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ABSTRACT. Homological algebra with focus on the functorial aspects.

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1. Algebraic Structures

In this introductory section we recall some basic concepts concerning rings. For details consult any book on ring theory, for example the notes [Th].

(1.1) **Commutative groups.** Let G be a set equipped with:

- a map $+\colon G\times G \to G$, that is, to any pair $(g, g') \in G\times G$ of elements, $+$ associates exactly one element $g+g' \in G$;
- a map $-\colon G \to G$, that is, to any element $g \in G$, it associates exactly one element $-g \in G$;
- an element $0 \in G$.

The set G is said to be a *group* exactly when the next three requirements are fulfilled.

(G1) $(g+g')+g'' = g+(g'+g'')$ for all $g, g', g'' \in G$;
(G2) $g+0 = g$ and $0+g = g$ for all $g \in G$;
(G3) $g+(-g) = 0$ and $(-g)+g = 0$ for all $g \in G$.

Here, (G1) is the *associativity* of $+$. (G2) states that 0 is a *neutral element*. (G3) means that $-g$ is an *inverse element* to g, and it follows that this is uniquely determined. We refer to this group as $(G, +)$, or just as the group G. The usage of the symbols $+$, $-$ and 0 is called *additive notation*.

Let $(G, +)$ be a group. It is said to be *commutative* (or Abelian) exactly when

(G4) $g+g' = g'+g$ for all $g, g' \in G$.

In this case $+$ is called the *addition*, 0 is the *zero element*, and $-g$ is the *opposite element* to g. For $g, h \in G$ we set $g-h =^{\text{def}} g+(-h)$ and note that $g-g = 0$ by (G3).

Consider a set $\{n\}$ consisting of exactly one element n; it is a commutative group with $n+n = n$, $0 = n$ and $-n = n$, and it is called the *zero group* and denoted 0, that is, $0 = \{0\}$.

(1.2) **Commutative rings.** Let $(R, +)$ be a commutative group. It is said to be a *ring* exactly when it (in addition to $+$, $-$ and 0) is equipped with:

- a so-called *multiplication*, that is, a map $R\times R \to R$ that to any pair $(r, r') \in R\times R$ associates exactly one element $rr' \in R$; and
- a so-called *identity element* $1 \in R$,

such that the next three requirements are fulfilled.

(R1) $(rr')r'' = r(r'r'')$ for all $r, r', r'' \in R$ (associativity);
(R2) $r(r'+r'') = rr'+rr''$ and $(r+r')r'' = rr''+r'r''$ for all $r, r', r'' \in R$ (distributivity);
(R3) $r1 = r$ and $1r = r$ for all $r \in R$.

It follows that

$$0r = 0 \quad \text{for all} \quad r \in R. \tag{1.2.1}$$

Namely: 0r=0r+0=0r+(0r-0r)=(0r+0r)-0r=(0+0)r-0r=0r-0r=0.

The ring R is said to be *commutative* exactly when

(R4) $rr' = r'r$ for all $r, r' \in R$.

Examples of commutative rings are the integers $\mathbb{Z}$, the complex numbers $\mathbb{C}$ and polynomial rings like $\mathbb{Z}[X]$, $\mathbb{C}[X]$. The ring of 2×2 matrices with complex entries is an example of a non-commutative ring, see also (1.3).

A ring R is said to be *non-trivial* exactly when $R \neq \{0\}$, that is, $1 \neq 0$ (if $r \in R\setminus\{0\}$ then $1r = r \neq 0 = 0r$ and hence $1 \neq 0$.)

(1.3) **Exercise: Endomorphism rings.** Let V be a vector space over the real numbers $\mathbb{R}$ (or any other field), and consider the set of endomorphisms $\Lambda =^{\text{def}} \operatorname{Hom}_{\mathbb{R}}(V,V)$ which consists of all linear maps: $V \to V$. Prove that Λ is a ring with the usual addition, and with composition of maps as the multiplication. Prove that it is commutative if and only if the dimension of the vector space V is at most 1.

(1.4) **Convention.** Throughout these notes

R denotes always non–trivial commutative ring.

(1.5) **Special ring elements.** For $r \in R$ we have the following definitions:

(1.5.1) r is *invertible* $\iff$ $\exists r^{-1} \in R\colon\ rr^{-1} = 1$;

(1.5.2) r is a *zero-divisor* $\iff$ $\exists a \in R\colon\ ra = 0 \ \wedge\ a \neq 0$;

(1.5.3) r is *non-zero-divisor* $\iff$ $\forall a \in R\colon\ ra = 0 \ \Rightarrow\ a = 0$.

Another word for an invertible element is a *unit*, and non-zero-divisors are often called *regular* elements.

(1.6) **Special rings.** We have the following two definitions and an observation.

(1.6.1) R is a *field* $\overset{\text{def}}{\iff}$ $\forall r \in R \setminus \{0\}\colon$ r is invertible; and

(1.6.2) R is a *integral domain* $\overset{\text{def}}{\iff}$ $\forall r \in R \setminus \{0\}\colon$ r is a non-zero-divisor

(1.6.3) $\iff$ $\forall r, r' \in R\colon\ rr' = 0 \ \Rightarrow\ r = 0 \ \vee\ r' = 0$.

Every field is a domain. The ring of complex numbers $\mathbb{C}$ is a field while the polynomial ring $\mathbb{C}[X]$ is an integral domain but not a field (for example, the element X is not invertible). The ring $\mathbb{Z}/(4)$ of residue classes modulo 4, that is, $\mathbb{Z}/(4) = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}\}$, is not an integral domain (since $\overline{2}\,\overline{2} = \overline{0}$ which is the zero element).

(1.7) **Ideals.** An *ideal* $\mathfrak{a}$ in R is a subset $\mathfrak{a} \subseteq R$ satisfying the next conditions.

(I1) $0 \in \mathfrak{a}$;

(I2) $a + a' \in \mathfrak{a}$ for all $a, a' \in \mathfrak{a}$;

(I3) $ra \in \mathfrak{a}$ for all $r \in R$ and $a \in \mathfrak{a}$.

The ideal $\mathfrak{a}$ is said to be *proper* exactly when $\mathfrak{a} \neq R$. Proper ideals contain no invertible elements: If $\mathfrak{a}$ is an ideal and $u \in \mathfrak{a}$ is invertible, then for any $r \in R$ we obtain $r = r1 = r(uu^{-1}) = r(u^{-1}u) = (ru^{-1})u \in \mathfrak{a}$; hence $\mathfrak{a} = R$.

The subsets $O = \{0\}$ and R are ideals in R and they are called the *trivial ideals*; O is called the *zero ideal*.

(1.8) **Principal ideals.** For $a \in R$ the notation Ra is used for the *principal ideal generated by* a, that is, $Ra =^{\text{def}} \{\, ra \mid r \in R \,\}$; often—when the name of the ring is clear from the context—we write (a) instead of Ra, that is, $(a) =^{\text{def}} \{\, ra \mid r \in R \,\}$. The trivial ideals are principal: $O = (0)$ and $R = (1)$.

The ring R is said to a *PID*, that is, a *principal ideal domain*, exactly when it is a domain and all its ideals are principal. The ring of integers $\mathbb{Z}$ and the polynomial ring $\mathbb{C}[X]$ over the complex numbers are principal ideal domains.

2. Modules

(2.1) **Definition: Module.** Let $(M,+)$ be a commutative group. It is said to be a *module over* R, or just an R–module, if it (in addition to $+$, $-$ and 0) is equipped with

- a (so-called) *R–multiplication*, that is, map $R \times M \to M$ that to any $(r,m) \in R \times M$ associates exactly one element $rm \in M$,

such that

(M1) $(rr')m = r(r'm)$ for all $r, r' \in R$ and $m \in M$ (associativity);
(M2) $(r+r')m = rm+r'm$ and $r(m+m') = rm+rm'$ for all $r, r' \in R$ and $m, m' \in M$ (distributivity);
(M3) $1m = m$ for all $m \in M$ (unitarity).

The ring R itself is an R–module with R–multiplication equal the ring-multiplication.

The zero group $O = \{0\}$ is always an R–module with the R–multiplication $r0 = 0$.

If R is a field then M is an R–module, if and only if M is a vector space over R.

Every commutative group G is a $\mathbb{Z}$–module with $\mathbb{Z}$–multiplication defined for $r \in \mathbb{Z}$ and $g \in G$ by $rg = 0$ for $r = 0$, by $rg = g + \cdots + g$, the sum of r copies of g, for $r > 0$, and for $r < 0$ by $rg = -(-r)g$ (which has already been defined as $-r > 0$). It follows by repeated applications of the distributive rule (M2) that this is a $\mathbb{Z}$–multiplication in the group M, and that it is only possible one. On the other hand, every module, and hence, in particular, every $\mathbb{Z}$–module, is, by definition, also a commutative group. So the concept "$\mathbb{Z}$–module" is the same as "commutative group".

For each element m in an R–module M the following hold.

$$0m = 0\,; \tag{2.1.1}$$

$$(-1)m = -m\,. \tag{2.1.2}$$

In the first equality the left hand zero is that of the ring while the right hand one is that of the module; in the second equality $-1 \in R$ and $-m \in M$. The first equality follows from: $0m = (0+0)m = 0m + 0m$, and hence $0m = 0m + (-0m) = 0$; the second one: $0 = 0m = (1+(-1))m = 1m + (-1)m = m + (-1)m$.

For $r, r' \in R$ and $m, m' \in M$ the next two equalities follow from the module rules (M1) and (M2) combined with (2.1.2).

$$(r-r')m = rm - r'm \quad \text{and} \quad r(m-m') = rm - rm'\,. \tag{2.1.3}$$

(2.2) **Definition: Submodule.** A *submodule* N of an R–module M is a subset $N \subseteq M$ satisfying:

(S1) $0 \in N$;
(S2) $n+n' \in N$ for all $n, n' \in N$;
(S3) $rn \in N$ for all $r \in R$ and $n \in N$.

Each submodule is itself an R–module with the restrictions of the addition and the multiplication. It is called *proper* exactly when $N \neq M$. The *trivial submodules* of M are M itself and $O = \{0\}$.

The submodules of the R–module R are exactly the ideals in R.

If R is a field and V is an R–module, that is, a vector-space over R, then the submodules of V are exactly the subspaces of V.

(2.3) **Intersection of submodules.** Let M be an R–module and let $N_1, \ldots, N_t$ be submodules of M. For the intersection $N_1 \cap \cdots \cap N_t$ we have

$$x \in N_1 \cap \cdots \cap N_t \iff x \in N_1\,, \ldots, x \in N_t\,,$$

and it follows that (S1)–(S3) are satisfied for the intersection, that is, $N_1 \cap \cdots \cap N_t$ is a submodule of M.

(2.4) **Exercise: Infinite intersection of submodules.** Let M be an R–module and let $(N_i)_{i\in I}$ be a family of submodules of M. For the intersection $\bigcap_{i\in I} N_i$ we have

$$x \in \bigcap_{i\in I} N_i \quad \Longleftrightarrow \quad x \in N_i \text{ for all } i \in I .$$

Prove that $\bigcap_{i\in I} N_i$ is a submodule of M.

(2.5) **Sum of submodules.** Let M be an R–module and let $N_1, \dots, N_t$ be submodules of M. The sum $N_1 + \cdots + N_t$ is defined by

$$N_1 + \cdots + N_t \stackrel{\text{def}}{=} \{ n_1 + \cdots + n_t \mid n_1 \in N_1, \dots, n_t \in N_t \} .$$

The sum $N_1 + \cdots + N_t$ is a submodule of M:

(S1) $0 = 0 + \cdots + 0 \in N_1 + \cdots + N_t$;
(S2) $(n_1 + \cdots + n_t) + (n_1' + \cdots + n_t') = (n_1 + n_1') \cdots + (n_t + n_t') \in N_1 + \cdots + N_t$ for $n_1, n_1' \in N_1, \dots, n_t, n_t' \in N_t$;
(S3) $r(n_1 + \cdots + n_t) = rn_1 + \cdots + rn_t \in N_1 + \cdots + N_t$ for $r \in R, n_1 \in N_1, \dots, n_t \in N_t$.

(2.6) **Exercise: Infinite sum of submodules 1.** Let M and $(N_i)_{i\in I}$ be as in (2.4). The sum $\sum_{i\in I} N_i$ of these submodules is a subset of M defined by

$$\sum_{i\in I} N_i \stackrel{\text{def}}{=} \{ n_{i_1} + \cdots + n_{i_t} \mid t \in \mathbb{N}, i_1, \dots, i_t \in I, n_{i_1} \in N_{i_1}, \cdots, n_{i_t} \in N_{i_t} \}$$

Prove that $\sum_{i\in I} N_i$ is a submodule of M.

(2.7) **Exercise: Submodule rule.** Let N be a subset of an R–module M. Prove that N is a submodule of M if and only if the following holds:

(2.7.1) $\qquad N \neq \varnothing \text{ and } rn + n' \in N \text{ for all } r \in R \text{ and } n, n' \in N .$

(2.8) **Linear combinations.** Let $n \in \mathbb{N}_0$ be a non-negative integer (possibly $n = 0$), let $r_1, \dots, r_n \in R$ be n elements in the ring R, and let $w_1, \dots, w_n \in M$ be n elements in an R–module M. The *linear combination* of the elements $w_1, \dots, w_n \in M$ with *coefficients* $r_1, \dots, r_n \in R$ is the sum $r_1w_1 + \cdots + r_nw_n$ which is an element in the module M; if $n = 0$ then the linear combination is, by definition, the zero element $0 \in M$.

(2.9) **Generated subset.** Let W be a (possibly empty) subset of the R–module M. The *subset generated by* W is then denoted by $\operatorname{span}_R W$ and defined as follows

(2.9.1) $\quad \operatorname{span}_R W \stackrel{\text{def}}{=} \{ r_1w_1 + \cdots + r_nw_n \mid n \in \mathbb{N}_0, r_1, \dots, r_n \in R, w_1, \dots, w_n \in W \},$

that is, $\operatorname{span}_R W$ is the set of all linear combinations of elements from W. It follows from the convention in (2.8) that $\operatorname{span}_R \varnothing = O$ (the zero submodule of M).

Note that $W \subseteq \operatorname{span}_R W$ since $w = 1w \in \operatorname{span}_R W$ for all $w \in W$.

(2.10) **Generated submodule.** Let M be an R–module with a subset W.

The set $\operatorname{span}_R W$ is then a submodule af M, and it is the *smallest* submodule of M containing W, that is, if N is a submodule of M then the following holds.

(2.10.1) $\qquad W \subseteq N \quad \Longleftrightarrow \quad \operatorname{span}_R W \subseteq N .$

If $W = \varnothing$ then these assertions are obvious, so assume that $W \neq \varnothing$. First we note that $\operatorname{span}_R W$ is a submodule of M:

(S1) $0 = 0w \in \operatorname{span}_R W$ for any $w \in W$;
(S2) the sum of two linear combinations of elements from W is again a linear combination af elements fra W;
(S3) $r(r_1w_1 + \cdots + r_nw_n) = (rr_1)w_1 + \cdots + (rr_n)w_n \in \operatorname{span}_R W$ by (M1)&(M2).

In the bi-implication, "$\Leftarrow$" follows from $W \subseteq \operatorname{span}_R W$, cf. (2.9), while "$\Rightarrow$" follows by (2.8). We say that $\operatorname{span}_R W$ is the *submodule generated* by W.

(2.11) **Exercise: Infinite sum of submodules 2.** Let M, $(N_i)_{i\in I}$, and $\sum_{i\in I} N_i$ be as in (2.6). Prove that $\sum_{i\in I} N_i = \operatorname{span}_R \left(\bigcup_{i\in I} N_i\right)$.

(2.12) **Finitely generated modules.** An R–module M is said to be *finitely generated* exactly when it has a *finite* subset W such that $M = \operatorname{span}_R W$, that is, there exist $n \in \mathbb{N}$ and $m_1, \dots, m_n \in M$ such that any element m in M is a linear combination of $m_1, \dots, m_n \in M$, and that is, there exist $r_1, \dots, r_n \in R$ such that $m = r_1m_1 + \cdots + r_nm_n$.

The zero module O er finitely generated; it can be generated by the element 0 (but it can also be generated by 0 elements).

The R–module R is finitely generated; it can be generated by the one element 1. Every principal ideal (a) in R is a finitely generated module, namely generated by a.

(2.13) **Cyclic modules 1.** An R–module M is said to be *cyclic* exactly when it is generated by *one* element: $M = \operatorname{span}_R\{m\}$ for an element $m \in M$.

The zero module is cyclic: $O = \operatorname{span}_R\{0\}$. The ring R is a cyclic R–module: $R = \operatorname{span}_R\{1\}$. An ideal is a cyclic module if and only if it is a principal ideal. We set $Rm = \operatorname{span}_R\{m\}$; this is in accordance the notation in (1.8).

(2.14) **Exercise: Sum of cyclic submodules.** Let W be a subset of an R–module M. Prove that $\operatorname{span}_R W = \sum_{w\in W} Rw$, cf. (2.11). In particular,

(2.14.1)
M is finitely generated $\iff$ $M = Rm_1 + \cdots + Rm_n$ for some $m_1, \dots, m_n \in M$.

(2.15) **Homomorphisms.** Let M and N be R–module, and let $\varphi\colon M \to N$ be a map. Then, φ is said to be a *homomorphism of R–modules*—or an *R–homomorphism*—or just a *homomorphism*—exactly when it satisfies

(H1) $\varphi(m + m') = \varphi(m) + \varphi(m')$ for all $m, m' \in M$;
(H2) $\varphi(rm) = r\varphi(m)$ for all $r \in R$ and $m \in M$.

Some use also the term *R–linear maps*. Here, (H1) means that φ is a group homomorphism, and this implies that

(2.15.1) $$\varphi(0_M) = 0_N$$

where 0_M and 0_N are the zero elements in M and N, respectively.

When we mention an R–homomorphism $\varphi\colon M \to N$, we alway assume that M and N are R–modules; M is said to be the *source* for φ, and N is said to be the *target* for φ.

If R is a field then φ is an R–homomorphism, if and only if it is a linear map of vector spaces $M \to N$ over R.

Furthermore, φ is a $\mathbb{Z}$–homomorphism, if and only if it just satisfies homomorphism rule (H1), that is, it is a homomorphism $M \to N$ of commutative groups.

If N is a submodule of an R–module M, then the *inclusion map* $\iota\colon N \to M$, $n \mapsto n$ is an R–homomorphism which often is displayed as $N \hookrightarrow M$.

The *identity map* on M is denoted $1_M\colon M \to M$; it is—of course—an R–homomorphism.

(2.16) **Multiplication homomorphisms.** Let $a \in R$ be an element and let M be an R–module. A map $a_M\colon M \to M$ is then defined by $a_M(m) = am$ for $m \in M$. This map is an R–homomorphism: For $m, n \in M$ and $r \in R$ we have the next equalities.

$$a_M(m+n) \overset{\text{by def}}{=} a(m+n) \overset{\text{(M2)}}{=} am + an \overset{\text{by def}}{=} a_M(m) + a_M(n) \quad \text{and}$$

$$a_M(rm) \overset{\text{by def}}{=} a(rm) \overset{\text{(M1)}}{=} (ar)m \overset{\text{(R4)}}{=} (ra)m \overset{\text{(M1)}}{=} r(am) \overset{\text{by def}}{=} ra_M(m)\,.$$

Here, (M1) and (M2) are from the definition of modules (2.1) while (R4) is the **commutativity** of R, cf. (1.2) and (1.4).

The R–homomorphism $a_M\colon M \to M$ is called *multiplication with a on M*. Note that a_M for $a = 1$ is the identity 1_M on M.

(2.17) **Isomorphisms.** If $\varphi\colon M \to N$ is a *bijective* R–homomorphism, then the inverse map $\varphi^{-1}\colon N \to M$ is also an R–homomorphism:

$$\begin{aligned}\varphi^{-1}(n+n') &= \varphi^{-1}\big(\varphi(\varphi^{-1}(n)) + \varphi(\varphi^{-1}(n'))\big) = \varphi^{-1}\big(\varphi(\varphi^{-1}(n) + \varphi^{-1}(n'))\big)\\ &= (\varphi^{-1}\varphi)(\varphi^{-1}(n) + \varphi^{-1}(n')) \quad = \varphi^{-1}(n) + \varphi^{-1}(n')\end{aligned}$$

for $n, n' \in N$, and similarly $\varphi^{-1}(rn) = r\varphi^{-1}(n)$ for $r \in R$ and $n \in N$.

A bijective R–homomorphism $\varphi\colon M \to N$ is said to be an *R–isomorphism*; we write $\varphi\colon M \xrightarrow{\cong} N$. The identity $1_M\colon M \to M$ is an R–isomorphism. The notation $M \cong N$ means that there exists an R–isomorphism $M \to N$, and M and N is said to be *isomorphic*.

(2.18) **Zero homomorphism.** Let M and N be R–modules. The *zero homomorphism* $o = o_{MN}\colon M \to N$ is defined by $o(m) = 0$ for all $m \in M$. When $M = N$ it is the multiplication $0_M\colon M \to M$ with 0. There exists exactly one R–homomorphism $O \to N$ and exactly one R–homomorphism $M \to O$; namely the zero homomorphisms.

(2.19) **Kernel and image.** Let $\varphi\colon M \to N$ be an R–homomorphism. Its *kernel* and *image*, respectively, are defined as follows.

$$\operatorname{Ker}\varphi \overset{\text{def}}{=} \varphi^{-1}(O) \overset{\text{def}}{=} \{\, m \in M \mid \varphi(m) = 0 \,\}\,; \text{ and} \tag{2.19.1}$$

$$\operatorname{Im}\varphi \overset{\text{def}}{=} \varphi(M) \overset{\text{def}}{=} \{\, \varphi(m) \mid m \in M \,\}\,, \tag{2.19.2}$$

The following hold.

$$\operatorname{Ker}\varphi \text{ is a submodule of } M\,; \tag{2.19.3}$$

$$\operatorname{Im}\varphi \text{ is a submodule of } N\,. \tag{2.19.4}$$

Although this follows from exercise (2.20) a proof is provided: For $m, m' \in \operatorname{Ker}\varphi$ and $r \in R$ we have:

$$\varphi(0) = 0\,,\ \varphi(m+m') = \varphi(m) + \varphi(m') = 0 + 0 = 0\,, \varphi(rm) = r\varphi(m) = r0 = 0\,,$$

Thus, $0, m+m', rm \in \operatorname{Ker}\varphi$. To $y, y' \in \operatorname{Im}\varphi$ choose $m, m' \in M$ such that $\varphi(m) = y$ and $\varphi(m') = y'$; for $r \in R$ the elements

$$0 = \varphi(0)\,,\ y + y' = \varphi(m) + \varphi(m') = \varphi(m+m')\,, ry = r\varphi(m) = \varphi(rm)$$

all belong to $\operatorname{Im}\varphi$.

For every R–homomorphism $\varphi\colon M \to N$ the following hold.

(2.19.5) $\qquad \varphi$ is injective $\iff \operatorname{Ker}\varphi = O$

(2.19.6) $\qquad \varphi$ is surjective $\iff \operatorname{Im}\varphi = N$.

The latter follows directly from the definition of surjectivity. Although the former follows from the special case $R = \mathbb{Z}$, that is, the well-known case of commutative groups, a proof is included here: Assume first that φ is injective and that m belongs to $\operatorname{Ker}\varphi$. Since $\varphi(m) = 0 = \varphi(0)$ we obtain $m = 0$ (as desired) by the injectivity of φ. Next assume that $\operatorname{Ker}\varphi = O$ and that $m, m' \in M$ have $\varphi(m) = \varphi(m')$. We get

$$\varphi(m - m') = \varphi(m + (-m')) = \varphi(m) + \varphi(-m') = \varphi(m) + (-\varphi(m')) = \varphi(m) - \varphi(m') = 0.$$

Thus, $m - m' \in \operatorname{Ker}\varphi$, that is, $m = m'$, and the injectivity of φ has been established.

(2.20) **Exercise: Images and inverse images of submodules.** Let $\varphi\colon M \to N$ be an R–homomorphism, and let M' and N' be submodules of M and N, respectively. Prove that

(2.20.1) $\qquad \varphi(M') \stackrel{\text{def}}{=} \{\, \varphi(m') \mid m' \in M' \,\}$ is a submodule of N;

(2.20.2) $\qquad \varphi^{-1}(N') \stackrel{\text{def}}{=} \{\, m \in M \mid \varphi(m) \in N' \,\}$ er en submodule of M.

(2.21) **Finite direct sums.** Let $M_1, \dots, M_t$ be R–modules, and consider the product:

$$\Sigma \stackrel{\text{def}}{=} M_1 \times \cdots \times M_t \stackrel{\text{def}}{=} \{\, (m_1, \dots, m_t) \mid m_1 \in M_1, \dots, m_t \in M_t \,\}.$$

This set Σ can be equipped with the addition

$$(m_1, \dots, m_1) + (m'_1, \dots, m'_t) \stackrel{\text{def}}{=} (m_1 + m'_1, \dots, m_1 + m'_t)$$

for $(m_1, \dots, m_t), (m'_1, \dots, m'_t) \in \Sigma$, and Σ becomes a commutative group with $(0, \dots, 0)$ as the zero element and with $-(m_1, \dots, m_1) = (-m_1, \dots, -m_1)$. The R–multiplication $r(m_1, \dots, m_1) = (rm_1, \dots, rm_1)$ for $r \in R$ and $(m_1, \dots, m_1) \in \Sigma$ makes Σ into an R–module which is denoted $M_1 \oplus \cdots \oplus M_t$ and called the *direct sum* of $M_1, \dots, M_t$. (The notation $M_1 \times \cdots \times M_t$ is sometimes used, in particular for rings, and it is called the *direct product.*)

For any $u \in \{\, 1, \dots, t \,\}$ there are two R–homomorphisms:

(2.21.1) The *inclusion* $\iota_u\colon M_u \to M_1 \oplus \cdots \oplus M_t$ given by $m_u \mapsto (0, \dots, 0, m_u, 0, \dots, 0)$ with the m_u in space number u.

(2.21.2) The *projection* $\pi_u\colon M_1 \oplus \cdots \oplus M_t \to M_u$ given by $(m_1, \dots, m_v) \mapsto m_u$.

Note that

(2.21.3) $\qquad \pi_u \circ \iota_u = 1_{M_u}$.

(2.22) **Inner direct sum.** Let $N_1, \dots, N_t$ be submodules of an R–module M. The map

$$\nu_{N_1 \cdots N_t}\colon N_1 \oplus \cdots \oplus N_t \to N_1 + \cdots + N_t \text{ given by } (n_1, \dots, n_t) \mapsto n_1 + \cdots + n_t$$

is a *surjective* R–homomorphism, cf. (2.5). The submodules $N_1, \dots, N_t$ are said to *form inner direct sum* exactly when $\nu_{N_1 \cdots N_t}$ is injective, that is, $\nu_{N_1 \cdots N_t}$ is an isomorphism:

$$N_1 \oplus \cdots \oplus N_t \cong N_1 + \cdots + N_t$$

For $t = 2$ we have:

(2.22.1) $\qquad \operatorname{Ker}\nu_{N_1 N_2} = \{\, (x, -x) \mid x \in N_1 \cap N_2 \,\}$

and hence:

(2.22.2) N_1 and N_2 form inner direct sum $\iff$ $N_1 \cap N_2 = O$.

(2.23) **Residues.** Let H be a submodule of an R-module M and let $x \in M$. The *residue class* and the *set of residue classes* are defined by, respectively,

$$[x]_H = \{ x + h \mid h \in H \} \tag{2.23.1}$$

$$M/H = \{ [m]_H \mid m \in M \}, \tag{2.23.2}$$

and they are pronounced x *modulo* H and M *modulo* H, respectively. Often we use the abbreviated notations $\overline{x} = [x]_H$ and $\overline{M} = M/H$, respectively, when it is obvious from the context that we are working modulo H. Note that for $m_1, m_2 \in M$ the next holds:

$$\overline{m_1} = \overline{m_2} \iff m_1 - m_2 \in H. \tag{2.23.3}$$

(2.24) **Residue module.** The set M/H has an *addition*, cf. [Th, GRP (4.15)]; it well-defined by $\overline{m} + \overline{n} = \overline{m+n}$. It has also an R-*multiplication* well-defined by $r\overline{m} = \overline{rm}$. (If $\overline{m} = \overline{m'}$ then $m - m' \in H$ and hence $rm - rm' = r(m - m') \in H$; use (2.23.3).)

With these M/H becomes an R-module. Note also:

$$-\overline{m} = \overline{-m}, \quad \overline{0} \text{ is the zero element, and } \overline{m} = \overline{0} \text{ if and only if } m \in H.$$

The *residue map*

$$\rho_{MH} \colon M \to M/H \text{ given by } m \mapsto \overline{m} = [m]_H \tag{2.24.1}$$

is a surjective R-homomorphism with $\operatorname{Ker} \rho_{MH} = H$.

(2.25) **Residue ring.** Let $\mathfrak{a}$ be a proper ideal in R and consider the residue set $\overline{R} \stackrel{\text{def}}{=} R/\mathfrak{a}$ which is an R-module by (2.24). It is actually a commutative non-trivial ring with multiplication well-defined by $\overline{r}\,\overline{s} =^{\text{def}} \overline{rs}$ for $r, s \in R$, cf. [Th, RNG (2.7)]. This uses that R is supposed to be commutative.

(2.26) **Over-lining.** Let $\mathfrak{a}$ be a fixed proper ideal, and consider the residue ring $\overline{R} \stackrel{\text{def}}{=} R/\mathfrak{a}$. For any R-module M we set $\mathfrak{a}M =^{\text{def}} \operatorname{span}_R\{ am \mid a \in \mathfrak{a} \text{ and } m \in M \}$, that is,

$$\mathfrak{a}M = \{ a_1m_1 + \cdots + a_nm_n \mid n \in \mathbb{N}_0, a_1, \ldots, a_n \in \mathfrak{a}, m_1, \ldots, m_n \in M \}.$$

by (2.9.1) since $R\mathfrak{a} = \mathfrak{a}$. Set $\overline{M} \stackrel{\text{def}}{=} M/\mathfrak{a}M$. (For $M = R$ this is in accordance with the earlier definition of $\overline{R}$.) It follows from (2.27.3) below that $\overline{M}$ is an $\overline{R}$-module with multiplication well-defined by $\overline{r}\,\overline{m} =^{\text{def}} \overline{rm}$, that is, $[r]_{\mathfrak{a}}[m]_{\mathfrak{a}M} =^{\text{def}} [rm]_{\mathfrak{a}M}$.

(2.27) **Annihilators.** Let M be an R-module, and let $x \in M$ be an element.

$$\operatorname{Ann}_R M \stackrel{\text{def}}{=} \{ r \in R \mid rm = 0 \text{ for all } m \in M \}$$

$$\operatorname{Ann}_R(x) \stackrel{\text{def}}{=} \{ r \in R \mid rx = 0 \}$$

Note that these are ideals in R. The next equality results from (2.9.1) and the commutativity of R, and it is followed by a special case, cf. (2.14.1).

$$\operatorname{Ann}_R M = \bigcap_{w \in W} \operatorname{Ann}_R(w) \text{ if } M = \operatorname{span}_R W. \tag{2.27.1}$$

$$\operatorname{Ann}_R M = \operatorname{Ann}_R(m_1) \cap \cdots \cap \operatorname{Ann}_R(m_n) \text{ if } M = Rm_1 + \cdots + Rm_n. \tag{2.27.2}$$

Let N be an R–module such that $\mathfrak{a} \subseteq \operatorname{Ann}_R N$. The R–module N is then also an $\overline{R}$–module with multiplication well-defined by

$$\overline{r}n =^{\text{def}} rn \text{ for } r \in R \text{ and } n \in N. \tag{2.27.3}$$

Namely, if $\overline{r} = \overline{r'}$ then $r - r' \in \mathfrak{a} \subseteq \operatorname{Ann}_R N$ and hence $rn - r'n = (r - r')n = 0$. In particular, $\overline{M}$ is an $\overline{R}$–module for any R–module M, cf. (2.26).

(2.28) **Homomorphism Theorem.** *If $\varphi\colon M \to N$ is an R–homomorphism and H is a submodule of $\operatorname{Ker}\varphi$, then there is one and only one R–homomorphism $\widehat{\varphi}\colon M/H \to N$ such that $\widehat{\varphi}([m]_H) = \varphi(m)$ for all $m \in M$.*

(2.29) **Proof.** It is known from [Th, GRP (5.6)] (applied to the surjective group homomorphism $M \to \operatorname{Im}\varphi$, $m \mapsto \varphi(m)$) that there exists a unique group homomorphism $\widehat{\varphi}\colon M/H \to N$ such that $\widehat{\varphi}([m]_H) = \varphi(m)$ for all $m \in M$; this is also an R–homomorphism:

$$\widehat{\varphi}(r[m]_H) = \widehat{\varphi}([rm]_H) = \varphi(rm) = r\varphi(m) = r\widehat{\varphi}([m]_H) \tag{2.29.1}$$

for all $m \in M$ and $r \in R$. □

(2.30) **Isomorphism Theorem.** *If $\varphi\colon M \to N$ is an R–homomorphism , then there is one and only one R–isomorphism $\widehat{\varphi}\colon M/\operatorname{Ker}\varphi \to \operatorname{Im}\varphi$ such that $\widehat{\varphi}([m]_{\operatorname{Ker}\varphi}) = \varphi(m)$ for all $m \in M$.*

(2.31) **Proof.** It is known from [Th, GRP (5.8)] that there exists a unique group isomorphism $\widehat{\varphi}\colon M/\operatorname{Ker}\varphi \to \operatorname{Im}\varphi$ such that $\widehat{\varphi}([m]_H) = \varphi(m)$ for all $m \in M$; this is also an R–isomorphism by the calculation (2.29.1). □

(2.32) **Cyclic modules 2.** Let M be an R–module. Recall from (2.13) that it is cyclic if and only if it is generated by one of its elements, that is, $M = Rw$ $(= \operatorname{span}_R\{w\})$ for some $w \in M$.

$$M \text{ is cyclic} \iff M \cong R/\operatorname{Ann}_R M. \tag{2.32.1}$$

If $M = Rw$ is cyclic, then the map $\varphi\colon R \to M$ defined by $\varphi(r) = rw$ is a surjective homomorphism of R–modules with $\operatorname{Ker}\varphi = \operatorname{Ann}_R(w) = \operatorname{Ann}_R M$. The Isomorphism Theorem (2.30) yields the desired isomorphism. If, on the other hand, $\psi\colon R/\operatorname{Ann}_R M \to M$ is an isomorphism, then M is generated by $\psi([1]_{\operatorname{Ann}_R M})$.

(2.33) **Finite Generation Theorem.** *Let H be a submodule of an R–module M. The following then hold.*

$$M \text{ is finitely generated} \implies M/H \text{ is finitely generated}; \tag{2.33.1}$$

$$M/H \text{ and } H \text{ are finitely generated} \implies M \text{ is finitely generated}. \tag{2.33.2}$$

(2.34) **Proof.** If $M = Rm_1 + \cdots + Rm_s$ then $\overline{M} = R\overline{m_1} + \cdots + R\overline{m_s}$; this proves (2.33.1).

Next assume that $\overline{M} = R\overline{m_1} + \cdots + R\overline{m_s}$ and $H = Rh_1 + \cdots + Rh_t$ [and we want to prove that $M = Rm_1 + \cdots + Rm_s + Rh_1 + \cdots + Rh_t$]. Consider $m \in M$. Choose first $r_1, \dots, r_s \in R$ such that $\overline{m} = r_1\overline{m_1} + \cdots + r_s\overline{m_s}$ $(\in \overline{M})$, that is, $m - (r_1m_1 + \cdots + r_sm_s) \in H$. Choose next $r'_1, \dots, r'_t \in R$ such that $m - (r_1m_1 + \cdots + r_sm_s) = r'_1h_1 + \cdots + r'_th_t$, that is, $m = r_1m_1 + \cdots + r_sm_s + r'_1h_1 + \cdots + r'_th_t$ as required. □

(2.35) **Noetherian modules.** Later we will present a class of R–modules M for which also the following holds:

$$M \text{ is finitely generated} \implies H \text{ is finitely generated}$$

holds. Cf. for (3.14) for an example of a finitely generated R–module M with a submodule H which is *not* finitely generated.

3. Free Modules

The modules in this section title are the building blocks for the homological theory of modules. We start by presenting the finitely generated free R-modules.

(3.1) **Finitely generated free modules.** Let $W = \{ w_1, \dots, w_n \}$ is a finite subset[1] of an R-module M.

The set $W = \{ w_1, \dots, w_n \}$ is then said to be *linearly independent* exactly when *every linear relation is trivial,* that is, for all $r_1, \dots, r_n \in R$ we have:

$$r_1 w_1 + \cdots + r_n w_n = 0 \quad \Longrightarrow \quad r_1 = 0, \dots, r_n = 0 . \tag{3.1.1}$$

The linear relation is at the left and the triviality is expressed at the right.

The set W is said to be *basis* for M exactly when

$$W = \{ w_1, \dots, w_n \} \text{ linearly independent and } \operatorname{span}_R W = M , \tag{3.1.2}$$

that is, W is a linearly independent set that generates M.

The R-module$M$ is said to be *finitely generated free module* exactly when it has a *finite basis.*

(3.2) R^n: For $n \in \mathbb{N}$ we set $R^n \stackrel{\text{def}}{=} R \oplus \cdots \oplus R$, the direct sum of n copies of R, cf. (2.21); the addition and R-multiplication on R^n are given by, respectively:

$$(r_1, \dots, r_n) + (r'_1, \dots, r'_n) = (r_1 + r'_1, \dots, r_n + r'_n) \text{ and } r(r_1, \dots, r_n) = (rr_1, \dots, rr_n) .$$

In addition, we set $R^0 \stackrel{\text{def}}{=} O$, the zero module.

The finitely generated modules R^n for $n \in \mathbb{N}_0$ are free:

O is free with basis $\varnothing$.
R is free with basis $\{ 1 \}$.
R^2 is free with basis $\{ (1,0), (0,1) \}$.
R^n is free with basis $\{ \delta_1, \dots, \delta_n \}$ where $\delta_i = (0, \dots, 0, 1, 0, \dots, 0)$ with the 1 in space number i.

(3.3) **Theorem: Finitely generated free modules.** *The modules R^n are the prototypes of finitely generated free modules: For any R-module M the following holds:*

$$M \text{ is finitely generated free} \quad \Longleftrightarrow \quad \exists\, n \in \mathbb{N}_0 \colon M \cong R^n .$$

(3.4) **Proof.** To prove " $\Longrightarrow$ " assume that $W = \{ w_1, \dots, w_n \}$ is a finite basis for M, and define the map $\sigma \colon R^n \to M$ by $\sigma(r_1, \dots, r_n) \stackrel{\text{def}}{=} r_1 w_1 + \cdots + r_n w_n$. This map is easily seen to be an R-homomorphism. It is injective since W is linearly independent, cf. (3.1.1), and it is surjective since $\operatorname{span}_R W = M$. Thus, σ is an isomorphism.

" $\Longleftarrow$ " follows from (3.5). □

(3.5) **Exercise: Free module and isomorphism 1.** Let F be a finitely generated free R-module with basis $W = \{ w_1, \dots, w_n \}$, let N be an R-module, and let $\varphi \colon F \to N$ be an isomorphism. Prove that N is a finitely generated free R-module with basis $\varphi(W)$.

(3.6) **Theorem: Finitely generated modules over a field.** *If R is a field and F is a finitely generated R-module, then F is a finitely generated free R-module.*

[1] In these notes, when we list a finite set, we assume tacitly that the list is without repetitions, that is, in this example, $w_i \neq w_j$ whenever $i \neq j$.

(3.7) **Proof.** Since F is a finitely generated vector space it has a (finite) basis. □

(3.8) **Theorem: Finitely generated free modules over a PID.** *If R is a PID, cf. (1.8), F is a finitely generated free R–module, and G is a submodule of F, then G is finitely generated free.*

(3.9) **Proof.** Choose by (3.3) $n \in \mathbb{N}_0$ and an isomorphism $\varphi\colon F \to R^n$, and set $H =^{\text{def}} \varphi(G)$ which is a submodule of R^n. By (3.3) it suffices to prove that H is a finitely generated free module. This will be done by induction on n.

For $n = 0$ we have that $H = O$ which is free.

Assume now $n > 0$, and consider the projection

$$\pi\colon R^n \to R^{n-1} \text{ given by } \pi(r_1, r_2, \ldots, r_n) = (r_2, \ldots, r_n).$$

The submodule $\pi(H)$ of R^{n-1} is finitely generated free by the inductive hypothesis. Let $(p_1, \ldots, p_v)$ be a basis for $\pi(H)$, and choose $h_1, \ldots, h_v \in H$ such that $\pi(h_i) = p_i$ for all i.

If $H \cap \operatorname{Ker}\pi = O$ then the restriction $\pi_{|H}\colon H \to \pi(H)$ is an isomorphism, and H is finitely generated free by (3.3).

Next assume that $H \cap \operatorname{Ker}\pi \neq O$. Since $\operatorname{Ker}\pi = \{(r_1, 0, \ldots, 0) \mid r_1 \in R\}$ there is an isomorphism $\psi\colon \operatorname{Ker}\pi \to R$. Since R is PID, $a \in R$ can be chosen such that $\psi(H \cap \operatorname{Ker}\pi) = (a)$. Now choose $h_0 \in H \cap \operatorname{Ker}\pi$ such that $\psi(h_0) = a$. It remains to be seen that $(h_0, h_1, \ldots, h_v)$ is a basis for H; this is verified by inspection. □

(3.10) **Exercise: Finitely generated free modules.** Let F be a finitely generated free R–module, that is, $F \cong R^n$ for some $n \in \mathbb{N}_0$, cf. (3.3). Prove that this n is unique, that is, if $R^n \cong R^{n'}$ then $n = n'$. [Use (without proof) that there exits a maximal ideal $\mathfrak{m}$ in R and consider the corresponding over-lining, cf. (2.26). Apply then (3.33) twice and (5.3).]

(3.11) **Exercise: Finitely generated free modules over a non-commutative ring.** Consider the vector-space $V = \mathbb{R}^{(\mathbb{N})}$ over the real numbers $\mathbb{R}$. Prove that the vector-space $\Lambda = \operatorname{Hom}_{\mathbb{R}}(V, V)$ is a (non-commutative) ring with composition as the multiplication. Prove that the left Λ–module $\Lambda \oplus \Lambda$ is isomorphic to the left Λ–module Λ.

(3.12) **Exercise: Finitely generated free modules over a PID.** Let R, F, and G be as in (3.8) and choose $m, n \in \mathbb{N}_0$ such that $F \cong R^m$ and $G \cong R^n$. Prove that $n \leq m$.

(3.13) **Module of functions.** Let W be a (possible empty) set. Let R^W denote the set of maps $\varphi\colon W \to R$; the elements of R^W are called *functions on* W. If $W = \varnothing$ then $R^W = R^{\varnothing}$ consists of one and only one map, the empty map. The set R^W becomes an R–module by the definitions:

$$(\varphi + \psi)(w) \stackrel{\text{def}}{=} \varphi(w) + \psi(w) \tag{3.13.1}$$

$$(r\varphi)(w) \stackrel{\text{def}}{=} r\varphi(w) \tag{3.13.2}$$

for $\varphi, \psi \in W$ and $r \in R$. If $W = \varnothing$ then $R^W = R^{\varnothing}$ is the zero module O. For $\varphi \in R^W$ we define its *support* by

$$\operatorname{supp}\varphi \stackrel{\text{def}}{=} \{\, w \in W \mid \varphi(w) \neq 0 \,\},$$

and we set

$$R^{(W)} \stackrel{\text{def}}{=} \{\, \varphi \in R^W \mid \operatorname{supp}\varphi \text{ is finite} \,\} \tag{3.13.3}$$

which is a submodule of R^W. The set W is finite if and only if $R^{(W)} = R^W$. We make the obvious identification

$$R^{(\{1,\dots,n\})} = R^n . \tag{3.13.4}$$

(3.14) **Exercise: Ring of functions.** Let R by any commutative non-trivial ring, and let W be any set. Prove that the R-module R^W becomes a commutative ring with the product $\varphi\psi \in R^W$ of $\varphi, \psi \in R^W$ defined by $(\varphi\psi)(w) =^{\text{def}} \varphi(w)\psi(w)$ for $w \in W$. Prove that $R^{(W)}$ is an ideal in R^W. Prove that the submodule $R^{(W)}$ of the R^W-module R^W is *not* finitely generated when W is infinite.

(3.15) **Spanning homomorphisms.** Any subset W of an R-module M induces a map

$$\sigma_{RWM}\colon R^{(W)} \to M \text{ given by } \sigma_{RWM}(\varphi) \stackrel{\text{def}}{=} \sum_{w\in W} \varphi(w)w . \tag{3.15.1}$$

As $\operatorname{supp}\varphi$ is a finite subset of W, this sum is finite in the sense that its terms $\varphi(w)w$ are zero for all $w \in W$ but finitely many; namely $\varphi(w)w = 0w = 0$ whenever $w \notin \operatorname{supp}\varphi$, cf. (3.13.3) and (2.1.1).

If $\{x_1, \dots, x_n\}$ be a finite subset of W then the next holds for any $\varphi \in R^{(W)}$:

$$\operatorname{supp}\varphi \subseteq \{x_1, \dots, x_n\} \implies \sigma_{RWM}(\varphi) = \varphi(x_1)x_1 + \cdots + \varphi(x_n)x_n . \tag{3.15.2}$$

It is straightforward to verify that σ_{RWM} is an R-homomorphism, it is called the *spanning homomorphism* for the subset W of the R-module M, cf. (3.16.1).

(3.16) **Surjectivity.** Let W be a subset of an R-module M. The first equality below follows from (2.9.1), and then the next follows by definition.

$$\operatorname{Im}\sigma_{RWM} = \operatorname{span}_R W ; \tag{3.16.1}$$

$$\sigma_{RWM} \text{ is surjective} \iff W \text{ generates } M . \tag{3.16.2}$$

(3.17) **Kronecker delta functions.** Let X be a set, and let $w \in X$ be an element. The *Kronecker delta function* $\delta_w\colon X \to R$ is defined by

$$\delta_w(x) = \begin{cases} 1 & \text{if } x = w \\ 0 & \text{otherwise.} \end{cases}$$

If φ belongs to $R^{(X)}$ and $\{x_1, \dots, x_n\}$ is a finite subset of X, then the following hold:

$$\operatorname{supp}\varphi \subseteq \{x_1, \dots, x_n\} \implies \varphi = \varphi(x_1)\delta_{x_1} + \cdots + \varphi(x_n)\delta_{x_n} ; \tag{3.17.1}$$

the function on either side maps any x_i to $\varphi(x_i)$ and any other $x \in X$ to 0.

(3.18) **Linear independence.** A subset W of an R-module M is said to be *linearly independent* exactly when *every linear relation of elements from* W *is trivial*, that is, if $n \in \mathbb{N}_0$, $r_1, \dots, r_n \in R$, and $w_1, \dots, w_n \in W$ are such that

$$r_1w_1 + \cdots + r_nw_n = 0 , \tag{3.18.1}$$

then

$$r_1 = 0, \dots, r_n = 0 . \tag{3.18.2}$$

It is (3.18.1) that is the linear relation, and (3.18.2) that expresses the triviality. The empty subset $W = \varnothing$ is (by definition) linearly independent.

(3.19) **Theorem: Linear independence.** *Let W be a subset of an R–module M. The following then holds.*

(3.19.1) $\qquad W$ *is linearly independent* $\iff$ σ_{RWM} *is injective,*

(3.20) **Proof.** " $\Longrightarrow$ ": Assume that W is linearly independent, and that φ is in $\operatorname{Ker}\sigma_{RWM}$. [In view of (2.19.5) we want to prove that $\varphi = 0$]. Choose a finite subset $\{x_1, \dots, x_n\}$ of X such that $\operatorname{supp}\varphi \subseteq \{x_1, \dots, x_n\}$. We have then by (3.15.2):

$$\varphi(x_1)x_1 + \cdots + \varphi(x_n)x_n = \sigma_{RWM}(\varphi) = 0\,.$$

This implies that $\varphi(x_i) = 0$ for all i, and hence that $\varphi = 0$ [as desired].

" $\Longleftarrow$ ": Assume that $\sigma_{RWM}(\varphi)$ is injective, and that we have a linear relation $r_1w_1 + \cdots + r_nw_n = 0$ [and we want to prove that $r_i = 0$ for all i].

Set $\varphi = r_1\delta_{w_1} + \cdots + r_n\delta_{w_n} \in R^{(W)}$. Note that $\sigma_{RWM}(\varphi) = r_1w_1 + \cdots + r_nw_n = 0$, so by (2.19.5) we obtain $\varphi = 0$, and hence $r_i = \varphi(w_i) = 0$ for all i [as desired]. □

(3.21) **Exercise: Transcendence.** Consider the complex numbers $\mathbb{C}$ as a module over the integers $\mathbb{Z}$, a number $\alpha \in \mathbb{C}$, and the subset

$$W_\alpha \stackrel{\text{def}}{=} \{1, \alpha, \alpha^2, \dots, \alpha^n, \dots\}$$

of the $\mathbb{Z}$–module $\mathbb{C}$. Prove the following:

$$W_\alpha \text{ is linearly independent} \iff \alpha \text{ is a transcendent number,}$$

cf. [Th, POL (3.17.8)]

(3.22) **Exercise: Annihilators and linear independence.** Let W be a linearly independent subset of the R–module M. Prove that $\operatorname{Ann}_R(w) = O$ for all $w \in W$.

(3.23) **Exercise: Subsets of the ring.** Let W be a non-empty subset of the ring. Prove that W is linearly independent if and only if W consist of exactly one element w and $\operatorname{Ann}_R(w) = O$.

(3.24) **Basis.** A subset W of an R–module M is a *basis for* M exactly when the spanning homomorphism σ_{RWM} is an isomorphism, that is, if and only if W is a linearly independent generating set for M, cf. (3.19.1) and (3.16.2).

(3.25) **Natural basis.** Set $\Delta_X \stackrel{\text{def}}{=} \{\delta_w \mid w \in X\}$ $(\subseteq R^{(X)})$. It is then straightforward to verify that the following holds.

(3.25.1) $\qquad$ The set Δ_X is a basis for $R^{(X)}$.

This basis Δ_X is called the *natural basis for* $R^{(X)}$.

If X is finite, say $X = \{1, \dots, n\}$ for an $n \in \mathbb{N}_0$, then R^n has natural basis $(\delta_1, \dots, \delta_n)$, cf. (3.2).

(3.26) **Free modules.** An R–module F is said to be *free* exactly when it has a basis, that is, there is a subset W of F such that $\sigma_{RWF}\colon R^{(W)} \to F$ is an isomorphism, cf. (3.24).

(3.27) **Free modules over a set.** For any set X, the R–module $R^{(X)}$ is free with basis Δ_X, cf. (3.25.1); it is called the *free module over* X.

(3.28) **Exercise: Free module and isomorphism 2.** Let F be a free R–module with basis W, let N be an R–module, and let $\varphi\colon F \to N$ be an isomorphism. Prove that N is a free R–module with basis $\varphi(W)$.

Prove that an R–module M is free if and only if there is a set X such that $M \cong R^{(X)}$.

(3.29) **Theorem: Module as homomorphic image of a free.** *Any R–module M is a homomorphic image of a free R–module, that is, there exists a free R–module F and a surjective homomorphism $F \to M$. If M is finitely generated by n elements, then M is a homomorphic image of R^n.*

(3.30) **Proof.** Choose any generating set W for M (for example, $W = M$), but choose it consisting of n elements, if M is finitely generated by n elements. Then $\sigma_{RWM}\colon R^{(W)} \to M$ is surjective by (3.16.2), and $R^{(W)}$ is free by (3.27). If W consists of $n \in \mathbb{N}_0$ elements, then $R^{(W)} = R^n$, cf. (3.13.4). □

(3.31) **Exercise: $\mathbb{Q}$ free over $\mathbb{Z}$** Determine whether the set of rational numbers $\mathbb{Q}$ is free as a $\mathbb{Z}$–module.

(3.32) **Exercise: Free modules over integral domains.** Let R be an integral domain and let B be its field of fractions, cf. [Th, RNG (4.4)]. Assume that R is not a field.

(a) Prove that $R \subset B$ [2]

(b) Prove that B is an R–module.

(c) Prove that B is not a free R–module.

(d) Let M be a free R–submodule of B. Prove that M is cyclic.

(e) Prove that B cannot be an R–submodule of a free R–module.

(3.33) **Theorem: Over-lining and modules of functions.** *Let $\mathfrak{a}$ be a proper ideal, and consider the corresponding over-lining, cf. (2.26). There is then an $\overline{R}$–isomorphism:*

$$\overline{R^{(X)}} \cong \overline{R}^{(X)}\,.$$

(3.34) **Proof.** Recall that the elements of $R^{(X)}$ are functions $\varphi\colon X \to R$ such that $\varphi(x) = 0$ for all but finitely many $x \in X$. Let the map $\Phi\colon R^{(X)} \to \overline{R}^{(X)}$ be defined by that for $\varphi \in R^{(X)}$ the image $\Phi(\varphi) \in \overline{R}^{(X)}$, that is, the function $\Phi(\varphi)\colon X \to \overline{R}$, is defined by $\big(\Phi(\varphi)\big)(x) =^{\text{def}} \overline{\varphi(x)}$ $(= [\varphi(x)]_{\mathfrak{a}})$ for $x \in X$.

It is easy to see that Φ is an R–homomorphism. To prove that Φ is surjective, let ψ belong to $\overline{R}^{(X)}$, that is, $\psi\colon X \to \overline{R}$ is a map. For a given $x \in \operatorname{supp}\psi$ choose any element $\varphi(x) \in R$ such that $\overline{\varphi(x)} = \psi(x) \in \overline{R}$. Set $\varphi(x) = 0$ for all $x \notin \operatorname{supp}\psi$. This defines $\varphi \in R^{(X)}$ such that $\overline{\varphi(x)} = \psi(x)$ for all $x \in X$, that is, $\Phi(\varphi) = \psi$. In the next chain of bi-implications the last one results from (3.17.1).

$$\varphi \in \operatorname{Ker}\Phi \iff \forall x \in X\colon\ \varphi(x) \in \mathfrak{a} \iff \varphi(X) \subseteq \mathfrak{a} \iff \varphi \in \mathfrak{a}R^{(X)}\,.$$

Thus, $\operatorname{Ker}\Phi = \mathfrak{a}R^{(X)}$, and the Isomorphism Theorem (2.30) yields the next isomorphism:

$$\overline{R^{(X)}} = R^{(X)}/\mathfrak{a}R^{(X)} \cong \overline{R}^{(X)}\,,$$

and this proves the desired assertion. □

[2] In these notes, the notation $\subset$ always means *proper* inclusion, that is, $X \subset Y$ if and only if $X \subseteq Y$ and $X \neq Y$.

4. Modules of Homomorphisms

(4.1) **Homomorphisms form an R–module.** Let M and N be R–modules, and set

$$\operatorname{Hom}_R(M,N) \stackrel{\text{def}}{=} \{\,\varphi\colon M \to N \mid \varphi \text{ is an } R\text{–homomorphism}\,\}.$$

This is an R–module: For $\varphi,\psi \in \operatorname{Hom}_R(M,N)$ and $r \in R$ define maps as follows:

$$\varphi+\psi\colon M \to N \text{ by } (\varphi+\psi)(m) \stackrel{\text{def}}{=} \varphi(m)+\psi(m)\,; \tag{4.1.1}$$

$$r\varphi\colon M \to N \text{ by } \quad (r\varphi)(m) \stackrel{\text{def}}{=} r(\varphi(m))\,. \tag{4.1.2}$$

Then $\varphi+\psi, r\varphi \in \operatorname{Hom}_R(M,N)$, that is, they are R–homomorphisms. For example, for $a \in R$ and $m \in M$ there are equalities:

$$(r\varphi)(am) \stackrel{(1)}{=} r(\varphi(am)) \stackrel{(2)}{=} r(a\varphi(m)) \tag{4.1.3}$$

$$\stackrel{(3)}{=} (ra)\varphi(m) \stackrel{(4)}{=} (ar)\varphi(m) \tag{4.1.4}$$

$$\stackrel{(5)}{=} a(r\varphi(m)) \stackrel{(6)}{=} a(r\varphi)(m)\,. \tag{4.1.5}$$

The reasons behind these six equalities are as follows.

(1) by the above definition of $r\varphi$.
(2) since φ is an R–homomorphism.
(3) since N is an R–module.
(4) since R is **commutative!** Cf. (1.4)
(5) since N is an R–module.
(6) by the definition of $r\varphi$.

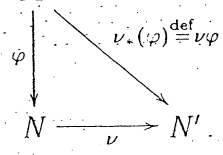
With the addition above and the R–multiplication the set $\operatorname{Hom}_R(M,N)$ becomes an R–module. The zero homomorphism, cf. (2.18), is the zero element, and the opposite element $-\varphi$ to $\varphi \in \operatorname{Hom}_R(M,N)$ is defined by $(-\varphi)(m) =^{\text{def}} -\varphi(m)$ for $m \in M$.

(4.2) **Induced homomorphism (covariant case).** Let M be an R–module, and let $\nu\colon N \to N'$ be a homomorphism of R–modules. There is then a *(covariantly) induced R–homomorphism:*

$$\nu_* \stackrel{\text{def}}{=} \operatorname{Hom}_R(M,\nu)\colon \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,N')$$

defined by

$$\nu_*(\varphi) \stackrel{\text{def}}{=} \nu\varphi \ (\stackrel{\text{def}}{=} \nu\circ\varphi)^{3},$$ that is, by the commutative diagram[4]



Thus, $\nu_*(\varphi)\colon M \to N'$ is defined by $\big(\nu_*(\varphi)\big)(m) = \nu(\varphi(m))$ for $m \in M$.

It is easy to verify that $\nu_*\colon \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,N')$ is a homomorphism of R–modules. For example, that $\nu_*(r\varphi) = r\nu_*(\varphi)$ for $r \in R$ and $\varphi \in \operatorname{Hom}_R(M,N)$ is verified as follows:

$$\begin{aligned}\big(\nu_*(r\varphi)\big)(m) &= \big(\nu\circ(r\varphi)\big)(m) = \nu\big((r\varphi)(m)\big) = \nu(r(\varphi(m)))\\ &= r(\nu(\varphi(m))) \quad = r(\nu\circ\varphi)(m) = r(\nu_*(\varphi))(m)\,.\end{aligned}$$

[3] Here—and very often below—the composition circle $\circ$ is omitted.

[4] And that is, whenever two compositions of homomorphisms in the diagram have the same source and the same target, then they are equal.

If $\nu'\colon N' \to N''$ is another R–homomorphism, then

(4.2.1) $$(\nu'\nu)_* = (\nu')_*\nu_*\colon \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,N)'',$$

since $(\nu'\nu)_*(\varphi) = (\nu'\nu)\varphi = \nu'(\nu\varphi) = (\nu')_*(\nu_*(\varphi))$.

For the identity homomorphism $1_N \in \operatorname{Hom}_R(N,N)$ we have the next formula.

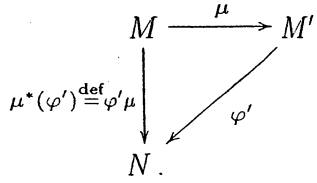
(4.2.2) $$(1_N)_* = 1'_{\operatorname{Hom}_R(M,N)}\colon \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,N)$$

(4.3) **Induced homomorphism (contravariant case).** Let $\mu\colon M \to M'$ be a homomorphism of R–modules, and let N be an R–module. There is a *(contravariantly) induced R–homomorphism:*

$$\mu^* \stackrel{\text{def}}{=} \operatorname{Hom}_R(\mu,N)\colon \operatorname{Hom}_R(M';N) \to \operatorname{Hom}_R(M,N)$$

defined by

$\mu^*(\varphi') \stackrel{\text{def}}{=} \varphi'\mu$ $(\stackrel{\text{def}}{=} \varphi' \circ \mu)$, that is, by the commutative diagram



Thus, $\mu^*(\varphi')\colon M \to N$ is defined by $\big(\mu^*(\varphi')\big)(m) = \varphi'(\mu(m))$ for $m \in M$.

It is easy to verify that $\mu^*\colon \operatorname{Hom}_R(M',N) \to \operatorname{Hom}_R(M,N)$ is a homomorphism of R–modules.

If $\mu^-\colon M^- \to M$ is another R–homomorphism, then

(4.3.1) $$(\mu\mu^-)^* = (\mu^-)^*\mu^*\colon \operatorname{Hom}_R(M',N) \to \operatorname{Hom}_R(M^-,N).$$

For the identity homomorphism $1_M \in \operatorname{Hom}_R(M,M)$ we have the next formula.

(4.3.2) $$(1_M)^* = 1_{\operatorname{Hom}_R(M,N)}\colon \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,N)$$

(4.4) **Homomorphisms from the ring.** Let N be an R–module. The map

$$\alpha\colon \operatorname{Hom}_R(R,N) \to N \quad \text{given by} \quad \varphi \mapsto \alpha(\varphi) \stackrel{\text{def}}{=} \varphi(1)$$

is then an isomorphism of R–modules; the inverse $\alpha^{-1}\colon N \to \operatorname{Hom}_R(R,N)$ is defined as follows: for $n \in N$ the homomorphism $\alpha^{-1}(n) \in \operatorname{Hom}_R(R,N)$ is given by $\big(\alpha^{-1}(n)\big)(r) = rn$ for $r \in R$.

5. FUNCTORS

(5.1) **Category of R–modules.** The class of all R–modules and of all R-homomorphisms is called the *category of R–modules* and denoted $\mathrm{Mod}(R)$; the modules in it are sometimes referred to as its *object* while the R–homomorphisms are referred to as its *morphisms*.

In this section we consider also another non-trivial commutative ring R' and the category $\mathrm{Mod}(R')$ of R'–modules and R'–homomorphisms.

(5.2) **Covariant functor.** Let $F\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R')$ be a map in the sense that

- to any R–module M, the map F associates a unique R'–module $F(M)$,
- to any R–homomorphism $\varphi\colon M \to N$, the map F associates a unique R'–homomorphism $F(\varphi)\colon F(M) \to F(N)$, that is, F induces a map $\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{R'}(F(M),F(N))$ given by $\varphi \mapsto F(\varphi)$.

The map F is said to be a *covariant functor:* $\mathrm{Mod}(R) \to \mathrm{Mod}(R')$, exactly when the next two requirements are fulfilled.

(5.2.1) $F(\varphi\psi) = F(\varphi)F(\psi)$ for all composable R–homomorphisms $L \xrightarrow{\psi} M \xrightarrow{\varphi} N$,

(5.2.2) $F(1_M) = 1_{F(M)}$ for the identity map of any R–module M.

(5.3) **Functor of isomorphism.** If $F\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R')$ is a covariant functor and $\varphi\colon M \to N$ is an R–isomorphism, then $F(\varphi)\colon F(M) \to F(N)$ is an R'–isomorphism and $\big(F(\varphi)\big)^{-1} = F(\varphi^{-1})$, since $F(\varphi^{-1})F(\varphi) = F(\varphi^{-1}\varphi) = F(1_M) = 1_{F(M)}$ by (5.2.1) and (5.2.2), and similarly $F(\varphi)F(\varphi^{-1}) = 1_{F(N)}$.

(5.4) **The functor $\mathrm{Hom}(H,-)$.** Let H be an R–module. The map:

$\mathrm{Hom}_R(H,-)\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R)$ given by $M \mapsto \mathrm{Hom}_R(H,M)$, $\varphi \mapsto \mathrm{Hom}_R(H,\varphi)$

is then a covariant functor, cf. (4.2.1) and (4.2.2).

(5.5) **Over-lining functor.** Let $\mathfrak{a}$ be a fixed proper ideal, and consider the over-lining from (2.26). Any R–homomorphism $\varphi\colon M \to N$ induces an $\overline{R}$–homomorphism $\overline{\varphi}\colon \overline{M} \to \overline{N}$ well-defined by $\overline{\varphi}(\overline{m}) =^{\mathrm{def}} \overline{\varphi(m)}$. This follows from the Homomorphism Theorem (2.28) applied to to the induced R–homomorphism $\widetilde{\varphi}\colon M \to \overline{N}$, $m \mapsto \overline{\varphi(m)}$ since $\mathfrak{a}M \subseteq \operatorname{Ker}\widetilde{\varphi}$. It is straightforward to verify that $\overline{\varphi\psi} = \overline{\varphi}\,\overline{\psi}$ for φ and ψ as in (5.2.1) and that $\overline{1_M} = 1_{\overline{M}}$ for all R–modules M.

Thus, over-lining $M \mapsto \overline{M}$, $\varphi \mapsto \overline{\varphi}$ is a covariant functor $\mathrm{Mod}(R) \to \mathrm{Mod}(\overline{R})$.

(5.6) **Exercise: Constant functor.** Let $F\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R')$ be a covariant functor. Assume that F is constant, that is, there exist an R'–module M'_0 and an R'–homomorphism $\varphi'_0\colon M'_0 \to M'_0$ such that $F(M) = M'_0$ for all R–modules M and $F(\varphi) = \varphi'_0$ for all R–homomorphisms $\varphi\colon M \to N$. What is φ'_0?

The next subsections are needed for the presentation of the fraction functor in (5.11).

(5.7) **Special multiplicative system.** A subset S of R is said to be a *special multiplicative system,* exactly when

(5.7.1) $1 \in S$, $ss' \in S$ for all $s,s' \in S$, and $0 \notin S$.

(It is the condition $0 \notin S$ that is "special".)

Examples of special multiplicative systems:

(5.7.2) $S \stackrel{\text{def}}{=} R \setminus \mathfrak{p}$ when $\mathfrak{p}$ is a prime ideal in R, cf. [Th, RNG (2.9)].

(5.7.3) $S \stackrel{\text{def}}{=} \{ a^n \mid n \in \mathbb{N}_0 \}$ when $a \in R$ is such that $a^n \neq 0$ for all $n \in \mathbb{N}_0$.

(5.8) **Set of fractions.** Let M be an R–module, let S be a special multiplicative system in R, and consider the product set

$$M \times S \stackrel{\text{def}}{=} \{ (m, s) \mid m \in M, \, s \in S \}$$

with the relation $\sim$ is defined by

$$(m, s) \sim (m', s') \iff \exists\, u \in S \colon us'm = usm'$$

for $(m, s), (m', s') \in M \times S$. If also (m'', s'') belongs to $M \times S$, then the following three assertions are easily verified:

(5.8.1) Reflexivity: $(m, s) \sim (m, s)$;

(5.8.2) Symmetry: $(m, s) \sim (m', s') \implies (m', s') \sim (m, s)$;

(5.8.3) Transitivity: $(m, s) \sim (m', s') \wedge (m', s') \sim (m'', s'') \implies (m, s) \sim (m'', s'')$.

For example, in order to prove the transitivity, assume that $u, u' \in S$ are such that $us'm = usm'$ and $u's''m' = u's'm''$. Setting $u'' = u'us'$ (which belongs to S) we get $u''s''m = u''sm''$ by an easy calculation which uses the commutativity of R.

It follows that $\sim$ is an equivalence relation on $M \times S$. For $(m, s) \in M \times S$ we define the corresponding *fraction* as the equivalence class:

$$m/s \stackrel{\text{def}}{=} \{ (n, t) \in M \times S \mid (n, t) \sim (m, s) \};$$

we call m and s the *numerator* and the *denominator,* respectively. We note that

(5.8.4) $$m/s = m'/s' \iff \exists\, u \in S \colon us'm = usm';$$

in particular

(5.8.5) $$m/s = 0/1 \iff \operatorname{Ann}_R(m) \cap S \neq \varnothing.$$

Next, we define the *set of fractions* as

$$S^{-1}M \stackrel{\text{def}}{=} \{ m/s \mid m \in M \wedge s \in S \}.$$

(5.9) **Ring and module of fractions.** Let M be an R–module, let S be a special multiplicative system in R, and consider the set of fractions $S^{-1}M$. It is then straightforward to verify that the addition and the R–multiplication are well-defined by

(5.9.1) $$m/s + m'/s' \stackrel{\text{def}}{=} (s'm + sm')/ss'$$

(5.9.2) $$r(m/s) \stackrel{\text{def}}{=} (rm)/s$$

for $m/s, m'/s' \in S^{-1}M$ and $r \in R$. These make $S^{-1}M$ into an R–module with $0/1$ as zero element and with $-(m/s) = (-m)/s$.

The R–module $S^{-1}R$ is now easily seen to be a commutative ring with multiplication well-defined by

$$(r/s)(r'/s') \stackrel{\text{def}}{=} (rr')/ss'$$

and with $1/1$ as the identity element; it is called the *ring of fractions* (with S as the set of denominators). This ring is non-trivial by(5.8.5) in view of the last condition in (5.7.1).

The R–module $S^{-1}M$ has a well-defined $S^{-1}R$–multiplication

$$(r/s)(m/s') \stackrel{\text{def}}{=} (rm)/ss' ,$$

and this makes $S^{-1}M$ into an $S^{-1}R$–module. If $\varphi\colon M \to N$ is an R–homomorphism, then an $S^{-1}R$–homomorphism $S^{-1}\varphi\colon S^{-1}M \to S^{-1}N$ is well-defined by

$$(5.9.3) \qquad \big(S^{-1}\varphi\big)(m/s) \stackrel{\text{def}}{=} \varphi(m)/s \quad \text{for } m/s \in S^{-1}M .$$

(5.10) **Field of fractions.** Let R be an integral domain and consider the special multiplicative system $S =^{\text{def}} R \setminus O$, cf. (5.7.2). Then $B =^{\text{def}} S^{-1}R$ is the field of fractions of R, cf. [Th, RNG (4.4)], and any R–module M induces a vector space $S^{-1}M$ over B.

(5.11) **Fraction functor.** Any special multiplicative system S in R induces a covariant functor:

$$S^{-1}\colon \operatorname{Mod}(R) \to \operatorname{Mod}(S^{-1}R) \text{ given by } M \mapsto S^{-1}M , \varphi \mapsto S^{-1}\varphi ,$$

cf. (5.9), in particular, (5.9.3).

(5.12) **Contravariant functor.** Let $F\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ be such that

- to any R–module M, F associates a unique R'–module $F(M)$,
- to any R–homomorphism $\varphi\colon M \to N$, F associates a unique R'–homomorphism $F(\varphi)\colon F(N) \to F(M)$, that is, F induces a map

$$\operatorname{Hom}_R(M,N) \to \operatorname{Hom}_{R'}(F(N),F(M)) , \ \varphi \mapsto F(\varphi) .$$

Note the change in directions of the arrows:

$$(5.12.1) \qquad M \xrightarrow{\varphi} N$$

$$(5.12.2) \qquad F(M) \xleftarrow{F(\varphi)} F(N) .$$

Then F is said to be a *contravariant functor:* $\operatorname{Mod}(R) \to \operatorname{Mod}(R')$ exactly when the next two requirements are fulfilled:

(5.12.3) $F(\varphi\psi) = F(\psi)F(\varphi)$ for all composable R–homomorphisms $L \xrightarrow{\psi} M \xrightarrow{\varphi} N$,

(5.12.4) $F(1_M) = 1_{F(M)}$ for the identity map of any R–module M.

For the former, note the direction of the arrows: $F(L) \xleftarrow{F(\psi)} F(M) \xleftarrow{F(\varphi)} F(N)$.

(5.13) **Functor of isomorphism.** If $F\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ is a contravariant functor and $\varphi\colon M \to N$ is an R–isomorphism, then $F(\varphi)\colon F(N) \to F(M)$ is an R'–isomorphism and $\big(F(\varphi)\big)^{-1} = F(\varphi^{-1})$.

(5.14) **The functor $\operatorname{Hom}(-,H)$.** Any R–module H induces a contravariant functor

$\operatorname{Hom}_R(-,H)\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R)$ given by $M \mapsto \operatorname{Hom}_R(M,H)$, $\varphi \mapsto \operatorname{Hom}_R(\varphi,H)$,

cf. (4.3.1) and (4.3.2).

6. Exactness

(6.1) **Sequences.** A family of composable R–homomorphisms (and their sources and targets)

$$(\dagger) \qquad \cdots \xrightarrow{\varphi_{\ell+2}} M_{\ell+1} \xrightarrow{\varphi_{\ell+1}} M_\ell \xrightarrow{\varphi_\ell} M_{\ell-1} \xrightarrow{\varphi_{\ell-1}} \cdots$$

is said to be a *sequence* (of R–homomorphisms). Note that the arrows point to the right, and that the index of each homomorphism correspond to that of its source. Sequences can be *bounded to the left:*

$$(6.1.1) \qquad M_s \xrightarrow{\varphi_s} M_{s-1} \xrightarrow{\varphi_{s-1}} \cdots$$

bounded to the right:

$$(6.1.2) \qquad \cdots \xrightarrow{\varphi_{t+2}} M_{t+1} \xrightarrow{\varphi_{t+1}} M_t\,,$$

or *bounded:*

$$(6.1.3) \qquad M_s \xrightarrow{\varphi_s} M_{s-1} \xrightarrow{\varphi_{s-1}} \cdots\cdots \xrightarrow{\varphi_{t+2}} M_t \xrightarrow{\varphi_{t+1}} M_t\,.$$

(6.2) **Exact sequences.** The (possibly bounded or partly bounded) sequence $(\dagger)$ is said to *exact at* the module M_ℓ exactly when $\operatorname{Im}\varphi_{\ell+1} = \operatorname{Ker}\varphi_\ell$, and it is said to be *exact* exactly when it is exact at all possible modules, that is, $\operatorname{Im}\varphi_\ell = \operatorname{Ker}\varphi_\ell$ for all possible integers ℓ. In sequences (6.1.1) and (6.1.3), respectively, (6.1.2) and (6.1.3), it is not possible talk about exactness at M_s, respectively, at M_t.

(6.3) **Exercise: Kernel–cokernel sequence.** For any R–homomorphism $\varphi\colon M \to N$ we define its *cokernel* by $\operatorname{Coker}\varphi =^{\text{def}} N/\operatorname{Im}\varphi$. Establish an exact sequence:

$$O \to \operatorname{Ker}\varphi \xrightarrow{\alpha} M \xrightarrow{\varphi} N \xrightarrow{\beta} \operatorname{Coker}\varphi \to O\,.$$

(6.4) **Short sequences.** A *short sequence* is a sequence of the form

$$O \to K \xrightarrow{\kappa} L \xrightarrow{\lambda} M \to O\,,$$

that is, a sequence consisting of four R–homomorphisms (and hence five modules) such that the leftmost has the zero module as source (and thus is the zero homomorphism) and such that the rightmost has the zero module as target (and thus is the zero homomorphism).

(6.5) **Exactness of short sequences.** A short sequence

$$(*) \qquad O \to K \xrightarrow{\kappa} L \xrightarrow{\lambda} M \to O$$

is said to be

- *half-exact* exactly when $K \xrightarrow{\kappa} L \xrightarrow{\lambda} M$ is exact, that is, $\operatorname{Im}\kappa = \operatorname{Ker}\lambda$;
- *left-exact* exactly when $O \to K \xrightarrow{\kappa} L \xrightarrow{\lambda} M$ is exact, that is, if and only if κ is injective and $\operatorname{Im}\kappa = \operatorname{Ker}\lambda$;
- *right-exact* exactly when $K \xrightarrow{\kappa} L \xrightarrow{\lambda} M \to O$ is exact, that is, if and only if $\operatorname{Im}\kappa = \operatorname{Ker}\lambda$ and λ is surjective;
- *short-exact* exactly when it is exact, that is, if and only if κ is injective, $\operatorname{Im}\kappa = \operatorname{Ker}\lambda$, and λ is surjective.

(6.6) **Special short-exact sequences.** A sequence of the form

$$O \to H \xrightarrow{\iota} L \xrightarrow{\rho} L/H \to O$$

where L is an R–module, H is a submodule of L, ι is the inclusion, and ρ is the residue homomorphism, is short-exact; it is called a *special short-exact sequence*.

(6.7) **Isomorphic short sequences.** Two short sequences

$$O \longrightarrow K \xrightarrow{\kappa} L \xrightarrow{\lambda} M \longrightarrow O \tag{6.7.1}$$

$$O \longrightarrow K' \xrightarrow{\kappa'} L' \xrightarrow{\lambda'} M' \longrightarrow O \tag{6.7.2}$$

are said to be *isomorphic* exactly when there exist isomorphisms

$$\alpha\colon K \xrightarrow{\cong} K', \beta\colon L \xrightarrow{\cong} L', \text{ and } \gamma\colon M \xrightarrow{\cong} M'$$

such that the diagram

$$\begin{array}{ccccc} K & \xrightarrow{\kappa} & L & \xrightarrow{\lambda} & M \\ {\scriptstyle\alpha}\downarrow{\scriptstyle\cong} & & {\scriptstyle\beta}\downarrow{\scriptstyle\cong} & & {\scriptstyle\gamma}\downarrow{\scriptstyle\cong} \\ K' & \xrightarrow{\kappa'} & L' & \xrightarrow{\lambda'} & M' \end{array} \tag{6.7.3}$$

is commutative, that is, $\beta\kappa = \kappa'\alpha$ and $\gamma\lambda = \lambda'\beta$.

(6.8) **Exercise.** Prove that, when two isomorphic short sequences are given, and the one is half-exact (respectively, left-exact, right-exact, short-exact), then the other has the corresponding property.

(6.9) **Isomorphic short sequences and functors.** Any (co- or contravariant) functor $F\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R')$ transform any two isomorphic short sequences into isomorphic sequences.

Let $O \to K \xrightarrow{\kappa} L \xrightarrow{\lambda} M \to O$ and $O \to K' \xrightarrow{\kappa'} L' \xrightarrow{\lambda'} M' \to O$ be isomorphic short sequences, and consider the commutative diagram (6.7.3). In the covariant case, the functor sends it into a diagram

$$\begin{array}{ccccc} F(K) & \xrightarrow{F(\kappa)} & F(L) & \xrightarrow{F(\lambda)} & F(M) \\ {\scriptstyle F(\alpha)}\downarrow{\scriptstyle\cong} & & {\scriptstyle F(\beta)}\downarrow{\scriptstyle\cong} & & {\scriptstyle F(\gamma)}\downarrow{\scriptstyle\cong} \\ F(K') & \xrightarrow{F(\kappa')} & F(L') & \xrightarrow{F(\lambda')} & F(M')\,; \end{array} \tag{6.9.1}$$

in the contravariant case all arrows should be reversed. $F(\alpha)$, $F(\beta)$, and $F(\gamma)$ are isomorphisms by (5.3) and (5.13). The diagram is commutative, since any functor sends a commutative diagram into a commutative one; for example, $F(\beta)F(\kappa) = F(\beta\kappa) = F(\kappa'\alpha) = F(\kappa')F(\alpha)$.

(6.10) **Lemma: Ubiquity of special short-exact sequences.**
Every short-exact sequence is isomorphic to a special short-exact sequence.

(6.11) **Proof.** Let $O \to K \xrightarrow{\kappa} L \xrightarrow{\lambda} M \to O$ be short-exact, and set $H =^{\mathrm{def}} \mathrm{Im}\,\kappa = \mathrm{Ker}\,\lambda$. Since H is the image of κ and since this map is injective, it induces an isomorphism $\widetilde{\kappa}\colon K \to H$, $k \mapsto \kappa(k)$. Setting $\alpha =^{\mathrm{def}} \widetilde{\kappa}^{-1}$ and $\beta =^{\mathrm{def}} 1_L$ (the identity) the left square in

the next diagram is commutative, when ι is the inclusion.

(6.11.1)
$$\begin{array}{ccccc} K & \xrightarrow{\kappa} & L & \xrightarrow{\lambda} & M \\ \cong\uparrow{\scriptstyle\alpha} & & \cong\uparrow{\scriptstyle\beta} & & \cong\uparrow{\scriptstyle\gamma} \\ H & \xrightarrow{\iota} & L & \xrightarrow{\rho} & L/H \end{array}$$

The isomorphism γ is induced by the Isomorphism Theorem (2.30) applied to the surjective homomorphism $\lambda\colon L \to M$. Letting ρ be the residue map, the right hand square is commutative, and the desired isomorphism of short sequences has been established. □

(6.12) **Half-exact functors.** A (co- or contravariant) functor $F\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ is said to be *half-exact* exactly when it sends any short-exact sequence into a half-exact sequence.

(6.13) **Exercise: Half-exact functor of O.** Let $F\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ be a (co- or contravariant) half-exact functor, and consider the zero module O. Prove that $F(O)$ is the zero module. [Prove that $M = O$ if and only if $1_M = 0$. Consider the sequence $O \to O \xrightarrow{1_O} O \xrightarrow{1_O} O \to O$. Prove that it is short-exact, and apply F.]

(6.14) **Left-exact functors.** A (co- or contravariant) functor $F\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ is said to be *left-exact* exactly when it sends any short-exact sequence into a left-exact sequence.

(6.15) **Right-exact functors.** A (co- or contravariant) functor $F\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ is said to be *right-exact* exactly when it sends any short-exact sequence into a right-exact sequence.

(6.16) **Exact functors.** A (co- or contravariant) functor $F\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ is said to be *exact* exactly when it sends any short-exact sequence into a short-exact sequence, that is, into an exact (short) sequence. Note that exact functors are **not** called short-exact functors.

(6.17) **Exactness of functors by special short-exact sequences.** A (co- or contravariant) functor $F\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ is, respectively, half-exact, left-exact, right-exact, or exact, if and only if it sends any *special* short-exact sequence into a, respectively, half-exact, left-exact, right-exact, or short-exact sequence. This follows from (6.10), (6.9), and (6.8).

(6.18) **Lemma: Left/right-exact functor of left/right exact sequences.** *Let $F\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ be a (co- or contravariant) functor, and consider the short sequence:*

$(*)$
$$O \to K \xrightarrow{\kappa} L \xrightarrow{\lambda} M \to O$$

as well as its image:

$F(*)$
$$O \to F(K) \xrightarrow{F(\kappa)} F(L) \xrightarrow{F(\lambda)} F(M) \to O\,.$$

Here shown in the covariant case; in the contravariant case all arrows should be reversed.

The following then hold.

(6.18.1) *F covariant and left-exact and $(*)$ left-exact* $\Longrightarrow$ *$F(*)$ left-exact.*

(6.18.2) *F covariant and right-exact and $(*)$ right-exact* $\Longrightarrow$ *$F(*)$ right-exact.*

(6.18.3) *F contravariant and left-exact and $(*)$ right-exact* $\Longrightarrow$ *$F(*)$ left-exact.*

(6.18.4) *F contravariant and right-exact and $(*)$ left-exact* $\Longrightarrow$ *$F(*)$ right-exact.*

(6.19) **Proof.** We prove (6.18.1) and leave to the reader the verification of the other three assertions.

Assume that F is covariant and left-exact, and that $(*)$ is left-exact [and we want to prove that $F(*)$ is left-exact]. The left-exact sequence $(*)$ induces the first of the next two short-exact sequences:

$$(\dagger) \qquad O \to K \xrightarrow{\kappa} L \xrightarrow{\widetilde{\lambda}} \operatorname{Im}\lambda \to O$$

$$(\ddagger) \qquad O \to \operatorname{Im}\lambda \xrightarrow{\iota} M \xrightarrow{\rho} M/\operatorname{Im}\lambda \to O\,.$$

Here $\widetilde{\lambda}(\ell) \stackrel{\text{def}}{=} \lambda(\ell)$ for $\ell \in L$, while ι is the inclusion, and ρ is the residue map. Note that

$$(\circ) \qquad \iota\widetilde{\lambda} = \lambda\,.$$

Since F is covariant and left-exact, $(\dagger)$ and $(\ddagger)$ induce exact sequences:

$$F(\dagger) \qquad O \to F(K) \xrightarrow{F(\kappa)} F(L) \xrightarrow{F(\widetilde{\lambda})} F(\operatorname{Im}\lambda)$$

$$F(\ddagger) \qquad O \to F(\operatorname{Im}\lambda) \xrightarrow{F(\iota)} F(M) \xrightarrow{F(\rho)} F(M/\operatorname{Im}\lambda)\,.$$

The proof of the desired exactness of

$$F(*) \qquad O \to F(K) \xrightarrow{F(\kappa)} F(L) \xrightarrow{F(\lambda)} F(M)$$

is divided into three parts:

1^o $F(\kappa)$ is injective by the exactness of $F(\dagger)$.

2^o $\operatorname{Im}F(\kappa) \subseteq \operatorname{Ker}F(\lambda)$: $F(\lambda)F(\kappa) = F(\lambda\kappa) \stackrel{(\circ)}{=} F((\iota\widetilde{\lambda})\kappa) = F(\iota\widetilde{\lambda}\kappa) = F(\iota)F(\widetilde{\lambda})F(\kappa) \stackrel{F(\dagger)}{=} F(\iota)0 = 0\,.$

3^o $\operatorname{Ker}F(\lambda) \stackrel{(\circ)}{=} \operatorname{Ker}\big(F(\iota)F(\widetilde{\lambda})\big) \overset{F(\ddagger)}{\subseteq} \operatorname{Ker}F(\widetilde{\lambda}) \stackrel{F(\dagger)}{=} \operatorname{Im}F(\kappa)\,.$ □

(6.20) **Overview of exactness of special functors.**

(1) The *over-lining functor* $\overline{}$: $\operatorname{Mod}(R) \to \operatorname{Mod}(\overline{R})$, cf. (5.5), is right-exact.
(2) The *covariant Hom functor* $\operatorname{Hom}_R(H, -)$: $\operatorname{Mod}(R) \to \operatorname{Mod}(R)$, cf. (5.4), is left-exact.
(3) The *fraction functor* S^{-1}: $\operatorname{Mod}(R) \to \operatorname{Mod}(S^{-1}R)$, cf. (5.11) is exact.
(4) The *contravariant Hom functor* $\operatorname{Hom}_R(-, H)$: $\operatorname{Mod}(R) \to \operatorname{Mod}(R)$, cf. (5.14), is left-exact.

Below, these propositions will be stated more precisely and proofs will be provided.

(6.21) **Theorem: Over-lining is right-exact.** *Let $\mathfrak{a}$ be a proper ideal and consider the ring $\overline{R} \stackrel{\text{def}}{=} R/\mathfrak{a}$. The over-lining functor $\overline{}$: $\operatorname{Mod}(R) \to \operatorname{Mod}(\overline{R})$ is then right-exact.*

(6.22) **Proof.** Recall from (5.5) that if M is an R–module then $\overline{M} \stackrel{\text{def}}{=} M/\mathfrak{a}M$, and if $\varphi\colon M \to N$ is an R–homomorphism then $\overline{\varphi}\colon \overline{M} \to \overline{N}$ is defined by $\overline{\varphi}(\overline{m}) =^{\text{def}} \overline{\varphi(m)}$. Let

$O \to H \xrightarrow{\iota} L \xrightarrow{\rho} L/H \to O$ be a special short-exact sequence, and we want to prove that the induced sequence

$$\overline{H} \xrightarrow{\overline{\iota}} \overline{L} \xrightarrow{\overline{\rho}} \overline{L/H} \to O$$

is exact. This is divided into three parts.

1^o $\operatorname{Im}\overline{\iota} \subseteq \operatorname{Ker}\overline{\rho}$, that is, $\overline{\rho}\,\overline{\iota} = 0$: This is very easy: $\overline{\rho}\,\overline{\iota} = \overline{\rho\iota} = \overline{0} = 0$.

2^o $\operatorname{Im}\overline{\iota} \supseteq \operatorname{Ker}\overline{\rho}$: Assume that $\overline{\ell} = [\,\ell\,]_{\mathfrak{a}L}$ belongs to $\operatorname{Ker}\overline{\rho}$, that is,

$$0 = \overline{\rho}(\overline{\ell}) = \overline{\rho(\ell)} = [\,\rho(\ell)\,]_{\mathfrak{a}(L/H)} = [\,[\,\ell\,]_H\,]_{\mathfrak{a}(L/H)}\,.$$

From this we get $[\,\ell\,]_H \in \mathfrak{a}(L/H)$, that is, there are $a_1, \dots, a_n \in \mathfrak{a}$ and $\ell_1, \dots, \ell_n \in L$ such that

$$[\,\ell\,]_H = a_1[\,\ell_1\,]_H + \cdots + a_n[\,\ell_n\,]_H\,,$$

that is, $[\,\ell\,]_H = [\,a_1\ell_1 + \cdots + a_n\ell_n\,]_H$ (in L/H), and hence $h \stackrel{\text{def}}{=} \ell - (a_1\ell_1 + \cdots + a_n\ell_n) \in H$. Thus

$$[\,\ell\,]_{\mathfrak{a}L} = [\,\ell - (a_1\ell_1 + \cdots + a_n\ell_n)\,]_{\mathfrak{a}L} = [\,h\,]_{\mathfrak{a}L} = \overline{\iota}([\,h\,]_{\mathfrak{a}H}) \in \operatorname{Im}\overline{\iota}\,,$$

as desired.

3^o $\overline{\rho}$ is surjective: Left to the reader. □

(6.23) **Example: Over-lining is not exact.** Consider the ring $R \stackrel{\text{def}}{=} \mathbb{Z}$, the ideal $\mathfrak{a} \stackrel{\text{def}}{=} (2)$, the corresponding over-lining, the exact sequence $O \to R \xrightarrow{2_R} R \xrightarrow{\rho} R/(2) \to O$, and the induced sequence $O \to \overline{R} \xrightarrow{\overline{2_R}} \overline{R} \xrightarrow{\overline{\rho}} \overline{R/(2)} \to O$. Since the homomorphism $\overline{2_R}$ is zero and its source is non-zero, it is not injective, and hence the induced sequence is not exact.

(6.24) **Theorem: S^{-1} is exact.** *If S is a special multiplicative system in R, then the fraction functor $S^{-1}\colon \operatorname{Mod}(R) \to \operatorname{Mod}(S^{-1}R)$ is exact.*

(6.25) **Proof.** For any R-homomorphism $\varphi\colon M \to N$ it is straightforward to verify that

$$\operatorname{Im} S^{-1}\varphi = S^{-1}\operatorname{Im}\varphi \quad \text{and} \quad \operatorname{Ker} S^{-1}\varphi = S^{-1}\operatorname{Ker}\varphi\,, \tag{6.25.1}$$

and the desired assertion follows as $S^{-1}O = O$. □

(6.26) **Exercise: S^{-1} is exact.** Do the verification of (6.25.1).

(6.27) **$\operatorname{Hom}(H,-)$ is left-exact.** *If H is an R-module, then the covariant homomorphism functor $\operatorname{Hom}_R(H,-)\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R)$ is left-exact.*

(6.28) **Proof.** Let $O \to K \xrightarrow{\iota} M \xrightarrow{\rho} M/K \to O$ be a special short-exact sequence. We want to prove that the induced sequence:

$$O \to \operatorname{Hom}_R(H,K) \xrightarrow{\iota_*} \operatorname{Hom}_R(H,M) \xrightarrow{\rho_*} \operatorname{Hom}_R(H,M/K)$$

is exact. This is divided into three parts.

1^o ι_* is injective: Assume that $\alpha \in \operatorname{Ker}\iota_*$ ($\subseteq \operatorname{Hom}_R(H,K)$), that is, $\iota\alpha = \iota_*(\alpha) = 0$. Since ι is injective, we get $\alpha = 0$, as desired.

2^o $\operatorname{Im}\iota_* \subseteq \operatorname{Ker}\rho_*$: This is easy: $\rho_*\iota_* = (\rho\iota)_* = 0_* = 0$.

3^o $\operatorname{Im}\iota_* \supseteq \operatorname{Ker}\rho_*$: Assume that $\beta \in \operatorname{Ker}\rho_*$ ($\subseteq \operatorname{Hom}_R(H,M)$), that is, $\rho\beta = \rho_*(\beta) = 0$ so $\operatorname{Im}\beta \subseteq \operatorname{Ker}\rho = K$, that is, β maps into K, that is, $\beta \in \operatorname{Hom}_R(H,K)$, and hence $\beta = \iota\beta = \iota_*(\beta) \in \operatorname{Im}\iota_*$ as desired. □

(6.29) **Example: Hom(H,–) is not exact.** Consider the ring $R \stackrel{\text{def}}{=} \mathbb{Z}$, the R-module $H \stackrel{\text{def}}{=} R/(2)$, the functor $\operatorname{Hom}_R(H,-)$, the sequence $O \to R \xrightarrow{2_R} R \xrightarrow{\rho} R/(2) \to O$ which is short-exact, and the induced sequence

$$(*) \qquad O \to \operatorname{Hom}_R(H,R) \to \operatorname{Hom}_R(H,R) \to \operatorname{Hom}_R(H,H) \to O\,.$$

Here, $\operatorname{Hom}_R(H,R) = O$: for $\varphi \in \operatorname{Hom}_R(H,R)$ and $h \in H$ we have $2\varphi(h) = \varphi(2h) = \varphi(0) = 0$ (in $R = \mathbb{Z}$), that is, $\varphi(h) = 0$ for all $h \in H$, and hence $\varphi = 0$. On the other hand, $\operatorname{Hom}_R(H,H) \neq O$ because $1_H \neq 0$, and hence $(*)$ is $O \to O \to O \to \operatorname{Hom}_R(H,H) \to O$ which is not exact.

(6.30) **Hom(–,H) is left-exact.** *If H is an R-module, then the contravariant homomorphism functor* $\operatorname{Hom}_R(-,H)\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R)$ *is left-exact.*

(6.31) **Proof.** Let $O \to K \xrightarrow{\iota} M \xrightarrow{\rho} M/K \to O$ be a special short-exact sequence. We want to prove that the induced sequence:

$$O \to \operatorname{Hom}_R(M/K,H) \xrightarrow{\rho^*} \operatorname{Hom}_R(M,H) \xrightarrow{\iota^*} \operatorname{Hom}_R(K,H)$$

is exact. This is divided into three parts.

1^o ρ^* is injective: Assume that $\alpha \in \operatorname{Ker}\rho^*$ ($\subseteq \operatorname{Hom}_R(M/K,H)$), that is, $\alpha\rho = \rho^*(\alpha) = 0$. Since ρ is surjective we obtain $\alpha = 0$ as desired.

2^o $\operatorname{Im}\rho^* \subseteq \operatorname{Ker}\iota^*$: This is easy: $\iota^*\rho^* = (\rho\iota)^* = 0^* = 0$.

3^o $\operatorname{Im}\rho^* \supseteq \operatorname{Ker}\iota^*$: Assume that $\beta \in \operatorname{Ker}\iota^*$, that is, $\beta\colon M \to H$ is an R-homomorphism such that $\beta\iota = \iota^*(\beta) = 0$, and this implies that $\operatorname{Ker}\beta \supseteq \operatorname{Im}\iota = K$. Thus the Homomorphism Theorem (2.28) yields an R-homomorphism $\overline{\beta}\colon M/K \to H$ such that $\overline{\beta}([m]_K) = \beta(m)$ for all $m \in M$, and hence $\beta = \overline{\beta}\rho = \rho^*(\overline{\beta}) \in \operatorname{Im}\rho^*$, as desired. □

(6.32) **Example: Hom(–,H) is not exact.** Consider the ring $R \stackrel{\text{def}}{=} \mathbb{Z}$, the R-module $H \stackrel{\text{def}}{=} R$, the functor $\operatorname{Hom}_R(-,H)$, the sequence $O \to R \xrightarrow{2_R} R \xrightarrow{\rho} R/(2) \to O$ which is short-exact, and the induced sequence which is the top row in the next commutative diagram.

$$\begin{array}{ccccccccc} O & \longrightarrow & \operatorname{Hom}_R(R/(2),R) & \xrightarrow{\rho^*} & \operatorname{Hom}_R(R,R) & \xrightarrow{(2_R)^*} & \operatorname{Hom}_R(R,R) & \longrightarrow & O\,. \\ & & & & \alpha\downarrow\cong & & \cong\downarrow\alpha & & \\ & & & & R & \xrightarrow[2_R]{} & R & & \end{array} \tag{6.32.1}$$

Here α is the isomorphism from (4.4) given by $\alpha(\beta) \stackrel{\text{def}}{=} \beta(1)$ for $\beta \in \operatorname{Hom}_R(R,R)$. Since 2_R is not surjective, $(2_R)^*$ cannot be it either, and hence the induced sequence is not exact.

(6.33) **Five Lemma.** *Assume that the next diagram is commutative and has exact rows.*

$$\begin{array}{ccccccccc} K & \xrightarrow{\alpha} & L & \xrightarrow{\beta} & M & \xrightarrow{\gamma} & N & \xrightarrow{\delta} & P \\ \kappa\downarrow & & \lambda\downarrow & & \mu\downarrow & & \nu\downarrow & & \pi\downarrow \\ K' & \xrightarrow[\alpha']{} & L' & \xrightarrow[\beta']{} & M' & \xrightarrow[\gamma']{} & N' & \xrightarrow[\delta']{} & P' \end{array}$$

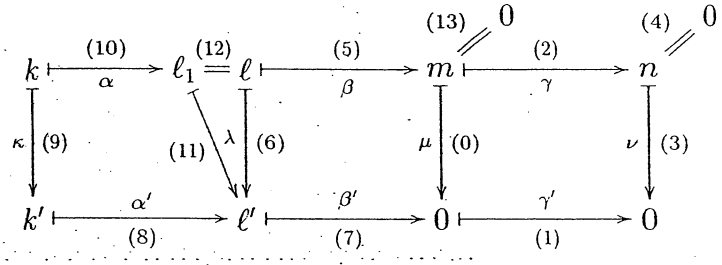
(a) *If λ and ν are injective and κ is surjective, then μ is injective.*

(b) *If λ and ν are surjective and π is injective, then μ is surjective.*

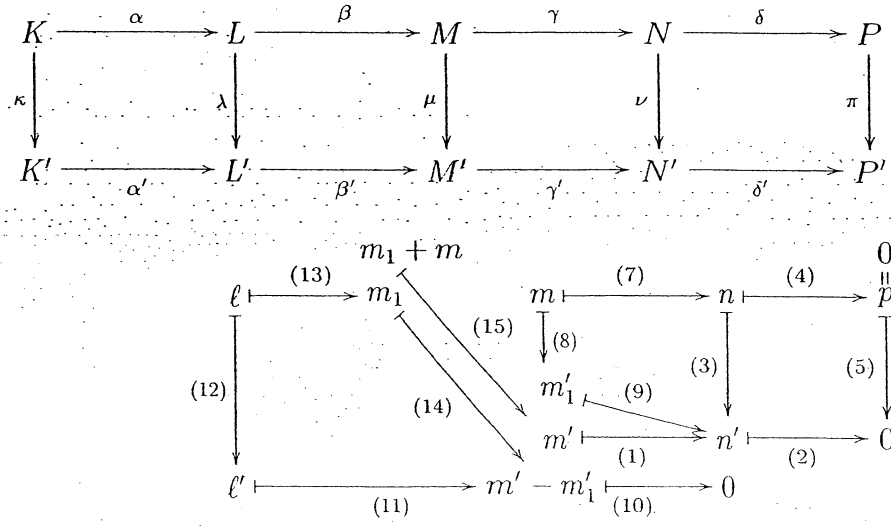
(c) *If λ and ν are bijective, κ is injective, and π is surjective, then μ is bijective.*
(d) *If κ, λ, ν, and π are isomorphisms, then so is μ.*

(6.34) **Proof.** (a)
(0) Assume that $m \in \operatorname{Ker}\mu$ [and we want to prove $m = 0$].
(1) $\gamma'\mu(m) = 0$.
(2) $n =^{\text{def}} \gamma(m)$.
(3) $\nu(n) = 0$ as $\nu\gamma = \gamma'\mu$.
(4) $n = 0$ as ν is injective.
(5) $m = \beta(\ell)$ for some $\ell \in L$ as $\operatorname{Ker}\gamma \subseteq \operatorname{Im}\beta$.
(6) $\ell' =^{\text{def}} \lambda(\ell)$.
(7) $\beta'(\ell') = 0$ as $\beta'\lambda = \mu\beta$.
(8) $\ell' = \alpha'(k')$ for some $k' \in K'$ as $\operatorname{Ker}\beta' \subseteq \operatorname{Im}\alpha'$.
(9) $k' = \kappa(k)$ for some $k \in K$ as κ is surjective.
(10) $\ell_1 =^{\text{def}} \alpha(k)$.
(11) $\lambda(\ell_1) = \ell'$ as $\lambda\alpha = \alpha'\kappa$.
(12) $\ell_1 = \ell$ as λ is injective.
(13) $m = \beta(\ell) = \beta\alpha(k) = 0(k) = 0$ [as desired].

This procedure is called a *diagram chase*, and can be illustrated as follows.



(b) is proved by the next diagram chase; the original diagram is repeated for convenience.



(6.35) **Additive functors.** A (co- or contravariant) functor $F\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ is said to be *additive* exactly when

$$F(\varphi + \psi) = F(\varphi) + F(\psi) \text{ and all } R\text{-homomorphisms } \varphi, \psi \in \operatorname{Hom}_R(M, N)\,,$$

that is, if and only if the next map

$$\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{R'}(F(M),F(N)), \varphi \mapsto F(\varphi) \text{ in the covariant case;}$$
$$\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{R'}(F(N),F(M)), \varphi \mapsto F(\varphi) \text{ in the covariant case}$$

is additive, that is, a $\mathbb{Z}$–homomorphism.

It follows that if F is additive, then (with notation as above):

(6.35.1) $$F(\varphi - \psi) = F(\varphi) - F(\psi)\,.$$

(6.36) **Examples: Additive functor.** The next functors are additive.

(1) *Over-lining functor* $\bar{\ }$: $\mathrm{Mod}(R) \to \mathrm{Mod}(\bar{R})$, cf. (5.5).
(2) *Covariant Hom functor* $\mathrm{Hom}_R(H,-)$: $\mathrm{Mod}(R) \to \mathrm{Mod}(R)$, cf. (5.4).
(3) *Fraction functor* S^{-1}: $\mathrm{Mod}(R) \to \mathrm{Mod}(S^{-1}R)$, cf. (5.11).
(4) *Contravariant Hom functor* $\mathrm{Hom}_R(-,H)$: $\mathrm{Mod}(R) \to \mathrm{Mod}(R)$, cf. (5.14).

(6.37) **Proposition: Additive functor of zero.** *Let F: $\mathrm{Mod}(R) \to \mathrm{Mod}(R')$ be a co- or contravariant additive functor. If 0: $M \to N$ is the zero homomorphism of R–modules and O is the zero module, then $F(0) = 0$ and $F(O) = O$.*

(6.38) **Proof.** We treat the covariant case only. In the group $\mathrm{Hom}_{R'}(F(M),F(N))$ we have by (6.35.1):

$$F(0) = F(0-0) = F(0) - F(0) = 0\,.$$

Furthermore, $1_{F(O)} = F(1_O) = F(0) = 0$; thus $F(O) = O$. □

(6.39) **Splitting sequences.** A short sequence

(†) $$O \longrightarrow M \xrightarrow{\mu} N \xrightarrow{\nu} P \longrightarrow O\,,$$

is said to *split* exactly when there are R–homomorphisms σ: $N \to M$ and τ: $P \to N$, that is,

(‡) $$O \longrightarrow M \underset{\sigma}{\overset{\mu}{\rightleftarrows}} N \underset{\tau}{\overset{\nu}{\rightleftarrows}} P \longrightarrow O\,,$$

such that

(*) $$\sigma\mu = 1_M\,,\ \mu\sigma + \tau\nu = 1_N\,,\ \text{ and } \ \nu\tau = 1_P\,.$$

We say also that (†) is *split-exact* because of the next result.

(6.40) **Lemma: Splitting sequences.** *If the sequence (†) splits then it is exact.*

(6.41) **Proof.** Divide into four parts.

1^o μ is injective since $\sigma\mu = 1_M$.

2^o $\mathrm{Im}\,\mu \subseteq \mathrm{Ker}\,\nu$: $\nu\mu = 0$ because:

$$\nu\mu = \nu 1_N \mu = \nu(\mu\sigma + \tau\nu)\mu = \nu\mu\sigma\mu + \nu\tau\nu\mu = \nu\mu 1_M + 1_P\nu\mu = 2\nu\mu\,.$$

3^o $\mathrm{Im}\,\mu \supseteq \mathrm{Ker}\,\nu$: For $n \in \mathrm{Ker}\,\nu$ we have:

$$n = 1_N(n) = (\mu\sigma + \tau\nu)(n) = \mu(\sigma(n)) + \tau(\nu(n)) = \mu(\sigma(n)) \in \mathrm{Im}\,\mu\,.$$

4^o ν is surjective since $\nu\tau = 1_P$. □

(6.42) **Proposition: Additive functor and split sequences.** *If F is an additive (co- or contravariant) functor, then F takes any split-exact sequence into a split-exact sequence.*

(6.43) **Proof.** We treat the covariant case only. Assume that F is an additive covariant functor, and consider the sequence (‡) satisfying (∗) of (6.39). Applying F yields a sequence

$$F(‡) \qquad O \longrightarrow F(M) \underset{F(\sigma)}{\overset{F(\mu)}{\rightleftarrows}} F(N) \underset{F(\tau)}{\overset{F(\nu)}{\rightleftarrows}} F(P) \longrightarrow O \ .$$

Since F is an additive functor, we obtain the next relations

$$F(*) \qquad F(\sigma)F(\mu) = 1_{F(M)} \ , \ F(\mu)F(\sigma) + F(\tau)F(\nu) = 1_{F(N)} \ , \text{ and } F(\nu)F(\tau) = 1_{F(P)} \ ,$$

and the desired assertion has been established. □

(6.44) **Direct summands.** A submodule H of an R–module M is said to be a *direct summand* in M exactly when there exists a submodule K of M such that $M = H + K$ and $H \cap K = O$, that is, $M = H \oplus K$ (inner direct sum), cf. (2.22.2).

(6.45) **Theorem: Split-exact sequences.** *Consider the short sequence:*

$$(†) \qquad O \longrightarrow M \xrightarrow{\mu} N \xrightarrow{\nu} P \longrightarrow O \ .$$

The following are the equivalent.

(*i*) *The sequence* (†) *is split-exact.*
(*ii*) *The sequence* (†) *is exact and* $\operatorname{Im}\mu$ *is a direct summand in* N.
(*iii*) *The sequence* (†) *is exact and* $\sigma\mu = 1_M$ *for some* R*–homomorphism* $\sigma\colon N \to M$.
(*iv*) *The sequence* (†) *is exact and* $\nu\tau = 1_P$ *for some* R*–homomorphism* $\tau\colon P \to N$.
(*v*) *The sequence* (†) *is isomorphic to the short sequence:*

$$(‡) \qquad \begin{array}{ccccccccc} O & \longrightarrow & M & \xrightarrow[\iota_M]{} & M \oplus P & \xrightarrow[\pi_P]{} & P & \longrightarrow & O \\ & & m & \longmapsto & (m,0) & & & & \\ & & & & (m,p) & \longmapsto & p \ . & & \end{array}$$

(∗) *Thus, if* (†) *is split-exact, then* $N \cong M \oplus P$.

(6.46) **Proof.** "$(i) \Rightarrow (iii) \wedge (iv)$" by (6.40).

"$(iii) \Rightarrow (v)$": Define $\varphi\colon N \to M \oplus P$ by $\varphi(n) \overset{\text{def}}{=} (\sigma(n), \nu(n))$ for $n \in N$. The next diagram is then commutative, and φ is an isomorphism by Five Lemma (6.33).

$$\begin{array}{ccccc} M & \underset{\sigma}{\overset{\mu}{\rightleftarrows}} & N & \xrightarrow{\nu} & P \\ {\scriptstyle 1_M}\downarrow\cong & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle 1_P}\cong \\ M & \xrightarrow[\iota]{} & M \oplus P & \xrightarrow[\pi]{} & P \end{array}$$

"$(v) \Rightarrow (i)$": By assumptions we have the next commutative diagram with vertical isomorphisms and split-exact lower row.

$$\begin{array}{ccccc} M & \xrightarrow{\mu} & N & \xrightarrow{\nu} & P \\ {\scriptstyle \alpha}\downarrow\cong & & \downarrow{\scriptstyle \beta}\cong & & \downarrow{\scriptstyle \gamma}\cong \\ M & \underset{\pi_M}{\overset{\iota_M}{\rightleftarrows}} & M \oplus P & \underset{\iota_P}{\overset{\pi_P}{\rightleftarrows}} & P \end{array}$$

The R–homomorphisms $\sigma \overset{\text{def}}{=} \alpha^{-1}\pi_M\beta$ and $\tau \overset{\text{def}}{=} \beta^{-1}\iota_P\gamma$ have then the desired properties.

For the remaining implications confer Exercise (6.47) □

(6.47) **Exercise: Split-exact sequences.** Prove the remaining implications in (6.46)

(6.48) **Proposition: Additive functor of direct sum.** *If F is an additive functor, and M and P are R–modules, then $F(M \oplus P) \cong F(M) \oplus F(P)$.*

(6.49) **Proof.** Due to (6.42), the additive functor F sends the split-exact sequence (‡) from (6.45) into the split-exact sequence $F(‡)$ from (6.43); (6.45.∗) yields the desired. □

(6.50) **Exercise: Functor and direct sum.** Let H be a non-zero R–module, and consider the constant functor, cf. (5.6):

$$C_H\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R)\,,\ M \mapsto H\,,\ \varphi \mapsto 1_H\,,$$

(a) Prove that C_H is not additive.
(b) Prove that $R^{(\mathbb{N})} \oplus R^{(\mathbb{N})} \cong R^{(\mathbb{N})}$.
(c) Prove that $C_H(M \oplus N) \cong C_H(M) \oplus C_H(N)$ provided $H \oplus H \cong H$.

(6.51) **Exercise: Gamma functor.** Let a be an element in the ring R. For any R–module M we set:

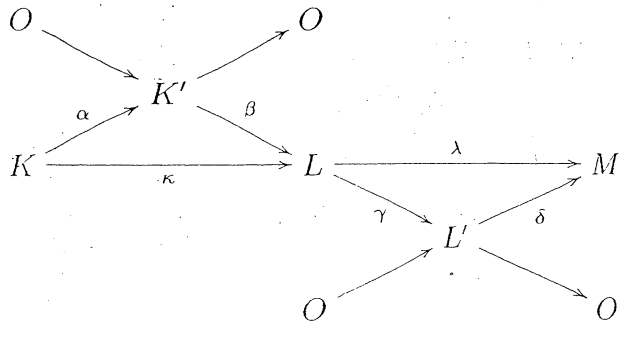
$$\Gamma_a M \stackrel{\text{def}}{=} \{\, m \in M \mid a^n m = 0 \ \text{ for some } \ n \in \mathbb{N}_0 \,\}\,.$$

(a) Prove that $\Gamma_a(M)$ is a submodule of M.
(b) Prove that $\Gamma_a(M/\Gamma_a M) = 0$.
(c) Prove that $\varphi(\Gamma_a(M)) \subseteq \Gamma_a N$ whenever $\varphi\colon M \to N$ is an R–homomorphism. Let $\Gamma_a(\varphi)\colon \Gamma_a(M) \to \Gamma_a(N)$ be the restriction.
(d) Prove that Γ_a is a functor.
(e) Prove that Γ_a is additive.
(f) Prove that Γ_a is left-exact.
(g) Prove that Γ_a is exact, if a is invertible or nilpotent, the latter is, $a^n = 0$ for some $n \in \mathbb{N}_0$.
(h) Prove that Γ_a is exact, if $R =^{\text{def}} \mathbb{Z}/(6)$ and $a =^{\text{def}} [3]_{(6)}$.
(i) Suppose that the ring R has exactly one maximal ideal. Prove that a is invertible or nilpotent, if Γ_a is exact.

(6.52) **Exercise: Half-exact is additive.** Let $F\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ be a (co- or contravariant) half-exact functor. Prove that F is additive.

(6.53) **Lemma: Exact tilted sequences.** *Let*

(6.53.1)

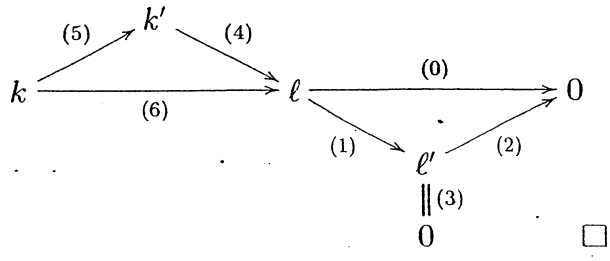


be a commutative diagram. If the three tilted sequences are exact, then so is the horizontal one.

(6.54) **Proof.** 1^o $\operatorname{Im}\kappa \subseteq \operatorname{Ker}\lambda$: $\lambda\kappa = \delta\gamma\beta\alpha = \delta 0\alpha = 0$.
2^o $\operatorname{Ker}\lambda \subseteq \operatorname{Im}\kappa$:
(0) Assume that $\ell \in \operatorname{Ker}\lambda$ [and we want to prove $\ell \in \operatorname{Im}\kappa$].

(1) $\ell' \stackrel{\text{def}}{=} \gamma(\ell) \in L'$.
(2) $\delta(\ell') = \delta(\gamma(\ell)) = 0$.
(3) $\ell' = 0$ (since δ is injective by assumption).
(4) Choose $k' \in K'$ such that $\beta(k') = \ell$ (using that $\operatorname{Ker}\gamma \subseteq \operatorname{Im}\beta$).
(5) Choose $k \in K$ such that $\alpha(k) = k'$ (since α is surjective).
(6) $\kappa(k) = \beta(\alpha(k)) = \beta(k') = \ell$.

This diagram chasing can be illustrated as follows.



□

(6.55) **Proposition: Exact functor of exact sequence.** *Let*

$$(\dagger) \qquad \cdots \to M_{\ell+1} \xrightarrow{\varphi_{\ell+1}} M_\ell \xrightarrow{\varphi_\ell} M_{\ell-1} \to \cdots$$

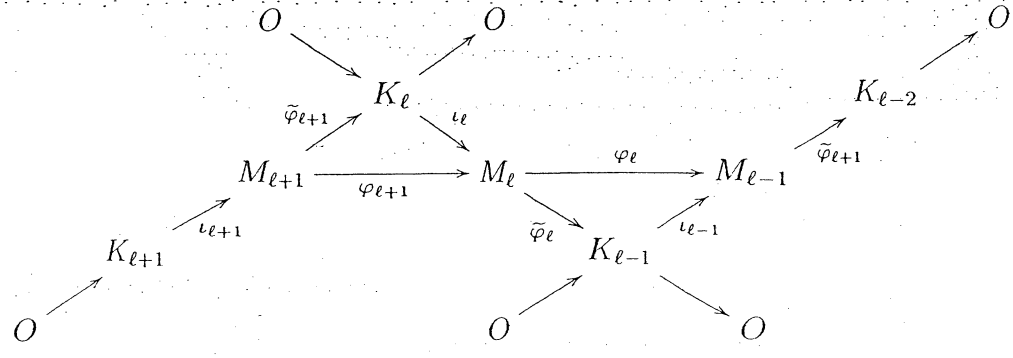
be a (possibly bounded or partly bounded) exact *sequence, and let F be a (co- or contravariant)* exact *functor. The induced sequence* [5]

$$\mathrm{F}(\dagger) \qquad \cdots \to F(M_{\ell+1}) \xrightarrow{F(\varphi_{\ell+1})} F(M_\ell) \xrightarrow{F(\varphi_\ell)} F(M_{\ell-1}) \to \cdots$$

is then exact.

(6.56) **Proof.** Set $K_n \stackrel{\text{def}}{=} \operatorname{Im}\varphi_{n+1} = \operatorname{Ker}\varphi_n$ for all (possible) n and consider the next commutative diagram in which the ι_n's are inclusions and the $\widetilde{\varphi}_n$'s are induced by the corresponding φ_n. In the next commutative diagram the tilted short sequences are exact.

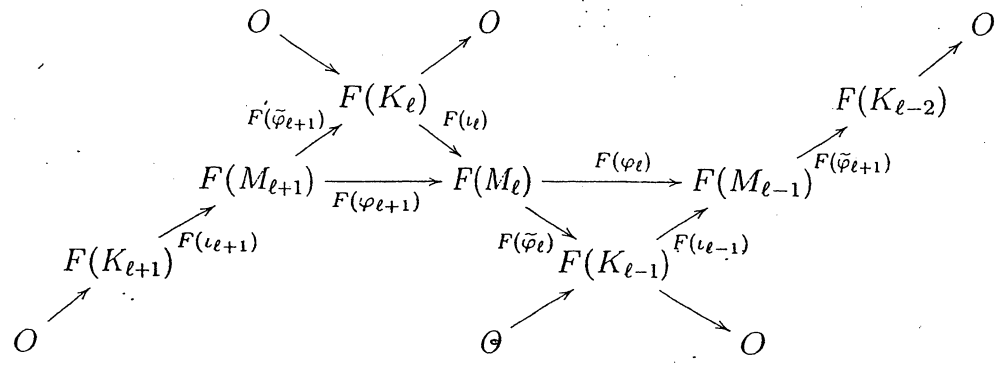
$(\triangle)$



[5] Here shown in the covariant case; in the contravariant case all arrows should be reversed.

Applying the exact functor F to the above diagram we obtain the next diagram with short-exact tilted sequences.

$F(\triangle)$



The exactness of the horizontal sequence follows from (6.53). □

7. Projective Modules

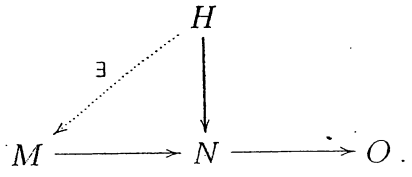
Recall that the covariant Hom functor $\operatorname{Hom}_R(H,-)$ is left-exact for all R–modules H.

(7.1) **Projective modules.** An R–module H is said to be *projective* exactly when the functor $\operatorname{Hom}_R(H,-)$ is exact, that is, if and only if the induced homomorphism

$$\operatorname{Hom}_R(H,\varphi)\colon \operatorname{Hom}_R(H,M) \to \operatorname{Hom}_R(H,N)$$

is *surjective* for any *surjective* R–homomorphism $\varphi\colon M \to N$, cf. (6.27).

(7.2) **Projectivity diagram.** Projectivity of the R–module H is often explained by the diagram:



This means: If the solid arrows are given and the row is exact, then the dotted arrow exists such the triangle commutes.

(7.3) **Not projective.** The $\mathbb{Z}$–module $\mathbb{Z}/(2)$ is not projective, cf. (6.29).

(7.4) $\boldsymbol{R}$ **is projective.** The R–module R is always projective: If $\varphi\colon M \to N$ is surjective, then it follows that $\operatorname{Hom}_R(R,\varphi)\colon \operatorname{Hom}_R(R,M) \to \operatorname{Hom}_R(R,N)$ is surjective from the next commutative diagram:

$$\begin{array}{ccc} \operatorname{Hom}_R(R,M) & \xrightarrow{\operatorname{Hom}_R(R,\varphi)} & \operatorname{Hom}_R(R,N) \\ \cong \downarrow \alpha_M & & \cong \downarrow \alpha_N \\ M & \xrightarrow[\varphi]{} & N\,, \end{array}$$

when $\alpha_M(\mu) \stackrel{\text{def}}{=} \mu(1)$ and $\alpha_N(\nu) \stackrel{\text{def}}{=} \nu(1)$, cf. (4.4).

(7.5) **Exercise:** $\boldsymbol{R}$ **is projective.** Prove that the R–module R is always projective by using the projectivity diagram (7.2).

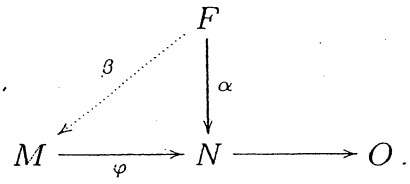
(7.6) **Isomorphic projectives.** If $\eta\colon H \xrightarrow{\cong} H'$ is an isomorphism of R–modules, then H is a projective module if and only if H' is so, because the next diagram is commutative

$$\begin{array}{ccc} \operatorname{Hom}_R(H,M) & \xrightarrow{\operatorname{Hom}_R(H,\varphi)} & \operatorname{Hom}_R(H,N) \\ \cong \uparrow \operatorname{Hom}_R(\eta,M) & & \cong \uparrow \operatorname{Hom}_R(\eta,N) \\ \operatorname{Hom}_R(H',M) & \xrightarrow[\operatorname{Hom}_R(H',\varphi)]{} & \operatorname{Hom}_R(H',N) \end{array}$$

for any R–homomorphism $\varphi\colon M \to N$.

(7.7) **Theorem: Free is projective.** *If F is a free R–module, then F is projective.*

(7.8) **Proof.** Assume that F is free with basis $(f_x)_{x\in X}$, and let the solid part of the next projectivity diagram with exact row, cf. (7.2), be given.



Use the surjectivity of φ to choose, for each $x \in X$, an element $m_x \in M$ such that $\varphi(m_x) = \alpha(f_x)$. Since every $f \in F$ is a (finite) unique sum $f = \sum_{x\in X} r_x f_x$ with all $r_x \in R$, we can define β by $\beta(f) = \sum_{x\in X} r_x m_x$. We then get the desired assertion:

$$\varphi(\beta(f)) = \varphi(\sum_{x\in X} r_x m_x) = \sum_{x\in X} r_x \varphi(m_x) = \sum_{x\in X} r_x \alpha(f_x) = \alpha(f)\,. \qquad \square$$

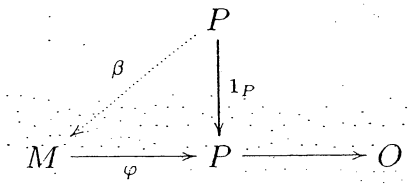
(7.9) **Projective but not free.** Consider the commutative group $R' \stackrel{\text{def}}{=} R^2$ ($\stackrel{\text{def}}{=} R \times R$); the addition is $(r_1, r_2)+(\widetilde{r}_1, \widetilde{r}_2) =^{\text{def}} (r_1+\widetilde{r}_1, r_2+\widetilde{r}_2)$. The group R' becomes a commutative ring with the multiplication $(r_1, r_2)(\widetilde{r}_1, \widetilde{r}_2) =^{\text{def}} (r_1\widetilde{r}_1, r_2\widetilde{r}_2)$; the identity element is $(1, 1)$. The subsets $\mathfrak{a}_1 =^{\text{def}} R \times O$ ($=^{\text{def}} \{\,(r, 0) \mid r \in R\,\}$) and $\mathfrak{a}_2 =^{\text{def}} O \times R$ are ideals in R', hence R'–modules. Since the R'–module R' is free and $R' = \mathfrak{a}_1 \oplus \mathfrak{a}_2$, the module $\mathfrak{a}_1$ is projective. For any element $(r, 0) \in \mathfrak{a}_1$ we have $(0, 1) \in \operatorname{Ann}_{R'}(r, 0)$, so $(r, 0)$ cannot be part of a basis for $\mathfrak{a}_1$, cf. (3.22). Thus, $\mathfrak{a}_1$ is a projective non-free R'–module.

(7.10) **Lemma: Surjections onto projective.** *If $\varphi\colon M \to P$ is a surjective R–homomorphism with the module P is projective, then the exact sequence*

$$(\dagger) \qquad O \to \operatorname{Ker}\varphi \xrightarrow{\iota} M \xrightarrow{\varphi} P \to O$$

splits, and P is isomorphic to a direct summand in M.

(7.11) **Proof.** Let the solid part of the next projectivity diagram with exact row, cf. (7.2), be given.



Choose β such that the triangle commutes, that is, $\varphi\beta = 1_P$. Thus, (iv) in Theorem: Split-exact sequences (6.45) is fulfilled, and hence the sequence (†) splits. The last assertion results from (v) in that theorem. $\square$

(7.12) **Lemma: Direct sum of projectives 1.** *If P and Q are R–modules, then the following holds.*

$$P \oplus Q \text{ is projective} \iff P \text{ and } Q \text{ are projective.}$$

(7.13) **Proof.** Set $S \stackrel{\text{def}}{=} P \oplus Q$ and consider the next split-exact sequence:

$$O \longrightarrow P \xrightarrow{\iota} S \xrightarrow{\pi} Q \longrightarrow O\,.$$

Let $\varphi\colon M \to N$ be any R-homomorphism, and consider the induced commutative diagram:

$$\begin{array}{ccccccccc} O & \longrightarrow & \operatorname{Hom}_R(Q,M) & \xrightarrow{\operatorname{Hom}_R(\pi,M)} & \operatorname{Hom}_R(S,M) & \xrightarrow{\operatorname{Hom}_R(\iota,M)} & \operatorname{Hom}_R(P,M) & \longrightarrow & O \\ & & \downarrow{\scriptstyle \operatorname{Hom}_R(Q,\varphi)} & & \downarrow{\scriptstyle \operatorname{Hom}_R(S,\varphi)} & & \downarrow{\scriptstyle \operatorname{Hom}_R(P,\varphi)} & & \\ O & \longrightarrow & \operatorname{Hom}_R(Q,N) & \xrightarrow[\operatorname{Hom}_R(\pi,N)]{} & \operatorname{Hom}_R(S,N) & \xrightarrow[\operatorname{Hom}_R(\iota,N)]{} & \operatorname{Hom}_R(P,N) & \longrightarrow & O \end{array}$$

The rows are exact as they result from application of an additive functor to a split-exact sequence, cf. (6.48).

Proof of "$\Rightarrow$". Assume that S is projective. By symmetry it suffices to prove that P is projective. Let $\varphi\colon M \to N$ be surjective [and we want to prove that $\operatorname{Hom}_R(P,\varphi)$ is surjective]. $\operatorname{Hom}_R(S,\varphi)$ is surjective by assumption, and $\operatorname{Hom}_R(\iota,N)$ is surjective by the exactness of the bottom row in the diagram. The commutativity of the diagram yields the equality

$$\operatorname{Hom}_R(\iota,N)\operatorname{Hom}_R(S,\varphi) = \operatorname{Hom}_R(P,\varphi)\operatorname{Hom}_R(\iota,M)\,.$$

We already know that the left hand side is surjective. Thus, so is the left factor of the right hand side, and the desired assertion has been established.

Proof of "$\Leftarrow$". Assume that P and Q are projective, assume that $\varphi\colon M \to N$ is surjective [and we want to prove that $\operatorname{Hom}_R(S,\varphi)$ is surjective]. This follows from the commutative diagram by using part (b) of Five Lemma (6.33) (with π the zero map: $O \to O$). □

(7.14) **Exercise: Direct sum of projectives 1.** Prove that the above lemma also follows from the projectivity diagram, cf. (7.2).

(7.15) **Theorem: Projective is direct summand in free.** *For any R-module P the following holds:*

$$P \text{ projective} \iff P \text{ direct summand in free.}$$

(7.16) **Proof.** "$\Rightarrow$": P is a homomorphic image of a free module by (3.29), so the assertion follows from (7.10).

"$\Leftarrow$": Use (7.7) and (7.12). □

(7.17) **Not projective.** Consider an integral domain R with field of fractions B, and assume that R is not a field. Part (e) in (3.32) asserts that the R-module B cannot be a submodule of a free R-module. Thus B is not projective by (7.15).

(7.18) **Zero-divisors.** The *set of zero-divisors* for an R-module M is defined as:

$$\operatorname{Zero}_R M \stackrel{\text{def}}{=} \{\, r \in R \mid \exists\, m \in M\colon m \neq 0 \wedge am = 0 \,\}\ (= \bigcup_{m \in M \setminus O} \operatorname{Ann}_R(m)).$$

(7.19) **Exercise: Zero-divisors.** Prove that $\operatorname{Zero}_R P \subseteq \operatorname{Zero}_R R$, if P is a projective R-module.

(7.20) **Proposition: Over-lining and projectivity.** *For a proper ideal $\mathfrak{a}$ in R and the corresponding over-lining functor $\overline{\ }\colon \operatorname{Mod}(R) \to \operatorname{Mod}(\overline{R})$ the following holds.*

$$P \text{ projective over } R \implies \overline{P} \text{ projective over } \overline{R}\,.$$

(7.21) **Proof.** Assume that P is projective. Thus P is a direct summand of a free R–module F, cf. (7.15), and F is isomorphic to $R^{(W)}$ for some set W, cf. (3.28). It follows that $\overline{P}$ is a direct summand of $\overline{R^{(W)}} \cong \overline{R}^{(W)}$, cf. (3.33). Now apply (7.15) once more. □

(7.22) **Proposition: Fractions and projectivity.** *For a special multiplicative system in R and the corresponding fraction functor $S^{-1}\colon \operatorname{Mod}(R) \to \operatorname{Mod}(S^{-1}R)$ the following holds.*

$$P \text{ projective over } R \implies S^{-1}P \text{ projective over } S^{-1}R.$$

(7.23) **Proof.** As in the previous proof (7.21), it suffices for any set W to establish an isomorphism $\Phi\colon S^{-1}(R^{(W)}) \xrightarrow{\cong} (S^{-1}R)^{(W)}$. For any $\varphi/s \in S^{-1}(R^{(W)})$ the element $\Phi(\varphi/s) \in (S^{-1}R)^{(W)}$, that is, the function $\Phi(\varphi/s)\colon W \to S^{-1}R$, is defined by $\Phi(\varphi/s)(w) =^{\text{def}} \varphi(w)/s$. It remains to be verified that Φ is a well-defined bijective $S^{-1}R$–homomorphism; this is straightforward—albeit tedious. □

(7.24) **Projective resolutions.** A *projective resolution* $P.$ of an R–module M is a sequence of R–homomorphisms:

$$P. = \cdots \to P_\ell \xrightarrow{\partial_\ell} P_{\ell-1} \xrightarrow{\partial_{\ell-1}} \cdots P_1 \xrightarrow{\partial_1} P_0 \to O$$

together with an R–homomorphism $P_0 \xrightarrow{\partial_0} M$ such that

- P_ℓ is a projective R–module for all $\ell \in \mathbb{N}_0$;
- $P. \to M \stackrel{\text{def}}{=} \cdots \to P_\ell \xrightarrow{\partial_\ell} P_{\ell-1} \xrightarrow{\partial_{\ell-1}} \cdots P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \to O$ is exact.

The sequence $P. \to M$ is then called the *augmented* projective resolution of M. Note that the index of the R–homomorphism ∂_ℓ in the projective resolution equals that of its *source*.

(7.25) **Free resolutions.** A *free resolution* of an R–module M is a projective resolution $F.$ of M in which each module F_ℓ is *free*. (Recall that every free module is projective, cf. (7.7).) The corresponding augmented free resolution is the exact sequence $F. \to M$.

If $x \in R \setminus \operatorname{Zero}_R R$, cf. (7.18), then $O \to R \xrightarrow{x} R \to O$ is a free resolution of the R–module $R/(x)$. The sequence $O \to R \xrightarrow{x} R \xrightarrow{\rho} R/(x) \to O$ is exact when ρ is the residue homomorphism; it is the corresponding augmented free resolution.

(7.26) **Lemma: Ubiquity of projective and free resolutions.** *All R–modules have a free, and hence projective, resolution.*

(7.27) **Proof.** To any R–module M we are required to construct an exact sequence

$$(*) \qquad \cdots \to F_{\ell+1} \xrightarrow{\partial_{\ell+1}} F_\ell \xrightarrow{\partial_\ell} F_{\ell-1} \xrightarrow{\partial_{\ell-1}} \cdots F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to O$$

with each F_ℓ free.

Start by choosing a surjective R–homomorphism $F_0 \xrightarrow{\partial_0} M$ with F_0 free, cf. (3.29). Now, $(*)$ is exact at M.

Assume next that that R–homomorphisms $\partial_0, \dots, \partial_\ell$ have been chosen such that $(*)$ is exact at $F_{\ell-1}, \dots, F_0, M$ [and we want $\partial_{\ell+1}$ such that it is exact at F_ℓ too]. Choose surjective R–homomorphism $\partial'_{\ell+1}\colon F_{\ell+1} \to \operatorname{Ker}\partial_\ell$ with free source, and let $\partial_{\ell+1}\colon F_{\ell+1} \to F_\ell$ be the induced homomorphism; it has $\operatorname{Im}\partial_{\ell+1} = \operatorname{Ker}\partial_\ell$ [as desired]. □

(7.28) **Projective dimension.** Let M be an R–module. It is said to have *finite projective dimension* exactly when it has a *bounded* projective resolution, that is, there is a bounded exact sequence

$$(\dagger_p) \qquad O \to P_p \xrightarrow{\partial_p} P_{p-1} \xrightarrow{\partial_{p-1}} \cdots P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \to O$$

with $P_0, \ldots, P_p$ projective modules.

When $M \neq O$ has finite projective dimension, then the *projective dimension* of M is the smallest $p \in \mathbb{N}_0$ such that there exists an exact sequence $(\dagger_p)$.

If $M \neq O$ has not finite projective dimension, then we set $\operatorname{pd}_R M = \infty$.

Finally, we set $\operatorname{pd}_R O = -\infty$.

Thus, for any R–module $M \neq O$ and any $n \in \mathbb{N}_0$ we have:

$$(7.28.1) \qquad \operatorname{pd}_R M \leq n \iff \begin{cases} \text{There exists an exact sequence:} \\ O \to P_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \to O \\ \text{with } P_0, \ldots, P_n \text{ projective.} \end{cases}$$

Note furthermore the next equivalence for an R–module M.

$$(7.28.2) \qquad \operatorname{pd}_R M = 0 \iff M \text{ is projective and non-zero.}$$

(7.29) **Projective dimension over a field.** If R be a field and M is any R–module, that is, vector-space over R, then $\operatorname{pd}_R M \leq 0$. Namely, M is a free R–module because it has a basis: if M is finitely generated then this is well-known from Linear Algebra. In general, it is a consequence of Zorn's Lemma: Let $\mathbb{W}$ be the set of linearly independent subsets W of the module M. This set $\mathbb{W}$ is then non-empty (as $\varnothing$ is in it), and it is also *inductively ordered* by inclusion, that is, if $\mathbb{Y}$ is a totally ordered[6] subset of $\mathbb{W}$, then $\mathbb{Y}$ has a majorant[7] in $\mathbb{W}$, namely the union $\bigcup_{Y \in \mathbb{Y}} Y$. By Zorn's Lemma $\mathbb{W}$ has a maximal element W_0[8]. Set $F =^{\text{def}} \operatorname{span}_R W_0$ which is free by (3.16.1), (3.19.1), (3.27). Thus it suffices to prove that $F = M$. Assume $F \subset M$ and look for a contradiction. Choose $x \in M \setminus F$ and divide into two cases.

1° $Rx \cap F \neq O$. Choose $r \in R$ such that $rx \in F$ and $rx \neq 0$. The latter yields $r \neq 0$, and hence, as R is a field, we have $x = r^{-1}(rx) \in F$ contradicting the choice of x.

2° $Rx \cap F = O$. In this case we will prove that $\{x\} \cup W_0$ in linearly independent (contradicting the maximality of W_0). Indeed, let $rx + r_1 w_1 + \cdots + r_n w_n = 0$ be a linear relation with $w_1, \ldots, w_n \in W_0$ and $r, r_1, \ldots, r_n \in R$, and note that then $rx \in Rx \cap \operatorname{span}_R W_0 = Rx \cap F = O$. Thus, $r = 0$ since otherwise $x = r^{-1}(rx) = 0 \in F$, and the remaining part of the linear relation is $r_1 w_1 + \cdots + r_n w_n = 0$ which yields $r_1 = \cdots = r_n = 0$.

Thus M is projective by (7.7), and hence $\operatorname{pd}_R M = 0$ if $M \neq O$, cf. (7.28.2), and $\operatorname{pd}_R M = -\infty$ otherwise.

(7.30) **Projective dimension over a PID.** If R be a PID, that is, a Principal Integral Domain, cf. (1.8), and M is any R–module, then $\operatorname{pd}_R M \leq 1$. If M is finitely generated, then it is a homomorphic image of a finitely generated free module P_0, cf. (3.29). Let P_1 be the kernel, that is, there is an exact sequence

$$O \to P_1 \to P_0 \to M \to O.$$

---

[6] that is, for $Y, Y' \in \mathbb{Y}$ either $Y \subseteq Y'$ or $Y' \subseteq Y$.

[7] that is, an element $W \in \mathbb{W}$ such that $Y \subseteq W$ for all $Y \in \mathbb{Y}$.

[8] that is, if $W \in \mathbb{W}$ and $W_0 \subseteq W$, then $W_0 = W$.

Here, P_1 is free by (3.8); thus $\operatorname{pd}_R M \leq 1$ by (7.28.1). In general, it follows from this special case by applying the derived functors that will be introduced later.

(7.31) **Exercise: Projective dimension of $R/(x)$.** Assume that $x \in R \setminus \operatorname{Zero}_R R$, cf. (7.18) and (7.25). Prove that the following hold.

$$\operatorname{pd}_R R/(x) = \begin{cases} -\infty & \text{if } (x) = R \\ 0 & \text{if } (x) \subset R \text{ and } R \cong (x) \oplus R/(x) \\ 1 & \text{otherwise.} \end{cases}$$

(7.32) **Proposition: Projective dimension and the fraction functor.** *If S is a special multiplicative system in R, and M is an R–module, then the next inequality holds.*

$$\operatorname{pd}_{S^{-1}R} S^{-1}M \leq \operatorname{pd}_R M\,.$$

(7.33) **Proof.** This is certainly true if $\operatorname{pd}_R M = \infty$ or $M = O$. Therefore, assume that $\operatorname{pd}_R M = p \in \mathbb{N}_0$ and choose an exact sequence

$$(\dagger_p) \qquad O \to P_p \to P_{p-1} \to \cdots \to P_1 \to P_0 \to M \to O$$

with $P_0, \ldots, P_p$ projective R–modules. The induced sequence

$$S^{-1}(\dagger_p) \qquad O \to S^{-1}P_p \to S^{-1}P_{p-1} \to \cdots \to S^{-1}P_1 \to S^{-1}P_0 \to S^{-1}M \to O\,,$$

since S^{-1} is an exact functor, cf. (6.24), cf. (6.55) (or just (6.25.1)). Since each $S^{-1}P_\ell$ is a projective $S^{-1}R$–module, cf. (7.22), we have proved that $\operatorname{pd}_{S^{-1}R} S^{-1}M \leq p$. □

(7.34) **Exercise: Projective dimension and special over-lining.** Let $\mathfrak{a} = (x)$ be a proper principal ideal in R and consider the corresponding over-lining.

(a) Let $O \to K \xrightarrow{\varphi} L \xrightarrow{\psi} M \to O$ be an exact sequence, and assume that $x \notin \operatorname{Zero}_R M$, cf. (7.18). Prove the exactness of the induced sequence

$$O \to \overline{K} \xrightarrow{\overline{\varphi}} \overline{L} \xrightarrow{\overline{\psi}} \overline{M} \to O\,.$$

(b) Prove that $\operatorname{pd}_{\overline{R}} \overline{M} \leq \operatorname{pd}_R M$ if $x \notin \operatorname{Zero}_R M \cup \operatorname{Zero}_R R$.

8. Complexes and Homology

(8.1) **Complexes.** An R–complex—or a complex of R–modules—is a sequence, cf. (6.1) of R–homomorphisms

$$K. = \cdots \to K_{n+1} \xrightarrow{\partial^{K.}_{n+1}} K_n \xrightarrow{\partial^{K.}_{n}} K_{n-1} \to \cdots \tag{8.1.1}$$

such that $\partial^{K.}_n \partial^{K.}_{n+1} = 0$ for all $n \in \mathbb{Z}$; the R–homomorphism $\partial^{K.}_n$ is said to be the n'th *differential*. The R–module K_n is said to be in *degree* n in the R–complex $K.$. An R–complex can be bounded to the left, to the right, or in both directions, cf. (6.1.1) – (6.1.3), and when this is the case, we leave out superfluous zero modules: we write, for example, an R–complex $K.$ with $K_n = O$ for $n < 0$ as $\cdots \to K_1 \xrightarrow{\partial_1} K_0 \to O$ (that is, not as $\cdots \to K_1 \xrightarrow{\partial_1} K_0 \to O \to \cdots$).

(8.2) **Boundary and cycles.** For any R–complex $K.$ and any $n \in \mathbb{Z}$ we define

$$\mathrm{B}_n(K.) \stackrel{\text{def}}{=} \operatorname{Im} \partial^{K.}_{n+1}, \text{ the } n\text{'th module of boundaries;}$$
$$\mathrm{Z}_n(K.) \stackrel{\text{def}}{=} \operatorname{Ker} \partial^{K.}_{n}, \text{ the } n\text{'th module of cycles; and}$$
$$\mathrm{C}_n(K.) \stackrel{\text{def}}{=} \operatorname{Coker} \partial^{K.}_{n} \ (\stackrel{\text{def}}{=} K_n/\mathrm{B}_n(K.)), \text{ the } n\text{'th cokernel.}$$

(8.3) **Homology.** Let $K.$ be an R–complex. As $\partial^{K.}_n \partial^{K.}_{n+1} = 0$ we have $\mathrm{B}_n(K.) \subseteq \mathrm{Z}_n(K.)$ and define

$$\mathrm{H}_n(K.) \stackrel{\text{def}}{=} \mathrm{Z}_n(K.)/\mathrm{B}_n(K.), \text{ the } n\text{'th homology module.}$$

The next four sequences are *short-exact* and provide relations between the homology modules, boundaries, cycles, and cokernels.

$$O \to \mathrm{B}_n(K.) \hookrightarrow \mathrm{Z}_n(K.) \twoheadrightarrow \mathrm{H}_n(K.) \to O \tag{8.3.1}$$

$$O \to \mathrm{B}_n(K.) \hookrightarrow K_n \twoheadrightarrow \mathrm{C}_n(K.) \to O \tag{8.3.2}$$

$$O \to \mathrm{Z}_n(K.) \hookrightarrow K_n \xrightarrow{\widetilde{\partial^{K.}_n}} \mathrm{B}_{n-1}(K.) \to O \tag{8.3.3}$$

$$O \to \mathrm{H}_n(K.) \hookrightarrow \mathrm{C}_n(K.) \xrightarrow{\widehat{\partial^{K.}_n}} \mathrm{B}_{n-1}(K.) \to O. \tag{8.3.4}$$

Here, the hooked arrows are inclusions, and the two-headed ones are inclusions. Furthermore, $\widetilde{\partial^{K.}_n}(x) = \partial^{K.}_n(x)$ for $x \in K_n$ (but the maps $\widetilde{\partial^{K.}_n}(x)$ and $\partial^{K.}_n(x)$ are not necessarily equal as their targets might be different). Finally, since $\mathrm{B}_n(K.) \subset \operatorname{Ker} \widetilde{\partial^{K.}_n}$, the Homomorphism Theorem (2.28) yields a well-defined R–homomorphism $\widehat{\partial^{K.}_n}$ given by $\widehat{\partial^{K.}_n}([x]_{\mathrm{B}_n(K.)}) = \partial^{K.}_n(x)$. It follows that $[x]_{\mathrm{B}_n(K.)} \in \operatorname{Ker} \widehat{\partial^{K.}_n}$ if and only if $x \in \operatorname{Ker} \partial^{K.}_n = \mathrm{Z}_n(K.)$; thus, $\operatorname{Ker} \widehat{\partial^{K.}_n} = \mathrm{Z}_n(K.)/\mathrm{B}_n(K.) = \mathrm{H}_n(K.)$, as desired.

(8.4) **Morphisms.** For R–complexes $K.$ and $L.$, an *R–morphism* $\varphi. : K. \to L.$ is a family $(\varphi_n : K_n \to L_n)_{n\in\mathbb{Z}}$ of R–homomorphisms such that $\partial^{L.}_n \varphi_n = \varphi_{n-1}\partial^{K.}_n$ for all

$n \in \mathbb{Z}$. The last condition is the commutativity of the diagram

(8.4.1)
$$\begin{array}{ccccccc} \cdots & \longrightarrow & K_n & \xrightarrow{\partial^K_n} & K_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \\ \cdots & \longrightarrow & L_n & \xrightarrow[\partial^L_n]{} & L_{n-1} & \longrightarrow & \cdots . \end{array}$$

This is often phrased as: "R–morphisms commute with the differentials." The R–homomorphism φ_n is said to be in *degree* n in the R–morphism $\varphi_\bullet$.

(8.5) **Special R–morphisms.** The following hold for R–complexes $K_\bullet$ and $L_\bullet$.
The *identity* $1_{K_\bullet} = \big(1_{K_n} \colon K_n \to K_n\big)_{n\in\mathbb{Z}}$ is an R–morphism.
The *zero-morphism* $o_{K_\bullet L_\bullet} = \big(o_{K_n L_n} \colon K_n \to L_n\big)_{n\in\mathbb{Z}}$, cf. (2.18), is an R–morphism.
For $a \in R$ there is the *multiplication morphism* $a_{K_\bullet} = \big(a_{K_n} \colon K_n \to K_n\big)_{n\in\mathbb{Z}}$, cf. (2.16).

(8.6) **Composition.** Let $\varphi_\bullet = \big(\varphi_n\big)_{n\in\mathbb{Z}} \colon K_\bullet \to L_\bullet$ and $\psi_\bullet = \big(\psi_n\big)_{n\in\mathbb{Z}} \colon L_\bullet \to M_\bullet$ be R–morphisms. The composite $\varphi_\bullet\psi_\bullet = \big(\varphi_n\psi_n\big)_{n\in\mathbb{Z}} \colon K_\bullet \to M_\bullet$ is then an R–morphism.

(8.7) **Isomorphisms.** An R–morphism $\varphi_\bullet = \big(\varphi_n\big)_{n\in\mathbb{Z}} \colon K_\bullet \to L_\bullet$ is said to be an *isomorphism* exactly when there exists an R–morphism $\varphi_\bullet^{-1} \colon L_\bullet \to K_\bullet$ such that $\varphi_\bullet^{-1}\varphi_\bullet = 1_{K_\bullet}$ and $\varphi_\bullet\varphi_\bullet^{-1} = 1_{L_\bullet}$. This is tantamount to that φ_n is an isomorphism of R–modules for all $n \in \mathbb{Z}$, and when the latter is the case, then $\varphi_\bullet^{-1} = \big(\varphi_n^{-1}\big)_{n\in\mathbb{Z}} \colon L_\bullet \to K_\bullet$ and this is an R–morphism.

Let $O \to K \to L \to M \to O$ and $O \to K' \to L' \to M' \to O$ be two short sequences, cf. (6.4). If they are complexes, then are isomorphic as such if and only if they are so in the sense of (6.7).

(8.8) **Module as complex.** Any R–module M can be viewed as the R–complex:

$$O \to M \to O$$

which has M in degree 0 and has the zero module in all other degrees. If an R–complex $N_\bullet$ is concentrated in degree 0, that is, $N_\ell = 0$ for $\ell \neq 0$, then $N_\bullet$ is equal to the R–module N_0 (considered as a complex). Thus, modules are identified with the corresponding complex concentrated in degree zero, and vice versa.

If $M_\bullet$ and $N_\bullet$ are R–complexes which are concentrated in degree 0, that is, they are modules under our identification, then $\varphi_\bullet \colon K_\bullet \to L_\bullet$ is an R–morphism, if and only if $\varphi_0 \colon K_0 \to L_0$ is an R–homomorphism.

(8.9) **Category of R–complexes.** The class of all R–complexes and all R–morphisms is called the *category of R–complexes* and is denoted $\mathrm{Com}(R)$. In view of (8.8), the category $\mathrm{Mod}(R)$ of R–modules and R–morphisms is contained in $\mathrm{Com}(R)$. Some says that $\mathrm{Mod}(R)$ is a *full* subcategory of $\mathrm{Com}(R)$ to emphasize that for any M and N in $\mathrm{Mod}(R)$, all morphisms $M \to N$ in $\mathrm{Com}(R)$ belong to $\mathrm{Mod}(R)$, cf. (8.8).

(8.10) **Short-exact sequence of complexes.** The next sequence of R–morphisms

$$O_\bullet \to K_\bullet \xrightarrow{\varphi_\bullet} L_\bullet \xrightarrow{\psi_\bullet} M_\bullet \to O_\bullet$$

is said to be *short exact* exactly when the sequence of R–homomorphisms

$$O \to K_n \xrightarrow{\varphi_n} L_n \xrightarrow{\psi_n} M_n \to O$$

is exact for each $n \in \mathbb{Z}$.

(8.11) **Homology homomorphisms.** Let $\varphi. : K. \to L.$ be an R–morphism. It follows then directly from the definition, cf. (8.4), that the next inclusions then hold for all $n \in \mathbb{Z}$.

$$\varphi_n(\mathrm{B}_n(K.)) \subseteq \mathrm{B}_n(L.) \quad \text{and} \quad \varphi_n(\mathrm{Z}_n(K.)) \subseteq \mathrm{Z}_n(L.)\,. \tag{8.11.1}$$

It follows that for each $n \in \mathbb{Z}$ an R–homomorphism $\mathrm{H}_n(\varphi.)$ is well-defined as follows:

$$\mathrm{H}_n(\varphi.)\colon \mathrm{H}_n(K.) \to \mathrm{H}_n(L.) \quad \text{by} \quad \mathrm{H}_n(\varphi.)([x]_{\mathrm{B}_n(K.)}) \stackrel{\text{def}}{=} [\varphi_n(x)]_{\mathrm{B}_n(L.)}\,. \tag{8.11.2}$$

Here, x belongs to $\mathrm{Z}_n(K.)$ so $\varphi_n(x) \in \mathrm{Z}_n(L.)$ by (8.11.1). Assume that $x' \in \mathrm{Z}_n(K.)$ and $[x]_{\mathrm{B}_n(K.)} = [x']_{\mathrm{B}_n(K.)}$. We then have $x - x' \in \mathrm{B}_n(K.)$, and hence $\varphi_n(x) - \varphi_n(x') = \varphi_n(x - x') \in \mathrm{B}_n(L.)$, cf. (8.11.1); thus, $[\varphi_n(x)]_{\mathrm{B}_n(L.)} = [\varphi_n(x')]_{\mathrm{B}_n(L.)}$. This proves that $\mathrm{H}_n(\varphi.)$ is well-defined. It fits into the next commutative diagram:

(8.11.3)

$$\begin{array}{ccccc}
\mathrm{B}_n(K.) & \hookrightarrow & \mathrm{Z}_n(K.) & \twoheadrightarrow & \mathrm{H}_n(K.) \\
\downarrow{\scriptstyle (\varphi_n)|_{\mathrm{B}_n(K.)}} & & \downarrow{\scriptstyle (\varphi_n)|_{\mathrm{Z}_n(K.)}} & & \downarrow{\scriptstyle \mathrm{H}_n(\varphi.)} \\
\mathrm{B}_n(L.) & \hookrightarrow & \mathrm{Z}_n(L.) & \twoheadrightarrow & \mathrm{H}_n(L.)
\end{array}$$

Here, the hooked arrows are inclusions, the double head ones are residue homomorphisms, and the left and the middle vertical ones are restrictions of φ_n.

(8.12) **Theorem: Homology is a functor.** *For any $n \in \mathbb{Z}$ the n'th homology $\mathrm{H}_n(-)$ is a covariant functor: $\mathrm{Com}(R) \to \mathrm{Mod}(R)$. That is,*

(1) *If $K.$ is an R–complex, then $\mathrm{H}_n(K.)$ is an R–module.*
(2) *If $\varphi. : K. \to L.$ is an R–morphism, then $\mathrm{H}_n(\varphi.)\colon \mathrm{H}_n(K.) \to \mathrm{H}_n(L.)$ is an R–homomorphism.*
(3) *$\mathrm{H}_n(\psi.\varphi.) = \mathrm{H}_n(\psi.)\,\mathrm{H}_n(\varphi.)$ for R–morphisms $\varphi. : K. \to L.$ and $\psi. : L. \to M.$.*
(4) *$\mathrm{H}_n(1_{K.}) = 1_{\mathrm{H}_n(K.)}$, if $K.$ is an R–complex.*

(8.13) **Proof.** (1) is by the definition of homology, cf. (8.3), and (2) is by that of the homology homomorphism, cf. (8.11.2).

(3) The definition of the homology homomorphism, cf. (8.11.2), is used several times in the next chain of equalities in which x belongs to $\mathrm{Z}_n(K.)$.

$$\begin{aligned}
\mathrm{H}_n(\psi.\varphi.)([x]_{\mathrm{B}_n(K.)}) &= [(\psi_n\varphi_n)(x)]_{\mathrm{B}_n(M.)} \\
&= [(\psi_n(\varphi_n(x))]_{\mathrm{B}_n(M.)} \\
&= \mathrm{H}_n(\psi.)\big([\varphi_n(x)]_{\mathrm{B}_n(L.)}\big) \\
&= \mathrm{H}_n(\psi.)\big(\mathrm{H}_n(\varphi.)([x]_{\mathrm{B}_n(K.)})\big) \\
&= \big(\mathrm{H}_n(\psi.)\,\mathrm{H}_n(\varphi.)\big)([x]_{\mathrm{B}_n(K.)})\,.
\end{aligned}$$

This proves the desired assertion.

(4) For $x \in \mathrm{Z}_n(K.)$ we have $\mathrm{H}_n(1_{K.})([x]_{\mathrm{B}_n(K.)}) = [1_{K_n}(x)]_{\mathrm{B}_n(K.)}) = [x]_{\mathrm{B}_n(K.)}$, that is, $\mathrm{H}_n(K.)(1_{K.}) = 1_{\mathrm{H}_n(K.)}$. □

(8.14) **Theorem: Homology is half-exact.** *Let n be an integer. The functor $\mathrm{H}_n(-)$ is then half-exact, that is, any short exact sequence of R–morphisms:*

$$O_\bullet \to K_\bullet \xrightarrow{\varphi_\bullet} L_\bullet \xrightarrow{\psi_\bullet} M_\bullet \to O_\bullet$$

induces an exact sequence of R–homomorphisms:

$$\mathrm{H}_n(K_\bullet) \xrightarrow{\mathrm{H}_n(\varphi_\bullet)} \mathrm{H}_n(L_\bullet) \xrightarrow{\mathrm{H}_n(\psi_\bullet)} \mathrm{H}_n(M_\bullet)\,.$$

(8.15) **Proof.** We are required to prove that $\operatorname{Im}\mathrm{H}_n(\varphi_\bullet) = \operatorname{Ker}\mathrm{H}_n(\psi_\bullet)$.

The inclusion $\operatorname{Im}\mathrm{H}_n(\varphi_\bullet) \subseteq \operatorname{Ker}\mathrm{H}_n(\psi_\bullet)$ follows from $\mathrm{H}_n(\psi_\bullet)\,\mathrm{H}_n(\varphi_\bullet) = \mathrm{H}_n(\psi_\bullet\varphi_\bullet) = \mathrm{H}_n(o_\bullet) = o$ where the first equality is from (8.12.3), the second one is by assumption, and the last one is obvious from the definition (8.11.2).

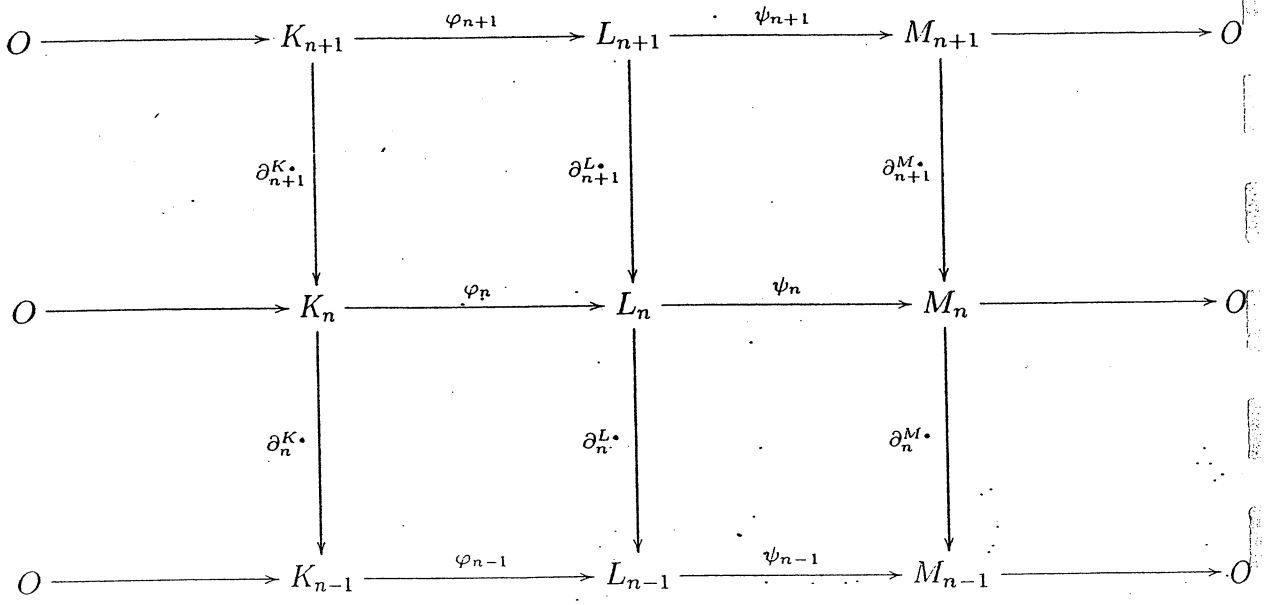
Next, we prove the inclusion $\operatorname{Im}\mathrm{H}_n(\varphi_\bullet) \supseteq \operatorname{Ker}\mathrm{H}_n(\psi_\bullet)$. To do this, we assume that $\ell \in \mathrm{Z}_n(L_\bullet)$ is such that the element $[\,\ell\,]_{\mathrm{B}_n(L_\bullet)}$ in $\mathrm{H}_n(L_\bullet)$ belongs to $\operatorname{Ker}\mathrm{H}_n(\psi_\bullet)$; thus, by the definition of $\mathrm{H}_n(\psi_\bullet)$, cf. (8.11.2), the *assumptions* are:

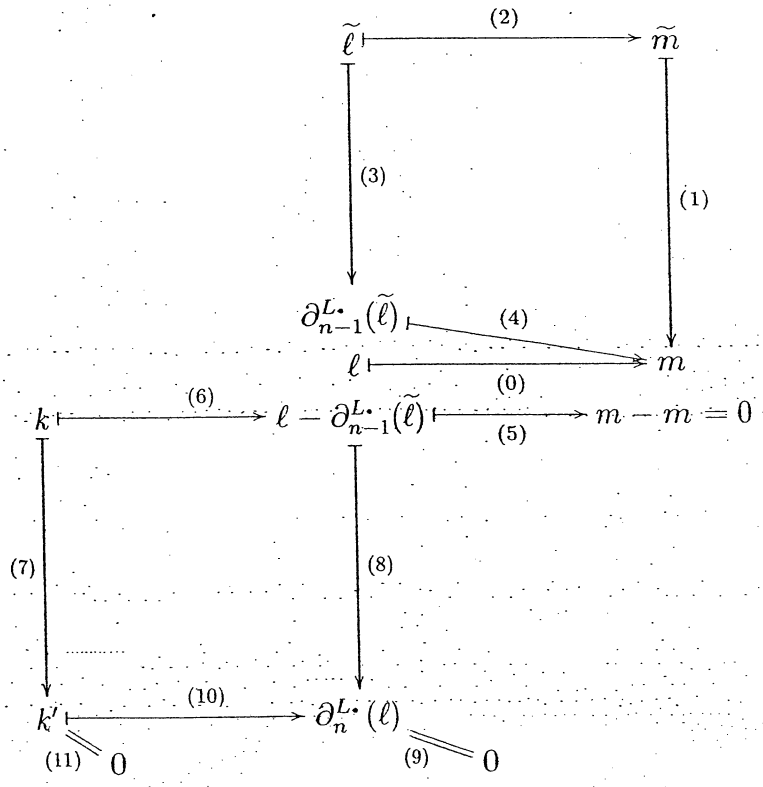
(8.15.1) $$\ell \in \mathrm{Z}_n(L_\bullet) \quad \text{and} \quad \psi_n(\ell) \in \mathrm{B}_n(M_\bullet) = \operatorname{Im}\partial^{M_\bullet}_{n+1}\,.$$

We *want to prove* that $[\,\ell\,]_{\mathrm{B}_n(L_\bullet)}$ belongs to $\operatorname{Im}\mathrm{H}_n(\varphi_\bullet)$, that is,

(8.15.2) $$[\,\ell\,]_{\mathrm{B}_n(L_\bullet)} = [\,\varphi_n(k)\,]_{\mathrm{B}_n(K_\bullet)} \quad \text{for some} \quad k \in \mathrm{Z}_n(K_\bullet)\,.$$

To achieve this, we chase—on the next page—a commutative diagram in which the rows are exact and the columns are complexes.





(0) $m =^{\text{def}} \psi_n(\ell)$.
(1) $m \in \operatorname{Im} \partial^{M_\bullet}_{n+1}$ by assumption, cf. (8.15.1).
(2) The row is exact.
(4) The square is commutative.
(6) The row is exact.
(9) By assumption, cf. (8.15.1).
(10) The square is commutative.
(11) The row is exact.

Thus, we have $k \in \mathrm{Z}_n(K_\bullet)$ such that $[\,\ell\,]_{\mathrm{B}_n(L_\bullet)} = [\,\ell - \partial^{L_\bullet}_{n-1}(\widetilde{\ell}\,)\,]_{\mathrm{B}_n(L_\bullet)} = [\,\varphi_n(k)\,]_{\mathrm{B}_n(K_\bullet)}$, and this was exactly what we desired, cf. (8.15.2). □

(8.16) **Theorem: Connecting Homomorphism.** *Let*

$$(\mathbb{S}) \qquad O_\bullet \to K_\bullet \xrightarrow{\varphi_\bullet} L_\bullet \xrightarrow{\psi_\bullet} M_\bullet \to O_\bullet$$

be a short-exact sequence of complexes, and let n be an integer. Furthermore, for any element w in $\mathrm{H}_n(M_\bullet)$, choose $m \in \mathrm{Z}_n(M_\bullet)$ such that $w = [\,m\,]_{\mathrm{B}_n(M_\bullet)} \in \mathrm{H}_n(M_\bullet)$, and let $k' \in K_{n-1}$ be determined by the next diagram chase:

$$\begin{array}{ccccccccc}
O & \longrightarrow & K_n & \xrightarrow{\varphi_n} & L_n & \xrightarrow{\psi_n} & M_n & \longrightarrow & O \\
 & & \downarrow{\scriptstyle \partial^{K_\bullet}_n} & & \downarrow{\scriptstyle \partial^{L_\bullet}_n} & & \downarrow{\scriptstyle \partial^{M_\bullet}_n} & & \\
O & \longrightarrow & K_{n-1} & \xrightarrow{\varphi_{n-1}} & L_{n-1} & \xrightarrow{\psi_{n-1}} & M_{n-1} & \longrightarrow & O
\end{array}$$

$$\begin{array}{ccccc}
 & & \ell & \xmapsto{(1)} & m \\
 & & \downarrow{\scriptstyle (2)} & & \downarrow{\scriptstyle (\mathfrak{d})} \\
k' & \xmapsto{(4)} & \ell' & \xmapsto{(3)} & 0
\end{array}$$

The following then hold.

(1) *The element k' belongs to $\mathrm{Z}_{n-1}(K_\bullet)$.*

(2) *The class $[\,k'\,]_{\mathrm{B}_{n-1}(K_\bullet)} \in \mathrm{H}_{n-1}(K_\bullet)$ does not depend on the choices made.*

(3) *The resulting map*

$$\delta_n^{(\mathbb{S})} \colon \mathrm{H}_n(M_\bullet) \to \mathrm{H}_{n-1}(K_\bullet) \quad \text{given by} \quad \delta_n^{(\mathbb{S})}(w) \stackrel{\mathrm{def}}{=} [\,k'\,]_{\mathrm{B}_{n-1}(K_\bullet)}$$

is an R–homomorphism.

(4) *The next sequence of R–homomorphisms is exact:*

$$\mathrm{H}_n(L_\bullet) \xrightarrow{\mathrm{H}_n(\psi_\bullet)} \mathrm{H}_n(M_\bullet) \xrightarrow{\delta_n^{(\mathbb{S})}} \mathrm{H}_{n-1}(K_\bullet) \xrightarrow{\mathrm{H}_{n-1}(\varphi_\bullet)} \mathrm{H}_{n-1}(L_\bullet)$$

(8.17) **Proof.** Each part results from enhancing the diagram chase. □

(8.18) **Long-exact sequence.** The previous theorem and the half-exactness of the homology functor, cf. (8.14), yield that every short-exact sequence $(\mathbb{S})$ of R–complexes, as in Theorem (8.16), induces a (so-called) *long-exact sequence:*

$$\begin{aligned}
(8.18.1) \qquad \cdots \longrightarrow \mathrm{H}_n(K_\bullet) &\xrightarrow{\mathrm{H}_n(\varphi_\bullet)} \mathrm{H}_n(L_\bullet) \xrightarrow{\mathrm{H}_n(\psi_\bullet)} \mathrm{H}_n(M_\bullet) \xrightarrow{\delta_n^{(\mathbb{S})}} \\
\mathrm{H}_{n-1}(K_\bullet) &\xrightarrow{\mathrm{H}_{n-1}(\varphi_\bullet)} \mathrm{H}_{n-1}(L_\bullet) \xrightarrow{\mathrm{H}_{n-1}(\psi_\bullet)} \mathrm{H}_{n-1}(M_\bullet) \xrightarrow{\delta_{n-1}^{(\mathbb{S})}} \cdots
\end{aligned}$$

(8.19) **Exercise: Snake Lemma.** Let the diagram

$$\begin{array}{ccccccccc}
O & \longrightarrow & K_1 & \xrightarrow{\varphi_1} & L_1 & \xrightarrow{\psi_1} & M_1 & \longrightarrow & O \\
 & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \mu} & & \\
O & \longrightarrow & K_0 & \xrightarrow{\varphi_0} & L_0 & \xrightarrow{\psi_0} & M_0 & \longrightarrow & O
\end{array}$$

be commutative and have exact rows. Prove that there is an induced exact sequence:

$$O \to \operatorname{Ker}\kappa \to \operatorname{Ker}\lambda \to \operatorname{Ker}\mu \xrightarrow{\delta} \operatorname{Coker}\kappa \to \operatorname{Coker}\lambda \to \operatorname{Coker}\mu \to O\,.$$

(8.20) **Theorem: Functoriality of Connecting homomorphism.** *Let*

$$\begin{array}{lccccccccc}
(\mathbb{S}) & O. & \longrightarrow & K. & \xrightarrow{\varphi.} & L. & \xrightarrow{\psi.} & M. & \longrightarrow & O. \\
 & & & \downarrow{\scriptstyle \kappa.} & & \downarrow{\scriptstyle \lambda.} & & \downarrow{\scriptstyle \mu.} & & \\
(\mathbb{S}') & O. & \longrightarrow & K'. & \xrightarrow{\varphi'.} & L'. & \xrightarrow{\psi'.} & M'. & \longrightarrow & O.
\end{array}$$

be a commutative diagram of complexes with exact rows. The next diagram involving the connecting homomorphisms of the rows is then commutative for each $n \in \mathbb{Z}$.

$$\begin{array}{ccc}
\mathrm{H}_n(M.) & \xrightarrow{\delta_n^{(\mathbb{S})}} & \mathrm{H}_{n-1}(K.) \\
\downarrow{\scriptstyle \mathrm{H}_n(\mu.)} & & \downarrow{\scriptstyle \mathrm{H}_{n-1}(\kappa.)} \\
\mathrm{H}_n(M.') & \xrightarrow[\delta_n^{(\mathbb{S}')}]{} & \mathrm{H}_{n-1}(K.')
\end{array}$$

(8.21) **Proof.** By inspection. □

(8.22) **Induced functor of complexes 1.** Let $T\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R')$ be a covariant functor and assume that it is *additive*, cf. (6.35). If $K.$ is an R-complex, then there is an induced sequence of R'-homomorphisms:

$$T(K.) \stackrel{\text{def}}{=} \cdots \longrightarrow T(K_{n+1}) \xrightarrow{T(\partial_{n+1}^{K.})} T(K_n) \xrightarrow{T(\partial_n^{K.})} T(K_{n-1}) \longrightarrow \cdots$$

This is indeed an R'-complex with $\partial_n^{T(K.)} = T\big(\partial_n^{K.}\big)$ because

$$T(\partial_{n+1}^{K.})T(\partial_n^{K.}) = T\big(\partial_{n+1}^{K.}\partial_n^{K.}\big) = T(0) = 0$$

since T is additive, see also (6.37).

If $\varphi. = (\varphi_n\colon K_n \to L_n)_{n\in\mathbb{Z}}$ is an R-morphism: $K. \to L.$, then the induced family $T(\varphi.) \stackrel{\text{def}}{=} \big(T(\varphi_n)\colon T(K_n) \to T(L_n)\big)_{n\in\mathbb{Z}}$ is an R'-morphism, because the commutativity of the left diagram below implies that of the right.

(8.22.1)

$$\begin{array}{ccc}
K_n & \xrightarrow{\varphi_n} & L_n \\
\downarrow{\scriptstyle \partial_n^{K.}} & & \downarrow{\scriptstyle \partial_n^{L.}} \\
K_{n-1} & \xrightarrow[\varphi_{n-1}]{} & L_{n-1}
\end{array}
\qquad\qquad
\begin{array}{ccc}
T(K_n) & \xrightarrow{T(\varphi_n)} & T(L_n) \\
\downarrow{\scriptstyle T(\partial_n^{K.})} & & \downarrow{\scriptstyle T(\partial_n^{L.})} \\
T(K_{n-1}) & \xrightarrow[T(\varphi_{n-1})]{} & T(L_{n-1})X
\end{array}$$

Thus, there is an induced map

(8.22.2)

$$T\colon \mathrm{Com}(R) \to \mathrm{Com}(R') \quad \text{given by} \quad K. \mapsto T(K.)\,, \quad \varphi.\colon K. \to L. \mapsto T(\varphi.]\,.$$

This is easily seen to be a functor, the functor of complexes *induced* by the module functor T.

(8.23) **Theorem: Homology and Exact Covariant Module Functor.** *Consider an exact additive*[9] *covariant functor* $T\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R')$. *There exists then for any R-complex $K.$ and any $n \in \mathbb{Z}$ an R'-isomorphism*

(8.23.1) $$\theta_n^{K.}\colon \mathrm{H}_n(T(K.)) \xrightarrow{\cong} T(\mathrm{H}_n(K.))\,.$$

[9] One can prove that all half-exact functors are additive; this is, however, not so important here.

Furthermore, if $\varphi_\bullet\colon K_\bullet \to L_\bullet$ is any R–morphism, then the next diagram is commutative.

(8.23.2)
$$\begin{array}{ccc} \mathrm{H}_n(T(K_\bullet)) & \xrightarrow[\cong]{\theta_n^{K_\bullet}} & T(\mathrm{H}_n(K_\bullet)) \\ \Big\downarrow{\scriptstyle \mathrm{H}_n(T(\varphi_\bullet))} & & \Big\downarrow{\scriptstyle T(\mathrm{H}_n(\varphi_\bullet))} \\ \mathrm{H}_n(T(L_\bullet)) & \xrightarrow[\theta_n^{L_\bullet}]{\cong} & T(\mathrm{H}_n(L_\bullet)) \,. \end{array}$$

(8.24) **Proof.** Chase the diagrams induced by the short-exact sequences in (8.3). □

(8.25) **Exercise: Mapping Cones.** Let $\varphi_\bullet\colon K_\bullet \to L_\bullet$ be an R–morphism. For $n \in \mathbb{Z}$ consider the next definitions.

(8.25.1)
$$M_n \stackrel{\text{def}}{=} L_n \oplus K_{n-1}\,;$$

(8.25.2)
$$\mu_n\colon M_n \to M_{n-1}\,,\ \mu_n(\ell_n, k_{n-1}) \stackrel{\text{def}}{=} \big(\partial_n^{L_\bullet}(\ell_n) + \varphi_{n-1}(k_{n-1})\,,\ -\partial_{n-1}^{K_\bullet}(k_{n-1})\big)$$

(a) Prove that $M_\bullet = \cdots \to M_{n+1} \xrightarrow{\mu_{n+1}} M_n \xrightarrow{\mu_n} M_{n-1} \to \cdots$ is an R–complex.
It is called the *mapping cone* of the R–morphism $\varphi_\bullet$.
Let $K_\bullet'$ be the R–complex $K_\bullet$ *shifted* one degree to the left, that is, it is given by

(8.25.3)
$$K_n' \stackrel{\text{def}}{=} K_{n-1} \quad \text{og} \quad \partial_n^{K_\bullet'} \stackrel{\text{def}}{=} -\partial_{n-1}^{K_\bullet}\,.$$

Furthermore, $\iota_\bullet\colon L_\bullet \to M_\bullet$ is the inclusion, that is, $\iota_n(\ell_n) =^{\text{def}} (\ell_n, 0) \in M_n$, and $\pi_\bullet\colon M_\bullet \to K_\bullet'$ is the projection, that is, $\pi_n(\ell_n, k_{n-1}) =^{\text{def}} k_{n-1} \in K_\bullet'$.
(b) Prove that

(𝕊)
$$O_\bullet \to L_\bullet \xrightarrow{\iota_\bullet} M_\bullet \xrightarrow{\pi_\bullet} K_\bullet' \to O_\bullet$$

is an exact sequence in $\mathrm{Com}(R)$.
(c) Make the identification $\mathrm{H}_{n+1}(K_\bullet') = \mathrm{H}_n(K_\bullet)$.
(d) Prove that the connecting homomorphism $\delta_{n+1}^{(\mathbb{S})}\colon \mathrm{H}_{n+1}(K_\bullet') \to \mathrm{H}_n(L_\bullet)$ is equal to $\mathrm{H}_n(\varphi_\bullet)\colon \mathrm{H}_n(L_\bullet)$ under the above identification.
(e) Describe $M_\bullet$ when $K_\bullet$ and $L_\bullet$ are modules, cf. (8.8).
(f) Prove that $M_\bullet$ is exact if and only if $\mathrm{H}_n(\varphi_\bullet)$ is an isomorphism for all $n \in \mathbb{Z}$.

(8.26) **Quasi-isomorphisms.** An R–morphism $\varphi_\bullet\colon K_\bullet \to L_\bullet$ is said to be a *quasi-isomorphisms* exactly when $\mathrm{H}_n(\varphi_\bullet)$ is an isomorphism for all $n \in \mathbb{Z}$, and the latter is the case if and only if the mapping cone of $\varphi_\bullet$ is an exact complex, cf. (8.25). When $\varphi_0\colon K_0 \to L_0$ is an R–homomorphism, then the corresponding R–morphism $\varphi_\bullet\colon K_\bullet \to L_\bullet$, cf. (8.8), is a quasi-isomorphism, if and only if φ_0 is an isomorphism in $\mathrm{Mod}(R)$.

(8.27) **Exercise: Hom and mapping cones.** Let $\varphi_\bullet\colon K_\bullet \to L_\bullet$ be an R–morphism with mapping cone $M_\bullet$, cf. (8.25), and let H be an R–module.
(a) Prove that the R–complex $\mathrm{Hom}_R(H, M_\bullet)$, cf. (8.22) is isomorphic to the mapping cone of of the R–morphism $\mathrm{Hom}_R(H, \varphi_\bullet)\colon \mathrm{Hom}_R(H, M_\bullet) \to \mathrm{Hom}_R(H, M_\bullet)$.

9. Homotopy and Projective Resolutions

(9.1) **Homotopic morphisms.** Two R–morphisms $\varphi_\bullet, \psi_\bullet : K_\bullet \to L_\bullet$ in $\mathrm{Com}(R)$ (with same source and same target) are said to be *homotopic,* exactly when there exists a family

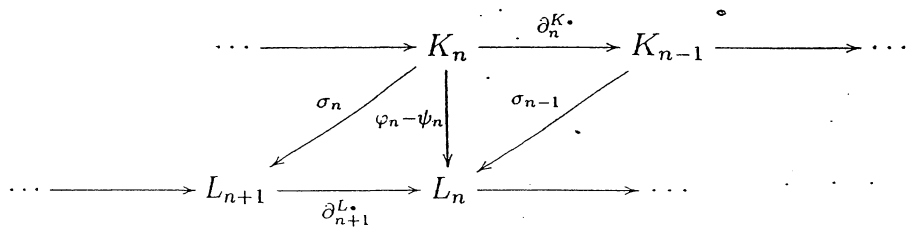
$$\big(\sigma_n \colon K_n \to L_{n+1}\big)_{n\in\mathbb{Z}}$$

of R–homomorphisms such that

$$\varphi_n - \psi_n = \sigma_{n-1}\partial^{K_\bullet}_n + \partial^{L_\bullet}_{n+1}\sigma_n \quad \text{for all} \quad n \in \mathbb{Z}.$$

When this is the case we write $\varphi_\bullet \sim \psi_\bullet$.

(9.2) **Illustration: Homotopy.**



Note, however, that there are NO requirement of commutativity on neither the rhomboid nor the triangles!

(9.3) **Proposition.** *If $\varphi_\bullet$ and $\psi_\bullet$ are homotopic R–morphisms, then the induced R–homomorphisms $\mathrm{H}_n(\varphi_\bullet)$ and $\mathrm{H}_n(\psi_\bullet)$ are equal for all $n \in \mathbb{Z}$.*

(9.4) **Proof.** Choose the family $\big(\sigma_n \colon K_n \to L_{n+1}\big)_{n\in\mathbb{Z}}$ such that

$$(*) \qquad \varphi_n - \psi_n = \sigma_{n-1}\partial^{K_\bullet}_n + \partial^{L_\bullet}_{n+1}\sigma_n \quad \text{for all} \quad n \in \mathbb{Z}.$$

For $x \in \mathrm{Z}_n(K_\bullet)$ we get $\varphi_n(x) - \psi_n(x) = \partial^{L_\bullet}_{n+1}(\sigma_n(x))$ which belongs to $\mathrm{B}_n(L_\bullet)$, that is, $[\varphi_n(x)]_{\mathrm{B}_n(L_\bullet)} = [\psi_n(x)]_{\mathrm{B}_n(L_\bullet)}$ in $\mathrm{H}_n(L_\bullet)$. By the definition (8.11.2) of homology homomorphisms we get for all $x \in \mathrm{Z}_n(K_\bullet)$ that $\mathrm{H}_n(\varphi_\bullet)([x]_{\mathrm{B}_n(K_\bullet)}) = \mathrm{H}_n(\psi_\bullet)([x]_{\mathrm{B}_n(K_\bullet)})$, that is, $\mathrm{H}_n(\varphi_\bullet) = \mathrm{H}_n(\psi_\bullet)$, as desired. □

(9.5) **Proposition.** *Let $T \colon \mathrm{Mod}(R) \to \mathrm{Mod}(R')$ be an additive covariant functor and consider the induced one $T \colon \mathrm{Com}(R) \to \mathrm{Com}(R')$, cf. (8.22). The following then holds.*

$$\varphi_\bullet \sim \psi_\bullet \implies T(\varphi_\bullet) \sim T(\psi_\bullet).$$

(9.6) **Proof.** Apply T to $(*)$ above. □

(9.7) **Split R–complexes.** An R–complex $K_\bullet$ is said to *split,* exactly if $1_{K_\bullet} \sim o_\bullet$, that is, there exists a family $\big(\sigma_n \colon K_n \to K_{n+1}\big)_{n\in\mathbb{Z}}$ of R–homomorphisms such that

$$(*) \qquad 1_{K_n} = \sigma_{n-1}\partial^{K_\bullet}_n + \partial^{K_\bullet}_{n+1}\sigma_n \quad \text{for all} \quad n \in \mathbb{Z}.$$

If $K_\bullet$ is short, say $K_\bullet = O \to K_2 \to K_1 \to K_0 \to O$, then it splits in this sense exactly when it does so in the sense of (6.39).

Let $T \colon \mathrm{Mod}(R) \to \mathrm{Mod}(R')$ be an additive covariant functor. If $K_\bullet$ splits, then so does $T(K_\bullet)$, cf. (8.22) and (9.5).

(9.8) **Proposition.** *If $K_\bullet$ splits, then it is exact, that is, $\mathrm{H}_n(K_\bullet) = 0$ for all $n \in \mathbb{Z}$.*

(9.9) **Proof.** We have $1_{\mathrm{H}_n(K.)} = \mathrm{H}_n(1_{K.}) = \mathrm{H}_n(o_{K.}) = o$ by, respectively, (8.12.4), (9.3), and the definition (8.11.2). Thus, $\mathrm{H}_n(K.) = O$. □

(9.10) **Exercise.** Consider the ring $R \stackrel{\text{def}}{=} \mathbb{Z}/(4)$ and prove that the next complex is exact but does not split:

$$K. \stackrel{\text{def}}{=} \cdots \to R \xrightarrow{2} R \xrightarrow{2} R \xrightarrow{2} R \to \cdots .$$

(9.11) **Exercise.** Let R be a field, and let $K.$ be an *exact* complex. Prove that $K.$ splits. [Divide $K.$ into tilted sequences as in (6.56) and apply (7.29) and (7.10).]

(9.12) **Exercise.** Let R be a field, and let T: $\mathrm{Mod}(R) \to \mathrm{Mod}(R')$ be an additive covariant functor. Prove that T is exact.

(9.13) **Exercise.** Assume that the R–complex

$$P. = \cdots \to P_n \to \cdots \to P_1 \to \cdots \to P_0 \to O$$

is exact, and that P_ℓ is projective for all $\ell \in \mathbb{N}_0$. Prove that $P.$ splits.

(9.14) **Continuation of homomorphisms.** Let μ: $M \to M'$ be an R–homomorphism of R–modules, and let $P.$ and $P'.$ be projective resolutions of M and M', respectively, cf. (7.24), and set $\partial_n = \partial_n^{P.}$ and $\partial'_n = \partial_n^{P'.}$ for all $n \in \mathbb{N}_0$.

A *continuation* of μ is an R–morphism $\mu.: P. \to P'.$ such that $\mu\partial_0 = \partial'_0\mu_0$, that is, such that the next diagram is commutative:

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & P_n & \xrightarrow{\partial_n} & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \xrightarrow{\partial_1} & P_0 & \xrightarrow{\partial_0} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mu_n} & & \downarrow{\scriptstyle \mu_{n-1}} & & & & \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu_0} & & \downarrow{\scriptstyle \mu} & & \\
\cdots & \longrightarrow & P'_n & \xrightarrow[\partial'_n]{} & P'_{n-1} & \longrightarrow & \cdots & \longrightarrow & P'_1 & \xrightarrow[\partial'_1]{} & P'_0 & \xrightarrow[\partial'_0]{} & M' & \longrightarrow & 0.
\end{array}$$

(The right hand square commutes because $\mu\partial_0 = \partial'_0\mu_0$, and the others commute since $\mu.$ is a morphism.)

(9.15) **Theorem.** *The following then hold.*

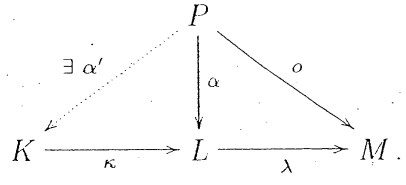
(∃) *There exists a continuation of μ.*

(!) *If $\mu.: P. \to P'.$ and $\widetilde{\mu}.: P. \to P'.$ both are continuations of μ, then $\mu. \sim \widetilde{\mu}.$.*

The proof of this theorem relies on the next auxiliary result.

(9.16) **Lemma.** *Let $K \xrightarrow{\kappa} L \xrightarrow{\lambda} M$ be an exact sequence of R–modules, let P be a projective R–module, and let α: $P \to L$ be an R–homomorphism such that $\lambda\alpha = o$. There exists then an R–homomorphism α': $P \to K$ such that $\kappa\alpha' = \alpha$. Illustration:*

(9.16.1)



This means: If the solid arrows are given, the row is exact, and the right triangle commutes, then the dotted arrow exists such the left triangle commutes. (For $M = 0$ then this is the projectivity diagram (7.2).)

(9.17) **Proof of Lemma.** As P is projective, the next sequence is exact, and the existence of the desired α' follows.

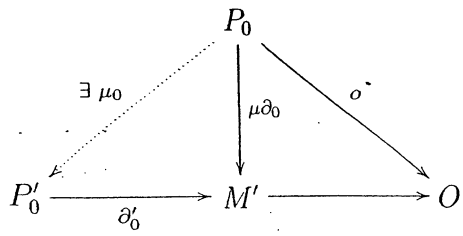
$$\begin{array}{ccccc} \operatorname{Hom}_R(P,K) & \xrightarrow{\operatorname{Hom}_R(P,\kappa)} & \operatorname{Hom}_R(P,L) & \xrightarrow{\operatorname{Hom}_R(P,\lambda)} & \operatorname{Hom}_R(P,M) \\ \exists\,\alpha' & \longmapsto & \alpha & \longmapsto & o\,. \end{array}$$

□

(9.18) **Proof of the theorem.**

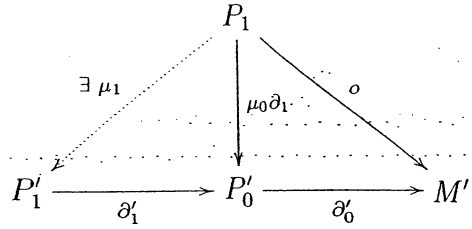
Part ($\exists$) is established by constructing $\mu_0, \mu_1, \dots, \mu_n, \dots$ recursively using the lemma.

$\underline{n=0.}$ The diagram



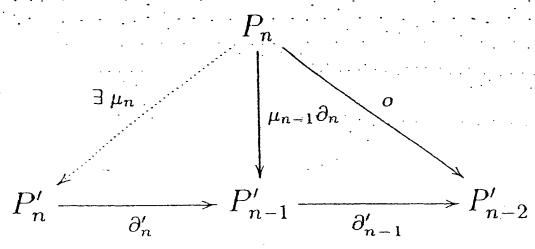
yields μ_0 such that $\partial'_0\mu_0 = \mu\partial_0$.

$\underline{n=1.}$ By the choice above we have $\partial'_0\mu_0\partial_1 = \mu\partial_0\partial_1 = \mu o = o$, so the next diagram



provides μ_1 such that $\partial'_1\mu_1 = \mu_0\partial_1$ as desired.

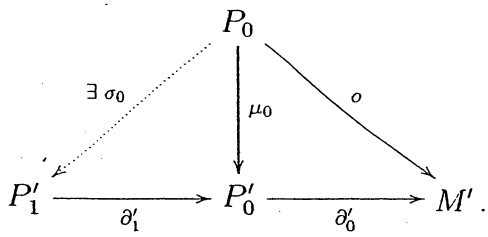
$\underline{n>1.}$ Assume that $\mu_0, \mu_1, \dots, \mu_{n-1}$ have been chosen appropriately; in particular, such that $\mu_{n-2}\partial_{n-1} = \partial'_{n-1}\mu_{n-1}$. We then have $\partial'_{n-1}\mu_{n-1}\partial_n = \mu_{n-2}\partial_{n-1}\partial_n = \mu_{n-2}o = o$, and the next diagram



provides μ_n such that $\partial'_n\mu_n = \mu_{n-1}\partial_n$ as desired.

Part (!). Assuming that $\mu.$ and $\widetilde{\mu}.$ both continues μ, and hence that $\mu. - \widetilde{\mu}.$ continues the zero homomorphism $o\colon M \to M'$, we want to prove that $\mu. \sim \widetilde{\mu}.$, that is, $\mu. - \widetilde{\mu}. \sim o.$. Thus, we may assume that $\mu = o$, that is, $\mu.$ continues o, and we want to prove that $\mu. \sim o.$, that is, there exists a family $(\sigma_n\colon P_n \to P'_{n+1})_{n\in\mathbb{Z}}$ such that $\mu_n = \partial'_{n+1}\sigma_n + \sigma_{n-1}\partial_n$ for all $n \in \mathbb{N}_0$.

$\underline{n = 0.}$

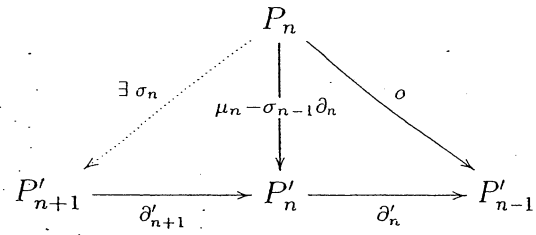


Thus, $\partial'_1\sigma_0 = \mu_0$ as required, since $\sigma_{-1} = o$ (as its source is zero).

$\underline{n > 0.}$ We have already $\mu_{n-1} = \partial'_n\sigma_{n-1} + \sigma_{n-2}\partial_{n-1}$ and hence the fourth equality in the next chain.

$$\begin{aligned}\partial'_n(\mu_n - \sigma_{n-1}\partial_n) &= \partial'_n\mu_n - \partial'_n\sigma_{n-1}\partial_n &&= \mu_{n-1}\partial_n - \partial'_n\sigma_{n-1}\partial_n\\ &= (\mu_{n-1} - \partial'_n\sigma_{n-1})\partial_n &&= \sigma_{n-2}\partial_{n-1}\partial_n\\ &= \sigma_{n-2}o &&= o\end{aligned}$$

The diagram



now yields $\mu_n = \partial'_{n+1}\sigma_n + \sigma_{n-1}\partial_n$ as desired. □

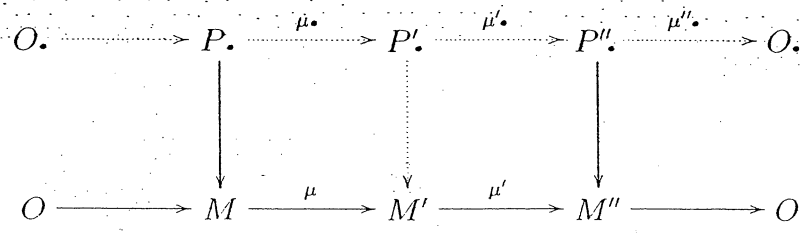
Without proof we mention the next useful result.

(9.19) **Horse Shoe Lemma.** *Let $O \to M \xrightarrow{\mu} M' \xrightarrow{\mu'} M'' \to O$ be an exact sequence in $\mathrm{Mod}(R)$, and let $P_\bullet$ and $P''_\bullet$ be projective resolutions of M and M'', respectively. There exists then a projective resolution $P'_\bullet$ of M' and continuations $\mu_\bullet : P_\bullet \to P'_\bullet$ and $\mu'_\bullet : P'_\bullet \to P''_\bullet$ of $\mu : M \to M'$ and $\mu' : M' \to M''$, respectively, such that*

$$O_\bullet \to P_\bullet \xrightarrow{\mu_\bullet} P'_\bullet \xrightarrow{\mu'_\bullet} P''_\bullet \to O$$

is an exact sequence in $\mathrm{Com}(R)$.

Illustration: There is the next exact sequence of augmented projective resolutions.

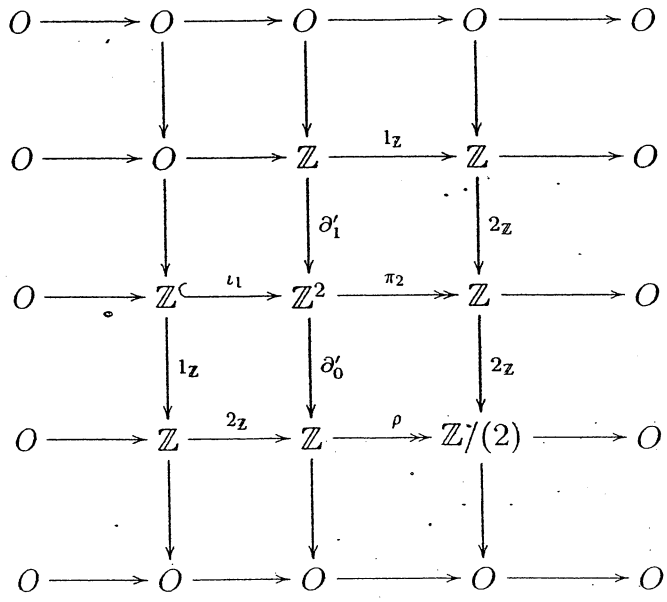


□

(9.20) **Example.** Consider the next exact sequence of $\mathbb{Z}$–modules:

$$O \to \mathbb{Z} \xrightarrow{2_{\mathbb{Z}}} \mathbb{Z} \xrightarrow{\rho} \mathbb{Z}/(2) \to O\,.$$

The next is a corresponding exact sequence of augmented projective resolutions:



Here, $\partial'_1(x) =^{\text{def}} (-x, 2x)$, $\partial'_0(x,y) =^{\text{def}} 2x + y$, $\iota_1(x) =^{\text{def}} (x,0)$, and $\pi_2(x,y) =^{\text{def}} y$.

10. Left Derived of Covariant Functors

There are two kinds of functors, the covariant ones and the contravariant ones. For each of these there are two kinds of derived functors: *left derived functors* and *right derived functors*; this makes four cases to consider. Two of these cases rely on the theory of *projective* resolutions, namely, that of left derived functors of covariant functors which is described in this section, and that of right derived functors of contravariant functors, confer Section 11. The two remaining ones rely on (so-called) injective resolutions and are deferred to the later section 13 and its exercise (13.15).

(10.1) **Convention.** Throughout this section

$$\boxed{T\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R') \text{ is an } \textit{additive covariant} \text{ functor.}}$$

We also consider its induced functor $T\colon \operatorname{Com}(R) \to \operatorname{Com}(R')$, cf. (8.22).

(10.2) **Definition.** For $n \in \mathbb{Z}$ the *n'th left derived functor* $\mathrm{L}_n T\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ of T is defined as follows:

(M) If M is an R–module, then choose a projective resolution $P_\bullet$ of M, and set

$$\mathrm{L}_n T(M) \stackrel{\text{def}}{=} \mathrm{H}_n(T(P_\bullet)) .$$

Dependence upon the *choice* of $P_\bullet$ will be discussed below.

(μ) If $\mu\colon M \to M'$ is an R–homomorphism, then choose projective resolutions $P_\bullet$ and $P'_\bullet$ of M and M', respectively, choose a continuation $\mu_\bullet : P_\bullet \to P'_\bullet$ of μ, and set

$$\mathrm{L}_n T(\mu) \stackrel{\text{def}}{=} \mathrm{H}_n(T(\mu_\bullet))\colon \mathrm{H}_n(T(P_\bullet)) \to \mathrm{H}_n(T(P'_\bullet)) .$$

Dependence upon the *choices* of $P_\bullet$, $P'_\bullet$, and $\mu_\bullet$ will be discussed below.

(10.3) **Theorem: Uniqueness.** *The following hold.*

(M) *For any R–module M, the R'–module $\mathrm{L}_n T(M)$ is uniquely determined up to unique isomorphism. That is, if $P_\bullet$ and $\widehat{P}_\bullet$ are projective resolutions of M, and $\varphi_\bullet : P_\bullet \to \widehat{P}_\bullet$ is any continuation of 1_M, then the induced map*

$$\mathrm{H}_n(T(\varphi_\bullet))\colon \mathrm{H}_n(T(P_\bullet)) \to \mathrm{H}_n(T(\widehat{P}_\bullet))$$

is an isomorphism, and it does not depend on the choice of $\varphi_\bullet$.

(μ) *For any R–homomorphism $\mu\colon M \to M'$, the induced R'–homomorphism $\mathrm{L}_n T(\mu)$ is uniquely determined up to isomorphism. That is, the diagram*

$$\begin{array}{ccc} \mathrm{H}_n(T(P_\bullet)) & \xrightarrow[\cong]{\mathrm{H}_n(T(\varphi_\bullet))} & \mathrm{H}_n(T(\widehat{P}_\bullet)) \\ \Big\downarrow{\scriptstyle \mathrm{H}_n(T(\mu_\bullet))} & & \Big\downarrow{\scriptstyle \mathrm{H}_n(T(\widehat{\mu}_\bullet))} \\ \mathrm{H}_n(T(P'_\bullet)) & \xrightarrow[\mathrm{H}_n(T(\varphi'_\bullet))]{\cong} & \mathrm{H}_n(T(\widehat{P}'_\bullet)) , \end{array}$$

is commutative, whenever $\varphi_\bullet$ is as in (M), $P'_\bullet$ and $\widehat{P}'_\bullet$ are projective resolutions of M', the morphisms $\mu_\bullet : P_\bullet \to P'_\bullet$ and $\widehat{\mu}_\bullet : \widehat{P}_\bullet \to \widehat{P}'_\bullet$ are continuations of μ, and $\varphi'_\bullet : P'_\bullet \to \widehat{P}'_\bullet$ is a continuation of $1_{M'}$.

(10.4) **Proof.** (M) Choose a continuation $\widehat{\varphi}_\cdot : \widehat{P}_\cdot \to P_\cdot$ of 1_M, and note that then the composite $\widehat{\varphi}_\cdot\varphi_\cdot : P_\cdot \to P_\cdot$ and the identity $1_{P_\cdot} : P_\cdot \to P_\cdot$ are both continuations of $1_M : M \to M$. Thus, $\widehat{\varphi}_\cdot\varphi_\cdot \sim 1_{\widehat{P}_\cdot}$ by (9.15.!), so (9.5) yields

$$T(\widehat{\varphi}_\cdot)T(\varphi_\cdot) \sim T(1_{P_\cdot}) = 1_{T(P_\cdot)} .$$

For any $n \in \mathbb{Z}$, apply now the homology functor to obtain

$$\mathrm{H}_n(T(\widehat{\varphi}_\cdot))\,\mathrm{H}_n(T(\varphi_\cdot)) = 1_{\mathrm{H}_n(T(P_\cdot))} .$$

Similarly we get

$$\mathrm{H}_n(T(\varphi_\cdot))\,\mathrm{H}_n(T(\widehat{\varphi}_\cdot)) = 1_{\mathrm{H}_n(T(\widehat{P}_\cdot))} .$$

Thus, $\mathrm{H}_n(T(\varphi_\cdot))$ is an isomorphism.

The last assertion in (M) is left for the reader to verify, and so is that of (μ). □

(10.5) **Theorem: Functoriality.** *For all $n \in \mathbb{Z}$ the derived functor $\mathrm{L}_n T$ is a functor:*

(1) $\mathrm{L}_n T(\mu'\mu) = \mathrm{L}_n T(\mu')\,\mathrm{L}_n T(\mu)$ *for all $n \in \mathbb{Z}$ when $\mu : M \to M'$ and $\mu' : M' \to M''$ are R–homomorphisms.*

(2) $\mathrm{L}_n T(1_M) = 1_{\mathrm{L}_n T(M)}$ *for all $n \in \mathbb{Z}$ when M is an R–module.*

(10.6) **Proof.** (1) Let $P_\cdot$, $P'_\cdot$, and $P''_\cdot$ be projective resolutions of M, M', and M'', respectively, and let $\mu_\cdot : P_\cdot \to P'_\cdot$ and $\mu'_\cdot : P'_\cdot \to P''_\cdot$ be continuations of μ and μ', respectively. Then $\mu'_\cdot\mu_\cdot : P_\cdot \to P''_\cdot$ is a continuation of $\mu'\mu : M \to M''$, and

$$\mathrm{L}_n T(\mu'\mu) = \mathrm{H}_n(T(\mu'_\cdot\mu_\cdot)) = \mathrm{H}_n(T(\mu'_\cdot))\,\mathrm{H}_n(T(\mu_\cdot)) = \mathrm{L}_n T(\mu')\,\mathrm{L}_n T(\mu)$$

by (10.2.μ), (8.22), (8.12.3), and (10.2.μ).

(2) Let $P_\cdot$ be a projective resolution of M. The identity $1_{P_\cdot}$ is then a continuation of $1_M : M \to M$, and hence

$$\mathrm{L}_n T(1_M) = \mathrm{H}_n(T(1_{P_\cdot})) = \mathrm{H}_n(1_{T(P_\cdot)}) = 1_{\mathrm{H}_n(T(P_\cdot))} = 1_{\mathrm{L}_n T(M)}$$

by (10.2.μ), (8.22), (8.12.4), and (10.2.M). □

(10.7) **Vanishing.** *The following hold.*

(1) $\mathrm{L}_n T = O$ *for all $n < 0$.*

(2) $\mathrm{L}_n T(M) = O$ *for all $n > \mathrm{pd}_R M$.*

(3) $\mathrm{L}_n T(M) = O$ *for all $n > 0$ if M is projective.*

(4) $\mathrm{L}_n T(O) = O$ *for all $n \in \mathbb{Z}$.*

(5) $\mathrm{L}_n T = O$ *for all $n \neq 0$, if R is a field.*

(6) $\mathrm{L}_n T(M') = O$ *for all $n > 1$, if R is PID, $n > 1$, and M' is finitely generated*[10].

(10.8) **Proof.** Let $P_\cdot$ be a projective resolution of M. If $n < 0$, then $P_n = O$. If $n > \mathrm{pd}_R M$, then we may assume $P_n = O$. In both cases, $\mathrm{L}_n T(M) = \mathrm{H}_n(T(P_\cdot)) = O/O = O$. (1) and (2) have been established, and (3) and (4) follow, while (5) and (6) result from (7.29) and (7.30), respectively. □

(10.9) **Derived over-lining.** Let $\mathfrak{a}$ be a proper ideal in R, and consider the corresponding over-lining functor $\overline{\ }\ : \mathrm{Mod}(R) \to \mathrm{Mod}(\overline{R})$, cf. (2.26), as well as its derivatives $\mathrm{L}_n\overline{\ }$ for $n \in \mathbb{Z}$. In some cases the following holds.

$$\mathrm{L}_n\overline{\ } = O \quad \text{for} \quad n > \mathrm{pd}_R \overline{R} .$$

[10] Later it will turn out that finite generation is superfluous here.

A first case is covered by the next proposition (in which $\mathrm{pd}_R \overline{R} \leq 1$), while another one will be treated in a later section.

(10.10) **Proposition: Derived special over-lining.** *Let $\mathfrak{a} = (x)$ be a proper principal ideal in R, and assume that $x \notin \mathrm{Zero}_R R$. The following then hold for all R–modules M.*
(1) $\mathrm{L}_n{}^{\overline{\ }}(M) = 0 \quad$ *for* $\quad n \geq 2$.
(2) $\mathrm{L}_1{}^{\overline{\ }}(M) = 0 \iff x \notin \mathrm{Zero}_R M$.

(10.11) **Proof.** Choose a projective resolution $P.$ of M and note that $\mathrm{L}_n{}^{\overline{\ }}(M) = \mathrm{H}_n(\overline{P.})$. By (7.19) there is the next short-exact sequence

$$O. \to P. \xrightarrow{x_{P.}} P. \xrightarrow{\rho.} \overline{P.} \to O.$$

in $\mathrm{Com}(R)$. Here, $\rho.$ is the residue map $P_n \to \overline{P_n}$ in each degree, and $x_{P.}$ is the multiplication morphism of (8.5); the latter induces in homology the R–homomorphism $x_{\mathrm{H}_n(P.)}$, confer the definition (8.11.2) of homology homomorphisms. In the induced long-exact sequence, cf. (8.18.1),

$$\begin{aligned}
&\cdots \longrightarrow \mathrm{H}_n(P.) \longrightarrow \mathrm{H}_n(\overline{P.}) \xrightarrow{\delta_n} \\
&\mathrm{H}_{n-1}(P.) \longrightarrow \cdots \\
&\cdots \\
&\cdots \longrightarrow \mathrm{H}_1(P.) \longrightarrow \mathrm{H}_1(\overline{P.}) \xrightarrow{\delta_1} \\
&\mathrm{H}_0(P.) \xrightarrow{x} \mathrm{H}_0(P.) \longrightarrow \mathrm{H}_0(\overline{P.}) \xrightarrow{\delta_0} \\
&\mathrm{H}_{-1}(P.) \longrightarrow \cdots
\end{aligned}$$

we have $\mathrm{H}_n(P.) = 0$ for $n \neq 0$. This yields first that $\mathrm{H}_n(\overline{P.}) = 0$ for $n \geq 2$ and (1) follows. Next it gives an exact sequence

$$O \to \mathrm{H}_1(\overline{P.}) \to \mathrm{H}_0(P.) \xrightarrow{x} \mathrm{H}_0(P.) \xrightarrow{\rho} \mathrm{H}_1(\overline{P.}) \to O,$$

and (2) follows since $\mathrm{H}_0(P.) \cong M$. □

(10.12) **Proposition: Additivity.** *The functor $\mathrm{L}_n T$ is additive for all $n \in \mathbb{N}_0$.*

(10.13) **Proof.** Let $\varphi, \psi \colon M \to N$ be two R–homomorphisms. Choose projective resolutions $P.$ and $Q.$ of M and N, respectively, and continuations $\varphi. \colon P. \to Q.$ and $\psi. \colon P. \to Q.$ of φ and ψ, respectively. The third equality in the next chain is by the definition (8.11.2) of homology homomorphisms:

$$\begin{aligned}
\mathrm{L}_n T(\varphi + \psi) &= \mathrm{H}_n(T(\varphi. + \psi.)) \\
&= \mathrm{H}_n((T(\varphi.) + T(\psi.)) \\
&= \mathrm{H}_n(T.(\varphi.)) + \mathrm{H}_n(T.(\psi.)) \\
&= \mathrm{L}_n T(\varphi) + \mathrm{L}_n T(\psi). \quad \square
\end{aligned}$$

(10.14) **Theorem: Long-exact sequence of derived functors.**
Let $T \colon \mathrm{Mod}(R) \to \mathrm{Mod}(R')$ be an additive covariant functor, and let

$$M. = O \to M \xrightarrow{\mu} M' \xrightarrow{\mu'} M'' \to O$$

be an exact sequence of R–modules. For all $n \in \mathbb{Z}$ there exists then an R'–homomorphism

$$\delta_n^{TM.} \colon \mathrm{L}_n T(M'') \to \mathrm{L}_{n-1} T(M)$$

such that the next sequence is exact:

$$\cdots \longrightarrow \mathrm{L}_{n+1}T(M') \xrightarrow{\mathrm{L}_{n+1}T(\mu')} \mathrm{L}_{n+1}T(M'') \xrightarrow{\delta^{TM\bullet}_{n+1}}$$
$$\mathrm{L}_n T(M) \xrightarrow{\mathrm{L}_n T(\mu)} \mathrm{L}_n T(M') \xrightarrow{\mathrm{L}_n T(\mu')} \mathrm{L}_n T(M'') \xrightarrow{\delta^{TM\bullet}_{n}}$$
$$\mathrm{L}_{n-1}T(M) \xrightarrow{\mathrm{L}_{n-1}T(\mu)} \mathrm{L}_{n-1}T(M') \xrightarrow{\mathrm{L}_{n-1}T(\mu')} \cdots$$

(10.15) **Proof.** The Horse Shoe Lemma (9.19) provides an exact sequence

$$O_\bullet \to P_\bullet \xrightarrow{\mu_\bullet} P'_\bullet \xrightarrow{\mu'_\bullet} P''_\bullet \to O$$

in $\mathrm{Com}(R)$ consisting of projective resolutions of M, M', and M'', respectively, and with $\mu_\bullet$ and $\mu_\bullet{}'$ being continuations of μ and μ', respectively. The exact sequence of R–modules

$$O \to P_n \xrightarrow{\mu_n} P'_n \xrightarrow{\mu'_n} P''_n \to O$$

splits since P''_n is projective, cf. (7.10), and hence, since T is an additive functor, the induced sequence

$$O \to T(P_n) \xrightarrow{T(\mu_n)} T(P'_n) \xrightarrow{T(\mu'_n)} T(P''_n) \to O$$

also splits, cf. (6.42), and hence it is exact. It follows that the sequence

$$O_\bullet \to T(P_\bullet) \xrightarrow{T(\mu_\bullet)} T(P'_\bullet) \xrightarrow{T(\mu'_\bullet)} T(P''_\bullet) \to O_\bullet$$

in $\mathrm{Com}(R')$ is exact. In view of the definition of the derived functor $\mathrm{L}_n T$, cf. (10.2), the desired long-exact sequence results from Theorem (8.16). □

(10.16) **Corollary 1.** *The functor $\mathrm{L}_n T$ is half-exact for all $n > 0$.* □

(10.17) **Corollary 2.** *The functor $\mathrm{L}_0 T$ is right-exact.*

(10.18) **Proof.** $\mathrm{L}_{-1}T = O$ by (10.7.1). □

(10.19) **Corollary 3.** *If $O \to K \to P \to M \to O$ is an exact sequence of R–modules and P is projective, then $\mathrm{L}_n T(M) \cong \mathrm{L}_{n-1}T(K)$ for $n > 1$.*

(10.20) **Proof.** The theorem provides the next exact sequence:

$$\mathrm{L}_n T(P) \to \mathrm{L}_n T(M) \xrightarrow{\delta_n} \mathrm{L}_{n-1}T(K) \to \mathrm{L}_{n-1}T(P).$$

The two outer modules vanish by (10.7.3), so δ_n is an isomorphism. □

(10.21) **Corollary 4.** *Assume that T is right-exact. To any R–module M there exists then an R'-isomorphism $\theta^M : T(M) \to \mathrm{L}_0 T(M)$ such that the next diagram is commutative for any R–homomorphism $\mu : M \to M'$:*

$$\begin{array}{ccc} T(M) & \xrightarrow{T(\mu)} & T(M') \\ {\scriptstyle\theta^M}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\downarrow{\scriptstyle\theta^{M'}} \\ \mathrm{L}_0 T(M) & \xrightarrow[\mathrm{L}_0 T(\mu)]{} & \mathrm{L}_0 T(M') \end{array}$$

Because of this commutative diagram, the isomorphism θ is often said to be *natural,* and the functors T and $\mathrm{L}_0 T$ are said to be *isomorphic.*

(10.22) **Proof.** Let $P.$ be a projective resolution of M, and consider the next diagram.

$$\begin{array}{ccccccc} T(P_1) & \xrightarrow{T(\partial_1^{P.})} & T(P_0) & \longrightarrow & T(M) & \longrightarrow & O \\ \| & & \| & & \vdots\,\theta^M & & \\ T(P_1) & \xrightarrow[T(\partial_1^{P.})]{} & T(P_0) & \longrightarrow & \mathrm{L}_0 T(M) & \longrightarrow & O \end{array}$$

The top row results from the exact sequence $P_1 \to P_0 \to M \to O$ by applying the functor T, and it is exact because T is right-exact, cf. (6.18.2). The lower row is exact as $\mathrm{L}_0 T(M) = \mathrm{H}_0(T(P.)) = \mathrm{Coker}(T(P_1) \to T(P_0))$. As the left square is commutative, the isomorphism θ^M has been provided. That $\theta^{M'} T(\mu) = \mathrm{L}_0 T(\mu)\theta^M$ is left for the reader to verify. □

(10.23) **Corollary 5.** *If T is exact, then $\mathrm{L}_n T = O$ for all $n \geq 1$.*

(10.24) **Proof.** For a given R–module M, use induction on n. Choose an exact sequence $O \to K \to P \to M \to O$ with P projective.

$\underline{n = 1.}$ The theorem and (10.7.3) give an exact sequence:

$$O \to \mathrm{L}_1 T(M) \xrightarrow{\delta_1} \mathrm{L}_0 T(K) \to \mathrm{L}_0 T(P) \to \mathrm{L}_0 T(M) \to O\,.$$

As T is right-exact, Corollary 4 (10.21) yields that the functor $\mathrm{L}_0 T$ is isomorphic to T and thus exact, and it follows from the exact sequence that $\mathrm{L}_1 T(M) = 0$.

$\underline{n > 1.}$ Corollary 3 (10.19) provides the isomorphism $\mathrm{L}_n T(M) \cong \mathrm{L}_{n-1} T(K)$. The latter module vanishes by the inductive hypothesis. □

(10.25) **Corollary 6.** *If T is right-exact and $\mathrm{L}_1 T = O$, then T is exact.* □

(10.26) **Standard Kernel Notation.** Let M be an R–module, let $P.$ be a projective resolution of M, consider the augmented resolution:

$$P. \to M = \cdots \to P_\ell \xrightarrow{\partial_\ell} P_{\ell-1} \xrightarrow{\partial_{\ell-1}} \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M \xrightarrow{\partial_{-1}} O\,,$$

and define the *standard kernel notation* for $P.$ as follows:

$$\mathrm{K}_n(P.) \stackrel{\text{def}}{=} \mathrm{Ker}\,\partial_n\,.$$

Note that $\mathrm{K}_n(P.) = \mathrm{Z}_n(P. \to M)$ for all $n \in \mathbb{Z}$, that is,

$$\mathrm{K}_n(P.) = \begin{cases} O & \text{if } n < -1 \\ M & \text{if } n = -1 \\ \mathrm{Ker}(P_0 \to M) & \text{if } n = 0 \\ \mathrm{Ker}(P_n \to P_{n-1}) & \text{if } n > 0 \end{cases} \tag{10.26.1}$$

Moreover, for each $n \geq 0$ there is the next short-exact sequence:

$$O \to \mathrm{K}_n(P.) \to P_n \to \mathrm{K}_{n-1}(P.) \to O\,. \tag{10.26.2}$$

Namely, the Isomorphism Theorem (2.30) yields the isomorphism $P_n/\mathrm{K}_n(P.) \cong \mathrm{Im}\,\partial_n$, and the latter module is equal $\mathrm{K}_{n-1}(P.)$, since the sequence $P. \to M$ is exact.

The next useful result is an improved version of Corollary 3 (10.19).

(10.27) **Corollary 3 Plus.** *If $P_\bullet$ is a projective resolution of an R–module M, then*

$$\mathrm{L}_{m+1}T(\mathrm{K}_{n-1}(P_\bullet)) \cong \mathrm{L}_m T(\mathrm{K}_n(P_\bullet)) \tag{10.27.1}$$

whenever $n \geq 0$ and $m \geq 1$. (Note that the sum of the indices on either side is $m+n$.) Furthermore, for any $k \geq 0$ there is the next isomorphism:

$$\mathrm{L}_{k+1}T(M) \cong \mathrm{L}_1 T(\mathrm{K}_{k-1}(P_\bullet))\,. \tag{10.27.2}$$

(10.28) **Proof.** The short-exact sequence (10.26.2) induces by (10.14) the next long-exact sequence:

$$\mathrm{L}_{m+1}T(P_n) \to \mathrm{L}_{m+1}T(\mathrm{K}_{n-1}(P_\bullet)) \to \mathrm{L}_m T(\mathrm{K}_n(P_\bullet)) \to \mathrm{L}_m T(P_n)\,.$$

Here, the outer modules vanish as $m \geq 1$, cf. (10.7.3), and the first isomorphism has been established. The second one is obvious for $k = 0$ (as $\mathrm{K}_{-1}(P_\bullet) = M$), and it follows from the next chain in which $1 \leq \ell \leq k+1$:

$$\mathrm{L}_{k+1}T(M) = \mathrm{L}_{k+1}T(\mathrm{K}_{-1}(P_\bullet)) \cong \cdots \cong \mathrm{L}_\ell T(\mathrm{K}_{k-\ell}(P_\bullet)) \cong \cdots \cong \mathrm{L}_1 T(\mathrm{K}_{k-1}(P_\bullet))\,. \qquad \square$$

11. Right Derived of Contravariant Functors

This section describes the theory of right derived functors of an additive contravariant functor. As this is dual to that of left derived functors of an additive covariant functor, no proofs are provided. At the end Ext–modules are presented and applied to projective dimensions.

We start by some useful notation.

(11.1) **Cohomological Notation.** Sometimes it is handy to use superscripts when describing complexes:

$$K^\bullet = \cdots \to K^{n-1} \xrightarrow{\partial_{K^\bullet}^{n-1}} K^n \xrightarrow{\partial_{K^\bullet}^{n}} K^{n+1} \to \cdots$$

which is subject to the condition $\partial^n_{K^\bullet}\partial^{n-1}_{K^\bullet} = 0$ for all $n \in \mathbb{Z}$. We set $\mathrm{B}^n(K^\bullet) \stackrel{\text{def}}{=} \operatorname{Im}\partial^{n-1}_{K^\bullet}$ and $\mathrm{Z}^n(K^\bullet) \stackrel{\text{def}}{=} \operatorname{Ker}\partial^n_{K^\bullet}$ and define the *n'th cohomology* by

$$\mathrm{H}^n(K^\bullet) \stackrel{\text{def}}{=} \mathrm{Z}^n(K^\bullet)/\mathrm{B}^n(K^\bullet)\,.$$

This $K^\bullet$ correspond to the usual complex $K_\bullet$ given by $K_n =^{\text{def}} K^{-n}$ and $\partial^{K_\bullet}_n =^{\text{def}} \partial^{-n}_{K^\bullet}$; note that $\mathrm{H}_n(K_\bullet) = \mathrm{H}^{-n}(K^\bullet)$.

(11.2) **Convention.** Throughout this section

$$\boxed{U\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R') \text{ is an } \textit{additive} \text{ and } \textit{contravariant} \text{ functor.}}$$

The functor U induces a contravariant functor $U\colon \mathrm{Com}(R) \to \mathrm{Com}(R')$: when we use homological notation in $\mathrm{Com}(R)$ and cohomological notation in $\mathrm{Com}(R')$ it is given as follows on an R–complex $K_\bullet$ and an R–morphism $\alpha_\bullet\colon K_\bullet \to L_\bullet$:

$$U(K_\bullet)^n \stackrel{\text{def}}{=} U(K_n) \quad \text{with} \quad \partial^n_{U(K_\bullet)} \stackrel{\text{def}}{=} (-1)^{n+1}U(\partial^{K_\bullet}_{n+1}); \tag{11.2.1}$$

$$U(\alpha_\bullet)\colon U(L_\bullet) \to U(K_\bullet) \quad \text{is defined by} \quad U(\alpha_\bullet)^n \stackrel{\text{def}}{=} U(\alpha_n)\,. \tag{11.2.2}$$

The sign is often handy—but not always needed. It is straightforward to verify that $U(P_\bullet)$ is an R'–complex and that $U(\alpha_\bullet)$ is an R'–morphism. The next result is dual to Theorem (8.23).

(11.3) **Theorem: Homology and Exact Contravariant Module Functor.** *Let $U\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R')$ be an exact additive contravariant functor. For any R–complex $K_\bullet$ and any $n \in \mathbb{Z}$ there exists then an R'–isomorphism*

$$\theta^n_{K_\bullet}\colon \mathrm{H}^n(U(K_\bullet)) \xrightarrow{\cong} U(\mathrm{H}_n(K_\bullet))\,, \tag{11.3.1}$$

such that if, furthermore, $\varphi_\bullet\colon K_\bullet \to L_\bullet$ is any R–morphism, then the next diagram is commutative.

$$\begin{array}{ccc} \mathrm{H}^n(U(L_\bullet)) & \xrightarrow[\cong]{\theta^n_{L_\bullet}} & U(\mathrm{H}_n(L_\bullet)) \\ \Big\downarrow{\scriptstyle \mathrm{H}^n(U(\varphi_\bullet))} & & \Big\downarrow{\scriptstyle U(\mathrm{H}_n(\varphi_\bullet))} \\ \mathrm{H}^n(U(K_\bullet)) & \xrightarrow[\theta^n_{K_\bullet}]{\cong} & U(\mathrm{H}_n(K_\bullet)) \end{array} \tag{11.3.2}$$

□

(11.4) **Definition.** For $n \in \mathbb{Z}$ the *n'th right derived functor* $\mathrm{R}^n U\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R')$ of U is defined as follows:

(M) If M is an R–module, then choose a projective resolution $P_\bullet$ of M, and set

$$\mathrm{R}^n U(M) \stackrel{\text{def}}{=} \mathrm{H}^n(U(P_\bullet)).$$

(μ) If $\mu\colon M \to M'$ is an R–homomorphism, then choose projective resolutions $P_\bullet$ and $P'_\bullet$ of M and M', respectively, choose a continuation $\mu_\bullet\colon P_\bullet \to P'_\bullet$ of μ, and set

$$\mathrm{R}^n U(\mu) \stackrel{\text{def}}{=} \mathrm{H}^n(U(\mu_\bullet))\colon \mathrm{H}^n(U(P'_\bullet)) \to \mathrm{H}^n(U(P_\bullet)).$$

(11.5) **Theorem: Uniqueness.** *The following hold.*

(M) *For any R–module M, the R'–module $\mathrm{R}^n U(M)$ is uniquely determined up to unique isomorphism.*

(μ) *For any R–homomorphism $\mu\colon M \to M'$, the induced R'–homomorphism $\mathrm{R}^n U(\mu)$ is uniquely determined up to unique isomorphism.* □

(11.6) **Theorem: Functoriality.** *For all $n \in \mathbb{Z}$ the derived functor $\mathrm{R}^n U$ is a functor:*

(1) $\mathrm{R}^n U(\mu'\mu) = \mathrm{R}^n U(\mu)\,\mathrm{R}^n U(\mu')$ *for all $n \in \mathbb{Z}$ when $\mu\colon M \to M'$ and $\mu'\colon M' \to M''$ are R–homomorphisms.*

(2) $\mathrm{R}^n U(1_M) = 1_{\mathrm{R}^n U(M)}$ *for all $n \in \mathbb{Z}$ when M is an R–module.* □

(11.7) **Vanishing.** *The following hold.*

(1) $\mathrm{R}^n U = O$ *for all $n < 0$.*

(2) $\mathrm{R}^n U(M) = O$ *for all $n > \mathrm{pd}_R M$.*

(3) $\mathrm{R}^n U(M) = O$ *for all $n > 0$ if M is projective.*

(4) $\mathrm{R}^n U(O) = O$ *for all $n \in \mathbb{Z}$.*

(5) $\mathrm{R}^n U = O$ *for all $n \neq 0$, if R is a field.*

(6) $\mathrm{R}^n U(M') = O$ *for all $n > 1$, if R is PID, $n > 1$, and M' is finitely generated*[11].

(11.8) **Proposition: Additivity.** *The functor $\mathrm{R}^n U$ is additive for all $n \in \mathbb{N}_0$.* □

(11.9) **Theorem: Long-exact sequence of derived functors.**
Let $U\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R')$ be an additive contravariant functor, and let

$$M^\bullet = O \to M \xrightarrow{\mu} M' \xrightarrow{\mu'} M'' \to O$$

be an exact sequence of R–modules. For all $n \in \mathbb{Z}$ there exists then an R'–homomorphism

$$\delta^n_{UM^\bullet}\colon \mathrm{R}^n U(M) \to \mathrm{R}^{n+1} U(M'')$$

such that the next sequence is exact:

$$\cdots \longrightarrow \mathrm{R}^{n-1} U(M') \xrightarrow{\mathrm{R}^{n-1} U(\mu)} \mathrm{R}^{n-1} U(M) \xrightarrow{\delta^{n-1}_{UM^\bullet}}$$

$$\mathrm{R}^n U(M'') \xrightarrow{\mathrm{R}^n U(\mu')} \mathrm{R}^n U(M') \xrightarrow{\mathrm{R}^n U(\mu)} \mathrm{R}^n U(M) \xrightarrow{\delta^n_{UM^\bullet}}$$

$$\mathrm{R}^{n+1} U(M'') \xrightarrow{\mathrm{R}^{n+1} U(\mu')} \mathrm{R}^{n+1} U(M') \xrightarrow{\mathrm{R}^{n+1} U(\mu)} \cdots$$ □

(11.10) **Corollary 1.** *The functor $\mathrm{R}^n U$ is half-exact for all $n > 0$.* □

[11] Later it will turn out that finite generation is superfluous here.

(11.11) **Corollary 2.** *The functor* $\mathrm{R}^0 U$ *is left-exact.* □

(11.12) **Corollary 3.** *If* $O \to K \to P \to M \to O$ *is an exact sequence of* R*–modules and* P *is projective, then* $\mathrm{R}^n U(M) \cong \mathrm{R}^{n-1} U(K)$ *for* $n > 1$. □

(11.13) **Corollary 3 Plus.** *If* $P_\bullet$ *is a projective resolution of an* R*–module* M*, and the standard kernel notation of* (10.26) *is used, then*

$$\mathrm{R}^{m+1} U(\mathrm{K}_{n-1}(P_\bullet)) \cong \mathrm{R}^m U(\mathrm{K}_n(P_\bullet)) \tag{11.13.1}$$

whenever $n \geq 0$ *and* $m \geq 1$. *(Note that the sum of the indices on either side is* $m+n$*.) Furthermore, for any* $k \geq 0$ *there is the next isomorphism:*

$$\mathrm{R}^{k+1} U(M) \cong \mathrm{R}^1 U(\mathrm{K}_{k-1}(P_\bullet)) . \tag{11.13.2}$$

□

(11.14) **Corollary 4.** *If* U *is left-exact, then the functors* U *and* $\mathrm{R}^0 U$ *are isomorphic (in the sense of* (10.21)*).* □

(11.15) **Corollary 5.** *If* U *is exact, then* $\mathrm{R}^n U = O$ *for all* $n \geq 1$. □

(11.16) **Corollary 6.** *If* U *is left-exact and* $\mathrm{R}^1 U = O$*, then* U *is exact.* □

(11.17) **Ext–modules.** Let N be an R-module. For any $n \in \mathbb{Z}$ we set

$$\mathrm{Ext}^n_R(-, N) \stackrel{\text{def}}{=} \mathrm{R}^n \mathrm{Hom}_R(-, N) .$$

These are called the *contravariant Ext–functors*. They are half-exact, additive, and contravariant. The functors $\mathrm{Ext}^0_R(-, N)$ and $\mathrm{Hom}_R(-, N)$ are isomorphic (in the sense of (10.23)). Ext will be studied in further detail in Section 16.

(11.18) **Long-exact Sequence for Contravariant Ext.** Let H be an R–module, and let $M^\bullet$ be the exact sequence $O \to M \stackrel{\mu}{\to} M' \stackrel{\mu'}{\to} M'' \to O$ of R–modules. There is then an induced long-exact sequence:

$$O \to \mathrm{Hom}_R(M'', H) \xrightarrow{\mathrm{Hom}_R(\mu', H)} \mathrm{Hom}_R(M', H) \xrightarrow{\mathrm{Hom}_R(\mu, H)} \mathrm{Hom}_R(M, H) \xrightarrow{\delta^0_{M^\bullet H}}$$

$$\mathrm{Ext}^1_R(M'', H) \xrightarrow{\mathrm{Ext}^1_R(\mu', H)} \mathrm{Ext}^1_R(M', H) \xrightarrow{\mathrm{Ext}^1_R(\mu, H)} \cdots$$

$$\cdots \longrightarrow \mathrm{Ext}^{n-1}_R(M', H) \xrightarrow{\mathrm{Ext}^{n-1}_R(\mu, H)} \mathrm{Ext}^{n-1}_R(M, H) \xrightarrow{\delta^{n-1}_{M^\bullet H}}$$

$$\mathrm{Ext}^n_R(M'', H) \xrightarrow{\mathrm{Ext}^n_R(\mu', H)} \mathrm{Ext}^n_R(M', H) \xrightarrow{\mathrm{Ext}^n_R(\mu, H)} \mathrm{Ext}^n_R(M, H) \xrightarrow{\delta^n_{M^\bullet H}}$$

$$\mathrm{Ext}^{n+1}_R(M'', H) \xrightarrow{\mathrm{Ext}^{n+1}_R(\mu', H)} \mathrm{Ext}^{n+1}_R(M', H) \xrightarrow{\mathrm{Ext}^{n+1}_R(\mu, H)} \cdots$$

(11.19) **PDT $\stackrel{\text{def}}{=}$ Projective Dimension Theorem.** *The following are equivalent for any* R*–module* M *and any* $n \in \mathbb{N}_0$.

(*i*) $\mathrm{pd}_R M \leq n$;
(*ii*) $\mathrm{Ext}^m_R(M, H) = O$ *for all* $m > n$ *and all* R*–modules* H;
(*iii*) $\mathrm{Ext}^{n+1}_R(M, H) = O$ *for and all* R*–modules* H;
(*iv*) *If* $P_\bullet$ *is a projective resolution of* M*, then* $\mathrm{Ext}^{n+1}_R(M, \mathrm{K}_n(P_\bullet)) = O$.
(*v*) *If* $P_\bullet$ *is a projective resolution of* M*, then* $\mathrm{K}_{n-1}(P_\bullet)$ *is projective.*

In items (iv) *and* (v) *the standard kernel notation of* (10.26) *is used.*

(11.20) **Proof.** The implications "$(ii) \Longrightarrow (iii)$", "$(iii) \Longrightarrow (iv)$", and "$(v) \Longrightarrow (i)$" are obvious, while "$(i) \Longrightarrow (ii)$" follows from the vanishing result (11.7).

"$(iv) \Longrightarrow (v)$" follows from the next lemma, since (11.13) yields

$$\operatorname{Ext}^1_R(\mathrm{K}_{n-1}(P.), \mathrm{K}_n(P.)) \cong \operatorname{Ext}^{n+1}_R(M, \mathrm{K}_n(P.)) \,. \quad \square$$

(11.21) **Lemma.** *Let* $O \to K \xrightarrow{\alpha} P \to K' \to O$ *be an exact sequence, and assume that* P *is a projective module. The following then holds:*

$$K' \text{ is projective} \iff \operatorname{Ext}^1_R(K', K) = O \,.$$

(11.22) **Proof.** The implication " $\Longrightarrow$ " is by the vanishing result (11.7). To prove the opposite one, assume that $\operatorname{Ext}^1_R(K', K) = O$, and consider the exact sequence

$$\operatorname{Hom}_R(P, K) \xrightarrow{\operatorname{Hom}_R(\alpha, K)} \operatorname{Hom}_R(K, K) \xrightarrow{\delta^1} \operatorname{Ext}^1_R(K', K)$$

from (11.18). Thus, our assumption yields that $\operatorname{Hom}_R(\alpha, K)$ is surjective. Choose $\sigma \in \operatorname{Hom}_R(P, K)$ such that $\operatorname{Hom}_R(\alpha, K)(\sigma) = 1_K$, that is, $\sigma\alpha = 1_K$. By Theorem (6.45) the short-exact sequence splits, and it follows that K' is projective, cf. (7.10), as desired. $\square$

(11.23) **PDC** $\stackrel{\text{def}}{=}$ **Projective Dimension Corollary.** *If* M *is an* R*–module, then*

$$\operatorname{pd}_R M = \sup\{\, n \in \mathbb{N}_0 \mid \operatorname{Ext}^n_R(M, H) \neq O \text{ for some } H \,\}$$

This number is in $\mathbb{N}_0^* \stackrel{\text{def}}{=} \{-\infty\} \cup \mathbb{N}_0 \cup \{\infty\}$. $\square$

12. Injective Modules

Recall that the contravariant functor $\operatorname{Hom}_R(-,H)$ is left-exact for all R–modules H.

(12.1) **Injective modules.** An R–module H is said to be *injective* exactly when the functor $\operatorname{Hom}_R(-,H)$ is exact, that is, if and only if the induced homomorphism

$$\operatorname{Hom}_R(\varphi,H)\colon \operatorname{Hom}_R(N,H) \to \operatorname{Hom}_R(M,H)$$

is *surjective* for any *injective* R–homomorphism $\varphi\colon M \to N$, cf. (6.30).

(12.2) **Injectivity diagram.** Injectivity of the R–module H is often explained by the diagram:

$$\begin{array}{ccccc} O & \longrightarrow & M & \longrightarrow & N \\ & & \downarrow & \swarrow_{\exists} & \\ & & H & & \end{array}$$

This means: If the solid arrows are given and the row is exact, then the dotted arrow exists such the triangle commutes.

(12.3) **Not injective.** The $\mathbb{Z}$–module $\mathbb{Z}/(2)$ is not injective, cf. (6.32).

The next three auxiliary results correspond to the results on projectivity (7.6), (7.10), and (7.12), and their proofs are dual, that is, they are obtained by reversing the arrows.

(12.4) **Isomorphic injectives.** If $\eta\colon H \xrightarrow{\cong} H'$ is an isomorphism of R–modules, then H is an injective module if and only if H' is so □

(12.5) **Lemma: Injective submodule splits off.** *If $\varphi\colon I \to N$ is an injective R–homomorphism with the source I injective, then the exact sequence*

$$O \to I \xrightarrow{\varphi} M \twoheadrightarrow M/\operatorname{Im}\varphi \to O$$

splits, and I is isomorphic to a direct summand in M. □

(12.6) **Lemma: Direct sum of injectives.** *If I and J are R–modules, then the following holds.*

$$I \oplus J \text{ is injective} \iff I \text{ and } J \text{ are injective.}$$ □

The next theorem is an important tool in the theory of injective modules.

(12.7) **Baer's Criterion.** *The following are equivalent for any R–module H.*

(*i*) *The module H is injective;*

(*ii*) *The restriction homomorphism $\operatorname{Hom}_R(R,H) \to \operatorname{Hom}_R(\mathfrak{a},H)$ is surjective for any ideal $\mathfrak{a}$ in R, that is,*

$$\begin{array}{ccccc} O & \longrightarrow & \mathfrak{a} & \hookrightarrow & R \\ & & \downarrow & \swarrow_{\exists} & \\ & & H & & \end{array} \quad ;$$

(*iii*) *To any ideal $\mathfrak{a}$ in R and any R–homomorphism $\alpha\colon \mathfrak{a} \to H$ there exists $h_0 \in H$ such that $\alpha(a) = ah_0$ for all $a \in \mathfrak{a}$.*

(12.8) **Proof.** The implication "(*i*) $\Longrightarrow$ (*ii*)" is by the definition of injectivity.

"$(ii) \Longrightarrow (iii)$": Let $\alpha \in \operatorname{Hom}_R(\mathfrak{a}, H)$ be given, and choose, by the assumption, $\beta\colon R \to H$ such that the restriction $\beta|_{\mathfrak{a}}$ is equal α, and set $h_0 =^{\text{def}} \beta(1)$. For any $a \in \mathfrak{a}$ we have then $\alpha(a) = \beta(a) = a\beta(1) = ah_0$ as desired.

"$(iii) \Longrightarrow (i)$": We assume that (iii) holds and want to prove that $\operatorname{Hom}_R(-, H)$ is exact, and to test this it is sufficient to consider a special short-exact sequence

$$O \to K \hookrightarrow M \twoheadrightarrow M/K \to O$$

and prove that the induced sequence

$$\operatorname{Hom}_R(M, H) \to \operatorname{Hom}_R(K, H) \to O$$

is exact. The latter is, to any R–homomorphism $\kappa\colon K \to H$ there exists an R–homomorphism $\mu\colon M \to H$ such that the restriction $\mu|_K$ equals κ. To achieve this, let $\mathbb{X}$ denote the set of pairs (L, λ) such that

- L is submodule of M and $L \supseteq K$;
- $\lambda\colon L \to H$ is an R–homomorphism with $\lambda|_K = \kappa$.

We will prove that $\mathbb{X}$ contains an element of the form (M, μ).

This set $\mathbb{X}$ is (partially) ordered by:

$$(L, \lambda) \preceq (L', \lambda') \quad \overset{\text{def}}{\Longleftrightarrow} \quad L \subseteq L' \quad \text{and} \quad \lambda'|_L = \lambda\,.$$

We will prove that $\mathbb{X}$ hereby is inductively ordered, cf. (7.29). If $\mathbb{Y}$ is a totally ordered subset of $\mathbb{X}$, then it has a majorant, namely $(\widetilde{L}, \widetilde{\lambda})$ where $\widetilde{L}$ is the union of the submodules L occurring in $\mathbb{Y}$ (and this is a submodule of M as $\mathbb{Y}$ is totally ordered), and where $\widetilde{\lambda}$ is defined by $\widetilde{\lambda}(x) = \lambda(x)$ whenever $(L, \lambda) \in \mathbb{Y}$ is such that $x \in L$ (and $\widetilde{\lambda}$ is hereby well-defined due to the definition of the ordering $\preceq$).

Thus, $\mathbb{X}$ is inductively ordered, so Zorn's Lemma gives a maximal element (N, ν) in $\mathbb{X}$.

If $N = M$ then $\nu\colon M \to H$ is the desired extension of κ.

Next assume that $N \subset M$ [and look for a contradiction]. Choose $m_0 \in M \setminus N$, and set $\mathfrak{a} =^{\text{def}} \{a \in R \mid am_0 \in N\}$, which is an ideal in R. Define $\alpha\colon \mathfrak{a} \to H$ by $\alpha(a) =^{\text{def}} \nu(am_0)$. Since α is an R–homomorphism, our assumption (iii) yields an element $h_0 \in H$ such that $\alpha(a) = ah_0$ for all $a \in \mathfrak{a}$. Set $N' =^{\text{def}} N + Rm_0$, which is a submodule of M, and define $\nu'\colon N' \to H$ by $\nu'(n + rm_0) =^{\text{def}} \nu(n) + rh_0$ for $n \in N$ and $r \in R$. To see that this well-defines ν', assume that $n_1 + r_1m_0 = n_2 + r_2m_0$, and conclude that $(r_1 - r_2)m_0 = n_2 - n_1 \in N$, so $r_1 - r_2 \in \mathfrak{a}$, and hence

$$(r_1 - r_2)h_0 = \alpha(r_1 - r_2) = \nu\big((r_1 - r_2)m_0\big) = \nu(n_2 - n_1) = \nu(n_2) - \nu(n_1)\,;$$

thus $\nu(n_1) + r_1h_0 = \nu(n_2) + r_2h_0$ (as required for the well-definition of ν'). Note that $\nu'|_N = \nu$, and hence $\nu'|_K = \kappa$.

Now it has been verified that (N', ν') belongs to $\mathbb{X}$. Since $(N, \nu) \preceq (N', \nu')$ and $(N, \nu) \neq (N', \nu')$, we have obtained the desired contradiction to the maximality of (N, ν). □

(12.9) **Injectives over a field.** If R is a field, then O and R are the only ideals, so (ii) i Baer's Criterion is always fulfilled, and all modules are injective.

(12.10) **All exact sequences split.** The following are equivalent.

- (i) All modules are injective.
- (ii) All modules are projective.
- (iii) All additive functors are exact.
- (iv) All short-exact sequences split.

Indeed, "$(i) \Longrightarrow (iv)$" and "$(ii) \Longrightarrow (iv)$" by (12.5) and (7.10), respectively. Furthermore, "$(iv) \Longrightarrow (iii)$" since additive functors preserve splitting sequences (6.42). Finally, "$(iii) \Longrightarrow (i)$" and "$(iii) \Longrightarrow (i)$" are obvious.

Later, in (18.36), we will prove that they are equivalent to

(v) The ring R is a finite product of fields.

These five conditions are equivalent also over non-commutative rings, when the fifth one is replaced by "The ring is a finite product of matrix algebras over division rings." Such rings are said to be *semisimple*.

(12.11) **Isomorphic Functors.** Let $F, G\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R')$ be two functor of the same variance. They are said to be *isomorphic*, exactly when there for each R–module M is an R'-isomorphism $\theta_M\colon F(M) \to G(M)$ such that the next diagram is commutative for each R–homomorphism $\varphi\colon M \to N$:

$$\begin{array}{ccc} F(M) & \xrightarrow[\cong]{\theta_M} & G(M) \\ {\scriptstyle F(\varphi)}\downarrow & & \downarrow{\scriptstyle G(\varphi)} \\ F(N) & \xrightarrow[\theta_N]{\cong} & G(N) \end{array}$$

The diagram is here shown in the covariant case; in the contravariant one, the vertical arrows should be reversed.

If F and G are isomorphic, and F is exact (respectively, left-exact, right-exact, half-exact), then G is the same.

If T is a right-exact covariant functor, then it is isomorphic to $\mathrm{L}_0 T$, cf. (10.21). If U is a left-exact contravariant functor, then it is isomorphic to $\mathrm{R}^0 U$, cf. (11.14).

(12.12) **Theorem: Swap isomorphism**[12]. *Let K, L, and M be R–modules. There is then an isomorphism*

$$\Omega_{KLM}\colon \mathrm{Hom}_R(K, \mathrm{Hom}_R(L, M)) \to \mathrm{Hom}_R(L, \mathrm{Hom}_R(K, M))$$

which is natural in K, in L, and in M, that is, there are the next three pairs of isomorphic functors:

$$\mathrm{Hom}_R(-, \mathrm{Hom}_R(L, M)) \cong \mathrm{Hom}_R(L, \mathrm{Hom}_R(-, M)) \tag{12.12.1}$$

$$\mathrm{Hom}_R(K, \mathrm{Hom}_R(-, M)) \cong \mathrm{Hom}_R(-, \mathrm{Hom}_R(K, M)) \tag{12.12.2}$$

$$\mathrm{Hom}_R(K, \mathrm{Hom}_R(L, -)) \cong \mathrm{Hom}_R(L, \mathrm{Hom}_R(K, -)) \tag{12.12.3}$$

(12.13) **Proof.** Let us start by analyzing the element of the source: Assume that κ belongs to it, that is, κ is an R–homomorphism $\kappa\colon K \to \mathrm{Hom}_R(L, M)$, so it associates to any $k \in K$ an element $\kappa(k)$ in $\mathrm{Hom}_R(L, M)$, that is, an R–homomorphism $\kappa(k)\colon L \to M$, and this sends each $\ell \in L$ into the element $(\kappa(k))(\ell) \in M$.

To determine $\Omega_{KLM}\colon \mathrm{Hom}_R(K, \mathrm{Hom}_R(L, M)) \to \mathrm{Hom}_R(L, \mathrm{Hom}_R(K, M))$ we want for any $\kappa \in \mathrm{Hom}_R(K, \mathrm{Hom}_R(L, M))$ to define $\Omega_{KLM}(\kappa)$ in $\mathrm{Hom}_R(L, \mathrm{Hom}_R(K, M))$, that is, the R–homomorphism $\Omega_{KLM}(\kappa)\colon L \to \mathrm{Hom}_R(K, M)$, and this is done by defining for each $\ell \in L$ the R–homomorphism $(\Omega_{KLM}(\kappa))(\ell)\colon K \to M$ by the assignment

$$\big((\Omega_{KLM}(\kappa))(\ell)\big)(k) \stackrel{\mathrm{def}}{=} (\kappa(k))(\ell)$$

[12] The name "swap" was introduced in the book *Gorenstein Dimensions* (1990) by Lars Winther Christensen.

for $k \in K$. It is seen by inspection that Ω_{KLM} is an R–homomorphism. Thus we have also the R–homomorphism

$$\Omega_{LKM}\colon \operatorname{Hom}_R(L, \operatorname{Hom}_R(K, M)) \longrightarrow \operatorname{Hom}_R(K, \operatorname{Hom}_R(L, M))\,.$$

In is seen by inspection that

$$\Omega_{LKM}\Omega_{KLM} = 1_{\operatorname{Hom}_R(K,\operatorname{Hom}_R(L,M))} \text{ and } \Omega_{KLM}\Omega_{LKM} = 1_{\operatorname{Hom}_R(L,\operatorname{Hom}_R(K,M))}\,.$$

Thus, Ω_{KLM} is an R–isomorphism.

That Ω_{KLM} is natural in K means that for any R–homomorphism $\varphi\colon K \to K'$ the next diagram is commutative:

$$\begin{array}{ccc} \operatorname{Hom}_R(K', \operatorname{Hom}_R(L, M)) & \xrightarrow{\Omega_{K'LM}} & \operatorname{Hom}_R(L, \operatorname{Hom}_R(K', M)) \\ \Big\downarrow{\scriptstyle \operatorname{Hom}_R(\varphi, \operatorname{Hom}_R(L,M))} & & \Big\downarrow{\scriptstyle \operatorname{Hom}_R(L, \operatorname{Hom}_R(\varphi, M))} \\ \operatorname{Hom}_R(K, \operatorname{Hom}_R(L, M)) & \xrightarrow[\Omega_{KLM}]{} & \operatorname{Hom}_R(L, \operatorname{Hom}_R(K, M))\,. \end{array}$$

An inspection shows that this indeed is the case. Naturality in L follows, and naturality in M can be verified. □

(12.14) **Corollary: Hom(P, I) is projective.** *If P is a projective R–module, and I is an injective R–module, then $\operatorname{Hom}_R(P, I)$ is an injective R–module.*

(12.15) **Proof.** By the theorem there is the following isomorphism of functors:

$$\operatorname{Hom}_R(-, \operatorname{Hom}_R(P, I)) \cong \operatorname{Hom}_R(P, \operatorname{Hom}_R(-, I))\,.$$

The right hand functor is exact as it the composite of the exact functors $\operatorname{Hom}_R(-, I)$ and $\operatorname{Hom}_R(P, -)$. Thus the left hand functor is exact, as desired. □

(12.16) **Divisible modules.** Let R be an integral domain, and let D be an R–module. We say that D is *divisible*, exactly when the multiplication map $r_D\colon D \to D$, $x \mapsto rx$ is surjective for all $r \in R \setminus O$. Thus,

$$D \text{ is divisible} \iff \forall r \in R \setminus O \forall d \in D \exists x \in D : rx = d\,.$$

(12.17) **Theorem: Injectivity vs. Divisibility.** *Let D be a module over an integral domain R, and let K be a submodule of D. The following then hold.*

(1) *If D is injective, then D is divisible.*
(2) *If R is PID and D is divisible, then D is injective.*
(3) *If D is divisible, then D/K is divisible.*
(4) *If R is PID and D is injective, then D/K is injective.*

(12.18) **Proof.** (1) Assume that D is injective, and let $r \in R \setminus O$ and $d \in D$ be given. By setting $\alpha(ar) =^{\text{def}} ad$ for $a \in R$ an R–homomorphism $\alpha\colon (r) \to D$ has been well-defined (as $r \neq 0$ and R is a domain). By (iii) in Baer's Criterion (12.7) we get $d_0 \in D$ such that $\alpha(ar) = (ar)d_0$ for all $a \in R$. With $x =^{\text{def}} d_0$ we have $rx = (1r)d_0 = \alpha(1r) = 1d = d$, as desired.

(2) Assuming that D is divisible, we will verify condition (iii) in Baer's Criterion (12.7). Let $\alpha\colon \mathfrak{a} \to D$ be any R–homomorphism with an ideal as source, and search for $d_0 \in D$ such that $\alpha(a) = ad_0$ for all $a \in \mathfrak{a}$. As R is PID we can choose $a_0 \in R$ such that $\mathfrak{a} = (a_0)$. If $a_0 = 0$ then $d_0 = 0$ will do the job. Therefore assume that $a_0 \neq 0$ and choose by the

divisibility of D an element $d_0 \in D$ such that $\alpha(a_0) = a_0 d_0$. Now, for $a \in \mathfrak{a}$ choose $r \in R$ such that $a = ra_0$, and obtain the desired assertion:

$$\alpha(a) = \alpha(ra_0) = r\alpha(a_0) = r(a_0 d_0) = (ra_0)d_0 = ad_0 .$$

(3) is obvious.

(4) If D is injective, then it is divisible by (1). Thus, D/K is divisible by (3) and hence injective by (2). □

(12.19) **Exercise: Field of Fractions.** Let R be a domain. Prove that its field of fractions B is an injective R–module.

(12.20) **Faithfully injective modules.** An R–module I is said to be *faithfully injective*, exactly when it is injective and $\operatorname{Hom}_R(M, I) \neq 0$ whenever M is a non-zero R–module.

(12.21) **Faithfully injective group.** The $\mathbb{Z}$–module $\Delta =^{\text{def}} \mathbb{Q}/\mathbb{Z}$ is faithfully injective. Indeed, the group $\mathbb{Q}$ of rational numbers is a divisible $\mathbb{Z}$–module, and $\mathbb{Z}$ is a submodule of it. Thus, the group Δ is divisible by (12.17.3) and hence injective by (12.17.4). If M is a non-zero $\mathbb{Z}$–module, choose a non-zero $m \in M$, and let $n \in \mathbb{N} \cup \infty$ be the order of m. If n is finite, then a $\mathbb{Z}$–homomorphism $\varphi\colon \mathbb{Z}m \to \Delta$ is well-defined by $\varphi(zm) =^{\text{def}} [\frac{z}{n}]_{\mathbb{Z}}$; if n is infinite, then a $\mathbb{Z}$–homomorphism $\varphi\colon \mathbb{Z}m \to \Delta$ is well-defined by $\varphi(zm) =^{\text{def}} [\frac{z}{2}]_{\mathbb{Z}}$. In both cases we have a non-zero $\mathbb{Z}$–homomorphism $\varphi\colon \mathbb{Z}m \to \Delta$. Since Δ is an injective $\mathbb{Z}$–module, φ can be extended to a $\mathbb{Z}$–homomorphism $\widetilde{\varphi}\colon M \to \Delta$. Thus, $\operatorname{Hom}_{\mathbb{Z}}(M, \Delta) \neq O$ as desired.

(12.22) **Induced structure.** As any R–module is a $\mathbb{Z}$–module, and any R–homomorphism is a $\mathbb{Z}$–homomorphism, we have $\operatorname{Mod}(R) \subseteq \operatorname{Mod}(\mathbb{Z})$. (However, if M and N are R–modules, then $\operatorname{Hom}_R(M, N)$ might be a proper subset of $\operatorname{Hom}_{\mathbb{Z}}(M, N)$.)

If M is an R–module, and G is a $\mathbb{Z}$–module, then the $\mathbb{Z}$–module $\operatorname{Hom}_{\mathbb{Z}}(M, G)$ becomes an R–module with the multiplication given as follows:

For $r \in R$ and $\mu \in \operatorname{Hom}_{\mathbb{Z}}(M, G)$ the product $r\mu\colon M \to G$ is defined by

$$(r\mu)(m) \stackrel{\text{def}}{=} \mu(rm) \quad \text{for} \quad m \in M .$$

A calculation shows that this endows $\operatorname{Hom}_{\mathbb{Z}}(M, G)$ with a structure of an R–module.

The definition above does not require the underlying ring to be commutative. Indeed, let Λ be any (not necessarily commutative) ring. A commutative group $(M, +)$ is then said to be a Λ–module, exactly when there is a multiplication $\Lambda \times M \to M$ such that (M1), (M2), and (M3) of (2.1) hold when $r, r' \in R$ are replaced by $\lambda, \lambda' \in \Lambda$. (Some refer to this as M is a *left* Λ–module.) The *opposite ring* Λ° is the group $(\Lambda, +)$ equipped with the opposite multiplication $\Lambda \times \Lambda \to \Lambda$, $(\lambda, \lambda') \mapsto \lambda'\lambda$. If N is an Λ°–module, then the associativity condition (M1) of (2.1) can be written $\lambda(\lambda' n) = (\lambda'\lambda)n$ (In this case some write the multiplication as $(\lambda, n) \mapsto n\lambda$; thus (M1) becomes $n(\lambda\lambda') = (n\lambda)\lambda'$, and some refer to N as a right Λ–module.)

If M is a Λ–module and G is a $\mathbb{Z}$–module, then $\operatorname{Hom}_{\mathbb{Z}}(M, G)$ is a Λ°–module.

(12.23) **Proposition: Swap isomorphism.** *Let K and L be R–modules and let G be an $\mathbb{Z}$–module. There is then an isomorphism*

$$\operatorname{Hom}_R(K, \operatorname{Hom}_{\mathbb{Z}}(L, G)) \cong \operatorname{Hom}_R(L, \operatorname{Hom}_{\mathbb{Z}}(K, G))$$

of R–modules, and it is natural in K, in L, and in G.

(12.24) **Proof.** This is a restriction of $\operatorname{Hom}_{\mathbb{Z}}(K, \operatorname{Hom}_{\mathbb{Z}}(L, G)) \cong \operatorname{Hom}_{\mathbb{Z}}(L, \operatorname{Hom}_{\mathbb{Z}}(K, G))$ from the Theorem (12.12) □

(12.25) **Pontryagin Module.** The *Pontryagin module* of R is $\Delta_R =^{\text{def}} \operatorname{Hom}_{\mathbb{Z}}(R,\Delta)$ where $\Delta = \mathbb{Q}/\mathbb{Z}$ is the faithfully injective $\mathbb{Z}$–module introduced in (12.21). By (12.22), Δ_R is an R–module.

The Pontryagin module Δ_R is *faithfully injective*, because of the next isomorphisms:

$$\operatorname{Hom}_R(-,\Delta_R) \cong \operatorname{Hom}_R(R,\operatorname{Hom}_{\mathbb{Z}}(-,\Delta)) \cong \operatorname{Hom}_{\mathbb{Z}}(-,\Delta)\,,$$

where the former results from (12.23) and the latter from (4.4).

(12.26) **Pontryagin Duality.** The *Pontryagin duality functor* is the contravariant exact functor $(-)^\vee =^{\text{def}} \operatorname{Hom}_R(-,\Delta_R)$.

For an R–module M the module $M^\vee = \operatorname{Hom}_R(M,\Delta)_R$ is said to be the *Pontryagin dual* R–module of M. If $\alpha\colon M \to N$ is an R–homomorphism, then the induced one $\alpha^\vee\colon N^\vee \to M^\vee$ is the *Pontryagin dual* R–homomorphism. Note that the faithful injectivity of Δ_R yield the next:

(12.26.1) $\quad M \neq O \iff M^\vee \neq O\,;$

(12.26.2) $\quad \alpha$ is injective $\iff \alpha^\vee$ is surjective;

(12.26.3) $\quad \alpha$ is surjective $\iff \alpha^\vee$ is injective;

(12.26.4) $\quad \alpha$ is bijective $\iff \alpha^\vee$ is bijective.

(12.27) **ExP** $\stackrel{\text{def}}{=}$ **Exactness Principle.** *The following are equivalent for R–complex $K_\cdot$.*

(i) $K_\cdot$ is exact;

(ii) $\operatorname{Hom}_R(K_\cdot, I)$ is exact for all injective R–modules I;

(iii) $(K_\cdot)^\vee$ is exact.

(12.28) **Proof.** "$(i) \Longrightarrow (ii)$" and "$(iii) \Longrightarrow (i)$" follow from (11.3), while "$(ii) \Longrightarrow (iii)$" is obvious. □

(12.29) **Evaluation Homomorphism.** The map $\varepsilon^M_R\colon M \to M^{\vee\vee}$ is given as follows: For $m \in M$, the image $\varepsilon^M_R(m) \in M^{\vee\vee} = \operatorname{Hom}_R(M^\vee,\Delta_R)$, that is, the R–homomorphism $\varepsilon^M_R(m)\colon M^\vee \to \Delta_R$, is defined by $\big(\varepsilon^M_R(m)\big)(\mu) =^{\text{def}} \mu(m)$ for $\mu \in M^\vee = \operatorname{Hom}_R(M,\Delta_R)$.

(12.30) **Injectivity.** *The evaluation homomorphism $\varepsilon^M_R\colon M \to M^{\vee\vee}$ is injective.*

(12.31) **Proof.** For $m \in M \setminus O$ we have $Rm \neq O$, and hence $\operatorname{Hom}_R(Rm,\Delta_R) \neq O$, so choose an R–homomorphism $\rho\colon Rm \to \Delta_R$ which is non-zero, that is, $\rho(m) \neq 0$. By the injectivity of Δ_R we can extend ρ to $\mu\colon M \to \Delta_R$. Thus, $\mu(m) = \rho(m) \neq 0$ as desired. □

(12.32) **Embedding into an injective module.** *Every R–module M can be embedded into an injective R–module I.*

(12.33) **Proof.** Choose a surjective R–homomorphism $\pi_M\colon P \to M^\vee$ with projective source. The induced homomorphism $M^{\vee\vee} \to P^\vee$ is then injective, and its target is an injective module by (12.14). Finally, M is embedded into $M^{\vee\vee}$ by (12.30). □

(12.34) **Injective resolutions.** An *injective resolution* $I^\cdot$ of an R–module M is a sequence of R–homomorphisms:

$$I^\cdot = O \to I^0 \xrightarrow{\partial^0} I^1 \xrightarrow{\partial^1} \cdots \to I^\ell \xrightarrow{\partial^\ell} I^{\ell+1} \xrightarrow{\partial^{\ell+1}} \cdots$$

together with an R–homomorphism $M \xrightarrow{\partial^{-1}} I^0$ such that

- I^ℓ is an injective R–module for all $\ell \in \mathbb{N}_0$;
- $M \to I^\cdot \stackrel{\text{def}}{=} O \to M \xrightarrow{\partial^{-1}} I^0 \xrightarrow{\partial^0} I^1 \xrightarrow{\partial^1} \cdots \to I^\ell \xrightarrow{\partial^\ell} I^{\ell+1} \xrightarrow{\partial^{\ell+1}} \cdots$ is exact.

The sequence $M \to I^\cdot$ is then called the *augmented* injective resolution of M. Note that the index of the R–homomorphism ∂^ℓ in the injective resolution equals that of its *source*.

(12.35) **Lemma: Ubiquity of injective resolutions.** *Each R–module M has an injective resolution.*

(12.36) **Proof.** First embed M into an injective R–module I^0, next embed $C^1 =^{\text{def}} I^0/M$ into an injective R–module I^1, then $C^2 =^{\text{def}} I^1/C^1$ into an injective R–module I^2, and so on. □

(12.37) **Injective dimension.** Let M be an R–module. It is said to have *finite injective dimension* exactly when it has a *bounded* injective resolution, that is, there is a bounded exact sequence

$$(\dagger^n) \qquad O \to M \xrightarrow{\partial^{-1}} I^0 \xrightarrow{\partial^0} I^1 \xrightarrow{\partial^1} \cdots \to I^\ell \xrightarrow{\partial^\ell} I^{\ell+1} \xrightarrow{\partial^{\ell+1}} \cdots \to I^n \to O$$

with $I^0, \ldots, I^n$ *injective* modules.

When $M \neq O$ has finite injective dimension, then the *injective dimension* of M is the smallest $n \in \mathbb{N}_0$ such that there exists an exact sequence $(\dagger^n)$.

If $M \neq O$ has not finite injective dimension, then we set $\operatorname{id}_R M = \infty$.

Finally, we set $\operatorname{id}_R O = -\infty$.

Thus, for any R–module $M \neq O$ and any $n \in \mathbb{N}_0$ we have:

$$(12.37.1) \qquad \operatorname{id}_R M \leq n \iff \begin{cases} \text{There exists an exact sequence:} \\ O \to M \xrightarrow{\partial^{-1}} I^0 \to \cdots \to I^n \to O \\ \text{with } I^0, \ldots, I^n \text{ injective.} \end{cases}$$

Furthermore,

$$(12.37.2) \qquad \operatorname{id}_R M = 0 \iff M \text{ is injective and non-zero.}$$

(12.38) **Injective dimension over a field.** If M is a module over a field R, then $\operatorname{id}_R M$ is zero if $M \neq 0$ and $-\infty$ otherwise, cf. (12.9).

(12.39) **Injective dimension over a PID.** If R is a PID and M is an R–module, then $\operatorname{id}_R M \leq 1$. Indeed, by (12.32) there is an exact sequence $O \to M \to I \to C \to O$, and (12.17.4) implies that C is injective.

(12.40) **Bounds on Homological Dimensions.** Consider for any $n \in \mathbb{N}_0$ the next two assertions concerning the ring R.

- (p^n_R) $\operatorname{pd}_R M \leq n$ for all R–modules M;
- (i^n_R) $\operatorname{id}_R M \leq n$ for all R–modules M.

If R is a field, then (p^0_R) and (i^0_R) hold by (7.29) and (12.38).

If R is a PID, then (i^1_R) holds by (12.39), and (7.30) yields the next:

$(\mathrm{p}^1_R)'$ $\operatorname{pd}_R M' \leq 1$ for all finitely generated R–modules M'.

Later, in Auslander's Theorem (17.6), we prove that (p^n_R) holds, if and only if (i^n_R) holds; this uses further properties of the Ext functor. It then follows that (p^1_R) actually does hold, when R is a PID.

13. Right Derived of Covariant Functors

(13.1) **Convention.** Throughout this section

$$\boxed{T\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R') \text{ is an } \textbf{\textit{additive}} \text{ and } \textbf{\textit{covariant}} \text{ functor.}}$$

We also consider its induced functor $T\colon \operatorname{Com}(R) \to \operatorname{Com}(R')$, cf. (8.22).

This section describes—without proofs—the theory of right derived functors of covariant functors; this relies on the theory of injective modules.

(13.2) **Definition.** For $n \in \mathbb{Z}$ the *n'th right derived functor* $\mathrm{R}^n T\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ of T is defined as follows:

(M) If M is an R–module, then choose an injective resolution $I^\cdot$ of M, and set

$$\mathrm{R}^n T(M) \stackrel{\text{def}}{=} \mathrm{H}^n(T(I^\cdot)).$$

(μ) If $\mu\colon M \to N$ is an R–homomorphism, then choose injective resolutions $I^\cdot$ and $J^\cdot$ of M and N, respectively, choose an R–morphism $\mu^\cdot : I^\cdot \to J^\cdot$ which is a *continuation* of μ, that is, $\mu_0 \partial_{I^\cdot}^{-1} = \partial_{J^\cdot}^{-1} \mu$, and set

$$\mathrm{R}^n T(\mu) \stackrel{\text{def}}{=} \mathrm{H}^n(T(\mu^\cdot))\colon \mathrm{H}^n(T(I^\cdot)) \to \mathrm{H}^n(T(J^\cdot)).$$

(13.3) **Theorem: Uniqueness.** *The following hold.*

(M) *For any R–module M, the R'–module $\mathrm{R}^n T(M)$ is uniquely determined up to unique isomorphism.*

(μ) *For any R–homomorphism $\mu\colon M \to M'$, the induced R'–homomorphism $\mathrm{R}^n T(\mu)$ is uniquely determined up to unique isomorphism.* □

(13.4) **Theorem: Functoriality.** *For all $n \in \mathbb{Z}$ the derived functor $\mathrm{R}^n T$ is a functor:*

(1) $\mathrm{R}^n T(\mu'\mu) = \mathrm{R}^n T(\mu')\,\mathrm{R}^n T(\mu)$ *for all $n \in \mathbb{Z}$ when $\mu\colon M \to M'$ and $\mu'\colon M' \to M''$ are R–homomorphisms.*

(2) $\mathrm{R}^n T(1_M) = 1_{\mathrm{R}^n T(M)}$ *for all $n \in \mathbb{Z}$ when M is an R–module.* □

(13.5) **Vanishing.** *The following hold.*

(1) $\mathrm{R}^n T = O$ *for all $n < 0$.*

(2) $\mathrm{R}^n T(M) = O$ *for all $n > \operatorname{id}_R M$.*

(3) $\mathrm{R}^n T(M) = O$ *for all $n > 0$ if M is injective.*

(4) $\mathrm{R}^n T(O) = O$ *for all $n \in \mathbb{Z}$.*

(5) $\mathrm{R}^n T = O$ *for all $n \neq 0$, if R is a field.*

(6) $\mathrm{R}^n T = O$ *for all $n > 1$, if R is PID and $n > 1$.* □

(13.6) **Proposition: Additivity.** *The functor $\mathrm{R}^n T$ is additive for all $n \in \mathbb{N}_0$.* □

(13.7) **Theorem: Long-exact sequence of derived functors.**
Let $T\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ be an additive covariant functor, and let

$$M^\cdot = O \to M \xrightarrow{\mu} M' \xrightarrow{\mu'} M'' \to O$$

be an exact sequence of R–modules. For all $n \in \mathbb{Z}$ there exists then an R'–homomorphism

$$\delta^n_{TM^\cdot}\colon \mathrm{R}^n T(M'') \to \mathrm{R}^{n+1} T(M)$$

such that the next sequence is exact:

$$\cdots \longrightarrow \mathrm{R}^{n-1}T(M') \xrightarrow{\mathrm{R}^{n-1}T(\mu)} \mathrm{R}^{n-1}T(M'') \xrightarrow{\delta^{n-1}_{TM^\bullet}}$$
$$\mathrm{R}^{n}T(M) \xrightarrow{\mathrm{R}^{n}T(\mu')} \mathrm{R}^{n}T(M') \xrightarrow{\mathrm{R}^{n}T(\mu)} \mathrm{R}^{n}T(M'') \xrightarrow{\delta^{n}_{TM^\bullet}}$$
$$\mathrm{R}^{n+1}T(M) \xrightarrow{\mathrm{R}^{n+1}T(\mu')} \mathrm{R}^{n+1}T(M') \xrightarrow{\mathrm{R}^{n+1}T(\mu)} \cdots \qquad \square$$

(13.8) **Corollary 1.** *The functor* $\mathrm{R}^n T$ *is half-exact for all* $n > 0$. $\square$

(13.9) **Corollary 2.** *The functor* $\mathrm{R}^0 T$ *is left-exact.* $\square$

(13.10) **Corollary 3.** *If* $O \to M \to I \to C \to O$ *is an exact sequence of* R*–modules and* I *is injective, then* $\mathrm{R}^n T(M) \cong \mathrm{R}^{n-1}T(C)$ *for* $n > 1$. $\square$

(13.11) **Corollary 4.** *If* T *is left-exact, then the functors* T *and* $\mathrm{R}^0 T$ *are isomorphic.* $\square$

(13.12) **Corollary 5.** *If* T *is exact, then* $\mathrm{R}^n T = O$ *for all* $n \geq 1$. $\square$

(13.13) **Corollary 6.** *If* T *is left-exact and* $\mathrm{R}^1 T = O$*, then* T *is exact.* $\square$

(13.14) **Exercise: Derived functors of Γ_a.** Consider the Gamma functor Γ_a for an element $a \in R$, cf. (6.51), and the set $S(a) = \{\, a^n \mid n \in \mathbb{N}_0 \,\}$. If $0 \notin S(a)$, then $S(a)$ is a special multiplicative system, and we set $M_a = S(a)^{-1}M$; otherwise we set $M_a = O$.

(a) Establish, for any R–module M an exact sequence $O \to \Gamma_a(M) \xrightarrow{\iota_a^M} M \xrightarrow{\rho_a^M} M_a$.
Assume from now on, that R is a domain, and that D is a divisible R–module.

(b) Prove that the next sequence is exact $O \to \Gamma_a(D) \xrightarrow{\iota_a^D} D \xrightarrow{\rho_a^D} D_a \to O$.
Let $O \to D \to J \to C \to O$ be an exact sequence with J injective.

(c) Establish the next commutative diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
 & & O & & O & & O & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
O & \longrightarrow & \Gamma_a(D) & \xrightarrow{\iota_a^D} & D & \xrightarrow{\rho_a^D} & D_a & \longrightarrow & O \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
O & \longrightarrow & \Gamma_a(J) & \xrightarrow{\iota_a^J} & J & \xrightarrow{\rho_a^J} & J_a & \longrightarrow & O \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
O & \longrightarrow & \Gamma_a(C) & \xrightarrow{\iota_a^C} & C & \xrightarrow{\rho_a^C} & C_a & \longrightarrow & O \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & O & & O & &
\end{array}$$

(d) Prove that $\mathrm{R}^1 \Gamma_a(D) = 0$.

(e) Prove that $\mathrm{R}^n \Gamma_a(D) = 0$ for all $n \geq 1$.

(f) Prove that $\mathrm{R}^n \Gamma_a(M) = 0$ for all $n \geq 2$ for any R–module M.

(Actually (f) holds over any ring—the proof is harder, though.)

(13.15) **Exercise: Left Derived of Contravariant Functors.** Describe this theory.

14. Direct Products and Sums

(14.1) **Family of Sets.** The notation $(M_x)_{x\in X}$ is a *family of sets*, when each M_x is a set and X is an index set (or class).

Example A. $M_x = R$ for all $x \in X$.

Example B. $X = \{1,\dots,n\}$. There are then n modules: $M_1, \dots, M_n$.

(14.2) **Product Set.** The *product set* $\prod_{x\in X} M_x$ of a family $(M_x)_{x\in X}$ of sets is the set $\{(m_x)_{x\in X} \mid \forall\, x \in X\colon m_x \in M_x\}$ consisting of all families $(m_x)_{x\in X}$ of elements such that $m_x \in M_x$.

Example A continued. Recall from (3.13) that $R^X = \{\varphi \mid \varphi\colon X \to R \text{ is a function}\}$. When $M_x = R$ for all $x \in X$, then the map $R^X \to \prod_{x\in X} M_x$, $\varphi \mapsto (\varphi(x))_{x\in X}$ is bijective, and we make the identification $R^X = \prod_{x\in X} M_x$.

Example B continued. When $X = \{1,\dots,n\}$, then $\prod_{x\in X} M_x = M_1 \times \cdots \times M_n$, the usual product set.

(14.3) **Direct Product of Modules.** The *direct product* of a family $(M_x)_{x\in X}$ of R–modules is the product set $\prod_{x\in X} M_x$ equipped with addition and multiplication defined as follows:

$$(m_x)_{x\in X} + (m'_x)_{x\in X} \stackrel{\text{def}}{=} (m_x + m'_x)_{x\in X} \qquad r(m_x)_{x\in X} \stackrel{\text{def}}{=} (rm_x)_{x\in X}.$$

Indeed, this makes it into an R–module.

Example A continued. In this case, the product module is R^X, the module of functions, cf. (3.13).

Example B continued. Here, the product module is the direct sum $M_1 \oplus \cdots \oplus M_n$, cf. (2.21).

(14.4) **Direct Sum of Modules.** For $\underline{m} = (m_x)_{x\in X} \in \prod_{x\in X} M_x$ we set $\operatorname{supp} \underline{m} \stackrel{\text{def}}{=} \{x \in X \mid m_x \neq 0\}$. The set $\coprod_{x\in X} M_x \stackrel{\text{def}}{=} \{\underline{m} \in \prod_{x\in X} M_x \mid \operatorname{supp} \underline{m} \text{ is finite}\}$ is a submodule of the R–module $\prod_{x\in X} M_x$, and it is called the *direct sum* (or coproduct) of the family $(M_x)_{x\in X}$.

Example A continued. The direct sum is the module of functions $R^{(X)}$, cf. (3.13).

Example B continued. It is again the finite direct sum $M_1 \oplus \cdots \oplus M_n$, cf. (2.21) and (14.3) with elements $(m_1,\dots,m_n)$, $m_x \in M_x$ for all x.

(14.5) **Families of Homomorphisms.** Let $(\varphi_x\colon M_x \to N_x)_{x\in X}$ be a family of R–homomorphisms.

(1) There is then the induced R–homomorphism: $\prod_{x\in X} \varphi_x\colon \prod_{x\in X} M_x \to \prod_{x\in X} N_x$ given by $(m_x)_{x\in X} \mapsto (\varphi_x(m_x))_{x\in X}$.

(2) As well as its restriction $\coprod_{x\in X} \varphi_x\colon \coprod_{x\in X} M_x \to \coprod_{x\in X} N_x$.

(3) $\operatorname{Im}\big(\prod_{x\in X} \varphi_x\big) = \prod_{x\in X} \operatorname{Im} \varphi_x$ and $\operatorname{Im}\big(\coprod_{x\in X} \varphi_x\big) = \coprod_{x\in X} \operatorname{Im} \varphi_x$.

(4) $\operatorname{Ker}\big(\prod_{x\in X} \varphi_x\big) = \prod_{x\in X} \operatorname{Ker} \varphi_x$ and $\operatorname{Ker}\big(\coprod_{x\in X} \varphi_x\big) = \coprod_{x\in X} \operatorname{Ker} \varphi_x$.

(5) For any family of R–homomorphisms $(\psi_x\colon N_x \to P_x)_{x\in X}$ the following are equivalent.

(i) The sequence $M_x \xrightarrow{\varphi_x} N_x \xrightarrow{\psi_x} P_x$ is exact for all $x \in X$.

(ii) The sequence $\prod_{x\in X} M_x \xrightarrow{\prod_{x\in X} \varphi_x} \prod_{x\in X} N_x \xrightarrow{\prod_{x\in X} \psi_x} \prod_{x\in X} P_x$ is exact.

(iii) The sequence $\coprod_{x\in X} M_x \xrightarrow{\coprod_{x\in X} \varphi_x} \coprod_{x\in X} N_x \xrightarrow{\coprod_{x\in X} \psi_x} \coprod_{x\in X} P_x$ is exact.

(14.6) **Direct Product and Sum of Functors.** Any family

$$\big(F_x\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')\big)_{x\in X}$$

of functors F_x of the same variance induces functors

$$\prod_{x\in X} F_x, \coprod_{x\in X} F_x\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$$

by setting $\big(\prod_{x\in X} F_x\big)(M) \stackrel{\text{def}}{=} \prod_{x\in X} F_x(M)$ and $\big(\prod_{x\in X} F_x\big)(\varphi) \stackrel{\text{def}}{=} \prod_{x\in X} F_x(\varphi)$ for any R–module M and any R–homomorphism φ. The sum $\coprod_{x\in X} F_x$ is defined similarly. If $X = \{1,\dots,n\}$ then $F_1 \oplus \cdots \oplus F_n$ denotes $\prod_{x\in X} F_x = \coprod_{x\in X} F_x$.

Exactness. The following are equivalent.

(*i*) F_x is exact, respectively, left-exact, right-exact, half-exact, for all $x \in X$;
(*ii*) $\prod_{x\in X} F_x$ has the same property;
(*iii*) $\coprod_{x\in X} F_x$ has the same property.

(14.7) **Morphisms of Functors.** Let F and G be two functors $\operatorname{Mod}(R) \to \operatorname{Mod}(R')$ of the same variance. A family $\big(\Theta_M\colon F(M) \to G(M)\big)_M$ indexed by the R–modules M is then said to be a *morphisms*[13] $\Theta\colon F \to G$, exactly when the next diagram

$$\begin{array}{ccc} F(M) & \xrightarrow{\Theta_M} & G(M) \\ {\scriptstyle F(\varphi)}\downarrow & & \downarrow{\scriptstyle G(\varphi)} \\ F(N) & \xrightarrow[\Theta_N]{} & G(N)\,. \end{array}$$

is commutative for each R–homomorphism $\varphi\colon M \to N$. The diagram is here shown in the covariant case; in the contravariant one, the vertical arrows should be reversed.

If $\Theta\colon F \to G$ is a morphism, then it is an isomorphism of functors, cf. (12.11), if and only if Θ_M is an isomorphism for all R–modules M.

If Θ_M is an isomorphism for all M in a certain subcategory of $\operatorname{Mod}(R)$, then we say the Θ is an *isomorphism on* that subcategory.

(14.8) **Exercise: Hom(R,–) is the Identity.** Prove that the functor $\operatorname{Hom}_R(R,-)$ is isomorphic to the identity functor.

(14.9) **Exercise: Isomorphic Hom functors.** Prove that the functors $\operatorname{Hom}_R(-,M)$ and $\operatorname{Hom}_R(-,N)$ are isomorphic if and only if $M \cong N$. Prove the same for the functors $\operatorname{Hom}_R(M,-)$ and $\operatorname{Hom}_R(N,-)$.

(14.10) **Isomorphism Theorem: Hom & Product.** *Let $\big(M_x\big)_{x\in X}$ be a family of R–modules. There is then the next isomorphism of functors:*

$$\Phi\colon \prod_{x\in X} \operatorname{Hom}_R(-,M_x) \xrightarrow{\cong} \operatorname{Hom}_R\Big(-,\prod_{x\in X} M_x\Big)\,.$$

(14.11) **Proof.** For any L, the map $\Phi^L\colon \prod \operatorname{Hom}_R(L,M_x) \to \operatorname{Hom}_R(L,\prod M_x)$ is given by, that for $\underline{\varphi} = \big(\varphi_x\big)_x$ in the source, the image $\Phi^L(\underline{\varphi})$ in $\operatorname{Hom}_R(L,\prod M_x)$, that is, the R–homomorphism $\Phi^L(\underline{\varphi})\colon L \to \prod M_x$, is defined by $\big(\Phi^L(\underline{\varphi})\big)(\ell) \stackrel{\text{def}}{=} \big(\varphi_x(\ell)_x\big)_x (\in \prod M_x)$. It is straightforward to verify that Φ^L is a morphisms of functors. It is an isomorphism with the inverse given by $(\Phi^L)^{-1}(\lambda) \stackrel{\text{def}}{=} \big(\pi_x\lambda_x\big)_x$ for the projections $\pi_x\colon \prod M_x \to M_x$. □

[13] Or we say that Θ_M is *functorial* in M.

(14.12) **Corollary: Product of Injectives.** *For any family* $(M_x)_{x\in X}$ *of* R*-modules the next holds:*

$$\prod_{x\in X} M_x \text{ is injective} \iff M_x \text{ is injective for all } x \in X. \qquad \square$$

(14.13) **Isomorphism Theorem: Hom & Sum.** *If* $(M_x)_{x\in X}$ *is a family of* R*-modules, then the restriction of the isomorphism* Φ *from Theorem* (14.10) *to direct sums*

$$\Phi|\colon \coprod_{x\in X} \operatorname{Hom}_R(-, M_x) \longrightarrow \operatorname{Hom}_R(-, \coprod_{x\in X} M_x)$$

is an isomorphism on the subcategory $\operatorname{Mod}'(R)$ *of finitely generated* R*-modules, that is, for any finitely generated* R*-module* L' *there is the next isomorphism:*

$$\Phi|^{L'}\colon \coprod_{x\in X} \operatorname{Hom}_R(L', M_x) \xrightarrow{\cong} \operatorname{Hom}_R(L', \coprod_{x\in X} M_x).$$

(14.14) **Proof.** We verify first, that the restriction maps into $\operatorname{Hom}_R(L', \coprod_{x\in X} M_x)$: For $\underline{\varphi} = (\varphi_x)_{x\in X} \in \coprod_{x\in X} \operatorname{Hom}_R(L', M_x)$ and $\ell' \in L'$ we have $\big(\Phi^{L'}(\underline{\varphi})\big)(\ell') = (\varphi_x(\ell'))_{x\in X}$, so the support of $(\varphi_x(\ell'))_{x\in X}$ is contained in $\operatorname{supp}\underline{\varphi}$ which is finite by assumption. The restriction is injective since $\Phi^{L'}$ itself is. To prove that it is surjective, let $\lambda \in \operatorname{Hom}_R(L', \coprod M_x)$ be given. From the previous theorem we know that there exists $\underline{\varphi} \in \prod \operatorname{Hom}_R(L', M_x)$ such that $\Phi^{L'}(\underline{\varphi}) = \lambda$. Thus, it suffices to prove that $\operatorname{supp}\underline{\varphi}$ is finite. Assume that $L' = \operatorname{span}_R\{\ell_1, \dots, \ell_n\}$, and set $Y =^{\text{def}} \operatorname{supp}\lambda(\ell_1) \cup \cdots \operatorname{supp}\lambda(\ell_n)$. This set is finite because it is a finite union of finite sets. If the element x belongs to $\operatorname{supp}\underline{\varphi}$, then $\varphi_x \neq 0$, that is, $\varphi_x(\ell_i) \neq 0$ for some i; thus $x \in Y$. $\square$

(14.15) **Noetherian Rings.** The ring R is said to be *Noetherian* [14], exactly when every ideal in R is finitely generated. It is easy to see, that R is Noetherian if and only if it has the *Ascending Chain Condition* on ideals: Every ascending chain $\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \cdots \subseteq \mathfrak{a}_n \subseteq \cdots$ of ideals is ultimately stable, that is, there is an $N \in \mathbb{N}_0$ such that $\mathfrak{a}_N = \mathfrak{a}_{N+1} \cdots$.

Fields and PIDs are Noetherian. If R is Noetherian, then the residue ring $R/\mathfrak{a}$, the ring $S^{-1}R$ of fractions, and the polynomial ring $R[X]$ are all Noetherian. (Proofs will be provided later.)

(14.16) **Corollary: Direct sum of Injectives.** *If* R *is Noetherian, then the next holds for any family* $(M_x)_{x\in X}$ *of* R*-modules:*

$$\coprod_{x\in X} M_x \text{ is injective} \iff M_x \text{ is injective for all } x \in X.$$

(14.17) **Proof.** The implication " $\Longrightarrow$ " follows from (12.6) as $\coprod_{x\in X} M_x = M_y \oplus \coprod_{x\neq y} M_x$.

To prove the other implication, assume that each M_x injective, any use Baer's Criterion (12.7) to prove that $M = \coprod_{x\in X} M_x$ is injective. Let $\mathfrak{a}$ be an ideal and consider the next commutative diagram:

$$\begin{array}{ccccc} \operatorname{Hom}_R(R, M) & \longrightarrow & \operatorname{Hom}_R(\mathfrak{a}, M) & \longrightarrow & O \\ \cong\downarrow{\scriptstyle \Phi^R} & & \cong\downarrow{\scriptstyle \Phi^{\mathfrak{a}}} & & \\ \coprod_{x\in X} \operatorname{Hom}_R(R, M_x) & \longrightarrow & \coprod_{x\in X} \operatorname{Hom}_R(\mathfrak{a}, M_x) & \longrightarrow & O \end{array}$$

[14] After Emmy Noether (German), 1882–1935, the founder of modern algebra.

The isomorphisms are by Theorem (14.13) as the modules R and $\mathfrak{a}$ are finitely generated. The lower row is exact by assumption, and hence the top row is exact, as desired. □

(14.18) **Exercise: Direct sum of Injectives.** Prove that the ring R is Noetherian, if the direct sum $\coprod_{x\in X} M_x$ is injective, whenever each M_x is injective. [Assume that $\mathfrak{a}_1 \subset \cdots \subset \mathfrak{a}_n \subset \cdots$ is an infinite strictly increasing chain of ideals in R. Embed each $R/\mathfrak{a}_n$ into an injective module M_n. Prove that $\coprod_{n\in\mathbb{N}} M_n$ is not injective.]

(14.19) **Isomorphism Theorem: Hom & Sum–Product.** *If $(M_x)_{x\in X}$ is a family of R–modules, then there is the next isomorphism of functors:*

$$\Psi\colon \operatorname{Hom}_R(\coprod_{x\in X} M_x, -) \xrightarrow{\cong} \prod_{x\in X} \operatorname{Hom}_R(M_x, -)\,.$$

(14.20) **Proof.** For $\varphi \in \operatorname{Hom}_R(\coprod M_x, L)$ we set $\Psi^L(\varphi) =^{\text{def}} (\varphi\iota_x)_{x\in X}$, which belongs to $\prod_{x\in X} \operatorname{Hom}_R(M_x, L)$, and where ι_x is the inclusion $M_x \hookrightarrow \coprod M_x$. Define

$$\Omega^L\colon \prod_{x\in X} \operatorname{Hom}_R(M_x, L) \to \operatorname{Hom}_R(\coprod_{x\in X} M_x, L)$$

by $\big(\Omega^L((\mu_x)_{x\in X})\big)\big((m_x)_{x\in X}\big) =^{\text{def}} \sum_{x\in X} \mu_x(m_x) \in L$; the sum is finite because $(m_x)_{x\in X}$ belongs to $\coprod M_x$. It is straightforward to verify that the maps $\Omega^L\Psi^L$ and $\Psi^L\Omega^L$ are the identities. □

(14.21) **Corollary: Direct sum of Projectives.** *If R is Noetherian and $(M_x)_{x\in X}$ is a family of R–modules, then the next holds:*

$$\coprod_{x\in X} M_x \text{ is projective} \iff M_x \text{ is projective for all } x \in X\,. \qquad \square$$

(14.22) **Exercise: Module of Maps.** Let M be an R–module, let X be a set, and define M^X to be the set of maps $X \to M$.

(a) Provide M^X with the structure of an R–module.
(b) Identify M^X with $\prod_{x\in X} M_x$ when $M_x = M$ for all $x \in X$.
(c) Establish an isomorphism $M^X \to \operatorname{Hom}_R(R^{(X)}, M)$ of R–modules.

15. Complexes of Homomorphisms

This section contains auxiliary results to be used in Section 16.

(15.1) **Complex of Homomorphisms.** For R–complexes $X.$ and $Y.$ we define an R–complex $\operatorname{Hom}_R(X.,Y.).$ as follows.

(M) The n'th *module* is:

$$\operatorname{Hom}_R(X.,Y.)_n \stackrel{\text{def}}{=} \prod_{\ell\in\mathbb{Z}} \operatorname{Hom}_R(X_\ell, Y_{\ell+n}) .$$

Thus, each element $\alpha.$ in this module is a family $\big(\alpha_\ell\colon X_\ell \to Y_{\ell+n}\big)_{\ell\in\mathbb{Z}}$ of R–homomorphisms, and $\alpha.$ is said to be a *homomorphism of complexes:* $X. \to Y.$ *of degree* n. (Examples: Any morphism $\varphi.\colon X. \to Y.$ is a homomorphism of degree zero. The family $\big(\partial^X_\ell\big)_{\ell\in\mathbb{Z}}$ of differentials in an R–complex $X.$ is a homomorphism: $X. \to X.$ of degree -1. Any family $\sigma. = \big(\sigma_\ell\colon X_\ell \to Y_{\ell+1}\big)_{\ell\in\mathbb{Z}}$ defining a homotopy between two morphisms: $X. \to Y.$ belongs to $\operatorname{Hom}_R(X.,Y.)_1$.)

(μ) The n'th *differential*

$$\partial_n^{\operatorname{Hom}_R(X.,Y.)}\colon \operatorname{Hom}_R(X.,Y.)_n \to \operatorname{Hom}_R(X.,Y.)_{n-1}$$

in degree n is defined as follows:

$$\partial_n^{\operatorname{Hom}_R(X.,Y.)}\big(\alpha.\big) \stackrel{\text{def}}{=} \big(\partial^Y_{\ell+n}\alpha_\ell - (-1)^n\alpha_{\ell-1}\partial^X_\ell\big)_{\ell\in\mathbb{Z}}$$

for $\alpha. = \big(\alpha_\ell\big)_{\ell\in\mathbb{Z}} \in \operatorname{Hom}_R(X.,Y.)_n$.

For any $\beta. \in \operatorname{Hom}_R(X.,Y.)_0$ the last definition yields:

(15.1.1) $\qquad \beta. \in \mathrm{Z}_0(\operatorname{Hom}_R(X.,Y.).) \iff \beta.$ is a morphism: $X. \to Y.$.

(15.1.2) $\qquad \beta. \in \mathrm{B}_0(\operatorname{Hom}_R(X.,Y.).) \iff \beta. \sim 0.$.

The last is that $\beta.$ is homotopic to the zero-morphism $0.\colon X. \to Y.$. Thus, the set of equivalence classes of morphisms $X. \to Y.$ with respect to homotopy is precisely $\mathrm{H}_0(\operatorname{Hom}_R(X.,Y.).)$.

As usual, cf. (8.8), we identify any complex concentrated i degree 0 with the module in degree 0, and vice versa. If M and N are R–modules, then the complex $\operatorname{Hom}_R(M,N).$ equals the module $\operatorname{Hom}_R(M,N)$, and if, furthermore, $X.$ is an R–complex, then the complex $\operatorname{Hom}_R(M,X.).$, respectively, $\operatorname{Hom}_R(X.,M).$, is equal to the result of applying the covariant functor $\operatorname{Hom}_R(M,-)$, respectively, the contravariant functor $\operatorname{Hom}_R(-,M)$, to the complex $X.$, cf. (8.22), respectively, (11.2.1) (and here the sign matters).

(15.2) **Induced Morphisms.** Let $\psi.\colon Y. \to Y'.$ be a morphism of R–complexes, and let $X.$ and $Z.$ be R–complexes. There are the next *induced morphisms:*

$$\operatorname{Hom}_R(X.,\psi.).\colon \operatorname{Hom}_R(X.,Y.). \to \operatorname{Hom}_R(X.,Y'.). \ , \quad \alpha \mapsto \psi\alpha$$

$$\operatorname{Hom}_R(\psi.,Z.).\colon \operatorname{Hom}_R(Y'.,Z.). \to \operatorname{Hom}_R(Y.,Z.). \ , \quad \alpha' \mapsto \alpha'\psi .$$

(15.3) **Shifted Complex.** Let $X.$ be an R–complex. The R–complex $\Sigma X.$ is defined as follows:

$$(\Sigma X.)_n \stackrel{\text{def}}{=} X_{n-1}, \quad \partial_n^{\Sigma X.} \stackrel{\text{def}}{=} -\partial^X_{n-1} ,$$

and it is said to be the complex X *shifted one degree to the left*[15].

Note that $\mathrm{H}_{n+1}(\Sigma X.) = \mathrm{H}_n(X.)$.

[15] The complex $\Sigma X.$ is also called the *suspension* of X, and it was introduced in (8.25.3) using a different notation.

(15.4) **Cone of Morphism.** Let $\alpha. : X. \to Y.$ be a morphism of R–complexes. The *mapping cone* $\mathrm{Cone}(\alpha.)$. of $\alpha.$ is the R–complex defined as follows:

$$\mathrm{Cone}(\alpha.)_n \stackrel{\text{def}}{=} Y_n \oplus X_{n-1} \ (= (Y. \oplus \Sigma X.)_n),$$
$$\partial_n^{\mathrm{Cone}(\alpha.)}(y_n, x_{n-1}) \stackrel{\text{def}}{=} (\partial_n^{Y.}(y_n) + \alpha_{n-1}(x_{n-1}), -\partial_{n-1}^{X.}(x_{n-1})) .$$

There is the next exact sequence of R–complexes:

$$O. \to Y. \xrightarrow{\iota.} \mathrm{Cone}(\alpha.). \xrightarrow{\pi.} \Sigma X. \to O. ,$$

where $\iota.$ and $\pi.$ degreewise are the inclusion and the projection, respectively. In the corresponding long-exact sequence, the connecting homomorphism $\mathrm{H}_{n+1}(\Sigma X.) \to \mathrm{H}_n(Y.)$ turns out to be $\mathrm{H}_n(\alpha.)$. Thus, setting $C. =^{\text{def}} \mathrm{Cone}(\alpha.).$, there is the next exact sequence: (15.4.1)

$$\mathrm{H}_{n+1}(X.) \xrightarrow{\mathrm{H}_{n+1}(\alpha.)} \mathrm{H}_{n+1}(Y.) \longrightarrow \mathrm{H}_{n+1}(C.) \longrightarrow \mathrm{H}_n(X.) \xrightarrow{\mathrm{H}_n(\alpha.)} \mathrm{H}_n(Y.) \longrightarrow \mathrm{H}_n(C.)$$

(15.5) **Cone and Homomorphisms.** Let $\psi: Y. \to Y'.$ be an R–morphism, and let $X.$ and $Z.$ be R–complexes. It is then not hard to establish the next identification and the next isomorphism:

$$\mathrm{Cone}(\mathrm{Hom}_R(X., \psi.).) = \mathrm{Hom}_R(X., \mathrm{Cone}(\psi.).) ;$$
$$\mathrm{Cone}(\mathrm{Hom}_R(\psi., Z.).) \cong \Sigma \mathrm{Hom}_R(\mathrm{Cone}(\psi.)., Z.).$$

(15.6) **Quasi-isomorphisms.** Let $\alpha: X. \to Y.$ be an R–morphism; it is said to be a *quasi-isomorphism*, exactly when it induces isomorphisms in homology, that is, the homomorphism $\mathrm{H}_n(\alpha.): \mathrm{H}_n(X.) \to \mathrm{H}_n(Y.)$ is an isomorphism for all $n \in \mathbb{Z}$.

If $\alpha: M \to N$ is an R–homomorphism, then it is also an R–morphism, cf. (8.8), and it is a quasi-isomorphism of complexes, if and only if it is an isomorphism of modules.

If $P.$ is a projective resolution of an R–module M, then the induced R–morphism $\pi. : P. \to M$ is a quasi-isomorphism.

If $I.$[16] is an injective resolution of an R–module N, then the induced R–morphism $\nu. : N \to I.$ is a quasi-isomorphism.

(15.7) **Cone and Quasi-isomorphisms.** By (15.4.1) we have the next:

(15.7.1) α is a quasi-isomorphism $\iff$ $\mathrm{Cone}(\alpha.)$ is exact.

(15.8) **Exact Complex of Homomorphisms.** Consider R–complexes $P.$, $I.$, and $K.$. The following then hold.

(P) If $P.$ is *bounded to the right* and each P_n is *projective*, then the next holds:

$$K. \text{ is exact} \implies \mathrm{Hom}_R(P., K.). \text{ is exact.}$$

(I) If $I.$ is *bounded to the left* and each I_n is *injective*, then the next holds:

$$K. \text{ is exact} \implies \mathrm{Hom}_R(K., I.). \text{ is exact.}$$

For (P), assume that $P_n = 0$ for $n < 0$, and prove that $\mathrm{H}_0(\mathrm{Hom}_R(P., K.).)$ vanishes, that is, that any morphism $P. \to K.$ is homotopic to the zero morphism. This is achieved as in the proof of the theorem (9.15.!) (which actually does not need to require the modules P'_n to be projective).

[16] Here written homologically.

(15.9) **Induced Quasi-isomorphisms.** Let $\psi. : Y. \to Y'.$ be an R–morphism, and let $P.$ and $I.$ be R–complexes. The following then results from (15.7), (15.8), (15.5), and (15.7) once more:

(P) If $P.$ is *bounded to the right* and each P_n is *projective*, then the next holds:

$$\psi. \text{ is a quasi-isomorphism} \implies \operatorname{Hom}_R(P., \psi.) \text{ is a quasi-isomorphism.}$$

(I) If $I.$ is *bounded to the left* and each I_n is *injective*, then the next holds:

$$\psi. \text{ is a quasi-isomorphism} \implies \operatorname{Hom}_R(\psi., I.) \text{ is a quasi-isomorphism.}$$

(15.10) **Resolutions and Induced Quasi-isomorphisms.** Let $P.$ be a projective resolution of an R–module M, and let $I.$[17] be an injective resolution of an R–module N. Consider the induced quasi-isomorphisms $\pi. : P. \to M$ and $\nu. : N \to I.$, cf. (15.6), as well as the next two quasi-isomorphisms, that they induce, cf. (15.9):

$$\operatorname{Hom}_R(P., N). \xrightarrow{\operatorname{Hom}_R(P., \nu.).} \operatorname{Hom}_R(P., I.). \xleftarrow{\operatorname{Hom}_R(\pi., I.).} \operatorname{Hom}_R(M, I.).$$

For each $n \in \mathbb{Z}$, these two quasi-isomorphisms induce the next two isomorphisms in *homological* degree $-n$:

(15.10.1)

$$\operatorname{Ext}^n_R(M, N) \xrightarrow[\cong]{\operatorname{H}_{-n}(\operatorname{Hom}_R(P., \nu.).)} \operatorname{H}_{-n}(\operatorname{Hom}_R(P., I.).) \xleftarrow[\cong]{\operatorname{H}_{-n}(\operatorname{Hom}_R(\pi., I.).)} \operatorname{H}_{-n}(\operatorname{Hom}_R(M, I.).).$$

Recall from (11.17) that $\operatorname{Ext}^n_R(M, N)$ is $\operatorname{R}^n \operatorname{Hom}_R(-, N)(M)$, that is, the n'th right derived of the contravariant functor $\operatorname{Hom}_R(-, N)$ applied to the module M, and hence $\operatorname{Ext}^n_R(M, N) = \operatorname{H}_{-n}(\operatorname{Hom}_R(P., N).)$.

By the way, the right hand module above is $\operatorname{R}^n \operatorname{Hom}_R(M, -)(N)$, that is, the n'th right derived of the covariant functor $\operatorname{Hom}_R(M, -)$ applied to the module N, cf. (13.2).

[17] Again written homologically.

16. Ext

For any R–module N and any $n \in \mathbb{Z}$ the contravariant functor $\operatorname{Ext}^n_R(-,N)$ was introduced in (11.17) as the n'th right derived of the contravariant functor $\operatorname{Hom}_R(-,N)$. In this section, this functor is extended to a bi-functor $\operatorname{Ext}^n_R(-,-)$ in *two* variables (and it is explained what this means). Furthermore, it is proved that $\operatorname{Ext}^n_R(M,-)$ for fixed M is isomorphic to the n-th right derived of the covariant functor $\operatorname{Hom}_R(M,-)$.

(16.1) **Hom is a bi-functor.** That is, if $\mu\colon M \to M'$ and $\nu\colon N \to N'$ are R–homomorphisms, then the next diagrams are commutative:

(16.1.1)
$$\begin{array}{ccc} \operatorname{Hom}_R(M,N) & \xrightarrow{\operatorname{Hom}_R(M,\nu)} & \operatorname{Hom}_R(M,N') \\ {\scriptstyle \operatorname{Hom}_R(\mu,N)}\uparrow & & \uparrow{\scriptstyle \operatorname{Hom}_R(\mu,N')} \\ \operatorname{Hom}_R(M',N) & \xrightarrow[\operatorname{Hom}_R(M',\nu)]{} & \operatorname{Hom}_R(M',N') \end{array} \qquad \begin{array}{ccc} \alpha\mu & \longmapsto & \nu\alpha\mu \\ \uparrow & & \uparrow \\ \alpha & \longmapsto & \nu\alpha \end{array}$$

We set $\operatorname{Hom}_R(\mu,\nu) =^{\text{def}} \operatorname{Hom}_R(M,\nu)\operatorname{Hom}_R(\mu,N)\colon \operatorname{Hom}_R(M',N) \to \operatorname{Hom}_R(M,N')$.

(16.2) **$\operatorname{Ext}^n_R(M,-)$ is a functor.** For a fixed R–module M we make $\operatorname{Ext}^n_R(M,-)$ into a functor as follows: First, choose a projective resolution $P.$ of M. Next, consider any R–homomorphism $\nu\colon N \to N'$, and the induced R–morphism

$$\operatorname{Hom}_R(P.,\nu)\colon \operatorname{Hom}_R(P.,N) \to \operatorname{Hom}_R(P.,N')\,.$$

The desired R–homomorphism is obtained by taking cohomology:

$$\operatorname{Ext}^n_R(M,\nu) \stackrel{\text{def}}{=} \mathrm{H}^n(\operatorname{Hom}_R(P.,\nu))\colon \operatorname{Ext}^n_R(M,N) \to \operatorname{Ext}^n_R(M,N')\,.$$

This makes $\operatorname{Ext}^n_R(M,-)$ into a functor: the composite of $\operatorname{Hom}_R(P.,-)\colon \operatorname{Mod}(R) \to \operatorname{Com}(R)$ and the cohomology functor $\mathrm{H}^n(-)\colon \operatorname{Com}(R) \to \operatorname{Mod}(R)$.

(16.3) **Ext is a bi-functor.** That is, if $\mu\colon M \to M'$ and $\nu\colon N \to N'$ are R–homomorphisms, then the next diagram is commutative:

$$\begin{array}{ccc} \operatorname{Ext}^n_R(M,N) & \xrightarrow{\operatorname{Ext}^n_R(M,\nu)} & \operatorname{Ext}^n_R(M,N') \\ {\scriptstyle \operatorname{Ext}^n_R(\mu,N)}\uparrow & & \uparrow{\scriptstyle \operatorname{Ext}^n_R(\mu,N')} \\ \operatorname{Ext}^n_R(M',N) & \xrightarrow[\operatorname{Ext}^n_R(M',\nu)]{} & \operatorname{Ext}^n_R(M',N')\,. \end{array}$$

Indeed, choose projective resolutions $P.$ and $P'.$ of M and M', respectively, and consider the next commutative diagram:

$$\begin{array}{ccc} \operatorname{Hom}_R(P.,N) & \xrightarrow{\operatorname{Hom}_R(P.,\nu)} & \operatorname{Hom}_R(P.,N') \\ {\scriptstyle \operatorname{Hom}_R(P.,N)}\uparrow & & \uparrow{\scriptstyle \operatorname{Hom}_R(P.,N')} \\ \operatorname{Hom}_R(P'.,N) & \xrightarrow[\operatorname{Hom}_R(P.,\nu)]{} & \operatorname{Hom}_R(P'.,N')\,. \end{array}$$

The desired assertion follows by taking n'th cohomology.

(16.4) **Ext as derived of covariant Hom.** By Definition (11.17), $\operatorname{Ext}^n_R(-,N)$ is the derived functor $\mathrm{R}^n \operatorname{Hom}_R(-,N)$ for $n \in \mathbb{Z}$ and an R–module N. Temporarily, we set $\widetilde{\operatorname{Ext}}^n_R(M,\sim) =^{\mathrm{def}} \mathrm{R}^n \operatorname{Hom}_R(M,\sim)$, cf. (13.2), for $n \in \mathbb{Z}$ and $M \in \operatorname{Mod}(R)$. Dually again, $\operatorname{Ext}^n_R(-,\sim)$ is a bi-functor, cf. (16.3). It follows from (15.10.1) that this bi-functor $\widetilde{\operatorname{Ext}}^n_R(-,\sim)$ is isomorphic to $\operatorname{Ext}^n_R(-,\sim)$. Now, the notation $\widetilde{\operatorname{Ext}}^n_R(-,\sim)$ is superfluous.

(16.5) **Theorem: Ext.** *For every $n \in \mathbb{Z}$ there exists a bi-functor*

$$\operatorname{Ext}^n_R(-,-)\colon \operatorname{Mod}(R)^{\mathrm{op}} \times \operatorname{Mod}(R) \to \operatorname{Mod}(R);$$

contravariant in the first variable[18] *and covariant in the second.*

(P1) *For any R–module N, the functor $\operatorname{Ext}^n_R(-,N)$ is the n'th right derived functor $\mathrm{R}^n \operatorname{Hom}_R(-,N)$ of the contravariant functor $\operatorname{Hom}_R(-,N)$.*

(I1) *For any R–module L, the functor $\operatorname{Ext}^n_R(L,-)$ is the n'th right derived functor $\mathrm{R}^n \operatorname{Hom}_R(L,-)$ of the covariant functor $\operatorname{Hom}_R(L,-)$.*

(P2) *$\operatorname{Ext}^n_R(M,N)$ depends on the choice of projective resolution of M, but is unique up to unique isomorphism.*

(I2) *$\operatorname{Ext}^n_R(L,M)$ depends on the choice of injective resolution of M, but is unique up to unique isomorphism.*

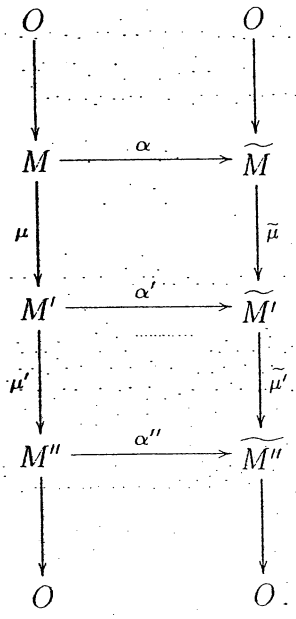
(P3) $\operatorname{Ext}^0_R(-,N) \cong \operatorname{Hom}_R(-,N)$.

(I3) $\operatorname{Ext}^0_R(L,-) \cong \operatorname{Hom}_R(L,-)$.

(P4) *N is injective, if and only if $\operatorname{Ext}^n_R(-,N) = O$ for all $n > 0$.*

(I4) *L is projective, if and only if $\operatorname{Ext}^n_R(L,-) = O$ for all $n > 0$.*

(H) *Consider the next* short-exact ladder[19]*:*



[18] As indicated by the superscript $^{\mathrm{op}}$.

[19] That is, a commutative diagram with exactly two columns (or rows) satisfying the mentioned exactness property.

(P5) *The short-exact ladder* (H) *induces the next long-exact ladder:*

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathrm{Ext}_R^{n-1}(\widetilde{M'},N) & \xrightarrow{\mathrm{Ext}_R^{n-1}(\alpha',N)} & \mathrm{Ext}_R^{n-1}(M',N) \\
{\scriptstyle \mathrm{Ext}_R^{n-1}(\widetilde{\mu},N)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ext}_R^{n-1}(\mu,N)} \\
\mathrm{Ext}_R^{n-1}(\widetilde{M},N) & \xrightarrow{\mathrm{Ext}_R^{n-1}(\alpha,N)} & \mathrm{Ext}_R^{n-1}(M,N) \\
{\scriptstyle \widetilde{\delta}_P^{n-1}}\downarrow & & \downarrow{\scriptstyle \delta_P^{n-1}} \\
\mathrm{Ext}_R^{n}(\widetilde{M''},N) & \xrightarrow{\mathrm{Ext}_R^{n}(\alpha'',N)} & \mathrm{Ext}_R^{n}(M'',N) \\
{\scriptstyle \mathrm{Ext}_R^{n}(\widetilde{\mu'},N)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ext}_R^{n}(\mu',N)} \\
\mathrm{Ext}_R^{n}(\widetilde{M'},N) & \xrightarrow{\mathrm{Ext}_R^{n}(\alpha',N)} & \mathrm{Ext}_R^{n}(M',N) \\
{\scriptstyle \mathrm{Ext}_R^{n}(\widetilde{\mu},N)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ext}_R^{n}(\mu,N)} \\
\mathrm{Ext}_R^{n}(\widetilde{M},N) & \xrightarrow{\mathrm{Ext}_R^{n}(\alpha,N)} & \mathrm{Ext}_R^{n}(M,N) \\
{\scriptstyle \widetilde{\delta}_P^{n}}\downarrow & & \downarrow{\scriptstyle \delta_P^{n}} \\
\vdots & & \vdots
\end{array}$$

(I5) *The short-exact ladder* (H) *induces the next long-exact ladder:*

$$\begin{array}{ccc}
\vdots & & \vdots \\
{\scriptstyle \delta_I^{n-1}}\downarrow & & \downarrow{\scriptstyle \widetilde{\delta}_I^{n-1}} \\
\mathrm{Ext}_R^{n}(L,M) & \xrightarrow{\mathrm{Ext}_R^{n}(L,\alpha)} & \mathrm{Ext}_R^{n}(L,\widetilde{M}) \\
{\scriptstyle \mathrm{Ext}_R^{n}(L,\mu)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ext}_R^{n}(L,\widetilde{\mu})} \\
\mathrm{Ext}_R^{n}(L,M') & \xrightarrow{\mathrm{Ext}_R^{n}(L,\alpha')} & \mathrm{Ext}_R^{n}(L,\widetilde{M'}) \\
{\scriptstyle \mathrm{Ext}_R^{n}(L,\mu')}\downarrow & & \downarrow{\scriptstyle \mathrm{Ext}_R^{n}(L,\widetilde{\mu'})} \\
\mathrm{Ext}_R^{n}(L,M'') & \xrightarrow{\mathrm{Ext}_R^{n}(L,\alpha'')} & \mathrm{Ext}_R^{n}(L,\widetilde{M''}) \\
{\scriptstyle \delta_I^{n}}\downarrow & & \downarrow{\scriptstyle \widetilde{\delta}_I^{n}} \\
\mathrm{Ext}_R^{n+1}(L,M) & \xrightarrow{\mathrm{Ext}_R^{n+1}(L,\alpha)} & \mathrm{Ext}_R^{n+1}(L,\widetilde{M}) \\
{\scriptstyle \mathrm{Ext}_R^{n+1}(L,\mu)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ext}_R^{n+1}(L,\widetilde{\mu})} \\
\mathrm{Ext}_R^{n+1}(L,M') & \xrightarrow{\mathrm{Ext}_R^{n+1}(L,\alpha')} & \mathrm{Ext}_R^{n+1}(L,\widetilde{M'}) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

(16.6) **Proof.** (P1) is by the definition (11.17), and (I1) is by (16.4). (P2) and (I2) are by (11.5) and (13.3), respectively.(P3) and (I3) are by (11.14) and (13.11), respectively. (P4) and (I4) follow from (11.16), (11.15), (13.13), and (13.12). The vertical exactness in (P5) is by (11.18); commutativity of diagrams involving connecting homomorphisms δ is by (8.20), while that of the other ones is by (16.3). The proof of (I5) is dual. □

17. Homological dimensions

Projective dimension of a module was introduced in (7.28), and the main results are **PDT** (11.19) and **PDC** (11.23). Injective dimension was defined in (12.37). This section brings the main results on injective dimension and relations to projective dimension.

(17.1) **Injectivity Lemma.** *The following are equivalent for any R–module M.*

(i) M *is injective;*
(ii) $\operatorname{Ext}^n_R(H, M) = O$ *for all* $n > 0$ *and all* R*–modules* H*;*
(iii) $\operatorname{Ext}^1_R(R/\mathfrak{a}, M) = O$ *for all ideals* $\mathfrak{a}$*.*

(17.2) **Proof.** "$(i) \Longrightarrow (ii)$" by (P4) in Theorem: Ext (16.5).
"$(ii) \Longrightarrow (iii)$" is obvious.
"$(iii) \Longrightarrow (i)$": The exact sequence $O \to \mathfrak{a} \to R \to R/\mathfrak{a} \to O$ induces the next one:

$$\operatorname{Hom}_R(R, M) \to \operatorname{Hom}_R(\mathfrak{a}, M) \xrightarrow{\delta^0} \operatorname{Ext}^1_R(R/\mathfrak{a}, M)\,,$$

and the desired assertion follows from Baer's Criterion (12.7). □

(17.3) **IDT** $\stackrel{\text{def}}{=}$ **Injective Dimension Theorem.** *The following are equivalent for any R–module M and any* $n \in \mathbb{N}_0$*.*

(i) $\operatorname{id}_R M \le n$*;*
(ii) $\operatorname{Ext}^m_R(H, M) = O$ *for all* $m > n$ *and all* R*–modules* H*;*
(iii) $\operatorname{Ext}^{n+1}_R(H, M) = O$ *for and all* R*–modules* H*;*
(iii') $\operatorname{Ext}^{n+1}_R(H', M) = O$ *for and all finitely generated* R*–modules* H'*;*
(iii") $\operatorname{Ext}^{n+1}_R(R/\mathfrak{a}, M) = O$ *for and all ideals* $\mathfrak{a}$*;*
(iv) If $I^\bullet$ *is a injective resolution of* M*, then the cokernel* $\mathrm{C}^{n-1}(I^\bullet)$ *is injective.*

(17.4) **Proof.** The proofs of the equivalence of (i), (ii), (iii), and (iv) are dual to the corresponding ones in the proof of PDT (11.19).
"$(iii) \Longrightarrow (iii')$" and "$(iii') \Longrightarrow (iii")$" are obvious.
"$(iii") \Longrightarrow (iv)$": By an argument dual to the one for (11.13) we get the isomorphism $\operatorname{Ext}^1_R(R/\mathfrak{a}, \mathrm{C}^{n-1}(I^\bullet)) \cong \operatorname{Ext}^{n+1}_R(R/\mathfrak{a}, M)$. Thus, by assumption, the Injectivity Lemma (17.1) yields that $\mathrm{C}^{n-1}(I^\bullet)$ is injective. □

(17.5) **IDC** $\stackrel{\text{def}}{=}$ **Injective Dimension Corollary.** *If M is an R–module, then*

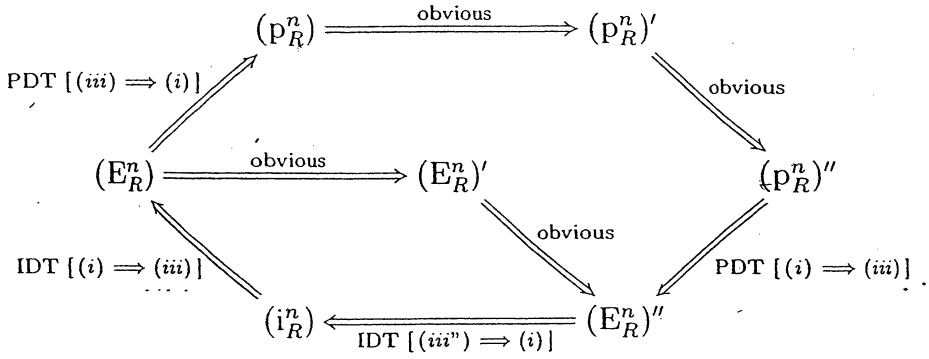
$$\begin{aligned}\operatorname{id}_R M &= \sup\{\, n \in \mathbb{N}_0 \mid \operatorname{Ext}^n_R(H, M) \neq O \text{ for some } H \,\}\\ &= \sup\{\, n \in \mathbb{N}_0 \mid \operatorname{Ext}^n_R(H', M) \neq O \text{ for some finitely generated } H' \,\}\\ &= \sup\{\, n \in \mathbb{N}_0 \mid \operatorname{Ext}^n_R(R/\mathfrak{a}, M) \neq O \text{ for some ideal } \mathfrak{a} \,\}\,.\end{aligned}$$

This number is in $\mathbb{N}^*_0$, cf. (11.23). □

(17.6) **Auslander's Theorem.** *The following are equivalent for the ring R and* $n \in \mathbb{N}_0$*.*

(P^n_R) $\operatorname{pd}_R M \le n$ *for all* R*–modules* M*;*
(I^n_R) $\operatorname{id}_R N \le n$ *for all* R*–modules* N*;*
$(\mathrm{P}^n_R)'$ $\operatorname{pd}_R M' \le n$ *for all finitely generated* R*–modules* M'*;*
$(\mathrm{P}^n_R)''$ $\operatorname{pd}_R R/\mathfrak{a} \le n$ *for all ideals* $\mathfrak{a}$*;*
(E^n_R) $\operatorname{Ext}^{n+1}_R(M, N) = O$ *for all* R*–modules* M *and* N*;*
$(\mathrm{E}^n_R)'$ $\operatorname{Ext}^{n+1}_R(M', N) = O$ *for all finitely generated* R*–modules* M' *&* R*–modules* N*;*
$(\mathrm{E}^n_R)''$ $\operatorname{Ext}^{n+1}_R(R/\mathfrak{a}, N) = O$ *for all ideals* $\mathfrak{a}$ *and all* R*–modules* N*.*

(17.7) **Proof.**



□

(17.8) **Global Dimension.** It is defined as follows:

$$\operatorname{gldim} R \stackrel{\text{def}}{=} \sup\{\operatorname{pd}_R M \mid M \text{ is an } R\text{–module}\} \ (\in \mathbb{N}_0 \cup \{\infty\}).$$

The next is a reformulation of Auslander's Theorem (17.6).

(17.9) **Auslander's Theorem for Global Dimension.**

$$\begin{aligned}\operatorname{gldim} R &= \sup\{\operatorname{id}_R N \mid N \textit{ is an } R\textit{–module}\}\\ &= \sup\{\operatorname{pd}_R M' \mid M' \textit{ is a finitely generated } R\textit{–module}\}\\ &= \sup\{\operatorname{pd}_R R/\mathfrak{a} \mid \mathfrak{a} \textit{ is an ideal}\}\\ &= \sup\{n \mid \operatorname{Ext}^n_R(M,-) \neq O \textit{ for an } R\textit{–module } M\}\\ &= \sup\{n \mid \operatorname{Ext}^n_R(M',-) \neq O \textit{ for a finitely generated } R\textit{–module } M'\}\\ &= \sup\{n \mid \operatorname{Ext}^n_R(R/\mathfrak{a},-) \neq O \textit{ for an ideal } \mathfrak{a}\}.\end{aligned}$$

□

(17.10) **Global Dimension over a field.** $\operatorname{gldim} R = 0$ if R is a field. This follows from the theorem, since every finitely generated R–module, that is, vector space over R, is free, that is, has a basis. Thus, every module is projective. Actually, every module is free, cf. (7.29).

(17.11) **Global Dimension over a PID.** $\operatorname{gldim} R = 1$ if R is a PID which is not a field. Indeed, if $r \in R \setminus O$ is not a unit, then $\operatorname{pd}_R R/(r) > 0$, see for example (7.19), so $\operatorname{gldim} R > 0$. On the other hand, $\operatorname{pd}_R M' \leq 1$ whenever M' is finitely generated, cf. (7.30). Alternatively, $\operatorname{id}_R N \leq 1$ by (12.39)

(17.12) **Dedekind Domains.** One can prove that if R is a domain, then $\operatorname{gldim} R = 1$, if and only if every ideal $\mathfrak{a} \neq O$ in R is a (finite) product of prime ideals; these rings are called *Dedekind domains* and are important in algebraic number theory, confer [Sc,2.1].

(17.13) **Ring of Fractions & Global Dimension.** If S is a special multiplicative system in R, then $\operatorname{gldim} S^{-1}R \leq \operatorname{gldim} R$, cf. (7.32).

(17.14) **Global Dimension over $\mathbb{Z}/(4)$.** $\operatorname{gldim} \mathbb{Z}/(4) = \infty$. Setting $R =^{\text{def}} \mathbb{Z}/(4)$ and $M =^{\text{def}} \mathbb{Z}/(2)$, the latter is a module over the former with multiplication well-defined by $[a]_{(4)}[x]_{(2)} =^{\text{def}} [ax]_{(2)}$. The sequence $O \to M \xrightarrow{\alpha} R \xrightarrow{\beta} M \to O$ is exact, when the homomorphisms are well-defined by $\alpha([x]_{(2)}) =^{\text{def}} [2x]_{(4)}$ and $\beta([a]_{(4)}) =^{\text{def}} [a]_{(2)}$. Since the sequence cannot split, M is not projective, that is, $\operatorname{pd}_R M > 0$, and the next lemma and the above exact sequence imply that $\operatorname{pd}_R M = \operatorname{pd}_R M + 1$; thus, $\operatorname{pd}_R M = \infty$.

(17.15) **Lemma: Projective Dimension & Short-exact Sequences.** *If the sequence* $O \to K \to L \to M \to O$ *is exact, and* $\operatorname{pd}_R L < \operatorname{pd}_R M$ $(\leq \infty)$, *then* $\operatorname{pd}_R M = \operatorname{pd}_R K + 1$.

(17.16) **Proof.** For any $n \in \mathbb{N}_0$ and any R–module N, we have the next exact sequence:

$$(\dagger_n) \qquad \operatorname{Ext}^n_R(L,N) \to \operatorname{Ext}^n_R(K,N) \xrightarrow{\delta^n} \operatorname{Ext}^{n+1}_R(M,N) \to \operatorname{Ext}^{n+1}_R(L,N)\,,$$

By hypothesis, $\ell =^{\text{def}} \operatorname{pd}_R L$ is finite. Thus, there is an isomorphism $\operatorname{Ext}^n_R(K,N) \cong \operatorname{Ext}^{n+1}_R(M,N)$ whenever $n > \ell$, and it follows that $\operatorname{pd}_R M = \operatorname{pd}_R K + 1$, if $\operatorname{pd}_R M > \ell + 1$ (and this includes $\operatorname{pd}_R M = \infty$). Next assume $\operatorname{pd}_R M = \ell + 1$, and consider the exact sequence

$$(\dagger_\ell) \qquad \operatorname{Ext}^\ell_R(L,N) \to \operatorname{Ext}^\ell_R(K,N) \xrightarrow{\delta^\ell} \operatorname{Ext}^{\ell+1}_R(M,N) \to O\,.$$

which yields $\operatorname{pd}_R K = \ell$ as desired in this case. □

(17.17) **Exercise: Injective Dimension & Short-exact Sequences.** *If the sequence* $O \to K \to L \to M \to O$ *is exact, and* $\operatorname{id}_R L < \operatorname{id}_R K$ $(\leq \infty)$, *prove that then* $\operatorname{id}_R K = \operatorname{id}_R M + 1$.

(17.18) **LHF 0$^{\text{def}}$= Lemma on Half-exact Functors 0.** *If* $T\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ *is a half-exact (co- or contravariant) functor such that* $T(R/\mathfrak{a}) = O$ *for all ideals* $\mathfrak{a}$ *in* R, *then* $T(M') = O$ *for all finitely generated* R*–modules* M'.

(17.19) **Proof.** Assume that $M' = Rm'_1 + \cdots + Rm'_n$ and use induction on n.

$\underline{n = 1}$: $M' = Rm'_1$ is cyclic, so it is isomorphic to $R/\mathfrak{a}$ for some ideal $\mathfrak{a}$, cf. (2.32), and $T(M') = O$ by assumption.

$\underline{n > 1}$: Set $M'' =^{\text{def}} Rm'_n$ and $N' =^{\text{def}} M'/M''$, and note that N' is generated by the $n-1$ elements $[m'_1]_{M''}, \ldots, [m'_{n-1}]_{M''}$, so $T(N') = O$ by the inductive hypothesis. Furthermore, $T(M'') = O$ by the already established case. As the functor T is half-exact, it sends the short-exact sequence $O \to M'' \to M' \to N' \to O$ into the exact sequence $T(M'') \to T(M') \to T(N')$. As the two outer modules in the latter sequence are trivial, so is the middle one, as desired. □

18. NOETHERIAN RINGS AND MODULES

(18.1) **Noetherian Modules.** An R–module M is said to be *Noetherian*, exactly when every submodule is finitely generated. The R–module R is Noetherian, if and only if the ring R is Noetherian in as defined in (14.15).

(18.2) **Exercise: ACC.** Prove that an R–module M is Noetherian, if and only if it satisfies the next *Ascending Chain Condition:*

(ACC) Every ascending chain $M_1 \subseteq \cdots \subseteq M_n \subseteq M_{n+1} \subseteq \cdots$ of submodules of M is eventually stationary, that is, there exists an $N \in \mathbb{N}$ such that $M_N = M_{N+i}$ for all $i \in \mathbb{N}$: $M_1 \subseteq \cdots \subseteq M_N = M_{N+1} = \cdots$.

(18.3) **Noetherian Rings.** These include fields, PIDs, homomorphic images of Noetherian rings, cf. (18.4), and rings of fractions of Noetherian rings with respect to special multiplicative systems, cf. (18.5).

(18.4) **Exercise: Homomorphic images.** Let $\mathfrak{a}$ be a proper ideal in a Noetherian ring R. Prove that the ring of residue classes $R/\mathfrak{a}$ is Noetherian. Justify the title of this exercise.

(18.5) **Exercise: Ring of fractions.** Let S be a special multiplicative system in a Noetherian ring R. Prove that the ring $S^{-1}R$ of fractions is Noetherian.

(18.6) **Polynomial Ring.** If R is a Noetherian ring, then the polynomial ring $R[X]$ in one variable X is Noetherian, cf. [Fo, ROM 4.2(5)].

(18.7) **Non–Noetherian Ring.** Let the set X be infinite, and consider the ring R^X. By (3.14) $R^{(X)}$ is an ideal that is not finitely generated. Thus, the ring R^X is not Noetherian.

(18.8) **Proposition: Noetherianness & Short-exact Sequences.** *For any exact sequence of R–modules $O \to K \xrightarrow{\alpha} L \xrightarrow{\beta} M \to O$ the following holds:*

$$L \text{ is Noetherian} \iff K \text{ and } M \text{ are Noetherian.}$$

(18.9) **Partial proof.** Assume that L is Noetherian, and we will prove that M is Noetherian. Let N be a submodule of M. By assumptions, the inverse image $\beta^{-1}(N)$ is finitely generated. Hence so is $\beta(\beta^{-1}(N)) = N$ (where the inclusion "$\supseteq$" holds as β is surjective. □

(18.10) **Exercise.** Carry out the remaining parts of the proof of (18.8).

(18.11) **Corollary.** *For any R–modules $M_1, \ldots, M_n$ the next holds:*

$$M_1 \oplus \cdots \oplus M_n \text{ is Noetherian} \iff M_i \text{ is Noetherian for all } i.$$

(18.12) **Proof.** Apply the proposition to the exact sequence:

$$O \to M_1 \oplus \cdots \oplus M_{n-1} \hookrightarrow M_1 \oplus \cdots \oplus M_n \twoheadrightarrow M_n \to O\,.$$

□

(18.13) **Finitely Generated over Noetherian Ring.** *If M is a finitely generated module over a Noetherian ring R, the M is a Noetherian module.*

(18.14) **Proof.** Choose a surjective R–homomorphism $R^n \to M$. Since R^n is Noetherian by the corollary (18.11), so is M by the proposition (18.8). □

(18.15) **FtP** $\stackrel{\text{def}}{=}$ **Finiteness Principle.** *Let* $F\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ *be an additive functor, and assume that*

(∗) $F(R)$ *is a finitely generated* R'*–module.*

Assume furthermore, **either**

(a) F *is right-exact and covariant;* **or**
(b) F *is left-exact and contravariant, and the ring* R' *is Noetherian.*

The following then holds:

(†) M' *is a finitely generated over* R $\implies$ $F(M')$ *is a finitely generated over* R'.

(18.16) **Proof.** (3.29) yields an exact sequence $R^n \to M' \to O$. As $F(R^n) \cong F(R)^n$, it induces the next exact sequence, respectively:

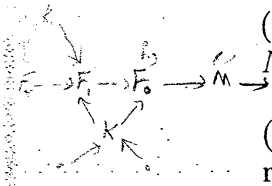
(a') $F(R)^n \to F(M') \to O$;

(b') $O \to F(M') \to F(R)^n$.

In case (a), the desired assertion (†) follows from the assumption (∗) and the exact sequence (a'). In case (b), we use that $F(R)^n$ is Noetherian by (18.13). □

(18.17) **Corollary.** *If* R *is Noetherian, and* M *and* N *are finitely generated* R*–modules, then the module* $\operatorname{Hom}_R(M, N)$ *is finitely generated.*

(18.18) **Proof.** The functor $\operatorname{Hom}_R(-, N)$ is left-exact and contravariant. The R–module $\operatorname{Hom}_R(R, N)$ is finitely generated, since it is isomorphic to N by (4.4). Thus, the desired assertion follows from **FtP**. □



(18.19) **Proposition.** *If* M *is a finitely generated module over a Noetherian ring* R*, then* M *has a projective resolution* $P_\cdot$ *consisting of finitely generated free modules.*

(18.20) **Proof.** As in (7.27), choose finitely generated free modules $P_0, P_1, \dots, P_n, \dots$ recursively. The first one, P_0, can be chosen finitely generated free as M is finitely generated, cf. (3.29). When finitely generated free modules $P_0, P_1, \dots, P_{n-1}$ have been chosen, there is the next exact sequence:

$$O \to K_{n-1} \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to O\,.$$

Since P_{n-1} is finitely generated, and R is Noetherian, the module P_{n-1} is Noetherian by (18.13), and hence the submodule K_{n-1} is finitely generated, so it is a homomorphic image of a finitely generated free P_n. □

(18.21) **Corollary.** *If* R *is Noetherian, and* M *and* N *are finitely generated* R*–modules, then the modules* $\operatorname{Ext}^n_R(M, N)$ *are finitely generated for all* n.

(18.22) **Proof.** Choose, by the proposition, a projective resolution $P_\cdot$ of M with each P_n finitely generated. The module $\operatorname{Hom}_R(P_n, N)$ is then Noetherian by (18.17), and hence its submodule $\mathrm{Z}^n(\operatorname{Hom}_R(P_\cdot, N))$ is finitely generated. Thus, so is $\operatorname{Ext}^n_R(M, N) = \mathrm{H}^n(\operatorname{Hom}_R(P_\cdot, N))$ as it is a homomorphic image of $\mathrm{Z}^n(\operatorname{Hom}_R(P_\cdot, N))$. □

(18.23) **IsoP** $\stackrel{\text{def}}{=}$ **Isomorphism Principle.** *Let $\Theta\colon F \to G$ be a morphism of additive functors $F, G\colon \mathrm{Mod}(R) \to \mathrm{Mod}(R')$. Assume that R is Noetherian, and that*

$$(*) \qquad \Theta^R\colon F(R) \to G(R) \text{ is an isomorphism.}$$

Assume furthermore, **either**

(a) *F and G are right-exact and covariant;* **or**

(b) *F and G are left-exact and contravariant.*

The following then holds:

(†) $\Theta^{M'}\colon F(M') \to G(M')$ *is an isomorphism for all finitely generated R-modules M'.*

(18.24) **Proof.** The proof is divided into two parts.

Part 1: Θ^{R^n} *is an isomorphism for all* $n \in \mathbb{N}$. Proof by induction on n.

$n = 1$ by assumption.

$n > 1$. The split-exact sequence $O \to R^{n-1} \hookrightarrow R^n \twoheadrightarrow R \to O$ induces, as Θ is a morphism of functors, the next commutative diagram:

$$\begin{array}{ccccccccc} O & \longrightarrow & F(R^{n-1}) & \longrightarrow & F(R^n) & \longrightarrow & F(R) & \longrightarrow & O \\ & & \downarrow{\scriptstyle \Theta^{R^{n-1}}}\ \cong & & \downarrow{\scriptstyle \Theta^{R^n}} & & \downarrow{\scriptstyle \Theta^{R}}\ \cong & & \\ O & \longrightarrow & G(R^{n-1}) & \longrightarrow & G(R^n) & \longrightarrow & G(R) & \longrightarrow & O \end{array}$$

The rows are exact, cf. (6.42), Θ^R is an isomorphism by assumption, and $\Theta^{R^{n-1}}$ is an isomorphism by the inductive hypothesis. Thus, by the Five Lemma, cf. (6.33), Θ^R is one too, as desired.

Part 2: (†) *holds.* Choose, by (18.19), an exact sequence $F_1 \to F_0 \to M \to O$ with F_0 and F_1 finitely generated free. This sequence induces in each of the two cases a commutative diagram with exact rows:

(a)
$$\begin{array}{ccccccc} F(F_1) & \longrightarrow & F(F_0) & \longrightarrow & F(M) & \longrightarrow & O \\ \downarrow{\scriptstyle \Theta^{F_1}}\ \cong & & \downarrow{\scriptstyle \Theta^{F_0}}\ \cong & & \downarrow{\scriptstyle \Theta^{M}} & & \\ G(F_1) & \longrightarrow & G(F_0) & \longrightarrow & G(M) & \longrightarrow & O \end{array}$$

(b)
$$\begin{array}{ccccccc} O & \longrightarrow & F(M) & \longrightarrow & F(F_0) & \longrightarrow & F(F_1) \\ & & \downarrow{\scriptstyle \Theta^{M}} & & \downarrow{\scriptstyle \Theta^{F_0}}\ \cong & & \downarrow{\scriptstyle \Theta^{F_1}}\ \cong \\ O & \longrightarrow & G(M) & \longrightarrow & G(F_0) & \longrightarrow & G(F_1) \end{array}$$

The isomorphisms result from the first part of the proof. The Five Lemma shows that Θ^M is an isomorphism in either case. □

(18.25) **Theorem: Hom & Fractions.** *Let R be Noetherian, let N be an R-module, and let S be a special multiplicative system. There is then a morphism of functors:*

$$\lambda^S_{-N}\colon S^{-1}\mathrm{Hom}_R(-, N) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}(-), S^{-1}N)\,.$$

Whenever M' is a finitely generated R-module, it induces an isomorphism:

$$S^{-1}\mathrm{Hom}_R(M', N) \xrightarrow[\cong]{\lambda^S_{M'N}} \mathrm{Hom}_{S^{-1}R}(S^{-1}M', S^{-1}N)$$

(18.26) **Proof.** For any R–module M an $S^{-1}R$–homomorphism

$$\lambda^S_{MN}\colon S^{-1}\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N)$$

is defined by that the $S^{-1}R$–homomorphism $\lambda^S_{MN}(\varphi/s)\colon S^{-1}M \to S^{-1}N$ for $\varphi/s \in S^{-1}\mathrm{Hom}_R(M,N)$ is well-defined by $\big(\lambda^S_{MN}(\varphi/s)\big)(m/t) =^{\mathrm{def}} \varphi(m)/(st)$ for $m/t \in S^{-1}M$. An inspection verifies that this is a morphism of functors.

Both functors are left-exact and contravariant. The next diagram is commutative:

$$\begin{array}{ccc} S^{-1}\mathrm{Hom}_R(R,N) & \xrightarrow{\lambda^S_{RN}} & \mathrm{Hom}_{S^{-1}R}(S^{-1}R, S^{-1}N) \\ & {\scriptstyle S^{-1}\alpha^N_R}\searrow \quad \swarrow{\scriptstyle \alpha^{S^{-1}N}_{S^{-1}R}} & \\ & S^{-1}N\,. & \end{array}$$

Consult (4.4) concerning the isomorphisms α. It follows that λ^S_{RN} is an isomorphism, and the Isomorphism Principle **IsoP** (18.23) yields that $\lambda^S_{M'N}$ is an isomorphism for all finitely generated R–module M', as desired. □

(18.27) **Corollary: Ext & Fractions.** *Let R be Noetherian, let N be an R–module, and let S be a special multiplicative system. For each finitely generated R–module M' there is an isomorphism:*

$$S^{-1}\mathrm{Ext}^n_R(M',N) \cong \mathrm{Ext}^n_{S^{-1}R}(S^{-1}M', S^{-1}N)\,.$$

(18.28) **Proof.** Choose, by Proposition (18.19), a projective resolution $P.$ of M' with each P_n finitely generated.

$$\begin{aligned} S^{-1}\mathrm{Ext}^n_R(M',N) &= S^{-1}\mathrm{H}^n(\mathrm{Hom}_R(P.,N)) \\ &\cong \mathrm{H}^n(S^{-1}\mathrm{Hom}_R(P.,N)) \\ &\cong \mathrm{H}^n(\mathrm{Hom}_{S^{-1}R}(S^{-1}P., S^{-1}N)) \\ &= \mathrm{Ext}^n_{S^{-1}R}(S^{-1}P., S^{-1}N)\,. \end{aligned}$$

The equalities are by the definition of Ext, cf. (11.17), the first isomorphism is by (8.22), and the second one is by the theorem. □

(18.29) **Corollary: Injective Dimension & Fractions.** *If R is Noetherian, and N is any R–module, then $\mathrm{id}_{S^{-1}R} S^{-1}N \le \mathrm{id}_R N$.*

(18.30) **Proof.** By **IDT** (17.3) it suffices to prove that $\mathrm{Ext}^n_{S^{-1}R}(S^{-1}R/\widetilde{\mathfrak{a}}, S^{-1}N) = O$ whenever $n > \mathrm{id}_R N$ and $\widetilde{\mathfrak{a}}$ is an ideal in $S^{-1}R$. The ideal $\widetilde{\mathfrak{a}}$ in $S^{-1}R$ induces an ideal $\mathfrak{a} =^{\mathrm{def}} \{a \in R \mid a/1 \in \widetilde{\mathfrak{a}}\}$ in R. Note that $S^{-1}\mathfrak{a} = \widetilde{\mathfrak{a}}$, so that the exactness of the fraction functor yields $S^{-1}(R/\mathfrak{a}) \cong S^{-1}R/\widetilde{\mathfrak{a}}$, cf. (6.24). This gives the first of the next two isomorphisms:

$$\mathrm{Ext}^n_{S^{-1}R}(S^{-1}R/\widetilde{\mathfrak{a}}, S^{-1}N) \cong \mathrm{Ext}^n_{S^{-1}R}(S^{-1}(R/\mathfrak{a}), S^{-1}N) \cong S^{-1}\mathrm{Ext}^n_R(R/\mathfrak{a}, N) = O$$

for $n > \mathrm{id}_R N$; the second isomorphism is by the previous corollary (18.27), and the equality is by **IDT** (17.3). □

(18.31) **Corollary: Injectivity & Fractions.** *If R is Noetherian, and N is an injective R–module, then $S^{-1}N$ is an injective module over $S^{-1}R$.* □

(18.32) **Theorem: Projective Dimension over Noetherian Rings.** *If R is Noetherian, and M' is a finitely generated R–module, then the next holds:*

$$\begin{aligned}\operatorname{pd}_R M' &= \sup\{\, n\in \mathbb{N}_0 \mid \operatorname{Ext}^n_R(M',N') \neq O \ \text{ for some finitely generated } \ N'\,\}\\ &= \sup\{\, n\in \mathbb{N}_0 \mid \operatorname{Ext}^n_R(M',R/\mathfrak{b}) \neq O \ \text{ for some ideal } \ \mathfrak{b}\,\}\,.\end{aligned}$$

(18.33) **Proof.** By **PDC** (11.23) it suffices to prove the next:

$$(*)\qquad \operatorname{pd}_R M' \leq \sup\{\, n\in \mathbb{N}_0 \mid \operatorname{Ext}^n_R(M',R/\mathfrak{b}) \neq O \ \text{ for some ideal } \ \mathfrak{b}\,\}\,.$$

Let w denote the number on the right, and assume that it is finite. As $\operatorname{Ext}^{w+1}_R(M',R/\mathfrak{b}) = O$ for all ideals $\mathfrak{b}$, **LHF 0** (17.18) yields $\operatorname{Ext}^{w+1}_R(M',N') = O$ for all finitely generated N'. Let $P.$ be a projective resolution of M' with P_n finitely generated—and hence Noetherian—for all n, cf. (18.19). The w'th kernel $\operatorname{K}_w(P.)$, in the standard notation of (10.26), is a submodule of the Noetherian module P_w, and hence $\operatorname{K}_w(P.)$ is finitely generated. It results that $\operatorname{Ext}^{w+1}_R(M',\operatorname{K}_w(P.)) = O$ by our assumption. Now, the desired $\operatorname{pd}_R M' \leq w$ follows from **PDT** (11.19). □

(18.34) **Theorem: Global Dimension over Noetherian Rings.** *If R is Noetherian, then the next holds:*

$$\begin{aligned}\operatorname{gldim} R &= \sup\{\operatorname{id}_R N' \mid N' \ \text{ is finitely generated }\}\\ &= \sup\{\operatorname{id}_R R/\mathfrak{b} \mid \mathfrak{b} \ \text{ is an ideal }\}\\ &= \sup\{\, n\in \mathbb{N}_0 \mid \operatorname{Ext}^n_R(R/\mathfrak{a},R/\mathfrak{b}) \neq O \ \text{ for some ideals } \ \mathfrak{a}\,,\,\mathfrak{b}\,\}\,.\end{aligned}$$

(18.35) **Proof.** In the first two lines the inequalities "$\geq$" are obvious, and "$\geq$" in the last line follows from **IDT** (17.3). Let s denote the last number in the theorem. Thus, it suffices to prove that $\operatorname{gldim} R \leq s$. Since $\operatorname{gldim} R = \sup\{\operatorname{pd}_R R/\mathfrak{a} \mid \mathfrak{a} \text{ is an ideal}\}$ by Auslander's Theorem (17.9), the previous theorem (18.32) yields $\operatorname{gldim} R \leq s$. □

(18.36) **Exercise: Global Dimension Zero.** Assume that $\operatorname{gldim} R = 0$. Prove that R is Noetherian. Prove that R is isomorphic to a finite product of fields.

19. Tensor Product

(19.1) **Maps.** For sets X and Y, the *set of maps* $\alpha\colon X \to Y$ is denoted $\operatorname{Maps}(X,Y)$. For maps of sets $\varphi\colon X \to X'$ and $\psi\colon Y \to Y'$ there are the next two induced maps:

$$\operatorname{Maps}(\varphi,Y)\colon \operatorname{Maps}(X',Y) \to \operatorname{Maps}(X,Y)\,,\ \alpha' \mapsto \alpha'\varphi\,;$$
$$\operatorname{Maps}(X,\psi)\colon \operatorname{Maps}(X,Y) \to \operatorname{Maps}(X,Y')\,,\ \beta \mapsto \psi\beta\,.$$

(19.2) **Hom.** For R–modules M and N, the set $\operatorname{Hom}_R(M,N)$ is a subset of $\operatorname{Maps}(M,N)$, that is, there is an inclusion:

$$\omega_{MN}\colon \operatorname{Hom}_R(M,N) \hookrightarrow \operatorname{Maps}(M,N)\,.$$

(19.3) **Maps of Product Sets.** Consider sets X, Y, and Z, and there product set:

$$X \times Y = \{\,(x,y) \mid x \in X \ \text{ and } \ y \in Y\,\}\,.$$

There are then bijective maps $\Phi = \Phi^{XYZ}$ and $\Psi = \Psi^{XYZ}$, each the inverses of the other:

$$(\Diamond) \qquad \operatorname{Maps}(X \times Y, Z) \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \operatorname{Maps}(X, \operatorname{Maps}(Y,Z))\,.$$

Indeed, for $\gamma \in \operatorname{Maps}(X \times Y, Z)$ the element $\Phi(\gamma) \in \operatorname{Maps}(X, \operatorname{Maps}(Y,Z))$, that is, the map $\Phi(\gamma)\colon X \to \operatorname{Maps}(Y,Z)$, is defined as follows: For $x \in X$ the element $\big(\Phi(\gamma)\big)(x) \in \operatorname{Maps}(Y,Z)$, that is, the map $\big(\Phi(\gamma)\big)(x)\colon Y \to Z$, is defined as:

$$\Big(\big(\Phi(\gamma)\big)(x)\Big)(y) \stackrel{\text{def}}{=} \gamma(x,y)\,.$$

Moreover, for $\beta \in \operatorname{Maps}(X, \operatorname{Maps}(Y,Z))$ and $(x,y) \in X \times Y$ we set:

$$\big(\Psi(\beta)\big)(x,y) \stackrel{\text{def}}{=} \big(\beta(x)\big)(y)\,.$$

It is now straightforward to verify that $\Psi\circ\Phi$ and $\Phi\circ\Psi$ are the identities on $\operatorname{Maps}(X\times Y,Z)$ and $\operatorname{Maps}(X, \operatorname{Maps}(Y,Z))$, respectively. (Sometimes $\operatorname{Maps}(U,W)$ is denoted by W^U, and $(\Diamond)$ is expressed as $Z^{X\times Y} \cong \big(Z^Y\big)^X$.)

(19.4) **Homomorphisms of Suitable Product Modules.** For R–modules M and N we want to construct a new module $M \otimes_R N$, the *tensor product*, aping $(\Diamond)$, that is, such that there, for any R–module P, are the top isomorphisms in the next commutative diagram:

$$(\square) \qquad \begin{array}{ccc} \operatorname{Hom}_R(M \otimes_R N, P) & \underset{\cong}{\overset{\cong}{\rightleftarrows}} & \operatorname{Hom}_R(M, \operatorname{Hom}_R(N,P)) \\ \Big\downarrow{\scriptstyle \omega_{M\otimes_R N,P}} & & \Big\downarrow{\scriptstyle \omega_{M,\operatorname{Hom}_R(N,P)}} \\ \operatorname{Maps}(M \otimes_R N, P) & & \operatorname{Maps}(M, \operatorname{Hom}_R(N,P)) \\ \Big\downarrow{\scriptstyle \operatorname{Maps}(\varepsilon_{M,N},P)} & & \Big\downarrow{\scriptstyle \operatorname{Maps}(M,\omega_{NP})} \\ \operatorname{Maps}(M \times N, P) & \underset{\Psi^{MNP}}{\overset{\Phi^{MNP}}{\rightleftarrows}} & \operatorname{Maps}(M, \operatorname{Maps}(N,P))\,. \end{array}$$

The lower left vertical map is induced by by a map $\varepsilon_{M,N}\colon M \times N \to M \otimes_R N$ that results from the construction below of the tensor product, cf. (19.12).

(19.5) **Bilinear Maps.** Let M, N, and P be R–modules.

The set of maps $\mathrm{Maps}(M \times N, P)$ is then an R–module, cf. (14.22), namely $P^{M\times N}$.

A submodule $\mathrm{Bil}_R(M, N; P)$ of it is then defined to consist exactly of the *R–bilinear maps* $\gamma\colon M \times N \to P$, that is, those γ satisfying the next four relations (in the module P) for all $m, m' \in M$, all $n, n' \in N$, and all $r \in R$:

$$\text{(a1)} \qquad \gamma(m+m', n) = \gamma(m,n) + \gamma(m',n)\,;$$

$$\text{(m1)} \qquad \gamma(rm, n) = r\gamma(m,n)\,;$$

$$\text{(a2)} \qquad \gamma(m, n+n') = \gamma(m,n) + \gamma(m,n')\,;$$

$$\text{(m2)} \qquad \gamma(m, rn) = r\gamma(m,n)\,.$$

(a1) and (m1) mean that, for a fixed $n \in N$, the map $M \to P$, $x \mapsto \gamma(x,n)$ is an R–homomorphism, that is, it belongs to $\mathrm{Hom}_R(M, P)$.

(a2) and (m2) mean that, for a fixed $m \in M$, the map $N \to P$, $x \mapsto \gamma(m,x)$ is an R–homomorphism, that is, it belongs to $\mathrm{Hom}_R(N, P)$.

The subset $\mathrm{Bil}_R(M, N; P)$ is a submodule of the R–module $\mathrm{Maps}(M \times N, P)$, it is called *the module of bilinear maps: $M \times N \to P$.*

(19.6) **Bil & Double Hom.** The diagram ($\square$) in (19.4) contains two bijective maps Φ^{MNP} and Ψ^{MNP}, and each one is the inverse of the other. The restrictions $\Phi|^{MNP}$ and $\Psi|^{MNP}$ of these, cf. the diagram below, are easily seen to be isomorphisms, and again each one is the inverse of the other:

$$\text{(†)} \qquad \begin{array}{ccc} \mathrm{Bil}_R(M,N;P) & \underset{\Psi|^{MNP}}{\overset{\Phi|^{MNP}}{\underset{\cong}{\rightleftarrows}}} & \mathrm{Hom}_R(M, \mathrm{Hom}_R(N,P)) \\ \downarrow & & \downarrow \\ \mathrm{Maps}(M\times N, P) & \underset{\Psi^{MNP}}{\overset{\Phi^{MNP}}{\rightleftarrows}} & \mathrm{Maps}(M, \mathrm{Maps}(N,P)) \end{array}.$$

(19.7) **Free Module over $M\times N$.** Let M and N be R–modules. Consider the product *set* $M \times N$ (thus forgetting its module structure for the time being), and the free R–module $R^{(M\times N)}$ with basis $\big(\delta_{(m,n)}\big)_{(m,n)\in M\times N}$, cf. (3.27).

Let K_{MN} be the submodule of $R^{(M\times N)}$ generated by the elements

$$\text{(a1)} \qquad \delta_{(m+m',n)} - \delta_{(m,n)} - \delta_{(m',n)}\,;$$

$$\text{(m1)} \qquad \delta_{(rm,n)} - r\delta_{(m,n)}\,;$$

$$\text{(a2)} \qquad \delta_{(m,n+n')} - \delta_{(m,n)} - \delta_{(m,n')}\,;$$

$$\text{(m2)} \qquad \delta_{(m,rn)} - r\delta_{(m,n)}\,.$$

(19.8) **Tensor Product of Modules.** The *tensor product* of M and N is the R–module

$$M \otimes_R N \overset{\text{def}}{=} R^{(M\times N)}/K_{MN} \quad \text{(confer (19.7))}.$$

(19.9) **Tensor Product of Elements.** Furthermore, for $m \in M$ and $n \in N$ we define the the *tensor product of the elements* $m \in M$ and $n \in N$ as follows:

$$m \otimes n \overset{\text{def}}{=} [\,\delta_{(m,n)}\,]_{K_{MN}}$$

(19.10) **Generators.** As $\{\, \delta_{(m,n)} \mid (m,n) \in M \times N \,\}$ generates $R^{(M\times N)}$ we have:

$$M \otimes_R N = \operatorname{span}_R\{\, m \otimes n \mid m \in M \text{ and } n \in N \,\},$$

that is, $M \otimes_R N$ is generated by $m \otimes n$ for $m \in M$ and $n \in N$.

(19.11) **Relations.** Since K_{MN}, cf. (19.7), contains the elements (a1), (m1), (a2), and (m2), there are the following relations among these generators.

(A1) $$(m+m') \otimes n = m \otimes n + m' \otimes n;$$

(M1) $$(rm) \otimes n = r(m \otimes n);$$

(A2) $$m \otimes (n+n') = m \otimes n + m \otimes n';$$

(M2) $$m \otimes (rn) = r(m \otimes n).$$

(19.12) **Bilinear Map into Tensor Product.** The map

$$\varepsilon_{M,N} \colon M \times N \to M \otimes_R N,\ (m,n) \mapsto m \otimes n$$

is *bilinear*, that is, $\varepsilon_{M,N} \in \operatorname{Bil}_R(M,N; M \otimes_R N)$. In general, it is not an R–homomorphism (and that was the reason for forgetting the module structure on $M \times N$ in (19.7)).

(19.13) **Theorem: Bilinear Map versus Homomorphism from Tensor Product.** *Let M, N, and P be R–modules. The following then hold.*

(a) *Any R–bilinear map $\gamma \colon M \times N \to P$ induces an R–homomorphism $\widetilde{\gamma} \colon M \otimes_R N \to P$ given on generators by $\widetilde{\gamma}(m \otimes n) =^{\text{def}} \gamma(m,n)$.*

(b) *Any R–homomorphism $\alpha \colon M \otimes_R N \to P$ induces an R–bilinear map $\widehat{\alpha} \colon M \times N \to P$ given by $\widehat{\alpha}(m,n) =^{\text{def}} \alpha(m \otimes n)$.*

(c) *There are R–isomorphisms that are inverse to each other:*

$$\operatorname{Bil}_R(M,N;P) \underset{\cong}{\overset{\cong}{\rightleftarrows}} \operatorname{Hom}_R(M \otimes_R N, P)$$
$$\gamma \longmapsto \widetilde{\gamma}$$
$$\widehat{\alpha} \longleftarrow \alpha .$$

(19.14) **Proof.** (a) Let the R–homomorphism $R^{(M\times N)} \to P$ be given on basis elements by $\delta_{(m,n)} \mapsto \gamma(m,n)$; its kernel contains K_{MN} since γ is bilinear. Thus, the Homomorphism Theorem (2.28) yields an R–homomorphism $\widetilde{\gamma} \colon M \otimes_R N \to P$ that is well-defined by $\widetilde{\gamma}(m \otimes n) = \gamma(m,n)$.

(b) is obvious, and (c) follows. □

(19.15) **Tensor Product is a Functor.** For any R–homomorphism $\mu \colon M \to M'$ and any R–module N there is an induced R–homomorphism

$$\mu \otimes_R N \colon M \otimes_R N \to M' \otimes_R N,\ m \otimes n \mapsto \mu(m) \otimes n,$$

corresponding to the bilinear map $M \times N \to M' \otimes_R N$, $(m,n) \mapsto \mu(m) \otimes n$. It is straightforward to verify that this makes

$$- \otimes_R N \colon \operatorname{Mod}(R) \to \operatorname{Mod}(R),\ M \mapsto M \otimes_R N,\ \mu \mapsto \mu \otimes_R N$$

into an additive, covariant functor. Similarly, there is an additive, covariant functor:

$$N \otimes_R - \colon \operatorname{Mod}(R) \to \operatorname{Mod}(R),\ M \mapsto N \otimes_R M,\ \mu \mapsto N \otimes_R \mu.$$

(19.16) Exercise. Prove that $- \otimes_R -$ is a bi-functor.

(19.17) **Theorem: Adjointness.** *If M, N, and P are R–modules, then there are inverse R–isomorphisms:*

$$\mathrm{Hom}_R(M \otimes_R N, P) \underset{\Gamma^{MNP}}{\overset{\Delta^{MNP}}{\underset{\cong}{\rightleftarrows}}} \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P)) .$$

These induce the next isomorphisms of functors[20]*:*

(19.17.1) $\mathrm{Hom}_R(- \otimes_R N, P) \cong \mathrm{Hom}_R(-, \mathrm{Hom}_R(N, P))$;

(19.17.2) $\mathrm{Hom}_R(M \otimes_R -, P) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(-, P))$;

(19.17.3) $\mathrm{Hom}_R(M \otimes_R N, -) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, -))$.

(19.18) **Proof.** Definition of the map Δ^{MNP}: For $\gamma \in \mathrm{Hom}_R(M \otimes_R N, P)$ the map $\Delta^{MNP}(\gamma)\colon M \to \mathrm{Hom}_R(N, P)$ is defined by that the map $\big(\Delta^{MNP}(\gamma)\big)(m)\colon N \to P$ for $m \in M$ is defined for $n \in N$ by:

$$\Big(\big(\Delta^{MNP}(\gamma)\big)(m)\Big)(n) \stackrel{\text{def}}{=} \gamma(m \otimes n) \ (\in P)$$

Definition of Γ^{MNP}: For $\beta \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, P))$ the map $\Gamma^{MNP}(\beta)\colon M \otimes_R N \to P$ is well-defined on a generator $m \otimes n \in M \otimes_R N$ by:

$$\big(\Gamma^{MNP}(\beta)\big)(m \otimes n) \stackrel{\text{def}}{=} \big(\beta(m)\big)(n) \ (\in P)$$

It follows from (19.6.†) and (19.13.c) that Δ^{MNP} and Γ^{MNP} are inverse isomorphisms. The three remaining assertions are verified by inspection. □

(19.19) **Theorem: Tensor Product is Right Exact.** *If $O \to M \to M' \to M'' \to O$ is exact, then the induced sequence $M \otimes_R N \to M' \otimes_R N \to M'' \otimes_R N \to O$ is exact.*

(19.20) **Proof.** We want to prove the exactness of the next complex:

$$(*) \qquad K_\bullet \stackrel{\text{def}}{=} M \otimes_R N \to M' \otimes_R N \to M'' \otimes_R N \to O .$$

Let I be any R–module, set $H \stackrel{\text{def}}{=} \mathrm{Hom}_R(N, I)$, and consider the next commutative diagram:

$$\begin{array}{ccccccc}
O & \longrightarrow & \mathrm{Hom}_R(M'' \otimes_R N, I) & \longrightarrow & \mathrm{Hom}_R(M' \otimes_R N, I) & \longrightarrow & \mathrm{Hom}_R(M \otimes_R N, I) \\
 & & \Big\downarrow{\scriptstyle \Delta^{M''NI}}\cong & & \Big\downarrow{\scriptstyle \Delta^{M'NI}}\cong & & \Big\downarrow{\scriptstyle \Delta^{MNI}}\cong \\
O & \longrightarrow & \mathrm{Hom}_R(M'', H) & \longrightarrow & \mathrm{Hom}_R(M', H) & \longrightarrow & \mathrm{Hom}_R(M, H)
\end{array}$$

The isomorphisms and the commutativity come from (19.17.1), and the lower row is exact as it results from applying the left-exact functor $\mathrm{Hom}_R(-, H)$ applied the the given exact sequence. Thus, the upper row is exact, and the Exactness Principle **ExP** (12.27) yields that the sequence $(*)$ is exact, as desired. □

(19.21) **Exercise.** Provide a direct, element-wise, proof of the theorem (19.19).

(19.22) **Theorem: Tensor Product & Over-lining.** *Let $\mathfrak{a}$ be a proper ideal in R. The corresponding over-lining functor is then isomorphic to the tensor product functor*

[20] That is, the isomorphisms Δ^{MNP} and Γ^{MNP} are *functorial* in M, N, and P.

$\overline{R}\otimes_R -$. *The isomorphisms are given as follows:*

(19.22.1)
$$\begin{array}{ccc} \overline{M} & \overset{\widetilde{\varphi}^M}{\underset{\widetilde{\psi}^M}{\rightleftarrows}} & \overline{R}\otimes_R M \\ \overline{m} & \longmapsto & \overline{1}\otimes m \\ \overline{rm} & \longmapsfrom & \overline{r}\otimes m\,, \end{array} \qquad \text{that is,} \qquad \begin{array}{ccc} M/\mathfrak{a}M & \overset{\widetilde{\varphi}^M}{\underset{\widetilde{\psi}^M}{\rightleftarrows}} & (R/\mathfrak{a})\otimes_R M \\ [m]_{\mathfrak{a}M} & \longmapsto & [1]_{\mathfrak{a}}\otimes m \\ [rm]_{\mathfrak{a}M} & \longmapsfrom & [r]_{\mathfrak{a}}\otimes m\,. \end{array}$$

(19.23) **Proof.** The R–homomorphism $\varphi\colon M \to (R/\mathfrak{a})\otimes_R M$ defined by $\varphi(m) \overset{\text{def}}{=} [1]_{\mathfrak{a}}\otimes m$ has $\operatorname{Ker}\varphi \supseteq \mathfrak{a}M$, so it induces well-defined $\widetilde{\varphi}^M$ as above, cf. (2.28): On the other hand, the map $\psi\colon (R/\mathfrak{a})\times M \to M/\mathfrak{a}M$ defined by $\psi([r]_{\mathfrak{a}},m) =^{\text{def}} [rm]_{\mathfrak{a}M}$ is well-defined and bilinear; thus, it induces $\widetilde{\psi}^M$ above. Finally, it remain to check that $\widetilde{\psi}^M\widetilde{\varphi}^M$ and $\widetilde{\varphi}^M\widetilde{\psi}^M$ are the identities on $\overline{M}$ and $\overline{R}\otimes_R M$, respectively. □

(19.24) **Corollary.** *The functor $R\otimes_R -$ is isomorphic to the identity functor $1_{\mathrm{Mod}(R)}$.* □

(19.25) **Theorem: Tensor Product & Fraction Functor.** *Let S be a special multiplicative system. The corresponding fraction functor $S^{-1}(-)$ is then isomorphic to the tensor product functor $S^{-1}R\otimes_R -$. The isomorphisms are given as follows:*

$$\begin{array}{ccc} S^{-1}M & \overset{\gamma^M}{\underset{\delta^M}{\rightleftarrows}} & S^{-1}R\otimes_R M \\ m/s & \longmapsto & (1/s)\otimes m \\ rm/s & \longmapsfrom & (r/s)\otimes m\,. \end{array}$$

(19.26) **Proof.** A bilinear map $\beta^M : S^{-1}R\times M \to S^{-1}M$ is well-defined by $\beta^M(r/s,m) \overset{\text{def}}{=} (rm)/s$, and it induces the R–homomorphism δ^M of the theorem. It is not hard to verify that the R–homomorphism γ^M is also well-defined, and it is easy to check that $\delta^M\gamma^M$ and $\gamma^M\delta^M$ are identities. □

(19.27) **Associativity.** *For R–modules M, N, and P there is the next isomorphism:*

$$\begin{array}{ccc} (M\otimes_R N)\otimes_R P & \xrightarrow{\cong} & M\otimes_R (N\otimes_R P) \\ (m\otimes n)\otimes p & \longmapsto & m\otimes(n\otimes p)\,. \end{array}$$

This is functorial in M, N, and P. □

(19.28) **Commutativity.** *For R–modules M and P there is the next isomorphism:*

$$\begin{array}{ccc} M\otimes_R P & \xrightarrow{\cong} & P\otimes_R M \\ m\otimes p & \longmapsto & p\otimes m\,. \end{array}$$

This is functorial in M and P. □

(19.29) **Exercise.** Obtain the Swap Isomorphism (12.12) from the associativity and the commutativity above.

(19.30) **Theorem: Tensor Product of Projectives.** *If P and P' are projective R–modules, then their tensor product $P\otimes_R P'$ is a projective R–module.*

(19.31) **Proof.** By Adjointness (19.17) we have the next isomorphism of functors:

$$(*) \qquad \mathrm{Hom}_R(P \otimes_R P', -) \cong \mathrm{Hom}_R(P, \mathrm{Hom}_R(P', -)) .$$

By assumptions, the functors $\mathrm{Hom}_R(P, -)$ and $\mathrm{Hom}_R(P', -)$ are exact. Hence so is their composition, that is, the right hand functor in $(*)$. Thus, $\mathrm{Hom}_R(P \otimes_R P', -)$ is exact, as desired. □

(19.32) **Dot—Slash Notation.** Setting $M{\cdot}N =^{\mathrm{def}} N \otimes_R M$ and $M/N =^{\mathrm{def}} \mathrm{Hom}_R(N, M)$ (and noting the order of M and N), we have the next five isomorphisms:

$(\cdot)$	$M{\cdot}N \cong M{\cdot}N$	commutativity, cf. (19.28).
$(\cdot\cdot)$	$(M{\cdot}N){\cdot}P \cong M{\cdot}(N{\cdot}P)$	associativity, cf. (19.27).
$(/\,/)$	$(M/N)/P \cong M/(N{\cdot}P)$	adjointness, cf. (19.17).
$(R\cdot)$	$R{\cdot}N \cong N$	unitarity, cf. (19.24).
$(/R)$	$M/R \cong M$	co-unitarity, cf. (4.4).

There are also the next two R-homomorphisms:

$(/\cdot)$	$(M/N){\cdot}P \to M/(N/P)$	evaluation, cf. (20.20).
$(\cdot/)$	$(M{\cdot}N)/P \leftarrow M{\cdot}(N/P)$	co–evaluation, cf. (20.15).

20. Flat Modules

The covariant functor $H \otimes_R -$ is right-exact for all R–modules H, cf. (19.19).

(20.1) **Flat modules.** An R–module H is said to be *flat*, exactly when the tensor functor $H \otimes_R -$ is *exact*, that is, if and only if the following holds for R–homomorphisms:

$$\mu\colon M \to M' \text{ is injective} \implies H \otimes_R \mu\colon H \otimes_R M \to H \otimes_R M' \text{ is injective.}$$

(20.2) **Examples: Flat modules.**

(a) The R–module R is flat as $R \otimes_R - \cong 1_{\mathrm{Mod}(R)}$, cf. (19.24), is exact.
(b) For any special multiplicative system S the R–module $S^{-1}R$ is flat as $S^{-1}R \otimes_R -$ is isomorphic to $S^{-1}(-)$, cf. (19.25), and the latter functor is exact by (6.24).
(c) If R is a domain but not a field, then the residue field B is flat by the above, but not projective, cf. (7.17).
(d) If $r \in R \setminus \mathrm{Zero}_R R$ is a non-invertible, then the R–module $R/(x)$ is not flat: Tensoring the injective homomorphism $R \xrightarrow{x} R$ by $R/(x)$ yields $R/(x) \xrightarrow{0} R/(x)$ which is not injective.
(e) If F is a flat R–module, then $\mathrm{Zero}_R F \subseteq \mathrm{Zero}_R R$. Indeed, for $x \in R \setminus \mathrm{Zero}_R R$ the injective homomorphism $R \xrightarrow{x} R$ induces by (a) the map $F \xrightarrow{x} F$. This map is injective due to the flatness of F, and hence $x \notin \mathrm{Zero}_R F$.

(20.3) **Proposition: Tensor Product of Flat Modules.** *If the R–modules F and F' are flat, then the module $F \otimes_R F'$ is flat.*

(20.4) **Proof.** The functor $(F \otimes_R F') \otimes_R -$ is isomorphic to $F \otimes_R (F' \otimes_R -)$, cf. (19.27), and the latter one is exact because it is the composition of two exact functors. □

(20.5) **Theorem: Flatness versus Injectivity.** *The following are equivalent for an R–module F.*

(i) F is flat;
(ii) $\mathrm{Hom}_R(F, I)$ is injective for all injective modules I;
(iii) The Pontryagin dual $F^\vee$ is injective.

(20.6) **Proof.** By Adjointness (19.17) we have the isomorphism:

$$(*) \qquad \mathrm{Hom}_R(- \otimes_R F, I) \cong \mathrm{Hom}_R(-, \mathrm{Hom}_R(F, I))$$

and "$(i) \implies (ii)$" follows.
"$(ii) \implies (iii)$" results from the definition of Pontryagin duals (12.26).
"$(iii) \implies (i)$": The isomorphism $(*)$ shows, in particular, that the functors $(- \otimes_R F)^\vee$ and $\mathrm{Hom}_R(-, F^\vee)$ are isomorphic. As the latter is exact, so is the former, and by the faithfulness, we conclude that $- \otimes_R F$ is exact too. □

(20.7) **Corollary: Projective is Flat.** *If P is a projective module, then P is flat.*

(20.8) **Proof.** The module $\mathrm{Hom}_R(P, I)$ is injective by (12.14), so the flatness of P results from the theorem (20.5). □

(20.9) **Corollary: Direct Sum of Flat Modules.** *If $(F_x)_{x \in X}$ is a family of flat R–modules, then the direct sum $\coprod_{x \in X} F_x$ is flat.*

(20.10) **Proof.** Theorem (14.19) yields $\operatorname{Hom}_R(\coprod_{x\in X} F_x, I) \cong \prod_{x\in X} \operatorname{Hom}_R(F_x, I)$ for any module I. If it is injective, then the latter module is injective by the theorem (20.5) and Corollary (14.12). Application of the theorem once more, yields the desired assertion. □

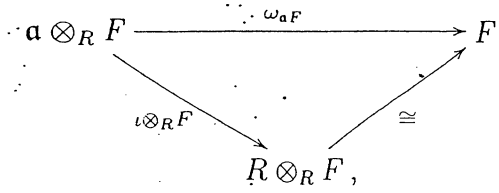
(20.11) **Exercise: Direct Sum and Tensor Product.** Let $(M_x)_{x\in X}$ be a family of R–modules. Establish an isomorphism $(\coprod_{x\in X} M_x)\otimes_R - \cong \coprod_{x\in X}(M_x \otimes_R -)$ of functors. Provide an alternative proof of the corollary above (20.9).

(20.12) **Tensor Product of Ideal and Module.** The next notation is needed for the Flatness Theorem (20.13) that follows below.

Let $\mathfrak{a}$ be an ideal in R, let $\iota\colon \mathfrak{a} \hookrightarrow R$ be the inclusion, and let F be an R–module.

The R–homomorphism $\omega_{\mathfrak{a}F}\colon \mathfrak{a}\otimes_R F \to F$ is then defined by the commutativity of the next diagram:

(∇)



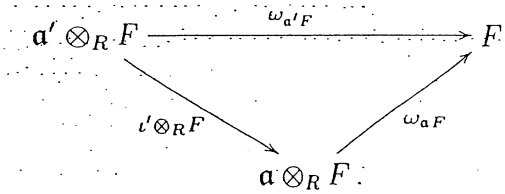
where the isomorphism is from (19.24). Thus, $\omega_{\mathfrak{a}F}$ is well-defined on a generator by the next formula:

$(*)$ $$\omega_{\mathfrak{a}F}(a\otimes f) \stackrel{\text{def}}{=} af$$

for $a\in\mathfrak{a}$ and $f\in F$.

If $\mathfrak{a}'$ is an ideal contained in $\mathfrak{a}$, and $\iota'\colon \mathfrak{a}' \hookrightarrow \mathfrak{a}$ is the inclusion, then there is the next commutative diagram:

(∇')



(20.13) **Flatness Theorem.** *The following are equivalent for any R–module F.*

(i) F is flat;

(ii) $\omega_{\mathfrak{a}F}\colon \mathfrak{a}\otimes_R F \to F$ is injective for all ideals $\mathfrak{a}$;

(ii') $\omega_{\mathfrak{a}'F}\colon \mathfrak{a}'\otimes_R F \to F$ is injective for all finitely generated ideals $\mathfrak{a}'$.

(20.14) **Proof.** "$(i) \Longrightarrow (ii')$": Assuming that F is flat, the injective R–homomorphism $\iota_{\mathfrak{a}'}\colon \mathfrak{a}' \hookrightarrow R$ induces an injective map $\iota_{\mathfrak{a}'}\otimes_R F\colon \mathfrak{a}'\otimes_R F \to R\otimes_R F$ in the diagram (∇) in (20.12); thus, $\omega_{\mathfrak{a}'F}$ is injective.

"$(ii') \Longrightarrow (ii)$": Assume that (ii') holds, and that w belongs to $\operatorname{Ker}\omega_{\mathfrak{a}F}$ [and we want to prove $w=0$]. As the module $\mathfrak{a}\otimes_R F$ is generated by $\{\, a\otimes f \mid a\in\mathfrak{a} \text{ and } f\in F\,\}$, cf. (19.10), $w = x_1(a_1\otimes f_1)+\cdots+x_n(a_n\otimes f_n)$ for some $n\in\mathbb{N}$, $x_1,\dots,x_n\in R$, $a_1,\dots,a_n\in\mathfrak{a}$, and $f_1,\dots,f_n\in F$. Consider the finitely generated ideal $\mathfrak{a}' \stackrel{\text{def}}{=} (a_1,\dots,a_n)$ which is contained in $\mathfrak{a}$, and the inclusion $\iota'\colon \mathfrak{a}' \hookrightarrow \mathfrak{a}$. To distinguish the elements in $\mathfrak{a}'\otimes_R F$ from those in $\mathfrak{a}\otimes_R F$, we use $\otimes'$ (instead of $\otimes$) when describing the generators of $\mathfrak{a}'\otimes_R F$, that is, $\mathfrak{a}'\otimes_R F$ is generated by $\{\, a'\otimes' f \mid a'\in\mathfrak{a}' \text{ and } f\in F\,\}$, and we have:

$$\iota'\otimes_R F\colon \mathfrak{a}'\otimes_R F \to \mathfrak{a}\otimes_R F,\ a'\otimes' f \mapsto a'\otimes f\,.$$

Furthermore, recall that $a_1, \dots, a_n$ belong to the ideal $\mathfrak{a}'$ (by the definition of it), and set:

$$w' \stackrel{\text{def}}{=} x_1(a'_1 \otimes' f_1) + \cdots + x_n(a'_n \otimes' f_n) \in \mathfrak{a}' .$$

We have then $(\iota' \otimes_R F)(w') = w$ and the commutative diagram (∇') in (20.12) yields $\omega_{\mathfrak{a}'F}(w') = \omega_{\mathfrak{a}F}(w) = 0$. Thus, by assumption, $w' = 0$, and hence $w = 0$, as desired.

"$(ii) \Longrightarrow (i)$": By Theorem (20.5) it suffices to prove that $\operatorname{Hom}_R(F, I)$ is injective, whenever I is an injective module. Let $\mathfrak{a}$ be any ideal, and consider the next commutative diagram:

$$\begin{array}{ccc}
\operatorname{Hom}_R(F, I) & & \\
\Big\downarrow \cong & \searrow^{\operatorname{Hom}_R(\omega_{\mathfrak{a}F}, I)} & \\
\operatorname{Hom}_R(R \otimes_R F, I) & \xrightarrow[\operatorname{Hom}_R(\iota \otimes_R F, I)]{} & \operatorname{Hom}_R(\mathfrak{a} \otimes_R F, I) \\
{\scriptstyle \Delta^{RFI}}\Big\downarrow \cong & & \cong \Big\downarrow {\scriptstyle \Delta^{\mathfrak{a}FI}} \\
\operatorname{Hom}_R(R, \operatorname{Hom}_R(F, I)) & \longrightarrow & \operatorname{Hom}_R(\mathfrak{a}, \operatorname{Hom}_R(F, I)) \\
\psi & \longmapsto & r|_{\mathfrak{a}} .
\end{array}$$

The top isomorphism is from (19.24), and the two other isomorphisms are by adjointness, cf. (19.17.1). Since, by assumptions, I is an injective module and $\omega_{\mathfrak{a}F}$ is an injective homomorphism, it follows that $\operatorname{Hom}_R(\omega_{\mathfrak{a}F}, I)$ is surjective. Thus, so is the lower horizontal map, and Baer's Criterion (12.7) yields the desired injectivity of $\operatorname{Hom}_R(F, I)$. □

(20.15) **Co–Evaluation Homomorphism.** For R–modules K, L, and M the R–homomorphism

$$\begin{array}{ccc}
\operatorname{Hom}_R(K, L) \otimes_R M & \xrightarrow{\sigma^{KLM}} & \operatorname{Hom}_R(K, L \otimes_R M) \\
\kappa \otimes m & \xmapsto{\text{def}} & (k \mapsto \kappa(k) \otimes m)
\end{array}$$

is well-defined, and induces three morphisms of functors[21]:

(20.15.1) $\quad \sigma^{(-)LM}\colon \operatorname{Hom}_R(-, L) \otimes_R M \to \operatorname{Hom}_R(-, L \otimes_R M)$

(20.15.2) $\quad \sigma^{K(-)M}\colon \operatorname{Hom}_R(K, -) \otimes_R M \to \operatorname{Hom}_R(K, - \otimes_R M)$

(20.15.3) $\quad \sigma^{KL(-)}\colon \operatorname{Hom}_R(K, L) \otimes_R - \to \operatorname{Hom}_R(K, L \otimes_R -)$.

The definition of σ^{KLM} can be explained further as follows: Consider a generator $\kappa \otimes m$ in the tensor product $\operatorname{Hom}_R(K, L) \otimes_R M$, that is, $\kappa \in \operatorname{Hom}_R(K, L)$ and $m \in M$. The element $\sigma^{KLM}(\kappa \otimes m)$ in $\operatorname{Hom}_R(K, L \otimes_R M)$, that is, the homomorphism $\sigma^{KLM}(\kappa \otimes m)$ from K to $L \otimes_R M$, is then defined by $\big(\sigma^{KLM}(\kappa \otimes m)\big)(k) =^{\text{def}} \kappa(k) \otimes m$.

To prove that, for example, $\sigma^{(-)LM}$ is a morphism of functors, one is required to prove that the next diagram is commutative for any homomorphism $\alpha\colon K \to K'$:

$$\begin{array}{ccc}
\operatorname{Hom}_R(K, L) \otimes_R M & \xrightarrow{\sigma^{KLM}} & \operatorname{Hom}_R(K, L \otimes_R M) \\
{\scriptstyle \operatorname{Hom}_R(\alpha, L) \otimes_R M}\Big\uparrow & & \Big\uparrow {\scriptstyle \operatorname{Hom}_R(\alpha, L \otimes_R M)} \\
\operatorname{Hom}_R(K', L) \otimes_R M & \xrightarrow[\sigma^{K'LM}]{} & \operatorname{Hom}_R(K', L \otimes_R M) .
\end{array}$$

[21] That is, the homomorphism σ^{KLM} is *functorial* in K, L, and M.

This can be done by inspection, for example by element wise considerations in the diagram:

$$\begin{array}{ccc} \kappa'\alpha \otimes m & \longmapsto & (k \mapsto (\kappa'\alpha)(k) \otimes m) \\ \uparrow & & \| \\ & & (k \mapsto \kappa'(\alpha(k)) \otimes m) \\ & & \uparrow \\ \kappa' \otimes m & \longmapsto & (k' \mapsto \kappa'(k') \otimes m) \end{array}$$

(20.16) **Theorem: Co–Evaluation.** *If the ring R is Noetherian, K is a finitely generated R–module, L is any R–module, and F is a flat R–module, then the co–evaluation*

$$\sigma^{KLF}\colon \operatorname{Hom}_R(K, L) \otimes_R F \to \operatorname{Hom}_R(K, L \otimes_R F)$$

is an isomorphism.

(20.17) **Proof.** Note first that the functors $\operatorname{Hom}_R(-, L) \otimes_R F$ and $\operatorname{Hom}_R(-, L \otimes_R F)$ are left-exact, the former because F is flat. In the next commutative diagram the isomorphisms are from (4.4).

$$\begin{array}{ccc} \operatorname{Hom}_R(R, L) \otimes_R F & \xrightarrow{\sigma^{RLF}} & \operatorname{Hom}_R(R, L \otimes_R F) \\ & {\scriptstyle \cong}\ \searrow \quad \swarrow\ {\scriptstyle \cong} & \\ & \alpha^{L \otimes_R F} \quad \alpha^{L \otimes_R F} & \\ & L \otimes_R F & \end{array}$$

Thus, σ^{RLF} is an isomorphism. **IsoP** (18.23) yields the desired assertion. □

(20.18) **Corollary: Tensor Product of Injective & Flat.** *If the ring R is Noetherian, I is an injective R–module, and F is a flat R–module, then the tensor product $I \otimes_R F$ is an injective module.*

(20.19) **Proof.** Let $\mathfrak{a}$ be any ideal, and let $\iota\colon \mathfrak{a} \to R$ be the inclusion. By (20.15.1) there is then the next commutative diagram:

$$\begin{array}{ccc} \operatorname{Hom}_R(R, I) \otimes_R F & \xrightarrow{\operatorname{Hom}_R(\iota, I) \otimes_R F} & \operatorname{Hom}_R(\mathfrak{a}, I) \otimes_R F \\ \downarrow {\scriptstyle \sigma^{RIF}}\ {\scriptstyle \cong} & & {\scriptstyle \cong}\ \downarrow {\scriptstyle \sigma^{\mathfrak{a}IF}} \\ \operatorname{Hom}_R(R, I \otimes_R F) & \longrightarrow & \operatorname{Hom}_R(\mathfrak{a}, I \otimes_R F) \\ \psi & \longmapsto & r|_{\mathfrak{a}} \end{array}$$

The two co–evaluation homomorphisms are isomorphisms by Theorem (20.16). The injectivity of the module I implies that the homomorphism $\operatorname{Hom}_R(\iota, I)\colon \operatorname{Hom}_R(R, I) \to \operatorname{Hom}_R(\mathfrak{a}, I)$ is surjective, and hence so is the top horizontal map in the diagram. It follows that the bottom one is surjective too, and Baer's Criterion (12.7) implies that $I \otimes_R F$ is injective, as desired. □

(20.20) **Exercise: Evaluation.** Let K, L, and M be R–modules.

(a) Establish a homomorphism $\tau^{KLM}\colon K \otimes_R \operatorname{Hom}_R(L, M) \to \operatorname{Hom}_R(\operatorname{Hom}_R(K, L), M)$ which is functorial in K, L, and M.

(b) Describe this in case $L = M = \Delta^R$, the Pontryagin module, cf. (12.25) and (12.26).

Assume, from now on, that the ring R is *Noetherian*.
(c) Prove that τ^{KLI} is an isomorphism, when K is finitely generated and I is injective.
(d) Prove that $\operatorname{Hom}_R(L, I)$ is flat when L and I are injective.

(20.21) **Theorem: Finitely Generated Flat is Projective.** *If R is Noetherian, and F is a finitely generated flat R–module, then F is projective.*

(20.22) **Proof.** Assume that the sequence of R–homomorphisms $L \xrightarrow{\lambda} L' \to O$ is exact, and prove exactness of the next induced complex:

$$K. \stackrel{\text{def}}{=} \operatorname{Hom}_R(F, L) \xrightarrow{\lambda_*} \operatorname{Hom}_R(F, L') \to O .$$

To accomplish this, by **ExP** (12.27), it suffices to prove that, for any injective R–module I, the top sequence in the next diagram is exact.

$$\begin{array}{ccccc} O & \longrightarrow & \operatorname{Hom}_R(\operatorname{Hom}_R(F, L'), I) & \xrightarrow{\operatorname{Hom}_R(\lambda_*, I)} & \operatorname{Hom}_R(\operatorname{Hom}_R(F, L), I) \\ & & \Big\downarrow {\scriptstyle \tau^{FL'I}} \cong & & \cong \Big\downarrow {\scriptstyle \tau^{FLI}} \\ O & \longrightarrow & F \otimes_R \operatorname{Hom}_R(L', I) & \xrightarrow{F \otimes_R \operatorname{Hom}_R(\lambda, I)} & F \otimes_R \operatorname{Hom}_R(L, I) . \end{array}$$

The two isomorphisms are from (b) in the Evaluation (20.20). The lower row is exact because contravariant Hom is always left-exact, and because F is flat. The diagram is commutative, so the top row is exact, as desired. □

(20.23) **Torsionfree Modules.** Let R be a domain. An R–module M is then said to be *torsionfree*, exactly when , for all $r \in R \setminus O$ the multiplication homomorphism $r_M \colon M \to M$, $m \mapsto rm$ is injective, that is, the next holds:

$$\forall r \in R \setminus O \; \forall m \in M \setminus O \colon rm \neq 0 .$$

Compare (12.16).

(20.24) **Exercise: Flatness versus Torsionfreeness.** Let F be a module over an integral domain R, and let K be a submodule of F. The following then hold.
(1) If F is flat, then F is torsionfree.
(2) If R is PID and F is torsionfree, then F is flat.
(3) If F is torsionfree, then K is torsionfree.
(4) If R is PID and F is flat, then K is flat.

21. Tor

For any R–module N and any $n \in \mathbb{Z}$ the covariant functor $\operatorname{Tor}_n^R(-, N)$ will be introduced as the n'th left derived of the covariant functor $- \otimes_R N$. This functor will be extended to a bi-functor $\operatorname{Tor}_n^R(-,-)$ in two variables.

(21.1) **Tensor Product as Bi-functor.** That is, if $\mu\colon M \to M'$ and $\nu\colon N \to N'$ are R–homomorphisms, then the next diagrams are commutative:

(21.1.1)
$$\begin{array}{ccc} M \otimes_R N & \xrightarrow{M\otimes_R\nu} & M \otimes_R N' \\ {\scriptstyle \mu\otimes_R N}\downarrow & & \downarrow{\scriptstyle \mu\otimes_R N'} \\ M' \otimes_R N & \xrightarrow[M'\otimes_R\nu]{} & M' \otimes_R N' \end{array} \qquad \begin{array}{ccc} m \otimes n & \longmapsto & m \otimes_R \nu(n) \\ \downarrow & & \downarrow \\ \mu(m) \otimes_R n & \longmapsto & \mu(m) \otimes_R \nu(n) \end{array}$$

We set $\mu \otimes_R \nu =^{\text{def}} (\mu \otimes_R N')(M \otimes_R \nu) = (M' \otimes_R \nu)(\mu \otimes_R N)\colon M \otimes_R N \to M' \otimes_R N'$.

The next result is analogous to Theorem: Ext (16.5).

(21.2) **Theorem: Tor.** *For every $n \in \mathbb{Z}$ there exists a functor in two variables:*

$$\operatorname{Tor}_n^R(-,-)\colon \operatorname{Mod}(R) \times \operatorname{Mod}(R) \to \operatorname{Mod}(R)\,;$$

it is covariant in both variables.

(F1) *For any R–module N, the functor $\operatorname{Tor}_n^R(-, N)$ is the n'th left derived functor $\mathrm{L}_n(- \otimes_R N)$ of the covariant functor $- \otimes_R N$, and for any R–module M, the functor $\operatorname{Tor}_n^R(M, -)$ is the n'th left derived functor $\mathrm{L}_n(M \otimes_R -)$ of the covariant functor $M \otimes_R -$. Thus, $\operatorname{Tor}_n^R(M, N) = \mathrm{H}_n(P_\bullet \otimes_R N) = \mathrm{H}_n(M \otimes_R Q_\bullet)$ when $P_\bullet$ and $Q_\bullet$ are projective resolutions of M and N, respectively.*

(F2) *$\operatorname{Tor}_n^R(M, N)$ depends on the choice of projective resolution of M or N, but is unique up to unique isomorphism.*

(F3) *$\operatorname{Tor}_0^R(-,-) \cong - \otimes_R -$ as a functor in two variables.*

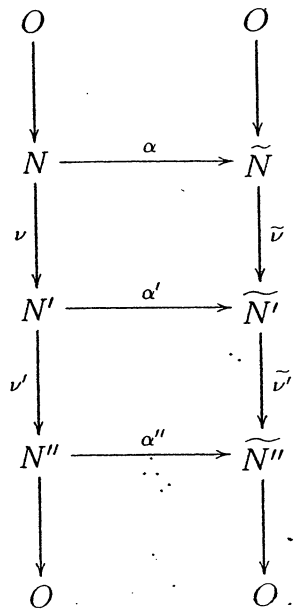
(F4) *$\operatorname{Tor}_n^R(M, N) \cong \operatorname{Tor}_n^R(N, M)$ functorial in both variables.*

(F5) *Tor is a bi-functor. That is, if $\mu\colon M \to M'$ and $\nu\colon N \to N'$ are R–homomorphisms, then the next diagram is commutative:*

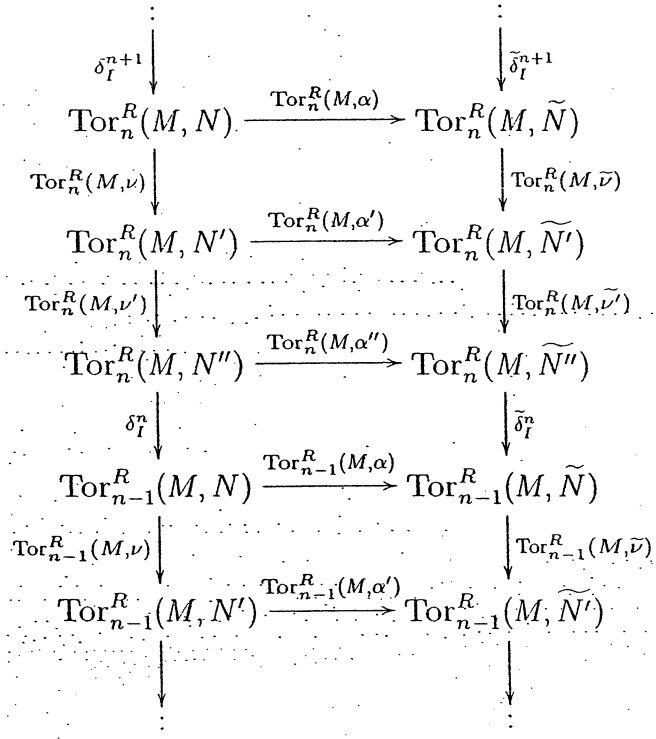
$$\begin{array}{ccc} \operatorname{Tor}_n^R(M, N) & \xrightarrow{\operatorname{Tor}_n^R(M,\nu)} & \operatorname{Tor}_n^R(M, N')\,. \\ {\scriptstyle \operatorname{Tor}_n^R(\mu,N)}\downarrow & & \downarrow{\scriptstyle \operatorname{Tor}_n^R(\mu,N')} \\ \operatorname{Tor}_n^R(M', N) & \xrightarrow[\operatorname{Tor}_n^R(M',\nu)]{} & \operatorname{Tor}_n^R(M', N') \end{array}$$

(F6) *F is flat, if and only if $\operatorname{Tor}_n^R(-, F) = O$ for all $n > 0$, if and only if $\operatorname{Tor}_n^R(F, -) = O$ for all $n > 0$.*

(F7) *The next short-exact ladder:*



induces the next long-exact ladder:



In particular, any exact sequence $O \to K \to F \to N \to O$ *with* F *flat induces an exact sequence:*

$$(*) \qquad O \to \operatorname{Tor}_1^R(L,N) \xrightarrow{\delta_1} L \otimes_R K \to L \otimes_R F \to L \otimes_R N \to O$$

and isomorphisms $\operatorname{Tor}_{n+1}^R(L,N) \cong \operatorname{Tor}_n^R(L,K)$ *for* $n \geq 1$. □

(21.3) **Flat resolutions.** A *flat resolution* $F.$ of an R–module M is a sequence of R–homomorphisms:

$$F. = \cdots \to F_\ell \xrightarrow{\partial_\ell} F_{\ell-1} \xrightarrow{\partial_{\ell-1}} \cdots F_1 \xrightarrow{\partial_1} F_0 \to O$$

together with an R–homomorphism $F_0 \xrightarrow{\partial_0} M$ such that

- F_ℓ is a flat R–module for all $\ell \in \mathbb{N}_0$;
- $F. \to M \stackrel{\text{def}}{=} \cdots \to F_\ell \xrightarrow{\partial_\ell} F_{\ell-1} \xrightarrow{\partial_{\ell-1}} \cdots F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to O$ is exact.

The sequence $F. \to M$ is then called the *augmented* flat resolution of M. Note that the index of the R–homomorphism ∂_ℓ in the flat resolution equals that of its *source*.

Any R–module admits a flat resolution as any projective resolution is a flat resolution.

(21.4) **Example.** Let R be a domain that is not a field, and let B be the field of fractions. The R–module B/R has then $O \to R \to B \to O$ as flat resolution, and this is not a projective resolution, cf. (20.2.c).

(21.5) **Exercise: Tor by Flat Resolution.** Let $F.$ be a flat resolution of M. Prove for any $n \in \mathbb{N}_0$ that the functors $\operatorname{Tor}_n^R(M, -)$ and $\operatorname{H}_n(F. \otimes_R -)$ are isomorphic.

(21.6) **Exercise: Flat Resolution & Injective Resolution.** Let $F.$ be a flat resolution of M, and let I be an injective module. Prove that $\operatorname{Hom}_R(F., I)$ is an injective resolution of $\operatorname{Hom}_R(M, I)$.

(21.7) **Lemma.** *Let K be a submodule of an R–module M, and let $\mathfrak{a}$ be an ideal. If the module M/K is flat, then $(\mathfrak{a}M) \cap K = \mathfrak{a}K$.*

(21.8) **Proof.** In the next diagram:

$$\begin{array}{ccccccc}
\operatorname{Tor}_1^R(R/\mathfrak{a}, M/K) & \xrightarrow{\delta_1} & (R/\mathfrak{a}) \otimes_R K & \longrightarrow & (R/\mathfrak{a}) \otimes_R M & & \\
\| & & & & & & \\
O & & \downarrow \tilde{\psi}^K \cong & & \cong \downarrow \tilde{\psi}^M & & \\
O & \longrightarrow \operatorname{Ker}\varphi & \longrightarrow & K/(\mathfrak{a}K) & \xrightarrow[\varphi]{} & M/(\mathfrak{a}M) &
\end{array}$$

the upper row is (part of) the long-exact sequence obtained by applying the functor $(R/\mathfrak{a}) \otimes_R -$ to the exact sequence $O \to K \hookrightarrow M \to M/K \to O$. The equality is because M/K is flat by assumption, confer also (F6) in Tor Theorem (21.2). The two isomorphisms are by (19.22). The homomorphism φ is well-defined by $\varphi([k]_{\mathfrak{a}K}) =^{\text{def}} [k]_{\mathfrak{a}M}$, and it is easy to verify that the rectangle commutes. It follows that $\operatorname{Ker}\varphi = O$. On the other hand, $\operatorname{Ker}\varphi = (\mathfrak{a}M \cap K)/\mathfrak{a}K$, and the desired assertion follows. □

(21.9) **von Neumann Regular Rings.** The ring R is said to be *von Neumann regular*, exactly when there to each $r \in R$ exists an $x \in R$ such that $rxr = r$, that is, $(r^2) = (r)$ for all $r \in R$.

(21.10) **Example: von Neumann Regular Ring.** Let L be a field , and consider the ring $R =^{\text{def}} L^X$, cf. (3.14). This ring is von Neumann regular: To each element φ in R, that is, function $\varphi\colon X \to R$, define $\psi \in R$ by $\psi(x) =^{\text{def}} \varphi(x)^{-1}$ when $\varphi(x) \neq 0$, and $\varphi(x) =^{\text{def}} 0$ otherwise, because then $\varphi(x)\psi(x)\varphi(x) = \varphi(x)$ for all $x \in X$, that is, $\varphi\psi\varphi = \varphi$.

(21.11) **Lemma: von Neumann Regular Rings.** *R is von Neumann regular, if the R–module $R/(r)$ is flat for all $r \in R$.*

(21.12) **Proof.** Letting, for an $r \in R$, $M =^{\text{def}} R$, $K =^{\text{def}} (r)$, and $\mathfrak{a} =^{\text{def}} (r)$, Lemma (21.7) yields that $(r) = (r)R \cap (r) = \mathfrak{a}M \cap K = \mathfrak{a}K = (r)(r) = (r^2)$, as desired. □

(21.13) **Exercise: von Neumann Regular Rings.** Assume that R is von Neumann regular. Prove the next assertions.

(a) To any principal ideal (a) there exists an *idempotent*[22] e generating (a).
(b) The R-module $R/(e)$ is projective whenever e is an idempotent.
(c) The R-module $R/\mathfrak{a}$ is projective whenever $\mathfrak{a}$ is an ideal generated by two elements.
(d) The R-module $R/\mathfrak{a}$ is projective whenever $\mathfrak{a}$ is a finitely generated ideal.
(e) The exact sequence $O \to \mathfrak{a} \to R \to R/\mathfrak{a} \to O$ splits whenever $\mathfrak{a}$ is a finitely generated ideal.
(f) Every R-module is flat.

The next characterization of von Neumann regular rings follows directly from Lemma (21.11) and Exercise (21.13).

(21.14) **Theorem: von Neumann Regular Rings.** *The ring R is von Neumann regular if and only if all R-modules are flat.* □

(21.15) **Exercise: Finite Generation of Tensor Product and Tor.** Let M and N be finitely generated R-modules.

(a) Prove that $M \otimes_R N$ is finitely generated.
(b) Prove that $\operatorname{Tor}_n^R(M, N)$ is finitely generated for all $n \in \mathbb{N}_0$, if R is Noetherian.

[22] That is, $e^2 = e$.

22. Flat Dimension

(22.1) **Flat dimension.** An R–module M is said to have *finite flat dimension* exactly when it has a *bounded* flat resolution, and when this is the case, the *flat dimension* of M is determined as smallest $f \in \mathbb{N}_0$ such that it has a flat resolution of length f. If M does not have finite flat dimension, then we set $\operatorname{fd}_R M = \infty$, provided the module is not the zero module: we set $\operatorname{fd}_R O = -\infty$.

Thus, for any R–module $M \neq O$ and any $n \in \mathbb{N}_0$ we have:

$$\text{(22.1.1)} \qquad \operatorname{fd}_R M \leq n \iff \begin{cases} \text{There exists an exact sequence:} \\ O \to F_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to O \\ \text{with } F_0, \dots, F_n \text{ flat.} \end{cases}$$

Note furthermore the next equivalence for an R–module M.

$$\text{(22.1.2)} \qquad \operatorname{fd}_R M = 0 \iff M \text{ is flat and non-zero.}$$

(22.2) **Example.** Let R be a domain that is not a field, and let B be its field of fractions. Then $\operatorname{fd}_R B = 0$ as it is a flat module, and $\operatorname{fd}_R(B/R) = 1$ by (21.4), since B/R is not flat by (20.2.e), since any non-invertible $r \in R \setminus O$ belongs to $\operatorname{Zero}_R(B/R)$ (as $r[\frac{1}{r}]_R = [1]_R[0]_R = 0$ and $[\frac{1}{r}]_R \neq 0$).

(22.3) **Example.** If M is a module over a PID R, then $\operatorname{fd}_R M \leq 1$ by (20.24.4).

As any projective module is flat by (20.7), we obtain the next inequality:

(22.4) **Lemma: Flat & Projective Dimension.** *For all R–modules M the inequality $\operatorname{fd}_R M \leq \operatorname{pd}_R M$ holds.* □

(22.5) **Example.** If R be a domain and not a field, and let B be its field of fractions. There is then a strict inequality $\operatorname{fd}_R B = 0 < \operatorname{pd}_R B$, cf. (20.2.c).

(22.6) **FDT $\overset{\text{def}}{=}$ Flat Dimension Theorem.** *The following are equivalent for any R–module M and any $n \in \mathbb{N}_0$.*

(i) $\operatorname{fd}_R M \leq n$;
(ii) $\operatorname{Tor}^R_m(H, M) = O$ *for all* $m > n$ *and all R–modules H;*
(iii) $\operatorname{Tor}^R_{n+1}(H, M) = O$ *for and all R–modules H;*
(iii˜) $\operatorname{Tor}^R_{n+1}(R/\mathfrak{a}', M) = O$ *for and all finitely generated ideals $\mathfrak{a}'$;*
(iv) If $F_\bullet$ is a flat resolution of M, then the kernel $\operatorname{K}_{n-1}(F_\bullet)$ is flat.

(22.7) **Proof.** With one exception, all parts of this proof are similar to the corresponding ones in the proof of IDT, cf. (17.4). Thus, we concentrate on the exceptional one:

"$(iii\tilde{}) \implies (iv)$": Assume that $(iii\tilde{})$ holds, and that $F_\bullet$ is a flat resolution of M, and consider the kernels $K_\ell =^{\text{def}} \operatorname{Ker}(F_\ell \to F_{\ell-1})$ for $\ell \in \mathbb{N}$, $K_0 =^{\text{def}} \operatorname{Ker}(F_0 \to M)$, and $K_{-1} =^{\text{def}} M$, as well as the resulting short-exact sequences:

$$(\dagger_\ell) \qquad O \to K_\ell \to F_\ell \to K_{\ell-1} \to O$$

for $\ell \in \mathbb{N}_0$. Let $\mathfrak{a}'$ be a finitely generated ideal, and consider the short-exact sequence

$$(*) \qquad O \to \mathfrak{a}' \overset{\iota'}{\hookrightarrow} R \twoheadrightarrow R/\mathfrak{a}' \to O.$$

By Flatness Theorem (20.13)—in view of the diagram (∇) in (20.12)—it suffices to prove that $K_{n-1} \otimes_R \iota' \colon K_{n-1} \otimes_R \mathfrak{a}' \to K_{n-1} \otimes_R R$ is *injective*. Because of the exact sequence

$$(*) \qquad \operatorname{Tor}_1^R(K_{n-1}, R/\mathfrak{a}') \xrightarrow{\delta_1} K_{n-1} \otimes_R \mathfrak{a}' \to K_{n-1} \otimes_R R$$

from (F7) in Theorem: Tor (21.2), it suffices to prove $\operatorname{Tor}_1^R(K_{n-1}, R/\mathfrak{a}') = 0$. This follows from the next chain (in which the sum of indices is n in each line):

$$\begin{aligned} \operatorname{Tor}_1^R(K_{n-1}, R/\mathfrak{a}') &\cong \operatorname{Tor}_2^R(K_{n-2}, R/\mathfrak{a}') \\ &\cong \cdots \\ &\cong \operatorname{Tor}_n^R(K_0, R/\mathfrak{a}') \\ &\cong \operatorname{Tor}_{n+1}^R(K_{-1}, R/\mathfrak{a}') \\ &= \operatorname{Tor}_{n+1}^R(M, R/\mathfrak{a}') = 0\,; \end{aligned}$$

the former equality is by convention and the latter is by assumption. □

(22.8) **FDC $\stackrel{\text{def}}{=}$ Flat Dimension Corollary.** *If M is an R–module, then*

$$\begin{aligned} \operatorname{fd}_R M &= \sup\{\, n \in \mathbb{N}_0 \mid \operatorname{Tor}_n^R(H, M) \neq O \ \text{ for some } \ H \,\} \\ &= \sup\{\, n \in \mathbb{N}_0 \mid \operatorname{Tor}_n^R(R/\mathfrak{a}', M) \neq O \ \text{ for some finitely generated ideal } \ \mathfrak{a}' \,\}\,. \end{aligned}$$

This number is in $\mathbb{N}_0^*$, cf. (11.23). □

(22.9) **Proposition: Over Noetherian Ring.** *If R is Noetherian, and M' is a finitely generated R–module, then* $\operatorname{pd}_R M' = \operatorname{fd}_R M'$.

(22.10) **Proof.** Since one inequality is known in general, cf. (22.4), it suffices to prove $\operatorname{pd}_R M' \leq \operatorname{fd}_R M'$. Therefore, assume that $n \stackrel{\text{def}}{=} \operatorname{fd}_R M' < \infty$ and $M' \neq O$, and choose a projective resolution of M' by finitely generated modules P_n, cf. (18.19). We want to see that the kernel $\mathrm{K}_{n-1}(P.)$ is projective. It is flat by **FDT** (22.6), and it is finitely generated, because it is a submodule of P_{n-1} which, as a finitely generated module over a Noetherian ring, is Noetherian by (18.13). It results from Theorem (20.21) that $\mathrm{K}_{n-1}(P.)$ is projective, as desired. □

(22.11) **Exercise: Flat Dimension & Fractions.** Let M be an R–module, and let S be a special multiplicative system. Prove that $\operatorname{fd}_{S^{-1}R} S^{-1}M \leq \operatorname{fd}_R M$.

(22.12) **Exercise: Tor & Fractions.** Let M be an R–module, and let S be a special multiplicative system. Establish an isomorphism of bi-functors

$$\operatorname{Tor}_n^{S^{-1}R}(S^{-1}(-), S^{-1}(\sim)) \cong S^{-1}\operatorname{Tor}_n^R(-, \sim)\,.$$

(22.13) **Exercise: Torsion.** Let R be a domain with B as field of fractions. For any R–module M define the *torsion submodule* of M as

$$M_T \stackrel{\text{def}}{=} \{\, m \in M \mid xm = 0 \ \text{ for some } \ x \in R \setminus O \,\}\,.$$

(a) Prove that the functors $(-)_T$ and $\operatorname{Tor}_1^R(-, B/R)$ are isomorphic.
(b) Prove that $\operatorname{Tor}_n^R(M, N)_T = \operatorname{Tor}_n^R(M, N)$ for $n > 0$.

The name Tor is an abbreviation of Torsion, and can be motivated by either of the above.

(22.14) **Theorem: Higher Adjointness.** *For any injective R–module I, any R–modules M and N, and any $n \in \mathbb{N}_0$ there is an isomorphism of functors:*

$$\operatorname{Ext}^n_R(M, \operatorname{Hom}_R(N, I)) \cong \operatorname{Hom}_R(\operatorname{Tor}^R_n(M, N), I)\,,$$

which is functorial in M and N.

(22.15) **Proof.** Let $P.$ be a projective resolution of M. In the next chain the two equalities are by the definitions: (21.2.(F1)) and (11.17), respectively.

$$\begin{aligned}\operatorname{Hom}_R(\operatorname{Tor}^R_n(M,N), I) = \operatorname{Hom}_R(\operatorname{H}_n(P. \otimes_R N), I) &\cong \operatorname{H}^n(\operatorname{Hom}_R(P. \otimes_R N, I)) \\ &\cong \operatorname{H}^n(\operatorname{Hom}_R(P., \operatorname{Hom}_R(N, I))) = \operatorname{Ext}^n_R(M, \operatorname{Hom}_R(N, I))\,.\end{aligned}$$

The first isomorphism is by (11.3); the second one is the adjunction (19.17.1). □

(22.16) **Corollary: Injective & Flat Dimensions.** *There is the next equality:*

$$\operatorname{fd}_R N = \sup\{\, \operatorname{id}_R \operatorname{Hom}_R(N, I) \mid I \text{ is an injective } R\text{–module} \,\}\,.$$

(22.17) **Proof.** The inequality "$\geq$" follows from the theorem in view of **FDT** (22.6) and **IDT** (17.3). Setting I equal the Pontryagin module Δ^R, cf. (12.25), the isomorphism of the theorem yields, in view of (12.26.1),

$$\operatorname{Ext}^n_R(M, \operatorname{Hom}_R(N, \Delta^R)) \neq O \iff \operatorname{Tor}^R_n(M, N) \neq O\,.$$

Thus, $\operatorname{fd}_R N = \operatorname{id}_R \operatorname{Hom}_R(N, \Delta^R)$, and the desired equality has been established. □

(22.18) **Exercise: Higher Associativity.** Prove that the modules $\operatorname{Tor}^R_n(M, N) \otimes_R F$ and $\operatorname{Tor}^R_n(M, N \otimes_R F)$ are isomorphic, whenever F is flat.

(22.19) **Exercise: Higher Evaluation.** Prove that the modules $\operatorname{Tor}^R_n(K, \operatorname{Hom}_R(L, I))$ and $\operatorname{Hom}_R(\operatorname{Ext}^n_R(K, L), I)$ are isomorphic, and that $\operatorname{fd}_R(\operatorname{Hom}_R(K, I)) \leq \operatorname{id}_R K$, whenever R is Noetherian, K is finitely generated, and I is injective.

(22.20) **Exercise: Higher Co–Evaluation.** Prove that the modules $\operatorname{Ext}^n_R(K, L) \otimes_R F$ and $\operatorname{Ext}^n_R(K, L \otimes_R F)$ are isomorphic, and that $\operatorname{id}_R(L \otimes_R F) \leq \operatorname{id}_R L$, whenever R is Noetherian, K is finitely generated, and F is flat.

(22.21) **Weak[23] Global Dimension.** It is defined as follows:

$$\operatorname{wgldim} R \stackrel{\text{def}}{=} \sup\{\operatorname{fd}_R M \mid M \text{ is an } R\text{–module}\}\ (\in \mathbb{N}_0 \cup \{\infty\}).$$

(22.22) **Remarks.** (1) $\operatorname{wgldim} R = 0$ if and only if R is von Neumann regular, cf. (21.14).
(2) $\operatorname{wgldim} R \leq \operatorname{gldim} R$ by (22.4). An example with strict inequality is provided by (22.23) below, and in (22.26) it is proved that equality holds always over Noetherian rings.

(22.23) **Weak Global Dim vs. Global Dim.** Let L be a field, let X be an *infinite* set, and set $R =^{\text{def}} L^X$ which is a von Neumann regular ring, cf. (21.10), and hence has $\operatorname{wgldim} R = 0$, cf. (21.14). We shall prove that $\operatorname{gldim} R > 0$ by proving that not all short-exact sequences split, cf. (12.10). Indeed, assume that the ideal $\mathfrak{a} =^{\text{def}} L^{(X)}$ is an direct summand in R, we get $R = \mathfrak{a} \oplus \mathfrak{b}$ (inner direct sum) for some ideal $\mathfrak{b}$ in R. In particular, $\mathfrak{ab} \subseteq \mathfrak{a} \cap \mathfrak{b} = O$. We will next prove that then $\mathfrak{b} = O$. Namely, assuming that

[23] In older literature, flat dimension is called "weak homological dimension".

$\varphi \in \mathfrak{b}$ and $\varphi \neq O$, choose $x \in X$ such that $\varphi(x) \neq 0$. The function $\delta_x\varphi$ is then non-zero (as it sends x into $\delta_x(x)\varphi(x) = \varphi(x) \neq 0$), and as it belongs to $\mathfrak{ab}$, this contradicts $\mathfrak{ab} = O$. Now $\mathfrak{b} = O$, and we have $L^{(X)} = \mathfrak{a} = R = L^X$ which contradicts the infinity of X. (Consulting Exercise (18.36) is an alternative to the above.)

(22.24) **Theorem: Weak Global Dim.**

$$\begin{aligned}\operatorname{wgldim} R &= \sup\{\operatorname{fd}_R M' \mid M' \textit{ finitely generated module}\}\\ &= \sup\{\operatorname{fd}_R R/\mathfrak{a}' \mid \mathfrak{a}' \textit{ finitely generated ideal}\}\\ &= \sup\{n \mid \operatorname{Tor}^R_n(R/\mathfrak{a}', -) \neq O \textit{ for a finitely generated ideal}\}.\end{aligned}$$

(22.25) **Proof.** For $n \in \mathbb{N}_0$ consider the next four assertions:

(f^n_R) $\operatorname{fd}_R M \leq n$ for all R–modules M;
$(\mathrm{f}^n_R)'$ $\operatorname{fd}_R M' \leq n$ for all finitely generated R–modules M';
$(\mathrm{f}^n_R)^\sim$ $\operatorname{fd}_R R/\mathfrak{a}' \leq n$ for all finitely generated ideals $\mathfrak{a}'$;
$(\mathrm{T}^n_R)^\sim$ $\operatorname{Tor}^R_{n+1}(R/\mathfrak{a}', -) = O$ for all finitely generated ideals $\mathfrak{a}'$.

$$\begin{array}{ccc}(\mathrm{f}^n_R) & \xrightarrow{\text{obvious}} & (\mathrm{f}^n_R)'\\ \Big\Uparrow{\scriptstyle \textbf{FDC (22.8)}} & & \Big\Downarrow{\scriptstyle \text{obvious}}\\ (\mathrm{T}^n_R)^\sim & \xleftarrow[\textbf{FDC (22.8)}]{} & (\mathrm{f}^n_R)^\sim.\end{array}$$

□

(22.26) **Theorem: Weak Global Dim over Noetherian Ring.** *If R is Noetherian, then* $\operatorname{wgldim} R = \operatorname{gldim} R$.

(22.27) **Proof.** By, respectively, Auslander's Theorem (17.9), Proposition: Over Noetherian Ring (22.9), Definition (22.21), and Remark (22.22.2) we obtain the next four relations:

$$\begin{aligned}\operatorname{gldim} R &= \sup\{\operatorname{pd}_R M' \mid M' \text{ finitely generated}\}\\ &= \sup\{\operatorname{fd}_R M' \mid M' \text{ finitely generated}\}\\ &\leq \operatorname{wgldim} R \leq \operatorname{gldim} R.\end{aligned}$$

□

23. KRULL DIMENSION

First we summarize aspects of the theory of commutative rings in general; later we specialize to the Noetherian ones.

(23.1) **Radical.** The set $\operatorname{Rad}(R) \stackrel{\text{def}}{=} \{\, x \in R \mid x^n = 0 \text{ for some } n \in \mathbb{N} \,\}$ is called the *radical*[24] of R. This is an ideal in R (because R is commutative; for example, the binomial formula gives $(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$.)

(23.2) **Prime Ideals & Maximal Ideals.** We use the next notation:

$$\operatorname{Spec} R \stackrel{\text{def}}{=} \{\, \mathfrak{p} \mid \mathfrak{p} \text{ is a prime ideal} \,\} \supseteq \operatorname{Max} R \stackrel{\text{def}}{=} \{\, \mathfrak{p} \mid \mathfrak{p} \text{ is a maximal ideal} \,\} \neq \varnothing .$$

$$\mathfrak{p} \in \operatorname{Spec} R \iff R/\mathfrak{p} \text{ is a domain.} \qquad \mathfrak{m} \in \operatorname{Max} R \iff R/\mathfrak{m} \text{ is a field.}$$

The existence of at least one maximal ideal is due to Zorn's Lemma: The set of proper ideals is non-empty (as R is non-trivial) and inductively ordered (as the union of a totally ordered set of ideals is again an ideal).

(23.3) **Minimal Prime Ideals.**

$$\mathfrak{p} \in \operatorname{Min} R \iff^{\text{def}} \mathfrak{p} \text{ is minimal in } \operatorname{Spec} R$$

Thus, $\mathfrak{p} \in \operatorname{Min} R$, if and only if $\mathfrak{p} \in \operatorname{Spec} R$ and $\mathfrak{p} = \mathfrak{q}$ whenever $\mathfrak{q} \subseteq \mathfrak{p}$ and $\mathfrak{q} \in \operatorname{Spec} R$.

(23.3.1) $$\operatorname{Min} R \neq \varnothing .$$

This follows by Zorn's Lemma applied to $\operatorname{Spec} R$ ordered by *inverse* inclusion: If $\mathbb{Y}$ is a totally ordered non-empty subset, then $\mathfrak{q} =^{\text{def}} \bigcap_{\mathfrak{p} \in \mathbb{Y}} \mathfrak{p}$ belongs to $\operatorname{Spec} R$. Indeed, if $a, b \in R$ has $ab \in \mathfrak{q}$ and $b \notin \mathfrak{q}$, then choose $\mathfrak{p}_0 \in \mathbb{Y}$ such that $b \notin \mathfrak{p}_0$, and note that $\mathfrak{q} = \bigcap_{\mathfrak{p}' \in \mathbb{Y}'} \mathfrak{p}'$ when $\mathbb{Y}' =^{\text{def}} \{\, \mathfrak{p}' \in \mathbb{Y} \mid \mathfrak{p}' \subseteq \mathfrak{p}_0 \,\}$ (as $\mathbb{Y}$ is totally ordered). Because $b \notin \mathfrak{p}'$ and $a \in \mathfrak{p}'$ for all $\mathfrak{p}' \in \mathbb{Y}'$, we conclude that $a \in \mathfrak{q}$, as desired.

Proof of the next useful result will be provided later.

(23.4) **Proposition.** *If R is Noetherian, then $\operatorname{Min} R$ is finite.*

(23.5) **Lemma.** *Each ring homomorphism $\varphi\colon R \to \widetilde{R}$ induces the next map:*

$$\operatorname{Spec} : \operatorname{Spec} \widetilde{R} \to \operatorname{Spec} R\,,\ \widetilde{\mathfrak{p}} \mapsto \varphi^{-1}(\widetilde{\mathfrak{p}}) .$$

(23.6) **Proof:** For $\widetilde{\mathfrak{p}} \in \operatorname{Spec} \widetilde{R}$ we are required to prove that

$$\mathfrak{p} =^{\text{def}} \varphi^{-1}(\widetilde{\mathfrak{p}}) \,(\stackrel{\text{def}}{=} \{\, r \in R \mid \varphi(r) \in \widetilde{\mathfrak{p}} \,\})$$

belongs to $\operatorname{Spec} R$. The induced ring homomorphism $R \to \widetilde{R}/\widetilde{\mathfrak{p}}$, $r \mapsto [\,\varphi(r)\,]_{\widetilde{\mathfrak{p}}}$, has kernel $\mathfrak{p}$. Thus, the Isomorphism Theorem (for rings) yields that $R/\mathfrak{p}$ is isomorphic to a subring of $\widetilde{R}/\widetilde{\mathfrak{p}}$. As the latter is a domain, so is the former, that is, $\mathfrak{p} \in \operatorname{Spec} R$. □

(23.7) Exercise: **Spec as Functor.** Describe Spec as a contravariant functor from the category of rings and ring homomorphisms to the category of sets and maps.

(23.8) **Variety of an Ideal.** For an ideal $\mathfrak{a}$ in R its *variety* $\operatorname{V}_R(\mathfrak{a})$ is defined as follows:

$$\operatorname{V}_R(\mathfrak{a}) \stackrel{\text{def}}{=} \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \supseteq \mathfrak{a} \,\} ,$$

[24] Or *nilradical*.

If W is any subset of R we set $\mathrm{V}_R(W) \stackrel{\text{def}}{=} \mathrm{V}_R(\operatorname{span}_R W)$, for example, for $a \in R$ we write $\mathrm{V}_R(a)$ instead of $\mathrm{V}_R((a))$.

There is a map $\Phi_{\mathfrak{a}}\colon \operatorname{Spec}(R/\mathfrak{a}) \to \mathrm{V}_R(\mathfrak{a})$ given by $\widetilde{\mathfrak{p}} \mapsto {\pi_{\mathfrak{a}}}^{-1}(\widetilde{\mathfrak{p}})$ when $\pi_{\mathfrak{a}}\colon R \to R/\mathfrak{a}$ is the residue homomorphism. The map $\Phi_{\mathfrak{a}}$ bijective, and it *respects inclusions*, that is, $\Phi_{\mathfrak{a}}(\widetilde{\mathfrak{p}}_1) \subseteq \Phi_{\mathfrak{a}}(\widetilde{\mathfrak{p}}_2)$ if $\widetilde{\mathfrak{p}}_1 \subseteq \widetilde{\mathfrak{p}}_2$, and its inverse in given by ${\Phi_{\mathfrak{a}}}^{-1}(\mathfrak{p}) = \mathfrak{p}/\mathfrak{a}$.

(23.9) **Prime Ideals in Ring of Fractions.** Let S be a special multiplicative system in R. The map

$$\Psi_S\colon \operatorname{Spec}(S^{-1}R) \to \{\, \mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap S = \varnothing \,\},\ \widetilde{\mathfrak{p}} \mapsto \{\, r \in R \mid r/1 \in \widetilde{\mathfrak{p}} \,\}$$

is the one induced by the ring homomorphism $R \to S^{-1}R$, $r \mapsto r/1$, cf. (23.5); Ψ_S is *bijective* and *respects inclusions*. To any $\mathfrak{p} \in \operatorname{Spec} R$ with $\mathfrak{p} \cap S = \varnothing$, the ideal $S^{-1}\mathfrak{p}$ belongs to $\operatorname{Spec}(S^{-1}R)$, and $\Psi_S^{-1}(\mathfrak{p}) = S^{-1}\mathfrak{p}$.

(23.10) **Proposition: Radical.** $\operatorname{Rad}(R) = \bigcap_{\mathfrak{p} \in \operatorname{Min} R} \mathfrak{p}$.

(23.11) **Proof.** The inclusion "$\subseteq$" follows from the fact that $a^n \in \mathfrak{p}$ implies $a \in \mathfrak{p}$ whenever $\mathfrak{p}$ is a prime ideal.

"$\supseteq$": Assume that $x \in R \setminus \operatorname{Rad}(R)$ [and we want $\mathfrak{p} \in \operatorname{Min} R$ with $x \notin \mathfrak{p}$]. Consider the special multiplicative system $S(x) =^{\text{def}} \{\, x^n \mid n \in \mathbb{N}_0 \,\}$, and the non-trivial ring $R_x =^{\text{def}} S(x)^{-1}R$. Choosing $\widetilde{\mathfrak{p}} \in \operatorname{Min} R_x$ by (23.3.1), the corresponding prime ideal $\mathfrak{p}$, cf. (23.9), then belongs to $\operatorname{Min} R$ and $x \notin \mathfrak{p}$ by (23.9) once more. □

(23.12) **Examples: Spec.**
(a) If L is a field, then $\operatorname{Max} L = \operatorname{Spec} L = \operatorname{Min} L = \{\, O \,\}$ and $\operatorname{Rad}(L) = O$.
(b) If R is a domain, then $\operatorname{Min} R = \{\, O \,\}$ and $\operatorname{Rad}(R) = O$
(c) If R is a PID, then $\operatorname{Max} R = \{\, (p) \mid p \text{ is a prime element} \,\}$, $\operatorname{Spec} R = \operatorname{Max} R \cup \{\, O \,\}$.
(d) $\operatorname{Spec} R/\mathfrak{a} = \{\, \mathfrak{p}/\mathfrak{a} \mid \mathfrak{p} \in \mathrm{V}_R(\mathfrak{a}) \,\}$, when $\mathfrak{a}$ is an ideal, cf. (23.8).
(e) $\operatorname{Spec} S^{-1}R = \{\, S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Spec} R,\ \mathfrak{p} \cap S = \varnothing \,\}$, when S is a special multiplicative system, cf. (23.9).
(f) $\operatorname{Spec} \mathbb{Z}/(4) = \{\, ([2]_{(4)}) \,\}$.
(g) $\operatorname{Spec} \mathbb{Z}/(6) = \{\, ([2]_{(6)}), ([3]_{(6)}) \,\}$.

(23.13) **Local Ring.** The ring R is said to be *local*, exactly when it is *Noetherian* and has *only one maximal ideal*. Recall that $\operatorname{Max} R \neq \varnothing$ by (23.2). We use the next terminology:

$$(R, \mathfrak{m}, k) \text{ is local} \iff^{\text{def}} R \text{ is Noetherian, } \operatorname{Max} R = \{\, \mathfrak{m} \,\}, \text{ and } k = R/\mathfrak{m}.$$

We say that k is the *residue field*.

(23.14) **Localization.** Let R be Noetherian. For $\mathfrak{p} \in \operatorname{Spec} R$, the set $R \setminus \mathfrak{p}$ is a special multiplicative system, and we set

$$M_{\mathfrak{p}} \stackrel{\text{def}}{=} (R \setminus \mathfrak{p})^{-1} M$$

for any R–module M. Then $M_{\mathfrak{p}}$ is a module over the ring $R_{\mathfrak{p}}$, which is local by (23.12.e):

$$(R_{\mathfrak{p}}, \mathfrak{p}_{\mathfrak{p}}, \kappa_R(\mathfrak{p})) \text{ is a local ring with } \kappa_R(\mathfrak{p}) \stackrel{\text{def}}{=} R_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}.$$

Here, $\mathfrak{p}_{\mathfrak{p}} = \{\, p/s \mid p \in \mathfrak{p} \text{ and } s \in R \setminus \mathfrak{p} \,\}$, and the field $\kappa_R(\mathfrak{p})$ is isomorphic to the field of fractions of the domain $R/\mathfrak{p}$.

(23.15) **Krull Dimension.** The *Krull dimension* of the ring R is denoted $\dim R$, and it is the supremum of $n \in \mathbb{N}_0$ such that exists a chain

$$(\dagger) \qquad \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$$

in $\operatorname{Spec} R$, that is, a chain of prime ideals. In the chain (†) there are exactly n strict inclusions; for $n = 0$ the chain is just $\mathfrak{p}_0$ (without inclusions). As $\operatorname{Spec} R \neq \varnothing$ by (23.2) we have:

$$\dim M \in \mathbb{N}_0 \cup \{\infty\}.$$

(23.16) **Examples: Dim.** Confer the corresponding part of Example (23.12).

(a) If L is a field, then $\dim L = 0$.
(b) If R is a domain and not a field, then $\dim R > 0$.
(c) If R is a PID and not a field, then $\dim R = 1$.
(d) $\dim R/\mathfrak{a} \leq \dim R$, when $\mathfrak{a}$ is an ideal.
(e) $\dim S^{-1}R \leq \dim R$, when S is a special multiplicative system.
(f) $\dim \mathbb{Z}/(4) = 0$.
(g) $\dim \mathbb{Z}/(6) = 0$.

In cases (d) and (e), more precise results will be provided later.

(23.17) **Infinite Dimension.** Let L be a field, and consider the next infinite chain of polynomial domains:

$$L[X_1] \subset L[X_1, X_2] \subset \cdots \subset L[X_1, \ldots, X_n] \subset \cdots$$

The union of these rings is easily seen to be a ring; it is denoted $L[X_1, X_2, \cdots]$ and said to be the polynomial ring over L in the infinitely many indeterminates $X_1, X_2, \ldots$, and it is a domain. In the infinite chain

$$O \subset (X_1) \subset (X_1, X_2) \subset \cdots \subset (X_1, \ldots, X_n) \subset \cdots$$

of ideals in $L[X_1, X_2, \cdots]$. These ideals are prime ideals as $L[X_1, X_2, \cdots]/(X_1, \ldots, X_n) \cong L[X_{n+1}, X_{n+2}, \cdots]$, and the latter is a domain. Thus $\dim L[X_1, X_2, \cdots] = \infty$.

(23.18) **Height of Prime Ideal.** For $\mathfrak{p} \in \operatorname{Spec} R$ the height $\operatorname{ht}_R \mathfrak{p}$ is defined as:

$$\operatorname{ht}_R \mathfrak{p} \stackrel{\text{def}}{=} \dim R_{\mathfrak{p}} \ (\in \mathbb{N}_0 \cup \{\infty\}).$$

By (23.12.e) applied with $S = R \setminus \mathfrak{p}$ we get:

$$\operatorname{ht}_R \mathfrak{p} = \sup\{ n \in \mathbb{N}_0 \mid \text{There is prime ideal chain: } \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n \subseteq \mathfrak{p} \}.$$

In particular,

$$\operatorname{ht}_R \mathfrak{p} = 0 \iff \mathfrak{p} \in \operatorname{Min} R.$$

(23.19) **Exercise: Height Inequality.** Let $\mathfrak{p}$ be a prime ideal. Prove the next inequality:

$$\operatorname{ht}_R \mathfrak{p} + \dim R/\mathfrak{p} \leq \dim R.$$

Provide an example with strict inequality.

(23.20) **Exercise: Height.** Let R be a UFD[25], and let $\mathfrak{p} \in \operatorname{Spec} R$ have $\operatorname{ht}_R \mathfrak{p} = 1$. Prove that $\mathfrak{p}$ is a principal ideal generated by a prime element.

The next two fundamental results are proved in [Ka] and [BH], respectively.

[25] That is, a Unique Factorization Domain

(23.21) **PIT** $\stackrel{\text{def}}{=}$ **Principal Ideal Theorem by Krull.** *Let x be an element in R, and assume that R is Noetherian. For any prime ideal $\mathfrak{p}$ that is minimal in $\mathrm{V}_R(x)$ (that is, $\mathfrak{p}/(x) \in \operatorname{Min} R/(x)$) the next inequality holds:*

$$\operatorname{ht}_R \mathfrak{p} \leq 1 .$$

(23.22) **PAL** $\stackrel{\text{def}}{=}$ **Prime Avoidance Lemma.** *Assume that $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are ideals in R, and that at most two of these are* not *prime ideals. If $\mathfrak{a}$ is an ideal that is* not *contained in any $\mathfrak{p}_i$, then*

$$\mathfrak{a} \nsubseteq \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n .$$

Thus, "the ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ can be avoided by an element in $\mathfrak{a}$".

(23.23) **Proposition: Spec R infinite.** *If R is Noetherian and $\dim R \geq 2$, then the set $\operatorname{Spec} R$ is infinite.*

(23.24) **Proof.** We will actually prove that $\mathbb{X} \stackrel{\text{def}}{=} \{ \mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{ht}_R \mathfrak{p} \leq 1 \}$ is infinite. Assume that $\mathbb{X} = \{ \mathfrak{p}_1, \dots, \mathfrak{p}_n \}$ is finite [and look for a contradiction]. As $\dim R \geq 2$ by assumption, $\mathfrak{q} \in \operatorname{Spec} R$ can be chosen such that $\operatorname{ht}_R \mathfrak{q} \geq 2$. Next, use **PAL** (23.22) to choose $x \in \mathfrak{q} \setminus \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$. By (23.3.1) $\operatorname{Min} R/(x)$ is non-empty, so $\mathfrak{p}'$ can be chosen minimal in $\mathrm{V}_R(x)$, cf. (23.12.d). By **PIT** (23.21) we have $\operatorname{ht}_R \mathfrak{p}' \leq 1$, and hence $\mathfrak{p}' \in \mathbb{X} = \{ \mathfrak{p}_1, \dots, \mathfrak{p}_n \}$ contradicting $x \in \mathfrak{p}' \setminus \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$. □

For the rest of this section we assume, cf. (23.13):

$$\boxed{(R, \mathfrak{m}, k) \text{ is a } \textit{local} \text{ ring.}}$$

(23.25) **GPIT** $\stackrel{\text{def}}{=}$ **General PIT.** *For $n \in \mathbb{N}_0$ the following are equivalent.*

(*i*) *There exist $a_1, \dots, a_n \in \mathfrak{m}$ such that $\mathrm{V}_R(a_1, \dots, a_n) = \{ \mathfrak{m} \}$.*
(*ii*) $\dim R \leq n$;

(23.26) **Proof.** "(*ii*) $\Longrightarrow$ (*i*)" by induction on n:

$\underline{n = 0}$: $\dim R \leq 0$ means that $\dim R = 0$, that is, $\operatorname{Spec} R = \{ \mathfrak{m} \}$. The ideal generated by 0 elements is $\operatorname{span}_R \varnothing = O$, the zero ideal, cf. (2.9.1). Thus, $\mathrm{V}_R(O) = \operatorname{Spec} R = \{ \mathfrak{m} \}$, as desired.

$\underline{n > 0}$: By Proposition (23.4) the set $\operatorname{Min} R$ is finite, say $\operatorname{Min} R = \{ \mathfrak{q}_1, \dots, \mathfrak{q}_m \}$. By **PAL** (23.22) we can choose $x \in \mathfrak{m}$ such that $x \notin \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_m$. Set $\overline{R} =^{\text{def}} R/(x)$ which is a local ring with maximal ideal $\mathfrak{m}' =^{\text{def}} \mathfrak{m}/(x)$. In order to be able to apply the inductive hypothesis, we want to prove:

$$(*) \qquad \dim \overline{R} \leq n - 1 .$$

To achieve this, let $\widetilde{\mathfrak{p}}_1 \subset \cdots \subset \widetilde{\mathfrak{p}}_\ell$ be any (strict) chain in $\operatorname{Spec} \overline{R}$, and consider the corresponding chain $\mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_\ell$ in $\operatorname{Spec} R$, cf. (23.12.d), that is, $x \in \mathfrak{p}_i$ and $\mathfrak{p}_i/(x) = \widetilde{\mathfrak{p}}_i$. As x is in $\mathfrak{p}_0$ and as it is not in any of the minimal prime ideals $\mathfrak{q}_1, \dots, \mathfrak{q}_n$, we conclude that $\mathfrak{p}_0$ is not minimal, that is, $\operatorname{ht}_R \mathfrak{p}_0 \geq 1$, so we may choose $\mathfrak{p}_{-1} \in \operatorname{Spec} R$ such that $\mathfrak{p}_{-1} \subset \mathfrak{p}_0$. Now, we have constructed a chain in $\operatorname{Spec} R$ of length $\ell + 1$, and hence $\ell + 1 \leq \dim R$. As the latter, by assumption, is at most n, we have established $(*)$.

The inductive hypothesis now provide elements $[a_1]_{(x)}, \dots, [a_{n-1}]_{(x)} \in \overline{R}$ such that $\mathrm{V}_{\overline{R}}([a_1]_{(x)}, \dots, [a_{n-1}]_{(x)}) = \{ \mathfrak{m}' \}$, and hence $\mathrm{V}_R(a_1, \dots, a_{n-1}, x) = \{ \mathfrak{m} \}$, as desired.

"$(i) \Longrightarrow (ii)$":

$\underline{n = 0}$: In this case $\mathrm{V}_R(a_1, \ldots, a_n) = \mathrm{V}_R(O) = \operatorname{Spec} R$ and hence $\operatorname{Spec} R = \{\mathfrak{m}\}$; thus, $\dim R = 0$.

$\underline{n = 1}$: As, by assumption, $\mathfrak{m}$ is minimal in $\mathrm{V}_R(a_1)$, **PIT** (23.21) yields $\operatorname{ht}_R \mathfrak{m} \leq 1$, that is, $\dim R \leq 1$ as desired.

$\underline{n > 1}$: This requires a special argument that will be provided later—if at all. □

(23.27) **Corollary 1.** *If $\mathfrak{m}$ can be generated by b ($\in \mathbb{N}_0$) elements, then $\dim R \leq b$.* □

(23.28) **Corollary 2.** $\dim R < \infty$. □

(23.29) **Corollary 3.** *Set $d \stackrel{\text{def}}{=} \dim R$. There exists then $a_1, \ldots, a_d \in \mathfrak{m}$ such that $\mathrm{V}_R(a_1, \ldots, a_d) = \{\mathfrak{m}\}$.* □

(23.30) **Exercise: Generating Ideal up to Radical.** Prove that there to any ideal $\mathfrak{a}$ in R exists $a_1, \ldots, a_d \in \mathfrak{a}$ such that $d = \dim R$ and $\mathrm{V}_R(a_1, \ldots, a_d) = \mathrm{V}_R(\mathfrak{a})$.

(23.31) **Theorem: Dimension of Ring Modulo Element.** *Set $d \stackrel{\text{def}}{=} \dim R$, assume that $x \in \mathfrak{m}$, and set $\overline{R} =^{\text{def}} R/(x)$. The following then hold.*

(a) $d - 1 \leq \dim \overline{R} \leq d$.

(b) *If $\dim \overline{R} = d$, then there exists $\mathfrak{p} \in \operatorname{Min} R$ such that $\dim(R/\mathfrak{p}) = d$ and $x \in \mathfrak{p}$.*

(c) *If $x \notin \bigcup_{\mathfrak{p} \in \operatorname{Min} R} \mathfrak{p}$, then $\dim \overline{R} = d - 1$.*

(d) *If $\dim R > 0$, then there x can be chosen in $\mathfrak{m}$ such that $\dim \overline{R} = d - 1$*

(23.32) **Proof.** First note that

$$(\dagger) \qquad \operatorname{Spec} \overline{R} = \{\, \mathfrak{p}/(x) \mid \mathfrak{p} \in \mathrm{V}_R(x) \,\}$$

by (23.12.d). Thus, for (a), only the inequality $\dim \overline{R} \geq d - 1$ needs explanation. Set $e = \dim \overline{R}$, and choose by Corollary 3 (23.29) $[a_1]_{(x)}, \ldots, [a_e]_{(x)} \in \mathfrak{m}' =^{\text{def}} \mathfrak{m}/(x)$ (the maximal ideal in $\overline{R}$) such that $\mathfrak{m}'$ is minimal in $\mathrm{V}_{\overline{R}}([a_1]_{(x)}, \ldots, [a_e]_{(x)})$. Thus, $\mathfrak{m}$ is minimal in $\mathrm{V}_R(a_1, \ldots, a_e, x)$, and $d \leq e + 1$ follows from **GPIT** (23.25).

(b) results from (†), and (c) follows.

(d): Choose $x \in \mathfrak{m} \setminus \bigcup_{\mathfrak{p} \in \operatorname{Min} R} \mathfrak{p}$ by **PAL** (23.22) in view of the finiteness of $\operatorname{Min} R$, cf. (23.4). Apply then (c). □

(23.33) **Non-local Noetherian Rings.**

(a) Let A be a Noetherian ring which need not be local. If $d =^{\text{def}} \dim A < \infty$ and $\mathfrak{a}$ is an ideal in $\mathfrak{a}$, then there exist $d + 1$ elements $a_1, \ldots, a_{d+1}$ such that $\mathrm{V}_A(a_1, \ldots, a_{d+1}) = \mathrm{V}_A(\mathfrak{a})$. This is a variation of Exercise (23.30).

(b) Later we will give an example of a Noetherian ring A with $\dim A = \infty$. Corollary 2 (23.28) implies that this ring is not local. (The ring $L[\, X_1, X_2, \cdots]$ in Example (23.17) is not Noetherian.) In any ring A with $\dim A = \infty$, there is an infinite sequence of *chains* in $\operatorname{Spec} A$: $\mathfrak{p}_{00}$; $\mathfrak{p}_{10} \subset \mathfrak{p}_{11}$; $\ldots$; $\mathfrak{p}_{n0} \subset \mathfrak{p}_{n1} \subset \cdots \subset \mathfrak{p}_{nn}$; $\ldots$ (but each chain is finite when the ring is Noetherian).

24. Minimal Generation

Throughout this section:

$$\boxed{(R,\mathfrak{m},k) \text{ is a local,}}$$

that is, R is Noetherian, $\operatorname{Max} R = \{\,\mathfrak{m}\,\}$, and $k = R/\mathfrak{m}$.

Over-lining will, in this section, always refer to the ideal $\mathfrak{m}$, that is, for an R–module N, an element $n \in N$, and an R–homomorphism $\varphi\colon N \to P$, we write $\overline{N} = N/\mathfrak{m}N$ (which is a vector space over k), $\overline{n} = [n]_{\mathfrak{m}N}$ (which is an element in $\overline{N}$), and $\overline{\varphi}\colon \overline{N} \to \overline{P}$ (which is the induced linear map of vector spaces). Thus, the structure of $\overline{N}$ as a vector space over k is given by $\overline{r}\,\overline{n} = \overline{rn}$ for $r \in R$ and $n \in N$, cf. (2.27.3). As already indicated, there is one exception to the general usage of over-lining: in general we write k, not $\overline{R}$.

Further notation in this section: N denotes an R–module, b is in $\mathbb{N}_0$, and $n_1, \dots, n_b$ is a set of elements in N (and it is empty if $b = 0$). For $a_1, \dots, a_b \in R$ we consider the linear combination:

$$\Sigma_i a_i n_i = a_1 n_1 + \cdots + a_b n_b \,.$$

When i is used as a lower index, its range will be $\{\,1, \dots, b\,\}$, and the same applies to the letter j. Thus, $1 \le i, j \le b$ (in this section).

(24.1) **Invertible elements.** For $x, u \in R$ the following hold.

(24.1.1) $$u \notin \mathfrak{m} \iff u \text{ is invertible in } R\,.$$

(24.1.2) $$x \in \mathfrak{m} \implies 1 - x \text{ is invertible in } R\,.$$

In the former, both sides are equivalent to $(u) = R$, and in the latter, x and $1 - x$ cannot both belong to $\mathfrak{m}$.

(24.2) **Linear independence over $\mathbf{k}$.** We shall often use that elements $\overline{n_1}, \dots, \overline{n_b}$ in $\overline{N}$ are *not* linearly independent if and only if there exists a number[26] j such that

(†) $$\overline{n_j} \in \Sigma_{i \ne j} k\overline{n_i}\,,$$

and this is tantamount to

(‡) $$n_j \in \Sigma_{i \ne j} R n_i + \mathfrak{m}N\,.$$

(24.3) **Theorem: Noether's Isomorphisms.** *The following hold for submodules N' and N'' of N.*

(24.3.1) $$N'/(N' \cap N'') \cong (N' + N'')/N''\,.$$

(24.3.2) $$(N/N'')/(N'/N'') \cong N/N' \quad \text{if } N'' \subseteq N'\,.$$

(24.4) **Proof.** (1): The map $N' \to (N' + N'')/N''$ given by $n' \mapsto [n']_{N''}$ is a surjective R–homomorphism with kernel $N' \cap N''$. Thus, the desired assertion results from Isomorphism Theorem (2.30).

(2): The map $N \to (N/N'')/(N'/N'')$ given by $n \mapsto [[n]_{N''}]_{N'/N''}$ is a surjective R–homomorphism with kernel N', so Isomorphism Theorem (2.30) yields again the desired. □

[26] In $\{\,1, \dots, b\,\}$ as always in this section

(24.5) **NAK $\stackrel{\text{def}}{=}$ Nakayama's Lemma.**[27] *If N is a finitely generated R–module and $N \neq O$, then $\overline{N} \neq O$.*

(24.6) **Proof.** As over-lining is right-exact, cf. (6.21), **LHF 0** (17.18) yields that it suffices to assume $N = R/\mathfrak{a}$ for an ideal $\mathfrak{a}$. As R is local, and $R/\mathfrak{a} \neq O$, we obtain that $\mathfrak{a} \subseteq \mathfrak{m}$ and hence an exact sequence $N \to k \to O$. As over-lining is right-exact we get another exact sequence $\overline{M} \to \overline{k} \to O$. Since $\overline{k} = k \neq O$ we get $\overline{N} \neq O$ as desired. □

(24.7) **NAK Corollary 1.** *Let N' be a submodule of a finitely generated R–module N. The following then holds:*

$$\mathfrak{m}N + N' = N \quad \Longrightarrow \quad N' = N\,.$$

(24.8) **Proof.** The second equality in the chain below results from the next Exercise (24.9).

$$\overline{N/N'} = (N/N')/\mathfrak{m}(N/N') = (N/N')/\big((\mathfrak{m}N + N')/N'\big) \cong N/(\mathfrak{m}N + N') = O\,.$$

The isomorphism is from Noether's Isomorphism (24.3.2). □

(24.9) **Exercise: Ideal times Residue Module.** Prove that $\mathfrak{a}(N/N') = (\mathfrak{a}N + N')/N'$ whenever $\mathfrak{a}$ is an ideal, and N' is a submodule of an R–module N.

(24.10) **NAK Corollary 2.** *For $n_1, \ldots, n_b \in N$, the following are equivalent.*

(i) $n_1, \ldots, n_b$ generates N;
(ii) $\overline{n_1}, \ldots, \overline{n_b}$ generates $\overline{N}$, and N is finitely generated.

(24.11) **Proof.** "$(i) \Longrightarrow (ii)$" is obvious.

"$(ii) \Longrightarrow (i)$": Consider the submodule $N' \stackrel{\text{def}}{=} \Sigma_i Rn_i$ of N. In the k–module $\overline{N}$ we have $(N' + \mathfrak{m}N)/\mathfrak{m}N = \Sigma_i k\overline{n_i} = \overline{N}$; the former by the definition of N'; the latter by the assumption. Thus, $N' + \mathfrak{m}N = N$ and NAK Corollary 1 (24.7) yields $N' = N$ as desired. □

(24.12) **Definition: Minimal Generators.** Any set of elements $n_1, \ldots, n_b \in N$ is a *minimal generating set for N*, exactly when it generates N, and no n_j is superfluous, the latter is, for all j we have:

$$(*) \qquad n_j \notin \Sigma_{i \neq j} Rn_i\,.$$

(24.13) **Proposition: Minimal Generators.** *For $n_1, \ldots, n_b \in N$, the following are equivalent.*

(i) $n_1, \ldots, n_b$ is a minimal generating set *for N;*
(ii) $\overline{n_1}, \ldots, \overline{n_b}$ is a basis *for $\overline{N}$ over k, and the R–module N is finitely generated.*

(24.14) **Proof.** "$(i) \Longrightarrow (ii)$": As $n_1, \ldots, n_b$ generates N, the residues $\overline{n_1}, \ldots, \overline{n_b}$ certainly generates $\overline{N}$. Assume that they are not linearly independent [and look for a contradiction]. By (24.2.‡) there is a j such that $n_j \in \Sigma_{i \neq j} Rn_i + \mathfrak{m}N$, that is, $n_j = \Sigma_{i \neq j} r_i n_i + x$ with all $r_i \in R$ and $x \in \mathfrak{m}N$. Since $n_1, \ldots, n_b$ generates the module N, we can find

[27] Late Hideyuki Matsumura wrote in [M1, page 11]: "This simple but important lemma is due to T. Nakayama, G. Azumaya, and W. Krull, and although it is usually called *Lemma of Nakayama*, late Prof. Nakayama did not like the name."

$m_1, \dots, m_b \in \mathfrak{m}$ such that $x = \Sigma_i m_i n_i$. Thus,

$$n_j = \Sigma_{i \neq j} r_i n_i + \Sigma_i m_i n_i \ ,$$

that is,

$$(1 - m_j) n_j = \Sigma_{i \neq j} (r_i + m_i) n_i \in \Sigma_{i \neq j} R m_i \ .$$

As $1 - m_j$ is invertible in R, cf. (24.1.2), n_j belongs to $\Sigma_{i \neq j} R m_i$ contradicting [as desired] the minimality of the generating set $n_1, \dots, n_b$.

"$(ii) \Longrightarrow (i)$": Under our assumptions, NAK Corollary 2 (24.10) yields that $n_1, \dots, n_b$ generate N, and we are required to prove the minimality. Assume that, for some j, we have n_j in $\Sigma_{i \neq j} R n_i$. We get $\overline{n_j} \in \Sigma_{i \neq j} k \overline{n_i}$ contradicting the linear independence of $\overline{n_1}, \dots, \overline{n_b}$.

(24.15) **Vector Space Dimension.** The dimension of a k–vector space V is denoted by $|V|_k$; it belongs to $\mathbb{N}_0 \cup \{\infty\}$. Thus, we reserve the operator name dim for the Krull dimension.

(24.16) **Definition; Zeroth Betti Number.** For any R–module N its *zeroth Betti number* is defined to be $\beta_0^R(N) \stackrel{\text{def}}{=} |\overline{N}|_k$ in $\mathbb{N}_0 \cup \{\infty\}$. (Later, in (24.30), higher order Betti numbers $\beta_n^R(N)$ will be introduced for $n \in \mathbb{N}$.)

From Proposition: Minimal Generators (24.13) we obtain the next:

(24.17) **Corollary: Minimal Generators.** *For any finitely generated R–module N the next are equivalent.*

(i) $n_1, \dots, n_b$ *is a* minimal generating set *for* N;
(ii) $n_1, \dots, n_b$ generates N, *and* $b = \beta_0^R(N)$. □

(24.18) **Minimal Number of Generators.** Let N be a finitely generated R–module. It follows from the corollary above, that all minimal generating sets for N have the number of elements, and that:

$$\beta_0^R(N) \ \text{ is the } \textit{minimal number of generators} \text{ for } \ N \ .$$

(24.19) **Lemma.** *If $O \to U \to V \to W \to O$ is an exact sequence of vector spaces over k, then*

$$|V|_k = |U|_k + |W|_k \ (\textit{in } \mathbb{N}_0 \cup \{\infty\}).$$

(24.20) **Proof.** Because W is a projective k–module, cf. (7.29) or (17.10), the sequence split, cf. (7.10), and hence $V \cong U \oplus W$, cf. (6.45.$*$), and the desired assertion follows. □

(24.21) **Lemma: Modulo an Element.** *If N is a finitely generated R–module, and x is an element in N, then the following holds:*

$$\beta_0^R(N/Rx) = \begin{cases} \beta_0^R(N) - 1 & \textit{if } \ x \notin \mathfrak{m}N \\ \beta_0^R(N) & \textit{if } \ x \in \mathfrak{m}N \ . \end{cases}$$

(24.22) **Proof.** We start by some calculations.

$$\overline{N/Rx} = (N/Rx)/\mathfrak{m}(N/Rx) = (N/Rx)/((\mathfrak{m}N + Rx)/Rx) \cong N/(\mathfrak{m}M + Rx).$$

$$(\mathfrak{m}N + Rx)/\mathfrak{m}N = R[\,x\,]_{\mathfrak{m}N} = k[\,x\,]_{\mathfrak{m}N} \ (\subseteq \overline{N})$$

In the first line, the first equality is by definition of over-lining (in this section), the second one is by Exercise (24.9), and the third one is by Noether's second isomorphism (24.3.2) In the second line, the left hand module is generated by the residue class of x, and this yields the first equlity; the second one is because both sides are submodules of the vector space $\overline{N}$. Noether's second isomorphism (24.3.2) yield also the next exact sequence:

$$O \to (\mathfrak{m}N + Rx)/\mathfrak{m}N \to N/\mathfrak{m}N \to N/(\mathfrak{m}N + Rx) \to O\,.$$

In view of the above, the last exact sequence gives the next exact sequence:

$$O \to k[\,x\,]_{\mathfrak{m}N} \to \overline{N} \to \overline{N/Rx} \to O\,.$$

Thus, by Definition (24.16) and Lemma (24.19) we have obtained the next:

$$\beta_0^R(N/Rx) = \beta_0^R(N) - |k[\,x\,]_{\mathfrak{m}N}|_k\,.$$

Finally, note that:

$$|k[\,x\,]_{\mathfrak{m}N}|_k = \begin{cases} 1 & \text{if } x \notin \mathfrak{m}N \\ 0 & \text{if } x \in \mathfrak{m}N\,. \end{cases}$$

The desired assertion has been proved. □

(24.23) **Lemma: Two Rings.** *Let N be a finitely generated R–module and let $\mathfrak{a}$ be an ideal such $\mathfrak{a} \subseteq \operatorname{Ann}_R N$. Then N is also a module over $R/\mathfrak{a}$ and:*

$$\beta_0^{R/\mathfrak{a}}(N) = \beta_0^R(N)\,.$$

(24.24) **Proof.** For the first assertion consult (2.27.3). It follows that $n_1, \ldots, n_b$ generates N as an R–module, if and only it does so as a module over $R/\mathfrak{a}$, and that it is a minimal generation set over R if and only if it is so over $R/\mathfrak{a}$. Now apply (24.18). □

(24.25) **Exercise: Zeroth Betti Number & Tensor Product.** Prove that the following hold for R–modules M and N.

(a) $\overline{M \otimes_R N} \cong \overline{M} \otimes_k \overline{N}$.

Assume from now on, that M and N are finitely generated.

(b) $\beta_0^R(M \otimes_R N) = \beta_0^R(M)\beta_0^R(N)$.

(c) $M \neq O$ and $N \neq O \implies M \otimes_R N \neq O$.

(24.26) **Lemma: Free Cover.** *Let N be a finitely generated R–module, and let $n_1, \ldots, n_b$ be a minimal generating set for N. There exists then an exact sequence:*

$$(*) \qquad O \to K \xrightarrow{\iota} R^b \xrightarrow{\sigma} N \to O$$

inducing an isomorphism $\overline{\sigma}\colon \overline{R}^b \xrightarrow{\cong} \overline{N}$, and such that $\overline{\iota} = o$ and $\operatorname{Tor}_1^R(N,k) \cong \overline{K}$.

(24.27) **Proof.** Define σ by $\sigma(r_1, \ldots, r_b) \stackrel{\text{def}}{=} \Sigma_i r_i n_i$. It is surjective as $n_1, \ldots, n_b$ generates N. Set $K =^{\text{def}} \operatorname{Ker}\sigma$. The exact sequence $(*)$ has been established. For $\overline{r_1}, \ldots, \overline{r_b} \in \overline{R}^b$ we have $\overline{\sigma}(\overline{r_1}, \ldots, \overline{r_b}) = \Sigma_i \overline{r_i}\,\overline{n_i}$ in $\overline{N}$. Furthermore, $\overline{\sigma}$ is an isomorphism as $\overline{n_1}, \ldots, \overline{n_b}$ is a basis for $\overline{N}$ over k, cf. (24.13). The remaining assertions now follow from the next long-exact sequence:

$$\operatorname{Tor}_1^R(R^b,k) \to \operatorname{Tor}_1^R(N,k) \xrightarrow{\delta_1} \overline{K} \xrightarrow{\overline{\iota}} \overline{R}^b \xrightarrow[\cong]{\overline{\sigma}} \overline{N} \to O$$

because $\operatorname{Tor}_1^R(R^b,k) = O$. □

Compare the next result to Theorem: Finitely Generated Flat is Projective (20.21) which does not require the ring to be local.

(24.28) **Theorem: Finitely Generated Flat is Free.** *Let N be a finitely generated R–module, and let $n_1, \dots, n_b$ be a minimal generating set for N. If the module N is flat, then it is free, and $n_1, \dots, n_b$ is a basis for it.*

(24.29) **Proof.** It follows from the lemma (24.26) that $\overline{K} \cong \operatorname{Tor}_1^R(N,k)$, and the latter module is O since N is flat. Thus, $\overline{K} = O$, so **NAK** (24.5) yields $K = O$, and $N \cong R^b$ by the lemma once more. □

(24.30) **Definition: Higher Betti Numbers.** For any R–module N, the n'th *Betti number* is $\beta_n^R(N) =^{\text{def}} |\operatorname{Tor}_n^R(N,k)|_k \ (\in \mathbb{N}_0 \cup \{\infty\})$.

If N is finitely generated, then $\beta_n^R(N) < \infty$ for all n, cf. (21.15.b).

(24.31) **Theorem: Minimal Free Resolutions.** *For any finitely generated R–module N there exists an exact sequence:*

$$(\dagger) \qquad \cdots \to R^{b_n} \xrightarrow{\partial_n} R^{b_{n-1}} \to \cdots \to R^{b_1} \xrightarrow{\partial_1} R^{b_0} \xrightarrow{\partial_0} N \to O$$

with $b_0, b_1, \dots, b_n \dots \in \mathbb{N}_0$ such that $\overline{\partial_0}$ is an isomorphism, and $\overline{\partial_n} = o$ for $n \geq 1$.

Furthermore, for any exact sequence (†) such that $\overline{\partial_n} = o$ for $n \in \mathbb{N}$, there are equalities:

$$(\times) \qquad b_n = \beta_n^R(N) \quad \text{for all} \quad n \in \mathbb{N}_0 \,.$$

(24.32) **Proof.** Set $K_{-1} \stackrel{\text{def}}{=} N$, and choose recursively for $\ell = 0, 1, \dots$ short-exact sequences:

$$O \to K_\ell \xrightarrow{\iota_\ell} R^{b_\ell} \xrightarrow{\sigma_\ell} K_{\ell-1} \to O$$

as in (24.26,$*$) such that $\overline{\sigma_\ell}$ is an isomorphism and $\overline{\iota_\ell} = o$. Setting $\partial_0 =^{\text{def}} \sigma_0$ and $\partial_n =^{\text{def}} \iota_{n-1}\sigma_n$ for $n \geq 1$, note that $\overline{\partial_0} = \overline{\sigma_0}$ is an isomorphism and $\overline{\partial_n} = \overline{\iota_{n-1}\sigma_n} = o\overline{\sigma_n} = o$.

To prove the last assertion, note that

$$P_\cdot \stackrel{\text{def}}{=} \cdots \to R^{b_n} \to R^{b_{n-1}} \to \cdots \to R^{b_1} \to R^{b_0} \to O$$

is a projective resolution of N, and that

$$P_\cdot \otimes_R k = \cdots \xrightarrow{o} k^{b_n} \xrightarrow{o} k^{b_{n-1}} \xrightarrow{o} \cdots \xrightarrow{o} k^{b_1} \xrightarrow{o} k^{b_0} \to O \,.$$

It follows that $\operatorname{Tor}_n^R(M,k) = \operatorname{H}_n(P_\cdot \otimes_R k) = k^{b_n}$ for $n \in \mathbb{N}_0$, so the definition (24.30) of Betti numbers yields the desired assertion. □

(24.33) **Corollary: Vanishing of Betti Numbers.** *The following hold, when N is a finitely generated R–module.*

(a) $\operatorname{pd}_R N = \sup\{ n \in \mathbb{N}_0 \mid \beta_n^R(N) \neq 0 \} \ (\leq \infty)$.

(b) $\beta_n^R(N) \neq 0$ *for all* $n \leq \operatorname{pd}_R N$.

(24.34) **Proof.** Both assertions follow from (†) in view of (×), cf. the theorem(24.31). □

(24.35) **Corollary: Modulo an Element.** *If N is a finitely generated R–module, and $x \in \mathfrak{m} \setminus \operatorname{Zero}_R N \cup \operatorname{Zero}_R R$, then $\beta_n^{R/(x)}(N/xN) = \beta_n^R(N)$ and $\operatorname{pd}_{R/(x)} N/xN = \operatorname{pd}_R N$.*

(24.36) **Remark.** By (7.34.b), $\operatorname{pd}_{R/(x)} N/xN \leq \operatorname{pd}_R N$ holds in further generality.

(24.37) **Proof of Corollary.** Let $\widetilde{\ }$ denote the functor $-\otimes_R R/(x)$ (as the usual overlining is occupied as denoting $-\otimes_R k$). Thus, $\widetilde{M} \cong M/xM$ for all R–modules M. Note that $(\widetilde{R}, \mathfrak{m}/(x), k)$ is a local ring.

The exact sequence of R–modules from the theorem (24.31):

$$\cdots \to R^{b_n} \to R^{b_{n-1}} \to \cdots \to R^{b_0} \to N \to O$$

induces, by Exercise (7.34), an exact sequence of $\widetilde{R}$–modules:

$$\cdots \to \widetilde{R}^{b_n} \to \widetilde{R}^{b_{n-1}} \to \cdots \to \widetilde{R}^{b_0} \to \widetilde{N} \to O\,.$$

Since $\overline{\widetilde{R}} = k$, the theorem (24.31) yields $\beta_n^{R/(x)}(M/xM) = \beta_n^R(N)$, and $\operatorname{pd}_{R/(x)} N/xN = \operatorname{pd}_R N$ follows by the first corollary (24.33). □

(24.38) **Exercise.** Provide a more direct prove of Lemma: Modulo an Element (24.21).

25. Regular Local Rings I

Throughout this section:

$$\boxed{(R,\mathfrak{m},k) \text{ is a local.}}$$

(25.1) **Minimal Generation of Maximal Ideal.** The minimal number of generators of the maximal ideal $\mathfrak{m}$ is $\beta_0^R(\mathfrak{m}) = |\mathfrak{m}/\mathfrak{m}^2|_k$, cf. (24.18) and (24.16), when $\mathfrak{m}^2 =^{\text{def}} \mathfrak{m}\mathfrak{m}$. By **GPIT** (23.25) and (24.18) we have the next

(FI) *Fundamental Inequality:* $\dim R \leq \beta_0^R(\mathfrak{m})$.

(25.2) **Modulo an Element.** For $x \in \mathfrak{m}$ we set[28]:

$$\overline{R} \stackrel{\text{def}}{=} R/(x) \text{ and } \mathfrak{m}' \stackrel{\text{def}}{=} \mathfrak{m}/(x). \tag{25.2.1}$$

Note that (cf. also the Noether isomorphism (24.3.2)):

$$(\overline{R},\mathfrak{m}',k) \text{ is a local ring.} \tag{25.2.2}$$

For $x \in \mathfrak{m} \setminus \bigcup_{\mathfrak{p}\in \operatorname{Min} R} \mathfrak{p}$, one deduces from (c), respectively, (24.21) and (24.23), the following:

$$\dim \overline{R} = \dim R - 1 \tag{25.2.3}$$

$$\beta_0^{\overline{R}}(\mathfrak{m}') = \begin{cases} \beta_0^R(\mathfrak{m}) - 1 & \text{if } x \notin \mathfrak{m}^2 \\ \beta_0^R(\mathfrak{m}) & \text{if } x \in \mathfrak{m}^2. \end{cases} \tag{25.2.4}$$

(25.3) **Definition: Regular Local Ring.** The local ring $(R,\mathfrak{m},k)$ is said to be *regular*, exactly when there is equality in the Fundamental Inequality(FI), that is,

$$R \text{ is regular} \iff^{\text{def}} \dim R = \beta_0^R(\mathfrak{m}).$$

(25.4) **Dimension Zero.**

$$R \text{ is regular and } \dim R = 0 \iff R \text{ is a field.}$$

Indeed, $\beta_0^R(\mathfrak{m}) = 0$ if and only if $\mathfrak{m} = O$.

(25.5) **Dimension One.**

$$R \text{ is regular and } \dim R = 1 \iff \mathfrak{m} \text{ is principal.}$$

Indeed, $\beta_0^R(\mathfrak{m}) = 1$ if and only if $\mathfrak{m}$ is principal. These rings turn out to local PIDs, cf. (25.10) and consult also (25.12).

(25.6) **Fundamental Lemma.** *Assume that $x \in \mathfrak{m} \setminus \bigcup_{\mathfrak{p}\in \operatorname{Min} R} \mathfrak{p}$, and consider the local ring $(\overline{R},\mathfrak{m}',k)$, cf. (25.2.1). The following then hold.*

(a) *If R is regular, then:*

$$\overline{R} \text{ is regular} \iff x \notin \mathfrak{m}^2.$$

(b) *If $\overline{R}$ is regular, then R is regular and $x \notin \mathfrak{m}^2$.*

[28] In this section—in contrast to the usage in Section 24.

(25.7) **Proof.** (a): As $\beta_0^R(\mathfrak{m}) - 1 = \dim R - 1 = \dim \overline{R}$ by the assumptions and (25.2.3), the desired equivalence follows from (25.2.4).

(b): Assume that $\overline{R}$ is regular, that is, the first equality in the next chain.

$$\dim \overline{R} = \beta_0^{\overline{R}}(\mathfrak{m}') \geq \beta_0^R(\mathfrak{m}) - 1 \geq \dim R - 1 = \dim \overline{R}.$$

Here, the inequalities result from (25.2.4) and Fundamental Inequality(FI), respectively, the first equality is our assumption, while the last one is by (25.2.3). Thus, equality holds in both cases, that is, respectively, $x \notin \mathfrak{m}^2$ and R is regular. □

(25.8) **Theorem.** *If R is regular, then R is a domain.*

(25.9) **Proof.** We use induction on $d \stackrel{\text{def}}{=} \dim R$.

$\underline{d = 0}$: In this case R is a field by (25.4) and hence a domain.

$\underline{d > 0}$: First, note that, in this case, $\mathfrak{m} \nsubseteq \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Min} R$. Second, note that $\mathfrak{m} \nsubseteq \mathfrak{m}^2$ by **NAK** (24.5) as, in this case, $\mathfrak{m} \neq O$. Thus, **PAL** (23.22) yields $\mathfrak{m} \nsubseteq (\bigcup_{\mathfrak{p} \in \operatorname{Min} R} \mathfrak{p}) \cup \mathfrak{m}^2$. Choose

$$(*) \qquad x \in \mathfrak{m} \setminus \left(\left(\bigcup_{\mathfrak{p} \in \operatorname{Min} R} \mathfrak{p} \right) \cup \mathfrak{m}^2 \right).$$

Now, $\overline{R} \stackrel{\text{def}}{=} R/(x)$ is regular by (a) in Fundamental Lemma (25.6), and hence $\overline{R}$ is a domain by the inductive hypothesis, that is, (x) is a prime ideal in R. By the choice of the element x, this prime ideal is not minimal. Consider the special multiplicative system $S =^{\text{def}} R \setminus (x)$ and the ring $S^{-1}R$, and choose, by (23.3.1) a minimal prime ideal in this ring; by (23.9), this minimal prime ideal is of the form $S^{-1}\mathfrak{p}$ for a prime ideal $\mathfrak{p}$ in R such that $S \cap \mathfrak{p} = \varnothing$, that is, $\mathfrak{p} \subset (x)$. Now we have the next:

$$(\times) \qquad x\mathfrak{p} = \mathfrak{p}.$$

Namely, as "$\subseteq$" is obvious, we prove the opposite inclusion. Assume that p belongs to $\mathfrak{p}$. As $\mathfrak{p} \subset (x)$ there exists $r \in R$ such that $p = rx$, which—as $x \notin \mathfrak{p}$—implies that $r \in \mathfrak{p}$ and hence $p \in x\mathfrak{p}$ as required for $(\times)$. Now **NAK** (24.5) yields $\mathfrak{p} = O$. Thus, R is a domain, as desired. □

(25.10) **Theorem: Dimension One.** *If $\dim R = 1$, then the next holds:*

$$R \text{ is regular} \iff R \text{ is PID.}$$

(25.11) **Proof.** The implication "$\Longleftarrow$" is obvious by (25.5).

"$\Longrightarrow$": Assume that $\mathfrak{m} = (t)$. Furthermore, assume that there exists an ideal that is *not* a principal ideal [and look for a contradiction]. As R is Noetherian, we can choose $\mathfrak{a}$ maximal among non-principal ideals, cf. (18.2). Set $\mathfrak{b} =^{\text{def}} \{ r \in R \mid tr \in \mathfrak{a} \}$. Since $\mathfrak{a} \subseteq \mathfrak{m} = (t)$ (as $\mathfrak{a}$ is not equal to R which is a principal ideal), we have $\mathfrak{a} = t\mathfrak{b}$, and **NAK** (24.5) yields $\mathfrak{a} \subset \mathfrak{b}$ which, in turn, implies that $\mathfrak{b}$ is a principal ideal, say $\mathfrak{b} = (b)$. Thus, $\mathfrak{a} = t\mathfrak{b} = (tb)$ which is the desired contradiction. □

(25.12) **DVR $\stackrel{\text{def}}{=}$ Discrete Valuation Ring.** The local ring R is said to by a *discrete valuation ring*, abbreviated as *DVR*, exactly when it is regular and $\dim R = 1$. Assume now that R is a DVR with maximal ideal $\mathfrak{m}$. It is then PID by the previous theorem (25.10), and hence UFD. If $\mathfrak{m} = (t)$, then for each $r \in R \setminus O$ there exists $v(r) \in \mathbb{N}_0$ and $u \in R \setminus \mathfrak{m}$ such that $r = ut^{v(r)}$, and $v(r)$ and u are uniquely determined. The map $r \mapsto v(r)$ can be extended to the field of fractions $w\colon B \to \mathbb{Z} \cup \{\infty\}$ by $w(r/s) =^{\text{def}} v(r) - v(s)$

and $w(0) = \infty$. This w satisfies $w(bc) = w(b) + w(c)$ and $w(b+c) \geq \min\{ w(b), w(c) \}$ for all $b, c \in B$, and it is said to be a *discrete valuation.*

(25.13) **Krull's Conjectures.** If R is a regular local ring, then the next holds.

(a) Any localization $R_{\mathfrak{p}}$ of R at a prime ideal $\mathfrak{p}$ is a regular local ring.
(b) The ring R is UFD.

These will be proved later by proving and using the next key result.

$$R \text{ is regular} \iff \operatorname{gldim} R < \infty .$$

(a) then follows directly by (17.13). (b) requires more—to be provided later.

(25.14) **Regular Sequence.** Let M be an R-module.

An element x in R is the said to be *M-regular*, exactly when $x \notin \operatorname{Zero}_R M$, that is, if and only if the sequence $O \longrightarrow M \xrightarrow{x} M$ is exact, and that is, if and only if $xm \neq 0$ for all $m \in M \setminus O$.

For $n \in \mathbb{N}_0$ a sequence $x_1, \ldots, x_n \in R$ is said to be *M-regular*, exactly when the next holds:

$$x_1 \notin \operatorname{Zero}_R M,\ x_2 \notin \operatorname{Zero}_R M/x_1M,\ \ldots,\ x_n \notin \operatorname{Zero}_R M/(x_1, \ldots, x_{n-1})M ,$$

that is, if and only if, x_i is $M/(x_1, \ldots, x_{i-1})M$-regular for all $i \in \{ 1, \ldots, n \}$. For $n = 0$ the sequence is empty, and it is regular.

Here, we *do not* require that $(x_1, \ldots, x_n)M \subset M$ which, however, is fulfilled if when M is finitely generated, $M \neq O$, and $x_1, \ldots, x_n \in \mathfrak{m}$, cf. **NAK** (24.5).

Furthermore, the above definition is concerning an *ordered* sequence $x_1, \ldots, x_n$.

(25.15) **Example.** Let $R \stackrel{\text{def}}{=} \mathbb{Z}_{(2)}$, the localization of the ring of integers $\mathbb{Z}$ at the prime ideal (2), cf. (23.14), that is,

$$R = \{ r/s \in \mathbb{Q} \mid r \in R,\ s \in \mathbb{N},\ s \text{ odd} \} ,$$

which is a local ring (and a subring of $\mathbb{Q}$). Consider the R-module $M =^{\text{def}} \mathbb{Q}$. The sequence $1, 0$ is M-regular (as $\operatorname{Zero}_R O = \varnothing$) but $0, 1$ is not.

(25.16) **Remark.** One can prove that if M is finitely generated, and $x_1, \ldots, x_n$ is an M-regular sequence, then any permutation $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$ of $x_1, \ldots, x_n$ is again an M-regular sequence.

The next result is important in our quest for proofs of the Krull conjectures.

(25.17) **Proposition: Maximal Ideal in Regular Local Ring.** *If R is regular, and $x_1, \ldots, x_d$ are minimal generators for $\mathfrak{m}$, then $x_1, \ldots, x_d$ is an R-regular sequence.*

(25.18) **Proof.** We use induction on d. Note first that, under our hypotheses, $d = \dim R$.

$\underline{d = 0}$: The sequence is empty in this case, cf. (25.4), and thus R-regular.

$\underline{d > 0}$: By (24.13) we have $x_1 \notin \mathfrak{m}^2$. Thus, as $\operatorname{Min} R = \{ O \}$ by (25.8), the local ring $\overline{R} \stackrel{\text{def}}{=} R/(x_1)$ is regular by Fundamental Lemma (25.6.a). The sequence $[x_2]_{(x_1)}, \ldots, [x_d]_{(x_1)}$ is a minimal set of generators for the maximal ideal $\mathfrak{m}/(x_1)$ in $\overline{R}$, and hence it is $\overline{R}$-regular by the inductive hypothesis, and our desired assertion results from the next lemma. □

(25.19) **Lemma.** *For $x_1, \dots, x_n \in \mathfrak{m}$, $n \geq 1$, and R–module M the following holds.*

$$x_1, \dots, x_n \text{ is } M\text{–regular} \iff \begin{cases} x_1 \notin \operatorname{Zero}_R M \text{ and} \\ x_2, \dots, x_n \text{ is } M/x_1M\text{–regular.} \end{cases}$$

(25.20) **Proof.** For $j \geq 2$ we have the next:

$$\begin{aligned} (M/x_1M)/(x_2, \dots, x_j)(M/x_1M) &= (M/x_1M)/(x_1, x_2 \dots, x_j)(M/x_1M) \\ &\cong M/(x_1, x_2 \dots, x_j)M\,, \end{aligned}$$

where the equality is (24.9) and the isomorphism is Noether's (24.3.2). The desired assertion follows by the definition (25.14). □

26. Half-exact Functors

Throughout this section:

R is a Noetherian ring.

(26.1) **Associated Primes & Support.** For any R-module$M$ we define:

$$\operatorname{Ass}_R M \stackrel{\text{def}}{=} \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{Ann}_R(x) = \mathfrak{p} \text{ for some } x \in M\}$$

$$\operatorname{Supp}_R M \stackrel{\text{def}}{=} \{\mathfrak{p} \in \operatorname{Spec} R \mid \operatorname{Ann}_R(x) \subseteq \mathfrak{p} \text{ for some } x \in M\}.$$

The elements of the former set are called *associated primes*, and the set is called the *assassinator*; the latter set is the *support* of M.

(26.2) **Proposition: Prime Ideals.** *The following hold for R-modules K, L, and M.*

(a) $\operatorname{Ass}_R M \subseteq \operatorname{Supp}_R M$.
(b) *If $M \neq O$, then* $\operatorname{Ass}_R M \neq \varnothing$.
(c) *If $\mathfrak{p} \in \operatorname{Spec} R$, then $M_{\mathfrak{p}} \neq O$, if and only if $\mathfrak{p} \in \operatorname{Supp}_R M$.*
(d) *If M is finitely generated, then the set $\operatorname{Ass}_R M$ is finite.*
(e) $\operatorname{Zero}_R M = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R M} \mathfrak{p}$.
(f) *If $O \to K \to L \to M \to O$ is an exact sequence, then there are the next inclusions:*

$$\operatorname{Ass}_R K \subseteq \operatorname{Ass}_R L \subseteq \operatorname{Ass}_R K \cup \operatorname{Ass}_R M .$$

(g) *If $O \to K \to L \to M \to O$ is an exact sequence, then there are the next equality:*

$$\operatorname{Supp}_R L = \operatorname{Supp}_R K \cup \operatorname{Supp}_R M .$$

(h) $\operatorname{Supp}_R M \subseteq \operatorname{V}_R(\operatorname{Ann}_R M)$ *with equality if M is finitely generated.*

(26.3) **Proof.** (a) is obvious from the definitions.

(b): The set $\mathbb{X} =^{\text{def}} \{\operatorname{Ann}_R(x) \mid x \in M \setminus O\}$ has—as R is Noetherian—a maximal element, and this is a prime ideal.

(c) follows from (23.14) and (5.8.5).

(d) follows from (26.5) below.

(e): The inclusion "$\supseteq$" is obvious. To prove the other one assume that $a \in R$ and $x \in M$ are such that $ax = 0$; choose $\mathfrak{p}$ in $\operatorname{Ass}_R Rx \subseteq \operatorname{Ass}_R M$.

(f): The first inclusion is obvious from the definition. For the second one, assume that $\mathfrak{p} = \operatorname{Ann}_R \ell \in \operatorname{Ass}_R L$ with $\ell \in L$, and assume $R\ell \cap K = O$, respectively, $R\ell \cap K \neq O$, and conclude $\mathfrak{p} \in \operatorname{Ass}_R M$ respectively, $\mathfrak{p} \in \operatorname{Ass}_R K$.

(g): The assertion follows directly from (c) in view of the exactness of the localization functor, cf. (23.14) and (6.24).

(h): The inclusion follows from the definitions, cf. (26.1) and (23.8), since $\operatorname{Ann}_R(x) \subseteq \operatorname{Ann}_R M$ for all $x \in M$. Next, assume that $M = Rm_1 + \cdots + RM_n$ and consider the R-homomorphism $\varphi\colon R \to M^n$ defined by $\varphi(r) =^{\text{def}} (rm_1, \ldots, rm_n)$. As $\operatorname{Ker}\varphi = \operatorname{Ann}_R M$, there is an exact sequence $O \to R/\operatorname{Ann}_R M \to M^n$ by Isomorphism Theorem (2.30), and hence $\operatorname{V}_R(\operatorname{Ann}_R M) = \operatorname{Supp}_R(R/\operatorname{Ann}_R M) \subseteq \operatorname{Supp}_R M^n = \operatorname{Supp}_R M$ by Definitions (23.8) and (26.1) together with (g). □

(26.4) **Definition: Prime Factor Chain.** A chain $O = M_0 \subset M_1 \cdots \subset M_n = M$ of submodules of an R-module M *together with* $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \operatorname{Spec} R$ such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for all $i \in \{1, \ldots, n\}$ is said to be a *prime factor chain* of M.

(26.5) **Proposition: Prime Factor Chain.** *If M is a finitely generated R-module, then M admits a prime factor chain, and there are inclusions:*

$$\operatorname{Ass}_R M \subseteq \{ \mathfrak{p}_1, \ldots, \mathfrak{p}_n \} \subseteq \operatorname{Supp}_R M$$

for any such prime factor chain.

(26.6) **Proof.** The zero module admits a prime factor chain, namely the empty one (with zero submodules and zero prime ideals). Choose M' maximal among the submodules admitting prime factor chains. Then, $M' = M$ since $\operatorname{Ass}_R M/M' \neq \varnothing$ otherwise by (26.2.b). For the second assertion consult (f) and (g) in (26.2). □

(26.7) **Special Prime Avoidance Lemma.** *If M is a finitely generated R-module, and $\mathfrak{a}$ is an ideal in R such that $\mathfrak{a} \subseteq \operatorname{Zero}_R M$, then $\mathfrak{a} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_R M$.*

(26.8) **Proof.** Use **PAL** (23.22) together with (d) and (e) in (26.2). □

(26.9) **LHF 1** $\stackrel{\text{def}}{=}$ **Lemma on Half-exact Functors 1.** *Let $T\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R')$ be a half-exact functor. To any finitely generated R-module M with $T(M) \neq O$ there exists then $\mathfrak{p} \in \operatorname{Supp}_R M$ such that $T(R/\mathfrak{p}) \neq O$.*

(26.10) **Proof.** Choose a prime factor chain $O = M_0 \subset M_1 \cdots \subset M_n = M$ with prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \operatorname{Supp}_R M$, cf. (26.4) and (26.5). Each of the short-exact sequences

$$O \to M_{i-1} \to M_i \to R/\mathfrak{p}_i \to O$$

induces an exact sequence

$$T(M_{i-1}) \to T(M_i) \to T(R/\mathfrak{p}_i)$$

(with the arrows reversed in the contravariant case). As $T(M_n) = T(M) \neq O$ and $T(M_0) = T(O) = O$, cf. (6.13), we conclude $T(R/\mathfrak{p}_i) \neq O$ for some i. □

(26.11) **Linear Functors.** Let $T\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R)$ be a functor with the same target as source. It is said to be *R-linear*, exactly if it additive, and it satisfies $T(r\mu) = rT(\mu)$ for all $r \in R$ and all R-homomorphism μ, that is, if and only if the induced map $\operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(T(M),T(N))$ is an R-homomorphism. If T is additive, then T is R-linear, if and only if $T(r_M) = r_{T(M)}$ for all $r \in R$ and all R-modules M, , cf. (2.16). Indeed, if $\mu\colon M \to M'$ is an R-homomorphism and $r \in R$, then $T(r\mu) = T(r_{M'} \circ \mu) = T(r_M) \circ T(\mu)$.

(26.12) **Lemma: Linear Functor & Annihilator.** *If T is an R-linear functor, then $\operatorname{Ann}_R T(M) \supseteq \operatorname{Ann}_R M$ for all R-modules M.*

(26.13) **Proof.** $r_M = o$ for $r \in \operatorname{Ann}_R M$, hence $r_{T(M)} = o$, that is, $r \in \operatorname{Ann}_R T(M)$. □

(26.14) **Proposition: Derivative of Linear Functor.** *If T is a linear functor, then so are its derivatives $\mathrm{L}_n T$ and $\mathrm{R}^n T$.*

(26.15) **Proof.** For an R-module M with projective resolution $P_\bullet$ and $r \in R$, the morphism $r_{P_\bullet}\colon P_\bullet \to P_\bullet$, cf. (8.5), is a continuation of $r_M\colon M \to M$; thus:

$$\mathrm{L}_n T(r_M) = \mathrm{H}_n(T(r_{P_\bullet})) = \mathrm{H}_n(r_{T(P_\bullet)}) = r_{\mathrm{H}_n(T(P_\bullet))} = r_{\mathrm{L}_n T(M)}.$$ □

(26.16) **Ext & Tor.** For any $b \in \mathbb{N}_0$ the functors $\operatorname{Ext}^n_R(-,-)$ and $\operatorname{Tor}^R_n(-,-)$ are linear in either variable. Thus,

(26.16.1) $\qquad \operatorname{Ann}_R \operatorname{Ext}^n_R(M,N) \supseteq \operatorname{Ann}_R M + \operatorname{Ann}_R N;$

(26.16.2) $\qquad \operatorname{Ann}_R \operatorname{Tor}^R_n(M,N) \supseteq \operatorname{Ann}_R M + \operatorname{Ann}_R N.$ □

(26.17) **Auxiliary Lemma.** *Let $T\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R)$ be a half-exact and R–linear functor. If M' is a finitely generated R–module with $T(M') \neq O$, then there exists $\mathfrak{p} \in \operatorname{Supp}_R M'$ such that:*

(†) $\qquad T(R/\mathfrak{p}) \neq O$ *and* $T(R/\mathfrak{a}) = O$ *whenever* $\mathfrak{a}$ *is an ideal with* $\mathfrak{a} \supset \mathfrak{p}$ . . .

(26.18) **Proof.** The set $\mathbb{X} \stackrel{\text{def}}{=} \{\, \mathfrak{q} \in \operatorname{Supp}_R M' \mid T(R/\mathfrak{q}) \neq O \,\}$ is non-empty by **LHF 1** (26.9), so choose $\mathfrak{p}$ maximal in it. If $T(R/\mathfrak{a}) \neq O$, then **LHF 1** (26.9) gives $\mathfrak{q} \in \operatorname{Supp}_R R/\mathfrak{a}$ with $T(R/\mathfrak{q}) \neq O$. Since $\operatorname{Supp}_R R/\mathfrak{a} = \operatorname{V}_R(\mathfrak{a}) \subseteq \operatorname{V}_R(\mathfrak{p}) \subseteq \operatorname{Supp}_R M'$, we have $\mathfrak{q} \in \operatorname{Supp}_R M'$. As $\mathfrak{p} \subset \mathfrak{a} \subseteq \mathfrak{q}$, this contradicts the choice of $\mathfrak{p}$ as being maximal in $\mathbb{X}$. Thus, (†) has been proved. □

(26.19) **LHF 2** $\stackrel{\text{def}}{=}$ **Lemma on Half-exact Functors 2.** *Let $T\colon \operatorname{Mod}(R) \to \operatorname{Mod}(R)$ be half-exact and R–linear, and assume either*

(a) *T is contravariant; or*

(b) *T is covariant and left-exact.*

If M' is a finitely generated R–module with $T(M') \neq O$, then there exists $\mathfrak{p} \in \operatorname{Supp}_R M'$ such that $T(R/\mathfrak{p})_{\mathfrak{p}} \neq O$.

(26.20) **Proof.** Auxiliary Lemma (26.17) yields $\mathfrak{p} \in \operatorname{Supp}_R M'$ such that $T(R/\mathfrak{p}) \neq O$, and $T(R/\mathfrak{a}) = O$ whenever $\mathfrak{a}$ is an ideal with $\mathfrak{a} \supset \mathfrak{p}$.

We want to prove that

(‡) $\qquad O \to T(R/\mathfrak{p}) \xrightarrow{T(r_{R/\mathfrak{p}})} T(R/\mathfrak{p})$

is exact for all $r \in R \setminus \mathfrak{p}$. Indeed, set $\mathfrak{a} =^{\text{def}} \mathfrak{p} + (r)$ $(\supset \mathfrak{p})$, and note that the exact sequence

$$O \to R/P \xrightarrow{r_{R/\mathfrak{p}}} R/\mathfrak{p} \to R/\mathfrak{a} \to O$$

in either case, (a) or (b), induces the exact sequence (‡). Now, it follows that the map $T(R/\mathfrak{p}) \to T(R/\mathfrak{p})_{\mathfrak{p}}$ is injective, cf. the next Exercise (26.21) □

(26.21) **Exercise: Injectivity of Fraction Homomorphism.** If S is a special multiplicative system in R, and M is an R–module such that $S \cap \operatorname{Zero}_R M = \varnothing$, then the fraction homomorphism $M \to S^{-1}M$, $m \mapsto m/1$, is injective. Formulate and prove a corresponding assertion concerning surjectivity.

(26.22) **Theorem: Measuring Injective & Flat Dimension.** *For any R–module N there are the next two equalities:*

(i) $\operatorname{id}_R N = \sup\{\, n \in \mathbb{N}_0 \mid \operatorname{Ext}^n_R(R/\mathfrak{p}, N)_{\mathfrak{p}} \neq O$ *for some* $\mathfrak{p} \in \operatorname{Spec} R \,\}$;

(f) $\operatorname{fd}_R N = \sup\{\, n \in \mathbb{N}_0 \mid \operatorname{Tor}^R_n(R/\mathfrak{p}, N)_{\mathfrak{p}} \neq O$ *for some* $\mathfrak{p} \in \operatorname{Spec} R \,\}$.

(26.23) **Proof.** (i): The inequality "$\geq$" follows directly from **IDC** (17.5). To prove the other one, assume that $n \leq \operatorname{id}_R N$ [and we want to find $\mathfrak{p} \in \operatorname{Spec} R$ such that $\operatorname{Ext}^n_R(R/\mathfrak{p}, N)_{\mathfrak{p}} \neq O$]. The functor $\operatorname{Ext}^n_R(-, N)$ is contravariant and R–linear, so **LHF 2** (26.19) yields the desired $\mathfrak{p}$ in view of **IDT** (17.3).

(f): The inequality "$\geq$" follows directly from **FDC** (22.8). If $n = \operatorname{fd}_R N < \infty$, then $\operatorname{Tor}_n^R(-, N)$ is left-exact and covariant, so **FDT** (22.6) and **LHF 2** (26.19) yield $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{Tor}_n^R(R/\mathfrak{p}, N)_\mathfrak{p}$ non-zero. In general, if the right hand side of (f) is finite, then equlity follows from the next exercise. □

(26.24) **Exercise: LHF 2 Révised.** Let T: $\operatorname{Mod}(R) \to \operatorname{Mod}(R)$ be a covariant functor. It is said to be *left-connected*, exactly if there to any short-exact sequence

$$O \to K \xrightarrow{\kappa} L \xrightarrow{\lambda} M \to O$$

exists an R–homomorphism Δ: $T(M) \to T(K)$ such that also the next sequence is exact:

$$T(M) \xrightarrow{\Delta} T(K) \xrightarrow{T(\kappa)} T(L) \xrightarrow{T(\lambda)} T(M)\,.$$

(a) Prove that T is left-connected, if it is left-exact.
(b) Prove that the functor $T_{gN} =^{\text{def}} \coprod_{i \geq g} \operatorname{Tor}_g^R(-, N)$ is left-connected for all $g \in \mathbb{N}_0$ and all R–modules N.
(c) Assume that T is a left-connected and R–linear covariant functor, and let M' be a finitely generated R–module such that $T(M') \neq O$. Prove that there exists $\mathfrak{p} \in \operatorname{Supp}_R M'$ such that $T(R/\mathfrak{p})_\mathfrak{p} \neq O$.
(d) Prove the remaining part of the equality (26.22.f).

(26.25) **LHF 3** $\stackrel{\text{def}}{=}$ **Lemma on Half-exact Functors 3.** *Suppose that $(R, \mathfrak{m}, k)$ is local, and that T: $\operatorname{Mod}(R) \to \operatorname{Mod}(R)$ is an R–linear, covariant, half-exact functor, such that $T(M')$ is finitely generated, whenever M' is finitely generated. If there is one finitely generated R–module M' with $T(M') \neq O$, then $T(k) \neq O$.*

(26.26) **Proof.** Auxiliary Lemma (26.17) yields $\mathfrak{p} \in \operatorname{Supp}_R M'$ such that $T(R/\mathfrak{p}) \neq O$, and $T(R/\mathfrak{a}) = O$ whenever $\mathfrak{a}$ is an ideal with $\mathfrak{a} \supset \mathfrak{p}$. For any $r \in R \setminus \mathfrak{p}$, set $\mathfrak{a} =^{\text{def}} \mathfrak{p} + (r)$, and note that the short-exact sequence:

$$O \to R/\mathfrak{p} \xrightarrow{r} R/\mathfrak{p} \to R/\mathfrak{a} \to O$$

induce the next exact sequence:

$$T(R/\mathfrak{p}) \xrightarrow{r} T(R/\mathfrak{p}) \to O\,.$$

Thus, **NAK** (24.5) yields $r \notin \mathfrak{m}$. This implies that $\mathfrak{p} = \mathfrak{m}$, and hence $T(k) = T(R/\mathfrak{p}) \neq O$ as desired. □

(26.27) **PDC: Finite Generated over Local Ring.** *If L' is a finitely generated module over a local ring $(R, \mathfrak{m}, k)$, then:*

$$\operatorname{pd}_R L' = \sup\{\, n \in \mathbb{N}_0 \mid \operatorname{Ext}_R^n(L', k) \neq O \,\}\,.$$

(26.28) **Proof.** "$\geq$" follows from **PDC** (11.23).

"$\leq$": Let $u \in \mathbb{N}_0$ be such that $u \leq \operatorname{pd}_R L'$ (the latter might be infinite), and we want to prove $\operatorname{Ext}_R^u(L', k) \neq O$. The functor $\operatorname{Ext}_R^u(L', -)$ is linear by (26.14), covariant, and half-exact. Moreover, it carries finitely generated R–modules into finitely generated ones by (18.21), and there is a finitely generated R–module M' such that $\operatorname{Ext}_R^u(L', M') \neq O$, cf. (18.32). Thus, the desired assertion follows from **LHF 3** (26.25). □

(26.29) **Exercise.** Deduce the result (26.27) above by taking a minimal free resolution of L', cf. (24.31). [Prove and use that $\operatorname{Hom}_R(-, k) \cong \operatorname{Hom}_R(- \otimes_R k, k)$.] Furthermore, prove that $\operatorname{Ext}_R^u(L', k) \neq O$ for all $u \leq \operatorname{pd}_R L'$.

(26.30) **Lemma.** *Let M be a finitely generated module over a local ring $(R,\mathfrak{m},k)$, and let x belong to $\mathfrak{m} \setminus \operatorname{Zero}_R M$. There is then an equality $\operatorname{pd}_R M/xM = \operatorname{pd}_R M + 1$.*

(26.31) **Proof.** The exact sequence $O \to M \xrightarrow{x} M \to \overline{M} \to O$ induces the next one:

$$\operatorname{Ext}_R^{\ell-1}(M,k) \xrightarrow{x} \operatorname{Ext}_R^{\ell-1}(M,k) \xrightarrow{\delta} \operatorname{Ext}_R^{\ell}(\overline{M},k) \longrightarrow \operatorname{Ext}_R^{\ell}(M,k) \xrightarrow{x} \operatorname{Ext}_R^{\ell}(M,k)\,.$$

Here we used that $\operatorname{Ext}_R^u(-,k)$ is a linear functor, cf. (26.16). By (26.16.1), $\operatorname{Ann}_R \operatorname{Ext}_R^u(M,k) \supseteq \mathfrak{m}$, so the multiplications by x are zero. Thus, for each $\ell \in \mathbb{N}_0$ there is an exact sequence:

$$O \to \operatorname{Ext}_R^{\ell-1}(M,k) \to \operatorname{Ext}_R^{\ell}(\overline{M},k) \to \operatorname{Ext}_R^{\ell}(M,k) \to O\,,$$

and we obtain:

$$\begin{aligned} \operatorname{Ext}_R^{\ell}(\overline{M},k) \neq O \quad &\Longrightarrow \quad \operatorname{Ext}_R^{\ell-1}(M,k) \neq O \text{ or } \operatorname{Ext}_R^{\ell}(M,k) \neq O; \\ \operatorname{Ext}_R^{\ell-1}(M,k) \neq O \quad &\Longrightarrow \quad \operatorname{Ext}_R^{\ell}(\overline{M},k) \neq O\,. \end{aligned}$$

In view of PDC: Finite Generated over Local Ring (26.27), the former yields the inequality $\operatorname{pd}_R \overline{M} \leq \operatorname{pd}_R M + 1$, and the latter gives the opposite inequality. □

(26.32) **Exercise.** Let M, R, and x be as in (26.30), and assume furthermore that $x \notin \operatorname{Zero}_R R$. Prove that on the category $\operatorname{Mod}(R/(x))$ the functors $\operatorname{Ext}_{R/(x)}^n(M/xM,-)$ and $\operatorname{Ext}_R^n(M,-)$ are isomorphic. Provide another proof of (24.35): $\operatorname{pd}_{R/(x)} M/xM = \operatorname{pd}_R M$.

(26.33) **Exercise.** Let $(R,\mathfrak{m},k)$ be a one-dimensional regular local ring (for example, $R =^{\text{def}} \mathbb{Z}_{(2)}$), let a be a generator for $\mathfrak{m}$, and let T denote the functor $\mathrm{R}^1\Gamma_a$, cf. (13.14). Prove that T is R–linear, that $T(R) \neq O$, and that $T(R/\mathfrak{p})_{\mathfrak{p}} = O$ for all $\mathfrak{p} \in \operatorname{Supp}_R R$.

27. DEPTH

Throughout this section:

$$\boxed{(R, \mathfrak{m}, k) \text{ is a local.}}$$

(27.1) **Depth of Module.** $\operatorname{depth}_R M \stackrel{\text{def}}{=} \inf\{ n \in \mathbb{N}_0 \mid \operatorname{Ext}^n_R(k, M) \neq O \} \in \mathbb{N}_0 \cup \{\infty\}$.

(27.2) **Example.** Let R be a local domain with field of fractions B. For $x \in \mathfrak{m} \setminus O$ the multiplication homomorphism $x_k \colon k \to k$ is zero while $x_B \colon B \to B$ is an isomorphism; thus, for $n \in \mathbb{N}_0$ the homomorphism $x_{\operatorname{Ext}^n_R(k,B)} = \operatorname{Ext}^n_R(x_k, B) = \operatorname{Ext}^n_R(k, x_B)$ is both zero and an isomorphism, and hence $\operatorname{Ext}^n_R(k, B) = O$. Thus, $\operatorname{depth}_R B = \inf \varnothing = \infty$.

(27.3) **Lemma: Depth Zero.** *For any R-module M the following are then equivalent.*

(*i*) $\operatorname{depth}_R M = 0$;
(*ii*) $\operatorname{Hom}_R(k, M) \neq O$;
(*iii*) $\mathfrak{m} \in \operatorname{Ass}_R M$.

If M is finitely generated, then these are equivalent to the next:

(*iv*) $\operatorname{Zero}_R M = \mathfrak{m}$.

(27.4) **Proof.** "(*i*) $\Longleftrightarrow$ (*ii*)" by the definition of depth (27.1).
"(*ii*) $\Longrightarrow$ (*iii*)": For $\varphi \in \operatorname{Hom}_R(k, M) \setminus O$ we have $\mathfrak{m} = \operatorname{Ann}_R(\varphi([1]_\mathfrak{m}) \in \operatorname{Ass}_R M$.
"(*iii*) $\Longrightarrow$ (*ii*)": If $\mathfrak{m} = \operatorname{Ann}_R(m) \in \operatorname{Ass}_R M$ and $m \in M$, then $\varphi \colon k \to M$, $[r]_\mathfrak{m} \mapsto rm$ is a well-defined R-homomorphism, which is non-zero.
"(*iii*) $\Longrightarrow$ (*iv*)" by (e) in Proposition: Prime Ideals (26.2)
"(*iv*) $\Longrightarrow$ (*iii*)" Assume from now on that M is finitely generated. (d) and (e) in Proposition: Prime Ideals (26.2) yield that $\operatorname{Zero}_R M$ is a *finite* union of prime ideals, namely the associated ones, so **PAL** (23.22) gives the desired. □

(27.5) **Proposition: Depth Modulo Regular Element.** *If M is an R-module and $x \in \mathfrak{m} \setminus \operatorname{Zero}_R M$, then* $\operatorname{depth}_R M/xM = \operatorname{depth}_R M - 1$ $(\in \mathbb{N}_0 \cup \{\infty\})$.

(27.6) **Proof.** The exact sequence $O \to M \xrightarrow{x} M \to \overline{M} \to O$ induces the next one:

$$\operatorname{Ext}^{\ell-1}_R(k, M) \xrightarrow{x} \operatorname{Ext}^{\ell-1}_R(k, M) \longrightarrow \operatorname{Ext}^{\ell-1}_R(k, \overline{M}) \xrightarrow{\delta} \operatorname{Ext}^{\ell}_R(k, M) \xrightarrow{x} \operatorname{Ext}^{\ell}_R(k, M) .$$

Here the multiplications by x are zero, and for each $\ell \in \mathbb{N}_0$ we obtain an exact sequence:

$$O \to \operatorname{Ext}^{\ell-1}_R(k, M) \to \operatorname{Ext}^{\ell-1}_R(k, \overline{M}) \to \operatorname{Ext}^{\ell}_R(k, M) \to O .$$

Thus:

$$\operatorname{Ext}^{\ell-1}_R(k, \overline{M}) \neq O \iff \operatorname{Ext}^{\ell-1}_R(k, M) \neq O \text{ or } \operatorname{Ext}^{\ell}_R(k, M) \neq O ,$$

and the desired equation follows. □

(27.7) **Remarks.** If $\mathfrak{a}$ is an ideal in R, and M is an R-module, then the following hold.

(a) $\mathfrak{a} \subseteq \operatorname{Ann}_R(M/\mathfrak{a}M)$.
(b) $\operatorname{Ann}_R M \subseteq \operatorname{Zero}_R M$ if $M \neq O$.
(c) If $x \notin \operatorname{Zero}_R(M/\mathfrak{a}M)$ and $\mathfrak{a}M \subset M$, then $x \notin \mathfrak{a}$ by (a) and (b), and hence $\mathfrak{a} \subset \mathfrak{a}+(x)$.
(d) If $x_1, \dots, x_n \in \mathfrak{m}$ is an M-regular sequence, cf. (25.14), and $(x_1, \dots, x_n)M$ is a proper subset of M, then there is the next strictly ascending finite chain:

$$(x_1) \subset \cdots \subset (x_1, \dots, x_i) \subset (x_1, \dots, x_{i+1}) \subset \cdots \subset (x_1, \dots, x_n) .$$

(27.8) **Maximal Regular Sequence.** Element $x_1, \dots, x_n \in \mathfrak{m}$ are said to form a *maximal M-regular sequence*, exactly when it is an M–regular sequence, and $\operatorname{Zero}_R(M/(x_1, \dots, x_n)M) = \mathfrak{m}$. The latter implies that $(x_1, \dots, x_n)M \subset M$ (as $\operatorname{Zero}_R O = \varnothing$), and that $x_1, \dots, x_n, w$ is *not* M–regular for any $w \in \mathfrak{m}$.

(27.9) **Lemma: Existence of Maximal Regular Sequence.** *If $\mathfrak{m}M \subset M$, then M admits a maximal regular sequence in $\mathfrak{m}$. In particular, if M is finitely generated and non-zero, then it admits a maximal regular sequence in $\mathfrak{m}$.*

(27.10) **Proof.** The first assertion follows from (d) in (27.7), since R is Noetherian, and then the second one results from **NAK** (24.5). □

(27.11) **Example.** Since $\operatorname{Zero}_R O = \varnothing$, any sequence $y_1, \dots, y_m$ is an O–regular sequence. Thus, if $x_1, \dots, x_n$ is an M–regular sequence such that $(x_1, \dots, x_n)M = M$, then $x_1, \dots, x_n, y_1, \dots, y_m$ is M–regular for any $y_1, \dots, y_m \in R$.

(27.12) **Proposition: Maximal Sequence & Depth.** *If $x_1, \dots, x_n \in \mathfrak{m}$ is an M–regular sequence, then the following hold.*

(a) $\operatorname{depth}_R M \geq n$.

(b) *If M is finitely generated, then the next holds:*

$$\operatorname{depth}_R M = n \iff x_1, \dots, x_n \text{ is a maximal } M\text{–regular sequence in } \mathfrak{m}.$$

(27.13) **Proof.** (a): Proposition: Depth Modulo Regular Element (27.5) yields the next sequence:

$$\operatorname{depth}_R M = \operatorname{depth}_R(M/x_1M) + 1 = \cdots = \operatorname{depth}_R(M/(x_1, \dots, x_n)M) + n \geq n.$$

(b): By Lemma: Depth Zero (27.3), $\operatorname{depth}_R(M/(x_1, \dots, x_n)M) = 0$ if and only if $\operatorname{Zero}_R(M/(x_1, \dots, x_n)M) = \mathfrak{m}$, that is, if and only if $x_1, \dots, x_n$ is a maximal M–regular sequence in $\mathfrak{m}$. □

(27.14) **Corollary: Depth of Finitely Generated Module.** *If M is finitely generated, then the next equality holds:*

$$\operatorname{depth}_R M = \sup\{\, n \in \mathbb{N}_0 \mid \text{There exists } M\text{–regular sequence } x_1, \dots, x_n \in \mathfrak{m} \,\}.$$ □

(27.15) **Lemma.** *Let M be a finitely generated R–module, and let $\mathfrak{p} \in \operatorname{Spec} R$ be such that $\dim(R/\mathfrak{p}) = 1$*[29].

(a) *For $\ell \in \mathbb{N}_0$ the following then holds:*

$$\operatorname{Ext}^{\ell}_R(R/\mathfrak{p}, M) \neq O \implies \operatorname{Ext}^{\ell+1}_R(k, M) \neq O.$$

(b) $\operatorname{depth}_R M \leq \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + 1$.

(27.16) **Proof.** (a): Choose $x \in \mathfrak{m} \setminus \mathfrak{p}$, and consider the short-exact sequence:

$$O \to R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \longrightarrow R/\mathfrak{a} \to O$$

where $\mathfrak{a} \stackrel{\text{def}}{=} \mathfrak{p} + (x)$, and the induced one:

$$\operatorname{Ext}^{\ell}_R(R/\mathfrak{p}, M) \xrightarrow{x} \operatorname{Ext}^{\ell}_R(R/\mathfrak{p}, M) \xrightarrow{\delta} \operatorname{Ext}^{\ell+1}_R(R/\mathfrak{a}, M).$$

[29] Such prime ideals are sometimes said to be sub-maximal.

As $\operatorname{Ext}^{\ell}_R(R/\mathfrak{p}, M)$ is finitely generated by (18.21), **NAK** (24.5) yields $\operatorname{Ext}^{\ell+1}_R(R/\mathfrak{a}, M) \neq O$. Thus, **LHF 1** (26.9) gives the desired: $\operatorname{Ext}^{\ell+1}_R(k, M) \neq O$ as $\operatorname{Supp}_R(R/\mathfrak{a}) = \operatorname{V}_R(\mathfrak{a}) = \{\mathfrak{m}\}$ (because $\dim(R/\mathfrak{p}) = 1$ and $\mathfrak{p} \subset \mathfrak{a}$).

(b): Set $\ell \stackrel{\text{def}}{=} \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p}$ and assume that $\ell < \infty$. Let $\kappa_R(\mathfrak{p})$ be the residue field of the local ring $R_\mathfrak{p}$, cf. (23.14), and note that $\kappa_R(\mathfrak{p}) \cong (R/\mathfrak{p})_\mathfrak{p}$. Thus, (18.27) yields the next isomorphism:

$$\operatorname{Ext}^{\ell}_R(R/\mathfrak{p}, M)_\mathfrak{p} \cong \operatorname{Ext}^{\ell}_{R_\mathfrak{p}}(\kappa_R(\mathfrak{p}), M_\mathfrak{p}) .$$

As the latter module is non-zero by our assumptions, $\operatorname{Ext}^{\ell}_R(R/\mathfrak{p}, M) \neq O$, and hence $\operatorname{Ext}^{\ell+1}_R(k, M) \neq O$ by (a). Thus, $\operatorname{depth}_R M \leq \ell + 1$, as desired: □

(27.17) **Proposition: Depth Inequality.** *If M is finitely generated, and $\mathfrak{p}$ belongs to $\operatorname{Supp}_R M$, then the next holds:*

$$\operatorname{depth}_R M \leq \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + \dim(R/\mathfrak{p}) .$$

(27.18) **Proof.** We use induction on $n \stackrel{\text{def}}{=} \dim(R/\mathfrak{p})$.

$\underline{n = 0}$: In this case $\mathfrak{p} = \mathfrak{m}$, so the inequality certainly holds (and is an equality).

$\underline{n > 0}$: Choose a chain of prime ideals:

$$(*) \qquad \mathfrak{p} = \mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n = \mathfrak{m} .$$

This is possible since $\operatorname{Spec}(R/\mathfrak{p}) = \{\, \mathfrak{q}/\mathfrak{p} \mid \mathfrak{q} \in \operatorname{V}_R(\mathfrak{p}) \,\}$ by (23.12.d). From the chain $(*)$ it follows that $\dim(R/\mathfrak{p}_1) = n - 1$, so the inductive hypothesis yields that

$$\operatorname{depth}_R M \leq \operatorname{depth}_{R_{\mathfrak{p}_1}} M_{\mathfrak{p}_1} + (n - 1) .$$

Now apply (b) in Lemma (27.15) to the $R_{\mathfrak{p}_1}$–module $M_{\mathfrak{p}_1}$ and obtain:

$$\operatorname{depth}_{R_{\mathfrak{p}_1}} M_{\mathfrak{p}_1} \leq \operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} + 1$$

because $\dim(R_{\mathfrak{p}_1}/\mathfrak{p}_{\mathfrak{p}_1}) = 1$ and $\big(M_{\mathfrak{p}_1}\big)_{\mathfrak{p}_{\mathfrak{p}_1}} \cong M_\mathfrak{p}$, consult also (23.12.e) and the next exercise. □

(27.19) **Exercise: Localization & Fractions.** Let S be a special multiplicative system in R, let $\mathfrak{p} \in \operatorname{Spec} R$ be such that $\mathfrak{p} \cap S = \varnothing$, and let M be an R–module. Provide an isomorphism $M_\mathfrak{p} \cong (S^{-1}M)_{S^{-1}\mathfrak{p}}$ of $R_\mathfrak{p}$–modules. Prove the next equalities:

$$(27.19.1) \qquad \operatorname{Ass}_{S^{-1}R} S^{-1}M = \{\, S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Ass}_R M \text{ and } \mathfrak{p} \cap S = \varnothing \,\}$$

$$(27.19.2) \qquad \operatorname{Supp}_{S^{-1}R} S^{-1}M = \{\, S^{-1}\mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R M \text{ and } \mathfrak{p} \cap S = \varnothing \,\} .$$

(27.20) **Proposition: Depth & Dim.** *If M is a finitely generated R–module, and $\mathfrak{p}$ belongs to $\operatorname{Ass}_R M$, then*

$$\operatorname{depth}_R M \leq \dim(R/\mathfrak{p}) .$$

(27.21) **Proof.** Since $\mathfrak{p} \in \operatorname{Ass}_R M$, (27.19.1) yields $\mathfrak{p}_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}} M_\mathfrak{p}$, that is, $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = 0$ by (27.3), and Proposition: Depth Inequality (27.17) yields the desired inequality. □

(27.22) **Exercise: Minimal Prime.** Let M be an R–module, and let $\mathfrak{p}$ be minimal in $\operatorname{Supp}_R M$. Prove that $\mathfrak{p}$ belongs to $\operatorname{Ass}_R M$.

(27.23) **Krull Dimension of Module.** The *Krull dimension* of an R–module M is:

$$\dim_R M \stackrel{\text{def}}{=} \sup\{\, \dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R M \,\} .$$

Note that, for any ideal $\mathfrak{a}$ in R, the Krull dimension $\dim_R(R/\mathfrak{a})$ of the R-module $R/\mathfrak{a}$ equals the Krull dimension $\dim(R/\mathfrak{a})$ of the ring $R/\mathfrak{a}$, cf. (23.8) and (23.12.d). Note also $\dim_R M \leq \dim R$, and that the latter if finite by Corollary 2 (23.28) to **GPIT** (23.25).

(27.24) **Theorem: Depth & Dim.** *If M is a finitely generated non-zero R-module, then* $\operatorname{depth}_R M \leq \dim_R M$.

(27.25) **Proof.** Choose $\mathfrak{p} \in \operatorname{Supp}_R M$ such that $\dim(R/\mathfrak{p}) = \dim_R M$, and note that this $\mathfrak{p}$ is necessarily minimal in $\operatorname{Supp}_R M$, and hence $\mathfrak{p} \in \operatorname{Ass}_R M$ by Exercise: Minimal Prime (27.22), and it follows from (27.19.1) that $\mathfrak{p}_\mathfrak{p} \in \operatorname{Ass}_{R_\mathfrak{p}} M_\mathfrak{p}$, which, in turn, yields $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} = 0$ by Lemma: Depth Zero (27.3). Thus, the desired inequality results from Proposition: Depth Inequality (27.17). □

(27.26) **Cohen–Macaulay Modules.** A finitely generated R-module M is said to be a *Cohen-Macaulay*, exactly when $\operatorname{depth}_R M = \dim_R M$. This is an interesting class of modules, see [BH].

28. AUSLANDER–BUCHSBAUM EQUATION

Throughout this section:

$$\boxed{(R, \mathfrak{m}, k) \text{ is a local.}}$$

(28.1) **Auslander–Buchsbaum Equation.** *If M is a finitely generated non-zero R–module with $\operatorname{pd}_R M$ finite, then*

$$\operatorname{depth} R = \operatorname{pd}_R M + \operatorname{depth}_R M .$$

Here we write $\operatorname{depth} R \stackrel{\text{def}}{=} \operatorname{depth}_R R$. The proof of ABE will be provided below after some preparations.

(28.2) **Lemma.** *The following are equivalent for any finitely generated R–module N.*

(*i*) *N is free;*
(*ii*) *N is projective;*
(*iii*) *N is flat.*

(28.3) **Proof.** "$(i) \Longrightarrow (ii)$" and "$(ii) \Longrightarrow (iii)$" hold always, while "$(iii) \Longrightarrow (i)$" is by Theorem: Finitely Generated Flat is Free (24.28). □

(28.4) **Lemma.** *If $\operatorname{depth} R = 0$ and M is a finitely generated R–module with $\operatorname{pd}_R M$ finite, then M is free.*

(28.5) **Proof.** Assume that $p \stackrel{\text{def}}{=} \operatorname{pd}_R M > 0$ [and look for a contradiction]. Choose by Theorem: Minimal Free Resolution (24.31) a minimal free resolution:

$$(\dagger) \qquad O \to F_p \xrightarrow{\partial_p} F_{p-1} \to \cdots \to F_1 \to F_0 \to M \to O$$

and set $B \stackrel{\text{def}}{=} \operatorname{Im} \partial_p$ which is isomorphic to the free non-zero module F_p. Since $\operatorname{depth} R = 0$, we have $\mathfrak{m} \in \operatorname{Ass}_R R$ by Lemma: Depth Zero (27.3), that is, $\mathfrak{m} = \operatorname{Ann}_R(x)$ for an $x \in R$. By the minimality of (†) we have $B \subseteq \mathfrak{m}F_{p-1}$ (as $\overline{\partial_p} = o$). Thus, $xB = O$, and this contradicts that B is free and non-zero. □

(28.6) **Depth over two Rings.** If M is a finitely generated R–module, and $\mathfrak{a}$ is en ideal contained in $\operatorname{Ann}_R M$, then $\operatorname{depth}_{R/\mathfrak{a}} M = \operatorname{depth}_R M$, since $[x_1]_\mathfrak{a}, \ldots, [x_n]_\mathfrak{a} \in \mathfrak{m}/\mathfrak{a}$ is M–regular, if and only if $x_1, \ldots, x_n \in \mathfrak{m}$ is M–regular (as $[x]_\mathfrak{a} m \stackrel{\text{def}}{=} xm$ for $x \in \mathfrak{m}$ and $m \in M$).

(28.7) **Depth of Free.** If F is a finitely generated free non-zero R–module, then $\operatorname{depth}_R F = \operatorname{depth} R$ (as any sequence is F–regular if and only if it is R–regular).

(28.8) **Proof of ABE.** We use induction on $s \stackrel{\text{def}}{=} \operatorname{depth} R$.

$\underline{s = 0}$: In this case, M is free by Lemma (28.4), and (28.7) yields $\operatorname{pd}_R M + \operatorname{depth}_R M = 0 + 0 = 0 = s = \operatorname{depth} R$.

$\underline{s > 0}$: Divide into two cases:

1° $\operatorname{depth}_R M > 0$: The assumptions on R and M imply that $\mathfrak{m} \notin \operatorname{Ass}_R R \cup \operatorname{Ass}_R M$, cf. (27.3). As the set of prime ideals $\operatorname{Ass}_R R \cup \operatorname{Ass}_R M$ is finite by (d) in (26.2), **PAL** (23.22) yields that $\mathfrak{m}$ is not contained in the union of these prime ideals, that is, $\mathfrak{m} \nsubseteq \operatorname{Zero}_R R \cup \operatorname{Zero}_R M$ by (f) in (26.2). Choosing $x \in \mathfrak{m} \setminus (\operatorname{Zero}_R R \cup \operatorname{Zero}_R M)$ and setting

$\overline{R} =^{\text{def}} R/(x)$ and $\overline{M} =^{\text{def}} M/xM$ we get the following:

$$\begin{aligned}
\operatorname{depth} R &= \operatorname{depth}_R R && \text{by convension}\\
&= \operatorname{depth}_R \overline{R} + 1 && \text{by (27.5)}\\
&= \operatorname{depth}_{\overline{R}} \overline{R} + 1 && \text{by (28.6)}\\
&= \operatorname{depth} \overline{R} + 1 && \text{by convension}\\
&= \operatorname{pd}_{\overline{R}} \overline{M} + \operatorname{depth}_{\overline{R}} \overline{M} + 1 && \text{by the inductive hypothesis}\\
&= \operatorname{pd}_R M + \operatorname{depth}_{\overline{R}} \overline{M} + 1 && \text{by (24.35)}\\
&= \operatorname{pd}_R M + \operatorname{depth}_R \overline{M} + 1 && \text{by (28.6)}\\
&= \operatorname{pd}_R M + \operatorname{depth}_R M && \text{by (27.5),}
\end{aligned}$$

as desired.

2° $\operatorname{depth}_R M = 0$: Choose an exact sequence $O \to K \to F \to M \to O$ with F finitely generated and free. As $\operatorname{depth} R > 0$, (28.2) and (28.7) yield $\operatorname{pd}_R M > 0$, and hence $\operatorname{pd}_R M = \operatorname{pd}_R K + 1$. The next is an induced long-exact sequence:

$$O \to \operatorname{Hom}_R(k,K) \to \operatorname{Hom}_R(k,F) \to \operatorname{Hom}_R(k,M) \xrightarrow{\delta} \operatorname{Ext}^1_R(k,K)\,.$$

As $\operatorname{Hom}_R(k,F) = 0$ by (28.7) and $\operatorname{Hom}_R(k,M) \neq O$ by assumption, $\operatorname{Hom}_R(k,K) = O$ and $\operatorname{Ext}^1_R(k,K) \neq O$ by the long-exact sequence, and hence $\operatorname{depth}_R K = 1$, which implies $\operatorname{pd}_R K = \operatorname{depth} R - 1$ by case 1°. Now we have $\operatorname{pd}_R M = \operatorname{pd}_R K + 1 = \operatorname{depth} R$ as desired—in this case. □

(28.9) **Range of Homological Dimensions.** If M is a finitely generated R–module with $\operatorname{pd}_R M$ finite, then **ABE** (28.1) yields $\operatorname{pd}_R M \in \{0, 1, \dots, \operatorname{depth} R\}$. On the other hand, to any $p \in \{0, 1, \dots, \operatorname{depth} R\}$ there exists a finitely generated R–module M with $\operatorname{pd}_R M = p$, namely $M =^{\text{def}} R/(x_1, \dots, x_p)$ where $x_1, \dots, x_p$ is an R–regular sequence in $\mathfrak{m}$, cf. (26.30).

H. Bass has proved that to any $p \in \{0, 1, \dots, \dim R\}$ there exists a R–module M with $\operatorname{pd}_R M = p$. Gruson, Jensen, and Raynaud have proved that $\operatorname{pd}_R N \leq \dim R$, if just $\operatorname{fd}_R N < \infty$. Auslander has proved:

$$\begin{aligned}
\sup\{\operatorname{id}_R N \mid \operatorname{id}_R N < \infty\} &= \sup\{\operatorname{fd}_R N \mid \operatorname{fd}_R N < \infty\}\\
&= \sup\{\operatorname{depth} R_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Spec} R\}\\
&= \begin{cases} \dim R & \text{if } \operatorname{depth} R = \dim R\\ \dim R - 1 & \text{otherwise.}\end{cases}
\end{aligned}$$

(28.10) **Exercise.** Prove some of the equalities above.

29. Regular Local Rings II

Throughout this section:

$$\boxed{(R, \mathfrak{m}, k) \text{ is a local.}}$$

(29.1) **Main Theorem: Regularity & Global Dimension.** *The following are equivalent.*

(*i*) *R is a regular local ring;*
(*ii*) $\operatorname{pd}_R k < \infty$;
(*ii'*) $\operatorname{pd}_R k = \dim R$;
(*iii*) $\operatorname{gldim} R < \infty$;
(*iii'*) $\operatorname{gldim} R \stackrel{\bullet}{=} \dim R$.

(29.2) **Corollary: Krull's First Conjecture holds.** *If R is regular, then $R_{\mathfrak{p}}$ is a regular local ring for all $\mathfrak{p} \in \operatorname{Spec} R$.*

(29.3) **Proof of Corollary.** Consult also (17.13). □

(29.4) **Proof of Main Theorem.** Set $d \stackrel{\text{def}}{=} \dim R$.

"$(i) \Longrightarrow (ii')$": Let $x_1, \dots, x_d$ be a minimal generating set for $\mathfrak{m}$, and note that it is an R–regular sequence by Proposition (25.17). Applying Lemma (26.30) d times yields:

$$d = \operatorname{pd}_R R + d = \operatorname{pd}_R R/(x_1) + d - 1 = \cdots = \operatorname{pd}_R R/(x_1, \dots, x_d) = \operatorname{pd}_R R/\mathfrak{m} = \operatorname{pd}_R k\,.$$

"$(ii') \Longrightarrow (iii')$": First, we note that:

$$(*) \qquad \operatorname{Ext}^n_R(R/\mathfrak{p}, k)_{\mathfrak{p}} = O \text{ for all } \mathfrak{p} \in \operatorname{Spec} R \setminus \{\mathfrak{m}\}\,, n \in \mathbb{N}_0\,.$$

Indeed, $\operatorname{Supp}_R \operatorname{Ext}^n_R(R/\mathfrak{p}, k) \subseteq \operatorname{V}_R(\operatorname{Ann}_R \operatorname{Ext}^n_R(R/\mathfrak{p}, k)) \subseteq \{\mathfrak{m}\}$ by (26.2.h), (26.16.1). In view of $(*)$, the formula for injective dimension (26.22.i) gives the next equality:

$$(\dagger) \qquad \operatorname{id}_R k = \sup\{ n \in \mathbb{N}_0 \mid \operatorname{Ext}^n_R(k, k) \neq O \} \leq d\,.$$

The inequality results from the hypothesis (ii') and **PDT** (11.19). Next, choose by Auslander's Theorem for Global Dimension (17.9) a finitely generated R–module N' such that the next equality holds:

$$(\ddagger) \qquad \operatorname{gldim} R = \operatorname{pd}_R N' = \sup\{ n \in \mathbb{N}_0 \mid \operatorname{Ext}^n_R(N', k) \neq O \}\,.$$

Here, the second equality is by PDC: Finitely Generated over Local Ring (26.27). Now **IDC** (17.5) yields the unlabeled inequality in the next chain:

$$\operatorname{gldim} R \stackrel{(\ddagger)}{=} \sup\{ n \in \mathbb{N}_0 \mid \operatorname{Ext}^n_R(N', k) \neq O \} \leq \operatorname{id}_R k \stackrel{(\dagger)}{\leq} d,.$$

This gives the desired assertion as $d = \operatorname{pd}_R k \leq \operatorname{gldim} R$.

"$(iii') \Longrightarrow (iii)$" and "$(iii) \Longrightarrow (ii)$" are obvious.

"$(ii) \Longrightarrow (i)$" by induction on $s \stackrel{\text{def}}{=} \operatorname{depth} R$.

$\underline{s = 0}$: Since $\operatorname{pd}_R k < \infty$, **ABE** (28.1) and Lemma (28.2) yield that k is a free R-module, and hence R is a field, that is, R is regular.

$\underline{s > 0}$: As $\mathfrak{m} \not\subseteq \mathfrak{m}^2$ by **NAK** (24.5) (since $\mathfrak{m} \neq O$), and as $\mathfrak{m} \not\subseteq \mathfrak{p}$ whenever $\mathfrak{p} \in \operatorname{Ass}_R R$ (since $\operatorname{depth} R = s > 0$), **PAL** (23.22) provides $x \in \mathfrak{m} \setminus (\mathfrak{m}^2 \cup \operatorname{Zero}_R R)$, cf. also (26.2.e). Set $\overline{R} =^{\text{def}} R/(x)$ and $\overline{\mathfrak{m}} =^{\text{def}} \mathfrak{m}/x\mathfrak{m}$. By the next lemma (29.5), the $\overline{R}$–module k is a direct summand in the $\overline{R}$–module $\overline{\mathfrak{m}}$. The latter has finite projective dimension, since $\operatorname{pd}_{\overline{R}} \overline{\mathfrak{m}} = \operatorname{pd}_R \mathfrak{m}$ by Corollary: Modulo an Element (24.35), and hence $\operatorname{pd}_{\overline{R}} k$ is finite.

Thus, the inductive hypothesis yields that $\overline{R}$ is regular, and Fundamental Lemma (25.6.b) shows that R is regular—as desired. □

(29.5) **Lemma.** *If $x \in \mathfrak{m} \setminus \mathfrak{m}^2$, then k is a direct summand in the $R/(x)$-module $\mathfrak{m}/x\mathfrak{m}$.*

(29.6) **Proof.** An R-homomorphism $\iota\colon k \to \mathfrak{m}/x\mathfrak{m}$ is well-defined by $\iota([r]_{\mathfrak{m}}) =^{\text{def}} [rx]_{x\mathfrak{m}}$.

Since $x_1 \stackrel{\text{def}}{=} x \notin \mathfrak{m}^2$, it can be extended to a minimal generating set $x_1, x_2, \dots, x_b$ for $\mathfrak{m}$ with $b =^{\text{def}} \beta_0^R(\mathfrak{m})$. Then $[x_2]_{(x)}, \dots, [x_b]_{(x)}$ is a minimal generating set for $\mathfrak{m}/(x)$, since it generates it and has the correct number, $\beta_0^R(\mathfrak{m}/(x)) = \beta_0^R(\mathfrak{m}) - 1 = b-1$, of elements.

Next we prove for $a_1, \dots, a_b, c_1, \dots, c_b \in R$ that the next holds:

$$[\Sigma_i a_i x_i]_{x\mathfrak{m}} = [\Sigma_i c_i x_i]_{x\mathfrak{m}} \ \text{(in } \mathfrak{m}/x\mathfrak{m}) \implies [a_1]_{\mathfrak{m}} = [c_1]_{\mathfrak{m}} \ \text{(in } k = R/\mathfrak{m}).$$

Indeed, if $\Sigma_i(a_i - b_i)x_i \in x\mathfrak{m}$, then

$$(a_1 - c_1 - m)x_1 + \Sigma_{i>0}(a_i - c_i)x_i = 0 \ \text{(in } \mathfrak{m})$$

for some $m \in \mathfrak{m}$, and hence $\Sigma_i [a_i - c_i]_{\mathfrak{m}}[x_i]_{\mathfrak{m}^2} = 0$ in the vector space $\mathfrak{m}/\mathfrak{m}^2$ over $k = R/\mathfrak{m}$. Since $([x_1]_{\mathfrak{m}^2}, \dots, [x_b]_{\mathfrak{m}^2})$ is a basis for $\mathfrak{m}/\mathfrak{m}^2$, cf. (24.13), in particular, $[a_1 - c_1]_{\mathfrak{m}} = 0$, that is, $[a_1]_{\mathfrak{m}} = [c_1]_{\mathfrak{m}}$ in k. Thus, an R-homomorphism $\rho\colon \mathfrak{m}/x\mathfrak{m} \to k$ is well-defined by $\rho([\Sigma_i a_i x_i]_{x\mathfrak{m}}) =^{\text{def}} [a_1]_{\mathfrak{m}}$. As $\rho(\iota([r]_{\mathfrak{m}})) = \rho([rx]_{x\mathfrak{m}}) = \rho([rx_1]_{\mathfrak{m}}) = [r]_{\mathfrak{m}}$, the sequence

$$O \longrightarrow k \underset{\rho}{\overset{\iota}{\rightleftarrows}} \mathfrak{m}/x\mathfrak{m} \longrightarrow \operatorname{Coker} \iota \longrightarrow O$$

splits, as desired. □

30. Regular Rings

Throughout this section:

R is a Noetherian ring.

(30.1) **Definition: Regular Ring.** The Noetherian ring R is said to be *regular*, exactly when $R_{\mathfrak{m}}$ is a regular local ring for each $\mathfrak{m} \in \operatorname{Max} R$. If this is the case, then $R_{\mathfrak{p}}$ is a regular local ring for any $\mathfrak{p} \in \operatorname{Spec} R$. Indeed, choose a maximal ideal $\mathfrak{m}$ containing $\mathfrak{p}$. Note that $\mathfrak{p}_{\mathfrak{m}} \in \operatorname{Spec} R_{\mathfrak{m}}$, cf. (23.9), and that $(R_{\mathfrak{m}})_{\mathfrak{p}_{\mathfrak{m}}} \cong R_{\mathfrak{p}}$, cf. (27.19). As $R_{\mathfrak{m}}$ is a regular local ring by assumption, the validity of Krull's First conjecture, cf. (29.2), ensures that so is $R_{\mathfrak{p}}$.

(30.2) **Regular Ring of Infinite Dimension.** Nagata has constructed a Noetherian ring with infinite Krull dimension, and this ring turns out to be regular, see [Na, page 203]. Let R be any regular ring with $\dim R = \infty$. There is then for each $n \in \mathbb{N}_0$ an $\mathfrak{m}_n \in \operatorname{Spec} R$ such that $\dim R_{\mathfrak{m}_n} \geq n$. For the residue field $\kappa_R(\mathfrak{m}_n) \cong (R/\mathfrak{m}_n)_{\mathfrak{m}_n}$ of $R_{\mathfrak{m}_n}$ we then have

$$\operatorname{pd}_R(R/\mathfrak{m}_n) \geq \operatorname{pd}_{R_{\mathfrak{m}_n}} \kappa_R(\mathfrak{m}_n) = \dim R_{\mathfrak{m}_n} \geq n\,,$$

confer Main Theorem (29.1). For $M \stackrel{\text{def}}{=} \coprod_{\mathfrak{m} \in \operatorname{Max} R} R/\mathfrak{m}$ we therefore have $\operatorname{pd}_R M \geq n$ for all $n \in \mathbb{N}_0$; thus, $\operatorname{pd}_R M = \infty$.

This shows that any regular ring of infinite dimension admits a non-zero module of *infinite* projective dimension. The next result shows that over a reular ring *finitely generated* non-zero modules have always *finite* projective dimension.

(30.3) **Theorem.** *If M is a finitely generated module over a regular ring R, then $\operatorname{pd}_R M$ is finite.*

(30.4) **Proof.** Choose by (18.19) a projective resolution $P_\bullet$ of M with each P_n finitely generated, and use the standard kernel notation of (10.26). For any $\mathfrak{p} \in \operatorname{Spec} R$ condition (iv) in **PDT** (11.19) yields the next:

$$(*) \qquad \operatorname{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} > n \iff \operatorname{Ext}^{n+1}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, (K_n)_{\mathfrak{p}}) \neq O\,.$$

Recall $\operatorname{Ext}^{n+1}_R(M, K_n)_{\mathfrak{p}} \cong \operatorname{Ext}^{n+1}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, (K_n)_{\mathfrak{p}})$, cf. (18.27). Set

$$\mathfrak{a}_n \stackrel{\text{def}}{=} \operatorname{Ann}_R \operatorname{Ext}^{n+1}_R(M, K_n)\,.$$

As $\operatorname{Ext}^{n+1}_R(M, K_n)$ is finitely generated by (18.21), we have $\operatorname{Supp}_R \operatorname{Ext}^{n+1}_R(M, K_n) = \operatorname{V}_R(\mathfrak{a}_n)$, cf. (26.2.h), and $(*)$ has been translated into the next:

$$(**) \qquad \operatorname{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} > n \iff \mathfrak{p} \in \operatorname{V}_R(\mathfrak{a}_n)\,.$$

In particular, we have the next:

$$(\dagger) \qquad \operatorname{V}_R(\mathfrak{a}_0) \supseteq \operatorname{V}_R(\mathfrak{a}_1) \supseteq \cdots \supseteq \operatorname{V}_R(\mathfrak{a}_n) \supseteq \operatorname{V}_R(\mathfrak{a}_{n+1}) \supseteq \cdots\,.$$

As $\operatorname{Min} R/\mathfrak{a}_n$ is finite, the set $\operatorname{V}_R(\mathfrak{a}_n)$ has a finite number minimal elements. Let $\mathfrak{b}_n$ denote there intersection, and note that $\operatorname{V}_R(\mathfrak{b}_n) = \operatorname{V}_R(\mathfrak{a}_n)$, and that $\mathfrak{b}_n$ equals the intersection of *all* prime ideals in $\operatorname{V}_R(\mathfrak{a}_n)$. Now $(\dagger)$ yields the next sequence:

$$(\ddagger) \qquad \mathfrak{b}_0 \subseteq \mathfrak{b}_1 \subseteq \cdots \subseteq \mathfrak{b}_n \subseteq \mathfrak{b}_{n+1} \subseteq \cdots\,.$$

Set $\mathfrak{b} \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{N}_0} \mathfrak{b}_n$ which is an ideal in R. Furthermore, $\operatorname{V}_R(\mathfrak{b}) = \bigcap_{n \in \mathbb{N}_0} \operatorname{V}_R(\mathfrak{b}_n) = \bigcap_{n \in \mathbb{N}_0} \operatorname{V}_R(\mathfrak{a}_n)$. As $\operatorname{pd}_{R_{\mathfrak{p}}} M_{\mathfrak{p}} \leq \dim A_{\mathfrak{p}} < \infty$ for all $\mathfrak{p} \in \operatorname{Spec} R$ we conclude from $(**)$

that $\bigcap_{n\in\mathbb{N}_0} \mathrm{V}_R(\mathfrak{a}_n) = \varnothing$, and hence $\mathfrak{b} = R$; thus, $1 \in \mathfrak{b}_m$ for some $m\in\mathbb{N}_0$, that is, $\mathrm{V}_R(\mathfrak{a}_m) = \mathrm{V}_R(\mathfrak{b}_m) = \varnothing$, and it follows that $\mathfrak{a}_m = R$, that is, $\operatorname{Ext}^{m+1}_R(M, K_m) = O$, and $\operatorname{pd}_R M \leq m$ follows from **PDT** (11.19). □

(30.5) **Remark.** Let A be a ring. The set $\operatorname{Spec} A$ is then a topological space with $\{\, \mathrm{V}_A(\mathfrak{a}) \mid \mathfrak{a} \text{ is an ideal in } A \,\}$ as the class of closed subsets. If $\big(F_n\big)_{n\in\mathbb{N}_0}$ is a family of closed sets such that

$$F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n \supseteq \cdots \quad \text{and} \quad \bigcap_{n\in\mathbb{N}_0} F_n = \varnothing\,,$$

then there exists $m\in\mathbb{N}_0$ such that $F_m = \varnothing$. This topology is quasi-compact but seldom Hausdorff. If A is Noetherian, then every closed subset is quasi-compact.

(30.6) **Main Theorem: Regular Ring.** *If R is a regular ring, then it is isomorphic to a finite product $R_1 \times \cdots \times R_n$ of regular domains, and the next holds:*

$$(\square) \qquad \operatorname{gldim} R = \max\{\operatorname{gldim} R_1, \ldots, \operatorname{gldim} R_n\}\,.$$

(30.7) **Proof.** As R is Noetherian, the set $\operatorname{Min} R$ is finite, cf. (23.4), say $\operatorname{Min} R = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, and set $R_i =^{\text{def}} R/\mathfrak{p}_i$ when $1 \leq i \leq n$. Let $\widetilde{R}$ denote the finite product $R_1 \times \cdots \times R_n$ of Noetherian integral domains, this is again a Noetherian ring.

For each $\mathfrak{m} \in \operatorname{Max} R$ there exists one and only one $\mathfrak{p} \in \operatorname{Min} R$ such that $\mathfrak{p} \subseteq \mathfrak{m}$. Indeed, the regular local ring $R_\mathfrak{m}$ is a domain, cf. (25.8), and the our assertion follows, because $\operatorname{Spec} R_\mathfrak{m} = \{\, \mathfrak{p}_\mathfrak{m} \mid \mathfrak{p} \in \operatorname{Spec} R \text{ and } \mathfrak{p} \subseteq \mathfrak{m} \,\}$, cf. (23.9).

The next map:

$$\varphi^R\colon R \to \widetilde{R},\ r \mapsto ([r]_{\mathfrak{p}_1}, \ldots, [r]_{\mathfrak{p}_n})$$

is a ring homomorphism as well as an R–homomorphism of R–modules.

For each $\mathfrak{m} \in \operatorname{Max} R$ the induced homomorphism $\varphi_\mathfrak{m}\colon R_\mathfrak{m} \to \widetilde{R}_\mathfrak{m}$ is an isomorphism. Indeed, choose i such that $\mathfrak{m} \supseteq \mathfrak{p}_i$. For $j \neq i$ we then have $\mathfrak{m} \not\supseteq \mathfrak{p}_j$, and hence $(R_j)_\mathfrak{m} = O$. Thus, $\widetilde{R}_\mathfrak{m} = (R_i)_\mathfrak{m} = (R/\mathfrak{p}_i)\mathfrak{m} \cong R_\mathfrak{m}/(\mathfrak{p}_i)_\mathfrak{m} = R_\mathfrak{m}/O = R_\mathfrak{m}$, and the desired assertion follows. We have now proved that $\operatorname{Max} R \cap \operatorname{Supp}_R \operatorname{Ker}\varphi = \varnothing$ and that $\operatorname{Max} R \cap \operatorname{Supp}_R \operatorname{Coker}\varphi = \varnothing$, and hence that $\operatorname{Supp}_R \operatorname{Ker}\varphi = \operatorname{Supp}_R \operatorname{Coker}\varphi = \varnothing$, that is, $\operatorname{Ker}\varphi = O$ and $\operatorname{Coker}\varphi = O$, so φ is an isomorphism as desired. Thus, the isomorphism $R \cong R_1 \times \cdots \times R_n$ has been established.

The remaining assertions, the regularity of each R_i and the equality $(\square)$, result from Exercise (30.10) below. □

(30.8) **Global Dimension Zero.** If A is a commutative ring with $\operatorname{gldim} A = 0$, then it is Noetherian by (14.18) and hence a finite product of fields by the theorem, but this is already known from (18.36).

(30.9) **Global Dimension One.** If $\operatorname{gldim} R = 1$ then R is a finite product of Dedekind domains, cf. (17.12), and fields.

(30.10) **Exercise: Direct Product of Rings.** Let the ring A be isomorphic to a direct product $A_1 \times \cdots \times A_n$ of rings.

(a) Prove that the assignment $N \mapsto A_i N$ provides an exact functor $\operatorname{Mod}(A) \to \operatorname{Mod}(A_i)$ that takes projective A–modules into projective A_i–modules .

(b) Prove that $\operatorname{pd}_R N = \max\{\operatorname{pd}_{A_1} A_1 N, \ldots, \operatorname{pd}_{A_n} A_n N\}$.

(c) Establish the remaining assertions in Main Theorem: Regular Ring (30.6).

31. Regular Local Rings III

Throughout this section:

$$\boxed{(R, \mathfrak{m}, k) \text{ is a local.}}$$

This section provides an affirmative answer to Krull's second conjecture.

(31.1) **Main Theorem: Regular Local is UFD.** *If R is a regular local ring, then it is UFD, that is, a Unique Factorization Domain.*

(31.2) **Remarks on Factoriality.** That a domain R is has unique factorization means that there to each non-invertible non-zero element r in R exist $n \in \mathbb{N}$ and *irreducible* elements $p_1, \dots, p_n$ in R such that $r = p_1 \cdots p_n$, and that this factorization is unique up to the order of the factors and up to multiplication with invertible elements. As this ring R is Noetherian, such factorization always exists, and uniqueness is tantamount to that each irreducible element is a *prime* element, that is, it generates a prime ideal.

Next, we introduce the properties P_2 and P_3 of the ring R and state three lemmas concerning these properties. The Main Theorem follows from Lemma 3 and Lemma 1, while Lemma 2 is used in the proof of Lemma 3.

(31.3) **Definition.** The ring R satisfies property P_2, respectively, property P_3, exactly if for all elements a, b in R the projective dimension $\operatorname{pd}_R R/(a,b)$ is at most 2, respectively, at most 3.

(31.4) **Lemma 1.** *If R satisfies P_3, then R is UFD.*

(31.5) **Lemma 2.** *If R is UFD, then R satisfies P_2.*

(31.6) **Lemma 3.** *If R is regular, then R satisfies P_3.*

(31.7) **Proof of Lemma 1.** Assume that a is an irreducible element in R [and we want to prove that (a) is a prime ideal.] To achieve this, consider elements $b \in R \setminus (a)$ and $c \in R$ such that $bc \in (a)$ [and we want to prove that $c \in (a)$]. Next, define two R-homomorphisms as follows:

$$\varphi\colon (a) \cap (b) \to R^2 \text{ by } \varphi(z) \stackrel{\text{def}}{=} (x, y) \text{ when } z = xa = yb$$
$$\psi\colon R^2 \to (a,b) \text{ by } \psi(x,y) \stackrel{\text{def}}{=} xa - yb.$$

Consider the following two exact sequences:

$$O \longrightarrow (a) \cap (b) \xrightarrow{\varphi} R^2 \xrightarrow{\psi} (a,b) \longrightarrow O$$

$$O \longrightarrow (a,b) \longrightarrow R \longrightarrow R/(a,b) \longrightarrow O.$$

As $\operatorname{pd}_R R/(a,b) \leq 3$ by assumption, we get $\operatorname{pd}_R (a,b) \leq 2$ and $\operatorname{pd}_R (a) \cap (b) \leq 1$. Set

$$\mathfrak{c} \stackrel{\text{def}}{=} (b) : (a) \stackrel{\text{def}}{=} \{ x \in R \mid xa \in (b) \}.$$

An isomorphism $\mathfrak{c} \xrightarrow{\cong} (a) \cap (b)$ is defined by $x \mapsto xa$.

Next we want to prove that $\mathfrak{c}$ is a principal ideal. *Assume* that $n \stackrel{\text{def}}{=} \beta_0^R(\mathfrak{c}) > 1$ [and look for a contradiction.] Note first that—under this assumption—the module $\mathfrak{c}$ is not free, and hence not projective, cf. (28.2). Thus, the above yield $\operatorname{pd}_R \mathfrak{c} = 1$.

Choose, by **NAK** (24.5), $x_1 \in \mathfrak{c} \setminus \mathfrak{m}\mathfrak{c}$, and next $y_1 \in R$ such that $x_1 a = y_1 b$; the latter is possible because $x_1 \in \mathfrak{c} = (b) : (a)$. It is then easy to see that $\mathfrak{c} = (x_1) : (y_1)$. Extend x_1 to a minimal generating set $x_1, \dots, x_n$ for $\mathfrak{c}$. Choose $y_2 \in R$ such that $x_2 y_1 = y_2 x_1$, which is possible since $x_2 \in \mathfrak{c} = (x_1) : (y_1)$. Let $(f_1, \dots, f_n)$ be a basis for $F \stackrel{\text{def}}{=} R^n$, let $\rho\colon F \to \mathfrak{c}$ be defined by $\rho(f_i) \stackrel{\text{def}}{=} x_i$, and set $G \stackrel{\text{def}}{=} \operatorname{Ker} \rho$. As $(x_1, \dots, x_n)$ is a minimal generating set for $\mathfrak{c}$, this G is a submodule of $\mathfrak{m}F$ and ρ is surjective.

As $\operatorname{pd}_R \mathfrak{c} = 1$, the exact sequence $O \to G \to F \to \mathfrak{c} \to O$ shows that G is free, cf. (28.2). Let $(g_1, \dots, g_k)$ be a basis for G. Since $\rho(y_2 f_1 - y_1 f_2) = y_2 x_1 - y_1 x_2 = 0$ and $\rho(x_2 f_1 - x_1 f_2) = 0$ there exist $r_1, \dots, r_k, s_1, \dots, s_k \in R$ such that $y_2 f_1 - y_1 f_2 = \Sigma_{i=1}^k r_i g_i$ and $x_2 f_1 - x_1 f_2 = \Sigma_{i=1}^k s_i g_i$. Because $x_1(y_2 f_1 - y_1 f_2) = y_1(x_2 f_1 - x_1 f_2)$, we get $x_1 r_i = y_1 s_i$, and hence $s_1, \dots, s_k \in (x_1) : (y_1)$. Since $G \subseteq \mathfrak{m}F$, there exist, for each $i \in \{1, \dots, k\}$, elements $m_{i1}, \dots, m_{in} \in \mathfrak{m}$ such that $g_i = m_{i1} f_1 + \cdots + m_{in} f_n$. Thus,

$$x_2 f_1 - x_1 f_2 = \Sigma_{i=1}^k s_i g_i = \Sigma_{i=1}^k s_i (\Sigma_{j=1}^n m_{ij} f_j) = \Sigma_{j=1}^n (\Sigma_{i=1}^k s_i m_{ij}) f_j$$

By comparison of coefficients on either side, we get $x_1 = -\Sigma_{i=1}^k s_i m_{i2} \in \mathfrak{m}\mathfrak{c}$ *contradicting* the choice of x_1.

Now we have proved that $\beta_0^R(\mathfrak{c})$ is *not* larger than 1, that is, $\mathfrak{c}$ is a principal ideal. Since $(a) \cap (b) \cong \mathfrak{c}$ we can choose $d \in R$ such that $(a) \cap (b) = (d)$. As $ab \in (d) \subseteq (b)$, choose $e, f \in R$ such that $ab = ed$ and $d = fb$, and hence $a = ef$. Here, $f \in \mathfrak{m}$, because otherwise we have $(b) = (d)$ and hence $b \in (d) \subseteq (a)$ contradicting our assumption $b \notin (a)$. Thus, as a is assumed to be an irreducible element, e must be invertible, and hence $(ab) = (d)$. The latter ideal equals $(a) \cap (b)$ by the choice of d. Since $bc \in (a)$ by assumption, we get $bc \in (a) \cap (b) = (ab)$, and $bc = gab$, which yields $c = ga \in (a)$, as desired. □

(31.8) **Proof of Lemma 2.** Assume that R is UFD, and that $\mathfrak{c} = (a, b)$ is an ideal generated by two elements [and we want to prove that $\operatorname{pd}_R \mathfrak{c} \leq 1$.] This is certainly the case if $\mathfrak{c}$ is principal, so we assume that this is not the case, in particular, $a, b \in \mathfrak{m}$. As R is UFD, we can choose a greatest common divisor d of a an b, that is, $a = da'$, $b = db'$, such that a' and b' have only invertible elements as common divisors. In particular, a', b' is an R–regular sequence, and hence $\operatorname{pd}_R R/(a', b') \leq 2$, that is, $\operatorname{pd}_R(a', b') \leq 1$. Thus, as $\mathfrak{c} = (a, b) = (a', b')d \cong (a', b')$ has projective dimension at most 1, as desired. □

(31.9) **Proof of Lemma 3.** Assume that R is regular, and that $a, b \in R$, and set $\mathfrak{c} \stackrel{\text{def}}{=} (a, b)$. We want to prove that $\operatorname{pd}_R(R/\mathfrak{c}) \leq 3$. We use induction on $d \stackrel{\text{def}}{=} \dim R$.

$\underline{d \leq 3}$: In this case we have $\operatorname{pd}_R(R/\mathfrak{c}) \leq 3$ since $\operatorname{gldim} R = \dim R$ by Main Theorem: Regularity & Global Dimension (29.1)

$\underline{d > 3}$: We want to prove $\operatorname{pd}_R \mathfrak{c} \leq 2$. We may assume that $a, b \in \mathfrak{m} \setminus O$. Setting $\mathfrak{c} : \mathfrak{a} \stackrel{\text{def}}{=} \{ x \in R \mid x\mathfrak{a} \subseteq \mathfrak{c} \}$ for any ideal $\mathfrak{a}$, we have an ascending chain:

$$\mathfrak{c} : \mathfrak{m} \subseteq \mathfrak{c} : \mathfrak{m}^2 \subseteq \cdots \subseteq \mathfrak{c} : \mathfrak{m}^s \subseteq \mathfrak{c} : \mathfrak{m}^{s+1} \subseteq \cdots .$$

Choose $s \in \mathbb{N}$ such that $\mathfrak{c} : \mathfrak{m}^s = \mathfrak{c} : \mathfrak{m}^{s+1}$, set $\mathfrak{d} \stackrel{\text{def}}{=} \mathfrak{c} : \mathfrak{m}^s$, and note that $\mathfrak{m} \notin \operatorname{Ass}_R(R/\mathfrak{d})$. Choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2 \cup \operatorname{Zero}_R(R/\mathfrak{d})$ by **PAL** (23.22) (in view of **NAK** (24.5)). By Fundamental Lemma (25.6) (since R is a domain by (25.8)) $\overline{R} \stackrel{\text{def}}{=} R/(x)$ is regular with $\dim \overline{R} = d - 1$ and hence it satisfies P_3 by the inductive hypothesis, so it is is UFD by the already established Lemma 1. Thus, $\overline{R}$ satisfies P_2 by Lemma 2, which has also been proved. Consider the ideal $\mathfrak{c}' \stackrel{\text{def}}{=} (\mathfrak{c} + (x))/(x)$ in $\overline{R}$, and note that it is generated by

the two elements $\overline{a}, \overline{b}$. It follows that $\mathrm{pd}_{\overline{R}}(\overline{R}/\mathfrak{c}') \leq 2$, and we obtain the next sequence:

$$\begin{aligned}
\operatorname{depth}_R(\overline{R}/\mathfrak{c}') &= \operatorname{depth}_{\overline{R}}(\overline{R}/\mathfrak{c}') && \text{by (28.6)} \\
&= \operatorname{depth} \overline{R} - \mathrm{pd}_{\overline{R}}(\overline{R}/\mathfrak{c}') && \text{by } \mathbf{ABE} \text{ (28.1)} \\
&\geq \operatorname{depth} \overline{R} - 2 && \text{by the above} \\
&= \dim \overline{R} - 2 && \text{by (25.17) and (27.14)} \\
&= d - 3 > 0 && \text{by assumption.}
\end{aligned}$$

Thus, $\mathfrak{m} \notin \operatorname{Ass}_R(\overline{R}/\mathfrak{c}') = \operatorname{Ass}_R(R/(\mathfrak{c} + (x)))$ where the equality is by Noether's Isomorphism (24.3.2). It follows that $(\mathfrak{c} + (x)) : \mathfrak{m} = \mathfrak{c} + (x)$, and hence

$$\mathfrak{d} = \mathfrak{c} : \mathfrak{m}^s \subseteq (\mathfrak{c} + (x)) : \mathfrak{m}^s = \mathfrak{c} + (x).$$

To each $d \in \mathfrak{d}$ there exists therefore $c \in \mathfrak{c}$ and $r \in R$ such that $d = c + rx$, that is, $rx = d - c$, and hence $r \in \mathfrak{d}$ as $x \notin \operatorname{Zero}_R(R/\mathfrak{d})$. This shows that $\mathfrak{d} \subseteq \mathfrak{c} + \mathfrak{d}x$, and hence $\mathfrak{d} = \mathfrak{c} + \mathfrak{d}x$. Thus, $x(\mathfrak{d}/\mathfrak{c}) = (x\mathfrak{d} + \mathfrak{c})/\mathfrak{c} = \mathfrak{d}/\mathfrak{c}$, and **NAK** (24.5) yields $\mathfrak{d} = \mathfrak{c}$. Now we have $x\mathfrak{c} = (x) \cap \mathfrak{c}$, because "$\subseteq$" is obvious, and "$\supseteq$" follows since $x \notin \operatorname{Zero}_R(R/\mathfrak{d}) = \operatorname{Zero}_R(R/\mathfrak{c})$. Thus $\overline{\mathfrak{c}} =^{\mathrm{def}} \mathfrak{c}/x\mathfrak{c} = \mathfrak{c}/(x) \cap \mathfrak{c} \cong (\mathfrak{c} + (x))/(x) = \mathfrak{c}'$ using the Noether isomorphism (24.3.1). From (24.35) we conclude that $\mathrm{pd}_R \mathfrak{c} = \mathrm{pd}_{\overline{R}} \overline{\mathfrak{c}} = \mathrm{pd}_{\overline{R}} \mathfrak{c}' \leq 1$ which implies the desired. □

(31.10) **Final Remark.** Consult [Ka] concerning another proof of Main Theorem (31.1).

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