of Y, and $f : X\to Y$ is a function such that $f^{-1}[E]$ is open for every $E\in\mathcal{E}$, then f is continuous. (The topology generated by $\mathcal{E}$ includes the given topology on Y.)

(iv) If $\mathcal{U}$ is a (sub-)base for a topology on X, and $Y\subseteq X$, then $\{Y\cap U : U\in\mathcal{U}\}$ is a (sub-)base for the subspace topology of Y. (Császár 78, 2.3.13(e)-(f).)

(v) If X is a topological space and $\mathcal{U}$ is a subbase for the topology of X, then a filter $\mathcal{F}$ on X converges to $x\in X$ iff $\{U : x\in U\in\mathcal{U}\}\subseteq\mathcal{F}$. (If the condition is satisfied, $\mathcal{F}\cup\{A : A\subseteq X,\ x\notin A\}$ is a topology on X including $\mathcal{U}$.)

(vi) If X and Y are topological spaces with subbases $\mathcal{U}$, $\mathcal{V}$ respectively, then $\{U\times Y : U\in\mathcal{U}\}\cup\{X\times V : V\in\mathcal{V}\}$ is a subbase for the product topology of $X\times Y$. (Kuratowski 66, §15.I.)

(vii) If X is a topological space, $\mathcal{E}$ is a network for the topology of X, and Y is a subset of X, then $\{E\cap Y : E\in\mathcal{E}\}$ is a network for the topology of Y.

(viii) If X is a topological space and $\mathcal{A}$ is a (σ-)isolated family of subsets of X, then $\{A\cap Y : A\in\mathcal{A}'\}$ is (σ-)isolated whenever $Y\subseteq X$ and $\mathcal{A}'\subseteq\mathcal{A}$.

(ix) If a topological space X has a σ-isolated network, so does every subspace of X.

(b) If $\langle H_i\rangle_{i\in I}$ is a partition of a topological space X into open sets and $F_i\subseteq H_i$ is closed (either in X or in H_i) for each $i\in I$, then $F=\bigcup_{i\in I}F_i$ is closed in X. ($X\setminus F=\bigcup_{i\in I}(H_i\setminus F_i)$.)

(c) If X is a topological space, $A\subseteq X$ and $x\in X$, then $x\in\overline{A}$ iff there is an ultrafilter on X, containing A, which converges to x. ($\{A\}\cup\{G : x\in G\subseteq X,\ G\text{ is open}\}$ has the finite intersection property; use 4A1Ia.)

(d) Semi-continuity Let X be a topological space.

(i) A function $f : X \to \mathbb{R}$ is lower semi-continuous iff $-f$ is upper semi-continuous. (Čech 66, 18D.8.)

(ii) If $f : X \to \mathbb{R}$ is lower semi-continuous, Y is another topological space, and $g : Y \to X$ is continuous, then $fg : Y \to \mathbb{R}$ is lower semi-continuous. ($\{y : (fg)(y) > \alpha\} = g^{-1}[\{x : f(x) > \alpha\}]$.) Similarly, if $f : X \to \mathbb{R}$ is upper semi-continuous and $g : Y \to X$ is continuous, then $fg : Y \to \mathbb{R}$ is upper semi-continuous.

(iii) If $f, g : X \to]-\infty, \infty]$ are lower semi-continuous so is $f + g : X \to]-\infty, \infty]$. (Čech 66, 18D.8.)

(iv) If $f, g : X \to [0, \infty]$ are lower semi-continuous so is $f \times g : X \to [0, \infty]$. (Čech 66, 18D.8.)

(v) If Φ is any non-empty set of lower semi-continuous functions from X to $[-\infty, \infty]$, then $x \mapsto \sup_{f \in \Phi} f(x) : X \to [-\infty, \infty]$ is lower semi-continuous.

(vi) $f : X \to \mathbb{R}$ is continuous iff f is both upper semi-continuous and lower semi-continuous iff f and $-f$ are both lower semi-continuous.

(e) Separable spaces (i) If $\langle A_i \rangle_{i \in I}$ is a countable family of separable subsets of a topological space X then $\overline{\bigcup_{i \in I} A_i}$ is separable. (If $D_i \subseteq A_i$ is countable and dense for each i, $\bigcup_{i \in I} D_i$ is countable and dense in $\bigcup_{i \in I} A_i$.)

(ii) If $\langle X_i \rangle_{i \in I}$ is a family of separable topological spaces and $\#(I) \leq \mathfrak{c}$, then $\prod_{i \in I} X_i$ is separable. (Engelking 89, 2.3.16.)

(iii) A continuous image of a separable topological space is separable. (Engelking 89, 1.4.11.)

(f)(i) Let $\langle X_i \rangle_{i \in I}$ be any family of topological spaces, with product X. If $J \subseteq I$ is any set, and we write X_J for $\prod_{i \in J} X_i$, then the canonical map $x \mapsto x{\restriction}J : X \to X_J$ is open. (Engelking 89, p. 79.)

(ii) Let X and Y be topological spaces and $f : X \to Y$ a continuous open map. Then $\operatorname{int} f^{-1}[B] = f^{-1}[\operatorname{int} B]$ and $\overline{f^{-1}[B]} = f^{-1}[\overline{B}]$ for every $B \subseteq Y$. **P** Because f is continuous, $f^{-1}[\operatorname{int} B]$ is an open set included in $f^{-1}[B]$, so is included in $\operatorname{int} f^{-1}[B]$. Because f is open, $f[\operatorname{int} f^{-1}[B]]$ is an open set included in $f[f^{-1}[B]] \subseteq B$, so $f[\operatorname{int} f^{-1}[B]] \subseteq \operatorname{int} B$, that is, $\operatorname{int} f^{-1}[B] \subseteq f^{-1}[\operatorname{int} B]$. Now apply this to $Y \setminus B$ and take complements. **Q**

(g) Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces with product X.

(i) If $A \subseteq X$ is determined by coordinates in $J \subseteq I$ in the sense of 254M, then $\overline{A}$ and $\operatorname{int} A$ are also determined by coordinates in J. **P** Let $\pi : X \to \prod_{i \in J} X_i$ be the canonical map. Then $A = \pi^{-1}[\pi[A]]$, so (f) tells us that $\operatorname{int} A = \pi^{-1}[\operatorname{int} \pi[A]]$ and $\overline{A} = \pi^{-1}[\overline{\pi[A]}]$; but these are both determined by coordinates in J. **Q**

(ii) If $F \subseteq X$ is closed, there is a smallest set $J^* \subseteq I$ such that F is determined by coordinates in J^*. **P** Let $\mathcal{J}$ be the family of all those sets $J \subseteq I$ such that F is determined by coordinates in $\mathcal{J}$. If J_1, $J_2 \in \mathcal{J}$, then $J_1 \cap J_2 \in \mathcal{J}$ (254Ta). Set $J^* = \bigcap \mathcal{J}$. **?** Suppose, if possible, that F is not determined by coordinates in J^*. Then there are $x \in F$, $y \in X \setminus F$ such that $x{\restriction}J^* = y{\restriction}J^*$. Because $X \setminus F$ is open, there is a finite set $K \subseteq I$ such that $z \notin F$ whenever $z \in X$ and $z{\restriction}K = y{\restriction}K$. Because $\mathcal{J}$ is closed under finite intersections, there is a $J \in \mathcal{J}$ such that $K \cap J = K \cap J^*$. Define $z \in X$ by setting $z(i) = x(i)$ for $i \in J$, $z(i) = y(i)$ for $i \in I \setminus J$. Then $z{\restriction}J = x{\restriction}J$, so $z \in F$, but $z{\restriction}K = y{\restriction}K$, so $z \notin F$. **X**

Thus $J^* \in \mathcal{J}$ and is the required smallest member of $\mathcal{J}$. **Q**

(h) Let X be a topological space.

(i) If $\mathcal{E}$ is a locally finite family of closed subsets of X, then $\bigcup \mathcal{E}'$ is closed for every $\mathcal{E}' \subseteq \mathcal{E}$. (Engelking 89, 1.1.11.)

(ii) If $\langle f_i \rangle_{i \in I}$ is a family in $C(X)$ such that $\langle \{x : f_i(x) \neq 0\} \rangle_{i \in I}$ is locally finite, then we have a continuous function $f : X \to \mathbb{R}$ defined by setting $f(x) = \sum_{i \in I} f_i(x)$ for every $x \in X$. **P** For any x, $\{i : f_i(x) \neq 0\}$ is finite, so f is well-defined. If $x_0 \in X$ and $\epsilon > 0$, there is a neighbourhood V of x_0 such that $J = \{i : i \in I, f_i(x) \neq 0 \text{ for some } x \in V\}$ is finite; now there is a neighbourhood W of x_0, included in V, such that $\sum_{i \in J} |f_i(x) - \sum_{i \in J} f_i(x_0)| < \epsilon$ for every $x \in W$, so that $|f(x) - f(x_0)| < \epsilon$ for every $x \in W$. As x_0 and ϵ are arbitrary, f is continuous. **Q**

(i) Let X be a topological space and A, B two subsets of X. Then $\partial(A * B) \subseteq \partial A \cup \partial B$, where $*$ is any of $\cup$, $\cap$, $\setminus$, $\triangle$. (Generally, if $F \subseteq X$, $\{A : \partial A \subseteq F\} = \{A : \overline{A} \setminus F \subseteq \operatorname{int} A\}$ is a subalgebra of $\mathcal{P}X$.)

4A2C G_δ, F_σ, zero and cozero sets Let X be a topological space.

(a) (i) The union of two G_δ sets in X is a G_δ set. (Engelking 89, p. 26; Kuratowski 66, §5.V.)

(ii) The intersection of countably many G_δ sets is a G_δ set. (Engelking 89, p. 26; Kuratowski 66, §5.V.)

(iii) If Y is another topological space, $f : X \to Y$ is continuous and $E \subseteq Y$ is G_δ in Y, then $f^{-1}[E]$ is G_δ in X. ($f^{-1}[\bigcap_{n\in\mathbb{N}} H_n] = \bigcap_{n\in\mathbb{N}} f^{-1}[H_n]$.)

(iv) If Y is a G_δ set in X and $Z \subseteq Y$ is a G_δ set for the subspace topology of Y, then Z is a G_δ set in X. (Kuratowski 66, §5.V.)

(v) A set $E \subseteq X$ is an F_σ set iff $X \setminus E$ is a G_δ set. (Kuratowski 66, §5.V.)

(b) (i) A zero set is closed. A cozero set is open.

(ii) The union of two zero sets is a zero set. (Császár 78, 4.2.36.) The intersection of two cozero sets is a cozero set.

(iii) The intersection of a sequence of zero sets is a zero set. (If $f_n : X \to \mathbb{R}$ is continuous for each n, $x \mapsto \sum_{n=0}^{\infty} \min(2^{-n}, |f_n(x)|)$ is continuous.) The union of a sequence of cozero sets is a cozero set.

(iv) If Y is another topological space, $f : X \to Y$ is continuous and $L \subseteq Y$ is a zero set, then $f^{-1}[L]$ is a zero set. (Čech 66, 28B.3.) If $f : X \to Y$ is continuous and $L \subseteq Y$ is a cozero set, then $f^{-1}[L]$ is a cozero set. (Čech 66, 28B.3.) If $K \subseteq X$, $L \subseteq Y$ are zero sets then $K \times L$ is a zero set in $X \times Y$. ($K \times L = \pi_1^{-1}[K] \cap \pi_2^{-1}[L]$.)

(v) If $H \subseteq X$ is a (co-)zero set and $Y \subseteq X$, then $H \cap Y$ is a (co-)zero set in Y. (Use (iv).)

(vi) A cozero set is the union of a non-decreasing sequence of zero sets. (If $f : X \to \mathbb{R}$ is continuous, $X \setminus f^{-1}[\{0\}]) = \bigcup_{n\in\mathbb{N}} g_n^{-1}[\{0\}]$, where $g_n(x) = \max(0, 2^{-n} - |f(x)|)$.) In particular, a cozero set is an F_σ set; taking complements, a zero set is a G_δ set.

(vii) If $\mathcal{G}$ is a partition of X into open sets, and $H \subseteq X$ is such that $H \cap G$ is a cozero set in G for every $G \in \mathcal{G}$, then H is a cozero set in X. (If $f_G : G \to \mathbb{R}$ is continuous for every $G \in \mathcal{G}$, then $f : X \to \mathbb{R}$ is continuous, where $f(x) = f_G(x)$ for $x \in G \in \mathcal{G}$.) Similarly, if $F \subseteq X$ is such that $F \cap G$ is a zero set in G for every $G \in \mathcal{G}$, then F is a zero set in X.

4A2D Weight Let X be a topological space.

(a) $w(Y) \leq w(X)$ for every subspace Y of X (4A2B(a-iv)).

(b) A disjoint family $\mathcal{G}$ of non-empty open sets in X has cardinal at most $w(X)$. (If $\mathcal{U}$ is a base for the topology of X, then every non-empty member of $\mathcal{G}$ includes a non-empty member of $\mathcal{U}$, so we have an injective function from $\mathcal{G}$ to $\mathcal{U}$.)

(c) A point-countable family $\mathcal{G}$ of open sets in X has cardinal at most $\max(\omega, w(X))$. **P** If $X = \emptyset$, this is trivial. Otherwise, let $\mathcal{U}$ be a base for the topology of X with $\#(\mathcal{U}) = w(X) > 0$. Choose a function $f : \mathcal{G} \to \mathcal{U}$ such that $\emptyset \neq f(G) \subseteq G$ whenever $G \in \mathcal{G} \setminus \{\emptyset\}$. Then $\mathcal{G}_U = \{G : f(G) = U\}$ is countable for every $U \in \mathcal{U}$, so there is an injection $h_U : \mathcal{G}_U \to \mathbb{N}$; now $G \mapsto (f(G), h_{f(G)}(G)) : \mathcal{G} \to \mathcal{U} \times \mathbb{N}$ is injective, so $\#(\mathcal{G}) \leq \#(\mathcal{U} \times \mathbb{N}) = \max(\omega, w(X))$. **Q**

(d) If X is a dyadic space then X is a continuous image of $\{0,1\}^{w(X)}$. **P** By definition, X is Hausdorff and there are a set I and a continuous surjection $f : \{0,1\}^I \to X$; because any power of $\{0,1\}$ is compact, so is X. If $w(X)$ is finite, $\#(X) = w(X) \leq \#(\{0,1\}^{w(X)})$ and the result is trivial; so we may suppose that $w(X)$ is infinite. Let $\mathcal{U}$ be a base for the topology of X with cardinality $w(X)$. Set $Z = \{0,1\}^I$ and let $\mathcal{E}$ be the algebra of subsets of Z determined by coordinates in finite sets, so that $\mathcal{E}$ is an algebra of subsets of Z and is a base for the topology of Z. For each pair U, V of members of $\mathcal{U}$ such that $\overline{U} \subseteq V$, $f^{-1}[V] \subseteq Z$ is open; the set $\{E : E \in \mathcal{E},\ E \subseteq f^{-1}[V]\}$ is upwards-directed and covers the compact set $f^{-1}[\overline{U}]$, so there is an $E_{UV} \in \mathcal{E}$ such that $f^{-1}[\overline{U}] \subseteq E_{UV} \subseteq f^{-1}[V]$. Let $J \subseteq I$ be a set of cardinal at most $\max(\omega, w(X))$ such that every E_{UV} is determined by coordinates in J. Fix any $w \in \{0,1\}^{I\setminus J}$ and define $g : \{0,1\}^J \to X$ by setting $g(z) = f(z, w)$ for every $z \in \{0,1\}^J$, identifying Z with $\{0,1\}^J \times \{0,1\}^{I\setminus J}$. Then g is continuous. **?** If g is not surjective, set $H = X \setminus g[\{0,1\}^J]$. Take $x \in H$; take $V \in \mathcal{U}$ such that $x \in V \subseteq H$; take an open set G such that $x \in G \subseteq \overline{G} \subseteq V$ (this must be possible because X, being

compact and Hausdorff, is regular); take $U \in \mathcal{U}$ such that $x \in U \subseteq G$, so that $x \in U \subseteq \overline{U} \subseteq V$. Because f is surjective, there is a $(u,v) \in \{0,1\}^J \times \{0,1\}^{I \setminus J}$ such that $f(u,v) = x$. Now $(u,v) \in f^{-1}[U] \subseteq E_{UV}$; as E_{UV} is determined by coordinates in J, $(u,w) \in E_{UV} \subseteq f^{-1}[V]$ and $g(u) = f(u,w) \in V$; but V is supposed to be disjoint from $g[\{0,1\}^J]$. **X** So g is surjective, and X is a continuous image of $\{0,1\}^J$. Since $\#(J) \leq \max(\omega, w(X)) = w(X)$, $\{0,1\}^J$ and X are continuous images of $\{0,1\}^{w(X)}$. **Q**

(e) If X is a dyadic space then X is separable iff it is a continuous image of $\{0,1\}^{\mathfrak{c}}$. **P** $\{0,1\}^{\mathfrak{c}}$ is separable (4A2B(e-ii)), so any continuous image of it is separable. If X is a separable dyadic space, let $A \subseteq X$ be a countable dense set. If G, $G' \subseteq X$ are distinct regular open sets, then $G \cap A \neq G \cap A'$. Thus X has at most $\mathfrak{c}$ regular open sets; since X is compact and Hausdorff, therefore regular, its regular open sets form a base, and $w(X) \leq \mathfrak{c}$. By (d), X is a continuous image of $\{0,1\}^{\max(\omega, w(X))}$ which is in turn a continuous image of $\{0,1\}^{\mathfrak{c}}$. **Q**

4A2E The countable chain condition (a)(i) Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces. If $X_J = \prod_{i \in J} X_i$ is ccc for every finite $J \subseteq I$, then $X = \prod_{i \in I} X_i$ is ccc. (Kunen 80, II.1.9; Fremlin 84, 12I.)

(ii) A separable topological space is ccc. (If D is a countable dense set and $\mathcal{G}$ is a disjoint family of non-empty open sets, we have a surjection from a subset of D onto $\mathcal{G}$.)

(iii) The product of any family of separable topological spaces is ccc. **P** By 4A2B(e-ii) and (ii) here, the product of finitely many separable spaces is separable, therefore ccc; so we can apply (i). **Q**

(b) Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces, and suppose that $X = \prod_{i \in I} X_i$ is ccc. For $J \subseteq I$ and $x \in X$ set $X_J = \prod_{i \in J} X_i$, $\pi_J(x) = x \restriction J$.

(i) If $G \subseteq X$ is open, there is an open set $W \subseteq G$ determined by coordinates in a countable subset of I such that $G \subseteq \overline{W}$. **P** Let $\mathcal{W}$ be the family of subsets of X determined by coordinates in countable sets. Then $\mathcal{W}$ is a σ-algebra (254Mb) including the standard base $\mathcal{U}$ for the topology of X. Let $\mathcal{U}_0$ be a maximal disjoint family in $\{U : U \in \mathcal{U}, U \subseteq G\}$. Then $\mathcal{U}_0$ is countable, so $W = \bigcup \mathcal{U}_0$ belongs to $\mathcal{W}$. No member of $\mathcal{U}$ can be included in $G \setminus W$, so $G \setminus \overline{W}$ must be empty, and we have a suitable set. **Q** So $\overline{G}$ and $\operatorname{int} \overline{G}$ are determined by coordinates in a countable set (4A2B(g-ii)); in particular, if G is a regular open set, then it is determined by coordinates in a countable set.

(ii) If $f : X \to \mathbb{R}$ is continuous, there are a countable set $J \subseteq I$ and a continuous function $g : X_J \to \mathbb{R}$ such that $f = g\pi_J$. **P** For each $q \in \mathbb{Q}$, set $F_q = \overline{\{x : f(x) < q\}}$. By (i), F_q is determined by coordinates in a countable set. Because $\mathbb{Q}$ is countable, there is a countable $J \subseteq I$ such that every F_q is determined by coordinates in J. Also $\{x : f(x) < \alpha\} = \bigcup_{q \in \mathbb{Q}, q < \alpha} F_q$ is determined by coordinates in J for every $\alpha \in \mathbb{R}$, so $f(x) = f(y)$ whenever $x \restriction J = y \restriction J$, and there is a $g : X_J \to \mathbb{R}$ such that $f = g\pi_J$. Now if $H \subseteq \mathbb{R}$ is open, $g^{-1}[H] = \pi_J[f^{-1}[H]]$ is open, so g is continuous. **Q**

4A2F Separation axioms (a) Hausdorff spaces (i) A Hausdorff space is T_1. (Čech 66, 27A.1.) Any subspace of a Hausdorff space is Hausdorff. (Engelking 89, 2.1.6; Čech 66, 27A.3; Kuratowski 66, §5.VIII; Császár 78, 2.5.21.)

(ii) If X is a Hausdorff space and $\langle x_n \rangle_{n \in \mathbb{N}}$ is a sequence in X, then a point x of X is a cluster point of $\langle x_n \rangle_{n \in \mathbb{N}}$ iff there is a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$ such that $x = \lim_{n \to \mathcal{F}} x_n$. (If x is a cluster point of $\langle x_n \rangle_{n \in \mathbb{N}}$, apply 4A1Ia to $\{\{n : n \geq n_0, x_n \in G\} : n_0 \in \mathbb{N}, G \subseteq X \text{ is open}, x \in G\}$.)

(b) Regular spaces A regular T_1 space is Hausdorff. (Čech 66, 27B.7; Gaal 64, p. 81.) Any subspace of a regular space is regular. (Engelking 89, 2.1.6; Kuratowski 66, §14.I.)

(c) Completely regular spaces In a completely regular space, the cozero sets form a base for the topology. (Čech 66, 28B.5.)

(d) Normal spaces (i) **Urysohn's Lemma** If X is normal and E, F are disjoint closed subsets of X, then there is a continuous function $f : X \to [0,1]$ such that $f(x) = 0$ for $x \in E$ and $f(x) = 1$ for $x \in F$. (Engelking 89, 1.5.11; Kuratowski 66, §14.IV.)

(ii) A regular normal space is completely regular.

(iii) A normal T_1 space is Hausdorff (Gaal 64, p. 86) and completely regular (Császár 78, 4.2.5; Gaal 64, p. 110).

(iv) If X is normal and E, F are disjoint closed sets in X there is a zero set including E and disjoint from F. (Take a continuous function f which is zero on E and 1 on F, and set $Z = \{x : \max(0, f(x) - \frac{1}{2}) = 0\}$.)

(v) In a normal space a closed G_δ set is a zero set. (Engelking 89, 1.5.12.)

(vi) If X is a normal space and $\langle G_i\rangle_{i\in I}$ is a point-finite cover of X by open sets, there is a family $\langle H_i\rangle_{i\in I}$ of open sets, still covering X, such that $\overline{H}_i \subseteq G_i$ for every i. (Engelking 89, 1.5.18; Čech 66, 29C.1; Gaal 64, p. 89.)

(vii) If X is a normal space and $\langle G_i\rangle_{i\in I}$ is a point-finite cover of X by open sets, there is a family $\langle H'_i\rangle_{i\in I}$ of cozero sets, still covering X, such that $H'_i \subseteq G_i$ for every i. (Take $\langle H_i\rangle_{i\in I}$ from (vi), and apply (iv) to the disjoint closed sets $\overline{H}_i$, $X \setminus G_i$ to find a suitable cozero set H'_i for each i.)

(viii) **Tietze's theorem** Let X be a normal space, F a closed subset of X and $f : F \to \mathbb{R}$ a continuous function. Then there is a continuous function $g : X \to \mathbb{R}$ extending f. (Engelking 89, 2.1.8; Kuratowski 66, §14.IV; Gaal 64, p. 203.) It follows that if $F \subseteq X$ is closed and $f : F \to [0,1]^I$ is a continuous function from F to any power of the unit interval, there is a continuous function from X to $[0,1]^I$ extending f. (Extend each of the functionals $x \mapsto f(x)(i)$ for $i \in I$.)

(ix) If X is a normal space and $\langle G_i\rangle_{i\in I}$ is a locally finite cover of X by open sets, there is a family $\langle g_i\rangle_{i\in I}$ of continuous functions from X to $[0,1]$ such that $g_i \leq \chi G_i$ for every $i \in I$ and $\sum_{i\in I} g_i(x) = 1$ for every $x \in X$. (Engelking 89, proof of 5.1.9.)

(e) Paracompact spaces A Hausdorff paracompact space is regular. (Engelking 89, 5.1.5.) A regular paracompact space is normal. (Engelking 89, 5.1.5; Gaal 64, p. 160.)

(f) Countably paracompact spaces A normal space X is countably paracompact iff whenever $\langle F_n\rangle_{n\in\mathbb{N}}$ is a sequence of closed subsets of X with empty intersection, there is a sequence $\langle G_n\rangle_{n\in\mathbb{N}}$ of open sets, also with empty intersection, such that $F_n \subseteq G_n$ for every $n \in \mathbb{N}$. (Engelking 89, 5.2.2; Császár 78, 8.3.56(f).)

(g) Metacompact spaces (i) A paracompact space is metacompact.

(ii) A closed subspace of a metacompact space is metacompact.

(iii) A normal metacompact space is countably paracompact. (Engelking 89, 5.2.6; Császár 78, 8.3.56(c).)

(h) Separating compact sets (i) If X is a Hausdorff space and K and L are disjoint compact subsets of X, there are disjoint open sets G, $H \subseteq X$ such that $K \subseteq G$ and $L \subseteq H$. (Császár 78, 5.3.18.)

(ii) If X is a regular space, $F \subseteq X$ is closed, and $K \subseteq X \setminus F$ is compact, there are disjoint open sets G, $H \subseteq X$ such that $K \subseteq G$ and $F \subseteq H$. (Engelking 89, 3.1.6.)

(iii) If X is a completely regular space, $G \subseteq X$ is open and $K \subseteq G$ is compact, there is a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ for $x \in K$ and $f(x) = 0$ for $x \in X \setminus G$. **P** For each $x \in K$ there is a continuous function $f_x : X \to [0,1]$ such that $f_x(x) = 1$ and $f_x(y) = 0$ for $y \in X \setminus G$. Set $H_x = \{y : f_x(y) > \frac{1}{2}\}$. Then $\bigcup_{x\in K} H_x \supseteq K$, so there is a finite set $I \subseteq K$ such that $K \subseteq \bigcup_{x\in I} H_x$. Set $f(y) = \min(1, 2\sum_{x\in I} f_x(y))$ for $y \in X$. **Q**

(iv) If X is a completely regular Hausdorff space and K and L are disjoint compact subsets of X, there are disjoint cozero sets G, $H \subseteq X$ such that $K \subseteq G$ and $L \subseteq H$. **P** By (i), there are disjoint open sets G', H' such that $K \subseteq G'$ and $L \subseteq H'$. By (iii), there is a continuous function $f : X \to [0,1]$ such that $f(x) = 1$ for $x \in K$ and $f(x) = 0$ for $x \in X \setminus G'$; set $G = \{x : f(x) \neq 0\}$, so that G is a cozero set and $K \subseteq G \subseteq G'$. Similarly there is a cozero set H including L and included in H'. **Q**

(v) If X is a completely regular space and $K \subseteq X$ is a compact G_δ set, then K is a zero set. **P** Let $\langle G_n\rangle_{n\in\mathbb{N}}$ be a sequence of open sets with intersection K. For each $n \in \mathbb{N}$ there is a continuous function $f_n : X \to [0,1]$ such that $f_n(x) = 1$ for $x \in K$, $f_n(x) = 0$ for $x \in X \setminus G_n$, by (iii). Now $K = \bigcap_{n\in\mathbb{N}}\{x : 1 - f_n(x) = 0\}$ is a zero set, by 4A2C(b-iii). **Q**

(vi) If $\langle X_n\rangle_{n\in\mathbb{N}}$ is a sequence of topological spaces with product X, $K \subseteq X$ is compact, $F \subseteq X$ is closed and $K \cap F = \emptyset$, there is some $n \in \mathbb{N}$ such that $x{\restriction}n \neq y{\restriction}n$ for any $x \in F$, $y \in K$. **P?** Otherwise, we can find for each $n \in \mathbb{N}$ points $x_n \in F$, $y_n \in K$ such that $x_n{\restriction}n = y_n{\restriction}n$. Let $y \in K$ be a cluster point of $\langle y_n\rangle_{n\in\mathbb{N}}$ (2A3Ob). Then $y \notin F$, so there must be an $m \in \mathbb{N}$ and open sets $G_i \subseteq X_i$, for $i \leq m$, such that $y \in G \subseteq X \setminus F$, where $G = \{z : z \in X, z(i) \in G_i \text{ for every } i \leq m\}$. Now there is an $n > m$ such that $y_n \in G$; but in this case $x_n \in G$ and $x_n \notin F$, which is impossible. **XQ**

(vii) If X is a compact Hausdorff space, $f : X \to \mathbb{R}$ is continuous, and $\mathcal{U}$ is a subbase for $\mathfrak{T}$, then there is a countable set $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $f(x) = f(y)$ whenever $\{U : x \in U \in \mathcal{U}\} = \{U : y \in U \in \mathcal{U}\}$. (Apply (i) to sets of the form $K = \{x : f(x) \leq \alpha\}$, $L = \{x : f(x) \geq \beta\}$.)

(i) Perfectly normal spaces A topological space X is perfectly normal iff every closed set is a zero set. (Engelking 89, 1.4.9.)

Consequently, every open set in a perfectly normal space is a cozero set (and, of course, an F_σ set).

4A2G Compact and locally compact spaces (a) In any topological space, the union of two compact subsets is compact.

(b) A compact Hausdorff space is normal. (Engelking 89, 3.1.9; Császár 78, 5.3.23; Gaal 64, p. 139.)

(c) If X is a compact Hausdorff space, $Y \subseteq X$ is a zero set and $Z \subseteq Y$ is a zero set in Y, then Z is a zero set in X. (By 4A2C(b-vi) and 4A2C(a-iv), Z is a G_δ set in X; now use 4A2F(d-v).)

(d) If X is a Hausdorff space, $\mathcal{V}$ is a downwards-directed family of compact neighbourhoods of a point x of X and $\bigcap \mathcal{V} = \{x\}$, then $\mathcal{V}$ is a base of neighbourhoods of $\{x\}$. **P** Let G be any open set containing x. Fix any $V_0 \in \mathcal{V}$. Note that because X is Hausdorff, every member of $\mathcal{V}$ is closed (3A3Dc). So $\{V_0 \cap V \setminus G : V \in \mathcal{V}\}$ is a family of (relatively) closed subsets of V_0 with empty intersection, cannot have the finite intersection property (3A3Da), and there is a $V \in \mathcal{V}$ such that $V_0 \cap V \setminus G = \emptyset$. Now there is a $V' \in \mathcal{V}$ such that $V' \subseteq V_0 \cap V$ and $V' \subseteq G$. As G is arbitrary, $\mathcal{V}$ is a base of neighbourhoods of x. **Q**

(e) Let $(X, \mathfrak{T})$ be a locally compact Hausdorff space.

(i) If $K \subseteq X$ is a compact set and $G \supseteq K$ is open, then there is a continuous $f : X \to [0, 1]$ with compact support such that $\chi K \leq f \leq \chi G$. (Let $\mathcal{V}$ be the family of relatively compact open subsets of X. Then $\mathcal{V}$ is upwards-directed and covers X, so there is a $V \in \mathcal{V}$ including K. By 3A3Bb, $\mathfrak{T}$ is completely regular; now 4A2F(h-iii) tells us that there is a continuous $f : X \to [0, 1]$ such that $\chi K \leq f \leq \chi(G \cap V)$, so that f has compact support.)

(ii) $\mathfrak{T}$ is the coarsest topology on X such that every $\mathfrak{T}$-continuous real-valued function with compact support is continuous. **P** Let Φ be the set of continuous functions of compact support for $\mathfrak{T}$. If $\mathfrak{S}$ is a topology on X such that every member of Φ is continuous, and $x \in G \in \mathfrak{T}$, then there is an $f \in \Phi$ such that $f(x) = 1$ and $f(y) = 0$ for $y \in X \setminus G$, by (i). Now f is $\mathfrak{S}$-continuous, by hypothesis, so $H = \{y : f(y) > \frac{1}{2}\}$ belongs to $\mathfrak{S}$ and $x \in H \subseteq G$. As x is arbitrary, $G = \mathrm{int}_{\mathfrak{S}} G$ belongs to $\mathfrak{S}$; as G is arbitrary, $\mathfrak{T} \subseteq \mathfrak{S}$. **Q**

(f) (i) A topological space X is countably compact iff every sequence in X has a cluster point in X, that is, X is relatively countably compact in itself. (Engelking 89, 3.10.3; Császár 78, 5.3.31(e); Gaal 64, p. 129.)

(ii) If X is a countably compact topological space and $\langle F_n \rangle_{n \in \mathbb{N}}$ is a sequence of closed sets such that $\bigcap_{i \leq n} F_i \neq \emptyset$ for every $n \in \mathbb{N}$, then $\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$. (Engelking 89, 3.10.3; Császár 78, 5.3.31(c).)

(iii) In any topological space, a relatively compact set is relatively countably compact (2A3Ob).

(iv) Let X and Y be topological spaces and $f : X \to Y$ a continuous function. If $A \subseteq X$ is relatively countably compact in X, then $f[A]$ is relatively countably compact in Y. **P** Let $\langle y_n \rangle_{n \in \mathbb{N}}$ be a sequence in $f[A]$. Then there is a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in A such that $f(x_n) = y_n$ for every $n \in \mathbb{N}$. Because A is relatively countably compact, $\langle x_n \rangle_{n \in \mathbb{N}}$ has a cluster point $x \in X$. If $n_0 \in \mathbb{N}$ and H is an open set containing $f(x)$, there is an $n \geq n_0$ such that $x_n \in f^{-1}[H]$, so that $y_n \in H$. Thus $f(x)$ is a cluster point of $\langle y_n \rangle_{n \in \mathbb{N}}$; as $\langle y_n \rangle_{n \in \mathbb{N}}$ is arbitrary, $f[A]$ is relatively countably compact. **Q**

(v) A relatively countably compact set in $\mathbb{R}$ must be bounded. (If $A \subseteq \mathbb{R}$ is unbounded there is a sequence $\langle x_n \rangle_{n \in \mathbb{N}}$ in A such that $|x_n| \geq n$ for every n.) So if X is a topological space, $A \subseteq X$ is relatively countably compact and $f : X \to \mathbb{R}$ is continuous, then $f[A]$ is bounded.

(vi) If X and Y are topological spaces and $f : X \to Y$ is continuous, then $f[A]$ is countably compact whenever $A \subseteq X$ is countably compact. (Engelking 89, 3.10.5.)

(g) (i) Let X and Y be topological spaces and $\phi : X \times Y \to \mathbb{R}$ a continuous function. Define $\theta : X \to C(Y)$ by setting $\theta(x)(y) = \phi(x, y)$ for $x \in X$, $y \in Y$. Then θ is continuous if we give $C(Y)$ the topology of uniform

convergence on compact subsets of Y. (As noted in ENGELKING 89, pp. 157-158, the topology of uniform convergence on compact sets is the 'compact-open' topology of $C(Y)$, so the result here is covered by ENGELKING 89, 3.4.1.)

(ii) In particular, if Y is compact then θ is continuous if we give $C(Y)$ its usual norm topology.

(h) (i) Suppose that X is a compact space such that there are no non-trivial convergent sequences in X, that is, no convergent sequences which are not eventually constant. If $\langle F_n\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of infinite closed subsets of X, then $F=\bigcap_{n\in\mathbb{N}}F_n$ is infinite. **P** Because X is compact, F cannot be empty (3A3Da). Choose a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ such that $x_n\in F_n\setminus\{x_i:i<n\}$ for each $n\in\mathbb{N}$. If $G\supseteq F$ is an open set, then $\bigcap_{n\in\mathbb{N}}F_n\setminus G=\emptyset$, so there must be some $n\in\mathbb{N}$ such that $F_n\subseteq G$, and $x_i\in G$ for $i\geq n$. **?** If $F=\{y_0,\dots,y_k\}$, let $l\leq k$ be the first point such that whenever $G\supseteq\{y_0,\dots,y_l\}$ is open, then $\{i:x_i\notin G\}$ is finite. Then there is an open set $G'\supseteq\{y_j:j<l\}$ such that $I=\{i:x_i\notin G'\}$ is infinite. But if H is any open set containing y_l, then $\{i:x_i\notin G'\cup H\}$ is finite, so $\{i:i\in I,\,x_i\notin H\}$ is finite. Thus if we re-enumerate $\langle x_i\rangle_{i\in I}$ as $\langle x'_n\rangle_{n\in\mathbb{N}}$, $\langle x'_n\rangle_{n\in\mathbb{N}}$ converges to y_l and is a non-trivial convergent sequence. **X** Thus F is infinite, as claimed. **Q**

(ii) If X is an infinite scattered compact Hausdorff space it has a non-trivial convergent sequence. **P** Let $\langle x_n\rangle_{n\in\mathbb{N}}$ be any sequence of distinct points in X. Set $F_n=\overline{\{x_i:i\geq n\}}$ for each n, so that $F=\bigcap_{n\in\mathbb{N}}F_n$ is a non-empty set. Because X is scattered, F has an isolated point z say; let G be an open set such that $F\cap G=\{z\}$, and H an open set such that $z\in H\subseteq\overline{H}\subseteq G$ (3A3Bc). In this case, $I=\{i:x_i\in H\}$ must be infinite; re-enumerate $\langle x_i\rangle_{i\in I}$ as $\langle x'_n\rangle_{n\in\mathbb{N}}$. **?** If $\langle x'_n\rangle_{n\in\mathbb{N}}$ does not converge to z, there is an open set H' containing z such that $\{n:x'_n\notin H'\}$ is infinite, that is, $\{i:i\in I,\,x_i\notin H'\}$ is infinite. In this case, $F_n\cap\overline{H}\setminus H'$ is non-empty for every $n\in\mathbb{N}$, but $F\cap\overline{H}\setminus H'=\emptyset$, which is impossible. **X** Thus $\langle x'_n\rangle_{n\in\mathbb{N}}$ is a non-trivial convergent sequence in X. **Q**

(iii) If X is an extremally disconnected Hausdorff space (definition: 3A3Ae), it has no non-trivial convergent sequence. **P?** Suppose, if possible, that there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ converging to $x\in X$ such that $\{n:x_n\neq x\}$ is infinite. Choose $\langle n_i\rangle_{i\in\mathbb{N}}$ and $\langle G_i\rangle_{i\in\mathbb{N}}$ inductively, as follows. Given that $x\notin\overline{G}_j$ for $j<i$, there is an n_i such that $x_{n_i}\neq x$ and $x_{n_i}\notin\overline{G}_j$ for every $j<i$; now let G_i be an open set such that $x_{n_i}\in G_i$ and $x\notin\overline{G}_i$, and continue.

Since all the n_i must be distinct, $\langle x_{n_i}\rangle_{i\in\mathbb{N}}\to x$. But consider

$$G=\bigcup_{i\in\mathbb{N}}G_{2i}\setminus\bigcup_{j<2i}\overline{G}_j,\quad H=\bigcup_{i\in\mathbb{N}}G_{2i+1}\setminus\bigcup_{j\leq 2i}\overline{G}_j.$$

Then G and H are disjoint open sets and $x_{n_{2i}}\in G$, $x_{n_{2i+1}}\in H$ for every i. So $x\in\overline{G}\cap\overline{H}$. But $\overline{G}$ is open (because X is extremally disconnected), and is disjoint from H, and now $\overline{H}$ is disjoint from $\overline{G}$; so they cannot both contain x. **XQ**

(i) (i) Let X be a non-empty compact Hausdorff space without isolated points. Then there are a closed set $F\subseteq X$ and a continuous surjection $f:F\to\{0,1\}^{\mathbb{N}}$. **P** For $\sigma\in\bigcup_{n\in\mathbb{N}}\{0,1\}^n$ choose closed sets $V_\sigma\subseteq X$ inductively, as follows. $V_\emptyset=X$. Given that V_σ is a closed set with non-empty interior, there are distinct points x, $y\in\operatorname{int}V_\sigma$ (because X has no isolated points); let G, H be disjoint open subsets of X such that $x\in G$ and $y\in H$; and let $V_{\sigma^\frown 0}$, $V_{\sigma^\frown 1}$ be closed sets such that

$$x\in\operatorname{int}V_{\sigma^\frown 0}\subseteq G\cap\operatorname{int}V_\sigma,\quad y\in\operatorname{int}V_{\sigma^\frown 1}\subseteq H\cap\operatorname{int}V_\sigma.$$

(This is possible because X is regular.) The construction ensures that $V_\tau\subseteq V_\sigma$ whenever $\tau\in\{0,1\}^n$ extends $\sigma\in\{0,1\}^m$, and that $V_\tau\cap V_\sigma=\emptyset$ whenever τ, $\sigma\in\{0,1\}^n$ are different. Set $F=\bigcap_{n\in\mathbb{N}}\bigcup_{\sigma\in\{0,1\}^n}V_\sigma$; then F is a closed subset of X and we have a continuous function $f:F\to\{0,1\}^{\mathbb{N}}$ defined by saying that $f(x)(i)=\sigma(i)$ whenever $n\in\mathbb{N}$, $\sigma\in\{0,1\}^n$, $i<n$ and $x\in V_\sigma$. Finally, f is surjective, because if $z\in\{0,1\}^{\mathbb{N}}$ then $\langle V_{z\restriction n}\rangle_{n\in\mathbb{N}}$ is a non-increasing sequence of closed sets in the compact space X, so has non-empty intersection V say, and $f(x)=z$ for any $x\in V$. **Q**

(ii) If X is a non-empty compact Hausdorff space without isolated points, then $\#(X)\geq\mathfrak{c}$. (Use (i).)

(iii) If X is a compact Hausdorff space which is not scattered, it has an infinite closed subset without isolated points with a countable π-base. **P** Because X is not scattered, it has a non-empty subset A without isolated points. Then $\overline{A}$ is compact and has no isolated points; by (i), there are a closed set $F_0\subseteq\overline{A}$ and a continuous surjection $f:F_0\to\{0,1\}^{\mathbb{N}}$. Let $\mathcal{F}$ be the family of closed sets $F\subseteq F_0$ such that $f[F]=\{0,1\}^{\mathbb{N}}$. If $\mathcal{F}'\subseteq\mathcal{F}$ is non-empty and downwards-directed, then $F'=\bigcap\mathcal{F}'$ is closed; also, for every $z\in\{0,1\}^{\mathbb{N}}$,

$\{F \cap f^{-1}[\{z\}] : F \in \mathcal{F}'\}$ is a downwards-directed family of closed sets in the compact space X, so has non-empty intersection, and $F' \cap f^{-1}[\{z\}] \neq \emptyset$; as z is arbitrary, $f[F'] = \{0,1\}^{\mathbb{N}}$, and $F' \in \mathcal{F}$. Thus every totally ordered subset of $\mathcal{F}$ has a lower bound in $\mathcal{F}$, and $\mathcal{F}$ has a minimal element F_1 say.

Let $\mathcal{U}$ be a countable base for the topology of $\{0,1\}^{\mathbb{N}}$, and set $\mathcal{V} = \{F_1 \cap f^{-1}[U] : U \in \mathcal{U}\}$, so that $\mathcal{V}$ is a countable family of relatively open subsets of F_1. If $H \subseteq F_1$ is relatively open and not empty, then $F_1 \setminus H$ is a proper closed subset of F_1; because F_1 is minimal in $\mathcal{F}$, $F_1 \setminus H \notin \mathcal{F}$ and $f[F_1 \setminus H] \neq \{0,1\}^{\mathbb{N}}$. But $F_1 \setminus H$ is compact, so $f[F_1 \setminus H]$ is compact and its complement in $\{0,1\}^{\mathbb{N}}$ is open; there is therefore a non-empty $U \in \mathcal{U}$ such that $U \cap f[F_1 \setminus H] = \emptyset$ and $f^{-1}[U] \cap F_1 \subseteq H$. But as $f[F_1] = \{0,1\}^{\mathbb{N}}$, $f^{-1}[U] \cap F_1$ is not empty, and is a non-empty member of $\mathcal{V}$ included in H. As H is arbitrary, $\mathcal{V}$ is a π-base for the topology of F_1, and F_1 is a closed subset of X with a countable π-base. **Q**

(iv) Let X be a compact Hausdorff space. Then there is a continuous surjection from X onto $[0,1]$ iff X is not scattered. **P** (α) Suppose that $f : X \to [0,1]$ is a continuous surjection. Let $\mathcal{E}$ be the family of closed sets $F \subseteq X$ such that $f[F] = [0,1]$. If $\mathcal{F} \subseteq \mathcal{E}$ is non-empty and downwards-directed, then for any $t \in [0,1]$ the family $\{F \cap f^{-1}[\{t\}] : F \in \mathcal{F}\}$ is a downwards-directed family of non-empty closed sets, so (because X is compact) has non-empty intersection; this shows that $\bigcap \mathcal{F} \in \mathcal{E}$. By Zorn's Lemma, $\mathcal{E}$ has a minimal element K say. **?** If $x \in K$ is an isolated point of K, then $K \setminus \{x\}$ is closed, therefore compact, and $f[K \setminus \{x\}] \supseteq [0,1] \setminus \{f(x)\}$ is compact and dense in $[0,1]$. But this means that $K \setminus \{x\} \in \mathcal{E}$ and K is not minimal. **X** Thus $K \subseteq X$ is a non-empty set with no isolated points and X is not scattered. (β) If X is not scattered, let $A \subseteq X$ be a non-empty set with no isolated points. Then $\overline{A}$ is a non-empty compact subset of X with no isolated points, so there is a continuous surjection $g : \overline{A} \to \{0,1\}^{\mathbb{N}}$ ((i) above). Now there is a continuous surjection $h : \{0,1\}^{\mathbb{N}} \to [0,1]$ (e.g., set $h(y) = \sum_{n=0}^{\infty} 2^{-n-1} y(n)$ for $y \in \{0,1\}^{\mathbb{N}}$), so we have a continuous surjection $hg : \overline{A} \to [0,1]$. By Tietze's theorem (4A2F(d-viii)), there is a continuous function $f_0 : X \to \mathbb{R}$ extending hg; setting $f(x) = \max(0, \min(f_0(x), 1))$ for $x \in X$, we have a continuous surjection $f : X \to [0,1]$. **Q**

(v) A Hausdorff continuous image of a scattered compact Hausdorff space is scattered. (Immediate from (iv).)

(vi) If X is an uncountable first-countable compact Hausdorff space, it is not scattered. **P** Let $\mathcal{G}$ be the family of countable open subsets of X, and G^* its union. No finite subset of $\mathcal{G}$ can cover X, so $X \setminus G^*$ is non-empty. **?** If x is an isolated point of $X \setminus G^*$, then $\{x\} \cup G^*$ is a neighbourhood of x; let $\langle U_n \rangle_{n \in \mathbb{N}}$ run over a base of open neighbourhoods of x with $\overline{U}_0 \subseteq \{x\} \cup G^*$. For each $n \in \mathbb{N}$, $F_n = \overline{U}_0 \setminus U_n$ is a compact set included in G^*, so is covered by finitely many members of $\mathcal{G}$, and is countable. But this means that $U_0 = \{x\} \cup \bigcup_{n \in \mathbb{N}} U_0 \setminus U_n$ is countable, and $x \in G^*$. **X** Thus $X \setminus G^*$ is a non-empty set with no isolated points, and X is not scattered. **Q**

It follows that there is a continuous surjection from X onto $[0,1]$, by (iv).

(j) A locally compact Hausdorff space is Čech-complete. (Engelking 89, p. 196.)

(k) If X is a topological space, $f : X \to \mathbb{R}$ is lower semi-continuous, and $K \subseteq X$ is compact and not empty, then there is an $x_0 \in K$ such that $f(x_0) = \inf_{x \in K} f(x)$. Similarly, if $g : X \to \mathbb{R}$ is upper semi-continuous, there is an $x_1 \in K$ such that $g(x_1) = \sup_{x \in K} g(x)$.

4A2H Lindelöf spaces (a) If X is a topological space, then a subset Y of X is Lindelöf (in its subspace topology) iff for every family $\mathcal{G}$ of open subsets of X covering Y there is a countable subfamily of $\mathcal{G}$ still covering Y.

(b) (i) A regular Lindelöf space X is paracompact, therefore normal and completely regular. (Engelking 89, 3.8.11 & 5.1.2.)

(ii) If X is a Lindelöf space and $\mathcal{A}$ is a locally finite family of subsets of X then $\mathcal{A}$ is countable. **P** The family $\mathcal{G}$ of open sets meeting only finitely many members of $\mathcal{A}$ is an open cover of X. If $\mathcal{G}_0 \subseteq \mathcal{G}$ is a countable cover of X then $\{A : A \in \mathcal{A}, A \text{ meets some member of } \mathcal{G}_0\} = \mathcal{A} \setminus \{\emptyset\}$ is countable. **Q**

(c) (i) A topological space X is hereditarily Lindelöf iff for any family $\mathcal{G}$ of open subsets of X there is a countable family $\mathcal{G}_0 \subseteq \mathcal{G}$ such that $\bigcup \mathcal{G}_0 = \bigcup \mathcal{G}$. **P** ($\alpha$) If X is hereditarily Lindelöf and $\mathcal{G}$ is a family of open subsets of X, then $\mathcal{G}$ is an open cover of $\bigcup \mathcal{G}$, so has a countable subcover. (β) If X is not hereditarily Lindelöf, let $Y \subseteq X$ be a non-Lindelöf subspace, and $\mathcal{H}$ a cover of Y by relatively open sets which has no

countable subcover; setting $\mathcal{G} = \{G : G \subseteq X \text{ is open}, G \cap Y \in \mathcal{H}\}$, there can be no countable $\mathcal{G}_0 \subseteq \mathcal{G}$ with union $\bigcup \mathcal{G}$. **Q**

(ii) Let X be a regular hereditarily Lindelöf space. Then every closed subset of X is a zero set, so X is perfectly normal. **P** Let $F \subseteq X$ be closed. Let $\mathcal{G}$ be the family of open sets $G \subseteq X$ such that $\overline{G} \cap F = \emptyset$; because X is regular, $\bigcup \mathcal{G} = X \setminus F$; because X is hereditarily Lindelöf, there is a sequence $\langle G_n \rangle_{n \in \mathbb{N}}$ in G such that $X \setminus F = \bigcup_{n \in \mathbb{N}} G_n$. This means that $F = \bigcap_{n \in \mathbb{N}} X \setminus \overline{G_n}$ is a G_δ set. But X is normal ((b) above), so F is a zero set (4A2F(d-v)). By 4A2Fi, X is perfectly normal. **Q**

(d) Any σ-compact topological space is Lindelöf. (Engelking 89, 3.8.5.)

4A2I Stone-Čech compactifications (a) Let X be a completely regular Hausdorff space. Then there is a compact Hausdorff space βX, the **Stone-Čech compactification** of X, in which X can be embedded as a dense subspace. If Y is another compact Hausdorff space, then every continuous function from X to Y has a unique continuous extension to a continuous function from βX to Y. (Engelking 89, 3.6.1; Császár 78, 6.4d; Čech 66, 41D.5.)

(b) Let I be any set, and write βI for its Stone-Čech compactification when I is given its discrete topology.

(i) Let Z be the Stone space of the Boolean algebra $\mathcal{P}I$. Then there is a canonical homeomorphism $\phi : \beta I \to Z$ defined by saying that $\phi(i)(a) = \chi a(i)$ for every $i \in I$, $a \subseteq Z$. **P** Recall that Z is the set of ring homomorphisms from $\mathcal{P}I$ onto $\mathbb{Z}_2$ (311E). If $i \in I$, let $\hat{i}$ be the corresponding member of Z defined by setting $\hat{i}(a) = \chi a(i)$ for every $a \subseteq I$. Then $i \mapsto \hat{i} : I \to Z$ is continuous, while Z is compact and Hausdorff (311I), so has a unique extension to a continuous function $\phi : \beta I \to Z$.

If $G \subseteq Z$ is open and not empty, it includes a set of the form $\widehat{a} = \{\theta : \theta \in Z, \theta(a) = 1\}$ where $a \subseteq I$ is not empty; if i is any member of a, $\hat{i} \in G$ so $G \cap \phi[\beta I] \neq \emptyset$. This shows that $\phi[\beta I]$ is dense in Z; as βI is compact, $\phi[\beta I]$ is compact, therefore closed, and is equal to Z. Thus ϕ is surjective.

If t, u are distinct points of βI, there is an open subset H of βI such that $t \in H$ and $u \notin \overline{H}$. Set $a = H \cap I$. Then $t \in \overline{a}$, the closure of a regarded as a subset of βI, so $\phi(t) \in \overline{\phi[a]}$ (3A3Cd). But $\phi[a] = \{\hat{i} : i \in a\} \subseteq \widehat{a}$, which is open-and-closed, so $\phi(t) \in \widehat{a}$. Similarly, setting $b = I \setminus \overline{H}$, $\phi(u) \in \widehat{b}$; since $\widehat{a} \cap \widehat{b} = \widehat{a \cap b}$ is empty, $\phi(t) \neq \phi(u)$. This shows that ϕ is injective, therefore a homeomorphism between βI and Z (3A3Dd). **Q**

(ii) $C(\beta I)$ is isomorphic, as Banach lattice, to $\ell^\infty(I)$. **P** Let Z be the Stone space of $\mathcal{P}I$. By 363Ha, we can identify $\ell^\infty(I)$, as Banach lattice, with $L^\infty(\mathcal{P}I) = C(Z)$. But (i) tells us that we have a canonical identification between $C(Z)$ and $C(\beta I)$. **Q**

(iii) We have a one-to-one correspondence between filters $\mathcal{F}$ on I and non-empty closed sets $F \subseteq \beta I$, got by matching $\mathcal{F}$ with $\bigcap\{\widehat{a} : a \in \mathcal{F}\}$, or F with $\{a : a \subseteq I, F \subseteq \widehat{a}\}$, where $\widehat{a} \subseteq \beta I$ is the open-and-closed set corresponding to $a \subseteq I$. **P** The identification of βI with the Stone space Z of $\mathcal{P}I$ ((i) above) means that we can regard the map $a \mapsto \widehat{a}$ as a Boolean isomorphism between $\mathcal{P}I$ and the algebra of open-and-closed subsets of βI (311I). For any filter $\mathcal{F}$ on I, set $H(\mathcal{F}) = \bigcap\{\widehat{a} : a \in \mathcal{F}\}$; because $\{\widehat{a} : a \in \mathcal{F}\}$ is a downwards-directed family of non-empty closed sets in the compact Hausdorff space βI, $H(\mathcal{F})$ is a non-empty closed set. If $F \subseteq \beta I$ is a non-empty closed set, then it is elementary to check that $\mathcal{H}(F) = \{a : F \subseteq \widehat{a}\}$ is a filter on I, and evidently $H(\mathcal{H}(F)) \supseteq F$. But if $t \in \beta I \setminus F$, then (because $\{\widehat{a} : a \subseteq I\}$ is a base for the topology of βI, see 311I again) there is an $a \subseteq I$ such that $t \in \widehat{a}$ and $F \cap \widehat{a} = \emptyset$, that is, $F \subseteq \widehat{I \setminus a}$; so $\widehat{I \setminus a} \in \mathcal{H}(F)$ and $H(\mathcal{H}(F)) \subseteq \widehat{I \setminus a}$ and $t \notin H(\mathcal{H}(F))$. Thus $H(\mathcal{H}(F)) = F$ for every non-empty closed set $F \subseteq \beta I$.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are filters on I and $a \in \mathcal{F}_1 \setminus \mathcal{F}_2$, then $\{\widehat{b \setminus a} : b \in \mathcal{F}_2\}$ is a downwards-directed family of non-empty closed sets in βI, so has non-empty intersection; if $t \in \widehat{b \setminus a} = \widehat{b} \setminus \widehat{a}$ for every $b \in \mathcal{F}_2$, then $t \in H(\mathcal{F}_2) \setminus H(\mathcal{F}_1)$. This shows that $\mathcal{F} \mapsto H(\mathcal{F})$ is injective. It follows that $\mathcal{F} \mapsto H(\mathcal{F})$, $F \mapsto \mathcal{H}(F)$ are the two halves of a bijection, as claimed. **Q**

(iv) There are no non-trivial convergent sequences in βI. **P** Because $\mathcal{P}I$ is Dedekind complete, its Stone space is extremally disconnected (314S), so, by the identification in (i), βI is extremally disconnected, and has no non-trivial convergent sequences (4A2G(h-iii)). **Q** (Compare Engelking 89, 3.6.15.)

4A2J Uniform spaces (See §3A4.) Let $(X, \mathcal{W})$ be a uniform space; give X the associated topology $\mathfrak{T}$ (3A4Ab).

(a) $\mathcal{W}$ is generated by a family of pseudometrics. (Engelking 89, 8.1.11; Bourbaki 66, IX.1.4; Császár 78, 4.2.32.) More precisely: if $\langle W_n\rangle_{n\in\mathbb{N}}$ is any sequence in $\mathcal{W}$, there is a pseudometric ρ on X such that (α) $\{(x,y) : \rho(x,y) \leq \epsilon\} \in \mathcal{W}$ for every $\epsilon > 0$ (β) whenever $n \in \mathbb{N}$ and $\rho(x,y) < 2^{-n}$ then $(x,y) \in W_n$ (Engelking 89, 8.1.10).

It follows that $\mathfrak{T}$ is completely regular, therefore regular (3A3Be). $\mathfrak{T}$ is defined by the bounded uniformly continuous functions, in the sense that it is the coarsest topology $\mathfrak{S}$ on X such that these are all continuous. **P** Let P be the family of pseudometrics compatible with $\mathcal{W}$ in the sense of (α) just above. If $x \in G \in \mathfrak{T}$, there is a $\rho \in$ P such that $\{y : \rho(x,y) < 1\} \subseteq G$; setting $f(y) = \rho(x,y)$, we see that f is uniformly continuous, therefore $\mathfrak{S}$-continuous, and that $x \in \operatorname{int}_{\mathfrak{S}} G$. As x is arbitrary, $G \in \mathfrak{S}$; as G is arbitrary, $\mathfrak{T} \subseteq \mathfrak{S}$; but of course $\mathfrak{S} \subseteq \mathfrak{T}$ just because uniformly continuous functions are continuous. **Q**

(b) If $W \in \mathcal{W}$ and $x \in X$ then $x \in \operatorname{int} W[\{x\}]$. (Engelking 89, 8.1.3.) If $A \subseteq X$ then $\overline{A} = \bigcap_{W\in\mathcal{W}} W[A]$. (Engelking 89, 8.1.4.)

(c) Any subset of a totally bounded set in X is totally bounded. (Engelking 89, 8.3.2; Császár 78, 3.2.70.) The closure of a totally bounded set is totally bounded. **P** If A is totally bounded and $W \in \mathcal{W}$, take $W' \in \mathcal{W}$ such that $W' \circ W' \subseteq W$. Then there is a finite set $I \subseteq X$ such that $A \subseteq W'[I]$. In this case

$$\overline{A} \subseteq W'[A] \subseteq W'[W'[I]] = (W' \circ W')[I] \subseteq W[I]$$

by (b). As W is arbitrary, $\overline{A}$ is totally bounded. **Q**

(d) A subset of X is compact iff it is complete (definition: 3A4F) (for its subspace uniformity) and totally bounded. (Engelking 89, 8.3.16; Čech 66, 41A.8; Császár 78, 5.2.22; Gaal 64, pp. 278-279.) So if X is complete, every closed totally bounded subset of X is compact, and the totally bounded sets are just the relatively compact sets. (A closed subspace of a complete space is complete.)

(e) If $f : X \to \mathbb{R}$ is a continuous function with compact support, it is uniformly continuous. **P** Set $K = \overline{\{x : f(x) \neq 0\}}$. Let $\epsilon > 0$. For each $x \in X$, there is a $W_x \in \mathcal{W}$ such that $|f(y) - f(x)| \leq \frac{1}{2}\epsilon$ whenever $y \in W_x[\{x\}]$. Let $W'_x \in \mathcal{W}$ be such that $W'_x \circ W'_x \subseteq W_x$. Set $G_x = \operatorname{int} W'_x[\{x\}]$; then $x \in G_x$, by (b). Because K is compact, there is a finite set $I \subseteq K$ such that $K \subseteq \bigcup_{x\in I} G_x$. Set $W = (X \times X) \cap \bigcap_{x\in I} W'_x \in \mathcal{W}$. Take any $(y,z) \in W \cap W^{-1}$. If neither y nor z belongs to K, then of course $|f(y) - f(z)| \leq \epsilon$. If $y \in K$, let $x \in I$ be such that $y \in G_x$. Then

$$y \in W'_x[\{x\}] \subseteq W_x[\{x\}], \quad z \in W[W'_x[\{x\}]] \subseteq W'_x[W'_x[\{x\}]] \subseteq W_x[\{x\}],$$

so

$$|f(y) - f(z)| \leq |f(y) - f(x)| + |f(z) - f(x)| \leq \epsilon.$$

The same idea works if $z \in K$. So $|f(y) - f(z)| \leq \epsilon$ for all y, $z \in W \cap W^{-1}$; as ϵ is arbitrary, f is uniformly continuous. **Q**

(f) (i) If $(Y, \mathfrak{S})$ is a completely regular space, there is a uniformity on Y compatible with $\mathfrak{S}$. (Engelking 89, 8.1.20.)

(ii) If $(Y, \mathfrak{S})$ is a compact Hausdorff space, there is exactly one uniformity on Y compatible with $\mathfrak{S}$; it is induced by the set of all those pseudometrics on Y which are continuous as functions from $Y \times Y$ to $\mathbb{R}$. (Engelking 89, 8.3.13.)

(iii) If $(Y, \mathfrak{S})$ is a compact Hausdorff space and $\mathcal{V}$ is the uniformity on Y compatible with $\mathfrak{S}$, then any continuous function from Y to X is uniformly continuous.

(g) The set U of uniformly continuous real-valued functions on X is a Riesz subspace of $\mathbb{R}^X$ containing the constant functions. If a sequence in U converges uniformly, the limit function again belongs to U.

4A2K First-countable, sequential and countably tight spaces (a) Let X be a countably tight topological space. If $\langle F_\xi\rangle_{\xi<\zeta}$ is a non-decreasing family of closed subsets of X indexed by an ordinal ζ, then $E = \bigcup_{\xi<\zeta} F_\xi$ is closed unless $\operatorname{cf}\zeta = \omega$, in which case E is an F_σ set. **P** If $\operatorname{cf}\zeta = 0$, that is, $\zeta = 0$, then $E = \emptyset$ is closed. If $\operatorname{cf}\zeta = 1$, that is, $\zeta = \xi + 1$ for some ordinal ξ, then $E = F_\xi$ is closed. If $\operatorname{cf}\zeta = \omega$, there is a sequence $\langle\xi_n\rangle_{n\in\mathbb{N}}$ in ζ with supremum ζ, so that $E = \bigcup_{n\in\mathbb{N}} F_{\xi_n}$ is F_σ. If $\operatorname{cf}\zeta > \omega$, take $x \in \overline{E}$. Then there is a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in E such that $x \in \overline{\{x_n : n \in \mathbb{N}\}}$. For each n there is a $\xi_n < \zeta$ such that $x_n \in F_{\xi_n}$, and now $\xi = \sup_{n\in\mathbb{N}} \xi_n < \zeta$ and $x \in \overline{F}_\xi = F_\xi \subseteq E$. As x is arbitrary, E is closed. **Q**

(b) If X is countably tight, any subspace of X is countably tight, just because if $A \subseteq Y \subseteq X$ then the closure of A in Y is the intersection of Y with the closure of A in X. If X is compact and countably tight, then any Hausdorff continuous image of X is countably tight. **P** Let $f : X \to Y$ be a continuous surjection, where Y is Hausdorff, B a subset of Y and $y \in \overline{B}$. Set $A = f^{-1}[B]$. Then $\overline{A}$ is compact, so $f[\overline{A}]$ is compact, therefore closed; because f is surjective, $y \in \overline{f[A]} \subseteq f[\overline{A}]$, and there is an $x \in \overline{A}$ such that $f(x) = y$. Now there is a countable set $A_0 \subseteq A$ such that $x \in \overline{A}_0$, in which case

$$y = f(x) \in f[\overline{A}_0] \subseteq \overline{f[A_0]},$$

while $f[A_0]$ is a countable subset of B. **Q**

(c) If X is a sequential space, it is countably tight. **P** Suppose that $A \subseteq X$ and $x \in \overline{A}$. Set $B = \bigcup\{\overline{C} : C \subseteq A \text{ is countable}\}$. If $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in B converging to $x \in X$, then for each $n \in \mathbb{N}$ we can find a countable set $C_n \subseteq A$ such that $x_n \in \overline{C}_n$, and now $C = \bigcup_{n\in\mathbb{N}} C_n$ is a countable subset of A such that $x \in \overline{C} \subseteq B$. So B is sequentially closed, therefore closed, and $x \in B$. As A and x are arbitrary, X is countably tight. **Q**

(d) If X is a sequential space, Y is a topological space and $f : X \to Y$ is sequentially continuous, then it is continuous. (Engelking 89, 1.6.15.)

(e) First-countable spaces are sequential. (Engelking 89, 1.6.14.)

(f) Let X be a locally compact Hausdorff space in which every singleton set is G_δ. Then X is first-countable. **P** If $\{x\} = \bigcap_{n\in\mathbb{N}} G_n$ where each G_n is open, then for each $n \in \mathbb{N}$ we can find a compact set F_n such that $x \in \operatorname{int} F_n \subseteq G_n$. By 4A2Gd, $\{\bigcap_{i\le n} F_i : n \in \mathbb{N}\}$ is a base of neighbourhoods of x. **Q**

4A2L (Pseudo-)metrizable spaces 'Pseudometrizable' spaces, as such, hardly appear in this volume, for the usual reasons; they surface briefly in §463. It is perhaps worth noting, however, that all the ideas, and very nearly all the results, in this paragraph apply equally well to pseudometrics and pseudometrizable topologies.

(a) Any subspace of a (pseudo-)metrizable space is (pseudo-)metrizable (2A3J). A topological space is metrizable iff it is pseudometrizable and Hausdorff (2A3L).

(b) Metrizable spaces are paracompact (Engelking 89, 5.1.3; Császár 78, 8.3.16; Čech 66, 30C.2; Gaal 64, p. 155), therefore hereditarily metacompact ((a) above and 4A2F(g-i)).

(c) A metrizable space is perfectly normal (Engelking 89, 4.1.13; Császár 78, 8.4.5.), so every closed set is a zero set and every open set is a cozero set (in particular, is F_σ).

(d) If X is a pseudometrizable space, it is first-countable. (If ρ is a pseudometric defining the topology of X, and x is any point of X, then $\{\{y : \rho(y,x) < 2^{-n}\} : n \in \mathbb{N}\}$ is a base of neighbourhoods of x.) So X is sequential and countably tight (4A2Ke, 4A2Kc), and if Y is another topological space and $f : X \to Y$ is sequentially continuous, then f is continuous (4A2Kd).

(e) Relative compactness Let X be a pseudometrizable space and A a subset of X. Then the following are equiveridical: (α) A is relatively compact; (β) A is relatively countably compact; (γ) every sequence in A has a subsequence with a limit in X. **P** Fix a pseudometric ρ defining the topology of X. (α)⇒(β) by 4A2G(f-iii). If $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in A with a cluster point $x \in X$, then we can choose $\langle n_i\rangle_{i\in\mathbb{N}}$ inductively

such that $\rho(x_{n_i}, x) \leq 2^{-i}$ and $n_{i+1} > n_i$ for every i; now $\langle x_{n_i}\rangle_{i\in\mathbb{N}} \to x$; it follows that $(\beta)\Rightarrow(\gamma)$. Now assume that (α) is false. Then there is an ultrafilter $\mathcal{F}$ on X containing A which has no limit in X (3A3Be, 3A3De). If $\mathcal{F}$ is a Cauchy filter, choose $F_n \in \mathcal{F}$ such that $\rho(x, y) \leq 2^{-n}$ whenever x, $y \in F_n$, and $x_n \in A \cap \bigcap_{i\leq n} F_i$ for each n; then it is easy to see that $\langle x_n\rangle_{n\in\mathbb{N}}$ is a sequence in A with no convergent subsequence. If $\mathcal{F}$ is not a Cauchy filter, let $\epsilon > 0$ be such that there is no $F \in \mathcal{F}$ such that $\rho(x, y) \leq \epsilon$ for every x, $y \in F$. Then $X \setminus U(x, \frac{1}{2}\epsilon) \in \mathcal{F}$ for ever $x \in X$, so we can choose $\langle x_n\rangle_{n\in\mathbb{N}}$ inductively such that $x_n \in A \setminus \bigcup_{i<n} U(x_i, \frac{1}{2}\epsilon)$ for every $n \in \mathbb{N}$, and again we have a sequence $\langle x_n\rangle_{n\in\mathbb{N}}$ in A with no convergent subsequence in X. Thus not-$(\alpha)\Rightarrow$ not-(γ) and the proof is complete. **Q**

(f) Compactness If X is a pseudometrizable space, it is compact iff it is countably compact iff it is sequentially compact. ((e) above, using 4A2G(f-i). Compare Engelking 89, 4.1.17, and Császár 78, 5.3.33 & 5.3.47.)

(g) (i) If (X, ρ) is a metric space, its topology has a base which is σ-metrically-discrete. **P** Enumerate X as $\langle x_\xi\rangle_{\xi<\kappa}$ where κ is a cardinal. Let $\langle(q_n, q'_n)\rangle_{n\in\mathbb{N}}$ be a sequence running over $\{(q, q') : q, q' \in \mathbb{Q}, 0 < q < q'\}$ in such a way that $q'_n - q_n \geq 2^{-n}$ for every $n \in \mathbb{N}$. For $n \in \mathbb{N}$, $\xi < \kappa$ set $G_{n\xi} = \{x : \rho(x, x_\xi) < q_n, \inf_{\eta<\xi} \rho(x, x_\eta) > q'_n\}$ (interpreting $\inf \emptyset$ as ∞). Then $\mathcal{U} = \langle G_{n\xi}\rangle_{\xi<\kappa,n\in\mathbb{N}}$ is a σ-metrically-discrete family of open sets. If $G \subseteq X$ is open and $x \in G$, let $\epsilon > 0$ be such that $U(x, 2\epsilon) \subseteq G$. Let $\xi < \kappa$ be minimal such that $\rho(x, x_\xi) < \epsilon$, and let $n \in \mathbb{N}$ be such that $\rho(x, x_\xi) < q_n < q'_n < \epsilon$; then $x \in G_{n\xi} \subseteq G$. As x and G are arbitrary, $\mathcal{U}$ is a base for the topology of X. **Q**

(ii) Consequently, any metrizable space has a σ-disjoint base. (Compare Engelking 89, 4.4.3; Császár 78, 8.4.5; Kuratowski 66, §21.XVII.)

(h) The product of a countable family of metrizable spaces is metrizable. (Engelking 89, 4.2.2; Császár 78, 7.3.27.)

(i) Let X be a metrizable space and $\kappa \geq \omega$ a cardinal. Then $w(X) \leq \kappa$ iff X has a dense subset of cardinal at most κ. (Engelking 89, 4.1.15.)

(j) If (X, ρ) is any metric space, then the balls $B(x, \delta) = \{y : \rho(y, x) \leq \delta\}$ are all closed sets (cf. 1A2G). In particular, in a normed space $(X, \|\,\|)$, the balls $B(x, \delta) = \{y : \|y - x\| \leq \delta\}$ are closed.

4A2M Complete metric spaces (a) Baire's theorem for complete metric spaces In a complete metric space, the intersection of a sequence of dense G_δ sets is dense. (Engelking 89, 4.3.36 & 3.9.4; Császár 78, 9.2.1 & 9.2.8; Gaal 64, p. 287; Kechris 95, 8.4.) So a non-empty complete metric space is not meager (cf. 3A3Ha).

(b) Let $\langle(X_i, \rho_i)\rangle_{i\in I}$ be a countable family of complete metric spaces. Then there is a complete metric on $X = \prod_{i\in I} X_i$ which defines the product topology on X. (Engelking 89, 4.3.12; Kuratowski 66, §33.III.)

(c) Let (X, ρ) be a complete metric space, and $E \subseteq X$ a G_δ set. Then there is a complete metric on E which defines the subspace topology of E. (Engelking 89, 4.3.23; Kuratowski 66, §33.VI; Kechris 95, 3.11.)

(d) Let (X, ρ) be a complete metric space. Then it is Čech-complete. (Engelking 89, 4.3.26.)

(e) A non-empty complete metric space without isolated points is uncountable. (If x is not isolated $\{x\}$ is nowhere dense.)

4A2N Countable networks: Proposition (a) If X is a topological space with a countable network, any subspace of X has a countable network.

(b) Let X be a space with a countable network. Then X is hereditarily Lindelöf. If it is regular, every closed set is a zero set.

(c) If X is a topological space, and $\langle A_n\rangle_{n\in\mathbb{N}}$ is a sequence of subsets of X each of which has a countable network (for its subspace topology), then $A = \bigcup_{n\in\mathbb{N}} A_n$ has a countable network.

(d) A continuous image of a space with a countable network has a countable network.

(e) Let $\langle X_i\rangle_{i\in I}$ be a countable family of topological spaces with countable networks, with product X. Then X has a countable network.

(f) If X is a Hausdorff space with a countable network, there is a countable family $\mathcal{G}$ of open sets such that whenever x, y are distinct points in X there are disjoint G, $H \in \mathcal{G}$ such that $x \in G$ and $y \in H$.

(g) If X is a regular topological space with a countable network, it has a countable network consisting of closed sets.

(h) A compact Hausdorff space with a countable network is metrizable.

(i) If a topological space X has a countable network, then any dense set in X includes a countable dense set; in particular, X is separable.

(j) If a topological space X has a countable network, then $C(X)$, with the topology of pointwise convergence inherited from the product topology of $\mathbb{R}^X$, has a countable network.

proof (a) If $\mathcal{E}$ is a countable network for the topology of X, and $Y \subseteq X$, then $\{Y \cap E : E \in \mathcal{E}\}$ is a countable network for the topology of Y.

(b) By Engelking 89, 3.8.12 X is Lindelöf. Since any subspace of X has a countable network ((a) above), it also is Lindelöf, and X is hereditarily Lindelöf. By 4A2H(c-ii), if X is regular, closed sets are zero sets.

(c) If $\mathcal{E}_n$ is a countable network for the topology of A_n for each n, then $\bigcup_{n\in\mathbb{N}} \mathcal{E}_n$ is a countable network for the topology of A.

(d) Let X be a topological space with a countable network $\mathcal{E}$, and Y a continuous image of X. Let $f : X \to Y$ be a continuous surjection. Then $\{f[E] : E \in \mathcal{E}\}$ is a network for the topology of Y. **P** If $H \subseteq Y$ is open and $y \in H$, then $f^{-1}[H]$ is an open subset of X and there is an $x \in X$ such that $f(x) = y$. Now there must be an $E \in \mathcal{E}$ such that $x \in E \subseteq f^{-1}[H]$, so that $y \in f[E] \subseteq H$. **Q** But $\{f[E] : E \in \mathcal{E}\}$ is countable, so Y has a countable network.

(e) For each $i \in I$ let $\mathcal{E}_i$ be a countable network for the topology of X_i. For each finite $J \subseteq I$, let $\mathcal{C}_J$ be the family of sets expressible as $\prod_{i\in I} E_i$ where $E_i \in \mathcal{E}_i$ for each $i \in J$ and $E_i = X_i$ for $i \in I \setminus J$; then $\mathcal{C}_J$ is countable because $\mathcal{E}_i$ is countable for each $i \in J$. Because the family $[I]^{<\omega}$ of finite subsets of I is countable (3A1Cd), $\mathcal{E} = \bigcup\{\mathcal{C}_J : J \in [I]^{<\omega}\}$ is countable. But $\mathcal{E}$ is a network for the topology of X. **P** If $G \subseteq X$ is open and $x \in G$, then there is a family $\langle G_i\rangle_{i\in I}$ such that every $G_i \subseteq X_i$ is open, $J = \{i : G_i \neq X_i\}$ is finite, and $x \in \prod_{i\in I} G_i$. For $i \in J$, there is an $E_i \in \mathcal{E}_i$ such that $x(i) \in E_i \subseteq G_i$; set $E_i = X_i$ for $i \in I \setminus J$. Then

$$E = \prod_{i\in I} E_i \in \mathcal{C}_J \subseteq \mathcal{E}$$

and $x \in E \subseteq G$. **Q**

So $\mathcal{E}$ is a countable network for the topology of X.

(f) By (b) and (e), $X \times X$ is hereditarily Lindelöf. In particular, $W = \{(x, y) : x \neq y\}$ is Lindelöf. Set

$$\mathcal{V} = \{G \times H : G, H \subseteq X \text{ are open}, G \cap H = \emptyset\}.$$

Because X is Hausdorff, $\mathcal{V}$ is a cover of W. So there is a countable $\mathcal{V}_0 \subseteq \mathcal{V}$ covering W. Set

$$\mathcal{G} = \{G : G \times H \in \mathcal{V}_0\} \cup \{H : G \times H \in \mathcal{V}_0\}.$$

Then $\mathcal{G}$ is a countable family of open sets separating the points of X.

(g) Let X be a regular space and $\mathcal{E}$ a countable network for the topology of X. Set $\mathcal{E}' = \{\overline{E} : E \in \mathcal{E}\}$. If $G \subseteq X$ is open and $x \in G$, then (because the topology is regular) there is an open set H such that $x \in H \subseteq \overline{H} \subseteq G$. Now there is an $E \in \mathcal{E}$ such that $x \in E \subseteq H$, in which case $\overline{E} \in \mathcal{E}'$ and $x \in \overline{E} \subseteq G$. So $\mathcal{E}'$ is a countable network for X consisting of closed sets.

(h) Engelking 89, 3.1.19.

(i) Let $D \subseteq X$ be dense, and $\mathcal{E}$ a countable network for the topology of X. Let $D' \subseteq D$ be a countable set such that $D' \cap E \neq \emptyset$ whenever $E \in \mathcal{E}$ and $D \cap E \neq \emptyset$. If $G \subseteq X$ is open and not empty, there is an $x \in D \cap G$; now there is an $E \in \mathcal{E}$ such that $x \in E \subseteq G$, and as $x \in D \cap E$ there must be an $x' \in D' \cap E$, so that $x' \in D' \cap G$. As G is arbitrary, D' is dense in X.

Taking $D = X$, we see that X has a countable dense subset.

(j) Let $\mathcal{E}$ be a countable network for the topology of X and $\mathcal{U}$ a countable base for the topology of $\mathbb{R}$ (4A2Ua). For $E \in \mathcal{E}$, $U \in \mathcal{U}$ set $H(E,U) = \{f : f \in C(X), E \subseteq f^{-1}[U]\}$. Then the set of finite intersections of sets of the form $H(E,U)$ is a countable network for the topology of pointwise convergence on $C(X)$. (Compare 4A2Oe.)

4A2O Second-countable spaces (a) Let $(X,\mathfrak{T})$ be a topological space and $\mathcal{U}$ a countable subbase for $\mathfrak{T}$. Then $\mathfrak{T}$ is second-countable. ($\{X\} \cup \{U_0 \cap U_1 \cap \ldots \cap U_n : U_0, \ldots, U_n \in \mathcal{U}\}$ is countable and is a base for $\mathfrak{T}$, by 4A2B(a-i).)

(b) Any base of a second-countable space includes a countable base. (Császár 78, 2.4.17.)

(c) Let $(X,\mathfrak{T})$ be a second-countable space. $\mathfrak{T}$ has a countable network (because a base is also a network), so is separable and hereditarily Lindelöf (Engelking 89, 1.3.8 & 3.8.1, 4A2Nb, 4A2Ni).

(d) The product of a countable family of second-countable spaces is second-countable. (Engelking 89, 2.3.14.)

(e) If X is a second-countable space then $C(X)$, with the topology of uniform convergence on compact sets, has a countable network. **P** (See Engelking 89, Ex. 3.4H.) Let $\mathcal{U}$ be a countable base for the topology of X and $\mathcal{V}$ a countable base for the topology of $\mathbb{R}$ (4A2Ua). For $U \in \mathcal{U}$, $V \in \mathcal{V}$ set $H(U,V) = \{f : f \in C(X), U \subseteq f^{-1}[V]\}$. Then the set of finite intersections of sets of the form $H(U,V)$ is a countable network for the topology of uniform convergence on compact subsets of X. **Q**

4A2P Separable metrizable spaces (a) (i) A metrizable space is second-countable iff it is separable. (Engelking 89, 4.1.16; Császár 78, 2.4.16; Gall p. 120.)

(ii) A compact metrizable space is separable (Engelking 89, 4.1.18; Császár 78, 5.3.35; Kuratowski 66, §21.IX), so is second-countable and has a countable network.

(iii) Any base of a separable metrizable space includes a countable base (4A2Ob), which is also a countable network, so the space is hereditarily Lindelöf (4A2Nb).

(iv) Any subspace of a separable metrizable space is separable and metrizable (4A2La, 4A2Na, 4A2Ni).

(v) A countable product of separable metrizable spaces is separable and metrizable (4A2B(e-ii), 4A2Lh).

(b) A topological space is separable and metrizable iff it is second-countable, regular and Hausdorff. (Engelking 89, 4.2.9; Császár 78, 7.1.57; Kuratowski 66, §22.II.)

(c) Let X be a compact metrizable space, Y a Hausdorff space and $f : X \to Y$ a continuous surjection. Then Y is metrizable. (It is a compact Hausdorff space, by 2A3N(b-ii), with a countable network, by 4A2Nd, so is metrizable, by 4A2Nh.)

(d) A metrizable space is separable iff it is ccc iff it is Lindelöf. (Engelking 89, 4.1.16.)

(e) If X is a compact metrizable space, then $C(X)$ is separable under its usual norm topology defined from the norm $\| \|_\infty$. (Engelking, 3.4.16.)

4A2Q Polish spaces: Proposition (a) A countable discrete space is Polish.

(b) A compact metrizable space is Polish.

(c) The product of a countable family of Polish spaces is Polish.

(d) A G_δ subset of a Polish space is Polish in its subspace topology; in particular, a set which is either open or closed is Polish.

(e) The disjoint union of countably many Polish spaces is Polish.

(f) If X is any set and $\langle \mathfrak{T}_n \rangle_{n\in\mathbb{N}}$ is a sequence of Polish topologies on X such that $\mathfrak{T}_m \cap \mathfrak{T}_n$ is Hausdorff for all $m, n \in \mathbb{N}$, then the topology $\mathfrak{T}_\infty$ generated by $\bigcup_{n\in\mathbb{N}} \mathfrak{T}_n$ is Polish.

(g) If X is a Polish space, it is homeomorphic to a G_δ set in a compact metric space.

proof (a) Any set X is complete under the discrete metric ρ defined by setting $\rho(x,y) = 1$ whenever x, $y \in X$ are distinct. This defines the discrete topology, and if X is countable it is separable, therefore Polish.

(b) By 4A2P(a-ii), it is separable; by 4A2Jd, any metric defining the topology is complete.

(c) If $\langle X_i\rangle_{i\in I}$ is a countable family of Polish spaces with product X, then surely X is separable (4A2B(e-ii)); and 4A2Mb tells us that its topology is defined by a complete metric.

(d) If X is Polish and E is a G_δ set in X, then E is separable, by 4A2P(a-iv), and its topology is defined by a complete metric, by 4A2Mc. So E is Polish. Any open set is of course a G_δ set, and any closed set is a G_δ set by 4A2Lc.

(e) Let $\langle X_i\rangle_{i\in I}$ be a countable disjoint family of Polish spaces, and $X = \bigcup_{i\in I} X_i$. For each $i \in I$ let ρ_i be a complete metric on X_i defining the topology of X_i. Define $\rho : X \times X \to [0,\infty[$ by setting $\rho(x,y) = \min(1, \rho_i(x,y))$ if $i \in I$ and x, $y \in X_i$, $\rho(x,y) = 1$ otherwise. It is easy to check that ρ is a complete metric on X defining the disjoint union topology on X. X is separable, by 4A2B(e-i), therefore Polish.

(f) This result is in Kechris 95, 13.3; but I spell out the proof because it is an essential element of some measure-theoretic arguments. On $X^{\mathbb{N}}$ take the product topology $\mathfrak{T}$ of the topologies $\mathfrak{T}_n$. This is Polish, by (c). Consider the diagonal $\Delta = \{x : x \in X^{\mathbb{N}}, x(m) = x(n) \text{ for all } m, n \in \mathbb{N}\}$. This is closed in $X^{\mathbb{N}}$. **P** If $x \in X^{\mathbb{N}} \setminus \Delta$, let m, $n \in \mathbb{N}$ be such that $x(m) \neq x(n)$. Because $\mathfrak{T}_m \cap \mathfrak{T}_n$ is Hausdorff, there are disjoint G, $H \in \mathfrak{T}_m \cap \mathfrak{T}_n$ such that $x(m) \in G$ and $x(n) \in H$. Now $\{y : y \in X^{\mathbb{N}}, y(m) \in G, y(n) \in H\}$ is an open set in $X^{\mathbb{N}}$ containing x and disjoint from Δ. As x is arbitrary, Δ is closed. **Q**

By (d), Δ, with its subspace topology, is a Polish space. Let $f : X \to \Delta$ be the natural bijection, setting $f(t) = x$ if $x(n) = t$ for every n, and let $\mathfrak{S}$ be the topology on X which makes f a homeomorphism. The topology on Δ is generated by $\{\{x : x \in \Delta, x(n) \in G\} : n \in \mathbb{N}, G \in \mathfrak{T}_n\}$, so $\mathfrak{S}$ is generated by $\{\{t : t \in X, t \in G\} : n \in \mathbb{N}, G \in \mathfrak{T}_n\} = \bigcup_{n\in\mathbb{N}} \mathfrak{T}_n$. Thus $\mathfrak{S} = \mathfrak{T}_\infty$ and $\mathfrak{T}_\infty$ is Polish.

(g) Kechris 95, 4.14.

4A2R Order topologies Let $(X, \leq)$ be a totally ordered set and $\mathfrak{T}$ its order topology.

(a) The set $\mathcal{U}$ of open intervals in X (definition: 4A2A) is a base for $\mathfrak{T}$.

(b) $[x,y]$, $[x,\infty[$ and $]-\infty,x]$ are closed sets for all x, $y \in X$.

(c) $\mathfrak{T}$ is Hausdorff and normal.

(d) If $A \subseteq X$ then $\overline{A}$ is the set of elements of X expressible as either suprema or infima of non-empty subsets of A.

(e) $F \subseteq X$ is closed iff it is order-closed.

(f) If $\langle x_n\rangle_{n\in\mathbb{N}}$ is a non-decreasing sequence in X with supremum x, then $x = \lim_{n\to\infty} x_n$.

(g) A set $K \subseteq X$ is compact iff $\sup A$ and $\inf A$ are defined in X and belong to K for every non-empty $A \subseteq K$.

(h) X is Dedekind complete iff $[x,y]$ is compact for all x, $y \in X$.

(i) X is compact iff it is either empty or Dedekind complete with greatest and least elements.

(j) If X is Dedekind complete, any open set $G \subseteq X$ is expressible as a union of disjoint open intervals.

(k) If X is well-ordered it is locally compact.

(l) In $X \times X$, $\{(x,y) : x < y\}$ is open and $\{(x,y) : x \leq y\}$ is closed.

(m) If $F \subseteq X$ is *either* compact *or* an interval, then the subspace topology on F is induced by the inherited order of F.

(n) If X is ccc it is hereditarily Lindelöf, therefore perfectly normal.

(o) If Y is another totally ordered set with its order topology, an order-preserving function from X to Y is continuous iff it is order-continuous.

proof (a) Put the definition of 'order topology' (4A2A) together with 4A2B(a-i).

(b) Their complements are either X, or members of $\mathcal{U}$, or unions of two members of $\mathcal{U}$.

(c) Fix a well-ordering $\preccurlyeq$ of X. If $x < y \in X$, define U_{xy}, U_{yx} as follows: if $]x,y[$ is empty, $U_{xy} =]-\infty,x] =]-\infty,y[$ and $U_{yx} = [y,\infty[=]x,\infty[$; otherwise, let z be the $\preccurlyeq$-least member of $]x,y[$ and set $U_{xy} =]-\infty,z[$, $U_{yx} =]z,\infty[$.

This construction ensures that if x, y are any distinct points of X, U_{xy} and U_{yx} are disjoint open sets containing x, y respectively, so $\mathfrak{T}$ is Hausdorff.

Now suppose that $F \subseteq X$ is closed and that $x \in X \setminus F$. Then $V_{xF} = \operatorname{int}(X \cap \bigcap_{y \in F} U_{xy})$ contains x. **P** There are u, $v \in X \cup \{-\infty, \infty\}$ such that $x \in]u, v[\subseteq X \setminus F$. If $]u, x[= \emptyset$, set $u' = u$; otherwise, let u' be the $\preccurlyeq$-least member of $]u, x[$. Similarly, if $]x, v[= \emptyset$, set $v' = v$; otherwise, let v' be the $\preccurlyeq$-least member of $]x, v[$. Then $u' < x < v'$. Now suppose that $y \in F$ and $y > x$. If $]x, v[= \emptyset$, then $U_{xy} \supseteq]-\infty, x] =]-\infty, v'[$. Otherwise, $v' \in]x, v[\subseteq]x, y[$, so $U_{xy} =]-\infty, z[$ where z is the $\preccurlyeq$-least member of $]x, y[$. But this means that $z \preccurlyeq v'$ and either $z = v'$ or $z \notin]x, v[$; in either case, $v' \leq z$ and $]-\infty, v'[\subseteq U_{xy}$.

Similarly, $]u', \infty[\subseteq U_{xy}$ whenever $y \in F$ and $y < x$. So $x \in]u', v'[\subseteq V_{xF}$. **Q**

Finally, suppose that E and F are disjoint closed sets. Set $G = \bigcup_{x \in E} V_{xF}$, $H = \bigcup_{y \in F} V_{yE}$. Then G and H are open sets including E, F respectively. If $x \in E$ and $y \in F$, then $V_{xF} \cap V_{yE} \subseteq U_{xy} \cap U_{yx} = \emptyset$, so $G \cap H = \emptyset$. As E and F are arbitrary, $\mathfrak{T}$ is normal.

(d) Let B be the set of such suprema and infima. For $x \in X$ set $A_x = A \cap]-\infty, x]$, $A'_x = A \cap [x, \infty[$. Then $x \in B$ iff either $x = \sup A_x$ or $x = \inf A'_x$, so

$$\begin{aligned} x \notin B &\iff x \neq \sup A_x \text{ and } x \neq \inf A'_x \\ &\iff \text{there are } u < x,\ v > x \text{ such that } A_x \subseteq]-\infty, u] \text{ and } A'_x \subseteq [u, \infty[\\ &\iff \text{there are } u,\ v \text{ such that } x \in]u, v[\subseteq X \setminus A \\ &\iff x \notin \overline{A}. \end{aligned}$$

Thus $B = \overline{A}$, as claimed.

(e) Because X is totally ordered, all its subsets are both upwards-directed and downwards-directed; so we have only to join the definition in 313Da to (d) above.

(f) If $x \in]u, v[$ then there is some $n \in \mathbb{N}$ such that $x_n \geq u$, and now $x_i \in]u, v[$ for every $i \geq n$.

(g)(α) If K is compact and $A \subseteq K$ is non-empty, let B be the set of upper bounds for A in $X \cup \{-\infty, \infty\}$, and set $\mathcal{G} = \{]-\infty, a[: a \in A\} \cup \{]b, \infty[: b \in B\}$. Then no finite subfamily of $\mathcal{G}$ can cover K; and if $c \in K \setminus \bigcup \mathcal{G}$ then $c = \sup A$. Similarly, any non-empty subset of K has an infimum in X which belongs to K.

(β) Now suppose that K satisfies the condition. By (d) above, it is closed. If it is empty it is certainly compact. Otherwise, $a_0 = \inf K$ and $b_0 = \sup K$ are defined in X and belong to K. Let $\mathcal{G}$ be an open cover of K. Set

$$A = \{x : x \in X,\ K \cap [a_0, x] \text{ is not covered by any finite } \mathcal{G}_0 \subseteq \mathcal{G}\}.$$

Note that A is bounded below by a_0. **?** If $b_0 \in A$, then $c = \inf A$ is defined and belongs to $[a_0, b_0]$, because X is Dedekind complete. If $c \notin K$ then there are u, v such that $c \in]u, v[\subseteq X \setminus K$; if $c \in K$ then there are u, v such that $c \in]u, v[\subseteq G$ for some $G \in \mathcal{G}$. In either case, $u \notin A$, so that $K \cap [a_0, v[\subseteq (K \cap [a_0, u]) \cup]u, v[$ is covered by a finite subset of $\mathcal{G}$, and A does not meet $[a_0, v[$, that is, $A \subseteq [v, \infty[$ and v is a lower bound of A. **X** Thus $b_0 \notin A$, and $K = K \cap [a_0, b_0]$ is covered by a finite subset of $\mathcal{G}$. As $\mathcal{G}$ is arbitrary, K is compact.

(h) Use (g).

(i) Use (h).

(j) Gaal 64, p. 36. (Compare 2A2I.)

(k) Use (h).

(l) Write W for $\{(x, y) : x < y\}$. If $x < y$, then either there is a z such that $x < z < y$, in which case $]-\infty, z[\times]z, \infty[$ is an open set containing (x, y) and included in W, or $]x, y[= \emptyset$, so $]-\infty, y[\times]x, \infty[$ is an open set containing (x, y) and included in W. Thus W is open.

Now $\{(x, y) : x \leq y\} = (X \times X) \setminus \{(x, y) : y < x\}$ is closed.

(m) The subspace topology $\mathfrak{T}_F$ on F is generated by sets of the form $F \cap]-\infty, x[$, $F \cap]x, \infty[$ where $x \in X$ (4A2B(a-iv)), while the order topology $\mathfrak{S}$ on F is generated by sets of the form $F \cap]-\infty, x[$, $F \cap]x, \infty[$ where $x \in F$. So $\mathfrak{S} \subseteq \mathfrak{T}_F$.

Now suppose that F is either compact or an interval, and that $x \in X$. If $x \in F$, or $F \cap]-\infty, x[$ is either F or $\emptyset$, then of course $F \cap]-\infty, x[\in \mathfrak{S}$. Otherwise, F meets both $]-\infty, x[$ and $[x, \infty[$ and does not contain

x, so is not an interval, and is compact. Accordingly $x' = \inf(F \cap [x, \infty[)$ is defined and belongs to F, by (g), and $F \cap]-\infty, x[= F \cap]-\infty, x'[\in \mathfrak{S}$.

Similarly, $F \cap]x, \infty[\in \mathfrak{S}$ for every $x \in X$. But this means that $\mathfrak{T}_F \subseteq \mathfrak{S}$, so the two topologies are equal, as stated.

(n)(α) Let $\mathcal{G}$ be a family of open subsets of X with union H. Set

$$\mathcal{A} = \{A : A \subseteq \bigcup \mathcal{G}_0 \text{ for some countable } \mathcal{G}_0 \subseteq \mathcal{G}\}.$$

(I seek to show that $H \in \mathcal{A}$.) Of course $\bigcup \mathcal{A}_0 \in \mathcal{A}$ for every countable subset $\mathcal{A}_0$ of $\mathcal{A}$.

(β) Let $\mathcal{C}$ be the family of members C of $\mathcal{A}$ which are order-convex (that is, $[x, y] \subseteq C$ for all x, $y \in C$), and $\mathcal{C}^*$ the family of maximal members of $\mathcal{C}$. If $C \in \mathcal{C}$ is not included in any member of $\mathcal{C}^*$, then there is a $C' \in \mathcal{C}$ such that $C \subseteq C'$ and $\operatorname{int}(C' \setminus C) \neq \emptyset$. **P** Since no member of $\mathcal{C}$ including C can be maximal, we can find $C' \in \mathcal{C}$ such that $C \subseteq C'$ and $\#(C' \setminus C) \geq 5$. Because C is order-convex, every point of $X \setminus C$ is either a lower bound or an upper bound of C, and there must be three points $x < y < z$ of $C' \setminus C$ on the same side of C. In this case,

$$y \in]x, z[\subseteq \operatorname{int}(C' \setminus C),$$

so we have an appropriate C'. **Q**

(γ) In fact, every member of $\mathcal{C}$ is included in a member of $\mathcal{C}^*$. **P?** Suppose, if possible, otherwise. Then we can choose a strictly increasing family $\langle C_\xi \rangle_{\xi < \omega_1}$ in $\mathcal{C}$ inductively, as follows. Start from any non-empty $C_0 \in \mathcal{C}$ not included in any member of $\mathcal{C}^*$. Given that $C_0 \subseteq C_\xi \in \mathcal{C}$, then C_ξ cannot be included in any member of $\mathcal{C}^*$, so by (β) above there is a $C_{\xi+1} \in \mathcal{C}$ such that $\operatorname{int}(C_{\xi+1} \setminus C_\xi)$ is non-empty. Given $\langle C_\eta \rangle_{\eta < \xi}$ where $\xi < \omega_1$ is a non-zero countable limit ordinal, set $C_\xi = \bigcup_{\eta < \xi} C_\eta$; then C_ξ is order-convex, because $\{C_\eta : \eta < \xi\}$ is upwards-directed, and belongs to $\mathcal{A}$, because $\mathcal{A}$ is closed under countable unions, so $C_\xi \in \mathcal{C}$ and the induction proceeds.

Now, however, $\langle \operatorname{int}(C_{\xi+1} \setminus C_\xi) \rangle_{\xi < \omega_1}$ is an uncountable disjoint family of non-empty open sets, and X is not ccc. **XQ**

(δ) Since $C \cup C'$ is order-convex whenever C, $C' \in \mathcal{C}$ and $C \cap C' \neq \emptyset$, $\mathcal{C}^*$ is a disjoint family. Moreover, if $x \in H$, there is some open interval containing x and belonging to $\mathcal{C}$, so $x \in \operatorname{int} C$ for some $C \in \mathcal{C}^*$; this shows that $\mathcal{C}^*$ is an open cover of H. Because X is ccc, $\mathcal{C}^*$ is countable, so $H = \bigcup \mathcal{C}^* \in \mathcal{A}$. Thus there is some countable $\mathcal{G}_0 \subseteq \mathcal{G}$ with union H; as $\mathcal{G}$ is arbitrary, X is hereditarily Lindelöf, by 4A2H(c-i).

(ϵ) By 4A2H(c-ii), X is perfectly normal.

(o)(α) Suppose that f is continuous. If $A \subseteq X$ is a non-empty set with supremum x in X, then $x \in \overline{A}$, by (d), so $f(x) \in \overline{f[A]}$ (3A3Cc) and $f(x)$ is less than or equal to any upper bound of $f[A]$; but $f(x)$ is an upper bound of $f[A]$, because f is order-preserving, so $f(x) = \sup f[A]$. Similarly, $f(\inf A) = \inf f[A]$ whenever A is non-empty and has an infimum, so f is order-continuous.

(β) Now suppose that f is order-continuous. Take any $y \in Y$ and consider $A = f^{-1}[\,]-\infty, y[\,]$, $B = X \setminus A$. If $x \in A$ then $f(x)$ cannot be $\inf f[B]$ so x cannot be $\inf B$ and there is an $x' \in X$ such that $x < x' \leq z$ for every $z \in B$; in which case $x \in]-\infty, x'[\subseteq A$. So A is open. Similarly, $f^{-1}[\,]y, \infty[\,]$ is open. By 4A2B(a-ii), f is continuous.

4A2S Order topologies on ordinals (a) Let ζ be an ordinal with its order topology.

(i) ζ is locally compact (4A2Rk); all the sets $[0, \eta] =]-\infty, \eta + 1[$, for $\eta < \zeta$, are open and compact (4A2Rh). If ζ is a successor ordinal, it is compact, being of the form $[0, \eta]$ where $\zeta = \eta + 1$.

(ii) For any $A \subseteq \zeta$, $\overline{A} = \{\sup B : \emptyset \neq B \subseteq A, \sup B < \zeta\}$. (4A2Rd, because $\inf B \in B \subseteq A$ for every non-empty $B \subseteq A$.)

(iii) If $\xi \leq \zeta$, then the subspace topology on ξ induced by the order topology of ζ is the order topology of ξ. (4A2Rm.)

(b) Give ω_1 its order topology.

(i) ω_1 is first-countable. **P** If $\xi < \omega_1$ is either zero or a successor ordinal, then $\{\xi\}$ is open so $\{\{\xi\}\}$ is a base of neighbourhoods of ξ. If ξ is a non-zero limit ordinal, there is a sequence $\langle \xi_n \rangle_{n \in \mathbb{N}}$ in ξ with supremum ξ, and $\{]\xi_n, \xi] : n \in \mathbb{N}\}$ is a base of neighbourhoods of ξ. **Q**

(ii) Singleton subsets of ω_1 are zero sets. (Assemble 4A2F(d-v), 4A2Rc and (i) above.)

(iii) If $f : \omega_1 \to \mathbb{R}$ is continuous, there is a $\xi < \omega_1$ such that $f(\eta) = f(\xi)$ for every $\eta \geq \xi$. **P?** Otherwise, we may define a strictly increasing family $\langle \zeta_\xi \rangle_{\xi<\omega_1}$ in ω_1 by saying that $\zeta_0 = 0$,

$$\zeta_{\xi+1} = \min\{\eta : \eta > \zeta_\xi,\, f(\eta) \neq f(\zeta_\xi)\}$$

for every $\xi < \omega_1$,

$$\zeta_\xi = \sup\{\zeta_\eta : \eta < \xi\}$$

for non-zero countable limit ordinals ξ. Now

$$\omega_1 = \textstyle\bigcup_{k\in\mathbb{N}} \{\xi : |f(\zeta_{\xi+1}) - f(\zeta_\xi)| \geq 2^{-k}\},$$

so there is a $k \in \mathbb{N}$ such that $A = \{\xi : |f(\zeta_{\xi+1}) - f(\zeta_\xi)| \geq 2^{-k}\}$ is infinite. Take a strictly increasing sequence $\langle \xi_n \rangle_{n\in\mathbb{N}}$ in A and set $\xi = \sup_{n\in\mathbb{N}} \xi_n = \sup_{n\in\mathbb{N}} (\xi_n + 1)$. Then $\langle \zeta_{\xi_n} \rangle_{n\in\mathbb{N}}$ and $\langle \zeta_{\xi_n+1} \rangle_{n\in\mathbb{N}}$ are strictly increasing sequences with supremum ζ_ξ, so both converge to ζ_ξ in the order topology of ω_1 (4A2Rf), and

$$f(\zeta_\xi) = \lim_{n\to\infty} f(\zeta_{\xi_n}) = \lim_{n\to\infty} f(\zeta_{\xi_n+1}).$$

But this means that

$$\lim_{n\to\infty} f(\zeta_{\xi_n}) - f(\zeta_{\xi_n+1}) = 0,$$

which is impossible, because $|f(\zeta_{\xi_n}) - f(\zeta_{\xi_n+1})| \geq 2^{-k}$ for every n. **XQ**

4A2T Upper limits of families of sets For the sake of §446 I outline some elementary ideas. Let X be a topological space, $\langle A_i \rangle_{i\in I}$ a family of subsets of X, and $\mathcal{F}$ a filter on I.

(a) $\limsup_{i\to\mathcal{F}} A_i$ is closed. If $F \subseteq X$ is closed and $A_i \subseteq F$ for every i, then $\limsup_{i\to\mathcal{F}} A_i \subseteq F$. If $\langle B_i \rangle_{i\in I}$ is another family of sets, and $B_i \subseteq A_i$ for every i, then $\limsup_{i\to\mathcal{F}} B_i \subseteq \limsup_{i\to\mathcal{F}} A_i$. If $x \in A_i$ for every i then $x \in \limsup_{i\to\mathcal{F}} A_i$. (Immediate from the definition in 4A2A.)

If X is Hausdorff, $x_i \in A_i$ for every i and $x = \lim_{i\to\mathcal{F}} x_i$ is defined in X, then $x \in \limsup_{i\to\mathcal{F}} A_i$. ($\limsup_{i\to\mathcal{F}} \{x_i\} = \{x\}$.)

If $f : X \to X$ is a homeomorphism then $f[\limsup_{i\to\mathcal{F}} A_i] = \limsup_{i\to\mathcal{F}} f[A_i]$.

(b) If $K \subseteq X$ is compact and $K \cap \limsup_{i\to\mathcal{F}} A_i = \emptyset$, then $\{i : A_i \cap K = \emptyset\} \in \mathcal{F}$. ($\{K \cap \overline{\bigcup_{i\in J} A_i} : J \in \mathcal{F}\}$ is a downwards-directed family of relatively closed subsets of K with empty intersection, so contains $\emptyset$.)

(c) Suppose that $\mathcal{F}$ is an ultrafilter. If $G \subseteq X$ is open and $G \cap \limsup_{i\to\mathcal{F}} A_i \neq \emptyset$ then $\{i : G \cap A_i \neq \emptyset\} \in \mathcal{F}$. (Otherwise, $J = \{i : G \cap A_i = \emptyset\} \in \mathcal{F}$ and

$$\limsup_{i\to\mathcal{F}} A_i \subseteq \overline{\textstyle\bigcup_{i\in J} A_i} \subseteq X \setminus G.)$$

4A2U Old friends (a) $\mathbb{R}$, with its usual topology, is metrizable (2A3Ff) and separable (the countable set $\mathbb{Q}$ is dense), so is second-countable (4A2P(a-i)). Every subset of $\mathbb{R}$ is separable (4A2P(a-iv)); in particular, every dense subset of $\mathbb{R}$ has a countable subset which is still dense.

(b) $\mathbb{N}^\mathbb{N}$ is Polish in its usual topology (4A2Qc), so has a countable network (4A2P(a-iii), or 4A2Ne), and is hereditarily Lindelöf (4A2Nb or 4A2Pd). Moreover, it is homeomorphic to $[0,1] \setminus \mathbb{Q}$(Kuratowski 66, §36.II; Kechris 95, 7.7).

(c) The map $x \mapsto \frac{2}{3} \sum_{j=0}^\infty 3^{-j} x(j)$ (cf. 134Gb) is a homeomorphism between $\{0,1\}^\mathbb{N}$ and the Cantor set $C \subseteq [0,1]$. (It is a continuous bijection.)

(d) If I is any set, then the map $A \mapsto \chi A : \mathcal{P}I \to \{0,1\}^I$ is a homeomorphism (for the usual topologies on $\mathcal{P}I$ and $\{0,1\}^I$, as described in 4A2A and 3A3K). So $\mathcal{P}I$ is zero-dimensional, compact (3A3K) and Hausdorff. If I is countable, then $\mathcal{P}I$ is metrizable, therefore Polish (4A2Qb).

4A3 Topological σ-algebras

I devote a section to some σ-algebras which can be defined on topological spaces. While 'measures' will not be mentioned here, the manipulation of these σ-algebras is an essential part of the technique of measure theory, and I will give proofs and exercises as if this were part of the main work. I look at Borel σ-algebras (4A3A-4A3J), Baire σ-algebras (4A3K-4A3P), Baire property algebras (4A3Q, 4A3R) and cylindrical σ-algebras on linear spaces (4A3T-4A3V).

4A3A Borel sets If $(X, \mathfrak{T})$ is a topological space, the **Borel σ-algebra** of X is the σ-algebra $\mathcal{B}(X)$ of subsets of X generated by $\mathfrak{T}$. Its elements are the **Borel sets** of X. If $(Y, \mathfrak{S})$ is another topological space with Borel σ-algebra $\mathcal{B}(Y)$, a function $f : X \to Y$ is **Borel measurable** if $f^{-1}[H] \in \mathcal{B}(X)$ for every $H \in \mathfrak{S}$, and is a **Borel isomorphism** if it is a bijection and $\mathcal{B}(Y) = \{F : F \subseteq Y, f^{-1}[F] \in \mathcal{B}(X)\}$, that is, f is an isomorphism between the structures $(X, \mathcal{B}(X))$ and $(Y, \mathcal{B}(Y))$.

4A3B (Σ, T)**-measurable functions** It is time I put the following idea into bold type.

(a) Let X and Y be sets, with σ-algebras $\Sigma \subseteq \mathcal{P}X$ and $\mathrm{T} \subseteq \mathcal{P}Y$. A function $f : X \to Y$ is (Σ, T)**-measurable** if $f^{-1}[F] \in \Sigma$ for every $F \in \mathrm{T}$.

(b) If Σ, T and Υ are σ-algebras of subsets of X, Y and Z respectively, and $f : X \to Y$ is (Σ, T)-measurable while $g : Y \to Z$ is (T, Υ)-measurable, then $gf : X \to Z$ is (Σ, Υ)-measurable. (If $H \in \Upsilon$, $g^{-1}[H] \in \mathrm{T}$ so $(gf)^{-1}[H] = f^{-1}[g^{-1}[H]] \in \Sigma$.)

(c) Let $\langle X_i \rangle_{i \in I}$ be a family of sets with product X, Y another set, and $f : X \to Y$ a function. If $\mathrm{T} \subseteq \mathcal{P}Y$, $\Sigma_i \subseteq \mathcal{P}X_i$ are σ-algebras for each i, then f is $(\mathrm{T}, \widehat{\bigotimes}_{i \in I} \Sigma_i)$-measurable iff $\pi_i f : Y \to X_i$ is (T, Σ_i)-measurable for every i, where $\pi_i : X \to X_i$ is the coordinate map. **P** π_i is $(\widehat{\bigotimes}_{j \in I} \Sigma_j, \Sigma_i)$-measurable, so if f is $(\mathrm{T}, \widehat{\bigotimes}_{j \in I} \Sigma_j)$-measurable then $\pi_i f$ must be (T, Σ_i)-measurable. In the other direction, if every $\pi_i f$ is measurable, then $\{H : H \subseteq X, f^{-1}[E] \in \mathrm{T}\}$ is a σ-algebra of subsets of X containing $\pi_i^{-1}[E]$ whenever $i \in I$ and $E \in \Sigma_i$, so includes $\widehat{\bigotimes}_{i \in I} \Sigma_i$, and f is measurable. **Q**

4A3C Elementary facts (a) If X is a topological space and Y is a subspace of X, then $\mathcal{B}(Y)$ is just the subspace σ-algebra $\{E \cap Y : E \in \mathcal{B}(X)\}$. **P** $\{E : E \subseteq X, E \cap Y \in \mathcal{B}(Y)\}$ and $\{E \cap Y : E \in \mathcal{B}(X)\}$ are σ-algebras containing all open sets, so include $\mathcal{B}(X)$, $\mathcal{B}(Y)$ respectively. **Q**

(b) If X is a set, Σ is a σ-algebra of subsets of X, $(Y, \mathfrak{S})$ is a topological space and $f : X \to Y$ is a function, then f is $(\Sigma, \mathcal{B}(Y))$-measurable iff $f^{-1}[H] \in \Sigma$ for every $H \in \mathfrak{S}$. **P** If f is $(\Sigma, \mathcal{B}(Y))$-measurable then $f^{-1}[H] \in \Sigma$ for every $H \in \mathfrak{S}$ just because $\mathfrak{S} \subseteq \mathcal{B}(Y)$. If $f^{-1}[H] \in \Sigma$ for every $H \in \mathfrak{S}$, then $\{F : F \subseteq Y, f^{-1}[F] \in \Sigma\}$ is a σ-algebra of subsets of Y (cf. 112E) including $\mathfrak{S}$, so includes $\mathcal{B}$, and f is $(\Sigma, \mathcal{B})$-measurable. **Q**

(c) If X and Y are topological spaces, and $f : X \to Y$ is a function, then f is Borel measurable iff it is $(\mathcal{B}(X), \mathcal{B}(Y))$-measurable. (Apply (b) with $\Sigma = \mathcal{B}(X)$.) So if X, Y and Z are topological spaces and $f : X \to Y$, $g : Y \to Z$ are Borel measurable functions, then $gf : X \to Z$ is Borel measurable. (Use 4A3Bb.)

(d) If X and Y are topological spaces and $f : X \to Y$ is continuous, it is Borel measurable. (Immediate from the definitions in 4A3A.)

(e) If X is a topological space and $f : X \to [-\infty, \infty]$ is lower semi-continuous, then it is Borel measurable. (The inverse image of a half-open interval $]\alpha, \beta]$ is a difference of open sets, so is a Borel set, and every open subset of $[-\infty, \infty]$ is a countable union of such half-open intervals.)

(f) If $\langle X_i \rangle_{i \in I}$ is a family of topological spaces with product X, then $\mathcal{B}(X) \supseteq \widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$. (Put (d) and 4A3Bc together.)

(g) Let X be a topological space.

(i) The algebra $\mathfrak{A}$ of subsets generated by the open sets is precisely the family of sets expressible as a disjoint union $\bigcup_{i\le n} G_i \cap F_i$ where every G_i is open and every F_i is closed. **P** Write $\mathcal{A}$ for the family of sets expressible in this form. Of course $\mathcal{A} \subseteq \mathfrak{A}$. In the other direction, observe that

$X \in \mathcal{A}$,

if E, $E' \in \mathcal{A}$ then $E \cap E' \in \mathcal{A}$,

if $E \in \mathcal{A}$ then $X \setminus E \in \mathcal{A}$

because if G_i is open and F_i is closed for $i \le n$, then (identifying $\{0, \dots, n\}$ with $n+1$)

$$\begin{aligned} X \setminus \bigcup_{i\le n} (G_i \cap F_i) &= \bigcap_{i\le n} (X \setminus G_i) \cup (G_i \setminus F_i) \\ &= \bigcup_{I \subseteq n+1} (\bigcap_{i\in I} (G_i \setminus F_i) \cap \bigcap_{i \in (n+1)\setminus I} (X \setminus G_i)) \end{aligned}$$

belongs to $\mathcal{A}$. So $\mathcal{A}$ is an algebra of sets and must be equal to $\mathfrak{A}$. **Q**

(ii) $\mathcal{B}(X)$ is the smallest family $\mathcal{E} \supseteq \mathfrak{A}$ such that $\bigcup_{n\in\mathbb{N}} E_n \in \mathcal{E}$ for every non-decreasing sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$ and $\bigcap_{n\in\mathbb{N}} E_n \in \mathcal{E}$ for every non-increasing sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in $\mathcal{E}$. (136G.)

4A3D Hereditarily Lindelöf spaces (a) Suppose that X is a hereditarily Lindelöf space and $\mathcal{U}$ is a subbase for the topology of X. Then $\mathcal{B}(X)$ is the σ-algebra of subsets of X generated by $\mathcal{U}$. **P** Write Σ for the σ-algebra generated by $\mathcal{U}$. Of course $\Sigma \subseteq \mathcal{B}(X)$ just because every member of $\mathcal{U}$ is open. In the other direction, set

$$\mathcal{V} = \{X\} \cup \{U_0 \cap U_1 \cap \dots \cap U_n : U_0, \dots, U_n \in \mathcal{U}\};$$

then $\mathcal{V} \subseteq \Sigma$ and $\mathcal{V}$ is a base for the topology of X (4A2B(a-i)). If $G \subseteq X$ is open, set $\mathcal{V}_1 = \{V : V \in \mathcal{V}, V \subseteq G\}$; then $G = \bigcup \mathcal{V}_1$. Because X is hereditarily Lindelöf, there is a countable set $\mathcal{V}_0 \subseteq \mathcal{V}_1$ such that $G = \bigcup \mathcal{V}_0$ (4A2H(c-i)), so that $G \in \Sigma$. Thus every open set belongs to Σ and $\mathcal{B}(X) \subseteq \Sigma$. **Q**

(b) Let X be a set, Σ a σ-algebra of subsets of X, Y a hereditarily Lindelöf space, $\mathcal{U}$ a subbase for the topology of Y, and $f : X \to Y$ a function. If $f^{-1}[U] \in \Sigma$ for every $U \in \mathcal{U}$, then f is $(\Sigma, \mathcal{B}(Y))$-measurable. **P** $\{F : F \subseteq Y, f^{-1}[F] \in \Sigma\}$ is a σ-algebra of subsets of Y including $\mathcal{U}$, so contains every open set, by (a). **Q**

(c) Let $\langle X_i\rangle_{i\in I}$ be a family of topological spaces with product X. Suppose that X is hereditarily Lindelöf.

(i) $\mathcal{B}(X) = \widehat{\bigotimes}_{i\in I} \mathcal{B}(X_i)$. **P** By 4A3Cf, $\mathcal{B}(X) \supseteq \widehat{\bigotimes}_{i\in I} \mathcal{B}(X_i)$. On the other hand, $\widehat{\bigotimes}_{i\in I} \mathcal{B}(X_i)$ is a σ-algebra including

$$\mathcal{U} = \{\pi_i^{-1}[G] : i \in I, G \subseteq X_i \text{ is open}\},$$

where $\pi_i(x) = x(i)$ for $i \in I$ and $x \in X$; since $\mathcal{U}$ is a subbase for the topology of X, (a) tells us that $\widehat{\bigotimes}_{i\in I} \mathcal{B}(X_i)$ includes $\mathcal{B}(X)$. **Q**

(ii) If Y is another topological space, then a function $f : Y \to X$ is Borel measurable iff $\pi_i f : Y \to X_i$ is Borel measurable for every $i \in I$, where $\pi_i : X \to X_i$ is the canonical map. (Use 4A3Bc.)

4A3E Applications Recall that any topological space with a countable network (in particular, any second-countable space, any separable metrizable space) is hereditarily Lindelöf (4A2Nb, 4A2Oc, 4A2P(a-iii)); and so is any ccc totally ordered space (4A2Rn). So 4A3Da-4A3Db will be applicable to these. As for product spaces, the product of a countable family of spaces with countable networks again has a countable network (4A2Ne), so for such spaces we shall be able to use 4A3Dc. For instance, $\mathcal{B}(\{0,1\}^{\mathbb{N}})$ is the σ-algebra of subsets of $\{0,1\}^{\mathbb{N}}$ generated by the sets $\{x : x(n) = 1\}$ for $n \in \mathbb{N}$.

4A3F Spaces with countable networks (a) Let X be a topological space with a countable network. Then $\#(\mathcal{B}(X)) \le \mathfrak{c}$. **P** Let $\mathcal{E}$ be a countable network for the topology of X and Σ the σ-algebra of subsets of X generated by $\mathcal{E}$. Then $\#(\Sigma) \le \mathfrak{c}$ (4A1O). If $G \subseteq X$ is open, there is a subset $\mathcal{E}'$ of $\mathcal{E}$ such that $\bigcup \mathcal{E}' = G$; but $\mathcal{E}'$ is necessarily countable, so $G \in \Sigma$. It follows that $\mathcal{B}(X) \subseteq \Sigma$ and $\#(\mathcal{B}(X)) \le \mathfrak{c}$. **Q**

(b) $\#(\mathcal{B}(\mathbb{N}^{\mathbb{N}})) = \mathfrak{c}$. **P** $\mathbb{N}^{\mathbb{N}}$ has a countable network (4A2Ub), so $\#(\mathcal{B}(\mathbb{N}^{\mathbb{N}})) \leq \mathfrak{c}$. On the other hand, $\mathcal{B}(\mathbb{N}^{\mathbb{N}})$ contains all singletons, so

$$\#(\mathcal{B}(\mathbb{N}^{\mathbb{N}})) \geq \#(\mathbb{N}^{\mathbb{N}}) \geq \#(\{0,1\}^{\mathbb{N}}) = \mathfrak{c}. \quad \mathbf{Q}$$

4A3G Second-countable spaces Suppose that X is a second-countable space and Y is any topological space. Then $\mathcal{B}(X \times Y) = \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$. **P** By 4A3Cf, $\mathcal{B}(X \times Y) \supseteq \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$. On the other hand, let $\mathcal{U}$ be a countable base for the topology of X. If $W \subseteq X \times Y$ is open, set

$$V_U = \bigcup\{H : H \subseteq Y \text{ is open}, U \times H \subseteq W\}$$

for $U \in \mathcal{U}$. Then $W = \bigcup_{U \in \mathcal{U}} U \times V_U$ belongs to $\mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$. As W is arbitrary, $\mathcal{B}(X \times Y) \subseteq \mathcal{B}(X)\widehat{\otimes}\mathcal{B}(Y)$. **Q**

4A3H Borel sets in Polish spaces: Proposition Let $(X, \mathfrak{T})$ be a Polish space and $E \subseteq X$ a Borel set. Then there is a Polish topology $\mathfrak{S}$ on X, including $\mathfrak{T}$, for which E is open.

proof Let $\mathcal{E}$ be the union of all the Polish topologies on X including $\mathfrak{T}$. Of course $X \in \mathcal{E}$. If $E \in \mathcal{E}$ then $X \setminus E \in \mathcal{E}$. **P** There is a Polish topology $\mathfrak{S} \supseteq \mathfrak{T}$ such that $E \in \mathfrak{S}$. Now both E and $X \setminus E$ are Polish in the subspace topologies $\mathfrak{S}_E$, $\mathfrak{S}_{X \setminus E}$ induced by $\mathfrak{S}$ (4A2Qd), so the disjoint union topology $\mathfrak{S}'$ of $\mathfrak{S}_E$ and $\mathfrak{S}_{X \setminus E}$ is also Polish (4A2Qe). Now $\mathfrak{S}' \supseteq \mathfrak{S} \supseteq \mathfrak{T}$ and $X \setminus E \in \mathfrak{S}'$, so $X \setminus E \in \mathcal{E}$. **Q** Moreover, the union of any sequence $\langle E_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{E}$ belongs to $\mathcal{E}$. **P** For each $n \in \mathbb{N}$ let $\mathfrak{S}_n \supseteq \mathfrak{T}$ be a Polish topology containing E_n. If m, $n \in \mathbb{N}$ then $\mathfrak{S}_m \cap \mathfrak{S}_n$ includes $\mathfrak{T}$, so is Hausdorff. By 4A2Qf, the topology $\mathfrak{S}$ generated by $\bigcup_{n \in \mathbb{N}} \mathfrak{S}_n$ is Polish. Of course $\mathfrak{S} \supseteq \mathfrak{T}$, and $\bigcup_{n \in \mathbb{N}} E_n \in \mathfrak{S}$, so $\bigcup_{n \in \mathbb{N}} E_n \in \mathcal{E}$. **Q**

Thus $\mathcal{E}$ is a σ-algebra. Since it surely includes $\mathfrak{T}$, it includes $\mathcal{B}(X, \mathfrak{T})$, as claimed.

4A3I Corollary If $(X, \mathfrak{T})$ is a Polish space and $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence of Borel subsets of X, then there is a Polish topology $\mathfrak{T}'$ on X, including $\mathfrak{T}$, for which every E_n is open-and-closed.

proof For each $n \in \mathbb{N}$ we can find a Polish topology $\mathfrak{T}_n \supseteq \mathfrak{T}$ containing E_k (if $n = 2k$ is even) or $X \setminus E_k$ (if $n = 2k+1$ is odd); now the topology $\mathfrak{T}'$ generated by $\bigcup_{n \in \mathbb{N}} \mathfrak{T}_n$ is Polish, by 4A2Qf, and every E_n is open-and-closed for $\mathfrak{T}'$.

4A3J Borel sets in ω_1: Proposition A set $E \subseteq \omega_1$ is a Borel set iff either E or its complement includes a closed cofinal set.

proof (a) Let Σ be the family of all those sets $E \subseteq \omega_1$ such that either E or $\omega_1 \setminus E$ includes a closed cofinal set. Of course Σ is closed under complements. Because the intersection of a sequence of closed cofinal sets is a closed cofinal set (4A1Bc), the union of any sequence in Σ belongs to Σ; so Σ is a σ-algebra. If E is closed, then either it is cofinal with ω_1, and is a closed cofinal set, or there is a $\xi < \omega_1$ such that $E \subseteq \xi$, in which case $\omega_1 \setminus E$ includes the closed cofinal set $\omega_1 \setminus \xi$; in either case, $E \in \Sigma$. Thus every open set belongs to Σ, and Σ includes $\mathcal{B}(\omega_1)$.

(b) Now suppose that $E \subseteq \omega_1$ is such that there is a closed cofinal set $F \subseteq \omega_1 \setminus E$. For each $\xi < \omega_1$ let $f_\xi : \xi \to \mathbb{N}$ be an injective function. Define $g : E \to \mathbb{N}$ by setting $g(\eta) = f_{\alpha(\eta)}(\eta)$, where $\alpha(\eta) = \min(F \setminus \eta)$ for $\eta \in E$. Set $A_n = g^{-1}[\{n\}]$ for $n \in \mathbb{N}$, so that $E = \bigcup_{n \in \mathbb{N}} A_n$. If $\xi \in \overline{A}_n \setminus A_n$ and $\xi' < \xi$, there must be η, $\eta' \in A_n$ such that $\xi' \leq \eta < \eta' < \xi$. But now, because $f_{\alpha(\eta')}$ is injective, while $g(\eta) = g(\eta') = n$, $\alpha(\eta) < \alpha(\eta')$, so $\alpha(\eta) \in F \cap]\xi', \xi[$. As ξ' is arbitrary, $\xi \in \overline{F} = F$. This shows that $\overline{A}_n \subseteq A_n \cup F$ and $A_n = \overline{A}_n \setminus F$ is a Borel set. This is true for every $n \in \mathbb{N}$, so $E = \bigcup_{n \in \mathbb{N}} A_n$ is a Borel set.

(c) If $E \subseteq \omega_1$ includes a closed cofinal set, then (b) tells us that $\omega_1 \setminus E$ and E belong to Σ. Thus $\Sigma \subseteq \mathcal{B}(\omega_1)$ and $\Sigma = \mathcal{B}(\omega_1)$, as claimed.

4A3K Baire sets When we come to study measures in terms of the integrals of continuous functions (§436), we find that it is sometimes inconvenient or even impossible to apply them to arbitrary Borel sets, and we need to use a smaller σ-algebra, as follows.

(a) Definition Let X be a topological space. The **Baire σ-algebra** $\mathcal{Ba}(X)$ of X is the σ-algebra generated by the zero sets. Members of $\mathcal{Ba}(X)$ are called **Baire** sets. (**Warning!** Do not confuse 'Baire sets' in this sense with 'sets with the Baire property' in the sense of 4A3Q, nor with 'sets which are Baire spaces in their subspace topologies'.)

(b) For any topological space X, $\mathcal{Ba}(X) \subseteq \mathcal{B}(X)$ (because every zero set is closed, therefore Borel). If $\mathfrak{T}$ is perfectly normal – for instance, if it is metrizable (4A2Lc), or is regular and hereditarily Lindelöf (4A2H(c-iii)) – then $\mathcal{Ba}(X) = \mathcal{B}(X)$ (because every closed set is a zero set, by 4A2Fi, so every open set belongs to $\mathcal{Ba}(X)$).

(c) Let X and Y be topological spaces, with Baire σ-algebras $\mathcal{Ba}(X)$, $\mathcal{Ba}(Y)$ respectively. If $f : X \to Y$ is continuous, it is $(\mathcal{Ba}(X), \mathcal{Ba}(Y))$-measurable. **P** Let T be the σ-algebra $\{F : F \subseteq Y, f^{-1}[F] \in \mathcal{Ba}(X)\}$. If $g : Y \to \mathbb{R}$ is continuous, then $gf : X \to \mathbb{R}$ is continuous, so

$$f^{-1}[\{y : g(y) = 0\}] = \{x : gf(x) = 0\} \in \mathcal{Ba}(X),$$

and $\{y : g(y) = 0\} \in \mathrm{T}$. Thus every zero set belongs to T, and $\mathrm{T} \supseteq \mathcal{Ba}(Y)$. **Q**

(d) In particular, if X is a subspace of Y, then $E \cap X \in \mathcal{Ba}(X)$ whenever $E \in \mathcal{Ba}(Y)$. More fundamentally, $F \cap X$ is a zero set in X for every zero set $F \subseteq Y$, just because $g \restriction X$ is continuous for any continuous $g : Y \to \mathbb{R}$.

(e) If X is a topological space and Y is a separable metrizable space, a function $f : X \to Y$ is **Baire measurable** if $f^{-1}[H] \in \mathcal{Ba}(X)$ for every open $H \subseteq Y$. Observe that f is Baire measurable in this sense iff it is $(\mathcal{Ba}(X), \mathcal{B}(Y))$-measurable iff it is $(\mathcal{Ba}(X), \mathcal{Ba}(Y))$-measurable.

4A3L Lemma Let $(X, \mathfrak{T})$ be a topological space. Then $\mathcal{Ba}(X)$ is just the smallest σ-algebra of subsets of X with respect to which every continuous real-valued function on X is measurable.

proof (a) Let $f : X \to \mathbb{R}$ be a continuous function and $\alpha \in \mathbb{R}$. Set $g(x) = \max(0, f(x) - \alpha)$ for $x \in X$; then g is continuous, so

$$\{x : f(x) \leq \alpha\} = \{x : g(x) = 0\}$$

is a zero set and belongs to $\mathcal{Ba}(X)$. As α is arbitrary, f is $\mathcal{Ba}(X)$-measurable.

(b) On the other hand, if Σ is any σ-algebra of subsets of X such that every continuous real-valued function on X is Σ-measurable, and $F \subseteq X$ is a zero set, then there is a continuous g such that $F = g^{-1}[\{0\}]$, so that $F \in \Sigma$; as F is arbitrary, $\Sigma \supseteq \mathcal{Ba}(X)$.

4A3M Product spaces Let $\langle X_i \rangle_{i \in I}$ be a family of topological spaces with product X.

(a) $\mathcal{Ba}(X) \supseteq \widehat{\bigotimes}_{i \in I} \mathcal{Ba}(X_i)$. (Apply 4A3Bc to the identity map from X to itself; compare 4A3Cf.)

(b) Suppose that X is ccc. Then every Baire subset of X is determined by coordinates in a countable set. **P** By 254Mb, the family $\mathcal{W}$ of sets determined by coordinates in countable sets is a σ-algebra of subsets of X. By 4A2E(b-ii), every continuous real-valued function is $\mathcal{W}$-measurable, so $\mathcal{W}$ contains every zero set and every Baire set. **Q**

4A3N Products of separable metrizable spaces: Proposition Let $\langle X_i \rangle_{i \in I}$ be a family of separable metrizable spaces, with product X.

(a) $\mathcal{Ba}(X) = \widehat{\bigotimes}_{i \in I} \mathcal{Ba}(X_i) = \widehat{\bigotimes}_{i \in I} \mathcal{B}(X_i)$.

(b) $\mathcal{Ba}(X)$ is the family of Borel subsets of X which are determined by coordinates in countable sets.

(c) A set $Z \subseteq X$ is a zero set iff it is closed and determined by coordinates in a countable set.

(d) If Y is a dense subset of X, then the Baire σ-algebra $\mathcal{Ba}(Y)$ of Y is just the subspace σ-algebra $\mathcal{Ba}(X)_Y$ induced by $\mathcal{Ba}(X)$.

(e) If Y is a set, T is a σ-algebra of subsets of Y, and $f : Y \to X$ is a function, then f is $(\mathrm{T}, \mathcal{Ba}(X))$-measurable iff $\pi_i f : Y \to X_i$ is $(\mathrm{T}, \mathcal{B}(X_i))$-measurable for every $i \in I$, where $\pi_i(x) = x(i)$ for $x \in X$, $i \in I$.

proof (a) X is ccc (4A2E(a-iii)), so if $f : X \to \mathbb{R}$ is continuous, there are a countable set $J \subseteq I$ and a continuous function $g : X_J \to \mathbb{R}$ such that $f = g\tilde{\pi}_J$, where $X_J = \prod_{i\in J} X_J$ and $\tilde{\pi}_J : X \to X_J$ is the canonical map (4A2E(b-ii)). Now X_J is separable and metrizable (4A2P(a-v)), therefore hereditarily Lindelöf (4A2P(a-iii)), so $\mathcal{B}(X_J) = \widehat{\bigotimes}_{i\in J}\mathcal{B}(X_i)$, by 4A3D(c-i). By 4A3Bc, $\tilde{\pi}_J$ is $(\widehat{\bigotimes}_{i\in I}\mathcal{B}(X_i), \widehat{\bigotimes}_{i\in J}\mathcal{B}(X_j))$-measurable, so f is $\widehat{\bigotimes}_{i\in J}\mathcal{B}(X_j)$-measurable. As f is arbitrary, $\mathcal{Ba}(X) \subseteq \widehat{\bigotimes}_{i\in I}\mathcal{B}(X_i)$ (4A3L). Also $\mathcal{B}(X_i) = \mathcal{Ba}(X_i)$ for every i (4A3Kb), so $\widehat{\bigotimes}_{i\in I}\mathcal{B}(X_i) = \widehat{\bigotimes}_{i\in I}\mathcal{Ba}(X_i)$. With 4A3Ma, this proves the result.

(b)(i) If $W \in \mathcal{Ba}(X)$ it is certainly a Borel set (4A3Kb), and by 4A3Mb it is determined by coordinates in a countable set.

(ii) If W is a Borel subset of X determined by coordinates in a countable subset J of X, then write $X_J = \prod_{i\in J} X_i$ and $X_{I\setminus J} = \prod_{i\in I\setminus J} X_i$; let $\tilde{\pi}_J : X \to X_J$ be the canonical map. We can identify W with $W' \times X_{I\setminus J}$, where W' is some subset of X_J. Now if $z \in X_{I\setminus J}$, $W' = \{w : w \in X_J, (w, z) \in W\}$ is a Borel subset of X_J, because $w \mapsto (w, z) : X_J \to X$ is continuous. (I am passing over the trivial case $X = \emptyset$.) Since X_J is metrizable (4A2P(a-v)), $W' \in \mathcal{Ba}(X_J)$ (4A3Kb) and $W = \tilde{\pi}_J^{-1}[W']$ is a Baire set (4A3Kc).

(c)(i) If Z is a zero set, it is surely closed; and it is determined by coordinates in a countable set by (b) above, or directly from 4A2E(b-ii).

(ii) If Z is closed and determined by coordinates in a countable set J, then (in the language of (b) above) it can be identified with $Z' \times X_{I\setminus J}$ for some $Z' \subseteq X_J$. As in the proof of (b), Z' is closed (at least, if $X_{I\setminus J} \neq \emptyset$), so is a zero set (4A2Lc), and $Z = \tilde{\pi}_J^{-1}[Z']$ is a zero set (4A2C(b-iv)).

(d)(i) $\mathcal{Ba}(Y) \supseteq \mathcal{Ba}(X)_Y$ by 4A3Kd.

(ii) Let $f : Y \to \mathbb{R}$ be any continuous function. For each $n \in \mathbb{N}$, there is an open set $G_n \subseteq X$ such that $G_n \cap Y = \{y : f(y) > 2^{-n}\}$. Now $\overline{G}_n$ is determined by coordinates in a countable set (4A2E(b-i)), so is a zero set, by (c) here. Because Y is dense in X, $\overline{G}_n = \overline{G_n \cap Y}$ does not meet $\{y : f(y) = 0\}$, and $\{y : f(y) > 0\} = Y \cap \bigcup_{n\in\mathbb{N}} \overline{G}_n$ belongs to $\mathcal{Ba}(X)_Y$. Thus $\mathcal{Ba}(X)_Y$ contains every cozero subset of Y and includes $\mathcal{Ba}(Y)$.

(e) Put (a) and 4A3Bc together.

4A3O Compact spaces (a) Let $(X, \mathfrak{T})$ be a topological space, $\mathcal{U}$ a subbase for $\mathfrak{T}$, and $\mathfrak{A}$ the algebra of subsets of X generated by $\mathcal{U}$. If $H \subseteq X$ is open and $K \subseteq H$ is compact, there is an $E \in \mathfrak{A}$ such that $K \subseteq E \subseteq H$. **P** Set $\mathcal{V} = \{X\} \cup \{U_0 \cap U_1 \cap \ldots \cap U_n : U_0, \ldots, U_n \in \mathcal{U}\}$, so that $\mathcal{V}$ is a base for $\mathfrak{T}$ and $\mathcal{V} \subseteq \mathfrak{A}$. $\{U : U \in \mathcal{V}, U \subseteq H\}$ is an open cover of the compact set K, so there is a finite set $\mathcal{U}_0 \subseteq \mathcal{V}$ such that $E = \bigcup \mathcal{U}_0$ includes K and is included in H; now $E \in \mathfrak{A}$. **Q**

(b) Let $(X, \mathfrak{T})$ be a compact space and $\mathcal{U}$ a subbase for $\mathfrak{T}$. Then every open-and-closed subset of X belongs to the algebra of subsets of X generated by $\mathcal{U}$. (If $F \subseteq X$ is open-and-closed, it is also compact; apply (a) here with $K = H = F$.)

(c) Let $(X, \mathfrak{T})$ be a compact space and $\mathcal{U}$ a subbase for $\mathfrak{T}$. Then $\mathcal{Ba}(X)$ is included in the σ-algebra of subsets of X generated by $\mathcal{U}$. **P** Let Σ be the σ-algebra generated by $\mathcal{U}$. If $Z \subseteq X$ is a zero set, there is a sequence $\langle H_n\rangle_{n\in\mathbb{N}}$ of open sets with intersection Z (4A2C(b-vi)); now we can find a sequence $\langle E_n\rangle_{n\in\mathbb{N}}$ in Σ such that $Z \subseteq E_n \subseteq H_n$ for every n, by (a), so that $Z = \bigcap_{n\in\mathbb{N}} E_n \in \Sigma$. This shows that every zero set belongs to Σ, so Σ must include $\mathcal{Ba}(X)$. **Q**

(d) If $\langle X_i\rangle_{i\in I}$ is a family of compact Hausdorff spaces with product X, then $\mathcal{Ba}(X) = \widehat{\bigotimes}_{i\in I}\mathcal{Ba}(X_i)$. **P** By 4A3Ma, $\mathcal{Ba}(X) \supseteq \widehat{\bigotimes}_{i\in I}\mathcal{Ba}(X_i)$. On the other hand, let $\mathcal{U}_i$ be the family of cozero sets in X_i for each i. Because X_i is completely regular (3A3Bb), $\mathcal{U}_i$ is a base for its topology (4A2Fc). Set

$$\mathcal{W} = \{\textstyle\prod_{i\in I} U_i : U_i \in \mathcal{U}_i \text{ for every } i \in I, \{i : U_i \neq X_i\} \text{ is finite}\},$$

so that $\mathcal{W} \subseteq \widehat{\bigotimes}_{i\in I}\mathcal{Ba}(X_i)$ is a base for the topology of X. By (c) above, $\mathcal{Ba}(X)$ is the σ-algebra generated by $\mathcal{W}$, and $\mathcal{Ba}(X) \subseteq \widehat{\bigotimes}_{i\in I}\mathcal{Ba}(X_i)$. **Q**

(e) In a compact Hausdorff zero-dimensional space the Baire σ-algebra is the σ-algebra generated by the open-and-closed sets. (Apply (c) with $\mathcal{U}$ the family of open-and-closed sets.)

(f) In particular, for any set I, $\mathcal{Ba}(\{0,1\}^I)$ is the σ-algebra generated by sets of the form $\{x : x(i) = 1\}$ as i runs over I.

4A3P Proposition The Baire σ-algebra $\mathcal{Ba}(\omega_1)$ of ω_1 is just the countable-cocountable algebra (211R).

proof We see from 4A2S(b-iii) that every continuous function is measurable with respect to the countable-cocountable algebra, so $\mathcal{Ba}(\omega_1)$ is included in the countable-cocountable algebra. On the other hand,

$$[0,\xi] = \{\eta : \eta \leq \xi\} = [0,\xi+1[= \omega_1 \setminus]\xi,\omega_1[$$

is an open-and-closed set (4A2S(a-i)), therefore a zero set, therefore belongs to $\mathcal{Ba}(\omega_1)$, for every $\xi < \omega_1$. Now if $\xi < \omega_1$, it is itself a countable set, so

$$[0,\xi[= \textstyle\bigcup_{\eta<\xi}[0,\eta] \in \mathcal{Ba}(\omega_1), \quad \{\xi\} = [0,\xi] \setminus [0,\xi[\in \mathcal{Ba}(\omega_1).$$

It follows that every countable set belongs to $\mathcal{Ba}(\omega_1)$ and the countable-cocountable algebra is included in $\mathcal{Ba}(\omega_1)$.

4A3Q Baire property If X is a topological space, a subset X has the **Baire property** if it is expressible in the form $G \triangle M$ where $G \subseteq X$ is open and $M \subseteq X$ is meager; that is, $A \subseteq X$ has the Baire property if there is an open set $G \subseteq X$ such that $G \triangle A$ is meager. (For $A = G \triangle M$ iff $M = G \triangle A$.) The family $\widehat{\mathcal{B}}(X)$ of all such sets is the **Baire property algebra** of X. (See 4A3R.) (**Warning!** do not confuse the 'Baire property algebra' $\widehat{\mathcal{B}}$ with the 'Baire σ-algebra' $\mathcal{Ba}$ as defined in 4A3K.)

4A3R Proposition Let X be a topological space.

(a) Let $A \subseteq X$ be any set.

(i) There is a unique regular open set $H \subseteq X$ such that $A \setminus H$ is meager and $H \cap G$ is empty whenever $G \subseteq X$ is open and $A \cap G$ is meager.

(ii) H is in itself a Baire space.

(iii) If A has the Baire property, $H \triangle A$ is meager.

(iv) If $A = X$, then H is the smallest comeager regular open set in X.

(b) $\widehat{\mathcal{B}}(X)$ is a σ-algebra of subsets of X including $\mathcal{B}(X)$.

(c) Write $\mathcal{M}$ for the family of meager subsets of X, so that $\mathcal{M}$ is a σ-ideal of $\widehat{\mathcal{B}}$. If X has a countable network, then the quotient algebra $\widehat{\mathcal{B}}/\mathcal{M}$ has a countable order-dense set (definition: 313J).

proof (a) (See Kechris 95, 8.29.)

(i) Set $\mathcal{G} = \{G : G \subseteq X \text{ is open, } A \cap G \text{ is meager}\}$. Let $\mathcal{G}_0 \subseteq \bigcup \mathcal{G}$ be a maximal disjoint set, and $G_0 = \bigcup \mathcal{G}_0$. Then $A \cap G_0$ is meager. **P** For each $G \in \mathcal{G}_0$, let $\langle F_{Gn}\rangle_{n\in\mathbb{N}}$ be a sequence of nowhere dense closed sets covering $A \cap G$. Set $A_n = \bigcup_{G\in\mathcal{G}_0} G \cap F_{Gn}$. If $U \subseteq X$ is any non-empty open set, either $U \cap A_n$ is empty or there is a $G \in \mathcal{G}_0$ such that $U \cap G \neq \emptyset$, in which case $U \cap G \setminus F_{Gn}$ is a non-empty open subset of U not meeting A_n. Thus A_n is nowhere dense. This is true for every n, so $A \cap G_0 \subseteq \bigcup_{n\in\mathbb{N}} A_n$ is meager. **Q**

Set $H = \operatorname{int}(X \setminus G_0)$, so that H is a regular open set (definition: 314O). $A \setminus H \subseteq (\overline{G}_0 \setminus G_0) \cup (A \cap G_0)$ is meager. If $G \in \mathcal{G}$ then $G \cap H = G \setminus \overline{G}_0$ belongs to $\mathcal{G}$ and is disjoint from every member of $\mathcal{G}_0$; by the maximality of $\mathcal{G}_0$, it is empty.

If $H' \neq H$ is another regular open set, either $H \setminus \overline{H'}$ is not empty and $A \setminus H' \supseteq A \cap H \setminus \overline{H'}$ is non-meager, or $H' \setminus \overline{H}$ is not empty and meets A in a meager set; so H is the only regular open set with the declared properties.

(ii) If now $\langle H_n\rangle_{n\in\mathbb{N}}$ is a sequence of open subsets of H which are dense in H, and $H' \subseteq H$ is any non-empty open set, $H' \setminus H_n$ is nowhere dense for every n, so $H' \setminus \bigcap_{n\in\mathbb{N}} H_n$ is meager. On the other hand, $A \cap H'$ is non-meager so H' also is, and H' must meet $\bigcap_{n\in\mathbb{N}} H_n$. As H' is arbitrary, $\bigcap_{n\in\mathbb{N}} H_n$ is dense in H; as $\langle H_n\rangle_{n\in\mathbb{N}}$ is arbitrary, H is a Baire space in its subspace topology.

(iii) If A has the Baire property, there is an open set G such that $A \triangle G$ is meager. In this case, $A \cap H \setminus \overline{G}$ must be meager, so $H \subseteq \overline{G}$ and $H \setminus A \subseteq (G \setminus A) \cup (\overline{G} \setminus G)$ is meager and $H \triangle A = (A \setminus H) \cup (H \setminus A)$ is meager.

(iv) If $A = X$, then H is comeager and is disjoint from every meager open set. So any comeager regular open set has complement disjoint from H and includes H; H is the smallest such set.

(b) (See Čech 66, §22C; Kuratowski 66, §11.III; Kechris 95, 8.22.) Of course $X = X \triangle \emptyset$ belongs to $\widehat{\mathcal{B}}(X)$. If $E \in \widehat{\mathcal{B}}(X)$, let $G \subseteq X$ be an open set such that $E \triangle G$ is meager. Then

$$(X \setminus \overline{G}) \triangle (X \setminus E) \subseteq (\overline{G} \setminus G) \cup (G \triangle E)$$

is meager, so $X \setminus E \in \widehat{\mathcal{B}}(X)$. If $\langle E_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\widehat{\mathcal{B}}(X)$, then for each $n \in \mathbb{N}$ we can find an open set G_n such that $G_n \triangle E_n$ is meager, and now

$$(\bigcup_{n \in \mathbb{N}} G_n) \triangle (\bigcup_{n \in \mathbb{N}} E_n) \subseteq \bigcup_{n \in \mathbb{N}} G_n \triangle E_n$$

is meager, so $\bigcup_{n \in \mathbb{N}} E_n \in \widehat{\mathcal{B}}(X)$.

This shows that $\widehat{\mathcal{B}}(X)$ is a σ-algebra of subsets of X. Since it contains every open set, it must include the Borel σ-algebra.

(c) Suppose that X has a countable network $\mathcal{A}$. For $A \in \mathcal{A}$, set $d_A = \overline{A}^\bullet$, the equivalence class of $\overline{A} \in \widehat{\mathcal{B}}$ in $\widehat{\mathcal{B}}/\mathcal{M}$. If $b \in \widehat{\mathcal{B}}/\mathcal{M}$ is non-zero, take $E \in \widehat{\mathcal{B}}$ such that $E^\bullet = b$, and an open set $G \subseteq X$ such that $E \triangle G \in \mathcal{M}$. Then G is not meager. Set $\mathcal{A}_1 = \{A : A \in \mathcal{A}, A \subseteq G\}$; then $\mathcal{A}_1$ is countable and $G = \bigcup \mathcal{A}_1$, so there is a non-meager $A \in \mathcal{A}_1$. In this case $\overline{A}$ is not meager, so $d_A \neq 0$, while $\overline{A} \setminus G \subseteq \overline{G} \setminus G$ is nowhere dense, so $d_A \subseteq b$. As b is arbitrary, $\{d_A : A \in \mathcal{A}\}$ is order-dense in $\widehat{\mathcal{B}}/\mathcal{M}$, and is countable because $\mathcal{A}$ is.

***4A3S** The following result will be useful in §424 and in Volume 5.

Lemma Let X and Y be sets, Σ a σ-algebra of subsets of X, T a σ-algebra of subsets of Y and $\mathcal{J}$ a σ-ideal of T. Suppose that the quotient Boolean algebra $\mathrm{T}/\mathcal{J}$ has a countable order-dense set.

(a) $\{x : x \in X, W[\{x\}] \cap A \in \mathcal{J}\}$ belongs to Σ for any $W \in \Sigma \widehat{\otimes} \mathrm{T}$ and $A \subseteq Y$.

(b) For every $W \in \Sigma \widehat{\otimes} \mathrm{T}$ there are sequences $\langle E_n \rangle_{n \in \mathbb{N}}$ in Σ, $\langle V_n \rangle_{n \in \mathbb{N}}$ in T such that $(W \triangle W_1)[\{x\}] \in \mathcal{J}$ for every $x \in X$, where $W_1 = \bigcup_{n \in \mathbb{N}} E_n \times V_n$.

proof (Compare Kechris 95, 16.1.)

(a) Let $D \subseteq \mathrm{T}/\mathcal{J}$ be a countable order-dense set. Let $\mathcal{V} \subseteq \mathrm{T}$ be a countable set such that $D = \{V^\bullet : V \in \mathcal{V}\}$. Let Λ be the family of subsets W of $X \times Y$ such that

$$W[\{x\}] \in \mathrm{T} \text{ for every } x \in X,$$

$$\{x : x \in X, W[\{x\}] \cap A \in \mathcal{J}\} \in \Sigma \text{ for every } A \subseteq Y.$$

(i) Of course $\emptyset \in \Lambda$, because if $W = \emptyset$ then $W[\{x\}] = \emptyset$ for every $x \in X$, and $\{x : W[\{x\}] \cap A \in \mathcal{J}\} = X$ for every $A \subseteq Y$.

(ii) Suppose that $W \in \Lambda$, and set $W' = (X \times Y) \setminus W$. Then

$$W'[\{x\}] = Y \setminus W[\{x\}] \in \mathrm{T}$$

for every $x \in X$. Now suppose that $A \subseteq Y$. Set $\mathcal{V}^* = \{V : V \in \mathcal{V}, A \cap V \notin \mathcal{J}\}$,

$$E = \{x : W'[\{x\}] \cap A \in \mathcal{J}\},$$

$$E' = \{x : V \cap W[\{x\}] \notin \mathcal{J} \text{ for every } V \in \mathcal{V}^*\}.$$

Then $E = E'$. **P** (α) If $x \in E$ and $V \in \mathcal{V}^*$, then $A \cap W'[\{x\}] = A \setminus W[\{x\}]$ belongs to $\mathcal{J}$, so $V \cap A \setminus W[\{x\}] \in \mathcal{J}$ and $V \cap W[\{x\}] \supseteq V \cap A \cap W[\{x\}]$ is not in $\mathcal{J}$. As V is arbitrary, $x \in E'$; as x is arbitrary, $E \subseteq E'$. (β) If $x \notin E$, then $W'[\{x\}] \cap A \notin \mathcal{J}$. Set $\mathcal{V}_1 = \{V : V \in \mathcal{V}, V \setminus W'[\{x\}] \in \mathcal{J}\}$, $D_1 = \{V^\bullet : V \in \mathcal{V}_1\}$. Then $D_1 = \{d : d \in D, d \subseteq W'[\{x\}]^\bullet\}$, so $W'[\{x\}]^\bullet = \sup D_1$ in $\mathrm{T}/\mathcal{J}$ (313K). Because $\mathcal{J}$ is a σ-ideal, the map $F \mapsto F^\bullet : \mathrm{T} \to \mathrm{T}/\mathcal{J}$ is sequentially order-continuous (313Qb), and $(\bigcup \mathcal{V}_1)^\bullet = \sup D_1$ (313Lc), that is,

$W'[\{x\}] \triangle \bigcup \mathcal{V}_1 \in \mathcal{J}$. There must therefore be a $V \in \mathcal{V}_1$ such that $V \cap A \notin \mathcal{J}$, that is, $V \in \mathcal{V}^*$. At the same time, $V \cap W[\{x\}] = V \setminus W'[\{x\}]$ belongs to $\mathcal{J}$, so V witnesses that $x \notin E'$. As x is arbitrary, $E' \subseteq E$. **Q**

Since $W \in \Lambda$,

$$E = E' = X \setminus \bigcup_{V \in \mathcal{V}^*} \{x : V \cap W[\{x\}] \in \mathcal{J}\}$$

belongs to T. As A is arbitrary, $W' \in \Lambda$. Thus the complement of any member of Λ belongs to Λ.

(iii) If $\langle W_n \rangle_{n \in \mathbb{N}}$ is any sequence in Λ with union W, then

$$W[\{x\}] = \bigcup_{n \in \mathbb{N}} W_n[\{x\}] \in \mathrm{T}$$

for every x, while

$$\{x : W[\{x\}] \cap A \in \mathcal{J}\} = \bigcap_{n \in \mathbb{N}} \{x : W_n[\{x\}] \cap A \in \mathcal{J}\} \in \Sigma$$

for every $A \subseteq Y$. So $W \in \Lambda$.

(iv) What this shows is that Λ is a σ-algebra of subsets of $X \times Y$. But if $W = E \times F$, where $E \in \Sigma$ and $F \in \mathrm{T}$, then

$$W[\{x\}] \in \{\emptyset, F\} \subseteq \mathrm{T} \text{ for every } x \in X,$$

$$\{x : W[\{x\}] \cap A \in \mathcal{J}\} \in \{X \setminus E, X\} \subseteq \Sigma$$

for every $A \subseteq Y$; so $W \in \Lambda$. Accordingly Λ must include $\Sigma \widehat{\otimes} \mathrm{T}$, as claimed.

(b) Let $\langle V_n \rangle_{n \in \mathbb{N}}$ be a sequence running over $\mathcal{V} \cup \{\emptyset\}$. For each $n \in \mathbb{N}$, set

$$E_n = \{x : V_n \setminus W[\{x\}] \in \mathcal{J}\};$$

by (a), $E_n \in \Sigma$. Set $W_1 = \bigcup_{n \in \mathbb{N}} E_n \times V_n$. Take any $x \in X$. Then $W_1[\{x\}] = \bigcup_{n \in I} V_n$ where $I = \{n : x \in E_n\}$. Since $V_n \setminus W[\{x\}] \in \mathcal{J}$ for every $n \in I$, $W_1[\{x\}] \setminus W[\{x\}] \in \mathcal{J}$. **?** If $W[\{x\}] \setminus W_1[\{x\}] \notin \mathcal{J}$, there is a $d \in D$ such that $0 \neq d \subseteq W[\{x\}]^\bullet \setminus W_1[\{x\}]^\bullet$. Let $n \in \mathbb{N}$ be such that $V_n^\bullet = d$; then $V_n \setminus W[\{x\}] \in \mathcal{J}$, so $n \in I$ and $V_n \subseteq W_1[\{x\}]$ and $d \subseteq W_1[\{x\}]^\bullet$, which is impossible. **X**

As x is arbitrary, W_1 has the required properties.

4A3T Cylindrical σ-algebras I end the section with a note on a particular type of Baire σ-algebra.

Definition Let X be a linear topological space. Then the **cylindrical σ-algebra** of X is the smallest σ-algebra Σ of subsets of X such that every continuous linear functional on X is Σ-measurable.

4A3U Proposition Let X be a linear topological space and $\mathfrak{T}_s = \mathfrak{T}_s(X, X^*)$ its weak topology. Then the cylindrical σ-algebra of X is just the Baire σ-algebra of $(X, \mathfrak{T}_s)$.

proof (a) Let $\langle f_i \rangle_{i \in I}$ be a Hamel basis for X^* (4A4Ab). For $x \in X$, set $Tx = \langle f_i(x) \rangle_{i \in I}$; then $T : X \to \mathbb{R}^I$ is a linear operator. Now $Y = T[X]$ is dense in $\mathbb{R}^I$. **P** Y is a linear subspace of $\mathbb{R}^I$, so its closure $\overline{Y}$ also is (2A5Ec). If $\phi \in (\mathbb{R}^I)^*$ is such that $\phi(Tx) = 0$ for every $x \in X$, there are a finite set $J \subseteq I$ and a family $\langle \alpha_i \rangle_{i \in J} \in \mathbb{R}^J$ such that $\phi(y) = \sum_{i \in J} \alpha_i y(i)$ for every $y \in \mathbb{R}^I$ (4A4Be). In this case, $\sum_{i \in J} \alpha_i f_i(x) = \phi(Tx) = 0$ for every $x \in X$; but $\langle f_i \rangle_{i \in I}$ is linearly independent, so $\alpha_i = 0$ for every $i \in J$ and $\phi = 0$. By 4A4Eb, $\overline{Y}$ must be the whole of $\mathbb{R}^I$. **Q**

(b)(i) Set $\mathfrak{T}'_s = \{T^{-1}[H] : H \subseteq Y \text{ is open}\}$. Then $\mathfrak{T}'_s = \mathfrak{T}_s$. **P** Because every f_i is continuous, T is continuous, so $\mathfrak{T}'_s \subseteq \mathfrak{T}_s$. On the other hand, any $f \in X^*$ is a linear combination of the f_i, so is $\mathfrak{T}'_s$-continuous, and $\mathfrak{T}_s \subseteq \mathfrak{T}'_s$. **Q**

(ii) If $g : X \to \mathbb{R}$ is $\mathfrak{T}_s$-continuous, there is a continuous $g_1 : Y \to \mathbb{R}$ such that $g = g_1 T$. **P** If $x, x' \in X$ are such that $Tx = Tx'$, then every $\mathfrak{T}_s$-open set containing one must contain the other, so $g(x) = g(x')$. This means that there is a function $g_1 : Y \to \mathbb{R}$ such that $g = g_1 T$. Next, if $U \subseteq \mathbb{R}$ is open, $T^{-1}[g_1^{-1}[U]] = g^{-1}[U]$ belongs to $\mathfrak{T}_s = \mathfrak{T}'_s$, so $g_1^{-1}[U]$ must be open in Y; as U is arbitrary, g_1 is continuous. **Q**

(c) Now let Σ be the cylindrical σ-algebra of X. Then every $f_i : X \to \mathbb{R}$ is Σ-measurable, so $T : X \to \mathbb{R}^I$ is $(\Sigma, \mathcal{B}\mathfrak{a}(\mathbb{R}^I))$-measurable, by 4A3Ne. Because Y is dense in $\mathbb{R}^I$, $\mathcal{B}\mathfrak{a}(Y)$ is the subspace σ-algebra induced by $\mathcal{B}\mathfrak{a}(\mathbb{R}^I)$ (4A3Nd). So $T : X \to Y$ is $(\Sigma, \mathcal{B}\mathfrak{a}(Y))$-measurable. Now if $g : X \to \mathbb{R}$ is a continuous function,

there is a continuous function $g_1 : Y \to \mathbb{R}$ such that $g = g_1 T$; g_1 is $\mathcal{Ba}(Y)$-measurable, so g is Σ-measurable. As this is true for every $\mathfrak{T}_s$-continuous $g : X \to \mathbb{R}$, $\mathcal{Ba}(X) \subseteq \Sigma$.

(d) On the other hand, $\Sigma \subseteq \mathcal{Ba}(X)$ just because every member of X^* is $\mathfrak{T}_s$-continuous. So $\Sigma = \mathcal{Ba}(X)$, as claimed.

4A3V Proposition Let $(X, \mathfrak{T})$ be a separable metrizable locally convex linear topological space, and $\mathfrak{T}_s = \mathfrak{T}_s(X, X^*)$ its weak topology. Then the cylindrical σ-algebra of X is also both the Baire σ-algebra and the Borel σ-algebra for both $\mathfrak{T}$ and $\mathfrak{T}_s$.

proof (a) For any linear topological space, we have

$$\Sigma \subseteq \mathcal{Ba}(X, \mathfrak{T}_s) \subseteq \mathcal{Ba}(X, \mathfrak{T}) \subseteq \mathcal{B}(X, \mathfrak{T}), \quad \mathcal{Ba}(X, \mathfrak{T}_s) \subseteq \mathcal{B}(X, \mathfrak{T}_s) \subseteq \mathcal{B}(X, \mathfrak{T}),$$

writing Σ for the cylindrical σ-algebra and $\mathcal{Ba}(X, \mathfrak{T}_s)$, $\mathcal{Ba}(X, \mathfrak{T})$, $\mathcal{B}(X, \mathfrak{T}_s)$ and $\mathcal{B}(X, \mathfrak{T})$ for the Baire and Borel σ-algebras of the two topologies. So all I have to do is to show that $\mathcal{B}(X, \mathfrak{T}) \subseteq \Sigma$.

Let $\langle \tau_n \rangle_{n \in \mathbb{N}}$ be a sequence of seminorms defining the topology of X (4A4Cf), and $D \subseteq X$ a countable dense set; for $n \in \mathbb{N}$, $\delta > 0$ and $x \in X$, set $U_n(x, \delta) = \{y : \tau_i(y - x) < \delta \text{ for every } i \leq n\}$. Then every $U_n(x, \delta)$ is a convex open set. Set

$$\mathcal{U} = \{U_n(z, 2^{-m}) : z \in D,\ m,\ n \in \mathbb{N}\}.$$

(b) If $G \subseteq X$ is any convex open set, $\overline{G} \in \Sigma$. **P** Set $\mathcal{U}_G = \{U : U \in \mathcal{U},\ U \cap G = \emptyset\}$. For each $U \in \mathcal{U}_G$, there are $f_U \in X^*$, $\alpha_U \in \mathbb{R}$ such that $f_U(x) < \alpha_U$ for every $x \in G$ and $f_U(x) > \alpha_U$ for every $x \in U$ (4A4Db). So $F = \{x : f_U(x) \leq \alpha_U \text{ for every } U \in \mathcal{U}_G\}$ belongs to Σ. Of course $F \supseteq \overline{G}$. On the other hand, if $x \notin \overline{G}$, there are m, $n \in \mathbb{N}$ such that $G \cap U_n(x, 2^{-m}) = \emptyset$; if we take $z \in D \cap U_n(x, 2^{-m-1})$, $U = U_n(z, 2^{-m-1})$ then $x \in U \in \mathcal{U}_G$, so $f_U(x) > \alpha_U$ and $x \notin F$. Thus $\overline{G} = F$ belongs to Σ. **Q**

(c) In particular, $\mathcal{V} = \{\overline{U} : U \in \mathcal{U}\}$ is included in Σ. But $\mathcal{V}$ is a countable network for $\mathfrak{T}$. So every member of $\mathfrak{T}$ is a union of countably many members of Σ and belongs to Σ. It follows at once that $\mathcal{B}(X, \mathfrak{T}) \subseteq \Sigma$, as required.

4A3X Basic exercises (a) (i) Let X be a regular space with a countable network and Y any topological space. Show that $\mathcal{B}(X \times Y) = \mathcal{B}(X) \widehat{\otimes} \mathcal{B}(Y)$. (*Hint*: 4A2Ng.) (ii) Set $X = [0, 1]$ with the topology generated by $\{X \setminus \{x\} : x \in X\}$ and $Y = [0, 1]$ with the discrete topology. Show that X has a countable network and that $\{(x, x) : x \in [0, 1]\}$ is a closed subset of $X \times Y$ not belonging to $\mathcal{B}(X) \times \mathcal{B}(Y)$.

(b) Let X be a topological space, and $E \subseteq X$. Show that the following are equiveridical: (i) $E \in \mathcal{Ba}(X)$; (ii) there are a continuous function $f : X \to [0, 1]^{\mathbb{N}}$ and $F \in \mathcal{B}([0, 1]^{\mathbb{N}})$ such that $E = f^{-1}[F]$; (iii) there are a metric space Z, a continuous function $f : X \to Z$ and an $F \in \mathcal{B}(Z)$ such that $E = f^{-1}[F]$.

>(c) Let X be a topological space, and $K \subseteq X$ a compact set such that $K \in \mathcal{Ba}(X)$. Show that K is a zero set. (*Hint*: 4A3Xb.)

(d) Let X be a compact Hausdorff space such that $\mathcal{Ba}(X) = \mathcal{B}(X)$. Show that X is perfectly normal.

(e) Let $\mathfrak{S}$ be the topology on $\mathbb{R}$ generated by the usual topology and $\{\{x\} : x \in \mathbb{R} \setminus \mathbb{Q}\}$. Show that $\mathfrak{S}$ is completely regular and Hausdorff and that $\mathbb{Q}$ is a closed Baire set which is not a zero set.

(f) Let $\langle X_i \rangle_{i \in I}$ be a family of spaces with countable networks, and X their product. Show that $\mathcal{Ba}(X) = \widehat{\bigotimes}_{i \in I} \mathcal{Ba}(X_i)$.

>(g) (i) Let I be an uncountable set with its discrete topology, and X the one-point compactification of I. Show that $\mathcal{Ba}(I)$ is not the subspace σ-algebra generated by $\mathcal{Ba}(X)$. (ii) Show that ω_1 has a subset I such that $\mathcal{Ba}(I)$ is not the subspace σ-algebra induced by $\mathcal{Ba}(\omega_1)$.

4A3Y Further exercises (a) Let X be a Čech-complete space and $\langle E_n \rangle_{n \in \mathbb{N}}$ a sequence of Borel sets in X. Show that there is a Čech-complete topology on X, finer than the given topology, with the same weight, and containing every E_n.

(b) Let $\langle X_i\rangle_{i\in I}$ be a family of regular spaces with countable networks, with product X. Show that (i) $\mathcal{B}\mathfrak{a}(X)$ is the family of Borel subsets of X which are determined by coordinates in countable sets; (ii) a set $Z \subseteq X$ is a zero set iff it is closed and determined by coordinates in a countable set; (iii) if $Y \subseteq X$ is dense, then $\mathcal{B}\mathfrak{a}(Y)$ is just the subspace σ-algebra $\mathcal{B}\mathfrak{a}(X)_Y$ induced by $\mathcal{B}\mathfrak{a}(X)$.

4A3 Notes and comments Much of this section consists of easy technicalities. It is however not always easy to guess at the exact results obtainable by these methods. It is important to notice that Baire σ-algebras on subspaces can give difficulties which do not arise with Borel σ-algebras (4A3Xg, 4A3Ca). I have expressed 4A3D in terms of 'hereditarily Lindelöf' spaces. Of course the separable metrizable spaces form by far the most important class of these, but there are others (the split interval, for instance) which are of great interest in measure theory. Similarly, there are important products of non-metrizable spaces which are ccc (e.g., 417Xu(vii)), so that 4A3Mb has something to say.

4A3H-4A3I are a most useful tool in studying Borel subsets of Polish spaces, especially in conjunction with the First Separation Theorem (422I); see 424G and 424H. I include 4A3Ya and 4A3Yb to show that some more of the arguments here can be adapted to non-separable or non-metrizable spaces.

You will note my caution in the definition of 'Baire measurable' function (4A3Ke). This is supposed to cover the case of functions taking values in $[-\infty, \infty]$ without taking a position on functions between general topological spaces.

4A4 Locally convex spaces

As in §3A5, all the ideas, and nearly all the results as stated below, are applicable to complex linear spaces; but for the purposes of this volume the real case will almost always be sufficient, and for definiteness you may take it that the scalar field is $\mathbb{R}$, except in 4A4J-4A4K. (Complex Hilbert spaces arise naturally in §445.)

4A4A Linear spaces (a) If U is a linear space, a **Hamel basis** for U is a maximal linearly independent family $\langle u_i\rangle_{i\in I}$ in U, so that every member of U is uniquely expressible as $\sum_{i\in J} \alpha_i u_i$ for some finite $J \subseteq I$, $\langle \alpha_i\rangle_{i\in J} \in (\mathbb{R} \setminus \{0\})^J$.

(b) Every linear space has a Hamel basis. (Schaefer 71, p. 10; Köthe 69, §7.3.)

(c) If U is a linear space, I write U' for the algebraic dual of U, the linear space of all linear functionals from U to $\mathbb{R}$.

4A4B Linear topological spaces (see §2A5) **(a)** If U is a linear topological space, and V is a linear subspace of U, then V, with the linear structure and topology induced by those of U, is again a linear topological space. (Bourbaki 87, I.1.3; Schaefer 71, §I.2; Köthe 69, §15.2.)

(b) If $\langle U_i\rangle_{i\in I}$ is any family of linear topological spaces, then $U = \prod_{i\in I} U_i$, with the product linear space structure and topology, is again a linear topological space. (Bourbaki 87, I.1.3; Schaefer 71, §I.2; Köthe 69, §15.4.) In particular, $\mathbb{R}^X$, with its usual linear and topological structures, is a linear topological space, for any set X.

(c) If U and V are linear topological spaces, the set of continuous linear operators from U to V is a linear subspace of the space $\mathrm{L}(U;V)$ of all linear operators from U to V. If U, V and W are linear topological spaces, and $T : U \to V$ and $S : V \to W$ are continuous linear operators, then $ST : U \to W$ is a continuous linear operator.

(d) If U is a linear topological space, I will write U^* for the **dual** of U, the space of all continuous linear functionals from U to $\mathbb{R}$ (compare 2A4H). U^* is a linear subspace of U' as defined in 4A4Ac. The **weak topology** on U, $\mathfrak{T}_s(U, U^*)$, is that defined by the method of 2A5B from the seminorms $u \mapsto |f(u)|$ as f runs through U^* (compare 2A5Ia). The **weak* topology** on U^*, $\mathfrak{T}_s(U^*, U)$, is that defined by the method of 2A5B from the seminorms $f \mapsto |f(u)|$ as u runs through U (compare 2A5Ig). By 2A5B, both are linear space topologies. If U and V are linear topological spaces, $T : U \to V$ is a continuous linear operator, and $g \in V^*$, then $gT \in U^*$ ((c) above).

(e) If $U = \prod_{i\in I} U_i$ is a product of linear topological spaces, then every element of U^* is of the form $u \mapsto \sum_{i\in J} f_i(u(i))$ where $J \subseteq I$ is finite and $f_i \in U_i^*$ for every $i \in J$. (Bourbaki 87, II.6.6; Schaefer 71, IV.4.3; Köthe 69, §22.5.)

(f) If U is a linear topological space, a set $A \subseteq U$ is **bounded** if for every neighbourhood G of 0 there is an $n \in \mathbb{N}$ such that $A \subseteq nG$. The closure of a bounded set is bounded. (Schaefer 71, I.5.1; Rudin 91, 1.13.)

(g) If U and V are linear topological spaces, $T : U \to V$ is a continuous linear operator and $A \subseteq U$ is a bounded set, then $T[A] \subseteq V$ is bounded. (Schaefer 71, I.5.4; Köthe 69, §15.6; Rudin 91, 1.32.)

(h) Let U be a linear topological space. If G is a neighbourhood of 0 in U, then its **polar** set in U^*, $G^\circ = \{f : f \in U^*, f(x) \leq 1 \text{ for every } x \in G\}$, is a $\mathfrak{T}_s(U^*, U)$-compact subset of U^* (compare 3A5F). (Schaefer 71, III.4.3; Köthe 69, §20.9; Rudin 91, 3.15.)

(i) Let U be a linear topological space. If $D \subseteq U$ is non-empty and closed under addition and multiplication by rationals, $\overline{D}$ is a linear subspace of U. **P** The linear span

$$V = \{\textstyle\sum_{i=0}^n \alpha_i u_i : u_0, \dots, u_n \in D, \alpha_0, \dots, \alpha_n \in \mathbb{R}\}$$

of

$$D = \{\textstyle\sum_{i=0}^n \alpha_i u_i : u_0, \dots, u_n \in D, \alpha_0, \dots, \alpha_n \in \mathbb{Q}\}$$

is included in $\overline{D}$, because addition and scalar multiplication are continuous; so $\overline{D} = \overline{V}$ is a linear subspace. **Q** If $A \subseteq U$ is separable, then the closed linear subspace generated by A is separable. **P** let $D_0 \subseteq A$ be a countable dense subset; then

$$D = \{\textstyle\sum_{i=0}^n \alpha_i u_i : u_0, \dots, u_n \in D_0, \alpha_0, \dots, \alpha_n \in \mathbb{Q}\}$$

is countable, and $\overline{D}$ is separable; but $\overline{D}$ is the closed linear subspace generated by A. **Q**

(j) If $\langle u_i\rangle_{i\in I}$ is an indexed family in a Hausdorff linear topological space U and $u \in U$, we say that $u = \sum_{i\in I} u_i$ if for every neighbourhood G of u there is a finite set $J \subseteq I$ such that $\sum_{i\in K} u_i \in G$ whenever $K \subseteq I$ is finite and $J \subseteq K$ (compare 226Ad).

If now V is another Hausdorff linear topological space and $T : U \to V$ is a continuous linear operator, $\sum_{i\in I} Tu_i = T(\sum_{i\in I} u_i)$ if the right-hand-side is defined. **P** Set $u = \sum_{i\in I} u_i$. If H is an open set containing Tu, then $T^{-1}[H]$ is an open set containing u, so there is a $J \in [I]^{<\omega}$ (notation: 3A1J) such that $\sum_{i\in K} u_i \in T^{-1}[H]$ and $\sum_{i\in K} Tu_i \in H$ whenever $J \subseteq K \in [I]^{<\omega}$. **Q**

(k) If U is a Hausdorff linear topological space, then any finite-dimensional linear subspace of U is closed. (Schaefer 71I.3.3; Taylor 64, 3.12-C; Rudin 91, 1.2.1.)

4A4C Locally convex spaces (a) A linear topological space U (2A5A) is **locally convex** if the convex open sets form a base for the topology.

(b) A linear topological space is locally convex iff its topology can be defined by a family of seminorms (2A5B, 2A5D). (Bourbaki 87, II.4.1; Schaefer 71, §II.4; Köthe 69, §18.1.)

(c) Let U be a linear space and τ a seminorm on U. Then $N_\tau = \{u : \tau(u) = 0\}$ is a linear subspace of X. On the quotient space U/N_τ we have a norm defined by setting $\|u^\bullet\| = \tau(u)$ for every $u \in U$. (Bourbaki 87, II.1.3; Schaefer 71, II.5.4; Rudin 91, 1.43.)

(d) Let U be a locally convex linear topological space, and T the family of continuous seminorms on U. For each $\tau \in \mathrm{T}$, write $N_\tau = \{u : \tau(u) = 0\}$, as in (c) above, and π_τ for the canonical map from U to $U_\tau = U/N_\tau$. Give each U_τ its norm, and set $\mathcal{G}_\tau = \{\pi_\tau^{-1}[H] : H \subseteq U_\tau \text{ is open}\}$. Then $\bigcup_{\tau\in\mathrm{T}} \mathcal{G}_\tau$ is a base for the topology of X closed under finite unions. (Schaefer 71, II.5.4.)

(e) A linear subspace of a locally convex linear topological space is locally convex. (Bourbaki 87, II.4.3; Köthe 69, §18.3.) The product of any family of locally convex linear topological spaces (4A4Bb) is locally convex. (Bourbaki 87, II.4.3; Köthe 69, §18.3.)

(f) If U is a metrizable locally convex linear topological space, its topology can be defined by a sequence of seminorms. (Bourbaki 87, II.4.1; Köthe 69, §18.2.)

(g) Let U be a linear space and V a linear subspace of the space U' of all linear functionals on U. Let $\mathfrak{T}_s(V,U)$ be the topology on V generated by the seminorms $f \mapsto |f(u)|$ as u runs over U (compare 4A4Bd), and let $\phi : V \to \mathbb{R}$ be a $\mathfrak{T}_s(V,U)$-continuous linear functional. Then there is a $u \in U$ such that $\phi(f) = f(u)$ for every $f \in V$. (Bourbaki 87, IV.1.1; Schaefer 71, IV.1.2; Köthe 69, §20.2; Rudin 91, 3.10; Dunford & Schwartz 57, II.3.9; Taylor 64, 3.81-A.)

(h) Grothendieck's theorem If U is a complete locally convex Hausdorff linear topological space, and ϕ is a linear functional on the dual U^* such that $\phi \restriction G^\circ$ is $\mathfrak{T}_s(U^*,U)$-continuous for every neighbourhood G of 0 in U (defining G° as in 4A4Bh), then ϕ is of the form $f \mapsto f(u)$ for some $u \in U$. (Bourbaki 87, III.3.6; Schaefer 71, IV.6.2; Köthe 69, §21.9.)

4A4D Hahn-Banach theorem (a) Let U be a linear space and $\theta : U \to [0,\infty[$ a seminorm.

(i) If $V \subseteq U$ is a linear subspace and $g : V \to \mathbb{R}$ is a linear functional such that $|g(v)| \leq \theta(v)$ for every $v \in V$, then there is a linear functional $f : U \to \mathbb{R}$, extending g, such that $|f(u)| \leq \theta(u)$ for every $u \in U$.

(ii) If $u_0 \in U$ then there is a linear functional $f : U \to \mathbb{R}$ such that $f(u_0) = \theta(u_0)$ and $|f(u)| \leq \theta(u)$ for every $u \in U$. (Bourbaki 87, II.3.2; Rudin 91, 3.3; Dunford & Schwartz 57, II.3.11; Taylor 64, 3.7-C.)

(b) Let U be a linear topological space and G, H two disjoint convex sets in U, of which one has non-empty interior. Then there are an $f \in U^*$ and an $\alpha \in \mathbb{R}$ such that $f(u) \leq \alpha \leq f(v)$ for every $u \in G$, $v \in H$. If G and H are both open, then $f(u) < \alpha < f(v)$ for every $u \in G$, $v \in H$. (Bourbaki 87, II.5.2; Schaefer 71, II.9.1; Köthe 69, §17.1.)

4A4E The Hahn-Banach theorem in locally convex spaces Let U be a locally convex linear topological space.

(a) If $V \subseteq U$ is a linear subspace, then every member of V^* extends to a member of U^* (compare 3A5Ab). (Bourbaki 87, II.4.1; Schaefer 71, II.4.2; Köthe 69, §20.1; Rudin 91, 3.6; Taylor 64, 3.8-D.)

Consequently $\mathfrak{T}_s(V,V^*)$ is just the subspace topology on V induced by $\mathfrak{T}_s(U,U^*)$.

(b) Let $C \subseteq U$ be a non-empty closed convex set. If $u \in U$ then $u \in C$ iff $f(u) \leq \sup_{v\in C} f(v)$ for every $f \in U^*$ iff $f(u) \geq \inf_{v\in C} f(v)$ for every $f \in U^*$. (Bourbaki 87, II.5.3; Schaefer 71, II.9.2; Köthe 69, §20.7; Dunford & Schwartz 57, V.2.12.)

If $V \subseteq U$ is a closed linear subspace and $u \in U \setminus V$ there is an $f \in U^*$ such that $f(u) \neq 0$ and $f(v) = 0$ for every $v \in V$. (Bourbaki 87, II.5.3; Köthe 69, §20.1; Rudin 91, 3.5; Taylor 64, 3.8-E.)

(c) If U is Hausdorff, U^* separates its points (compare 3A5Ae). (Bourbaki 87, II.4.1; Rudin 91, 3.4.)

(d) If $u \in U$ belongs to the $\mathfrak{T}_s(U,U^*)$-closure of a convex set $C \subseteq U$, it belongs to the closure of C (compare 3A5Ee). (Schaefer 71, II.9.2; Köthe 69, §20.7; Rudin 91, 3.12.) In particular, if C is closed, it is $\mathfrak{T}_s(U,U^*)$-closed. (Dunford & Schwartz 57, V.2.13.)

(e) If C, $C' \subseteq U$ are disjoint non-empty closed convex sets, of which one is compact, there is an $f \in U^*$ such that $\sup_{u\in C} f(u) < \inf_{u\in C'} f(u)$. (Apply (b) to $C - C'$. See Bourbaki 87, II.5.3; Schaefer 71, II.9.2; Köthe 69, §20.7; Rudin 91, 3.4; Dunford & Schwartz 57, V.3.13.)

(f) Let V be a linear subspace of U'. Let $K \subseteq U$ be a non-empty $\mathfrak{T}_s(U,V)$-compact convex set, and $\phi_0 : V \to \mathbb{R}$ a linear functional such that $\phi_0(f) \leq \sup_{u\in K} f(u)$ for every $f \in V$. Then there is a $u_0 \in K$ such that $\phi_0(f) = f(u_0)$ for every $f \in V$. **P** Give U the topology $\mathfrak{T}_s(U,V)$ and V' the topology $\mathfrak{T}_s(V',V)$

(4A4Cg). For $u \in U$, $f \in V$ set $\hat{u}(f) = f(u)$; then $u \mapsto \hat{u}$ is a continuous linear operator from U to V' (use 2A3H), so $\hat{K} = \{\hat{u} : u \in K\}$ is a compact convex subset of V'.

? Suppose, if possible, that $\phi_0 \notin \hat{K}$. By (b), there is a continuous linear functional $\boldsymbol{\theta} : V' \to \mathbb{R}$ such that $\boldsymbol{\theta}(\phi) > \sup_{u \in K} \boldsymbol{\theta}(\hat{u})$. But there is an $f \in V$ such that $\boldsymbol{\theta}(\phi) = \phi(f)$ for every $\phi \in V'$ (4A4Cg), so that $\phi_0(f) > \sup_{u \in K} f(u)$, contrary to hypothesis. **X** So there is a $u_0 \in K$ such that $\phi_0 = \hat{u}_0$, as claimed. **Q**

(g) The Bipolar Theorem Let $A \subseteq U'$ be a non-empty set. Set $A^\circ = \{u : u \in U, f(u) \leq 1 \text{ for every } f \in A\}$ (compare 4A4Bh). If $g \in U'$ is such that $g(u) \leq 1$ for every $u \in A^\circ$, then g belongs to the $\mathfrak{T}_s(U', U)$-closed convex hull of $A \cup \{0\}$. **P** Put 4A4Cg and (b) above together, as in (f). See Bourbaki 87, II.6.3; Schaefer 71, IV.1.5; Köthe 69, 20.8. **Q**

4A4F The Mackey topology Let U be a linear space and V a linear subspace of U'. The **Mackey topology** $\mathfrak{T}_k(V, U)$ on V is the topology of uniform convergence on convex $\mathfrak{T}_s(U, V)$-compact subsets of U. Every $\mathfrak{T}_k(V, U)$-continuous linear functional on V is of the form $f \mapsto f(u)$ for some $u \in U$ (use 4A4Ef). So every $\mathfrak{T}_k(V, U)$-closed convex set is $\mathfrak{T}_s(V, U)$-closed, by 4A4Ed. (See Bourbaki 87, IV.1.1; Schaefer 71, IV.3.2; Köthe 69, 21.4.)

4A4G Extreme points (a) Let X be a real linear space, and $C \subseteq X$ a convex set. An element of C is an **extreme** point of C if it is not expressible as a convex combination of two other members of C; equivalently, if it is not expressible as $\frac{1}{2}(x + y)$ where $x, y \in C$ are distinct.

(b) The Kreĭn-Mil'man theorem Let U be a Hausdorff locally convex linear topological space and $K \subseteq U$ a compact convex set. Then K is the closed convex hull of the set of its extreme points. (Bourbaki 87, II.7.1; Schaefer 71, II.10.4; Köthe 69, §25.1; Rudin 91, 3.22.)

(c) Let U and V be Hausdorff locally convex linear topological spaces, $T : U \to V$ a continuous linear operator, $K \subseteq X$ a compact convex set and v any extreme point of $T[K] \subseteq V$. Then there is an extreme point u_0 of K such that $Tu_0 = v$. **P** Set $K_1 = \{u : u \in K, Tu = v\}$. Then K_1 is a compact convex set so has an extreme point u_0. **?** If u_0 is not an extreme point of K, it is expressible as $\alpha u_1 + (1-\alpha)u_2$ where u_1, u_2 are distinct points of K and $\alpha \in]0, 1[$. So $v = \alpha Tu_1 + (1-\alpha)Tu_2$, and we must have $Tu_1 = Tv = Tu_2$, because v is an extreme point of $T[K]$; but this means that u_1, $u_2 \in K_1$ and u_0 is not an extreme point of K_1. **X** So u_0 has the required properties. **Q**

4A4H Proposition Let I be a set, W a closed linear subspace of $\mathbb{R}^I$, U a linear topological space and V a Hausdorff linear topological space. Let $K \subseteq U$ be a compact set and $T : U \times \mathbb{R}^I \to V$ a continuous linear operator. Then $T[K \times W]$ is closed.

proof Take $v_0 \in \overline{T[K \times W]}$. Let $J \subseteq I$ be a maximal set such that

> whenever $L \subseteq J$ is finite and $H \subseteq V$ is an open set containing v_0, there are a $u \in K$ and an $x \in W$ such that $x(i) = 0$ for every $i \in L$ and $T(u, x) \in H$.

Let $\mathcal{F}$ be the filter on $U \times \mathbb{R}^I$ generated by the closed set $K \times W$, the sets $\{(u, x) : x(i) = 0\}$ for $i \in J$, and the sets $T^{-1}[H]$ for open sets H containing v_0. Then for any $j \in I$ there is an $F \in \mathcal{F}$ such that $\{x(j) : (u, x) \in F\}$ is bounded. **P?** Suppose, if possible, otherwise. Then, in particular, $j \notin J$. So there must be a finite set $L \subseteq J$ and an open set H containing v_0 such that $x(j) \neq 0$ whenever $u \in K$, $x \in W$, $x(i) = 0$ for every $i \in L$ and $T(u, x) \in H$. By 2A5C, or otherwise, there is a neighbourhood G_0 of 0 in V such that $v_0 + G_0 - G_0 \subseteq H$ and $\alpha v \in G_0$ whenever $v \in G_0$ and $|\alpha| \leq 1$. Fix $u^* \in K$, $x^* \in W$ such that $x^*(i) = 0$ for every $i \in L$ and $T(u^*, x^*) \in v_0 + G_0$. If $x \in W$ and $x(i) = 0$ for every $i \in L$ and $x(j) = x^*(j)$, then $T(u^*, x^* - x) \notin v_0 + G_0 - G_0$ so $T(0, x) \notin G_0$. It follows that $T(0, x) \notin G_0$ whenever $x \in W$ and $x(i) = 0$ for every $i \in L$ and $|x(j)| \geq |x^*(j)|$.

Let G be a neighbourhood of 0 in V such that $G + G - G - G \subseteq G_0$. We are supposing that $\{x(j) : (u, x) \in F\}$ is unbounded for every $F \in \mathcal{F}$. So for every $n \in \mathbb{N}$ there are $u_n \in K$ and $x_n \in W$ such that $x_n(i) = 0$ for $i \in L$, $T(u_n, x_n) \in v_0 + G$ and $|x_n(j)| \geq n$. Let $u \in K$ be a cluster point of $\langle u_n \rangle_{n \in \mathbb{N}}$. Then $T(u, 0)$ is a cluster point of $\langle T(u_n, 0) \rangle_{n \in \mathbb{N}}$, so $M = \{n : T(u_n, 0) \in T(u, 0) + G\}$ is infinite. For $n \in M$, $T(0, x_n) = T(u_n, x_n) - T(u_n, 0) \in v_0 - T(u, 0) + G - G$; so if $m, n \in M$, $T(0, x_m - x_n) \in G - G - (G - G) \subseteq G_0$.

But note now that $(x_m - x_n)(i) = 0$ for all m, $n \in \mathbb{N}$ and $i \in L$, and that because M is infinite there are certainly m, $n \in M$ such that $|x_m(j) - x_n(j)| \geq |x^*(j)|$; which contradicts the last paragraph. **XQ**

Now let $\mathcal{G}$ be any ultrafilter on $U \times \mathbb{R}^I$ including $\mathcal{F}$. Then for every $i \in I$ there is a $\gamma_i < \infty$ such that $\mathcal{G}$ contains $\{(u, x) : |x(i)| \leq \gamma_i\}$. It follows that $\mathcal{G}$ has a limit $(\hat{u}, \hat{x})$ in $K \times W$. Now the image filter $T[[\mathcal{G}]]$ (2A1Ib) converges to $T(\hat{u}, \hat{x})$; since $T[[\mathcal{F}]] \to v_0$, and the topology of V is Hausdorff, $v_0 = T(\hat{u}, \hat{x}) \in T[K \times W]$. As v_0 is arbitrary, $T[K \times W]$ is closed.

4A4I Normed spaces (a) Two norms $\| \|$, $\| \|'$ on a linear space U give rise to the same topology iff they are **equivalent** in the sense that, for some $M \geq 0$,

$$\|x\| \leq M\|x\|', \quad \|x\|' \leq M\|x\|$$

for every $x \in U$. (Köthe 69, §14.2; Taylor 64, 3.1-D; Jameson 74, 2.8.)

(b) If U and V are normed spaces, $T : U \to V$ is a linear operator and $gT : X \to \mathbb{R}$ is continuous for every $g \in V^*$, then T is a bounded operator. (Jameson 74, 27.6.)

(c) If U is any normed space, its dual U^*, under its usual norm (2A4H), is a Banach space. (Rudin 91, 4.1; Dunford & Schwartz 57, II.3.9; Köthe 69, §14.5.)

(d) If U is a separable normed space, its dual U^* (regarded as a normed space, as in 2A4H) is isometrically isomorphic to a closed linear subspace of ℓ^∞. **P** Let $\langle x_n \rangle_{n \in \mathbb{N}}$ run over a dense subset of the unit ball of U (4A2P(a-iv)); define $T : U^* \to \ell^\infty$ by setting $(Tf)(n) = f(x_n)$ for every n. T is a linear isometry between U^* and $T[U^*]$, which is closed because U^* is complete (4A4Ic, 3A4Fd). **Q**

(e) Let U be a Banach space. Suppose that $\langle u_i \rangle_{i \in I}$ is a family in U such that $\gamma = \sum_{i \in I} \|u_i\| < \infty$.

(i) $\sum_{i \in I} u_i$ is defined in the sense of 4A4Bj. **P** For $J \in [I]^{<\omega}$, set $v_J = \sum_{i \in J} u_i$. For each $n \in \mathbb{N}$, there is a $J_n \in [I]^{<\omega}$ such that $\gamma - \sum_{i \in K} \|u_i\| \leq 2^{-n}$ whenever $J_n \subseteq K \in [I]^{<\omega}$. Now

$$\|v_{J_m} - v_{J_n}\| \leq \sum_{i \in J_m \triangle J_n} \|u_i\| \leq 2^{-m} + 2^{-n}$$

for all m, $n \in \mathbb{N}$, so $\langle v_{J_n} \rangle_{n \in \mathbb{N}}$ is a Cauchy sequence and has a limit v say. If now $n \in \mathbb{N}$ and $J_n \subseteq K \in [I]^{<\omega}$,

$$\begin{aligned} \|v - \sum_{i \in K} u_i\| &= \lim_{m \to \infty} \|v_{J_m} - \sum_{i \in K} u_i\| \leq \limsup_{m \to \infty} \sum_{i \in J_m \triangle K} \|u_i\| \\ &\leq \lim_{m \to \infty} 2^{-m} + 2^{-n} = 2^{-n}, \end{aligned}$$

so $v = \sum_{i \in I} u_i$. **Q**

(ii) Now if $\langle I_j \rangle_{j \in J}$ is any partition of I, $w_j = \sum_{i \in I_j} u_i$ is defined for every j, and $\sum_{j \in J} w_j$ is defined and equal to $\sum_{i \in I} u_i$.

(f) Let U be a normed space. For $u \in U$, define $\hat{u} \in U^{**} = (U^*)^*$ by setting $\hat{u}(f) = f(u)$ for every $f \in U^*$. Then $\{\hat{u} : u \in U, \|u\| \leq 1\}$ is weak*-dense in $\{\phi : \phi \in U^{**}, \|\phi\| = 1\}$. (Apply 4A4Eg with $A = \{\hat{u} : \|u\| \leq 1\}$.)

4A4J Inner product spaces (a) Let U be an inner product space over $\begin{matrix}\mathbb{R}\\ \mathbb{C}\end{matrix}$ (3A5L). An **orthonormal basis** in U is a family $\langle e_i \rangle_{i \in I}$ in U such that (α) $(e_i|e_j) = 0$ if $i \neq j$, 1 if $i = j$ (β) the closed linear subspace of U generated by $\{e_i : i \in I\}$ is U itself.

(b) If U, V are inner product spaces over $\mathbb{R}$ and $T : U \to V$ is an isometry such that $T(0) = 0$, then $(Tu|Tv) = (u|v)$ for all u, $v \in U$ and T is linear. **P** (α)

$$(Tu|Tv) = \tfrac{1}{2}(\|Tu\|^2 + \|Tv\|^2 - \|Tu - Tv\|^2) = \tfrac{1}{2}(\|u\|^2 + \|v\|^2 - \|u - v\|^2) = (u|v).$$

(β) For any u, $v \in U$,

$$\begin{aligned}\|T(u+v)-Tu-Tv\|^2 &= \|T(u+v)\|^2+\|Tu\|^2+\|Tv\|^2\\ &\qquad -2(T(u+v)|Tu)-2(T(u+v)|Tv)+2(Tu|Tv)\\ &= \|u+v\|^2+\|u\|^2+\|v\|^2\\ &\qquad -2(u+v|u)-2(u+v|v)+2(u|v)\\ &= 0.\end{aligned}$$

So T is additive. (γ) Consequently $T(qu)=qTu$ for every $u\in U$ and $q\in\mathbb{Q}$; as T is continuous, it is linear. **Q**

(c) If U is an inner product space over $\begin{matrix}\mathbb{R}\\ \mathbb{C}\end{matrix}$, a linear operator $T:U\to U$ is **self-adjoint** if $(Tu|v)=(u|Tv)$ for all u, $v\in U$.

(d) If U is a finite-dimensional inner product space over $\mathbb{R}$, it is isomorphic to Euclidean space $\mathbb{R}^r$, where $r=\dim U$. (Taylor 64, 3.21-A.) In particular, any finite-dimensional inner product space is a Hilbert space.

(e) If U is an inner product space over $\begin{matrix}\mathbb{R}\\ \mathbb{C}\end{matrix}$ and $V\subseteq U$ is a linear subspace of U, then $V^\perp=\{x : x\in U, (x|y)=0 \text{ for every } y\in V\}$ is a linear subspace of U, and $\|x+y\|^2=\|x\|^2+\|y\|^2$ for $x\in V$, $y\in V^\perp$. If V is complete (in particular, if V is finite-dimensional), then $U=V\oplus V^\perp$. (Rudin 91, 12.4; Bourbaki 87, V.1.6; Taylor 64, 4.82-A.)

(f) If U is an inner product space over $\mathbb{R}$ and v_1, $v_2\in U$ are such that $\|v_1\|=\|v_2\|=1$, there is a linear operator $T:U\to U$ such that $Tv_1=v_2$ and $\|Tu\|=\|u\|$ and $\|Tu-u\|\le\|v_1-v_2\|\|u\|$ for every $u\in U$. **P** If v_2 is a multiple of v_1, say $v_2=\alpha v_1$, take $Tu=\alpha u$ for every u. Otherwise, set $w=v_2-(v_2|v_1)v_1$ and $w_1=\frac{1}{\|w\|}w$, so that $v_2=\cos\theta v_1+\sin\theta w_1$, where $\theta=\arccos(v_2|v_1)$. Let V be the two-dimensional linear subspace of U generated by v_1 and w_1, so that $U=V\oplus V^\perp$. Define a linear operator $T:U\to U$ by saying that $Tv_1=v_2$, $Tw_1=\cos\theta w_1-\sin\theta v_1$ and $Tu=u$ for $u\in V^\perp$. Then T acts on V as a simple rotation through an angle θ, so $\|Tv\|=1$ and $\|Tv-v\|=\|v_2-v_1\|$ whenever $v\in V$ and $\|v\|=1$; generally, if $u\in U$, then $\|Tu\|=\|u\|$ and

$$\|Tu-u\|=\|T(Pu)-Pu\|=\|v_2-v_1\|\|Pu\|\le\|v_2-v_1\|\|u\|,$$

where P is the orthogonal projection of U onto V. **Q**

(g) Let U be an inner product space over $\begin{matrix}\mathbb{R}\\ \mathbb{C}\end{matrix}$, and $\langle u_i\rangle_{i\in I}$ a countable family in U. Then there is a countable orthonormal family $\langle v_j\rangle_{j\in J}$ in U such that $\{v_j : j\in J\}$ and $\{u_i : i\in I\}$ span the same linear subspace of U. **P** We can suppose that $I\subseteq\mathbb{N}$; set $u_i=0$ for $i\in\mathbb{N}\setminus I$. Define $\langle v_n\rangle_{n\in\mathbb{N}}$ inductively by setting $v_n'=u_n-\sum_{i<n}(u_n|v_i)v_i$, $v_n=0$ if $v_n'=0$, $\frac{1}{\|v_n'\|}v_n'$ otherwise. Set $J=\{n : v_n\neq 0\}$. **Q**

4A4K Hilbert spaces (a) If U is a real or complex Hilbert space, its unit ball is compact in the weak topology $\mathfrak{T}_s(U,U^*)$; any bounded set is relatively compact for $\mathfrak{T}_s(U,U^*)$. (Bourbaki 87, V.1.7; Dunford & Schwartz 57, IV.4.6.)

(b) If U is a real or complex Hilbert space, any norm-bounded sequence in U has a weakly convergent subsequence. (462D; Dunford & Schwartz 57, IV.4.7.)

(c) If U is a real or complex Hilbert space and $\langle u_i\rangle_{i\in I}$ is any orthonormal family in U, then it can be extended to an orthonormal basis. (Dunford & Schwartz 57, IV.4.10; Taylor 64, 3.2-I.) In particular, U has an orthonormal basis.

4A4L Compact operators (see 3A5Ka) **(a)** Let U, V and W be Banach spaces. If $T\in\mathrm{B}(U;V)$ and $S\in\mathrm{B}(V;W)$ and either S or T is a compact operator, then ST is compact. (Dunford & Schwartz 57, VI.5.4; Jameson 74, 34.2.)

(b) If U is a Banach space, $T \in \mathrm{B}(U;U)$ is a compact linear operator and $\gamma \neq 0$ then $\{u : Tu = \gamma u\}$ is finite-dimensional. (Rudin 91, 4.18; Taylor 64, 5.5-C; Dunford & Schwartz 57, VII.4.5; Jameson 74, 34.8.)

4A4M Self-adjoint compact operators If U is a Hilbert space and $T : U \to U$ is a self-adjoint compact linear operator, then $T[U]$ is the closed linear span of $\{Tv : v \text{ is an eigenvector of } T\}$. (Taylor 64, 6.4-B.)

4A4N Max-flow Min-cut Theorem (Ford & Fulkerson 56) Let (V, E, γ) be a (finite) transportation network, that is,

V is a finite set of 'vertices',

$E \subseteq \{(v, v') : v, v' \in V, v \neq v'\}$ is a set of (directed) 'edges',

$\gamma : E \to [0, \infty[$ is a function;

we regard a member $e = (v, v')$ of E as 'starting' at v and 'ending' at v', and $\gamma(e)$ is the 'capacity' of the edge e. Suppose that v_0, $v_1 \in V$ are distinct vertices such that no edge ends at v_0 and no edge starts at v_1. Then we have a 'flow' $\phi : E \to [0, \infty[$ and a 'cut' $X \subseteq E$ such that

(i) for every $v \in V \setminus \{v_0, v_1\}$,

$$\sum_{e \in E, e \text{ starts at } v} \phi(e) = \sum_{e \in E, e \text{ ends at } v} \phi(e),$$

(ii) $\phi(e) \leq \gamma(e)$ for every $e \in E$,

(iii) there is no path from v_0 to v_1 using only edges in $E \setminus X$,

(iv) $\sum_{e \in E, e \text{ starts at } v_0} \phi(e) = \sum_{e \in E, e \text{ ends at } v_1} \phi(e) = \sum_{e \in X} \gamma(e)$.

(Bollobás 79, §III.1; Anderson 87, 12.3.1.)

4A5 Topological groups

For Chapter 44 we need a variety of facts about topological groups. Most are essentially elementary, and all the non-trivial ideas are covered by at least one of Császár 78 and Hewitt & Ross 63. In 4A5A-4A5C and 4A5I I give some simple definitions concerning groups and group actions. Topological groups, properly speaking, appear in 4A5D. Their simplest properties are in 4A5E-4A5G. I introduce 'right' and 'bilateral' uniformities in 4A5H; the latter are the more interesting (4A5M-4A5O), but the former are also important (4A5P). 4A5J-4A5L deal with quotient spaces, including spaces of cosets of non-normal subgroups. I conclude with notes on metrizable groups (4A5Q-4A5S).

4A5A Notation If X is a group, $x_0 \in X$, and A, $B \subseteq X$ I write

$$x_0 A = \{x_0 x : x \in A\}, \quad Ax_0 = \{xx_0 : x \in A\},$$

$$AB = \{xy : x \in A, y \in B\}, \quad A^{-1} = \{x^{-1} : x \in A\}.$$

A is **symmetric** if $A = A^{-1}$. Observe that $(AB)C = A(BC)$, $(AB)^{-1} = B^{-1}A^{-1}$ for any A, B, $C \subseteq X$.

4A5B Group actions (a) If X is a group and Z is a set, an **action** of X on Z is a function $(x, z) \mapsto x \bullet z : X \times Z \to Z$ such that

$$(xy) \bullet z = x \bullet (y \bullet z) \text{ for all } x, y \in X \text{ and } z \in Z,$$

$$e \bullet z = z \text{ for every } z \in Z$$

where e is the identity of X.

In this context I may say that 'X acts on Z', taking the operation $\bullet$ for granted.

(b) An action $\bullet$ of a group X on a set Z is **transitive** if for every w, $z \in Z$ there is an $x \in X$ such that $x \bullet w = z$.

(c) If $\bullet$ is an action of a group X on a set Z, I write $x \bullet A = \{x \bullet z : z \in A\}$ whenever $x \in X$ and $A \subseteq Z$.

(d) If $\bullet$ is an action of a group X on a set Z, then $z \mapsto x\bullet z : Z \to Z$ is a bijection for every $x \in X$. (For it has an inverse $z \mapsto x^{-1}\bullet z$.) So if Z is a topological space and $z \mapsto x\bullet z$ is continuous for every x, it is a homeomorphism for every x.

(e) An action $\bullet$ of a group X on a set Z is **faithful** if whenever x, $y \in X$ are distinct there is a $z \in Z$ such that $x\bullet z \neq y\bullet z$; that is, the natural homomorphism from X to the group of bijections from Z to itself is injective. An action of X on Z is faithful iff for any $x \in X$ which is not the identity there is a $z \in Z$ such that $x\bullet z \neq z$.

(f) If $\bullet$ is an action of a group X on a set Z, then $Y_z = \{x : x \in X, x\bullet z = z\}$ is a subgroup of X (the **stabilizer** of z) for every $z \in Z$. If $\bullet$ is transitive, then Y_w and Y_z are conjugate subgroups for all w, $z \in Z$. (If $x\bullet w = z$, then $Y_z = xY_wx^{-1}$.)

4A5C Examples Let X be any group.

(a) Write

$$x\bullet_l y = xy, \quad x\bullet_r y = yx^{-1}, \quad x\bullet_c y = xyx^{-1}$$

for x, $y \in X$. These are all actions of X on itself, the **left**, **right** and **conjugacy** actions.

(b) If $A \subseteq X$, we have an action of X on the set $\{xA : x \in X\}$ defined by setting $x\bullet(yA) = xyA$ for x, $y \in X$.

4A5D Definitions (a) A **topological group** is a group X endowed with a topology such that the operations $(x, y) \mapsto xy : X \times X \to X$ and $x \mapsto x^{-1} : X \to X$ are continuous.

(b) A **Polish group** is a topological group in which the topology is Polish (definition: 4A2A).

4A5E Elementary facts Let X be any topological group.

(a) For any $x \in X$, the functions $y \mapsto xy$, $y \mapsto yx$ and $y \mapsto y^{-1}$ are all homeomorphisms from X to itself. (Hewitt & Ross 63, 4.2; Folland 95, 2.1.)

(b) The maps $(x, y) \mapsto x^{-1}y$, $(x, y) \mapsto xy^{-1}$ and $(x, y) \mapsto xyx^{-1}$ from $X \times X$ to X are continuous.

(c) $\{G : G \text{ is open}, e \in G, G^{-1} = G\}$ is a base of neighbourhoods of the identity e of X. (Hewitt & Ross 63, 4.6; Folland 95, 2.1.)

(d) If $G \subseteq X$ is an open set, then AG and GA are open for any set $A \subseteq X$. (Hewitt & Ross 63, 4.4.)

(e) If $F \subseteq X$ is closed and $x \in X \setminus F$, there is a neighbourhood U of e such that $UxUU \cap FUU = \emptyset$. **P** Set $U_1 = X \setminus x^{-1}F$. Let U_2 be a neighbourhood of e such that $U_2U_2U_2U_2^{-1}U_2^{-1} \subseteq U_1$. Let U be a neighbourhood of e such that $U \subseteq U_2 \cap xU_2x^{-1}$; this works. **Q**

(f) If $K \subseteq X$ is compact and $F \subseteq X$ is closed then KF and FK are closed. If K, $L \subseteq X$ are compact so is KL. (Hewitt & Ross 63, 4.4.)

(g) If there is any compact set $K \subseteq X$ such that $\operatorname{int} K$ is non-empty, then X is locally compact.

(h) If $K \subseteq X$ is compact and $\mathcal{F}$ is a downwards-directed family of closed subsets of X with intersection F_0, then $KF_0 = \bigcap_{F\in\mathcal{F}} KF$ and $F_0K = \bigcap_{F\in\mathcal{F}} FK$. **P** Of course $KF_0 \subseteq \bigcap_{F\in\mathcal{F}} KF$. If $x \in X \setminus KF_0$, then $K^{-1}x \cap F_0$ is empty; because $K^{-1}x$ is compact, there is some $F \in \mathcal{F}$ such that $K^{-1}x \cap F = \emptyset$ (3A3Db), so that $x \notin KF$. Accordingly $KF_0 = \bigcap_{F\in\mathcal{F}} KF$. Similarly, $F_0K = \bigcap_{F\in\mathcal{F}} FK$. **Q**

(i) If $K \subseteq X$ is compact and $G \subseteq X$ is open, then $\{x : xK \subseteq G\}$ and $\{x : Kx \subseteq G\}$ are open. **P** $\{x : xK \not\subseteq G\} = (X \setminus G)K^{-1}\}$, which by (e) above is closed. Similarly, $\{x : Kx \not\subseteq G\} = K^{-1}(X \setminus G)$ is closed. **Q**

(j) If X is Hausdorff, $K \subseteq X$ is compact and U is a neighbourhood of e, there is a neighbourhood V of e such that $xy \in U$ whenever x, $y \in K$ and $yx \in V$; that is, $y^{-1}zy \in U$ whenever $z \in V$ and $y \in K$. **P** If U is open, then $\{xy : x, y \in K, yx \notin U\}$ is a compact set not containing e. Compare 4A5Oc below. **Q**

(k) Any open subgroup of X is also closed. (Császár 78, 11.2.12; Hewitt & Ross 63, 5.5; Folland 95, 2.1.)

(l) If X is locally compact, it has an open subgroup which is σ-compact. (Hewitt & Ross 63, 5.14; Folland 95, 2.3.)

4A5F Proposition (a) Let $(X, \mathfrak{T})$ and $(Y, \mathfrak{S})$ be topological groups. If $\phi : X \to Y$ is a group homomorphism which is continuous at the identity of X, it is continuous. (Császár 78, 11.2.17; Hewitt & Ross 63, 5.40.)

(b) Let X be a group and $\mathfrak{S}$, $\mathfrak{T}$ two topologies on X both making X a topological group. If every $\mathfrak{S}$-neighbourhood of the identity is a $\mathfrak{T}$-neighbourhood of the identity, then $\mathfrak{S} \subseteq \mathfrak{T}$. (Apply (a) to the identity map from $(X, \mathfrak{T})$ to $(X, \mathfrak{S})$.)

4A5G Proposition If $\langle X_i \rangle_{i \in I}$ is any family of topological groups, then $\prod_{i \in I} X_i$, with the product topology and the product group structure, is again a topological group. (Hewitt & Ross 63, 6.2.)

4A5H The uniformities of a topological group Let $(X, \mathfrak{T})$ be a topological group. Write $\mathcal{U}$ for the set of open neighbourhoods of the identity e of X.

(a) For $U \in \mathcal{U}$, set $W_U = \{(x, y) : xy^{-1} \in U\} \subseteq X \times X$. The family $\{W_U : U \in \mathcal{U}\}$ is a filter base, and the filter on $X \times X$ which it generates is a uniformity on X, the **right uniformity** of X. (**Warning!!** Some authors call this the 'left uniformity'.) This uniformity induces the topology $\mathfrak{T}$ (Császár 78, 11.2.7). It follows that $\mathfrak{T}$ is completely regular, therefore regular (4A2Ja, or Hewitt & Ross 63, 8.4).

(b) For $U \in \mathcal{U}$, set $\tilde{W}_U = \{(x, y) : xy^{-1} \in U, x^{-1}y \in U\} \subseteq X \times X$. The family $\{\tilde{W}_U : U \in \mathcal{U}\}$ is a filter base, and the filter on $X \times X$ which it generates is a uniformity on X, the **bilateral uniformity** of X. This uniformity induces the topology $\mathfrak{T}$. (Császár 78, 11.3.c.)

4A5I Definitions If X is a topological group and Z a topological space, an action of X on Z is 'continuous' or 'Borel measurable' if it is continuous, or Borel measurable, when regarded as a function from $X \times Z$ to Z.

Of course the left, right and conjugacy actions of a topological group on itself are all continuous.

4A5J Quotient groups: Proposition Let X be a topological group, Y a subgroup of X, and Z the set of left cosets of Y in X. Set $\pi x = xY$ for $x \in X$.

(a) We have a topology on Z defined by saying that $V \subseteq Z$ is open iff $\pi^{-1}[V]$ is open in X. The canonical map $\pi : X \to Z$ is continuous and open.

(b) Z is Hausdorff iff Y is closed. If X is locally compact, so is Z.

(c) We have a continuous action of X on Z defined by saying that $x \bullet \pi x' = \pi(xx')$ for any x, $x' \in X$.

(d) If Y is a normal subgroup of X, then the group operation on Z renders it a topological group.

proof (a) By Hewitt & Ross 63, 5.15-5.16 the given formula defines a topology on Z for which π is continuous and open.

(b) By Hewitt & Ross 63, 5.21, Z is Hausdorff iff Y is closed. By Hewitt & Ross 63, 5.22 or Folland 95, 2.2, Z is locally compact if X is.

(c) I have noted in 4A5Cb that the formula given defines an action. If $V \subseteq Z$ is open and $x_0 \in X$, $z_0 \in Z$ are such that $x_0 \bullet z_0 \in V$, take $x_0' \in X$ such that $\pi x_0' = z_0$, and observe that $x_0 x_0' \in \pi^{-1}[V]$, which is open. So there are open neighbourhoods V_0, V_0' of x_0, x_0' respectively such that $V_0 V_0' \subseteq \pi^{-1}[V]$, and $x \bullet z \in V$ whenever $x \in V_0$ and $z \in \pi[V_0']$. Since $\pi[V_0']$ is an open neighbourhood of z_0, this is enough to show that $\bullet$ is continuous at (x_0, z_0).

(d) Császár 78, 11.2.15; Hewitt & Ross 63, 5.26; Folland 95, 2.2.

4A5K Proposition Let X be a topological group with identity e.
(a) $Y = \overline{\{e\}}$ is a closed normal subgroup of X.
(b) Writing $\pi : X \to X/Y$ for the canonical map,
(i) a subset of X is open iff it is the inverse image of an open subset of X/Y,
(ii) a subset of X is closed iff it is the inverse image of a closed subset of X/Y,
(iii) $\pi[G]$ is a regular open set in X/Y for every regular open set $G \subseteq X$,
(iv) $\pi[F]$ is nowhere dense in X/Y for every nowhere dense set $F \subseteq X$,
(v) $\pi^{-1}[V]$ is nowhere dense in X for every nowhere dense $V \subseteq X/Y$.

proof (a) Császár 78, 11.2.13; Hewitt & Ross 63, 5.4; Folland 95, 2.3.

(b)(i)-(ii) If $G \subseteq X$ is open and $x \in G$, then $x\overline{\{e\}} = \overline{\{x\}} \subseteq G$, because X is regular (4A5Ha). So $G = GY = \pi^{-1}[\pi[G]]$. Since π is an open map (4A5Ja), $\pi[G]$ is open and G is the inverse image of an open set. If $F \subseteq X$ is closed,

$$F = X \setminus \pi^{-1}[\pi[X \setminus F]] = \pi^{-1}[(X/Y) \setminus \pi[X \setminus F]]$$

is the inverse image of a closed set, and $\pi[F] = (X/Y) \setminus \pi[X \setminus F]$ is closed.

Conversely, because π is continuous, the inverse image of an open or closed set is again open or closed.

(iii) If $A \subseteq X$, then $\pi[\overline{A}]$ is a closed set included in $\overline{\pi[A]}$ (because π is continuous), so $\overline{\pi[A]} = \pi[\overline{A}]$. If $G \subseteq X$ is a regular open set, then $\pi^{-1}[\operatorname{int}\pi[\overline{G}]]$ is an open subset of $\pi^{-1}[\pi[\overline{G}]] = \overline{G}$, so is included in $\operatorname{int}\overline{G} = G$. But this means that the open set $\pi[G]$ includes $\operatorname{int}\pi[\overline{G}] = \operatorname{int}\overline{\pi[G]}$, and $\pi[G] = \operatorname{int}\overline{\pi[G]}$ is a regular open set.

(iv) If $F \subseteq X$ is nowhere dense, then its closure is of the form $\pi^{-1}[V]$ for some closed set $V \subseteq X/Y$. Now if $H \subseteq X/Y$ is a non-empty open set, $\pi^{-1}[H]$ is a non-empty open subset of X, so is not included in F, and H cannot be included in V. Thus V is nowhere dense; but $V \supseteq \pi[F]$, so $\pi[F]$ is nowhere dense.

(v) If $V \subseteq X/Y$ is nowhere dense, and $G \subseteq X$ is open and not empty, then $G = \pi^{-1}[H]$ for some non-empty open $H \subseteq X/Y$. In this case, $H \setminus \overline{V}$ is non-empty, so $\pi^{-1}[H \setminus \overline{V}]$ is a non-empty open subset of G disjoint from $\pi^{-1}[V]$. As G is arbitrary, $\pi^{-1}[V]$ is nowhere dense.

4A5L Theorem Let X be a topological group and Y a normal subgroup of X. Let $\pi : X \to X/Y$ be the canonical homomorphism.
(a) If X' is another topological group and $\phi : X \to X'$ a continuous homomorphism with kernel included in Y, then we have a continuous homomorphism $\psi : X/Y \to X'$ defined by the formula $\psi\pi = \phi$; ψ is injective iff Y is the kernel of ϕ.
(b) Suppose that K_1, K_2 are two subgroups of X/Y such that $K_2 \lhd K_1$. Set $Y_1 = \pi^{-1}[K_1]$ and $Y_2 = \pi^{-1}[K_2]$. Then $Y_2 \lhd Y_1$ and Y_1/Y_2 and K_1/K_2 are isomorphic as topological groups.

proof (a) This is elementary group theory, except for the claim that ψ is continuous. But if $H \subseteq X'$ is open, then $\psi^{-1}[H] = \pi[\phi^{-1}[H]]$ is open because ϕ is continuous and π is open (4A5Ja); so ψ is continuous.

(b) See Hewitt & Ross 63, 5.35.

4A5M Proposition Let X be a topological group.
(a) Let Y be any subgroup of X. If X is given its bilateral uniformity, then the subspace uniformity on Y is the bilateral uniformity of Y. (Császár 78, 11.3.13.)
(b) If X is locally compact it is complete under its right uniformity. (Császár 78, 11.3.21.) If X is complete under its right uniformity it is complete under its bilateral uniformity. (Császár 78, 11.3.10.)
(c) Suppose that X is Hausdorff and that Y is a subgroup of X which is locally compact in its subspace topology. Then Y is closed in X. **P** Putting (a) and (b) together, we see that Y is complete in its subspace uniformity; by 3A4Fd, it is closed. **Q**

4A5N Theorem Let X be a Hausdorff topological group. Then its completion $\widehat{X}$ under its bilateral uniformity can be endowed (in exactly one way) with a group structure rendering it a Hausdorff topological group in which the natural embedding of X in $\widehat{X}$ represents X as a dense subgroup of $\widehat{X}$. (Császár 78, 11.3.15.) If X has a neighbourhood of the identity which is totally bounded for the bilateral uniformity, then $\widehat{X}$ is locally compact. (Császár 78, 11.3.24.)

4A5O Proposition Let X be a topological group.

(a) If $A \subseteq X$, then the following are equiveridical: (i) A is totally bounded for the bilateral uniformity of X; (ii) for every neighbourhood U of the identity there is a finite set $I \subseteq X$ such that $A \subseteq IU \cap UI$.

(b) If A, $B \subseteq X$ are totally bounded for the bilateral uniformity of X, so are $A \cup B$, A^{-1} and AB. In particular, $\bigcup_{i \leq n} x_i B$ is totally bounded for any $x_0, \ldots, x_n \in X$.

(c) If $A \subseteq X$ is totally bounded for the bilateral uniformity, and U is any neighbourhood of the identity, then $\{y : xyx^{-1} \in U \text{ for every } x \in A\}$ is a neighbourhood of the identity.

(d) If X is the product of a family $\langle X_i \rangle_{i \in I}$ of topological groups, a subset A of X is totally bounded for the bilateral uniformity of X iff it is included in a product $\prod_{i \in K} A_i$ where $A_i \subseteq X_i$ is totally bounded for the bilateral uniformity of X_i for every $i \in I$.

(e) If X is locally compact, a subset of X is totally bounded for the bilateral uniformity iff it is relatively compact.

proof (a)(i)⇒(ii) Suppose that A is totally bounded, and that U is a neighbourhood of the identity e of X. Set

$$W = \{(x, y) : xy^{-1} \in U^{-1},\ x^{-1}y \in U\} = \{(x, y) : y \in Ux \cap xU\};$$

then W belongs to the uniformity, so there is a finite set $I \subseteq X$ such that $A \subseteq W[I]$. But $W[I] \subseteq UI \cap IU$, so $A \subseteq UI \cap IU$.

(ii)⇒(i) Now suppose that A satisfies the condition, and that W belongs to the uniformity. Then there is a neighbourhood U of e such that $\{(x, y) : xy^{-1} \in U,\ x^{-1}y \in U\} \subseteq W$. Let V be a neighbourhood of e such that $VV^{-1} \subseteq U$ and $V^{-1}V \subseteq U$. Let $I \subseteq X$ be a finite set such that $A \subseteq VI \cap IV$. For w, $z \in I$ set $A_{wz} = A \cap Vw \cap zV$. If x, $y \in A_{wz}$, $xy^{-1} \in Vww^{-1}V^{-1} \subseteq U$ and $x^{-1}y \in V^{-1}z^{-1}zV \subseteq U$. But this means that $A_{wz} \times A_{wz} \subseteq W$. So if we take a finite set J which meets every non-empty A_{wz}, $A \subseteq W[J]$. As W is arbitrary, A is totally bounded.

(b) Of course $A \cup B$ is totally bounded; this is immediate from the definition of 'totally bounded'. If U is a neighbourhood of e, so is U^{-1}, so there is a finite set $I \subseteq X$ such that $A \subseteq IU^{-1} \cap U^{-1}I$ and $A^{-1} \subseteq UI^{-1} \cap I^{-1}U$; as U is arbitrary, A^{-1} is totally bounded.

To see that AB is also totally bounded, let U be a neighbourhood of e, and take a neighbourhood V of e such that $VV \subseteq U$. Then there is a finite set $I \subseteq X$ such that $A \subseteq VI$ and $B \subseteq IV$. Let W be a neighbourhood of e such that $zWz^{-1} \cup z^{-1}Wz \subseteq V$ for every $z \in I$, and J a finite set such that $B \subseteq WJ$ and $A \subseteq JW$. Then

$$zW \subseteq Vz, \quad Wz \subseteq zV$$

for every $z \in I$, so

$$IW \subseteq VI, \quad WI \subseteq IV$$

and

$$AB \subseteq VIWJ \subseteq VVIJ \subseteq UK, \quad AB \subseteq JWIV \subseteq JIVV \subseteq KU$$

where $K = IJ \cup JI$ is finite. As U is arbitrary, AB is totally bounded.

(c) Let V be a neighbourhood of e such that $VVV^{-1} \subseteq U$. Let I be a finite set such that $A \subseteq VI$. Let W be a neighbourhood of e such that $zWz^{-1} \subseteq V$ for every $z \in I$. If now $y \in W$ and $x \in A$, there is a $z \in I$ such that $x \in Vz$, so that

$$xyx^{-1} \in VzWz^{-1}V^{-1} \subseteq VVV^{-1} \subseteq U.$$

Turning this round, $\{y : xyx^{-1} \in U \text{ for every } x \in A\}$ includes W and is a neighbourhood of e.

(d)(i) Suppose that A is totally bounded. Set $A_k = \pi_k[A]$ for each $k \in K$, where $\pi_k(x) = x(k)$ for $x \in X$. If U is a neighbourhood of the identity in X_k, then $V = \pi_k^{-1}[U]$ is a neighbourhood of the identity in X, so there is a finite set $I \subseteq X$ such that $A \subseteq IV \cap VI$; now $A_k \subseteq JU \cap UJ$, where $J = \pi_k[I]$ is finite. As U is arbitrary, A_k is totally bounded. This is true for every k, and $A \subseteq \prod_{k \in K} A_k$.

(ii) Suppose that $A \subseteq \prod_{k \in K} A_k$ where $A_k \subseteq X_k$ is totally bounded for each $k \in K$. If A is empty, of course it is totally bounded; assume that $A \neq \emptyset$. If $K = \emptyset$, then $X = \{\emptyset\}$ is the trivial group, and again

A is totally bounded; so assume that K is non-empty. Let V be a neighbourhood of the identity in X. Then there are a non-empty finite set $L \subseteq K$ and a family $\langle U_k \rangle_{k \in L}$ such that U_k is a neighbourhood of the identity in X_k for each $k \in L$, and $V \supseteq \bigcap_{k \in L} \pi_k^{-1} U_k$. For each $k \in L$, let I_k be a finite subset of X_k such that $A_k \subseteq I_k U_k \cap U_k I_k$. Set

$$I = \{x : x \in X, \, x(k) \text{ is the identity for } k \in K \setminus L, \, x(k) \in I_k \text{ for } k \in L\}.$$

Then I is finite and $A \subseteq IV \cap VI$. As V is arbitrary, A is totally bounded.

(e) By 4A5Mb and 4A2Jd, the totally bounded sets are the relatively compact sets.

4A5P Lemma Let X be a locally compact Hausdorff topological group. Take $f \in C_k(X)$, the space of continuous real-valued functions on X with compact supports.

(a) Let $K \subseteq X$ be a compact set. Then for any $\epsilon > 0$ there is a neighbourhood W of the identity E of X such that $|f(xay) - f(xby)| \leq \epsilon$ whenever $x \in K$, $y \in X$ and $ab^{-1} \in W$.

(b) For any $x_0 \in X$, there is a non-negative $f^* \in C_k(X)$ such that for every $\epsilon > 0$ there is an open set G containing x_0 such that $|f(xy) - f(x_0 y)| \leq \epsilon f^*(y)$ for every $x \in G$ and $y \in X$.

proof (a) By 4A2Je and 4A5Ha, f is uniformly continuous for the right uniformity of X. There is therefore a symmetric neighbourhood U of e, included in V, such that $|f(y_1) - f(y_2)| \leq \epsilon$ whenever y_1, $y_2 \in X$ and $y_1 y_2^{-1} \in U$. By 4A5Oc, there is a symmetric neighbourhood W of e such that $xzx^{-1} \in U$ whenever $x \in K$ and $z \in W$.

Now suppose that $x \in K$, $y \in X$ and $ab^{-1} \in W$. Then $(xay)(xby)^{-1} = xab^{-1}x^{-1} \in U$, so $|f(xay) - f(xby)| \leq \epsilon$, as required.

(b) We need a trifling refinement of the ideas above.

(i) Suppose for the moment that $x_0 = e$. Set $L = \overline{\{x : f(x) \neq 0\}}$ and let V be a compact symmetric neighbourhood of the identity e, so that L and VL are compact. Let $f^* \in C_k(X)$ be such that $f^* \geq \chi(VL)$ (4A2G(e-i)). Given $\epsilon > 0$, take U as in (a), so that U is a symmetric neighbourhood of e and $|f(y_1) - f(y_2)| \leq \epsilon$ whenever $y_1 y_2^{-1} \in U$; this time arrange further that $U \subseteq V$. Then if $x \in U$ and $y \in X$,

either y and xy belong to VL, while $(xy)y^{-1} \in U$, so $|f(xy) - f(y)| \leq \epsilon \leq \epsilon f^*(y)$

or neither y nor xy belongs to L, so $|f(xy) - f(y)| = 0 \leq \epsilon f^*(y)$.

(ii) For the general case, set $f_0(x) = f(x_0 x)$ for $x \in X$. Because $x \mapsto x_0 x$ is a homeomorphism, $f_0 \in C_k(X)$. By (i), we have a non-negative $f^* \in C_k(X)$ such that for every $\epsilon > 0$ there is a neighbourhood G_ϵ of e such that $|f_0(xy) - f_0(y)| \leq \epsilon f^*(y)$ whenever $x \in G_\epsilon$ and $y \in X$. Now, given $\epsilon > 0$, $G' = x_0 G_\epsilon$ is a neighbourhood of x_0 and $|f(xy) - f(x_0 y)| \leq \epsilon f^*(y)$ whenever $x \in G'$ and $y \in X$. So f^* witnesses that the result is true.

4A5Q Metrizable groups: Proposition Let $(X, \mathfrak{T})$ be a topological group. Then the following are equiveridical:

(i) X is metrizable;

(ii) the identity e of X has a countable neighbourhood base;

(iii) there is a metric ρ on X, inducing the topology $\mathfrak{T}$, which is **right-translation-invariant**, that is, $\rho(x_1, x_2) = \rho(x_1 y, x_2 y)$ for all x_1, x_2, $y \in X$;

(iv) there is a right-translation-invariant metric on X which induces the right uniformity of X.

proof Császár 78, 11.2.10; Hewitt & Ross 63, 8.3.

Warning! A Polish group (4A5Db) is of course metrizable, so has a right-translation-invariant metric inducing its topology. At the same time, it has a complete metric inducing its topology. But there is no suggestion that these two metrics should be the same, or even induce the same uniformity.

4A5R Corollary If X is a locally compact topological group and $\{e\}$ is a G_δ set in X, then X is metrizable. (Put 4A5Q and 4A2Kf together, or see Hewitt & Ross 63, 8.5.)

4A5S Lemma Let X be a σ-compact locally compact Hausdorff topological group and $\langle U_n\rangle_{n\in\mathbb{N}}$ any sequence of neighbourhoods of the identity in X. Then X has a compact normal subgroup $Y \subseteq \bigcap_{n\in\mathbb{N}} U_n$ such that X/Y is metrizable.

proof Let $\langle K_n\rangle_{n\in\mathbb{N}}$ be a sequence of compact sets covering X. Choose inductively a sequence $\langle V_n\rangle_{n\in\mathbb{N}}$ of compact neighbourhoods of e such that, for each $n \in \mathbb{N}$,

(α) $V_{n+1} \subseteq V_n^{-1}$, $\quad V_{n+1}V_{n+1} \subseteq V_n$, $\quad V_n \subseteq U_n$,

(β) $xyx^{-1} \in V_n$ whenever $y \in V_{n+1}$ and $x \in \bigcup_{i\le n} K_i$.

(When we come to choose V_{n+1}, we can achieve (α) because inversion and multiplication are continuous, and (β) by 4A5Oc; and we can then shrink V_{n+1} to a closed neighbourhood of e because X is regular, by 4A5Ha.) Set $Y = \bigcap_{n\in\mathbb{N}} V_n$. Then ($\alpha$) is enough to ensure that Y is a compact subgroup of X included in $\bigcap_{n\in\mathbb{N}} U_n$, while ($\beta$) ensures that Y is normal, because for any $x \in X$ there is an $n \in \mathbb{N}$ such that $xV_{m+1}x^{-1} \subseteq V_m$ for every $m \ge n$.

Let $\pi : X \to X/Y$ be the canonical map. Then $C = \bigcap_{n\in\mathbb{N}} \pi[\operatorname{int} V_n]$ is a G_δ subset of X/Y, because π is open. But

$$\pi^{-1}[C] \subseteq \bigcap_{n\in\mathbb{N}} \pi^{-1}[\pi[V_{n+1}]] = \bigcap_{n\in\mathbb{N}} V_{n+1}Y \subseteq \bigcap_{n\in\mathbb{N}} V_n = Y,$$

so $C = \{e_{X/Y}\}$, writing $e_{X/Y}$ for the identity of X/Y. Thus $\{e_{X/Y}\}$ is a G_δ set; as X/Y is locally compact and Hausdorff (4A5Jb), it is metrizable (4A5R).

***4A5T** I shall not rely on the following fact, but it will help you to make sense of some of the results of this volume.

Theorem A compact Hausdorff topological group is dyadic.

proof Uspenskiĭ 88.

4A6 Banach algebras

I give results which are needed for Chapter 44. Those down to 4A6K should be in any introductory text on normed algebras; 4A6L-4A6N, as expressed here, are a little more specialized. As with normed spaces or linear topological spaces, Banach algebras may be defined over either $\mathbb{R}$ or $\mathbb{C}$. In §445 we need complex Banach algebras, but in §446 I think the ideas are clearer in the context of real Banach algebras. Accordingly, as in §2A4, I express as much as possible of the theory in terms applicable equally to either, speaking of 'normed algebras' or 'Banach algebras' without qualification, and using the symbol $\begin{smallmatrix}\mathbb{R}\\ \mathbb{C}\end{smallmatrix}$ to represent the field of scalars. Since (at least, if you keep to the path indicated here) the ideas are independent of which field we work with, you will have no difficulty in applying the arguments given in Folland 95 or Hewitt & Ross 63 for the complex case to the real case. In 4A6B and 4A6I-4A6K, however, we have results which apply only to 'complex' Banach algebras, in which the underlying field is taken to be $\mathbb{C}$.

4A6A Definition (a) I repeat a definition from §2A4. A **normed algebra** is a normed space U together with a multiplication, a binary operator $\times$ on U, such that

$$u \times (v \times w) = (u \times v) \times w,$$

$$u \times (v + w) = (u \times v) + (u \times w), \quad (u + v) \times w = (u \times w) + (v \times w),$$

$$(\alpha u) \times v = u \times (\alpha v) = \alpha(u \times v),$$

$$\|u \times v\| \le \|u\|\|v\|$$

for all u, v, $w \in U$ and $\alpha \in \begin{smallmatrix}\mathbb{R}\\ \mathbb{C}\end{smallmatrix}$. A normed algebra is **commutative** if its multiplication is commutative.

(b) A **Banach algebra** is a normed algebra which is a Banach space. A **unital Banach algebra** is a Banach algebra with a multiplicative identity e such that $\|e\| = 1$. (**Warning:** some authors reserve the term 'Banach algebra' for what I call a 'unital Banach algebra'.)

In a unital Banach algebra I will always use the letter e for the identity.

4A6B Stone-Weierstrass Theorem: fourth form Let X be a locally compact Hausdorff space, and $C_0 = C_0(X;\mathbb{C})$ the complex Banach algebra of continuous functions $f : X \to \mathbb{C}$ such that $\{x : |f(x)| \geq \epsilon\}$ is compact for every $\epsilon > 0$. Let $A \subseteq C_0$ be such that

A is a linear subspace of C_0;

$f \times g \in A$ for every f, $g \in A$;

the complex conjugate $\bar{f}$ of f belongs to A for every $f \in A$;

for every $x \in X$ there is an $f \in A$ such that $f(x) \neq 0$:

whenever x, y are distinct points of X there is an $f \in A$ such that $f(x) \neq f(y)$.

Then A is $\| \ \|_\infty$-dense in C_0.

proof Let $X_\infty = X \cup \{\infty\}$ be the one-point compactification of X (3A3O). For $f \in C_0$ write $f^\#$ for the extension of f to $X \cup \{\infty\}$ with $f^\#(\infty) = 0$, so that $f^\# \in C_b(X_\infty;\mathbb{C})$. Let $B \subseteq C_b(X_\infty;\mathbb{C})$ be the set of all functions of the form $f^\# + \alpha\chi X_\infty$ where $f \in A$ and $\alpha \in \mathbb{C}$. Then B is a subalgebra of $C_b(X \cup \{\infty\})$ which contains complex conjugates of its members and constant functions and separates the points of X_∞.

Take any $h \in C_0$ and $\epsilon > 0$. By the 'third form' of the Stone-Weierstrass theorem (281G), there is a $g \in B$ such that $\|g - h^\#\|_\infty \leq \frac{1}{2}\epsilon$. Express g as $f^\# + \alpha\chi X_\infty$ where $f \in A$ and $\alpha \in \mathbb{C}$. Then

$$|\alpha| = |g(\infty)| = |g(\infty) - h^\#(\infty)| \leq \tfrac{1}{2}\epsilon,$$

so

$$\|h - f\|_\infty = \|h^\# - f^\#\|_\infty \leq \|h^\# - g\|_\infty + \|g - f^\#\|_\infty \leq \tfrac{1}{2}\epsilon + |\alpha| \leq \epsilon.$$

As h and ϵ are arbitrary, A is dense in C_0.

4A6C Proposition If U is any Banach space other than $\{0\}$, then the space $\mathbb{B}(U;U)$ of bounded linear operators from U to itself is a unital Banach algebra. (Köthe 69, 14.6.)

4A6D Proposition Any normed algebra U can be embedded as a subalgebra of a unital Banach algebra V, in such a way that if U is commutative so is V. (Folland 95, §1.3; Hewitt & Ross 63, C.3.)

4A6E Proposition Let U be a unital Banach algebra and $W \subseteq U$ a closed proper ideal. Then U/W, with the quotient linear structure, ring structure and norm, is a unital Banach algebra. (Hewitt & Ross 63, C.2.)

4A6F Proposition If U is a Banach algebra and $\phi : U \to \begin{matrix}\mathbb{R}\\ \mathbb{C}\end{matrix}$ is a multiplicative linear functional, then $|\phi(u)| \leq \|u\|$ for every $u \in U$.

proof ? Otherwise, there is a u such that $|\phi(u)| > \|u\|$; set $v = \frac{1}{\phi(u)}u$, so that $\phi(v) = 1$ and $\|v\| < 1$. Since $\|v^n\| \leq \|v\|^n$ for every $n \geq 1$, $w = \sum_{n=1}^\infty v^n$ is defined in U (4A4Ie), and $w = vw + v$. But this means that $\phi(w) = \phi(v)\phi(w) + \phi(v) = \phi(w) + 1$, which is impossible. **X**

4A6G Definition Let U be a normed algebra and $u \in U$.

(a) For any $u \in U$, $\lim_{n\to\infty} \|u^n\|^{1/n}$ is defined and equal to $\inf_{n\geq 1} \|u^n\|^{1/n}$. (Hewitt & Ross 63, C.4.)

(b) This common value is called the **spectral radius** of u.

4A6H Theorem If U is a unital Banach algebra, then the set R of invertible elements is open, and $u \mapsto u^{-1}$ is a continuous function from R to itself. Writing e for the identity of U, $v \in R$ and $\|v^{-1} - e\| \leq \frac{\|v-e\|}{1-\|v-e\|}$ whenever $\|v - e\| < 1$. (Folland 95, 1.4; Hewitt & Ross 63, C.8 & C.10; Rudin 91, 10.7 & 10.12.)

4A6I Theorem Let U be a complex unital Banach algebra and $u \in U$. Write r for the spectral radius of u.

(a) If $\zeta \in \mathbb{C}$ and $|\zeta| > r$ then $\zeta e - u$ is invertible.

(b) There is a ζ such that $|\zeta| = r$ and $\zeta e - u$ is not invertible.

proof Folland 95, 1.8; Hewitt & Ross 63, C.24; Rudin 91, 1.13.

4A6J Theorem Let U be a commutative complex unital Banach algebra, and $u \in U$. Then for any $\zeta \in \mathbb{C}$ the following are equiveridical:

(i) there is a non-zero multiplicative linear functional $\phi : U \to \mathbb{C}$ such that $\phi(u) = \zeta$;

(ii) $\zeta e - u$ is not invertible.

proof Folland 95, 1.13; Hewitt & Ross 63, C.20; Rudin 91, 11.5.

4A6K Corollary Let U be a commutative complex Banach algebra and $u \in U$. Then its spectral radius $r(u)$ is $\max\{|\phi(u)| : \phi \text{ is a multiplicative linear functional on } U\}$. (Folland 95, 1.13; Hewitt & Ross 63, C.20; Rudin 91, 11.9.)

4A6L Exponentiation Let U be a unital Banach algebra. For any $u \in U$, $\sum_{k=0}^{\infty} \|\frac{1}{k!}u^k\| \leq \sum_{k=0}^{\infty} \frac{1}{k!}\|u\|^k$ is finite, so

$$\exp(u) = \sum_{k=0}^{\infty} \frac{1}{k!}u^k$$

is defined in U (4A4Ie). (In this formula, interpret u^0 as e for every u.)

4A6M Lemma Let U be a unital Banach algebra.

(a) If u, $v \in U$ and $\max(\|u\|, \|v\|) \leq \gamma$ then $\|\exp(u) - \exp(v) - u + v\| \leq \|u - v\|(\exp(\gamma) - 1)$. So if $\max(\|u\|, \|v\|) \leq \frac{1}{2}$ and $\exp(u) = \exp(v)$ then $u = v$.

(b) If $\|u - e\| \leq \frac{1}{6}$ then there is a v such that $\exp(v) = u$ and $\|v\| \leq 2\|u - e\|$.

(c) If u, $v \in U$ and $uv = vu$ then $\exp(u + v) = \exp(u)\exp(v)$.

proof (a) Note first that if $k \geq 1$ then

$$\begin{aligned}\|u^k - v^k\| &= \|\sum_{i=0}^{k-1} u^{k-i}v^i - u^{k-i-1}v^{i+1}\| = \|\sum_{i=0}^{k-1} u^{k-i-1}(u - v)v^i\| \\ &\leq \sum_{i=0}^{k-1} \|u\|^{k-i-1}\|u - v\|\|v\|^i \leq \sum_{i=0}^{k-1} \gamma^{k-1}\|u - v\| = k\gamma^{k-1}\|u - v\|.\end{aligned}$$

So

$$\begin{aligned}\|\exp(u) - \exp(v) - u + v\| &= \|\sum_{k=2}^{\infty} \frac{1}{k!}(u^k - v^k)\| \leq \sum_{k=2}^{\infty} \frac{1}{k!}\|u^k - v^k\| \\ &\leq \sum_{k=2}^{\infty} \frac{k}{k!}\gamma^{k-1}\|u - v\| = \|u - v\|(\exp(\gamma) - 1).\end{aligned}$$

Now if $\exp(u) = \exp(v)$ and $\gamma \leq \frac{1}{2}$, $\exp(\gamma) - 1 < 1$ so $\|u - v\| = 0$ and $u = v$.

(b) Set $\gamma = \|u - e\|$. Define $\langle v_n \rangle_{n \in \mathbb{N}}$ in U by setting $v_0 = 0$, $v_{n+1} = v_n + u - \exp(v_n)$ for $n \in \mathbb{N}$. Then

$$\|v_{n+1} - v_n\| = \|u - \exp(v_n)\| \leq 2^{-n}\gamma, \quad \|v_n\| \leq 2(1 - 2^{-n})\gamma$$

for every $n \in \mathbb{N}$. **P** Induce on n. The induction starts with $\|v_0\| = 0$ and $\|u - \exp(v_0)\| = \|u - e\| = \gamma$. Given that $\|v_n\| \leq 2(1 - 2^{-n})\gamma$ and $\|u - \exp(v_n)\| \leq 2^{-n}\gamma$, then

$$\|v_{n+1}\| \le \|v_n\| + \|v_{n+1} - v_n\| = \|v_n\| + \|u - \exp(v_n)\|$$
$$\le 2(1 - 2^{-n})\gamma + 2^{-n}\gamma = 2(1 - 2^{-n-1})\gamma.$$

Now $\max(\|v_{n+1}\|, \|v_n\|) \le 2\gamma \le \frac{1}{3}$, so

$$\|u - \exp(v_{n+1})\| = \|v_{n+1} - v_n + \exp(v_n) - \exp(v_{n+1})\| \le \|v_{n+1} - v_n\|(\exp(\tfrac{1}{3}) - 1)$$
$$\le \tfrac{1}{2}\|v_{n+1} - v_n\| \le 2^{-n-1}\gamma,$$

and the induction continues. **Q**

Since $\sum_{n=0}^{\infty} \|v_{n+1} - v_n\|$ is finite, $v = \lim_{n\to\infty} v_n$ is defined in U, and $\|v\| = \lim_{n\to\infty} \|v_n\| \le 2\gamma$. Accordingly

$$\|\exp(v) - \exp(v_n)\| \le \|v - v_n\| + \|\exp(v) - \exp(v_n) - v + v_n\|$$
$$\le \|v - v_n\|(1 + \exp(\tfrac{1}{3}) - 1) \to 0$$

as $n \to \infty$, and $\exp(v) = \lim_{n\to\infty} \exp(v_n) = u$.

(c) Because $uv = vu$, $(u+v)^m = \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} u^j v^{m-j}$ for every $m \in \mathbb{N}$ (induce on m; the point is that $uv^j = v^j u$ for every $j \in \mathbb{N}$). Next, $\sum_{j,k\in\mathbb{N}} \frac{1}{j!k!}\|u\|^j\|v\|^k$ is finite. So

$$\exp(u+v) = \sum_{m=0}^{\infty} \frac{1}{m!}(u+v)^m$$
$$= \sum_{m=0}^{\infty} \Big(\sum_{j+k=m} \frac{1}{j!k!} u^j v^k \Big) = \sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} \frac{1}{j!k!} u^j v^k$$

(using 4A4I(e-ii))

$$= \sum_{j\in\mathbb{N}} \Big(\sum_{k\in\mathbb{N}} \frac{1}{j!k!} u^j v^k \Big)$$

(4A4I(e-ii) again)

$$= \sum_{j\in\mathbb{N}} \Big(\frac{1}{j!} u^j \sum_{k\in\mathbb{N}} \frac{1}{k!} v^k \Big)$$

(by 4A4Bj, because $w \mapsto \frac{1}{j!}u^j w$ is a continuous linear operator for each j)

$$= \sum_{j\in\mathbb{N}} \frac{1}{j!} u^j \exp(v) = \Big(\sum_{j\in\mathbb{N}} \frac{1}{j!} u^j \Big) \exp(v)$$

(4A4Bj again)

$$= \exp(u)\exp(v),$$

as claimed.

4A6N Lemma If U is any unital Banach algebra, $u \in U$ and $u \ne e$, there is some $n \in \mathbb{N}$ such that $\|u^n - e\| > \frac{1}{6}$.

proof ? Otherwise, for every $n \in \mathbb{N}$ take v_n such that $\exp(v_n) = u^{2^n}$ and $\|v_n\| \le \frac{1}{3}$ (4A6Mb). Then $\exp(v_{n+1}) = \exp(v_n)^2 = \exp(2v_n)$ (4A6Mc) and $\|v_{n+1}\| \le \frac{1}{3}$, $\|2v_n\| \le \frac{1}{3}$ so $v_{n+1} = 2v_n$ for every n (4A6Ma). Inducing on n, $v_n = 2^n v_0$ for every n, so that $\|v_0\| \le 2^{-n}\|v_n\| \to 0$ as $n \to \infty$, and $u = \exp(v_0) = e$. **X**

Concordance

I list here the section and paragraph numbers which have (to my knowledge) appeared in print in references to this volume, and which have since been changed.

411I Completion regular measures The definition of 'completion regular' measure, referred to in FREMLIN 00, has been moved to 411J.

414N Density topologies The note on 'density topologies', referred to in the 2001 edition of Volume 2, has been moved to 414P.

415Yd Sorgenfrey topology This exercise, referred to in the 2001 edition of Volume 2, has been moved to 415Ye.

416P The algebra of open-and-closed subsets This paragraph, referred to in the 2002 edition of Volume 3, has been moved to 416Q.

416T Kakutani's theorem The description of the usual measure on $\{0,1\}^\kappa$, referred to in FREMLIN & PLEBANEK 03, has been moved to 416U.

417E τ-additive product measures The reference in FREMLIN 00 to Kakutani's theorem that the product measure on $\{0,1\}^I$ is completion regular should be directed to 415E or 416U rather than 417E.

§**445 Convolutions** The material mentioned in the 2001 edition of Volume 2 has been moved to §444. The particular result referred to as '445K' is now 444R.

445Xq The exercise 445Xq, referred to in the 2002 edition of Volume 3, has been moved to 445Xp.

452I In FREMLIN 00 I quote Pachl's result that if (X,Σ,μ) is countably compact, (Y,T,ν) is strictly localizable and $f:X\to Y$ is inverse-measure-preserving, then ν is countably compact; this is now in 452R.

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Index to volumes 1-4

Principal topics and results

The general index below is intended to be comprehensive. Inevitably the entries are voluminous to the point that they are often unhelpful. I am therefore preparing a shorter, better-annotated, index which will, I hope, help readers to focus on particular areas. It does not mention definitions, as the bold-type entries in the main index are supposed to lead efficiently to these; and if you draw blank here you should always, of course, try again in the main index. Entries in the form of mathematical assertions frequently omit essential hypotheses and should be checked against the formal statements in the body of the work.

absolutely continuous real functions §225
—— as indefinite integrals 225E
absolutely continuous additive functionals §232
—— characterization 232B
—— on measure algebras 327B
actions of groups on topological spaces
—— and invariant measures 441C, 443P, 443Q, 443T, 449A
—— inducing actions on measure algebras and function spaces 443C, 443G, 444F
additive functionals on Boolean algebras §326, §327, §362
—— dominating or subordinate to a given functional 391E, 391F
—— on $\mathcal{P}I$ §464
almost continuous functions §418
—— and measurable functions 418E, 418J, 451S
—— and image measures of Radon measures 418I
amenable group §449
analytic spaces §423, §433
—— K-analytic spaces with countable networks 423C
—— and Souslin's operation 423E
angelic spaces §462
Archimedean Riesz spaces §353
associate norm (on a dual function space) §369
atomless measure spaces
—— have elements of all possible measures 215D
automorphisms and automorphism groups
—— of Boolean algebras §381, §382, §384
—— —— automorphisms as products of involutions 382M
—— —— normal subgroups of automorphism groups 382R, 383I
—— —— isomorphisms of automorphism groups 384D, 384E, 384J, 384M
—— of measure algebras §383, §385, §386, §387, §388, 493H

Baire measures §435
Banach lattices §354
—— order-bounded linear operators are continuous 355C
—— duals 356D
Banach-Ulam problem 363S
band algebra in a Riesz space 352Q
barycenter of a measure on a locally convex space §461
—— and convex functions 461C
—— existence of barycenters of given measures 461D, 461E, 461G
—— existence of measures with given barycenters 461H, 461K, 461M, 461N
Becker-Kechris theorem (on Borel measurable actions of Polish groups) 424H
Bernoulli shifts §387
—— calculation of entropy 385R
—— as factors of given homomorphisms (Sinaĭ's theorem) 387E, 387L
—— specified by entropy (Ornstein's theorem) 387I, 387K

Besicovitch's Covering Lemma (for balls in $\mathbb{R}^r$) 472B, 472C
Besicovitch's Density Theorem (for Radon measures on $\mathbb{R}^r$) 472D
biduals of Riesz spaces 356F, 356I
Bochner's theorem (positive definite functions are Fourier-Stieltjes transforms/characteristic functions) 445N
Boolean algebras Chap. 31
Boolean homomorphisms §312
—— between measure algebras §324
—— induced by measurable functions 324A
—— continuity and uniform continuity 324F
—— when measure-preserving 324K
—— represented by measurable functions 343B, 344B, 344E
Boolean rings §311
Borel measures §434
Borel sets in $\mathbb{R}^r$ 111G
—— and Lebesgue measure 114G, 115G, 134F
bounded variation, real functions of §224
—— as differences of monotonic functions 224D
—— integrals of their derivatives 224I
—— Lebesgue decomposition 226C
Brownian motion 455C
—— as a Radon measure on $C([0,\infty[)$ 455D
Cantor set and function 134G, 134H
capacity 432I, 432J
Carathéodory's construction of measures from outer measures 113C
Carleson's theorem (Fourier series of square-integrable functions converge a.e.) §286
Cauchy's Perimeter Theorem 475S
Central Limit Theorem (sum of independent random variables approximately normal) §274
—— Lindeberg's condition 274F, 274G
change of measure in the integral 235M
change of variable in the integral §235
—— $\int J \times g\phi \, d\mu = \int g \, d\nu$ 235A, 235E, 235L
—— finding J 235O;
—— —— $J = |\det T|$ for linear operators T 263A; $J = |\det \phi'|$ for differentiable operators ϕ 263D
—— —— —— when the measures are Hausdorff measures 265B, 265E
—— when ϕ is inverse-measure-preserving 235I
—— $\int g\phi \, d\mu = \int J \times g \, d\nu$ 235T
characteristic function of a probability distribution §285
—— sequences of distributions converge in vague topology iff characteristic functions converge pointwise 285L
Choquet's theorem (analytic sets are capacitable) 432J
Choquet's theorem (on measures on sets of extreme points) 461K
closed subalgebra of a measure algebra 323H, 323J
—— classification theorem 333H, 333K
—— defined by a group of automorphisms 333R
compact measure space §342, §343, §451
—— and representation of homomorphisms between measure algebras 343B
Compactness Theorem (for sets with bounded perimeter) 474S
complete measure space §212
completion of a measure 212C *et seq.*
concentration of measure 264H, §476, §492
conditional expectation
—— of a function §233
—— as operator on $L^1(\mu)$ 242J

construction of measures
—— image measures 112E
—— from outer measures (Carathéodory's method) 113C
—— subspace measures 131A, 214A
—— product measures 251C, 251F, 251W, 254C
—— from inner measures 413C
—— extending given functionals or measures 413I, 413J, 413O
—— —— yielding a Borel measure 435C
—— —— yielding a quasi-Radon measure 415K
—— —— yielding a Radon measure 416J, 416M, 432D, 435B
—— dominating given functionals 413S
—— τ-additive product measures 417C, 417E
—— as pull-backs 132G, 418L
—— as projective limits of Radon measures 418M
—— from integrals §436
—— —— from sequentially smooth functionals 436D
—— —— from smooth functionals 436H
—— —— yielding Radon measures (Riesz' theorem) 436J, 436K
—— of invariant measures 441C, 441E, 441H, 448P
continued fractions 372M *et seq.*
Control Measure Problem §393
convergence theorems (B.Levi, Fatou, Lebesgue) §123
convergence in measure (linear space topology on L^0)
—— on $L^0(\mu)$ §245
—— on $L^0(\mathfrak{A})$ §367
—— when Hausdorff/complete/metrizable 245E, 367N
—— and positive linear operators 367P
—— and pointwise compact sets of measurable functions §463, 465G
convex functions 233G *et seq.*, §461
Convex Isoperimetric Theorem 475T
convolution of functions
—— (on $\mathbb{R}^r$) §255, 473D, 473E
—— (on general topological groups) §444
—— $\int h \times (f * g) = \int h(x+y)f(x)g(y)dxdy$ 255G, 444O
—— $f * (g * h) = (f * g) * h$ 255J, 444O
convolution of measures
—— (on $\mathbb{R}^r$) §257
—— (on general topological groups) §444
—— $\int h\, d(\nu_1 * \nu_2) = \int h(x+y)\nu_1(dx)\nu_2(dy)$ 257B, 444C
—— of absolutely continuous measures 257F, 444Q
convolution of measures and functions 444H, 444J
countable sets 111F, 1A1C *et seq.*
countable-cocountable measure 211R
countably compact measures §451, §452
—— and disintegrations 452I, 452M
—— and inverse-measure-preserving functions 452R
counting measure 112Bd

Dedekind (σ)-complete Boolean algebras §314
—— Riesz spaces §353
Dedekind completion of a Boolean algebra 314T
differentiable functions (from $\mathbb{R}^r$ to $\mathbb{R}^s$) §262, §263
direct sum of measure spaces 214K

disintegration of measures §452
—— of one Radon measure over another, using a strong lifting to concentrate on fibers 453K
—— of Poisson point processes 495I
—— of symmetric measures on product space 459E, 459G
distribution of a finite family of random variables §271
—— as a Radon measure 271B
—— of $\phi(\boldsymbol{X})$ 271J
—— of an independent family 272G
—— determined by characteristic functions 285M
—— of an infinite family of random variables 454J
distributive laws in Boolean algebras 313B
—— in Riesz spaces 352E
divergence of a vector field §474
Divergence Theorem 475N, 484N
Doob's Martingale Convergence Theorem 275G
duals
—— of topological groups §445
—— of Riesz spaces §356, §362, 369C, 369J
—— of Banach lattices 356D
Duality Theorem (for abelian locally compact Hausdorff groups) 445U
Dye's theorem (on the full subgroup generated by a measure-preserving automorphism) 388K, 388L

effectively regular measure 491N
entropy (of a partition or a function) §385, §386
—— calculation (Kolmogorov-Sinaĭ theorem) 385P; (Bernoulli shifts) 385R
equidistributed sequences §491
Ergodic Theorem §372
—— (Pointwise Ergodic Theorem) 372D, 372E
—— (Maximal Ergodic Theorem) 372C
essential boundary (of a set in $\mathbb{R}^r$) §475
exchangeable random variables 459B, 459C
exhaustion, principle of 215A
extended real line §135
extension of measures 413K, 413O, 433J, 435C, 442Yc, 464D
—— yielding quasi-Radon measures 415L, 415M, 415N
—— yielding Radon measures 416F, 416M, 416N, 416O, 416P, 416Q, 432D, 432F, 433I, 435B
extension of order-continuous positive linear operators 355F, 368B, 368M
extremely amenable topological group §493

Fatou's Lemma ($\int \liminf \leq \liminf \int$ for sequences of non-negative functions) 123B
Federer exterior normal 474Q
Féjer sums (running averages of Fourier sums) converge to local averages of f 282H
—— uniformly if f is continuous 282G
de Finetti's theorem 459C
Fourier series §282
—— norm-converge in L^2 282J
—— converge at points of differentiability 282L
—— converge to midpoints of jumps, if f of bounded variation 282O
—— and convolutions 282Q
—— converge a.e. for square-integrable function 286V
Fourier transforms
—— on $\mathbb{R}$ §283, §284
—— on abelian topological groups §445
—— formula for $\int_c^d f$ in terms of $\hat{f}$ 283F
—— and convolutions 283M, 445G

—— as a multiplicative operator 445K
—— in terms of action on test functions 284H *et seq.*
—— of square-integrable functions 284O, 286U, 445R
—— inversion formulae for differentiable functions 283I; for functions of bounded variation 283L; for positive definite functions 445P
—— $\hat{f}(y) = \lim_{\epsilon\downarrow 0} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iyx} e^{-\epsilon x^2} f(x)dx$ a.e. 284M
free products of Boolean algebras §315
—— universal mapping theorem 315I
free products of measure algebras §325
—— universal mapping theorems 325D, 325J
Fremlin's Alternative (for sequences of measurable functions) 463H
Fubini's theorem §252
—— $\int f d(\mu \times \nu) = \iint f(x,y)dxdy$ 252B
—— when both factors σ-finite 252C, 252H
—— for characteristic functions 252D, 252F
—— for τ-additive product measures 417H
function spaces §369, §374, §376
Fundamental Theorem of the Calculus ($\frac{d}{dx}\int_a^x f = f(x)$ a.e.) §222, 483I; ($\int_a^b F'(x)dx = F(b) - F(a)$) 225E, 483R

Gagliardo-Nirenberg-Sobolev inequality 473H
gauge, gauge integral §481, §482
Gaussian distributions §456
—— are τ-additive 456K

Haar measure (on topological groups) Chap. 44
—— construction 441E
—— uniqueness 442B
—— action of the group on the measure algebra 443C
—— —— and on L^0 443G
—— derivable from Haar measure on locally compact groups 443L
—— completion regular 443M
—— has a translation-invariant lifting 447J
—— —— which is a strong lifting 453B
Hahn decomposition of a countably additive functional 231E
—— on a Boolean algebra 326I
Hahn-Banach property *see* Nachbin-Kelley theorem (363R)
Hardy-Littlewood Maximal Theorem 286A
Hausdorff measures (on $\mathbb{R}^r$) §264
—— are topological measures 264E
—— r-dimensional Hausdorff measure on $\mathbb{R}^r$ a multiple of Lebesgue measure 264I
—— on general metric spaces §471
—— Increasing Sets Lemma 471G
—— inner regularity properties 471D, 471I, 471S
—— density theorem 471P
—— need not be semi-finite 439H
—— $(r-1)$ dimensional measure on $\mathbb{R}^r$ 265F-265H, Chap. 47
Henry's theorem (on extending measures to Radon measures) 416M
Henstock integral §483

image measures 112E
—— of Radon measures 418I
—— of countably compact measures 452R
indefinite integrals
—— differentiate back to original function 222E, 222H, 483I

—— to construct measures §234
—— Saks-Henstock indefinite integrals 482B-482D, 484J
independent random variables §272
—— joint distributions are product measures 272G
—— limit theorems §273, §274
inner measures 413A *et seq.*
inner regularity of measures §412
—— (with respect to closed sets) Baire measures 412D; Borel measures on metrizable spaces 412E
—— (with respect to compact sets) Lebesgue measure 134F; Radon measures §256, §416
—— in complete locally determined spaces 412J
—— in product spaces 412S, 412U, 412V
—— and Souslin's operation 431D
integral operators §376
—— represented by functions on a product space 376E
—— characterized by action on weakly convergent sequences 376H
—— into M-spaces or out of L-spaces 376M
—— and weak compactness 376P
—— disintegrated 376S
integration of real-valued functions, construction §122
—— as a positive linear functional 122O
—— —— acting on $L^1(\mu)$ 242B
—— by parts 225F
—— characterization of integrable functions 122P, 122R
—— over subsets 131D, 214E
—— functions and integrals with values in $[-\infty,\infty]$ §133
—— measures derived from integrals §436
integration of families of measures §452
invariant means on topological groups 449E, 449I
invariant measures §441, §448
—— on measure algebras 394P
—— on locally compact spaces, under continuous group actions 441C, 443P, 443Q, 443T
—— on locally compact groups (Haar measures) 441E
—— on locally compact metric spaces, under isometries 441H
—— on Polish groups 448P
isoperimetric theorems 474L, 475T

Jensen's inequality 233I-233J
—— expressed in $L^1(\mu)$ 242K

Kakutani's theorem (on the representation of L-spaces as L^1-spaces) 369E
Kolmogorov's extension theorem (for a consistent family of probability measures on finite subproducts) 454D
Kolmogorov-Sinaĭ theorem (on the entropy of a measure-preserving homomorphism) 385P
Komlós' subsequence theorem 276H
Kwapien's theorem (on maps between L^0 spaces) 375I

Lebesgue's Density Theorem (in $\mathbb{R}$) §223
—— $\lim_{h\downarrow 0}\frac{1}{h}\int_x^{x+h} f = f(x)$ a.e. 223A
—— $\lim_{h\downarrow 0}\frac{1}{2h}\int_{x-h}^{x+h}|f(x-y)|dy = 0$ a.e. 223D
—— (in $\mathbb{R}^r$) 261C, 261E
—— (for Hausdorff measure) 471P
—— (in topological groups with B-sequences) 447D
Lebesgue measure, construction of §114, §115
—— further properties §134
—— characterized as measure space 344I

Lebesgue's Dominated Convergence Theorem ($\int \lim = \lim \int$ for dominated sequences of functions) 123C, 367J
B.Levi's theorem ($\int \lim = \lim \int$ for monotonic sequences of functions) 123A; ($\int \sum = \sum \int$) 226E
lifting (of a measure)
—— complete strictly localizable spaces have liftings 341K
—— translation-invariant liftings on $\mathbb{R}^r$ and $\{0,1\}^I$ §345
—— respecting product structures §346
—— strong liftings §453
—— Haar measures have translation-invariant liftings 447J
Lipschitz functions §262
—— differentiable a.e. 262Q
localizable measure space
—— assembling partial measurable functions 213N
—— which is not strictly localizable 216E
locally finite perimeter (for sets in $\mathbb{R}^r$) §474, §475
locally uniformly rotund norm §467
Loomis-Sikorski representation of a Dedekind σ-complete Boolean algebra 314M
lower density (on a measure space)
—— strictly localizable spaces have lower densities 341H
—— respecting a product structure 346G
Lusin's theorem (measurable functions are almost continuous), 418J
—— (on $\mathbb{R}^r$) 256F
—— (on Radon measure spaces) 451S
Lusin's theorem (injective continuous images of Polish spaces are Borel) 423I

Maharam's theorem
—— (homogeneous probability algebras of Maharam type κ are isomorphic) 331I
—— —— to measure algebra of $\{0,1\}^\kappa$) 331L
—— classification of localizable measure algebras 332J
Maharam type
—— (calculation of) 332S
—— —— for measure algebras of qproduct spaces 334A, 334C
—— relative to a subalgebra 333E
Markov processes §455
—— as Baire measures on $\prod_{t\in T} X_t$ defined from conditional distributions 455A
martingales §275
—— L^1-bounded martingales converge a.e. 275G, 367K
—— when of form $\mathbb{E}(X|\Sigma_n)$ 275H, 275I
measurable algebra §391
—— characterized by weak (σ,∞)-distributivity and existence of strictly positive additive functional 391D; by existence of uniformly exhaustive strictly positive Maharam submeasure 392J
—— Control Measure Problem §393
measurable envelopes
—— elementary properties 132E
—— existence 213L
measurable functions
—— (real-valued) §121
—— —— sums, products and other operations on finitely many functions 121E
—— —— limits, infima, suprema 121F
—— into general topological spaces §418
—— and almost continuous functions 418E, 418J, 451S
—— as selectors 423M, 433F-433G
measurable sets
—— and Souslin's operation 431A

measure algebras
—— properties related to those of measure spaces 322B
—— relationships between elementary properties 322C
—— satisfying the countable chain condition 322G
—— topologies and uniformities on §323, §324; uniqueness of topology and uniformity 324H
measure-free cardinals §438
measure-preserving Boolean homomorphisms
—— extension from a closed subalgebra 333C
modular function (of a topological group carrying Haar measures) 442I
—— and calculation of integrals 442K
—— of a subgroup 443Q
—— of a quotient group 443S
Monotone Class Theorem 136B
monotonic functions
—— are differentiable a.e. 222A

Nadkarni-Becker-Kechris theorem (on measures invariant under Polish group actions) 448P
narrow topologies on spaces of signed measures 437J *et seq.*
von Neumann-Jankow selection theorem 423N
non-measurable set (for Lebesgue measure) 134B
non-paradoxical group (of automorphisms of a Boolean algebra) §394, §448
—— characterizations 394D, 448D
—— invariant functionals 394O, 394P, 395B, 448P

order-bounded linear operators §355
order-closed ideals and order-continuous Boolean homomorphisms 313P
order-continuous linear operators §355
order*-convergence of a sequence in a lattice §367
order-dense sets in Boolean algebras 313K
—— in Riesz spaces 352N
Ornstein's theorem (on isomorphism of Bernoulli shifts with the same entropy) 387I, 387K
outer measures constructed from measures §132
—— elementary properties 132A
outer regularity of Lebesgue measure 134F

partially ordered linear spaces §351
perfect measure space 342K *et seq.*, §451
—— and pointwise compact sets of measurable functions 463I
perimeter measure (for sets in $\mathbb{R}^r$) §474, 475G
Pfeffer integral §484
Plancherel Theorem (on Fourier series and transforms of square-integrable functions) 282K, 284O, 445R
Poincaré's inequality 473K
pointwise convergence (in a space of functions) §462, §463, 465D, 465G
—— (in an isometry group) 441G
Poisson point processes §495
—— and disintegrations 495I
Poisson's theorem (a count of independent rare events has an approximately Poisson distribution) 285Q
product of two measure spaces §251
—— basic properties of c.l.d. product measure 251I
—— Lebesgue measure on $\mathbb{R}^{r+s}$ as a product measure 251M
—— more than two spaces 251W
—— Fubini's theorem 252B, 417H
—— Tonelli's theorem 252G, 417H
—— and L^1 spaces §253
—— continuous bilinear maps on $L^1(\mu) \times L^1(\nu)$ 253F

—— conditional expectation on a factor 253H
—— and measure algebras 325A *et seq.*
—— of τ-additive measures 417C; of quasi-Radon measures 417N; of Radon measures 417P
—— of compact, countably compact and perfect measures 451I
product of any number of probability spaces §254
—— basic properties of the (completed) product 254F
—— characterization with inverse-measure-preserving functions 254G
—— products of subspaces 254L
—— products of products 254N
—— determination by countable subproducts 254O
—— subproducts and conditional expectations 254R
—— and measure algebras 325I *et seq.*
—— having homogeneous measure algebras 334E
—— of separable metrizable spaces 415E; of compact metrizable spaces 416U
—— of τ-additive measures 417E; associative law 417J; of quasi-Radon measures 415E, 417O; of Radon measures 416U, 417Q
—— of compact, countably compact and perfect measures 451J
Prokhorov's theorem (on projective limits of Radon measures) 418M

quasi-Radon measures §415
—— are strictly localizable 415A
—— subspaces are quasi-Radon 415B
—— specification and uniqueness 415H
—— construction 415K
—— —— from smooth linear functionals 436H
—— products 415E, 417N, 417O
—— on locally convex spaces §466

Rademacher's theorem (Lipschitz functions are differentiable a.e.) 262Q
Radon measures, §416
—— on $\mathbb{R}^r$ §256
—— as completions of Borel measures 256C, 433C
—— indefinite-integral measures 256E, 416S
—— image measures 256G, 418I
—— product measures 256K, 416U, 417P, 417Q
—— extending given functionals or measures 416J, 416M-416Q, 432D
—— subspaces 416R
—— and Stone spaces of measure algebras 416V
—— and measurable functions 451S
—— as pull-backs 418L, 433D
—— and disintegrations 453K
—— Haar measures 441E; other invariant measures 441H, 448P
—— on locally convex spaces §466
Radon-Nikodým theorem (truly continuous additive set-functions have densities) 232E
—— in terms of $L^1(\mu)$ 242I
—— on a measure algebra 327D, 365E
rearrangements (monotonic rearrangements of measurable functions) §373, §374
regular open algebras 314P, §513
representation
—— of Boolean algebras *see* Stone's theorem (311E)
—— of Dedekind σ-complete Boolean algebras 314M
—— of measure algebras 321J
—— of homomorphisms between measure algebras §343, §344
—— of linear functionals as integrals §436
—— of partially ordered linear spaces 351Q

—— of Riesz spaces 352L, 353M, 368E *et seq.*, 369A
—— of Riesz space duals 369C
—— of L-spaces 369E
—— of M-spaces 354L
representations of topological groups 446B
Riesz representation theorem (construction of a Radon measure from an integral) 436J, 436K
Riesz spaces (vector lattices) §352
—— identities 352D, 352F, 352M
Saks-Henstock lemma 482B
—— indefinite integrals 482D, 484J
Shannon-McMillan-Breiman theorem (entropy functions of partitions generated by a measure-preserving homomorphism converge a.e.) 386E
signed measures 437C, 437H
simple product of Boolean algebras
—— universal mapping theorem 315B
—— of measure algebras 322K
Sinaĭ's theorem (on existence of Bernoulli partitions for a measure-preserving automorphism) 387E, 387L
Souslin's operation §421
—— on measurable sets §431
split interval 343J, 419L
stable set of functions §465
—— is relatively pointwise compact 465D
—— pointwise convergence and convergence in measure 465G
—— uniform strong law of large numbers 465M
—— in L^0 465P
—— in L^1 465R
—— R-stable set 465S
Stone space of a Boolean ring or algebra 311E *et seq.*
—— as topological space 311I
—— and Boolean homomorphisms 312P, 312Q
—— of a Dedekind complete algebra 314S
—— and the countable chain condition 316B
—— and weak (σ, ∞)-distributivity 316I
—— of a measure algebra 321J, 322N, 322Q
—— of a Radon measure space 416V
Stone-Weierstrass theorem §281
—— for Riesz subspaces of $C_b(X)$ 281A
—— for subalgebras of $C_b(X)$ 281E
—— for *-subalgebras of $C_b(X;\mathbb{C})$ 281G
—— for *-subalgebras of $C_0(X;\mathbb{C})$ 4A6B
strictly localizable measures
—— sufficient condition for strict localizability 213O
—— complete locally determined effectively locally finite τ-additive topological measures are strictly localizable 414J
strong law of large numbers ($\lim_{n\to\infty} \frac{1}{n+1}\sum_{i=0}^{n}(X_i - \mathbb{E}(X_i)) = 0$ a.e.) §273
—— when $\sum_{n=0}^{\infty} \frac{1}{(n+1)^2}\operatorname{Var}(X_n) < \infty$ 273D
—— when $\sup_{n\in\mathbb{N}} \mathbb{E}(|X_n|^{1+\delta}) < \infty$ 273H
—— for identically distributed X_n 273I
—— for martingale difference sequences 276C, 276F
—— convergence of averages for $\|\ \|_1$, $\|\ \|_p$ 273N
—— and functions on product spaces 465H
—— nearly uniformly on stable sets 465M
strong liftings §453

submeasure §392
subspace measures
—— for measurable subspaces §131
—— for arbitrary subspaces §214
surface measure in $\mathbb{R}^r$ §265
symmetric group 449Xc, 492H, 493Xc
symmetric measures on product spaces 459E, 459G

tagged-partition structures §481, §482
Talagrand's measure §464
Tamanini-Giacomelli theorem 484B
tensor products of L^1 spaces §253, 376C
Tonelli's theorem (f is integrable if $\iint |f(x,y)|dxdy < \infty$) 252G, 417H
topological groups chap. 44
—— amenable groups §449
—— duality theorem §445
—— extremely amenable groups §493
—— Haar measure §441, §442, §443
—— structure theory §446

uniformly exhaustive submeasures
—— and additive functionals 392F
uniformly integrable sets in L^1 §246
—— criteria for uniform integrability 246G
—— in L-spaces 354P *et seq.*
—— and convergence in measure 246J
—— and weak compactness 247C, 356Q
—— in dual spaces 356O, 362E
universally measurable sets and functions §434

vague topologies on spaces of signed measures 437J *et seq.*
Vitali's theorem (for coverings by intervals in $\mathbb{R}$) 221A
—— (for coverings by balls in $\mathbb{R}^r$) 261B
—— (in general metric spaces) 471O
—— (in topological groups with B-sequences) 447C

weak compactness in $L^1(\mu)$ §247
—— in L-spaces 356Q
—— in $C_0(X)$ 462E
weak topologies of locally convex spaces §466
weakly compactly generated normed spaces 467L
weakly (σ,∞)-distributive Boolean algebras §316
—— and measure algebras 322F, 391D
—— Riesz spaces 368Q
Weyl's Equidistribution Theorem 281N

c.l.d. version 213D *et seq.*
$C(X)$ §436, §462
$C_b(X)$ §436, §437
K-analytic spaces §422, §432
—— basic properties 422G
—— and Souslin-F sets 422H
—— are capacitable 432B, 432J
K-countably determined spaces 467H *et seq*
L-space 354N *et seq.*, §371
$L^0(\mu)$ (space of equivalence classes of measurable functions) §241
—— as Riesz space 241E

$L^0(\mathfrak{A})$ (based on a Boolean algebra) §364, §368, §369, §375
—— as a quotient of a space of functions 364D
—— algebraic operations 364E
—— calculation of suprema and infima 364M
—— linear operators induced by Boolean homomorphisms 364R
—— *when $\mathfrak{A}$ is a regular open algebra 364U
—— —— *of a compact extremally disconnected space 364W
—— as the domain of a linear operator 375A, 375C
—— positive linear operators between L^0 spaces can be derived from Riesz homomorphisms 375I
$L^1(\mu)$ (space of equivalence classes of integrable functions) §242
—— norm-completeness 242F
—— density of simple functions 242M
—— (for Lebesgue measure) density of continuous functions and step functions 242O
$L^1(\mathfrak{A}, \bar{\mu})$ (based on a measure algebra) §365
—— related to $L^1(\mu)$ 365B
—— as L-space 365C, 369E
—— linear operators induced by Boolean homomorphisms 365H, 365O, 365P
—— duality with L^∞ 365M
—— universal mapping theorems 365I, 365K
—— stable sets 465R
$L^p(\mu)$ (space of equivalence classes of pth-power-integrable functions, where $1 < p < \infty$) §244
—— is a Banach lattice 244G
—— has dual $L^q(\mu)$, where $\frac{1}{p} + \frac{1}{q} = 1$ 244K
—— and conditional expectations 244M
$L^p(\mathfrak{A})$ (based on a Boolean algebra) §366
—— related to $L^p(\mu)$ 366B
—— as Banach lattice 366C
—— linear operators induced by Boolean homomorphisms 366H
$L^\infty(\mu)$ (space of equivalence classes of bounded measurable functions) §243
—— duality with L^1 243F-243G
—— norm-completeness 243E
—— order-completeness 243H
$L^\infty(\mathfrak{A})$ (where $\mathfrak{A}$ is a Boolean algebra) §363
—— universal mapping theorems 363E
—— linear operators induced by Boolean homomorphisms 363F
—— order-completeness 363M
—— Nachbin-Kelley theorem (generalized Hahn-Banach theorem) 363R
ℓ^∞ and its dual §464
$\mathrm{L}^\sim(U; V)$ (space of order-bounded linear operators) §355, 371B, 371C
$\mathrm{L}^\times(U; V)$ (space of differences of order-continuous linear operators) §355, 376E
M-space 354G *et seq.*
$S(\mathfrak{A})$ (space of 'simple' functions corresponding to a Boolean ring $\mathfrak{A}$) §361
—— universal mapping theorems 361F-361I
—— linear operators induced by ring homomorphisms 361J
—— dual spaces §362
$\mathcal{T}$, $\mathcal{T}^{(0)}$, $\mathcal{T}^\times$ (classes of operators which are norm-reducing for every $\| \|_p$) 371F-371G, §372, §373, §374
usco-compact relations §422

σ-algebras of sets §111
—— generated by given families 136B, 136G
σ-finite measures 215B
τ-additive measures §414, §417
—— and semi-continuous functions 414A, 414B
—— products

—— —— of two τ-additive measures 417C
—— —— of many τ-additive probability measures 417E
—— on quasi-dyadic spaces 434Q

$\{0,1\}^\kappa$
—— usual measure is (Maharam) homogeneous with Maharam type κ 331K; is homogeneous in strict sense 344L; is a completion regular Radon measure 416U

General index

References in **bold** type refer to definitions; references in *italics* are passing references. Definitions marked with > are those in which my usage is dangerously at variance with that of some other author.

Abel's theorem 224Yi
Abel-Poisson kernel **282Yg**
abelian topological group 441Ia, 444D, 444Og, *444Sb*, 444Xd, 444Xk, §445, 446Xb, 449Cf, 491Xn
absolute summability *226Ac*
absolutely continuous additive functional **232Aa**, 232B, 232D, 232F-232I, 232Xa, 232Xb, 232Xd, 232Xf, 232Xh, *234Ce*, *256J*, 257Xf, **327A**, 327B, 327C, 362C, 362Xh, *362Xi*, 363S, 414D, 414Xd, 438Xb
absolutely continuous function **225B**, 225C-225G, 225K-225O, 225Xa-225Xh, 225Xn, 225Xo, 225Ya, 225Yc, *232Xb*, *233Xc*, *244Yh*, 252Yj, 256Xg, 262Bc, 263I, **264Yp**, 265Ya, 282R, 282Yf, *283Ci*
absolutely continuous submeasure **393D**, 393E, 393H, 393Yg
Absoluteness Theorem *see* Shoenfield's Absoluteness Theorem
abstract integral operator **376D**, 376E, 376F, 376H, 376J, 376K, 376M, 376P, 376Xc, 376Xk, 376Yh-376Yj, 376Yl, 376Yn
accumulation point *see* cluster point (**3A3La**)
action of a group on a set **441A**, 441C, 441Ya, 442Xe, 444Xe, 448Xi, 449J, 449L, 449Xr, **4A5B**, 4A5C
—— Borel measurable action 424H, 448P, 448Xf, **4A5I**
—— continuous action 424H, 441B, 441Ga, 441Xa, 441Yc, 441Yj, 443C, 443G, 443O-443Q, 443T, 443Xd, 443Yb, 444F, 449A, 449B, 449D, 461Yc, 471Xc, 471Xj, **4A5I**, 4A5Jc
—— of a semigroup **449Ya**
—— *see also* conjugacy action (**4A5Ca**), left action (**4A5Ca**), right action (**4A5Ca**), transitive action (**4A5Bb**)
additive functional on an algebra of sets *see* finitely additive (**136Xg**, **231A**), countably additive (**231C**)
additive function(al) on a Boolean algebra *see* finitely additive (**326A**, **361B**), countably additive (**326E**), completely additive (**326J**)
additive*see also* τ-additive (**411C**)
adherence point *see* cluster point (**3A3La**)
adjoint operator 243Fc, 243 *notes*, 371Xe, 371Yd, 373S-373U, 373Xu, 373Yg, **3A5Ed**, 437Xd
affine subspace *476Xk*
algebra (over $\mathbb{R}$) **361Xb**, 361Xf; *see also* algebra of sets (**231A**), Baire property algebra (**314Yd**), Banach algebra (**2A4Jb**, **4A6Ab**), Boolean algebra (**311A**), category algebra (**314Yd**), Jordan algebra (**491Ye**), normed algebra (**2A4J**, **4A6Aa**), f-algebra (**352W**), regular open algebra (**314Q**)
algebra of sets 113Yi, **136E**, 136F, 136G, 136Xg, 136Xh, 136Xk, 136Ya, 136Yb, **231A**, 231B, 231Xa, 311Bb, 311Xb, 311Xh, 312B, 315G, 315L, 362Xg, 363Yf, 381Xa, 475M, 4A3Cg; *see also* Boolean algebra (**311A**), σ-algebra (**111A**)
algebraic dual 4A4Ac, *4A4Cg*
Alexandroff compactification *see* one-point compactification (**3A3O**)
almost continuous function 256F, **411M**, 411Ya, 418D-418K, 418Nb, 418P, 418Q, 418Sb, 418Tb, 418Xc-418Xn, 418Xp, 418Xq, 418Xs-418Xu, 418Ye-418Yh, *419Xe*, 419Xh, 432Xe, 433E, 434Yb, 438F, 438G, 443Pf, 451S, 452Xe, 453K, 454N
almost every, almost everywhere **112Dd**; *see also* Haar almost everywhere (**443Ae**)
almost isomorphic (inverse-measure-preserving functions) **385U**, 385V, 385Xn-385Xp, 385Yf
almost Lindelöf *see* measure-compact (**435D**)

almost strong lifting **453A**, 453D-453G, 453K, 453Xf, 453Xi, 453Ya, 453Zb; *see also* strong lifting (**453A**)
almost surely **112De**
ambit *see* greatest ambit (**449D**)
amenable group *394 notes*, **>449A**, 449C, 449E-449G, 449I, 449K, 449L 449Xa-449Xe, 449Xg-449Xl, 449Xn-449Xt, 449Yb, *449Yf*, 461Yc, 493Ya; *see also* extremely amenable (**493A**)
amenable semigroup **449Ya**, 449Yh
analytic (complex) function 133Xc
analytic (topological) space **423A**, 423B-423O, 423Xa-423Xh, 423Yb-423Yd, 424Xb, 424Xd, 433C-433I, 433Xb, 433Xc, 433Yb, 433Yd, 434B, 434Dc, 434Ka, 434Xd, 434Xi, 434Ya, 471F, 471I, 471S, 471Xc, 471Xh, 471Yc, 439Yc, 452N, 452P, 453G, 454O, 454P, 494A; *see also* K-analytic (**422F**)
angelic topological space **2A5J**, **462Aa**, 462B-462D, 462Xa, 462Yb
antichain 316 *notes*
aperiodic Boolean homomorphism **381Ad**, 381H, 381Nf, 381P, 381Xc, 381Xj, 381Xl, 382J, 386C, 386D, 386Ya, 388F, 388H, 388J, 388K, 388Yb; *see also* nowhere aperiodic (**381Xk**)
approximately continuous Henstock integral 481Q
Archimedean partially ordered linear space **351R**, 351Xe, 361Gb
Archimedean Riesz space **241F**, 241Yb, 242Xc, 353A-353F, 353Ha, 353L-353P, 353Xa, 353Xb, 353Xd, 353Ya, 353Yd-353Yh, 354Ba, 354F, *354I-354K*, 354Yi, 355Xd, 356G-356I, 361Ee, 367C, 367F, 367Xh, *367Ya*, 367Yf, 368B, 368E-368G, 368I, 368J, 368M, 368O, 368P, 368R, 368Ya, 368Yb
area *see* surface measure
arrow ('double arrow space', 'two arrows space') *see* split interval (**343J**)
associate extended Fatou norm **369H**, 369I-369L, 369O, 369R, 369Xd, 374B, 374C, 374Xh, *376S*
asymptotic density
asymptotic density **273G**, 464Jb, **491A**, 491I, 491Ka, 491Xa, 491Xb, 491Xd, *491Xh*, 491Xy, 491Ya, 491Ye; *see also* upper asymptotic density (**491A**)
asymptotic density ideal **491A**, 491I, 491J; *see also* $\mathfrak{Z}$
asymptotically equidistributed *see* equidistributed (**281Yi**, **491B**)
atom (in a Boolean algebra) **316L**, 316M, 316Xr-316Xt, 316Xy, 316Yq, *322Bf*, *324Ye*, *331H*, *333Xb*, *343Xe*, *362Xe*; *see also* relative atom (**331A**)
atom (in a measure space) **211I**, *211Xb*, *246G*, 322Bf, 414G, 415Xa, 416Xa
atom (in a Riesz space) 362Xe
atomic *see* purely atomic (**211K**, **316Lc**)
atomless Boolean algebra **316Lb**, 316Mb, 316Xr, 316Xs, 316Xu-316Xw, 316Yl-316Yn, 316Yp, 316Yq, *322Bg*, *322Kc*, 324Kf, 324Ye, 331Xk, 332I, 332P, *332Ya*, 375B, 375Xb, 375Yb, 381P, 382Q, 384E, 384F, 384Xb, 386D, 386M, *387Cd*, *387C-387H*, *387L*, 392Xg; *see also* relatively atomless (**331A**)
atomless additive functional **326Ya**, 326Yb-326Yd, 326Yf, 362Xf, 362Xg, 362Yi
atomless measure algebra 331C, 369Xh, 374Xl, 383G, 383H, 383J, 383Xg, 383Xh, *383Ya*, 384Ld, 384M, 384O, 384P, *384Xd*, 493E, 493H, 493Ya, 493Yb
atomless measure (space) **211J**, 211Md, *211Q*, *211Xb*, 211Yd, 211Ye, *212G*, 213He, 214Xe, 215D, 215Xe, 215Xf, *216A*, 216Ya, 234Ff, *234Yd*, 251T, 251Wo, 251Xq, 251Xs, 251Yc, 252Yp, 252Yr, 252Yt, 254V, 254Yf, 256Xd, 264Yg, *322Bg*, 325Ye, 342Xc, 343Cb, 343Xi, 344I, 415Xa, 417Xn, 419Xc, *433Xf*, *433Yb*, 439D, 439Xh, 451Xb, 471Dg, 493F, 494B, 494C, 494Xb-494Xd, 494Ya, 495D, 495F-495I, 495L-495N, 495Xa-495Xd
atomless vector measure 326Ye
automorphism *see* Banach lattice automorphism, Boolean automorphism, Borel automorphism,cyclic automorphism,ergodic automorphism, group automorphism (**3A6B**), induced automorphism (**381M**), inner automorphism (**3A6B**), measure-preserving Boolean automorphism, measure-preserving ring automorphism, measure space automorphism, outer automorphism (**387B**)
automorphism group of a Boolean algebra 381A, 381B, 381Xa, 382O-382S, 382Xg, 382Yb, 383Xh, 383Ya, 383Yb, §384, §394; of a measure algebra 366Xh, 374J, 381Xj, §§383-384, 394F, 394R, 493G, 493H, 493Xe, 493Yc, 493Yd
axiom *see* Banach-Ulam problem, choice (**2A1J**), constructibility, continuum hypothesis (**4A1Ad**), countable choice, Jensen's $\diamondsuit$, Ostaszewski's $\clubsuit$ (**4A1M**)
axiom of countability (for topological spaces) *see* first-countable (**4A2A**), second-countable (**4A2A**)

Baire measurable function 417Be, 437Yg, **4A3Ke**
Baire measure **411K**, 411Xf, 411Xh, 412D, 415N, 416Xk, 432F, §435, 436E, 436F, 436Xj, *436Yc*, 437E, 452Xc, 454B, 454M, 454P, 455Xa, 461M, 491Mc, 491Yd; *see also* signed Baire measure (**437G**)
Baire property (for subsets of a topological space) **314Yd**, **4A3Q**, 4A3Ra
Baire property algebra **314Yd**, 316Yi, 341Yb, 367Yj, 414Yd, 431F, 466Xh, **4A3Q**, 4A3R
Baire property envelope 431Fa
Baire set 412Yd, 417Be, 417Xu, 421L, *422Xd*, 422Xe, 434Hc, 434Pd, 435H, **4A3K**
Baire space **314Yd**, 341Yb, 364Yi, 364Yl, 367Yh, 367Yj, 414Yd, 4A3Ra
Baire's theorem 3A3G, 4A2Ma
Baire σ-algebra 254Xs, **341Yc**, 341Zb, 343Xc, 344E, 344Ya, 344Yc-344Ye, 411R, 415Xp, 417U, 417V, 421Xh, 421Yb, 423Db, *434Yf*, 435Xa, 435Xb, 437E, 439A, 443Yg, 452N, 461Xg, **4A3K**, 4A3L-4A3P, 4A3U, 4A3V, 4A3Xb-4A3Xd, 4A3Xf, 4A3Xg, 4A3Yb
ball (in $\mathbb{R}^r$) 252Q, 252Xh, *474L*; (in other metric spaces) 466 *notes*, 4A2Lj; *see also* sphere
Banach algebra **>2A4Jb**, 363Xa, 444E, 444S, 444Xu, 444Yb, 444Yc, *445H*, 446A, **4A6Aa**, 4A6F; *see also* commutative Banach algebra (**4A6Aa**), unital Banach algebra (**4A6Ab**)
Banach function space §369, §374
Banach lattice **242G**, 242Xc, 242Yc, 242Ye, 243E, 243Xb, 326Yj, **354A**, 354C, 354Ee, *354L*, 354Xa, 354Xb, 354Xe-354Xj, 354Xn-354Xp, 354Yd, 354Yi-354Yl, 355C, 355K, 355Xb, 355Ya, 356D, 356M, 356Xc, 356Xk, 356Yg, 356Yh, 356Yi, 363E, 365K, 366C, 366Xd, 367D, 367P, 369B, 369G, 369Xe, 371B-371E, 371Xa, *371Xc*, 375Xb, 376C, 376M, 376Xg, 376Yj, 467N; *see also* extended Fatou norm, L-space (**354M**), L^p, M-space (**354Gb**), Orlicz space
Banach lattice automorphism 366Xh
Banach-Mazur game *451U*
Banach space *231Yh*, *262Yi*, **2A4D**, 2A4E, 2A4I, 326Yk, 354Yk, 3A5H, 3A5I, 466B, 466F, 466L, 466M, 467F, 467G, 467Ic, 467M, 467Xg, 467Xh, *483Yg*, 4A4I; *see also* Banach algebra (**2A4Jb**), Banach lattice (**242G**), separable Banach space
Banach-Tarski paradox 449Yg
Banach-Ulam problem 232Hc, 326 *notes*, 363S, 376Yg, 439J, 439L; *see also* measure-free cardinal (**438A**)
band in a Riesz space **352O**, 352P-352V, 352Xe, 352Xf, 352Xj, 353B, 353C, 353E, 353H, 353I, 354Bd, 354O, 355H, 355I, 356B, 356L, 362B, 362C, 362Xf-362Xi, 364Xn, *364Xo*, 414Yb, 436L, 436Xr, 437A, 437F, 437Xq; *see also* complemented band, complement of a band, principal band, projection band
band algebra (of an Archimedean Riesz space) **353B**, 353D, 356Yc, 356Yd, 361Yc, 362Ya, 362Yb, 365S, *365Xm*, 366Xb, 368E, 368R; *see also* complemented band algebra (**352Q**), projection band algebra (**352S**)
band projection **352R**, 352Sb, 352Xm, 355Yh, 356C, 356Xe, 356Xf, 356Yb, 362B, 362D, 362Ye, 362Yi
barycenter **461Aa**, 461B-461E, 461G, 461H, 461J, 461K, 461M, 461N, 461Xb-461Xd, 461Xj, 461Yb, *464Xb*
base for a filter **4A1Ia**
base for a topology **3A3M**, 4A2Ba, 4A2Fc, 4A2Lg, 4A2O, 4A2Pa
—— *see also* π-base (**4A2A**)
base of neighbourhoods **4A2A**, 4A2Gd
basically disconnected topological space 314Yf
basis *see* Hamel basis (**4A4Aa**), orthonormal basis (**4A4Ja**)
Becker-Kechris theorem 424H; *see also* Nadkarni-Becker-Kechris theorem
A.Bellow's problem 463Za
Bernoulli partition **387A**, 387B, 387D-387H, 387J
—— shift **385Q**, 385R, 385S, 385Xi-385Xm, 386Xc, 387B, 387I, 387K, 387L, 387Xa, 387Xc, 387Ya
Berry-Esséen theorem 274Hc, 285 *notes*
Besicovitch's Covering Lemma 472A-472C, 472Yb
Besicovitch's Density Theorem 472D, 472Ye
bidual of a normed space (U^{**}) 342Yc, 356Xh, *3A5Ga*, 437Ib, 4A4If; *see also* order-continuous bidual
bidual of a topological group 445E, 445O, 445U
Bienaymé's equality 272R
bilateral uniformity on a topological group 443H, 443I, 443K, 443Xk, 443Yd, 444Xs, *445Ab*, 445E, 445Ya, 449Xh, 493Xb, **4A5Hb**, 4A5M-4A5O

bilinear map §253 (**253A**), 255Xb, 363L, 376B, 376C, 376Ya-376Yc, 436Yf, 437M, 437Xn, 437Ym, 444Sa, *482Xa*

Bipolar Theorem 4A4Eg

Bochner integral **253Yf**, 253Yg, 253Yi, 354Yl, 451Yf, 463Yb, *463Yc*, 483Yg

Bochner's theorem 285Xr, 445N, 445Xh, 445Xo

Boolean algebra Chap. 31 (**311Ab**), 363Xf; *see also* algebra of sets, complemented band algebra (**352Q**), Dedekind (σ-)complete Boolean algebra, measure algebra (**321A**), projection band algebra (**352S**), regular open algebra (**314Q**)

Boolean automorphism 363Ye, 366Yh, *374Yd*, 381C-381E, 381G, 381I, 381J, 381L, 381Q, 381Xc, 381Xf, 381Xg, 381Xn, 381Xo, 381Yb, 382A, 382B, 382D, 382E, 382G-382M, 382Xa, 382Xc, 382Xd, 384A, 388D, 388Xa, 388Ya, 394Ge, 394Ya, 395A, 424Xk, *442H*, 443Af; *see also* Borel automorphism, cyclic automorphism (**381R**), induced automorphism (**381M**), involution, measure-preserving Boolean automorphism, periodic automorphism (**381Bc**), von Neumann automorphism (**388D**), weakly von Neumann automorphism (**388D**)

Boolean homomorphism **312F**, 312G-312K, 312N, 312P-312T, 312Xe, 312Xg, 312Xj, *312Ya*, 313L-313N, 313P-313R, 313Xp, 313Yb, 314I, 314K, 314Xh, 316Yo, §324, *326Be*, *332Yc*, 343A, *352Ta*, 361Xd, 363F, 363Xb, 363Xc, 381B, 381H, 381Xj, *385K*; *see also* aperiodic Boolean homomorphism (**381Bd**), Boolean isomorphism, measure-preserving Boolean homomorphism, (sequentially) order-continuous Boolean homomorphism, periodic Boolean homomorphism (**381Bc**)

Boolean-independent subalgebras **315Xn**

Boolean isomorphism 312L, 314I, 332Xj, 344Yb, 366Yh

Boolean ring §311 (**311Aa**), §361

Boolean subalgebra *see* subalgebra of a Boolean algebra (**312A**)

'Boolean value' (of a proposition) *364Bb*

Borel algebra *see* Borel σ-algebra

Borel automorphism 419Xh

Borel-Cantelli lemma 273K

Borel constituent *see* constituent (**423P**)

Borel isomorphism 254Xs, 419Xh, 423Yc, 424C, 424D, **4A3A**

Borel lifting *345F*, *345Xg*, *345Yc*

Borel measurable function **121C**, 121D, 121Eg, 121H, 121K, **121Yf**, 134Fd, 134Xd, 134Yt, **135Ef**, 135Xc, 135Xe, 225H, 225J, *225Yf*, 233Hc, 241Be, 241I, 241Xd, 256M, 364I, 364J, 364Xg, 364Yd, 367Yo, *417Bb*, *418A*, 423G, 423Ib, 423O, 423Xi, 423Yc, *423Ye*, 433D, 434Xi, 437Jd, 437Yg, 471Xf, *443Jb*, *444F*, *444G*, *444Xg*, 466Xh, 476Xe, 493Yd, **4A3A**, 4A3C, 4A3Dc; *see also* Borel isomorphism

Borel measure **>411K**, 411Xf-411Xh, 412E, *412Xf*, 414L, 414Xj, 415C, 415D, 416F, 416H, 417Xc, 417Xd, 432Xc, 433C, §434, 435C, 435Xa-435Xd, 435Xf, 435Xg, 451Xi, 452Xa, 471S, 438I, 438S, 438Yc, 438Yj, 444E, 466H, 491Xw; (on $\mathbb{R}$) 211P, 216A, 341Lg, 343Xd, 345F; *see also* signed Borel measure (**437G**)

Borel-measure-compact (topological) space **434Ga**, 434H, 434I, 434Na, 434Xj, 434Xk-434Xn, 434Xu, 434Yg, 434Yj, 435Fd, 435Xk, 438J, 438Xk, 438Xo, 438Yd, *439P*

Borel-measure-complete (topological) space **434Gb**, 434I, 434Ka, 434N, 434Xj, 434Xk-434Xn, 434Xr, 434Xs, 434Xu, 438Jc, 438M, 438Xk, 438Xo, 438Ye, *439P*, 439Xh, *466H*

Borel sets in $\mathbb{R}$, $\mathbb{R}^r$ **111G**, *111Yd*, 114G, 114Yd, 115G, 115Yb, *115Yd*, *121Ef*, *121K*, 134F, 134Xc, *135C*, 136D, 136Xj, *212Xc*, *212Xd*, 254Xs, 225J, 264E, 264F, 264Xb, *342Xi*, 364H, 364Yc, 474Xb, 475Cc;

—— in other topological spaces *411A*, *414Ye*, *418A*, 421Xj, 421Xk, 422J, 422Xf, *422Yc*, *422Yd*, 423E, 423I, 423K, *423L*, 423Xe, *424F*, *434Fc*, 443Jb, 443Xn, *444F*, *444Xg*, *451N*, **4A3A**, 4A3H-4A3J, 4A3Ya

Borel space *see* standard Borel space (**424A**)

Borel σ-algebra (of subsets of $\mathbb{R}^r$) **111Gd**, *114Yg*, *114Yh*, *114Yi*, 121J, 121Xd, 121Xe, 121Yb, *216A*, 251L, 364G, 382Yb, 439A;

—— (of other spaces) **135C**, 135Xb, **256Ye**, 271Ya, *314Ye*, *333Ya*, *411K*, 421H, 421Xe-421Xg, 421Xk, 421Xm, 423N, 423Xd, 423Yc, 424Xb, 424Xd-424Xf, 424Yb, *431Xa*, 433I, 434Eb, 435Xb, 435Xd, 437H, 443Jb, 443Yh, 452N, 461Xi, 466E, 466Yb, **4A3A**, 4A3C-4A3G, 4A3Kb, 4A3N, 4A3R, 4A3V, 4A3Xa, 4A3Xd; *see also* standard Borel space (**424A**), Effros Borel structure (**424Ya**)

boundary of a set in a topological space 411Gi, 411Yc, 474Xc, 475C, *475J*, 475S, 475T, 475Xa, **4A2A**, 4A2Bi; *see also* essential boundary (**475B**), reduced boundary (**474G**)

bounded bilinear map **253Ab**
bounded linear operator 253Ab, 253F, 253Gc, 253L, 253Xc, 253Yf, 253Yj, 253Yk, **2A4F**, 2A4G-2A4I, 2A5If, 355C, 3A5Ed, 466L, 466M, *466Yc*, 4A4Ib, *4A4La*; *see also* (weakly) compact linear operator (**3A5K**), order-bounded linear operator (**355A**), $\mathrm{B}(U;V)$ (**2A4F**)
bounded set (in $\mathbb{R}^r$) **134E**; (in a normed space) **2A4Bc**, 3A5H, 4A4Ka; (in a linear topological space) **245Yf**, 367Xw, **4A4B**; *see also* order-bounded (**2A1Ab**)
bounded support, function with *see* compact support (**4A2A**)
bounded variation, function of §224 (**224A**, **224K**), 225Cb, 225M, *225Oc*, *225Xh*, *225Xn*, 225Yc, 226Bc, 226C, 264Yp, 264Yb, 282M, 282O, 283L, 283Xj, 283Xk, *283Xm*, *283Xn*, *283Xq*, 284Xk, *284Yd*, 343Yc, 354Xt, 438Xp, 438Yi, 483Xg, 483Yd
bounding *see* ω^ω-bounding (**316Yg**)
box product of tagged-partition structures 481P
Breiman *see* Shannon-McMillan-Breiman theorem (386E)
Brownian motion **455C**, 455D, 455Yc, 456Xg; *see also* fractional Brownian motion (**456Ye**)

canonical outward-normal function **474G**, 474J, 474Q, 475O, 475P; *see also* Federer exterior normal (**474O**)
Cantor function **134H**, 134I, 222Xd, 225N, 226Cc, 262K, 264Xf, 264Yn
Cantor measure **256Hc**, 256Ia, 256Xk, 264Ym
Cantor set **134G**, 134H, 134I, 134Xe, 256Hc, 256Xk, 264J, 264Ym, 264Yn, 412Ya, 471Xg, 4A2Uc; *see also* $\{0,1\}^I$
cap (in a sphere) **476M**
capacity **432I**, 432J, 432Xf-432Xh; *see also* Hausdorff capacity (**471H**)
Carathéodory complete measure space *see* complete (**211A**)
Carathéodory's method (of constructing measures) 113C, 113D, 113Xa, 113Xd, 113Xg, 113Yc, 114E, 114Xa, *121Yc*, 132Xc, 136Ya, 212A, *212Xg*, 213C, 213Xa, 213Xb, 213Xd, 213Xf, *213Xg*, 213Xj, 213Ya, 214H, 214Xc, 216Xb, *251C*, *251Wa*, *251Xd*, *264C*, *264K*, 413Xd, 413Xl, 431C, *436Xc*, *436Xk*, 471C, 471Ya, 452Xi
cardinal **2A1Kb**, 2A1L, 3A1B-3A1E, 4A1A, 4A1B; *see also* measure-free cardinal (**438A**), regular cardinal (**4A1Aa**)
cardinal function (of a Boolean algebra) *see* cellularity (**332D**, Maharam type (**331F**)
—— (of a partially ordered set) *see* cofinality (**3A1F**)
—— (of a topological space) *see* cellularity (**332Xd**, density (**331Yf**), weight (**4A2A**)
Carleson's theorem 282K *remarks*, 282 *notes*, 284Yg, 286U, 286V
carry Haar measures (in 'topological group carrying Haar measures') **442D**, 442Xb, 442Xc, 443A, 443C-443F, 443H, 443J, 443K, 443N, 443Xa, 443Xd-443Xf, 443Xi-443Xk, 443Xn, 443Ya, 443Ye, 444J, 444L, 444Xi-444Xk, 444Xm, 444Yi, 445J, 445O, 445Xc, 445Yi, 445Yj, 447A, 447B, 447J, 449Yd, 493Xh; *see also* Haar measure
category algebra (of a topological space) **314Yd**, 4A3Rc
Cauchy distribution **285Xm**, 367Xw, 455Xd, 455Yf, 495Ya
Cauchy filter **2A5F**, 2A5G, **3A4F**
Cauchy's inequality 244E
Cauchy's Perimeter Theorem 475S
Cauchy sequence *242Yc*, **2A4D**, 354Ed, 356Yf, 356Yg, 3A4Fe
Čech-complete space 434Jg, 437Wf, 437Yl, **4A2A**, 4A2Gj, 4A2Md, 4A3Ya
Čech-Stone compactification *see* Stone-Čech compactification (**4A2I**)
cellularity(of a topological space) **332Xd**, 332Xe, 438Ye, 438Yj, ;
—— (of a Boolean algebra) §332 (**332D**), 365Ya, 438Xc, 443Ya;
—— *see also* ccc (**316A**), magnitude (**332Ga**)
centered Gaussian distribution *see* Gaussian distribution (**456Ab**)
Central Limit Theorem 274G, 274I-274K, 274Xc-274Xg, 285N, 285Xn, 285Ym
Cesaro sums 273Ca, 282Ad, 282N, 282Xn
Chacon-Krengel theorem 371 *notes*, 465 *notes*

chain condition (in Boolean algebras or topological spaces) *see* ccc (**316A**), σ-linked (**391L**), σ-m-linked (**391Yh**)
chain rule for differentiation 473Bc
chain rule for Radon-Nikodým derivatives 235Xi
change of variable in integration §235, 263A, 263D, 263F, 263G, 263I, 263Xc, 263Xe, 263Yc
character (on a topological group) **445Aa**, 491Xn
characteristic function (of a set) **122Aa**
—— (of a probability distribution) §285 (**285Aa**), 445Xh, 454M, 454Xk, 466J, 466K; *see also* Fourier-Stieltjes transform (**445C**)
—— (of a random variable) §285 (**285Ab**), 495M
charge **326A**; *see* finitely additive functional
chargeable Boolean algebra 391D, 391J, *391N*, **391X**, 391Xa-391Xf, 391Yb, 391Yc, *391Yf*, 392F, 394Xd; topological space **417Xu**
Chebyshev's inequality 271Xa
choice, axiom of 134C, 1A1G, 254Da, **2A1J**, 2A1K-2A1M, *361 notes*, 441 *notes*; *see also* countable choice
choice function **2A1J**
Choquet capacity *see* capacity
Choquet's theorem (analytic sets are capacitable) 432J
Choquet's theorem (on measures on sets of extreme points) 461K
circle group 255M, *255Yh*, 345Xb, 345Xg, *443Xt*, 445B
clopen algebra 311I, 311J, 311Xh, 437Xl, 481Xh
closed convex hull 253G, **2A5E**, 461H, 461I, 461Xf, 461Xj, 462J, 463Ye, 4A4Eg, 4A4Gb
Closed Graph Theorem *466 notes*
closed interval (in $\mathbb{R}$ or $\mathbb{R}^r$) 114G, 115G, 1A1A; (in general partially ordered spaces) **2A1Ab**; (in totally ordered spaces) **4A2A**, 4A2Rb
closed set (in a topological space) 134Fb, 134Xc, **1A2E**, 1A2F, 1A2G, 2A2A, **2A3A**, 2A3D, *2A3Nb*; (in an ordered space) *see* order-closed (**313D**); (spaces of closed sets) *see* Hausdorff metric (**476A**), Vietoris topology (**476C**)
closed subalgebra of a measure algebra 323H-323K (**323I**), 323Xb, 323Xd, 323Yd, *325Xa*, 327F, *331B*, *331D*, 332P, 332T, §333, 334Xb, *364Xe*, 366J, *372G*, *372J*, *385D*, *385G*, *385Xq*, 386L-386N, 386Xf, 386Yb, 386Yc, 387Be, 388I; *see also* order-closed subalgebra
closed subgroup of a topological group 443O-443S, 443Xt, 443Xy, 443Yj, 443Yk, 4A5Jb, 4A5Mc; *see also* compact subgroup
closure of a set **2A2A**, 2A2B, **2A3D**, 2A3Kb, 2A5Ec, 3A5Eb, *475Ca*, *475R*, 4A2Bg, 4A2Rd, 4A2Sa, 4A4Bi; *see also* essential closure (**475B**)
club *see* Ostaszewski's ♣
cluster point (of a filter) **3A3L**; (of a sequence) **2A3O**, 4A2Fa, 4A2Gf
coanalytic set 423P, 423Q, 423Yc
Coarea Theorem *265 notes*
coarser topology **4A2A**
cofinal repetitions **4A1Ab**
cofinal set (in a partially ordered set) **3A1F**, 4A1B, *4A1Ca*
cofinality (of a partially ordered set) **3A1F**, 424Yc; (of a cardinal) 4A1Ac; (of the Lebesgue null ideal) 521B, 521E, 521F, 521P, 521T-521V, 521Yb, 522O, 523Oc, 523Qf, 523S, 526H, 527Bd, 528Pc, 529Xb, 529Xc; *see also* uncountable cofinality
cofinite *see* finite-cofinite algebra (**231Xa**, **316Yk**)
comeager set **3A3Fb, 3A3G**
commutative (ring, algebra) **2A4Jb**, 363Xf, **3A2A**, 3A2Ff, 3A2Hc, **4A6Aa**
commutative Banach algebra 224Yb, 243D, 255Xc, 257Ya, 363B, 444Ed, 444Sb, 444Xd, 4A6B, 4A6D, 4A6J, 4A6K
commutative group *see* abelian group
commutator in a group **382Xe**, 382Xf, *382Yc*
commuting Boolean homomorphisms 381E, 381Sc

compact class of sets **342Ab**, 342D, 342Yc, 356Yi, **451Aa**, 451Ha, 451Xh

compact Hausdorff space 364V, 364W, 364Yj, 364Yl, 364Ym, 3A3D, 3A3Ha, *3A3K*, 422Gc, *434B*, 437O, 437Wa, *452N*, *462Z*, 463J, 463Zc, 476Cd, 481Xh, 4A2Fh, 4A2G, 4A2Nh, 4A2Ri, 4A2Ud, 4A3Xd

compact linear operator 371Ya, 371Yb, 376Yn, **3A5Ka**, 444V, 444Xt, 4A4L, 4A4M; *see also* weakly compact operator (**3A5Kb**)

compact measure **342A**, 342E-342H, 342J, *342M*, *342N*, 342Xd-342Xh, 342Xl, 342Xm, 342Xo, 342Ya, 342Yb, 343J, 343K, *343Xa*, 343Xb, 343Ye, 415Ym, 416W, 416Xx, 433Xe, *439Xa*, 451Ab, 451Da, 451F, 451Ga, 451Ia, 451Ja, 451L, 451P-451R, 451Xi, 451Xj, 451Xm, 451Ya, 451Yd, 451Ye, 454Ab, 454B, 454F; *see also* locally compact measure (**342Ad**)

compact metrizable space 423Dc, 424Xf, 434Kb, 437R, 441Xl, 441Ya, 441Yg, 441Gb, 441Xm, 449Xu, *461Xf*, *461Xi*, *462H*, 463C-463E, 463I, 463Xb, 463Xd, 463Xk, 463Za, *465Xm*, 476Ad, 476G, 476Yc, 481P, 4A2Nh, 4A2P, 4A2Q, 4A2Ud

compact-open topology **441Ye**, 441Yf; *see also* uniform convergence on compact sets

compact set (in a topological space) 247Xc, **2A2D**, 2A2E-2A2G, **>2A3N**, 2A3R, 3A3D, *421M*, *431D*, 421Xo, *461H*, *461I*, 461Xe, 461Xj, §*463*, 4A2Fh, 4A2G, 4A2Jd, 4A2Rh, 4A2Sa, *4A2Tb*, 4A3Xc, 4A4Bh, 4A4H, 4A4Ka, 4A5E; (spaces of compact subsets) 476Db, 476Eb;*see also* convex compact, relatively compact (**2A3N**), relatively weakly compact (**2A5Id**), weakly compact (**2A5Ic**)

compact subgroup of a topological group *443R*, 443Xw, 443Yl

compact support, function with 242O, 242Pd, 242O, 242Xh, 244H, 244Yi, 256Be, *256D*, 256Xh, 262Yd-262Yg, 416I, 416Xj, 436J, 436Xo, 443O, 444Xr, *473Be*, *473Db*, 473Ed, 473H, *475K*, *475N*, 495Xl, **4A2A**, 4A2Ge, 4A5P

compact support, measure with 284Yi

compact topological group 441Xd, 441Xk, *443Ag*, 444Xf, 445A, 445Xi, 445Xm, 446B, 446C, 446Yb, *447Ya*, 491H, 491Xn, 491Yi, 4A5T

compact topological space **>2A3N**, 3A3Dd, 3A3J, *415Yb*, 437Md, 462Yd, 4A2G, 4A2Jf, 4A2Kb, 4A2Lf, 4A3O; *see also* compact Hausdorff space, countably compact (**4A2A**), Eberlein compactum **467O**, locally compact (**3A3Ag**), metacompact (**4A2A**), paracompact (**4A2A**)

compactification *see* one-point compactification (**3A3O**), Stone-Čech compactification (**4A2I**)

compactly generated *see* weakly compactly generated (**467L**)

Compactness Theorem (for sets with bounded perimeter) 474S, *476 notes*

compatible set of tagged partitions **481F**, 481Hf, 481Xi, 481Ya

complement of a band (in a Riesz space) **352P**, 352Xh

complementary Young's functions **369Xc**, 369Xd, 369Xr

complemented band (in a Riesz space) **352P**, *352Xf*, 352Xh, 352Xm, 353B, 353Xa, *464M*, *464O*, *464R*; *see also* complemented band algebra (**352Q**)

complemented band algebra **352Q**, 352S, 352Xg, *353B*, 356Yc

complete generation of a Boolean algebra *see* Maharam type (**331F**)

complete lattice 314Bf

complete linear topological space 245Ec, **2A5F**, 2A5H, 367Nc, 461E, 461I, 461Xe, 461Xj, 466A, 4A4Ch

complete locally determined measure (space) 213C, 213D, 213H, 213J, 213L, 213O, 213Xg, 213Xi, 213Xl, 213Yd, 213Ye, 214I, 214Xc, 216D, 216E, 234Fb, 251Yb, 252B, 252D, 252E, 252N, 252Xb, 252Yh, 252Yj, 252Yr-252Yt, 253Yj, 253Yk, *325B*, 341M, 341Pb, *342N*, 364Xm, *376S*, *376Xj*, 411Xd, 412J-412L, 412Xh-412Xk, 413F, 413G, 413I-413K, 413M, 413O, 413Xh, 413Xj, 414I, 414J, *414P*, *414R*, 414Xf, 414Xh, 414Xr, *416Dd*, 416R, *417C*, *417D*, 417H, 418E, 431A, 431B, 431D, 431Xa, 431Xc, *432A*, 434Db, 434S, 438E, 438Xf, 438Xi, 439Fb, *439Xc*, *451Rb*, *451Yq*, 463Xc, 465D, 482E; *see also* c.l.d. version (**213E**)

complete measure (space) *112Df*, *113Xa*, *122Ya*, **211A**, 211M, 211N, 211R, *211Xc*, *211Xd*, §212, 214I, *214J*, *216A*, 216C, *216Ya*, *234A*, *234Dd*, *254Fd*, 254G, *254J*, *264Dc*, 321K, 341J, 341K, 341Mb, *341Xc*, 341Yd, *343B*, *343Xi*, *344C*, *344I*, *344Xb-344Xd*, 385V, *412M*, 413C, *413S*, *431Ya*, *415Xd*, 417A, *418T*, *451N*, 463Xf; *see also* complete locally determined measure

complete metric space *224Ye*, *323Yd*, *393Xb*, 3A4Fe, 3A4Hc, 434Jg, 438H, 438Xf, 441Ya, 476Ab, 4A2L; *see also* Banach space (**2A4D**), Polish space (**4A2A**)

complete normed space *see* Banach space (**2A4D**), Banach lattice (**242G**, **354Ab**)

complete Riesz space *see* Dedekind complete (**241Fc**)

complete uniform space 323G, **3A4F**, 3A4G, 3A4H, 4A2Jd, 4A5Mb; *see also* completion (**3A4H**)

complete (in 'κ-complete filter') 368F, 438Yb, **4A1Ib**, 4A1J, 4A1K
complete *see also* Dedekind complete (**314Aa**), Dedekind σ-complete (**314Ab**), uniformly complete (**354Yi**)
completed indefinite-integral measure *see* indefinite-integral measure (**234B**)
completely additive functional on a Boolean algebra **326J**, 326K-326P, *326Xe*, 326Xf, 326Xh, 326Xi, 326Yp-326Ys, 327B-327E, 327Ya, 332Xo, 362Ad, 362B, 362D, 362Xe-362Xg, 363K, *363S,* 365Ea, 394Xe, 438Xc; *see also* M_τ
—— (on $\mathcal{P}I$) 464Fb
completely additive part (of an additive functional on a Boolean algebra) 326Yq, 362Bd; (of a linear functional on a Riesz space) 355Yh
completely continuous linear operator *see* compact linear operator (**3A5Ka**)
completely generate *see* τ-generate (**331E**)
completely regular topological space *353Yb*, *367Yh*, **>3A3Ac**, 3A3B, 411Gi, 414Yf, 415I, 415P, 415Xp, *415Yk*, 416T, 416Xi, 416Xk, 416Xv, *422Ya*, *434Jg*, *434Yc*, *434Yd*, *436Xl*, *437Kc*, *437L*, *453Cb*, 459F, 4A2F, 4A2Hb, 4A2J
completion (of a Boolean algebra) *see* Dedekind completion (**314U**)
—— (of a measure (space)) §212 (**212C**), 213Fa, 213Xa, 213Xb, 213Xk, 214Xb, 214Xj, *232Xe*, *234Db*, *235D*, 235Hc, 241Xb, 242Xb, 243Xa, 244Xa, 245Xb, 251S, 251Wn, 251Xp, 252Ya, 254I, *256C*, 322Da, 342Gb, 342Ib, 342Xn, 343A, 411Xc, 411Xd, 412H, 412Xh, 412Xi, 413E, *416Xm*, 433Cb, 451G, 451Ym, 452A-452C, 452Xg, *456Xg*, 465Cj, 458Xl, 491Xt
—— (of a metric space) *393B*
—— (of a normed space) 3A5Ib
—— (of a normed Riesz space) 354Xh, 354Yg
—— (of a topological group) 443K, 445Ya, 449Xh, 493Xb, 4A5N
—— (of a uniform space) *325Ea*, 325Yb, **3A4H**
completion regular measure **411Jb**, 411Pb, 411Yb, 415D, 415E, 415Xj, 416U, 417Xv, 417Xw, *419C*, *419D*, 419E, 434A, 434Q, 434Xa, *434Xq*, *434Ya*, *435A*, *435Xb*, 439J, 443M, 451T, 453N, 491Md; *see also* inner regular with respect to zero sets
complex measure 437Yb
complexification of a Riesz space **354Yk**, 355Yk, 356Yh, 362Yj, *363Xj*, 364Yn, 437Yb, 444Yc
complex-valued function §133
component (in a Riesz space) **352Rb**, 352Sb, 356Yb, *368Yj*, 437Ya; (in a topological space) *111Ye*; *see also* Maharam-type-κ component (**332Gb**)
composition of relations **422Df**
compound Poisson distribution 495M
concave function **385A**
concentration of measure 476J-476L, 476O, 476P, 476Xh, 476Xj, 492D, 492E, 492H, 492I, 492Xb-492Xd, 493I
concentration by partial reflection 476H, 476I, 476Xg
conditional distributions *454H*, *455A*, *455B*
conditional entropy **385D**, 385E, 385G, 385N, 385Xb-385Xd, 385Xq, 385Ya, 385Yb, 386L
conditional expectation **233D**, 233E, 233J, 233K, 233Xg, 233Yc, 235Yc, 242J, 246Ea, 253H, 253Le, 275Ba, 275H, 275I, 275K, 275Ne, 275Xi, 275Ya, 275Yk, *275Yl*, 365R, *369Xq*, 372K, 452Qb, 452Xr, 458A, 458Xt, 459A, 459D, 465M, 465Xq
conditional expectation operator **242Jf**, 242K, 242L, 242Xe, 242Yk, 243J, 244M, 244Yk, 246D, 254R, 254Xp, 275Xd, 275Xe, 327Xb, 365R, 365Xl, 366J, 372G, 385B, *385D*, *385E*, 465Xr
conditional probability *see* regular conditional probability (452E)
conditionally (order)-complete *see* Dedekind complete (**314A**)
cone **3A5B**, 3A5D; *see also* positive cone (**351C**)
conegligible set **112Dc**, 214Cc
connected set **222Yb**, *434Ya*
conjugacy action of a group 441Xd, **4A5Ca**, 4A5I
—— class in a group *441Xd*

conjugate Boolean homomorphisms 381Gd, 381Sb, 385T
conjugate operator *see* adjoint operator (**3A5Ed**)
consistent disintegration **452E**, 452G, 452I, *452K*, 452O, 452Q, 452Xl, 452Xm, *452Xq*, 452Yb, 452Ye, 454Xf, 495I; *see also* strongly consistent (**452E**)
consistent lifting 346I, **346J**, *346L*, 346Xe, 346Yd
constituent (of a coanalytic set) **421Ng**, 421O, 421Yd, 422K, 423P, 423Q, 423Ye, *439Xe*, 439Yg
constructibility, axiom of *4A1M*
continued fractions 372M-372O; (continued-fraction approximations) **372M**, 372Xj, 372Yf; (continued-fraction coefficients) **372M**, 372O, 372Xh-372Xj, 372Xt, 372Ye
continuity, points of *224H*, *224Ye*, 225J
continuous function 121D, 121Yf, *262Ia*, **2A2C**, 2A2G, **2A3B**, 2A3H, 2A3Nb, 2A3Qb, 3A3C, *3A3Eb*, 3A3Ib, 3A3Nb, *418D*, *418L*, 422Dc, 434Xe, 434Xi, *473Db*, 4A2B, 4A2C, 4A2F, 4A2G, 4A2Ia, 4A2Kd, *4A2Ld*, 4A2Ro, 4A2Sb, 4A3Cd, 4A3Kc, 4A3L, 4A5Fa
continuous at a point **3A3C**
continuous image 493Ba, 4A2Gi, 4A2Kb, 4A2Nd, 4A2Pc
continuous linear functional *284Yj*; *see also* dual linear space (**2A4H**, **4A4Bd**)
continuous linear operator 2A4Fc, 456B, 456Eb, 456G, 461B, *461L*, 4A4B, 4A4Gc, 4A4H; *see also* bounded linear operator (**2A4F**)
continuous outer measure *see* Maharam submeasure (**392G**)
continuous *see also* almost continuous (**411M**), order-continuous (**313H**), semi-continuous (**225H**, **4A2A**), uniformly continuous (**3A4C**)
continuum *see* $\mathfrak{c}$ (**2A1L**)
continuum hypothesis *463Yd*, 463Ye, *463Za*, **4A1Ad**; *see also* generalized continuum hypothesis
control measure **393O**, 393P, 393S, 393Xg, 393Yh, 393Yi
Control Measure Problem §393, *493 notes*
convergence in mean (in $\mathcal{L}^1(\mu)$ or $L^1(\mu)$) **245Ib**
convergence in measure (in $L^0(\mu)$) §245 (**>245A**), 246J, 246Yc, 247Ya, 416Xj, 418R, 418S, 418Yj, 444F, 445I, 465Xi, 465Xp, 493F
—— (in $\mathcal{L}^0(\mu)$) §245 (**>245A**), 271Yd, 274Yd, 285Xs, §463, *464E*, 464Yb, 465G, 465Xi, 465Xj
—— (in the algebra of measurable sets) **232Ya**, 474S
—— (in $L^0(\mathfrak{A}, \bar{\mu})$) **367M**, 367N-367W, 367Xo, 367Xp, 367Xr, 367Xt, 367Xv-367Xx, 367Yo, 367Yr, 369M, 369Xf, 369Ye, *393M*, 443Ad, 443G
—— (of sequences) §245, 246J, 246Xh, 246Xi, 253Xe, 255Yf, 271L, 273Ba, 275Xk, 275Yp, 276Yf, 367Na, 367Q, *369Yf*, 376Xf
convergent almost everywhere 245C, 245K, 273Ba, 276G, 276H, 463G-463I, 463Xb, 463Xj, 473Ee; *see also* ergodic theorem, strong law
convergent filter **2A3Q**, 2A3S, 2A5Ib, 3A3Ic, 3A3Lb, 4A2B;
—— sequence **135D**, 245Yi, **2A3M**, 2A3Sg, 3A3Lc, 4A2Gh, 4A2Ib, 4A2Le, 4A2Rf; *see also* convergence in measure (**245Ad**), order*-convergent (**367A**), pointwise convergence
convex function **233G**, 233H-233J, 233Xb-233Xf (**233Xd**), 233Xh, 233Ya, 233Yc, 233Yd, 242K, 242Yi, 242Yj, 242Yk, *244Xm*, *244Yg*, *255Yk*, *275Yg*, *365Rb*, **367Xx**, 369Xb, 369Xc, 369Xd, 373Xb, **461A**, 461C, 461Xa; *see also* mid-convex (**233Ya**)
convex hull **2A5E**, 462Xd, 465N, 465Xo; *see also* closed convex hull (**2A5E**)
Convex Isoperimetric Theorem 475T
convex polytope **481O**
convex set **233Xd**, **244Yj**, *262Xh*, **2A5E**, *326Ye*, *351Ce*, 3A5C, 3A5Ee, 3A5Ld, 461Ac, *461C*, *465Xb*, 475R-475T, 475Xa, 475Xk, *484Xc*, 4A4Db, 4A4E, *4A4F*, 4A4Ga
—— compact set 437S, 438Xl, 461D, 461I-461N, 461Xh, 461Xi, 461Yb, 463D, 463Xd, 4A4E-4A4G
—— *see also* cone (**3A5B**), order-convex (**351Xb**)
convex structure **373Xv**, 373Yf, 437P, 459Xd, 459Xe
convolution in L^0 255Fc, 255Xc, 255Yf, 255Yk, 444S, 444V, 444Xt, 444Xu, 444Yk
convolution of functions **255E**, 255F-255K, **255O**, 255Xa-255Xc, 255Xf-255Xj, 255Ya, 255Yb, 255Yd, 255Ye, 255Yi, 255Yl, 255Ym, **255Yn**, 262Xj, 262Yd, 262Ye, 262Yh, 263Ya, 282Q, 282Xt, 283M, 283Wd,

283Wf, 283Wj, 283Xl, 284J, 284K, 284O, 284Wf, 284Wi, 284Xb, 284Xd, 373Xg, 376Xa, 376Xd, **444O**, 444P-444S, 444Xq-444Xs, 444Yi, 444Yl, 445G, 445Sc, 445Xn, 449H; *see also* smoothing by convolution
convolution of measures §257 (**257A**), 272S, 285R, 285Yn, 364Xf, **444A**, 444B-444E, 444Q, 444Xa-444Xc, 444Xe, 444Xf, 444Xp, 444Yb, 444Yi, 445D, 455Yd, 458Yc, 495Ab
convolution of measures and functions *257Xe*, 284Xo, 284Yi, **444H**, 444I, **444J**, 444K, 444M, 444P, 444T, 444Xh-444Xk, 444Xn, 444Xp, 444Yi, 449Yc
convolution of sequences 255Xe, *255Yo*, 282Xq, 352Xl, 376Ym
countable (set) **111F**, 114G, 115G, §1A1, 226Yc, 4A1P
countable antichain condition 316 *notes* (*see* ccc (**316A**))
countable chain condition *see* ccc (**316A**)
countable choice (axiom of) *134C*
countable-cocountable algebra **211R**, 211Ya, *232Hb*, *316Yk*, *368Xa*, 454Xg, 4A3P
countable-cocountable measure **211R**, *232Hb*, 252L, 326Xf, 342M, 356Xc, 411R, 435Xb, 451B, 463Xh
countable network 415Xj, 417T, 417Yh, 417Yi, 418Yb, 418Yc, 418Yg, 418Yh, 419Xe, 423Be, 423C, 423Ib, 423Ya, 423Yb, 423Yd, 433A, 433B, 433Xa, 433Ya, 434Pc, 434Xr, 434Xt, 434Yb, 451M, 451N, 453Fa, 453I, 453J, 491Eb, 491Xx, 491Ym, 4A2N-4A2P, 4A3E, 4A3F, 4A3Rc, 4A3Xa, 4A3Xf, 4A3Yb
countable separation property *see* σ-interpolation property (**466G**)
countable sup property (in a Riesz space) **241Yd**, 242Yd, 242Ye, 244Yb, 353Ye, 354Yc, 355Yi, 356Ya, 363Yb, 364Yb, 366Yb, 368Yh, 376Ye, 376Yf
countably additive functional (on a σ-algebra of sets) §231 (**231C**), §232, 246Yg, 246Yi, 327C, *362Xg*, 363S, 416Ya; (on a Boolean algebra) **>326E**, 326F-326I, 326L, 326Xb-326Xi, 327B, 327C, 327F, 327G, 327Xa, 327Xd, 327Xe, 362Ac, 362B, 363S, 363Yh, 392Xf, 392Yb, 438Xc; *see also* completely additive (**326J**), countably subadditive (**392H**), M_σ
countably additive part (of an additive functional on a Boolean algebra) 326Yn, 362Bc, 362Ye, 393Yg
countably compact class of sets **413L**, 413M, 413O, 413R, 413S, 413Ye, 451Ac, 451B, 451Hb, 451Xd, 451Xh, *452H*, *452I*, *452M*, 457E
countably compact measure (space) **451B**, 451C, 451Db, 451Gb, 451Ib, 451Jb, *451T*, 451Xc-451Xe, 451Yb, 451Yj, 452H, 452I, 452M, 452R, 452S, 452Xk, 452Xl, 452Xn, 452Xr, 452Yb, 452Ye, 454Ab, 454Xe, 459E
countably compact measure property **454Xe**, 454Xf, 454Xg, 454Xi
countably compact set, topological space 413Ye, 421Ya, 422Yc, 434M, 434N, 436Yc, 462B, 462C, 462F-462J, 462Xb, 462Ya, 463I, 463Xb, **>4A2A**, 4A2Gf, 4A2Lf
countably full family of gauges **482Ia**, 482K, 482L
countably full group (of automorphisms of a Boolean algebra) **381Af**, 381I, 381Nh, 381Yb, 382H-382M
countably full local semigroup (of partial automorphisms of a Boolean algebra) §448 (**448A**)
countably generated σ-algebra 415Xi, 419Yb, 423Xd, 424Ba, 424Xb-424Xd, 424Xh-424Xj, 433I, 433J, 451F, 452Xl, 452Xn
countably K-determined *see* K-countably determined (**467H**)
countably metacompact (topological) space **434Yf**, 435Ya
countably paracompact (topological) space 435C, 435Xg, 435Xh, 435Ya, **4A2A**, 4A2Ff
countably separated (measure space, σ-algebra of sets) **343D**, 343E-343H, 343K, 343L, 343Xe, 343Xh, 343Xi, 343Yb, 343Ye, 343Yf, 344B, 344C, 344I, 344Xb-344Xd, 381Xm, 382Xd, 383Xa, 385V, 388A, 388B, 424E, 433B, 433Xe, *451Ad*, 452I, 453Xf
countably subadditive functional **392H**
countably tight (topological) space 423Ya, 434N, *434Yg*, 463E, 463Xd, **4A2A**, 4A2K, 4A2Ld
counting measure **112Bd**, *122Xd*, 122 *notes*, 211N, 211Xa, *213Xb*, 226A, 241Xa, 242Xa, 243Xl, 244Xi, 244Xn, 245Xa, 246Xc, 251Xb, 251Xh, *252K*, 255Yo, *264Db*, *324Xe*, 416Xb, *417Xi*, 442Ie, 445Xi, 445Xj, *481Xc*
covariance matrix (of a Gaussian distribution) **456Ac**, 456B, 456F, 456Xa, 456Xg
covariant measure *441 notes*
cover *see* measurable envelope (**132D**)
covering theorem 221A, 261B, 261F, *261Xc*, 261Ya, 261Yi, 261Yk, 472A-472C, 472Xb, 472Ya-472Yc
cozero set **3A3Pa**, 412Xg, *416Xi*, 421Xh, 443N, *443Yi*, 4A2Cb, 4A2F, 4A2Lc

cycle notation (for automorphisms of Boolean algebras) **381R**, 381S, 381T, 382Fb; *see also* pseudo-cycle (388Xc)

cyclic automorphism 381R; *see also* exchanging involution (**381R**)

cylinder (in 'measurable cylinder') **254Aa**, 254F, *254G*, *254Q*, 254Xa

cylindrical σ-algebra (in a locally convex space) *461D*, 461Xg, 461Xi, 464R, 466J, 466Xc, 466Xd, 466Xf, 491Yc, **4A3T**, 4A3U, 4A3V

Daniell integral 436Ya, 436Yb

Davies R.O. 325 *notes*

decimal expansions 273Xf

decomposable distribution *see* infinitely divisible

decomposable measure (space) *see* strictly localizable (**211E**)

decomposition (of a measure space) **211E**, *211Ye*, 213O, *213Xh*, 214Ia, 214K, 214M, *214Xi*, *322L*, 412I, 416Xe, 417Xe; *see also* disintegration (**452E**), Hahn decomposition of an additive functional (231F), Jordan decomposition of an additive functional (231F), Lebesgue decomposition of a countably additive functional (232I), Lebesgue decomposition of a function of bounded variation (§226)

decreasing rearrangement (of a function) **252Yp**; (of an element of $M^{0,\infty}(\mathfrak{A})$) §373 (**373C**), 374B-374D, 374J, 374L, 374Xa, 374Xj, 374Ya-374Yd

Dedekind complete Boolean algebra 314B, 314Ea, 314Fa, 314G, 314Ha, 314I, 314Ja, 314K, 314P, 314S, 314T, *314Va*, 314Xa, 314Xb, 314Xf, 314Xg, 314Xh, 314Yd, 314Yg, 314Yh, 315F, 315Yf, 316Fa, *316K*, 316Xt, 316Yi, *316Yn*, *331Yd*, 332Xa, *333Bc*, 352Q, 363Mb, 363P-363S, 364O, 368A-368D, 368K, 368M, *368Xe*, 375C, 375D, 375Xa, 375Xc, 375Ya, 381C, 381D, 381F, 381Ib, 381Xb, 381Xf, 381Xk, 381Yc, 382E, 382F, 382N, 382Q-382S, 382Xe-382Xj, 384D, 384J, 393A-393C, 393J, 393Xc-393Xf, 393Ya, 393Yc-393Yf 438Xc, *448Ya*

Dedekind complete partially ordered set *135Ba*, **314Aa**, 314B, 314Ya, 315De, 343Yc, 3A6C, 419L, 419Xf, 4A2R*see also* Dedekind complete Boolean algebra, Dedekind complete Riesz space

Dedekind complete Riesz space **241Fc**, 241G, 241Xf, 242H, 242Yc, 243H, 243Xj, 244L, 353G, 353I, 353Jb, 353K, 353Yb, 354Ee, §355, 356B, *356K*, *356Xj*, *356Ye*, 361H, 363Mb, 363P, 363S, 364O, 366C, 366G, 368H-368J, 371B-371D, 371Xd, 371Xe, 375J, 375Ya, 437Yj, 438Xd

Dedekind completion of a Boolean algebra 314T, **314U**, 314Xe, 314Yh, 315Yd, 322O, 361Yc, 365Xm, 366Xk, 368Xb, 384Lf, 384Xc, 391Xc; *see also* localization (**322P**)

Dedekind completion of a Riesz space 368I, **368J**, 368Xb, 368Yb, 369Xp

Dedekind σ-complete Boolean algebra 314C-314F, 314Hb, 314Jb, 314M, 314N, 314Xa, 314Xc, 314Ye, 314Yf, *315O*, *316C*, 316Fa, 316Hb, *316J*, *316Xg*, 316Yi, *316Yk*, *321A*, 324Yb, 324Yc, *326Fg*, 326I, 326O, 326P, 326Xc, 326Xi, 326Yf, 326Yo, 344Yb, 362Xa, 363Ma, 363P, 363Xh, 363Yh, §364, 367Xk, 368Yd, 375Yb, 381C, 381D, 381Ia, 381K-381N, 381Xb, 381Xk, 381Xp, 381Yb, 382D, 382E, 382G-382M, 382S, 382Xk, 382Xl, 382Ya, 382Yc, 392H, 392Xd, 392Xf, 392Xg, 393A, 393E, 393M, 393O-393Q, 393S, 393Xb, 393Yg-393Yi, 424Xd, *491Xo*

Dedekind σ-complete partially ordered set **314Ab**, 315De, 367Bf, 367Xb, 466G; *see also* Dedekind σ-complete Boolean algebra, Dedekind σ-complete Riesz space

Dedekind σ-complete Riesz space **241Fb**, 241G, 241Xe, 241Yb, 241Yh, 242Yg, 243H, 243Xb, 353G, 353H, 353J, 353Xb, 353Yc, *353Yd*, 354Xn, 354Yi, *354Yl*, 356Xc, 356Xd, 363M-363P, 364C, 364E

delta function *see* Dirac's delta function (**284R**)

delta system *see* Δ-system (**4A1D**)

De Morgan's laws 311Xf; *see also* distributive laws

dense set in a topological space 242Mb, 242Ob, 242Pd, 242Xi, 243Ib, 244H, 244Ob, 244Yi, 254Xo, 281Yc, **2A3U**, 2A4I, *313Xj*, 323Dc, 367O, **3A3E**, 3A3G, 3A3Ie, *3A4Ff*, 412N, 412Yc, 417Xu, *449Xg*, *493Xa*, *4A2Ma*, *4A2Ni*, *4A2Ua*; *see also* nowhere dense (**3A3Fa**), order-dense (**313J**, **352Na**), quasi-order-dense (**352Na**)

density (of a topological space) 331Ye, **331Yf**, 365Ya, 393Yf

density function (of a random variable) **271H**, 271I-271K, 271Xc-271Xe, 272T, 272Xd, 272Xj, 495P; *see also* Radon-Nikodým derivative (**232If**, **234B**)

density point **223B**, 223Xi, 223Yb, 481Q

density topology **223Yb**, 223Yc, 223Yd, **261Yf**, **414P**, 414R, 414Xk-414Xo, 414Xq, 414Yd-414Yh, 418Xl, 451Yq, 453Xg, 475Xe

density *see also* asymptotic density (**273G**, **491A**), Besicovitch's Density Theorem (472D), Lebesgue's Density Theorem (223A), lower density (**341C**)
derivate *see* lower derivate (**483H**), upper derivate (**483H**)
derivative of a function (of one variable) 222C, 222E, 222F, 222G, 222H, 222I, 222Yd, 225J, *225L*, *225Of*, *225Xc*, *226Be*, 282R; (of many variables) **262F**, 262G, 262P, 473B, 473Ya;*see also* gradient, partial derivative
derived set (of a set of ordinals) 4A1Bb; derived tree **421N**, 421O
determinant of a matrix 2A6A
determined *see* locally determined measure space (**211H**), locally determined negligible sets (**213I**)
determined by coordinates (in 'W is determined by coordinates in J') **254M**, 254O, 254R-254T, 254Xp, 254Xr, 311Xh, 325N, 325Xh, 415E, 417M, 417Xu, 456H, *481P*, 4A2Bg, 4A2Eb, 4A3Mb, 4A3N, 4A3Yb
Devil's Staircase *see* Cantor function (**134H**)
diagonal intersection (of a family of sets) **4A1Bd**, 4A1Ic
diamond *see* Jensen's $\diamondsuit$
Dieudonné's measure **411Q**, *411Xj*, 434Xf, 434Xx, 491Xm
differentiability, points of 222H, 225J
differentiable function (of one variable) 123D, *222A*, *224I*, *224Kg*, *224Yc*, *225L*, *225Of*, *225Xc*, *225Xn*, 233Xc, 252Yj, 255Xg, 255Xh, 262Xk, 265Xd, *274E*, 282L, 282Xs, 283I-283K, *283Xm*, 284Xc, 284Xk, 483Xh; (of many variables) **262Fa**, 262Gb, 262I, 262Xg, 262Xi, 262Xj, 473B, 473Cd, 484N, 484Xf; *see also* derivative
'differentiable relative to its domain' **262Fb**, 262I, 262M-262Q, 262Xd-262Xf, 262Yc, 263D, 263Xc, 263Xd, 263Yc, 265E, 282Xk
diffused measure *see* atomless measure (**211J**)
dilation **286C**
dimension *see* Hausdorff dimension (**264 notes**, **471Xe**)
Dini's theorem 436Ic
Dirac's delta function 257Xa, **284R**, 284Xn, 284Xo, 285H, 285Xp
direct image (of a set under a function or relation) 1A1B
direct sum of measure spaces **214K**, 214L, 214Xi-214Xl, 241Xg, 242Xd, 243Xe, *244Xg*, *245Yh*, 251Xh, 251Xi, 322Kb, 332C, 342Gd, 342Ic, 342Xn, *343Yb*, 411Xh, 412Xl, 414Xb, 415Xb, 416Xc, 451Xe, 451Yh
direct sum of σ-algebras 424Xa
direct sum of topological spaces **4A2A**, 4A2Qe; *see also* disjoint union topology (**4A2A**)
directed set *see* downwards-directed (**2A1Ab**), upwards-directed (**2A1Ab**)
Dirichlet kernel **282D**; *see also* modified Dirichlet kernel (**282Xc**)
disconnected *see* extremally disconnected (**3A3Ae**), basically disconnected
discrete *see* σ-metrically-discrete (**4A2A**)
discrete topology (**3A3Ah**), *416Xb*, *417Xi*, 436Xg, 439Cd, 439Xh, 442Ie, 445A, 445Xj, 449G, 449K, 449Xj, 449Xr, 449Xs, 449Yb, 4A2Ib, 4A2Qa
disintegration of a measure **452E**, 452F-452K, 452M-452Q, 452Xf-452Xh, 452Xk-452Xn, 453K, 453Xi, 453Ya, 455A, 455Ya, 459E-459G, 459Xc, 495H; *see also* consistent disintegration (**452E**)
disjoint family (of sets) **112Bb**; (in a Boolean ring) **311Gb**; (in a Riesz space) **352C**, 352Fb
disjoint sequence theorem 246G, *246Ha*, 246Yd, 246Ye, 246Yf, 246Yj, 354Rb, 356O, 356Xm
disjoint set (in a Boolean ring) **311Gb**; (in a Riesz space) **352C**
disjoint union topology **315Xe**, 411Xh, 412Xl, 414Xb, 415Xb, 416Xc, **4A2A**, 4A2Bb, 4A2Cb; *see also* direct sum of topological spaces
distribution *see* Schwartzian distribution, tempered distribution
distribution of a random variable 241Xc, 271E, 271F, 271Ga, 272G, 272S, 272Xe, 272Yc, 272Yf, 272Yg, 285H, 285Xg, 415Fb, *415Xq*, 454L; (of an element of $L^0(\mathfrak{A},\bar{\mu})$) **364Xd**, 364Xf, 367Xv, 373Xh; *see also* Cauchy distribution (**285Xm**), empirical distribution (**273 notes**), exponential distribution (**495P**), gamma distribution (**455Xe**), Poisson distribution (**285Xo**), relative distribution (**458C**)
—— of a finite family of random variables 271B, **271C**, 271D-271G, 272G, 272Ye, 272Yf, 285Ab, 285C, 285Mb, 454J, 454K, 456M, 459C
—— of an infinite family of random variables 454J-454M (**454K**), 454Xj; *see also* Gaussian distribution (**456Ab**
distribution function of a random variable **>271G**, 271L, 271Xb, 271Yb, 271Yc, 271Yd, 272Xe, 272Yc, 273Xg, 273Xh, 274F-274L, 274Xd, 274Xg, 274Xh, 274Ya, 274Yc, 285P, 495P

distributive lattice *367Yc*, 382Xh, **3A1Ic**; *see also* $(2,\infty)$-distributive (**367Yd**)
distributive laws in Boolean algebras 313B; *see also* De Morgan's laws, weakly σ-distributive (**316Yg**), weakly (σ,∞)-distributive (**316G**)
distributive laws in Riesz spaces 352E; *see also* weakly σ-distributive, weakly (σ,∞)-distributive (**368N**)
divergence (of a vector field) **474B**, 474C-474E, 475N, 475Xg, 484N
Divergence Theorem 475N, 475Xg, 484N, 484Xf
divisible distribution *see* infinitely divisible
Dominated Convergence Theorem *see* Lebesgue's Dominated Convergence Theorem (**123C**)
Doob's Martingale Convergence Theorem **275G**
double arrow space *see* split interval (**343J**)
doubly recurrent Boolean automorphism **381Ag**, 381M, 381N, 381Xk, 381Xp, 381Xo, 382J
doubly stochastic matrix *373Xe*
Dowker space **439O**
downwards-directed partially ordered set **2A1Ab**
dual group (of a topological group) §445 (**445A**)
dual linear operator *see* adjoint operator (**3A5Ed**)
dual linear space (of a linear topological space) *461Aa*, *461F*, *461G*, **4A4B**, 4A4C-4A4F
dual normed space 243G, 244K, **2A4H**, 356D, 356N, 356O, 356P, 356Xg, 356Yh, 365L, 365M, 366C, 366D, 369K, 3A5A, 3A5C, 3A5E-3A5H, 436Ib, 437I, 4A4I
—— *see also* algebraic dual, bidual, order-bounded dual (**356A**), order-continuous bidual, order-continuous dual (**356A**), sequentially order-continuous dual (**356A**), sequentially smooth dual (**437Aa**), smooth dual (**437Ab**)
Duality Theorem (Pontryagin-van Kampen) 445U
Dunford's theorem 376N
dyadic cube **484A**
dyadic cycle system (for a Boolean automorphism) **388D**, 388H, 388J
dyadic numbers 445Xp
dyadic (topological) space 434 *notes*, 491Xz, **4A2A**, 4A2D, 4A5T; *see also* quasi-dyadic (**434O**)
Dye's theorem 388L
dynamical system 437S, 461P
Dynkin class **136A**, 136B, 136Xb, 452A, 495C

Eberlein compactum **467O**, 467P
Eberlein's theorem *2A5J*, 356 *notes*, 462D
effectively locally finite measure **411Fb**, 411G, 411P, 411Xd, 411Xh, 411Xi, 412F, 412G, 412Se, 412Xe, 412Xf, 414A-414M, 414P, 414R, 414Xd-414Xh, 414Xj, 415C, 415D, 415M-415O, 416H, 417B-417D, 417H, 417I, 417S, 417T, 417V, 417Xc, 417Xe, 417Xh, 417Xj, 417Yb, 417Yc, 418Ye, *419A*, 433Xb, *434Ib*, *434Ja*, *434R*, 435Xe, 481N, 482Xd; *see also* locally finite (**411Fa**)
effectively regular measure **491L**, 491M-491O, 491Xr-491Xv, 491Xx
Effros Borel structure **424Ya**
Egorov's theorem 131Ya, 215Yb, 412Xr, 418Xc
Egorov property (of a Boolean algebra) **316Yg**, 316Yh
eigenvector 4A4Lb, 4A4M
embedding *see* regular embedding (**313N**)
empirical distribution 273Xh, **273 notes**, 465H, 465I, 465M, 465Xn, 465Ya
entropy §385; (of a partition of unity) **385C**, 386E, 386I, 386L, 386O, 387B-387D, 387F-387H; (of a measure-preserving Boolean homomorphism) **385M**, 386L, 386Xe, 386Yb, 386Yc, 387C-387E, 387I-387L, 388Yb, 493Yd; *see also* conditional entropy (**385D**), entropy metric (**385Xq**)
entropy metric **385Xq**, 385Yg
enumerate, enumeration **3A1B**
envelope *see* Baire property envelope, measurable envelope (**132D**), $\operatorname{upr}(a,\mathfrak{C})$ (**314V**)
enveloped *see* regularly enveloped (**491L**)
equidecomposable (in 'G-equidecomposable') **395Ya**; (in 'G-τ-equidecomposable') §394 (**394A**); (in 'G-σ-equidecomposable') §448 (**448A**)

equidistributed sequence (in a topological probability space) 281N, **281Yi**, *281Yj*, 281Yk, **491B**, 491C-491H, 491N, 491O, 491Q, 491Xe-491Xg, 491Xi-491Xn, 491Xw, 491Xx, 491Xz, 491Yb-491Yi, 491Ym, 491Z

Equidistribution Theorem *see* Weyl's Equidistribution Theorem (281N)

equivalent norms *355Xb*, 369Xd, **4A4Ia**

equiveridical **121B**, **212B**, **312B**, **411P**

ergodic automorphism (of a Boolean algebra) 363F, 381Xp, 382Xk, 385Se, 385Ta, 387C, 387D, 387E, 387J, 387Xb, 388Xg, 388Yc, 394Ge

ergodic Boolean homomorphism **372Pa**, 381P, 386D, 386F

ergodic group of automorphisms (of a Boolean algebra) **394Ge**, 394Q, 394R, 394Xa, 394Xg, 394Yd

ergodic inverse-measure-preserving function **372P**, 372Q, 372Xo, 372Xs, 372Yi, 372Yj, 372Yl, 388Yd, 461O, 461P, 461Yd, 491Xx

ergodic measure-preserving Boolean homomorphism (of a measure algebra) 372Q, 372Xk, 372Xl, 372Yl, 385Se, 386D, 386F, 387C-387E, 387J, 387Xb, 388Xg, 388Yc, 395Xc, 493Yc, 493Yd

Ergodic Theorem 372D, 372E, 372Ya, 386Xc, 449Xu; *see also* Maximal Ergodic Theorem (372C), Mean Ergodic Theorem (372Xa), Wiener's Dominated Ergodic Theorem (372Yb)

essential boundary **475B**, 475C, 475D, 475G, 475J-475L, 475N-475Q, 475Xb, 475Xc, 475Xg, 475Xh, 475Xj, 476Ie, 484Ra

essential closure **475B**, 475C, 475Xb, 475Xc, 475Xe, 475Xf, 484B, *484K*, 484Ra, 484Ya

essential interior **475B**, 475C, *475J*, 475Xb-475Xf

essential supremum of a family of measurable sets **211G**, 213K, 215B, 215C; of a real-valued function **243D**, 243I, 376S, 376Xn, 443Gb, 444R, *492G*, *492Xa*, *492Xd*

essentially bounded function **243A**

Etemadi's lemma 272U

Euclidean metric (on $\mathbb{R}^r$) **2A3Fb**, *443Xx*

Euclidean topology §1A2, §2A2, **2A3Ff**, 2A3Tc

even function 255Xb, 283Yb, 283Yc

exchangeable family of random variables **276Xe**, *459B*, **459C**, 459Xa, 459Xb, 459Ya

exchanging involution **381Ra**, 381S, 382C, 382E, 382H, 382K, 382L, 382M, 382P, *382Ya*

exhaustion, principle of 215A, 215C, 215Xa, 215Xb, *232E*, 246Hc, 342B, 365 *notes*

exhaustive submeasure **392Bb**, 392C, 392Hc, 392Xb-392Xd, 393A-393C, 393H; *see also* uniformly exhaustive (**392Bc**)

expectation of a random variable **271Ab**, 271E, 271F, 271I, *271Xa*, 272Q, 272Xb, *272Xi*, 285Ga, *285Xo*; *see also* conditional expectation (**233D**)

exponential distribution **495P**, 495Xj, 495Xo, 495Xp

exponentially bounded group **449Xp**

exponentiation (in a Banach algebra) **4A6L**, 4A6M

extended Fatou norm §369 (**369F**), 376S, 376Xn, 376Yl, 415Yl; *see also* $\mathcal{T}$-invariant extended Fatou norm (**374Ab**), rearrangement-invariant extended Fatou norm (**374Eb**)

extended real line *121C*, §135

exterior *see* Federer exterior normal (**474O**)

extension of finitely additive functionals 391G, 413K, 413N, 413Q, 413S, 413Yd, 449L, 449Xt, 457A, 457C, 457D, 457Hc, 457Xb-457Xe, 457Ya

extension of measures 132Yd, 212Xk, 327Xf, 413O, 415L-415N, 415Yi, 416F, 416M, *416O*, 416P, *416Q*, 416Yb, 416Yc, 417A, *417C*, *417E*, 417Xa, 417Ya, 417Ye-417Yg, 432D, 432F, *432Ya*, 433I, 433J, 433Xf, *434A*, *434Ha*, *434Ib*, 434R, 435B, 435C, 435Xa-435Xd, 435Xg, 435Xj, *439A*, *439M*, *439O*, 449M, 454C-454F, *454Xa-454Xc*, 457E, 457G, *457Hc*, *457J*, 457Xa, 457Xf, 457Xg, 457Xk, *457Yc*, 458Xr, 464D, *466H*, 466Xd, *466Za*; *see also* completion (**212C**), c.l.d. version (**213E**)

extremally disconnected topological space 314S, 314Xj, 353Yb, 363Yc, 364W, 364Yk-364Ym, 368G, 368Yc, **3A3Ae**, 3A3Bd, 4A2Gh, 491Yg

extreme point of a convex set **353Yf**, 437P, 437Xp, 438Xl, 459Xd, 459Xe, 461O, 461P, 461Xb, 461Xe, 461Xk, 461Xl, 461Yd, 461Ye, **4A4G**; (of a compact set in a Hausdorff linear topological space) **461Xj**

—— set of extreme points 461K-461N, 461Xh-461Xi, 461Yb

extremely amenable topological group §493 (**493A**)

factor (of an automorphism of a Boolean algebra) **387Ac**, 387L, 387Xb
fair-coin probability 254J
faithful action 449De, **4A5Be**
faithful representation **446A**, 446N
Fatou's Lemma **123B**, 133K, 135G, 135Hb, 365Xc, 365Xd, 367Xf
Fatou norm on a Riesz space 244Yf, **354Da**, 354Eb, 354J, 354Xk, 354Xn, 354Xo, 354Ya, 354Ye, 356Da, 367Xg, 369G, 369Xe, 371B-371D, 371Xc, 371Xd; *see also* extended Fatou norm (**369F**)
favourable *see* weakly α-favourable (**451U**)
Federer exterior normal **474O**, 474P, 474Q, 475N, 475Xd, 475Xg, *484M*, 484N, 484Xf; *see also* canonical outward-normal function (**474G**)
Féjer integral **283Xf**, 283Xh-283Xj
Féjer kernel **282D**
Féjer sums **282Ad**, 282B-282D, 282G-282I, 282Yc
Feller, W. Chap. 27 *intro.*
fiber product **458La**, 458Xf
field (of sets) *see* algebra (**136E**)
filter **2A1I**, 2A1N, 2A1O, 2A5F, 464C, *464D*, *464Fc*, 4A1I, 4A2Ib; *see also* complete filter (**4A1Ib**), convergent filter (**2A3Q**), normal filter (**4A1Ic**), ultrafilter (**2A1N**)
filter base **4A1Ia**
fine (in 'δ-fine tagged partition') **481E**, 481G
finer topology **4A2A**
de Finetti's theorem 459C
finite-cofinite algebra **231Xa**, 231Xc, **316Yk**, 392Xc
finite-dimensional linear subspace 4A4Bk, 4A4Je
finite-dimensional representation **446A**, 446B, 446C, 446N, 446Xa, 446Xb, 446Ya, 446Yb, 491Yi
finite group 445Ye
finite intersection property **3A3D**, *434M*, 4A1Ia
finite perimeter (for sets in $\mathbb{R}^r$) **474D**, 475Mb, 475Q, 484B, 484C, 484F, 484Xf
finite rank operators *see* abstract integral operators (**376D**)
finitely additive functional on an algebra of sets 136Xg, 136Ya, 136Yb, **231A**, 231B, 231Xb-231Xe, 231Ya-231Yh, 232A, 232B, 232E, 232G, 232Ya, 232Ye, 232Yg, 243Xk, 327C, 413H, 413K, 413N, 413P, 413Q, 413S, 413Xl, 413Xm, 413Xp, 416N-416Q, 416Xl, 416Xm, 416Xr, 471Q, 449J, 449Xr, 457A-457D, *482G*, *482H*, *493D*; *see also* countably additive, completely additive
finitely additive function(al) on a Boolean algebra **326A**, 326B-326D, 326Fa, 326G, 326Kg, 326Xa-326Xc, 326Xe, 326Xg, 326Xi, 326Yt-326Yd, 326Yg-326Yj, **326Yk**, 326Yl-326Yn, 326Yq, 326Yr, 327B, 327C, 331B, 361Xa-361Xc, 391D-391G, 391I, 391J, *391N*, 393Be, 393C, 393Xi, 394N, 394O, 413Xo, 416Q, 481Xh, 491Xy; *see also* chargeable Boolean algebra (**391X**), completely additive functional (**326J**), countably additive functional (**326E**)
finitely additive function(al) on a Boolean ring **361B**, 361C, 361E-361I, 365Eb
finitely additive measure **326A**
finitely full subgroup **381Xg**
first category *see* meager (**3A3Fb**)
first-countable topological space 434R, *434Xx*, 434Yk, 438S, 439K, 439Xf, 452Xa, *462Xa*, 462Xc, **4A2A**, 4A2K, 4A2Ld, 4A2Sb, 4A5Q
First Separation Theorem (of descriptive set theory) 422I, 422Xd, 422Yc
fixed-point subalgebra (in a Boolean algebra) **312T**, 313Xq, 333Q, 333R, 361Xk, 364Xv, 372Pa, 386B, 386Xa, 381P, 381Xj, 381Xp, 388L, **394G**, 394H, 394I, 394K-394N, 394P, 394Xb-394Xf, 394Yb, 394Yc
fixed point on compacta property *see* extremely amenable (**493A**)
Fodor's theorem *see* Pressing-Down Lemma (**4A1Cc**)
Fourier coefficients **282Aa**, 282B, 282Cb, 282F, 282I, 282J, 282M, 282Q, 282R, 282Xa, 282Xg, 282Xq, 282Xt, 282Ya, 283Xu, 284Ya, 284Yg
Fourier's integral formula **283Xm**
Fourier series *121G*, §282 (**282Ac**)

Fourier sums **282Ab**, 282B-282D, 282J, 282L, 282O, 282P, 282Xi-282Xk, 282Xp, 282Xt, 282Yd, 286V, 286Xb
Fourier transform **133Xd**, **133Yc**, §283 (**283A**, **283Wa**), §284 (**284H**, **284Wd**), 285Ba, 285D, 285Xd, 285Ya, 286U, 286Ya, §445 (**445F**)
Fourier-Stieltjes transform **445C**, 445D, 445Ec, 445Xc, 445Xd, 445Xo, 445Yf; *see also* characteristic function (**285A**)
fractional Brownian motion **456Ye**
Fréchet filter **2A3Sg**
Fréchet-Urysohn topological space **462Aa**
free group 449G, 449Yf, 449Yg
free product of Boolean algebras §315 (**315H**), 316Yl, §325, 326Q, 326Yt, 326Yu, 361Yf, 384Le, 391Xb, 391Ye; *see also* localizable measure algebra free product (**325Ea**), probability algebra free product (**325K**), relative free product (**458I**)
free set 4A1E
Fremlin's Alternative 463H
Frolík's theorem 382E
Frostman's lemma 471Xb
Fubini's theorem 252B, 252C, 252H, 252R, *393Td*, 417H, *434R*, *434Xv*, *452E*, *444N*, *444Xo*, 482M, 482Yc
full family of gauges **482Ia**, 482J; *see also* countably full (**482Ia**)
full local semigroup (of partial automorphisms of a Boolean algebra) **394A**, 394B, 394C, 394Gc, 394Xc, 394Ya, 394Yd, 448Ya; *see also* countably full (**448A**)
full outer measure **132F**, 132G, 132Xk, 132Yd, 133Yf, 134D, 134Yt, 214F, 254Yf, 322Jb, 324Cb, *343Xa*, *415J*, 415Qc, 419Xd, 432G, 433Xf, 443Yj, *456D*, *495G*; full outer Haar measure **443Ac**, 443K
full subgroup of Aut $\mathfrak{A}$ **381Ae**, 381Ib, 381J, 381Qc, 381Xe, 381Xf, 381Xn, 381Yb, 381Yc, 382N, 382Q, 382Xe, 382Xh, 382Xi, 382Xj, 383C, 388A-388C, 388G, 388H, 388J-388L, 388Xa, 388Ya; *see also* countably full (**381Bf**), finitely full (**381Xg**)
—— subsemigroup of Aut $\mathfrak{A}$ **381Yd**
fully non-paradoxical group of automorphisms of a Boolean algebra **394E**, 394F, 394H, 394I, 394K-394R, 394Xd-394Xf, 394Xh, 394Ya-394Yc, 395B, 395Ya
fully symmetric extended Fatou norm *see* $\mathcal{T}$-invariant (**374Ab**)
function 1A1B
function space *see* Banach function space (§369)
functionally closed *see* zero set (**3A3Pa**)
functionally open *see* cozero set (**3A3Pa**)
Fundamental Theorem of Calculus 222E, 222H, *222I*, 225E, 482J, 483I, 484O; *see also* Divergence Theorem
Fundamental Theorem of Statistics 273Xh, 273 *notes*

Gagliardo-Nirenberg-Sobolev inequality 473H
Gaifman's example 391N, 391Yf-391Yh,
game *see* infinite game, Banach-Mazur game
gamma distribution *see* standard gamma distribution (**455Xe**)
Gamma function *see* Γ-function (**225Xj**)
gauge **481E**, 481G, 481Q, 481Xf, 481Xg; *see also* neighbourhood gauge (**481E**), uniform metric gauge (**481E**)
gauge integral **481D**, 481Xa-481Xh, 481Ya, 482B, 482D-482H, 482J-482M, 482Xa-482Xg, 482Ya-482Yc, 483Xo, 483Yb, 484G, 484Xc; *see also* Henstock integral (**483A**), Pfeffer integral (**484G**)
Gauss C.F. *372Yg*
Gauss-Green theorem *see* Divergence Theorem (475N)
Gaussian distribution §456 (**456A**), 495 *notes*; *see also* standard normal distribution (**274Aa**)
Gaussian process 456M, **456N**, 456Xg, 456Xh, 456Ya, 456Ye
Gaussian quasi-Radon measure **456Xf**
Gaussian random variable *see* normal random variable (**274Ad**)
generated filter **4A1Ia**
generated set of tagged partitions **481Ba**

generated topology **3A3Ma**
generated (σ-)algebra of sets **111Gb**, 111Xe, 111Xf, *121J*, 121Xd, 136B, 136C, 136G, 136Xc, 136X, 136Xl, 136Yb, 421F, 421Xi
generated (σ)-subalgebra of a Boolean algebra **313F**, 313M, 314G, 314Ye, 331E, 4A1O
generating Bernoulli partition **387Ab**, 387Bg
generating set in a Boolean algebra 331E; *see also* σ-generating (**331E**), τ-generating (**331E**)
Giacomelli *see* Tamanini-Giacomelli theorem (484B)
Glivenko-Cantelli class *465 notes*
Glivenko-Cantelli theorem *273 notes*
Gödel K. *4A1M*
gradient (of a scalar field) **473B**, 473C, 473Dd, 473H-473L, 474Bb
graph of a function 264Xf, 265Yb
greatest ambit (of a topological group) **449D**, 449E, 493C
Green's theorem *see* Divergence Theorem (475N)
Grothendieck's theorem 4A4Ch
group 255Yn, 255Yo; *see also* abelian group, amenable group, automorphism group, circle group, ergodic group, finite group, free group, isometry group, locally compact group, orthogonal group, Polish group (**4A5Db**), simple group, topological group (**4A5Da**)
group automorphism **3A6B**; *see also* inner automorphism (**3A6B**), outer automorphism (**3A6B**)
group homomorphism 442Xi, *445Xe*, 4A5Fa, 4A5L

'Haar almost everywhere' **443Ae**
Haar measurable envelope **443Ab**, 443Xg, 443Xn
Haar measurable function **443Ae**, 443G, *444Xo*
Haar measurable set **442H**, 442Xd, 443A, 443D, *443F*, 443J, 443Pc, 447A, 447B, 447D-447H
Haar measure Chap. 44 (**441D**), 452Xj, 491H, 491Xm, 491Xn, 491Yg, 492H, 492I
Haar measure algebra **442H**, 442Xe, 443A, 443C, 443Jb, 443Xd, 443Xe, 443Xj, 447A, 447Xa
Haar negligible set **442H**, 443A, *443F*, 443Jb, 443Pc, 444L, 444Xm, 444Ye
Haar null set **444Ye**
Hahn-Banach theorem 363R, 373Yd, 3A5A, 3A5C, 4A4D, 4A4E
Hahn decomposition of an additive functional 231Eb, 231F, 326I, 326O
—— *see also* Vitali-Hahn-Saks theorem (246Yg)
Hajian-Ito theorem 395B
half-open interval (in $\mathbb{R}$ or $\mathbb{R}^r$) **114Aa**, 114G, *114Xe*, *114Yj*, **115Ab**, 115Xa, *115Xc*, *115Yd*; (in general totally ordered spaces) **4A2A**
Hall's Marriage Lemma 3A1K, 4A1H
Halmos-Rokhlin-Kakutani lemma 386C
Hamel basis **4A4A**
Hamming metric on a product space **492D**, 492E, 492Xd
Hardy-Littlewood Maximal Theorem 286A
Hausdorff capacity **471H**, 471Xb
Hausdorff dimension **264 notes**, **471Xe**
Hausdorff measure §264 (**264C**, **264Db**, **264K**, **264Yo**), 265Yb, 343Ye, *345Xb*, *345Xg*, 411Yb, 439H, 439Xk, 441Yd, 441Yh, 441Yi, 442Ya, 443Ym, 456Xb, §471 (**471A**, **471Ya**), 476M; *see also* normalized Hausdorff measure (**265A**)
Hausdorff metric (on a space of closed subsets) **246Yb**, 441Ya, 441Yb, **476A**, 476B, 476D, 476F, 476Xa, 476Xb, 476Xc, 476Xe, 476Ya, 495Xk, 495Yc
Hausdorff outer measure §264 (**264A**, **264K**, **264Yo**), **471A**, 471Dc, 471J
Hausdorff topology **2A3E**, 2A3L, 2A3Mb, 2A3S, **3A3Aa**, 3A3D, 3A3Id, *3A3K*, 3A4Ac, 3A4Fd, 476Cb, 4A2F, 4A2La, *4A2Nf*, *4A2Qf*, *4A2Rc*, 4A4Ec
Hausdorff uniformity **3A4Ac**
Helly space **438Qb**, 438R, 438Xl
Henry's theorem 416M, 432 *notes*

Henstock integrable function §483 (**483A**, **483Yg**)
—— integral 481J, 481K, §483 (**483A**, **483Yg**); *see also* approximately continuous Henstock integral, indefinite Henstock integral, Saks-Henstock lemma
Henstock-Stieltjes integral 483Xn
hereditarily Lindelöf (topological space) 411Xd, 414O, 418Ya, 419L, 419Xf, 422Xe, 423Da, 433Xb, 434Ie, 434Ke, 438Lc, 439D, **4A2A**, 4A2Hc, 4A2N-4A2P, 4A2Rn, 4A3D, 4A3E, 4A3Kb
hereditarily metacompact 438Lb, **4A2A**, 4A2Lb
hereditarily separable **423Ya**
hereditarily weakly submetacompact *see* hereditarily weakly θ-refinable (**438K**)
hereditarily weakly θ-refinable **438K**, 438L-438N, 438P, 438Q, 438Xm, 466Eb, 467Pb
hereditarily σ-relatively metacompact *see* hereditarily weakly θ-refinable (**438K**)
Hilbert space 244N, 244Yj, **3A5L**, 456F, 456Xa, 456Xd, 456Yb, 466Xk, 493Xg, 495Xg, 4A4K, 4A4M; *see also* inner product space (**3A5L**)
Hilbert transform **376Ym**
Hölder's inequality **244Eb**
homeomorphism **3A3Ce**
homogeneous Boolean algebra **331M**, 331N, *331Xj*, 331Xk, 331Yg-331Yj, 381D, 382Q, 382S, 384E-384G, 384La, 384Xb, 424Xk; *see also* Maharam-type-homogeneous (**331Fc**), Martin-number-homogeneous, relatively Maharam-type-homogeneous (**333Ac**)
—— measure algebra 331N, 331Xk, 332M, 373Yb, 374H, 374Yc, 375Kb, 383E, 383F, 383I, 394R; *see also* quasi-homogeneous (**374G**)
—— measure space 344L
—— probability algebra 333P, 333Yc, 334E, 372Xk
homomorphic image (of a Boolean algebra) 314M; (of a group) 449Ca, 493Ba
homomorphism *see* Boolean homomorphism (**312F**), group homomorphism, lattice homomorphism (**3A1I**), ring homomorphism (**3A2D**)
—— of topological groups 446I; *see also* finite-dimensional representation
hull *see* convex hull (**2A5E**), closed convex hull (**2A5E**)
Humpty Dumpty's privilege 372 *notes*
Huntingdon's metric **437Yi**
hypo-radonian space *see* pre-Radon (**434Gc**)

ideal in an algebra of sets **232Xc**, 312C, *363Yf*; *see also* σ-ideal (**112Db**)
—— in a Banach algebra 4A6E
—— in a Boolean ring or algebra 311D, 312C, 312K, 312O, 312Xd, 312Xi, 312Xk, 382R, *392Xd*; *see also* principal ideal (**312D**), σ-ideal (**313E**)
—— in a Riesz space *see* solid linear subspace (352J)
—— in a ring 352Xm, **3A2E**, 3A2F, 3A2G
—— *see also* asymptotic density
idempotent **363Xf**, 363Xg
identically distributed random variables **273I**, 273Xh, 274 *notes*, 276Yg, 285Xn, 285Yc, 372Xg; *see also* exchangeable family (**276Xe**, **459C**)
image filter **2A1Ib**, 2A3Qb, 2A3S, 4A1K
image measure 112E, **112F**, 112Xd, 112Xg, *123Ya*, 132G, 132Xk, 132Yb, 132Yf, 211Xd, 212Bd, 212Xg, 235L, 254Oa, 256G, 342Xh, 342Xj, *343Xb*, 343Yf, 387Xd, *413Xc*, 415Rb, 418I, 418L, 418Xa, *418Xd*, 418Xo, 418Xp, 418Yf, 432G, 433D, 437Q, 443Pd, 444Xa, 451N, 451R, 451Xg, 454J, 456B, 456Xb, 461B, 474H, 491Ea
image measure catastrophe 235J, 343 *notes*, 418 *notes*
Increasing Sets Lemma 471G
indecomposable transformation *see* ergodic inverse-measure-preserving function (**372Pb**)
indefinite Henstock integral 483D, **483E**, 483F, 483I, 483K, 483Mc, 483Pa, 483R, 483Xg, 483Xh, 483Yc
indefinite integral 131Xa, 222D-222F, 222H, *222I*, 222Xa-222Xc, 222Yc, 224Xg, 225E, *225Od*, 225Xh, **232D**, 232E, 232Yf, 232Yi; *see also* indefinite Henstock integral, indefinite-integral measure, Pettis integral (**463Ya**), Saks-Henstock indefinite integral (**482C**, **484I**)

indefinite-integral measure §234 (**234A**), 235M, *235P*, 235Xi, 253I, 256E, 256J, 256L, 256Xe, 256Yd, 257F, 257Xe, 263Ya, 275Yi, 275Yj, 285Dd, 285Xe, 285Ya, 322Xf, 342Xd, 342Xn, 342Yd, *365T*, 412Q, 414H, 414Xe, 415O, 416S, 416Xt, 416Yf, 471F, 442L, 444K, 444P, 444Q, 445F, 445Q, 451Xc, 451Yi, 452Xf, 452Xh, 453Xe, 465Ck, *475G*, 491Xt; *see also* uncompleted indefinite-integral measure

independence §272 (**272A**)

independent family in a probability algebra **325Xe**, 325Yf, 371Yc, 495J

independent family in $L^0(\mathfrak{A})$ **364Xe**, 364Xf, 364Xt, **367W**, 367Xw, 367Xy, 495K, 495L, 495Xg, 495Xh, 495Ya, 495Yb

independent increments 455C, 455Xc-455Xe, 455Yd

independent random variables **272Ac**), 272D-272I, 272L, 272M, 272P-272U, 272Xb, 272Xd, 272Xh-272Xj, 272Ya-272Yd, 272Yf, 272Yg, 273B, 273D, 273E, 273H, 273I, 273L-273N, 273Xh, 273Xi, 273Xk, 274B-274D, 274F-274K, 274Xc, 274Xd, 274Xg, *275B*, 275Yh, *276Af*, 285I, 285Xf, 285Xg, 285Xm-285Xo, 285Xs, 285Yc, 285Yk, 285Yl, 454Xj, 456Xe, 456Xh, 495Ab, 495B, 495D, 495Xf; *see also* independent increments, relatively independent (**458A**)

independent sets **272Aa**, *272Bb*, 272F, 272N, 273F, 273K, *464Qb*

independent subalgebras of a probability algebra **325L**, 325Xe, 325Xf, 325Xg, *327Xe*, 364Xe, 385Q, 385Sf, *385Xd-385Xf*, 387B; *see also* Boolean-independent (**315Xn**)

independent σ-algebras **272Ab**, 272B, 272D, 272F, 272J, 272K, 272M, 272O, 275Ym

indiscrete topology **4A2A**

Individual Ergodic Theorem *see* Ergodic Theorem (372D, 372E)

induced automorphism (on a principal ideal of a Boolean algebra) **381M**, 381N, 381Qe, 381Xk, 381Xp, 381Xo, 382J, 382K, 388F, 388G, 388Xb

induced topology *see* subspace topology (**2A3C**)

inductive definitions 2A1B

inequality *see* Cauchy, Chebyshev, Gagliardo-Nirenberg-Sobolev, Hölder, isodiametric, Jensen, Poincaré, Wirtinger, Young

infimum in a Boolean algebra 313A-313C

infinite game 451U

infinitely decomposable *see* infinitely divisible

infinitely divisible distribution 455Xc-455Xe, 455Yc, *455 notes*, 495 *notes*

infinity 112B, 133A, §135

information *385Cb*

initial ordinal **2A1E**, 2A1F, 2A1K

injective ring homomorphism 3A2Da, 3A2G

inner automorphism of a group **3A6B**

inner measure **113Yh**, 212Yc, 213Xe, 213Yc, *324Cb*, **413A**, 413C, 413Xa-413Xc, 413Ya

—— —— defined by a measure **413D**, 413E, 413F, 413Xd-413Xg, 413Yb, 418Xt, 431Xe, 443Xa, 451Ob, *463F*

inner product space 244N, 253Xe, **3A5L**, 456Xd, 476H, 476I, 476M-476P, 476Xi-476Xk, 493I, 493J, 493Xf, 4A4J; *see also* Hilbert space (**3A5L**)

inner regular finitely additive functional 413N, 413Q

inner regular measure **256A**, **342Aa**, 342C, *342Xa*, **411B**, 412A, 412C, 412H, 412I, 412J, 412K, 412L, 412N, 412O, 412Q, 412R, 412T, 412Xc, 412Xd, 412Xh, 412Xk, 412Xr, 412Ya, 413E-413G, 413I-413K, 413M, 413O, 413S, 413Xh, 413Xj, 414M, 414Xg, 416Yb, 416Yc, 417Ye-417Yg, 418Yi, 418Yj, 431Xb, 434L, *443L*, *443Xr*, 451K, *452I*, *452M*, *452Xd*, 454A, 454N, 454Yd, 462H, *462Yc*, 463Yd, *465Xf*, 475F, 475Yb

—— —— with respect to Borel sets 412B, 412Pb, 412Sd, 412Uc, 412Xi, 412Xl, 412Xn, 412Yb, §417, 419H, 432Ya, 433A, 433Xa, 433Xf, 434Xl, 459F

—— —— with respect to closed sets 256Bb, 256Ya, 342F, 412B, 412E, 412G, 412Pc, 412S, 412Ua, 412Wb, *412Xa*, 412Xf, 412Xg, 412Xl, 412Xn, 412Xo, 412Yb, 414M, 414N, 414Xc, 414Ya, *415Xe*, 416P, 416Xi, 417C, 417E, 418J, 418Xr, 418Yg-418Yi, 419A, 431Xb, 432D, 432G, 432Xc, 432Xf, 433E, 434A, *434Ga*, *434Ha*, 434M, 434Yb, 434Yf, 435C, 471De, 471Xh, 471Yb, 438F, 454N, 454Yd, *466H*, 482F, 482Xd, 491Xx; *see also* quasi-Radon measure (**411Ha**)

—— —— with respect to compact sets 256Bc; *see also* tight (**411Ja**)

—— —— with respect to zero sets 412B, 412D, 412Pd, 412Sc, 412Ub, 412Xl, 412Xn, 412Xs, 413Yc, 454J; *see also* completion regular (**411Jb**)

integrable function §122 (**122M**), *123Ya*, **133B**, **133Db**, **133Dc**, 133F, 133J, 133Xa, *135Fa*, 212B, 212F, 213B, 213G, *412Wa*, 436Yb, 482F, 482Xc-482Xe; *see also* Bochner integrable function (**253Yf**), Henstock integrable function (**483A**), Pfeffer integrable function (**484G**), $\mathcal{L}^1(\mu)$ (**242A**)

integral §122 (**122E**, **122K**, **122M**), **363L**, 436A; (on an L-space) 356P; *see also* Henstock integral (**483A**), integrable function, Lebesgue integral (**122Nb**), lower integral (**133I**), McShane integral, Perron integral, Pettis integral (**463Ya**), Pfeffer integral (**484G**), Riemann integral (**134K**), symmetric Riemann-complete integral, upper integral (**133I**)

integral-geometric measure *475Q*

integral operator *see* kernel operator, abstract integral operator (§376)

integration by parts 225F, *225Oe*, 252Xi, 483Yc, 483Ye

integration by substitution *see* change of variable in integration

integration of measures 452A-452D, 452Xa-452Xe, 452Ya, 453Xk

interior of a set **2A3D**, 2A3Ka, *475Ca*, *475R*, 4A2Bg; *see also* essential interior (**475B**)

interpolation *see* σ-interpolation property (**466G**), Riesz Convexity Theorem

intersection number (of a family in a Boolean algebra) **391H**, 391I-391K, 391Xh, 391Xi, 392E

interval (in a totally ordered set) **4A2A**; *see also* closed interval (**4A2A**), half-open interval (**114Aa**, **115Ab**, **4A2A**), open interval (**111Xb**, **4A2A**), split interval (**343J**)

invariant function (in 'G-invariant function') **394A**, 394N-394P, 394Xd, 394Xe, 394Z, 395B, 395Xd

—— additive functional 449J, 449Xq-449Xs

—— ideal **382R**

—— integral 441J, 441L

—— lower density 346Xd

—— mean 449E, 449H, 449I, 449Xi, 449Xo, 449Yc

—— measure *436Xs*, **441Ad**, 441B, 441C, 441H-441L, 441Xa-441Xd, 441Xk, 441Xp, 441Yj, 441Yk, 441Yb, 441Yc, 442Ya, 442Z, 443P, 443Q, 443T, 443Xs, 443Xu, *443Xv*, 443Xz, 447E, §448, 449E, 461O, 461P, 461Xl, 461Xm, 461Ye, 461Ye, 476G, 476I, 476M, 476Yc; *see also* Haar measure (**441D**), symmetric measure

—— *see also* rearrangement-invariant (**374E**), $\mathcal{T}$-invariant (**374A**), translation-invariant

inverse (of an element in an f-algebra) 353Pc, 353Yh

inverse Fourier transform **283Ab**, 283B, **283Wa**, 283Xb, **284I**, 445P, 445Yg; *see also* Fourier transform

inverse image (of a set under a function or relation) **1A1B**

inverse-measure-preserving function 132G, 134Yl, 134Ym, 134Yn, **235G**, 235H, 235I, 235Xe, 235Ya, 235Yb, 241Xh, 242Xf, 243Xn, 244Xo, 246Xf, 254G, 254H, 254Ka, 254O, 254Xc-254Xf, 254Xh, 254Yb, 324M, 324N, 341P, 341Xd, 341Yc, 341Yd, 343C, *343J*, 343Xd, 343Xh, 343Yd, 365Xi, 372H-372L, 372Xe, 372Xf, 372Yh, 372Ym, 386Xb, 385S-385V, 411Ne, 412K, 412M, 413Eh, *413Yc*, 414Xa, 414Xq, 414Xs, 416W, 417Xg, 418H, 418M, 418P, 418Q, 418Xf, 418Xg, 418Xn, 418Xs, 418Xt, 418Yl, 419L, 419Xf, 432G, 433K, 433Xg, 433Yd, 436Xs, 437S, 441Xa, 442Xa, 443L, 451E, 452E, 452I, 452O-452R, 452Xl, 452Xm, 452Yb, 453K, *454G*, 454J, 456Ba, 456Eb, 456G, 456Yb, 457F, 457Xf-457Xj, 457Yd, 457Za, 457Zb, 458Xo, 459A, 461O, 461P, 461Xl-461Xm, 461Yd, 461Ye, *464B*, *465Xe*, 491Ea, 495G, 495I, 495Xd, 495Xi; isomorphic inverse-measure-preserving functions **385Tb**; *see also* almost isomorphic (**385U**); *see also* ergodic inverse-measure-preserving function (**372P**), image measure (**112E**), mixing inverse-measure-preserving function (**372P**)

Inversion Theorem (for Fourier series and transforms) 282G-282I, 282L, 282O, 282P, 283I, 283L, 284C, 284M, 445P; *see also* Carleson's theorem

invertible element (in a Banach algebra) 4A6H-4A6J;(in an f-algebra) 353Pc, 353Yh

involution in a group **3A6A**

involution in Aut $\mathfrak{A}$ 382C, 382N, 382Q, 382Xc-382Xe, 382Xg, 382Xi, 382Ya, *382Yc*, 383D, 383G, 383Xa, 385Xl, *388Ye*; *see also* exchanging involution (**381R**), 'many involutions' (**382O**)

irreducible continuous function **313Ye**

isodiametric inequality 264H

isolated family of sets 438Nd, 466D, **4A2A**, 4A2Ba

isolated point *316M*, *316Yj*, *316Yq*, **3A3Pb**

isometry 324Yh, **3A4Ff**, 3A4G, 441Xl, 448Xf, 448Xi, 474C, 474H, 476N, 476Xi, 4A4Jb
—— group (of a metric space) **441F**, 441G, 441H, 441Xm-441Xp, 441Ya, 441Yb, 441Ye, 441Yg, 441Yj, 443Xx, 443Xz, 443Ym, 448Xk, 449Xc, 449Xd, 476G, 476M, 476Xb, 476Xd, 476Xi, 476Yc, 493J, 493Xc, 493Xf, 493Xg, 493Yb
isomorphic *see* almost isomorphic (**385U**)
isomorphism *see* Boolean isomorphism, Borel isomorphism (**4A3A**), measure space isomorphism, order-isomorphism
Isoperimetric Theorem *475 notes*, 476L; *see also* Convex Isoperimetric Theorem

Jacobian **263Ea**, 484P, 484S
Jensen's inequality 233I, 233J, 242Yi, 365Rb
Jensen's $\diamondsuit$ *4A1M*
joint distribution *see* distribution (**271C**)
Jordan algebra **>411Yc**, 491Ye
Jordan decomposition of an additive functional 231F, 231Ya, *232C*, 326D, 326H, 326M

Kadec norm **466C**, 466D-466F, 467B
Kadec-Klee norm *see* Kadec norm (**466C**)
Kakutani's theorem 369E, *369Yb*; *see also* Halmos-Rokhlin-Kakutani lemma (386C)
Kalton-Roberts theorem 392F, 392J
Kampen, E.R.van *see* Duality Theorem (445U)
Kawada's theorem 394Q
Kechris A.S. *see* Becker-Kechris theorem, Nadkarni-Becker-Kechris theorem
Kelley's criterion 391L; *see also* Nachbin-Kelley theorem
kernel (of a ring homomorphism) **3A2D**, 3A2Eb, *3A2G*; (of a Souslin scheme) **421B**, 421C, 421I, 421Ng, 421O, 421Q; *see also* Abel-Poisson kernel (**282Yg**), Dirichlet kernel (**282D**), Féjer kernel (**282D**), modified Dirichlet kernel (**282Xc**)
kernel operator 373Xf, §376
Kirzbraun's theorem 262C
Kolmogorov's extension theorem 454D-454G
Kolmogorov's Strong Law of Large Numbers 273I, 275Yn, 372Xg
Kolmogorov-Sinaĭ theorem 385P
Komlós' theorem 276H, 276Yh, 367Xn, 463Xe
Kreĭn's theorem 461I
Kreĭn-Mil'man theorem 4A4Gb
Kronecker's lemma 273Cb
Kwapien's theorem 375I

Lacey-Thiele Lemma 286M
Laplace's central limit theorem 274Xe
Laplace transform **123Xc**, *123Yb*, **133Xc**, 225Xe
laterally complete Riesz space **368L**, 368M
lattice **2A1Ad**, 311L, 311Xi, 311Yb, 315Xb, *315Yg*, 367A, 367B, 367Xa, 3A1I; *see also* Riesz space (=vector lattice) (**352A**), Banach lattice (**242G**, **354Ab**)
—— homomorphism **3A1Ia**
—— norm *see* Riesz norm (**242Xg**, **354Aa**)
—— of normal subgroups 382Xh, 382Xj, 383H, 383J, 383Xg, 383Xh, 383Ya, 3A6C
law of a random variable *see* distribution (**271C**)
law of large numbers *see* strong law (§273)
law of rare events 285Q
Lebesgue, H. Vol. 1 *intro.*, Chap. 27 *intro.*
Lebesgue Covering Lemma *2A2Ed*
Lebesgue decomposition of a countably additive functional 232I, 232Yb, 232Yg, *256Ib*
Lebesgue decomposition of a function of bounded variation **226C**, 226Dc, *226Ya*, 232Yb
Lebesgue density *see* lower Lebesgue density (**341E**)

Lebesgue's Density Theorem §223, 261C, *275Xg*, 471P, 447D, 447Xc, *475Yb*; *see also* Besicovitch's Density Theorem (472D)
Lebesgue's Dominated Convergence Theorem 123C, 133G, 363Yh, 367J, *367Xf*, 436Yb
Lebesgue extension *see* completion (**212C**)
Lebesgue integrable function **122Nb**, *122Yb*, *122Ye*, *122Yf*, 483Bb, 483C, 483G, *483Q*, 483Xf, 483Xg, 483Xl, 483Yf, 484H
Lebesgue integral **122Nb**
Lebesgue measurable function **121C**, 121D, 134Xd, *225H*, 233Yd, 262K, 262P, 262Yc, 466Xh, 483Ba, 484Hd
Lebesgue measurable set **114E**, 114F, 114G, *114Xe*, 114Ye, **115E**, 115F, 115G, 115Yc
Lebesgue measure (on $\mathbb{R}$) §114 (**114E**), 131Xb, 133Xc, 133Xd, 134G-134L, 212Xc, *216A*, Chap. 22, 242Xi, 246Yd, 246Ye, *252N*, *252O*, *252Xf*, *252Xg*, *252Yj*, *252Yp*, §255, *342Xo*, 343H, 344K, 345Xc, 346Xe, 372Xd, *383Xf*, 384Q, 412Ya, 413Xe, 413Xk, 415Xc, 419J, 419Xf, 453Xb, 439E, 439F, 445Xk, *454Yb*, 481Q, 482Ya, 483A
—— —— (on $\mathbb{R}^r$) §115 (**115E**), 132C, *132Ef*, 133Yc, §134, 211M, 212Xd, 245Yj, 251M, 251Wi, 252Q, 252Xh, 252Yu, 254Xk, 255A, 255K, 255L, 255Xd, 255Yc, 255Yd, 256Ha, 256J-256L, 264H, 264I, 342Ja, 344K, 345B, 345D, 345Xf, 411O, 411Ya, 415Ye, 419Xd, *441Ia*, 441Xg, 442Xf, *443Yj*, *453B*, 449M, 471Yi, 476J-476L, 495Xp
—— —— (on $[0,1]$, $[0,1[$) 211Q, *216A*, 252Yq, 254K, 254Xh, 254Xj-254Xl, 324Yg, 343Cb, *343J*, 343Xd, 344K, 385Xj, 385Xk, 385Yd, *415F*, *417Xi*, *417Xq*, *418Xd*, 419L, *419Xg*, 436Xm, 438Ce, 439Xa, 448Xk, *451Ad*, *454Xa*, 457G-457I, 457Xg-457Xj, 457Yd, 457Za, 457Zb, 463Xj, 463Yc, 463Ye, 491Xf, 491Yf, 491Z
—— —— (on other subsets of $\mathbb{R}^r$) 242O, 244Hb, 244I, 244Yh, 246Yf, *246Yl*, 251Q, 252Ym, 255M, 255N, 255O, 255Ye, 255Yh, 343Cc, 343H, 343M, 344J, 345Xd, 346Xb, 372Yj, 372Yk, 388Yd, 433Xc, *433Yb*, 442Xc, 463Zb, 495P, 495Xi, 495Xk, 495Xn
—— —— algebra 331Xd, 331Xe, 373C, 374C, 374D, 374Xa, *374Xf*, 374Xh, 374Ya, 374Yd, 375Xe, 493Yd
Lebesgue negligible set **114E**, **115E**, 134Yk
Lebesgue outer measure **114C**, 114D, 114Xc, *114Yd*, **115C**, 115D, 115Xb, 115Xd, 115Yb, 132C, 134A, 134D, 134Fa
Lebesgue set of a function **223D**, *223Xf*, 223Xg, 223Xh, 223Yg, **261E**, 261Ye, 282Yg
Lebesgue-Stieltjes measure **114Xa**, *114Xb*, 114Yb, 114Yc, 114Yf, 131Xc, 132Xg, 134Xc, 211Xb, 212Xd, 212Xi, 225Xf, 232Xb, *232Yb*, 235Xb, 235Xg, 235Xh, 252Xi, 256Xg, 271Xb, *224Yh*, 342Xe
left action (of a group on itself) **4A5Ca**, 4A5I
left Haar measure **441D**; *see* Haar measure
left modular function (of a topological group) *see* modular function (**442I**)
left-translation-invariant *see* translation-invariant
length of a curve 264Yl, 265Xd, 265Ya
length of a finite function **421A**
length of an interval **114Ab**
B.Levi's theorem **123A**, 123Xa, 133K, 135G, 135Hb, 226E, 242E, 365Df, 365Dh, 482K
Levi property of a normed Riesz space 242Yb, 244Ye, **354Db**, 354J, 354N, *354Xi*, 354Xn, 354Xo, 354Yi, 356Da, 356L, 356Xk, 356Yi, 365C, 366D, 369G, 369Xe, 369Yg, 371C, 371D, 371Xc, 371Xd
Lévy's martingale convergence theorem 275I, 367Kb
Lévy's metric **274Ya**, *285Yd*, 437Yi
lexicographic ordering **351Xa**
Liapounoff's central limit theorem 274Xg
Liapounoff's vector measure theorem 326Ye
lifting of a measure (algebra) Chap. 34 (**341A**, **341Ya**), 363Xe, 363Yg, 414Xp; *see also* almost strong lifting (**453A**), lifting topology (**414Q**), linear lifting (**363Xe**), strong lifting (**453A**), translation-invariant lifting (**345A**, **447A**)
Lifting Theorem 341K, 363Yg
lifting topology **414Q**, 414R, 414Xo-414Xt, 414Yh, 418Xm, 453Xd
limit of a filter **2A3Q**, 2A3R, 2A3S
limit of a sequence **2A3M**, 2A3Sg
limit ordinal **2A1Dd**

Lindeberg's central limit theorem 274F-274H, 285Ym
Lindeberg's condition **274H**
Lindelöf space 411Ge, 412Xo, 422De, 422Gg, 422Xb, 434Hb, 435Fb, 435Xd, 435Xj, *439Xm*, *439Yg*, *462Yd*, 466Xd, 467Xa, **>4A2A**, 4A2H, 4A2Ub; *see also* hereditarily Lindelöf (**4A2A**)
Lindelöf-Σ space *see* K-countably determined (**467H**)
linear functional 3A5D, *4A4Ac*; *see also* positive linear functional, tight linear functional (**436Xn**)
linear lifting 341Xe, 341Ye, *345Yb*, **363Xe**, 363Yg
linear operator *262Gc*, 263A, 265B, 265C, §*2A6*, 355D, 3A5E, 456Xd, 466Xi, 4A4J; *see also* bounded linear operator (**2A4F**), (weakly) compact linear operator (**3A5K**), continuous linear operator, order-bounded linear operator (**355A**), positive linear operator (**351F**), self-adjoint linear operator (**4A4Jc**)
linear order *see* totally ordered set (**2A1Ac**)
linear space 4A4A
linear space topology *see* linear topological space (**2A5A**), weak topology (**2A5I**), weak* topology (**2A5I**), $\mathfrak{T}_s(U,V)$ (**3A5E**)
linear subspace (of a normed space) 2A4C; (of a partially ordered linear space) 351E, 351Rc; (of a linear topological space) 2A5Ec, 4A4B-4A4E
linear topological space 245D, *284Ye*, **2A5A**, 2A5B, 2A5C, 2A5Eb, 2A5F, 2A5G, 2A5H, 2A5I, 367N, 367Ym, 367Yn, 393L, 393M, 393O-393Q, 393Yh, 393Yi, 3A4Ad, 3A4Bd, 3A4Fg, 418Xj, 445Yc, 463A, 4A3T, 4A3U, 4A4A, 4A4B, 4A4H; *see also* locally convex space (**4A4Ca**)
linked set(in a Boolean algebra) **391L**; (in κ-linked set, $<\kappa$-linked set) **391Yh**; *see also* σ-linked (**391L**), σ-m-linked (**391Yh**)
lipeomorphism **484Q**, 484S
Lipschitz constant **262A**, 262C, 262Yi, 471J
Lipschitz function 225Yc, **262A**, 262B-262E, 262N, 262Q, 262Xa-262Xc, 262Xh, 262Yi, 263F, 264G, **264Yj**, *282Yb*, 437Yi, **471J**, **471Xc**, 473C-473E, 473H-473J, 474B, *474D*, *474E*, *474K*, *474M*, 475K, 475N, 475Xg, 492H, 492Xb-492Xd; *see also* lipeomorphism (**484Q**)
local convergence in measure *see* convergence in measure (**245A**)
local semigroup *see* full local semigroup (**394A**), countably full local semigroup (**448A**)
local type (lifting) *see* strong lifting (**453A**)
localizable measure algebra **322Ae**, 322Be, 322C, 322K-322O, 323Gc, 323H, 323J, 323K, *323Xc*, *323Yb*, *323Z*, *324Yf*, 325C, 325D, 332B, 332C, 332J, 332O, 332S, 332T, 332Xo, 332Ya, 332Yc, 333H, 333Ia, *364Xk*, 365M, 365N, 365Sa, 365T, 366Xb, *366Xf*, 367Nc, 367V, 369A, 369B, 369D, 369E, *369R*, 373L-373N, 373R, *373T*, *373U*, 375Yd, 381Xn, 383B-383H, 383Xb, 383Xc, 383Xe, 383Xh, 384M, 384N, 384Pa, 384Ya-384Yd, 391Cb, 394Xg, 394Xh, 394Yd, *395A*, 411Pd, 434T; *see also* localization (**322P**)
localizable measure algebra free product 325D, **325Ea**, 325H, 325Xa, 325Xb, 325Xc, 327Ya, 333E-333H, 333K, 333Q, 333R, 334B, 334Xb, 376B, 376E, 376F, 376Xb, 376Yb, 438S, 465O; *see also* probability algebra free product (**325K**)
localizable measure (space) **211G**, 211L, *211Ya*, 211Yb, 212G, 213Hb, 213L-213N, *213Xl*, 213Xm, 214Id, 214J, 214Xa, 214Xd, 214Xf, 216C, 216E, *216Ya*, 216Yb, 234Fc, 234G, 234Ye, 241G, 241Ya, 243G, 243H, 245Ec, 245Yf, 252Yq, *252Yr*, 252Yt, 254U, 322Be, 322N, *325B*, *342N*, *342Yb*, 363S, 364Xm, *419C*, *419D*, 433A, 438Xb, 463Xm; *see also* strictly localizable (**211E**)
localization of a semi-finite measure algebra **322P**, 322Yb, 332Yb, 365Xm, 366Xe, 366Xf, 369Xp
locally compact Hausdorff space 416G-416I, 416L, 416T, 416Xj, 416Xv, 416Yd, 419Xf, 434Jg, 435Xe, 436J, 436K, 436Xo, 436Xq, 437I, 441C, 441H, 441Xp, 462E, 476Cd, 476Xd, 495N, 495O, 495Xm, 4A2Ge, 4A2Kf, 4A2Rk, 4A2Sa
locally compact measure (space) **342Ad**, 342H, 342I, 342L-342N, 342Xc, 342Xd, 343B, 343Xa, 343Xg, 343Ya, 344H
locally compact topological group 434 *notes*, 441E, 441Xh, *441Xn*, §442, 443E, 443K, 443L, 443O-443S, 443Xs, 443Xt, 443Xw, 443Xx, 443Yb, 443Yc, 443Yk, 443Yl, 445Ac, 445J, 445U, §446, 447E-447I, 447Xa, 449H, 449I, 449Xo, 449Xq, 493K, 4A5E, 4A5Jb, 4A5M-4A5P, 4A5R, 4A5S
locally compact topological space **3A3Ag**, 3A3B, 3A3G, 3A3Hb, 414Xj, *415Ym*, *422Xb*, 441Yc; *see also* locally compact Hausdorff space, locally compact topological group
locally convex linear topological space **393Yi**, §461, 462Yb, 463A, 466A, 466K, 466Xa, 466Xc, 466Xd, 466Xj, 466Ya, 4A3V, **4A4C**, 4A4D-4A4G

locally determined measure (space) **211H**, 211L, *211Ya*, *216Xb*, *216Ya*, 216Yb, 251Xc, 252Ya, 322N, *465Xi*; *see also* complete locally determined measure

locally determined negligible sets **213I**, 213J-213L, 213Xj-213Xl, 214Ib, 214Xg, 214Xh, 216Yb, 234Yb, 252Yb, 376Yk, 431Xd

locally finite family of sets **4A2A**, 4A2Bh, 4A2Fd, 4A2Hb

locally finite measure **256A**, *256C*, 256G, 256Xa, *256Ya*, **411Fa**, 411G, 411Pe, 411Xc, 411Xg-411Xi, 412Xo, *414Xj*, 415G, 416C, 416Dd, 416F-416H, 416M, 416P, 416S, 416T, 416Xi, 417Xb, 419A, *432C-432G*, *432Xf*, *433Xf*, *434J*, 435B, 441Xj, 442Aa, *451N*, *451Rb*, *466A*; *see also* effectively locally finite (**411Fb**)

locally finite perimeter **474D**, 474E-474N, 474Q, 474R, 474Xa-474Xd, 474Ya, 475D-475G, 475J, 475L-475P, 475Xh, 484L-484N, 484Rb

locally finite set 495Xc

locally integrable function 242Xi, **255Xh**, 255Xi, **256E**, 261Xa, 262Yg, **411Fc**, 411G, 411Xb, 411Xe, 414Xf, 415Pb, 415Xo, 447Xc, 472D, 473D, 473Ee

locally uniformly convex *see* locally uniformly rotund (**467A**)

locally uniformly rotund norm **467A**, 467B-467G, 467K, 467N, 467Yb

Loeb measure 413Xn, 451Xd

Loève, M. Chap. 27 *intro.*

logistic map *see* quadratic map (**372Xn**)

Loomis-Sikorski theorem 314M

Lorentz norm **374Xj**, 374Yb

Losert's example 453N, 453Xk

lower density on a measure space §341 (**341C**, **341Ya**), 343Yc, §345, 346F, 346G, 346Xa, 346Xc, 346Xd, 346Ye, 346Yf, 346Zb, 363Xe, 414Yf, 447Yb, 453Xh, 472Xc; *see also* density topology (**414P**), lower Lebesgue density (**341E**), translation-invariant lower density (**447A**)

lower derivate **483H**, 483J, 483Xm

lower integral **133I**, 133J, 133Xe, 135H, *364Xm*

lower Lebesgue density 223Yf, **341E**, 414Xm, 414Ye, 414Yg, 414Yh, *475B*, 475Xe

lower limit topology *see* Sorgenfrey topology (**415Xc**)

lower Riemann integral **134Ka**

lower semi-continuous function **225H**, 225I, 225Xl, 225Xm, 225Yd, 225Ye, 323Cb, 367Xx, 412W, 412Xq, 414A, 414B, 417B, *437Jd*, *443Xc*, 444Fa, 452C, 452D, 452Xa, 461C, 461Xa, 472F, 476F, **4A2A**, 4A2Bd, 4A2Gk, 4A3Ce

Lusin measurable function *see* almost continuous (**411M**)

Lusin's theorem 134Yc, 256F, 451S

Luxemburg, W.A.J. 363 *notes*

Mackey topology (on a linear space) *466Ya*, **4A4F**

magnitude of (an element in) a measure algebra **332Ga**, *332J*, *332O*, 383Xd, *383Xe*, 383Xh

magnitude of a measure space **332Ga**, 343Yb, 344I, 438Xb

Maharam algebra *see* localizable measure algebra (**322Ae**)

Maharam measure (space) *see* localizable (**211G**)

Maharam submeasure **392G**, 392H-392J, 392Xc, 392Xd, 392Xf, 392Xg, 392Ya, 392Yb, 393A-393C, 393E, 393M, 393Xc-393Xf, 393Ya, 393Yf, 393Yg, 438Ya, 491Id

Maharam's theorem 331I, **332B**

Maharam type of a Boolean algebra **331F**, 331H, 331Xc, 331Xg, 331Xh, 331Xj, 331Yd, 332H, 365Ya, 373Yb, 382Xl, 393Xe, 393Xf, 393Ya, 393Yf, 448L; *see also* relative Maharam type (**333Aa**)

—— —— of a measure algebra 331I-331K, 331Xd-331Xf, 331Xi, 332M, 332N, 332R-332T, 333D, §334, 369Xg, 375Kb, 387K, 387L, 387Xc, 388K, 388L, 438S, 438Xh, *448Xj*, *493Xe*

—— —— of a measure (space) **331Fc**, 365Xp, 367Xr, *416Yg*, 433A, 433Xa, 433Ya, 434Yl, 438Yc, 438Yj, 439A, 453Zb, 456Yd, 464Qb

Maharam-type-homogeneous Boolean algebra **331Fb**, 331H, 331Xh, 331Xj, 332A, 332H, 332Xa; *see also* homogeneous (**331M**)

—— —— measure algebra 331I, 331K, 331L, 331N, 331Xd, 331Xe, 331Yj, 344Xe; *see also* relatively Maharam-type-homogeneous (**333Ac**)

—— —— measure (space) **331Fc**, 334Xe, 334Xf, 334Ya, 341Yc, 341Yd, 346E
Maharam-type-κ component in a Boolean algebra **332Gb**, 332H, 332J, 332O, 332P, 332Xj, 332Ya, 384N, *394Xh*
many involutions (group with many involutions) **382O**, 382P-382R, 382Xe, 382Xg-382Xj, 383G, 384A, 384C, 384D, 384I, 394Xc
Marczewski functional **343E**
marginal measure **452L**, *452M*, *452N*, 459G
Mařík's theorem 435C, *439O*
Markov process 455B
Markov time *see* stopping time (**275L**)
Marriage Lemma see Hall's Marriage Lemma (3A1K)
martingale §275 (**275A**, **275Cc**, **275Cd**, **275Ce**); *see also* reverse martingale
martingale convergence theorems 275G-275I, 275K, 275Xf, 367K, 369Xq
martingale difference sequence **276A**, 276B, 276C, 276E, 276Xd, 276Ya, 276Yb, 276Ye, 276Yg
martingale inequalities 275D, 275F, 275Xb, 275Yc-275Ye, 276Xb, *372 notes*, 492G, 492Xa
Max-flow Min-cut Theorem *332Xk*, 4A4N
maximal element in a partially ordered set **2A1Ab**
Maximal Ergodic Theorem 372C
maximal ideal space of a Banach algebra 445H
maximal theorems 275D, 275Yc, 275Yd, 276Xb, 286A, 286T, 372C, 372Yb, 472E, 472F, 472Yd
Mazur S. *see* Banach-Mazur game
McMillan *see* Shannon-McMillan-Breiman theorem (386E)
McShane integral 481M, 481N, 482Xh
meager set 314L, *314M*, 314Yd, *316I*, 316Yf, 316Yi, *341Yb*, **3A3F**, 3A3Ha, *433Yc*, 434Ye, 4A2Ma
mean (of a random variable) *see* expectation (**271Ab**)
Mean Ergodic Theorem 372Xa
—— *see also* convergence in mean (**245Ib**)
measurable additive functional (on $\mathcal{P}I$) **464I**, 464J, 464K, 464M, 464Q
measurable algebra **391B**, 391C, 391D, 391K, 391Xf, 391Z, 392J, 393A, 393J, *393Ye*, 394P, 394Q, 394Xf, 395B, 395Xd
measurable cover *see* measurable envelope (**132D**)
measurable envelope **132D**, 132E, 132F, 132Xf, 132Xg, 134Fc, 134Xc, 213K-213M, 214G, 216Yc, 322I, 322J, 413Ei, 419J, *414Xl*, *414Xo*, *414Xp*; *see also* Haar measurable envelope (**443Ab**), full outer measure (**132F**)
measurable envelope property **213Xl**, 214Xl, 364Xm, 431Ya, 433Xa
measurable function (taking values in $\mathbb{R}$) §121 (**121C**), *122Ya*, *212B*, *212F*, 213Yd, 214La, 214Ma, 235C, 235K, 252O, 252P, 256F, 256Yb, 256Yc, 316Yi, 322Yf, 413G, 414Xf, 414Xk, 414Xr, 418Xl, 418Xm, 418Xr, 463K, 482E, 482Ya
—— —— (taking values in $\mathbb{R}^r$) **121Yf**), 256G, 411Ya
—— —— (taking values in other spaces) **133Da**, 133E, 133Yb, **135E**, *135Xd*, 135Yf, **411L**, 418A-418C, 418E, 418J, 418K, 418R-418T, 418Xa-418Xd, 418Xx, 418Ya-418Yc, 418Yf-418Yh, 419Xe, 419Xg, 431Xa, 433E, 438D-438G, 438Xf-438Xi, 451N, 451O, 451Q-451S, 451Xn, 451Xp, 451Yf, 452Xb, 452Xc, 4A3Cb, 4A3Db, 4A3Ke
—— —— ((Σ, T)-measurable function) **121Yb**, 235Xc, 251Ya, 251Yc, 343A, 411L, 423O, 424B, 424Xh, 434De, 443Yg, 451Yl, 459B, **4A3B**, 4A3C, 4A3Db, 4A3K, 4A3Ne
—— —— *see also* Borel measurable, Haar measurable (**443Ae**), Lebesgue measurable, measurable additive functional (**464I**), measurable selector, scalarly measurable (**463Ya**), universally measurable (**434Dd**), universally Radon-measurable (**434Ec**)
measurable neighbourhood gauge **483Ya**
measurable selector 423M-423O, 423Xf, 423Xg, 424Xg, 424Xh, 433F, 433G
measurable set **112A**; μ-measurable set **212Cd**; *see also* Haar measurable (**442H**), relatively measurable (**121A**), universally measurable (**434D**), universally Radon-measurable (**434E**)
measurable space **111Bc**
measurable space with negligibles 431Yb, 431Yc

measurable transformation §235, *365H*; *see also* inverse-measure-preserving function

measure **112A**; *see also* control measure (**393O**), vector measure (**393O**)

—— (in 'μ measures E', 'E is measured by μ') **>112Be**

measure algebra 211Yb, 211Yc, vol. 3 (**321A**), 448Xj, 448Xk, 493H, 495J, 495Xf, 495Xh; isomorphic measure algebras 331I, 331L, 332B, 332C, 332J, 332K, 332P, 332Q, 332Ya, 332Yb

measure algebra of a measure space 321H-321K (**321I**), 322B, 322N, *327C*, 412N, 414A, 415L, *416C*, *416W*, 418T, 441Ka, 441Yk, 474 *notes*, 493G; *see also* Haar measure algebra (**442H**), usual topology on a measure algebra (**323A**)

measure-algebra topology §323 (**323A**), 324F-324H, *324K*, 324Xb, *324Xc*, 327B, 327C, 365Ea, 367R, 367Xs, 367Yo, 411Yc, 412N, 443Aa, 443C, 444Fc, 444Yd, 448Xj, 493E, 494A; *see also* strong measure-algebra topology (**323Ae**)

measure-algebra uniformity **323A**, 323B-323D, 323G, 324F, 324H, 327B, 443Aa

measure-compact (topological) space **435D**, 463F-463H, 435Xb, 435Xh, 435Xi, 435Xk, 435Yb, 436Xf, 436Xg, 438Jb, 438Xk, *439P*, 439Q, 454Pa, 454Xm; *see also* strongly measure-compact (**435Xk**), Borel-measure-compact (**434Gb**)

measure-free cardinal **438A**, 438B, 438C, 438I, 438R, 438S, 438Xa-438Xe, 438Xh, 438Xj, 438Xn-438Xp, 438Yb-438Yf, 438Yi, 439Cd, *439J*, *439L*, 452Yb, 454Yb, 466F; *see also* Banach-Ulam problem

measure-free weight 438D-438H, 438J, 438M, 438S, 438Xf, 438Xg, 438Xi, 438Xk, 448Xi, 467Pb

measure metric (on a measure algebra) **323A**, *323Xg*, 324Yh, 448Xk, 493G, 493Yb

measure-preserving Boolean automorphism 332L, 333Gb, 333P-333R, 333Yc, 343Jc, 344C, 344E-344G, 344Xf, 364Xt, 372Xk, 372Xl, 372Yi, 373Yb, *374E*, 374G, 374L, 374Yc, 383D, 383Yc, 385P, 385Sb, 385Sf, 385Xg, 385Yd, 386N, 387B-387H, 387J, 387L, 388A, 388C, 388E-388H, 388J-388L, 388Xa-388Xe, 388Yd, 395Xc; *see also* automorphism group of a measure algebra, two-sided Bernoulli shift (**385Qb**)

measure-preserving Boolean homomorphism **324I**, 324J-324P, 324Xd, 324Xe, 324Yf, 324Yh, 325C, 325D, 325I, 325J, 325Xd, *327Xa*, 331D, 332L-332O, 332Q, 332Xm, 332Xn, 333C, 333D, 333F, 333Gb, 333Xc, 333Yd, 343B, 366Xf, 369Xm, 369Xn, 372G, *372H*, 372Xc, 373Bd, 373U, 373Xi, 383Yc, 385L-385Q, *385T*, 385V, 385Xf-385Xi, 385Xm, 385Xp, 385Yb-385Ye, 386A-386F, 386L, 386Xe, 386Xf, 386Ya-386Yc, 387A, 387B, *387F*, 458K, 491N, 491Xy; isomorphic measure-preserving Boolean homomorphisms **385Ta**; *see also* automorphism of a measure algebra, Bernoulli shift (**385Q**), measure-preserving ring homomorphism

measure-preserving function *see* inverse-measure-preserving function (**235G**), measure space automorphism, measure space isomorphism

measure-preserving ring automorphism 366L, 366Xh

measure-preserving ring homomorphism **361Ad**, 365O, 365Q, 366H, 366K, 366L, 366Xg, 366Xh, 366Yg, 369Xl, 372F; *see also* measure-preserving Boolean homomorphism

measure space §112 (**112A**), 113C, *113Yi*

measure space automorphism 255A, 255Ca, 255N, 255Ya, *255 notes*, 344C, 344E-344G, 344Xf, 345Ab, 345Xa, 346Xd, 385Sb, 419Xh, 443Xb, *482 notes*

measure space isomorphism 254K, 254Xj-254Xl, 255Ca, 255Mb, 344I, 344J, 344K, 344L, 344Xa-344Xc, 344Xe-344Xg, 344Yc, 433Xf, 433Yb

measure zero, cardinal of *see* measure-free cardinal (**438A**)

metacompact (topological) space 435Xj, 438J, 438Yd-438Yf, 451Ye, **>4A2A**, 4A2Fg; *see also* countably metacompact (**434Yf**), hereditarily metacompact (**4A2A**)

metric **2A3F**, 2A4Fb; *see also* entropy metric (**385Xq**), Euclidean metric (**2A3Fb**), Hamming metric (**492D**), Hausdorff metric (**246Yb**, **476A**), Huntingdon's metric **437Yi**, Lévy's metric (**274Ya**), measure metric (**323Ad**), pseudometric (**2A3F**), right-translation-invariant metric (**4A5Q**)

metric gauge *see* uniform metric gauge (**481E**)

metric outer measure 264Xb, **264Yc**, **471B**, 471C, 471Ya

metric space *224Ye*, *261Yi*, 316Yj, 3A4Fe, 3A4Ff, 3A4Hc, 434L, §471, 448Xi, 4A2Lg, 482E, *484Q*; *see also* complete metric space, isometry group (**441F**), metrizable space (**2A3Ff**)

metrically discrete *see* σ-metrically-discrete (**4A2A**)

metrically transitive transformation *see* ergodic inverse-measure-preserving function (**372Pb**)

metrizable (topological) space **2A3Ff**, 2A3L, 412E, 418B, 418F, 418G, 418Xc, *421Yb*, 423D, 434Ye, 437Yj, 437Yn, 438D-438G, 438I, 438Jc, 438Xk, 438Yc, 439Yd, 441Xm, 451Q-451S, 451Xm, 461K, 461N,

462Xa, *491Cg*, 491Xk, 4A2L, 4A2Nh, 4A3Kb, 4A4Cf; *see also* compact metrizable space, metric space, separable metrizable space, Polish space (**4A2A**)
—— linear topological space 462Yb, 466A, 466Xa, 466Xc, 466Xd, 4A3V; *see also* normed space
—— topological group 446Ec, 4A5Q-4A5S
—— uniformity **3A4Bc**, 445Ya
mid-convex function **233Ya**, 233Yd
minimal element in a partially ordered set **2A1Ab**
mixing inverse-measure-preserving function **372P**, 372Q, 372R, 372Xk, 372Xm-372Xo, 372Xq, 372Xr, 372Xu, 372Xv, 372Yh, 372Yl
mixing measure-preserving homomorphism 333P, 333Yc, **372P**, 372Q, 372R, 372Xk, 372Xl, 372Xp, 372Yl, 372Yn, 385Se, 387Xb, *388Yc*
mixture of distributions *459C*; *see also* disintegration (**452E**)
moderated set function **482Ib**, 482K, 482M, 482Xj, 482Xk, 482Yc
modified Dirichlet kernel **282Xc**
modular function (of a group carrying Haar measures) **442I**, 442J, 442K, 442Xf, 442Xh, 443G, 443Q-443S, 443Xo, 443Yo, *444J*, 444M, 444O, *444Xn*, *447E*, *447F*
modular functional on a lattice **413Xp**; *see also* supermodular (**413P**)
modulation **286C**
monocompact class of sets **413Yd**
Monotone Class Theorem 136B, 312Xc, 313G, 313Xd
Monotone Convergence Theorem *see* B.Levi's theorem (133A)
monotonic function 121D, 222A, 222C, 222Yb, *224D*, **313Xh**; (space of monotonic functions) *463Xg*, *463Xh*
Monte Carlo integration 273J, 273Ya, 465H, 465M
multilinear map **253Xc**
multiplicative identity (in a Banach algebra) 4A6Ab; (in an f-algebra) 353P, 353Yg
multiplicative linear functional on a Banach algebra 445H, 4A6F, 4A6J, 4A6K
multiplicative Riesz homomorphism 352Xm, 353Pd, 361J, 363F, 364R

Nachbin-Kelley theorem 363R, 363Yj
Nadkarni-Becker-Kechris theorem 448P
narrow topology (on a space of measures) **>437J**, 437K-437M, 437O, 437Q-437S, 437Ub, 437V, 437Xi-437Xk, 437Xo, 437Xs, 437Yk, 437Yl, 437Yn, 438Xl, 444Xc, 449Yc, 452Xb, 452Xe, 458Yf, 459F, 459G, 459Xc-459Xe, 495O, 495Xm, 495Xn
natural extension (of an inverse-measure-preserving function) *383Yc*
negligible set **112D**, 131Ca, 214Cb, 344H, 344Yf, 412Jb, 412Xi, 413Xh, 436Yb, 482L; *see also* Haar negligible (**442H**), Lebesgue negligible (**114E**, **115E**), null ideal (**112Db**), universally negligible (**439B**)
neighbourhood (of a point in a topological space) **4A2A**; *see also* base of neighbourhoods (**4A2A**)
neighbourhood gauge **481E**, *481J-481P*, *481Xh*, 482E-482I, 482Xc-482Xf, 482Xj, 482Xk, 482Ya-482Yc, 483A, 483Ya; *see also* uniform metric gauge (**481E**)
network in a topological space 438Ld, 466D, 467Pb, **4A2A**, 4A2Ba; *see also* countable network
Neumann, J. von 341 *notes*; von Neumann-Jankow selection theorem 423O, 423Xg, 423Yd, 424Xh, 433F, 433G
Nikodým *see* Radon-Nikodým
Nirenberg *see* Gagliardo-Nirenberg-Sobolev inequality (473H)
no small subgroups (in 'topological group with no small subgroups') **446E**, 446H-446K, 446M, 446O
non-decreasing sequence of sets **112Ce**
non-increasing sequence of sets **112Cf**
non-measurable set 134B, 134D, 134Xg, 419J
non-principal ultrafilter 464Ca, 464Jc
non-scattered compact set 494B, 494C, 494Xa-494Xc
non-stationary set **4A1C**
norm **2A4B**, 4A4Cc, 4A4Ia; (of a linear operator) **2A4F**, 2A4G, 2A4I; (of a matrix) **262H**, 262Ya; (norm topology) 242Xg, **2A4Bb**; *see also* Fatou norm (**354Da**), Kadec norm (**466C**), Riesz norm (**354Aa**)

normal density function 274A, 283N, 283We, 283Wf
normal distribution **274Ad**, 495Xg; *see also* standard normal (**274A**)
normal distribution function **274Aa**, 274F-274K, 274M, 274Xe, 274Xg
normal filter **4A1Ic**, 4A1J-4A1L
normal random variable **274A**, 274B, 285E, 285Xm, 285Xn, 455C, 456Ab, 456M
normal subgroup 382R, 382Xg, 382Xi, 382Yb, 383Ic, 383Xe, 383Yb; *see also* lattice of normal subgroups
—— —— of a topological group 443R, 443S, 443Xv-443Xx, *443Yc*, 443Yk, *449C*, *493Bc*, 4A5J-4A5L, 4A5S
normal subspace (of a Riesz space) *see* band (**352O**)
normal topological space 411Yc, 412Xg, 412Yd, *414Yg*, 415Yj, 418Xr, 435C, 435Xg, 435Xh, 438Jb, 438Yf, 476Cc, 476Yb, *491Cd*, 491Xs, **>4A2A**, 4A2F-4A2H, 4A2Rc; *see also* perfectly normal (**4A2A**)
—— *see also* canonical outward-normal function (**474G**), Federer exterior normal (**474O**), outward-normal function (**474F**)
normalized Haar measure **442Ie**
normalized Hamming metric **492D**
normalized Hausdorff measure 264 *notes*, §265 (**265A**), Chap. 47
normalizer of a subgroup **443Oc**
normed algebra **2A4J**, 3A5J, **4A6Aa**, 4A6D
normed space *224Yf*, §2A4 (**2A4Ba**), *451Xp*, *461F*, 462D, 466C-466E, §467, 4A4I; *see also* Banach space (**2A4D**)
Novikov's Separation Theorem *see* Second Separation Theorem
nowhere aperiodic Boolean homomorphism **381Xk**
nowhere dense set *313R*, *313Yb*, 315Yb, *316I*, *316Yc*, *316Yf*, *316Yh*, **3A3F**, 443N, 4A5Kb
nowhere rigid Boolean algebra **384H**, 384I-384L, 384Xa-384Xc
null ideal **112Db**; *see also* Lebesgue null ideal
null set *see* negligible (**112Da**), Haar null (**444Ye**)

odd function 255Xb, 283Yd
odometer transformation **388E**, 388Ye, 445Xp
Ogasawara's representation theorem for Riesz spaces 368 *notes*
one-parameter subgroup (in a topological group) 446I
one-point compactification *311Ya*, **3A3O**, *423Xi*, 435Xb, 462Xa, 467Xi
open interval *111Xb*, 114G, 115G, 1A1A, 2A2I, **4A2A**, 4A2Ra
open map 314Xj, **4A2A**, 4A2Bf, 4A5Ja
open set (in $\mathbb{R}^r$) **111Gc**, *111Yc*, *114Yd*, *115G*, *115Yb*, **133Xb**, 134Fa, 134Yj, **135Xa**, **1A2A**, 1A2B, 1A2D, **256Ye**, **2A3A**, 2A3G; (in $\mathbb{R}$) 111Gc, 111Ye, *114G*, 134Xc, 2A2I; *see also* topology (**2A3A**)
operation $\mathcal{A}$ *see* Souslin's operation (**421B**)
operator topology *see* strong operator topology (**3A5M**), very weak operator topology (**373K**)
optional time *see* stopping time (**275L**)
orbit of a bijection 382Xa, 388A, 388Xa, 388Xb, 381Qc, 388Ya
—— of a group action *441C*, *441Xc*, 441Yg
order (in a Boolean ring) **311H**
order-bounded dual Riesz space ($U^\sim$) §356 (**356Aa**), 362A, 363K, 365L, 365M, 366D, 366Ya, 368P, 371Xe, 375Yb, 436I, 437A
order-bounded set (in a partially ordered space) **2A1Ab**, 354Xd, 368Xe
order-bounded linear operator (between Riesz spaces) §355 (**355A**); *see also* order-bounded dual (**356A**), $\mathrm{L}^\sim$ (**355A**)
order-closed set in a partially ordered space **313Da**, 313Fa, 313Id, 313Xb, 313Xc, 314Ya, 316Fb, 323D, *352Oa*, 354Xp, 4A2Re; (ideal in a Boolean algebra) 313Eb, 313Pa, 313Qa, 313Xo, 314Xb, *314Yh*, *316Xh*, *316Xi*, *316Xn*, *316Xu*; (Riesz subspace of a Riesz space) 353J, 354Xp, 375C; (subalgebra of a Boolean algebra) 313E-313G, 314E-314H, 314Ja, 314Xg, 314Xh, *314Yc*, 315Xm, *322Md*, *322Xh*, 323H-323J, *323Yc*, 331E, 331G, 331Yb, 331Yc, *333Bc*, *393Xd*; *see also* sequentially order-closed (**313Db**)
order-complete *see* Dedekind complete (**241Ec**, **314Aa**)
order-continuous bidual Riesz space ($U^{\times\times}$) 356I, 356J, 369D

order-continuous Boolean homomorphism 313L-313N, 313Xl-313Xn, 313Xp, 313Xq, 313Yf, 314Fa, 314G, 314R, 314T, 314Xj, *315Jc*, *315P*, 315Ya, 316Fd, 316Xy, 322Yd, 324E-324G, 324K, 324Ya, *324Ye*, 324Yf, *325Aa*, 325C, 325D, 325H, *326Kf*, 331H, 331J, *332Xm*, *332Xn*, 334Xb, 343B, 344A, 344C, 344E, 344F, *352Xi*, *363Ff*, *364Rc*, 364Yh, *366Xf*, 369Xm, 369Xn, 373Bd, 373U, *375Ya*, 381Ec, 381F, 395Xa, 416W, 491N*see also* regular embedding (**313N**)

order-continuous dual Riesz space ($U^\times$) §356 (**356Ac**), 362A, 363K, *363S*, 365L, 365M, 366D, 366Ya, 366Yc, 367Xf, 368Pc, 368Yj, 369A, 369C, 369D, 369K, 369Q, 369Xa, 369Xe, 369Yh, 371Xe, 375B, *375J*, *375Yd*, 437A, *437C*, 438Xd, *464R*

order-continuous norm (on a Riesz space) **242Yc**, 242Ye, 244Yd, *313Yd*, 326Yj, **354Dc**, 354E, 354N, 354Xi, *354Xj*, 354Xl, 354Xm, 354Xo, 354Xp, 354Yc-354Yh, 355K, 355Yg, 356D, 356M, 356Yg, 356Yi, 365C, 366D, 367E, 367Xu, 367Ym, *367Yq*, 369B, 369Xf, 369Xg, 369Yc-369Yf, 371Xa-371Xc, 371Ya, 374Xe, 374Xg, 374Yb, 376L, 376M, 415Yl, 444Yj, 467N

order-continuous order-preserving function **313Ha**, 313I, 313Xi, 313Yc, *313Yd*, 315D, *315Yg*, 316Fc, *326Kc*, 361Cf, 361Gb, 363Eb, 363Ff, *366Yd*, 367Xb, 367Yb, 385Ya, 386Yb, 386Yc, 392I, 392Xb, 394N, 4A2Ro; *see also* sequentially order-continuous

order-continuous positive linear operator 351Ga, 355G, 355H, 355K, 364Rc, 365P, 366H, 368I, 375Xa, 375Xb, 375Yd; *see also* order-continuous dual (**356A**), order-continuous Riesz homomorphism, $\mathrm{L}^\times$ (**355G**)

order-continuous Riesz homomorphism 327Yb, 351Xc, 352N, 352Oe, 352Rb, 352Ub, 352Xe, *353O*, 353Xc, 356I, 361Je, 355F, 365O, 366H, 367Xh, 368B, 375C, 375Xc, *375Ya*

order-continuous ring homomorphism (between Boolean rings) 361A, 361Je, 365P

order*-convergent sequence in a partially ordered set 245Xc, 356Xd, **367A**, 367B-367K, 367Xa-367Xn, 367Xs-367Xu, 367Ya-367Yk, 367Yq, 368Yg, 368Yh, 369Xq, 376G, 376H, 376Xe, 376Xg, 376Ye, 376Yh, 393Xa, 393Yd, 393Ye; (in $C(X)$) 367L, 367Yh-367Yj; (in $L^0(\mu)$) **245C**, 245K, 245L, 245Xc, 245Xd, 376J; (in $L^0(\mathfrak{A})$) 367Na, 367Q, 367Xq, 367Xt, 367Yg, 367Yp, 368Yi, 372D-372G, 372Xb, 386E, 386F

order-convex set in a partially ordered set **351Xb**; *see also* interval (**4A2A**)

order-dense set in a Boolean algebra **313J**, 313K, 313Xm, *314Yg*, 332A, 412N, 4A3Rc, 4A3S

order-dense Riesz subspace of a Riesz space **352N**, 353A, 353D, 354Ef, 354I, 354Ya, 355F, 355J, *356I*, 363C, 363Xd, 364L, 365F, 365G, 367Ob, 367Ye, 368B-368E, 368G-368I, 368M, 368P, 368S, 368Ya, 368Ye, 369A, 369B, 369C, 369D, 369G, 375D, 375Xc; *see also* quasi-order-dense (**352Na**)

order-dense subalgebra of a Boolean algebra 313O, *313Xj*, 313Xn, *313Ye*, 314I, 314T, 314Xf, 316Xf, 316Xo, 316Xr, *316Yk*, *316Yn*, 323Dc, *363Xd*, *391Xj*

order-isomorphism *312L*

order*-limit *see* order*-convergent (**367A**)

order-preserving function **313H**, 313I, *313La*, 315D, 324Yg, *326Bf*, 361Ce

order topology (on a partially ordered set) 313Xb, 313Xj, 313Yc, 326Yr, 367Yk, 393Xa, 393Yd; (on a totally ordered set) 434Yl, 438Yj, **4A2A**, 4A2R, 4A2S

order unit (in a Riesz space) 243C, **353L**, 353M, 363N, 368Ya; *see also* order-unit norm (**354Ga**), standard order unit (**354Gc**), weak order unit (**353L**)

order-unit norm 354F, 354G (**354Ga**), 354I-354K, 354Yi, 353Yf, 355Xc, 356N, 356O; *see also* M-space (**354Gb**)

ordered set *see* partially ordered set (**2A1Aa**), totally ordered set (**2A1Ac**), well-ordered set (**2A1Ae**)

ordinal **2A1C**, 2A1D-2A1F, 2A1K

ordinate set **252N**, 252Yg, 252Yh

Orlicz norm **369Xd**, 369Xi, 369Xj, 369Xn, 369Yc, 369Yd, 369Ye, 369Yg, 374Xc

Orlicz space **369Xd**, 369Xi, 369Xj, 373Xm

Ornstein's theorem 387I, 387K

orthogonal complement 4A4Je

orthogonal group on $\mathbb{R}^r$ 441Xk, 441Yi, 445Yd

orthogonal matrix **2A6B**, 2A6C

orthogonal projection in Hilbert space 244Nb, 244Yj, 244Yk, 366K

—— in $\mathbb{R}^r$ *475H*, *475P*, *475Q*, *475S*

orthonormal basis 416Yg, 444Yk, 456Yb, **4A4Ja**

orthonormal vectors **2A6B**, 4A4Jg

oscillation of a function **483O**, 483P, 483Q
Ostaszewski's ♣ 439O, **4A1M**, 4A1N
outer automorphism of a group *384G*, *384O*, 384Pb, 384Q, 384Yc, 384Yd, **3A6B**
outer measure §113 (**113A**), 114Xd, **132B**, *132Xg*, 136Ya, 212Ea, *212Xa*, 212Xb, *212Xg*, 213C, 213Xa, 213Xg, 213Xk, 213Ya, 251B, 251Wa, *251Xd*, 254B, *264B*, *264Xa*, *264Ya*, *264Yo*, 393T, 393Xh, 393Yh, 413Xd, 452Xi, *463F*, *463Xa*; *see also* Lebesgue outer measure (**114C**, **115C**), metric outer measure (**264Yc**, **471B**), regular outer measure (**132Xa**), submeasure (**392A**)
—— —— defined from a measure 113Ya, §132 (**132B**), 213C, 213F, 213Xa, 213Xg-213Xj, 213Xk, 213Yd, 214Cd, *215Yc*, 235Ya, 251O, 251R, 251Wk, 251Wm, 251Xm, 251Xo, 252Yh, *254G*, 254L, 254S, 254Xb, 254Xq, 254Yd, 264Fb, 264Ye, 412Ib, 412Jc, 413E-413G, 413Xd, 417Xj, 431E, 431Xd, 432Xf, 471Dc, 471Yb, 451Oc, 451Xk, *464C*, *464D*
outer regular measure 256Xi, **411D**, 411O, 411Pe, 411Xa, 411Yb, 412W, 412Xo-412Xq, 412Yf, 415Xh, 415Yc, *416Yd*, *419C*, 432Xb, 482F, 482Xd, 482Xh
outward-normal function 474E, **474F**, 474H, 474I, 474K, 474M, 474Ya; *see also* canonical outward-normal function (**474G**)

paracompact (topological) space 415Xo, **>4A2A**, 4A2F, 4A2Hb, 4A2Lb; *see also* countably paracompact (**4A2A**)
Parseval's identity **284Qd**
partial derivative *123D*, *252Yj*, 262I, 262J, 262Xh, 262Yb, 262Yc
partial lower density on a measure space **341Dc**, 341N, 447Ab, 447B, 447F-447H
partial order 441Yj; *see also* partially ordered set (**2A1Aa**)
partially ordered linear space **241E**, 241Yg, 326C, §351 (**351A**), 355Xa, 361C, 361G, 362Aa; *see also* Riesz space (**352A**)
partially ordered set **2A1Aa**, 313D, 313F, 313H, 313I, 313Xb, 313Xg, 313Xh, 313Yc, 315C, 315D, 315Xb, *315Yg*, *466G*
partition **1A1J**
partition of unity in a Boolean ring or algebra **311G**, 313K, *313L*, 313Xk, 315E, 315F, 315Xk, 315Xq, *316Xp*, 322E, 332E, 332I, 332Xi, *332Yb*, *352T*, *375H*, *375I*, *381C*, *381D*, *381H*, *381Ia*, *381N*, *381Xf*, *381Xg*, *382Fb*, *383F*, 385C, 385D, 385G-385P, 385R, 385Xa-385Xe, 385Xi, 385Xq, 385Xr, 385Ya-385Yc, 385Yg, *386E*, 386I, 386J, 386L, 386M, *386O*, 386Xd, 386Xe, 388Ib, 392Xg; *see also* Bernoulli partition (**387A**)
pathological submeasure **393Tc**
Peano curve 134Yl-134Yo
perfect measure (space) **342K**, 342L, 342M, 342Xh-342Xo, 343K-343M, 343Xg, 343Xh, 344C, 344I, 344Xb-344Xd, 344Yf, 385V, 416W, **451Ad**, 451C-451G, 451I-451O, 451Xa, 451Xc, 451Xe, 451Xf, 451Xi, 451Xl, 451Yj, 454Ab, 454C-454E, 454G, 454J, 457F, 463F-463I, 463Xl, 494Ya, 495F, 495Xe; *see also* perfect measure property (**454Xd**)
perfect measure property 454I, **454Xd**, 454Xe, 454Xh, 454Yb, 454Yc, 455Ba
perfect Riesz space **356J**, 356K-356M, 356P, 356Xg, 356Xi-356Xk, 356Ye, 356Yg, 365C, 365N, 366D, 369D, 369K, 369Q, 369Ya, 374M, 374Xk, 374Xl, 374Ya, 376P, 376Xl, 437Yj
perfectly normal topological space 412E, 422Xe, 434Xj, 434Yj, 438Jc, **>4A2A**, 4A2Fi, 4A2Hc, 4A2Lc, 4A2Rn, 4A3Kb, 4A3Xd
perimeter **474D**, 475Mb, 475Xj, 475Xk, 475Yc, 484B, 484C, 484Ya, 484Yb; *see also* Cauchy's Perimeter Theorem (475S), locally finite perimeter (**474D**)
perimeter measure 474E-474R (**474F**), 474Xa, 474Ya, 475E-475G
periodic Boolean automorphism **381Ac**, 381H, 381R, 381Xj, 381Xl, 381Xm, 382F, 493Yc
periodic extension of a function on $]-\pi,\pi]$ **282Ae**
permutation group *see* symmetric group
Perron integral 483J
Pettis integrable function **463Ya**, 463Yb, 463Yc
Pettis integral **463Ya**, 464Yb
Pfeffer integrable function **484G**, 484H-484J, 484L, 484O, 484Xe
Pfeffer integral **484G**, 484H, 484N, 484S, 484Xd
Pfeffer's Divergence Theorem 484N

Plancherel Theorem (on Fourier series and transforms of square-integrable functions) 282K, 284O, 284Qd, 445R

Poincaré's inequality 473K

point process *see* Poisson point process (**495E**)

point-countable family of sets **4A2A**, 4A2Dc

point-finite family of sets 438B, 438Xj, 438Ya, 451Yd, **4A2A**, 4A2Fd

point-supported measure **112Bd**, 112Xg, **211K**, 211O, 211Qb, *211Rc*, 211Xb, 211Xf, 213Xo, 234Xc, 215Xr, 256Hb, 343Xi, 416Xb, 419G, 465Yd, 491D, 491Yg

pointwise compact set of functions 462E, 462F, 462J, 462Xd, 462Ya, 463C, 463D, 463F, 463G, *463K*, 463N, 463Xb-463Xe, *463Xg*, 463Xi, 463Xj, 463Xl-463Xn, 463Ya, 463Yc-463Ye, 463Za, 463Zb, 464E, 464Yb, 465Db, *465V*, 465Xg, 465Xj, 465Xm *see also* Eberlein compactum (**467O**)

pointwise convergence (topology on a space of functions) 281Yf, 437Xl, 438P, 438Xp, 438Yh, 438Yi, 443Yn, 454Pa, **462Ab**, 462B, 462E-462H, 462J, 462Xd, 462Ya, 462Yd, 462Z, §463, 464E, 464Yb, 465Cb, 465G, 465Xi, 465Xj, 467Pa, 4A2Nj; *see also* pointwise compact

—- (on an isometry group) 441G, 441Xm-441Xo, 441Ye, 441Yj, 443Xx, 443Ym, 448Xk, 449Xc, 449Xd, 476Xi, 493G, 493J, 493Xc, 493Xf, 493Xg, 493Yb

pointwise convergent (sequence or filter) **462Ab**, 462Xb; *see also* order*-convergent (**245Cb**, **367A**)

pointwise topology *see* pointwise convergence

Poisson distribution 285Q, 285Xo, 455Xc, 455Ye, 495A, 495B, 495D, *495M*, 495Xf; *see also* compound Poisson distribution

Poisson point process **495E**, 495F-495I, 495L-495N, 495P, 495Xa-495Xe, 495Xi-495Xk, 495Xn, 495Xp, 495Yc

Poisson's theorem 285Q

—— *see also* Abel-Poisson kernel (**282Yg**)

Pol, R. 463 *notes*

polar coordinates 263G, 263Xf

polar set **4A4Bh**, 4A4Ch, 4A4Eg

Polish group 424H, 441Xm, 441Xn, 441Yf, 444Ye, 448P, 448Xg, 448Xj, 448Xk, *449G*, 493Xe, 493Xg, **4A5Db**, *4A5Q*

Polish (topological) space 284Ye, 418C, 423Ba, 423I, 423Xa, 424F, 424H, 424Ya, 424Yb, 433Yc, 434Kb, 437Rb, 437Wg, 437Yo, 438Yh, 441Xm, 454B, *454O*, 454Xl, 454Xm, *461Xi*, **4A2A**, 4A2Q, 4A2Ud, 4A3H, 4A3I; *see also* standard Borel space (**424A**)

Pólya's urn scheme 275Xc

polynomial (on $\mathbb{R}^r$) 252Yu

polytope *see* convex polytope (**481O**)

Pontryagin Duality Theorem *see* Duality Theorem (445U)

porous set **223Ye**, **261Yg**, 262L

positive cone 253G, 253Xi, 253Yd, **351C**

positive definite function 283Xt, 285Xr, **445L**, 445M, 445N, 445P, 445Xe-445Xh, 445Xo

positive linear functional (on a Riesz space) 436D, 436H, 436J, 436K, 436Xe, 436Xp, 436Yc

positive linear operator (between partially ordered linear spaces) **351F**, 355B, 355E, 355K, 355Xa, 361G, 367P, 375I-375K, 375Xa, 375Xe, 375Xg, 376Cc, *452H*, *481C*, 495K, 495Lb, 495Xh, 495Yb; *see also* (sequentially) order-continuous positive linear operator, Riesz homomorphism (**351H**)

positive measure everywhere *see* self-supporting (**411Na**)

predictable sequence **276Ec**

pre-Radon (topological) space **434Gc**, 434J, 434Ka, 434Xn, *434Xo*, 434Xs, 434Yd, 435Xk, *439Xl*, 462Z, 466B, 466Xf; *see also* Radon space (**434C**)

presque partout **112De**

Pressing-Down Lemma **4A1Cc**

primitive product measure **251C**, 251E, 251F, 251H, 251K, 251Wa, 251Xa-251Xc, 251Xe, 251Xf, 251Xj, 251Xl-251Xp, 252Yc, 252Yd, 252Yg, 253Ya-253Yc, 253Yg, 325Ya, 412Ye

principal band (in a Riesz space) 352V, 353C, 353H, 353Xb, 362Yd, 437Yj

principal ideal in a Boolean ring or algebra **312D**, 312E, 312J, 312S, 312Xf, 312Yf, 314E, 314Xb, 314Xi, 315E, 315Xl, *316Xb*, *316Xk*, *316Xv*, *321Xa*, 322H-322J, 322Xf, 323Xe, 325Xb, *331Fb*, 331H, *331M*, 332A, *332L*, *332P*, 332Xh, *352Sc*, *352Xg*, 364Xo, 384Lb, 391Xe, 391Xj, 491Xq

principal projection property (of a Riesz space) **353Xb**, 353Yd, 354Yi

principal ultrafilter **2A1N**

probability algebra **322Aa**, 322Ba, 322Ca, 322G, *322Mb*, §385, 386E, 386H-386J, 386L-386O, 386Xe, 386Yb, 386Yc, §*387*, 391B, 491Ke, 491P

probability algebra free product 325F, 325G, 325I-325M (**325K**) 325Xd, 334D, 334Xc, 372Yl, 385Xm, *388Yc*, 458Xe

probability density function *see* density (**271H**)

probability space **211B**, 211L, 211Q, *211Xb*, *211Xc*, *211Xd*, *212G*, *213Ha*, 215B, *243Xi*, 253H, 253Xh, §254, Chap. 27, *322Ba*, *441Xe*, *444Xu*

process *see* Gaussian process (**456N**), Markov process, Poisson point process (**495E**)

product Boolean algebra *see* simple product (**315A**), free product (**315H**)

product f-algebra **352Wc**, 364S

product linear topological space 467Xc, 4A4B

product measure Chap. 25, 434R, 434Xv, 434Xw, 434Xx, 434Yk, 436F, 436Xh, 436Yd, 436Ye, 438S, 439L, 439R; *see also* c.l.d. product measure (**251F**, **251W**), primitive product measure (**251C**), product probability measure (**254C**), quasi-Radon product measure (**417R**), Radon product measure (**417R**), τ-additive product measure (**417G**)

product partial order **315C**, 315D, 315Xb

product partially ordered linear space **351L**, 351Rd, 351Xc, 351Xd; *see also* product Riesz space

product probability measure §254 (**254C**), 272G, 272J, 272M, 275J, 275Yi, 275Yj, *281Yk*, 325I, 334C, 334E, 334Xf, 334Xg, 334Ya, 342Gf, 342Xn, 343H, §346, 372Xf, *385S*, 412T-412V 413Yb, 415E, 415F, 415Xj, 416U, 417E, 417Xt, 417Xw, 417Yf, 417Yg, 433H, 443Xq, 451J, 451K, 451Yo, 453I, 453J, 454Xj, 456Aa, 456B, 456Xb, 458Xk, 459E, 465H-465M, 491Eb, 491Xu, 494Xc, 494Ya, 495P, 495Xb, 495Xj, 495Xp; *see also* quasi-Radon product measure (**417R**), Radon product measure (**417R**), relative product measure (**458Lb**), τ-additive product measure (**417G**)

product Riesz space 352K, 352T, 354Xb

product ring **3A2H**

product tagged-partition structure 481P, 481Yb

product topological group 442Xb, 442Xh, 443Xb, 443Xd, 443Xq, 443Xt, 445Bd, 445Xl, 445Yb, 449Ce, 493Bd, 4A5G, 4A5Od

product topology 281Yc, **2A3T**, 315Xf, 315Yb, *315Yd*, *323L*, 391Yc, **3A3I**, 3A3J, 3A3K, 417Xu, 418B, 418Dd, 418Xb, 418Xd, 418Yb, 419Xg, *422Dd*, 422Ge, 423Bc, 434Jh, 434Pb, 434Xc, 434Xe, 434Xr-434Xu, 435Xk, 436Xg, 437Wd, 437Yo, *438E*, 438Xm, 438Xn, 439Xg, *439Yg*, *463J*, *463K*, 467Xa, 4A2B, 4A2Cb, 4A2E, 4A2Fh, 4A2H, 4A2L, 4A2O-4A2R, 4A3C-4A3E, 4A3G, 4A3M-4A3O, 4A3Xa, 4A3Yb

product uniformity **3A4E**

—— *see also* inner product space

projection (on a subalgebra of a Boolean algebra) *see* upper envelope (**314Va**); (on a Riesz space) *see* band projection (**352Rb**)

projection band (in a Riesz space) **352R**, 352S, 352T, *352Xh*, 353E, 353H, 353I, 353Xb, 361Xh, 361Ye, *362B*

projection band algebra 352S, 352T, 361K, 363J, 363N, 363O, 364Q

projection pair (of linear subspaces) **467Hc**, 467J

projectional resolution of the identity (on a Banach space) **467G**, *467K*

projective system of probability spaces 418M-418Q, 418Xu, 418Xv, 418Yk, 418Yl, 433K, 433Yd, 451Yb, 451Yc

Prokhorov space **437V**, 437W, 437Xs, 437Xt, 437Yn, 437Yo, 439S, 439Yi, 452Xe; *see also* strongly Prokhorov (**437Yo**)

Prokhorov's theorem 418M, 418N, 418P, 418Q, 433K, 433Yd

pseudo-cycle (for a Boolean automorphism) 388Xc

pseudometric **2A3F**, 2A3G, 2A3H, 2A3I, 2A3J, 2A3K, 2A3L, 2A3Mc, 2A3S, 2A3T, 2A3Ub, *2A5B*, *323A*, 3A3Be, 3A4B, 3A4D-3A4F, 458Yf, 4A2Ja

pseudometrizable topology 463A, **4A2A**, 4A2L
pseudo-simple function **122Ye**, 133Ye
pull-back measures 132G, 418L, 432G, 433D
purely atomic Boolean algebra **316Lc**, 316M, 316Xr-316Xx, 316Yk, 316Yl, 316Yq, *322Bh*, *322Kc*, 324Kg, 331Yd, *332Xb*, 332Xc, *362Xf*, *385J*, *385Xq*, 394Xf
purely atomic measure (space) **211K**, 211N, 211R, *211Xb, 211Xc, 211Xd*, *212G*, 213He, 214Xe, *234Xb*, 251Xr, *322Bh*, 342Xn, 343Xi, 375Ya, 416Xb, *438Ce*, 451Xa, 451Xo, 451Yg
purely infinite measurable set **213 notes**
purely non-measurable additive functional **464I**, 464Jc, 464L, 464M, 464P, 464Xa
push-forward measure *see* image measure (**112F**)

quadratic map **372Xn**
quasi-dyadic space **434O**, 434P, 434Q, 434Yh, *434Yi*, 434Yj; *see also* dyadic (**434** *notes*)
quasi-homogeneous measure algebra **374G**, 374H-374M, 374Xl, 374Ye, 394Xg
quasi-order-dense Riesz subspace (of a Riesz space) **352N**, 353A, 353K, 353Ya
quasi-Radon measure (space) 256Ya, 263Ya, **411Ha**, 411Pd, 411Yc, 414Xj, §415, 416A, 416C, 416G, 416R, 416T, 416Xu, 418Hb, 418K, 418Xi, 418Xn, 432E, *434A*, 434H-434J, 434L, *435D*, 436H, 436Xl, 437N, 471Dh, 438G, 439Xh, 441B, 441Xc, §444, 445D, *445Ec*, *445M*, *445Xc*, *445Yf*, 451T, 453Ga, 453J, 453Xd, 453Xg, 453Xh, 454Xj, 454Xk, 456Aa, 456L, 456Xf, 459E, 466A, 466B, 466K, 466Xa-466Xd, 466Xj, 482Xk, 491Mb, 491Xk, 491Xv; *see also* Haar measure (**441D**), Radon measure (**411Hb**)
—— product measure (of finitely many spaces) 417N, **417R**, 417Xv, 418Xf, 418Xh, 436Xh, 441Xi, 443Xb, 443Xp, *444A*, 444N, 444O, 444Ya, 444Yl; (of infinitely many probability spaces) 417O, **417R**, 417Xq, 417Xw, 417Yh, 417Yi, 433H, 441Xi, 443Xq, 453J, 456Xf, 458Xk, 459G; *see also* Radon product measure (**417R**), τ-additive product measure (**417G**)
quasi-simple function **122Yd**, 133Yd
quasi-Stonian topological space **314Yf**
quotient Banach algebra 4A6E
quotient Boolean algebra (or ring) 312K, 312Xk, 313Q, 313Xo, 314C, 314D, 314M, 314N, *314Yd*, 316C, 316D, 316Xh, 316Xi, 316Xn, 316Xu, *316Ye*, 316Yp, 341Xf, 361M, 363Gb, 391Xe, 391Yb, 392Xd, 424Xk, 491I
quotient partially ordered linear space 241Yg, **351J**, 351K, 351Xb; *see also* quotient Riesz space
quotient Riesz space 241Yg, 241Yh, 242Yg, 352J, 352U, 354Yi, 361M, 364C, 364U
quotient ring **3A2F**, 3A2G
quotient topological group 443S, 443Xw-443Xy, 443Yk, 446Ab, 446O, 446P, 447E, 447F, 449Cc, 493Bc, 4A5J-4A5L, 4A5S
quotient topology *245Ba*, *443O-443R*, *443Xs*, *443Yl*

Rademacher's theorem 262Q
Radon, J. 416 *notes*
Radon measure **>411Hb**, 414Xj, §416, 418I, 418L, 418M, 418P, 418Q, 418Xo, 418Xp, 418Xs, 432C-432H, 432Xe, 433Cb, 433D, 433Ha, 433I, 434Ea, 434Fa, 434J, 434Ya, 435B, 435Xa-435Xd, *435Xk*, *435Yb*, 436J, 436K, 436Xn, 436Xp, 437N-437P, 437S, 471F, 471Q, 439Xh, 441Bb, 441E, 441H, 441L, 444Xa, *444Xl*, *444Ye*, *445N*, *445Xb*, 451Ab, 451N, 451Rb, 451S, 451Xk, 451Xn-451Xp, 453Gb, 453K, 453Xf, *453Za*, *453Zb*, 454J, *454K*, 454O, 454Pb, 455D, 455Xe, 455Ye, 456Aa, 457Xf, 458O, 458Xr, 458Yf, 459Xc-459Xe, 462G-462I, 463J, 463K, 463Xk, 466A, 466Xk, 466Xl, 466Ya, 482Xc, 482Xe, 482Xg, 491Ma, 491Xz, 491Yg, 494B, 494C, 494Xa-494Xc, 495N(on $\mathbb{R}$ or $\mathbb{R}^r$) §256 (**256A**), *284R*, 284Yi, 342Jb, 342Xj, 411O, 419C, 419E, 452D, 452O, 452P, 452Xe, 472C-472F, 483Yc; (on $]-\pi,\pi]$) **257Yb**; *see also* complex Radon measure, quasi-Radon measure (**411Ha**), tight Borel measure
Radon-Nikodým derivative **232Hf**, 232Yj, **234B**, 234Ca, 234Yd, *234Yf*, 235Xi, 256J, 257F, 257Xe, 257Xf, 272Xe, 272Yc, 275Ya, 275Yi, 285Dd, 285Xe, 285Ya, *363S*, 452Qa, 458Lb, 458P, 472Ye
Radon-Nikodým theorem 232E-232G, 234G, 235Xk, 242I, *244Yk*, 327D, 365E, 365Xf
Radon probability measure (on $\mathbb{R}$ or $\mathbb{R}^r$) 271B, 271C, 271Xb, 285Aa, 285M, 364Xd, 372Xf, 415Fb; (on other spaces) **256Ye**, 271Ya, 416U, 417Q, 418M, 461H, 461J, 461K, 461N, *461Xc*, 461Xj, *461Yb*

Radon product measure (of finitely many spaces) 256K, 417P, **417R**, 417Xi, 417Xo, 419E, 419Yb, 436Xo, 436Yf, 453Za; (of infinitely many probability spaces) 416U, 417Q, **417R**, 417Xp, 417Xr, *418Na*, 433Ha, 453J, 459Xc-459Xe, 461Ye, 494C, 494Xd

Radon (topological) space **434C**, 434F, 434K, 434Nd, 434Xh, *434Xm*, 434Xp-434Xu, 434Yc, 434Za, 434Zb, 438H, 438R, 438Xn, 438Xp, 438Yi, 439Ca, *439K*, *439P*, 439Xb, 454Pa, 454Xh, 454Xm, 455Ye, 466F, *466H*, 466Xe, 467Pb; *see also* pre-Radon (**434Gc**)

Ramsey's theorem 4A1G, *4A1L*

random variable **271Aa**

rank (of a tree) **421N**, 421O, 421Q, 423Yh

rapidly decreasing test function §284 (**284A**, **284Wa**), 285Dc, 285Xd, 285Ya

rare *see* nowhere dense (**3A3Fa**)

realcompact (topological) space **436Xg**, 439Xm

rearrangement 373Ya; *see also* decreasing rearrangement (**252Yp**, **373C**)

rearrangement-invariant extended Fatou norm **374Eb**, 374F, 374K, *441Yk*

rearrangement-invariant set **374Ea**, 374F, 374M, 374Xk, 374Xl, 374Ye

rectifiable *475 notes*

Recurrence Theorem (for a measure-preserving Boolean homomorphism) 386Xa, 386Xb, 386A

recurrent Boolean homomorphism **381Ag**, 381L, 381O, 381P, 381Q, 381Xh, 381Xi

recursion 2A1B

reduced boundary **474G**, 474N, 474Q, 474R, 474Xb, 474Xc, 474Xd, 475D, 475E, 475Xi; *see also* essential boundary (**475B**)

reduced power of $\mathbb{R}$ **351M**, 351Q, 351Yd, 352L, 368F

reduced product 413Xn

refinable *see* hereditarily weakly θ-refinable (**438K**)

refine **311Gd**, **>4A2A**

refinement **>4A2A**

refinement property *see* σ-refinement property (**448K**)

reflexive Banach space 372A, **3A5G**, 461G, 467Xf

regular cardinal **4A1Aa**, 4A1Bd, 4A1Cc, 4A1I-4A1L

regular conditional probability *see* disintegration (**452E**)

regular embedding of Boolean algebras **313N**

regular measure *see* completion regular (**411Jb**), effectively regular (**491L**), inner regular (**256Ac**, **411B**), outer regular (**411D**)

regular open algebra (of a topological space) 314P-314S (**314Q**), 314Xi, 314Xj, 314Yd, 315Xe, 315Yd, 316Xe, 316Yb, 316Yd, 316Yf, 316Yj, 316Yq, 332Xe, 363Yc, 364U-364W, 364Yh-364Yl, 368Ya, 391Yc, 391Yf, 417Xu; (of $\mathbb{R}$) 316K, 316Yb, 316Yn, 316Yo, 331Xh, 331Yi, 367Yp, 375Xe, 375Yf, 375Z, 376Yi, 391Xd, 392Yb

regular open set **314O**, 314P, 314Q, 417Xu, 443N, *443Yi*, 4A2Eb, 4A3Ra, 4A5Kb

regular operator (between Riesz spaces) 355 *notes*

regular outer measure 132C, **132Xa**, *213C*, 214Hb, 251Xm, 254Xb, 264Fb, 471Dc, 471Yb

regular Souslin scheme **421Xn**, 431Xc

regular topological space **2A5J**, *316Yf*, **>3A3Ab**, 3A3Ba, 3A3De, 414M, 414Xj, 414Ya, 415C, 415D, 415J, 415Qf, 415Rc, *416K*, *418H*, *418Xn*, 422E, 422K, *422Xe*, 434Ib, 434Jc, *462Aa*, *462C*, 476C, 4A2F, 4A2H, 4A2Ja, 4A2N, 4A2Pb; *see also* completely regular (**3A3Ac**)

regularly embedded (subalgebra of a Boolean algebra) **313N**, 313O, 313P, 316Xm, 316Xs, *326Kf*, 367Xi, 491Ke, 491P; (Riesz subspace) **352Ne**, 352Xe, 354Xk, 354Xm, 367F, 368Pa, 368S

regularly enveloped set **491L**

relation 1A1B

relative atom in a Boolean algebra **331A**

relative distribution **458C**, 458D, 458E, 459C, 458Xb, 458Ya, 458Yc, 459C

relative free product of probability algebras **458I**, 458J, 458K, 458M, 458Xe

relative Maharam type of a Boolean algebra over a subalgebra **333A**, 333B, 333C, 333E, 333F, 333Yb

relative product measure **>458Lb**, 458M-458P, 458Xg-458Xr, 458Ye

relatively atomless (Boolean algebra) **331A**, 381P, 386C, 388I, 394Xc
relatively compact set **2A3Na**, 2A3Ob, 3A3De, 3A5M, 4A2Le
relatively countably compact set (in a topological space) *462Aa*, 462Yc, 463I, 463Xf, **4A2A**, 4A2Gf, 4A2Le
relatively independent family (in a measure algebra) **458H**; (of measurable sets) **458Aa**
relatively independent closed subalgebras **458H**, 458Xd
relatively independent random variables **458A**, 458E, 459C, 458Xt, 458Ya, 458Yc, 458Yd, 459C
relatively independent σ-algebras **458A**, 458B, 458F, 458G, *458Xa*, 458Xc, 458Xj, 458Yb, 459B, 459D
relatively invariant measure *441 notes*
relatively Maharam-type-homogeneous **333Ac**, 333Bb
relatively measurable set **121A**
relatively von Neumann transformation **388Da**, 388K, 388Xg, 388Yb, 388Yc, 388Yf
relatively weakly compact set (in a normed space) 247C, **2A5I**, 356Q, 356Xl, *3A5Gb*, 3A5Hb, *3A5Kb*; (in other linear spaces) 376O, 376P, 376Xl
repeated integral §252 (**252A**), 434R, 436F, 436Xo; *see also* Fubini's theorem, Tonelli's theorem
representation (of a group) *see* finite-dimensional representation (**446A**), action (**441A**)
representation of homomorphisms (between Boolean algebras) 344Ya, 344Yd, 364Xr, 424Xk; (between measure algebras) 324A, 324B, 343A, 343B, 343G, 343J, 343M, 343Xc, 343Xf, 343Yd, 344A-344C, 344E-344G, 344Xf, 344Yc, 416W, *451Ab*
residual family **481F**, 481G, 481H, 481K, 481L, 481N, 481O, *481Q*, 481Xb, 481Xc, 481Xf, 481Yb, 484D
resolution of the identity 4A2Fd; *see also* projectional resolution of the identity (**467G**)
'respects coordinates' (said of a lifting) **346A**, 346C, 346E, 346Xg, 346Yc, 346Za
reverse martingale 275K
Riemann-complete integral *see* Henstock integral, symmetric Riemann-complete integral
Riemann integrable function **134K**, 134L, 281Yh, 281Yi, 491Xw
Riemann integral **134K**, 242 *notes*, 363Yi, 436Xm, *481E*, 481I, 481Xe, 491Xj
Riemann-Lebesgue lemma 282E, 445Ka
Riemann sum **481A**
Riesz Convexity Theorem 244 *notes*
Riesz homomorphism (between partially ordered linear spaces) **351H**, 351J, 351L, 351Q, 351Xc, 351Ya, 352G; (between Riesz spaces) 352G-352J, 352U, *352W*, 352Xb, 352Xe, 353Pd, 353Yf, 355Xe, 356Xh, 361Gc, 361J, *361Xg*, 362Xe, 363Ec, 363F, 363Xb, 363Xc, 364R, 365K, 375I, 375J, 375Xg, 375Ya, 376Cc, 436Xg; *see also* order-continuous Riesz homomorphism
Riesz norm **242Xg**, **354A**, 354B, 354D, 354F, 354M, 354Xc-354Xf, 354Xh, 354Yb, 354Yf, 354Yk, 355Xc, 356D, 356Xg, 356Xh, *438Xi*, 466H, 467Yb; *see also* Fatou norm (**354Da**), order-continuous norm (**354Dc**), order-unit norm (**354Ga**)
Riesz Representation Theorem (for positive linear functionals) 436J, 436K
Riesz space (= vector lattice) 231Yc, **241Ed**, 241F, 241Yc, 241Yg, chap. 35 (**352A**), 361Gc, 367C, 367E, 367Xc, 367Yn; *see also* Archimedean Riesz space (§352), Banach lattice (**354Ab**), Riesz norm (**354A**)
Riesz subspace (of a partially ordered linear space) **352I**; (of a Riesz space) 352I, 352J, 352L, 353A, *354O*, *354Rc*, 4A2Jg; *see also* band (**352O**), (quasi-)order-dense Riesz subspace (**352N**), solid linear subspace (**351I**)
right action (of a group on itself) **4A5Ca**, 4A5I
right Haar measure **441D**, 441Xe, 442C, 442E, 442F, 442H, 442I, 442L, 442Xd, 442Xf, 442Xg, 443Xh, 443Xi, 444Yn
right modular function (on a topological group) **442Ib**
right-translation-invariant lifting *447Xa*, *447Ya*
right-translation-invariant metric 441Xn, **4A5Q**
right uniformity (of a topological group) 449B, 449D, 449E, 449H, 449Yc, **>4A5Ha**, 4A5Mb, *4A5Q*
rigid Boolean algebra **384Ha**, *384L*; *see also* nowhere rigid (**384Hb**)
ring §3A2 (**3A2A**); *see also* Boolean ring (**311Aa**)
ring homomorphism **3A2D**, 3A2F-3A2H
ring homomorphism between Boolean rings 311D, *312Xf*, 312Xg, 312Xh, 312Yc, 312Yd, 312Ye, 312Yf, 361A, 361Cc, 361J, 361Xe, 361Xg, 375H

Rokhlin *see* Halmos-Rokhlin-Kakutani lemma (386C)
root (of a Δ-system) **4A1Da**
root algebra (of a Bernoulli shift) **385Q**, 385R, 385S, 387B, 387Ya
rotund *see* locally uniformly rotund (**467A**)

Saks *see* Vitali-Hahn-Saks theorem (246Yg)
Saks-Henstock indefinite integral 482B, **482C**, 482D, 482G, 482Xa, 482Xb, **484I**, 484J, 484L, 484O
Saks-Henstock lemma 482A, 482B, 483F, 483Xe, 484Hc
saltus function **226B**, **226Db**, 226Xa
scalarly measurable **463Ya**
scattered topological space 439Ca, 439Xf, 439Xh, **4A2A**, 4A2G; *see also* non-scattered
Schröder-Bernstein theorem 2A1G, 332 *notes*, 344D, 344Xa
Schwartz function *see* rapidly decreasing test function (**284A**)
Schwartzian distribution 284R, 284 *notes*; *see also* tempered distribution (**284** *notes*)
second-countable topological space 434Ya, 454Yd, **4A2A**, 4A2O, 4A2P, 4A2Ua, 4A3G
Second Separation Theorem (of descriptive set theory) 422Yd
selection theorem 423M-423O, 423Xf, 423Xg, 424Xg, 424Xh, 433F, 433G
selector *see* measurable selector
self-adjoint linear operator 444V, **4A4Jc**, 4A4M
self-supporting set (in a topological measure space) **256Xf**, **411N**, 414F, 415E, 415Xk, 416Dc, 416Xe, 417Ma, 443Xa, *443Xl*, 443Yf, 456D
semicompact paving *see* countably compact class (**413L**)
semi-continuous function *see* lower semi-continuous (**225H**, **4A2A**), upper semi-continuous (**4A2A**)
semi-finite measure algebra §322 (**322Ad**), 323Dd, 323Ga, 323Xa, 324K, *324Xb*, 325Ae, 325D, 327B, 331C, 331Xl, 332E, 332F, 332I, *332R*, 332Xi, 332Yb, 364L, 365E, 365G, 365M, 365P, *365Sb*, 366E, *366Xe*, *366Xf*, *366Xk*, 367Nb, 368S, *369H*, 369Xa, *371Xc*, *373R*, 375I, 383E, 383F, 383Ga, 383I, *384Ld*, 384P, 384Xe, 391Ca
semi-finite measure (space) **211F**, 211L, 211Xf, *211Ya*, *212G*, 213A, 213B, 213Hc, 213Xc, 213Xd, 213Xj, 213Xl, 213Xm, 213Ya-213Yc *214Xe*, 214Xh, *215B*, *216Xa*, *216Yb*, 234Fa, *235O*, *235Xd*, 235Xe, *241G*, *241Ya*, *241Yd*, 243G, 245Ea, 245J, *245Xd*, 245Xj, 245Xl, 246J, 246Xh, 251J, 251Xc, 252P, *252Yf*, 253Xf, 253Xg, *322Bd*, 322Yd, 327C, 327D, *342L*, 342Xa, *342Xc*, *342Xn*, *343B*, *344H*, *365Xp*, *367Xr*, 411Gd, *412Xb*, 413E, 413Xf, *414N*, *418F*, *431Xb*, *431Xe*, 471S, *438B*, *438Ce*, *438I*, *463Cb*, 481Xf
semi-finite version of a measure **213Xc**, 213Xd, 322Xb, 413Xf
semigroup *see* amenable semigroup (**449Ya**), countably full local semigroup (**448A**), full local semigroup (**394A**), topological semigroup (**444Yb**)
semi-martingale *see* submartingale (**275Yf**)
seminorm **2A5D**, 4A4C, 4A4Da
semi-radonian *see* pre-Radon (**434Gc**)
semi-ring of sets **115Ye**
separable (topological) space **2A3Ud**, 316Xd, 316Yb, 316Yj, 367Xr, 391Yc, **3A3E**, 417Xu, 491H, 491Xz, 4A2Be, 4A2De, 4A2Ea, 4A2Ni, 4A2Oc, 4A4Bi; *see also* hereditarily separable (**423Ya**)
separable Banach space 244I, 254Yc, 365Xp, 366Xc, 369Xg, 424Xe, 456Yd, 466M, 466Xe, 493Xg
separable metric space 476Xd, 491Xw
separable metrizable space 245Yj, 264Yb, 284Ye, 415E, 415Xk, 417T, 418B, *418G*, 418J, 418K, 421Xg, 423Yc, 434Ie, *434O*, 471Df, 471Xf, 438D, *439Xm*, 441Xm, 451O, 454N-454P, 454Yd, 4A2P, 4A2Ua, 4A3E, 4A3N, 4A3V, 4A4Id; *see also* Polish space (**4A2A**)
separable normed space 467E, 467Xe
separate zero sets **473J**
separated*see* countably separated (**343D**)
separately continuous function *436Ye*, 437Mc, 462I, 463J, 463K, 463Zc, **4A2A**
separation *see* First Separation Theorem, Second Separation Theorem; (in 'countable separation property') *see* σ-interpolation property (**466G**)
separator (for a Boolean automorphism) **382Aa**, 382B-382E, 382I, 382J, 382L, 382M, 382Xa, 382Xd, 382Xl

sequential (topological) space 436F, 436Yd, 436Ye, **4A2A**, 4A2K, 4A2Ld; *see also* Fréchet-Urysohn (**462Aa**)
sequentially closed set **4A2A**
sequentially compact set, topological space 413Ye, 434Za, 462C, *463Cd*, **>4A2A**, 4A2Lf
sequentially continuous function 463B, **4A2A**, 4A2Kd, 4A2Ld
sequentially order-closed set in a partially ordered space **313Db**, 313Xg, 313Yc, 316Fb, 353Ja, 364Xm, 367Yb; *see also* σ-ideal (**313Ec**), σ-subalgebra (**313Ec**)
sequentially order-continuous additive function (on a Boolean algebra) 326Gc, 363Eb
—— —— Boolean homomorphism 313Lc, 313Pb, 313Qb, 313Xq, 313Yb, 314Fb, 314Hb, 314Xc, 314Ye, 315Ya, 316Fd, 324A, 324B, 324Kd, 324Xa, *324Xe*, 324Yc, *326Ff*, *343Ab*, 363Ff, 364G, 364H, 364R, *364Xp*, 364Xv, 364Yc, 364Yf, 364Yn, 365H, 365Xg, 375Ya, 375Ye, 381K, 424Xk
—— —— dual (of a Riesz space)($U_c^\sim$) **356Ab**, 356B, 356D, 356L, 356Xa, 356Xb, 356Xc, 356Xd, 356Xf, 356Ya, 362Ac, 363K, 363S, 437Aa, 437B, 438Xd
—— —— function **313Hb**, 313Ic, 313Xg, 313Yc, 315D, 316Fc, 361Cf, 361Gb, 367Xb, 367Yb, 375Xd, 392H
—— —— positive linear operator or functional 351Gb, 355G, 355I, 361Gb, 363Eb, 363Ff, 364R, 375A, 436A, 436Xi; *see also* sequentially order-continuous dual (**356A**), $\mathsf{L}_c^\sim$ (**355G**)
—— —— Riesz homomorphism 361Jf, 375Ya, 437C-437E, 437Hb, 437Xc, 437Xe, *437Xn*, 437Yd
—— —— ring homomorphism between Boolean rings 361Ac, 361Jf, 375Ka
sequentially smooth dual (of a Riesz subspace of $\mathbb{R}^X$, $U_\sigma^\sim$) **437Aa**, 437B-437E, 437Xa, 437Xc, 437Ya, 437Yd, 437Yi
—— —— linear operator 437D
—— —— positive linear functional **436A**, 436C-436E, 436G, 436Xc, 436Xe, 436Xf, 436Ya, 436Yc; *see also* sequentially smooth dual (**437Aa**)
Shannon-McMillan-Breiman theorem 386E, 386Xc
shift operators (on function spaces based on topological groups) 286C, 443G, 444F, 444Of, 444Xg, 444Xp, 444Yk; *see also* Bernoulli shift (**385Q**)
Shoenfield's Absoluteness Theorem §393 *notes*
Sierpiński Class Theorem *see* Monotone Class Theorem (136B)
signed measure 416Ya, 437B, 437E, 437G, 437I, 444E, 444Sb, 444Xd, 444Yb, 445Yi, 445Yj; *see also* countably additive functional (**231C**), M_σ
signed Baire measure 437E, **437G**
signed Borel measure 437F, **437G**
signed tight Borel measure **437G**; *see* M_t
signed τ-additive Borel measure **437G**; *see* M_τ
simple function §122 (**>122A**), 242M, 361D
simple group 382S, 383I, 383Xb-383Xd
simple product of Boolean algebras §315 (**315A**), 316Xc, 316Xl, 316Xw, 332Xa, 332Xg, 364S, 384Lc, 391Xb, 391Xj
—— of measure algebras **322K**, 323L, 325Xc, 332B, 332Xm, 332Xn, *333H*, *333Ia*, 333K, 333R, 366Xi
simple product property **417Yi**, 419Yc
Sinaĭ's theorem 387E, 387L
singular additive functional **232Ac**, 232I, **232Yg**
singular measures 231Yf, **232Yg**, 362Xa, 461Xl
small subgroups *see* 'no small subgroups' (**446E**)
smooth dual (of a Riesz subspace of $\mathbb{R}^X$)($U_\tau^\sim$) **437Ab**, 437Xa, 437Xc, 437Xd, 437Xq, 437Ya; *see also* M_τ
smooth function (on $\mathbb{R}$ or $\mathbb{R}^r$) **242Xi**, 255Xi, 262Yd-262Yg, **284A**, **284Wa**, **473Bf**, 473D, 473E
smooth positive linear functional **436G**, 436H, 436Xf, 436Xi-436Xl, 439I; *see also* sequentially smooth (**436A**), smooth dual (**437Ab**)
smoothing by convolution 261Ye, 473D, 473E
Sobolev space 282Yf, *473 notes*; *see also* Gagliardo-Nirenberg-Sobolev inequality (473H)
solid hull (of a subset of a Riesz space) 247Xa, **352Ja**
solid set (in a partially ordered linear space) **351I**; (in a Riesz space) *354Xg*
—— linear subspace (of a partially ordered linear space) 351J, 351K, 351Yb; (of a Riesz space) 352J, 353J, 353K, *353N*, 355F, *355J*, *355Yj*, 368Ye, *383J*, *411Xe*, *464K*, *464L*; *see also* band (**352O**)

Sorgenfrey topology on $\mathbb{R}$ **415Xc**, 415Yd, 415Ye, 416Xx, 417Yi, 418Yf, 419Xf, 423Xe, 434Xm, 434Yi, 438Xm, 438Xo, **439Q**, 453Xb, 462Xa, 482Ya
Souslin scheme **421B**
—— set *see* Souslin-F set (**421K**)
—— space *see* analytic space (**423A**)
Souslin-F set 421J, **421K**, 421L, 421Xe, 421Xf, 421Xj, 421Xl, 421Xm, 421Yd, 422H, 422K, 422Ya, 422Yc, 423E, 423O, 423Xf, 423Ye, 431B, 431E, 431Xb, 434Dc, 434Hc, 435G, 435Xk, 436Xg, 466M, 467Xa; σ-algebra generated by Souslin-F sets 423O, 423Xg, 423Xi
Souslin property *see* ccc (**316Ab**)
Souslin's operation §421 (**>421B**), 422Hc, 422Xc, 423E, 423M, 423N, 423Yb, 424Xg, 424Xh, 424Yc, §431, 434Dc, 434Eb, 434Fd, 434Xn, 471Dd, 454Xh
space-filling curve 134Yl
spectral radius (of an element in a Banach algebra) 445Kd, 445Yk, **4A6G**, 4A6I, 4A6K
spectrum (of an M-space) **354L**
sphere, surface measure on 265F-265H, 265Xa-265Xc, 265Xe, 456Xb, 457Xi
—— isometry group of 441Xo
spherical polar coordinates 263Xf, 265F
split interval (= 'double arrow space') **343J**, 343Xf, 343Yc, 344Xf, 412Yb, 419L, 419Xg, 419Xh, 419Yc, 424Yd, 433Xg, 434Ke, 434Yi, 438Qa, 438R, 438Xl, 438Xn, 453Xc, 463Xg, 491Xg
—— line **419Xf**
square-integrable function **244Na**; *see also* $\mathcal{L}^2$
stabilizer subgroup **4A5Bf**
stable set of functions §465 (**465B**); (in L^0) **465O**, 465P-465R; *see also* R-stable (**465S**)
standard Borel space §424 (**>424A**), 433J, 433K, 433Yd, 451L, 452N, 452Xm, 454F, 454H, 454Xg, 455A
standard extension of a countably additive functional 327F, **327G**, 327Xa, 327Xd, 327Xe, 327Yc
standard gamma distribution **455Xe**, 495Xh, 495Xo
standard normal distribution, standard normal random variable **274A**, 456Aa, 456M, 456Xe
standard order unit (in an M-space) **354Gc**, 354H, *354L*, *356N*, 356P, 363Ba, 363Ye
stationary set **4A1C**
Steiner symmetrization 264H, 476 notes
step-function **226Xa**
Stieltjes measure *see* Henstock-Stieltjes integral, Lebesgue-Stieltjes measure (**114Xa**)
Stirling's formula **252Yn**
stochastic *see* doubly stochastic matrix
stochastically independent *see* independent (**272A**, **325L**, **325Xe**)
Stone-Čech compactification 418Yk, 422Ya, 434Yc, 434Yd, 435Yb, 464P, **4A2I**; (of $\mathbb{N}$) 416Yd, 434Yg, 463M
Stone's condition *see* truncated Riesz space (**436B**)
Stone representation of a Boolean ring or algebra 311E, 352 *notes*; *see also* Stone space
Stone space of a Boolean ring or algebra 311E, **311F**, 311I-311K, 311Xg, 311Ya, 312O-312S, 312Xi-312Xk, 312Yc-312Yf, 313C, 313R, 313Xp, 313Yb, 314M, 314S, 314T, 314Yd, 315H, *315Xc*, 316B, 316I, 316Yb, 316Yc, 316Yi, 363A, 363Yf, 381Q, 381Xl, 382Xa, 413Yc, 416Q, 436Xp, 4A2Ib
Stone space of a measure algebra 321J, **321K**, 322N, 322Q, 322Xi, 322Yb, 322Yf, 341O, 341P, 342Jc, 343B, 344A, 344Xe, 346K, 346L, 346Xf, 411P, 414Xs, 415Q, 415R, 416V, 416Xw, 419E, 453M, 453Xa, 453Za, 463Ye, 491Xm
Stone-Weierstrass theorem 281A, 281E, 281G, 281Ya, 281Yg, 4A6B
stopping time **275L**, 275M-275O, 275Xi, 275Xj
straightforward set of tagged partitions **481Ba**, 481G
Strassen's theorem 457D
strategy (in an infinite game) 451U
strictly localizable measure (space) **211E**, 211L, *211N*, 211Xf, *211Ye*, 212G, 213Ha, 213J, 213L, 213O, 213Xa, 213Xh, 213Xn, 213Ye, 214Ia, 214J, 215Xf, *216E*, 234Fd, *235P*, 251N, 251P, 251Xn, 252B, 252D, 252E, 252Ys, *252Yt*, 322Kd, 322N, 322Qb, 322Xi, 325He, 341H, 341K, 342Hb, 343B, *343Xi*, 344C, 344I,

344Xb, 344Xc, *346Xd*, 385V, 412I, *412Yf*, 414J, 414Xt, 415A, 416B, *416W*, 417C, *417Ya*, 451Xa, *452O*, *452R*, 452Yb, *452Yc*, *453E*, *453F*, *453H*, *465Pb*, *465Xp*

strictly positive additive functional (on a Boolean algebra) **391D**, 391J, *391N*, *395Xd*, 417Xu; *see also* chargeable Boolean algebra (**391X**)

—— —— measure (on a topological space) **411N**, 411O, 411Pc, 411Xh, *415E*, *415Fb*, *415Xj*, *416U*, 417M, 417Sc, *417Yd*, *417Yh*, *418Xg*, 433Hb, 435Xj, 441Xc, 441Xh, 441Yj, 442Aa, 443Td, *443Yi*, 444Xm, 453D, 453I, 453J, 463E, 463J, 463Xd, 463Xk

—— —— submeasure (on a Boolean algebra) **392B**, 392F, 392I, 392J, 392Xc, 392Xg, 392Ya, 393Xa, 393Xb, 393Yd, 491Ia

strong law of large numbers 273D, 273H, 273I, 273Xh, 275Yn, 276C, 276F, 276Ye, *276Yg*, 458Yd, 459Xa, 465H, *465M*

strong lifting **453A**, 453B-453D, 453H-453J, 453M, *453N*, 453Xa-453Xe, 453Za; *see also* almost strong lifting (**453A**)

strong measure-algebra topology **323Ae**, 323Xg, 366Yi, 412Yc

strong operator topology 366Yi, 372Yn, 388Yf, **3A5M**

strongly consistent disintegration **452E**, 452G, 452P, 458N, 458Xp, 458Xq, 495I

strongly measure-compact (topological) space **435Xk**

strongly mixing *see* mixing (**372P**)

strongly Prokhorov topological space **437Yo**

subalgebra of a Boolean algebra **312A**, 312B, 312M, 312N, 312Xb, 312Xc, 312Xj, 313Fc, 313G, 313Xd, 313Xe, *315Xn*, 315Xp, *316Xa*, 331E, 331G, 332Xf, 363G, *391Xb*, *391Xf*, *391Xj*, *457A*, *457C*, *457D*; *see also* Boolean-independent subalgebras (**315Xn**), closed subalgebra (**323I**), fixed-point subalgebra (**394Ga**), independent subalgebras (**325L**), order-closed subalgebra, order-dense subalgebra, regularly embedded subalgebra (**313N**),σ-subalgebra (**233A**, **313E**)

subbase (of a topology) **>4A2A**, 4A2Ba, 4A2Fh, 4A2Hc, 4A2Oa, 4A3D, 4A3O

subdivision *see* tagged-partition structure allowing subdivisions (**481G**)

subgroup of a topological group 443Xg, 4A5E, 4A5J; *see also* closed subgroup, compact subgroup, normal subgroup, one-parameter subgroup

subhomomorphism *see* σ-subhomomorphism (**375E**)

sublattice **3A1Ib**

submartingale **275Yf**, 275Yg

submeasure (on a Boolean algebra) §392 (**392A**), 393B, 393T, 393U, 393Xi, 393Yb, 491Aa; *see also* Maharam submeasure (**392G**), outer measure (**113A**), pathological submeasure (**393Tc**), strictly positive submeasure

submodular functional **413Xq**

subring **3A2C**

—— of a Boolean ring 311Xd, 312Xa, 312Xg, 481Hd

subspace measure 113Yb, 214A, **214B**, 214C, 214H, 214I, 214Xb-214Xh, 216Xa, 216Xb, 241Ye, 242Yf, 243Ya, 244Yc, 245Yb, 251P, 251Q, 251Wl, 251Xn, 251Yb, 254La, 254Ye, 264Yf, 322I, 322J, 322Xg, 322Yd, 343H, *343M*, 343Xa, 411Xg, 412O, 412P, 412Xp, *413Xc*, 414K, 414Xn, 414Xp, 415B, 415J, 415Yb, 416R, 416T, 417I, 417Xf, *435Xb*, 471E, 443K, 443Xg, 451Xf, 451Ya, 453E; (on a measurable subset) 131A, **131B**, 131C, 132Xb, 214J, 214K, 214Xa, 214Xi, 241Yf, 247A, 342Ga, 342Ia, 342Xn, 343L, 344J, 344Xa, 344Xe, 344Xf, 414Yc, 415Xd, *415Yh*, 416Xo, 416Xu, 416Xv, 419A, 443F, 451D, 451Yk, *453Dd*, 454Pb, 491Xt, *495Xb*; (integration with respect to a subspace measure) **131D**, 131E-131H, 131Xa-131Xc, *133Dc*, 133Xa, **214D**, 214E-214G, 214M, 214Xm, 412J

subspace of a normed space 2A4C

subspace topology **2A3C**, 2A3J, 3A4Db, 422Gf, *434F*, *434Id*, *434J*, 438La, 462C, 4A2B-4A2D, 4A2F, 4A2Ha, 4A2Kb, 4A2L, 4A2N, 4A2P-4A2S, 4A2Ua, 4A3Ca, 4A3Kd

subspace uniformity **3A4D**, 4A5Ma

subspace σ-algebra **121A**, *214Ce*, *418A*, 424Bd, 424E, 424G, 454Xd, 454Xe, 461Xg, 461Xi, 4A3Ca, 4A3Kd, 4A3Nd, 4A3Xg, 4A3Yb

substitution *see* change of variable in integration

successor cardinal 2A1Fc, 4A1Aa

—— ordinal **2A1Dd**

sum over arbitrary index set 112Bd, 226A
sum of measures **112Xe**, **112Ya**, 212Xe, 212Xh, 212Xi, 212Xj, 212Yd, 212Ye, 334Xb, 334Xd, 415Ya, 416Xd, 437Yj
sum topology *see* disjoint union topology (**4A2A**)
summable family of real numbers **226A**, 226Xf
supermodular functional on a lattice **413P**, 413Xo
support of an additive functional on a Boolean algebra **326Xi**
support of a Boolean homomorphism **381B**, 381G, 381H, 381Xm, 382E, 382I, 382K, 382N, 382P-382R, 384B, *493Yc*
support of a topological measure **256Xf**, 257Xd, **>411N**, 415Qb, 417C, 417E, 417Xq, *434Ha*, *434Ia*, 444Xb, 456D, 456L, 456Xf, *464Xa*, 491Xz; (of a Gaussian distribution) 456D, **456E**, 456L
support of a submeasure on a Boolean algebra **392Xe**
support *see also* bounded support, compact support (**4A2A**)
supported *see* point-supported (**112Bd**)
supporting (element in a Boolean algebra supporting an automorphism) **381Ba**, 381E, 381Jb, 381Qb, 381Sa, 381Xk, 382D, 382N, 382O, 384B; *see also* self-supporting set (**256Xf**, **411Na**), support
supremum **2A1Ab**
—— in a Boolean algebra 313A-313C
surface measure *see* normalized Hausdorff measure (**265A**)
symmetric distribution 272Ye
symmetric difference (in a Boolean algebra) **311G**, 448Xj, 493E
symmetric group 449Xc, 492H, 492I, 493Xc
symmetric measure on a product space 459D, 459E, 459G, 459H, 459Xc, 459Xd, 459Xe, 459Ya
symmetric Riemann-complete integral 481L
symmetric set (in a group) **4A5A**; (in X^n) **465J**
symmetrization *see* Steiner symmetrization

tag **481A**
tagged partition §481 (**481A**)
tagged-partition structure allowing subdivisions **481G**, 481H-481K, 481M-481Q, 481Xb-481Xd, 481Xf, 481Xf, 481Yb, *481Yc*, 482A, 482B, 482E, 482G, 482H, 482K, 484F
Talagrand's measure **464D**, 464E, 464N, 464R, 464Xb, 464Z, 466Xm
Tamanini-Giacomelli theorem 484B, 484Ya
Tarski's theorem §394 *notes*, 449J, 449Ye; *see also* Banach-Tarski paradox
tempered distribution **284 notes**
tempered function §284 (**284D**), 286D
tempered measure **284Yi**
tensor product of linear spaces 253 *notes*, 376Ya-376Yc
tent map **372Xm**, 385Xk, 461Xn
test function 242Xi, 284 *notes*; *see also* rapidly decreasing test function (**284A**)
thick set *see* full outer measure (**132F**)
thin set **484A**, 484E, *484N*
Tietze's theorem 4A2Fd
tight functional **436Xn**, **437Tb**, 437Xq
—— measure 342Xh, 343Yc, 411E, 411Gf, **411Ja**, 411Yb, 412B, 412Sb, 412V, *412Xa*, 412Xe, 412Xl, 412Xn, 412Ye, 414Xj, 416D, 416M, 417C, 417E, 419D, 432B, 432Xc, 433Ca, 434A, *434Gc*, *434Ja*, 434Xb, 435Xb-435Xe, 451M, 451Ra, 471I, 471S, 471Yc, 471Yc; *see also* Radon measure (**411Hb**), signed tight Borel measure (**437G**)
—— *see also* countably tight (**4A2A**), uniformly tight (**285Xj**, **437T**)
Tonelli's theorem 252G, 252H, 252R, 417Hc
topological group chap. 44, §4A5 (**4A5Da**); *see also* abelian topological group, compact Hausdorff group, locally compact group, Polish group (**4A5Db**), unimodular group (**442I**)
topological measure (space) **256A**, **>411A**, 411G, 411Jb, 411N, 411Xg, 411Xh, 411Xj, 411Yb, 411Yc, 414B, 414C, 414E-414K, 414P, 414R, 414Xh, 414Ya, *419H*, *431E*, *432B*, *433A*, *433B*, 434D, *461C*, 463E,

463Xd, *464Z*, 471C, 471Da, 471Xf, 476D, 476E, 476I, 476Xe, 481N; *see also* Borel measure (**411K**), quasi-Radon measure space (**411Ha**), Radon measure space (**411Hb**)

topological semigroup **444Yb**, 449Ya

topological space §2A3 (**2A3A**), §3A3, §4A2

topological vector space *see* linear topological space (**2A5A**)

topology §2A2, §2A3 (**2A3A**); *see also* convergence in measure (**245A**, **367M**), linear space topology (**2A5A**), measure-algebra topology (**323Ab**), order topology (**4A2A**), pointwise convergence (**462Ab**), uniform convergence (**4A2A**)

total order 418Xw; *see also* totally ordered set (**2A1Ac**)

total variation (of an additive functional) **231Yh**; (of a function) *see* variation (**224A**)

totally bounded set (in a metric space) 434L, 441Ya, *448Xi*, 476Ac, 495Yc, **4A2A**; (in a uniform space) 443H, 443I, 443Xk, 443Yd, 463Xb, **4A2A**, 4A2J, 4A5O

totally finite measure algebra **322Ab**, 322Bb, 322C, *322Mb*, *322Qc*, *323Ad*, 323Ca, *324Kb*, *324P*, 327B, *331B*, *331D*, *332M*, *332P*, *332Q*, *333C-333G*, *333J-333N*, *333Q*, *333R*, *366Yh*, *373Xj*, *373Xn*, *373Yb*, 375Fb, 375H, *383Fb*, 383I, 383J, 383Xg, *383Ya*, 386Xa, 384O, *384Xd*, 386A-386D, 386K, 386Xd, 386Xf, 391B, 394R, *395Xa*

totally finite measure (space) **211C**, 211L, *211Xb, 211Xc, 211Xd*, *212G*, *213Ha*, 214Ia, 214Ja, *215Yc*, *232Bd*, *232G*, *243I*, *243Xk*, *245Fd*, *245Xe*, *245Ye*, 246Xi, 246Ya, *322Bb*, *386Xb*, *412Xg*, *441Xe*, *441Xh*, *442Ie*, *471Dh*, 481Xg, 495Xa

totally ordered set *135Ba*, **2A1Ac**, 3A1F

tower (in $\mathcal{P}\mathbb{N}$) 4A1Fb

trace (of a σ-algebra) *see* subspace σ-algebra (**121A**)

transfinite recursion 2A1B

transitive action 442Z, 443T, 443Xz, 448Xi, 476G, 476Yc, **4A5B**

translation-invariant inverse-measure-preserving function 387Xd

translation-invariant lifting **345A**, 345B-345D, 345Ya-345Yc, 346C, **447Aa**, 447I, 447J, *447Xa*, *447Ya*, 453B

translation-invariant lower density 345Xf, 345Ya, **447Ab**, 447B, 447F-447H

translation-invariant measure 114Xf, 115Xd, 134A, 134Ye, 134Yf, 255A, *255Ba*, 255Yn, 345A, 345Xb, 345Ya; *see also* Haar measure (**441D**)

translation-invariant σ-ideal 444Ye

transportation network **4A4N**

transversal (for a Boolean automorphism) **382Ab**, 382B, 382C, 382G-382I, 382L, 382Xa, 382Xk

tree (of sequences) **>421N**

truly continuous additive functional **232Ab**, 232B-232E, 232H, 232I, 232Xa, 232Xb, *232Xf*, 232Xh, 232Ya, 232Ye, *234Ce*, 327C, 362Xh, *363S*, 414D, 438Xb, 444Xl

truncated Riesz subspace of $\mathbb{R}^X$ **436B**, 436C, 436D, 436H

two-sided invariant mean 449Xs

Tychonoff space *see* completely regular (**3A3Ac**)

Tychonoff's theorem 3A3J

type *see* Maharam type (**331F**)

Ulam S. *see* Banach-Ulam problem

ultrafilter 254Yd, **2A1N**, 2A1O, 2A3R, 2A3Se, 351Yd, *3A3De*, *3A3Lc*, 4A1Ia, 4A1K, 4A1L, *4A2Bc*, *4A2Tc*; *see also* non-principal ultrafilter, principal ultrafilter (**2A1N**)

Ultrafilter Theorem 2A1O

uncompleted indefinite-integral measure **234Cc**, 465Ck

uncountable cofinality, cardinal of 4A1A-4A1C

uniferent homomorphism 312F

Uniform Boundedness Theorem 3A5H

uniform convergence, topology of **4A2A**

uniform convergence on compact sets, topology of 454N-454P, 454Xl, 454Xm, 455D, *456Ye*, 4A2Gg, 4A2Oe; *see also* Mackey topology (**4A4F**)

uniform metric gauge **481E**, 481I, 482E, 482Xg

uniform space *2A5F*, §3A4 (**3A4A**), 441Yc, 4A2J

uniformity §3A4 (**3A4A**), 3A5I, 4A2J; (of a linear topological space) 463Xb; *see also* bilateral uniformity (**4A5Hb**), measure-algebra uniformity (**323Ab**), right uniformity (**4A5Ha**), uniform space (**3A4A**)
uniformly complete Riesz space **354Yi**, 354Yk
uniformly continuous function *224Xa*, 255K, **3A4C**, 3A4Ec, 3A4G, *444Xs*, *449B*, *473Da*, 473Ed, 476Ya, 4A2J
uniformly convergent (sequence of functions) **3A3N**, 4A2Jg
uniformly distributed sequence *see* equidistributed (**281Yi**, **491B**)
uniformly exhaustive submeasure **392Bc**, 392C, 392E, 392F, 392J, 392Xc, 393A, 438Ya
uniformly integrable set (in $\mathcal{L}^1$) §246 (**>246A**), 252Yp, 272Yd, 273Na, 274J, 275H, 275Xi, *275Yl*, 276Xd, 276Yb; (in $L^1(\mu)$) §246 (**246A**), 247C, 247D, 247Xe, 253Xd, 354Q; (in an L-space) **354P**, 354Q, 354R, 356O, 356Q, 356Xm, 362E, 362Yf-362Yh, 371Xf; (in $L^1(\mathfrak{A},\bar{\mu})$) 367Xm, 373Xj, 373Xn
uniformly rotund *see* locally uniformly rotund (**467A**)
uniformly smooth (set of linear functionals) 437Ye
uniformly tight (set of linear functionals) **437Ta**, 437Ua, 437Yo;(set of measures) **285Xj**, 285Xk, **285Ye**, 285Yf, **437Tb**, 437U, 437V, 437Xr-437Xt, 437Yf, 437Ym, 452D, 452Xd
unimodular topological group **442I**, *442Xf*, *442Xg*, 442Yb, *443Ag*, 443Xw-443Xy, 443Yk, *443Yn*, 444R, 444Yh, *449Xe*, 449Xl, 449Xo
unit ball in $\mathbb{R}^r$ 252Q
unital (Banach) algebra **4A6Ab**, 4A6C-4A6E, 4A6I, 4A6J, 4A6L-4A6N
universal Gaussian distribution **456F**, 456G, 456H, 456Yb, 456Yc
universal mapping theorems 253F, 254G, 315B, 315I, 315 *notes*, 325C, 325H, 325J, 369Xk
universally complete Riesz space *see* laterally complete (**368L**)
universally measurable set 432A, **>434D**, 434Eb, 434F, 434S, 434T, 434Xc, 434Xd, 434Xf, 434Xg, 434Xy, 434Yc, 434Yb, 438Xj, 439Xd, 439Xl, 444Ye, 451Xi, 454Xi; *see also* $\Sigma_{\rm um}$ (**434Dc**)
—— —— function **>434D**, 434S, 434T, 437Xe, 437Yd, 463Zc, 466Xi
—— —— *see also* universally Radon-measurable (**434E**)
universally negligible topological space **439B**, 439C, 439F, 439G, 439Xb-439Xh, 439Ya-439Yd; *see also* universally τ-negligible (**439Xh**)
universally Radon-measurable set **>434E**, 434Fc, 434J, 434Xe-434Xg, 434Xi, 434Yd, 437Ib; *see also* $\Sigma_{\rm uRm}$ (**434Eb**)
—— —— function **434Ec**, 437Ib, 466L
universally τ-negligible topological space **439Xh**, 439Yd
up-crossing 275E, 275F
upper asymptotic density **491A**, 491I, 491Xb, 491Xd
upper derivate of a real function **483H**, 483J, 483Xm
upper envelope *see* $\mathrm{upr}(a,\mathfrak{C})$ (**314V**)
upper integral **133I**, 133J, 133K, 133Xe, 133Yf, 135H, 214Xm, 252Ye, 252Yh, 252Yi, 253J, 253K, 463Xj, *464Ab*, *464H*, *464Kb*, *465M*, *465Xn*
upper Riemann integral **134Ka**
upper semi-continuous function 414A, 471Xf, 476D, 476E, **4A2A**, 4A2Bd
—— —— relation *422A*
upwards-directed partially ordered set **2A1Ab**
Urysohn's lemma 4A2Fd
—— *see also* Fréchet-Urysohn (**462Aa**)
usual measure on $\{0,1\}^I$ **254J**; *see under* $\{0,1\}^I$
usual measure on $[0,1]^I$ **416U**; *see under* $[0,1]^I$
usual measure on $\mathcal{P}X$ **254J**; *see under* $\mathcal{P}X$
usual topology on $\mathcal{P}X$ **4A2A**; *see under* $\mathcal{P}X$

vague topology (on a space of signed measures) **274Ld**, 274Xh, 274Ya-274Yd, 275Yp, 285K, 285L, 285S, 285U, 285Xk, 285Xq, 285Xs, 285Yd, 285Yg-285Yi, 285Yn, 367Xv, **>437J**, 437L, 437M, 437Xf-437Xh, 437Xl, 437Xo, 437Xt, 437Ye-437Yh, 437Ym, 437Yo, 444Xf, 445Yh, 452Xc, 491 *notes*, *495Xl*; *see also* narrow topology (**437Jd**)

variance of a random variable **271Ac**, 271Xa, 272R, 272Xf, 285Gb, 285Xo
variation of a function §224 (**224A**, **224K**, **224Yd**, **224Ye**), 226B, 226Db, 226Xc, 226Xd, 226Yb, 264Xf, 265Yb, 463Xi, 463Xj, 463Yc, 465Xc; *see also* bounded variation (**224A**)
—— of a measure *see* total variation (**231Yh**)
vector integration *see* Bochner integral (**253Yf**), Henstock integral (**483Yg**), Pettis integral (**463Ya**)
vector lattice *see* Riesz space (**241E**, **352A**)
vector measure 326Ye, 361Gb, **393O**, 393P, 393Q, 393S, 393Xg, 393Yh, 393Yi, *474Ya*
very weak operator topology **373K**, 373L, 373Xp, 373Xv, 373Yg
Vietoris topology **476C**, 476E, 476Xb, 476Xf, 476Yb; *see also* k-Vietoris (**476Cd**)
virtually measurable function **122Q**, 122Xe, 122Xf, 212Bb, 212F, 241A, 252E
Vitali cover 261Ya
Vitali's theorem 221A, 221Ya, 221Yc-221Ye, 261B, 261Yk, 447C, 471N, 471O; *see also* Besicovitch's Covering Lemma (472A-472C)
Vitali-Hahn-Saks theorem 246Yg, 362Yh
volume **115Ac**
—— of a ball in $\mathbb{R}^r$ 252Q, 252Xh
von Neumann transformation **388D**, 388E, 388Xd, 388Xd, 388Xg *see also* relatively von Neumann (**388D**), weakly von Neumann (**388D**)

Wald's equation 272Xh
wandering *see* weakly wandering (**395C**)
weak operator topology *see* very weak operator topology (**373K**)
weak order unit (in a Riesz space) **353L**, 353P, 353Yg, *368Yd*
weak topology (of a linear topological space) 461I, 462Yb, 466A, 466Xa, 466Xb, 466Xd, 466Ya, 466Yb, 4A3U, 4A3V, **4A4Bd**, 4A4Cg, 4A4E
weak topology (of a normed space) 247Ya, **2A5I**, 356Yf, 356Yg, 3A5Eb, 436Xq, 462D, 462E, 462Xb, *464Z*, *465E*, 466B-466F, 466H, 466Xe, 466Xf, *466Xm*, 466Yb, 467Xg, 467Xh, 4A4K
—— *see also* very weak operator topology (**373K**), (relatively) weakly compact, weakly compactly generated, weakly convergent, weakly K-countably determined
weak* topology on a dual space 253Yd, 285Yg, **2A5Ig**, 3A5E, 3A5F, 437K, **4A4Bd**, 4A4If; *see also* vague topology (**274Ld**, **437J**)
weakly compact linear operator 371Xb, 376Q, **3A5Kb**; *see also* compact linear operator (**3A5Ka**)
weakly compact set (in a linear topological space) 247C, 247Xa, 247Xc, 247Xd, **2A5I**, *376Yj*, 461I, 462E, 462F, *4A4F*, 4A4Ka; *see also* Eberlein compactum (**467O**), relatively weakly compact (**2A5Id**)
weakly compactly generated normed space **467L**, 467M, 467Xd-467Xg
weakly convergent sequence in a normed space 247Yb, 367Xy, 4A4Kb
weakly K-countably determined normed space **467Hb**, 467I-467K, 467M, 467Xc, 467Xh, 467Ya
weakly measurable function *see* scalarly measurable (**463Ya**)
weakly von Neumann transformation **388D**, 388F, 388H, 388Xf, 388Yc
weakly wandering element **395C**
weakly α-favourable measure space **451U**, 451Yg-451Yq, 454Ya
—— —— topological space 451Yq, **4A2A**
weakly θ-refinable *see* hereditarily weakly θ-refinable (**438K**)
weakly σ-distributive Boolean algebra **316Yg**, 316Yh, 362Ye, 375Yc, 391Ya, 392Ya
weakly (σ,∞)-distributive Boolean algebra **316G**, 316H-316K, 316Xk-316Xq, 316Xx, 316Yf, 316Yg, 316Yi, 316Yj, 316Yp, 322F, 325Yd, 362D, 367Yk, 368Q, 368R, 368Xe, 368Yg, 368Yi, 376Yf, 391D, 391K, 391Ya, *391Yg*, 391Z, 392I, *393Yd*, 394Yc, 395Ya
weakly (σ,∞)-distributive Riesz space **368N**, 368O-368S, 368Xd, 368Xe, 368Ye, *368Yf*, 368Yh, 368Yj 376H
Weierstrass' approximation theorem 281F; *see also* Stone-Weierstrass theorem
weight of a topological space 437Yk, 491Xz, **4A2A**, 4A2D, 4A2Li; *see also* measure-free weight
well-distributed sequence *281Xh*
well-ordered set **2A1Ae**, 2A1B, 2A1Dg, 2A1Ka, 4A1Ad, 4A2Rk; *see also* ordinal (**2A1C**)

Well-ordering Theorem 2A1Ka
Weyl's Equidistribution Theorem 281M, 281N, 281Xh, *372Xo*, *491Xf*
Wiener's Dominated Ergodic Theorem 372Yb
Wiener measure *see* Brownian motion (**455C**)
Wirtinger's inequality 282Yf

Young's function **>369Xc**, 369Xd, 369Xr, 369Yc, 369Yd, 373Xm
Young's inequality 255Ym, 444Yh

Zermelo's Well-ordering Theorem 2A1Ka
Zermelo-Fraenkel set theory 3A1A
zero-dimensional topological space 311I-311K, 315Xf, 316Xo, 316Yd, 353Yc, **3A3Ad**, 3A3Bd, *414R*, 416Q, *419Xa*, 437Xl, 481Xh, 4A2Ud, 4A3Oe, 482Xc, 491Ch
zero-one law 254S, 272O, 272Xf, 272Xg, 325Xg, 417Xx, 458Yb, *464Ac*
zero set in a topological space *313Yb*, *316Yh*, *324Yb*, **3A3Pa**, 411Jb, 412Xg, *416Xi*, 421Xh, 423Db, 443N, 443Yf, *491C*, 4A2Cb, 4A2F, 4A2Gc, 4A2Hc, 4A2Lc, 4A2Nb, 4A2Sb, 4A3Kd, 4A3Nc, 4A3Xc, 4A3Yb
Zorn's lemma 2A1M, 3A1G

$\mathcal{A}$-operation *see* Souslin's operation (**421B**)
AC_* function **483O**, 483Pb, 483Q
ACG_* function **483Oc**, 483R
a.e. ('almost everywhere') **112Dd**
a.s. ('almost surely') **112De**
Aut (in $\mathrm{Aut}\,\mathfrak{A}$) *see* automorphism group of a Boolean algebra (**381A**); (in $\mathrm{Aut}_\mu\,\mathfrak{A}$) *see* automorphism group of a measure algebra (**383A**)
$\mathfrak{A}_L$ (Lebesgue measure algebra) **373C**

B (in $B(x,\delta)$, closed ball) **261A**, 2A2B
B (in $\mathrm{B}(U;V)$, space of bounded linear operators) 253Xb, 253Yj, 253Yk, **2A4F**, 2A4G, 2A4H, 371B-371D, 371G, 371Xd, 371Yc, 376M, 3A5H, 444Yg; ($\mathrm{B}(U;U)$) 395Xb, 4A6C; ($\mathrm{B}(\mathbb{R}^r,\mathbb{R}^r)$) 446A
$\mathcal{B}$ (in $\mathcal{B}(X)$) *see* Borel σ-algebra (**4A3A**)
$\widehat{\mathcal{B}}$ (in $\widehat{\mathcal{B}}(X)$) *see* Baire property algebra (**4A3Q**)
B-sequence (in a topological group) **446L**, 446M, 446N, 446P, 446Xc, 447C, 447D, 447F, 447Xb, 447Xc
$\mathcal{B}\mathfrak{a}$ (in $\mathcal{B}\mathfrak{a}(X)$) *see* Baire σ-algebra (**4A3K**)

c (in $c(\mathfrak{A})$, where $\mathfrak{A}$ is a Boolean algebra) *see* cellularity (**332D**)
c (in $c(X)$, where X is a topological space) *see* cellularity (**332Xd**)
$\mathfrak{c}$ (the cardinal of $\mathbb{R}$ or $\mathcal{P}\mathbb{N}$) 2A1H, **2A1L**, *343I*, *343Yb*, *344H*, *344Yf*, *383Xd*, 391Yd, 391Ye, *416Yg*, 419I, 419Xd, 421Xd, 421Xi, 421Yc, 423K, 423Xh, *423Ye*, 424Db, 434Xg, *436Xg*, 438C, 438R, 438Xn-438Xp, 438Yb, 438Yc, 438Yi, 439P, *439Xm*, 454Yb, 491F, 491G, 491P, 491Q, 491Xz, 4A1A, 4A1O, 4A2Be, 4A2De, 4A2Gi, 4A3F
$\mathfrak{c}^+$ (the successor of $\mathfrak{c}$) 523P, 531Xa; *see also* $2^{\mathfrak{c}^+}$
$\mathbb{C}\setminus\{0\}$ (the multiplicative group) 441Xf, 441Yh
C (in $C(X)$, where X is a topological space) 243Xo, 281Yc, 281Ye, 281Yf, 352Xk, **353M**, 353Xd, 354L, *354Yf*, 353Yc, 363A, 367L, 367Yh-367Yj, 368Ya, 416Xi, 416Yg, 424Xf, 436Xe, 436Xg, 437Yk, 454N-454P, 454Xl, 454Xm, 455D, 456Ye, 462B, 462G, 462H, 462J, 462Xd, 462Ya, 462Yd, 462Z, 463Xd, 466Xf, 481Xh, 4A2Ib, 4A2N-4A2P; (in $C(X;\mathbb{C})$) 437Yb
$C([0,1])$ 242 *notes*, 352Xh, 356Xb, 368Yf, 436Xi, 436Xm, 437Yc
C_b (in $C_b(X)$, where X is a topological space) **281A**, 281E, 281G, 281Ya, 281Yd, 281Yg, 285Yg, 352Xk, **354Hb**, 436E, 436Ia, 436L, *436Xf*, 436Xl, 436Yc, 437J, 437Xd, 437Yi, 462Xb, 483Mc, 491C
C_k (in $C_k(X)$, where X is a topological space) *see* compact support (**4A2A**)
C_0 (in $C_0(X)$, where X is a topological space) **436I**, 436K, 436Xq, 437I, 443Yb, *445K*, 449Ba, 462E, 4A6B
C^∞ (in $C^\infty(X)$, for extremally disconnected X) **364W**, 364Yl, 368G
$\boldsymbol{c}$ **354Xq**, 354Xs, 355Ye
$\boldsymbol{c}_0$ 354Xa, 354Xd, 354Xi, 371Yc, 461Xc; *see also* $\ell^\infty/\boldsymbol{c}_0$

cac ('countable antichain condition') 316 *notes*
ccc Boolean algebra **316Aa**, 316C-316F, 316Xa-316Xj, 316Yc-316Ye, 316Yg, *316Yp*, 322G, 324Yd, 325Yd, 326L, *326Xi*, 331Ge, 332D, *332H*, 363Yb, 364Yb, 367Yk, 368Yg, 368Yi, 381Yc, 391M, 391Xa, 392Ca, 392I, 393J, 393Yd, 393Ye, *394Xf*, 394Yb, *448Ya*
ccc topological space **316Ab**, 316B, 316Xd, 316Xe, 316Ya, 316Yd, 411Ng, 418Ye, 443Xm, 444Xm, 4A2E, 4A2Pd, 4A2Rn, 4A3E, 4A3Mb
cf (in $\operatorname{cf} P$) *see* cofinality (**3A1Fb**)
c.l.d. product measure §§251-253 (**251F**, **251W**), 254Db, 254U, 254Ye, 256K, 256L, 325A, 325B, 325C, 325H, 334A, 334Xa, 342Ge, 342Id, 342Xn, 343H, 354Yl, 376J, 376R, 376S, 376Yc, 411Xi, 412R, 412S, 413Xg, 417C, 417D, 417S, 417T, 417V, 417Xb, 417Xj, 417Xs, 417Xv, 417Ye, 417Yi, 418S, 418T, 419E, 419Ya, 434R, 436Yd, 438Xg, 438Xh, 439Xj, 443Xp, 451I, 451Yn, *464A*, *464B*, *464Hd*, *464Qc*, *464Yb*, 471Yi, 491Xu, 494Xb, 495Xj
c.l.d. version of a measure (space) **213E**, 213F-213H, 213M, 213Xb-213Xe, 213Xg, 213Xj, 213Xk, 213Xn, 213Xo, 213Yb, *214Xf*, 214Xj, *232Ye*, 234Yf, 241Ya, 242Yh, *244Ya*, 245Yc, 251Ic, 251S, 251Wn, 251Xd, 251Xj, 251Xk, 252Ya, 322D, 322Qb, 322Xc, 322Xi, 322Yb, 324Xc, 324Xe, 342Gb, 342Ib, 342Xn, 343H, 343Ye, 411Xc, 411Xd, 412H, 413Eg, 413Xi, 413Xl, *414Xj*, *415Xn*, 416F, 416H, 436Xa, *436Xc*, *436Xk*, *436Yd*, 471Xh, 451G, 451Ym, 465Cj, 471Yi, 491Xt
cl* *see* essential closure (475B)
CM_* (formulations of the Control Measure Problem) 393A, 393H, 393J, 393L, 393P, 393Xe, 393Xg

d (in $d(X)$) *see* density (**331Yf**)
D (in $D_n(A,\pi)$, where A is a subset of a Boolean algebra, and π is a homomorphism) **385K**, 385L, *385M*; (in $D_n(A)$, where A is a subset of a topological group) **446D**; (in $D_k(A,E,\alpha,\beta)$, where E is a set and A is a set of functions) **465Ae**
D^* (in D^*F) *see* upper derivate (**483H**)
D_* (in D_*F) *see* lower derivate (**483H**)
$\mathcal{D}$ (smooth functions with compact support) **473Be**, 473C-473E, 474Ba, *474K*
$\mathcal{D}_r$ (smooth functions with compact support) **474A**, 474Ba
∂ (in ∂A, where A is a subset of a topological space) *see* boundary (**4A2A**); (in ∂T, $\partial^\xi T$, where T is a tree) *see* derived tree (**421N**)
∂^* *see* essential boundary (**475B**)
$\partial^\$$ *see* reduced boundary (**474G**)
diam (in $\operatorname{diam} A$) = diameter
div (in $\operatorname{div}\phi$) *see* divergence (**474B**)
dom (in $\operatorname{dom} f$): the domain of a function f

ess sup *see* essential supremum (**243Da**)
exp *see* exponentiation (**4A6L**)
$\mathbb{E}$ (in $\mathbb{E}(X)$, expectation of a random variable) **271Ab**; (in $\mathbb{E}(X|\mathrm{T})$, conditional expectation) *465M*

^{f} (in $\mathfrak{A}^f$) **361Ad**
F_σ set *412Xf*, 414Ye, *443Jb*, *466Yc*, **4A2A**, 4A2C, 4A2Fi, *4A2Ka*, *4A2Lc*
$\mathcal{F}$ (in $\mathcal{F}(B\uparrow)$, $\mathcal{F}(B\downarrow)$) **323D**
f-algebra 241H, 241 *notes*, **352W**, 352Xk-352Xn, 353O, 353P, 353Xd, 353Yg, 353Yh, 361Eh, 363B, 364C-364E, 367Yf

G_δ set **264Xd**, 419B, *419Xb*, *434Xb*, 437We, 437Yo, *443Jb*, 461Xi, 461Xj, 471Db, *493Yd*, **4A2A**, 4A2C, 4A2Fd, 4A2Kf, 4A2L, 4A2Q, 4A5R
$GL(r,\mathbb{R})$ (the general linear group) **446A**
grad f *see* gradient (**473B**), 473C, 473Dd, 473H, 473I, 473J, 473K, 473L

h (in $h(\pi)$) *see* entropy (**385M**); (in $h(\pi,A)$) **385M**, 385N-385P, 385Xq, 385Yb, 387C
H (in $H(A)$) *see* entropy of a partition (**385C**); (in $H(A|\mathfrak{B})$) *see* conditional entropy (**385D**)
$\mathcal{HL}^1$ **483M**
$\mathcal{HL}^1_V$ **483Yg**
HL^1 **483M**, 483Xj

I (in I_σ, where $\sigma \in \mathbb{N}^k$) **421A**; (in $I_\nu(f)$ *see* gauge integral (**481C**)
$I^\parallel$ *see* split interval (**343J**)
int* *see* essential interior (**474B**)

JKR-space *439K*

K-analytic set, topological space **422F**, 422G-422K, 422Xb, 422Xd, 422Xf, 422Ya, 422Yb, 422Yd-422Yf, 423C-423E, §432, 434B, 434Dc, 434Hb, 434Jf, 434Ke, 434Xp, 434Xq, *435Fb*, 438P, 438Q, 455Ye, 467Xg
K-countably determined topological space **467H**, 467Xa, 467Xb, 467Xh; *see also* weakly K-countably determined (**467Hb**)
k-space **462J**
k-Vietoris topology **476Cd**, 476Xc, 476Xd, 495N, 495O, 495Xm, 495Xn
K_σ set 451Xl, **4A2A**
$\mathrm{K}_{\sigma\delta}$ set **422Yb**

ℓ^1 (in $\ell^1(X)$) **242Xa**, 243Xl, 246Xd, 247Xc, 247Xd, 354Xa, 356Xc, *464R*
ℓ^1 ($= \ell^1(\mathbb{N})$) 246Xc, 354M, 354Xd, 356Xl
ℓ^2 **244Xn**, 282K, 282Xg, 355Yb, 371Ye, 376Yh, 376Ym, 376Yn
ℓ^p (in $\ell^p(X)$) **244Xn**, 354Xa, 466C
ℓ^∞ (in $\ell^\infty(X)$) **243Xl**, **281B**, 281D, 354Ha, 354Xa, 361D, 361L, 461Xd, 464F, 464R, 464Xb, 464Z, 466I, 466Xl, 466Xm, 483Yg, 4A2Ib
ℓ^∞ ($= \ell^\infty(\mathbb{N})$) 243Xl, 354Xj, 356Xa, 371Yd, 383J, 4A4Id
ℓ^∞-complemented subspace **363Yd**
$\ell^\infty/\boldsymbol{c}_0$ 466Ib
L-space **354M**, 354N-354P, 354R, 354Xt, 354Yj, 356N, 356P, 356Q, 356Xm, 356Yf, 362A, 362B, 362Yj, 365C, 365Xc, 365Xd, 367Xn, 369E, 371A-371E, 371Xa, 371Xb, 371Xf, 371Ya, 376M, 376P, 376Yi, 376Yj, 436Ib, 436Yb, 437B, 437C, 437E, 437F, 437H, 437I, 437Yc, 437Yj, *444E*, 461Xl, 467Yb, 495K, 495Xh, 495Ya
$\mathcal{L}^0$ (in $\mathcal{L}^0(\mu)$) 121Xb, 121Ye, §241 (**241A**), §245, 253C, 253Ya, 443Ae, 443G; (in $\mathcal{L}^0(\Sigma)$) 345Yb, **364C**, 364D, 364E, 364J, 364Yi, 463A-463D, 463I, 463Xa-463Xc, 463Xa, 463Xb, 463Xc, 463Xf, 463Xl, 463Xm, 463Yd, 463Za; *see also* L^0 (**241C**), $\mathcal{L}^0_{\rm strict}$ (**241Yh**), $\mathcal{L}^0_{\mathbb{C}}$ (**241J**)
$\mathcal{L}^0_{\rm strict}$ **241Yh**
$\mathcal{L}^0_{\mathbb{C}}$ (in $\mathcal{L}^0_{\mathbb{C}}(\mu)$) **241J**, 253L
L^0 (in $L^0(\mu)$) §241 (**241A**), 242B, 242J, *243A*, *243B*, *243D*, *243Xe*, 243Xj, §245, 253Xe, 253Xf, 253Xg, 271De, 272H, 323Xf, 345Yb, *352Xk*, 364Jc, 376Yc, 416Xj, 418R, 418S, 418Xx, 438Xg, 441Kb, 442Xg, *466Xl*, 493F; (in $L^0_{\mathbb{C}}(\mu)$) **241J**; (in $L^0(\mathfrak{A})$) §364 (**364A**), 368A-368E, 368H, 368K, 368M, 368Qb, 368R, 368S, 368Xa, 368Xe, 368Ya, 368Yd, 368Yi, §369, *372H*, §375, 376B, 376Yb, 393M, 394I, 443A, 443G, 443Jb, 443Xi; (in $L^0_{\mathbb{C}}(\mathfrak{A})$) **364Yn**, 495Yb; *see also* $\mathcal{L}^0$ (**241A**, **364C**)
$\mathcal{L}^1$ (in $\mathcal{L}^1(\mu)$) *122Xc*, 242A, 242Da, 242Pa, 242Xb, 443P, 444P-444R; (in $\mathcal{L}^1_{\rm strict}(\mu)$ **242Yg**, 341Ye; (in $\mathcal{L}^1_{\mathbb{C}}(\mu)$ **242P**, *255Yn*; (in $\mathcal{L}^1_V(\mu)$) **253Yf**; *see also* L^1, $\|\ \|_1$
L^1 (in $L^1(\mu)$) §242 (**242A**), *243De*, 243F, 243G, 243J, 243Xf, *243Xg*, *243Xh*, 245H, 245J, 245Xh, 245Xi, §246, §247, §253, 254R, 254Xp, 254Ya, 254Yc, 255Xc, 257Ya, *282Bd*, 327D, 341Ye, 354M, 354Q, 354Xa, 365B, 376N, 376Q, 376S, 376Yl, 443Pf, 444S, 467Yb, 483Mb, 495L; (in $L^1_V(\mu)$) **253Yf**, 253Yi, 354Yl; (in $L^1(\mathfrak{A},\bar\mu)$ or $L^1_{\bar\mu}$) §365 (**365A**), 366Yc, 367J, 367K, 367U, *367Ys*, 369E, *369N*, *369O*, *369P*, 371Xc, 371Yb, 371Yc, 371Yd, 372B, 372C, 372E, 372G, 372Xc, 376C, 386E, 386F, 386H, 465R, 495Yb; *see also* $\mathcal{L}^1$, $L^1_{\mathbb{C}}$, $\|\ \|_1$
$L^1_{\mathbb{C}}(\mu)$ **242P**, 243K, 246K, *246Yl*, *247E*, 255Xc; (as Banach algebra, when μ is a Haar measure) 445H, 445I, 445K, 445Yk; *see also* convolution of functions
$\mathcal{L}^2$ (in $\mathcal{L}^2(\mu)$) 244Ob, 253Yj, §286, 465E, 465F; (in $\mathcal{L}^2_{\mathbb{C}}(\mu)$) 284N, 284O, 284Wh, 284Wi, 284Xi, 284Xk-284Xm, 284Yg; *see also* L^2, $\mathcal{L}^p$, $\|\ \|_2$
L^2 (in $L^2(\mu)$) 244N, 244Yk, *247Xe*, 253Xe, 355Ye, 372 *notes*, 416Yg, 444V, 444Xt, 444Xu, 444Yk, 445R, 445S, 445Xm, 445Xn, *456J*, 456Yd, *465E*; (in $L^2_{\mathbb{C}}(\mu)$) 282K, 282Xg, 284P; (in $L^2(\mathfrak{A},\bar\mu)$) 366K, 366L, 366Xh, 395A, 395Xb; *see also* $\mathcal{L}^2$, L^p, $\|\ \|_2$
$\mathcal{L}^p$ (in $\mathcal{L}^p(\mu)$) §244 (**244A**), 246Xg, 252Ym, 253Xh, 255K, *255Og*, 255Yc, 255Yd, 255Yl, 255Ym, 261Xa, 263Xa, 273M, 273Nb, 281Xd, 282Yc, 284Xj, 286A, 411Gh, 412Xc, 415Pa, 415Yj, 415Yk, 416I, 443G, 444R-444U, 444Xs, 444Yh, 444Ym, 472F, 473Ef; *see also* L^p, $\mathcal{L}^2$, $\|\ \|_p$

L^p (in $L^p(\mu)$, $1 < p < \infty$) §244 (**244A**), 245G, 245Xj, 245Xk, 245Yg, 246Xh, 247Ya, 253Xe, 253Xi, 253Yk, 255Yf, 354Xa, 354Yk, 366B, 376N, *411Xe*, 418Yj, 441Kc, 442Xg, 443A, 443G, 443Xi, 444M; (in $L^p(\mathfrak{A},\bar\mu) = L^p_{\bar\mu}$, $1 < p < \infty$) §366 (**366A**), 369L, 371Gd, 372Xp, 372Xq, 372Yb, 373Bb, 373F, 376Xb;(in $L^p_{\mathbb{C}}(\mu)$, $1 < p < \infty$) 354Yk;(in $L^p(\mathfrak{A},\bar\mu)$, $0 < p < 1$) **366Ya**, 366Yg, 366Yi;*see also* $\mathfrak{L}^p$, $\|\ \|_p$

$\mathfrak{L}^\infty$ (in $\mathfrak{L}^\infty(\mu)$) **243A**, 243D, *243I*, 243Xa, 243Xl, 243Xn, 481Xg; (in $\mathfrak{L}^\infty(\Sigma)$) 341Xe, 363H, 414Xt, 437B-437E, 437H, 437Ib, 437Xe, 437Yd;*see also* L^∞

$\mathfrak{L}^\infty_{\mathbb{C}}$ **243K**, 437Yb

$\mathfrak{L}^\infty_{\text{strict}}$ **243Xb**, 363I

L^∞ (in $L^\infty(\mu)$) §243 (**243A**), 253Yd, 341Xe, *352Xk*, 354Hc, 354Xa, 363I, 376Xn, 418Yi, 442Xg, 463Yc; (in $L^\infty(\mathfrak{A})$) §363 (**363A**), 364K, 364Xh, 365L, 365M, 365N, *365Xk*, 367Oc, 368Q, 394N, 436Xp, 437B, 437J, 443Ad, 443Jb, 447Yb, 457A; *see also* $\mathfrak{L}^\infty$, $L^\infty_{\mathbb{C}}$, $\|\ \|_\infty$

$L^\infty_{\mathbb{C}}$ **243K**, 243Xm

L^τ (where τ is an extended Fatou norm) 369G, 369J, 369K, 369M, 369O, 369R, 369Xi, 374Xd, 374Xi; *see also* Orlicz space (**369Xd**), L^p, $M^{1,\infty}$ (**369N**), $M^{\infty,1}$ (**369N**)

L (in $\mathrm{L}(U;V)$, space of linear operators) 253A, 253Xa, 351F, 351Xd, 351Xe, *4A4Bc*

L$^\sim$ (in $\mathrm{L}^\sim(U;V)$, space of order-bounded linear operators) §355 (**355A**), 356Xi, 361H, 361Xc, 361Yb, 363Q, 365K, 371B-371E, 371Gb, 371Xb-371Xe, 371Ya, 371Yc-371Ye, 375Kb, 376J, 376Xe, *376Ym*; *see also* order-bounded dual (**356A**)

L$^\sim_c$ (in $\mathrm{L}^\sim_c(U;V)$) **355G**, 355I, 355Yi, 376Yf; *see also* sequentially order-continuous dual (**356A**)

L$^\times$ (in $\mathrm{L}^\times(U;V)$) **355G**, 355H, 355J, 355K, 355Yg, 355Yi, 355Yj, 371B-371D, 371Gb, 376D, 376E, 376H, 376K, 376Xj, 376Yf; *see also* order-continuous dual (**356A**)

lim (in $\lim_{\mathcal{F}}$) **2A3S**; (in $\lim_{x\to\mathcal{F}}$) **2A3S**

lim inf (in $\liminf_{n\to\infty}$) §1A3 (**1A3Aa**), 2A3Sg; (in $\liminf_{\delta\downarrow 0}$) **2A2H**; (in $\liminf_{x\to\mathcal{F}}$) **2A3S**

lim sup (in $\limsup_{n\to\infty}$) §1A3 (**1A3Aa**), 2A3Sg; (in $\limsup_{\delta\downarrow 0}$) **2A2H**, 2A3Sg; (in $\limsup_{x\to\mathcal{F}} f(x)$) **2A3S**; (in $\limsup_{i\to\mathcal{F}} A_i$) 446G, **4A2A**, 4A2T

$\ln^+$ **275Yd**

M (in $M(\mathfrak{A})$, space of bounded finitely additive functionals) 362B, 362E, 363K, 436Xr, 437J; (when $\mathfrak{A} = \mathcal{P}X$) 464G-464M, 464O-464Q

M-space **354Gb**, 354H, 354L, 354Xq, 354Xr, 356P, 356Xj, 363B, 363O, 371Xd, 376M, *449D*; *see also* order-unit norm (**354Ga**)

M^0 (in $M^0(\mathfrak{A},\bar\mu) = M^0_{\bar\mu}$) **366F**, 366G, 366H, 366Yb, 366Yd, 366Yg, 373D, 373P, 373Xk

$\mathfrak{M}^{0,\infty}$ **252Yp**

$M^{0,\infty}$ (in $M^{0,\infty}(\mathfrak{A},\bar\mu) = M^{0,\infty}_{\bar\mu}$) **373C**, 373D, 373E, 373F, *373I*, *373Q*, *373Xo*, 374B, 374J, 374L

$M^{1,0}$ (in $M^{1,0}(\mathfrak{A},\bar\mu) = M^{1,0}_{\bar\mu}$) **366F**, 366G, 366H, 366Ye, 369P, 369Q, 369Yh, 371F, 371G, 372D, 372Ya, 373G, 373H, 373J, 373S, 373Xp, 373Xr, 374Xe

$M^{1,\infty}$ (in $M^{1,\infty}(\mu)$) *234Yd*, **244Xl**, 244Xm, 244Xo, 244Yc; (in $M^{1,\infty}(\mathfrak{A},\bar\mu) = M^{1,\infty}_{\bar\mu}$) **369N**, 369O-369Q, 369Xi-369Xk, 369Xm, 369Xq, §373, 374A, 374B, *374M*

$M^{\infty,0}$ (in $M^{\infty,0}(\mathfrak{A},\bar\mu)$) **366Xd**, 366Yc

$M^{\infty,1}$ (in $M^{\infty,1}(\mathfrak{A},\bar\mu) = M^{\infty,1}_{\bar\mu}$) **369N**, 369O, 369P, 369Q, 369Xi, 369Xj, 369Xk, 369Xl, 369Yh, *373K*, *373M*, 374B, *374M*, 374Xa, 374Ya

M_{m} *see* measurable additive functional (**464I**)

M_{pnm} *see* purely non-measurable additive functional (**464I**)

M_r (space of $r \times r$ matrices) 446A

M_t (space of signed tight Borel measures) 437Fb, **437G**, 437Ia, 437Xh, 437Xl, 437Xt, 437Yb, 437Yi, 437Ym, 437Yo

M_σ (in $M_\sigma(\mathfrak{A})$, space of countably additive functionals) 362B, 362Xd, 362Xh, 362Xi, 362Ya, 362Yb, 363K, 437C-437F, 437Xe, 438Xa

M_τ (in $M_\tau(\mathfrak{A})$, space of completely additive functionals) 326Yp, 327D, 362B, 362D, 362Xd, 362Xg, 362Xi, 362Ya, 362Yb, 363K, 438Xa, *464H*, 464Ja, *464R*; (in $M_\tau(X)$, space of signed τ-additive measures) 437F, **437G**, 437H, 437L, 437M, 437Xf, 437Xh, 437Yg-437Yi, 444E, 444S, 444Xd-444Xf, 444Yb, 444Yc, 444Yg, 445Yi, 445Yj

$\tilde M^+$ **437Jd**, 437K

med (in $\operatorname{med}(u,v,w)$) **352Xd**

$\mathbb{N}$ 3A1H; *see also* $\mathcal{P}\mathbb{N}$
$\mathbb{N} \times \mathbb{N}$ 111Fb
$\mathbb{N}^{\mathbb{N}}$ 372Xi, 421A, 424Cb, 4A2Ub
—— as topological space 421A, *421H-421K*, *421M*, *421Xf*, *421Xo*, *431D*, 4A3Fb

On (the class of ordinals) **3A1E**

p (in $p(t)$) **386G**, 386H
$\mathcal{P}$ (in $\mathcal{P}X$) 311Ba, 311Xe, *312B*, *312C*, *313Ec*, *313Xf*, 363S, 382Xc, 383Yb, *438A*, *438Xa*;(usual measure on $\mathcal{P}X$) **254J**, 254Xf, 254Xq, 254Yd, 441Xg, 464A-464D, 464H-464L, 464O-464Q, 464Ya; (usual topology on $\mathcal{P}X$) 423Ye, 463A, 463F, 463G, **4A2A**, 4A2Ud
$\mathcal{P}\mathbb{N}$ 1A1Hb, 2A1Ha, 2A1Lb, 315O, 316Yo, 324Yg, *326Yg*, 374Xk; (usual measure on) 273G, 273Xd, 273Xe
$\mathcal{P}\mathbb{N}/[\mathbb{N}]^{<\omega}$ 517Yd
$\mathcal{P}(\mathbb{N}^{\mathbb{N}})$ 316Yg
$\mathcal{P}\omega_1$ 419F, 419G, 419Yb
p.p. ('presque partout') **112De**
per (in per E, the distributional perimeter of E) **474D**
$\Pr(X > a)$, $\Pr(\boldsymbol{X} \in E)$ etc. **271Ad**, 434Xy

$\mathbb{Q}$ (the set of rational numbers) 111Eb, 1A1Ef, 364Yg, 439S, 442Xc; (as topological group) 445Xa
q (in $q(t)$) **385A**, 386M; (in $q_p(x,y)$, $q_{\|\,\|}(x,y)$) **467C**

$\mathbb{R}$ (the set of real numbers) 111Fe, 1A1Ha, 2A1Ha, 2A1Lb, 352M, 4A1Ac, 4A2Gf, 4A2Ua; (as topological group) 442Xc, 445Ba, 445Xa, 445Xk
$\mathbb{R}^X$ 245Xa, 256Ye, 352Xk, 375Ya, 3A3K, 435Xc, 4A4Bb; (as linear topological space) 4A4H;*see also* Euclidean metric, Euclidean topology, pointwise convergence
$\mathbb{R}^X|\mathcal{F}$ *see* reduced power (**351M**)
$\frac{\mathbb{R}}{\mathbb{C}}$ 2A4A
$\overline{\mathbb{R}}$ *see* extended real line (§135)
$\mathbb{R} \setminus \{0\}$ (the multiplicative group) 441Xf
r (in $r(u)$, where u is in a Banach algebra) *see* spectral radius (**4A6G**); (in $r(T)$, where T is a tree) *see* rank (**421N**)
RO (in $\mathrm{RO}(X)$) *see* regular open algebra (**314Q**)
R-stable set of functions **465S**, 465T-465V, 465Yc, 465Yd

S (in $S(\mathfrak{A})$) 243I, §361 (**361D**), 363C, 363Xg, 364K, 364Xh, *365F*, 367Oc, 368Q, *369O*; (in $S^f \cong S(\mathfrak{A}^f)$) *242M*, *244H*, 365F, 365Gb, *369O*, *369P*; (in $S(\mathfrak{A})^\sim$) 362A; (in $S(\mathfrak{A})^\sim_c$) 362Ac; (in $S(\mathfrak{A})^\times$) 362Ad; (in $S_{\mathbb{C}}(\mathfrak{A})$) **361Xj**, 361Yd; (in $S(\mathfrak{A}; G)$ **493Ya**; (in $S_{\boldsymbol{t}}(f,\nu)$ **481Bc**
$\mathcal{S}$ (in $\mathcal{S}(\mathcal{E})$) *see* Souslin's operation (**421B**)
$\mathcal{S}$ *see* rapidly decreasing test function (**284A**)
S^1 (the unit circle, as topological group) *see* circle group
S^{r-1} (the unit sphere in $\mathbb{R}^r$) *see* sphere
S_6 (the group of permutations of six elements) 384 *notes*
$_{sf}$ (in μ_{sf}) *see* semi-finite version of a measure (**213Xc**); (in μ^*_{sf}) **213Xf**, 213Xg, 213Xk

T_0 topology 437Xi, *451M*, *476Xf*, **4A2A**
T_1 topology **4A2A**, 4A2F
T_2 topology *see* Hausdorff (**2A3E**, **3A3Aa**)
$\mathcal{T}^{(0)}$ (in $\mathcal{T}^{(0)}_{\bar\mu,\bar\nu}$) **371F**, 371G, 371H, 372D, 372Xb, 372Yb, 372Yc, 373B, 373G, 373J, 373R, 373S, 373Xp, 373Xq, 373Xr, 373Xu, 373Xv
$\mathcal{T}$ (in $\mathcal{T}_{\bar\mu,\bar\nu}$) 244Xm, 244Xo, 244Yc, 246Yc, §373 (**373A**), 444Ym; *see also* $\mathcal{T}$-invariant (**374A**)
$\mathcal{T}^\times$ (in $\mathcal{T}^\times_{\bar\mu,\bar\nu}$) §373 (**373Ab**), 376Xa, 376Xh
$\mathcal{T}$-invariant extended Fatou norm **374Ab**, 374B-374D, 374Fa, 374Xb, 374Xd-374Xj, 374Yb, 444Yf, 444Yj
$\mathcal{T}$-invariant set **374Aa**, 374M, 374Xa, 374Xi, *374Xk*, 374Xl, 374Ya, 374Ye
$\mathfrak{T}_m$ *see* convergence in measure (**245A**)
$\mathfrak{T}_p$ *see* pointwise convergence (**462Ab**)

$\mathfrak{T}_s$ (in $\mathfrak{T}_s(U,V)$) 373M, 373Xq, 376O, **3A5E**, 465E, *465F*; *see also* weak topology (**2A5I**), weak* topology (**2A5Ig**)

U (in $U(x,\delta)$) **1A2A**; (in $U(A,\delta)$) **476F**
upr (in $\text{upr}(a,\mathfrak{C})$) **314V**, 314Xg, 333Xa, 365Rc, 386B, 394G, 394I, 394K-349N, 458Hc
usco-compact relation **422A**, 422B, 422C, 422D, 422E, *422F*, 422G, 422Xa, 432Xh, 443Yl, 467Ha

Var (in $\text{Var}(X)$) *see* variance (**271Ac**); (in $\text{Var}_D f$, $\text{Var} f$) *see* variation (**224A**)

w (in $w(X)$) *see* weight (**4A2A**)
w^*-topology *see* weak* topology **2A5Ig**
W (in $W_{\boldsymbol{t}}$) **481Bb**

$\mathbb{Z}$ (the set of integers) 111Eb, 1A1Ee; (as topological group) 255Xe, 441Xa, 445B
$\mathbb{Z}_2$ (the group $\{0,1\}$) **311Bc**, 311D
$\mathcal{Z}$ *see* asymptotic density zero ideal (**491A**)
$\mathfrak{Z} = \mathcal{P}\mathbb{N}/\mathcal{Z}$ **491I**, 491J, 491K, 491N-491P, 491Xo-491Xq, 491Yj-491Yl
ZFC *see* Zermelo-Fraenkel set theory

α-favourable *see* weakly α-favourable (**451U**, **4A2A**)

β_r (volume of unit ball in $\mathbb{R}^r$) 252Q, 252Xh, *265F*, *265H*, *265Xa*, *265Xb*, *265Xe*, *474N*, *474R*, *474Xe*
βX *see* Stone-Čech compactification (**4A2I**)

Γ-function **225Xj**, *225Xk*, 252Xh, 252Yk, 252Yn, 255Xj

Δ (the modular function of a topological group) **442I**; (in $\Delta(\theta)$) **464G**
Δ-system *2A1Pa*, **4A1D**

θ-refinable *see* hereditarily weakly θ-refinable

λ^∂ (in λ^∂_E) *see* perimeter measure (**474F**)

μ_G (standard normal distribution) **274Aa**
$\bar{\mu}_L$ (in §373) **373C**

$\nu_{\boldsymbol{X}}$ *see* distribution of a random variable (**271C**)

π-base for a topology 411Ng, **4A2A**, 4A2Gi
π-λ Theorem *see* Monotone Class Theorem (136B)

σ-additive *see* countably additive (**231C**, **326E**)
σ-algebra of sets §111 (**111A**), 136Xb, 136Xi, 212Xk, *314D*, 314M, *314N*, 314Yd, *316D*, 322Ya, *326Ys*, *343D*, 344D, *362Xg*, 363H, 382Xd; *see also* Baire property algebra (**4A3Q**), Baire σ-algebra (**4A3K**), Borel σ-algebra (**111G**, **4A3A**), cylindrical σ-algebra (**4A3T**), standard Borel space (**424A**)
σ-algebra defined by a random variable **272C**, 272D
σ-compact topological space 422Xb, *441Xh*, 467Xb, 495N, 495O, 495Xm, **4A2A**
—— locally compact group 443P, 443Xm, 443Yg, *447E-447G*, 4A5El, 4A5S
σ-complete *see* Dedekind σ-complete (**241Fb**, **314Ab**)
σ-discrete *see* σ-metrically-discrete (**4A2A**)
σ-disjoint family of sets **4A2A**, 4A2Lg
σ-field *see* σ-algebra (**111A**)
σ-finite measure algebra **322Ac**, 322Bc, 322C, 322G, *322M*, 323Gb, *323Ya*, 324K, 325Eb, 327B, 331N, 331Xl, 362Xd, 367Nd, 367Q, 367Xq, 367Xs, 369Xg, 383E, 448Xj, 493Xe
σ-finite measure (space) **211D**, 211L, 211M, 211Xe, *212G*, *213Ha*, 213Ma, 214Ia, 214Ja, 215B, 215C, 215Xe, 215Ya, 215Yb, *216A*, *232B*, *232F*, 234Fe, 235O, 235R, 235Xe, 235Xk, 241Yd, *243Xi*, 245Eb, 245K, 245L, *245Xe*, 251K, 251Wg, 252B-252E, 252H, 252P, 252R, 252Xc, *252Yb*, 252Yl, *322Bc*, *342Xi*, 362Xh, 365Xp, 367Xr, *376I*, *376J*, *376N*, *376S*, 411Ge, 411Ng, 411Xd, 412Xh, 412Xi, 412Xo, *414D*, 415D, *415Xg*, 415Xh, *415Xi*, 415Xo, 415Xp, *416Yd*, 417Xh, 417Xu, 418G, *418R-418T*, *418Xh*, 418Ye, 433Xb, *434R*, *436Yd*, *438Bc*, *438S*, *438Yc*, *452I*, *441Xe*, *441Xh*, 443Xm, *451Oc*, 451Xl, 463C-463E, 463H, 463I, 463L, 463Xb, 463Xd, 463Xf, 463Xk, 463Yd, 495H, 495I, 495Xa
σ-generating set in a Boolean algebra **331E**

σ-ideal (in a Boolean algebra) **313E**, 313Pb, 313Qb, *314C*, *314D*, 314L, *314N*, *314Yd*, *316C*, *316D*, *316Xi*, *316Ye*, 321Ya, 322Ya, *392Xd*

—— (of sets) **112Db**, 211Xc, 212Xf, 212Xk, *313Ec*, 322Ya, *363H*, *464Pa*, 4A1Cb; *see also* translation-invariant σ-ideal

σ-interpolation property (in a partially ordered set) **466G**, 466H, 491Yl

σ-isolated family of sets 438K, 438Ld, *438N*, 466D, 466Eb, 467Pb, **4A2A**

vthreeσ-linked Boolean algebra **391L**, 391M, 391N, 391Xj, 391Yb, 391Yd, 391Ye, 391Z, 393Xf; σ-m-linked Boolean algebra **391Yh**, 393Ya

σ-metrically-discrete family of sets **4A2A**, 4A2Lg

σ-order complete *see* Dedekind σ-complete (**314Ab**)

σ-order-continuous *see* sequentially order-continuous (**313H**)

σ-refinement property (for subgroups of $\operatorname{Aut}\mathfrak{A}$) **448K**, 448L-448O

σ-subalgebra of a Boolean algebra **313E**, 313F, 313G, 313Xd, 313Xe, 313Xq, 314Eb, 314Fb, 314Hb, 314Jb, 314Xc, 315Yc, *321G*, 321Xb, 322M, *323Z*, 324Xb, 326Fg, 331E, 331G, 364Xc, 364Xu, 366I, 4A1O; *see also* order-closed subalgebra

σ-subalgebra of sets §233 (**233A**), 321Xb, 323Xb, *412Ab*, 465Xg

σ-subhomomorphism between Boolean algebras **375E**, 375F-375H, 375Xd, 375Yc, 375Ye

(σ,∞)-distributive *see* weakly (σ,∞)-distributive (**316G**)

$\Sigma_{\rm um}$ (algebra of universally measurable sets) **434D**, 434T

$\Sigma_{\rm uRm}$ (algebra of universally Radon-measurable sets) **434E**, 437Ib

$\sum_{i\in I} a_i$ **112Bd**, *222Ba*, **226A**, **4A4Bj**, 4A4Ie

τ (in $\tau(\mathfrak{A})$) *see* Maharam type (**331Fa**); (in $\tau_{\mathfrak{C}}(\mathfrak{A})$) *see* relative Maharam type (**333Aa**)

τ-additive functional on a Boolean algebra *see* completely additive (**326J**)

τ-additive measure *256M*, 256Xb, 256Xc, **411C**, 411E, §414, 415C, 415L-415M, 415Xn, §417, 418Ha, 418Xi, 418Ye, 419A, 419D, 419H, 432D, 432Xc, 434G, 434Ha, 434Ja, 434Q, 434R, 434Xa, 434Yl, *435D*, 435E, 435Xa, 435Xc-435Xe, 436Xg, 436Xj, 437Kc, 439Xh, 444Yb, 451Xm, 452C, 453Dc, 453H, 454Pb, 456J, 456K, 459F, 461E, *462Yc*, 465S, 465T, *466H*, 466Xc, *466Xm*, 476F, 481N, 482Xd, 491Ce; *see also* quasi-Radon measure (**411Ha**), signed τ-additive measure (**437G**)

—— positive linear operator 437Xc

—— product measure §417 (**417G**), 418Xg, 434R, 437L, 453I, 459G, 465S, 465T, 491F, 491Ym, 494B, 494Xa; *see also* quasi-Radon product measure (**417R**), Radon product measure (**417R**)

τ-generating set in a Boolean algebra 313Fb, 313M, **331E**, 331F, 331G, 331Yb, 331Yc

τ-negligible *see* universally τ-negligible (**439Xh**)

τ-regular *see* τ-additive (**411C**)

Φ *see* normal distribution function (**274Aa**)

χ (in χA, where A is a set) **122Aa**; (in χa, where a belongs to a Boolean ring) **361D**, 361Ef, 361L, 361M, 364K; (the function $\chi:\mathfrak{A}\to L^0(\mathfrak{A})$) 364Kc, 367R

ψ (in ψ_E) *see* canonical outward-normal function (**474G**)

ω (the first infinite ordinal) **2A1Fa**, 3A1H; (in $[X]^{<\omega}$) 3A1Cd, 3A1J

ω^ω-bounding Boolean algebra *see* weakly σ-distributive (**316Yg**)

ω_1 (the first uncountable ordinal) **2A1Fc**, 419F, 419G, 419Yb, 421P, 435Xb, 435Xi, 435Xk, *438C*, 463Xh, 463Yd, 463Ye, 4A1A, 4A1Bb, 4A1Eb, 4A1M, 4A1N; *see also* $\mathcal{P}\omega_1$

—— (with its order topology) 411Q, 411R, 412Yb, 434Kc, 434Yi, 435Xi, 437Xt, 439N, 439Yf, 439Yh, 454Yc, 4A2Sb, 4A3J, 4A3P, 4A3Xg

ω_1-saturated ideal in a Boolean algebra **316C**, 316D, 341Lh, *344Yd*, *344Ye*, 431Yc

ω_1+1 (with its order topology) 434Kc, 434Xb, 434Xf, 434Xk, 434Xl, 434Yi, 435Xa, 435Xc, 435Xi, 437Xk, 463D

—— *see also* $\mathcal{P}\omega_1$

ω_2 **2A1Fc**, 4A1Ea

ω_ξ (the ξth uncountable initial ordinal) **3A1E**, 438Xe

ω (in $\omega(F\restriction A)$) *see* oscillation (**483Oa**)

$\overline{\ }$ *see* closure (**2A2A**, **2A3Db**)

$^-$ (in $\bar{h}(u)$, where h is a Borel function and $u \in L^0$) **241I**, 241Xd, 241Xi, 245Dd, **364I**, 364J, 364Xg, 364Xq, 364Yd, 364Ye, 367I, 367S, 367Xl, 367Yr, 495Yb; (when h is universally measurable) **434T**

$=_{\text{a.e.}}$ **112Dg**, 112Xh, **222E**, 241C, **444Ob**

$\leq_{\text{a.e.}}$ **112Dg**, 112Xh, **212B**

$\geq_{\text{a.e.}}$ **112Dg**, 112Xh, **233I**

$'$ (in U') see algebraic dual; (in T') *see* adjoint operator (**3A5Ed**)

$*$ (in $f * g$, $u * v$, $\lambda * \nu$, $\nu * f$, $f * \nu$) *see* convolution (**255E**, **255O**, **255Xe**, **255Yn**, **444A**, **444H**, **444K**, **444O**, **444S**)

* (in weak*) *see* weak* topology (**2A5Ig**); (in $U^* = \mathrm{B}(U;\mathbb{R})$, linear topological space dual) *see* dual (**2A4H**, **4A4Bd**); (in u^*) *see* decreasing rearrangement (**373C**); (in μ^*) *see* outer measure defined by a measure (**132B**)

$_*$ (in μ_*) *see* inner measure defined by a measure (**113Yh**, **413D**)

$^\sim$ (in $U^\sim$) *see* order-bounded dual (**356A**); (local usage in §471) **471M**

$^\sim_c$ (in $U^\sim_c$) *see* sequentially order-continuous dual (**356A**)

$^\sim_\sigma$ (in $U^\sim_\sigma$) *see* sequentially smooth dual (**437Aa**)

$^\sim_\tau$ (in $U^\sim_\tau$) *see* smooth dual (**437Ab**)

$\tilde{}$ (in $\tilde{a}$, $\tilde{f}$, $\tilde{u}$) **443Af**, 443Xe, 443Xi, 444R, 444V, 444Xs, 444Xt

$^\times$ (in $U^\times$) *see* order-continuous dual (**356A**); (in $U^{\times\times}$) *see* order-continuous bidual

$\setminus$ (in $E \setminus F$, 'set difference') **111C**

$\triangle$ (in $E \triangle F$, 'symmetric difference') **111C**, 311Ba

$\cup$, $\cap$ (in a Boolean ring or algebra) **311Ga**, 313Xi, 323B

$\setminus$, $\triangle$ (in a Boolean ring or algebra) **311Ga**, 323B

$\subseteq$, $\supseteq$ (in a Boolean ring or algebra) **311H**, *323Xa*

$\bigcup$ (in $\bigcup_{n\in\mathbb{N}} E_n$) **111C**; (in $\bigcup \mathcal{A}$) **1A1F**

$\bigcap$ (in $\bigcap_{n\in\mathbb{N}} E_n$) **111C**; (in $\bigcap \mathcal{E}$) **1A2F**

$\vee$, $\wedge$ (in a lattice) 121Xa, **2A1Ad**; (in $A \vee B$, where A, B are partitions of unity in a Boolean algebra) **385F**

$\int$ (in $\int f$, $\int f d\mu$, $\int f(x)\mu(dx)$) **122E**, **122K**, **122M**, 122Nb, **>363L**; (in $\int_A f$) **131D**, **214D**, 235Xf, 482G, 482H, *482Yb*; *see also* subspace measure; (in $\int u$) **242Ab**, 242B, 242D, **363L**, **365D**, 365Xa; (in $\int_A u$) **242Ac**; (in $\int_a u$) **365D**, 365Xb; *see also* upper integral, lower integral (**133I**)

$\overline{\int}$ *see* upper integral (**133I**)

$\underline{\int}$ *see* lower integral (**133I**)

$\text{H}\!\int$, $\text{H}\!\int_\alpha^\beta$ *see* Henstock integral (**483A**)

$\text{Pf}\!\int$ *see* Pfeffer integral (**484G**)

$\text{R}\!\int$ *see* Riemann integral (**134K**)

$\restriction$ (in $f \restriction A$, the restriction of a function to a set) **121Eh**

$|\ |$ (in a Riesz space) 241Ee, **242G**, **352C**, 352D, 352Fb, 352K, 354Aa, 354Bb

$\|\ \|_e$ *see* order-unit norm (**354Ga**)

$\|\ \|_1$ (on $L^1(\mu)$) §242 (**242D**), 246F, 253E, 275Xd, 282Ye, 483Mb; (on $\mathcal{L}^1(\mu)$) **242D**, 242Yg, 273Na, 273Xi, 415Pb, 473Da; (on $L^1(\mathfrak{A},\bar{\mu})$) **365A**, 365B, 365C, 386E, 386F; (on the ℓ^1-sum of Banach lattices) **354Xb**, 354Xo

$\|\ \|_2$ **244Da**, 273Xj, 282Yf, 366Yh, 473Xa; *see also* L^2, $\|\ \|_p$

$\|\ \|_p$ (for $1 < p < \infty$) §244 (**244Da**), 245Xm, 246Xb, 246Xh, 246Xi, 252Ym, 252Yp, 253Xe, 253Xh, 273M, 273Nb, 275Xe, 275Xf, 275Xh, 276Ya, **366A**, 366C, 366D, 366H, 366J, 366Xa, 366Xi, 366Yf, 367Xo, 369Oe, 372Xb, 372Yb, 374Xb, 415Pa, 415Yj, 415Yk, 416I, 443G, 444M, 444R, 444T, 444U, 444Ym, 473H, 473I, 473K; *see also* $\mathcal{L}^p$, L^p, $\|\ \|_{p,q}$

$\|\ \|_{p,q}$ (the Lorentz norm) 374Yb

$\|\ \|_\infty$ **243D**, **243Xb**, **243Xo**, 244Xh, 273Xk, 281B, 354Xb, 354Xo, 356Xc, 361D, 361Ee, 361I, 361J, 361L, 361M, 363A, 364Xh, 436Ic, *463Xi*, 473Da, 4A6B; *see also* essential supremum (**243D**), L^∞, $\mathcal{L}^\infty$, ℓ^∞

$\|\ \|_{1,\infty}$ **369O**, 369P, 369Xh-369Xj, 371Gc, 372D, 372F, 373F, 373Xl; *see also* $M^{1,\infty}$, $M^{1,0}$

$\|\ \|_{\infty,1}$ **369N**, 369O, 369Xi, 369Xj, 369Xl; *see also* $M^{\infty,1}$

$\|\ \|_H$ **483L**, 483M, 483N, 483Xi, **483Yg**

$\otimes$ (in $f \otimes g$) **253B**, 253C, 253J, 253L, 253Ya, 253Yb; (in $u \otimes v$) §253 (**253E**); (in $\mathfrak{A} \otimes \mathfrak{B}$, $a \otimes b$) *see* free product (**315M**); (in $\Sigma \otimes \mathrm{T}$) *457Fa*

$\bigotimes$ (in $\bigotimes_{i \in I} \mathfrak{A}_i$) *see* free product (**315H**); (in $\bigotimes_{i \in I} \Sigma_i$) *457Fb*; (in $\bigotimes_I \Sigma$) **465Ad**

$\widehat{\otimes}$ (in $\Sigma \widehat{\otimes} \mathrm{T}$) **251D**, 251K, 251L, 251Xk, 251Ya, 252P, 252Xd, 252Xe, 253C, 418R, 418T, 419F, 421H, 424Yd, 443Yg, 452Bb, 452M, 452N, *454C*, 4A3S, 4A3Xa

$\widehat{\bigotimes}$ (in $\widehat{\bigotimes}_{i \in I} \Sigma_i$) **251Wb**, 251Wf, **254E**, 254F, 254Mc, 254Xc, 254Xi, 254Xs, 343Xb, 424Bb, *454A*, *454D*, 454Xd, 454Xe, 4A3Cf, 4A3Dc, 4A3M-4A3O, 4A3Xf; (in $\widehat{\bigotimes}_I \Sigma$) 459D, 459E, **465Ad**, 465I, *465K*

$\prod$ (in $\prod_{i \in I} \alpha_i$) **254F**; (in $\prod_{i \in I} X_i$) **254Aa**

$\#$ (in $\#(X)$, the cardinal of X) **2A1Kb**

$\lhd$ (in $I \lhd R$) *see* ideal (**3A2Ea**)

$(\overleftarrow{\ })$ (in $(\overleftarrow{a\ \pi\ b})$, $(\overleftarrow{a\ \pi\ b\ \phi\ c})$ etc.) *see* cycle notation (**381R**), cyclic automorphism, exchanging involution (**381R**)

$^+$ (in κ^+, successor cardinal) **2A1Fc**, 438Cd; (in f^+, where f is a function) **121Xa**, **241Ef**; (in u^+, where u belongs to a Riesz space) **241Ef**, **352C**, 352D, 352Fc; (in U^+, where U is a partially ordered linear space) **351C**; (in $F(x^+)$, where F is a real function) **226Bb**

$^-$ (in f^-, where f is a function) **121Xa**, **241Ef**; (in u^-, in a Riesz space) **241Ef**, **352C**, 352D; (in $F(x^-)$, where F is a real function) **226Bb**

2 (in 2^κ) **3A1D**, 438Cf, 4A1Ac

$\perp$ (in $A^\perp$, in a Boolean algebra) 313Xo; (in $A^\perp$, in a Riesz space) **352O**, 352P-352R, *352Xh*; (in $V^\perp$, in a Hilbert space) *see* orthogonal complement; *see also* complement of a band (**352P**)

$^\circ$ (in A°) *see* polar set (**4A4Bh**)

$^\frown$ (in $z^\frown i$) 3A1H, **421A**

$\bullet_c$ (in $a \bullet_c f$) **441Ac**; (in $x \bullet_c a$) **443C**; (in $a \bullet_c u$) **443G**

$\bullet_l$ (in $a \bullet_l f$) **441Ac**, **444F**; (in $x \bullet_l a$) **443C**; (in $a \bullet_l u$) **443G**

$\bullet_r$ (in $a \bullet_r f$) **441Ac**; (in $x \bullet_r a$) **443C**; (in $a \bullet_r u$) **443G**

$^\wedge$, $^\vee$ (in $\hat{f}$, $\check{f}$) *see* Fourier transform, inverse Fourier transform (**283A**)

$\{0,1\}^I$ (usual measure on) **254J**, 254Xd, 254Xe, 254Yc, 272N, 273Xb, 331J-331L, 332B, 332C, 332N, *332Xm*, *332Xn*, 341Yc, 341Yd, 341Zb, 342Jd, 343Ca, 343I, 343Xc, 343Yd, 344G, 344L, 344Xg, 345Ab, 345C-345E, 345Xa, 346C, 383Xc, 416U, 441Xg, *453B*, 491G, 491Xm, 493G;

—— —— (when $I = \mathbb{N}$) 254K, 254Xj, 254Xq, 256Xk, 261Yd, 341Xb, 343Cb, 343H, *343M*, 345Yc, 346Zb, 388E, 471Xa; *see also* $\mathcal{P}X$

—— (usual topology of) 311Xh, **3A3K**, 434Kd, 434Xb, 491G, 491Q, 4A2Ud, 4A3Of; (when $I = \mathbb{N}$) *314Ye*, 423J, 4A2Gi, 4A2U, 4A3E

—— (open-and-closed algebra of) 311Xh, 315Xh, 316Xq, 316Yj, 316Ym, 331Yg, 391Xd, 393A, 393F

—— (regular open algebra of) 316Yj

—— (and Hausdorff measures) 471Xa, 471Yh

$[0,1]^I$ (usual measure on) **416U**, 419B, 491Yh

—— (usual topology of) 412Yb, 434Kd

$]0,1[^I$ (usual measure on) 415F

—— (usual topology of) 434Xo

$(2,\infty)$-distributive lattice **367Yd**

$\ll$ (in $\nu \ll \mu$) *see* absolutely continuous (**232Aa**)

$\preccurlyeq_G^\sigma$ (in $a \preccurlyeq_G^\sigma b$) §448 (**448A**)

$\preccurlyeq_G^\tau$ (in $a \preccurlyeq_G^\tau b$) **394A**, 394G, 394I, 394K, 394Ma, 394Xb

∞ *see* infinity

$[\ \ ]$ (in $[a,b]$) *see* closed interval (**115G**, **1A1A**, **2A2Ab**, **4A2A**); (in $f[A]$, $f^{-1}[B]$, $R[A]$, $R^{-1}[B]$) **1A1B**; (in $[X]^\kappa$, $[X]^{<\kappa}$, $[X]^{\leq\kappa}$) **3A1J**; (in $[X]^{<\omega}$) 3A1Cd, 3A1J

$[[\ \ ]]$ (in $f[[\mathcal{F}]]$) *see* image filter (**2A1Ib**)

$[\![\ \ ]\!]$ (in $[\![u > \alpha]\!]$, $[\![u \geq \alpha]\!]$, $[\![u \in E]\!]$ etc.) 363Xh, **364A**, 364B, **364H**, 364Xa, 364Xc, 364Yc, **434T**; (in $[\![\mu > \nu]\!]$) **326O**, **326P**

$[\ \ [$ (in $[a,b[$) *see* half-open interval (**115Ab**, **1A1A**, **4A2A**)

$]\ \ ]$ (in $]a,b]$) *see* half-open interval (**1A1A**)

$]\ \ [$ (in $]a,b[$) *see* open interval (**115G**, **1A1A**, **4A2A**)

⌈ ⌉ (in $\lceil b:a\rceil$) 394I, **394J**, 394K-394M, 394Xa, **448I**, 448J
⌊ ⌋ (in $\lfloor b:a\rfloor$) 394I, **394J**, 394K-394M, 394Xa, **448I**, 448J
$\llcorner$ (in $\mu\llcorner E$) **234E**, 235Xf, 475G
♢ *see* Jensen's ♢
♣ *see* Ostaszewski's ♣ (**4A1M**)