Homological Algebra Notes

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Contents

Introduction September 8, 2009 v
Format of these notes vii
Acknowledgements viii
Notation and conventions viii

Chapter I. Universal Constructions September 8, 2009 1
I.1. Finitely Generated Free Modules 1
I.2. Products of Modules 3
I.3. Coproducts of Modules 5
I.4. Localization 7
I.5. Hom: Functoriality and Localization 10

Chapter II. Tensor Products September 8, 2009 21
II.1. Existence and Uniqueness 21
II.2. Functoriality and Base-Change 25
II.3. Commutativity and Associativity 31
II.4. Right-Exactness 35
II.5. Hom-Tensor Adjointness 40

Chapter III. Injective, Projective, and Flat Modules September 8, 2009 43
III.1. Injective and Projective Modules 43
III.2. Flat Modules 51
III.3. Faithfully Flat Modules 58
III.4. Power Series Rings 60
III.5. Flat Ring Homomorphisms 65
III.6. Completions: A Survey 69
III.7. Completions: Some Details 69

Chapter IV. Homology, Resolutions, Ext and Tor September 8, 2009 71
IV.1. Chain Complexes and Homology 71
IV.2. Resolutions 76
IV.3. Ext-Modules 79
IV.4. Tor-Modules 84
IV.5. Epilogue 86

Chapter V. Depth September 8, 2009 89
V.1. Assumptions 89
V.2. Associated Primes and Supports of Modules 91
V.3. Prime Filtrations 96
V.4. Prime Avoidance and Nakayama's Lemma 100
V.5. Regular Sequences and Ext 105

V.6. Four Lemmas 112

Chapter VI. Chain Maps and Induced Maps on Ext and Tor September 8, 2009 115
VI.1. Chain Maps 115
VI.2. Isomorphisms for Ext and Tor 117
VI.3. Liftings of Resolutions 121
VI.4. Induced Chain Maps 125
VI.5. Ext-maps via projective resolutions 131
VI.6. Ext-maps via injective resolutions 135
VI.7. Tor-maps 137

Chapter VII. Ext, Tor, and Homological Dimensions September 8, 2009 143
VII.1. Assumptions 143
VII.2. Depth and Dimension 143
VII.3. Ext and Projective Dimension 146
VII.4. Tor and Projective Dimension 152
VII.5. Ext and Injective Dimension 154
VII.6. Tor and Flat Dimension 157

Chapter VIII. Long Exact Sequences September 8, 2009 161
VIII.1. General Long Exact Sequences 161
VIII.2. Long Exact Sequences in Ext and Tor 167
VIII.3. Horseshoe Lemmas 170
VIII.4. Mapping Cones 174
VIII.5. Ext, Tor, and Resolutions 179
VIII.6. Koszul Complexes 183
VIII.7. Epilogue: Tor and Torsion 192

Chapter IX. Depth and Homological Dimensions September 8, 2009 195
IX.1. Projective Dimension and Regular Sequences 195
IX.2. The Auslander-Buchsbaum Formula 197
IX.3. Depth and Flat Ring Homomorphisms 200
IX.4. Injective Dimension and Regular Sequences 205

Chapter X. Regular Local Rings September 8, 2009 207
X.1. Background from Dimension Theory 207
X.2. Definitions and Basic Properties 209
X.3. Theorems of Auslander, Buchsbaum and Serre 212
X.4. Regularity and Flat Local Homomorphisms 216

Index 221

Bibliography 225

Introduction September 8, 2009

An important question to ask (and re-ask) when one is learning a new subject is, "What does this subject do for me?" A complete answer to this question is usually hard to give, especially because the answer almost certainly depends on the interests of the person asking it. Here are a couple of motivating answers for the (commutative) algebraist who is thinking about learning some homological algebra.

Let R be a commutative ring (with identity).

Ext and Tor. Given an R-module N and an exact sequence of R-modules

$$0 \to M' \to M \to M'' \to 0 \tag{\&}$$

the operators $\operatorname{Hom}_R(-,-)$ and $- \otimes_R -$ give rise to three exact sequences

$$0 \to \operatorname{Hom}_R(N, M') \to \operatorname{Hom}_R(N, M) \to \operatorname{Hom}_R(N, M'') \tag{$*$}$$
$$0 \to \operatorname{Hom}_R(M'', N) \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M', N) \tag{$\dagger$}$$
$$M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0. \tag{\#}$$

One may be tempted to feel cheated by the loss of zeroes. When N is projective, we get to add "$\to 0$" onto the first sequence, and we get to add "$0 \to$" onto the last sequence. And when N is injective, we get to add "$\to 0$" onto the second sequence. But why are the last maps in the first two sequences not surjective in general? And why is the first map in the last sequence not injective? The answers to these questions are given in terms of Ext and Tor.

There are two sequences of operators

$$\{\operatorname{Ext}^n_R(-,-) \mid n = 1, 2, \ldots\} \qquad \text{and} \qquad \{\operatorname{Tor}^R_n(-,-) \mid n = 1, 2, \ldots\}$$

that satisfy the following properties.

(a) An R-module$N$ is projective if and only if $\operatorname{Ext}^n_R(N, -) = 0$ for all $n \geqslant 1$.
(b) Given an R-module N and an exact sequence of R-modules (&) there is a "long exact sequence"

$$\begin{aligned} 0 \to \operatorname{Hom}_R(N, M') &\to \operatorname{Hom}_R(N, M) \to \operatorname{Hom}_R(N, M'') \\ \longrightarrow \operatorname{Ext}^1_R(N, M') &\longrightarrow \operatorname{Ext}^1_R(N, M) \longrightarrow \operatorname{Ext}^1_R(N, M'') \to \cdots \\ \cdots \longrightarrow \operatorname{Ext}^n_R(N, M') &\longrightarrow \operatorname{Ext}^n_R(N, M) \longrightarrow \operatorname{Ext}^n_R(N, M'') \to \cdots . \end{aligned}$$

(c) An R-module N is injective if and only if $\operatorname{Ext}^n_R(-, N) = 0$ for all $n \geqslant 1$.

(d) Given an R-module N and an exact sequence of R-modules (&) there is a "long exact sequence"

$$\begin{aligned} 0 \to \operatorname{Hom}_R(M'',N) \to \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M',N) \\ \longrightarrow \operatorname{Ext}^1_R(M'',N) \longrightarrow \operatorname{Ext}^1_R(M,N) \longrightarrow \operatorname{Ext}^1_R(M',N) \to \cdots \\ \cdots \longrightarrow \operatorname{Ext}^n_R(M'',N) \longrightarrow \operatorname{Ext}^n_R(M,N) \longrightarrow \operatorname{Ext}^n_R(M',N) \to \cdots . \end{aligned}$$

(e) An R-module N is flat if and only if $\operatorname{Tor}^R_n(-,N) = 0$ for all $n \geqslant 1$.

(f) Given an R-module N and an exact sequence of R-modules (&) there is a "long exact sequence"

$$\begin{aligned} \cdots \to \operatorname{Tor}^R_n(M',N) \to \operatorname{Tor}^R_n(M,N) \to \operatorname{Tor}^R_n(M'',N) \to \cdots \\ \cdots \to \operatorname{Tor}^R_1(M',N) \to \operatorname{Tor}^R_1(M,N) \to \operatorname{Tor}^R_1(M'',N) \\ \longrightarrow M' \otimes_R N \longrightarrow M \otimes_R N \longrightarrow M'' \otimes_R N \to 0. \end{aligned}$$

The sequence in (b) shows exactly what is missing from $(*)$. Furthermore, when N is projective, item (a) explains exactly why we can add "$\to 0$" onto the sequence $(*)$. Similar comments hold for the sequence (†); also for (#) once one knows that every projective R-module is flat.

The constructions of Ext and Tor are homological in nature. So, the first answer to the question of what homological algebra gives you is: it shows you what has been missing and gives a full explanation for some special-case behaviors.

Another thing homological algebra gives you is invariants for studying rings and modules. Consider the following example. How do you distinguish between the vector spaces $\mathbb{R}^2$ and $\mathbb{R}^3$? Answer: by looking at the dimensions. The first one has dimension 2 and the second one has dimension 3. Therfore, they are not isomorphic.

In the study of modules over a commutative ring, even when there is a reasonable vector space dimension, it may not be enough to distinguish between non-isomorphic R-modules. Take for example the ring $\mathbb{R}[X,Y]$ and the modules $\mathbb{R}[X,Y]/(X,Y^2)$ and $\mathbb{R}[X,Y]/(X^2,Y)$. Each has vector space dimension 2 (over $\mathbb{R}$) but they are not isomorphic as R-modules.

Homological algebra gives you new invariants (numbers, functors, categories, etc.) to attach to an R-module that give you the power to detect (sometimes) when two modules are non-isomorphic. Of course, in the last example, one doesn't need to work very hard to see why the modules are not isomorphic. But in other situations, these homological invariants can be extremely powerful tools for the study of rings and modules. And these tools are so useful that many of them have become indispensable, almost unavoidable, items for the ring theorists' toolbox.

Regular sequences. Assume that R is noetherian and local with maximal ideal $\mathfrak{m}$. A sequence $x_1,\dots,x_n \in \mathfrak{m}$ is *R-regular* if (1) the element x_1 is a non-zero-divisor on R, and (2) for $i = 1,\dots,n-1$ the element x_{i+1} is a non-zero-divisor on the quotient $R/(x_1,\dots,x_i)$. The fact that R is noetherian implies that every R-regular sequence can be extended to a maximal one, that is, to one that cannot be further extended. It is not obvious, though, whether two maximal R-regular sequences have the same length. The fact that this works is a consequence of the following Ext-characterization: For each integer $n \geqslant 1$, the following conditions are equivalent:

(i) We have $\operatorname{Ext}^i_R(R/\mathfrak{m}, M) = 0$ for all $i < n$;
(ii) Every R-regular sequence in $\mathfrak{m}$ of length $\leqslant n$ can be extended to an R-regular sequence in $\mathfrak{m}$ of length n; and
(iii) There exists an R-regular sequence of length n in $\mathfrak{m}$.

The *depth* of a R is the length of a maximal R-regular sequence in $\mathfrak{m}$. It is the subject of Chapter V.

This is a handy invariant for induction arguments because when $x \in \mathfrak{m}$ is an R-regular element, the rings R and R/xR are homologically very similar, but we have $\operatorname{depth}(R/xR) = \operatorname{depth}(R) - 1$. Hence, if one is proving a result by induction on $\operatorname{depth}(R)$, one can often apply the induction hypothesis to R/xR and then show that the desired conclusion for R/xR implies the desired conclusion for R. One example of such an argument is found in the proof of the Auslander-Buchsbaum formula; see Chapter IX.

Regular local rings. Assume that R is noetherian and local with unique maximal ideal $\mathfrak{m} = (x_1, \dots, x_n)R$. The *Krull dimension* of R, denoted $d = \dim(R)$, is the supremum of the lengths of chains of prime ideals of R. A theorem of Wolfgang Krull implies that $d \leqslant n$, and R is *regular* if $\mathfrak{m}$ can be generated by a sequence of length d. Geometrically, this corresponds to a smoothness condition.

The following question was open for several years: If R is regular and $P \subsetneq R$ is a prime ideal, is the localization R_P also regular? It was solved by Maurice Auslander, David Buchsbaum and Jean-Pierre Serre using homological algebra, specifically, using the notion of the *projective dimension:* an R-module M has finite projective dimension if there is an exact sequence of R-module homomorphisms

$$0 \to P_t \to P_{t-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

such that each P_i is projective. They gave the following characterization of regular local rings: The following conditions are equivalent:

(i) The local ring R is regular;
(ii) Every R-module has finite projective dimension over R;
(iii) The residue field $R/\mathfrak{m}$ has finite projective dimension over R.

From this, they were able to deduce an affirmative answer to the localization question. This is an amazing result, not only because it answered an important open question, but also because the proof is relatively accessible. See Chapter X.

Format of these notes

When you buy a car, you don't necessarily want to rip the engine out and understand how every component works. You want to see how the car drives. Are the seats comfortable? Will this car suit your needs? Will it make your life better in some way? Once you buy the car and drive it for a while, you might then want to understand some of its inner-workings, to rip out the guts and really understand how the car works.

My approach to Homological Algebra is similar. These notes alternate between applications and the "guts". We begin with a certain amount of "guts" in Chapters I–III because they are necessary. (Many readers will be able to skip parts of Chapters I–III, though, since much of the material therein should be covered in a first year graduate algebra course.) As soon as it is reasonable, we focus on an

application. Specifically, the subject of Chapter V is depth. This chapter uses properties that are not proved until later chapters, if at all. These properties are clearly specified at the beginning of the chapter. The idea is to provide enough information in Chapters I–IV about the guts so that the proofs in Chapter V are accessible. In turn, the applications in Chapter V are supposed to motivate students to pursue a deeper understanding of the guts.

The remainder of the text alternates between guts and applications. Chapter VI explains more of the guts (how maps are induced on Ext and Tor), and Chapter VII contains (sort of) an application (the description of homological dimensions in terms of vanishing of Ext and Tor). Chapter VIII explains more of the guts (how we build long exact sequences including mapping cones and Koszul complexes), and Chapter IX contains an application (the Auslander-Buchsbaum formula which connects projective dimension (a homological invariant) in terms of depth (an elemental invariant)). Chapter X contains an application (Auslander, Buchsbaum and Serre's homological characterization of regular local rings and the solution of the localization problem for regular local rings).

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Notation and conventions

Throughout these notes, the term "ring" means "ring with identity", and "module" means "unital (or unitary) module". The term "ring homomorphism" means "homomorphism of rings with identity" in the sense that we assume that our ring homomorphisms respect the multiplicative identities. A ring is "local" if it has a unique maximal ideal. (Note that local rings are not assumed to be noetherian.) When we say that $(R, \mathfrak{m})$ is a local ring, we mean that R is a local ring with unique maximal ideal $\mathfrak{m}$. When we say that $(R, \mathfrak{m}, k)$ is a local ring, we mean that R is a local ring with unique maximal ideal $\mathfrak{m}$ and that $k = R/\mathfrak{m}$.

The symbol $\cong$ designates isomorphisms of modules. We let $\mathbb{1}_M \colon M \to M$ denote the identity function on a set M.

CHAPTER I

Universal Constructions September 8, 2009

I.1. Finitely Generated Free Modules

Finitely generated modules are build from finitely generated free modules, so we start with the basic properties of these.

Definition I.1.1. Let R be a commutative ring. Let M be an R-module, and fix a subset $\emptyset \neq X \subseteq M$. We say that X *generates* M if, for each $m \in M$, there exist an integer n and elements $r_1, \dots, r_n \in R$ and $x_1, \dots, x_n \in X$ such that $m = \sum_i r_i x_i$. When X generates M, we write $M = RX$. In addition, the empty set generates the zero module. If M is generated by a finite set, we say that M is *finitely generated*.

The set X is *linearly independent* if, for each integer $n \geqslant 1$, for each sequence $r_1, \dots, r_n \in R$ and for each sequence of distinct elements $x_1, \dots, x_n \in X$, if $\sum_i r_i x_i = 0$, then $r_i = 0$ for each $i = 1, \dots, n$. A *basis* for M is a linearly independent generating set for M.

The next example contains some of our favorite modules.

Example I.1.2. Let R be a commutative ring. The ring R is an R-module. More generally, the set

$$R^n = \left\{ \left(\begin{matrix} r_1 \\ \vdots \\ r_n \end{matrix} \right) \middle| \, r_1, \dots, r_n \in R \right\}$$

is an R-module. For $i = 1, \dots, n$ we set

$$\mathbf{e}_i = \left(\begin{matrix} \delta_{1,i} \\ \vdots \\ \delta_{n,i} \end{matrix} \right)$$

the ith *standard basis vector*. Here, $\delta_{i,j}$ is the Kronecker delta. The set $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is a basis for R^n.

These modules satisfy our first universal mapping property which defines them up to isomorphism. The proof of part (b) highlights the importance of the universal mapping property.

Proposition I.1.3. *Let R be a commutative ring, and let n be a positive integer. Let M be an R-module, and let $m_1, \dots, m_n \in M$.*

(a) *There exists a unique R-module homomorphism $f \colon R^n \to M$ such that $f(\mathbf{e}_i) = m_i$ for each $i = 1, \dots, n$.*

(b) *Assume that M satisfies the following: for every R-module P and for every sequence $p_1, \dots, p_n \in P$, there exists a unique R-module homomorphism $f \colon M \to P$ such that $f(m_i) = p_i$ for each $i = 1, \dots, n$. Then $M \cong R^n$.*

PROOF. (a) For the existence, let $f\colon R^n \to M$ be given by $\sum_i r_i\mathbf{e}_i \mapsto \sum_i r_i m_i$. The fact that $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ is a basis for R^n shows that f is well-defined. It is straightforward to show that f is an R-module homomorphism such that $f(\mathbf{e}_i) = m_i$ for each $i = 1,\dots,n$.

For the uniqueness, assume that $g\colon R^n \to M$ is an R-module homomorphism such that $g(\mathbf{e}_i) = m_i$ for each $i = 1,\dots,n$. Since g is R-linear, we have

$$g(\textstyle\sum_i r_i\mathbf{e}_i) = \sum_i r_i g(\mathbf{e}_i) = \sum_i r_i m_i = f(\sum_i r_i\mathbf{e}_i).$$

Since $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ generates R^n, this shows $g = f$.

(b) By assumption, there exists an R-module homomorphism $f\colon M \to R^n$ such that $f(m_i) = \mathbf{e}_i$ for each $i = 1,\dots,n$. By part (a), there exists an R-module homomorphism $g\colon R^n \to M$ such that $g(\mathbf{e}_i) = m_i$ for each $i = 1,\dots,n$.

We claim that $gf = \mathbb{1}_M$ and $fg = \mathbb{1}_{R^n}$. (Once this is shown, we will have $M \cong R^n$ via f.) The map $gf\colon M \to M$ is an R-module homomorphism such that

$$gf(m_i) = g(f(m_i)) = g(\mathbf{e}_i) = m_i \qquad \text{for } i = 1,\dots,n.$$

The identity map $\mathbb{1}_M\colon M \to M$ is an R-module homomorphism such that

$$\mathbb{1}_M(m_i) = m_i \qquad \text{for } i = 1,\dots,n.$$

Hence, the uniqueness condition in our assumption implies $gf = \mathbb{1}_M$. The equality $fg = \mathbb{1}_{R^n}$ is verified similarly using the uniqueness from part (a). □

Here is a useful restatement of Proposition I.1.3(a) in terms of commutative diagrams.

Remark I.1.4. Let R be a commutative ring. Let $j\colon \{\mathbf{e}_1,\dots,\mathbf{e}_n\} \to R^n$ denote the inclusion (of sets). For every function (map of sets) $f_0\colon \{\mathbf{e}_1,\dots,\mathbf{e}_n\} \to M$ there exists a unique R-module homomorphism $f\colon R^n \to M$ making the following diagram commute:

$$\begin{array}{ccc} \{\mathbf{e}_1,\dots,\mathbf{e}_n\} & \xrightarrow{\ j\ } & R^n \\ & \searrow^{f_0} & \big\downarrow \exists! f \\ & & M. \end{array}$$

Here is some notation from linear algebra.

Remark I.1.5. Let R be a commutative ring. Fix integers $n, k \geqslant 1$ and let $h\colon R^k \to R^n$ be an R-module homomorphism. We can represent h by an $n \times k$ matrix with entries in R as follows. Write elements of R^k and R^n as column vectors with entries in R. Let $\mathbf{e}_1,\dots,\mathbf{e}_k \in R^k$ be the standard basis. For $j = 1,\dots,k$ write

$$h(\mathbf{e}_j) = \begin{pmatrix} a_{1,j} \\ \vdots \\ a_{i,j} \\ \vdots \\ a_{n,j} \end{pmatrix}.$$

Then h is represented by the $n \times k$ matrix

$$[h] = (a_{i,j}) = \begin{pmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,k} \\ \vdots & & \vdots & & \vdots \\ a_{i,1} & \cdots & a_{i,j} & \cdots & a_{i,k} \\ \vdots & & \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,j} & \cdots & a_{n,k} \end{pmatrix}$$

in the following sense: For each vector

$$\begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix} \in R^k$$

we have

$$h\begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix} = h(\textstyle\sum_j r_j \mathbf{e}_j) = \sum_j r_j h(\mathbf{e}_j) = \sum_j r_j \begin{pmatrix} a_{1,j} \\ \vdots \\ a_{n,j} \end{pmatrix} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,k} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,k} \end{pmatrix} \begin{pmatrix} r_1 \\ \vdots \\ r_k \end{pmatrix}.$$

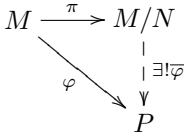
In particular, the image of h is generated by the columns of the matrix $(a_{i,j})$.

Exercises.

Exercise I.1.6. Let R be a commutative ring, and let M be an R-module.

(a) Prove that M is finitely generated if and only if there exists an integer $n \geqslant 1$ and a surjective R-module homomorphism $R^n \to M$.
(b) Prove that $M \cong R^n$ for some integer $n \geqslant 0$ if and only if M has a finite basis.

Exercise I.1.7. (Universal mapping property for R-module quotients) Let R be a commutative ring. Let M be an R-module, and fix an R-submodule $N \subseteq M$. Prove that, if $\varphi\colon M \to P$ is an R-module homomorphism such that $N \subseteq \operatorname{Ker}(\varphi)$, then there exists a unique R-module homomorphism $\overline{\varphi}\colon M/N \to P$ making the following diagram commute



that is, such that $\overline{\varphi}(\overline{m}) = \varphi(m)$ for all $m \in M$.

Exercise I.1.8. Let R be a commutative ring. Fix integers $m, n, p \geqslant 1$, and fix R-module homomorphisms $f\colon R^m \to R^n$ and $g\colon R^n \to R^p$. Prove that the matrix $[gf]$ representing the composition gf is the product $[g][f]$ of the matrices representing g and f.

I.2. Products of Modules

Products of modules will allow for constructions of more modules.

Remark/Definition I.2.1. Let R be a commutative ring. Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules. The *product* $\prod_{\lambda\in\Lambda} M_\lambda$ is the Cartesian product of the modules in this set, that is, the set of all sequences (m_λ) with $m_\lambda \in M_\lambda$ for each $\lambda \in \Lambda$. The set $\prod_\lambda M_\lambda$ has a well-defined R-module structure given by acting "coordinate-wise":

$$(m_\lambda) + (m'_\lambda) = (m_\lambda + m'_\lambda) \qquad\qquad r(m_\lambda) = (rm_\lambda)$$

For each $\mu \in \Lambda$, the function $\pi_\mu\colon \prod_\lambda M_\lambda \to M_\mu$ given by $(m_\lambda) \mapsto m_\mu$ is a well-defined surjective R-module homomorphism.

For each R-module N, set $N^\Lambda = \prod_{\lambda\in\Lambda} N_\lambda$, with $N_\lambda = N$ for each $\lambda \in \Lambda$. (This is also written $\prod_\lambda N$.) Note that N^Λ can be identified with the set of functions $f\colon \Lambda \to N$.

Before giving the universal mapping property for products, we discuss module-structures on Hom-sets.

Remark/Definition I.2.2. Let R be a commutative ring. Let M and N be R-modules, and set

$$\mathrm{Hom}_R(M,N) = \{R\text{-module homomorphisms } M \to N\}.$$

The set $\mathrm{Hom}_R(M,N)$ is an R-module under the following operations: for every $f, g \in \mathrm{Hom}_R(M,N)$ and $r \in R$, we have

$$\begin{aligned} f+g\colon M \to N \qquad & (f+g)(m) = f(m) + f(n) \\ rf\colon M \to N \qquad & (rf)(m) = r(f(m)) = rf(m) = f(rm). \end{aligned}$$

The product of R-modules comes with a universal mapping property which determines it up to isomorphism.

Proposition I.2.3. *Let R be a commutative ring. Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules, and let N be an R-module.*

(a) *Let $\{\psi_\lambda\colon N \to M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-module homomorphisms. There exists a unique R-module homomorphism $\Psi\colon N \to \prod_{\lambda\in\Lambda} M_\lambda$ making each of the following diagrams commute*

$$\begin{array}{ccc} N & \overset{\exists!\Psi}{\dashrightarrow} & \prod_\lambda M_\lambda \\ & {\scriptstyle\psi_\mu}\searrow & \downarrow{\scriptstyle\pi_\mu} \\ & & M_\mu \end{array}$$

that is, such that $\pi_\mu\Psi = \psi_\mu$ for each $\mu \in \Lambda$.

(b) *Assume that N and $\{\psi_\lambda\}$ satisfy the following: for every R-module P and every set of R-module homomorphisms $\{\phi_\lambda\colon P \to M_\lambda\}_{\lambda\in\Lambda}$, there exists a unique R-module homomorphism $\Phi\colon P \to N$ making each of the following diagrams commute*

$$\begin{array}{ccc} P & \overset{\exists!\Phi}{\dashrightarrow} & N \\ & {\scriptstyle\phi_\mu}\searrow & \downarrow{\scriptstyle\psi_\mu} \\ & & M_\mu \end{array}$$

that is, such that $\psi_\mu\Phi = \phi_\mu$ for each $\mu \in \Lambda$. Then $N \cong \prod_{\lambda\in\Lambda} M_\lambda$.

(c) *There exists an isomorphism of R-modules*

$$\theta\colon \mathrm{Hom}_R(N, \textstyle\prod_\lambda M_\lambda) \to \prod_\lambda \mathrm{Hom}_R(N, M_\lambda)$$

given by $\Psi \mapsto (\pi_\lambda\Psi)$.

Proof. (a) Existence: The rule $\Psi(n) = (\psi_\lambda(n))$ describes a well-defined homomorphism of R-modules $\Psi\colon N \to \prod_\lambda M_\lambda$ such that $\pi_\mu\Psi = \psi_\mu$ for each $\mu \in \Lambda$.

Uniqueness: Assume that $\Psi'\colon N \to \prod_\lambda M_\lambda$ is a second R-module homomorphism such that $\pi_\mu\Psi' = \psi_\mu$ for each $\mu \in \Lambda$. Fix an element $n \in N$, and write $\Psi'(n) = (m'_\lambda)$. For each $\mu \in \Lambda$, we have

$$m'_\mu = \pi_\mu(m'_\lambda) = \pi_\mu(\Psi'(n)) = \psi_\mu(n)$$

and hence

$$\Psi'(n) = (m'_\lambda) = (\psi_\lambda(n)) = \Psi(n).$$

Since n was chosen arbitrarily, this shows $\Psi' = \Psi$.

(b) By assumption, there exists an R-module homomorphism $\Phi\colon \prod_\lambda M_\lambda \to N$ such that $\psi_\mu\Phi = \pi_\mu$ for each $\mu \in \Lambda$. By part (a), there exists an R-module homomorphism $\Psi\colon N \to \prod_\lambda M_\lambda$ such that $\pi_\mu\Psi = \psi_\mu$ for each $\mu \in \Lambda$.

We claim that $\Phi\Psi = \mathbb{1}_N$ and $\Psi\Phi = \mathbb{1}_{\prod_\lambda M_\lambda}$. (Then we have $N \cong \prod_\lambda M_\lambda$ via Ψ.) The map $\Phi\Psi\colon N \to N$ is an R-module homomorphism such that

$$\psi_\mu\Phi\Psi = \pi_\mu\Psi = \psi_\mu \qquad \text{for all } \mu \in \Lambda.$$

The identity map $\mathbb{1}_N\colon N \to N$ is an R-module homomorphism such that

$$\psi_\mu\mathbb{1}_N = \psi_\mu \qquad \text{for all } \mu \in \Lambda.$$

Hence, the uniqueness condition in our assumption implies $\Phi\Psi = \mathbb{1}_N$. The equality $\Psi\Phi = \mathbb{1}_{\prod_\lambda M_\lambda}$ is verified similarly, using the uniqueness from part (b).

(c) It is straightforward so show that the map θ is a well-defined abelian group homomorphism. The existence statement in part (a) shows that θ is surjective, and the uniqueness statement in part (a) shows that θ is injective. □

I.3. Coproducts of Modules

Coproducts give yet another way to build new R-modules out of old ones.

Remark/Definition I.3.1. Let R be a commutative ring. Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules. The *coproduct* $\coprod_{\lambda\in\Lambda} M_\lambda$ is the subset of $\prod_{\lambda\in\Lambda} M_\lambda$ consisting of all sequences (m_λ) such that $m_\mu = 0$ for all but finitely many $\mu \in \Lambda$. The set $\coprod_{\lambda\in\Lambda} M_\lambda$ has a well-defined R-module structure given by acting "coordinate-wise":

$$(m_\lambda) + (m'_\lambda) = (m_\lambda + m'_\lambda) \qquad\qquad r(m_\lambda) = (rm_\lambda)$$

This module structure makes $\coprod_{\lambda\in\Lambda} M_\lambda$ into an R-submodule of $\prod_{\lambda\in\Lambda} M_\lambda$. We sometimes denote $\coprod_{\lambda\in\Lambda} M_\lambda$ using the "direct sum" notation $\oplus_{\lambda\in\Lambda} M_\lambda$. For each $\mu \in \Lambda$, the function $\varepsilon_\mu\colon M_\mu \to \coprod_{\lambda\in\Lambda} M_\lambda$ given by $m_\mu \mapsto (m_\lambda)$, where $m_\lambda = 0$ for all $\lambda \neq \mu$, is a well-defined injective R-module homomorphism.

Note that, if each $M_\mu \neq 0$, then $\coprod_{\lambda\in\Lambda} M_\lambda = \prod_{\lambda\in\Lambda} M_\lambda$ if and only if Λ is finite.

For each R-module N, set $N^{(\Lambda)} = \coprod_{\lambda\in\Lambda} N_\lambda$, with $N_\lambda = N$ for each $\lambda \in \Lambda$. (This is commonly written $\coprod_{\lambda\in\Lambda} N$.) Note that $N^{(\Lambda)}$ can be identified with the set of functions $f\colon \Lambda \to N$ such that $f(\lambda) = 0$ for all but finitely many $\lambda \in \Lambda$. When Λ is a finite set with cardinality r, we often write $N^r = N^{(\Lambda)} = N^\Lambda$.

The *free R-module on Λ* is $R^{(\Lambda)}$. For each $\mu \in \Lambda$ we set $\mathbf{e}_\mu = (\delta_{\lambda,\mu})$ the μth *standard basis vector*. Here, $\delta_{\lambda,\mu}$ is the Kronecker delta. The set $\{\mathbf{e}_\lambda\}_\lambda$ is a basis for $R^{(\Lambda)}$. An R-module M is *free* if there exists a set Λ such that $M \cong R^{(\Lambda)}$. Let $\varepsilon\colon \Lambda \to R^{(\Lambda)}$ be given by $\varepsilon(\mu) = \mathbf{e}_\mu$ for each $\mu \in \Lambda$.

Here is some useful notation for the future.

Remark I.3.2. Let R be a commutative ring, and let $M_1, \ldots, M_n$ be a R-modules. Given a sequence

$$(m_1, m_2, \ldots, m_n) \in \coprod_{i=1}^n M_n$$

we can write

$$(m_1, m_2, \ldots, m_n) = (m_1, 0, \ldots, 0) + (0, m_2, \ldots, 0) + \cdots + (0, 0, \ldots, m_n).$$

The analogue of this formula for infinite coporoducts goes like this. Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules. For each $\mu \in \Lambda$ let $\varepsilon_\mu\colon M_\mu \to \coprod_{\lambda\in\Lambda} M_\lambda$ denote the canonical inclusion. Then we have

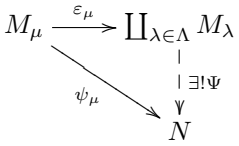
$$(m_\lambda) = \textstyle\sum_{\mu\in\Lambda} \varepsilon_\mu(m_\mu)$$

for each sequence $(m_\lambda) \in \coprod_{\lambda\in\Lambda} M_\lambda$. Notice that this sum is finite.

Exercises.

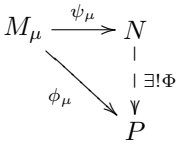
Exercise I.3.3. (Universal mapping property for coproducts.) Let R be a commutative ring. Let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules.

(a) Let $\{\psi_\lambda\colon M_\lambda \to N\}_{\lambda\in\Lambda}$ be a set of R-module homomorphisms. Prove that there is a unique R-module homomorphism $\Psi\colon \coprod_{\lambda\in\Lambda} M_\lambda \to N$ making each of the following diagrams commute



that is, such that $\Psi\varepsilon_\mu = \psi_\mu$ for each $\mu \in \Lambda$.

(b) Assume that N and $\{\psi_\lambda\}$ satisfy the following: for each ${}_RP$ and each set of R-module homomorphisms $\{\phi_\lambda\colon M_\lambda \to P\}_\lambda$, there is a unique left R-module hom $\Phi\colon N \to P$ making each of the next diagrams commute



that is, such that $\Phi\psi_\mu = \phi_\mu$ for each $\mu \in \Lambda$. Prove that $N \cong \coprod_{\lambda\in\Lambda} M_\lambda$.

(c) Prove that there exists an isomorphism of R-modules

$$\omega\colon \operatorname{Hom}_R(\textstyle\coprod_\lambda M_\lambda, N) \to \prod_\lambda \operatorname{Hom}_R(M_\lambda, N)$$

given by $\Psi \mapsto (\Psi\varepsilon_\lambda)$.

Exercise I.3.4. (Universal mapping property for free modules.) Let R be a commutative ring. Fix an R-module N and a subset $\{n_\lambda\}_{\lambda\in\Lambda} \subseteq N$.

(a) Prove that there is a unique R-module homomorphism $\Psi\colon R^{(\Lambda)} \to N$ such that $\Psi(\mathbf{e}_\lambda) = n_\lambda$, for each $\lambda \in \Lambda$.

(b) Assume that N and $\{n_\lambda\}$ satisfy the following: for every R-module P and every subset $\{p_\lambda\}_{\lambda\in\Lambda} \subseteq P$, there exists a unique R-module homomorphism $\Phi\colon N \to P$ such that $\Phi(n_\lambda) = p_\lambda$, for each $\lambda \in \Lambda$. Prove that $N \cong R^{(\Lambda)}$.

(c) Prove that there exists an isomorphism of R-modules

$$\omega\colon \operatorname{Hom}_R(R^{(\Lambda)}, N) \to N^\Lambda$$

given by $\Psi \mapsto (\Psi(\mathbf{e}_\lambda))$.

Exercise I.3.5. Let R be a commutative ring, and let M be an R-module.

(a) Prove that M is generated by a subset $S \subseteq M$ if and only if it is a homomorphic image of $R^{(S)}$.

(b) Prove that M is free if and only if it possesses a basis.

(c) Prove that $M \cong R^n$ for some integer $n \geqslant 0$ if and only if it is finitely generated and free.

Exercise I.3.6. Let R be a commutative ring, and let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules. Fix subsets $S_\lambda \subseteq M_\lambda$ and set $S = \cup_\lambda \varepsilon_\lambda(S_\lambda) \subseteq \coprod_\lambda M_\lambda \subseteq \prod_\lambda M_\lambda$.

(a) Prove that the R-module $\coprod_\lambda M_\lambda$ is generated by S if and only if M_λ is generated by S_λ for each $\lambda \in \Lambda$.
(b) Assume $M_\lambda \neq 0$ for all $\lambda \in \Lambda$. Prove that the module $\prod_\lambda M_\lambda$ is generated by S if and only if the set Λ is finite and M_λ is generated by S_λ for each $\lambda \in \Lambda$.
(c) Assume $M_\lambda \neq 0$ for all $\lambda \in \Lambda$. Prove that the following conditions are equivalent:
   (i) The R-module $\coprod_\lambda M_\lambda$ is finitely generated;
   (ii) The R-module $\prod_\lambda M_\lambda$ is finitely generated;
   (iii) The set Λ is finite and M_λ is finitely generated for each $\lambda \in \Lambda$.

I.4. Localization

In this section, we recall the definition and basic properties of localizations.

Definition I.4.1. Let R be a commutative ring. A subset $U \subseteq R$ is *multiplicatively closed* if $1 \in U$ and, for all $u, v \in U$ we have $uv \in U$.

Example I.4.2. Let R be a commutative ring. If $\mathfrak{p} \subsetneq R$ is a prime ideal, then the set $R \smallsetminus \mathfrak{p}$ is multiplicatively closed. If $s \in R$, then the set $\{1, s, s^2, s^3, \ldots\}$ is multiplicatively closed.

Definition I.4.3. Let R be a commutative ring, and let U be a multiplicatively closed subset of R. Define a relation on $R \times U$ as follows: $(r, u) \sim (s, v)$ provided that there is an element $w \in U$ such that $wrv = wsu$.

Fact I.4.4. Let R be a commutative ring, and let U be a multiplicatively closed subset of R. The relation from Definition I.4.3 is an equivalence relation.

Definition I.4.5. Let R be a commutative ring, and let U be a multiplicatively closed subset of R. Let $U^{-1}R$ denote the set of equivalence classes under the relation from Definition I.4.3, with the equivalence class of (r, u) in $U^{-1}R$ denoted r/u or $\frac{r}{u}$. Define $0_{U^{-1}R} = 0_R/1_R \in U^{-1}R$ and $1_{U^{-1}R} = 1_R/1_R \in U^{-1}R$. For all $r/u, s/v \in U^{-1}R$, define

$$\frac{r}{u} + \frac{s}{v} = \frac{rv + su}{uv} \qquad \text{and} \qquad \frac{r}{u}\frac{s}{v} = \frac{rs}{uv}.$$

Define $\psi\colon R \to U^{-1}R$ by the formula $\psi(r) = r/1_R$. The set $U^{-1}R$ is the *localization* of R at the set U.

Fact I.4.6. Let R be a commutative ring, and let U be a multiplicatively closed subset of R. The localization $U^{-1}R$ has the structure of a commutative ring (with identity) under the operations from Definition I.4.5 Furthermore, the map $\psi\colon R \to U^{-1}R$ given by $r \mapsto r/1$ is a well-defined ring homomorphism.

Example I.4.7. Let R be a commutative ring. If R is an integral domain, then the localization $K = (R \smallsetminus \{0\})^{-1}R$ is (isomorphic to) the field of fractions of R, and every non-zero localization $U^{-1}R$ is (isomorphic to) a subring of K that contains R; in particular, every non-zero localization of an integral domain is an integral domain. If $\mathfrak{p} \subsetneq R$ is a prime ideal, then the localization $(R \smallsetminus \mathfrak{p})^{-1}R$ is denoted $R_\mathfrak{p}$.

Here is the *universal mapping property for localization.*

Fact I.4.8. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings, and let U be a multiplicatively closed subset of R. If $\varphi(U)$ consists of units of S, then there is a unique ring homomorphism $\varphi'\colon U^{-1}R \to S$ making the following diagram commute

$$\begin{array}{ccc} R & \xrightarrow{\psi} & U^{-1}R \\ & \searrow^{\varphi} & \downarrow{\scriptstyle \exists!\varphi'} \\ & & S. \end{array}$$

Here, ψ is the natural map, and $\varphi'(r/u) = \varphi(r)\varphi(u)^{-1}$.

Definition I.4.9. Let R be a commutative ring, and let U be a multiplicatively closed subset of R. Let M be an R-module. Define a relation on $M \times U$ as follows: $(m, u) \sim (n, v)$ provided that there is an element $w \in U$ such that $wvm = wun$.

Fact I.4.10. Let R be a commutative ring, and let U be a multiplicatively closed subset of R. Let M be an R-module. The relation from Definition I.4.9 is an equivalence relation.

Definition I.4.11. Let R be a commutative ring, and let U be a multiplicatively closed subset of R. Let M be an R-module. Let $U^{-1}M$ denote the set of equivalence classes under the relation from Definition I.4.9, with the equivalence class of (m, u) in $U^{-1}M$ denoted m/u or $\frac{m}{u}$. Define $0_{U^{-1}M} = 0_M/1_R \in U^{-1}R$. For all $m/u, n/v \in U^{-1}M$, define

$$\frac{m}{u} + \frac{n}{v} = \frac{vm + un}{uv} \qquad \text{and} \qquad \frac{r}{u}\frac{m}{v} = \frac{rm}{uv}.$$

Define $\psi_M\colon M \to U^{-1}M$ by the formula $\psi_M(m) = m/1_R$. The set $U^{-1}M$ is the *localization* of M at the set U.

Fact I.4.12. Let R be a commutative ring, and let U be a multiplicatively closed subset of R. Let M be an R-module. Th set $U^{-1}M$ has the structure of an $U^{-1}R$-module under the operations from Definition I.4.11. In particular, $U^{-1}M$ is an R-module by restriction of scalars along φ, that is, by the scalar multiplication $r(m/u) = (rm)/u$. Under this module structure, the map $\psi_M\colon M \to U^{-1}M$ given by $m \mapsto m/1_R$ is a well-defined R-module homomorphism.

Let $f\colon M \to M'$ be an R-module homomorphism. It follows that the function $U^{-1}f\colon U^{-1}M \to U^{-1}M'$ given by $(U^{-1}f)(m/u) = f(m)/u$ is a well-defined $U^{-1}R$-module homomorphism. Furthermore, there is a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ {\scriptstyle \varphi_M}\downarrow & & \downarrow{\scriptstyle \varphi_{M'}} \\ U^{-1}M & \xrightarrow{U^{-1}f} & U^{-1}M' \end{array}$$

where the vertical maps are from the previous paragraph.

It is straightforward to show that localization is exact: Given an exact sequence of R-module homomorphisms

$$\cdots \xrightarrow{f_{i+2}} N_{i+1} \xrightarrow{f_{i+1}} N_i \xrightarrow{f_i} \cdots$$

the localized sequence

$$\cdots \xrightarrow{U^{-1}f_{i+2}} U^{-1}N_{i+1} \xrightarrow{U^{-1}f_{i+1}} U^{-1}N_i \xrightarrow{U^{-1}f_i} \cdots$$

is exact. In particular, if $N \subseteq M$ is a submodule, then the localization $U^{-1}N$ is naturally identified with a submodule of $U^{-1}M$; under this identification, there is a $U^{-1}R$-module isomorphism $(U^{-1}M)/(U^{-1}N) \cong U^{-1}(M,N)$.

Example I.4.13. Let R be a commutative ring, and let M be an R-module. If $\mathfrak{p} \subsetneq R$ is a prime ideal, then the localization $(R \smallsetminus \mathfrak{p})^{-1}M$ is denoted $M_{\mathfrak{p}}$.

Here is the *prime correspondence for localization*.

Fact I.4.14. Let R be a commutative ring, and let U be a multiplicatively closed subset of R. Let $\psi\colon R \to U^{-1}R$ denote the natural map. The following maps

$$\{\text{prime ideals } P \subsetneq U^{-1}R\} \longleftrightarrow \{\text{prime ideals } \mathfrak{q} \subsetneq R \mid \mathfrak{q} \cap U = \emptyset\}$$

$$P \longmapsto P(U^{-1}R) \cong U^{-1}P$$

$$\psi^{-1}(\mathfrak{q}) \longmapsfrom \mathfrak{q}$$

are inverse bijections. For each prime ideal $\mathfrak{q} \subsetneq R$ such that $\mathfrak{q} \cap U = \emptyset$, we have $U^{-1}R/U^{-1}\mathfrak{q} \cong U^{-1}(R/\mathfrak{q})$.

In particular, when $U = R \smallsetminus \mathfrak{p}$ for some prime ideal $\mathfrak{p} \subsetneq R$, the maps

$$\{\text{prime ideals } P \subsetneq R_{\mathfrak{p}}\} \longleftrightarrow \{\text{prime ideals } \mathfrak{q} \subsetneq R \mid \mathfrak{p} \cap \mathfrak{q} \subseteq \mathfrak{p}\}$$

$$P \longmapsto PR_{\mathfrak{p}} \cong P_{\mathfrak{p}}$$

$$\psi^{-1}(\mathfrak{q}) \longmapsfrom \mathfrak{q}$$

are inverse bijections. In particular, the ring $R_{\mathfrak{p}}$ is local with maximal ideal $\mathfrak{p}R_{\mathfrak{p}} \cong \mathfrak{p}_{\mathfrak{p}}$. For each prime ideal $\mathfrak{q} \subsetneq R$ such that $\mathfrak{q} \subseteq \mathfrak{p}$, we have $R_{\mathfrak{p}}/\mathfrak{q}_{\mathfrak{p}} \cong (R/\mathfrak{q})_{\mathfrak{p}}$.

Exercises.

Exercise I.4.15. Verify the statements from Example I.4.2.

Exercise I.4.16. Verify the statements from Fact I.4.4.

Exercise I.4.17. Verify the statements from Fact I.4.6.

Exercise I.4.18. Verify the statements from Example I.4.7.

Exercise I.4.19. Verify the statements from Fact I.4.8.

Exercise I.4.20. Verify the statements from Fact I.4.10.

Exercise I.4.21. Verify the statements from Fact I.4.12.

Exercise I.4.22. Verify the statements from Example I.4.13.

Exercise I.4.23. Verify the statements from Fact I.4.14.

Exercise I.4.24. Let R be a commutative ring, and let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules. Prove that for each multiplicatively closed subset $U \subseteq R$, there is a $U^{-1}R$-module isomorphism $U^{-1}(\coprod_{\lambda\in\Lambda} M_\lambda) \cong \coprod_{\lambda\in\Lambda} U^{-1}M_\lambda$.

Exercise I.4.25. Let R be a commutative ring, and let M be an R-module. Prove that the following conditions are equivalent:

(i) $M = 0$;
(ii) $U^{-1}M = 0$ for every multiplicatively closed subset $U \subseteq R$;

(iii) $M_{\mathfrak{p}} = 0$ for every prime ideal $\mathfrak{p} \subsetneq R$; and
(iv) $M_{\mathfrak{m}} = 0$ for every maximal ideal $\mathfrak{m} \subsetneq R$.

This says that being zero is a *local property*.

Exercise I.4.26. Let R be a commutative ring, and let $f\colon M \to N$ be an R-module homomorphism. Prove that the following conditions are equivalent:

(i) the map f is injective;
(ii) the localization $U^{-1}f\colon U^{-1}M \to U^{-1}N$ is injective for every multiplicatively closed subset $U \subseteq R$;
(iii) the localization $f_{\mathfrak{p}}\colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective for every prime ideal $\mathfrak{p} \subsetneq R$; and
(iv) the localization $f_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective for every maximal ideal $\mathfrak{m} \subsetneq R$.

Exercise I.4.27. Let R be a commutative ring, and let $f\colon M \to N$ be an R-module homomorphism. Prove that the following conditions are equivalent:

(i) the map f is surjective;
(ii) the localization $U^{-1}f\colon U^{-1}M \to U^{-1}N$ is surjective for every multiplicatively closed subset $U \subseteq R$;
(iii) the localization $f_{\mathfrak{p}}\colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is surjective for every prime ideal $\mathfrak{p} \subsetneq R$; and
(iv) the localization $f_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is surjective for every maximal ideal $\mathfrak{m} \subsetneq R$.

Exercise I.4.28. Let R be a commutative ring, and let $f\colon M \to N$ be an R-module homomorphism. Prove that the following conditions are equivalent:

(i) the map f is bijective;
(ii) the localization $U^{-1}f\colon U^{-1}M \to U^{-1}N$ is bijective for every multiplicatively closed subset $U \subseteq R$;
(iii) the localization $f_{\mathfrak{p}}\colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is bijective for every prime ideal $\mathfrak{p} \subsetneq R$; and
(iv) the localization $f_{\mathfrak{m}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is bijective for every maximal ideal $\mathfrak{m} \subsetneq R$.

I.5. Hom: Functoriality and Localization

This section deals with the basic properties of Hom. The modules $\operatorname{Hom}_R(M,N)$ are defined in I.2.2. We start this section with induced maps.

Definition I.5.1. Let R be a commutative ring, and consider R-module homomorphisms $f\colon M \to M'$ and $g\colon N \to N'$. Let

$$\operatorname{Hom}_R(M,g)\colon \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,N') \qquad \text{given by } \phi \mapsto g\phi$$
$$\operatorname{Hom}_R(f,N)\colon \operatorname{Hom}_R(M',N) \to \operatorname{Hom}_R(M,N) \qquad \text{given by } \psi \mapsto \psi f.$$

Example I.5.2. Let R be a commutative ring, and let M, N and N' be R-modules. Let $0^N_{N'}\colon N \to N'$ be the zero map. Each of the following maps is the zero-map:

$$\operatorname{Hom}_R(M,0^N_{N'})\colon \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,N')$$
$$\operatorname{Hom}_R(0^N_{N'},M)\colon \operatorname{Hom}_R(N',M) \to \operatorname{Hom}_R(N,M).$$

Remark I.5.3. Let R be a commutative ring, and let N be an R-module. There is an R-module isomorphism

$$\psi\colon \operatorname{Hom}_R(R,N) \xrightarrow{\cong} N \qquad \text{given by} \qquad \phi \mapsto \phi(1).$$

The inverse of ψ is given by $\psi^{-1}(n) = \phi_n \colon R \to N$ where $\phi_n(r) = rn$. If $f \colon N \to N'$ is an R-module homomorphism, then there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_R(R,N) & \xrightarrow{\operatorname{Hom}_R(R,f)} & \operatorname{Hom}_R(R,N') \\ \psi \downarrow \cong & & \psi' \downarrow \cong \\ N & \xrightarrow{\qquad f \qquad} & N'. \end{array}$$

Example I.5.4. Let R be a commutative ring, and let M and N be R-modules. Let $r \in R$, and let $\mu_r^N \colon N \to N$ be given by $n \mapsto rn$. Each of the following maps is given by multiplication by r:

$$\operatorname{Hom}_R(M, \mu_r^N) \colon \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,N)$$
$$\operatorname{Hom}_R(\mu_r^N, M) \colon \operatorname{Hom}_R(N,M) \to \operatorname{Hom}_R(N,M).$$

Indeed, for each $\phi \in \operatorname{Hom}_R(M,N)$ and each $m \in M$, we have

$$(\operatorname{Hom}_R(M, \mu_r^N)(\phi))(m) = (\mu_r^N \phi)(m) = \mu_r^N(\phi(m)) = r(\phi(m)) = (r\phi)(m)$$

hence $\operatorname{Hom}_R(M, \mu_r^N)(\phi) = r\phi$. Similarly, for all $\psi \in \operatorname{Hom}_R(N,M)$, we have $\operatorname{Hom}_R(\mu_r^N, M)(\psi) = r\psi$.

Here is the "functoriality" of Hom.

Fact I.5.5. Let R be a commutative ring. Let M be an R-module, and consider R-module homomorphisms $g \colon N \to N'$ and $g' \colon N' \to N''$. Then the following diagrams commute

$$\begin{array}{ccc} \operatorname{Hom}_R(M,N) & \xrightarrow{\operatorname{Hom}_R(M,g)} & \operatorname{Hom}_R(M,N') \\ & \searrow^{\operatorname{Hom}_R(M,g'g)} & \downarrow{\scriptstyle \operatorname{Hom}_R(M,g')} \\ & & \operatorname{Hom}_R(M,N'') \end{array}$$

$$\begin{array}{ccc} \operatorname{Hom}_R(N'',M) & \xrightarrow{\operatorname{Hom}_R(g',M)} & \operatorname{Hom}_R(N',M) \\ & \searrow^{\operatorname{Hom}_R(g'g,M)} & \downarrow{\scriptstyle \operatorname{Hom}_R(g,M)} \\ & & \operatorname{Hom}_R(N,M) \end{array}$$

that is, we have

$$\operatorname{Hom}_R(M, g'g) = \operatorname{Hom}_R(M,g') \operatorname{Hom}_R(M,g)$$
$$\operatorname{Hom}_R(g'g, M) = \operatorname{Hom}_R(g',M) \operatorname{Hom}_R(g,M).$$

See Exercise I.5.13.

Here is the "left-exactness" of Hom.

Fact I.5.6. Let R be a commutative ring. Let M be an R-module, and consider an exact sequence of R-module homomorphisms:

$$0 \to N' \xrightarrow{g'} N \xrightarrow{g} N''.$$

Then the induced sequence

$$0 \to \operatorname{Hom}_R(M,N') \xrightarrow{\operatorname{Hom}_R(M,g')} \operatorname{Hom}_R(M,N) \xrightarrow{\operatorname{Hom}_R(M,g)} \operatorname{Hom}_R(M,N'')$$

is also exact.

Let N be an R-module, and consider an exact sequence of R-module homomorphisms:

$$M' \xrightarrow{f'} M \xrightarrow{f} M'' \to 0.$$

Then the induced sequence

$$0 \to \operatorname{Hom}_R(M'', N) \xrightarrow{\operatorname{Hom}_R(f,N)} \operatorname{Hom}_R(M, N) \xrightarrow{\operatorname{Hom}_R(f',N)} \operatorname{Hom}_R(M, N')$$

is also exact. See Exercise I.5.14.

The next fact explains some of the interplay between module structures over different rings.

Fact I.5.7. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Fix an R-module homomorphism $f\colon M \to M'$ and an S-module homomorphism $g\colon N \to N'$.

(a) The R-module $\operatorname{Hom}_R(N, M)$ is also an S-module via the following scalar multiplication: for all $s \in S$ and all $\phi \in \operatorname{Hom}_R(N, M)$, define $s\phi \in \operatorname{Hom}_R(N, M)$ by the formula $(s\phi)(n) = \phi(sn)$ for all $n \in N$. The induced maps

$$\operatorname{Hom}_R(N, f)\colon \operatorname{Hom}_R(N, M) \to \operatorname{Hom}_R(N, M')$$
$$\operatorname{Hom}_R(g, M)\colon \operatorname{Hom}_R(N', M) \to \operatorname{Hom}_R(N, M)$$

are S-module homomorphisms.

(b) In particular, the R-module $\operatorname{Hom}_R(S, M)$ is also an S-module via the following action: for all $s \in S$ and all $\phi \in \operatorname{Hom}_R(S, M)$, define $s\phi \in \operatorname{Hom}_R(S, M)$ by the formula $(s\phi)(t) = \phi(st)$ for all $t \in S$. The induced map

$$\operatorname{Hom}_R(S, f)\colon \operatorname{Hom}_R(S, M) \to \operatorname{Hom}_R(S, M')$$

is an S-module homomorphism.

(c) The R-module $\operatorname{Hom}_R(M, N)$ is also an S-module via the following scalar multiplication: for all $s \in S$ and all $\phi \in \operatorname{Hom}_R(M, N)$, define $s\phi \in \operatorname{Hom}_R(M, N)$ by the formula $(s\phi)(m) = s\phi(m)$ for all $m \in M$. The induced maps

$$\operatorname{Hom}_R(M, g)\colon \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M, N')$$
$$\operatorname{Hom}_R(f, N)\colon \operatorname{Hom}_R(M', N) \to \operatorname{Hom}_R(M, N)$$

are S-module homomorphisms.

The next result shows that Hom localizes, sometimes.

Proposition I.5.8. *Let R be a commutative ring. Let M and N be R-modules, and let $U \subseteq R$ be a multiplicatively closed subset.*

(a) *For each element $\phi/u \in U^{-1}\operatorname{Hom}_R(M, N)$, the map $\phi_u\colon U^{-1}M \to U^{-1}N$ given by $\phi_u(m/v) = \phi(m)/(uv)$ is a well-defined $U^{-1}R$-module homomorphism.*

(b) *The rule of assignment*

$$\Theta_{U,M,N}\colon U^{-1}\operatorname{Hom}_R(M, N) \to \operatorname{Hom}_{U^{-1}R}(U^{-1}M, U^{-1}N)$$

given by $\phi/u \mapsto \phi_u$ is a well-defined $U^{-1}R$-module homomorphism.

(c) *Assume that M is finitely presented, that is, that there is an exact sequence of R-module homomorphisms*

$$R^m \xrightarrow{f} R^n \xrightarrow{g} M \to 0. \tag{I.5.8.1}$$

Then $\Theta_{U,M,N}$ is an isomorphism, so we have

$$U^{-1}\operatorname{Hom}_R(M,N) \cong \operatorname{Hom}_{U^{-1}R}(U^{-1}M, U^{-1}N).$$

(d) *If R is noetherian and M is finitely generated, then the map $\Theta_{U,M,N}$ is an isomorphism, so we have $U^{-1}\operatorname{Hom}_R(M,N) \cong \operatorname{Hom}_{U^{-1}R}(U^{-1}M, U^{-1}N)$.*

Proof. (a) We first show that, given elements $\phi \in \operatorname{Hom}_R(M,N)$ and $u \in U$, the ϕ_u map $\phi_u \colon U^{-1}M \to U^{-1}N$ given by $\phi_u(m/v) = \phi(m)/(uv)$ is a well-defined function. Let $m'/v' = m/v$ in $U^{-1}M$. By definition, there is an element $v'' \in U$ such that $vv''m' = v'v''m$ in M. Since ϕ is an R-module homomorphism, we have

$$vv''\phi(m') = \phi(vv''m') = \phi(v'v''m) = v'v''\phi(m)$$

in N. It follows that, in $U^{-1}N$, we have

$$\frac{\phi(m')}{uv'} = \frac{vv''\phi(m')}{vv''uv'} = \frac{v'v''\phi(m)}{v'v''uv} = \frac{\phi(m)}{uv}$$

as desired.

We next show that ϕ_u is independent of the choice of ϕ and u. Assume that $\phi'/u' = \phi/u$ in $U^{-1}\operatorname{Hom}_R(M,N)$. By definition, there is an element $u'' \in U$ such that $uu''\phi' = u'u''\phi$ in $\operatorname{Hom}_R(M,N)$, that is, for all $m \in M$, we have $uu''\phi'(m) = u'u''\phi(m)$. Thus, for all $m/v \in U^{-1}M$ we have the following equalities in $U^{-1}N$:

$$\phi'_{u'}\left(\frac{m}{v}\right) = \frac{\phi'(m)}{u'v} = \frac{uu''\phi'(m)}{uu''u'v} = \frac{u'u''\phi(m)}{u'u''uv} = \frac{\phi(m)}{uv} = \phi_u\left(\frac{m}{v}\right).$$

It follows that $\phi'_{u'} = \phi_u$, as desired.

The fact that ϕ_u is an $U^{-1}R$-module homomorphism is now straightforward to verify. For instance, we have

$$\begin{aligned}\phi_u\left(\frac{m}{v} + \frac{m'}{v'}\right) &= \phi_u\left(\frac{v'm + vm'}{vv'}\right) = \frac{\phi(v'm+vm')}{uvv'} = \frac{v'\phi(m) + v\phi(m')}{uvv'} \\ &= \frac{v'\phi(m)}{uvv'} + \frac{v\phi(m')}{uvv'} = \frac{\phi(m)}{uv} + \frac{\phi(m')}{uv'} = \phi_u\left(\frac{m}{v}\right) + \phi_u\left(\frac{m'}{v'}\right).\end{aligned}$$

The equality $\phi_u\left(\frac{r}{w}\frac{m}{v}\right) = \frac{r}{w}\phi_u\left(\frac{m}{v}\right)$ is verified similarly. See Exercise I.5.16.

(b) The fact that $\Theta_{U,M,N}$ is a well-defined function is established in part (a). It remains to show that $\Theta_{U,M,N}$ is an $U^{-1}R$-module homomorphism. We prove that $\Theta_{U,M,N}$ respects sums. The fact that $\Theta_{U,M,N}$ respects scalar multiplication is verified similarly; see Exercise I.5.16.

Let $\phi/u, \phi'/u' \in U^{-1}\operatorname{Hom}_R(M,N)$. We need to show that the maps

$$\Theta_{U,M,N}\left(\frac{\phi}{u} + \frac{\phi'}{u'}\right) \qquad \text{and} \qquad \Theta_{U,M,N}\left(\frac{\phi}{u}\right) + \Theta_{U,M,N}\left(\frac{\phi'}{u'}\right)$$

are the same maps $U^{-1}M \to U^{-1}N$. Using the equality $\frac{\phi}{u} + \frac{\phi'}{u'} = \frac{u'\phi + u\phi'}{uu'}$, this means that we need to show that

$$(u'\phi + u\phi')_{uu'} \qquad \text{and} \qquad \phi_u + \phi'_{u'}$$

are the same maps $U^{-1}M \to U^{-1}N$. Evaluating the first map at an arbitrary element $m/v \in U^{-1}M$, we have

$$(u'\phi + u\phi')_{uu'}\left(\frac{m}{v}\right) = \frac{(u'\phi + u\phi')(m)}{uu'v} = \frac{u'\phi(m) + u\phi'(m)}{uu'v}.$$

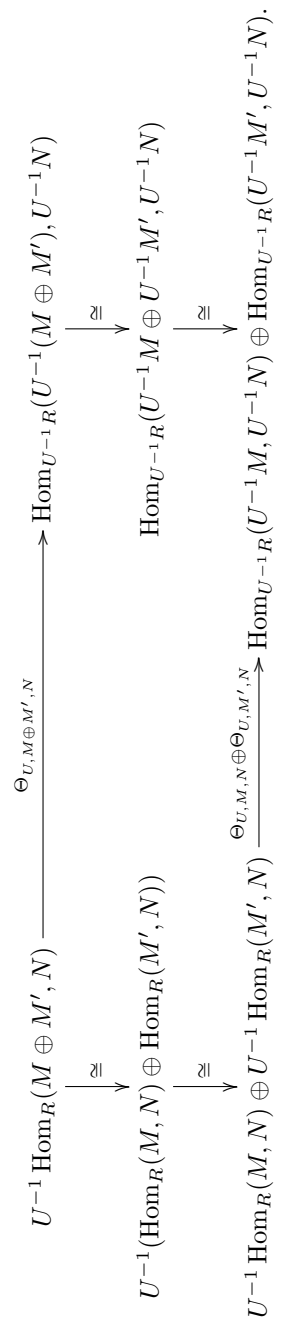
Evaluating the second map at the same element, we have

$$\begin{aligned}(\phi_u + \phi'_{u'})\left(\frac{m}{v}\right) = \phi_u\left(\frac{m}{v}\right) + \phi'_{u'}\left(\frac{m}{v}\right) = \frac{\phi(m)}{uv} + \frac{\phi'(m)}{u'v} = \frac{u'v\phi(m) + uv\phi'(m)}{uu'vv} \\ = \frac{v(u'\phi(m) + u\phi'(m))}{uu'vv} = \frac{u'\phi(m) + u\phi'(m)}{uu'v} = (u'\phi + u\phi')_{uu'}\left(\frac{m}{v}\right).\end{aligned}$$

Since this is true for every element $m/v \in U^{-1}M$, we conclude that $(u'\phi+u\phi')_{uu'} = \phi_u + \phi'_{u'}$, as desired.

(c) We prove this in four steps.

Step 1: If M' is another R-module, then $\Theta_{U,M\oplus M',N}$ is an isomorphism if and only if $\Theta_{U,M,N}$ and $\Theta_{U,M',N}$ are both isomorphisms. Indeed, there is a commutative diagram of $U^{-1}R$-module homomorphisms



Here, the unlabeled vertical isomorphisms are (induced by) the natural ones from Exercises I.3.3(c) and I.4.24. It follows that $\Theta_{U,M\oplus M',N}$ is an isomorphism if and only if $\Theta_{U,M,N} \oplus \Theta_{U,M',N}$ is an isomorphism, that is, if and only if $\Theta_{U,M,N}$ and $\Theta_{U,M',N}$ are both isomorphisms.

Step 2: If $M_1, \dots, M_n$ are R-modules, then $\Theta_{U,\coprod_{i=1}^n M_i,N}$ is an isomorphism if and only if $\Theta_{U,M_i,N}$ is an isomorphism for $i = 1, \dots, n$. Argue by induction on n, using Step 1 as the base case.

Step 3. We show that $\Theta_{U,R^n,N}$ is an isomorphism for $n = 1, 2, \dots$. By Step 2, we need only show that $\Theta_{U,R,N}$ is an isomorphism. Consider the Hom cancellation isomorphisms

$$f\colon \operatorname{Hom}_R(R,N) \xrightarrow{\cong} N \qquad F\colon \operatorname{Hom}_{U^{-1}R}(U^{-1}R, U^{-1}N) \xrightarrow{\cong} U^{-1}N$$

given by $\psi \mapsto \psi(1)$ in each case. There is a commutative diagram of $U^{-1}R$-module homomorphisms

$$\begin{array}{ccc} U^{-1}\operatorname{Hom}_R(R,N) & \xrightarrow{\Theta_{U,R,N}} & \operatorname{Hom}_{U^{-1}R}(U^{-1}R, U^{-1}N) \\ {\scriptstyle\cong}\big\downarrow{\scriptstyle U^{-1}f} & \swarrow_{F}^{\cong} & \\ U^{-1}N. & & \end{array}$$

It follows that $\Theta_{U,R,N}$ is an isomorphism.

Step 4. We verify the general case. (For the sake of our margins, we use the notation $(-)_U$ in place of $U^{-1}(-)$ in this case.) The operator $\operatorname{Hom}_R(-,N)$ is left-exact, so the following sequence is exact:

$$0 \to \operatorname{Hom}_R(M,N) \xrightarrow{\operatorname{Hom}_R(g,N)=g^*} \operatorname{Hom}_R(R^n,N) \xrightarrow{\operatorname{Hom}_R(f,N)=f^*} \operatorname{Hom}_R(R^m,N).$$

Thus, the localized sequence is also exact:

$$0 \to \operatorname{Hom}_R(M,N)_U \xrightarrow{(g^*)_U} \operatorname{Hom}_R(R^n,N)_U \xrightarrow{(f^*)_U} \operatorname{Hom}_R(R^m,N)_U.$$

On the other hand, localizing the sequence (I.5.8.1) yields the next exact sequence of R_U-module homomorphisms

$$R_U^m \xrightarrow{f_U} R_U^n \xrightarrow{g_U} M_U \to 0.$$

Apply the left-exact operator $\operatorname{Hom}_{R_U}(-,N_U) = (-)^\dagger$ to produce the following exact sequence

$$0 \to \operatorname{Hom}_{R_U}(M_U,N_U) \xrightarrow{(g_U)^\dagger} \operatorname{Hom}_{R_U}(R_U^n,N_U) \xrightarrow{(f_U)^\dagger} \operatorname{Hom}_{R_U}(R_U^m,N_U).$$

This explains why the rows in the next diagram are exact:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}_R(M,N)_U & \xrightarrow{(g^*)_U} & \operatorname{Hom}_R(R^n,N)_U & \xrightarrow{(f^*)_U} & \operatorname{Hom}_R(R^m,N)_U \\ & & \Big\downarrow{\scriptstyle\cong}{\scriptstyle\Theta_{U,M,N}} & & \Big\downarrow{\scriptstyle\cong}{\scriptstyle\Theta_{U,R^n,N}} & & \Big\downarrow{\scriptstyle\cong}{\scriptstyle\Theta_{U,R^m,N}} \\ 0 & \longrightarrow & \operatorname{Hom}_{R_U}(M_U,N_U) & \xrightarrow{(g_U)^\dagger} & \operatorname{Hom}_{R_U}(R_U^n,N_U) & \xrightarrow{(f_U)^\dagger} & \operatorname{Hom}_{R_U}(R_U^m,N_U). \end{array}$$

Check that this diagram commutes; see Exercise I.5.16. Case 1 shows that $\Theta_{U,R^n,N}$ and $\Theta_{U,R^m,N}$ are isomorphisms. Chase the diagram to show that this implies that $\Theta_{U,M,N}$ is an isomorphism; see Exercise I.5.16.

(d) Assume that R is noetherian and that M is finitely generated. Since M is finitely generated, there is an integer $n \geqslant 0$ and an R-module epimorphism $g\colon R^n \twoheadrightarrow M$. The kernel $\operatorname{Ker}(g)$ is a submodule of the noetherian module R^n, so it is finitely generated. Thus, there is an integer $m \geqslant 0$ and an R-module epimorphism

$f_1\colon R^m \twoheadrightarrow \operatorname{Ker}(g)$. Let $\epsilon\colon \operatorname{Ker}(g) \to R^n$ be the inclusion map, and check that the following sequence is exact:

$$R^m \xrightarrow{\epsilon f_1} R^n \xrightarrow{g} M \to 0.$$

The desired conclusion now follows from part (c). □

The next result augments Proposition I.2.3.

Proposition I.5.9. *Let R be a commutative ring, and let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules. Let N be a finitely generated R-module. Let $\epsilon\colon \coprod_{\lambda\in\Lambda} M_\lambda \to \prod_{\lambda\in\Lambda} M_\lambda$ denote the canonical inclusion. For each $\mu \in \Lambda$, let $\pi_\mu\colon \prod_{\lambda\in\Lambda} M_\lambda \to M_\mu$ be the canonical surjection.*

(a) *Let $\{\psi_\lambda\colon N \to M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-module homomorphisms such that $\psi_\lambda = 0$ for all but finitely many $\lambda \in \Lambda$. There exists a unique R-module homomorphism $\Psi\colon N \to \coprod_{\lambda\in\Lambda} M_\lambda$ making each of the following diagrams commute*

$$\begin{array}{ccc} N & \xrightarrow{\exists!\Psi} & \coprod_\lambda M_\lambda \\ & \searrow^{\psi_\mu} & \downarrow{\scriptstyle \pi_\mu\epsilon} \\ & & M_\mu \end{array}$$

that is, such that $\pi_\mu\Psi = \psi_\mu$ for each $\mu \in \Lambda$.

(b) *For each $\Psi \in \operatorname{Hom}_R(N, \coprod_\lambda M_\lambda)$, there is a subset $\Lambda' \subseteq \Lambda$ such that $\Lambda \smallsetminus \Lambda'$ is finite and, for all $n \in N$ and all $\lambda \in \Lambda'$ one has $\pi_\lambda(\epsilon(\Psi(n))) = 0$.*

(c) *There exists an isomorphism of R-modules*

$$\theta\colon \operatorname{Hom}_R(N, \coprod_\lambda M_\lambda) \to \coprod_\lambda \operatorname{Hom}_R(N, M_\lambda)$$

given by $\Psi \mapsto (\pi_\lambda\epsilon\Psi)$.

Proof. (a) For each $n \in N$, the sequence $(\psi_\lambda(n))$ has only finitely many non-zero terms since all but finitely many of the ψ_λ are non-zero. Hence, the sequence $(\psi_\lambda(n))$ is in $\coprod_\lambda M_\lambda$. Define $\Psi\colon N \to \coprod_{\lambda\in\Lambda} M_\lambda$ by the formula $\Psi(n) = (\psi_\lambda(n))$. It is straightforward to verify that Ψ is an R-module homomorphism making the desired diagram commute; hence, the existence.

For the uniqueness, one can argue as in the proof of Proposition I.2.3(a). Alternately, let $\Psi\colon N \to \coprod_{\lambda\in\Lambda} M'_\lambda$ be another R-module homomorphism making each of the following diagrams commute

$$\begin{array}{ccc} N & \xrightarrow{\Psi'} & \coprod_\lambda M_\lambda \\ & \searrow^{\psi_\mu} & \downarrow{\scriptstyle \pi_\mu\epsilon} \\ & & M_\mu. \end{array}$$

For each $\mu \in \Lambda$, this yields two commutative diagrams

$$\begin{array}{ccc} N & \xrightarrow{\epsilon\Psi} & \prod_\lambda M_\lambda \\ & \searrow^{\psi_\mu} & \downarrow{\scriptstyle \pi_\mu} \\ & & M_\mu \end{array} \qquad \begin{array}{ccc} N & \xrightarrow{\epsilon\Psi'} & \prod_\lambda M_\lambda \\ & \searrow^{\psi_\mu} & \downarrow{\scriptstyle \pi_\mu} \\ & & M_\mu. \end{array}$$

The uniqueness statement in Proposition I.2.3(a) implies that $\epsilon\Psi = \epsilon\Psi'$, and the fact that ϵ is injective implies that $\Psi = \Psi'$.

(b) Let $\Psi \in \operatorname{Hom}_R(N, \coprod_\lambda M_\lambda)$ be given. Let $n_1, \dots, n_t \in N$ be a generating sequence for N. For $i = 1, \dots, t$ there are only finitely many $\lambda \in \Lambda$ such that the λ-coordinate of $\Psi(n_i)$ is non-zero. That is, there are only finitely many $\lambda \in \Lambda$ such that $\pi_\lambda(\epsilon(\Psi(n_i))) \neq 0$. It follows that there is a subset $\Lambda' \subseteq \Lambda$ such that $\Lambda \smallsetminus \Lambda'$ is finite and, for $i = 1, \dots, t$ and all $\lambda \in \Lambda'$ one has $\pi_\lambda(\epsilon(\Psi(n_i))) = 0$. Because each element $n \in N$ is of the form $\sum_{i=1}^t r_i n_i$, it follows that, for all $n \in N$ and all $\lambda \in \Lambda'$ one has $\pi_\lambda(\epsilon(\Psi(n_i))) = 0$.

(c) Part (b) shows that, for each $\Psi \in \operatorname{Hom}_R(N, \coprod_\lambda M_\lambda)$, there are only finitely many $\lambda \in \Lambda$ such that $\pi_\lambda \epsilon \Psi$ is non-zero. It follows that the map θ is well-defined. It is straightforward to show that θ si an R-module homomorphism. The existence statement in part (a) shows that θ is surjective, and the uniqueness statement in part (a) shows that θ is injective. □

We close this section with a discussion of various module structures.

Remark I.5.10. Let R be a commutative ring, and let M be an R-module. Let $I \subseteq R$ be an ideal such that $IM = 0$. Then M has a well-defined R/I-module structure defined by the formula $\overline{r}m = rm$. Furthermore, M is finitely generated over R if and only if it is finitely generated over R/I.

Let N be a second R-module such that $IN = 0$. Then a function $f\colon M \to N$ is an R/I-module homomorphism if and only if it is an R-module homomorphism. In other words, there is an equality $\operatorname{Hom}_{R/I}(M, N) = \operatorname{Hom}_R(M, N)$.

Remark I.5.11. Let R be a commutative ring, and let M and N be R-modules. Let $I, J \subseteq R$ be ideals such that $IM = 0$ and $JN = 0$. Then $(I+J)\operatorname{Hom}_R(M, N) = 0$. To show this, it suffices to show that $I \operatorname{Hom}_R(M, N) = 0$ and $J \operatorname{Hom}_R(M, N) = 0$. Let $a \in I$ and $b \in J$ and $f \in \operatorname{Hom}_R(M, N)$. For each $m \in M$, we have

$$(af)(m) = f(am) = f(0) = 0$$
$$(bf)(m) = b(f(m)) = 0$$

and it follows that $af = 0 = bf$. This gives the desired result.

Because of this, Remark I.5.10 implies that $\operatorname{Hom}_R(M, N)$ has the structure of an $R/(I+J)$-module, the structure of an R/I-module, and the structure of an R/J-module via the formula $\overline{r}f = rf$. Furthermore, $\operatorname{Hom}_R(M, N)$ is finitely generated over R if and only if it is finitely generated over R/I, and similarly over R/J and $R/(I + J)$.

Exercises.

Exercise I.5.12. Complete the verification of the claims of Example I.5.4.

Exercise I.5.13. Verify the claims of Fact I.5.5.

Exercise I.5.14. Verify the claims of Fact I.5.6.

Exercise I.5.15. Verify the claims of Fact I.5.7.

Exercise I.5.16. Let R be a commutative ring, and let $U \subseteq R$ be a multiplicatively closed subset. Let $g\colon M \to M'$ and $h\colon N \to N'$ be R-module homomorphisms.

(a) Prove that the following diagram commutes:

$$\begin{array}{ccc} U^{-1}\operatorname{Hom}_R(M',N) & \xrightarrow{\Theta_{U,M',N}} & \operatorname{Hom}_{U^{-1}R}(U^{-1}M',U^{-1}N) \\ \Big\downarrow{\scriptstyle U^{-1}\operatorname{Hom}_R(g,N)} & & \Big\downarrow{\scriptstyle \operatorname{Hom}_{U^{-1}R}(U^{-1}g,U^{-1}N)} \\ U^{-1}\operatorname{Hom}_R(M,N) & \xrightarrow{\Theta_{U,M,N}} & \operatorname{Hom}_{U^{-1}R}(U^{-1}M,U^{-1}N). \end{array}$$

(b) Prove that the following diagram commutes:

$$\begin{array}{ccc} U^{-1}\operatorname{Hom}_R(M,N) & \xrightarrow{\Theta_{U,M,N}} & \operatorname{Hom}_{U^{-1}R}(U^{-1}M,U^{-1}N) \\ \Big\downarrow{\scriptstyle U^{-1}\operatorname{Hom}_R(M,h)} & & \Big\downarrow{\scriptstyle \operatorname{Hom}_{U^{-1}R}(U^{-1}M,U^{-1}h)} \\ U^{-1}\operatorname{Hom}_R(M,N') & \xrightarrow{\Theta_{U,M,N'}} & \operatorname{Hom}_{U^{-1}R}(U^{-1}M,U^{-1}N'). \end{array}$$

(c) Complete the proof of Proposition I.5.8.

Exercise I.5.17. Let R be a commutative ring. Let M be an R-module, and let $g\colon N \to N'$ be an R-module isomorphism. Prove that the following maps are isomorphisms:

$$\operatorname{Hom}_R(g,M)\colon \operatorname{Hom}_R(N',M) \to \operatorname{Hom}_R(N,M)$$
$$\operatorname{Hom}_R(M,g)\colon \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,N').$$

Exercise I.5.18. Let R be a commutative ring. Let $F\colon M \to M'$ and $g\colon N \to N'$ be R-module homomorphisms. Verify the following equalities

$$\operatorname{Hom}_R(g,M')\operatorname{Hom}_R(N',F) = \operatorname{Hom}_R(N,F)\operatorname{Hom}_R(g,M)$$
$$\operatorname{Hom}_R(F,N')\operatorname{Hom}_R(M',g) = \operatorname{Hom}_R(M,g)\operatorname{Hom}_R(F,N)$$

and rewrite each one in terms of a commutative diagram.

Exercise I.5.19. Complete the proof of Proposition I.5.9.

Exercise I.5.20. Let R be a commutative ring, and let M and N be R-modules. Prove that, if M is finitely presented and N is finitely generated, then $\operatorname{Hom}_R(M,N)$ is finitely generated.

Exercise I.5.21. Verify the statements in Remark I.5.10.

CHAPTER II

Tensor Products September 8, 2009

II.1. Existence and Uniqueness

This section is devoted to the basic properties of tensor products.

Remark II.1.1. Let R be a commutative ring. The function $\mu\colon R\times R\to R$ given by $\mu(r,s)=rs$ is not as well-behaved as one might like. For instance, it is not an R-module homomorphism:

$$\mu((1,0)+(0,1))=\mu(1,1)=1\neq 0=\mu(1,0)+\mu(0,1).$$

In a sense, the tensor product fixes this problem.

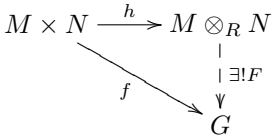
Definition II.1.2. Let R be a commutative ring. Let M, N, and G be R-modules. A function $f\colon M\times N\to G$ is *R-bilinear* if

$$\begin{aligned} f(m+m',n)&=f(m,n)+f(m',n)\\ f(m,n+n')&=f(m,n)+f(m,n')\\ f(rm,n)&=rf(m,n)=f(m,rn)\end{aligned}$$

for all $m,m'\in M$ all $n,n'\in N$ and all $r\in R$.

Example II.1.3. Let R be a commutative ring. The function $\mu\colon R\times R\to R$ given by $\mu(r,s)=rs$ is the prototype of an R-bilinear function.

Definition II.1.4. Let R be a commutative ring, and let M and N be R-modules. A *tensor product* of M and N over R is an R-module $M\otimes_R N$ equipped with an R-bilinear function $h\colon M\times N\to M\otimes_R N$ satisfying the following universal mapping property: For every R-module G and every R-bilinear function $f\colon M\times N\to G$, there exists a unique R-module homomorphism $F\colon M\otimes_R N\to G$ making the following diagram commute



that is, such that $Fh=f$. A *simple tensor* in $M\otimes_R N$ is an element of the form $m\otimes n=h(m,n)$.

Here is the existence of the tensor product.

Theorem II.1.5. *Let R be a commutative ring. If M and N are R-modules, then $M\otimes_R N$ exists.*

PROOF. Consider $R^{(M\times N)}$, the free R-module with basis $M \times N$. For $m \in M$ and $n \in N$, let $(m, n) \in R^{(M\times N)}$ denote the corresponding basis vector. Set

$$H = \left\langle \begin{array}{c} (m+m',n)-(m,n)-(m',n) \\ (m,n+n')-(m,n)-(m,n') \\ (rm,n)-r(m,n) \\ (m,rn)-r(m,n) \end{array} \middle| \begin{array}{c} m,m' \in M \\ n,n' \in N \\ r \in R \end{array} \right\rangle \subseteq R^{(M\times N)}.$$

Set $M \otimes_R N = R^{(M\times N)}/H$ and, for $m \in M$ and $n \in N$ write

$$m \otimes n = [(m, n)] = (m, n) + H \in R^{(M\times N)}/H = M \otimes_R N.$$

Define $h\colon M \times N \to M \otimes_R N$ to be the composition

$$M \times N \xrightarrow{\varepsilon} R^{(M\times N)} \xrightarrow{\pi} R^{(M\times N)}/H = M \otimes_R N$$

that is, by the rule $h(m, n) = m \otimes n$.

It is straightforward to show that h is well-defined and R-bilinear. For example:

$$\begin{aligned} h(m+m',n) &= (m+m') \otimes n \\ &= [(m+m',n)] \\ &= [(m,n)] + [(m',n)] \\ &= m \otimes n + m' \otimes n \\ &= h(m,n) + h(m',n). \end{aligned}$$

In terms of tensors, the R-bilinearity of h reads as

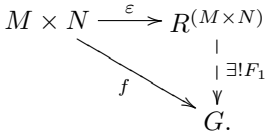
$$\begin{aligned} (m+m') \otimes n &= m \otimes n + m' \otimes n \\ m \otimes (n+n') &= m \otimes n + m \otimes n' \\ (rm) \otimes n &= r(m \otimes n) = m \otimes (rn) \end{aligned}$$

Note also that elements of $M \otimes_R N$ are of the form

$[\sum_i r_i(m_i, n_i)] = \sum_i r_i[(m_i, n_i)] = \sum_i[(r_i m_i, n_i)] = \sum_i((r_i m_i)\otimes n_i) = \sum_i(m'_i \otimes n_i)$

with $r_i \in R$ and $m_i \in M$ and $n_i \in N$; here $m'_i = r_i m_i$.[1]

To see that $M \otimes_R N$ satisfies the desired universal mapping property, let G be an R-module and let $f\colon M \times N \to G$ be an R-bilinear function. Use Exercise I.3.4(a) to see that there is a unique abelian group homomorphism $F_1\colon R^{(M\times N)} \to G$ such that $F_1(m, n) = f(m, n)$ for all $m \in M$ and all $n \in N$, that is, such that the following diagram commutes



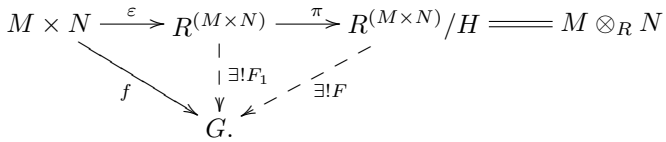
From the proof of Exercise I.3.4(a), we have

$$F_1(\textstyle\sum_i r_i(m_i, n_i)) = \sum_i r_i f(m_i, n_i).$$

Use this formula to check that each generator of H is in $\mathrm{Ker}(F_1)$; this will use the R-bilinearity of f. It follows that $H \subseteq \mathrm{Ker}(F_1)$, so Exercise I.1.7 implies that

[1] We'll see later that, usually, there are elements of $M \otimes_R N$ that cannot be written as simple tensors, that is, are not of the form $m \otimes n$.

there exists a unique R-module homomorphism $F\colon R^{(M\times N)}/H \to G$ making the right-hand triangle in the next diagram commute



Thus, we see that the desired homomorphism F exists. To see that it is unique, suppose that $F'\colon M \otimes_R N \to G$ is a second R-module homomorphism such that $F'h = f$. Each element of $M \otimes_R N$ is of the form $\xi = \sum_i m_i \otimes n_i$ for some elements $m_i \in M$ and $n_i \in N$. It follows that

$$\begin{aligned} F'(\xi) &= F'(\textstyle\sum_i m_i \otimes n_i) = \sum_i F'(m_i \otimes n_i) = \sum_i F'(h(m_i, n_i)) \\ &= \textstyle\sum_i F(h(m_i, n_i)) = \sum_i F(m_i \otimes n_i) = F(\sum_i m_i \otimes n_i) = F(\xi) \end{aligned}$$

as desired. □

Example II.1.6. Let R be a commutative ring, and let M and N be R-modules. The computations in the proof of Theorem II.1.5 show that

$$(\textstyle\sum_i r_i m_i) \otimes n = \sum_i (r_i m_i) \otimes n = \sum_i m_i \otimes (r_i n)$$

for all $m_i \in M$, all $r_i \in R$ and all $n \in N$. Other formulas hold similarly. In particular, for $r_i \in \mathbb{Z}$, we have

$$\textstyle\sum_i r_i(m_i \otimes n_i) = \sum_i ((r_i m_i) \otimes n_i) = \sum_i m_i' \otimes n_i$$

where $m_i' = r_i m_i$. In particular, $M \otimes_R N$ is generated as an R-module by the set of simple tensors $\{m \otimes n \mid m \in M, n \in N\}$.

The additive identity in $M \otimes_R N$ is $0_{M\otimes N} = 0_M \otimes 0_N$. This can be written several (seemingly) different ways. For instance, for each $n \in N$, we have

$$0_M \otimes n = (0_M 0_R) \otimes n = 0_M \otimes (0_R n) = 0_M \otimes 0_N.$$

Similarly, for all $m \in M$, we have $m \otimes 0_N = 0_M \otimes 0_N$.

Remark II.1.7. Let R be a commutative ring, and let M and N be R-modules. It should be reiterated that there are more elements in $M \otimes_R N$ than the simple tensors $m \otimes n$. General elements of $M \otimes_R N$ are of the form $\sum_i m_i \otimes n_i$, as was shown in Example II.1.6. However, certain properties of $M \otimes_R N$ are determined by their restrictions to the simple tensors, as we see next.

Lemma II.1.8. *Let R be a commutative ring. Let M, N and G be R-modules, and let $\gamma, \delta\colon M \otimes_R N \to G$ be R-module homomorphisms.*

(a) *$M \otimes_R N = 0$ if and only if $m \otimes n = 0$ for all $m \in M$ and all $n \in N$.*
(b) *$\gamma = \delta$ if and only if $\gamma(m \otimes n) = \delta(m \otimes n)$ for all $m \in M$ and all $n \in N$.*
(c) *If $G = M \otimes_R N$, then $\gamma = \mathbb{1}_{M\otimes_R N}$ if and only if $\gamma(m \otimes n) = m \otimes n$ for all $m \in M$ and all $n \in N$.*
(d) *$\gamma = 0$ if and only if $\gamma(m \otimes n) = 0$ for all $m \in M$ and all $n \in N$.*

Proof. Part (a) follows from the fact that every element of $M \otimes_R N$ is of the form $\sum_i m_i \otimes n_i$.

Part (b) can be proved similarly, or by using the uniqueness statement in the universal property.

Part (c) can be proved similarly, or by using the uniqueness statement in the universal property, or as the special case $\delta = \mathbb{1}_{M\otimes_R N}$ of part (b).

Part (d) can be proved similarly, or by using the uniqueness statement in the universal property, or as the special case $\delta = 0$ of part (b). □

When proving properties about tensor products, we very rarely use the construction. Usually, we use the universal property, as in the following example. The following properties are sometimes referred to as tensor cancellation.

Example II.1.9. Let R be a commutative ring, and let M and N be R-modules. There are R-module isomorphisms

$$F\colon M\otimes_R R \xrightarrow{\cong} M \qquad \text{and} \qquad G\colon R\otimes_R N \xrightarrow{\cong} N$$

such that $F(m\otimes r) = mr$ and $G(r\otimes n) = rn$. (The inverses are given by $F^{-1}(m) = m\otimes 1$ and $G^{-1}(n) = 1\otimes n$.) In particular, we have $M\otimes_R R\cong M$ and $R\otimes_R N\cong N$ and $R\otimes_R R\cong R$.

We will verify the claim for $M\otimes_R R$. The map $f\colon M\times R\to M$ given by $f(m,r) = mr$ is R-bilinear. Hence, the universal property yields a unique R-module homomorphism $F\colon M\otimes_R R\to M$ such that $F(m\otimes r) = rm$ for all $m\in M$ and $r\in R$. We will show that F is bijective. The main point is the following computation in $M\otimes_R R$

$$\textstyle\sum_i(m_i\otimes r_i) = \sum_i(m_i\otimes(r_i1)) = \sum_i((r_im_i)\otimes 1) = (\sum_i r_im_i)\otimes 1$$

which shows that every element of $M\otimes_R R$ is of the form $m\otimes 1$.

The map F is surjective because $m = F(m\otimes 1)$.

The map F is injective because $0 = F(m\otimes 1)$ implies $0 = F(m\otimes 1) = m\cdot 1 = m$ implies $0 = 0\otimes 1 = m\otimes 1$.

The map $F'\colon M\to M\otimes_R R$ given by $F'(m) = m\otimes 1$ is well-defined because it is the composition hf where $f\colon M\to M\times R$ is given by $m\mapsto(m,1)$ and $h\colon M\times R\to M\otimes_R R$ is the universal bilinear map. It is straightforward to show that F' is an R-module homomorphism. Also, for $m\in M$ we have

$$F(F'(m)) = F(m\otimes 1) = m \qquad \text{and} \qquad F'(F(m\otimes 1)) = F'(m) = m\otimes 1.$$

It follows that $F' = F^{-1}$.

Remark II.1.10. Let R be a commutative ring. It should be noted that other tensor products of R with itself, like $R\otimes_{\mathbb{Z}} R$ are not usually so simple.

Proposition II.1.11. *Let R be a commutative ring, and let M and N be R-modules. Fix subsets $A\subseteq M$ and $B\subseteq N$, and set $C = \{a\otimes b\in M\otimes_R N \mid a\in A \text{ and } b\in B\}$.*

(a) *If $RA = M$ and $RB = N$, then $RC = M\otimes_R N$.*
(b) *If M and N are finitely generated, then so is $M\otimes_R N$.*

Proof. (a) Fix $m\in M$ and $n\in N$. Since $RB = N$, we can write $n = \sum_j r_jb_j$ for some $r_j\in R$ and $b_j\in B$. For each j, use the condition $RA = M$ to write $r_jm = \sum_i s_{ij}a_{ij}$ for some $s_{ij}\in R$ and $a_{ij}\in A$. (Since all the sums are finite, we

can take the same index set for i for each j.) This yields

$$\begin{aligned} m \otimes n &= m \otimes (\textstyle\sum_j r_j b_j) \\ &= \textstyle\sum_j ((r_j m) \otimes b_j) \\ &= \textstyle\sum_j ((\sum_i s_{ij} a_{ij}) \otimes b_j) \\ &= \textstyle\sum_j \sum_i s_{ij} (a_{ij} \otimes b_j) \end{aligned}$$

Since each $a_{ij} \otimes b_j \in C$, we have each simple tensor in RC, so $M \otimes_R N \subseteq RC$. The reverse containment is clear.

Part (b) follows from part (a): if A and B are finite generating sets for M and N, then C is a finite generating set for $M \otimes_R N$. □

Exercises.

Exercise II.1.12. (Uniqueness of the tensor product.) Let R be a commutative ring. If M and N are R-modules, then $M \otimes_R N$ is unique up to R-module isomorphism.

Exercise II.1.13. Let R be a commutative ring. Let Λ be a set, and let M and N be R-modules.

(a) Prove that there are unique R-module homomorphisms $F\colon M \otimes_R R^{(\Lambda)} \to M^{(\Lambda)}$ and $G\colon M^{(\Lambda)} \to M \otimes_R R^{(\Lambda)}$ such that $F(m \otimes (r_\lambda)) = (mr_\lambda)$ and $G(m_\lambda) = \sum_\lambda m_\lambda \otimes \mathbf{e}_\lambda$. Prove that F and G are inverse isomorphisms, and hence we have $M \otimes_R R^{(\Lambda)} \cong M^{(\Lambda)}$.
(b) Formulate and prove the analogous result for $R^{(\Lambda)} \otimes_R N$ and $N^{(\Lambda)}$.

II.2. Functoriality and Base-Change

Here is the functoriality of tensor product.

Proposition II.2.1. *Let R be a commutative ring, and consider R-module homomorphisms $\alpha\colon M \to M'$ and $\alpha'\colon M' \to M''$ and $\beta\colon N \to N'$ and $\beta'\colon N' \to N''$.*

(a) *There exists a unique R-module homomorphism $\alpha \otimes_R \beta\colon M \otimes_R N \to M' \otimes_R N'$ such that $(\alpha \otimes_R \beta)(m \otimes n) = \alpha(m) \otimes_R \beta(n)$ for all $m \in M$ and all $n \in N$.*
(b) *The following diagram commutes*

$$\begin{array}{ccc} M \otimes_R N & \xrightarrow{\alpha \otimes_R \beta} & M' \otimes_R N' \\ & \searrow^{(\alpha'\alpha) \otimes_R (\beta'\beta)} & \downarrow{\scriptstyle \alpha' \otimes_R \beta'} \\ & & M'' \otimes_R N'' \end{array}$$

that is, we have $(\alpha' \otimes_R \beta')(\alpha \otimes_R \beta) = (\alpha'\alpha) \otimes_R (\beta'\beta)$.

Proof. (a) Use the universal mapping property to show that $\alpha \otimes_R \beta$ exists. Use Lemma II.1.8(b) to show that $\alpha \otimes_R \beta$ is unique.

(b) This follows from direct computation using Lemma II.1.8(b). □

Notation II.2.2. Continue with the notation of Proposition II.2.1. We write

$$M \otimes_R \beta = \mathbb{1}_M \otimes_R \beta\colon M \otimes_R N \to M \otimes_R N'$$
$$\alpha \otimes_R N = \alpha \otimes_R \mathbb{1}_N\colon M \otimes_R N \to M' \otimes_R N.$$

In other words, we have

$$(M \otimes_R \beta)(m \otimes n) = m \otimes \beta(n) \qquad (\alpha \otimes_R N)(m \otimes n) = \alpha(m) \otimes n$$

for all $m \in M$ and all $n \in N$. Part (b) of the proposition then reads as

$$(\alpha' \otimes_R N)(\alpha \otimes_R N) = (\alpha'\alpha) \otimes_R N \qquad (M \otimes_R \beta')(M \otimes_R \beta) = M \otimes_R (\beta'\beta).$$

Example II.2.3. Continue with the notation of Proposition II.2.1. The following diagram commutes

$$\begin{array}{ccc} M \otimes_R N & \xrightarrow{\alpha \otimes_R N} & M' \otimes_R N \\ {\scriptstyle M \otimes_R \beta}\downarrow & \searrow^{\alpha \otimes_R \beta} & \downarrow{\scriptstyle M' \otimes_R \beta} \\ M \otimes_R N' & \xrightarrow{\alpha \otimes_R N'} & M' \otimes_R N'. \end{array}$$

We verify the commutativity of the lower triangle:

$$\alpha \otimes_R \beta = (\alpha \mathbb{1}_M) \otimes_R (\mathbb{1}_{N'} \beta) = (\alpha \otimes_R \mathbb{1}_{N'})(\mathbb{1}_M \otimes_R \beta) = (\alpha \otimes_R N')(M \otimes_R \beta).$$

The lower triangle is dealt with similarly.

Using Lemma II.1.8(c) we have $\mathbb{1}_M \otimes_R \mathbb{1}_N = \mathbb{1}_{M \otimes_R N}$.

Using Example II.1.6 and Lemma II.1.8(d) we have $\alpha \otimes_R 0 = 0$ and $0 \otimes_R \beta = 0$.

Fix an element $r \in R$. Let $\mu_r^M \colon M \to M$ be given by $m \mapsto rm$. Such a "multiplication-map" is a *homothety*. Using Example II.1.6 (or Lemma II.1.8(b)) we have

$$\mu_r^M \otimes_R \mu_s^N = \mu_{rs}^{M \otimes_R N} \colon M \otimes_R N \to M \otimes_R N$$

that is, the tensor product of homotheties is a homothety. In particular, we have

$$\mu_r^M \otimes_R N = M \otimes_R \mu_r^N = \mu_r^{M \otimes_R N} \colon M \otimes_R N \to M \otimes_R N.$$

Remark II.2.4. Let R be a commutative ring, and let M and N be R-modules. Let $I, J \subseteq R$ be ideals such that $IM = 0$ and $JN = 0$. Then $(I+J)(M \otimes_R N) = 0$. To show this, it suffices to show that $I(M \otimes_R N) = 0$ and $J(M \otimes_R N) = 0$. Let $a \in I$, and let $\mu_a^M \colon M \to M$ denote the homothety $m \mapsto am$. Our assumption implies that $\mu_a^M = 0 = \mu_0^M$. Example II.2.3 implies that the induced map $M \otimes_R N \to M \otimes_R N$ is given by multiplication by a, and by multiplication by 0. That is, multiplication by a on $M \otimes_R N$ is 0. This implies that $a(M \otimes_R N) = 0$, and hence $I(M \otimes_R N) = 0$. The proof that $J(M \otimes_R N) = 0$ is similar.

Because of this, Remark I.5.10 implies that $M \otimes_R N$ has the structure of an $R/(I+J)$-module, the structure of an R/I-module, and the structure of an R/J-module via the formula $\overline{r}\xi = r\xi$. Furthermore, $M \otimes_R N$ is finitely generated over R if and only if it is finitely generated over R/I, and similarly over R/J and $R/(I+J)$.

Next we talk about base-change. First, we describe restriction of scalars.

Remark II.2.5. Let $\varphi \colon R \to S$ be a homomorphism of commutative rings. Every S-module N has a natural R-module structure defined as $rn = \varphi(r)n$. (We say that this R-module structure is given by *restriction of scalars* along φ.[2]) In particular, the ring S is an R-module by the action $rs = \varphi(r)s$.

[2]The terminology is explained as follows: If R is a subring of S and φ is the inclusion map, then the R-module structure on N is obtained by restricting the S-module structure to the smaller ring R. Since the elements of S are called scalars when they are multiplied against elements of N, we are restricting the class of scalars from S to the smaller ring R.

Under this operation, every S-module homomorphism $\alpha\colon N \to N'$ is also an R-module homomorphism. Also, the map φ is an R-module homomorphism.

Base-change is a special case of the following; see Proposition II.2.7.

Proposition II.2.6. *Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Let M be an R-module, and let N be an S-module.*

(a) *The tensor product $N \otimes_R M$ has a well-defined S-module structure given by $s(\sum_i n_i \otimes m_i) = \sum_i (sn_i) \otimes m_i$.*
(b) *Furthermore, this S-module structure is compatible with the R-module structure on $N \otimes_R M$ via restriction of scalars: for all $r \in R$ and all $n \in N$ and all $m \in M$, we have $r(n \otimes m) = \varphi(r)(n \otimes m)$.*
(c) *Let $f\colon M \to M'$ be an R-module homomorphism. Then the induced homomorphism $N \otimes_R f\colon N \otimes_R M \to N \otimes_R M'$ is an S-module homomorphism.*
(d) *Let $g\colon N \to N'$ be an S-module homomorphism. Then the induced homomorphism $g \otimes_R M\colon N \otimes_R M \to N' \otimes_R M$ is an S-module homomorphism.*

Proof. (a) First, we use the universal property to show that the operation

$$s(\textstyle\sum_i n_i \otimes m_i) = \sum_i (sn_i) \otimes m_i$$

is well-defined. Fix an element $s \in S$. Let $\mu_s\colon N \to N$ be the homothety given by $\mu_s(n) = sn$. This is a well-defined S-module homomorphism. Considering N as an R-module by restriction of scalars, the map μ_s is also an R-module homomorphism; see Remark II.2.5. The map $\mu_s \otimes_R M\colon N \otimes_R M \to N \otimes_R M$ is a well-defined R-module homomorphism by Proposition II.2.1(a). It is given on simple tensors by

$$(\mu_s \otimes_R M)(n \otimes m) = (\mu_s(n)) \otimes m = (sn) \otimes m$$

so the fact that this map is an R-module homomorphism implies that

$$(\mu_s \otimes_R M)(\textstyle\sum_i s_i \otimes m_i) = \sum_i (\mu_s \otimes_R M)(n_i \otimes m_i) = \sum_i (sn_i) \otimes m_i.$$

This shows that the desired action is well-defined.

The fact that $\mu_s \otimes_R M$ is additive implies $s(\xi+\zeta) = s\xi + s\zeta$ for all $\xi, \zeta \in N \otimes_R M$. The verification of the fact that this action satisfies the axioms for an S-module is tedious. For instance, for $s, s' \in S$, we have

$$\begin{aligned}
&(s+s')\textstyle\sum_i n_i \otimes m_i \\
&\quad = \textstyle\sum_i ((s+s')n_i) \otimes m_i && \text{(definition of the action on } N \otimes_R M) \\
&\quad = \textstyle\sum_i (sn_i + s'n_i) \otimes m_i && \text{(distributivity in } N) \\
&\quad = \textstyle\sum_i [(sn_i) \otimes m_i + (s'n_i) \otimes m_i] && \text{(distributivity in } N \otimes_R M) \\
&\quad = \textstyle\sum_i (sn_i) \otimes m_i + \sum_i (s'n_i) \otimes m_i && \text{(associativity in } N \otimes_R M) \\
&\quad = s\textstyle\sum_i n_i \otimes m_i + s'\sum_i n_i \otimes m_i && \text{(definition of the action on } N \otimes_R M)
\end{aligned}$$

so $(s+s')\xi = s\xi + s'\xi$ for all $s, s' \in S$ and all $\xi \in N \otimes_R M$.

(b) We have

$$\begin{aligned}
\varphi(r)(n \otimes m) &= (\varphi(r)n) \otimes m && \text{(definition of the } S\text{-action on } N \otimes_R M) \\
&= (rn) \otimes m && \text{(definition of the } R\text{-action on } N) \\
&= r(n \otimes m) && \text{(definition of the } R\text{-action on } N \otimes_R M)
\end{aligned}$$

(c) We have

$$\begin{aligned}(N \otimes_R f)(s(n \otimes m)) &= (N \otimes_R f)((sn) \otimes m) && \text{(definition of } S\text{-action on } N \otimes_R M)\\ &= (sn) \otimes f(m) && \text{(definition of } N \otimes_R f)\\ &= s(n \otimes f(m)) && \text{(definition of } S\text{-action on } N \otimes_R M)\\ &= s((N \otimes_R f)(n \otimes m)) && \text{(definition of } N \otimes_R f)\end{aligned}$$

(d) Similar to part (c). □

Base-change is, in a sense, reverse to the notion of restriction of scalars. This is also known as *extension of scalars*.

Proposition II.2.7. *Let $\varphi\colon R \to S$ be a homomorphism of commutative rings, and let M be an R-module.*

(a) *The tensor product $S \otimes_R M$ has a well-defined S-module structure given by $s(\sum_i s_i \otimes m_i) = \sum_i (ss_i) \otimes m_i$.*

(b) *Furthermore, this S-module structure is compatible with the R-module structure on $S \otimes_R M$ via restriction of scalars: for all $r \in R$ and all $s \in S$ and all $m \in M$, we have $r(s \otimes m) = \varphi(r)(s \otimes m)$.*

(c) *The function $\varphi_M\colon M \to S \otimes_R M$ given by $m \mapsto 1_S \otimes m$ is a well-defined R-module homomorphism making the following diagram commute*

$$\begin{array}{ccccc} M & \xrightarrow{\cong} & R \otimes_R M & \xrightarrow{\cong} & M \\ & {\scriptstyle \varphi_M}\searrow & \downarrow{\scriptstyle \varphi \otimes_R M} & \swarrow{\scriptstyle \varphi_M} & \\ & & S \otimes_R M & & \end{array} \qquad \text{(II.2.7.1)}$$

where the unspecified isomorphisms are from Example II.1.9.

(d) *Let $f\colon M \to M'$ be an R-module homomorphism. Then the induced map on tensor products $S \otimes_R f\colon S \otimes_R M \to S \otimes_R M'$ is an S-module homomorphism making the following diagram commute:*

$$\begin{array}{ccc} M & \xrightarrow{f} & M' \\ {\scriptstyle \varphi_M}\downarrow & & \downarrow{\scriptstyle \varphi_{M'}} \\ S \otimes_R M & \xrightarrow{S \otimes_R f} & S \otimes_R M' \end{array}$$

We say that the S-module $S \otimes_R M$ is obtained from the R-module M by base-change *or* extension of scalars *along φ.*[3]

Proof. (a) This is the special case $N = S$ in Proposition II.2.6(a).

(b) This is the special case $N = S$ in Proposition II.2.6(b).

(c) The map $H\colon M \to R \otimes_R M$ given by $m \mapsto 1 \otimes m$ is an isomorphism of R-modules. It is routine to show that the composition $(\varphi \otimes_R M) \circ H\colon M \to S \otimes_R M$ is given by $m \mapsto 1_S \otimes m$. This is exactly the rule describing φ_M. Since the maps $\varphi \otimes_R M$ and H are well-defined R-module homomorphisms, it follows that the

[3]The terminology is explained as follows. Since M is an R-module, one sometimes refers to R as the base for M. Since the tensored module $S \otimes_R M$ is a module over the different ring S, we have changed the base of the module M; in other words, we have performed a base-change. The terminology "extension of scalars" is explained similarly: the original module M has scalars in the smaller ring R, while the new module $S \otimes_R M$ has scalars in the larger ring S, so we have extended the range of scalars from the smaller ring to the larger ring.

composition φ_M is a well-defined R-module homomorphism making the left-most triangle in the diagram (II.2.7.1) commute. It is straightforward to show that the right-most triangle in the diagram (II.2.7.1) also commutes: If $F\colon R\otimes_R M\to M$ is the natural isomorphism, then

$$\begin{aligned}\varphi_M(F(r\otimes m)) &= \varphi_M(rm) = 1_S\otimes(rm) = (r1_S)\otimes m\\ &= (\varphi(r)1_S)\otimes m = \varphi(r)\otimes m = (\varphi\otimes_R M)(r\otimes m)\end{aligned}$$

as desired.

(d) The fact that $S\otimes_R f\colon S\otimes_R M\to S\otimes_R M'$ is an S-module homomorphism follows from Proposition II.2.6(c) using $N=S$. The commutativity of the diagram is a straightforward consequence of the definitions. □

Next we talk about the connection between tensor products and localization. First a definition.

Definition II.2.8. Let R be a commutative ring. A sequence of R-module homomorphism

$$N_\bullet = \cdots \xrightarrow{f_{i+1}} N_i \xrightarrow{f_i} N_{i-1} \xrightarrow{f_{i-1}} \cdots$$

is *exact* if $\operatorname{Im}(f_{i+1}) = \operatorname{Ker}(f_i)$ for all $i\in\mathbb{Z}$.

An R-module M is *flat* if, for every exact sequence of R-module homomorphisms

$$N_\bullet = \cdots \xrightarrow{f_{i+1}} N_i \xrightarrow{f_i} N_{i-1} \xrightarrow{f_{i-1}} \cdots$$

the tensored sequence

$$M\otimes_R N_\bullet = \cdots \xrightarrow{M\otimes_R f_{i+1}} M\otimes_R N_i \xrightarrow{M\otimes_R f_i} M\otimes_R N_{i-1} \xrightarrow{M\otimes_R f_{i-1}} \cdots$$

is exact.

We'll talk about flatness more once we have the right-exactness of tensor product. For now, we show that every localization is a tensor product.

Proposition II.2.9. *Let R be a commutative ring. Let $U\subseteq R$ be a multiplicatively closed subset, and let M be an R-module.*

(a) *Every element of $(U^{-1}R)\otimes_R M$ is of the form $\frac{1}{u}\otimes m$ for some $u\in U$ and $m\in M$.*

(b) *There is an $U^{-1}R$-module isomorphism $F\colon (U^{-1}R)\otimes_R M\to U^{-1}M$ given by $F\big(\frac{1}{u}\otimes m\big) = \frac{m}{u}$ and such that $F^{-1}(\frac{m}{u}) = \frac{1}{u}\otimes m$.*

(c) *For each R-module homomorphism $g\colon M\to M'$, there is a commutative diagram*

$$\begin{array}{ccc}(U^{-1}R)\otimes_R M & \xrightarrow{(U^{-1}R)\otimes_R g} & (U^{-1}R)\otimes_R M'\\ \cong\big\downarrow F & & \cong\big\downarrow F'\\ U^{-1}M & \xrightarrow{U^{-1}g} & U^{-1}M'\end{array}$$

where the vertical maps are the isomorphisms from part (b).

(d) *$U^{-1}R$ is a flat R-module.*

(e) *$\mathbb{Q}$ is a flat $\mathbb{Z}$-module.*

PROOF. (a) Fix an element $\sum_i \frac{r_i}{u_i} \otimes m_i$. Set $u = \prod_i u_i$ and $u'_i = \prod_{j \neq i} u_j$. Then $u = u'_i u_i$, so

$$\textstyle\sum_i \frac{r_i}{u_i} \otimes m_i = \sum_i \frac{u'_i r_i}{u'_i u_i} \otimes m_i = \sum_i \frac{1}{u} \otimes (u'_i r_i m_i) = \frac{1}{u} \otimes \left(\sum_i u'_i r_i m_i\right).$$

(b) The universal mapping property for tensor products shows that the map $F\colon (U^{-1}R) \otimes_R M \to U^{-1}M$ given by $F\big(\frac{r}{u} \otimes m\big) = \frac{rm}{u}$ is a well-defined R-module homomorphism. In fact, Proposition II.2.7(a) shows that $(U^{-1}R) \otimes_R M$ is an $U^{-1}R$-module. Also, $U^{-1}M$ is an $U^{-1}R$-module, and it is straightforward to show that the map F is an $U^{-1}R$-module homomorphism.

The map F is surjective because $\frac{m}{u} = F\big(\frac{1}{u} \otimes m\big)$.

To see that F is injective, fix $\frac{1}{u} \otimes m \in \operatorname{Ker}(F)$. (This uses part (a).) Then $0 = F\big(\frac{1}{u} \otimes m\big) = \frac{m}{u}$ implies that there exists an element $u' \in U$ such that $u'm = 0$. Hence, we have

$$\frac{1}{u} \otimes m = \frac{u'}{uu'} \otimes m = \frac{1}{uu'} \otimes (u'm) = \frac{1}{uu'} \otimes (0) = 0.$$

(c) We have $U^{-1}g\big(\frac{m}{u}\big) = \frac{g(m)}{u}$, so

$$\begin{aligned} F'((R_U \otimes_R g)(\tfrac{r}{u} \otimes m)) &= F'(\tfrac{r}{u} \otimes g(m)) \\ &= \frac{rg(m)}{u} \\ &= \frac{g(rm)}{u} \\ &= U^{-1}g(\tfrac{rm}{u}) \\ &= U^{-1}g(F(\tfrac{r}{u} \otimes m)) \end{aligned}$$

(d) Let $N_\bullet$ be an exact sequence of R-module homomorphisms as in Definition II.2.8. Since localization is exact, we know that $U^{-1}N_\bullet$ is exact. Parts (b) and (c) show $U^{-1}R \otimes_R N_\bullet = U^{-1}N_\bullet$, so $U^{-1}R \otimes_R N_\bullet$ is exact. Since $N_\bullet$ was chosen arbitrarily, we conclude that $U^{-1}R$ is a flat R-module.

(e) Since $\mathbb{Q} = \mathbb{Z}_{(0)}$, this follows from part (d). □

Exercises.

Exercise II.2.10. Continue with the notation of Proposition II.2.1. Prove that, if α and β are isomorphisms, then so is $\alpha \otimes_R \beta$. In particular, if α and β are isomorphisms, then so are $\alpha \otimes_R N$ and $M \otimes_R \beta$.

Exercise II.2.11. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Prove that, if M is a finitely generated R-module, then $S \otimes_R M$ is finitely generated as an S-module.

Exercise II.2.12. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Let M be an R-module, and let N be an S-module.

(a) Prove that the tensor product $N \otimes_R M$ has a well-defined S-module structure given by $s(\sum_i n_i \otimes m_i) = \sum_i (sn_i) \otimes m_i$.
(b) Prove that this S-module structure is compatible with the R-module structure on $N \otimes_R M$ via restriction of scalars: for all $r \in R$ and all $n \in N$ and all $m \in M$, we have $r(n \otimes m) = \varphi(r)(n \otimes m)$.

Exercise II.2.13. Let R be a commutative ring. Let M and N be R-modules, and let $U \subseteq R$ be a multiplicatively closed subset. Prove that there is an $U^{-1}R$-module isomorphism $(U^{-1}M) \otimes_{U^{-1}R} (U^{-1}N) \cong U^{-1}(M \otimes_R N)$. See also Corollary II.3.7.

Exercise II.2.14. Let R be a commutative ring, and let $U \subseteq R$ be a multiplicatively closed subset. Prove that every ideal $\mathfrak{a} \subseteq U^{-1}R$ is isomorphic to $U^{-1}\mathfrak{b}$ for some ideal $\mathfrak{b} \subseteq R$.

Exercise II.2.15. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings such that S is flat as an R-module. Let M and N be R-modules such that M is finitely presented. Prove that there is an S-module isomorphism

$$S \otimes_R \operatorname{Hom}_R(M,N) \cong \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N).$$

(Hint: Follow the proof of Proposition I.5.8.)

II.3. Commutativity and Associativity

The theme of this section is the following: the class of all R-modules behaves like a commutative ring under the operations of direct sum and tensor product, with additive identity 0 and multiplicative identity R; see Example II.1.9. We start the section by proving the commutativity of tensor product.

Proposition II.3.1. *Let R be a commutative ring, and let M and N be R-modules. Then there exists an R-module isomorphism $F\colon M \otimes_R N \to N \otimes_R M$ such that $F(m \otimes n) = n \otimes m$ for all $m \in M$ and all $n \in N$. Thus, we have $M \otimes_R N \cong N \otimes_R M$.*

Proof. Use the universal mapping property to show that there exist R-module homomorphisms $F\colon M \otimes_R N \to N \otimes_R M$ and $G\colon N \otimes_R M \to M \otimes_R N$ such that $F(m \otimes n) = n \otimes m$ and $G(n \otimes m) = m \otimes n$ for all $m \in M$ and all $n \in N$. Use Lemma II.1.8(c) to show that $FG = \mathbb{1}_{N \otimes_R M}$ and $GF = \mathbb{1}_{M \otimes_R N}$ so that F and G are inverse isomorphisms. □

Here is the distributive property for tensor products. Recall the notation from Remark I.3.2.

Theorem II.3.2. *Let R be a commutative ring. Let M be an R-module, and let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules. There is an R-module isomorphism*

$$F\colon M \otimes_R \left(\coprod_{\lambda\in\Lambda} N_\lambda\right) \xrightarrow{\cong} \coprod_{\lambda\in\Lambda}(M \otimes_R N_\lambda)$$

such that $F(m \otimes (n_\lambda)) = (m \otimes n_\lambda)$ for all $m \in M$ and all $(n_\lambda) \in \coprod_{\lambda\in\Lambda} N_\lambda$.

Proof. As in Proposition II.3.1, use the universal mapping property to show that there is a well-defined R-module homomorphism

$$F\colon M \otimes_R \left(\coprod_{\lambda\in\Lambda} N_\lambda\right) \to \coprod_{\lambda\in\Lambda}(M \otimes_R N_\lambda)$$

such that $F(m \otimes (n_\lambda)) = (m \otimes n_\lambda)$ for all $m \in M$ and all $(n_\lambda) \in \coprod_{\lambda\in\Lambda} N_\lambda$. Note that this implies that

$$F(m \otimes \varepsilon^N_\mu(n_\mu)) = \varepsilon^{M\otimes_R N}_\mu(m \otimes n_\mu)$$

for all $m \in M$ and $n_\mu \in N_\mu$; here

$$\varepsilon^N_\mu\colon N_\mu \to \coprod_{\lambda\in\Lambda} N_\lambda \qquad \text{and} \qquad \varepsilon^{M\otimes_R N}_\mu\colon M \otimes_R N_\mu \to \coprod_{\lambda\in\Lambda}(M \otimes_R N_\lambda)$$

are the natural inclusions.

We now construct an inverse for F.

First, we set

$$g_\mu = M \otimes_R \varepsilon^N_\mu\colon M \otimes_R N_\mu \to M \otimes_R \left(\coprod_{\lambda\in\Lambda} N_\lambda\right)$$

so that $g_\mu(m \otimes n_\mu) = m \otimes \varepsilon^N_\mu(n_\mu)$ for all $m \in M$ and all $n_\mu \in N_\mu$.

Second, Exercise I.3.3 yields a unique R-module homomorphism

$$G\colon \coprod_{\lambda\in\Lambda}(M\otimes_R N_\lambda)\to M\otimes_R(\coprod_{\lambda\in\Lambda}N_\lambda)$$

making each of the following diagrams commute:

$$\begin{array}{ccc} M\otimes_R N_\mu & \xrightarrow{\varepsilon_\mu^{M\otimes_R N}} & \coprod_{\lambda\in\Lambda}(M\otimes_R N_\lambda) \\ & \searrow^{g_\mu} & \downarrow \exists! G \\ & & M\otimes_R(\coprod_{\lambda\in\Lambda}N_\lambda). \end{array}$$

The commutativity of the diagram says

$$G(\varepsilon_\mu^{M\otimes_R N}(m\otimes n_\mu))=g_\mu(m\otimes n_\mu)=m\otimes\varepsilon_\mu^N(n_\mu)$$

for all $m\in M$ and all $n_\mu\in N_\mu$. Hence, we have

$$\begin{aligned} G((m_\lambda\otimes n_\lambda)) &= G\left(\textstyle\sum_{\mu\in\Lambda}\varepsilon_\mu^{M\otimes_R N}(m_\mu\otimes n_\mu)\right)\\ &= \textstyle\sum_{\mu\in\Lambda}G(\varepsilon_\mu^{M\otimes_R N}(m_\mu\otimes n_\mu))\\ &= \textstyle\sum_{\mu\in\Lambda}m_\mu\otimes\varepsilon_\mu^N(n_\mu)\end{aligned}$$

for all $(m_\lambda)\in M^{(\Lambda)}$ and all $(n_\lambda)\in\coprod_{\lambda\in\Lambda}N_\lambda$. Notice that each of these sums is finite.

It follows that we have

$$\begin{aligned} F(G((m_\lambda\otimes n_\lambda))) &= F\left(\textstyle\sum_{\mu\in\Lambda}m_\mu\otimes\varepsilon_\mu^N(n_\mu)\right)\\ &= \textstyle\sum_{\mu\in\Lambda}F\left(m_\mu\otimes\varepsilon_\mu^N(n_\mu)\right)\\ &= \textstyle\sum_{\mu\in\Lambda}\varepsilon_\mu^{M\otimes_R N}(m_\mu\otimes n_\mu)\\ &= (m_\lambda\otimes n_\lambda)\end{aligned}$$

so FG is the identity on $\coprod_{\lambda\in\Lambda}(M\otimes_R N_\lambda)$. On the other hand, we have

$$\begin{aligned} G(F(m\otimes(n_\lambda))) &= G((m\otimes n_\lambda))\\ &= \textstyle\sum_{\mu\in\Lambda}m\otimes\varepsilon_\mu^N(n_\mu)\\ &= m\otimes\left(\textstyle\sum_{\mu\in\Lambda}\varepsilon_\mu^N(n_\mu)\right)\\ &= m\otimes(n_\lambda)\end{aligned}$$

so Lemma II.1.8(c) implies that GF is the identity on $M\otimes_R\coprod_{\lambda\in\Lambda}N_\lambda$. Hence, F and G are inverse isomorphisms. □

The next example shows that tensor product does not commute with non-finite direct products.

Example II.3.3. We show that

$$\textstyle\prod_{n\in\mathbb{N}}(\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}/p^n\mathbb{Z})=0 \qquad\text{and}\qquad \mathbb{Q}\otimes_{\mathbb{Z}}\prod_{n\in\mathbb{N}}\mathbb{Z}/p^n\mathbb{Z}\neq 0$$

and hence $\prod_{n\in\mathbb{N}}(\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}/p^n\mathbb{Z})\ncong\mathbb{Q}\otimes_{\mathbb{Z}}\prod_{n\in\mathbb{N}}\mathbb{Z}/p^n\mathbb{Z}$.

Fix a natural number n and consider the following computation in $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}/p^n\mathbb{Z}$:

$$q\otimes\overline{r}=\left(\tfrac{1}{p^n}qp^n\right)\otimes\overline{r}=\left(\tfrac{1}{p^n}q\right)\otimes\overline{p^nr}=\left(\tfrac{1}{p^n}q\right)\otimes\overline{0}=0.$$

It follows that $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}/p^n\mathbb{Z}=0$ for each n and hence $\prod_{n\in\mathbb{N}}(\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}/p^n\mathbb{Z})=0$.

The function $f\colon \mathbb{Z} \to \prod_n \mathbb{Z}/p^n\mathbb{Z}$ given by $f(m) = (\overline{m}, \overline{m}, \overline{m}, \dots)$ is a well-defined $\mathbb{Z}$-module monomorphism. The $\mathbb{Z}$-module $\mathbb{Q}$ is flat by Proposition II.2.9(e), so we have $\mathbb{Q}\otimes_{\mathbb{Z}} f\colon \mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z} \hookrightarrow \mathbb{Q}\otimes_{\mathbb{Z}}\prod_{n\in\mathbb{N}}\mathbb{Z}/p^n\mathbb{Z}$. The isomorphism $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z} \cong \mathbb{Q} \neq 0$ shows $\mathbb{Q} \otimes_{\mathbb{Z}} \prod_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} \neq 0$

Here is a consequence of the distributive properties in Theorem II.3.2.

Proposition II.3.4. *Let R be a commutative ring, and let M be an R-module. For sets Γ and Λ, we have $R^{(\Gamma)} \otimes_R M \cong M^{(\Gamma)}$ and $R^{(\Gamma)} \otimes_R R^{(\Lambda)} \cong R^{(\Gamma\times\Lambda)}$. For integers m and n, we have $R^m \otimes_R M \cong M^m$ and $R^m \otimes_R R^n \cong R^{mn}$.* □

The next result contains a generalized version of the associativity property for tensor products. See also Corollary II.3.6.

Theorem II.3.5. *Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Let L and M be S-modules, and let N be an R-module. There is an S-module isomorphism*

$$\Psi\colon L \otimes_S (M \otimes_R N) \xrightarrow{\cong} (L \otimes_S M) \otimes_R N$$

given by $l \otimes (m \otimes n) \mapsto (l \otimes m) \otimes n$.

Proof. We complete the proof in four steps.

Step 1. Proposition II.2.6(a) implies that $M \otimes_R N$ is an S-module via the following action

$$s(m \otimes n) = (sm) \otimes n.$$

Hence, the tensor product $L \otimes_S (M \otimes_R N)$ is a well-defined S-module via the action

$$s(l \otimes (m \otimes n)) = (sl) \otimes (m \otimes n) = l \otimes ((sm) \otimes n).$$

Also, since $L \otimes_S M$ is an S-module, Proposition II.2.6(a) implies that $(L \otimes_S M) \otimes_R N$ is an S-module via the action

$$s((l \otimes m) \otimes n) = ((sl) \otimes m) \otimes n = (l \otimes (sm)) \otimes n.$$

Step 2. We show that, for each $l \in L$, the map

$$\Phi_l\colon M \otimes_R N \to (L \otimes_S M) \otimes_R N$$

given by $m \otimes n \mapsto (l \otimes m) \otimes n$ is a well-defined S-module homomorphism. To this end, let $l \in L$. It is straightforward to show that the map

$$\phi_l\colon M \times N \to (L \otimes_S M) \otimes_R N$$

given by $(m, n) \mapsto (l \otimes m) \otimes n$ is well-defined and R-bilinear. Thus, the universal mapping property for $M \otimes_R N$ shows that Φ_l is a well-defined R-module homomorphism. The following computation shows (essentially) that Φ_l is also an S-module homomorphism:

$$\Phi_l(s(m \otimes n)) = \Phi_l((sm) \otimes n) = (l \otimes (sm)) \otimes n = s((l \otimes m) \otimes n) = s\Phi_l(m \otimes n).$$

Step 1 explains the equalities in this sequence.

Step 3. We show that the map

$$\Psi\colon L \otimes_S (M \otimes_R N) \to (L \otimes_S M) \otimes_R N$$

given by $l \otimes (m \otimes n) \mapsto (l \otimes m) \otimes n$ is a well-defined S-module homomorphism. Step 2 shows that the map

$$\Phi\colon L \times (M \otimes_R N) \to (L \otimes_S M) \otimes_R N$$

given by $(l, m \otimes n) \mapsto (l \otimes m) \otimes n$ is well-defined. It is straightforward to show that it is S-bilinear, so the desired result follows directly from the universal mapping property for $L \otimes_S (M \otimes_R N)$.

Step 4. An argument similar to Steps 1–3 shows that the map

$$\Theta\colon (L \otimes_S M) \otimes_R N \to L \otimes_S (M \otimes_R N)$$

given by $(l \otimes m) \otimes n \mapsto l \otimes (m \otimes n)$ is a well-defined S-module homomorphism. It is straightforward to check that Ψ and Θ are inverse isomorphisms. □

Corollary II.3.6. *Let R be a commutative ring. Let L, M and N be R-modules. There is an R-module isomorphism*

$$\Psi\colon L \otimes_R (M \otimes_R N) \xrightarrow{\cong} (L \otimes_R M) \otimes_R N$$

given by $l \otimes (m \otimes n) \mapsto (l \otimes m) \otimes n$.

PROOF. This is the case of Theorem II.3.5 where $\varphi = \mathbb{1}_R\colon R \to R$. □

Corollary II.3.7. *Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Let M and N be R-modules. There is an S-module isomorphism*

$$\Psi\colon (S \otimes_R M) \otimes_S (S \otimes_R N) \xrightarrow{\cong} S \otimes_R (M \otimes_R N)$$

given by $(s \otimes m) \otimes (t \otimes n) \mapsto (st) \otimes (m \otimes n)$.

PROOF. We have the following sequence of homomorphisms:

$$\begin{array}{ll}
(S \otimes_R M) \otimes_S (S \otimes_R N) & (s \otimes m) \otimes (t \otimes n) \\
\downarrow\cong & \downarrow \\
((S \otimes_R M) \otimes_S S) \otimes_R N & ((s \otimes m) \otimes t) \otimes n \\
\downarrow\cong & \downarrow \\
(S \otimes_R M) \otimes_R N & ((st) \otimes m) \otimes n \\
\downarrow\cong & \downarrow \\
S \otimes_R (M \otimes_R N) & (st) \otimes (m \otimes n).
\end{array}$$

The first one is the S-module isomorphism from Theorem II.3.5. The second one follows from the S-module isomorphism $(S \otimes_R M) \otimes_S S \cong S \otimes_R M$ from Example II.1.9; check that the displayed isomorphism is also S-linear. The third isomorphism is from Corollary II.3.6; check that this isomorphism is also S-linear. □

Exercises.

Exercise II.3.8. (Alternate proofs of Theorem II.3.2.) Continue with the notation of Theorem II.3.2.

(a) Show that $M \otimes_R (\coprod_\lambda N_\lambda)$ satisfies the universal property for $\coprod_\lambda (M \otimes_R N_\lambda)$, and conclude from this that $M \otimes_R (\coprod_\lambda N_\lambda) \cong \coprod_\lambda (M \otimes_R N_\lambda)$.

(b) Show that $\coprod_\lambda (M \otimes_R N_\lambda)$ satisfies the universal property for $M \otimes_R (\coprod_\lambda N_\lambda)$, and conclude from this that $M \otimes_R (\coprod_\lambda N_\lambda) \cong \coprod_\lambda (M \otimes_R N_\lambda)$.

Exercise II.3.9. Let R be a commutative ring, and let $\{N_\lambda\}_{\lambda \in \Lambda}$ be a set of R-modules. Prove that $\coprod_\lambda N_\lambda$ is flat if and only if each N_λ is flat.

Exercise II.3.10. Let R be a commutative ring.

(a) Prove that every free R-module is flat.

(b) Prove that every projective R-module is flat.
(c) Show that the converses of parts (a) and (a) fail by proving that $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module

Exercise II.3.11. Let R be a commutative ring, and let $U \subseteq R$ be a multiplicatively closed subset. Prove that if M is a flat R-module, then $U^{-1}M$ is a flat $U^{-1}R$-module and a flat R-module.

Exercise II.3.12. Complete the proof of Theorem II.3.5.

Exercise II.3.13. Complete the proof of Corollary II.3.7.

Exercise II.3.14. (Alternate proof of Corollary II.3.6.) Let R be a commutative ring, and let M, N, P, and G be R-modules.

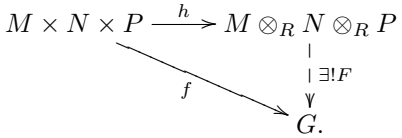
A function $f\colon M \times N \times P \to G$ is *R-trilinear* if it satisfies the following:

$$\begin{aligned} f(m+m',n,p) &= f(m,n,p) + f(m',n,p) \\ f(m,n+n',p) &= f(m,n,p) + f(m,n',p) \\ f(m,n,p+p') &= f(m,n,p) + f(m,n,p') \\ f(rm,n,p) &= rf(m,n,p) = f(m,rn,p) = f(m,n,rp) \end{aligned}$$

for all $m, m' \in M$ all $n, n' \in N$ all $p, p' \in P$ and all $r \in R$.

For example, the functions $f\colon M \times N \times P \to (M \otimes_R N) \otimes_R P$ given by $f(m,n,p) = (m \otimes n) \otimes p$ and $g\colon M \times N \times P \to M \otimes_R (N \otimes_R P)$ given by $g(m,n,p) = m \otimes (n \otimes p)$ are R-trilinear.

A *tensor product* of M, N and P over R is an R-module $M \otimes_R N \otimes_R P$ equipped with an R-trilinear function $h\colon M \times N \times P \to M \otimes_R N \otimes_R P$ satisfying the following universal mapping property: For every R-module G and every R-trilinear function $f\colon M \times N \times P \to G$, there exists a unique R-module homomorphism $F\colon M \otimes_R N \otimes_R P \to G$ making the following diagram commute



For each $m \in M$ and $n \in N$ and $p \in P$, set $m \otimes n \otimes p = h(m,n,p)$.

(a) Show that $M \otimes_R N \otimes_R P$ exists.
(b) Show that there are R-module isomorphisms

$$\begin{aligned} F&\colon M \otimes_R N \otimes_R P \to (M \otimes_R N) \otimes_R P \\ G&\colon M \otimes_R N \otimes_R P \to M \otimes_R (N \otimes_R P) \end{aligned}$$

given by $F(m \otimes n \otimes p) = (m \otimes n) \otimes p$ and $G(m \otimes n \otimes p) = m \otimes (n \otimes p)$. In particular, we have $(M \otimes_R N) \otimes_T P \cong M \otimes_R (N \otimes_T P)$.

II.4. Right-Exactness

Next, we go for exactness properties.

Proposition II.4.1. *Let R be a commutative ring. Let $f\colon M \to M'$ and $g\colon N \to N'$ be R-module epimorphisms.*

(a) *The map $f \otimes_R g\colon M \otimes_R N \to M' \otimes_R N'$ is surjective.*

(b) *The module* $\operatorname{Ker}(f \otimes_R g)$ *is generated as an* R*-module by the set*

$$L = \{m \otimes n \in M \otimes_R N \mid f(m) = 0 \text{ or } g(n) = 0\} \subseteq M \otimes_R N.$$

PROOF. (a) $\sum_i m_i' \otimes n_i' = \sum_i f(m_i) \otimes g(n_i) = (f \otimes_R g)(\sum_i m_i \otimes n_i)$.

(b) Let K denote the submodule of $M \otimes_R N$ generated by the set L. Each generator of L is in $\operatorname{Ker}(f \otimes_R g)$, so $L \subseteq \operatorname{Ker}(f \otimes_R g)$. Exercise I.1.7 provides a well-defined R-module epimorphism $\phi\colon (M \otimes_R N)/K \to M' \otimes_R N'$ such that $\phi(\overline{m \otimes n}) = f(m) \otimes g(n)$. To show that $K = \operatorname{Ker}(f \otimes_R g)$, it suffices to show that ϕ is injective.

Define a map $h\colon M' \times N' \to M \otimes_R N/K$ as follows: for $(m', n') \in M' \times_R N'$, fix $m \in M$ and $n \in N$ such that $f(m) = m'$ and $g(n) = n'$, and set $h(m', n') = \overline{m \otimes n}$. We need to show this is a well-defined function. Assume $f(m_1) = m' = f(m)$ and $g(n_1) = n' = g(n)$. Then $m_1 - m \in \operatorname{Ker}(f)$ and $n_1 - n \in \operatorname{Ker}(g)$ so in $M \otimes_R N$ we have

$$\begin{aligned} m_1 \otimes n_1 &= (m_1 - m) \otimes (n_1 - n) \\ &= \underbrace{(m_1 - m) \otimes (n_1 - n) + (m_1 - m) \otimes n + m \otimes (n_1 - n)}_{\in K} + m \otimes n. \end{aligned}$$

It follows that, in $(M \otimes_R N)/K$, we have $\overline{m_1 \otimes n_1} = \overline{m \otimes n}$ so h is well-defined.

We check that h is R-bilinear. For instance, we want $h(m_1' + m_2', n') = h(m_1', n') + h(m_2', n')$. Fix $m_1, m_2 \in M$ and $n \in N$ such that $f(m_1) = m_1'$, $f(m_2) = m_2'$ and $g(n) = n'$. Then $f(m_1 + m_2) = m_1' + m_2'$ so

$$h(m_1' + m_2', n') = \overline{(m_1 + m_2) \otimes n} = \overline{m_1 \otimes n} + \overline{m_2 \otimes n} = h(m_1', n') + h(m_2', n').$$

The other parts of bilinearity are verified similarly.

Since h is R-bilinear, the universal property for tensor products yields a well-defined R-module homomorphism $H\colon M' \otimes_R N' \to (M \otimes_R N)/K$ satisfying the following: for $m' \otimes n' \in M' \otimes_R N'$, fix $m \in M$ and $n \in N$ such that $f(m) = m'$ and $g(n) = n'$; then $H(m' \otimes n') = \overline{m \otimes n}$). It follows readily that the composition $H\phi\colon (M \otimes_R N)/K \to (M \otimes_R N)/K$ is the identity on $(M \otimes_R N)/K$, so ϕ is injective as desired. □

In general, the tensor product of injective maps is not injective:

Example II.4.2. The maps $\mu_2^{\mathbb{Z}}\colon \mathbb{Z} \to \mathbb{Z}$ and $\mathbb{1}_{\mathbb{Z}/2\mathbb{Z}}\colon \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ are both injective. (Recall that the notation for the homothety maps $\mu_2^{\mathbb{Z}}$ are in Example II.2.3.) From Example II.2.3 we know that

$$\mu_2^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{1}_{\mathbb{Z}/2\mathbb{Z}} = \mu_2^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = \mu_2^{\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}}\colon \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}.$$

Tensor cacellation II.1.9 implies that $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$, so $\mu_2^{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{1}_{\mathbb{Z}/2\mathbb{Z}} = 0$. This map is not injective.

Here is the right-exactness of the tensor product.

Proposition II.4.3. *Let* R *be a commutative ring, and let* M *be an* R*-module. For each an exact sequence of* R*-module homomorphisms* $N' \xrightarrow{g'} N \xrightarrow{g} N'' \to 0$ *the associated sequence*

$$M \otimes_R N' \xrightarrow{M \otimes_R g'} M \otimes_R N \xrightarrow{M \otimes_R g} M \otimes_R N'' \to 0$$

is exact.

PROOF. Because g is surjective, Proposition II.4.1(a) implies that $M \otimes_R g$ is surjective. Proposition II.2.1(b) shows that

$$(M \otimes_R g)(M \otimes_R g') = M \otimes_R (gg') = M \otimes_R 0 = 0$$

so $\operatorname{Im}(M \otimes_R g') \subseteq \operatorname{Ker}(M \otimes_R g)$. To show $\operatorname{Im}(M \otimes_R g') \supseteq \operatorname{Ker}(M \otimes_R g)$, it suffices to show that every generator of $\operatorname{Ker}(M \otimes_R g)$ is in $\operatorname{Im}(M \otimes_R g')$. By Proposition II.4.1(b), $\operatorname{Ker}(M \otimes_R g)$ is generated by $\{m \otimes n \mid g(n) = 0\}$. For each $m \otimes n \in M \otimes_R N$ such that $g(n) = 0$, there exists $n' \in N'$ such that $g'(n') = n$, so $m \otimes n = (M \otimes_R g')(m \otimes n') \in \operatorname{Im}(M \otimes_R g')$. □

Definition II.4.4. Let R be a commutative ring. Let ${}_R\mathcal{M}$ denote the class of all R-modules and all R-module homomorphisms. This is the *category* of all R-modules.

Definition II.4.5. Let R and S be commutative rings. A *(covariant) functor* $F\colon {}_R\mathcal{M} \to {}_S\mathcal{M}$ is a rule that

(1) assigns to each R-module M an S-module $F(M)$;
(2) assigns to each R-module homomorphism $\phi\colon M \to N$ an S-module homomorphism $F(\phi)\colon F(M) \to F(N)$ such that
   (a) the rule F respects identities: for every R-module M we have $F(\mathbb{1}_M) = \mathbb{1}_{F(M)}$, and
   (b) the rule F respects compositions: for each pair of R-module homomorphisms $\phi\colon M \to N$ and $\psi\colon N \to P$, we have $F(\psi\phi) = F(\psi)F(\phi)$.

A functor should be thought of as a homomorphism from ${}_R\mathcal{M}$ to ${}_S\mathcal{M}$. The property $F(\psi\phi) = F(\psi)F(\phi)$ is sometimes referred to as the "functoriality" of F.

Example II.4.6. Let R be a commutative ring. If M is an R-module, the operators $M \otimes_R -$ and $\operatorname{Hom}_R(M, -)$ are covariant functors ${}_R\mathcal{M} \to {}_R\mathcal{M}$. If $\varphi\colon R \to S$ is a homomorphism of commutative rings, then the operators $S \otimes_R -$ and $\operatorname{Hom}_R(S, -)$ are covariant functors ${}_R\mathcal{M} \to {}_S\mathcal{M}$.

Definition II.4.7. Let R and S be commutative rings, and let $F\colon {}_R\mathcal{M} \to {}_S\mathcal{M}$ be a covariant functor.

(a) F is *left-exact* if, for every exact sequence $0 \to M \xrightarrow{\phi} N \xrightarrow{\psi} P$ of R-module homomorphisms, the resulting sequence $0 \to F(M) \xrightarrow{F(\phi)} F(N) \xrightarrow{F(\psi)} F(P)$ of S-module homomorphisms is exact;
(b) F is *right-exact* if, for every exact sequence $M \xrightarrow{\phi} N \xrightarrow{\psi} P \to 0$ of R-module homomorphisms, the resulting sequence $F(M) \xrightarrow{F(\phi)} F(N) \xrightarrow{F(\psi)} F(P) \to 0$ of S-module homomorphisms is exact;
(c) F is *exact* if, for every exact sequence $M \xrightarrow{\phi} N \xrightarrow{\psi} P$ of R-module homomorphisms, the resulting sequence $F(M) \xrightarrow{F(\phi)} F(N) \xrightarrow{F(\psi)} F(P)$ of S-module homomorphisms is exact.

Example II.4.8. Let R be a commutative ring. Given an R-module M, the functor $M \otimes_R -$ is right-exact, and the functor $\operatorname{Hom}_R(M, -)$ is left-exact. If $\varphi\colon R \to S$ is a homomorphism of commutative rings, then the functor $S \otimes_R -$ is right-exact, and the functor $\operatorname{Hom}_R(S, -)$ is left-exact.

Definition II.4.9. Let R and S be commutative rings. A *contravariant functor* $F\colon {}_R\mathcal{M} \to {}_S\mathcal{M}$ is a rule that

(1) assigns to each R-module M an S-module $F(M)$;
(2) assigns to each R-module homomorphism $\phi\colon M \to N$ an S-module homomorphism $F(\phi)\colon F(N) \to F(M)$ such that
   (a) the rule F respects identities: for every R-module M we have $F(\mathbb{1}_M) = \mathbb{1}_{F(M)}$, and
   (b) the rule F respects compositions: for each pair of R-module homomorphisms $\phi\colon M \to N$ and $\psi\colon N \to P$, we have $F(\psi\phi) = F(\phi)F(\psi)$.

Note that a contravariant functor reverses arrows. This is the reason for the name "contravariant": the prefix "contra" means "against", signifying that F goes against the arrows. Contrast this with the term "covariant" which identifies functors that go with the arrows.

Example II.4.10. Let R be a commutative ring. Given an R-module M, the operator $\mathrm{Hom}_R(-, M)$ is a contravariant functor ${}_R\mathcal{M} \to {}_R\mathcal{M}$.

Definition II.4.11. Let R and S be commutative rings, and let $F\colon {}_R\mathcal{M} \to {}_S\mathcal{M}$ be a contravariant functor.

(a) F is *left-exact* if, for every exact sequence $M \xrightarrow{\phi} N \xrightarrow{\psi} P \to 0$ of R-module homomorphisms, the resulting sequence $0 \to F(P) \xrightarrow{F(\psi)} F(N) \xrightarrow{F(\phi)} F(M)$ of S-module homomorphisms is exact;
(b) F is *right-exact* if, for every exact sequence $0 \to M \xrightarrow{\phi} N \xrightarrow{\psi} P$ of R-module homomorphisms, the resulting sequence $F(P) \xrightarrow{F(\psi)} F(N) \xrightarrow{F(\phi)} F(M) \to 0$ of S-module homomorphisms is exact;
(c) F is *exact* if, for every exact sequence $M \xrightarrow{\phi} N \xrightarrow{\psi} P$ of R-module homomorphisms, the resulting sequence $F(P) \xrightarrow{F(\psi)} F(N) \xrightarrow{F(\phi)} F(M)$ of S-module homomorphisms is exact.

Example II.4.12. Let R be a commutative ring. Given an R-module M, the functor $\mathrm{Hom}_R(-, M)$ is left-exact.

Exercises.

Exercise II.4.13. Let R be a commutative ring, and consider two exact sequences of R-module homomorphisms

$$M' \xrightarrow{f'} M \xrightarrow{f} M'' \to 0 \qquad\qquad N' \xrightarrow{g'} N \xrightarrow{g} N'' \to 0.$$

(a) Prove that there is a well-defined R-module homomorphism

$$h\colon (M' \otimes_R N) \oplus (M \otimes_R N') \to M \otimes_R N$$

such that $h(m' \otimes n, m \otimes n') = f'(m') \otimes n + m \otimes g'(n')$.
(b) Prove that the following sequence is exact

$$(M' \otimes_R N) \oplus (M \otimes_R N') \xrightarrow{h} M \otimes_R N \xrightarrow{f\otimes g} M'' \otimes_R N'' \to 0.$$

Exercise II.4.14. Let R be a commutative ring. Let M be an R-module, and let $I \subseteq R$ be an ideal. Prove that $(R/I) \otimes_R M \cong M/IM$. (Sketch of proof: Start with the exact sequence $0 \to I \to R \to R/I \to 0$. Apply the functor $- \otimes_R M$ to this

sequence to obtain the top-most exact row in the following diagram

$$\begin{array}{ccccccccc} & & I\otimes_R M & \longrightarrow & R\otimes_R M & \longrightarrow & (R/I)\otimes_R M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & IM & \longrightarrow & M & \longrightarrow & M/IM & \longrightarrow & 0. \end{array}$$

The vertical maps are given by $f(i\otimes m) = im$ and $g(r\otimes m) = rm$ and $h(\overline{r}\otimes m) = \overline{rm}$. Show that these maps are well-defined and make the diagram commute. The map f is an epimorphism, and g is an isomorphism. Show that this implies that h is an isomorphism.)

Exercise II.4.15. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Let M be an R-module, and let N be an S-module.

(a) Prove that if S is flat over R and N is flat over S, then N is flat over R. (Hint: Use the isomorphism $(A \otimes_R S) \otimes_S M \cong A \otimes_R M$.)
(b) Prove that if M is flat over R, then $S \otimes_R M$ is flat over S. (Hint: Use the isomorphism $B \otimes_S (S \otimes_R M) \cong B \otimes_R M$.)

Exercise II.4.16. Let R and S be commutative rings, and let $F\colon {}_R\mathcal{M} \to {}_S\mathcal{M}$ be a functor, either covariant or contravariant. Prove that the following conditions are equivalent:

(i) The functor F is exact and $F(0) = 0$;
(ii) The functor F is left-exact and right-exact;
(iii) The functor F transforms every short exact sequence of R-module homomorphisms into a short exact sequence of S-module homomorphisms.

Exercise II.4.17. Let R be a commutative ring. Let M be an R-module, and prove that the following conditions are equivalent:

(i) The R-module M is a flat;
(ii) The functor $M\otimes_R -\colon {}_R\mathcal{M} \to {}_R\mathcal{M}$ transforms every R-module monomorphism into a monomorphism;
(iii) The functor $M \otimes_R -\colon {}_R\mathcal{M} \to {}_R\mathcal{M}$ is left-exact;
(iv) The functor $M \otimes_R -\colon {}_R\mathcal{M} \to {}_R\mathcal{M}$ is exact;
(v) The functor $M \otimes_R -\colon {}_R\mathcal{M} \to {}_R\mathcal{M}$ transforms every short exact sequence of R-module homomorphisms into a short exact sequence of R-module homomorphisms.

Exercise II.4.18. Let R be a commutative ring, and let M be an R-module. Recall that M is *projective* if, for every R-module homomorphism $f\colon M \to N'$ and every R-module epimorphism $g\colon N \to N'$, there exists an R-module homomorphism $h\colon M \to N$ such that $f = gh$. Prove that the following conditions are equivalent:

(i) The R-module M is projective;
(ii) The functor $\mathrm{Hom}_R(M, -)\colon {}_R\mathcal{M} \to {}_R\mathcal{M}$ transforms every R-module epimorphism into an epimorphism;
(iii) The functor $\mathrm{Hom}_R(M, -)\colon {}_R\mathcal{M} \to {}_R\mathcal{M}$ is right-exact;
(iv) The functor $\mathrm{Hom}_R(M, -)\colon {}_R\mathcal{M} \to {}_R\mathcal{M}$ is exact;
(v) The functor $\mathrm{Hom}_R(M, -)$ transforms every short exact sequence of R-module homomorphisms into a short exact sequence of R-module homomorphisms.

Exercise II.4.19. Let R be a commutative ring, and let M be an R-module. Recall that M is *injective* if, for every R-module homomorphism $f\colon N' \to M$ and every R-module monomorphism $g\colon N' \to N$, there exists an R-module homomorphism $h\colon N \to M$ such that $f = hg$. Prove that the following conditions are equivalent:

(i) The R-module M is injective;
(ii) The functor $\mathrm{Hom}_R(-, M)\colon {}_R\mathcal{M} \to {}_R\mathcal{M}$ transforms every R-module monomorphism into an epimorphism;
(iii) The functor $\mathrm{Hom}_R(-, M)\colon {}_R\mathcal{M} \to {}_R\mathcal{M}$ is left-exact;
(iv) The functor $\mathrm{Hom}_R(-, M)\colon {}_R\mathcal{M} \to {}_R\mathcal{M}$ is exact;
(v) The functor $\mathrm{Hom}_R(-, M)$ transforms every short exact sequence of R-module homomorphisms into a short exact sequence of R-module homomorphisms.

II.5. Hom-Tensor Adjointness

Next up: Hom-tensor adjointness and an alternative proof of right-exactness of tensor products.

Proposition II.5.1. *Let R be a commutative ring, and let M, N and P be R-modules. There are R-module isomorphisms*

$$\mathrm{Hom}_R(N, \mathrm{Hom}_R(M, P)) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathrm{Hom}_R(M \otimes_R N, P)$$

$$[\alpha\colon N \to \mathrm{Hom}_R(M, P)] \longmapsto \begin{bmatrix} M \otimes_R N \to P \\ m \otimes n \mapsto \alpha(n)(m) \end{bmatrix}$$

$$\begin{bmatrix} N \to \mathrm{Hom}_R(M, P) \\ n \mapsto [m \mapsto \beta(m \otimes n)] \end{bmatrix} \longmapsfrom [\beta\colon M \otimes_R N \to P].$$

Proof. It is straightforward to show that the map Φ is well-defined. Use the universal property for $M \otimes_R N$ to show that Ψ is well-defined. It is tedious (but routine) to show that Ψ and Φ are R-module homomorphisms and to show that the compositions $\Phi\Psi$ and $\Psi\Phi$ are the appropriate identities. □

Remark II.5.2. The isomorphisms in Proposition II.5.1 are natural in all three variables. For example, if $f\colon M \to M'$ is an R-module homomorphism, then there is a commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}(M \otimes N, P) & \xrightarrow{\Phi} & \mathrm{Hom}(N, \mathrm{Hom}(M, P)) & \xrightarrow{\Psi} & \mathrm{Hom}(M \otimes N, P) \\ \uparrow{\scriptstyle \mathrm{Hom}(f \otimes N, P)} & & \uparrow{\scriptstyle \mathrm{Hom}(f \otimes N, P)} & & \uparrow{\scriptstyle \mathrm{Hom}(f \otimes N, P)} \\ \mathrm{Hom}(M' \otimes N, P) & \xrightarrow{\Phi'} & \mathrm{Hom}(N, \mathrm{Hom}(M', P)) & \xrightarrow{\Psi'} & \mathrm{Hom}(M' \otimes N, P). \end{array}$$

There are similar diagrams for homomorphisms $N \to N'$ and $P \to P'$.

We conclude with an alternate proof of right-exactness of tensor product II.4.3.

Proof. Start with an exact sequence $N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$ of R-module homomorphisms and an R-module M. We want to show that the sequence

$$M \otimes N' \xrightarrow{M \otimes f} M \otimes N \xrightarrow{M \otimes g} M \otimes N'' \to 0$$

is exact. By Exercise II.5.5 it suffices to show that, for every R-module P, the following sequence is exact:

$$0 \to \operatorname{Hom}(M \otimes N'', P) \xrightarrow{\operatorname{Hom}(M\otimes g, P)} \operatorname{Hom}(M \otimes N, P) \xrightarrow{\operatorname{Hom}(M\otimes f, P)} \operatorname{Hom}(M \otimes N', P).$$

The left-exactness of $\operatorname{Hom}(-, P)$ implies that the following sequence is exact:

$$0 \to \operatorname{Hom}(N'', P) \xrightarrow{\operatorname{Hom}(g, P)} \operatorname{Hom}(N, P) \xrightarrow{\operatorname{Hom}(f, P)} \operatorname{Hom}(N', P).$$

The left-exactness of $\operatorname{Hom}(-, \operatorname{Hom}(M, P))$ implies that the bottom row of the following diagram is exact:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Hom}(M \otimes N'', P) & \longrightarrow & \operatorname{Hom}(M \otimes N, P) & \longrightarrow & \operatorname{Hom}(M \otimes N', P) \\
 & & \cong \downarrow \Phi'' & & \cong \downarrow \Phi & & \cong \downarrow \Phi' \\
0 & \longrightarrow & \operatorname{Hom}(N'', \operatorname{Hom}(M, P)) & \longrightarrow & \operatorname{Hom}(N, \operatorname{Hom}(M, P)) & \longrightarrow & \operatorname{Hom}(N', \operatorname{Hom}(M, P)).
\end{array}$$

Remark II.5.2 tell us that the diagram commutes, and Proposition II.5.1 says that the vertical maps are isomorphisms. Hence, the top row is exact. □

Exercises.

Exercise II.5.3. Fill in the details for the proof of Proposition II.5.1.

Exercise II.5.4. Show that all six diagrams in Remark II.5.2 commute.

Exercise II.5.5. Let R be a commutative ring. Prove that a sequence

$$M' \to M \to M'' \to 0$$

of R-module homomorphisms is exact if and only if for all R-modules N the associated sequence

$$0 \to \operatorname{Hom}_R(M'', N) \to \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(M', N)$$

is exact.

Exercise II.5.6. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Let N be an R-module, and let M and P be S-modules. Prove that there are S-module isomorphisms

$$\begin{array}{ccc}
\operatorname{Hom}_R(N, \operatorname{Hom}_S(M, P)) & \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} & \operatorname{Hom}_S(M \otimes_R N, P) \\
{[\alpha\colon N \to \operatorname{Hom}_S(M, P)]} & \longmapsto & \left[\begin{array}{l} M \otimes_R N \to P \\ m \otimes n \mapsto \alpha(n)(m) \end{array}\right] \\
\left[\begin{array}{l} N \to \operatorname{Hom}_S(M, P) \\ n \mapsto [m \mapsto \beta(m \otimes n)] \end{array}\right] & \longmapsfrom & {[\beta\colon M \otimes_R N \to P]}.
\end{array}$$

Verify the version of Remark II.5.2 in this situation. [Note: Proposition II.2.6(a) shows that $M \otimes_R N$ is an S-module.]

Exercise II.5.7. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Let P be an R-module, and let M and N be S-modules. Prove that there are S-module

isomorphisms

$$\begin{array}{c} \mathrm{Hom}_S(N, \mathrm{Hom}_R(M,P)) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathrm{Hom}_R(M \otimes_S N, P) \\ [\alpha\colon N \to \mathrm{Hom}_S(M,P)] \longmapsto \left[\begin{array}{l} M \otimes_S N \to P \\ m \otimes n \mapsto \alpha(n)(m) \end{array}\right] \\ \left[\begin{array}{l} N \to \mathrm{Hom}_R(M,P) \\ n \mapsto [m \mapsto \beta(m \otimes n)] \end{array}\right] \longmapsto [\beta\colon M \otimes_S N \to P]. \end{array}$$

Verify the version of Remark II.5.2 in this situation. [Note: Fact I.5.7 shows that $\mathrm{Hom}_R(M,P)$ is an S-module.]

Exercise II.5.8. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings, and let M and N be R-modules. Prove that there is an S-module isomorphism

$$\begin{array}{c} \mathrm{Hom}_R(S, \mathrm{Hom}_R(M,N)) \underset{\Gamma}{\overset{\Theta}{\rightleftarrows}} \mathrm{Hom}_S(S \otimes_R M, \mathrm{Hom}_R(S,N)) \\ [\psi\colon S \to \mathrm{Hom}_R(M,N)] \longmapsto \left[\begin{array}{l} S \otimes_S M \to \mathrm{Hom}_R(S,N) \\ s \otimes m \mapsto [t \mapsto \psi(st)(m)] \end{array}\right] \\ \left[\begin{array}{l} S \to \mathrm{Hom}_R(M,N) \\ s \mapsto [m \mapsto \phi(1 \otimes m)(s)] \end{array}\right] \longmapsto [\phi\colon S \otimes_R M \to \mathrm{Hom}_R(S,N)]. \end{array}$$

Verify the version of Remark II.5.2 in this situation. [Note: Fact I.5.7 shows that $\mathrm{Hom}_R(M,P)$ is an S-module.]

CHAPTER III

Injective, Projective, and Flat Modules September 8, 2009

III.1. Injective and Projective Modules

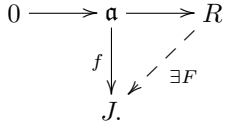
The definition of "projective module" is given in Exercise II.4.18. We begin with a reminder of some properties.

Remark III.1.1. Let R be a commutative ring. Every free R-module is projective. More specifically, an R-module P is projective if and only if there is an R-module Q such that $P \oplus Q$ is free. Given a set of R-modules $\{M_\lambda\}_{\lambda\in\Lambda}$, the coproduct $\coprod_{\lambda\in\Lambda} M_\lambda$ is a projective R-module if and only if M_λ is a projective R-module for each $\lambda \in \Lambda$. An R-module M is projective if and only if every short exact sequence $0 \to M' \to M'' \to M \to 0$ splits. If M is a projective R-module and $0 \to M' \to M'' \to M \to 0$ is an exact sequence of R-module homomorphisms, then M' is projective as an R-module if and only if M'' is projective as an R-module. Given an R-module N, there exists a projective R-module P and an R-module epimorphism $P \to N$.

The definition of "injective module" is given in Exercise II.4.19. It is dual to the definition of "projective module". We begin with a reminder of some properties.

Remark III.1.2. Let R be a commutative ring. Given a set $\{M_\lambda\}_{\lambda\in\Lambda}$ of R-modules, the product $\prod_{\lambda\in\Lambda} M_\lambda$ is an injective R-module if and only if M_λ is an injective R-module for each $\lambda \in \Lambda$. If M is injective as an R-module, then every short exact sequence $0 \to M \to M'' \to M' \to 0$ splits. If M is an injective R-module and $0 \to M \to M'' \to M' \to 0$ is an exact sequence of R-module homomorphisms, then M' is injective as an R-module if and only if M'' is injective as an R-module.

The following result is very useful in practice. Condition (ii) is summarized in the following diagram:



In other words, to check whether a given R-module is injective, one need only verify the definition for exact sequences of the form $0 \to \mathfrak{a} \to R$.

Theorem III.1.3 (Baer's criterion)**.** *Let R be a commutative ring, and let J be an R-module. The following conditions are equivalent:*

(i) *J is injective as an R-module;*
(ii) *for each ideal $\mathfrak{a} \subseteq R$ and each R-module homomorphism $f\colon \mathfrak{a} \to J$, there exists an R-module homomorphism $F\colon R \to J$ such that $F|_{\mathfrak{a}} = f$.*

PROOF. The implication (i) $\Longrightarrow$ (ii) follows by the definition in Exercise II.4.19.

(ii) $\Longrightarrow$ (i). Assume that J satisfies condition (ii), and consider a diagram of R-module homomorphisms with exact top row

$$\begin{array}{ccccc} 0 & \longrightarrow & M' & \xrightarrow{f} & M \\ & & {\scriptstyle g}\downarrow & \swarrow_{\exists h} & \\ & & J. & & \end{array}$$

We need to find an R-module homomorphism $h\colon M \to J$ making the diagram commute. For this, we use Zorn's Lemma. Set

$$S = \{R\text{-module homomorphisms } h\colon C \to J \mid \mathrm{Im}(f) \subseteq C \subseteq M \text{ and } hf = g\}.$$

Partially order S as follows: $(h_1\colon C_1 \to J) \leqslant (h_2\colon C_2 \to J)$ if and only if $C_1 \subseteq C_2$ and $h_2|_{C_1} = h_1$. Check that this is a partial order on S.

Claim: S satisfies the hypotheses of Zorn's Lemma. Let $\mathcal{C}$ be a non-empty chain in S. Define $D = \cup_{(h\colon C \to J) \in \mathcal{C}} C$. Since $\mathcal{C}$ is a chain in S, it follows that D is a submodule of M such that $\mathrm{Im}(f) \subseteq D$. Define $k\colon D \to J$ as follows. For each $d \in D$, there exists $(h\colon C \to J) \in \mathcal{C}$ such that $d \in C$; set $k(d) = h(d)$. Since $\mathcal{C}$ is a chain, it follows that $k(d)$ is independent of the choice of $(h\colon C \to J) \in \mathcal{C}$. Since $\mathcal{C}$ is a chain and each $(h\colon C \to J) \in \mathcal{C}$ is an R-module homomorphism, it is straightforward to show that k is an R-module homomorphism and that $kf = g$. In other words, $k\colon D \to J$ is in S. By construction, $(h\colon C \to J) \leqslant (k\colon D \to J)$ for each $(h\colon C \to J) \in \mathcal{C}$, so $(k\colon D \to J)$ is an upper bound for $\mathcal{C}$ in S.

Zorn's Lemma implies that S has a maximal element $(h\colon C \to J)$. We will use the maximality to show that $C = M$. It will then follow that $(h\colon M \to J) \in S$, so $h\colon M \to J$ makes the desired diagram commute.

Suppose that $C \subsetneq M$ and let $m \in M \smallsetminus C$. Set

$$\mathfrak{a} = \{r \in R \mid rm \in C\}.$$

Check that this is an ideal of R. Define $\phi\colon \mathfrak{a} \to J$ by the formula $\phi(r) = h(rm)$. Check that this is an R-module homomorphism. Condition (ii) yields an R-module homomorphism $\psi\colon R \to J$ making the following diagram commute

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathfrak{a} & \xrightarrow{\subseteq} & R \\ & & {\scriptstyle \phi}\downarrow & \swarrow_{\psi} & \\ & & J. & & \end{array}$$

Define $C' = C + Rm$ which is a submodule of M such that $\mathrm{Im}(g) \subseteq C \subsetneq C' \subseteq M$. We will construct an R-module homomorphism $h'\colon C' \to J$ such that $h'f = g$ and $h'|_C = h$; this will show that $(h'\colon C' \to J) \in S$ and $(h\colon C \to J) < (h'\colon C' \to J)$, thus contradicting the maximality of $(h\colon C \to J)$ in S.

Define $h'\colon C' \to J$ by the formula $h'(c + rm) = h(c) + \psi(r)$. We need to show that this is well-defined, so assume that $c + rm = c_1 + r_1 m$. It follows that $(r - r_1)m = rm - r_1 m = c_1 - c \in C$, and hence $r - r_1 \in I$. The next computation follows directly

$$h(c_1) - h(c) = h(c_1 - c) = h((r - r_1)m) = \phi(r - r_1) = \psi(r - r_1) = \psi(r) - \psi(r_1)$$

and hence $h(c_1) + \psi(r_1) = h(c) + \psi(r)$. Thus, h' is well-defined.

It is straightforward to show that h' is an R-module homomorphism, because h and ψ are R-module homomorphisms. For $m' \in M'$ we have $f(m') \in \operatorname{Im}(f) \subseteq C$, so (using $c = f(m')$ and $r = 0$) we have

$$h'(f(m')) = h(f(m')) = g(m')$$

because $hf = g$. It follows that $h'f = g$ as well. A similar argument shows that $h'|_C = h$, as desired. □

Corollary III.1.4. *Let R be a commutative noetherian ring, and let J be an R-module. The following conditions are equivalent:*

(i) *J is injective as an R-module;*
(ii) *for each monomorphism $\alpha\colon M \to N$ between finitely generated R-modules, the induced map $\operatorname{Hom}_R(\alpha, J)\colon \operatorname{Hom}_R(N, J) \to \operatorname{Hom}_R(M, J)$ is an epimorphism.*

PROOF. The implication (i) $\implies$ (ii) follows from Exercise II.4.19. For the converse, assume that for each R-module monomorphism $\alpha\colon M \to N$, the induced homomorphism $\operatorname{Hom}_R(\alpha, J)\colon \operatorname{Hom}_R(N, J) \to \operatorname{Hom}_R(M, J)$ is an epimorphism. We use Baer's criterion to show that J is injective. Let $\mathfrak{a}$ be an ideal of R, and let $i\colon \mathfrak{a} \to R$ denote the inclusion. Since R is noetherian, the ideal $\mathfrak{a}$ is finitely generated. Since R is also finitely generated, our assumption implies that the map $\operatorname{Hom}_R(i, J)\colon \operatorname{Hom}_R(R, J) \to \operatorname{Hom}_R(\mathfrak{a}, J)$ is surjective. Hence, there is an element $F \in \operatorname{Hom}_R(R, J)$ satisfying the first equality in the next sequence

$$f = \operatorname{Hom}_R(i, J)(F) = F \circ i = F|_{\mathfrak{a}}.$$

The other equalities are by definition. Baer's criterion implies that J is injective. □

Injective modules are harder to construct than projective ones. Here are a few examples.

Example III.1.5. If R is a field, then every R-module is injective. (The converse of this statement also holds when R is local or an integral domain.) See Exercise III.1.24.

Proposition III.1.6. *If R is an integral domain with field of fractions K, then K is injective as an R-module.*

PROOF. Assume that R is an integral domain. We use Baer's Criterion to prove that the field of fractions K is an injective R-module. Let $\mathfrak{a} \subseteq R$ be an ideal, and let $f\colon \mathfrak{a} \to K$ be an R-module homomorphism.

Claim: There exists an element $u \in K$ such that $f(a) = au$ for all $a \in \mathfrak{a}$. (Once this is shown we are done because we may define $F\colon R \to K$ by the formula $F(a) = au$ for all $a \in R$.)

Proof of claim: This is easy if $\mathfrak{a} = 0$, so assume $\mathfrak{a} \neq 0$. Fix an element $0 \neq b \in \mathfrak{a}$. Then, for all $a \in \mathfrak{a}$ we have

$$f(a) = \tfrac{b}{b} f(a) = \tfrac{1}{b} f(ba) = \tfrac{a}{b} f(b) = a\tfrac{f(b)}{b}$$

so the fraction $u = f(b)/b$ has the desired property. □

We next prove that every R-module is isomorphic to a submodule of an injective R-module. This requires some preparation, beginning with the case $R = \mathbb{Z}$.

Definition III.1.7. An abelian group D is *divisible* if, for each $d \in D$ and for each $0 \neq n \in \mathbb{Z}$, there exists $e \in D$ such that $ne = d$.

Remark III.1.8. These groups are "divisible" because you can always solve the division problem $d \div n$ in D.

Example III.1.9. The additive group $\mathbb{Q}$ is divisible.

Remark III.1.10. If D is a divisible abelian group and $D' \subseteq D$ is a subgroup, then the quotient D/D' is divisible. Every direct sum and direct product of divisible abelian groups is divisible.

Lemma III.1.11. *An abelian group G is divisible if and only if it is injective as a $\mathbb{Z}$-module.*

PROOF. $\Longrightarrow$: Assume that G is divisible. We use Baer's criterion to prove that G is injective as a $\mathbb{Z}$-module. Let $\mathfrak{a} \subseteq \mathbb{Z}$ be an ideal, and let $g\colon \mathfrak{a} \to G$ be a $\mathbb{Z}$-module homomorphism. Then $\mathfrak{a} = n\mathbb{Z}$ for some $n \geqslant 0$. We need to find a $\mathbb{Z}$-module homomorphism $h\colon \mathbb{Z} \to G$ making the following diagram commute

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathfrak{a} & \xrightarrow{\subseteq} & \mathbb{Z} \\ & & {\scriptstyle g}\downarrow & \swarrow_{\exists h} & \\ & & G. & & \end{array}$$

The case $n = 0$ is straightforward using $h = 0$, so assume that $n > 0$. Since G is divisible, there exists $a \in G$ such that $na = g(n)$. It follows that $g(mn) = mg(n) = mna$ for all $m \in \mathbb{Z}$. Define $h\colon \mathbb{Z} \to G$ as $h(m) = ma$ for all $m \in \mathbb{Z}$. This is a well-defined $\mathbb{Z}$-module homomorphism such that $h|_{\mathfrak{a}} = g$, as desired.

$\Longleftarrow$: Assume that G is injective as a $\mathbb{Z}$-module. To show that G is divisible, let $0 \neq n \in \mathbb{Z}$ and let $b \in G$. We need to find an element $c \in G$ such that $nc = b$. Define $g\colon n\mathbb{Z} \to G$ by the formula $g(nm) = mb$. This is a well-defined $\mathbb{Z}$-module homomorphism, so the fact that G is injective provides a $\mathbb{Z}$-module homomorphism $h\colon \mathbb{Z} \to G$ making the following diagram commute

$$\begin{array}{ccccc} 0 & \longrightarrow & I & \xrightarrow{\subseteq} & \mathbb{Z} \\ & & {\scriptstyle g}\downarrow & \swarrow_{\exists h} & \\ & & G. & & \end{array}$$

In particular, the element $c = h(1)$ satisfies

$$nc = nh(1) = h(n) = b$$

as desired. □

Proposition III.1.12. *The $\mathbb{Z}$-modules $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective.*

PROOF. It is straightforward to show that $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are divisible, so they are injective by Lemma III.1.11. □

Proposition III.1.13. *Let G be a $\mathbb{Z}$-module.*

(a) *For each non-zero element $0 \neq g \in G$, there is a $\mathbb{Z}$-module homomorphisms $\phi\colon G \to \mathbb{Q}/\mathbb{Z}$ such that $\phi(g) \neq 0$.*

(b) *The natural $\mathbb{Z}$-module homomorphism $\delta_G\colon G \to \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})$ given by $\delta_G(g)(\psi) = \psi(g)$ is a monomorphism.*

PROOF. (a) Let $\mathbb{Z}g \subseteq G$ denote the $\mathbb{Z}$-submodule of G generated by g, and set

$$\operatorname{Ann}_{\mathbb{Z}}(g) = \{n \in \mathbb{Z} \mid ng = 0\}.$$

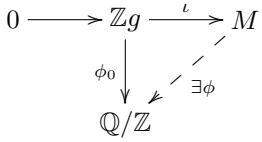
It is straightforward to show that $\operatorname{Ann}_{\mathbb{Z}}(g)$ is an ideal of $\mathbb{Z}$. In fact, $\operatorname{Ann}_{\mathbb{Z}}(g)$ is the kernel of the natural epimorphism $\tau\colon \mathbb{Z} \to \mathbb{Z}g$ given by $\tau(n) = ng$. Let $\overline{\tau}\colon \mathbb{Z}/\operatorname{Ann}_{\mathbb{Z}}(g) \to \mathbb{Z}g$ be the induced isomorphism, which is given by $\overline{\tau}(\overline{n}) = ng$.

Since $g \neq 0$, we have $\operatorname{Ann}_{\mathbb{Z}}(g) \subsetneq \mathbb{Z}$. In particular, there is a prime number $p \in \mathbb{Z}$ such that $\operatorname{Ann}_{\mathbb{Z}}(g) \subseteq p\mathbb{Z}$. Let $\alpha\colon \mathbb{Z}/p\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$ be the $\mathbb{Z}$-module homomorphism given by $\alpha(\overline{n}) = \overline{n/p}$. It is straightforward to show that α is a monomorphism. Let $\phi_0\colon \mathbb{Z}g \to \mathbb{Q}/\mathbb{Z}$ be the composition of the following maps

$$\mathbb{Z}g \xrightarrow{\overline{\tau}^{-1}} \mathbb{Z}/\operatorname{Ann}_{\mathbb{Z}}(g) \xrightarrow{\pi} \mathbb{Z}/p\mathbb{Z} \xrightarrow{\alpha} \mathbb{Q}/\mathbb{Z}$$

where π is the natural surjection. By definition, we have $\phi_0(g) = \overline{1/p} \neq 0$.

Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, there exists a $\mathbb{Z}$-module homomorphism $\phi\colon G \to \mathbb{Q}/\mathbb{Z}$ making the following diagram commute



where ι is the natural inclusion. It follows that $\phi(g) = \phi_0(g) \neq 0$, so ϕ has the desired properties.

(b) It is straightforward to show that δ_G is a $\mathbb{Z}$-module homomorphism, so it remains to show that it is injective. Let $0 \neq g \in G$, and let $\phi \in \operatorname{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$ be a $\mathbb{Z}$-module homomorphism such that $\phi(g) \neq 0$. It follows that $\delta_G(g)(\phi) = \phi(g) \neq 0$, and hence $\delta_G(g) \neq 0$. That is, we have $g \notin \operatorname{Ker}(\delta_G)$. Since g was chosen as an arbitrary non-zero element of G, it follows that δ_G is injective. □

Lemma III.1.14. *Let G be an abelian group. Then there is a divisible abelian group D and an abelian group monomorphism $f\colon G \hookrightarrow D$.*

PROOF. Let $\tau\colon F \to G$ be an epimorphism such that F is a free abelian group. Let $K = \operatorname{Ker}(\tau)$ so that we have $G \cong F/K$. Write $F \cong \mathbb{Z}^{(\Lambda)}$ for some set Λ and set $D_1 = \mathbb{Q}^{(\Lambda)}$. Remark III.1.10 shows that D is divisible. It is straightforward to construct an abelian group monomorphism $i\colon \mathbb{Z}^{(\Lambda)} \hookrightarrow \mathbb{Q}^{(\Lambda)}$, i.e., $i\colon F \hookrightarrow D_1$. Since i is a monomorphism, it follows that

$$G \cong F/K \cong i(F)/i(K) \subseteq D_1/i(K).$$

Since D_1 is divisible, so is the quotient $D_1/i(K)$, by Remark III.1.10. Thus, we have the desired monomorphism. □

Here is a way to construct injective R-modules.

Lemma III.1.15. *Let R be a commutative ring. If D is a divisible abelian group, then $\operatorname{Hom}_{\mathbb{Z}}(R, D)$ is an injective R-module.*

PROOF. This follows from Lemma III.1.11 and Exercise III.1.25(a), using the natural ring homomorphism $\mathbb{Z} \to R$. □

Theorem III.1.16. *Let R be a commutative ring, and let M be an R-module. Then there exists an R-module monomorphism $M \hookrightarrow J$ where J is an injective R-module.*

PROOF. M is an additive abelian group, so Lemma III.1.14 yields an abelian group monomorphism $f\colon M \hookrightarrow D$ where D is a divisible abelian group. The induced map $\operatorname{Hom}_{\mathbb{Z}}(R, f)\colon \operatorname{Hom}_{\mathbb{Z}}(R, M) \to \operatorname{Hom}_{\mathbb{Z}}(R, D)$ is an R-module homomorphism. It is a monomorphism because $\operatorname{Hom}_{\mathbb{Z}}(R, -)$ is left exact. This yields a sequence

$$M \xrightarrow{\cong} \operatorname{Hom}_R(R, M) \subseteq \operatorname{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \operatorname{Hom}_{\mathbb{Z}}(R, D)$$

where the inclusion is from the fact that every R-module homomorphism $R \to M$ is a $\mathbb{Z}$-module homomorphism. The composition of these maps is an R-module monomorphism. The R-module $J = \operatorname{Hom}_{\mathbb{Z}}(R, D)$ is injective by Lemma III.1.15, giving the desired result. □

Proposition III.1.17. *Let R be a commutative ring, and let I be an R-module. The following conditions are equivalent:*

(i) *I is an injective R-module;*
(ii) *Every exact sequence of the form $0 \to I \to M \to M'' \to 0$ splits.*

PROOF. The implication (i) $\Longrightarrow$ (ii) is contained in Remark III.1.2.

(ii) $\Longrightarrow$ (i) Theorem III.1.16 shows that there is an R-module monomorphism $f\colon I \to J$ such that J is injective. Condition (ii) implies that the resulting short exact sequence splits

$$0 \to I \xrightarrow{f} J \to J/I \to 0$$

and hence $J \cong I \oplus J/I$. The fact that J is injective implies that I and J/I are injective by Remark III.1.2. □

Proposition III.1.18. *If R is a commutative noetherian ring, then every coproduct of injective R-modules is injective.*

PROOF. Assume that R is noetherian, and let $\{I_\lambda\}_{\lambda\in\Lambda}$ be a set of injective R-modules. Let $\epsilon\colon \coprod_{\lambda\in\Lambda} M_\lambda \to \prod_{\lambda\in\Lambda} M_\lambda$ denote the canonical inclusion. For each $\mu \in \Lambda$, let $\pi_\mu\colon \prod_{\lambda\in\Lambda} M_\lambda \to M_\mu$ be the canonical surjection.

We use Baer's Criterion to show that $\coprod_\lambda I_\lambda$ is injective. Let $\mathfrak{a}$ be a non-zero ideal of R, and let $i\colon \mathfrak{a} \to R$ be the natural inclusion. Let $f\colon \mathfrak{a} \to \coprod_\lambda I_\lambda$ be an R-module homomorphism. Since R is noetherian, the ideal $\mathfrak{a}$ is finitely generated. Proposition I.5.9 shows that there are isomorphism of R-modules

$$\theta_{\mathfrak{a}}\colon \operatorname{Hom}_R(\mathfrak{a}, \coprod_\lambda I_\lambda) \xrightarrow{\cong} \coprod_\lambda \operatorname{Hom}_R(\mathfrak{a}, I_\lambda)$$
$$\theta_R\colon \operatorname{Hom}_R(R, \coprod_\lambda I_\lambda) \xrightarrow{\cong} \coprod_\lambda \operatorname{Hom}_R(R, I_\lambda)$$

given by $\Psi \mapsto (\pi_\lambda \epsilon \Psi)$. It is straightforward to show that the following diagram of R-module homomorphisms commutes

$$\begin{array}{ccccc}
\operatorname{Hom}_R(R, \coprod_\lambda I_\lambda) & \xrightarrow{\operatorname{Hom}_R(i, \coprod_\lambda I_\lambda)} & \operatorname{Hom}_R(\mathfrak{a}, \coprod_\lambda I_\lambda) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \theta_R}\cong & & \downarrow{\scriptstyle \theta_{\mathfrak{a}}}\cong & & \\
\coprod_\lambda \operatorname{Hom}_R(R, I_\lambda) & \xrightarrow{\coprod_\lambda \operatorname{Hom}_R(i, I_\lambda)} & \coprod_\lambda \operatorname{Hom}_R(\mathfrak{a}, I_\lambda) & \longrightarrow & 0.
\end{array}$$

Since each I_λ is injective, it follows that the bottom row of this diagram is exact. A straightforward diagram chase shows that this implies that the top row is also exact. It follows that there exists an R-module homomorphism $F\colon R \to \coprod_\lambda I_\lambda$ such that $F|_{\mathfrak{a}} = f$. □

Compare the following result to Corollary VII.5.13(c).

Proposition III.1.19. *Let R be a commutative noetherian ring, and let I be an R-module. The following conditions are equivalent:*

(i) *I is an injective R-module;*
(ii) *the localization $U^{-1}I$ is an injective $U^{-1}R$-module for each multiplicatively closed subset $U \subseteq R$;*
(iii) *the localization $I_{\mathfrak{p}}$ is an injective $R_{\mathfrak{p}}$-module for each prime ideal $\mathfrak{p} \subset R$; and*
(iv) *the localization $I_{\mathfrak{m}}$ is an injective $R_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m} \subset R$.*

PROOF. The implications (ii) $\implies$ (iii) $\implies$ (iv) are routine.

(i) $\implies$ (ii). Let I be an injective R-module. We use Baer's Criterion to show that $U^{-1}I$ is an injective $U^{-1}R$-module. Let $\mathfrak{a}$ be a non-zero ideal of $U^{-1}R$, and let $i\colon \mathfrak{a} \to U^{-1}R$ be the natural inclusion. It suffices to show that the induced map

$$\operatorname{Hom}_{U^{-1}R}(i, U^{-1}I)\colon \operatorname{Hom}_{U^{-1}R}(U^{-1}R, U^{-1}I) \to \operatorname{Hom}_{U^{-1}R}(\mathfrak{a}, U^{-1}I)$$

is surjective.

The ideal $\mathfrak{a}$ is isomorphic to $U^{-1}\mathfrak{b}$ for some ideal $\mathfrak{b} \subseteq R$; see Exercise II.2.14. Let $j\colon \mathfrak{b} \to R$ be the natural inclusion. Identify $U^{-1}\mathfrak{b}$ with its image in $U^{-1}R$, and identify the inclusion $i\colon \mathfrak{a} \to U^{-1}R$ with the induced map $U^{-1}j\colon U^{-1}\mathfrak{b} \to U^{-1}R$. Then it suffices to show that the map

$$\operatorname{Hom}_{U^{-1}R}(U^{-1}j, U^{-1}I)\colon \operatorname{Hom}_{U^{-1}R}(U^{-1}R, U^{-1}I) \to \operatorname{Hom}_{U^{-1}R}(U^{-1}\mathfrak{b}, U^{-1}I)$$

is surjective.

Since I is an injective R-module, the following sequence is exact:

$$\operatorname{Hom}_R(R, I) \xrightarrow{\operatorname{Hom}_R(j,I)} \operatorname{Hom}_R(\mathfrak{b}, I) \to 0.$$

The exactness of localization implies that the induced sequence

$$U^{-1}\operatorname{Hom}_R(R, I) \xrightarrow{U^{-1}\operatorname{Hom}_R(j,I)} U^{-1}\operatorname{Hom}_R(\mathfrak{b}, I) \to 0$$

is also exact, that is, the map $U^{-1}\operatorname{Hom}_R(j, I)$ is surjective.

Since R is noetherian, the ideal $\mathfrak{b}$ is finitely presented. Proposition I.5.8 yields the vertical isomorphisms in the following diagram:

$$\begin{array}{ccc} U^{-1}\operatorname{Hom}_R(R, I) & \xrightarrow{U^{-1}\operatorname{Hom}(j,I)} & U^{-1}\operatorname{Hom}_R(\mathfrak{b}, I) \\ \cong\downarrow & & \cong\downarrow \\ \operatorname{Hom}_{U^{-1}R}(U^{-1}R, U^{-1}I) & \xrightarrow{\operatorname{Hom}_{U^{-1}R}(U^{-1}j, U^{-1}I)} & \operatorname{Hom}_{U^{-1}R}(U^{-1}\mathfrak{b}, U^{-1}I). \end{array}$$

It is straightforward to verify that this diagram commutes. Since $U^{-1}\operatorname{Hom}_R(j, I)$ is surjective, we conclude that $\operatorname{Hom}_{U^{-1}R}(U^{-1}j, U^{-1}I)$ is surjective, as desired.

(iv) $\implies$ (i). Assume that $I_{\mathfrak{m}}$ is an injective $R_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m} \subset R$. Let $\mathfrak{b} \subseteq R$ be an ideal, and let $\epsilon\colon \mathfrak{b} \to R$ be the natural inclusion. We need to show that the induced map $\operatorname{Hom}_R(\epsilon, I)\colon \operatorname{Hom}_R(R, I) \to \operatorname{Hom}_R(\mathfrak{b}, I)$ is

surjective. Using Exercise I.4.27, it suffices to show that $\operatorname{Hom}_R(\epsilon, I)_{\mathfrak{m}}$ is surjective for each maximal ideal $\mathfrak{m} \subset R$.

Consider the exact sequence $0 \to \mathfrak{b}_{\mathfrak{m}} \xrightarrow{\epsilon_{\mathfrak{m}}} R_{\mathfrak{m}}$. Since $I_{\mathfrak{m}}$ is an injective $R_{\mathfrak{m}}$-module, the bottom row of the following diagram is exact:

$$\begin{array}{ccc} \operatorname{Hom}_R(R, I)_{\mathfrak{m}} & \xrightarrow{\operatorname{Hom}_R(\epsilon, I)_{\mathfrak{m}}} & \operatorname{Hom}_R(\mathfrak{b}, I)_{\mathfrak{m}} \\ \Theta_{R\smallsetminus\mathfrak{m}, R, I}\downarrow \cong & & \Theta_{R\smallsetminus\mathfrak{m}, \mathfrak{b}, I}\downarrow \cong \\ \operatorname{Hom}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}, I_{\mathfrak{m}}) & \xrightarrow{\operatorname{Hom}_{R_{\mathfrak{m}}}(\epsilon_{\mathfrak{m}}, I_{\mathfrak{m}})} & \operatorname{Hom}_R(\mathfrak{b}_{\mathfrak{m}}, I_{\mathfrak{m}}) \longrightarrow 0. \end{array}$$

Exercise I.5.16 implies that the diagram commutes. A straightforward diagram-chase shows that $\operatorname{Hom}_R(\epsilon, I)_{\mathfrak{m}}$ is surjective, as desired. □

Exercises.

Exercise III.1.20. Verify the properties from Remark III.1.1.

Exercise III.1.21. Let R be a commutative ring. Let M be an R-module, and let $\varphi\colon R \to S$ be a homomorphism of commutative rings.

(a) If M is projective as an R-module, then $S \otimes_R M$ is projective as an S-module.
(b) Show that the converse of part (a) fails in general.
(c) Prove that if M is a projective R-module, then the localization $U^{-1}M$ is a projective $U^{-1}R$-module for each multiplicatively closed subset $U \subseteq R$.
(d) Assume that M is finitely presented. Prove that the following conditions are equivalent:
   (i) M is a projective R-module;
   (ii) the localization $U^{-1}M$ is a projective $U^{-1}R$-module for each multiplicatively closed subset $U \subseteq R$;
   (iii) the localization $M_{\mathfrak{p}}$ is a projective $R_{\mathfrak{p}}$-module for each prime ideal $\mathfrak{p} \subset R$; and
   (iv) the localization $M_{\mathfrak{m}}$ is a projective $R_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m} \subset R$.

   Compare this to Corollaries VII.3.11 and VII.4.3.

Exercise III.1.22. (Schanuel's Lemma) Let R be a commutative ring. Consider two exact sequences of R-module homomorphisms

$$0 \longrightarrow K \xrightarrow{i} P_t \xrightarrow{f_t} P_{t-1} \xrightarrow{f_{t-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{f_1} P_0 \xrightarrow{\pi} M \longrightarrow 0$$
$$0 \longrightarrow L \xrightarrow{j} Q_t \xrightarrow{g_t} Q_{t-1} \xrightarrow{g_{t-1}} \cdots \xrightarrow{g_2} Q_1 \xrightarrow{g_1} Q_0 \xrightarrow{\tau} M \longrightarrow 0$$

where each P_i and Q_j is projective.

(a) Prove that $K \oplus Q_t \oplus P_{t-1} \oplus \cdots \cong L \oplus P_t \oplus Q_{t-1} \oplus \cdots$. (Note that each direct sum contains K and L and P_i's and Q_j's. It does not contain M. For instance, when $t = 0$, the isomorphism is $K \oplus Q_0 \cong L \oplus P_0$.)
(b) Prove that K is projective if and only if L is projective.

(See also Lemma VIII.4.10.)

Exercise III.1.23. Verify the properties from Remark III.1.2.

Exercise III.1.24. Let R be a commutative ring that is either local or an integral domain. Prove that the following conditions are equivalent:

(i) The ring R is a field;
(ii) Every R-module is free;
(iii) Every R-module is projective;
(iv) Every R-module is injective.

Provide examples showing that this fails if R is neither local nor an integral domain.

Exercise III.1.25. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings, and let M be an R-module.

(a) If M is injective as an R-module, then $\operatorname{Hom}_R(S, M)$ is injective as an S-module. (Make sure to specify the S-module structure on $\operatorname{Hom}_R(S, M)$.)
(b) Show that the converse of part (a) fails in general.

Exercise III.1.26. Finish the proof of Theorem III.1.3.

Exercise III.1.27. Verify the fact in Remark III.1.10.

Exercise III.1.28. Finish the proof of Proposition III.1.12.

Exercise III.1.29. Finish the proof of Proposition III.1.13.

Exercise III.1.30. State and prove the analogues of III.1.7–III.1.13 for modules over a principal ideal domain.

Exercise III.1.31. Finish the proof of Theorem III.1.15.

Exercise III.1.32. Finish the proof of Proposition III.1.18.

Exercise III.1.33. Finish the proof of Proposition III.1.19.

Exercise III.1.34. (Schanuel's Lemma) Let R be a commutative ring. Consider two exact sequences of R-module homomorphisms

$$0 \longrightarrow M \xrightarrow{i} I_t \xrightarrow{f_t} I_{t-1} \xrightarrow{f_{t-1}} \cdots \xrightarrow{f_2} I_1 \xrightarrow{f_1} I_0 \xrightarrow{\pi} C \longrightarrow 0$$
$$0 \longrightarrow M \xrightarrow{j} J_t \xrightarrow{g_t} J_{t-1} \xrightarrow{g_{t-1}} \cdots \xrightarrow{g_2} J_1 \xrightarrow{g_1} J_0 \xrightarrow{\tau} D \longrightarrow 0$$

where each I_j and J_i is injective.

(a) Prove that $C \oplus J_0 \oplus I_1 \oplus \cdots \cong D \oplus I_0 \oplus J_1 \oplus \cdots$. (Note that each direct sum contains C and D and I_j's and J_i's. It does not contain M. For instance, when $t = 0$, the isomorphism is $C \oplus J_0 \cong D \oplus I_0$.)
(b) Prove that C is injective if and only if D is injective.

III.2. Flat Modules

The goal of this section is to prove that, over a noetherian ring, every product of flat modules is flat. Much of this material is taken from Matsumura [**3**] and Rotman [**4**].

Lemma III.2.1. *Let R be a commutative ring. A sequence of R-module homomorphisms*

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \qquad (*)$$

is exact if and only if the induced sequence

$$\operatorname{Hom}_{\mathbb{Z}}(M'', \mathbb{Q}/\mathbb{Z}) \xrightarrow{\operatorname{Hom}_{\mathbb{Z}}(g,\mathbb{Q}/\mathbb{Z})} \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\operatorname{Hom}_{\mathbb{Z}}(f,\mathbb{Q}/\mathbb{Z})} \operatorname{Hom}_{\mathbb{Z}}(M', \mathbb{Q}/\mathbb{Z}) \quad (\dagger)$$

is exact.

PROOF. First note that every R-module is an additive abelian group, that is, a $\mathbb{Z}$-module. Hence, the sequence (†) is well-defined.

One implication is straightforward: If the sequence $(*)$ is exact, then the sequence (†) is exact because $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module; see Proposition III.1.12.

For the converse, assume that the sequence (†) is exact.

To show that $\operatorname{Im}(f) \subseteq \operatorname{Ker}(g)$, let $m = f(m') \in \operatorname{Im}(f)$, and suppose that $m \notin \operatorname{Ker}(g)$. It follows that $0 \neq g(m) \in M''$, so Proposition III.1.13(a) provides a $\mathbb{Z}$-module homomorphism $\phi\colon M'' \to \mathbb{Q}/\mathbb{Z}$ such that

$$0 \neq \phi(g(m)) = \phi(g(f(m'))) = \operatorname{Hom}_{\mathbb{Z}}(f, \mathbb{Q}/\mathbb{Z})(\operatorname{Hom}_{\mathbb{Z}}(g, \mathbb{Q}/\mathbb{Z})(\phi))(m').$$

This implies that $\operatorname{Hom}_{\mathbb{Z}}(f, \mathbb{Q}/\mathbb{Z}) \circ \operatorname{Hom}_{\mathbb{Z}}(g, \mathbb{Q}/\mathbb{Z}) \neq 0$, contradicting the exactness of the sequence (†).

To show that $\operatorname{Ker}(g) \subseteq \operatorname{Im}(f)$, let $m \in \operatorname{Ker}(g)$, and suppose that $m \notin \operatorname{Im}(f)$. Then the element $\overline{m} \in M/\operatorname{Im}(f)$ is non-zero, so Proposition III.1.13(a) provides a homomorphism $\psi\colon M/\operatorname{Im}(f) \to \mathbb{Q}/\mathbb{Z}$ such that $\psi(\overline{m}) \neq 0$. Let $\tau\colon M \to M/\operatorname{Im}(f)$ denote the natural surjection. Then we have

$$\psi(\tau(f(M'))) = \psi(\tau(\operatorname{Im}(f))) = \psi(0) = 0$$

that is

$$0 = \psi \circ \tau \circ f = \operatorname{Hom}_{\mathbb{Z}}(f, \mathbb{Q}/\mathbb{Z})(\psi \circ \tau).$$

This means that

$$\psi \circ \tau \in \operatorname{Ker}(\operatorname{Hom}_{\mathbb{Z}}(f, \mathbb{Q}/\mathbb{Z})) = \operatorname{Im}(\operatorname{Hom}_{\mathbb{Z}}(g, \mathbb{Q}/\mathbb{Z}))$$

where the last equality comes from the exactness of the sequence (†). This implies that there is a homomorphism $\beta \in \operatorname{Hom}_{\mathbb{Z}}(M'', \mathbb{Q}/\mathbb{Z})$ such that

$$\psi \circ \tau = \operatorname{Hom}_{\mathbb{Z}}(g, \mathbb{Q}/\mathbb{Z})(\beta) = \beta \circ g.$$

From this, we have the second equality in the next sequence

$$0 \neq \psi(\overline{m}) = \psi(\tau(m)) = \beta(g(m)) = \beta(0) = 0.$$

The non-vanishing is from our choice of ψ. The first equality is by definition, and the third equality is from the assumption $m \in \operatorname{Ker}(g)$. The displayed sequence is absurd, so we much have $m \in \operatorname{Im}(f)$. □

Remark III.2.2. Let R be a commutative ring, and let M be an R-module. The abelian group $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ has the structure of an R-module by the following action: for each $\phi \in \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ and each $r \in R$, we let $r\phi\colon M \to \mathbb{Q}/\mathbb{Z}$ be given by $(r\phi)(m) = \phi(rm)$. (This is a special case of Fact I.5.7(a) using the natural ring homomorphism $\mathbb{Z} \to R$.) The R-module $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is sometimes called the *character module* of M or the *Pontryagin dual* of M or the *Pontrjagin dual* of M.

Lemma III.2.3. *Let R be a commtuative ring. An R-module M is flat if and only if its character module* $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ *is injective.*

PROOF. Assume first that M is flat. To show that $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is injective, consider an exact sequence

$$0 \to L \xrightarrow{f} N.$$

It suffices to show that the induced sequence

$$\operatorname{Hom}_R(N, \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \xrightarrow{\operatorname{Hom}_R(f, \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}))} \operatorname{Hom}_R(L, \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) \to 0$$

is exact.

Since M is a flat R-module, the following sequence

$$0 \to M \otimes_R L \xrightarrow{M\otimes_R f} M \otimes_R N$$

is exact. Since $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module, the bottom row of the following diagram is exact:

$$\begin{array}{ccccc}
\operatorname{Hom}_R(N, \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) & \xrightarrow{\operatorname{Hom}_R(f, \operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}))} & \operatorname{Hom}_R(L, \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\operatorname{Hom}_{\mathbb{Z}}(M \otimes_R N, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\operatorname{Hom}_{\mathbb{Z}}(M\otimes_R f, \mathbb{Q}/\mathbb{Z})} & \operatorname{Hom}_{\mathbb{Z}}(M \otimes_R L, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0.
\end{array}$$

The vertical isomorphisms are Hom-tensor adjointness II.5.6. It is straightforward to show that the diagram commutes. (This is actually part of II.5.6.) A straightforward diagram chase shows that the top row is exact, as desired.

Conversely, assume that $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is injective. To show that M is flat, it suffices to start with an exact sequence

$$0 \to L \xrightarrow{f} N$$

and show that the induced sequence

$$0 \to M \otimes_R L \xrightarrow{M\otimes_R f} M \otimes_R N \tag{III.2.3.1}$$

is exact. Since $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is an injective R-module, the top row of the following commutative diagram is exact

$$\begin{array}{ccccc}
\operatorname{Hom}_R(N, \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) & \xrightarrow{\operatorname{Hom}_R(f, \operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}))} & \operatorname{Hom}_R(L, \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})) & \longrightarrow & 0 \\
\downarrow \cong & & \downarrow \cong & & \\
\operatorname{Hom}_{\mathbb{Z}}(M \otimes_R N, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\operatorname{Hom}_{\mathbb{Z}}(M\otimes_R f, \mathbb{Q}/\mathbb{Z})} & \operatorname{Hom}_{\mathbb{Z}}(M \otimes_R L, \mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0.
\end{array}$$

where the vertical isomorphisms are Hom-tensor adjointness II.5.6. A straightforward diagram chase shows that the bottom row is exact. Lemma III.2.1 implies that the sequence (III.2.3.1) is exact, as desired. □

Lemma III.2.4 (Baer's Criterion)**.** *Let R be a commtuative ring, and let M be an R-module. The following conditions are equivalent:*

(i) *the R-module M is flat; and*
(ii) *for every ideal $\mathfrak{a} \subseteq R$, the sequence $0 \to M \otimes_R \mathfrak{a} \xrightarrow{M\otimes_R i} M \otimes_R R$ is exact, where $i\colon \mathfrak{a} \to R$ is the inclusion.*

Proof. The implication (i) $\implies$ (ii) is by definition. For the converse, assume that for every ideal $\mathfrak{a} \subseteq R$, the sequence $0 \to M \otimes_R \mathfrak{a} \xrightarrow{M\otimes_R i} M \otimes_R R$ is exact, where $i\colon \mathfrak{a} \to R$ is the inclusion. We use Baer's criterion III.1.3 to show that $\operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is injective. (Then Lemma III.2.3 implies that M is flat.)

Let $\mathfrak{a} \subseteq R$ be an ideal and let $i\colon \mathfrak{a} \to R$ be the inclusion. Our assumption implies that the sequence $0 \to M \otimes_R \mathfrak{a} \xrightarrow{M\otimes_R i} M \otimes_R R$ is exact, so the fact that $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module implies that the bottom row of the following commutative

diagram is exact

$$\begin{array}{ccccc}
\operatorname{Hom}_R(R,\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})) & \xrightarrow{\operatorname{Hom}_R(i,\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z}))} & \operatorname{Hom}_R(\mathfrak{a},\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})) & \longrightarrow & 0 \\
\cong\downarrow & & \cong\downarrow & & \\
\operatorname{Hom}_{\mathbb{Z}}(M\otimes_R R,\mathbb{Q}/\mathbb{Z}) & \xrightarrow{\operatorname{Hom}_{\mathbb{Z}}(M\otimes_R i,\mathbb{Q}/\mathbb{Z})} & \operatorname{Hom}_{\mathbb{Z}}(M\otimes_R \mathfrak{a},\mathbb{Q}/\mathbb{Z}) & \longrightarrow & 0.
\end{array}$$

The vertical isomorphisms are Hom-tensor adjointness II.5.6. A straightforward diagram chase shows that the top row is exact. Since $\mathfrak{a}$ was chosen arbitrarily, it follows that $\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Q}/\mathbb{Z})$ is injective, as desired. □

The next result is proved like Corollary III.1.4.

Corollary III.2.5. *Let R be a commutative noetherian ring, and let F be an R-module. The following conditions are equivalent:*

(i) *the R-module F is flat; and*
(ii) *for each monomorphism $\alpha\colon M \to N$ between finitely generated R-modules, the induced map $F\otimes_R M \xrightarrow{M\otimes_R i} F\otimes_R N$ is a monomorphism.* □

Lemma III.2.6. *Let R be a commutative ring, let M be an R-module, and let n be a positive integer. Consider the free R-module R^n with basis $e_1,\dots,e_n$. If $\sum_{i=1}^n m_i\otimes e_i = \sum_{i=1}^n m_i'\otimes e_i$ in $M\otimes_R R^n$, then $m_i = m_i'$ for $i = 1,\dots,n$.*

Proof. We employ the following isomorphism from Theorem II.3.2

$$\alpha\colon M\otimes_R R^n \to M^n \qquad m\otimes\begin{pmatrix} r_1\\ \vdots\\ r_n\end{pmatrix} = m\otimes(\textstyle\sum_{i=1}^n r_ie_i)] \mapsto \begin{pmatrix} r_1m\\ \vdots\\ r_nm\end{pmatrix}.$$

The equation $\sum_{i=1}^n m_i\otimes e_i = \sum_{i=1}^n m_i'\otimes e_i$ in $M\otimes_R R^n$ implies that

$$\begin{pmatrix} m_1\\ \vdots\\ m_n\end{pmatrix} = \alpha(\textstyle\sum_{i=1}^n m_i\otimes e_i) = \alpha(\sum_{i=1}^n m_i'\otimes e_i) = \begin{pmatrix} m_1'\\ \vdots\\ m_n'\end{pmatrix}$$

in M^n and hence the desired equalities. □

Theorem III.2.7. *If R is a commutative noetherian ring, then every product of flat R-modules is flat.*

Proof. Let $\{F_\lambda\}_{\lambda\in\Lambda}$ be a set of flat R-modules. We use Baer's criterion III.2.4 to show that $\prod_\lambda F_\lambda$ is flat.

Let $\mathfrak{a}\subseteq R$ be an ideal and let $i\colon \mathfrak{a}\to R$ be the inclusion. Assume without loss of generality that $\mathfrak{a}\neq 0$. Since R is noetherian, the ideal $\mathfrak{a}$ is finitely generated, say $\mathfrak{a} = (a_1,\dots,a_n)R$ with $n\geqslant 1$. Let $e_1,\dots,e_n\in R^n$ be the standard basis, and let $g\colon R^n\to\mathfrak{a}$ be the epimorphism given by $e_i\mapsto a_i$. Set $K = \operatorname{Ker}(\tau)$ and let $\iota\colon K\to R^n$ denote the inclusion. Since R is noetherian, the module K is finitely generated, so there is an epimorphism $\tau\colon R^m\to K$ for some integer m. Set $h = \iota\circ\tau\colon R^m\to R^n$. It is straightforward to show that the following diagram has

exact row and column:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ R^m & \xrightarrow{h} & R^n & \xrightarrow{g} & \mathfrak{a} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle i} & & \\ & & & & R. & & \end{array}$$

Since each F_λ is flat, the next diagram has exact top row and column:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ F_\lambda \otimes_R R^m & \xrightarrow{F_\lambda \otimes_R h} & F_\lambda \otimes_R R^n & \xrightarrow{F_\lambda \otimes_R g} & F_\lambda \otimes_R \mathfrak{a} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle F_\lambda \otimes_R i} & & \\ & & & & F_\lambda \otimes_R R & \xrightarrow[\cong]{\alpha_\lambda} & F_\lambda. \end{array}$$

The map α_λ is the natural isomorphism given by $z_\lambda \otimes r \mapsto r z_\lambda$. The right exactness of tensor product implies that the next diagram has exact top row

$$\begin{array}{ccccccc} [\prod_\lambda F_\lambda] \otimes_R R^m & \xrightarrow{[\prod_\lambda F_\lambda] \otimes_R h} & [\prod_\lambda F_\lambda] \otimes_R R^n & \xrightarrow{[\prod_\lambda F_\lambda] \otimes_R g} & [\prod_\lambda F_\lambda] \otimes_R \mathfrak{a} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle [\prod_\lambda F_\lambda] \otimes_R i} & & \\ & & & & (\prod_\lambda F_\lambda) \otimes_R R & \xrightarrow[\cong]{\alpha} & (\prod_\lambda F_\lambda) \end{array}$$

where the map α is the natural isomorphism given by $(z_\lambda) \otimes r \mapsto r(z_\lambda)$. We need to show that the vertical map $[\prod_\lambda F_\lambda] \otimes_R i$ is injective, so let

$$\zeta \in \mathrm{Ker}([\textstyle\prod_\lambda F_\lambda] \otimes_R i) \subseteq [\prod_\lambda F_\lambda] \otimes_R \mathfrak{a}.$$

Write ζ as a finite sum of simple tensors in $[\prod_\lambda F_\lambda] \otimes_R \mathfrak{a}$. Then use the assumption $\mathfrak{a} = (a_1, \ldots, a_n)R$ to rewrite ζ as

$$\zeta = \textstyle\sum_{i=1}^n \zeta_i \otimes a_i$$

where each $\zeta_i = (\zeta_{i,\lambda}) \in \prod_\lambda F_\lambda$.

By assumption, we have

$$0 = ([\textstyle\prod_\lambda F_\lambda] \otimes_R i)(\zeta) = ([\prod_\lambda F_\lambda] \otimes_R i)(\sum_{i=1}^n \zeta_i \otimes a_i)) = \sum_{i=1}^n \zeta_i \otimes a_i$$

in $[\prod_\lambda F_\lambda] \otimes_R R$, and hence

$$0 = \alpha(([\textstyle\prod_\lambda F_\lambda] \otimes_R i)(\zeta)) = \sum_{i=1}^n a_i\zeta_i = \sum_{i=1}^n a_i(\zeta_{i,\lambda}) = (\sum_{i=1}^n a_i\zeta_{i,\lambda})$$

in $\prod_\lambda F_\lambda$. Thus, for each $\lambda \in \Lambda$ we have

$$0 = \textstyle\sum_{i=1}^n a_i\zeta_{i,\lambda} = \alpha_\lambda(\sum_{i=1}^n \zeta_{i,\lambda} \otimes a_i) = \alpha_\lambda(F_\lambda(\sum_{i=1}^n \zeta_{i,\lambda} \otimes a_i))$$

in F_λ. Since α_λ and F_λ are both monomorphisms, it follows that $\sum_{i=1}^n \zeta_{i,\lambda} \otimes a_i = 0$ in $F_\lambda \otimes_R \mathfrak{a}$ for each $\lambda \in \Lambda$. By construction, we have

$$0 = \textstyle\sum_{i=1}^n \zeta_{i,\lambda} \otimes a_i = (F_\lambda \otimes_R g)(\sum_{i=1}^n \zeta_{i,\lambda} \otimes e_i)$$

that is

$$\textstyle\sum_{i=1}^n \zeta_{i,\lambda} \otimes e_i \in \mathrm{Ker}(F_\lambda \otimes_R g) = \mathrm{Im}(F_\lambda \otimes_R h)$$

for each $\lambda \in \Lambda$, say that

$$\textstyle\sum_{i=1}^{n} \zeta_{i,\lambda} \otimes e_i = (F_\lambda \otimes_R h)(\nu_\lambda) \tag{III.2.7.1}$$

where $\nu_\lambda \in F_\lambda \otimes_R R^m$.

Let $\epsilon_1, \dots, \epsilon_m \in R^m$ be the standard basis. Write each ν_λ as a finite sum of simple tensors in $F_\lambda \otimes_R R^m$. Then use the fact that R^m is generated by $\epsilon_1, \dots, \epsilon_m$ to rewrite ν_λ as

$$\nu_\lambda = \textstyle\sum_{j=1}^{m} \nu_{\lambda,j} \otimes \epsilon_j$$

where each $\nu_{\lambda,j} \in F_\lambda$. Then equation (III.2.7.1) reads

$$\textstyle\sum_{i=1}^{n} \zeta_{i,\lambda} \otimes e_i = (F_\lambda \otimes_R h)(\sum_{j=1}^{m} \nu_{\lambda,j} \otimes \epsilon_j) = \sum_{j=1}^{m} \nu_{\lambda,j} \otimes h(\epsilon_j). \tag{III.2.7.2}$$

For $j = 1, \dots, m$ we write

$$h(\epsilon_j) = \textstyle\sum_{i=1}^{n} r_{j,i} e_i \tag{III.2.7.3}$$

for some elements $r_{j,i} \in R$. Then equation (III.2.7.2) reads

$$\textstyle\sum_{i=1}^{n} \zeta_{i,\lambda} \otimes e_i = \sum_{j=1}^{m} \nu_{\lambda,j} \otimes (\sum_{i=1}^{n} r_{j,i} e_i) = \sum_{i=1}^{n} (\sum_{j=1}^{m} r_{j,i} \nu_{\lambda,j}) \otimes e_i.$$

From Lemma III.2.6 we conclude that $\zeta_{i,\lambda} = \sum_{j=1}^{m} r_{j,i} \nu_{\lambda,j}$ in F_λ for $i = 1, \dots, n$, and hence

$$\zeta_i = (\zeta_{i,\lambda}) = (\textstyle\sum_{j=1}^{m} r_{j,i} \nu_{\lambda,j}) \tag{III.2.7.4}$$

in $\prod_\lambda F_\lambda$. Set

$$\omega = \textstyle\sum_{j=1}^{m} [(\nu_{\lambda,j}) \otimes \epsilon_j] \in [\prod_\lambda F_\lambda] \otimes_R R^m. \tag{III.2.7.5}$$

We show that

$$\zeta = ([\textstyle\prod_\lambda F_\lambda] \otimes_R h)(([\prod_\lambda F_\lambda] \otimes_R g)(\omega)).$$

(Then the exactness of the sequence

$$[\textstyle\prod_\lambda F_\lambda] \otimes_R R^m \xrightarrow{[\prod_\lambda F_\lambda] \otimes_R h} [\prod_\lambda F_\lambda] \otimes_R R^n \xrightarrow{[\prod_\lambda F_\lambda] \otimes_R g} [\prod_\lambda F_\lambda] \otimes_R \mathfrak{a} \to 0$$

implies that $\zeta = 0$, as desired.)

We compute directly:

$$\begin{aligned}
([\textstyle\prod_\lambda F_\lambda] \otimes_R h)(\omega) &= ([\prod_\lambda F_\lambda] \otimes_R h)(\textstyle\sum_{j=1}^{m} [(\nu_{\lambda,j}) \otimes \epsilon_j]) && \text{(by (III.2.7.5))} \\
&= \textstyle\sum_{j=1}^{m} [(\nu_{\lambda,j}) \otimes h(\epsilon_j)] && \text{(defn. of } [\textstyle\prod_\lambda F_\lambda] \otimes_R h) \\
&= \textstyle\sum_{j=1}^{m} [(\nu_{\lambda,j}) \otimes (\sum_{i=1}^{n} r_{j,i} e_i)] && \text{(by (III.2.7.3))} \\
&= \textstyle\sum_{j=1}^{m} \sum_{i=1}^{n} r_{j,i} [(\nu_{\lambda,j}) \otimes e_i] \\
&= \textstyle\sum_{i=1}^{n} [\sum_{j=1}^{m} r_{j,i} (\nu_{\lambda,j}) \otimes e_i] \\
&= \textstyle\sum_{i=1}^{n} [(\sum_{j=1}^{m} r_{j,i} \nu_{\lambda,j}) \otimes e_i] \\
&= \textstyle\sum_{i=1}^{n} \zeta_i \otimes e_i && \text{(by (III.2.7.4))}
\end{aligned}$$

and thus

$$\begin{aligned}
([\textstyle\prod_\lambda F_\lambda] \otimes_R g)(([\prod_\lambda F_\lambda] \otimes_R h)(\omega)) &= ([\textstyle\prod_\lambda F_\lambda] \otimes_R g)(\textstyle\sum_{i=1}^{n} \zeta_i \otimes e_i) \\
&= \textstyle\sum_{i=1}^{n} \zeta_i \otimes g(e_i) \\
&= \textstyle\sum_{i=1}^{n} \zeta_i \otimes a_i \\
&= \zeta
\end{aligned}$$

as claimed. □

Proposition III.2.8. *Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Let $P \subset S$ be a prime ideal and set $\mathfrak{p} = \varphi^{-1}(P)$.*

(a) *The ideal $\mathfrak{p} \subset R$ is prime, and there is a well-defined homomorphism of commutative rings $\varphi_P \colon R_\mathfrak{p} \to S_P$ given by $r/s \mapsto \varphi(r)/\varphi(s)$ that makes the following diagram commute*

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ \downarrow & & \downarrow \\ R_\mathfrak{p} & \xrightarrow{\varphi_P} & S_P \end{array}$$

where the unspecified vertical maps are the natural ones. The ring $R_\mathfrak{p}$ is local with maximal ideal $\mathfrak{p}R_\mathfrak{p}$, and S_P is local with maximal ideal PS_P, and one has $\varphi_P(\mathfrak{p}R_\mathfrak{p}) \subseteq PS_P$.

(b) *If S is flat as an R-module, then S_P is flat as an $R_\mathfrak{p}$-module.*

PROOF. (a) It is straightforward to show that $\mathfrak{p}$ is prime. The existence of the map φ_P follows from the universal mapping property for localization; see Fact I.4.8. It is straightforward to show that the diagram commutes. The fact that $R_\mathfrak{p}$ and S_P are local with maximal ideals as described is a standard fact. The containment $\varphi(\mathfrak{p}) \subseteq P$ implies that $\varphi_P(\mathfrak{p}R_\mathfrak{p}) \subseteq PS_P$ by the definitions.

(b) Assume that S is flat as an R-module. To show that S_P is flat as an $R_\mathfrak{p}$-module, let $L \xrightarrow{f} M \xrightarrow{g} N$ be an exact sequence of $R_\mathfrak{p}$-module homomorphisms. We will be done once we show that the following sequence is exact

$$L \otimes_{R_\mathfrak{p}} S_P \xrightarrow{f \otimes_{R_\mathfrak{p}} S_P} M \otimes_{R_\mathfrak{p}} S_P \xrightarrow{g \otimes_{R_\mathfrak{p}} S_P} N \otimes_{R_\mathfrak{p}} S_P. \tag{III.2.8.1}$$

Proposition II.2.9(b) implies that for every R-module A, there is an $R_\mathfrak{p}$-module isomorphism $\psi_A \colon A \otimes_R R_\mathfrak{p} \xrightarrow{\cong} A_\mathfrak{p}$ such that $a \otimes (r/s) \mapsto (ra)/s$. It is straightforward to show that, when A is an $R_\mathfrak{p}$-module, the natural map $A \to A_\mathfrak{p}$ is an $R_\mathfrak{p}$-module isomorphism; hence the map $i_A \colon A \to A \otimes_R R_\mathfrak{p}$ given by $a \mapsto a \otimes 1$ is an $R_\mathfrak{p}$-module isomorphism.

This yields the following commutative diagram with exact rows

$$\begin{array}{ccccc} L & \xrightarrow{f} & M & \xrightarrow{g} & N \\ {\scriptstyle i_L}\downarrow{\scriptstyle \cong} & & {\scriptstyle i_N}\downarrow{\scriptstyle \cong} & & {\scriptstyle i_M}\downarrow{\scriptstyle \cong} \\ L \otimes_R R_\mathfrak{p} & \xrightarrow{f \otimes_R R_\mathfrak{p}} & M \otimes_R R_\mathfrak{p} & \xrightarrow{g \otimes_R R_\mathfrak{p}} & N \otimes_R R_\mathfrak{p}. \end{array}$$

Applying the functor $- \otimes_{R_\mathfrak{p}} S_P$, we have the top half of the next commutative diagram

$$\begin{array}{ccccc} L \otimes_{R_\mathfrak{p}} S_P & \xrightarrow{f \otimes_{R_\mathfrak{p}} S_P} & M \otimes_{R_\mathfrak{p}} S_P & \xrightarrow{g \otimes_{R_\mathfrak{p}} S_P} & N \otimes_{R_\mathfrak{p}} S_P \\ {\scriptstyle i_L \otimes_{R_\mathfrak{p}} S_P}\downarrow{\scriptstyle \cong} & & {\scriptstyle i_N \otimes_{R_\mathfrak{p}} S_P}\downarrow{\scriptstyle \cong} & & {\scriptstyle i_M \otimes_{R_\mathfrak{p}} S_P}\downarrow{\scriptstyle \cong} \\ L \otimes_R R_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_P & \xrightarrow{f \otimes_R R_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_P} & M \otimes_R R_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_P & \xrightarrow{g \otimes_R R_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_P} & N \otimes_R R_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_P \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ L \otimes_R S_P & \xrightarrow{f \otimes_R S_P} & M \otimes_R S_P & \xrightarrow{g \otimes_R S_P} & N \otimes_R S_P. \end{array}$$

The bottom half comes from the cancellation isomorphism from Example II.1.9. Exercise III.2.11(b) implies that S_P is flat as an R-module, so the bottom row of this

diagram is exact. Since the diagram commutes and each vertical map is an isomorphism, we conclude that the top row is also exact. That is, the sequence (III.2.8.1) is exact, as desired. □

Exercises.

Exercise III.2.9. Let R be a commutative noetherian ring. Prove that, for every set Λ the R-module R^Λ is flat.

Exercise III.2.10. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Prove that, if M is a flat R-module, then $S \otimes_R M$ is a flat S-module.

Exercise III.2.11. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings such that S is flat as an R-module.

(a) Prove that, if N is a flat S-module, then N is flat as an R-module.
(b) Prove that the localization $U^{-1}S$ is flat as an R-module for every multiplicatively closed subset $U \subseteq S$.
(c) Prove that the polynomial ring $R[X_1, \dots, X_n]$ is flat as an R-module, and that the localization $U^{-1}R[X_1, \dots, X_n]$ is flat as an R-module for each multiplicatively closed subset $U \subseteq R[X_1, \dots, X_n]$.

Exercise III.2.12. Let R be a commutative ring, and let N be an R-module. Prove that the following conditions are equivalent:

(i) N is a flat R-module;
(ii) the localization $U^{-1}N$ is a flat $U^{-1}R$-module for each multiplicatively closed subset $U \subseteq R$;
(iii) the localization $N_\mathfrak{p}$ is a flat $R_\mathfrak{p}$-module for each prime ideal $\mathfrak{p} \subset R$; and
(iv) the localization $N_\mathfrak{m}$ is a flat $R_\mathfrak{m}$-module for each maximal ideal $\mathfrak{m} \subset R$.

Compare this to Exercise II.3.11 and Corollary VII.6.15.

Exercise III.2.13. Prove Corollary III.2.5.

Exercise III.2.14. Complete the proof of Proposition III.2.8.

Exercise III.2.15. Let R be a commutative ring. Let F be a flat R-module, and let $\phi\colon A \to B$ be an R-module homomorphism.

(a) Prove that there are R-module isomorphisms $F \otimes \operatorname{Im}(\phi) \cong \operatorname{Im}(F \otimes_R \phi)$ and $(F \otimes_R B)/\operatorname{Im}(F \otimes_R \phi) \cong F \otimes_R (B/\operatorname{Im}(\phi))$.
(b) Prove that there are R-module isomorphisms $F \otimes \operatorname{Ker}(\phi) \cong \operatorname{Ker}(F \otimes_R \phi)$ and $(F \otimes_R A)/\operatorname{Ker}(F \otimes_R \phi) \cong F \otimes_R (A/\operatorname{Ker}(\phi))$.
(c) Prove that, if A is a submodule of B, then $F \otimes_R A$ is naturally isomorphic to a submodule of $F \otimes_R B$ in such a way that $(F \otimes_R B)/(F \otimes_R A) \cong F \otimes_R (B/A)$.

III.3. Faithfully Flat Modules

Much of the material for this section comes from [**3**].

Definition III.3.1. Let R be a commutative ring. An R-module M is *faithfully flat* provided that, for every sequence $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ of R-module homomorphisms, the induced sequence $M \otimes_R A \xrightarrow{M\otimes_R\phi} M \otimes_R B \xrightarrow{M\otimes_R\psi} M \otimes_R C$ is exact if and only if the sequence $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ is exact.

Example III.3.2. Let R be a commutative ring. Every nonzero free R-module is faithfully flat.

Example III.3.3. The $\mathbb{Z}$-module $\mathbb{Q}$ is flat but is not faithfully flat. To see this, first recall that $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) = 0$. Next, note that the sequence $0 \to \mathbb{Z}/2\mathbb{Z} \to 0$ is not exact, while the induced sequence $\mathbb{Q} \otimes_{\mathbb{Z}} 0 \to \mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{Z}/2\mathbb{Z}) \to \mathbb{Q} \otimes_{\mathbb{Z}} 0$ is exact because it has the form $0 \to 0 \to 0$.

The following is a useful characterization of faithfully flat modules.

Theorem III.3.4. *Let R be a commutative ring, and let M be an R-module. The following conditions are equivalent:*

(i) *M is faithfully flat;*
(ii) *M is flat, and $M \otimes_R N \neq 0$ for each nonzero R-module N; and*
(iii) *M is flat, and $\mathfrak{m}M \neq M$ for each maximal ideal $\mathfrak{m} \subset R$.*

PROOF. (i) $\Longrightarrow$ (ii) Assume that M is faithfully flat, and let N be an R-module such that $M \otimes_R N = 0$. Applying the functor $M \otimes_R -$ to the sequence

$$0 \to N \to 0$$

yields a second sequence

$$0 \to M \otimes_R N \to 0.$$

The condition $M \otimes_R N = 0$ implies that the second sequence is exact. Since M is faithfully flat, the first sequence is also exact, and we conclude that $N = 0$.

(ii) $\Longrightarrow$ (iii) Assume that M is flat, and that $M \otimes_R N \neq 0$ for each nonzero R-module N. For each maximal ideal $\mathfrak{m} \subset R$, this yields the non-vanishing in the next sequence

$$M/\mathfrak{m}M \cong M \otimes_R R/\mathfrak{m} \neq 0$$

while the isomorphism is from Exercise II.4.14. It follows that $\mathfrak{m}M \neq M$, as desired.

(iii) $\Longrightarrow$ (ii) Assume that M is flat, and that $\mathfrak{m}M \neq M$ for each maximal ideal $\mathfrak{m} \subset R$. Let N be a nonzero R-module and fix a nonzero element $n \in N$. The following set is an ideal in R:

$$\operatorname{Ann}_R(n) = \{r \in R \mid rn = 0\}$$

and the fact that $n \neq 0$ implies that $\operatorname{Ann}_R(n) \subsetneq R$. In particular, there is a maximal ideal $\mathfrak{m} \subset R$ such that $\operatorname{Ann}_R(n) \subseteq \mathfrak{m}$. It follows that there is an R-module epimorphism $\tau\colon R/\operatorname{Ann}_R(n) \twoheadrightarrow R/\mathfrak{m}$.

Let $Rn \subseteq N$ denote the R-submodule of N generated by n. It is straightforward to show that $Rn \cong R/\operatorname{Ann}_R(n)$, so we have the first isomorphism in the following sequence

$$M \otimes_R (Rn) \cong M \otimes_R (R/\operatorname{Ann}_R(n)) \twoheadrightarrow M \otimes_R R/\mathfrak{m} \neq 0.$$

The epimorphism comes from the right-exactness of $M \otimes_R -$, and the non-vanishing is by assumption. It follows that $M \otimes_R (Rn) \neq 0$.

Let $i\colon Rn \to N$ denote the natural inclusion. Since M is flat, the induced map $M \otimes_R i\colon M \otimes_R (Rn) \to M \otimes_R N$ is a monomorphism. It follows that $M \otimes_R N$ contains the nonzero R-module $M \otimes_R (Rn)$, so we have $M \otimes_R N \neq 0$.

(ii) $\Longrightarrow$ (i) Assume that M is flat, and that $M \otimes_R N \neq 0$ for each nonzero R-module N. Consider a sequence

$$A \xrightarrow{\phi} B \xrightarrow{\psi} C \tag{III.3.4.1}$$

of R-module homomorphisms such that the induced sequence

$$M \otimes_R A \xrightarrow{M\otimes_R \phi} M \otimes_R B \xrightarrow{M\otimes_R \psi} M \otimes_R C \tag{III.3.4.2}$$

is exact.

Since M is flat, Exercise III.2.15(a) explains the isomorphism in the next sequence

$$M \otimes_R (\operatorname{Im}(\psi\phi)) \cong \operatorname{Im}(M \otimes_R (\psi\phi)) = 0.$$

The vanishing follows from the exactness of (III.3.4.2). Our assumption implies that $\operatorname{Im}(\psi\phi) = 0$, and hence $\psi\phi = 0$. That is, we have $\operatorname{Im}(\phi) \subseteq \operatorname{Ker}(\psi)$.

Since M is flat, Exercise III.2.15(c) explains the isomorphisms in the next sequence

$$\begin{aligned} M \otimes_R (\operatorname{Ker}(\psi)/\operatorname{Im}(\phi)) &\cong (M \otimes_R \operatorname{Ker}(\psi))/(M \otimes_R \operatorname{Im}(\phi)) \\ &\cong \operatorname{Ker}(M \otimes_R \psi)/\operatorname{Im}(M \otimes_R \phi) = 0 \end{aligned}$$

and the vanishing is from the exactness of (III.3.4.2). Our assumption implies that $\operatorname{Ker}(\psi)/\operatorname{Im}(\phi) = 0$, and hence $\operatorname{Im}(\phi) = \operatorname{Ker}(\psi)$, as desired. □

Exercises.

Exercise III.3.5. Justify the statement of Example III.3.2.

Exercise III.3.6. Complete the proof of Theorem III.3.4.

Exercise III.3.7. Let R be a commutative ring. Prove that the polyonomial ring $R[X_1, \dots, X_n]$ is faithfully flat as an R-module.

Exercise III.3.8. Let R be a commutative ring, and let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules.

(a) Prove that if each module M_λ is faithfully flat, then the coproduct $\coprod_\lambda M_\lambda$ is faithfully flat.
(b) Does the converse of part (a) hold? Justify your answer.
(c) Assume that R is noetherian. Prove that if each module M_λ is faithfully flat, then the product $\prod_\lambda M_\lambda$ is faithfully flat.

Exercise III.3.9. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Let M be an R-module, and let N be an S-module.

(a) Prove that if S is faithfully flat over R and N is faithfully flat over S, then N is faithfully flat over R.
(b) Prove that if N is flat over R and N is faithfully flat over S, then S is faithfully flat over R.
(c) Prove that if N is faithfully flat over R and N is faithfully flat over S, then S is faithfully flat over R.
(d) Prove that if M is faithfully flat over R, then $S \otimes_R M$ is faithfully flat over S.

(Hint: See Exercise II.4.15.)

III.4. Power Series Rings

The goal of this section is to familiarize the reader with the basic notions of power series rings.

Definition III.4.1. Let R be a commutative ring. The *ring of formal power series* in one variable is the ring $R[\![X]\!]$ defined as follows. The elements of $R[\![X]\!]$ are formal sums $\sum_{i=0}^{\infty} a_i X^i$ with each $a_i \in R$. The *constant term* of $\sum_{i=0}^{\infty} a_i X^i$ is a_0, and the *coefficients* of $\sum_{i=0}^{\infty} a_i X^i$ are the elements $a_0, a_1, a_2, \ldots \in R$. Addition and multiplication in $R[\![X]\!]$ are defined as in the polynomial ring $R[X]$:

$$\textstyle\sum_{i=0}^{\infty} a_i X^i + \sum_{i=0}^{\infty} b_i X^i = \sum_{i=0}^{\infty} (a_i + b_i) X^i$$
$$\textstyle(\sum_{i=0}^{\infty} a_i X^i)(\sum_{i=0}^{\infty} b_i X^i) = \sum_{i=0}^{\infty} c_i X^i$$

where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$. The additive and multiplicative identities are the same as for the polynomial ring:

$$0_{R[\![X]\!]} = 0_R = 0_R + 0_R X + 0_R X^2 + \cdots$$
$$1_{R[\![X]\!]} = 1_R = 1_R + 0_R X + 0_R X^2 + \cdots .$$

The ring of formal power series in n variables is defined inductively by the formula $R[\![X_1, \ldots, X_{n-1}, X_n]\!] = R[\![X_1, \ldots, X_{n-1}]\!][\![X_n]\!]$.

Remark III.4.2. Let R be a commutative ring. Our definition of $R[\![X]\!]$ is ad hoc in the same sense that the usual definition of the polynomial ring $R[X]$ is ad hoc. One constructs $R[X]$ as the coproduct $R[X] = \prod_{i=0}^{\infty} R$ of countably many copies of R. (In particular, the ring $R[X]$ is free as an R-module.) Similarly, one constructs $R[\![X]\!]$ as the product $R[\![X]\!] = \prod_{i=0}^{\infty} R$, of countably many copies of R.

Remark III.4.3. Let R be a commutative ring, and let n be a positive integer. Then $R[\![X_1, \ldots, X_n]\!]$ is a commutative ring with identity that contains the polynomial ring $R[X_1, \ldots, X_n]$ as a subring. In particular, it contains R as the subring of constant power series. Every element of $R[\![X_1, \ldots, X_n]\!]$ has a unique expression as a formal sum $\sum_{\mathbf{i} \in \mathbb{N}^n} a_{\mathbf{i}} X_1^{i_1} \cdots X_n^{i_n}$ where $\mathbf{i} = \{i_1, \ldots, i_n\}$. For each permutation σ of the set $\{1, \ldots, n\}$, there is an isomorphism of commutative rings $\varphi_\sigma \colon R[\![X_1, \ldots, X_n]\!] \xrightarrow{\cong} R[\![X_{\sigma(1)}, \ldots, X_{\sigma(n)}]\!]$ such that $\varphi_\sigma(r) = r$ for all $r \in R$ and $\varphi_\sigma(X_i) = X_i$ for $i = 1, \ldots, n$.

Proposition III.4.4. *Let R be a commutative ring, and let n be a positive integer.*

(a) *The ring R is an integral domain if and only if $R[\![X_1, \ldots, X_t]\!]$ is an integral domain for some (equivalently, for every) positive integer t.*

(b) *The variables $X_1, \ldots, X_n$ are non-zero-divisors on $R[\![X_1, \ldots, X_n]\!]$.*

Proof. (a) Every (non-zero) subring of an integral domain is an integral domain. Hence, if $R[\![X_1, \ldots, X_n]\!]$ is an integral domain, then so is $R \subseteq R[\![X_1, \ldots, X_n]\!]$.

We prove the converse by induction on n.

Base case: $n = 1$. Assume that R is an integral domain, and let f and g be non-zero elements of $R[\![X]\!]$. Since f and g are non-zero, they have the form $f = \sum_{i=r}^{\infty} a_i X^i$ and $g = \sum_{j=s}^{\infty} b_j X^j$ where a_r and b_s are non-zero elements of R. Since R is an integral domain, we have $a_r b n_s \neq 0$. A direct computation yields

$$fg = (a_r b_s) X^{r+s} + (a_r b_{s+1} + a_{r+1} b_s) X^{r+s+1} + \cdots .$$

Hence, the product fg has a non-zero coefficient, so $fg \neq 0$.

The induction step is a straightforward exercise.

(b) Fix a non-zero element $h = \sum_{\mathbf{i} \in \mathbb{N}^n} c_{\mathbf{i}} X_1^{i_1} \cdots X_n^{i_n} \in R[\![X_1, \ldots, X_n]\!]$. A direct computation yields

$$X_j h = \textstyle\sum_{\mathbf{i} \in \mathbb{N}^n} c_{\mathbf{i}} X_1^{i_1} \cdots X_j^{i_j + 1} \cdots X_n^{i_n}.$$

Since h is non-zero, it has a non-zero coefficient. The displayed equality shows that the non-zero coefficients of X_jh are the same as the non-zero coefficients of h. Hence, X_jh has a non-zero coefficient, that is, $X_jh \neq 0$. By definition, this means that X_j is a non-zero-divisor on $R[\![X_1,\dots,X_n]\!]$. □

Proposition III.4.5. *Let R be a commutative ring, and let n be a positive integer.*
(a) *A power series $\sum_{\mathbf{i}\in\mathbb{N}^n} a_{\mathbf{i}}X_1^{i_1}\cdots X_n^{i_n}$ is a unit in $R[\![X_1,\dots,X_n]\!]$ if and only if its constant term $a_{\mathbf{0}}$ is a unit in R.*
(b) *The ring $R[\![X_1,\dots,X_n]\!]$ is not a field.*

Proof. (a) We prove the case $n = 1$. The general case follows by induction on n. Fix an element $f = \sum_{i=0}^{\infty} a_iX^i \in R[\![X]\!]$.

Assume that $f = \sum_{i=0}^{\infty} a_iX^i$ is a unit in $R[\![X]\!]$ with $f^{-1} = \sum_{j=0}^{\infty} b_jX^j$. It follows that

$$1 = ff^{-1} = (a_0b_0) + (a_0b_1 + a_1b_0)X + \cdots .$$

This implies that $a_0b_0 = 1$, hence the constant term a_0 is a unit with inverse b_0.

For the converse, assume that a_0 is a unit in R. Set $b_0 = a_0^{-1}$. By induction on m, we may solve the following infinite system of equations for $b_0, b_1, b_2, \dots$:

$$\begin{aligned} a_0b_0 &= 1 \\ a_0b_1 + a_1b_0 &= 0 \\ a_0b_2 + a_1b_1 + a_2b_0 &= 0 \qquad\qquad \dots \end{aligned}$$

It follows readily that the series $\sum_{j=0}^{\infty} b_jX^j$ is a multiplicative inverse for f.

(b) The nonzero element $X_1 \in R[\![X_1,\dots,X_n]\!]$ is not a unit because its constant term is 0, which is not a unit in R. □

Proposition III.4.6. *Let R be a commutative ring, and let n be a positive integer.*
(a) *The ideal $\mathfrak{X} = (X_1,\dots,X_n)R[\![X_1,\dots,X_n]\!]$ consists of all formal power series in $R[\![X_1,\dots,X_n]\!]$ with constant term 0.*
(b) *One has $R[\![X_1,\dots,X_n]\!]/\mathfrak{X} \cong R$.*

Proof. (a) Let I denote the set of all formal power series in $R[\![X_1,\dots,X_n]\!]$ with constant term 0. The containment $\mathfrak{X} \subseteq I$ is straightforward since each generator X_i of $\mathfrak{X}$ is in I.

For the containment $\mathfrak{X} \supseteq I$, let $f = \sum_{\mathbf{i}\in\mathbb{N}^n} a_{\mathbf{i}}X_1^{i_1}\cdots X_n^{i_n} \in I$. Since f is in I, its constant term is 0, so we can rewrite

$$\begin{aligned} f &= \sum_{i_1\geqslant 1} a_{\mathbf{i}}X_1^{i_1}\cdots X_n^{i_n} + \sum_{\substack{i_1=0\\ i_2\geqslant 1}} a_{\mathbf{i}}X_2^{i_2}\cdots X_n^{i_n} + \cdots \\ &\quad + \sum_{\substack{i_1=i_2=\cdots=i_{n-2}=0\\ i_{n-1}\geqslant 1}} a_{\mathbf{i}}X_{n-1}^{i_{n-1}}X_n^{i_n} + \sum_{\substack{i_1=i_2=\cdots=i_{n-1}=0\\ i_n\geqslant 1}} a_{\mathbf{i}}X_n^{i_n} \\ &= X_1\Big(\sum_{i_1\geqslant 1} a_{\mathbf{i}}X_1^{i_1-1}\cdots X_n^{i_n}\Big) + X_2\Big(\sum_{\substack{i_1=0\\ i_2\geqslant 1}} a_{\mathbf{i}}X_2^{i_2-1}\cdots X_n^{i_n}\Big) + \cdots \\ &\quad + X_{n-1}\Big(\sum_{\substack{i_1=i_2=\cdots=i_{n-2}=0\\ i_{n-1}\geqslant 1}} a_{\mathbf{i}}X_{n-1}^{i_{n-1}-1}X_n^{i_n}\Big) + X_n\Big(\sum_{\substack{i_1=i_2=\cdots=i_{n-1}=0\\ i_n\geqslant 1}} a_{\mathbf{i}}X_n^{i_n-1}\Big) \end{aligned}$$

and this shows that $f \in \mathfrak{X}$.

(b) Let $\varphi\colon R[\![X_1,\dots,X_n]\!] \to R$ be given by $\sum_{\mathbf{i}\in\mathbb{N}^n} a_{\mathbf{i}}X_1^{i_1}\cdots X_n^{i_n} \mapsto a_{(0,\dots,0)}$. In other words, $\varphi(f)$ is the constant term of f. It is straightforward to show that φ is a well-defined ring epimomorphism. By definition, the kernel of φ is the set of all formal power series in $R[\![X_1,\dots,X_n]\!]$ with constant term 0, that is, $\operatorname{Ker}(\varphi) = \mathfrak{X}$. Hence, we have

$$R = \operatorname{Im}(\varphi) \cong R[\![X_1,\dots,X_n]\!]/\operatorname{Ker}(\varphi) = R[\![X_1,\dots,X_n]\!]/\mathfrak{X}$$

as desired. □

Proposition III.4.7. *Let R be a commutative ring, and let n be a positive integer.*

(a) *For each ideal $I \subseteq R$, the set*

$$I[\![X_1,\dots,X_n]\!] = \{\textstyle\sum_{\mathbf{i}\in\mathbb{N}^n} a_{\mathbf{i}}X_1^{i_1}\cdots X_n^{i_n} \in R[\![X_1,\dots,X_n]\!] \mid a_{\mathbf{i}} \in I \text{ for all } \mathbf{i}\in\mathbb{N}^n\}$$

is an ideal in $R[\![X_1,\dots,X_n]\!]$ such that

$$R[\![X_1,\dots,X_n]\!]/I[\![X_1,\dots,X_n]\!] \cong (R/I)[\![X_1,\dots,X_n]\!]$$

and $I[\![X_1,\dots,X_n]\!] \supseteq (I)R[\![X_1,\dots,X_n]\!]$.

(b) *If I is a finitely generated ideal of R, then $I[\![X_1,\dots,X_n]\!] = (I)R[\![X_1,\dots,X_n]\!]$.*

(c) *An ideal I of R is prime if and only if $I[\![X_1,\dots,X_n]\!]$ is a prime ideal of $R[\![X_1,\dots,X_n]\!]$.*

Proof. (a) Let $\varphi\colon R[\![X_1,\dots,X_n]\!] \to (R/I)[\![X_1,\dots,X_n]\!]$ be given by the rule of assignment $\sum_{\mathbf{i}\in\mathbb{N}^n} a_{\mathbf{i}}X_1^{i_1}\cdots X_n^{i_n} \mapsto \sum_{\mathbf{i}\in\mathbb{N}^n} \overline{a_{\mathbf{i}}}X_1^{i_1}\cdots X_n^{i_n}$. In other words, $\varphi(f)$ is the power series over R/I obtained by reducing the coefficients of f module I. It is straightforward to show that φ is a well-defined ring epimomorphism. By definition, the kernel of φ is the set of all formal power series in $R[\![X_1,\dots,X_n]\!]$ with coefficients in I, that is, $\operatorname{Ker}(\varphi) = I[\![X_1,\dots,X_n]\!]$. Hence, the set $I[\![X_1,\dots,X_n]\!]$ is an ideal of $R[\![X_1,\dots,X_n]\!]$ such that

$$(R/I)[\![X_1,\dots,X_n]\!] \cong R[\![X_1,\dots,X_n]\!]/\operatorname{Ker}(\varphi) = R[\![X_1,\dots,X_n]\!]/I[\![X_1,\dots,X_n]\!].$$

The generators of $(I)R[\![X_1,\dots,X_n]\!]$ are the elements of I, which are elements of $I[\![X_1,\dots,X_n]\!]$ by definition. This implies that $I[\![X_1,\dots,X_n]\!] \supseteq (I)R[\![X_1,\dots,X_n]\!]$.

(b) Let $b_1,\dots,b_m$ be a generating sequence for I. Because of part (a), we need only verify the containment $I[\![X_1,\dots,X_n]\!] \supseteq (I)R[\![X_1,\dots,X_n]\!]$. Fix an element $f = \sum_{\mathbf{i}\in\mathbb{N}^n} a_{\mathbf{i}}X_1^{i_1}\cdots X_n^{i_n} \in I[\![X_1,\dots,X_n]\!]$. Each $a_{\mathbf{i}}$ is in $I = (b_1,\dots,b_m)R$, so we can write $a_{\mathbf{i}} = \sum_{j=1}^m c_{\mathbf{i},j}b_j$ with $c_{\mathbf{i}} \in R$. It follows that

$$\begin{aligned} f &= \textstyle\sum_{\mathbf{i}\in\mathbb{N}^n} a_{\mathbf{i}}X_1^{i_1}\cdots X_n^{i_n} \\ &= \textstyle\sum_{\mathbf{i}\in\mathbb{N}^n} (\sum_{j=1}^m c_{\mathbf{i},j}b_j)X_1^{i_1}\cdots X_n^{i_n} \\ &= \textstyle\sum_{j=1}^m b_j(\sum_{\mathbf{i}\in\mathbb{N}^n} c_{\mathbf{i},j}X_1^{i_1}\cdots X_n^{i_n}). \end{aligned}$$

In other words, we have $f = \sum_{j=1}^m b_jf_j$ for some $f_j \in R[\![X_1,\dots,X_n]\!]$. Since the b_j are in I, this implies that $f \in (I)R[\![X_1,\dots,X_n]\!]$, as desired.

(c) Proposition III.4.4(a) implies that the ring

$$R[\![X_1,\dots,X_n]\!]/I[\![X_1,\dots,X_n]\!] \cong (R/I)[\![X_1,\dots,X_n]\!]$$

is an integral domain if and only if R/I is an integral domain. □

Proposition III.4.8. *Let R be a commutative ring, and let n be a positive integer. Set $\mathfrak{X} = (X_1,\dots,X_n)R[\![X_1,\dots,X_n]\!]$, and let I be an ideal of R.*

(a) *There is an equality* $\mathfrak{X} + I[\![X_1, \dots, X_n]\!] = (X_1, \dots, X_n, I)R[\![X_1, \dots, X_n]\!]$*, and an isomorphism* $R[\![X_1, \dots, X_n]\!]/(\mathfrak{X} + I[\![X_1, \dots, X_n]\!]) \cong R/I$*.*
(b) *The ideal* I *is prime if and only if* $\mathfrak{X} + I[\![X_1, \dots, X_n]\!]$ *is a prime ideal of* $R[\![X_1, \dots, X_n]\!]$*.*
(c) *The ideal* I *is maximal if and only if* $\mathfrak{X} + I[\![X_1, \dots, X_n]\!]$ *is a maximal ideal of* $R[\![X_1, \dots, X_n]\!]$*.*
(d) *If* $\mathfrak{M} \subsetneq R[\![X_1, \dots, X_n]\!]$ *is a maximal ideal, then* $\mathfrak{m} = \mathfrak{M} \cap R$ *is a maximal ideal of* R *and* $\mathfrak{M} = \mathfrak{X} + \mathfrak{m}[\![X_1, \dots, X_n]\!] = (X_1, \dots, X_n, \mathfrak{m})R[\![X_1, \dots, X_n]\!]$*.*
(e) *The set of maximal ideals of* $R[\![X_1, \dots, X_n]\!]$ *is in bijection with the set of maximal ideals of* R*. Thus, the ring* $R[\![X_1, \dots, X_n]\!]$ *is local if and only if* R *is local.*

PROOF. (a) The containment

$$\mathfrak{X} + I[\![X_1, \dots, X_n]\!] \supseteq (X_1, \dots, X_n, I)R[\![X_1, \dots, X_n]\!]$$

follows from the conditions

$$X_1, \dots, X_n \in \mathfrak{X} \subseteq \mathfrak{X} + I[\![X_1, \dots, X_n]\!]$$

and

$$I \subseteq I[\![X_1, \dots, X_n]\!] \subseteq \mathfrak{X} + I[\![X_1, \dots, X_n]\!].$$

For the containment $\mathfrak{X}+I[\![X_1, \dots, X_n]\!] \subseteq (X_1, \dots, X_n, I)R[\![X_1, \dots, X_n]\!]$, fix an element $f \in \mathfrak{X}+I[\![X_1, \dots, X_n]\!]$. Then there exist $g_1, \dots, g_n \in R[\![X_1, \dots, X_n]\!]$ and $h \in I[\![X_1, \dots, X_n]\!]$ such that $f = h + \sum_{i=1}^n X_i g_i$. The condition $h \in I[\![X_1, \dots, X_n]\!]$ implies that the constant term c_0 of h is in I. As in the proof of Proposition III.4.6(a), write $h = c_0 + \sum_{i=1}^n X_i h_i$ for some $h_1, \dots, h_n \in R[\![X_1, \dots, X_n]\!]$. Then we have

$$\begin{aligned} f &= h + \textstyle\sum_{i=1}^n X_i g_i = c_0 + \sum_{i=1}^n X_i h_i + \sum_{i=1}^n X_i g_i \\ &= c_0 + \textstyle\sum_{i=1}^n X_i (h_i + g_i) \in \mathfrak{X} + I[\![X_1, \dots, X_n]\!] \end{aligned}$$

as desired.

The desired isomorphism follows from the next sequence, which begins with the third isomorphism theorem:

$$\begin{aligned} \frac{R[\![X_1, \dots, X_n]\!]}{\mathfrak{X} + I[\![X_1, \dots, X_n]\!]} &\cong \frac{R[\![X_1, \dots, X_n]\!]/I[\![X_1, \dots, X_n]\!]}{(X_1, \dots, X_n)(R[\![X_1, \dots, X_n]\!]/I[\![X_1, \dots, X_n]\!])} \\ &\cong \frac{(R/I)[\![X_1, \dots, X_n]\!]}{(X_1, \dots, X_n)(R/I)[\![X_1, \dots, X_n]\!]} \\ &\cong R/I. \end{aligned}$$

The other isomorphisms are from Propositions III.4.7(a) and III.4.6(b).

(b) The isomorphism $R[\![X_1, \dots, X_n]\!]/(\mathfrak{X} + I[\![X_1, \dots, X_n]\!]) \cong R/I$ shows that $R[\![X_1, \dots, X_n]\!]/(\mathfrak{X} + I[\![X_1, \dots, X_n]\!])$ is an integral domain if and only if R/I is an integral domain.

(c) The isomorphism $R[\![X_1, \dots, X_n]\!]/(\mathfrak{X} + I[\![X_1, \dots, X_n]\!]) \cong R/I$ shows that $R[\![X_1, \dots, X_n]\!]/(\mathfrak{X} + I[\![X_1, \dots, X_n]\!])$ is a field if and only if R/I is a field.

(d) Let $\mathfrak{M} \subsetneq R[\![X_1, \dots, X_n]\!]$ be a maximal ideal, and set $\mathfrak{m} = \mathfrak{M} \cap R$. We first show that each variable X_j is in $\mathfrak{M}$. Suppose that $X_j \notin \mathfrak{M}$. Then we have $(\mathfrak{M}, X_j)R[\![X_1, \dots, X_n]\!] = R[\![X_1, \dots, X_n]\!]$ so there are power series $f \in \mathfrak{M}$ and $g \in R[\![X_1, \dots, X_n]\!]$ such that $1 = f + X_j g$. It follows that the constant term of f is 1, so f is a unit by Proposition III.4.5(a). This contradicts the assumption $f \in \mathfrak{M}$.

The containment $\mathfrak{M} \supseteq (X_1, \dots, X_n, \mathfrak{m})R[\![X_1, \dots, X_n]\!]$ follows from the inclusions $X_1, \dots, X_n \in (X_1, \dots, X_n, \mathfrak{m})R[\![X_1, \dots, X_n]\!]$ and from the containment $\mathfrak{m} \subseteq (X_1, \dots, X_n, \mathfrak{m})R[\![X_1, \dots, X_n]\!]$.

To prove the containment $\mathfrak{M} \subseteq (X_1, \dots, X_n, \mathfrak{m})R[\![X_1, \dots, X_n]\!]$, let $f \in \mathfrak{M}$ and write $f = c + \sum_{i=1}^n X_i f_i$ where $c \in R$ and $f_1, \dots, f_n \in R[\![X_1, \dots, X_n]\!]$. Since each X_i is in $\mathfrak{M}$, it follows that $c = f - \sum_{i=1}^n X_i f_i \in \mathfrak{M} \cap R = \mathfrak{m}$, so we have $f = c + \sum_{i=1}^n X_i f_i \in (X_1, \dots, X_n, \mathfrak{m})R[\![X_1, \dots, X_n]\!]$, as desired.

The fact that $\mathfrak{m}$ is a maximal ideal of R now follows from part (c).

(e) This follows from parts (c) and(d). □

Here is the reason for including this material in this chapter.

Proposition III.4.9. *Let R be a commutative ring, and let n be a positive integer. If R is noetherian, then $R[\![X_1, \dots, X_n]\!]$ is faithfully flat as an R-module.*

Proof. To show that $R[\![X_1, \dots, X_n]\!]$ is flat, either argue by induction on n, using Theorem III.2.7 and Exercise III.2.11, or apply Exercise III.2.9 directly with the set $\Lambda = \mathbb{N}^n$. To show that $R[\![X_1, \dots, X_n]\!]$ is faithfully flat, observe that Proposition III.4.8(c) implies that, for every maximal ideal $\mathfrak{m} \subset R$, we have $\mathfrak{m}R[\![X_1, \dots, X_n]\!] \neq R[\![X_1, \dots, X_n]\!]$; now, invoke Theorem III.3.4. □

Proposition III.4.10 (Hilbert Basis Theorem)**.** *Let R be a commutative ring, and let n be a positive integer. If R is noetherian, then so is $R[\![X_1, \dots, X_n]\!]$.*

Proof. Argue by induction on n. The base case $n = 1$ is a reading exercise; see Hungerford [**2**, (VIII.4.10)]. □

Exercises.

Exercise III.4.11. Verify the facts from Remark III.4.3.

Exercise III.4.12. Complete the proof of Proposition III.4.4.

Exercise III.4.13. Complete the proof of Proposition III.4.5.

Exercise III.4.14. Complete the proof of Proposition III.4.6.

Exercise III.4.15. Complete the proof of Proposition III.4.7.

Exercise III.4.16. Complete the proof of Proposition III.4.9.

III.5. Flat Ring Homomorphisms

We have seen that, given a commutative ring R, the ring of polynomials $R[X_1, \dots, X_n]$ is a (faithfully) flat R-module. When R is noetherian, the same is true of the power series ring $R[\![X_1, \dots, X_n]\!]$. In this section, we study the general properties of flat ring extensions.

Definition III.5.1. A homomorphism of commutative rings $\varphi\colon R \to S$ is *flat* if S is flat as an R-module. It is *faithfully flat* if S is faithfully flat as an R-module.

Example III.5.2. Let R be a commutative ring. Then the natural inclusion $R \to R[X_1, \dots, X_n]$ is faithfully flat by Exercise III.2.11(c). If R is noetherian, then the natural inclusion $R \to R[\![X_1, \dots, X_n]\!]$ is faithfully flat by Proposition III.4.9.

Proposition III.5.3. *Let $\varphi\colon R \to S$ be a faithfully flat ring homomorphism.*

(a) *The map φ is a monomorphism.*

(b) *For each prime ideal $\mathfrak{p} \subset R$, there is a prime ideal $P \subset S$ such that $\varphi^{-1}(P) = \mathfrak{p}$.*

PROOF. (a) Let $r \in R$ be a nonzero element. We need to show that $\varphi(r) \neq 0$. The submodule $Rr \subseteq R$ is non-zero. Since S is faithfully flat as an R-module, we have $(Rr) \otimes_R S \neq 0$. As an S-module, this is generated by $r \otimes 1$, so we have $0 \neq r \otimes 1 \in (Rr) \otimes_R S$.

Let $i\colon Rr \to R$ denote the natural inclusion. Since S is flat, the first map in the following sequence is a monomorphism

$$(Rr) \otimes_R S \xhookrightarrow{} R \otimes_R S \xrightarrow{\cong} S$$

$$r \otimes 1 \longmapsto r \otimes 1 \longmapsto \varphi(r).$$

The second map is from Example II.1.9. Since $0 \neq r \otimes 1 \in (Rr) \otimes_R S$, it follows that $0 \neq \varphi(r) \in S$.

(b) The R-module $(R/\mathfrak{p})_\mathfrak{p}$ is nonzero, so the faithful flatness of S implies the non-vanishing in the next sequence

$$\begin{aligned} 0 \neq S \otimes_R (R/\mathfrak{p})_\mathfrak{p} &\cong S \otimes_R (R/\mathfrak{p} \otimes_R R_\mathfrak{p}) \cong (S \otimes_R R/\mathfrak{p}) \otimes_R R_\mathfrak{p} \\ &\cong (S/\mathfrak{p}S) \otimes_R R_\mathfrak{p} \cong (S/\mathfrak{p}S)_\mathfrak{p} \cong U^{-1}(S/\mathfrak{p}S) \end{aligned}$$

where $U = \varphi(R \smallsetminus \mathfrak{p})$. In particular, it follows that $S/\mathfrak{p}S \neq 0$. The set $U^{-1}(S/\mathfrak{p}S)$ is a commutative ring, so there is a maximal ideal $\mathfrak{m} \subset U^{-1}(S/\mathfrak{p}S)$. The prime correspondences for localizations and quotients imply that $\mathfrak{m} = U^{-1}(P/\mathfrak{p}S)$ for some prime ideal $P \subset S$ such that $P \supseteq \mathfrak{p}S$ and $P \cap U = \emptyset$. The condition $P \supseteq \mathfrak{p}S$ implies that

$$\varphi^{-1}(P) \supseteq \varphi^{-1}(\mathfrak{p}S) \supseteq \varphi^{-1}(\varphi(\mathfrak{p})) \supseteq \mathfrak{p}.$$

The conditions $P \cap U = \emptyset$ and $U = \varphi(R \smallsetminus \mathfrak{p})$ imply that $\varphi^{-1}(P) \cap (R \smallsetminus \mathfrak{p}) = \emptyset$, so

$$\varphi^{-1}(P) \subseteq \mathfrak{p}.$$

Combining the two displays, we have $\varphi^{-1}(P) = \mathfrak{p}$, as desired. □

We next discuss how to construct some faithfully flat ring homomorphisms.

Definition III.5.4. A homomorphism of commutative rings $\varphi\colon R \to S$ is *local* if S is local with maximal ideal $\mathfrak{n}$, and R is local with maximal ideal $\mathfrak{m}$ where $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$.

Example III.5.5. Let $(R, \mathfrak{m})$ be a commutative local ring. Proposition III.4.8 shows that the natural inclusion $R \to R[\![X_1, \dots, X_n]\!]$ is local.

Letting $\mathfrak{M} \subseteq R[X_1, \dots, X_n]$ denote the maximal ideal $\mathfrak{M} = (\mathfrak{m}, X_1, \dots, X_n)$, the composition of natural maps $R \to R[X_1, \dots, X_n] \to R[X_1, \dots, X_n]_\mathfrak{M}$ is local.

Example III.5.6. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings. Let $P \subset S$ be a prime ideal and set $\mathfrak{p} = \varphi^{-1}(P)$. Proposition III.2.8(a) shows that the induced ring homomorpism $\varphi_P\colon R_\mathfrak{p} \to S_P$ given by $r/s \mapsto \varphi(r)/\varphi(s)$ is local.

Proposition III.5.7. *Let $\varphi\colon R \to S$ be a ring homomorphism such that S is local with maximal ideal $\mathfrak{n}$, and R is local with maximal ideal $\mathfrak{m}$. Then φ is local if and only if $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$.*

PROOF. See Exercise III.5.15. □

Proposition III.5.8. *Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a ring homomorphism. The following conditions are equivalent:*

(i) *φ is faithfully flat;*
(ii) *φ is flat and local;*
(iii) *φ is flat and $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$.*

PROOF. (i) $\iff$ (ii) Theorem III.3.4 implies that S is faithfully flat as an R-module if and only if S is flat as an R-module and $\mathfrak{m}S \neq S$. Since S is local with maximal ideal $\mathfrak{n}$, we have $\mathfrak{m}S \neq S$ if and only if $\mathfrak{m}S \subseteq \mathfrak{n}$, that is, if and only if φ is local.

(ii) $\iff$ (iii) This follows from Proposition III.5.7. □

Proposition III.5.9. *Let $\varphi\colon R \to S$ be a flat ring homomorphism. Let $P \subset S$ be a prime ideal, and set $\mathfrak{p} = \varphi^{-1}(P)$. The induced map $\varphi_P\colon R_\mathfrak{p} \to S_P$ is flat and local, i.e., faithfully flat.*

PROOF. Proposition III.2.8(b) says that φ_P is flat. Proposition III.2.8(a) implies that φ_P is local, so Proposition III.5.8 guarantees that φ_P is faithfully flat. □

The next result says that flat ring homomorphisms satisfy the *going-down property.*

Theorem III.5.10. *Let $\varphi\colon R \to S$ be a flat ring homomorphism. Let $P \subset S$ be a prime ideal, and set $\mathfrak{p} = \varphi^{-1}(P)$. If there is a chain $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n = \mathfrak{p}$ of prime ideals of R, then there is a chain $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n = P$ of prime ideals of S such that $\varphi^{-1}(P_i) = \mathfrak{p}_i$ for $i = 1, \ldots, n$.*

PROOF. By induction on n, it suffices to consider the case $n = 1$. The prime correspondence for localization implies that the ideal $(\mathfrak{p}_0)_\mathfrak{p} \subset R_\mathfrak{p}$ is prime. The induced map $\varphi_P\colon R_\mathfrak{p} \to S_P$ is faithfully flat by Proposition III.5.9, so Proposition III.5.3b implies that there is a prime ideal $Q \subset S_P$ such that $(\varphi_P)^{-1}(Q) = (\mathfrak{p}_0)_\mathfrak{p}$. Set $P_0 = \beta^{-1}(Q)$ where $\beta\colon S \to S_P$ is the natural map. The prime correspondence for localization implies that P_0 is a prime ideal of S such that $P_0 \subseteq P$.

Proposition III.2.8(b) says that the following diagram commutes

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & S \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ R_\mathfrak{p} & \xrightarrow{\varphi_P} & S_P \end{array}$$

where the vertical maps are the natural ones. The commutativity of the diagram yields the second equality in the following sequence

$$\phi^{-1}(P_0) = \phi^{-1}(\beta^{-1}(Q)) = \alpha^{-1}((\phi_P)^{-1}(Q)) = \alpha^{-1}((\mathfrak{p}_0)_\mathfrak{p}) = \mathfrak{p}_0.$$

The first equality is by the definition of P_0, and the third equality is by the definition of Q. The fourth equality is from the prime correspondence for localization. □

Corollary III.5.11. *Let R be a commutative ring.*

(a) *Let $P \subset R[X_1, \ldots, X_n]$ be a prime ideal, and set $\mathfrak{p} = P \cap R$. If there is a chain $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n = \mathfrak{p}$ of prime ideals of R, then there is a chain $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n = P$ of prime ideals of $R[X_1, \ldots, X_n]$ such that $P_i \cap R = \mathfrak{p}_i$ for $i = 1, \ldots, n$.*

(b) *Let $P \subset R[\![X_1,\dots,X_n]\!]$ be a prime ideal, and set $\mathfrak{p} = P \cap R$. If there is a chain $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \cdots \subseteq \mathfrak{p}_n = \mathfrak{p}$ of prime ideals of R, then there is a chain $P_0 \subseteq P_1 \subseteq \cdots \subseteq P_n = P$ of prime ideals of $R[\![X_1,\dots,X_n]\!]$ such that $P_i \cap R = \mathfrak{p}_i$ for $i = 1,\dots,n$.*

PROOF. The natural inclusions $R \to R[X_1,\dots,X_n]$ and $R \to R[\![X_1,\dots,X_n]\!]$ are flat, so the result follows from Theorem III.5.10. □

Lemma III.5.12. *Let $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ be a flat local ring homomorphism between commutative noetherian rings, and let M be a finitely generated R-module. There is an equality*

$$\dim_l \operatorname{Hom}_S(l, S\otimes_R M) = \dim_k \operatorname{Hom}_R(k,M)\cdot \dim_l \operatorname{Hom}_S(l, S/\mathfrak{m}S).$$

In particular, we have $\operatorname{Hom}_S(l, S\otimes_R M) = 0$ if and only if $\operatorname{Hom}_R(k,M) = 0$ or $\operatorname{Hom}_S(l,S/\mathfrak{m}S) = 0$.

PROOF. The R-modules k and M are finitely generated, so Exercise I.5.20 implies that $\operatorname{Hom}_R(k,M)$ is finitely generated over R. Also, the definition $k = R/\mathfrak{m}$ implies that $\mathfrak{m}k = 0$. Remark I.5.11 implies that $\operatorname{Hom}_R(k,M)$ has the structure of a k-module that is compatible with its R-module structure via the natural surjection $R \to k$ and that $\operatorname{Hom}_R(k,M)$ is finitely generated over k. So, we have

$$\operatorname{Hom}_R(k,M) \cong k^a \tag{III.5.12.1}$$

for some integer $a \geqslant 0$. Similarly, since $S/\mathfrak{m}S$ and $S\otimes_R M$ are finitely generated S-modules, there are integers $b, c \geqslant 0$ such that

$$\operatorname{Hom}_S(l, S/\mathfrak{m}S) \cong l^b \qquad \text{and} \qquad \operatorname{Hom}_S(l, S\otimes_R M) \cong l^c. \tag{III.5.12.2}$$

The definition $l = S/\mathfrak{n}$ yields the first and last steps in the next sequence

$$(S/\mathfrak{m}S)\otimes_S l = (S/\mathfrak{m}S)\otimes_S (S/\mathfrak{n}) \cong \frac{S/\mathfrak{m}S}{\mathfrak{n}(S/\mathfrak{m}S)} = \frac{S/\mathfrak{m}S}{\mathfrak{n}/\mathfrak{m}S} \cong S/\mathfrak{n} = l. \tag{III.5.12.3}$$

The second and fourth steps (both S-module isomorphisms) follow from Exercise II.4.14 and the third isomorphism theorem. The third step is from the containment $\mathfrak{n} \supset \mathfrak{m}S$. The definition $k = R/\mathfrak{m}$ yields the equality in the next sequence

$$S/\mathfrak{m}S \cong S\otimes_R R/\mathfrak{m} = S\otimes_R k \tag{III.5.12.4}$$

and the isomorphism is from Exercise II.4.14. It is straightforward to show that this is an S-module isomorphism.

The definitions in (III.5.12.1) and (III.5.12.2) explain steps (1), (6), and (10) in the next sequence

$$\begin{aligned} l^c &\overset{(1)}{\cong} \operatorname{Hom}_S(l, S\otimes_R M) \overset{(2)}{\cong} \operatorname{Hom}_S((S/\mathfrak{m}S)\otimes_S l, S\otimes_R M)\\ &\overset{(3)}{\cong} \operatorname{Hom}_S(l, \operatorname{Hom}_S(S/\mathfrak{m}S, S\otimes_R M)) \overset{(4)}{\cong} \operatorname{Hom}_S(l, \operatorname{Hom}_S(S\otimes_R k, S\otimes_R M))\\ &\overset{(5)}{\cong} \operatorname{Hom}_S(l, S\otimes_R \operatorname{Hom}_R(k,M)) \overset{(6)}{\cong} \operatorname{Hom}_S(l, S\otimes_R k^a) \overset{(7)}{\cong} \operatorname{Hom}_S(l, (S\otimes_R k)^a)\\ &\overset{(8)}{\cong} \operatorname{Hom}_S(l, S\otimes_R k)^a \overset{(9)}{\cong} \operatorname{Hom}_S(l, S/\mathfrak{m}S)^a \overset{(10)}{\cong} (l^b)^a \overset{(11)}{\cong} l^{ab}. \end{aligned}$$

The sequence (III.5.12.3) explains step (2), and (3) is Hom-tensor adjointness II.5.1. Steps (4) and (9) are from (III.5.12.4). Step (5) is due to Exercise II.2.15, and (7) is by Theorem II.3.2. Step (8) follows from Proposition I.2.3(c), and (11) is standard.

The previous sequence explains the second equality in the next sequence

$$\dim_l \operatorname{Hom}_S(l, S \otimes_R M)c = ab = \dim_k \operatorname{Hom}_R(k, M) \cdot \dim_l \operatorname{Hom}_S(l, S/\mathfrak{m}S)$$

and the others are from (III.5.12.1) and (III.5.12.2). This explains the desired equality, and the final statement of the lemma follows immediately. □

Lemma III.5.13. *Let $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a flat local ring homomorphism between commutative noetherian rings. There is an equality*

$$\dim_l \operatorname{Hom}_S(l, S) = \dim_k \operatorname{Hom}_R(k, R) \cdot \dim_l \operatorname{Hom}_S(l, S/\mathfrak{m}S).$$

Thus, $\operatorname{Hom}_S(l, S) = 0$ *if and only if* $\operatorname{Hom}_R(k, R) = 0$ *or* $\operatorname{Hom}_S(l, S/\mathfrak{m}S) = 0$.

Proof. This is the special case $M = R$ in Lemma III.5.12. □

Exercises.

Exercise III.5.14. Complete the proof of Proposition III.5.3.

Exercise III.5.15. Prove Proposition III.5.7.

Exercise III.5.16. Complete the proof of Theorem III.5.10.

Exercise III.5.17. Complete the proof of Lemma III.5.12.

III.6. Completions: A Survey

Probably only want to discuss the properties of the completion of R here.

Exercises.

Exercise III.6.1.

III.7. Completions: Some Details

How much can we reasonably do here? Can we prove flatness without inverse limits?

Exercises.

Exercise III.7.1.

CHAPTER IV

Homology, Resolutions, Ext and Tor September 8, 2009

Homological algebra is based on the notions of homology and cohomology. We start with some general ideas about homology and then proceed to the specific examples of Ext and Tor, which are defined in terms of resolutions.

IV.1. Chain Complexes and Homology

Here are the foundational notions for homological algebra.

Definition IV.1.1. Let R be a commutative ring. A sequence of R-module homomorphisms

$$M_\bullet = \cdots \xrightarrow{\partial^M_{i+1}} M_i \xrightarrow{\partial^M_i} M_{i-1} \xrightarrow{\partial^M_{i-1}} \cdots$$

is a *chain complex* or an *R-complex* if $\partial^M_{i-1}\partial^M_i = 0$ for all i. We say that M_i is the module in *degree i* in the R-complex $M_\bullet$. The *ith homology module* of an R-complex $M_\bullet$ is the R-module

$$\mathrm{H}_i(M_\bullet) = \mathrm{Ker}(\partial^M_i)/\operatorname{Im}(\partial^M_{i+1}).$$

Remark IV.1.2. Let R be a commutative ring. An R-complex $M_\bullet$ is exact if and only if $\mathrm{H}_i(M_\bullet) = 0$ for all i.

In the following example, and throughout these notes, we employ the linear algebra protocols described in Remark I.1.5.

Example IV.1.3. Consider the following sequence of $\mathbb{Z}$-modules

$$M_\bullet = \quad 0 \to \mathbb{Z} \xrightarrow{\left(\begin{smallmatrix} 9 \\ -6 \end{smallmatrix}\right)} \mathbb{Z}^2 \xrightarrow{(2\ 3)} \underbrace{\mathbb{Z}}_{\text{degree } 0} \to 0.$$

To show that this is a chain complex, we need only show that the products of the pairs of adjacent matrices are zero; see Exercise I.1.8: $(2\ 3)\left(\begin{smallmatrix} 9 \\ -6 \end{smallmatrix}\right) = (0)$. We compute the homology modules in each degree.

$$\mathrm{H}_0(M_\bullet) = \frac{\mathrm{Ker}(\mathbb{Z} \to 0)}{\operatorname{Im}\left(\mathbb{Z}^2 \xrightarrow{(2\ 3)} \mathbb{Z}\right)} = \frac{\mathbb{Z}}{\langle 2,3\rangle\mathbb{Z}} = \frac{\mathbb{Z}}{\mathbb{Z}} = 0$$

$$\mathrm{H}_1(M_\bullet) = \frac{\mathrm{Ker}\left(\mathbb{Z}^2 \xrightarrow{(2\ 3)} \mathbb{Z}\right)}{\operatorname{Im}\left(\mathbb{Z} \xrightarrow{\left(\begin{smallmatrix} 9 \\ -6 \end{smallmatrix}\right)} \mathbb{Z}^2\right)} = \frac{\left\langle \left(\begin{smallmatrix} 3 \\ -2 \end{smallmatrix}\right)\right\rangle \mathbb{Z}}{\left\langle \left(\begin{smallmatrix} 9 \\ -6 \end{smallmatrix}\right)\right\rangle \mathbb{Z}} = \frac{\left\langle \left(\begin{smallmatrix} 3 \\ -2 \end{smallmatrix}\right)\right\rangle \mathbb{Z}}{3\left\langle \left(\begin{smallmatrix} 3 \\ -2 \end{smallmatrix}\right)\right\rangle \mathbb{Z}} \cong \frac{\mathbb{Z}}{3\mathbb{Z}}$$

$$\mathrm{H}_2(M_\bullet) = \frac{\operatorname{Ker}\left(\mathbb{Z} \xrightarrow{\binom{9}{-6}} \mathbb{Z}^2\right)}{\operatorname{Im}(0 \to \mathbb{Z})} = \frac{0\mathbb{Z}}{0\mathbb{Z}} = 0.$$

The remaining homology modules are 0 because $M_i = 0$ when $i \neq 0, 1, 2$; see Exercise IV.1.11.

Example IV.1.4. We work over the ring $R = \mathbb{Z}/12\mathbb{Z}$. Here is an R-complex:

$$M_\bullet = \cdots \xrightarrow{6} \underbrace{\mathbb{Z}/12\mathbb{Z}}_{\text{degree } 2} \xrightarrow{4} \underbrace{\mathbb{Z}/12\mathbb{Z}}_{\text{degree } 1} \xrightarrow{6} \underbrace{\mathbb{Z}/12\mathbb{Z}}_{\text{degree } 0} \xrightarrow{4} \cdots .$$

We have

$$\operatorname{Ker}(\mathbb{Z}/12\mathbb{Z} \xrightarrow{4} \mathbb{Z}/12\mathbb{Z}) = 3\mathbb{Z}/12\mathbb{Z}$$

$$\operatorname{Ker}(\mathbb{Z}/12\mathbb{Z} \xrightarrow{6} \mathbb{Z}/12\mathbb{Z} = 2\mathbb{Z}/12\mathbb{Z}.$$

So, one of the homology modules is

$$\mathrm{H}_1(M_\bullet) = \frac{\operatorname{Ker}(\mathbb{Z}/12\mathbb{Z} \xrightarrow{6} \mathbb{Z}/12\mathbb{Z})}{\operatorname{Im}(\mathbb{Z}/12\mathbb{Z} \xrightarrow{4} \mathbb{Z}/12\mathbb{Z})} = \frac{2\mathbb{Z}/12\mathbb{Z}}{4\mathbb{Z}/12\mathbb{Z}} \cong \frac{2\mathbb{Z}}{4\mathbb{Z}} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$$

and another one is

$$\mathrm{H}_0(M_\bullet) = \frac{\operatorname{Ker}(\mathbb{Z}/12\mathbb{Z} \xrightarrow{4} \mathbb{Z}/12\mathbb{Z})}{\operatorname{Im}(\mathbb{Z}/12\mathbb{Z} \xrightarrow{6} \mathbb{Z}/12\mathbb{Z})} = \frac{3\mathbb{Z}/12\mathbb{Z}}{6\mathbb{Z}/12\mathbb{Z}} \cong \frac{3\mathbb{Z}}{6\mathbb{Z}} \cong \frac{\mathbb{Z}}{2\mathbb{Z}}.$$

The periodic nature of the complex $M_\bullet$ implies that $\mathrm{H}_n(M_\bullet) \cong \mathbb{Z}/2\mathbb{Z}$ for all $n \in \mathbb{Z}$.

Definition IV.1.5. Let R be a commutative ring. Given an R-complex $M_\bullet$ and an R-module N we define the following:

$$M_\bullet \otimes_R N = \cdots \xrightarrow{\partial^M_{i+1}\otimes_R N} \underbrace{M_i \otimes_R N}_{\text{degree } i} \xrightarrow{\partial^M_i \otimes_R N} \underbrace{M_{i-1} \otimes_R N}_{\text{degree } i-1} \xrightarrow{\partial^M_{i-1}\otimes_R N} \cdots$$

$$N \otimes_R M_\bullet = \cdots \xrightarrow{N\otimes_R \partial^M_{i+1}} \underbrace{N \otimes_R M_i}_{\text{degree } i} \xrightarrow{N\otimes_R \partial^M_i} \underbrace{N \otimes_R M_{i-1}}_{\text{degree } i-1} \xrightarrow{N\otimes_R \partial^M_{i-1}} \cdots$$

$\operatorname{Hom}_R(N, M_\bullet) =$

$$\cdots \xrightarrow{\operatorname{Hom}_R(N,\partial^M_{i+1})} \underbrace{\operatorname{Hom}_R(N, M_i)}_{\text{degree } i} \xrightarrow{\operatorname{Hom}_R(N,\partial^M_i)} \underbrace{\operatorname{Hom}_R(N, M_{i-1})}_{\text{degree } i-1} \xrightarrow{\operatorname{Hom}_R(N,\partial^M_{i-1})} \cdots$$

$\operatorname{Hom}_R(M_\bullet, N) =$

$$\cdots \xrightarrow{\operatorname{Hom}_R(\partial^M_i,N)} \underbrace{\operatorname{Hom}_R(M_i, N)}_{\text{degree } -i} \xrightarrow{\operatorname{Hom}_R(\partial^M_{i+1},N)} \underbrace{\operatorname{Hom}_R(M_{i+1}, N)}_{\text{degree } -(i+1)} \xrightarrow{\operatorname{Hom}_R(\partial^M_{i+2},N)} \cdots .$$

Proposition IV.1.6. *Let R be a commutative ring. Let $M_\bullet$ be an R-complex, and let N be an R-module. Then the following sequences are R-complexes: $M_\bullet \otimes_R N$, $N \otimes_R M_\bullet$, $\operatorname{Hom}_R(N, M_\bullet)$, and $\operatorname{Hom}_R(M_\bullet, N)$.*

Proof. The functoriality of $- \otimes_R N$ from Proposition II.2.1(b) provides the first equality in the next sequence

$$(\partial^M_i \otimes_R N)(\partial^M_{i+1} \otimes_R N) = (\partial^M_i \partial^M_{i+1}) \otimes_R N = 0 \otimes_R N = 0.$$

The second equality is from the fact that $M_\bullet$ is a chain complex, and the third equality is standard. The others are verified similarly; see Exercise IV.1.13. □

Example IV.1.7. Consider the following $\mathbb{Z}$-complex from Example IV.1.3:

$$\begin{aligned} M_\bullet = \qquad 0 \longrightarrow \mathbb{Z} &\xrightarrow{\left(\begin{smallmatrix}9\\-6\end{smallmatrix}\right)} \mathbb{Z}^2 \xrightarrow{(2\ 3)} \mathbb{Z} \longrightarrow 0 \\ a &\longmapsto \left(\begin{smallmatrix}9a\\-6a\end{smallmatrix}\right) \\ \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right) &\longmapsto (2a+3b). \end{aligned}$$

The complex $M_\bullet \otimes_{\mathbb{Z}} \mathbb{Z}^2$ has the following form:

$$\begin{aligned} M_\bullet \otimes \mathbb{Z}^2 = \qquad 0 \longrightarrow \mathbb{Z}\otimes\mathbb{Z}^2 &\xrightarrow{\left(\begin{smallmatrix}9\\-6\end{smallmatrix}\right)\otimes\mathbb{Z}^2} \mathbb{Z}^2\otimes\mathbb{Z}^2 \xrightarrow{(2\ 3)\otimes\mathbb{Z}^2} \mathbb{Z}\otimes\mathbb{Z}^2 \longrightarrow 0 \\ a\otimes\left(\begin{smallmatrix}c\\d\end{smallmatrix}\right) &\longmapsto \left(\begin{smallmatrix}9a\\-6a\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}c\\d\end{smallmatrix}\right) \\ \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}c\\d\end{smallmatrix}\right) &\longmapsto (2a+3b)\otimes\left(\begin{smallmatrix}c\\d\end{smallmatrix}\right). \end{aligned}$$

Recall that there are isomorphisms

$$\phi\colon \mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}^2 \xrightarrow{\cong} \mathbb{Z}^2 \qquad \text{given by} \qquad a\otimes\left(\begin{smallmatrix}c\\d\end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix}ac\\ad\end{smallmatrix}\right)$$

and

$$\psi\colon \mathbb{Z}^2\otimes_{\mathbb{Z}}\mathbb{Z}^2 \xrightarrow{\cong} \mathbb{Z}^4 \qquad \text{given by} \qquad \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right)\otimes\left(\begin{smallmatrix}c\\d\end{smallmatrix}\right) \mapsto \begin{pmatrix}ac\\bc\\ad\\bd\end{pmatrix}.$$

It is straightforward to check that these isomorphisms make the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}\otimes\mathbb{Z}^2 & \xrightarrow{\left(\begin{smallmatrix}9\\-6\end{smallmatrix}\right)\otimes\mathbb{Z}^2} & \mathbb{Z}^2\otimes\mathbb{Z}^2 & \xrightarrow{(2\ 3)\otimes\mathbb{Z}^2} & \mathbb{Z}\otimes\mathbb{Z}^2 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi}\cong & & \downarrow{\scriptstyle\psi}\cong & & \downarrow{\scriptstyle\phi}\cong & & \\ 0 & \longrightarrow & \mathbb{Z}^2 & \xrightarrow[\left(\begin{smallmatrix}9&0\\-6&0\\0&9\\0&-6\end{smallmatrix}\right)]{} & \mathbb{Z}^4 & \xrightarrow[\left(\begin{smallmatrix}2&3&0&0\\0&0&2&3\end{smallmatrix}\right)]{} & \mathbb{Z}^2 & \longrightarrow & 0. \end{array}$$

In other words, the complex is "isomorphic to" the bottom row of this diagram.

Definition IV.1.8. Let R be a commutative ring, and let $\{M_\bullet^\lambda\}_{\lambda\in\Lambda}$ be a set of R-complexes. The *product* of these complexes is the sequence

$$\begin{aligned} \textstyle\prod_{\lambda\in\Lambda} M_\bullet^\lambda = \qquad \cdots \longrightarrow \prod_{\lambda\in\Lambda} M_{i+1}^\lambda &\longrightarrow \prod_{\lambda\in\Lambda} M_i^\lambda \longrightarrow \cdots \\ (m_i^\lambda) &\longmapsto (\partial_i^{M^\lambda}(m_i^\lambda)). \end{aligned}$$

The *coproduct* (or *direct sum*) of these complexes is the sequence

$$\begin{aligned} \textstyle\coprod_{\lambda\in\Lambda} M_\bullet^\lambda = \qquad \cdots \longrightarrow \coprod_{\lambda\in\Lambda} M_{i+1}^\lambda &\longrightarrow \coprod_{\lambda\in\Lambda} M_i^\lambda \longrightarrow \cdots \\ (m_i^\lambda) &\longmapsto (\partial_i^{M^\lambda}(m_i^\lambda)). \end{aligned}$$

Remark IV.1.9. Let R be a commutative ring, and let $\{M_\bullet^\lambda\}_{\lambda\in\Lambda}$ be a set of R-complexes. It is straightforward to show that the product $\prod_{\lambda\in\Lambda} M_\bullet^\lambda$ and coproduct $\coprod_{\lambda\in\Lambda} M_\bullet^\lambda$ are both R-complexes. Also, for each index i, there are isomorphisms

$$\mathrm{H}_i\left(\prod_{\lambda\in\Lambda} M_\bullet^\lambda\right) \cong \prod_{\lambda\in\Lambda} \mathrm{H}_i(M_\bullet^\lambda) \qquad \mathrm{H}_i\left(\coprod_{\lambda\in\Lambda} M_\bullet^\lambda\right) \cong \coprod_{\lambda\in\Lambda} \mathrm{H}_i(M_\bullet^\lambda).$$

We sometimes write $\oplus_{\lambda\in\Lambda} M_\bullet^\lambda$ in place of $\coprod_{\lambda\in\Lambda} M_\bullet^\lambda$.

When $\Lambda = \{1,2\}$ we sometimes write $M_\bullet^1 \oplus M_\bullet^2$ in place of $\coprod_{i=1}^2 M_\bullet^i$. This complex has the following form:

$$\begin{array}{c} M_\bullet^1 \\ \oplus \\ M_\bullet^2 \end{array} = \cdots \to \begin{array}{c} M_{i+1}^1 \\ \oplus \\ M_{i+1}^2 \end{array} \xrightarrow{\begin{pmatrix} \partial_{i+1}^{M^1} & 0 \\ 0 & \partial_{i+1}^{M^1} \end{pmatrix}} \begin{array}{c} M_i^1 \\ \oplus \\ M_i^2 \end{array} \to \cdots.$$

This is the same as the product $M_\bullet^1 \times M_\bullet^2 = \prod_{i=1}^2 M_\bullet^i$. Similar comments hold when Λ is any finite set.

Theorem IV.1.10. *Let R be a commutative ring. Let $M_\bullet$ be an R-complex, and let N be an R-module.*

(a) *If N is flat, then $\mathrm{H}_i(M_\bullet \otimes_R N) \cong \mathrm{H}_i(M_\bullet) \otimes_R N$ for all $i \in \mathbb{Z}$.*
(b) *If N is flat, then $\mathrm{H}_i(N \otimes_R M_\bullet) \cong N \otimes_R \mathrm{H}_i(M_\bullet)$ for all $i \in \mathbb{Z}$.*
(c) *If N is projective, then $\mathrm{H}_i(\mathrm{Hom}_R(N, M_\bullet)) \cong \mathrm{Hom}_R(N, \mathrm{H}_i(M_\bullet))$ for all $i \in \mathbb{Z}$.*
(d) *If N is injective, then $\mathrm{H}_i(\mathrm{Hom}_R(M_\bullet, N)) \cong \mathrm{Hom}_R(\mathrm{H}_i(M_\bullet), N)$ for all $i \in \mathbb{Z}$.*

Proof. (a) Note that we have already shows that $M_\bullet \otimes_R N$ is an R-complex. Fix an integer i and consider the following exact sequence

$$0 \to \mathrm{Im}(\partial_{i+1}^M) \xrightarrow{\epsilon} \mathrm{Ker}(\partial_i^M) \xrightarrow{\tau} \mathrm{H}_i(M_\bullet) \to 0$$

where ϵ and τ are, respectively, the inclusion and the natural surjection.

Plan: We show that there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Im}(\partial_{i+1}^M) \otimes_R N & \xrightarrow{\epsilon\otimes_R N} & \mathrm{Ker}(\partial_i^M) \otimes_R N & \xrightarrow{\tau\otimes_R N} & \mathrm{H}_i(M_\bullet) \otimes_R N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle F}\cong & & \downarrow{\scriptstyle G}\cong & & \downarrow{\scriptstyle H} & & \\ 0 & \longrightarrow & \mathrm{Im}(\partial_{i+1}^M \otimes_R N) & \xrightarrow{\alpha} & \mathrm{Ker}(\partial_i^M \otimes_R N) & \xrightarrow{\pi} & \mathrm{H}_i(M_\bullet \otimes_R N) & \longrightarrow & 0 \end{array} \qquad (*)$$

wherein the maps F and G are isomorphisms, and the maps α and π are, respectively, the inclusion and the natural surjection. The Snake Lemma implies that H is an isomorphism, completing the proof.

Step 1: We build F using the universal mapping property for tensor products. Let $m = \partial_{i+1}^M(m') \in \mathrm{Im}(\partial_{i+1}^M) \subseteq M_i$, and let $n \in N$. In $M_i \otimes_R N$, we have

$$m \otimes n = \partial_{i+1}^M(m') \otimes n = (\partial_{i+1}^M \otimes_R N)(m' \otimes n) \in \mathrm{Im}(\partial_{i+1}^M \otimes_R N) \subseteq M_i \otimes_R N.$$

Hence, the map $f\colon \mathrm{Im}(\partial_{i+1}^M) \times N \to \mathrm{Im}(\partial_{i+1}^M \otimes_R N)$ given by $f(m,n) = m \otimes n$ is well-defined. It is straightforward to show that f is R-bilinear, so it induces a well-defined R-module homomorphism $F\colon \mathrm{Im}(\partial_{i+1}^M) \otimes_R N \to \mathrm{Im}(\partial_{i+1}^M \otimes_R N)$ such that $F(\sum_j m_j \otimes n_j) = \sum_j m_j \otimes n_j$.

Step 2: The map F is surjective. Every element of $\mathrm{Im}(\partial^M_{i+1} \otimes_R N)$ is of the form $\zeta = (\partial^M_{i+1} \otimes_R N)(\sum_j m_j \otimes n_j)$ for some $\sum_j m_j \otimes n_j \in M_{i+1} \otimes_R N$, so we have

$$\zeta = (\partial^M_{i+1} \otimes_R N)(\textstyle\sum_j m_j \otimes n_j) = \sum_j \partial^M_{i+1}(m_j) \otimes n_j = F(\sum_j \partial^M_{i+1}(m_j) \otimes n_j)$$

as desired.

Step 3: The map F is injective. Let $\gamma\colon \mathrm{Im}(\partial^M_{i+1}) \to M_i$ denote the inclusion. As N is flat and γ is injective, the induced map $\gamma \otimes_R N\colon \mathrm{Im}(\partial^M_{i+1}) \otimes_R N \to M_i \otimes_R N$ is injective. This map is given by $(\gamma \otimes_R N)(\sum_j m_j \otimes n_j) = \sum_j m_j \otimes n_j$. It follows that there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Im}(\partial^M_{i+1}) \otimes_R N & \xhookrightarrow{\gamma \otimes_R N} & M_i \otimes_R N \\ & {\scriptstyle F} \searrow \quad \nearrow & \\ & \mathrm{Im}(\partial^M_{i+1} \otimes_R N) & \end{array}$$

where the unlabeled map is the inclusion. Since $\gamma \otimes_R N$ is injective, it follows that F is injective.

Step 4: We construct G using the universal mapping property for tensor products. Let $m \in \mathrm{Ker}(\partial^i_M) \subseteq M_i$. For each $n \in N$, we have

$$(\partial^M_i \otimes_R N)(m \otimes n) = \partial^M_i(m) \otimes n = 0 \otimes n = 0$$

so $m \otimes n \in \mathrm{Ker}(\partial^M_i \otimes_R N)$. Hence, the map $g\colon \mathrm{Ker}(\partial^i_M) \times N \to \mathrm{Ker}(\partial^M_i \otimes_R N)$ given by $g(m,n) = m \otimes n$ is well-defined. It is straightforward to show that g is R-bilinear, so it induces an R-module homomorphism $G\colon \mathrm{Ker}(\partial^i_M) \otimes_R N \to \mathrm{Ker}(\partial^M_i \otimes_R N)$ such that $G(\sum_j m_j \otimes n_j) = \sum_j m_j \otimes n_j$.

Step 5: The map G is bijective. Consider the exact sequence

$$0 \to \mathrm{Ker}(\partial^M_i) \xrightarrow{\delta} M_i \xrightarrow{\partial^M_i} M_{i-1}$$

wherein δ is the inclusion. Apply the functor $- \otimes_R N$ to obtain the top row of the next diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ker}(\partial^M_i) \otimes_R N & \xrightarrow{\delta \otimes_R N} & M_i \otimes_R N & \xrightarrow{\partial^M_i \otimes_R N} & M_{i-1} \otimes_R N \\ & & \downarrow{\scriptstyle G} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\ 0 & \longrightarrow & \mathrm{Ker}(\partial^M_i \otimes_R N) & \xrightarrow{\sigma} & M_i \otimes_R N & \xrightarrow{\partial^M_i \otimes_R N} & M_{i-1} \otimes_R N \end{array}$$

wherein σ is the inclusion. It is straightforward to show that this diagram commutes. The top row of this diagram is exact because N is flat, and the bottom row is exact by construction. A straightforward diagram-chase now shows that G is an isomorphism.

Step 6: The left-most square in diagram $(*)$ commutes. We check the commutativity on simple tensors $m \otimes n \in \mathrm{Im}(\partial^M_{i+1}) \otimes_R N$:

$$G((\epsilon \otimes_R N)(m \otimes n)) = G(m \otimes n) = m \otimes n = \alpha(m \otimes n) = \alpha(F(m \otimes n)).$$

The general commutativity follows directly.

Step 7: The existence of the map H now follows from Exercise IV.1.17.

The proofs of parts (b)–(d) are similar. $\square$

Exercises.

Exercise IV.1.11. Let R be a commutative ring, and let $M_\bullet$ be an R-complex. Prove that, if $M_i = 0$, then $\mathrm{H}_i(M_\bullet) = 0$.

Exercise IV.1.12. Let R be a commutative noetherian ring, and let $M_\bullet$ be an R-complex. Prove that, if M_i is finitely generated over R, then $\mathrm{H}_i(M_\bullet)$ is finitely generated over R.

Exercise IV.1.13. Complete the proof of Proposition IV.1.6.

Exercise IV.1.14. Continue with the notation of Example IV.1.7, and compute the complexes $\mathbb{Z}^2 \otimes_{\mathbb{Z}} M_\bullet$, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^2, M_\bullet)$, and $\mathrm{Hom}_{\mathbb{Z}}(M_\bullet, \mathbb{Z}^2)$.

Exercise IV.1.15. Verify the facts from Remark IV.1.9.

Exercise IV.1.16. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings.

(a) Let $N_\bullet$ be an R-complex, and let M be an S-module. Prove that the following sequences are S-complexes: $N_\bullet \otimes_R M$ and $M \otimes_R N_\bullet$ and $\mathrm{Hom}_R(N_\bullet, M)$ and $\mathrm{Hom}_R(M, N_\bullet)$.

(b) Let $N_\bullet$ be an S-complex, and let M be an R-module. Prove that the following sequences are S-complexes: $N_\bullet \otimes_R M$ and $M \otimes_R N_\bullet$ and $\mathrm{Hom}_R(N_\bullet, M)$ and $\mathrm{Hom}_R(M, N_\bullet)$.

Exercise IV.1.17. Let R be a commutative ring, and consider the following commutative diagram of R-module homomorphisms with exact rows:

$$\begin{array}{ccccccc} M & \xrightarrow{f} & M' & \xrightarrow{f'} & M'' & \longrightarrow & 0 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} & & & & \\ N & \xrightarrow{h} & N' & \xrightarrow{h'} & N'' & \longrightarrow & 0. \end{array}$$

Prove that there is a well-defined R-module homomorphism $g''\colon M'' \to N''$ making the following diagram commute

$$\begin{array}{ccccccc} M & \xrightarrow{f} & M' & \xrightarrow{f'} & M'' & \longrightarrow & 0 \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g''} & & \\ N & \xrightarrow{h} & N' & \xrightarrow{h'} & N'' & \longrightarrow & 0. \end{array}$$

Exercise IV.1.18. Complete the proof of Theorem IV.1.10.

IV.2. Resolutions

Resolutions are special kinds of chain complexes. We use them to build Ext and Tor.

Projective resolutions.

Definition IV.2.1. Let R be a commutative ring, and let M be an R-module. A *projective resolution* of M over R or an *R-projective resolution* of M is an exact sequence of R-module homomorphisms

$$P^+_\bullet = \cdots \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \xrightarrow{\tau} \underbrace{M}_{\text{degree } -1} \to 0$$

such that each P_i is a projective R-module. The resolution $P^+_\bullet$ is a *free resolution of M over R* or an *R-free resolution of M* if each P_i is a free R-module. The *truncated projective (or free) resolution of M associated to $P^+_\bullet$* is the R-complex

$$P_\bullet = \cdots \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \to 0.$$

Example IV.2.2. Let R be a commutative ring. If P is a projective R-module, then projective and truncated projective resolutions of P over R are, respectively,

$$P^+_\bullet : 0 \to \underbrace{P}_{\text{degree } 0} \xrightarrow{\mathbb{1}_P} \underbrace{P}_{\text{degree } -1} \to 0 \qquad \text{and} \qquad P_\bullet : 0 \to \underbrace{P}_{\text{degree } 0} \to 0.$$

Example IV.2.3. Fix an integer $n \geqslant 2$. Projective and truncated projective resolutions of $\mathbb{Z}/n\mathbb{Z}$ over $\mathbb{Z}$ are given by

$$P^+_\bullet : 0 \to \underbrace{\mathbb{Z}}_{\text{degree } 1} \xrightarrow{n} \underbrace{\mathbb{Z}}_{\text{degree } 0} \xrightarrow{\pi} \underbrace{\mathbb{Z}/n\mathbb{Z}}_{\text{degree } -1} \to 0 \qquad \text{and} \qquad P_\bullet : 0 \to \underbrace{\mathbb{Z}}_{\text{degree } 1} \xrightarrow{n} \underbrace{\mathbb{Z}}_{\text{degree } 0} \to 0$$

respectively. When $r \in R$ is not a zero divisor, resolutions of $R/(r)$ over R are produced similarly.

The Fundamental Theorem for Finitely Generated Abelian Groups shows the following: If G is a finitely generated abelian group, then there are integers $r, s \geqslant 0$ and a $\mathbb{Z}$-projective resolution

$$0 \to \mathbb{Z}^s \to \mathbb{Z}^r \to G \to 0.$$

Resolutions may never stop.

Example IV.2.4. Fix integers $m, n \geqslant 2$. Projective and truncated projective resolutions of $\mathbb{Z}/n\mathbb{Z}$ over $\mathbb{Z}/mn\mathbb{Z}$ are given by

$$P^+_\bullet = \cdots \xrightarrow{m} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 3} \xrightarrow{n} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 2} \xrightarrow{m} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 1} \xrightarrow{n} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 0} \xrightarrow{\pi} \underbrace{\mathbb{Z}/n\mathbb{Z}}_{\text{degree } -1} \to 0$$

$$P_\bullet = \cdots \xrightarrow{m} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 3} \xrightarrow{n} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 2} \xrightarrow{m} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 1} \xrightarrow{n} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 0} \to 0$$

respectively. When $r, s \in R$ are not zero divisors, resolutions of $R/(s)$ over $R/(rs)$ are produced similarly.

Resolutions are not unique.

Example IV.2.5. Let R be a commutative ring. Let $P^+_\bullet$ be a projective resolution of an R-module M. Given any projective R-module Q, and an integer $i \geqslant 0$, the following sequence

$$\widehat{P}^+_\bullet = \cdots \xrightarrow{\partial^P_{i+3}} P_{i+2} \xrightarrow{\begin{pmatrix} \partial^P_{i+2} \\ 0 \end{pmatrix}} P_{i+1} \oplus Q \xrightarrow{\begin{pmatrix} \partial^P_{i+1} & 0 \\ 0 & \mathbb{1}_Q \end{pmatrix}} P_i \oplus Q \xrightarrow{(\partial^P_i \ 0)} P_{i-1} \xrightarrow{\partial^P_{i-1}} \cdots$$

is also a projective resolution of M.

The existence of projective resolutions is given in Exercise IV.2.9. This is essentially a consequence of the last property in Remark III.1.1.

Injective resolutions.

Definition IV.2.6. Let R be a commutative ring. Let M be an R-module. An *injective resolution* of M over R or an *R-injective resolution* of M is an exact sequence of R-module homomorphisms

$$ {}^{+}I_\bullet = \quad 0 \to \underbrace{M}_{\text{degree } 1} \xrightarrow{\epsilon} I_0 \xrightarrow{\partial^I_0} I_{-1} \xrightarrow{\partial^I_{-1}} I_{-2} \xrightarrow{\partial^I_{-1}} \cdots $$

such that each I_j is an injective R-module. The *truncated injective resolution of M associated to ${}^{+}I_\bullet$* is the R-complex

$$ I_\bullet = \quad 0 \to I_0 \xrightarrow{\partial^I_0} I_{-1} \xrightarrow{\partial^I_{-1}} I_{-2} \xrightarrow{\partial^I_{-1}} \cdots . $$

Example IV.2.7. Considering $\mathbb{Z}$ as a $\mathbb{Z}$-module, we have the following injective resolution

$$ 0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0. $$

The fact that $\mathbb{Q}/\mathbb{Z}$ is injective follows from the fact that a $\mathbb{Z}$-module is injective if and only if it is divisible. The analogous result also holds over any principal ideal domain R, which then has an injective resolution of the form

$$ 0 \to R \to K \to K/R \to 0 $$

where K is the field of fractions of R. In particular, this works when R is the polynomial ring $k[X]$ in one variable over a field k.

The existence of injective resolutions is given in Exercise IV.2.12. This is essentially a consequence of Theorem III.1.16.

Exercises.

Exercise IV.2.8. Let R be a commutative ring, and let $P^+_\bullet$ be a projective resolution of an R-module M. Prove that the truncated resolution $P_\bullet$ is an R-complex and that the homology of $P_\bullet$ is

$$ \mathrm{H}_i(P_\bullet) \cong \begin{cases} M & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases} $$

Conversely, let $Q_\bullet$ be a complex of projective R-modules such that $\mathrm{H}_i(Q_\bullet) = 0$ for all $i \geqslant 1$ and $Q_j = 0$ for all $j < 0$. Prove that $Q_\bullet$ is a projective resolution of $\mathrm{H}_0(Q_\bullet)$.

Exercise IV.2.9. Let R be a commutative ring, and let M be an R-module.

(a) Prove that M admits a free (hence projective) resolution over R.
(b) If R is noetherian and M is finitely generated as an R-module, then M admits a free (hence projective) resolution $P^+_\bullet$ over R such that each P_i is finitely generated over R.

Exercise IV.2.10. Let R be a commutative ring, and let $\{M^\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules. For each $\lambda \in \Lambda$, let $P^\lambda_\bullet$ be a projective resolution of M^λ.

(a) Prove that the coproduct complex $\coprod_{\lambda\in\Lambda} P^\lambda_\bullet$ is a projective resolution of the coproduct $\coprod_{\lambda\in\Lambda} M^\lambda$.
(b) Prove that, if each $P^\lambda_\bullet$ be a free resolution of M^λ, then $\coprod_{\lambda\in\Lambda} P^\lambda_\bullet$ is a free resolution of $\coprod_{\lambda\in\Lambda} M^\lambda$.

Exercise IV.2.11. Let R be a commutative ring. Let ${}^{+}I_\bullet$ be an injective resolution of an R-module M. Prove that the truncated resolution $I_\bullet$ is an R-complex and that the homology of $I_\bullet$ is

$$\mathrm{H}_j(I_\bullet) \cong \begin{cases} M & \text{if } j = 0 \\ 0 & \text{if } j \neq 0. \end{cases}$$

Conversely, let $J_\bullet$ be a complex of injective R-modules such that $\mathrm{H}_i(J_\bullet) = 0$ for all $i < 0$ and $J_i = 0$ for all $i > 0$. Prove that $J_\bullet$ is an injective resolution of $\mathrm{H}_0(J_\bullet)$.

Exercise IV.2.12. Let R be a commutative ring, and let M be an R-module. Prove that M admits an injective resolution over R.

Exercise IV.2.13. Let R be a commutative ring, and let $\{M^\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules. For each $\lambda \in \Lambda$, let $I^\lambda_\bullet$ be an injective resolution of M^λ.

(a) Prove that the product complex $\prod_{\lambda\in\Lambda} I^\lambda_\bullet$ is an injective resolution of $\prod_{\lambda\in\Lambda} M^\lambda$.
(b) Prove that, if R is noetherian, then the coproduct complex $\coprod_{\lambda\in\Lambda} I^\lambda_\bullet$ is an injective resolution of $\coprod_{\lambda\in\Lambda} M^\lambda$.

IV.3. Ext-Modules

Ext-modules via projective resolutions.

Definition IV.3.1. Let R be a commutative ring. Let M be an R-module, and fix a projective resolution of M over R

$$P_\bullet = \cdots \xrightarrow{\partial^P_{i+2}} P_{i+1} \xrightarrow{\partial^P_{i+1}} P_i \xrightarrow{\partial^P_i} P_{i-1} \xrightarrow{\partial^P_{i-1}} \cdots \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \to 0.$$

For each R-module N the sequence

$$\mathrm{Hom}(P_\bullet, N) =$$

$$0 \to \underbrace{\mathrm{Hom}(P_0, N)}_{\text{degree } 0} \xrightarrow{\mathrm{Hom}(\partial^P_1, N)} \underbrace{\mathrm{Hom}(P_1, N)}_{\text{degree } -1} \xrightarrow{\mathrm{Hom}(\partial^P_2, N)} \cdots$$

$$\cdots \to \underbrace{\mathrm{Hom}(P_{i-1}, N)}_{\text{degree } 1-i} \xrightarrow{\mathrm{Hom}(\partial^P_i, N)} \underbrace{\mathrm{Hom}(P_i, N)}_{\text{degree } -i} \xrightarrow{\mathrm{Hom}(\partial^P_{i+1}, N)} \underbrace{\mathrm{Hom}(P_{i+1}, N)}_{\text{degree } -1-i} \to \cdots$$

is an R-complex by Proposition IV.1.6. For each $i \in \mathbb{Z}$ set

$$\mathrm{Ext}^i_R(M, N) = \mathrm{H}_{-i}(\mathrm{Hom}_R(P_\bullet, N)) = \mathrm{Ker}(\mathrm{Hom}_R(\partial^P_{i+1}, N))/\,\mathrm{Im}(\mathrm{Hom}_R(\partial^P_i, N))$$

which is an R-module.

To aid us in computations of examples, we recall some facts from Remark I.5.3 and Example I.5.4.

Remark IV.3.2. Let R be a commutative ring, and let N be an R-module. There is an R-module isomorphism

$$\psi\colon \mathrm{Hom}_R(R, N) \xrightarrow{\cong} N \qquad \text{given by} \qquad \phi \mapsto \phi(1).$$

The inverse of ψ is given by $\psi^{-1}(n) = \phi_n\colon R \to N$ where $\phi_n(r) = rn$.

Let $r \in R$, and let $\mu^R_r\colon R \to R$ be the map given by $s \mapsto rs$. Then the map $\mathrm{Hom}_R(\mu^R_r, N)\colon \mathrm{Hom}_R(R, N) \to \mathrm{Hom}_R(R, N)$ is given by $\phi \mapsto r\phi$. Combining this with the isomorphism from the previous paragraph, this map is equivalent to

the map $\mu_r^N\colon N \to N$ given by $n \mapsto rn$; equivalent in the sense that there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_R(R,N) & \xrightarrow{\operatorname{Hom}_R(\mu_r^R,N)} & \operatorname{Hom}_R(R,N) \\ \psi \downarrow \cong & & \psi \downarrow \cong \\ N & \xrightarrow{\mu_r^N} & N. \end{array}$$

Here are some computations of Ext.

Example IV.3.3. Let m and n be integers such that $m, n \geqslant 2$. We compute $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ for all integers i. We start with the projective resolution of $\mathbb{Z}/m\mathbb{Z}$ over $\mathbb{Z}$ from Example IV.2.3

$$P^+_\bullet : 0 \to \underbrace{\mathbb{Z}}_{\text{degree } 1} \xrightarrow{m} \underbrace{\mathbb{Z}}_{\text{degree } 0} \xrightarrow{\pi} \underbrace{\mathbb{Z}/m\mathbb{Z}}_{\text{degree } -1} \to 0 \qquad \text{and} \qquad P_\bullet : 0 \to \underbrace{\mathbb{Z}}_{\text{degree } 1} \xrightarrow{m} \underbrace{\mathbb{Z}}_{\text{degree } 0} \to 0.$$

The complex $\operatorname{Hom}_{\mathbb{Z}}(P_\bullet, \mathbb{Z}/n\mathbb{Z})$ then has the following form

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) & \xrightarrow{m} & \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) & \longrightarrow & 0 \\ & & \text{degree } 0 & & \text{degree } -1 & & \\ 0 & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \xrightarrow{m} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0. \end{array}$$

It follows that

$$\operatorname{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \operatorname{Ker}(\mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z}) = \frac{(l/m)\mathbb{Z}}{n\mathbb{Z}} \cong \frac{\mathbb{Z}}{(mn/l)\mathbb{Z}} \cong \frac{\mathbb{Z}}{g\mathbb{Z}}$$

where $l = \operatorname{lcm}(m,n)$ and $g = \gcd(m,n)$, and

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \frac{\mathbb{Z}/n\mathbb{Z}}{\operatorname{Im}(\mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z})} = \frac{\mathbb{Z}/n\mathbb{Z}}{m(\mathbb{Z}/n\mathbb{Z})} \cong \frac{\mathbb{Z}}{(m,n)\mathbb{Z}} \cong \frac{\mathbb{Z}}{g\mathbb{Z}}$$

and $\operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$ for all $i \neq 0, 1$.

(For the kernel K of the map $\mathbb{Z}/n\mathbb{Z} \xrightarrow{m} \mathbb{Z}/n\mathbb{Z}$, argue as follows. Given an integer a, the element $\overline{a} \in \mathbb{Z}/n\mathbb{Z}$ is in K if and only if $am \in n\mathbb{Z}$. If $a \in (l/m)\mathbb{Z}$, then $ma \in l\mathbb{Z} \subseteq n\mathbb{Z}$. Conversely, if $ma \in n\mathbb{Z}$, then $n \mid am$. Since $m \mid am$ also, it follows that $l \mid am$. Hence, we have $(l/m) \mid a$, so $a \in (l/m)\mathbb{Z}$.)

Example IV.3.4. Fix integers $m, n \geqslant 2$. We compute $\operatorname{Ext}^i_{\mathbb{Z}/mn\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, N)$ for some modules N. Example IV.2.4 gives a projective and truncated projective resolutions of $\mathbb{Z}/n\mathbb{Z}$ over $\mathbb{Z}/mn\mathbb{Z}$ as

$$\begin{aligned} P^+_\bullet &= \cdots \xrightarrow{m} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 3} \xrightarrow{n} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 2} \xrightarrow{m} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 1} \xrightarrow{n} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 0} \xrightarrow{\pi} \underbrace{\mathbb{Z}/n\mathbb{Z}}_{\text{degree } -1} \to 0 \\ P_\bullet &= \cdots \xrightarrow{m} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 3} \xrightarrow{n} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 2} \xrightarrow{m} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 1} \xrightarrow{n} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\text{degree } 0} \to 0 \end{aligned}$$

respectively. The complex $\operatorname{Hom}_{\mathbb{Z}/mn\mathbb{Z}}(P_\bullet, N)$ has the following form:

$$\operatorname{Hom}_{\mathbb{Z}/mn\mathbb{Z}}(P_\bullet, N) = \qquad 0 \to \underbrace{N}_{\deg 0} \xrightarrow{n} \underbrace{N}_{\deg -1} \xrightarrow{m} \underbrace{N}_{\deg -2} \xrightarrow{n} \underbrace{N}_{\deg -3} \xrightarrow{m} \cdots .$$

(1) With $N = \mathbb{Z}/mn\mathbb{Z}$, we have

$$\mathrm{Hom}_{\mathbb{Z}/mn\mathbb{Z}}(P_\bullet, \mathbb{Z}/mn\mathbb{Z}) = \\ 0 \to \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\deg\ 0} \xrightarrow{n} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\deg\ -1} \xrightarrow{m} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\deg\ -2} \xrightarrow{n} \underbrace{\mathbb{Z}/mn\mathbb{Z}}_{\deg\ -3} \xrightarrow{m} \cdots .$$

As in Example IV.2.4, we see that this is exact in all degrees except degree 0. In degree 0, we have

$$\mathrm{Ker}(\mathbb{Z}/mn\mathbb{Z} \xrightarrow{n} \mathbb{Z}/mn\mathbb{Z}) = m\mathbb{Z}/mn\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}.$$

And so, we have

$$\mathrm{Ext}^i_{\mathbb{Z}/mn\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/mn\mathbb{Z}) \cong \begin{cases} 0 & \text{if } i \neq 0 \\ \mathbb{Z}/n\mathbb{Z} & \text{if } i = 0. \end{cases}$$

(2) With $N = \mathbb{Z}/m\mathbb{Z}$, we have

$$\mathrm{Hom}_{\mathbb{Z}/mn\mathbb{Z}}(P_\bullet, \mathbb{Z}/m\mathbb{Z}) = \\ 0 \to \underbrace{\mathbb{Z}/m\mathbb{Z}}_{\deg\ 0} \xrightarrow{n} \underbrace{\mathbb{Z}/m\mathbb{Z}}_{\deg\ -1} \xrightarrow{m=0} \underbrace{\mathbb{Z}/m\mathbb{Z}}_{\deg\ -2} \xrightarrow{n} \underbrace{\mathbb{Z}/m\mathbb{Z}}_{\deg\ -3} \xrightarrow{m=0} \cdots .$$

The image and kernel of the zero-map are easy to compute. For the other map, let $g = \gcd(m, n)$ and note that $m/g \in \mathbb{Z}$. For the image, we have

$$\mathrm{Im}(\mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z}) = (n)(\mathbb{Z}/m\mathbb{Z}) = (m, n)\mathbb{Z}/m\mathbb{Z} = g\mathbb{Z}/m\mathbb{Z}.$$

The kernel was computed in the previous example.

$$\mathrm{Ker}(\mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/g\mathbb{Z}.$$

And so, we have

$$\mathrm{Ext}^0_{\mathbb{Z}/mn\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \mathrm{Ker}(\mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/g\mathbb{Z}.$$

$$\mathrm{Ext}^1_{\mathbb{Z}/mn\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \frac{\mathrm{Ker}(\mathbb{Z}/m\mathbb{Z} \xrightarrow{0} \mathbb{Z}/m\mathbb{Z})}{\mathrm{Im}(\mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z})} = \frac{\mathbb{Z}/m\mathbb{Z}}{g\mathbb{Z}/m\mathbb{Z}} \cong \mathbb{Z}/g\mathbb{Z}.$$

$$\mathrm{Ext}^2_{\mathbb{Z}/mn\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = \frac{\mathrm{Ker}(\mathbb{Z}/m\mathbb{Z} \xrightarrow{n} \mathbb{Z}/m\mathbb{Z})}{\mathrm{Im}(\mathbb{Z}/m\mathbb{Z} \xrightarrow{0} \mathbb{Z}/m\mathbb{Z})} = \frac{(m/g)\mathbb{Z}/m\mathbb{Z}}{(0)} \cong \mathbb{Z}/g\mathbb{Z}.$$

And similarly, we have

$$\mathrm{Ext}^i_{\mathbb{Z}/mn\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/g\mathbb{Z}$$

for all $i \geqslant 0$.

(3) With $N = \mathbb{Z}/n\mathbb{Z}$, we have

$$\mathrm{Hom}_{\mathbb{Z}/mn\mathbb{Z}}(P_\bullet, \mathbb{Z}/n\mathbb{Z}) = \quad 0 \to \underbrace{\mathbb{Z}/n\mathbb{Z}}_{\deg\ 0} \xrightarrow{n=0} \underbrace{\mathbb{Z}/n\mathbb{Z}}_{\deg\ -1} \xrightarrow{m} \underbrace{\mathbb{Z}/n\mathbb{Z}}_{\deg\ -2} \xrightarrow{n=0} \underbrace{\mathbb{Z}/n\mathbb{Z}}_{\deg\ -3} \xrightarrow{m} \cdots .$$

This implies

$$\mathrm{Ext}^0_{\mathbb{Z}/mn\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$$

and the computation in part (2) shows

$$\mathrm{Ext}^i_{\mathbb{Z}/mn\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/g\mathbb{Z}$$

for all $i > 0$.

When $r, s \in R$ are not zero divisors, the computations of the R-modules $\operatorname{Ext}^i_{R/rsR}(R/sR, R/rsR)$, $\operatorname{Ext}^i_{R/rsR}(R/sR, R/rR)$, and $\operatorname{Ext}^i_{R/rsR}(R/sR, R/sR)$ are similar.

The following theorem contains a very important fact about Ext that we prove later; see Theorem VIII.5.2.

Theorem IV.3.5. *Let R be a commutative ring, and let M and N be R-modules. The modules $\operatorname{Ext}^i_R(M,N)$ are independent of the choice of projective resolution of M. In other words, if $P^+_\bullet$ and $Q^+_\bullet$ are projective resolutions of M, then there is an R-module isomorphism $\operatorname{H}_{-i}(\operatorname{Hom}_R(P_\bullet, N)) \cong \operatorname{H}_{-i}(\operatorname{Hom}_R(Q_\bullet, N))$ for each index i.*

Assuming this fact, we will prove some properties of Ext from the introduction.

Proposition IV.3.6. *Let R be a commutative ring, and let M and N be R-modules.*

(a) *We have $\operatorname{Ext}^i_R(M,N) = 0$ for all $i < 0$.*
(b) *We have $\operatorname{Ext}^i_R(M,0) = 0$ for all $i \in \mathbb{Z}$.*
(c) *We have $\operatorname{Ext}^i_R(0,N) = 0$ for all $i \in \mathbb{Z}$.*

PROOF. Let $P^+_\bullet$ be a projective resolution of M.

(a) As we have seen in Definition IV.3.1, we have $\operatorname{Hom}_R(P_\bullet, N)_j = 0$ for all $j > 0$. For $i < 0$, we then have $\operatorname{Ext}^i_R(M,N) = \operatorname{H}_{-i}(\operatorname{Hom}_R(P_\bullet, N)) = 0$; see Exercise IV.1.11.

(b) For each index i, we have $\operatorname{Hom}_R(P_i, 0) = 0$. This implies that $\operatorname{Hom}_R(P_\bullet, N)$ is the zero complex $0_\bullet$, so we have $\operatorname{Ext}^i_R(M,N) = \operatorname{H}_{-i}(\operatorname{Hom}_R(P_\bullet, N)) = \operatorname{H}_{-i}(0_\bullet) = 0$ for each index i.

(c) The zero complex $0_\bullet$ is a projective resolution of 0, so we may take $P_\bullet = 0_\bullet$. With this choice, we have $\operatorname{Hom}_R(P_\bullet, N) = 0_\bullet$, so for each index $i \in \mathbb{Z}$ we have $\operatorname{Ext}^i_R(M,N) = \operatorname{H}_{-i}(\operatorname{Hom}_R(P_\bullet, N)) = \operatorname{H}_{-i}(0_\bullet) = 0$. □

Proposition IV.3.7. *Let R be a commutative ring, and let M and N be R-modules. There is an R-module isomorphism $\operatorname{Ext}^0_R(M,N) \cong \operatorname{Hom}_R(M,N)$.*

PROOF. Let $P^+_\bullet$ be a projective resolution of M. From Definition IV.3.1, we see that $\operatorname{Ext}^0_R(M,N)$ is the kernel of the map

$$\operatorname{Hom}(P_0, N) \xrightarrow{\operatorname{Hom}(\partial^P_1, N)} \operatorname{Hom}(P_1, N).$$

On the other hand, the following sequence is exact

$$P_1 \xrightarrow{\partial^P_1} P_0 \xrightarrow{\tau} M \to 0$$

so the left-exactness of $\operatorname{Hom}_R(-,N)$ implies that the next sequence is also exact

$$0 \to \operatorname{Hom}_R(M,N) \xrightarrow{\operatorname{Hom}_R(\tau,N)} \operatorname{Hom}_R(P_0,N) \xrightarrow{\operatorname{Hom}_R(\partial^P_1,N)} \operatorname{Hom}_R(P_1,N).$$

It follows that

$$\operatorname{Hom}_R(M,N) \cong \operatorname{Im}(\operatorname{Hom}_R(\tau,N)) = \operatorname{Ker}(\operatorname{Hom}_R(\partial^P_1,N)) \cong \operatorname{Ext}^0_R(M,N)$$

as desired. □

Proposition IV.3.8. *Let R be a commutative ring, and let M and N be R-modules.*

(a) *If M is projective, then $\operatorname{Ext}^i_R(M,N) = 0$ for all $i \neq 0$.*

(b) *If N is injective, then $\operatorname{Ext}^i_R(M,N)=0$ for all $i\neq 0$.*

PROOF. (a) Assume that M is projective. From Example IV.2.2, we know that projective and truncated projective resolutions of M over R are, respectively,

$$P^+_\bullet : 0 \to \underbrace{M}_{\text{degree } 0} \xrightarrow{\mathbb{1}_M} \underbrace{M}_{\text{degree } -1} \to 0 \qquad \text{and} \qquad P_\bullet : 0 \to \underbrace{M}_{\text{degree } 0} \to 0.$$

Thus, we have

$$\operatorname{Hom}_R(P_\bullet, N) = \qquad 0 \to \underbrace{\operatorname{Hom}_R(M,N)}_{\text{degree } 0} \to 0$$

Thus, for $i \neq 0$ we have $\operatorname{Ext}^i_R(M,N) = \operatorname{H}_{-i}(\operatorname{Hom}_R(P_\bullet, N)) = 0$; see Exercise IV.1.11.

(b) Assume that N is injective. Let $P^+_\bullet$ be a projective resolution of M. In particular, $P^+_\bullet$ is exact. Since N is injective, the sequence $\operatorname{Hom}_R(P^+_\bullet, N)$ is exact, so for $i \geqslant 1$, we have

$$\begin{aligned}\operatorname{Ext}^i_R(M,N) &= \operatorname{Ker}(\operatorname{Hom}_R(\partial^P_{i+1}, N))/\operatorname{Im}(\operatorname{Hom}_R(\partial^P_i, N)) \\ &= \operatorname{Ker}(\operatorname{Hom}_R(\partial^{P^+}_{i+1}, N))/\operatorname{Im}(\operatorname{Hom}_R(\partial^{P^+}_i, N)) = 0\end{aligned}$$

as desired. □

Proposition IV.3.9. *Let R be a commutative noetherian ring, and let M and N be finitely generated R-modules. For each index i, the R-module $\operatorname{Ext}^i_R(M,N)$ is finitely generated.*

PROOF. Exercise IV.2.9(b) shows that M has a free resolution $F_\bullet$ such that each F_i is finitely generated, say $F_i \cong R^{b_i}$. It follows that we have

$$\operatorname{Hom}_R(F_i, N) \cong \operatorname{Hom}_R(R^{b_i}, N) \cong \operatorname{Hom}_R(R,N)^{b_i} \cong N^{b_i}$$

so each of the modules in $\operatorname{Hom}_R(F_\bullet, N)$ is finitely generated. Exercise IV.1.12 implies that each homology module $\operatorname{Ext}^i_R(M,N) \cong \operatorname{H}_{-i}(\operatorname{Hom}_R(P_\bullet, N))$ is finitely generated. □

Ext-modules via injective resolutions. Here is another result that we do not have the tools to prove yet. It says that Ext is "balanced". See Theorem VIII.5.4 for part of the result.

Theorem IV.3.10. *Let R be a commutative ring. Let M and N be R-modules, and let ${}^+I_\bullet$ be an injective resolution of N. For each integer i, there is an isomorphism*

$$\operatorname{H}_{-i}(\operatorname{Hom}_R(M, I_\bullet)) \cong \operatorname{Ext}^i_R(M,N).$$

In other words, the modules $\operatorname{Ext}^i_R(M,N)$ can be computed using an injective resolution of N, and this is independent of the choice of injective resolution of N.

Exercises.

Exercise IV.3.11. Compute $\operatorname{Ext}^i_{\mathbb{Z}/12\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$ and $\operatorname{Ext}^i_{\mathbb{Z}/12\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z})$ for all $i \geqslant 0$.

Exercise IV.3.12. Let G be a finitely generated $\mathbb{Z}$-module, and let H be a $\mathbb{Z}$-module. Prove that $\operatorname{Ext}^i_{\mathbb{Z}}(G,H) = 0$ for all $i > 1$.

IV.4. Tor-Modules

Tor is to tensor product as Ext is to Hom.

Definition IV.4.1. Let R be a commutative ring. Let M be an R-module, and fix a projective resolution of M over R

$$P_\bullet = \cdots \xrightarrow{\partial^P_{i+2}} P_{i+1} \xrightarrow{\partial^P_{i+1}} P_i \xrightarrow{\partial^P_i} P_{i-1} \xrightarrow{\partial^P_{i-1}} \cdots \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \to 0.$$

For each R-module N the sequence

$$P_\bullet \otimes N = \cdots \xrightarrow{\partial^P_{i+2}\otimes N} P_{i+1} \otimes N \xrightarrow{\partial^P_{i+1}\otimes N} P_i \otimes N \xrightarrow{\partial^P_i \otimes N} P_{i-1} \otimes N \xrightarrow{\partial^P_{i-1}\otimes N} \cdots$$
$$\cdots \xrightarrow{\partial^P_2 \otimes N} P_1 \otimes N \xrightarrow{\partial^P_1 \otimes N} P_0 \otimes N \to 0$$

is an R-complex by Proposition IV.1.6. For each $i \in \mathbb{Z}$ set

$$\operatorname{Tor}^R_i(M,N) = \operatorname{H}_i(P_\bullet \otimes_R N)$$

which is an R-module.

To aid us in computations, we recall some facts from Examples II.1.9 and II.2.3.

Remark IV.4.2. Let R be a commutative ring, and let N be an R-module. There is an R-module isomorphism

$$\psi\colon R \otimes_R N \xrightarrow{\cong} N \qquad \text{given by} \qquad r \otimes n \mapsto rn.$$

The inverse of ψ is given by $\psi^{-1}(n) = 1 \otimes n$.

Let $r \in R$, and let $\mu^R_r\colon R \to R$ be the map given by $s \mapsto rs$. Then the map $\mu^R_r \otimes_R N\colon R \otimes_R N \to R \otimes_R N$ is given by $s \otimes n \mapsto r(s \otimes n) = (rs) \otimes n$. Combining this with the isomorphism from the previous paragraph, this map is equivalent to the map $\mu^N_r\colon N \to N$ given by $n \mapsto rn$; equivalent in the sense that there is a commutative diagram

$$\begin{array}{ccc} R \otimes_R N & \xrightarrow{\mu^R_r \otimes_R N} & R \otimes_R N \\ {\scriptstyle\psi}\downarrow{\scriptstyle\cong} & & {\scriptstyle\psi}\downarrow{\scriptstyle\cong} \\ N & \xrightarrow{\mu^N_r} & N. \end{array}$$

Here are some computations of Tor.

Example IV.4.3. Let m and n be integers such that $m, n \geqslant 2$. We compute $\operatorname{Tor}^{\mathbb{Z}}_i(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ for all integers i. We start with the projective resolution of $\mathbb{Z}/m\mathbb{Z}$ over $\mathbb{Z}$ from Example IV.2.3

$$P^+_\bullet : 0 \to \underbrace{\mathbb{Z}}_{\text{degree } 1} \xrightarrow{m} \underbrace{\mathbb{Z}}_{\text{degree } 0} \xrightarrow{\pi} \underbrace{\mathbb{Z}/m\mathbb{Z}}_{\text{degree } -1} \to 0 \qquad \text{and} \qquad P_\bullet : 0 \to \underbrace{\mathbb{Z}}_{\text{degree } 1} \xrightarrow{m} \underbrace{\mathbb{Z}}_{\text{degree } 0} \to 0.$$

The complex $P_\bullet \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ then has the following form

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} & \xrightarrow{m} & \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0 \\ & & \text{degree } 1 & & \text{degree } 0 & & \\ 0 & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \xrightarrow{m} & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0. \end{array}$$

As in Example IV.3.3, it follows that

$$\operatorname{Tor}_0^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z},\mathbb{Z}/n\mathbb{Z}) \cong \frac{\mathbb{Z}}{g\mathbb{Z}} \cong \operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z},\mathbb{Z}/n\mathbb{Z})$$

and $\operatorname{Tor}_i^{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z},\mathbb{Z}/n\mathbb{Z}) = 0$ for all $i \neq 0, 1$.

The following theorem contains a very important fact about Tor that we do not have time to prove.

Theorem IV.4.4. *Let R be a commutative ring, and let M and N be R-modules. The modules $\operatorname{Tor}_i^R(M,N)$ are independent of the choice of projective resolution of M. In other words, if $P_\bullet^+$ and $Q_\bullet^+$ are projective resolutions of M, then there is an R-module isomorphism $\operatorname{H}_i(P_\bullet \otimes_R N) \cong \operatorname{H}_i(Q_\bullet \otimes_R N)$ for every integer i.*

The next three results are proved like Propositions IV.3.6–IV.3.8.

Proposition IV.4.5. *Let R be a commutative ring, and let M and N be R-modules.*

(a) *We have $\operatorname{Tor}_i^R(M,N) = 0$ for all $i < 0$.*
(b) *We have $\operatorname{Tor}_i^R(0,N) = 0$ for all i.*
(c) *We have $\operatorname{Tor}_i^R(M,0) = 0$ for all i.*

Proof. Exercise IV.4.13 □

Proposition IV.4.6. *Let R be a commutative ring, and let M and N be R-modules. We have $\operatorname{Tor}_0^R(M,N) \cong M \otimes_R N$.*

Proof. Exercise IV.4.14 □

Proposition IV.4.7. *Let R be a commutative ring, and let M and N be R-modules.*

(a) *If M is projective, then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \neq 0$.*
(b) *If N is flat, then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \neq 0$.*

Proof. Exercise IV.4.15 □

Here is another result that we do not have the tools to prove yet. It says that Tor is "balanced". See Theorem VIII.5.7 for part of the result.

Theorem IV.4.8. *Let R be a commutative ring. Let M and N be R-modules, and let $Q_\bullet$ be a projective resolution of N. For each integer i, there is an isomorphism*

$$\operatorname{H}_i(M \otimes_R Q_\bullet) \cong \operatorname{Tor}_i^R(M,N).$$

In other words, the modules $\operatorname{Tor}_i^R(M,N)$ can be computed using a projective resolution of N, and this is independent of the choice of projective resolution of N.

Corollary IV.4.9. *Let R be a commutative ring, and let M and N be R-modules. If M is flat, then $\operatorname{Tor}_i^R(M,N) = 0$ for all $i \neq 0$.*

Proof. Use Theorem IV.4.8 as in the proof of Proposition IV.4.7(b). □

Exercises.

Exercise IV.4.10. Let m, n be integers with $m, n \geqslant 2$. Compute the $\mathbb{Z}$-modules $\operatorname{Tor}_i^{\mathbb{Z}/mn\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/mn\mathbb{Z})$, $\operatorname{Tor}_i^{\mathbb{Z}/mn\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$, $\operatorname{Tor}_i^{\mathbb{Z}/mn\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ for all $i \geqslant 0$.

Exercise IV.4.11. Compute $\operatorname{Tor}_i^{\mathbb{Z}/12\mathbb{Z}}(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/3\mathbb{Z})$ and $\operatorname{Tor}_i^{\mathbb{Z}/12\mathbb{Z}}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/6\mathbb{Z})$ for all $i \geqslant 0$.

Exercise IV.4.12. Let G be a finitely generated $\mathbb{Z}$-module, and let H be a $\mathbb{Z}$-module. Prove that $\operatorname{Tor}_i^{\mathbb{Z}}(G, H) = 0$ for all $i > 1$.

Exercise IV.4.13. Prove Proposition IV.4.5

Exercise IV.4.14. Prove Proposition IV.4.6

Exercise IV.4.15. Prove Proposition IV.4.7

Exercise IV.4.16. Prove Corollary IV.4.9

Exercise IV.4.17. Let R be a commutative noetherian ring, and let M and N be finitely generated R-modules. Prove that the R-module $\operatorname{Tor}_i^R(M, N)$ is finitely generated for each index i.

IV.5. Epilogue

"Ext" is short for "extension", and "Tor" is short for "torsion". We give a brief discussion of these connections, without proofs. This section is not needed for the sequel.

Definition IV.5.1. Let R be a commutative ring, and let M and N be R-modules. An *extension* of N by M is a short exact sequence

$$0 \to N \to T \to M \to 0.$$

Given two extensions of N by M

$$\begin{aligned} \xi = \quad & 0 \longrightarrow N \xrightarrow{f} T \xrightarrow{g} M \longrightarrow 0 \\ \xi' = \quad & 0 \longrightarrow N \xrightarrow{f'} T' \xrightarrow{g'} M \longrightarrow 0 \end{aligned}$$

we say that ξ and ξ' are *equivalent* if there exists an R-module homomorphism $h\colon T \to T'$ making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \xrightarrow{f} & T & \xrightarrow{g} & M & \longrightarrow & 0 \\ & & \Big\| & & \Big\downarrow{\scriptstyle h} & & \Big\| & & \\ 0 & \longrightarrow & N & \xrightarrow{f'} & T' & \xrightarrow{g'} & M & \longrightarrow & 0. \end{array}$$

Remark IV.5.2. Let R be a commutative ring, and let M and N be R-modules. Let ξ and ξ' be equivalent extensions of N by M, with $h\colon T \to T'$ as in Definition IV.5.1. A straightforward diagram chase (or the Snake Lemma) shows that h is an isomorphism. Similarly, the inverse $h^{-1}\colon T' \to T$ shows that ξ' and ξ are equivalent. Moreover, the relation "equivalence" is an equivalence relation on the class of all extensions of N by M. The extension ξ is split (as a short exact sequence) if and only if it is equivalent to the sequence

$$0 \to N \xrightarrow{i} N \oplus M \xrightarrow{\tau} M \to 0$$

where i and τ are the canonical injection and surjection.

The following theorem provides the connection between Ext and extensions. We will not be proving it here.

Theorem IV.5.3. *Let R be a commutative ring, and let M and N be R-modules. Let $\mathrm{e}_R(M,N)$ denote the class of all equivalence classes of extensions of N by M. Then there is a bijection $\phi\colon \operatorname{Ext}^1_R(M,N) \xrightarrow{\approx} \mathrm{e}_R(M,N)$ such that $\phi(0)$ is the equivalence class of the split extension. In particular, the set $\mathrm{e}_R(M,N)$ has the structure of an R-module.*

In Lemma VII.3.4, we give a proof of the following useful corollary that does not use Theorem IV.5.3.

Corollary IV.5.4. *Let R be a commutative ring, and consider the following exact sequence of R-module homomorphisms:*

$$0 \to N \to T \to M \to 0.$$

If $\operatorname{Ext}^1_R(M,N) = 0$, then the displayed sequence splits.

Proof. The displayed sequence is an extension of N by M. By assumption, we have $\mathrm{e}_R(M,N) \approx \operatorname{Ext}^1_R(M,N) = 0$. Hence, the given sequence is equivalent to the split sequence; see Remark IV.5.2 □

Next, we describe a connection between Tor and torsion.

Definition IV.5.5. Let R be an integral domain, and let M be an R-module. The *torsion submodule* of M is

$$\mathrm{t}(M) = \{m \in M \mid \text{there exists } 0 \neq r \in R \text{ such that } rm = 0\}.$$

The module M is *torsion-free* if $\mathrm{t}(M) = 0$. The module M is *torsion* if $\mathrm{t}(M) = M$.

Remark IV.5.6. Let R be an integral domain with field of fractions K, and let M be an R-module. Then $\mathrm{t}(M)$ is a submodule of M. Moreover, it is the unique largest torsion R-submodule of M. The quotient $M/\mathrm{t}(M)$ is torsion free, that is $\mathrm{t}(M/\mathrm{t}(M)) = 0$.

The following theorem provides the connection between Tor and torsion. See Theorem VIII.7.2

Theorem IV.5.7. *Let R be an integral domain with field of fractions K, and let M be an R-module. There is an R-module isomorphism $\psi\colon \operatorname{Tor}^R_1(K/R,M) \xrightarrow{\cong} \mathrm{t}(M)$.*

Exercises.

Exercise IV.5.8. Verify the facts from Remark IV.5.2.

Exercise IV.5.9. Verify the facts from Remark IV.5.6.

Exercise IV.5.10. Let G be a finitely generated $\mathbb{Z}$-module.

(a) Prove that G is torsion if and only if it is finite.
(b) Prove that G is torsion-free if and only if it is free.
(c) Prove that $G \cong \mathrm{t}(G) \oplus \mathbb{Z}^n$ for some integer n.

CHAPTER V

Depth September 8, 2009

Here we assume that Ext has the properties described in the introduction and show how it yields non-homological information about rings and modules. We begin by stating explicitly the as-of-yet unexplained facts we are assuming for this chapter. These fact will be explained in later chapters.

V.1. Assumptions

Fact V.1.1. Let R be a commutative ring. Let $f\colon M \to M'$ and $f'\colon M' \to M''$ be R-module homomorphisms. Let $g\colon N \to N'$ and $g'\colon N' \to N''$ be R-module homomorphisms. For each $i \in \mathbb{Z}$, there are well-defined R-module homomorphisms

$$\operatorname{Ext}^i_R(M,g)\colon \operatorname{Ext}^i_R(M,N) \to \operatorname{Ext}^i_R(M,N')$$
$$\operatorname{Ext}^i_R(f,N)\colon \operatorname{Ext}^i_R(M',N) \to \operatorname{Ext}^i_R(M,N).$$

Each of the operators $\operatorname{Ext}^i_R(M,-)$ and $\operatorname{Ext}^i_R(-,N)$ is functorial: For each $i \in \mathbb{Z}$, there are commutative diagrams

$$\begin{array}{ccc} \operatorname{Ext}^i_R(M,N) & \xrightarrow{\operatorname{Ext}^i_R(M,g)} & \operatorname{Ext}^i_R(M,N') \\ & \searrow^{\operatorname{Ext}^i_R(M,g'g)} & \downarrow{\scriptstyle \operatorname{Ext}^i_R(M,g')} \\ & & \operatorname{Ext}^i_R(M,N'') \end{array}$$

$$\begin{array}{ccc} \operatorname{Ext}^i_R(M'',N) & \xrightarrow{\operatorname{Ext}^i_R(f',N)} & \operatorname{Ext}^i_R(M',N) \\ & \searrow^{\operatorname{Ext}^i_R(f'f,N)} & \downarrow{\scriptstyle \operatorname{Ext}^i_R(f,N)} \\ & & \operatorname{Ext}^i_R(M,N) \end{array}$$

that is, we have

$$\operatorname{Ext}^i_R(M,g'g) = \operatorname{Ext}^i_R(M,g')\operatorname{Ext}^i_R(M,g)$$
$$\operatorname{Ext}^i_R(f'f,N) = \operatorname{Ext}^i_R(f,N)\operatorname{Ext}^i_R(f',N).$$

See Propositions VI.5.4 and VI.5.8.

Remark V.1.2. Let R be a commutative ring. Let M, N and N' be R-modules, and consider the zero-map $0^N_{N'}\colon N \to N'$. We claim that the induced homomorphism $\operatorname{Ext}^i_R(M,0^N_{N'})\colon \operatorname{Ext}^i_R(M,N) \to \operatorname{Ext}^i_R(M,N')$ is the corresponding zero-map

$0^{\operatorname{Ext}^i_R(M,N)}_{\operatorname{Ext}^i_R(M,N')}$. Indeed, there is a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{0^N_0} & 0 \\ & \searrow^{0^N_{N'}} & \downarrow{\scriptstyle 0^0_{N'}} \\ & & N' \end{array}$$

so Fact V.1.1 yields a second commutative diagram

$$\begin{array}{ccc} \operatorname{Ext}^i_R(M,N) & \xrightarrow{\operatorname{Ext}^i_R(M,0^N_0)=0} & \operatorname{Ext}^i_R(M,0)=0 \\ & \searrow^{\operatorname{Ext}^i_R(M,0^N_{N'})} & \downarrow{\scriptstyle \operatorname{Ext}^i_R(M,0^0_{N'})=0} \\ & & \operatorname{Ext}^i_R(M,N'). \end{array}$$

Similarly, if $0^M_{M'}\colon M \to M'$ is the zero-map, then the induced homomorphism $\operatorname{Ext}^i_R(0^M_{M'},N)\colon \operatorname{Ext}^i_R(M',N) \to \operatorname{Ext}^i_R(M,N)$ is the zero-map $0^{\operatorname{Ext}^i_R(M',N)}_{\operatorname{Ext}^i_R(M,N)}$. In other words, we have $\operatorname{Ext}^i_R(0,N)=0$ and $\operatorname{Ext}^i_R(M,0)=0$, regardless of whether the given 0 represents the zero-module or the zero-map. See Examples VI.5.2 and VI.5.6.

Fact V.1.3. Let R be a commutative ring. Let M and N be R-modules, and let $r \in R$. Let $\mu^M_r\colon M \to M$ be given by $m \mapsto rm$, and let $\mu^N_r\colon N \to N$ be given by $n \mapsto rn$. For each $i \in \mathbb{Z}$, the induced maps

$$\operatorname{Ext}^i_R(M,\mu^N_r)\colon \operatorname{Ext}^i_R(M,N) \to \operatorname{Ext}^i_R(M,N)$$
$$\operatorname{Ext}^i_R(\mu^M_r,N)\colon \operatorname{Ext}^i_R(M,N) \to \operatorname{Ext}^i_R(M,N)$$

are given by $\xi \mapsto r\xi$. See Examples VI.5.2 and VI.5.6.

Remark V.1.4. Let R be a commutative ring. Let M and N be R-modules, and consider the identity maps $\mathbb{1}_N\colon N \to N$ and $\mathbb{1}_M\colon M \to M$. Since $\mathbb{1}_N$ and $\mathbb{1}_M$ are given by multiplication by 1, Fact V.1.3 implies that the induced maps

$$\operatorname{Ext}^i_R(M,\mathbb{1}_N)\colon \operatorname{Ext}^i_R(M,N) \to \operatorname{Ext}^i_R(M,N)$$
$$\operatorname{Ext}^i_R(\mathbb{1}_M,N)\colon \operatorname{Ext}^i_R(M,N) \to \operatorname{Ext}^i_R(M,N)$$

are given by $\xi \mapsto 1\xi = \xi$. That is, these maps are the respective identities; see Examples VI.5.2 and VI.5.6.

Remark V.1.5. Let R be a commutative ring. Let $\mathbf{f}\colon M \to M'$ and $g\colon N \to N'$ be R-module isomorphisms. We claim that the induced maps

$$\operatorname{Ext}^i_R(M,g)\colon \operatorname{Ext}^i_R(M,N) \to \operatorname{Ext}^i_R(M,N')$$
$$\operatorname{Ext}^i_R(f,N)\colon \operatorname{Ext}^i_R(M',N) \to \operatorname{Ext}^i_R(M,N)$$

are isomorphisms. We will prove this for $\operatorname{Ext}^i_R(f,N)$; the proof for $\operatorname{Ext}^i_R(M,g)$ is similar. We have $f^{-1} \circ f = \mathbb{1}_M$, so Fact V.1.1 and Remark V.1.4 imply the following:

$$\operatorname{Ext}^i_R(f,N) \circ \operatorname{Ext}^i_R(f^{-1},N) = \operatorname{Ext}^i_R(f^{-1} \circ f,N) = \operatorname{Ext}^i_R(\mathbb{1}_M,N) = \mathbb{1}_{\operatorname{Ext}^i_R(M,N)}.$$

Similarly, we have $\operatorname{Ext}^i_R(f^{-1},N) \circ \operatorname{Ext}^i_R(f,N) = \mathbb{1}_{\operatorname{Ext}^i_R(M',N)}$, so $\operatorname{Ext}^i_R(f,N)$ is an isomorphism with inverse $\operatorname{Ext}^i_R(f^{-1},N)$. See Examples VI.5.2 and VI.5.6.

Fact V.1.6. Let R be a commutative ring. Given an R-module N and an exact sequence of R-modules

$$0 \to M' \xrightarrow{f'} M \xrightarrow{f} M'' \to 0$$

there are two long exact sequences: the first one is for $\operatorname{Ext}^i_R(N,-)$

$$\begin{aligned} 0 \to \operatorname{Hom}_R(N,M') &\xrightarrow{\operatorname{Hom}_R(N,f')} \operatorname{Hom}_R(N,M) \xrightarrow{\operatorname{Hom}_R(N,f)} \operatorname{Hom}_R(N,M'') \\ \to \operatorname{Ext}^1_R(N,M') &\xrightarrow{\operatorname{Ext}^1_R(N,f')} \operatorname{Ext}^1_R(N,M) \xrightarrow{\operatorname{Ext}^1_R(N,f)} \operatorname{Ext}^1_R(N,M'') \to \cdots \\ \cdots \to \operatorname{Ext}^i_R(N,M') &\xrightarrow{\operatorname{Ext}^i_R(N,f)} \operatorname{Ext}^i_R(N,M) \xrightarrow{\operatorname{Ext}^1_R(N,f)} \operatorname{Ext}^i_R(N,M'') \to \cdots \end{aligned}$$

the second one is for $\operatorname{Ext}^i_R(-,N)$

$$\begin{aligned} 0 \to \operatorname{Hom}_R(M'',N) &\xrightarrow{\operatorname{Hom}_R(f,N)} \operatorname{Hom}_R(M,N) \xrightarrow{\operatorname{Hom}_R(f',N)} \operatorname{Hom}_R(M',N) \\ \to \operatorname{Ext}^1_R(M'',N) &\xrightarrow{\operatorname{Ext}^1_R(f,N)} \operatorname{Ext}^1_R(M,N) \xrightarrow{\operatorname{Ext}^1_R(f',N)} \operatorname{Ext}^1_R(M',N) \to \cdots \\ \cdots \to \operatorname{Ext}^i_R(M'',N) &\xrightarrow{\operatorname{Ext}^i_R(f,N)} \operatorname{Ext}^i_R(M,N) \xrightarrow{\operatorname{Ext}^i_R(f',N)} \operatorname{Ext}^i_R(M',N) \to \cdots . \end{aligned}$$

See Theorems VIII.2.1 and VIII.2.2.

V.2. Associated Primes and Supports of Modules

Before getting to depth, we need some preliminaries from commutative ring theory, namely, the notion of associated prime ideals.

Definition V.2.1. The *prime spectrum* of a commutative ring R is the set

$$\operatorname{Spec}(R) = \{\text{prime ideals of } R\}.$$

For each ideal $I \subseteq R$, we set

$$V(I) = \{P \in \operatorname{Spec}(R) \mid P \supseteq I\}.$$

The *radical* of I is the set

$$\operatorname{rad}(I) = \{r \in R \mid r^n \in I \text{ for some integer } n \geqslant 1\}.$$

Example V.2.2. Consider distinct positive prime integers $p_1 < \cdots < p_n$ and positive integers $e_1, \dots, e_n$. For the ideal $p_1^{e_1} \cdots p_n^{e_n}\mathbb{Z} \subseteq \mathbb{Z}$ we have

$$\begin{aligned} V(p_1^{e_1} \cdots p_n^{e_n}\mathbb{Z}) &= \{p_1\mathbb{Z}, \dots, p_n\mathbb{Z}\} \\ \operatorname{rad}(p_1^{e_1} \cdots p_n^{e_n}\mathbb{Z}) &= p_1 \cdots p_n\mathbb{Z}. \end{aligned}$$

Similar results hold for any non-zero ideal in a principal ideal domain.

Remark V.2.3. Let R be a commutative ring. Let $I \subseteq R$ be an ideal. The radical of I is an ideal of R such that $I \subseteq \operatorname{rad}(I)$ and $\operatorname{rad}(\operatorname{rad}(I)) = \operatorname{rad}(I)$. If J is another ideal of R such that $J \subseteq I$, then $\operatorname{rad}(J) \subseteq \operatorname{rad}(I)$. In general, we have

$$\operatorname{rad}(I) = \cap_{P \in V(I)} P.$$

If I is finitely generated and $I \subseteq \operatorname{rad}(J)$, then $I^n \subseteq J$ for all $n \gg 0$.

Lemma V.2.4. *Let R be a commutative ring. Let I and J be ideals in R such that $V(J) \subseteq V(I)$. Then $I \subseteq \operatorname{rad}(J)$. If I is finitely generated, then $I^n \subseteq J$ for all $n \gg 0$.*

PROOF. The ideal $\operatorname{rad}(J)$ is the intersection of the prime ideals of R that contain J. The condition $V(J) \subseteq V(I)$ implies that $I \subseteq P$ for all $P \in V(J)$, so

$$I \subseteq \cap_{P \in V(J)} P = \operatorname{rad}(J).$$

The final statement follows from Remark V.2.3. □

Definition V.2.5. Let R be a commutative ring, and let M be an R-module. The *annihilator ideal* of an element $m \in M$ is the set

$$\operatorname{Ann}_R(m) = \{r \in R \mid rm = 0\}.$$

The *annihilator ideal* of M is the set

$$\operatorname{Ann}_R(M) = \{r \in R \mid rM = 0\} = \cap_{m \in M} \operatorname{Ann}_R(m).$$

The *support* of M is the set

$$\operatorname{Supp}_R(M) = \{P \in \operatorname{Spec}(R) \mid M_P \neq 0\}.$$

Remark V.2.6. Let R be a commutative ring. Let M be an R-module, and let $m \in M$. The sets $\operatorname{Ann}_R(m) \subseteq R$ and $\operatorname{Ann}_R(M) \subseteq R$ are ideals of R. We have $\operatorname{Supp}_R(R) = \operatorname{Spec}(R)$ and $\operatorname{Supp}_R(0) = \emptyset$. If $I \subseteq R$ is an ideal, then

$$\operatorname{Supp}_R(R/I) = V(I) = \operatorname{Supp}_R(R/\operatorname{rad}(I)).$$

Similarly, one has $\operatorname{Supp}_R(M) = V(\operatorname{Ann}_R(M))$.

Example V.2.7. Let k be a field, and set $R = k[X, Y]$.

(a) If $f \in R$ is a non-zero non-unit, then

$$\operatorname{Supp}_R(R/fR) = \{P \in \operatorname{Spec}(R) \mid f \in P\}.$$

(b) For $m, n \geqslant 1$, we have

$$\operatorname{Supp}_R(R/(X^m, Y^n)R) = \{(X, Y)R\} = \operatorname{Supp}_R(R/((X, Y)R)^n).$$

(c) For $M = R/(X^2, XY)R$, we have

$$\operatorname{Supp}_R(R/(X^2, XY)R) = \operatorname{Supp}_R(R/(X)R) = \{P \in \operatorname{Spec}(R) \mid X \in P\}.$$

Definition V.2.8. Let R be a commutative ring. Let M be an R-module. A prime ideal $P \in \operatorname{Spec}(R)$ is an *associated prime ideal* of M if there is an element $m \in M$ such that $P = \operatorname{Ann}_R(m)$. The set of associated primes of M is denoted $\operatorname{Ass}_R(M)$.

Example V.2.9. Let R be a commutative ring. If $P \in \operatorname{Spec}(R)$, then one has $\operatorname{Ass}_R(R/P) = \{P\}$. More examples are given in Example V.3.6.

Remark V.2.10. Let R be a commutative ring, and let M be an R-module. A prime ideal $P \in \operatorname{Spec}(R)$ is an associated prime of M if and only if there is an injective R-module homomorphism $R/P \hookrightarrow M$, that is, if and only if M has a submodule $N \cong R/P$.

Proposition V.2.11. *Let R be a commutative ring. Assume that R is noetherian, and let M be a non-zero R-module.*

(a) *The set of ideals*

$$A_R(M) = \{\operatorname{Ann}_R(m) \mid 0 \neq m \in M\}$$

has maximal elements; each maximal element in $A_R(M)$ is an associated prime ideal of M. In particular, the set $\operatorname{Ass}_R(M)$ is non-empty, and every ideal of the form $\operatorname{Ann}_R(m)$ for some non-zero element $m \in M$ is contained in an associated prime ideal of M.

(b) *The set of zero-divisors for M is the union of the associated prime ideals of M.*
(c) *Every associated prime of M is in the support of M, so there is a containment* $\operatorname{Ass}_R(M) \subseteq \operatorname{Supp}_R(M)$.

PROOF. (a) The set $A_R(M)$ is non-empty because $M \neq 0$. Hence, $A_R(M)$ has maximal elements because R is noetherian. Let I be a maximal element of $A_R(M)$, say $I = \operatorname{Ann}_R(m)$. We show that I is prime. (Then $I \in \operatorname{Ass}_R(M)$.)

Fix $a, b \in R$ such that $ab \in I$, and assume $a \notin I$. Then $abm = 0$ and $am \neq 0$. Then $I = \operatorname{Ann}_R(m) \subseteq \operatorname{Ann}_R(am)$ and $\operatorname{Ann}_R(am) \in A_R(M)$, so the maximality of I in $A_R(M)$ implies $I = \operatorname{Ann}_R(am)$. The fact that $abm = 0$ implies $b \in \operatorname{Ann}_R(am) = I$, so I is prime.

The remaining conclusions follow from what we have just established.

(b) By definition, the set $\cup_{P \in \operatorname{Ass}_R(M)} P$ is contained in the set of zero-divisors for M. One the other hand, if x is a zero-divisor for M, then there is a non-zero $m \in M$ such that $xm = 0$; hence, $x \in \operatorname{Ann}_R(m)$ which is contained in some associated prime of M.

(c) For each $P \in \operatorname{Ass}_R(M)$, there is an exact sequence

$$0 \to R/P \to M.$$

Localizing this sequence yields a second exact sequence

$$0 \to (R/P)_P \to M_P.$$

Since $(R/P)_P \neq 0$, it follows that $M_P \neq 0$, that is, $P \in \operatorname{Supp}_R(M)$. □

The next example shows the necessity of the noetherian hypothesis in Proposition V.2.11.

Example V.2.12. Let k be a field. The cartesian product $A = \prod_{i=1}^{\infty} k$ is a commutative ring with coordinatewise operations

$$(a_1, a_2, \ldots) + (b_1, b_2, \ldots) = (a_1 + b_1, a_2 + b_2, \ldots)$$
$$(a_1, a_2, \ldots)(b_1, b_2, \ldots) = (a_1 b_1, a_2 b_2, \ldots)$$

and has the following additive and multiplicative identities:

$$0_A = (0_k, 0_k, \ldots) \qquad\qquad 1_A = (1_k, 1_k, \ldots).$$

The direct sum

$$I = \bigoplus_{i=1}^{\infty} k \subseteq \prod_{i=1}^{\infty} k = A$$

is an ideal of A that is not finitely generated. Hence, the ring A is not noetherian.

We claim that the ring $R = A/I$ does not have an associated prime ideal. (It will then follow from Proposition V.2.11 that R not noetherian.) We need to show that, for every non-zero element $\alpha \in R$, the ideal $\operatorname{Ann}_R(\alpha) \subseteq R$ is not prime.

The element $\alpha \in R$ is of the form $\alpha = a + I$ where $a = (a_1, a_2, \ldots) \in A$. The condition $\alpha \neq 0$ implies that $a \notin I$, that is, the set

$$|a| = \{i \in \mathbb{Z}_+ \mid a_i \neq 0\}$$

is an infinite set. Note that the ring R is not an integral domain. For instance, the elements $(1, 0, 1, 0, \ldots) + I, (0, 1, 0, 1, \ldots) + I$ are non-zero, and their product is $(0, 0, \ldots) + I = 0_R$.

Case 1: $|a| = \mathbb{Z}_+$. In this case, α is a unit with inverse $(a_1^{-1}, a_2^{-1}, \ldots) + I$, so we have $\operatorname{Ann}_R(\alpha) = 0$. Since R is not an integral domain, this ideal is not prime.

Case 2: the set $\mathbb{Z}_+ \smallsetminus |a|$ is finite. For $i = 1, 2, \ldots$ set

$$b_i = \begin{cases} 0_k & \text{if } i \in |a| \\ 1_k & \text{if } i \notin |a| \end{cases}$$

and set $b = (b_1, b_2, \ldots) \in A$. As the set $\mathbb{Z}_+ \smallsetminus |a|$ is finite, we have $b \in I$, and hence

$$\alpha = \alpha + (b + I) = (a_1 + b_1, a_2 + b_2, \ldots) + I$$

in R. By construction, we have $|(a_1 + b_1, a_2 + b_2, \ldots)| = \mathbb{Z}_+$. As in Case 1, the element α is a unit, so we have $\operatorname{Ann}_R(\alpha) = 0$ which is not prime.

Case 3: the set $\mathbb{Z}_+ \smallsetminus |a|$ is infinite. Write

$$R_1 = \frac{\prod_{i \in |a|} k}{\bigoplus_{i \in |a|} k} \qquad\qquad R_2 = \frac{\prod_{i \in \mathbb{Z}_+ \smallsetminus |a|} k}{\bigoplus_{i \in \mathbb{Z}_+ \smallsetminus |a|} k}.$$

It is straightforward to show that there is a ring isomorphism $f\colon R \xrightarrow{\cong} R_1 \times R_2$ such that $f(\alpha) = (\alpha_1, 0_{R_2})$ where α_1 is a unit in R_1. It follows that there are ring isomorphisms

$$\frac{R}{\operatorname{Ann}_R(\alpha)} \cong \frac{R_1 \times R_2}{\operatorname{Ann}_{R_1 \times R_2}(\alpha_1, 0_{R_2})} \cong \frac{R_1}{\operatorname{Ann}_{R_1}(\alpha_1)} \times \frac{R_2}{\operatorname{Ann}_{R_2}(0_{R_2})} \cong \frac{R_1}{0} \times \frac{R_2}{R_2} \cong R_1.$$

Since $|a|$ is infinite, the ring R_1 is not an integral domain, so the ideal $\operatorname{Ann}_R(\alpha)$ is not prime.

Proposition V.2.13. *Let R be a commutative ring, and consider the following exact sequence of R-module homomorphisms $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$.*

(a) *We have* $\operatorname{Supp}_R(M) = \operatorname{Supp}_R(M') \cup \operatorname{Supp}_R(M'')$.

(b) *We have* $\operatorname{Ass}_R(M') \subseteq \operatorname{Ass}_R(M) \subseteq \operatorname{Ass}_R(M') \cup \operatorname{Ass}_R(M'')$.

Proof. (a) Let $P \in \operatorname{Spec}(R)$. If $M_P = 0$, then the exact sequence

$$0 \to M'_P \to M_P \to M''_P \to 0 \tag{V.2.13.1}$$

shows $M'_P = 0 = M''_P$. In other words, we have

$$\operatorname{Spec}(R) - \operatorname{Supp}_R(M) \subseteq (\operatorname{Spec}(R) - \operatorname{Supp}_R(M')) \cap (\operatorname{Spec}(R) - \operatorname{Supp}_R(M''))$$

so DeMorgan's Law implies $\operatorname{Supp}_R(M) \supseteq \operatorname{Supp}_R(M') \cup \operatorname{Supp}_R(M'')$.

Assume now $M_P \neq 0$. The exact sequence (V.2.13.1) shows that either $M'_P \neq 0$ or $M''_P \neq 0$: otherwise the sequence has the form

$$0 \to 0 \to M_P \to 0 \to 0$$

which would imply $M_P = 0$. So, we have $\operatorname{Supp}_R(M) \subseteq \operatorname{Supp}_R(M') \cup \operatorname{Supp}_R(M'')$.

(b) Let $Q \in \operatorname{Ass}_R(M')$. Then there is a monomorphism $R/Q \hookrightarrow M'$. Composing this with the monomorphism $M' \hookrightarrow M$, we find a monomorphism $R/Q \hookrightarrow M$. Hence, we have $Q \in \operatorname{Ass}_R(M)$, so $\operatorname{Ass}_R(M') \subseteq \operatorname{Ass}_R(M)$.

For the final containment, let $P \in \operatorname{Ass}_R(M)$. Then there exists a submodule $R/P \cong N \subseteq M$. Note that, because f is a monomorphism, for each $0 \neq m' \in M'$, we have $\operatorname{Ann}_R(m') = \operatorname{Ann}_R(f(m'))$. Furthermore, if $f(m') \in N$, then Example V.2.9 implies $\operatorname{Ann}_R(f(m')) = P$.

If $f^{-1}(N) \neq 0$, fix an element $0 \neq m' \in f^{-1}(N)$. From the previous paragraph, we have

$$\operatorname{Ann}_R(m') = \operatorname{Ann}_R(f(m')) = P$$

so $P \in \operatorname{Ass}_R(M')$.

If $f^{-1}(N) = 0$, then the exactness of the given sequence implies

$$M'' \supseteq g(N) \cong N \cong R/P$$

so $P \in \operatorname{Ass}_R(M'')$. □

Lemma V.2.14. *Let R be a commutative ring. Let M be an R-module, and assume that there is a chain of submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$.*
(a) We have $\operatorname{Supp}_R(M) = \cup_{i=1}^n \operatorname{Supp}_R(M_i/M_{i-1})$.
(b) We have $\operatorname{Ass}_R(M) \subseteq \cup_{i=1}^n \operatorname{Ass}_R(M_i/M_{i-1})$.

PROOF. We prove part (b) by induction on n and leave part (a) as an exercise. The base case $n = 1$ is straightforward.

For the induction step, assume that $n > 1$ and that the result holds for all R-modules L such that there is a chain of submodules $0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_{n-1} = L$. Because of the exact sequence

$$0 \to M_{n-1} \to M \to M/M_{n-1} \to 0$$

Proposition V.2.13(b) provides the first containment in the following sequence

$$\begin{aligned}\operatorname{Ass}_R(M) &\subseteq \operatorname{Ass}_R(M_{n-1}) \cup \operatorname{Ass}_R(M/M_{n-1}) \\ &\subseteq [\cup_{i=1}^n \operatorname{Ass}_R(M_i/M_{i-1})] \cup \operatorname{Ass}_R(M_n/M_{n-1}) \\ &= \cup_{i=1}^n \operatorname{Ass}_R(M_i/M_{i-1})\end{aligned}$$

The second containment comes from our induction hypothesis because the module M_{n-1} has a chain of submodules $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{n-1}$; this also uses the assumption $M = M_n$. The equality is trivial. □

Note the equality in part (b) of the next result, contrasting with the containments in the previous two results.

Lemma V.2.15. *Let R be a commutative ring. Let $M_1, \ldots, M_n$ be R-modules.*
(a) We have $\operatorname{Supp}_R(\coprod_{i=1}^n M_i) = \cup_{i=1}^n \operatorname{Supp}_R(M_i)$.
(b) We have $\operatorname{Ass}_R(\coprod_{i=1}^n M_i) = \cup_{i=1}^n \operatorname{Ass}_R(M_i)$.

PROOF. The equality $\operatorname{Supp}_R(\coprod_{i=1}^n M_i) = \cup_{i=1}^n \operatorname{Supp}_R(M_i)$ and the containment $\operatorname{Ass}_R(\coprod_{i=1}^n M_i) \subseteq \cup_{i=1}^n \operatorname{Ass}_R(M_i)$ follow from Lemma V.2.14 via the filtration

$$0 \subseteq M_1 \subseteq \coprod_{i=1}^2 M_i \subseteq \cdots \subseteq \coprod_{i=1}^n M_i.$$

For the containment $\operatorname{Ass}_R(\coprod_{i=1}^n M_i) \supseteq \cup_{i=1}^n \operatorname{Ass}_R(M_i)$, let $\mathfrak{p} \in \operatorname{Ass}_R(M_j)$. Then there is a monomorphism $R/\mathfrak{p} \hookrightarrow M_j$. Compose this with the natural inclusion $M_j \hookrightarrow \coprod_{i=1}^n M_i$ to yield a monomorphism $R/\mathfrak{p} \hookrightarrow \coprod_{i=1}^n M_i$, and hence $\mathfrak{p} \in \operatorname{Ass}_R(\coprod_{i=1}^n M_i)$. □

Exercises.

Exercise V.2.16. Verify the facts in Example V.2.2.

Exercise V.2.17. Verify the facts in Remark V.2.3.

Exercise V.2.18. Let R be a commutative ring, and let M be an R-module. Let $U \subseteq R$ be a multiplicatively closed subset.

(a) Let $m \in M$. Prove that $m/1 = 0$ in $U^{-1}M$ if and only if there exists an element $u \in U$ such that $um = 0$ in M, i.e., if and only if $U \cap \operatorname{Ann}_R(m) \neq \emptyset$.

(b) Assume that M is finitely generated. Prove that $U^{-1}M = 0$ if and only if there exists an element $u \in U$ such that $uM = 0$, i.e., if and only if $U \cap \operatorname{Ann}_R(M) \neq \emptyset$.
(c) Provide an example showing that the finitely-generated assumption in part (b) is necessary.
(d) Verify the facts in Remark V.2.6.

Exercise V.2.19. Verify the facts in Example V.2.7.

Exercise V.2.20. Verify the facts in Example V.2.9.

Exercise V.2.21. Verify the conclusions of Remark V.2.10.

Exercise V.2.22. Let R be a commutative ring, and let M be an R-module.
(a) Prove that, if M is noetherian as an R-module, then the quotient $R/\operatorname{Ann}_R(M)$ is a noetherian ring.
(b) Verify the conclusions of Proposition V.2.11 when R is not necessarily noetherian but M is noetherian as an R-module.

Exercise V.2.23. Verify the facts in Example V.2.12.

Exercise V.2.24. Give examples of exact sequences as in Proposition V.2.13 such that $\operatorname{Ass}_R(M) \neq \operatorname{Ass}_R(M') \cup \operatorname{Ass}_R(M'')$ and $\operatorname{Ass}_R(M') \neq \operatorname{Ass}_R(M)$.

Exercise V.2.25. Let R be a commutative ring. Assume that R is noetherian, and let $r \in R$ be an R-regular element, that is, a non-unit that is not a zero-divisor on R. Prove that $\operatorname{Ass}_R(R/r^nR) = \operatorname{Ass}_R(R/rR)$ for all $n \geqslant 1$. [Hint: Verify that the following sequence is exact:

$$0 \to R/rR \xrightarrow{r^{n-1}} R/r^nR \to R/r^{n-1}R \to 0$$

and use induction on n.]

Exercise V.2.26. Complete the proof of Lemma V.2.14.

V.3. Prime Filtrations

The following result is a fundamental tool for the study of finitely generated modules over noetherian commutative rings. In applications, it allows one to reduce problems about finitely generated modules to the case where $M = R/P$. The chain of submodules in this result is called a *prime filtration* of M.

Theorem V.3.1. *Let R be a commutative ring. Assume that R is noetherian, and let M be a finitely generated R-module. There is a chain of submodules $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ such that, for $i = 1, \dots, n$ there exists $P_i \in \operatorname{Spec}(R)$ such that $M_i/M_{i-1} \cong R/P_i$.*

Proof. Let $P_1 \in \operatorname{Ass}_R(M)$, and fix a submodule $R/P_1 \cong M_1 \subseteq M$. If $M_1 = M$, then stop. If $M_1 \neq M$, then repeat the process with M/M_1 to find a prime ideal $P_2 \in \operatorname{Ass}(M/M_1)$ and a submodule $R/P_2 \cong M_2/M_1 \subseteq M/M_1$. Continue repeating. Since M is a noetherian R-module, the process must terminate in a finite number of steps. □

Theorem V.3.2. *Let R be a commutative ring. Let M be an R-module, and assume that there is a chain of submodules $0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$ such that, for $i = 1, \dots, n$ there exists $P_i \in \operatorname{Spec}(R)$ such that $M_i/M_{i-1} \cong R/P_i$.*
(a) *We have $\operatorname{Ass}_R(M) \subseteq \{P_1, \dots, P_n\} \subseteq \operatorname{Supp}_R(M)$.*

(b) *A prime ideal $P \in \operatorname{Spec}(R)$ is in $\operatorname{Supp}_R(M)$ if and only if $P \supseteq P_i$ for some i.*

PROOF. (a) The first containment is from Lemma V.2.14 using the equality $\operatorname{Ass}_R(R/P_i) = \{P_i\}$ for each index i; see Example V.2.9. For the second containment, note that we have

$$(M_i)_{P_i}/(M_{i-1})_{P_i} \cong (M_i/M_{i-1})_{P_i} \cong (R/P_i)_{P_i} \neq 0.$$

Since $(M_i)_{P_i}$ surjects onto the non-zero module $(M_i)_{P_i}/(M_{i-1})_{P_i}$, it follows that $(M_i)_{P_i} \neq 0$. Hence, $0 \neq (M_i)_{P_i} \subseteq M_{P_i}$ which implies that $M_{P_i} \neq 0$, so $P_i \in \operatorname{Supp}_R(M)$.

(b) Assume first that $P \supseteq P_i$. The localization $(P_i)_P \subseteq R_P$ is a prime ideal, and we have $(M_P)_{(P_i)_P} \cong M_{P_i} \neq 0$. It follows that $M_P \neq 0$, so $P \in \operatorname{Supp}_R(M)$.

Conversely, assume that $P \in \operatorname{Supp}_R(M)$. Localize the given filtration to find a filtration of M_P

$$0 = (M_0)_P \subseteq (M_1)_P \subseteq \cdots \subseteq (M_n)_P = M_P$$

such that, for $i = 1, \dots, n$ we have

$$(M_i)_P/(M_{i-1})_P \cong (M_i/M_{i-1})_P \cong (R/P_i)_P.$$

Since $M_P \neq 0$ by assumption, one of these quotients must be non-zero as well. In other words, there is an index i such that $P \in \operatorname{Supp}_R(R/P_i)$. Remark V.2.6 implies that $P \supseteq P_i$. □

Corollary V.3.3. *Let R be a commutative ring. Assume that R is noetherian, and let M be a finitely generated R-module. Then the set $\operatorname{Ass}_R(M)$ is finite.*

PROOF. The module M has a prime filtration by Theorem V.3.1, so the desired conclusion follows from Proposition V.3.2. □

Remark V.3.4. The conclusion of Corollary V.3.3 fails in general for modules that are not finitely generated: if $U = \{P_1, P_2, \dots\}$ is an infinite collection of distinct prime ideals of R, then $U \subseteq \operatorname{Ass}_R(\oplus_i R/P_i)$.

Example V.3.5. Let R be a unique factorization domain, and let $0 \neq r \in R$ be a non-unit. Write $r = p_1 \cdots p_n$ with each p_i prime. We show $\operatorname{Ass}_R(R/rR) = \{p_1R, \dots, p_nR\}$.

First, check that the following is a prime filtration of R/rR

$$(0) = (p_1 \cdots p_n R)/rR \subsetneq (p_1 \cdots p_{n-1} R)/rR \subsetneq \cdots \subsetneq p_1 R/rR \subsetneq R/rR$$

by showing that

$$\frac{(p_1 \cdots p_i R)/rR}{(p_1 \cdots p_{i+1} R)/rR} \cong \frac{p_1 \cdots p_i R}{p_1 \cdots p_{i+1} R} \cong \frac{R}{p_{i+1} R}.$$

The isomorphism $R/p_{i+1}R \to (p_1 \cdots p_i R)/(p_1 \cdots p_{i+1} R)$ is given by $\overline{x} \mapsto \overline{p_1 \cdots p_i x}$. From Theorem V.3.2, we have $\operatorname{Ass}_R(R/rR) \subseteq \{p_1R, \dots, p_nR\}$.

For the reverse containment, set $p_i' = \prod_{j \neq i} p_i$ for each i, and define a function $R/p_iR \to R/rR$ given by $\overline{x} \mapsto \overline{p_i' x}$. This is a well-defined monomorphism. Hence, we have $\operatorname{Ass}_R(R/rR) \supseteq \{p_1R, \dots, p_nR\}$.

Example V.3.6. Let k be a field, and set $R = k[X, Y]$.

(a) For $M = R/(X,Y)R$ and $m, n \geqslant 1$, we have

$$\operatorname{Ass}_R(R/(X^m, Y^n)R) = \{(X,Y)R\} = \operatorname{Ass}_R(R/((X,Y)R)^n).$$

This follows from the fact that each of the quotient modules $R/(X^m, Y^n)R$ and $R/((X,Y)R)^n$ has a prime filtration such that each quotient M_i/M_{i-1} is isomorphic to $R/(X,Y)R$.

(b) For $M = R/(X^2, XY)R$, we have

$$\operatorname{Ass}_R(R/(X^2, XY)R) = \{(X)R, (X,Y)R\}.$$

This follows from the fact that there is a filtration of $R/(X^2, XY)R$ where the quotients M_i/M_{i-1} are isomorphic to $R/(X)$ and $R/(X,Y)$.

Here is a version of the prime correspondence under localization for support and associated primes.

Proposition V.3.7. *Let R be a commutative ring, and let M be an R-module. Let $U \subseteq R$ be a multiplicatively closed subset.*

(a) *We have*

$$\operatorname{Supp}_{U^{-1}R}(U^{-1}M) = \{U^{-1}P \subsetneq U^{-1}R \mid P \in \operatorname{Supp}_R(M), P \cap U = \emptyset\}.$$

(b) *We have*

$$\operatorname{Ass}_{U^{-1}R}(U^{-1}M) \supseteq \{U^{-1}P \subsetneq U^{-1}R \mid P \in \operatorname{Ass}_R(M), P \cap U = \emptyset\}.$$

(c) *If R is noetherian, then*

$$\operatorname{Ass}_{U^{-1}R}(U^{-1}M) = \{U^{-1}P \subsetneq U^{-1}R \mid P \in \operatorname{Ass}_R(M), P \cap U = \emptyset\}.$$

Proof. Recall that

$$\operatorname{Spec}(U^{-1}R) = \{U^{-1}P \subsetneq U^{-1}R \mid P \in \operatorname{Spec}(R), P \cap U = \emptyset\}$$

and furthermore, for $U^{-1}P \in \operatorname{Spec}(U^{-1}R)$, we have $(U^{-1}M)_{U^{-1}P} \cong M_P$.

(a) This follows directly from the previous paragraph as $(U^{-1}M)_{U^{-1}P} = 0$ if and only if $M_P = 0$.

(b) Assume $P \in \operatorname{Ass}_R(M)$ and $P \cap U = \emptyset$. Then there exists a monomorphism $R/P \hookrightarrow M$, so the exactness of localization yields

$$U^{-1}R/U^{-1}P \cong U^{-1}(R/P) \hookrightarrow U^{-1}M.$$

Since $U^{-1}P \in \operatorname{Spec}(U^{-1}R)$, this implies that $U^{-1}P \in \operatorname{Ass}_{U^{-1}R}(U^{-1}M)$.

(c) Assume that R is noetherian. By part (b), it suffices to verify the containment "$\subseteq$". Let $U^{-1}P \in \operatorname{Ass}_{U^{-1}R}(U^{-1}M)$, and fix an element $m/u \in U^{-1}M$ such that $U^{-1}P = \operatorname{Ann}_{U^{-1}R}(m/u)$. Write $P = (x_1, \dots, x_n)R$. Then $x_i/1 \in U^{-1}P$, so we have $(x_i/1)(m/u) = 0$. Thus, there is an element $u_i \in U$ such that $u_i x_i m = 0$. Set $u' = u_1 \cdots u_n$. It follows that

$$P = (x_1, \dots, x_n)R \subseteq \operatorname{Ann}_R(u'm).$$

In particular, the map $g\colon R/P \to M$ given by $\overline{r} \mapsto ru'm$ is well-defined. Since $u/1$ and $u'/1$ are units in $U^{-1}R$, we conclude

$$U^{-1}P = \operatorname{Ann}_{U^{-1}R}(m/u) = \operatorname{Ann}_{U^{-1}R}(u'm/1).$$

In particular, the map $g'\colon U^{-1}R/U^{-1}P \to U^{-1}M$ given by $r/u'' \mapsto ru'm/u''$ is a well-defined monomorphism.

Let $f\colon R/P \to U^{-1}R/U^{-1}P$ and $f'\colon M \to U^{-1}M$ be the natural maps. Because R/P is an integral domain and $U \cap P = \emptyset$, we know that f is injective. The following diagram commutes

$$\begin{array}{ccc} R/P & \xrightarrow{\quad g \quad} & M \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f'} \\ U^{-1}R/U^{-1}P & \xrightarrow{\quad g' \quad} & U^{-1}M. \end{array}$$

It follows that g is injective, so $P \in \operatorname{Ass}_R(M)$. □

Corollary V.3.8. *Let R be a commutative ring, and let M be an R-module. Let $Q \in \operatorname{Spec}(R)$.*

(a) *We have* $\operatorname{Supp}_{R_Q}(M_Q) = \{P_Q \subsetneq R_Q \mid P \in \operatorname{Supp}_R(M), P \subseteq Q\}$.
(b) *We have* $\operatorname{Ass}_{R_Q}(M_Q) \supseteq \{P_Q \subsetneq R_Q \mid P \in \operatorname{Ass}_R(M), P \subseteq Q\}$.
(c) *If R is noetherian, then* $\operatorname{Ass}_{R_Q}(M_Q) = \{P_Q \subsetneq R_Q \mid P \in \operatorname{Ass}_R(M), P \subseteq Q\}$.

Proof. This is immediate from Proposition V.3.7 using $U = R \smallsetminus Q$. □

Proposition V.3.9. *Let R be a commutative noetherian ring, and let M be a non-zero finitely generated R-module. Consider a prime filtration*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

such that, for $i = 1, \ldots, n$ we have $M_i/M_{i-1} \cong R/P_i$. Then the minimal elements of $\operatorname{Supp}_R(M)$ *(with respect to inclusion) are the same as the minimal elements of* $\operatorname{Ass}_R(M)$*, and these are the same as the minimal elements of the set $\{P_1, \ldots, P_n\}$.*

Proof. Let P be a minimal element of $\operatorname{Supp}_R(M)$; we show that P is minimal in $\operatorname{Ass}_R(M)$. In particular, we have $M_P \neq 0$. Since R_P is noetherian, there exists $Q_P \in \operatorname{Ass}_{R_P}(M_P)$. By the previous result, we have $Q \in \operatorname{Ass}_R(M) \subseteq \operatorname{Supp}_R(M)$ and $Q \subseteq P$. The minimality of P in $\operatorname{Supp}_R(M)$ implies that $Q = P$ and thus $P = Q \in \operatorname{Ass}_R(M)$. Now, the containment $\operatorname{Ass}_R(M) \subseteq \operatorname{Supp}_R(M)$ implies P must be minimal in $\operatorname{Ass}_R(M)$.

Next, let P_j be minimal in $\{P_1, \ldots, P_n\}$; we show that P_j is minimal in $\operatorname{Supp}_R(M)$. Theorem V.3.2(a) implies that $P_j \in \operatorname{Supp}_R(M)$. To show that P_j is minimal in $\operatorname{Supp}_R(M)$, we take an element $Q \in \operatorname{Supp}_R(M)$ such that $Q \subseteq P_j$ and show $Q = P_j$. Theorem V.3.2(b) implies that $Q \supseteq P_i$ for some i, and hence $P_j \supseteq P_i$. The minimality of P_j implies that $P_j = P_i$, so $Q = P_j$.

Finally, let P be a minimal element of $\operatorname{Ass}_R(M)$; we show that P is minimal in $\{P_1, \ldots, P_n\}$. From Theorem V.3.2(a) we know that $P = P_i$ for some index i. Since the set $\{P_1, \ldots, P_n\}$ is finite, there is an index j such that P_j is minimal in $\{P_1, \ldots, P_n\}$ and such that $P_j \subseteq P_i = P$. It suffices to show that $P = P_j$. By the previous paragraph, we know that P_j is minimal in $\operatorname{Supp}_R(M)$, so the paragraph before that shows that P_j is minimal in $\operatorname{Ass}_R(M)$. Since $P_j \subseteq P$, the minimality of P in $\operatorname{Ass}_R(M)$ implies that $P = P_j$. □

Definition V.3.10. Let R be a commutative ring. Assume that R is noetherian, and let M be a non-zero finitely generated R-module. The minimal elements of $\operatorname{Ass}_R(M)$ are the *minimal associated prime ideals* of M, or simply the *minimal primes* of M. We set

$$\operatorname{Min}_R(M) = \{\text{minimal primes of } M\}$$

and $\operatorname{Min}(R) = \operatorname{Min}_R(R)$. The primes in $\operatorname{Ass}_R(M) \smallsetminus \operatorname{Min}_R(M)$ are the *embedded primes* of M.

Corollary V.3.11. *Let R be a commutative noetherian ring, and let M be a finitely generated R-module.*

(a) *The set $\operatorname{Min}(R)$ is finite, and every element of $\operatorname{Spec}(R)$ contains an element of $\operatorname{Min}(R)$.*

(b) *The set $\operatorname{Min}_R(M)$ is finite, and every element of $\operatorname{Supp}_R(M)$ contains an element of $\operatorname{Min}_R(M)$.*

PROOF. This is immediate from Proposition V.3.9 and Definition V.3.10. □

Exercises.

Exercise V.3.12. Verify the facts from Remark V.3.4.

Exercise V.3.13. Verify the facts from Example V.3.5.

Exercise V.3.14. Verify the facts from Example V.3.6.

Exercise V.3.15. Let k be a field. Set $R = k[X, Y]$ and $M = R/(X^2, XY)$. For each integer $n \geqslant 1$, show that there is a prime filtration of M over R such that the prime ideal (X, Y) occurs exactly n times in the filtration. In particular, this shows that the number of "links" in a prime filtration is dependent on the choice of prime filtration. (Compare this to the Jordan-Hölder Theorem.)

V.4. Prime Avoidance and Nakayama's Lemma

This section deals with two handy tools. Here is the first one.

Lemma V.4.1 (Prime Avoidance)**.** *Let R be a commutative ring, and fix ideals $I_1, \dots, I_n, J \subseteq R$. Assume that one of the following conditions holds:*

(1) *The ring R contains an infinite field k as a subring; or*

(2) *The ideals $I_1, \dots, I_{n-2}$ are prime.*

If $J \subseteq \cup_{j=1}^n I_j$, then $J \subseteq I_j$ for some j.

PROOF. (1) Assume first that the ring R contains an infinite field k as a subring, and suppose that $J \not\subseteq I_j$ for all j. Then $J \cap I_j \subsetneq J$ for each j. Since J is contained in $\cup_j I_j$, we have $J = J \cap (\cup_j I_j) = \cup_j (J \cap I_j)$. Each ideal $J \cap I_j$ and J is a vector space over k. Because k is infinite and $J \cap I_j \subsetneq J$ for each j we have $\cup_j (J \cap I_j) \subsetneq J$, a contradiction.

(2) Assume now that the ideals $I_1, \dots, I_{n-2}$ are prime. We prove the result by induction on n. The case $n = 1$ is straightforward.

Assume $n \geqslant 2$ and that the result holds for each list $I'_1, \dots, I'_{n-1}$. If $J \subseteq \cup_{j \neq l} I_j$ for some l, then we are done by induction. So, we assume that $J \not\subseteq \cup_{j \neq l} I_j$ for each l and fix $x_l \in J - \cup_{j \neq l} I_j$. In particular, $x_l \in I_l$ for each l. Notice that $x_1 + x_2 \in J$ and more generally $x_1 + x_2 \cdots x_n \in J$.

When $n = 2$, the element $x_1 + x_2$ is not in I_1: if it were, then $x_1 \in I_1$ would imply $x_2 = (x_1 + x_2) - x_1 \in I_1$, a contraditcion. Similarly, we have $x_1 + x_2 \notin I_2$, so $x_1 + x_2 \in J - (I_1 \cup I_2)$, contradicting the fact that $J \subseteq I_1 \cup I_2$.

When $n > 2$, we know that I_1 is prime. It suffices to show $x_1 + x_2 \cdots x_n \notin I_l$ for each l. If $x_1 + x_2 \cdots x_n \in I_1$, then the fact that $x_1 \in I_1$ implies $x_2 \cdots x_n \in I_1$, so $x_l \in I_1$ for some $l \neq 1$; this contradicts the choice of x_l. If $x_1 + x_2 \cdots x_n \in I_l$ for some $l \geqslant 2$, then the fact that $x_l \in I_l$ implies $x_1 \in I_l$, another contradiction. □

Here is a standard application of prime avoidance.

Corollary V.4.2. *Let R be a commutative ring. Assume that R is noetherian, and let M be a non-zero finitely generated R-module. If I is an ideal of R consisting of zero-divisors on M, then I is contained in an associated prime of M.*

PROOF. The set of associated primes $\mathrm{Ass}_R(M)$ is finite and non-empty, say $\mathrm{Ass}_R(M) = \{P_1, \dots, P_n\}$. Our assumption on I implies that $I \subseteq \cup_j P_j$, so prime avoidance implies that $I \subseteq P_j$ for some j. □

Corollary V.4.3. *Let R be a commutative ring. Assume that R is noetherian, and let M be a non-zero finitely generated R-module. Let $\mathfrak{m} \subsetneq R$ be a maximal ideal. Then $\mathfrak{m}$ contains a non-zero-divisor on M if and only if $\mathfrak{m} \notin \mathrm{Ass}_R(M)$.*

PROOF. The set of associated primes $\mathrm{Ass}_R(M)$ is finite and non-empty, say $\mathrm{Ass}_R(M) = \{P_1, \dots, P_n\}$. If $\mathfrak{m} \in \mathrm{Ass}_R(M)$, then $\mathfrak{m} = \mathrm{Ann}_R(m)$ for some element $m \neq 0$, so $\mathfrak{m}$ consists of zero-divisors on M. Conversely, if $\mathfrak{m}$ consists of zero-divisors on M, then the previous corollary implies that $\mathfrak{m} \subseteq P_j$ for some j, and the fact that $\mathfrak{m}$ is maximal implies $\mathfrak{m} = P_j \in \mathrm{Ass}_R(M)$. □

Here is another really useful tool.

Lemma V.4.4 (Nakayama's Lemma)**.** *Let R be a commutative ring. Assume that R is local with unique maximal ideal $\mathfrak{m} \subsetneq R$. Let M be a finitely generated R-module. If $M/\mathfrak{m}M = 0$, then $M = 0$.*

PROOF. Assume that $M/\mathfrak{m}M = 0$ and suppose that $M \neq 0$. Let $m_1, \dots, m_n$ be a generating sequence for M, and assume that this generating sequence is minimal, in the sense that no sequence of elements from M with $n-1$ elements also generates M. The assumption $M/\mathfrak{m}M = 0$ implies that $M = \mathfrak{m}M$. in particular, the element $m_1 \in M = \mathfrak{m}M$ then has the form $m_1 = \sum_{i=1}^n r_i m_i$ for some elements $r_i \in \mathfrak{m}$. This implies that

$$(1 - r_1)m_1 = \textstyle\sum_{j=2}^n r_i m_i$$

where the sum is 0 when $n = 1$. Since $\mathfrak{m}$ is the unique maximal ideal of R and $r_1 \in \mathfrak{m}$, the element $1 - r_1$ is a unit in R. Hence, the element

$$m_1 = (1 - r_1)^{-1} \textstyle\sum_{j=2}^n r_i m_i$$

is in the submodule $M' = R(m_2, \dots, m_n) \subseteq M$. Since the other generators of M are also in M', we have $M \subseteq M' \subseteq M$. This implies that $M = M'$, which is generated by $n - 1$ elements, a contradiction. □

Here are some consequences of Nakayama's Lemma, each of which may be referred to as Nakayama's Lemma.

Corollary V.4.5. *Let R be a commutative ring. Assume that R is local with unique maximal ideal $\mathfrak{m} \subsetneq R$, and let M be an R-module. Let $N \subseteq M$ be an R-submodule such that the quotient M/N is finitely generated. (For instance, this holds when M is finitely generated.) If $M = N + \mathfrak{m}M$, then $M = N$.*

PROOF. If $M = N + \mathfrak{m}M$, then we have

$$\mathfrak{m}(M/N) = (\mathfrak{m}M + N)/N = M/N$$

so Nakayama's Lemma implies that $M/N = 0$. □

Definition V.4.6. Let R be a commutative ring, and let M be a finitely generated R-module. A generating sequence $m_1, \dots, m_n \in M$ is *minimal* if no proper subsequence generates M.

Example V.4.7. Let A be a commutative ring, and let $R = A[X, Y]$ be a polynomial ring in two variables. The sequence X, Y is a minimal generating sequence for the ideal $(X, Y)R$.

Corollary V.4.8. *Let R be a commutative local ring with unique maximal ideal $\mathfrak{m} \subsetneq R$, and set $k = R/\mathfrak{m}$. Let M be a finitely generated R-module, and let $m_1, \dots, m_n \in M$.*

(a) *Then $M/\mathfrak{m}M$ is a finite-dimensional vector space over the field $k = R/\mathfrak{m}$, via the action $\overline{r}\ \overline{m} = \overline{rm}$.*
(b) *Then $\overline{m_1}, \dots, \overline{m_n} \in M/\mathfrak{m}M$ spans $M/\mathfrak{m}M$ over k if and only if $m_1, \dots, m_n$ generates M over R.*
(c) *The sequence $\overline{m_1}, \dots, \overline{m_n} \in M/\mathfrak{m}M$ is a basis of $M/\mathfrak{m}M$ over k if and only if $m_1, \dots, m_n$ is a minimal generating sequence for M over R.*

In particular, every minimal generating sequence for M has the same number of elements, namely $\dim_k(M/\mathfrak{m}M)$.

PROOF. (a) See Exercise V.4.13(a).

(b) One implication is in Exercise V.4.13(b). For the reverse implication assume that $\overline{m_1}, \dots, \overline{m_n} \in M/\mathfrak{m}M$ spans $M/\mathfrak{m}M$ over k. It follows that

$$M = R(m_1, \dots, m_n) + \mathfrak{m}M$$

so the previous corollary implies that $M = R(m_1, \dots, m_n)$, as desired.

(c) Assume first that $\overline{m_1}, \dots, \overline{m_n} \in M/\mathfrak{m}M$ is a basis of $M/\mathfrak{m}M$ over k. Part (b) implies that $m_1, \dots, m_n$ generates M over R. Suppose that this generating sequence is not minimal. Rearranging the sequence, if necessary, we assume that $m_1, \dots, m_{n-1}$ generates M over R. It follows that $\overline{m_1}, \dots, \overline{m_{n-1}} \in M/\mathfrak{m}M$ spans $M/\mathfrak{m}M$ over k, contradicting the fact that $\dim_k(M/\mathfrak{m}M) = n$.

Conversely, assume that $m_1, \dots, m_n$ is a minimal generating sequence for M over R. Part (b) implies that $\overline{m_1}, \dots, \overline{m_n} \in M/\mathfrak{m}M$ spans $M/\mathfrak{m}M$ over k. Suppose that this spanning sequence is not linearly independent. Rearranging the sequence, if necessary, we assume that $\overline{m_1}, \dots, \overline{m_{n-1}} \in M/\mathfrak{m}M$ spans $M/\mathfrak{m}M$ over k. Part (b) implies that $m_1, \dots, m_{n-1}$ generates M over R contradicting the minimality of the original generating sequence. □

Corollary V.4.9. *Let R be a commutative local ring with unique maximal ideal $\mathfrak{m} \subsetneq R$, and set $k = R/\mathfrak{m}$. Let P be a finitely generated projective R-module. Then $P \cong R^n$ where $n = \dim_k(P/\mathfrak{m}P)$.*

PROOF. The previous corollary implies that every minimal generating sequence for P is of the form $p_1, \dots, p_n$. Such a sequence yields an R-module epimorphism $\tau\colon R^n \to P$ such that $\tau(\mathbf{e}_i) = p_i$ for $i = 1, \dots, n$.

We claim that τ is also a monomorphism. To prove this, let $K = \operatorname{Ker}(\tau)$. We need to show that $K = 0$.

Since P is projective, the following exact sequence splits

$$0 \to K \xrightarrow{\epsilon} R^n \xrightarrow{\tau} P \to 0$$

where ϵ is the inclusion. Let $f\colon P \to R^n$ be a splitting homomorphism, that is, an R-module homomorphism such that $\tau f = \mathbb{1}_P$. It follows that $R^n = K \oplus f(P)$, specifically, that $R^n = K + f(P)$ and $K \cap f(P) = 0$.

Splitting the displayed exact sequence on the left yields a surjection $R^n \to K$, so K is finitely generated. Thus, to prove that $K = 0$, it suffices by Nakayama's Lemma to show that $K = \mathfrak{m}K$, that is, that $K \subseteq \mathfrak{m}K$. Let $x \in K$, and write $x = \sum_i r_i \mathbf{e}_i$ for some elements $r \in R$. We first show that each $r_i \in \mathfrak{m}$. Since $x \in K$, we have

$$0 = \tau(x) = \textstyle\sum_i r_i \tau(\mathbf{e}_i) = \sum_i r_i p_i.$$

In $P/\mathfrak{m}P$ we then have $0 = \sum_i \overline{r_i}\overline{p_i}$. The previous corollary implies that the sequence $\overline{p_1}, \dots, \overline{p_n}$ is a basis for $P/\mathfrak{m}P$. It follows that each $\overline{r_i} = 0$ in $R/\mathfrak{m}$, so we have each $r_i \in \mathfrak{m}$, as desired.

Now write $\mathbf{e}_i = k_i + f(q_i)$ with $k_i \in K$ and $q_i \in P$. We then have

$$x = \sum_i r_i \mathbf{e}_i = \left(\sum_i r_i k_i\right) + \left(\sum_i r_i f(q_i)\right)$$

and hence

$$x - \left(\sum_i r_i k_i\right) = \left(\sum_i r_i f(q_i)\right) \in K \cap f(P) = 0.$$

Thus, we have $x = \sum_i r_i k_i \in \mathfrak{m}K$, as desired. □

We close the section with some ideas that are useful for the next sections.

Lemma V.4.10. *Let R be a commutative noetherian ring, and let M be a non-zero finitely generated R-module. Let $I \subseteq R$ be an ideal, and let N a finitely generated R-module such that* $\operatorname{Supp}_R(N) = V(I)$. *If I consists of zero-divisors on M, then* $\operatorname{Hom}_R(N, M) \neq 0$.

Proof. The ideal I consists of zero-divisors on M, so Corollary V.4.2 yields an associated prime $P \in \operatorname{Ass}_R(M)$ such that $I \subseteq P$. Hence, there is an injective homomorphism $f\colon R/P \hookrightarrow M$. Localizing at P yields $f_P\colon (R/P)_P \hookrightarrow M_P$. Since $P \in V(I) = \operatorname{Supp}_R(N)$, we have $N_P \neq 0$. The module N is finitely generated over R, so N_P is finitely generated over R_P. The ring R_P has a unique maximal ideal, namely P_P, so Nakayama's Lemma implies $0 \neq N_P/PN_P$. That is, N_P/PN_P is a non-zero vector space over the field R_P/PR_P. In particular, there is a surjective R-module homomorphism $g\colon N_P/PN_P \twoheadrightarrow R_P/PR_P$. The natural surjection $h\colon N_P \twoheadrightarrow N_P/PN_P$ fits into the following composition

$$N_P \twoheadrightarrow N_P/PN_P \twoheadrightarrow R_P/PR_P \hookrightarrow M_P.$$

This composition is non-zero, and hence

$$0 \neq \operatorname{Hom}_{R_P}(N_P, M_P) \cong \operatorname{Hom}_R(N, M)_P$$

where the isomorphism is from Proposition I.5.8(d). It follows that $\operatorname{Hom}_R(N, M) \neq 0$, as desired. □

Corollary V.4.11. *Let R be a commutative noetherian ring, and let M be a non-zero finitely generated R-module. Let $\mathfrak{m} \subsetneq R$ be a maximal ideal such that $\mathfrak{m} \notin \operatorname{Ass}_R(M)$. Assume that $\mathfrak{m} \neq \mathfrak{m}^2$. (For instance, this occurs when R is local and not a field.) Then $\mathfrak{m} \smallsetminus \mathfrak{m}^2$ contains a non-zero-divisor on M.*

PROOF. If R is local and not a field, then $\mathfrak{m} \neq 0$. Hence, Nakayama's Lemma implies that $\mathfrak{m}^2 \subsetneq \mathfrak{m}$.

The set $\operatorname{Ass}_R(M)$ is finite and non-empty, say $\operatorname{Ass}_R(M) = \{P_1, \dots, P_n\}$. The condition $\mathfrak{m} \notin \operatorname{Ass}_R(M)$ implies that $P_i \subsetneq \mathfrak{m}$ for each i. From prime avoidance, we conclude that $\mathfrak{m}^2 \cup [\cup_{i=1}^n P_i] \subsetneq \mathfrak{m}$. thus, there is an element in $\mathfrak{m} \smallsetminus [\mathfrak{m}^2 \cup [\cup_{i=1}^n P_i]]$; any such element is a non-zero-divisor in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$. □

Lemma V.4.12. *Let R be a commutative noetherian ring, and let M and N be R-modules. If M is finitely generated, then*

$$\operatorname{Ass}_R(\operatorname{Hom}_R(M,N)) = \operatorname{Supp}_R(M) \cap \operatorname{Ass}_R(N).$$

PROOF. Since M is finitely generated, there is an integer $t \geqslant 0$ and an R-module epimorphism $R^t \twoheadrightarrow M$. The left-exactness of $\operatorname{Hom}_R(-,N)$ yields the monomorphism in the next sequence

$$\operatorname{Hom}_R(M,N) \hookrightarrow \operatorname{Hom}_R(R^t,N) \cong N^t.$$

The isomorphism is from Exercise I.3.4(c). Using this, the containment in the next display follows from Proposition V.2.13(b)

$$\operatorname{Ass}_R(\operatorname{Hom}_R(M,N)) \subseteq \operatorname{Ass}_R(N^t) = \operatorname{Ass}_R(N) \tag{V.4.12.1}$$

and the equality is from Lemma V.2.15(b).

Next, let $\mathfrak{p} \in \operatorname{Spec}(R) \smallsetminus \operatorname{Supp}_R(M)$. It follows that $M_\mathfrak{p} = 0$, and hence the second isomorphism in the next sequence

$$\operatorname{Hom}_R(M,N)_\mathfrak{p} \cong \operatorname{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \cong \operatorname{Hom}_{R_\mathfrak{p}}(0, N_\mathfrak{p}) = 0.$$

The first isomorphism is from Proposition I.5.8(d), and the equality is straightforward. From this, we deduce the second containment in the next sequence

$$\operatorname{Ass}_R(\operatorname{Hom}_R(M,N)) \subseteq \operatorname{Supp}_R(\operatorname{Hom}_R(M,N)) \subseteq \operatorname{Supp}_R(M) \tag{V.4.12.2}$$

while the first containment is from Proposition V.2.11(c).

Combining (V.4.12.1) and (V.4.12.2), we have

$$\operatorname{Ass}_R(\operatorname{Hom}_R(M,N)) \subseteq \operatorname{Supp}_R(M) \cap \operatorname{Ass}_R(N).$$

For the reverse containment, let $\mathfrak{p} \in \operatorname{Supp}_R(M) \cap \operatorname{Ass}_R(N)$.

Claim 1: We have $\operatorname{Hom}_R(M, R/\mathfrak{p}) \neq 0$. To see this, recall that Remark V.2.6 implies that $\mathfrak{p} \in \operatorname{Supp}_R(M) = V(\operatorname{Ann}_R(M))$. Hence, we have $\operatorname{Ann}_R(M) \subseteq \mathfrak{p}$, and thus $\operatorname{Ann}_R(M) \cdot R/\mathfrak{p} = 0$. The desired non-vanishing now follows from Lemma V.4.10.

Claim 2: For each non-zero element $\alpha \in \operatorname{Hom}_R(M, R/\mathfrak{p})$, we have $\operatorname{Ann}_R(\alpha) = \mathfrak{p}$. To see this, argue as in the first paragraph of this proof to find a monomorphism

$$\operatorname{Hom}_R(M, R/\mathfrak{p}) \hookrightarrow (R/\mathfrak{p})^t.$$

Since $\mathfrak{p}$ is prime, the annihilator of any non-zero element of $(R/\mathfrak{p})^t$ is $\mathfrak{p}$. Hence, the annihilator of α is $\mathfrak{p}$.

Claim 3: We have $\mathfrak{p} \in \operatorname{Ass}_R(\operatorname{Hom}_R(M, R/\mathfrak{p}))$. Indeed, Claim 1 yields a nonzero element $\alpha \in \operatorname{Hom}_R(M, R/\mathfrak{p})$, and Claim 2 says that $\operatorname{Ann}_R(\alpha) = \mathfrak{p}$. Hence, the map $R/\mathfrak{p} \to \operatorname{Hom}_R(M, R/\mathfrak{p})$ given by $\overline{r} \mapsto r\alpha$ is a well-defined R-module monomorphism. This implies the desired conclusion.

Claim 4: We have $\mathfrak{p} \in \operatorname{Ass}_R(\operatorname{Hom}_R(M,N))$. (Once we show this, the proof is complete.) Since $\mathfrak{p} \in \operatorname{Ass}_R(N)$, there is an R-module monomorphism $R/\mathfrak{p} \hookrightarrow N$. The induced map

$$\operatorname{Hom}_R(M, R/\mathfrak{p}) \hookrightarrow \operatorname{Hom}_R(M,N)$$

is a monomorphism. Claim 3 implies that $\mathfrak{p} \in \operatorname{Ass}_R(\operatorname{Hom}_R(M, R/\mathfrak{p}))$, so we conclude from Proposition V.2.13(b) that $\mathfrak{p} \in \operatorname{Ass}_R(\operatorname{Hom}_R(M, N))$. □

Exercises.

Exercise V.4.13. Let R be a commutative ring. Let M be an R-module, and let $I \subseteq R$ be an ideal.

(a) Prove that the quotient M/IM has a well-defined R/I-module structure given by $\overline{r}\ \overline{m} = \overline{rm}$.
(b) Prove that, if $m_1, \dots, m_n \in M$ generates M as an R-module, then the sequence $\overline{m_1}, \dots, \overline{m_n} \in M/IM$ generates M/IM over R and over R/I.
(c) Prove that the isomoprhism $(R/I) \otimes_R M \cong M/IM$ from Exercise II.4.14 is an R/I-module isomorphism.

Exercise V.4.14. Let k be a field.

(a) Set $R = k \times k$ and $\mathfrak{m} = 0 \times k \subsetneq R$ and $M = k \times 0$. Prove that $\mathfrak{m}$ is a maximal ideal ideal of R such that $\mathfrak{m}M = 0$ and that $M \neq 0$. It follows that the local assumption in Nakayama's Lemma is essential. Prove also that M is projective and not free, so the local assumption in Corollary V.4.9 is also essential.
(b) Let S be a local integral domain that is not a field, with maximal ideal $\mathfrak{m}$ and field of fractions of K. (For instance, we may take the localization $S = k[X]_{(X)}$ with field of fractions $K = k(X)$, or the localization $Z = \mathbb{Z}_{p\mathbb{Z}}$ with field of fractions $K = \mathbb{Q}$.) Prove that $K \neq 0$ and that $K = \mathfrak{m}K$. Conclude that K is not finitely generated as an R-module and that M needs to be finitely generated in Nakayama's Lemma.

Exercise V.4.15. Let R be a commutative local ring. Let M and N be finitely generated R-modules.

(a) Prove that, if M and N are non-zero, then so is $M \otimes_R N$. [Hint: Use the right-exactness of tensor product with Nakayama's Lemma.]
(b) Provide an example showing that the statement in part (a) can be false if the ring R has more than one maximal ideal.
(c) Provide an example of a ring R with a unique maximal ideal and non-zero finitely generated R-modules M and N such that $\operatorname{Hom}_R(N, M) = 0$.

Exercise V.4.16. Let R be a commutative noetherian ring, and fix finitely generated R-modules M and N. Prove that $\operatorname{Supp}_R(M \otimes_R N) = \operatorname{Supp}_R(M) \cap \operatorname{Supp}_R(N)$.

Exercise V.4.17. State and prove versions of Lemma V.4.4 and Corollary V.4.5 where R is not necessarily local and the ideal $\mathfrak{m}$ is replaced by the Jacobson radical of R.

V.5. Regular Sequences and Ext

Definition V.5.1. Let R be a commutative ring, and let M be an R-module. An element $a \in R$ is *M-regular* if it is not a zero-divisor on M and $M \neq aM$.

A sequence $a_1, \dots, a_n \in R$ is *M-regular* or is an *M-sequence* if a_1 is M-regular, and a_{i+1} is regular on $M/(a_1, \dots, a_i)M$ for $i = 1, \dots, n-1$.

Let I be an ideal, and assume that $a_1, \dots, a_n \in I$. Then $a_1, \dots, a_n$ is a *maximal M-regular sequence* in I if $a_1, \dots, a_n$ is an M-regular sequence and, for all $b \in I$, the sequence $a_1, \dots, a_n, b$ is not M-regular.

Remark V.5.2. Let R be a commutative noetherian ring, and let M be an R-module. Proposition V.2.11(b) shows that a is not a zero-divisor for M if and only if $a \notin \cup_{P\in \operatorname{Ass}_R(M)} P$.

If $a_1, \dots, a_n \in R$ is an M-regular sequence, then $M \neq (a_1, \dots, a_n)M$. In particular, the zero-module does not admit a regular sequence.

Example V.5.3. Let k be a field.

In the polynomial ring $P = k[X_1, \dots, X_n]$, the sequence $X_1, \dots, X_n$ is P-regular.

In $\mathbb{Z}$ if m, n are non-zero non-units, then m is $\mathbb{Z}$-regular and m, n is not a $\mathbb{Z}$-regular sequence: If $\gcd(m, n) = 1$ then $\overline{n}$ is a unit in $\mathbb{Z}/m\mathbb{Z}$ so $\overline{n}$ is not $\mathbb{Z}/m\mathbb{Z}$-regular. If $\gcd(m, n) > 1$ then $\overline{n}$ is a zero-divisor in $\mathbb{Z}/m\mathbb{Z}$ so $\overline{n}$ is not $\mathbb{Z}/m\mathbb{Z}$-regular.

The field k does not have a regular element because every non-zero element is a unit.

The ring $R = k[X]/(X^2)$ does not have a regular element: The only non-units of R are the non-zero constant multiples of $\overline{X}$, which are all zero-divisors since they are annihilated by $\overline{X}$.

The ring $S = k[X, Y]_{(X,Y)}/(XY)$ has a regular element $\overline{X}+\overline{Y}$: If $(\overline{X}+\overline{Y})\overline{f} = 0$ then $XY \mid (X + Y)f$. Since $X \nmid X + Y$, we have $X \mid f$, and similarly $Y \mid f$. So $XY \mid f$ and $\overline{f} = 0$. Note that $S/(\overline{X} + \overline{Y}) \cong k[X]/(X^2)$; since this has no regular elements, the S-sequence $\overline{X} + \overline{Y}$ cannot be extended to an S-sequence $\overline{X} + \overline{Y}, g$. We will see below that S does not have an S-sequence of length 2.

The ring $T = k[X, Y]_{(X,Y)}/(X^2, XY)$ does not have a regular element. To see this, suppose that $f \in T$ were T-regular. Then $fT \neq T$ implies that f is not a unit, so $f \in (X, Y)T$. Write $f = g\overline{X} + h\overline{Y}$. Then $\overline{X}f = 0$ so that f is a zero-divisor.

Remark V.5.4. Let R be a commutative ring. If R is noetherian and M is an R-module, then, for each ideal $I \subseteq R$, there is a maximal M-regular sequence in I. To see this, note that an M-sequence $a_1, \dots, a_n$ gives a strictly increasing chain of submodules

$$(a_1)M \subset (a_1, a_2)M \subsetneq \cdots \subset (a_1, \dots, a_n)M$$

and hence a strictly increasing chain of ideals

$$(a_1)R \subset (a_1, a_2)R \subsetneq \cdots \subset (a_1, \dots, a_n)R.$$

Since R is noetherian, this chain must stabilize.

A similar argument shows that every M-regular sequence in I can be extended to a maximal M-regular sequence in I.

Here is an algorithm for finding maximal regular sequences for finitely generated modules over local rings.

Remark V.5.5. Let R be a commutative ring. Assume that R is noetherian and local with maximal ideal $\mathfrak{m} \subsetneq R$, and let M be a non-zero finitely generated R-module.

Step 1. If $\mathfrak{m} \in \operatorname{Ass}_R(M)$, then $\mathfrak{m}$ consists of zero-divisors on M by Corollary V.4.3. Hence, the empty sequence is a maximal M-regular sequence.

Step 2. Assume that $\mathfrak{m} \notin \operatorname{Ass}_R(M)$. Corollary V.4.3 implies that $\mathfrak{m}$ contains an M-regular element x_1. Moreover, by Remark V.5.2, we know that any element $x_1 \in \mathfrak{m} - \cup_{P\in \operatorname{Ass}_R(M)} P$ is M-regular.

Step 3. If $\mathfrak{m} \in \operatorname{Ass}_R(M/x_1M)$, then $\mathfrak{m}$ consists of zero-divisors on M/x_1M by Corollary V.4.3. Hence, the sequence x_1 is a maximal M-regular sequence.

Step 4. Assume that $\mathfrak{m} \notin \operatorname{Ass}_R(M/x_1M)$. Corollary V.4.3 implies that $\mathfrak{m}$ contains an M/x_1M-regular element x_2. Moreover, by Remark V.5.2, we know that any element $x_2 \in \mathfrak{m} - \cup_{P\in\operatorname{Ass}_R(M/x_1M)}P$ is M/x_1M-regular.

Step 5. Repeat this process with $M/(x_1, x_2)M$, and so on. Remark V.5.4 shows that the process terminates in a finite number of steps.

The next lemma will be helpful for computing regular sequences in practice.

Lemma V.5.6. *Let R be a commutative noetherian ring, and fix an ideal $I \subsetneq R$.*

(a) One has $\operatorname{rad}(I) = \cap_{P\in V(I)}P = \cap_{P\in\operatorname{Ass}_R(R/I)}P = \cap_{P\in\operatorname{Min}_R(R/I)}P$.

(b) If I is an intersection of a finite number of prime ideals then $\operatorname{Ass}_R(R/I)$ *consists of the minimal elements among those primes, and* $\operatorname{Ass}_R(R/I) = \operatorname{Min}_R(R/I)$.

(c) If I is an intersection of prime ideals, then it is an intersection of a finite number of prime ideals.

Proof. (a) Remark V.2.3 explains the equality in the next sequence

$$\operatorname{rad}(I) = \cap_{P\in V(I)}P \subseteq \cap_{P\in\operatorname{Ass}_R(R/I)}P \subseteq \cap_{P\in\operatorname{Min}_R(R/I)}P \subseteq \cap_{P\in V(I)}P.$$

The first and second containments follow from the conditions

$$V(I) = \operatorname{Supp}_R(R/I) \supseteq \operatorname{Ass}_R(R/I) \supseteq \operatorname{Min}_R(R/I);$$

see Remark V.2.6, Proposition V.2.11(c), and Definition V.3.10. The third containment follows from the fact that $\operatorname{Min}_R(R/I)$ consists of all the minimal elements of $V(I)$, and that every element of $V(I)$ is contained in a minimal element.

(b) Let $P_1, \dots, P_n \in \operatorname{Spec}(R)$ such that $I = \cap_{i=1}^n P_i$. Reorder the P_i if necessary to assume that $P_1, \dots, P_j$ are minimal and $P_{j+1}, \dots, P_n$ are not minimal. It then follows that $I = \cap_{i=1}^j P_i$.

We show that $\{P_1, \dots, P_j\} \subseteq \operatorname{Min}_R(R/I)$. Proposition V.3.9 implies that $\operatorname{Min}_R(R/I)$ is the set of minimal elements of $\operatorname{Supp}_R(R/I)$, that is, the minimal elements of $V(I)$; see Remark V.2.6. So, we need to show that P_k is minimal in $V(I)$ for $k = 1, \dots, j$. Assume that $P \in V(I)$ such that $P \subseteq P_k$; we need to show that $P_k = P$. The condition $P \in V(I)$ means that P is prime and $P \supseteq I$. The condition $P \subseteq P_k$ implies that $\cap_{i=1}^n P_i = I \subseteq P_k$, hence the following sequence

$$\cap_{i=1}^n P_i = I \subseteq P \subseteq P_k \subseteq \cap_{i=1}^n P_i$$

and the desired equality $P_k = P$.

We already know that $\operatorname{Min}_R(R/I) \subseteq \operatorname{Ass}_R(R/I)$ by definition.

We show that $\operatorname{Ass}_R(R/I) \subseteq \{P_1, \dots, P_j\}$. (Once this is done, the proof of part (b) is complete.) Let $P \in \operatorname{Ass}_R(R/I)$. By definition, there is an element $x \in R$ such that the coset $\overline{x} \in R/I$ is non-zero and such that $P = \operatorname{Ann}_R(\overline{x})$. That is, we have $x \in R \smallsetminus I$ and $P = \{r \in R \mid xr \in I\}$. The first of these conditions yields

$$x \in R \smallsetminus I = R \smallsetminus \cap_{i=1}^j P_i = \cup_{i=1}^j (R \smallsetminus P_i)$$

so there is an index k such that $1 \leqslant k \leqslant j$ and $x \in R \smallsetminus P_k$. From the condition $P = \{r \in R \mid xr \in I\}$ we have

$$xP \subseteq I = \cap_{i=1}^j P_i \subseteq P_k$$

so the condition $x \notin P_k$ implies that $P \subseteq P_k$, since P_k is prime.

Also, we have $P \in \operatorname{Ass}_R(R/I) \subseteq \operatorname{Supp}_R(R/I) = V(I)$, which implies that $P \supseteq I = \cap_{i=1}^j P_i$. Since P is prime, we have $P \supseteq P_l$ for some index l with $1 \leqslant l \leqslant j$.

In other words, we have $P_l \subseteq P \subseteq P_k$. Since P_k and P_l are both minimal among the P_i, we have $P_k = P_l$ and hence $P = P_k \in \{P_1, \dots, P_j\}$, as desired.

(c) Assume that $\Lambda \subseteq \operatorname{Spec}(R)$ and $I = \cap_{P\in\Lambda} P$. It is straightforward to show that this implies that $I = \operatorname{rad}(I)$. Part (a) implies that

$$I = \operatorname{rad}(I) = \cap_{P\in \operatorname{Min}_R(R/I)} P$$

which is an intersection of finitely many prime ideals by Proposition V.3.9. □

We use Remark V.5.5 and Lemma V.5.6 to further analyze one of the examples from V.5.3.

Example V.5.7. Let k be a field. Set $R = k[X,Y]_{(X,Y)}$ and $I = (XY)R \subseteq R$ and $S = R/I$. It is straightforward to show that $I = \operatorname{rad}(I)$, in fact, we have $I = (X)R \cap (Y)R$. This is an intersection of prime ideals because $R/(X)R \cong k[Y]_{(Y)}$ and $R/(Y)R \cong k[X]_{(X)}$. Thus, we have

$$\operatorname{Ass}_R(S) = \{(X)R, (Y)R\}.$$

To find an S-regular element in R, we need only find an element $f \in (X,Y)R$ such that $f \notin (X)R$ and $f \notin (Y)R$. The element $f = X + Y$ satisfies these properties, as does any element $aX + bY$ where a, b are non-zero elements of k.

Now, we show that the length of a maximal M-regular sequence in I is independent of the choice of such a sequence. This is where Ext comes into play.

Lemma V.5.8. *Let R be a commutative ring, and let M and N be R-modules. Then $\operatorname{Ann}_R(M) \cup \operatorname{Ann}_R(N) \subseteq \operatorname{Ann}_R(\operatorname{Ext}^i_R(M,N))$ for all i.*

Proof. Let $x \in \operatorname{Ann}_R(M) \cup \operatorname{Ann}_R(N)$, and let $\mu^N_x \colon N \to N$ by given by $n \mapsto xn$. According to Fact V.1.3, the induced map

$$\operatorname{Ext}^i_R(M, \mu^N_x) \colon \operatorname{Ext}^i_R(M,N) \to \operatorname{Ext}^i_R(M,N)$$

is given by multiplication by x.

Assume now that $x \in \operatorname{Ann}_R(N)$ Then the map μ^N_x is the zero map, so Remark V.1.2 implies that the induced map $\operatorname{Ext}^i_R(M, \mu^N_x)$ is the zero-map. In other words, multiplication by x on $\operatorname{Ext}^i_R(M,N)$ is zero, as desired.

The case where $x \in \operatorname{Ann}_R(M)$ is handled similarly, using the map μ^M_x. □

Example V.5.9. From Example IV.3.4 we have

$$\operatorname{Ext}^i_{\mathbb{Z}/mn\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) \cong \mathbb{Z}/g\mathbb{Z}$$

for all $i \geqslant 0$ where $g = \gcd(m,n)$. In particular, we have

$$m \operatorname{Ext}^i_{\mathbb{Z}/mn\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}) = 0 = n \operatorname{Ext}^i_{\mathbb{Z}/mn\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$$

which agrees with the previous result.

Remark V.5.10. Let R be a commutative ring, and let M and N be R-modules. Let $I, J \subseteq R$ be ideals such that $IM = 0$ and $JN = 0$. Lemma V.5.8 implies that $(I+J)\operatorname{Ext}^i_R(M,N) = 0$. Because of this, Remark I.5.10 implies that $\operatorname{Ext}^i_R(M,N)$ has the structure of an $R/(I+J)$-module, the structure of an R/I-module, and the structure of an R/J-module via the formula $\overline{r}z = rz$. Furthermore, $\operatorname{Ext}^i_R(M,N)$ is finitely generated over R if and only if it is finitely generated over R/I, and similarly over R/J and $R/(I+J)$.

The next result is the main point of this chapter. Note that the case $n = 0$ is vacuous because $\operatorname{Ext}^i_R(M, N) = 0$ for all $n < 0$.

Theorem V.5.11. *Let R be a commutative ring, and assume that R is noetherian. Let $I \subseteq R$ be an ideal, and let M be a finitely generated R-module such that $IM \neq M$. For each integer $n \geqslant 1$, the following conditions are equivalent:*

(i) *We have $\operatorname{Ext}^i_R(N, M) = 0$ for all $i < n$ and for each finitely generated R-module N such that $\operatorname{Supp}_R(N) \subseteq V(I)$;*
(ii) *We have $\operatorname{Ext}^i_R(R/I, M) = 0$ for all $i < n$;*
(iii) *We have $\operatorname{Ext}^i_R(N, M) = 0$ for all $i < n$ for some finitely generated R-module N such that $\operatorname{Supp}_R(N) = V(I)$;*
(iv) *Every M-sequence in I of length $\leqslant n$ can be extended to an M-sequence in I of length n;*
(v) *There exists an M-sequence of length n in I.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) follow from Remark V.2.6 which contains the equality $\operatorname{Supp}_R(R/I) = V(I)$.

(iii) $\Longrightarrow$ (iv) Assume that $\operatorname{Ext}^i_R(N, M) = 0$ for all $i < n$ for some finitely generated R-module N such that $\operatorname{Supp}_R(N) = V(I)$. We prove that every M-sequence in I of length $\leqslant n$ can be extended to an M-sequence in I of length n, by induction on n.

Note that I contains an M-regular element: If I consisted entirely of zero-divisors for M, then we would have $\operatorname{Ext}^0_R(N, M) \cong \operatorname{Hom}_R(N, M) \neq 0$, a contradiction. Thus, I contains a non-zero-divisor $a_1 \in I$ for M. Note that $a_1 M \neq M$ because $a_1 \in I$ and $IM \neq M$. (Similarly, we have $I[M/a_1M] \neq M/a_1M$.) In particular, this shows that the empty sequence can be extended to a sequence with at least one element. Hence, we may assume that we are starting with an M sequence $a_1, \dots, a_k \in I$ such that $1 \leqslant k \leqslant n$.

Now, if $n = 1$, we are done: the sequence a_1 is an M-sequence of length 1 in I. This is the base case for our induction.

Assume that $n > 1$ and that the result holds for all R-modules M' such that $IM' \neq M'$ and such that $\operatorname{Ext}^i_R(N, M') = 0$ for all $i < n - 1$. We show that $\operatorname{Ext}^i_R(N, M/a_1M) = 0$ for all $i < n - 1$; the induction hypothesis then yields a regular sequence $a_2, \dots, a_n \in I$ for M/a_1M and it will then follow that the sequence $a_1, \dots, a_n \in I$ is M-regular.

Consider the exact sequence

$$0 \to M \xrightarrow{a_1} M \to M/a_1M \to 0.$$

The long exact sequence in $\operatorname{Ext}_R(N, -)$ has a piece of the following form

$$\operatorname{Ext}^i_R(N, M) \to \operatorname{Ext}^i_R(N, M/a_1M) \to \operatorname{Ext}^{i+1}_R(N, M)$$

for each $i \geqslant 0$. When $i < n - 1$, we have $\operatorname{Ext}^i_R(N, M) = 0 = \operatorname{Ext}^{i+1}_R(N, M)$ by hypothesis, so the exactness of the sequence implies that $\operatorname{Ext}^i_R(N, M/a_1M) = 0$.

(iv) $\Longrightarrow$ (v) Condition (iv) implies that the empty sequence can be extended to an M-sequence of length n.

(v) $\Longrightarrow$ (i) Assume that $a_1, \dots, a_n \in I$ is an M-regular sequence. Let N be a finitely generated R-module such that $\operatorname{Supp}_R(N) \subseteq V(I)$. We prove that $\operatorname{Ext}^i_R(N, M) = 0$ for all $i < n$, by induction on n.

Our assumptions on N imply that

$$V(\operatorname{Ann}_R(N)) = \operatorname{Supp}_R(N) \subseteq V(I).$$

Lemma V.2.4 implies $I^t \subseteq \operatorname{Ann}_R(N)$ for $t \gg 0$, and hence $a_j^t N = 0$ for $t \gg 0$. Lemma V.4.10 implies $a_j^t \operatorname{Ext}_R^i(N, M) = 0$ for $t \gg 0$. Consider the exact sequence

$$0 \to M \xrightarrow{a_1} M \to M/a_1M \to 0. \tag{V.5.11.1}$$

This is exact because a_1 is M-regular.

Base case: $n = 1$. Applying $\operatorname{Hom}_R(N, -)$ to the sequence (V.5.11.1) yields an exact sequence

$$0 \to \operatorname{Hom}_R(N, M) \xrightarrow{a_1} \operatorname{Hom}_R(N, M).$$

In other words, the map $\operatorname{Hom}_R(N, M) \xrightarrow{a_1} \operatorname{Hom}_R(N, M)$ is injective, so its t-fold composition $\operatorname{Hom}_R(N, M) \xrightarrow{a_1^t} \operatorname{Hom}_R(N, M)$ is also injective. For $t \gg 0$, we have $a_1^t \operatorname{Hom}_R(N, M) = a_1^t \operatorname{Ext}_R^0(N, M) = 0$. The injectivity of the multiplication map by a_1^t then implies $\operatorname{Hom}_R(N, M) = 0$, as desired.

Induction step. Assume $n > 1$ and assume the following: if M' is a finitely generated R-module such that $IM' \neq M'$ and I contains an M'-regular sequence of length $n-1$, then $\operatorname{Ext}_R^i(N, M') = 0$ for all $i < n-1$. Since I contains an M-sequence of length $n-1$, namely the sequence $a_1, \dots, a_{n-1}$, we know $\operatorname{Ext}_R^i(N, M) = 0$ for all $i < n-1$, and it remains to show $\operatorname{Ext}_R^{n-1}(N, M) = 0$. Since I contains an M/a_1M-sequence of length $n-1$, namely the sequence $a_2, \dots, a_n$, we know $\operatorname{Ext}_R^i(N, M/a_1M) = 0$ for all $i < n-1$. Consider the following piece of the long exact sequence in $\operatorname{Ext}_R(N, -)$ associated to (V.5.11.1):

$$\underbrace{\operatorname{Ext}_R^{n-2}(N, M/a_1M)}_{=0} \to \operatorname{Ext}_R^{n-1}(N, M) \xrightarrow{a_1} \operatorname{Ext}_R^{n-1}(N, M).$$

The argument of the base case now shows that $\operatorname{Ext}_R^{n-1}(N, M) = 0$. □

Corollary V.5.12. *Let R be a commutative noetherian ring, and let M be a finitely generated R-module. If I is an ideal of R such that $IM \neq M$, then each maximal M-sequence in I has the same length, namely*

$$\inf\{i \geqslant 0 \mid \operatorname{Ext}_R^i(R/I, M) \neq 0\}.$$

Proof. Use Lemma V.4.10 and Theorem V.5.11. □

Definition V.5.13. Let R be a commutative noetherian ring, and let M be a finitely generated R-module. If I is an ideal of R such that $IM \neq M$, then

$$\operatorname{depth}_R(I; M) = \inf\{i \geqslant 0 \mid \operatorname{Ext}_R^i(R/I, M) \neq 0\}.$$

If $IM = M$, then set $\operatorname{depth}_R(I; M) = \infty$.

Here are some of the examples from V.5.3.

Example V.5.14. Let A be a commutative noetherian ring, and consider the polynomial ring $R = A[X_1, \dots, X_n]$. The sequence $X_1, \dots, X_n$ is R-regular. In fact, this is a maximal R-regular sequence in $(X_1, \dots, X_n)R$, so we have

$$\operatorname{Ext}_R^i(R/(X_1, \dots, X_n)R, R) = 0 \qquad \text{for } i = 0, \dots, n-1 \qquad \text{and}$$
$$\operatorname{Ext}_R^n(R/(X_1, \dots, X_n)R, R) \neq 0.$$

(Theorem VIII.6.16 below shows that we have $\operatorname{Ext}^n_R(R/(X_1,\dots,X_n)R,R) \cong A$ and $\operatorname{Ext}^i_R(R/(X_1,\dots,X_n)R,R) = 0$ for all $i \neq n$.) In particular, this implies that $\operatorname{depth}_R((X_1,\dots,X_n)R;R) = n$. Similar computations hold for the power series ring $A[\![X_1,\dots,X_n]\!]$ and the localized polynomial ring $A[X_1,\dots,X_n]_{(X_1,\dots,X_n)}$.

Example V.5.15. Fix an integer $n \geqslant 2$. A maximal $\mathbb{Z}$-sequence in (n) is n, so

$$\operatorname{Ext}^0_{\mathbb{Z}}(\mathbb{Z}/n,\mathbb{Z}) \cong \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}/n,\mathbb{Z}) = 0 \qquad \text{and} \qquad \operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) \neq 0.$$

In fact, using the projective resolution $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/(n) \to 0$ we see that

$$\operatorname{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z} \qquad \text{and} \qquad \operatorname{Ext}^i_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z},\mathbb{Z}) = 0$$

for all $i \neq 1$. In particular, we have $\operatorname{depth}_{\mathbb{Z}}(n\mathbb{Z};\mathbb{Z}) = 1$.

Exercises.

Exercise V.5.16. Finish the proof of Lemma V.5.6.

Exercise V.5.17. Verify the facts from Example V.5.7.

Exercise V.5.18. Prove Corollary V.5.12.

Exercise V.5.19. Let R be a commutative noetherian ring, and let M be a finitely generated R-module. Let $I = (a_1,\dots,a_n)R$ be an ideal of R such that $IM \neq M$. Prove that if $a_1,\dots,a_n$ is M-regular, then $a_1,\dots,a_n$ is a maximal M-regular sequence in I.

Exercise V.5.20. Let A be a commutative noetherian ring.

(a) Verify the conclusion of Corollary V.5.12 for the ring $R = k[X]/(X^2)$, the module $M = R$, and the ideal $I = XR$ by showing that the ideal I does not contain an R-regular element, and

$$\operatorname{Ext}^i_R(R/I,R) \cong \begin{cases} R/I & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

(b) Verify the conclusion of Corollary V.5.12 for the ring $R = k[X,Y]/(XY)$, the module $M = R$, and the ideal $I = (X,Y)R$ by showing that the ideal I contains a maximal R-regular sequence of length 1, and

$$\operatorname{Ext}^i_R(R/I,R) \cong \begin{cases} R/I & \text{if } i = 1 \\ 0 & \text{if } i \neq 1. \end{cases}$$

(c) Verify the conclusion of Corollary V.5.12 for the ring $R = k[X,Y]/(X^2,XY)$, the module $M = R$, and the ideal $I = (X,Y)R$ by showing that the ideal I does not contain an R-regular element, and

$$\operatorname{Hom}_R(R/I,R) \cong R/I.$$

Exercise V.5.21 (Depth Lemma)**.** Let R be a commutative noetherian ring, and let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated R-modules. Let $I \subseteq R$ be an ideal, and prove the following inequalities:

$$\operatorname{depth}_R(I;M) \geqslant \inf\{\operatorname{depth}_R(I;M'), \operatorname{depth}_R(I;M'')\}$$
$$\operatorname{depth}_R(I;M') \geqslant \inf\{\operatorname{depth}_R(I;M), \operatorname{depth}_R(I;M'') + 1\}$$
$$\operatorname{depth}_R(I;M'') \geqslant \inf\{\operatorname{depth}_R(I;M') - 1, \operatorname{depth}_R(I;M)\}.$$

V.6. Four Lemmas

The results of this section are for use in Section IX.3. We begin with a lemma that compliments Theorem V.5.11.

Lemma V.6.1. *Let R be a commutative noetherian ring, and let M and N be non-zero finitely generated R-modules. If $\mathbf{x} = x_1, \dots, x_n$ is an M-regular sequence in $\operatorname{Ann}_R(N)$, then $\operatorname{Ext}^n_R(N, M) \cong \operatorname{Hom}_R(N, M/\mathbf{x}M)$.*

PROOF. We proceed by induction on n. The base case $n = 0$ is straightforward.

Assume that $n \geqslant 1$ and that the result holds for sequences of length $n-1$. Since $\operatorname{Ann}_R(N)$ contains an M-regular sequence of length n, and $\operatorname{Supp}_R(N) = V(\operatorname{Ann}_R(N))$, Theorem V.5.11 implies that $\operatorname{Ext}^{n-1}_R(N, M) = 0$.

Consider the exact sequence

$$0 \to M \xrightarrow{x_1} M \to M/x_1M \to 0.$$

The vanishing $\operatorname{Ext}^{n-1}_R(N, M) = 0$ implies that a piece of the long exact sequence in $\operatorname{Ext}_R(N, -)$ has the following form

$$0 \to \operatorname{Ext}^{n-1}_R(N, M/x_1M) \xrightarrow{\eth} \operatorname{Ext}^n_R(N, M) \xrightarrow[=0]{x_1} \operatorname{Ext}^n_R(N, M).$$

The last map in this sequence is 0 because $x_1N = 0$; see Lemma V.5.8. The exactness of this sequence implies that $\eth$ is an isomorphism, hence the first isomorphism in the following sequence

$$\operatorname{Ext}^n_R(N, M) \cong \operatorname{Ext}^{n-1}_R(N, M/x_1M) \cong \operatorname{Hom}_R(N, M/(x_1, \dots, x_n)M).$$

The second isomorphism follows from the inductive hypothesis, since $x_2, \dots, x_n$ is an M/x_1M-regular sequence in $\operatorname{Ann}_R(M)$ of length $n-1$. □

Lemma V.6.2. *Let $\varphi\colon R \to S$ be a flat ring homomorphism between commutative rings, and let M be a finitely generated non-zero R-module. If $\mathbf{x} = x_1, \dots, x_n \in R$ is an M-regular sequence such that $\mathbf{x}(S \otimes_R M) \neq S \otimes_R M$, then the sequences $\mathbf{x}$ and $\varphi(\mathbf{x}) = \varphi(x_1), \dots, \varphi(x_n) \in S$ are both $S \otimes_R M$-regular, and there are isomorphisms*

$$(S \otimes_R M)/\mathbf{x}(S \otimes_R M) \cong S \otimes_R (M/\mathbf{x}M) \cong (S \otimes_R M)/\varphi(\mathbf{x})(S \otimes_R M).$$

PROOF. First, note that the action of x_i on $S \otimes_R M$ is the same as the action of $\varphi(x_i)$ because the action of x_i on S is defined to be the same as the action of $\varphi(x_i)$. Thus, we need only show that $x_1, \dots, x_n \in R$ is $S \otimes_R M$-regular. We proceed by induction on n.

Base case: $n = 1$. Start with the exact sequence

$$0 \to M \xrightarrow{x_1} M \to M/x_1M \to 0.$$

Since S is flat over R, the induced sequence

$$0 \to S \otimes_R M \xrightarrow{x_1} S \otimes_R M \to S \otimes_R (M/x_1M) \to 0$$

is also exact. This shows that x_1 is $S \otimes_R M$-regular, and that

$$S \otimes_R (M/x_1M) \cong (S \otimes_R M)/x_1(S \otimes_R M).$$

The induction step is left as an exercise. □

The next two results identify cases where the hypotheses of Lemma V.6.2 are satisfied. The definition of a local ring homomorphism is in III.5.4.

Lemma V.6.3. *Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a local ring homomorphism between commutative rings, and let M be a finitely generated non-zero R-module. If $\mathbf{x} = x_1,\dots,x_n \in \mathfrak{m}$, then $\mathbf{x}(S\otimes_R M) \neq S\otimes_R M$.*

PROOF. The first and third isomorphisms in the following sequence are from Exercise II.4.14

$$\frac{S\otimes_R M}{\mathbf{x}(S\otimes_R M)} \cong (S\otimes_R M)\otimes_R (R/\mathbf{x}R) \cong S\otimes_R (M\otimes_R (R/\mathbf{x}R)) \cong S\otimes_R (M/\mathbf{x}M) \neq 0.$$

The second isomorphism is associativity II.3.6. For the non-vanishing, note that Nakayama's Lemma implies that $M/\mathbf{x}M \neq 0$, so the non-vanishing follows from the fact that φ is faithfully flat; see Theorem III.3.4 and Proposition III.5.8. □

Lemma V.6.4. *Let $\varphi\colon R \to S$ be a flat ring homomorphism between commutative rings. If $\mathbf{x} = x_1,\dots,x_n \in R$ is an R-regular sequence such that $\mathbf{x}S \neq S$ (e.g., if φ is a local homomorphism and $\mathbf{x}$ is in the maximal ideal of R), then the sequences $x_1,\dots,x_n \in R$ and $\varphi(x_1),\dots,\varphi(x_n) \in S$ are both S-regular.*

PROOF. This is the special case of Lemma V.6.2 with $M = R$. □

Exercises.

Exercise V.6.5. Complete the proof of Lemma V.6.2.

CHAPTER VI

Chain Maps and Induced Maps on Ext and Tor
September 8, 2009

Chain maps are essentially homomorphisms of chain complexes. In other words, they are the morphisms in the category of chain complexes. We discuss the basic properties of chain maps and show how they induce homomorphisms on Ext and Tor-modules.

VI.1. Chain Maps

Definition VI.1.1. Let R be a commutative ring, and let $M_\bullet$ and $N_\bullet$ be R-complexes. A *chain map* $F_\bullet\colon M_\bullet \to N_\bullet$ is a sequence $\{F_i\colon M_i \to N_i\}_{i\in\mathbb{Z}}$ making the next "ladder-diagram" commute.

$$\begin{array}{ccccccccc} M_\bullet & \cdots & \xrightarrow{\partial^M_{i+1}} & M_i & \xrightarrow{\partial^M_i} & M_{i-1} & \xrightarrow{\partial^M_{i-1}} & \cdots \\ {\scriptstyle F_\bullet}\downarrow & & & \downarrow{\scriptstyle F_i} & & \downarrow{\scriptstyle F_{i-1}} & & \\ N_\bullet & \cdots & \xrightarrow{\partial^N_{i+1}} & N_i & \xrightarrow{\partial^N_i} & N_{i-1} & \xrightarrow{\partial^N_{i-1}} & \cdots \end{array}$$

Chain maps are also called "morphisms of R-complexes".

An *isomorphism* from $M_\bullet$ to $N_\bullet$ is a chain map $F_\bullet\colon M_\bullet \to N_\bullet$ such that each map $F_i\colon M_i \to N_i$ is an isomorphism.

Example VI.1.2. Here is a chain map over the ring $R = \mathbb{Z}/12\mathbb{Z}$.

$$\begin{array}{ccccccccc} M_\bullet & \cdots & \xrightarrow{6} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{4} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{6} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{4} \cdots \\ {\scriptstyle F_\bullet}\downarrow & & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 2} & \\ N_\bullet & \cdots & \xrightarrow{4} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{6} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{4} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{6} \cdots \end{array}$$

The next result states that a chain map induces maps on homology.

Proposition VI.1.3. *Let R be a commutative ring, and let $F_\bullet\colon M_\bullet \to N_\bullet$ be a chain map.*

(a) *For each i, we have $F_i(\operatorname{Ker}(\partial^M_i)) \subseteq \operatorname{Ker}(\partial^N_i)$.*
(b) *For each i, we have $F_i(\operatorname{Im}(\partial^M_{i+1})) \subseteq \operatorname{Im}(\partial^N_{i+1})$.*
(c) *For each i, the map $\mathrm{H}_i(F_\bullet)\colon \mathrm{H}_i(M_\bullet) \to \mathrm{H}_i(N_\bullet)$ given by $\mathrm{H}_i(F_\bullet)(\overline{m}) = \overline{F_i(m)}$ is a well-defined R-module homomorphism.*

Proof. (a) and (b): Chase the diagram in Definition VI.1.1.

(c) The map $\mathrm{H}_i(F_\bullet)$ is well-defined by parts (a) and (b). It is straightforward to show that it is R-linear. □

Example VI.1.4. Consider the chain map from Example VI.1.2:

$$\begin{array}{ccccccccccccc}
M_\bullet & \cdots & \xrightarrow{6} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{4} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{6} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{4} & \cdots \\
{\scriptstyle F_\bullet}\downarrow & & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 2} & & \\
N_\bullet & \cdots & \xrightarrow{4} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{6} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{4} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{6} & \cdots .
\end{array}$$

The homology modules are computed in Example IV.1.4:

$$\mathrm{H}_1(M_\bullet) \cong \mathrm{H}_0(N_\bullet) \cong 2\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$$
$$\mathrm{H}_0(M_\bullet) \cong \mathrm{H}_1(N_\bullet) \cong 3\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}.$$

In degree 1, the map induced on homology is induced by multiplication by 3:

$$\mathrm{H}_1(F_\bullet)\colon \frac{(2)\mathbb{Z}}{(4)\mathbb{Z}} \to \frac{(3)\mathbb{Z}}{(6)\mathbb{Z}} \qquad \overline{2} \mapsto \overline{6} = 0.$$

In degree 0, the map induced on homology is induced by multiplication by 2:

$$\mathrm{H}_0(F_\bullet)\colon \frac{(3)\mathbb{Z}}{(6)\mathbb{Z}} \to \frac{(2)\mathbb{Z}}{(4)\mathbb{Z}} \qquad \overline{3} \mapsto \overline{6} = \overline{2} \neq 0.$$

This example shows that you have to be careful. Just because $\mathrm{H}_i(M_\bullet) \cong \mathrm{H}_i(N_\bullet) \cong \mathbb{Z}/(2)$ and F_i is multiplication by 2, it does not follow that $\mathrm{H}_i(F_\bullet) = 0$, and similarly for multiplication by 3. On the other hand, see Exercise VI.1.7.

Exercises.

Exercise VI.1.5. Complete the proof of Proposition VI.1.3.

Exercise VI.1.6. Let R be a commutative ring. Let $M_\bullet$ be an R-complex, and let $r \in R$. Let $\mu^M_\bullet\colon M_\bullet \to M_\bullet$ be given by $\mu^r_i(m) = rm$. Show that $\mu^M_\bullet$ is a chain map and that the induced map $\mathrm{H}_i(\mu^M_\bullet)\colon \mathrm{H}_i(M_\bullet) \to \mathrm{H}_i(M_\bullet)$ is given by $\overline{m} \mapsto r\overline{m}$ for all $i \in \mathbb{Z}$.

Exercise VI.1.7. Let R be a commutative ring. Consider chain maps of R-complexes $F_\bullet\colon L_\bullet \to M_\bullet$ and $G_\bullet\colon M_\bullet \to N_\bullet$.

(a) Show that, if $F_\bullet$ is an isomorphism, then so is $\mathrm{H}_i(F_\bullet)$ for each $i \in \mathbb{Z}$.
(b) Show that, if $F_i = 0$, then $\mathrm{H}_i(F_\bullet) = 0$.
(c) Show that, $\mathrm{H}_i(G_\bullet F_\bullet) = \mathrm{H}_i(G_\bullet)\,\mathrm{H}_i(F_\bullet)$ for each i.

Exercise VI.1.8. Let R be a commutative ring. Let $M_\bullet$ and $M'_\bullet$ be chain complexes, and let N be an R-module.

(a) Prove that a chain map $F_\bullet\colon M_\bullet \to M'_\bullet$ is an isomorphism if and only if it has a two-sided inverse, that is, if and only if there is a chain map $G_\bullet\colon M'_\bullet \to M_\bullet$ such that $F_\bullet G_\bullet$ is the identity on $M'_\bullet$ and $G_\bullet F_\bullet$ is the identity on $M_\bullet$.
(b) Prove that there is an isomorphism of R-complexes $\theta_\bullet\colon N \otimes_R M_\bullet \to M_\bullet \otimes_R N$.
(c) Prove that there are isomorphisms $R \otimes_R M_\bullet \xrightarrow{\cong} M_\bullet$ and $M_\bullet \otimes_R R \xrightarrow{\cong} M_\bullet$ and $\mathrm{Hom}_R(R, M_\bullet) \xrightarrow{\cong} M_\bullet$.

Exercise VI.1.9. (Hom-tensor adjointness) Let $\varphi\colon R \to S$ be a homomorphism of commutative rings.

(a) Let $N_\bullet$ be an R-complex, and let M and P be S-modules. Prove that there is an isomorphism of S-complexes

$$\mathrm{Hom}_R(N_\bullet, \mathrm{Hom}_S(M, P)) \cong \mathrm{Hom}_S(M \otimes_R N_\bullet, P).$$

(b) Let N be an R-module, let M be an S-complex, and let P be an S-module. Prove that there is an isomorphism of S-complexes

$$\mathrm{Hom}_R(N, \mathrm{Hom}_S(M_\bullet, P)) \cong \mathrm{Hom}_S(M_\bullet \otimes_R N, P).$$

(c) Let N be an R-module, let M be an S-module, and let P be an S-complex. Prove that there is an isomorphism of S-complexes

$$\mathrm{Hom}_R(N, \mathrm{Hom}_S(M, P_\bullet)) \cong \mathrm{Hom}_S(M \otimes_R N, P_\bullet).$$

Exercise VI.1.10. (Hom-tensor adjointness) Let $\varphi\colon R \to S$ be a homomorphism of commutative rings.

(a) Let P be an R-complex, and let M and N be S-modules. Prove that there is an isomorphism of S-complexes

$$\mathrm{Hom}_S(N, \mathrm{Hom}_R(M, P_\bullet)) \cong \mathrm{Hom}_R(M \otimes_S N, P_\bullet).$$

(b) Let P be an R-module, let M be an S-complex, and let N be an S-module. Prove that there is an isomorphism of S-complexes

$$\mathrm{Hom}_S(N, \mathrm{Hom}_R(M_\bullet, P)) \cong \mathrm{Hom}_R(M_\bullet \otimes_S N, P).$$

(c) Let P be an R-module, let M be an S-module, and let N be an S-complex. Prove that there is an isomorphism of S-complexes

$$\mathrm{Hom}_S(N_\bullet, \mathrm{Hom}_R(M, P)) \cong \mathrm{Hom}_R(M \otimes_S N_\bullet, P).$$

Exercise VI.1.11. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings.

(a) Let $M_\bullet$ be an R-complex, and let N be an R-module. Prove that there is an isomorphism of S-complexes

$$\mathrm{Hom}_R(S, \mathrm{Hom}_R(M_\bullet, P)) \cong \mathrm{Hom}_S(S \otimes_R M_\bullet, \mathrm{Hom}_R(S, N)).$$

(b) Let M be an R-module, and let $N_\bullet$ be an R-complex. Prove that there is an isomorphism of S-complexes

$$\mathrm{Hom}_R(S, \mathrm{Hom}_R(M, N_\bullet)) \cong \mathrm{Hom}_S(S \otimes_R M, \mathrm{Hom}_R(S, N_\bullet)).$$

VI.2. Isomorphisms for Ext and Tor

In this section, we describe how Ext and Tor localize, and how they behave with respect to some other natural operations. Most of the details are left as useful exercises for the reader. The first result shows that Tor is commutative.

Lemma VI.2.1. *Let R be a commutative ring, and let M and N be R-modules. For each integer i, there is an isomorphism $\mathrm{Tor}_i^R(M, N) \cong \mathrm{Tor}_i^R(N, M)$.*

Proof. Let $P_\bullet$ be a projective resolution of M. The first isomorphism in the following sequence is by definition:

$$\mathrm{Tor}_i^R(M, N) \cong \mathrm{H}_i(P_\bullet \otimes_R N) \cong \mathrm{H}_i(N \otimes_R P_\bullet) \cong \mathrm{Tor}_i^R(N, M).$$

The second isomorphism comes from Exercise VI.1.8(b), which says that $P_\bullet \otimes_R N \cong N \otimes_R P_\bullet$, and Exercise VI.1.7(a) which says that an isomorphism of complexes induces an isomorphism on homology. The third isomorphism is from Theorem IV.4.8. □

Definition VI.2.2. Let R be a commutative ring, let $U \subseteq R$ be a multiplicatively closed subset, and let $M_\bullet$ be an R-complex. The *localized complex* $U^{-1}M_\bullet$ is the sequence

$$U^{-1}M_\bullet = \cdots \xrightarrow{U^{-1}\partial^M_{i+1}} U^{-1}M_i \xrightarrow{U^{-1}\partial^M_i} U^{-1}M_{i-1} \xrightarrow{U^{-1}\partial^M_{i-1}} \cdots .$$

There is an isomorphism of $U^{-1}R$-complexes $U^{-1}M_\bullet \cong (U^{-1}R) \otimes_R M_\bullet$.

Let $F_\bullet \colon M_\bullet \to N_\bullet$ be a chain map of R-complexes. Define

$$U^{-1}F_\bullet \colon U^{-1}M_\bullet \to U^{-1}N_\bullet$$

to be the sequence of maps $\{U^{-1}F_i \colon U^{-1}M_i \to U^{-1}N_i\}$.

Remark VI.2.3. Let R be a commutative ring, let $U \subseteq R$ be a multiplicatively closed subset, and let $M_\bullet$ be an R-complex. The sequence $U^{-1}M_\bullet$ is a $U^{-1}R$-complex. The natural maps $M_i \to U^{-1}M_i$ form a chain map $M_\bullet \to U^{-1}M_\bullet$. If $F_\bullet \colon M_\bullet \to N_\bullet$ is a chain map of R-complexes, then the sequence

$$U^{-1}F_\bullet \colon U^{-1}M_\bullet \to U^{-1}N_\bullet$$

is a chain map of $U^{-1}R$-complexes that makes the following diagram commute

$$\begin{array}{ccc} M_\bullet & \xrightarrow{F_\bullet} & N_\bullet \\ \downarrow & & \downarrow \\ U^{-1}M_\bullet & \xrightarrow{U^{-1}F_\bullet} & U^{-1}N_\bullet \end{array}$$

where the unlabeled vertical maps are the natural ones.

The natural isomorphisms $(U^{-1}R) \otimes_R M_i \to U^{-1}M_i$ from Proposition II.2.9(b) form an isomorphism of $U^{-1}R$-complexes $(U^{-1}R) \otimes_R M_\bullet \xrightarrow{\cong} U^{-1}M_\bullet$ making the next diagram commute

$$\begin{array}{ccc} (U^{-1}R) \otimes_R M_\bullet & \xrightarrow{(U^{-1}R)\otimes_R F_\bullet} & (U^{-1}R) \otimes_R N_\bullet \\ \cong\downarrow & & \downarrow\cong \\ U^{-1}M_\bullet & \xrightarrow{U^{-1}F_\bullet} & U^{-1}N_\bullet. \end{array}$$

Lemma VI.2.4. *Let R be a commutative ring, let $U \subseteq R$ be a multiplicatively closed subset, and let M be an R-module.*

(a) *If $P^+_\bullet$ is a projective resolution of M over R, then $U^{-1}(P^+_\bullet) \cong (U^{-1}P_\bullet)^+$ is a projective resolution of $U^{-1}M$ over $U^{-1}R$.*

(b) *Assume that R is noetherian. If ${}^+I_\bullet$ is an injective resolution of M over R, then $U^{-1}({}^+I_\bullet) \cong {}^+(U^{-1}I_\bullet)$ is an injective resolution of $U^{-1}M$ over $U^{-1}R$.*

Proof. (a) The resolution $P^+_\bullet$ is an exact sequence of R-module homomorphisms:

$$P^+_\bullet = \cdots \xrightarrow{\partial^P_2} P_1 \xrightarrow{\partial^P_1} P_0 \xrightarrow{\tau} M \to 0.$$

The exactness of localization implies that the localized sequence is exact:

$$U^{-1}(P^+_\bullet) = \cdots \xrightarrow{U^{-1}\partial^P_2} U^{-1}P_1 \xrightarrow{U^{-1}\partial^P_1} U^{-1}P_0 \xrightarrow{U^{-1}\tau} U^{-1}M \to 0.$$

Each $U^{-1}P_i$ is a projective $U^{-1}R$-module by Exercise III.1.21(c). The desired conclusion is immediate.

The proof of part (b) is similar, using Proposition III.1.19. □

Lemma VI.2.5. *Let R be a commutative ring, let $U \subseteq R$ be a multiplicatively closed subset, and let $M_\bullet$ be an R-complex. For each index i, there is an isomorphism $\mathrm{H}_i(U^{-1}M_\bullet) \cong U^{-1}\,\mathrm{H}_i(U^{-1}M_\bullet)$.*

PROOF. The isomorphism $U^{-1}M_\bullet \cong (U^{-1}R)\otimes_R M_\bullet$ from Remark VI.2.3 yields the first isomorphism in the following sequence

$$\mathrm{H}_i(U^{-1}M_\bullet) \cong \mathrm{H}_i((U^{-1}R) \otimes_R M_\bullet) \cong (U^{-1}R) \otimes_R \mathrm{H}_i(M_\bullet) \cong U^{-1}\,\mathrm{H}_i(U^{-1}M_\bullet).$$

The second isomorphism is by Theorem IV.1.10(b) because $U^{-1}R$ is a flat R-module by Proposition II.2.9(d). The third isomorphism is from Proposition II.2.9(b). □

For the next result, recall that an R-module N is *finitely presented* if there is an exact sequence $R^m \to R^n \to N \to 0$. For instance, a finitely generated module over a noetherian ring is finitely presented.

Proposition VI.2.6. *Let R be a commutative ring, and let $U \subseteq R$ be a multiplicatively closed subset. Let $M_\bullet$ be an R-complex, and let N be an R-module.*

(a) *If N is finitely presented, then there is an isomorphism of $U^{-1}R$-complexes*

$$\mathrm{Hom}_{U^{-1}R}(U^{-1}N, U^{-1}M_\bullet) \cong U^{-1}\,\mathrm{Hom}_R(N, M_\bullet).$$

(b) *If each M_i is finitely presented, there is an isomorphism of $U^{-1}R$-complexes*

$$\mathrm{Hom}_{U^{-1}R}(U^{-1}M_\bullet, U^{-1}N) \cong U^{-1}\,\mathrm{Hom}_R(M_\bullet, N).$$

(c) *There is an isomorphism of $U^{-1}R$-complexes*

$$(U^{-1}M_\bullet) \otimes_{U^{-1}R} (U^{-1}N) \cong U^{-1}(M_\bullet \otimes_R N).$$

(d) *There is an isomorphism of $U^{-1}R$-complexes*

$$(U^{-1}N) \otimes_{U^{-1}R} (U^{-1}M_\bullet) \cong U^{-1}(N \otimes_R M_\bullet).$$

PROOF. (a) The natural isomorphisms

$$U^{-1}\,\mathrm{Hom}_R(N, M_i) \xrightarrow{\cong} \mathrm{Hom}_{U^{-1}R}(U^{-1}N, U^{-1}M_i)$$

from Proposition I.5.8 form a chain map, and hence the desired isomorphism.

(c) The natural isomorphisms

$$(U^{-1}M_i) \otimes_{U^{-1}R} (U^{-1}N) \xrightarrow{\cong} U^{-1}(M_i \otimes_R N)$$

from Exercise II.2.13 form a chain map, and hence the desired isomorphism. See also Corollary II.3.7.

The proofs of parts (b) and (d) are similar. □

Theorem VI.2.7. *Let R be a commutative ring, and let $U \subseteq R$ be a multiplicatively closed subset. Let M and N be R-modules.*

(a) *There are isomorphisms of $U^{-1}R$-modules*

$$\mathrm{Tor}_i^{U^{-1}R}(U^{-1}M, U^{-1}N) \cong U^{-1}\,\mathrm{Tor}_R^i(M, N).$$

(b) *If R is noetherian and N is finitely generated, then there are isomorphisms of $U^{-1}R$-modules*

$$\mathrm{Ext}^i_{U^{-1}R}(U^{-1}N, U^{-1}M) \cong U^{-1}\,\mathrm{Ext}^i_R(N, M).$$

PROOF. (b) The assumptions on R and N imply that N has a projective resolution $P_\bullet$ such that each P_i is a finitely generated free R-module. Lemma VI.2.4(a) implies that the localization $U^{-1}P_\bullet$ is a projective resolution of $U^{-1}M$ over $U^{-1}R$. Hence, the first isomorphism in the following sequence is by definition:

$$\begin{aligned}\operatorname{Ext}^i_{U^{-1}R}(U^{-1}N, U^{-1}M) &\cong \operatorname{H}_{-i}(\operatorname{Hom}_{U^{-1}R}(U^{-1}P_\bullet, U^{-1}M)\\ &\cong \operatorname{H}_{-i}(U^{-1}\operatorname{Hom}_R(P_\bullet, M)\\ &\cong U^{-1}\operatorname{H}_{-i}(\operatorname{Hom}_R(P_\bullet, M)\\ &\cong U^{-1}\operatorname{Ext}^i_R(N, M).\end{aligned}$$

The second isomorphism is by Proposition VI.2.6(b). The third isomorphism follows from Lemma VI.2.5, and the fourth isomorphism is by definition.

The proof of part (a) is similar. □

Exercises.

Exercise VI.2.8. Verify the facts from Remark VI.2.3.

Exercise VI.2.9. Complete the proof of Lemma VI.2.5.

Exercise VI.2.10. Complete the proof of Proposition VI.2.6.

Exercise VI.2.11. Complete the proof of Theorem VI.2.7.

Exercise VI.2.12. Let R be a commutative ring, let M be an R-module, and let $\{N_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules.

(a) Prove that there are R-module isomorphisms

$$\begin{aligned}\operatorname{Ext}^i_R(M, \textstyle\prod_\lambda N_\lambda) &\cong \textstyle\prod_\lambda \operatorname{Ext}^i_R(M, N_\lambda)\\ \operatorname{Ext}^i_R(\coprod_\lambda N_\lambda, M) &\cong \textstyle\prod_\lambda \operatorname{Ext}^i_R(N_\lambda, M)\\ \operatorname{Tor}^R_i(\coprod_\lambda N_\lambda, M) &\cong \coprod_\lambda \operatorname{Tor}^R_i(N_\lambda, M).\end{aligned}$$

(b) Prove that if R is noetherian and M is finitely generated, then there are R-module isomorphisms

$$\operatorname{Ext}^i_R(M, \coprod_\lambda N_\lambda) \cong \coprod_\lambda \operatorname{Ext}^i_R(M, N_\lambda).$$

Exercise VI.2.13. Let R be a commutative noetherian local ring.

(a) let $M_1, \dots, M_n$ be non-zero finitely generated R-modules. Show that one has $\operatorname{depth}_R(\oplus_{i=1}^n M_i) = \max\{\operatorname{depth}_R(M_i) \mid i = 1, \dots, n\}$. [Hint: Exercise VI.2.12.]

(b) Let M be a non-zero finitely generated projective R-module. Show that M is free and that $\operatorname{depth}_R(M) = \operatorname{depth}(R)$. [Hint: Corollary V.4.9.]

Exercise VI.2.14. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings such that S is flat as an R-module. Let M and N be R-modules.

(a) Prove that there are isomorphisms of S-modules

$$\operatorname{Tor}^S_i(S \otimes_R M, S \otimes_R N) \cong S \otimes_R \operatorname{Tor}^i_R(M, N).$$

(b) Prove that, if R is noetherian and N is finitely generated, then there are isomorphisms of S-modules

$$\operatorname{Ext}^i_S(S \otimes_R N, S \otimes_R M) \cong S \otimes_R \operatorname{Ext}^i_R(N, M).$$

(Hint: See Exercise II.2.15 and Corollary II.3.7.)

VI.3. Liftings of Resolutions

Note that Q is not required to be projective in the following lemma.

Lemma VI.3.1. *Let R be a commutative ring, and consider the following diagram of R-module homomorphisms with exact rows:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{\alpha} & P & \xrightarrow{\gamma} & M & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & N & \xrightarrow{\delta} & Q & \xrightarrow{\sigma} & N & \longrightarrow & 0. \end{array}$$

If P is projective, then there exist R-module homomorphisms $F\colon P \to Q$ and $f'\colon M' \to N'$ making the next diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{\alpha} & P & \xrightarrow{\gamma} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & N & \xrightarrow{\delta} & Q & \xrightarrow{\sigma} & N & \longrightarrow & 0. \end{array}$$

Proof. Apply $\mathrm{Hom}_R(P, -)$ to the bottom row of the given diagram. Since P is projective, this yields an exact sequence

$$0 \to \mathrm{Hom}_R(P, N') \xrightarrow{\mathrm{Hom}_R(P,\delta)} \mathrm{Hom}_R(P, Q) \xrightarrow{\mathrm{Hom}_R(P,\sigma)} \mathrm{Hom}_R(P, N) \to 0.$$

In particular, the map $\mathrm{Hom}_R(P, \sigma)\colon \mathrm{Hom}_R(P, Q) \to \mathrm{Hom}_R(P, N)$ is surjective. Since we have $f\gamma \in \mathrm{Hom}_R(P, N)$, this implies that there exists $F \in \mathrm{Hom}_R(P, Q)$ such that $\mathrm{Hom}_R(P, \sigma)(F) = f\gamma$, that is, such that $\sigma F = f\gamma$. In other words, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M' & \xrightarrow{\alpha} & P & \xrightarrow{\gamma} & M & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle f} & & \\ 0 & \longrightarrow & N & \xrightarrow{\delta} & Q & \xrightarrow{\sigma} & N & \longrightarrow & 0. \end{array}$$

In particular, we have

$$\sigma F \alpha = f \gamma \alpha = 0$$

and therefore

$$\mathrm{Im}(F\alpha) \subseteq \mathrm{Ker}(\sigma) = \mathrm{Im}(\delta) = N.$$

Hence, the image of the restriction $F|_{M'}\colon M' \to Q$ is contained in N', and thus F induces a well-defined homomorphism $f'\colon M' \to N'$ making the desired diagram commute. □

The following lifting property is the basis for many of the properties of Ext.

Proposition VI.3.2. *Let R be a commutative ring, and let M and N be R-modules. Let $P_\bullet^+$ be an R-projective resolution of M, and let $Q_\bullet^+$ be any "left resolution" of N, that is, an exact sequence of the following form:*

$$\cdots \xrightarrow{\partial^Q_{i+1}} Q_i \xrightarrow{\partial^Q_i} Q_{i-1} \xrightarrow{\partial^Q_{i-1}} \cdots \xrightarrow{\partial^Q_1} Q_0 \xrightarrow{\pi} N \to 0.$$

(For example, $Q^+_\bullet$ may be a projective or flat resolution of N.) Given an R-module homomorphism $f\colon M \to N$, there exist R-module homomorphisms $F_i\colon P_i \to Q_i$ making the following diagram commute

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{\partial^P_{i+1}} & P_i & \xrightarrow{\partial^P_i} & P_{i-1} & \xrightarrow{\partial^P_{i-1}} & \cdots & \xrightarrow{\partial^P_1} & P_0 & \xrightarrow{\tau} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \exists F_i} & & \downarrow{\scriptstyle \exists F_{i-1}} & & & & \downarrow{\scriptstyle \exists F_0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \xrightarrow{\partial^Q_{i+1}} & Q_i & \xrightarrow{\partial^Q_i} & Q_{i-1} & \xrightarrow{\partial^Q_{i-1}} & \cdots & \xrightarrow{\partial^Q_1} & Q_0 & \xrightarrow{\pi} & N & \longrightarrow & 0.
\end{array}$$

Proof. For each $i \geqslant 1$, let $M_i = \operatorname{Im}(\partial^P_i)$ and $N_i = \operatorname{Im}(\partial^Q_i)$. Set $M_0 = M$ and $N_0 = N$. Then, for $i \geqslant 0$ we have exact sequences

$$\begin{aligned}
& 0 \longrightarrow M_{i+1} \xrightarrow{\alpha_{i+1}} P_i \xrightarrow{\gamma_i} M_i \longrightarrow 0 \\
(*_i) \qquad & 0 \longrightarrow N_{i+1} \xrightarrow{\delta_{i+1}} Q_i \xrightarrow{\sigma_i} N_i \longrightarrow 0
\end{aligned}$$

where α_{i+1} and δ_{i+1} are the natural inclusions, and γ_i and σ_i are induced by the corresponding differential in $P^+_\bullet$ or $Q^+_\bullet$. In particular, the following diagrams commute for each $i \geqslant 0$:

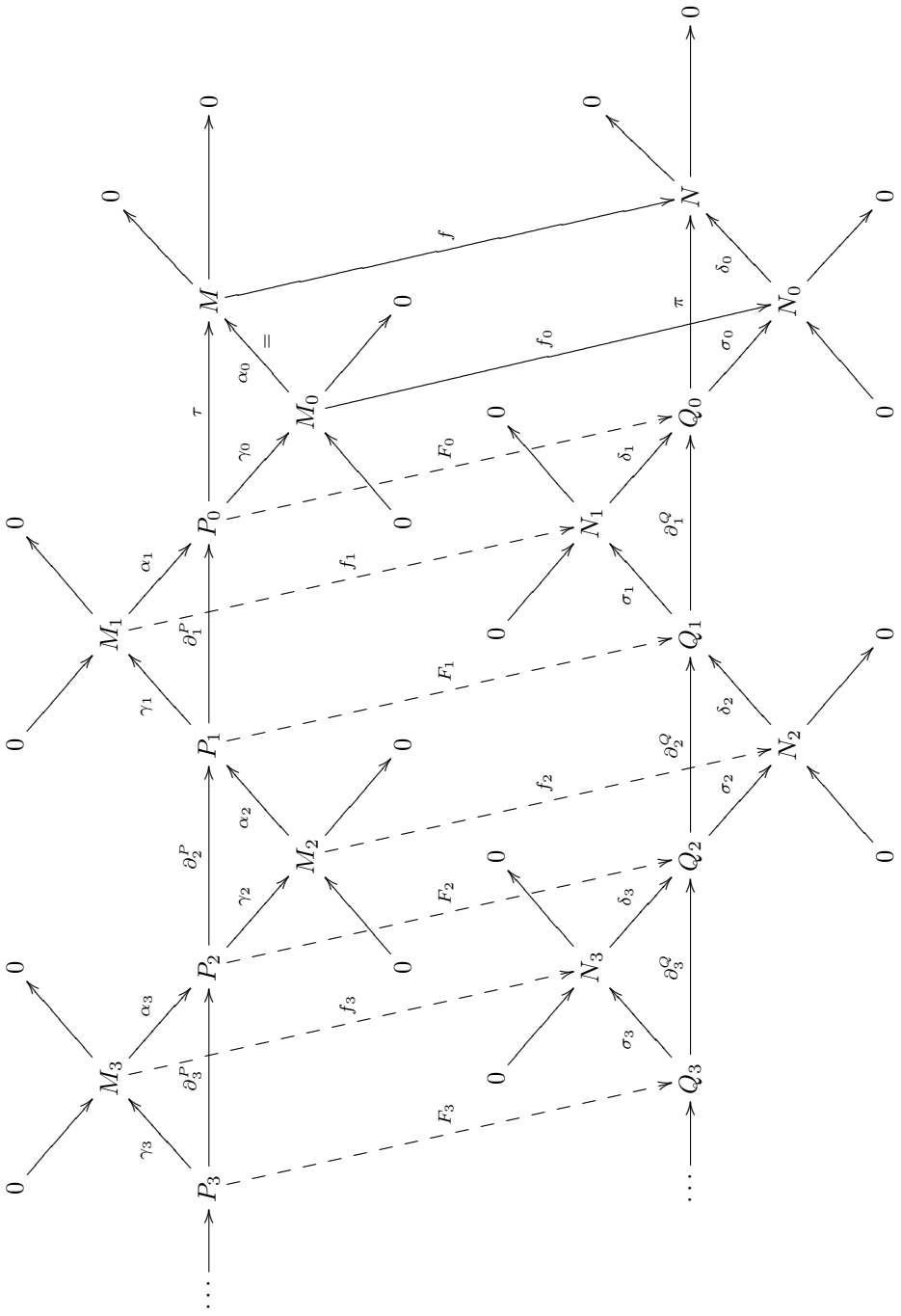
$$\begin{array}{ccc}
P_{i+1} & \xrightarrow{\gamma_i} & M_{i+1} \\
 & {\scriptstyle \partial^P_{i+1}}\searrow & \downarrow{\scriptstyle \alpha_{i+1}} \\
 & & P_i
\end{array}
\qquad\qquad
\begin{array}{ccc}
Q_{i+1} & \xrightarrow{\sigma_i} & N_{i+1} \\
 & {\scriptstyle \partial^Q_{i+1}}\searrow & \downarrow{\scriptstyle \delta_{i+1}} \\
 & & Q_i.
\end{array}$$

Set $f_0 = f\colon M_0 \to N_0$. By induction on i, given $f_i\colon M_i \to N_i$, Lemma VI.3.1 yields R-module homomorphisms $F_i\colon P_i \to Q_i$ and $f_{i+1}\colon M_{i+1} \to N_{i+1}$ making the following diagram commute:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M_{i+1} & \xrightarrow{\alpha_{i+1}} & P_i & \xrightarrow{\gamma_i} & M_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \exists f_{i+1}} & & \downarrow{\scriptstyle \exists F_i} & & \downarrow{\scriptstyle f_i} & & \\
0 & \longrightarrow & N_{i+1} & \xrightarrow{\delta_{i+1}} & Q_i & \xrightarrow{\sigma_i} & N_i & \longrightarrow & 0.
\end{array}$$

It is straightforward to show that the maps F_i make the desired diagram commute. (Note that the base case of our induction is the special case $i = 0$ of our inductive step.) □

Remark VI.3.3. Here is a diagrammatic version of the proof of Proposition VI.3.2. (Work through the diagram from right to left, i.e., top to bottom.)



Example VI.3.4. We work over the ring $R = \mathbb{Z}/12\mathbb{Z}$. Consider the natural R-module epimorphism $f\colon \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z}$. From Example IV.2.4, we know that

projective resolutions of $\mathbb{Z}/(6)$ and $\mathbb{Z}/(3)$ over $\mathbb{Z}/(12)$ are given by

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{2} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{6} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{6} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{\tau} & \mathbb{Z}/6\mathbb{Z} & \longrightarrow & 0 \\
 & & & & & & & & & & \downarrow f & & \\
\cdots & \xrightarrow{4} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{3} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{4} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{3} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/3\mathbb{Z} & \longrightarrow & 0.
\end{array}$$

One commutative diagram satisfying the conclusion of Proposition VI.3.2 is

$$\begin{array}{ccccccccccccc}
\cdots & \xrightarrow{2} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{6} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{2} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{6} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{\tau} & \mathbb{Z}/6\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow 2 & & \downarrow 1 & & \downarrow 2 & & \downarrow 1 & & \downarrow f & & \\
\cdots & \xrightarrow{4} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{3} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{4} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{3} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{\pi} & \mathbb{Z}/3\mathbb{Z} & \longrightarrow & 0.
\end{array}$$

Example VI.3.5. In the notation of Proposition VI.3.2, the sequence of homomorphisms $\{F_i\colon P_i \to Q_i\}$ gives chain maps $F_\bullet\colon P_\bullet \to Q_\bullet$ and $F_\bullet^+\colon P_\bullet^+ \to Q_\bullet^+$.

The next lifting property is proved like Proposition VI.3.2; see Exercise VI.3.11.

Proposition VI.3.6. *Let R be a commutative ring, and let M and N be R-modules. Let ${}^+J_\bullet$ be an R-injective resolution of N, and Let ${}^+I_\bullet$ be any "right resolution" of M, that is, ${}^+I_\bullet$ is an exact sequence of the following form:*

$$0 \to M \xrightarrow{\epsilon} I_0 \xrightarrow{\partial_0^I} \cdots \xrightarrow{\partial_{i+1}^I} I_i \xrightarrow{\partial_i^I} I_{i-1} \xrightarrow{\partial_{i-1}^I} \cdots .$$

(For example, ${}^+I_\bullet$ may be an R-injective resolution of M.) Given an R-module homomorphism $f\colon M \to N$, there exist R-module homomorphisms $G_i\colon I_i \to J_i$ making the following diagram commute

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & M & \xrightarrow{\epsilon} & I_0 & \xrightarrow{\partial_0^I} & \cdots & \xrightarrow{\partial_{i+1}^I} & I_i & \xrightarrow{\partial_i^I} & I_{i-1} & \xrightarrow{\partial_{i-1}^I} & \cdots \\
 & & \downarrow f & & \vdots\ \exists G_0 & & & & \vdots\ \exists G_i & & \vdots\ \exists G_{i-1} & & \\
0 & \longrightarrow & N & \xrightarrow{\iota} & J_0 & \xrightarrow{\partial_0^J} & \cdots & \xrightarrow{\partial_{i+1}^J} & J_i & \xrightarrow{\partial_i^J} & J_{i-1} & \xrightarrow{\partial_{i-1}^J} & \cdots .
\end{array}$$

Example VI.3.7. In the notation of Proposition VI.3.6, the sequence of homomorphisms $\{G_i\colon I_i \to J_i\}$ gives chain maps $G_\bullet\colon I_\bullet \to J_\bullet$ and ${}^+G_\bullet\colon {}^+I_\bullet \to {}^+J_\bullet$.

Exercises.

Exercise VI.3.8. Complete the proof of Proposition VI.3.2.

Exercise VI.3.9. Let R be a commutative ring, and let M be an R-module with projective resolution $P_\bullet^+$. Let $r \in R$, and let $\mu_r^M\colon M \to M$ be given by $m \mapsto rm$. Construct maps $F_i\colon P_i \to P_i$ satisfying the conclusion of Proposition VI.3.2 for $N = M$, $f = \mu_r^M$ and $Q_\bullet = P_\bullet$. Repeat this exercise with an injective resolution of M in Proposition VI.3.6.

Exercise VI.3.10. Let R be a commutative ring, and let $f\colon M \to N$ be an R-module homomorphism. Let $P_\bullet^+$ be an R-projective resolution of M, and let $Q_\bullet^+$ be an R-projective resolution of N. Let $\{F_i\colon P_i \to Q_i\}$ be as in the conclusion of Proposition VI.3.2, and let $F_\bullet\colon P_\bullet \to Q_\bullet$ be the corresponding chain map;

see Example VI.3.5. Prove that there are isomorphisms $\alpha\colon \mathrm{H}_0(P_\bullet) \xrightarrow{\cong} M$ and $\beta\colon \mathrm{H}_0(Q_\bullet) \xrightarrow{\cong} N$ making the following diagram commute:

$$\begin{array}{ccc} \mathrm{H}_0(P_\bullet) & \xrightarrow[\cong]{\alpha} & M \\ {\scriptstyle \mathrm{H}_0(F_\bullet)}\downarrow & & \downarrow{\scriptstyle f} \\ \mathrm{H}_0(Q_\bullet) & \xrightarrow[\cong]{\beta} & N. \end{array}$$

State and prove the analogous result for Proposition VI.3.6.

Exercise VI.3.11. Prove Proposition VI.3.6.

VI.4. Induced Chain Maps

In this section, we show how functors induce chain maps.

Definition VI.4.1. Let R be a commutative ring. Let $M_\bullet$ be an R-complex, and let $g\colon N \to N'$ be an R-module homomorphism.

(a) Recall from Definition IV.1.5 and Proposition IV.1.6 that $M_\bullet \otimes_R N$ is the R-complex

$$M_\bullet \otimes_R N = \cdots \xrightarrow{\partial^M_{i+1}\otimes_R N} \underbrace{M_i \otimes_R N}_{\text{degree } i} \xrightarrow{\partial^M_i \otimes_R N} \underbrace{M_{i-1} \otimes_R N}_{\text{degree } i-1} \xrightarrow{\partial^M_{i-1}\otimes_R N} \cdots .$$

For each $i \in \mathbb{Z}$, define $(M_\bullet \otimes_R g)_i\colon (M_\bullet \otimes_R N)_i \to (M_\bullet \otimes_R N')_i$ by the formula

$$(M_\bullet \otimes_R g)_i = M_i \otimes_R g\colon M_i \otimes_R N \to M_i \otimes_R N'.$$

This yields a sequence $M_\bullet \otimes_R g\colon M_\bullet \otimes_R N \to M_\bullet \otimes_R N'$ as in the following ladder diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial^M_{i+1}\otimes_R N} & M_i \otimes_R N & \xrightarrow{\partial^M_i \otimes_R N} & M_{i-1} \otimes_R N & \xrightarrow{\partial^M_{i-1}\otimes_R N} & \cdots \\ & & {\scriptstyle M_i\otimes_R g}\downarrow & & {\scriptstyle M_{i-1}\otimes_R g}\downarrow & & \\ \cdots & \xrightarrow[\partial^M_{i+1}\otimes_R N']{} & M_i \otimes_R N' & \xrightarrow[\partial^M_i \otimes_R N']{} & M_{i-1} \otimes_R N' & \xrightarrow[\partial^M_{i-1}\otimes_R N']{} & \cdots . \end{array}$$

(b) Recall from Definition IV.1.5 and Proposition IV.1.6 that $N \otimes_R M_\bullet$ is the R-complex

$$N \otimes_R M_\bullet = \cdots \xrightarrow{N\otimes_R \partial^M_{i+1}} \underbrace{N \otimes_R M_i}_{\text{degree } i} \xrightarrow{N\otimes_R \partial^M_i} \underbrace{N \otimes_R M_{i-1}}_{\text{degree } i-1} \xrightarrow{N\otimes_R \partial^M_{i-1}} \cdots .$$

For each $i \in \mathbb{Z}$, define $(g \otimes_R M_\bullet)_i\colon (N \otimes_R M_\bullet)_i \to (N' \otimes_R M_\bullet)_i$ by the formula

$$(g \otimes_R M_\bullet)_i = g \otimes_R M_i\colon N \otimes_R M_i \to N' \otimes_R M_i.$$

This yields a sequence $g \otimes_R M_\bullet\colon N \otimes_R M_\bullet \to N' \otimes_R M_\bullet$ as in the following ladder diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{N\otimes_R \partial^M_{i+1}} & N \otimes_R M_i & \xrightarrow{N\otimes_R \partial^M_i} & N \otimes_R M_{i-1} & \xrightarrow{N\otimes_R \partial^M_{i-1}} & \cdots \\ & & {\scriptstyle g\otimes_R M_i}\downarrow & & {\scriptstyle g\otimes_R M_{i-1}}\downarrow & & \\ \cdots & \xrightarrow[N'\otimes_R \partial^M_{i+1}]{} & N' \otimes_R M_i & \xrightarrow[N'\otimes_R \partial^M_i]{} & N' \otimes_R M_{i-1} & \xrightarrow[N'\otimes_R \partial^M_{i-1}]{} & \cdots . \end{array}$$

(c) Recall from Definition IV.1.5 and Proposition IV.1.6 that $\mathrm{Hom}_R(N, M_\bullet)$ is the R-complex

$$\cdots \xrightarrow{\mathrm{Hom}_R(N,\partial^M_{i+1})} \underbrace{\mathrm{Hom}_R(N, M_i)}_{\text{degree } i} \xrightarrow{\mathrm{Hom}_R(N,\partial^M_i)} \underbrace{\mathrm{Hom}_R(N, M_{i-1})}_{\text{degree } i-1} \xrightarrow{\mathrm{Hom}_R(N,\partial^M_{i-1})} \cdots .$$

For each $i \in \mathbb{Z}$, define $\mathrm{Hom}_R(g, M_\bullet)_i \colon \mathrm{Hom}_R(N', M_\bullet)_i \to \mathrm{Hom}_R(N, M_\bullet)_i$ by the formula

$$\mathrm{Hom}_R(g, M_\bullet)_i = \mathrm{Hom}_R(g, M_i) \colon \mathrm{Hom}_R(N', M_i) \to \mathrm{Hom}_R(N, M_i)$$

This yields a sequence $\mathrm{Hom}_R(g, M_\bullet) \colon \mathrm{Hom}_R(N', M_\bullet) \to \mathrm{Hom}_R(N, M_\bullet)$ as in the following ladder diagram

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\mathrm{Hom}_R(N',\partial^M_{i+1})} & \mathrm{Hom}_R(N', M_i) & \xrightarrow{\mathrm{Hom}_R(N',\partial^M_i)} & \mathrm{Hom}_R(N', M_{i-1}) & \xrightarrow{\mathrm{Hom}_R(N',\partial^M_{i-1})} & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{Hom}_R(g,M_i)} & & \downarrow{\scriptstyle \mathrm{Hom}_R(g,M_{i-1})} & & \\
\cdots & \xrightarrow[\mathrm{Hom}_R(N,\partial^M_{i+1})]{} & \mathrm{Hom}_R(N, M_i) & \xrightarrow[\mathrm{Hom}_R(N,\partial^M_i)]{} & \mathrm{Hom}_R(N, M_{i-1}) & \xrightarrow[\mathrm{Hom}_R(N,\partial^M_{i-1})]{} & \cdots .
\end{array}$$

(d) Recall from Definition IV.1.5 and Proposition IV.1.6 that $\mathrm{Hom}_R(M_\bullet, N)$ is the R-complex

$$\cdots \xrightarrow{\mathrm{Hom}_R(\partial^M_i,N)} \underbrace{\mathrm{Hom}_R(M_i, N)}_{\text{degree } -i} \xrightarrow{\mathrm{Hom}_R(\partial^M_{i+1},N)} \underbrace{\mathrm{Hom}_R(M_{i+1}, N)}_{\text{degree } -(i+1)} \xrightarrow{\mathrm{Hom}_R(\partial^M_{i+2},N)} \cdots .$$

For each $i \in \mathbb{Z}$, define $\mathrm{Hom}_R(M_\bullet, g)_i \colon \mathrm{Hom}_R(M_\bullet, N)_i \to \mathrm{Hom}_R(M_\bullet, N')_i$ by the formula

$$\mathrm{Hom}_R(M_\bullet, g)_i = \mathrm{Hom}_R(M_{-i}, g) \colon \mathrm{Hom}_R(M_{-i}, N) \to \mathrm{Hom}_R(M_{-i}, N')$$

This yields a sequence $\mathrm{Hom}_R(M_\bullet, g) \colon \mathrm{Hom}_R(M_\bullet, N) \to \mathrm{Hom}_R(M_\bullet, N')$ as in the following ladder diagram

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\mathrm{Hom}_R(\partial^M_i,N)} & \mathrm{Hom}_R(M_i, N) & \xrightarrow{\mathrm{Hom}_R(\partial^M_{i+1},N)} & \mathrm{Hom}_R(M_{i+1}, N) & \xrightarrow{\mathrm{Hom}_R(\partial^M_{i+2},N)} & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{Hom}_R(M_i,g)} & & \downarrow{\scriptstyle \mathrm{Hom}_R(M_{i+1},g)} & & \\
\cdots & \xrightarrow[\mathrm{Hom}_R(\partial^M_i,N')]{} & \mathrm{Hom}_R(M_i, N') & \xrightarrow[\mathrm{Hom}_R(\partial^M_{i+1},N')]{} & \mathrm{Hom}_R(M_{i+1}, N') & \xrightarrow[\mathrm{Hom}_R(\partial^M_{i+2},N')]{} & \cdots .
\end{array}$$

Proposition VI.4.2. *Let R be a commutative ring. Let $M_\bullet$ be an R-complex, and let $g \colon N \to N'$ be an R-module homomorphism. Then the following sequences are chain maps: $M_\bullet \otimes_R g$, $g \otimes_R M_\bullet$, $\mathrm{Hom}_R(g, M_\bullet)$, and $\mathrm{Hom}_R(M_\bullet, g)$.*

Proof. We need to check that the appropriate ladder diagrams from Definition VI.4.1 commute. In each case, this is a consequence of the appropriate functoriality. For instance, for $M_\bullet \otimes_R g$, we have

$$(M_{i-1} \otimes_R g)(\partial^M_i \otimes_R N) = \partial^M_i \otimes_R g = (\partial^M_i \otimes_R N')(M_i \otimes_R g)$$

by Example II.2.3. (One can also check this by hand using simple tensors.) The other three sequences are checked similarly; see Exercise VI.4.7. □

Example VI.4.3. Consider the following $\mathbb{Z}$-complex from Example IV.1.7

$$\begin{array}{rccccccccc} M_\bullet = & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\left(\begin{smallmatrix}9\\-6\end{smallmatrix}\right)} & \mathbb{Z}^2 & \xrightarrow{(2\ 3)} & \mathbb{Z} & \longrightarrow & 0 \\ & & & a & \longmapsto & \left(\begin{smallmatrix}9a\\-6a\end{smallmatrix}\right) & & & & \\ & & & & & \left(\begin{smallmatrix}a\\b\end{smallmatrix}\right) & \longmapsto & (2a+3b). & & \end{array}$$

and the R-module homomorphism

$$g = \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right) : \mathbb{Z} \to \mathbb{Z}^2.$$

The chain map $M_\bullet \otimes_{\mathbb{Z}} g\colon M_\bullet \otimes_{\mathbb{Z}} \mathbb{Z} \to M_\bullet \otimes_{\mathbb{Z}} \mathbb{Z}^2$ has the following form:

$$\begin{array}{rccccccccc} M_\bullet \otimes \mathbb{Z} = & 0 & \longrightarrow & \mathbb{Z}\otimes\mathbb{Z} & \xrightarrow{\left(\begin{smallmatrix}9\\-6\end{smallmatrix}\right)\otimes\mathbb{Z}} & \mathbb{Z}^2\otimes\mathbb{Z} & \xrightarrow{(2\ 3)\otimes\mathbb{Z}} & \mathbb{Z}\otimes\mathbb{Z} & \longrightarrow & 0 \\ {\scriptstyle M_\bullet\otimes g}\downarrow & & & {\scriptstyle \mathbb{Z}\otimes\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)}\downarrow & & {\scriptstyle \mathbb{Z}^2\otimes\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)}\downarrow & & {\scriptstyle \mathbb{Z}\otimes\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)}\downarrow & & \\ M_\bullet \otimes \mathbb{Z}^2 = & 0 & \longrightarrow & \mathbb{Z}\otimes\mathbb{Z}^2 & \xrightarrow[\left(\begin{smallmatrix}9\\-6\end{smallmatrix}\right)\otimes\mathbb{Z}^2]{} & \mathbb{Z}^2\otimes\mathbb{Z}^2 & \xrightarrow[(2\ 3)\otimes\mathbb{Z}^2]{} & \mathbb{Z}\otimes\mathbb{Z}^2 & \longrightarrow & 0. \end{array}$$

Recall from Example IV.1.7 that there is an isomorphism

$$\begin{array}{rccccccccc} M_\bullet \otimes \mathbb{Z}^2 = & 0 & \longrightarrow & \mathbb{Z}\otimes\mathbb{Z}^2 & \xrightarrow{\left(\begin{smallmatrix}9\\-6\end{smallmatrix}\right)\otimes\mathbb{Z}^2} & \mathbb{Z}^2\otimes\mathbb{Z}^2 & \xrightarrow{(2\ 3)\otimes\mathbb{Z}^2} & \mathbb{Z}\otimes\mathbb{Z}^2 & \longrightarrow & 0 \\ {\scriptstyle \alpha_\bullet}\downarrow{\scriptstyle \cong} & & & {\scriptstyle \phi}\downarrow{\scriptstyle \cong} & & {\scriptstyle \psi}\downarrow{\scriptstyle \cong} & & {\scriptstyle \phi}\downarrow{\scriptstyle \cong} & & \\ M'_\bullet = & 0 & \longrightarrow & \mathbb{Z}^2 & \xrightarrow[\left(\begin{smallmatrix}9&0\\-6&0\\0&9\\0&6\end{smallmatrix}\right)]{} & \mathbb{Z}^4 & \xrightarrow[\left(\begin{smallmatrix}2&3&0&0\\0&0&2&3\end{smallmatrix}\right)]{} & \mathbb{Z}^2 & \longrightarrow & 0. \end{array}$$

Also, Exercise VI.1.8(c) shows that the following diagram is an isomorphism

$$\begin{array}{rccccccccc} M_\bullet \otimes \mathbb{Z} = & 0 & \longrightarrow & \mathbb{Z}\otimes\mathbb{Z} & \xrightarrow{\left(\begin{smallmatrix}9\\-6\end{smallmatrix}\right)\otimes\mathbb{Z}} & \mathbb{Z}^2\otimes\mathbb{Z} & \xrightarrow{(2\ 3)\otimes\mathbb{Z}} & \mathbb{Z}\otimes\mathbb{Z} & \longrightarrow & 0 \\ {\scriptstyle \beta_\bullet}\downarrow{\scriptstyle \cong} & & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \\ M_\bullet = & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow[\left(\begin{smallmatrix}9\\-6\end{smallmatrix}\right)]{} & \mathbb{Z}^2 & \xrightarrow[(2\ 3)]{} & \mathbb{Z} & \longrightarrow & 0. \end{array}$$

Using these two isomorphisms, the chain map $M_\bullet\otimes g$ is "equivalent" to the following:

$$\begin{array}{rccccccccc} M_\bullet = & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\left(\begin{smallmatrix}9\\-6\end{smallmatrix}\right)} & \mathbb{Z}^2 & \xrightarrow{(2\ 3)} & \mathbb{Z} & \longrightarrow & 0 \\ {\scriptstyle \gamma_\bullet}\downarrow & & & {\scriptstyle \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)}\downarrow & & {\scriptstyle \left(\begin{smallmatrix}1&0\\0&1\\2&0\\0&2\end{smallmatrix}\right)}\downarrow & & {\scriptstyle \left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)}\downarrow & & \\ M'_\bullet = & 0 & \longrightarrow & \mathbb{Z}^2 & \xrightarrow[\left(\begin{smallmatrix}9&0\\-6&0\\0&9\\0&6\end{smallmatrix}\right)]{} & \mathbb{Z}^4 & \xrightarrow[\left(\begin{smallmatrix}2&3&0&0\\0&0&2&3\end{smallmatrix}\right)]{} & \mathbb{Z}^2 & \longrightarrow & 0. \end{array}$$

(These chain maps are equivalent in the sense that there is a three-dimensional rectangular commutative diagram whose faces are the four previous commtuative

diagrams. In other words, there is a commutative diagram of chain maps

$$\begin{array}{ccc} M_\bullet \otimes \mathbb{Z} & \xrightarrow{M_\bullet \otimes g} & M_\bullet \otimes \mathbb{Z}^2 \\ {\scriptstyle \beta_\bullet}\downarrow{\scriptstyle \cong} & & {\scriptstyle \alpha_\bullet}\downarrow{\scriptstyle \cong} \\ M_\bullet & \xrightarrow{\gamma_\bullet} & M'_\bullet. \end{array}$$

The interested reader is encouraged to check this.)

Definition VI.4.4. Let R be a commutative ring. Let N be an R-module, and let $F_\bullet \colon M_\bullet \to M'_\bullet$ be a chain map of R-complexes.

(a) For each $i \in \mathbb{Z}$, define $(F_\bullet \otimes_R N)_i \colon (M_\bullet \otimes_R N)_i \to (M'_\bullet \otimes_R N)_i$ by the formula

$$(F_\bullet \otimes_R N)_i = F_i \otimes_R N \colon M_i \otimes_R N \to M'_i \otimes_R N.$$

This yields a sequence $F_\bullet \otimes_R N \colon M_\bullet \otimes_R N \to M'_\bullet \otimes_R N$ as in the following ladder diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial^M_{i+1}\otimes_R N} & M_i \otimes_R N & \xrightarrow{\partial^M_i \otimes_R N} & M_{i-1} \otimes_R N & \xrightarrow{\partial^M_{i-1}\otimes_R N} & \cdots \\ & & {\scriptstyle F_i\otimes_R N}\downarrow & & {\scriptstyle F_{i-1}\otimes_R N}\downarrow & & \\ \cdots & \xrightarrow[\partial^{M'}_{i+1}\otimes_R N]{} & M'_i \otimes_R N & \xrightarrow[\partial^{M'}_i \otimes_R N]{} & M'_{i-1} \otimes_R N & \xrightarrow[\partial^{M'}_{i-1}\otimes_R N]{} & \cdots . \end{array}$$

(b) For each $i \in \mathbb{Z}$, define $(N \otimes_R F_\bullet)_i \colon (N \otimes_R M_\bullet)_i \to (N \otimes_R M'_\bullet)_i$ by the formula

$$(N \otimes_R F_\bullet)_i = N \otimes_R F_i \colon N \otimes_R M_i \to N \otimes_R M'_i.$$

This yields a sequence $N \otimes_R F_\bullet \colon N \otimes_R M_\bullet \to N \otimes_R M'_\bullet$ as in the following ladder diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{N\otimes_R \partial^M_{i+1}} & N \otimes_R M_i & \xrightarrow{N\otimes_R \partial^M_i} & N \otimes_R M_{i-1} & \xrightarrow{N\otimes_R \partial^M_{i-1}} & \cdots \\ & & {\scriptstyle N\otimes_R F_i}\downarrow & & {\scriptstyle N\otimes_R F_{i-1}}\downarrow & & \\ \cdots & \xrightarrow[N\otimes_R \partial^{M'}_{i+1}]{} & N \otimes_R M'_i & \xrightarrow[N\otimes_R \partial^{M'}_i]{} & N \otimes_R M'_{i-1} & \xrightarrow[N\otimes_R \partial^{M'}_{i-1}]{} & \cdots . \end{array}$$

(c) For each $i \in \mathbb{Z}$, define $\mathrm{Hom}_R(N, F_\bullet)_i \colon \mathrm{Hom}_R(N, M_\bullet)_i \to \mathrm{Hom}_R(N, M'_\bullet)_i$ by the formula

$$\mathrm{Hom}_R(N, F_\bullet)_i = \mathrm{Hom}_R(N, F_i) \colon \mathrm{Hom}_R(N, M_i) \to \mathrm{Hom}_R(N, M'_i)$$

This yields a sequence $\mathrm{Hom}_R(N, F_\bullet) \colon \mathrm{Hom}_R(N, M_\bullet) \to \mathrm{Hom}_R(N, M'_\bullet)$ as in the following ladder diagram

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\mathrm{Hom}_R(N,\partial^M_{i+1})} & \mathrm{Hom}_R(N, M_i) & \xrightarrow{\mathrm{Hom}_R(N,\partial^M_i)} & \mathrm{Hom}_R(N, M_{i-1}) & \xrightarrow{\mathrm{Hom}_R(N,\partial^M_{i-1})} & \cdots \\ & & {\scriptstyle \mathrm{Hom}_R(N,F_i)}\downarrow & & {\scriptstyle \mathrm{Hom}_R(N,F_{i-1})}\downarrow & & \\ \cdots & \xrightarrow[\mathrm{Hom}_R(N,\partial^{M'}_{i+1})]{} & \mathrm{Hom}_R(N, M'_i) & \xrightarrow[\mathrm{Hom}_R(N,\partial^{M'}_i)]{} & \mathrm{Hom}_R(N, M'_{i-1}) & \xrightarrow[\mathrm{Hom}_R(N,\partial^{M'}_{i-1})]{} & \cdots . \end{array}$$

(d) For each $i \in \mathbb{Z}$, define $\mathrm{Hom}_R(F_\bullet, N)_i \colon \mathrm{Hom}_R(M'_\bullet, N)_i \to \mathrm{Hom}_R(M_\bullet, N)_i$ by the formula

$$\mathrm{Hom}_R(F_\bullet, N)_i = \mathrm{Hom}_R(F_{-i}, N) \colon \mathrm{Hom}_R(M'_{-i}, N) \to \mathrm{Hom}_R(M_{-i}, N)$$

This yields a sequence $\mathrm{Hom}_R(F_\bullet, N) \colon \mathrm{Hom}_R(M'_\bullet, N) \to \mathrm{Hom}_R(M_\bullet, N)$ as in the following ladder diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\mathrm{Hom}_R(\partial_i^{M'},N)} & \mathrm{Hom}_R(M'_i,N) & \xrightarrow{\mathrm{Hom}_R(\partial_{i+1}^{M'},N)} & \mathrm{Hom}_R(M'_{i+1},N) & \xrightarrow{\mathrm{Hom}_R(\partial_{i+2}^{M'},N)} & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{Hom}_R(F_i,N)} & & \downarrow{\scriptstyle \mathrm{Hom}_R(F_{i+1},N)} & & \\
\cdots & \xrightarrow[\mathrm{Hom}_R(\partial_i^{M},N)]{} & \mathrm{Hom}_R(M_i,N) & \xrightarrow[\mathrm{Hom}_R(\partial_{i+1}^{M},N)]{} & \mathrm{Hom}_R(M_{i+1},N) & \xrightarrow[\mathrm{Hom}_R(\partial_{i+2}^{M},N)]{} & \cdots .
\end{array}$$

Proposition VI.4.5. *Let R be a commutative ring. Let N be an R-module, and let $F_\bullet \colon M_\bullet \to M'_\bullet$ be a chain map of R-complexes. Then the following sequences are chain maps: $F_\bullet \otimes_R N$, $N \otimes_R F_\bullet$, $\mathrm{Hom}_R(N, F_\bullet)$, and $\mathrm{Hom}_R(F_\bullet, N)$.*

Proof. We need to check that the appropriate ladder diagrams from Definition VI.4.4 commute. In each case, this is a consequence of the appropriate functoriality, since $F_\bullet$ is a chain map. For instance, for $F_\bullet \otimes_R N$, we have

$$\begin{aligned}
(F_{i-1} \otimes_R N)(\partial_i^M \otimes_R N) &= (F_{i-1}\partial_i^M) \otimes_R N = (\partial_i^{M'} F_i) \otimes_R N \\
&= (\partial_i^{M'} \otimes_R N)(F_i \otimes_R N)
\end{aligned}$$

by Proposition II.2.1(b). (One can also check this by hand using simple tensors.) The other three sequences are checked similarly; see Exercise VI.4.12. □

Example VI.4.6. We consider the chain map over $\mathbb{Z}/12\mathbb{Z}$ from Example VI.1.2:

$$\begin{array}{ccccccccc}
M_\bullet & \cdots & \xrightarrow{6} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{4} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{6} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{4} \cdots \\
{\scriptstyle F_\bullet}\downarrow & & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 3} & & \downarrow{\scriptstyle 2} & \\
N_\bullet & \cdots & \xrightarrow{4} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{6} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{4} & \mathbb{Z}/12\mathbb{Z} & \xrightarrow{6} \cdots .
\end{array}$$

We first tensor with the module $N = (\mathbb{Z}/12\mathbb{Z})/3(\mathbb{Z}/12\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$ to obtain a chain map which is equivalent to the following one:

$$\begin{array}{ccccccccc}
M_\bullet \otimes \mathbb{Z}/3\mathbb{Z} & \cdots & \xrightarrow{6=0} & \mathbb{Z}/3\mathbb{Z} & \xrightarrow{4=1} & \mathbb{Z}/3\mathbb{Z} & \xrightarrow{6=0} & \mathbb{Z}/3\mathbb{Z} & \xrightarrow{4=1} \cdots \\
{\scriptstyle F_\bullet\otimes\mathbb{Z}/3\mathbb{Z}}\downarrow & & & \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle 3=0} & & \downarrow{\scriptstyle 2} & \\
N_\bullet \otimes \mathbb{Z}/3\mathbb{Z} & \cdots & \xrightarrow{4=0} & \mathbb{Z}/3\mathbb{Z} & \xrightarrow{6=0} & \mathbb{Z}/3\mathbb{Z} & \xrightarrow{4=1} & \mathbb{Z}/3\mathbb{Z} & \xrightarrow{6=0} \cdots .
\end{array}$$

We next tensor with the module $N' = (\mathbb{Z}/12\mathbb{Z})^2$ to obtain a chain map which is equivalent to the following one:

$$\begin{array}{ccccccccc}
M_\bullet \otimes (\mathbb{Z}/12\mathbb{Z})^2 & \cdots & \xrightarrow{\left(\begin{smallmatrix}6&0\\0&6\end{smallmatrix}\right)} & (\mathbb{Z}/12\mathbb{Z})^2 & \xrightarrow{\left(\begin{smallmatrix}4&0\\0&4\end{smallmatrix}\right)} & (\mathbb{Z}/12\mathbb{Z})^2 & \xrightarrow{\left(\begin{smallmatrix}6&0\\0&6\end{smallmatrix}\right)} & (\mathbb{Z}/12\mathbb{Z})^2 & \xrightarrow{\left(\begin{smallmatrix}4&0\\0&4\end{smallmatrix}\right)} \cdots \\
{\scriptstyle F_\bullet\otimes(\mathbb{Z}/12\mathbb{Z})^2}\downarrow & & & \downarrow{\scriptstyle \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}3&0\\0&3\end{smallmatrix}\right)} & & \downarrow{\scriptstyle \left(\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right)} & \\
N_\bullet \otimes (\mathbb{Z}/12\mathbb{Z})^2 & \cdots & \xrightarrow{\left(\begin{smallmatrix}4&0\\0&4\end{smallmatrix}\right)} & (\mathbb{Z}/12\mathbb{Z})^2 & \xrightarrow{\left(\begin{smallmatrix}6&0\\0&6\end{smallmatrix}\right)} & (\mathbb{Z}/12\mathbb{Z})^2 & \xrightarrow{\left(\begin{smallmatrix}4&0\\0&4\end{smallmatrix}\right)} & (\mathbb{Z}/12\mathbb{Z})^2 & \xrightarrow{\left(\begin{smallmatrix}6&0\\0&6\end{smallmatrix}\right)} \cdots .
\end{array}$$

Exercises.

Exercise VI.4.7. Complete the proof of Proposition VI.4.2.

Exercise VI.4.8. Let R be a commutative ring. Let $M_\bullet$ be an R-complex, and let $g\colon N \to N'$ be an R-module isomorphism. Prove that the following chain maps are isomorphisms:

$$\begin{aligned} M_\bullet \otimes_R g &\colon M_\bullet \otimes_R N \to M_\bullet \otimes_R N' \\ g \otimes_R M_\bullet &\colon N \otimes_R M_\bullet \to N' \otimes_R M_\bullet \\ \operatorname{Hom}_R(g, M_\bullet) &\colon \operatorname{Hom}_R(N', M_\bullet) \to \operatorname{Hom}_R(N, M_\bullet) \\ \operatorname{Hom}_R(M_\bullet, g) &\colon \operatorname{Hom}_R(M_\bullet, N) \to \operatorname{Hom}_R(M_\bullet, N'). \end{aligned}$$

Exercise VI.4.9. Let R be a commutative ring. Let $M_\bullet$ be an R-complex, and let $g\colon N \to N'$ and $g'\colon N' \to N''$ be R-module homomorphisms. Verify the following equalities

$$\begin{aligned} M_\bullet \otimes_R (g'g) &= (M_\bullet \otimes_R g')(M_\bullet \otimes_R g) \\ (g'g) \otimes_R M_\bullet &= (g' \otimes_R M_\bullet)(g \otimes_R M_\bullet) \\ \operatorname{Hom}_R(g'g, M_\bullet) &= \operatorname{Hom}_R(g', M_\bullet) \operatorname{Hom}_R(g, M_\bullet) \\ \operatorname{Hom}_R(M_\bullet, g'g) &= \operatorname{Hom}_R(M_\bullet, g') \operatorname{Hom}_R(M_\bullet, g) \end{aligned}$$

and rewrite each one in terms of a commutative diagram.

Exercise VI.4.10. Continue with the notation of Example VI.4.3 and compute the following chain maps: $g \otimes_R M_\bullet$ and $\operatorname{Hom}_R(g, M_\bullet)$ and $\operatorname{Hom}_R(M_\bullet, g)$.

Exercise VI.4.11. Let R be a commutative ring. Let $M_\bullet$ be an R-complex, and let N be an R-module. Let $r \in R$, and let $\mu_r^N\colon N \to N$ be given by $n \mapsto rn$. Prove that each of the following maps is given by multiplication by r:

$$\begin{aligned} M_\bullet \otimes_R \mu_r^N &\colon M_\bullet \otimes_R N \to M_\bullet \otimes_R N \\ \mu_r^N \otimes_R M_\bullet &\colon N \otimes_R M_\bullet \to N \otimes_R M_\bullet \\ \operatorname{Hom}_R(\mu_r^N, M_\bullet) &\colon \operatorname{Hom}_R(N, M_\bullet) \to \operatorname{Hom}_R(N, M_\bullet) \\ \operatorname{Hom}_R(M_\bullet, \mu_r^N) &\colon \operatorname{Hom}_R(M_\bullet, N) \to \operatorname{Hom}_R(M_\bullet, N). \end{aligned}$$

Exercise VI.4.12. Complete the proof of Proposition VI.4.5.

Exercise VI.4.13. Let R be a commutative ring. Let $M_\bullet$ and $M'_\bullet$ be R-complexes, and let N be an R-module. Prove that, if $F_\bullet\colon M_\bullet \to M'_\bullet$ is an isomorphism, then so are the following:

$$\begin{aligned} F_\bullet \otimes_R N &\colon M_\bullet \otimes_R N \to M'_\bullet \otimes_R N \\ N \otimes_R F_\bullet &\colon N \otimes_R M_\bullet \to N \otimes_R M'_\bullet \\ \operatorname{Hom}_R(N, F_\bullet) &\colon \operatorname{Hom}_R(N, M_\bullet) \to \operatorname{Hom}_R(N, M'_\bullet) \\ \operatorname{Hom}_R(F_\bullet, N) &\colon \operatorname{Hom}_R(M_\bullet, N) \to \operatorname{Hom}_R(M'_\bullet, N). \end{aligned}$$

Exercise VI.4.14. Let R be a commutative ring, and let N be an R-module. Let $F_\bullet\colon M_\bullet \to M'_\bullet$ and $F'_\bullet\colon M'_\bullet \to M''_\bullet$ be chain maps. Verify the following equalities

$$\begin{aligned}(F'_\bullet F_\bullet) \otimes_R N &= (F'_\bullet \otimes_R N)(F_\bullet \otimes_R N)\\ N \otimes_R (F'_\bullet F_\bullet) &= (N \otimes_R F'_\bullet)(N \otimes_R F_\bullet)\\ \operatorname{Hom}_R(N, F'_\bullet F_\bullet) &= \operatorname{Hom}_R(N, F'_\bullet)\operatorname{Hom}_R(N, F_\bullet)\\ \operatorname{Hom}_R(F'_\bullet F_\bullet, N) &= \operatorname{Hom}_R(F_\bullet, N)\operatorname{Hom}_R(F'_\bullet, N)\end{aligned}$$

and rewrite each one in terms of a commutative diagram.

Exercise VI.4.15. Continue with the notation of Example VI.4.6. Compute the following chain maps: $N \otimes_R F_\bullet$ and $N' \otimes_R F_\bullet$ and $\operatorname{Hom}_R(N, F_\bullet)$ and $\operatorname{Hom}_R(N', F_\bullet)$ and $\operatorname{Hom}_R(F_\bullet, N)$ and $\operatorname{Hom}_R(F_\bullet, N')$.

Exercise VI.4.16. Let R be a commutative ring. Let $M_\bullet$ be an R-complex, and let N be an R-module. Let $r \in R$, and let $\mu^M_\bullet\colon M_\bullet \to M_\bullet$ be given by $m \mapsto rm$. Prove that each of the following maps is given by multiplication by r:

$$\begin{aligned}\mu^M_\bullet \otimes_R N&\colon M_\bullet \otimes_R N \to M_\bullet \otimes_R N\\ N \otimes_R \mu^M_\bullet&\colon N \otimes_R M_\bullet \to N \otimes_R M_\bullet\\ \operatorname{Hom}_R(N, \mu^M_\bullet)&\colon \operatorname{Hom}_R(N, M_\bullet) \to \operatorname{Hom}_R(N, M_\bullet)\\ \operatorname{Hom}_R(\mu^M_\bullet, N)&\colon \operatorname{Hom}_R(M_\bullet, N) \to \operatorname{Hom}_R(M_\bullet, N)\end{aligned}$$

Exercise VI.4.17. Let R be a commutative ring. Let $F_\bullet\colon M_\bullet \to M'_\bullet$ be a chain map, and let $g\colon N \to N'$ be an R-module homomorphism. Verify the following equalities

$$\begin{aligned}(M'_\bullet \otimes_R g)(F_\bullet \otimes_R N) &= (F_\bullet \otimes_R N')(M_\bullet \otimes_R g)\\ (g \otimes_R M'_\bullet)(N \otimes_R F_\bullet) &= (N' \otimes_R F_\bullet)(g \otimes_R M_\bullet)\\ \operatorname{Hom}_R(g, M'_\bullet)\operatorname{Hom}_R(N', F_\bullet) &= \operatorname{Hom}_R(N, F_\bullet)\operatorname{Hom}_R(g, M_\bullet)\\ \operatorname{Hom}_R(F_\bullet, N')\operatorname{Hom}_R(M'_\bullet, g) &= \operatorname{Hom}_R(M_\bullet, g)\operatorname{Hom}_R(F_\bullet, N)\end{aligned}$$

and rewrite each one in terms of a commutative diagram.

VI.5. Ext-maps via projective resolutions

If M is an R-module, then the operator $\operatorname{Hom}_R(M, -)$ is a functor. In particular, this means that, not only does it transform modules to modules, but it also transforms maps to maps. Similar comments hold for the operators $\operatorname{Hom}_R(-, M)$ and $M \otimes_R -$ and $- \otimes_R M$. We have seen how the Ext and Tor operators transform modules to modules. The point of the next three sections is to show how they transform maps to maps.

Definition VI.5.1. Let R be a commutative ring. Let M be an R-module, and let $P^+_\bullet$ be an R-projective resolution of M. Let $g\colon N \to N'$ be an R-module homomorphism, and consider the chain map

$$\operatorname{Hom}_R(P_\bullet, g)\colon \operatorname{Hom}_R(P_\bullet, N) \to \operatorname{Hom}_R(P_\bullet, N')$$

from Definition VI.4.1(d) and Proposition VI.4.2. Since this is a chain map, it induces R-module homomorphisms on corresponding homology modules

$$\mathrm{H}_{-i}(\mathrm{Hom}_R(P_\bullet, g))\colon \underbrace{\mathrm{H}_{-i}(\mathrm{Hom}_R(P_\bullet, N))}_{\mathrm{Ext}^i_R(M,N)} \to \underbrace{\mathrm{H}_{-i}(\mathrm{Hom}_R(P_\bullet, N'))}_{\mathrm{Ext}^i_R(M,N')}$$

so we set

$$\mathrm{Ext}^i_R(M, g) = \mathrm{H}_{-i}(\mathrm{Hom}_R(P_\bullet, g))\colon \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(M, N').$$

In general, these maps are a pain to compute. However, the next example is always a winner.

Example VI.5.2. Let R be a commutative ring. Let M and N be R-modules, and let $P^+_\bullet$ be an R-projective resolution of M. Let $r \in R$, and let $\mu^N_r\colon N \to N$ be given by $n \mapsto rn$. We claim that the induced map

$$\mathrm{Ext}^i_R(M, \mu^N_r)\colon \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(M, N)$$

is given by multiplication by r. Indeed, Exercise VI.4.11 shows that the chain map

$$\mathrm{Hom}_R(P_\bullet, \mu^N_r)\colon \mathrm{Hom}_R(P_\bullet, N)) \to \mathrm{Hom}_R(P_\bullet, N))$$

is given by multiplication by r, so Exercise VI.1.6 shows that the induced map

$$\mathrm{H}_{-i}(\mathrm{Hom}_R(P_\bullet, \mu^N_r))\colon \underbrace{\mathrm{H}_{-i}(\mathrm{Hom}_R(P_\bullet, N))}_{\mathrm{Ext}^i_R(M,N)} \to \underbrace{\mathrm{H}_{-i}(\mathrm{Hom}_R(P_\bullet, N))}_{\mathrm{Ext}^i_R(M,N)}$$

is also given by multiplication by r, as claimed.

In particular, the special case $r = 1$ shows that

$$\mathrm{Ext}^i_R(M, \mathbb{1}_N) = \mathbb{1}_{\mathrm{Ext}^i_R(M,N)}\colon \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(M, N)$$

for all $i \in \mathbb{Z}$.

Here is a result that we do not have time to prove.

Theorem VI.5.3. *Let R be a commutative ring. Let M be an R-module, and let $g\colon N \to N'$ be an R-module homomorphism. For each integer i, the map*

$$\mathrm{Ext}^i_R(M, g)\colon \mathrm{Ext}^i_R(M, N) \to \mathrm{Ext}^i_R(M, N')$$

is independent of the choice of projective resolution of M. In other words, if $P^+_\bullet$ and $Q^+_\bullet$ are R-projective resolutions of M then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_{-i}(\mathrm{Hom}_R(P_\bullet, N)) & \xrightarrow{\mathrm{H}_{-i}(\mathrm{Hom}_R(P_\bullet, g))} & \mathrm{H}_{-i}(\mathrm{Hom}_R(P_\bullet, N')) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{H}_{-i}(\mathrm{Hom}_R(Q_\bullet, N)) & \xrightarrow{\mathrm{H}_{-i}(\mathrm{Hom}_R(Q_\bullet, g))} & \mathrm{H}_{-i}(\mathrm{Hom}_R(Q_\bullet, N')) \end{array}$$

where the unspecified vertical isomorphisms are from Theorem IV.3.5.

Here is the functoriality of this version of Ext.

Proposition VI.5.4. *Let R be a commutative ring. Let M be an R-module, and let $g\colon N \to N'$ and $g'\colon N' \to N''$ be R-module homomorphisms. Then the following diagram commutes*

$$\begin{array}{ccc} \operatorname{Ext}^i_R(M,N) & \xrightarrow{\operatorname{Ext}^i_R(M,g)} & \operatorname{Ext}^i_R(M,N') \\ & \searrow^{\operatorname{Ext}^i_R(M,g'g)} & \downarrow{\scriptstyle \operatorname{Ext}^i_R(M,g')} \\ & & \operatorname{Ext}^i_R(M,N'') \end{array}$$

for each integer i, that is, we have $\operatorname{Ext}^i_R(M,g'g) = \operatorname{Ext}^i_R(M,g')\operatorname{Ext}^i_R(M,g)$.

Proof. Exercise VI.4.9 implies that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Hom}_R(P_\bullet,N) & \xrightarrow{\operatorname{Hom}_R(P_\bullet,g)} & \operatorname{Hom}_R(P_\bullet,N') \\ & \searrow^{\operatorname{Hom}_R(P_\bullet,g'g)} & \downarrow{\scriptstyle \operatorname{Hom}_R(P_\bullet,g')} \\ & & \operatorname{Hom}_R(P_\bullet,N''). \end{array}$$

Hence, Exercise VI.1.7(c) implies that the next diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_{-i}(\operatorname{Hom}_R(P_\bullet,N)) & \xrightarrow{\mathrm{H}_{-i}(\operatorname{Hom}_R(P_\bullet,g))} & \mathrm{H}_{-i}(\operatorname{Hom}_R(P_\bullet,N')) \\ & \searrow^{\mathrm{H}_{-i}(\operatorname{Hom}_R(P_\bullet,g'g))} & \downarrow{\scriptstyle \mathrm{H}_{-i}(\operatorname{Hom}_R(P_\bullet,g'))} \\ & & \mathrm{H}_{-i}(\operatorname{Hom}_R(P_\bullet,N'')). \end{array}$$

By definition, this is the desired diagram. □

Definition VI.5.5. Let R be a commutative ring. Let $f\colon M \to M'$ be an R-module homomorphism, and let N be an R-module. Let $P^+_\bullet$ be an R-projective resolution of M, and let $Q^+_\bullet$ be an R-projective resolution of M'. Let $F_\bullet\colon P_\bullet \to Q_\bullet$ be a lifting of f, that is, a chain map as in Proposition VI.3.2; see Example VI.3.5. Consider the chain map

$$\operatorname{Hom}_R(F_\bullet,N)\colon \operatorname{Hom}_R(Q_\bullet,N) \to \operatorname{Hom}_R(P_\bullet,N)$$

from Definition VI.4.4(d) and Proposition VI.4.5. Since this is a chain map, it induces R-module homomorphisms on corresponding homology modules

$$\mathrm{H}_{-i}(\operatorname{Hom}_R(F_\bullet,N))\colon \underbrace{\mathrm{H}_{-i}(\operatorname{Hom}_R(Q_\bullet,N))}_{\operatorname{Ext}^i_R(M',N)} \to \underbrace{\mathrm{H}_{-i}(\operatorname{Hom}_R(P_\bullet,N))}_{\operatorname{Ext}^i_R(M,N)}$$

so we set

$$\operatorname{Ext}^i_R(f,N) = \mathrm{H}_{-i}(\operatorname{Hom}_R(F_\bullet,N))\colon \operatorname{Ext}^i_R(M',N) \to \operatorname{Ext}^i_R(M,N).$$

In general, these maps are a pain to compute. However, the next example is always a winner.

Example VI.5.6. Let R be a commutative ring. Let M and N be R-modules, and let $P^+_\bullet$ be an R-projective resolution of M. Let $r \in R$, and let $\mu^M_r\colon M \to M$ be given by $m \mapsto rm$. We claim that the induced map

$$\operatorname{Ext}^i_R(\mu^M_r,N)\colon \operatorname{Ext}^i_R(M,N) \to \operatorname{Ext}^i_R(M,N)$$

is given by multiplication by r. Indeed, Exercise VI.3.9 shows that the multiplication map

$$\mu_{\bullet}^{P}\colon P_{\bullet} \to P_{\bullet} \qquad \text{given by} \qquad p \mapsto rp$$

is a lifting of μ_r^M. Exercise VI.4.16 shows that the chain map

$$\operatorname{Hom}_R(\mu_{\bullet}^{P}, N)\colon \operatorname{Hom}_R(P_{\bullet}, N)) \to \operatorname{Hom}_R(P_{\bullet}, N))$$

is given by multiplication by r, so Exercise VI.1.6 shows that the induced map

$$\operatorname{H}_{-i}(\operatorname{Hom}_R(\mu_{\bullet}^{P}, N))\colon \underbrace{\operatorname{H}_{-i}(\operatorname{Hom}_R(P_{\bullet}, N))}_{\operatorname{Ext}_R^i(M,N)} \to \underbrace{\operatorname{H}_{-i}(\operatorname{Hom}_R(P_{\bullet}, N))}_{\operatorname{Ext}_R^i(M,N)}$$

is also given by multiplication by r, as claimed.

In particular, the special case $r = 1$ shows that

$$\operatorname{Ext}_R^i(\mathbb{1}_M, N) = \mathbb{1}_{\operatorname{Ext}_R^i(M,N)}\colon \operatorname{Ext}_R^i(M, N) \to \operatorname{Ext}_R^i(M, N)$$

for all $i \in \mathbb{Z}$.

Here is a result that we do not have time to prove.

Theorem VI.5.7. *Let R be a commutative ring. Let $f\colon M \to M'$ be an R-module homomorphism, and let N be an R-module. For each integer i, the map*

$$\operatorname{Ext}_R^i(f, N)\colon \operatorname{Ext}_R^i(M', N) \to \operatorname{Ext}_R^i(M, N)$$

is independent of (1) the choice of projective resolutions of M and M' and (2) the choice of lifting of f. In other words, assume that $P_{\bullet}^{+}$ and $\widetilde{P}_{\bullet}^{+}$ are R-projective resolutions of M, that $Q_{\bullet}^{+}$ and $\widetilde{Q}_{\bullet}^{+}$ are R-projective resolutions of M', and that $F_{\bullet}\colon P_{\bullet} \to Q_{\bullet}$ and $\widetilde{F}_{\bullet}\colon \widetilde{P}_{\bullet} \to \widetilde{Q}_{\bullet}$ are liftings of f; then there is a commutative diagram where the unspecified vertical isomorphisms are from Theorem IV.3.5

$$\begin{array}{ccc}
\operatorname{H}_{-i}(\operatorname{Hom}_R(Q_{\bullet}, N)) & \xrightarrow{\operatorname{H}_{-i}(\operatorname{Hom}_R(F_{\bullet}, N))} & \operatorname{H}_{-i}(\operatorname{Hom}_R(P_{\bullet}, N)) \\
\cong\downarrow & & \cong\downarrow \\
\operatorname{H}_{-i}(\operatorname{Hom}_R(\widetilde{Q}_{\bullet}, N)) & \xrightarrow{\operatorname{H}_{-i}(\operatorname{Hom}_R(\widetilde{F}_{\bullet}, N))} & \operatorname{H}_{-i}(\operatorname{Hom}_R(\widetilde{P}_{\bullet}, N)).
\end{array}$$

Here is the functoriality of this version of Ext.

Proposition VI.5.8. *Let R be a commutative ring, and let N be an R-module. Let $f\colon M \to M'$ and $f'\colon M' \to M''$ be R-module homomorphisms. Then the following diagram commutes*

$$\begin{array}{ccc}
\operatorname{Ext}_R^i(M'', N) & \xrightarrow{\operatorname{Ext}_R^i(f', N)} & \operatorname{Ext}_R^i(M', N) \\
& \searrow^{\operatorname{Ext}_R^i(f'f, N)} & \downarrow{\scriptstyle \operatorname{Ext}_R^i(f, N)} \\
& & \operatorname{Ext}_R^i(M, N)
\end{array}$$

for each integer i, that is, we have $\operatorname{Ext}_R^i(f'f, N) = \operatorname{Ext}_R^i(f, N) \operatorname{Ext}_R^i(f', N)$.

Proof. Let $P_{\bullet}^{+}$ be a projective resolution of M. Let $\widetilde{P}_{\bullet}^{+}$ be a projective resolution of M'. Let $\widehat{P}_{\bullet}^{+}$ be a projective resolution of M''. Let $F_{\bullet}\colon P_{\bullet} \to \widetilde{P}_{\bullet}$ be a lifting of f, and let $F'_{\bullet}\colon \widetilde{P}_{\bullet} \to \widehat{P}_{\bullet}$ be a lifting of f'. It is straightforward to show that the chain map $F'_{\bullet}F_{\bullet}\colon P_{\bullet} \to \widehat{P}_{\bullet}$ is a lifting of the composition $f'f$.

Exercise VI.4.14 implies that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_R(\widehat{P}_\bullet, N) & \xrightarrow{\mathrm{Hom}_R(F'_\bullet, N)} & \mathrm{Hom}_R(\widetilde{P}_\bullet, N) \\ & \searrow^{\mathrm{Hom}_R(F'_\bullet F_\bullet, N)} & \downarrow{\scriptstyle \mathrm{Hom}_R(F_\bullet, N)} \\ & & \mathrm{Hom}_R(P_\bullet, N). \end{array}$$

Hence, Exercise VI.1.7(c) implies that the next diagram commutes:

$$\begin{array}{ccc} \mathrm{H}_{-i}(\mathrm{Hom}_R(\widehat{P}_\bullet, N)) & \xrightarrow{\mathrm{H}_{-i}(\mathrm{Hom}_R(F'_\bullet, N))} & \mathrm{H}_{-i}(\mathrm{Hom}_R(\widetilde{P}_\bullet, N)) \\ & \searrow^{\mathrm{H}_{-i}(\mathrm{Hom}_R(F'_\bullet F_\bullet, N))} & \downarrow{\scriptstyle \mathrm{H}_{-i}(\mathrm{Hom}_R(F_\bullet, N))} \\ & & \mathrm{H}_{-i}(\mathrm{Hom}_R(P_\bullet, N)). \end{array}$$

By definition, this is the desired diagram. □

Remark VI.5.9. Let R be a commutative ring. As a consequence of Propositions VI.5.4 and VI.5.8, we have $\mathrm{Ext}^i_R(0, N) = 0$ and $\mathrm{Ext}^i_R(M, 0) = 0$, whenever 0 is a zero-map. See Remark V.1.2.

Exercises.

Exercise VI.5.10. Let R be a commutative ring. Let M be an R-module, and let $g\colon N \to N'$ be an R-module homomorphism.

(a) Without using Proposition VI.5.4, prove that, if $g = 0$, then $\mathrm{Ext}^i_R(M, g) = 0$ for all $i \in \mathbb{Z}$.
(b) Prove that, if g is an isomorphism, then $\mathrm{Ext}^i_R(M, g)$ is an isomorphism for all $i \in \mathbb{Z}$.

Exercise VI.5.11. Let R be a commutative ring. Let $f\colon M \to M'$ be an R-module homomorphism, and let N be an R-module.

(a) Without using Proposition VI.5.8, prove that, if $f = 0$, then $\mathrm{Ext}^i_R(f, N) = 0$ for all indices $i \in \mathbb{Z}$.
(b) Prove that, if f is an isomorphism, then $\mathrm{Ext}^i_R(f, N)$ is an isomorphism for all indices $i \in \mathbb{Z}$.

Exercise VI.5.12. Let R be a commutative ring. Let $f\colon M \to M'$ and $g\colon N \to N'$ be R-module homomorphisms. Prove that the following diagram commutes for each index $i \in \mathbb{Z}$:

$$\begin{array}{ccc} \mathrm{Ext}^i_R(M', N) & \xrightarrow{\mathrm{Ext}^i_R(M', g)} & \mathrm{Ext}^i_R(M', N') \\ {\scriptstyle \mathrm{Ext}^i_R(f, N)}\downarrow & & \downarrow{\scriptstyle \mathrm{Ext}^i_R(f, N')} \\ \mathrm{Ext}^i_R(M, N) & \xrightarrow{\mathrm{Ext}^i_R(M, g)} & \mathrm{Ext}^i_R(M, N'). \end{array}$$

VI.6. Ext-maps via injective resolutions

In this section, we show how the Ext-maps from Section VI.5 can be computed via injective resolutions.

Remark VI.6.1. Let R be a commutative ring. Let N be an R-module, and let ${}^{+}J_{\bullet}$ be an R-injective resolution of N. Let $f\colon M \to M'$ be an R-module homomorphism, and consider the chain map

$$\operatorname{Hom}_R(f, J_\bullet)\colon \operatorname{Hom}_R(M'J_\bullet) \to \operatorname{Hom}_R(M, J_\bullet)$$

from Definition VI.4.1(c) and Proposition VI.4.2. Since this is a chain map, it induces R-module homomorphisms on corresponding homology modules

$$\operatorname{H}_{-i}(\operatorname{Hom}_R(f, J_\bullet))\colon \underbrace{\operatorname{H}_{-i}(\operatorname{Hom}_R(M'J_\bullet))}_{\operatorname{Ext}^i_R(M',N)} \to \underbrace{\operatorname{H}_{-i}(\operatorname{Hom}_R(M, J_\bullet))}_{\operatorname{Ext}^i_R(M,N)}.$$

Here is a result that we do not have time to prove.

Theorem VI.6.2. *Let R be a commutative ring. Let N be an R-module, and let $f\colon M \to M'$ be an R-module homomorphism. For each integer i, the map $\operatorname{Ext}^i_R(M', N) \to \operatorname{Ext}^i_R(M, N)$ from Remark VI.6.1 is independent of the choice of injective resolution of N, and it is equivalent to the map $\operatorname{Ext}^i_R(f, N)$ from Definition VI.5.5. That is, if ${}^{+}I_{\bullet}$ is an R-injective resolution of N then there is a commutative diagram*

$$\begin{array}{ccc} \operatorname{H}_{-i}(\operatorname{Hom}_R(M', I_\bullet)) & \xrightarrow{\operatorname{H}_{-i}(\operatorname{Hom}_R(f, I_\bullet))} & \operatorname{H}_{-i}(\operatorname{Hom}_R(M, I_\bullet)) \\ \cong\downarrow & & \cong\downarrow \\ \operatorname{Ext}^i_R(M', N) & \xrightarrow{\operatorname{Ext}^i_R(f,N)} & \operatorname{Ext}^i_R(M, N) \end{array}$$

where the unspecified vertical isomorphisms are from Theorem IV.3.10.

Remark VI.6.3. Let R be a commutative ring. Let $g\colon N \to N'$ be an R-module homomorphism, and let M be an R-module. Let ${}^{+}I_{\bullet}$ be an R-injective resolution of N, and let ${}^{+}J_{\bullet}$ be an R-injective resolution of N'. Let $G_\bullet\colon I_\bullet \to J_\bullet$ be a lifting of g, that is, a chain map as in Proposition VI.3.6; see Example VI.3.7. Consider the chain map

$$\operatorname{Hom}_R(M, G_\bullet)\colon \operatorname{Hom}_R(M, I_\bullet) \to \operatorname{Hom}_R(M, J_\bullet)$$

from Definition VI.4.4(c) and Proposition VI.4.5. Since this is a chain map, it induces R-module homomorphisms on corresponding homology modules

$$\operatorname{H}_{-i}(\operatorname{Hom}_R(M, G_\bullet))\colon \underbrace{\operatorname{H}_{-i}(\operatorname{Hom}_R(M, I_\bullet))}_{\operatorname{Ext}^i_R(M,N)} \to \underbrace{\operatorname{H}_{-i}(\operatorname{Hom}_R(M, J_\bullet))}_{\operatorname{Ext}^i_R(M,N')}.$$

Here is another result that we do not have time to prove.

Theorem VI.6.4. *Let R be a commutative ring. Let $g\colon N \to N'$ be an R-module homomorphism, and let M be an R-module. For each integer i, the homomorphism $\operatorname{Ext}^i_R(M, N) \to \operatorname{Ext}^i_R(M, N')$ from Remark VI.6.3 is independent of the choice of injective resolutions of N and N'; it is independent of the lifting of g; and it is equivalent to the map $\operatorname{Ext}^i_R(M, g)$ from Definition VI.5.1. That is, let ${}^{+}I_{\bullet}$ be an R-injective resolution of N, and let ${}^{+}J_{\bullet}$ be an R-injective resolution of N'. If*

$G_\bullet\colon I_\bullet \to J_\bullet$ is a lifting of g, then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_{-i}(\mathrm{Hom}_R(M, I_\bullet)) & \xrightarrow{\mathrm{H}_{-i}(\mathrm{Hom}_R(M,G_\bullet))} & \mathrm{H}_{-i}(\mathrm{Hom}_R(M, J_\bullet)) \\ \cong\downarrow & & \cong\downarrow \\ \mathrm{Ext}^i_R(M,N) & \xrightarrow{\mathrm{Ext}^i_R(M,g)} & \mathrm{Ext}^i_R(M,N') \end{array}$$

where the unspecified vertical isomorphisms are from Theorem IV.3.10.

Exercises.

Exercise VI.6.5. Use the definitions of this section to give another verification of Example VI.5.2.

Exercise VI.6.6. Use the definitions of this section to give another proof of Proposition VI.5.4.

Exercise VI.6.7. Use the definitions of this section to give another verification of Example VI.5.6.

Exercise VI.6.8. Use the definitions of this section to give another proof of Proposition VI.5.8.

Exercise VI.6.9. Use the definitions of this section to give another solution to Exercise VI.5.10.

Exercise VI.6.10. Use the definitions of this section to give another solution to Exercise VI.5.11.

Exercise VI.6.11. Use the definitions of this section to give another solution to Exercise VI.5.12.

VI.7. Tor-maps

Definition VI.7.1. Let R be a commutative ring. Let M be an R-module, and let $P^+_\bullet$ be an R-projective resolution of M. Let $g\colon N \to N'$ be an R-module homomorphism, and consider the chain map

$$P_\bullet \otimes_R g\colon P_\bullet \otimes_R N \to P_\bullet \otimes_R N'$$

from Definition VI.4.1(a) and Proposition VI.4.2. Since this is a chain map, it induces R-module homomorphisms on corresponding homology modules

$$\mathrm{H}_i(P_\bullet \otimes_R g)\colon \underbrace{\mathrm{H}_i(P_\bullet \otimes_R N)}_{\mathrm{Tor}^R_i(M,N)} \to \underbrace{\mathrm{H}_i(P_\bullet \otimes_R N')}_{\mathrm{Tor}^R_i(M,N')}$$

so we set

$$\mathrm{Tor}^R_i(M,g) = \mathrm{H}_i(P_\bullet \otimes_R g)\colon \mathrm{Tor}^R_i(M,N) \to \mathrm{Tor}^R_i(M,N').$$

In general, these maps are a pain to compute. However, the next example is always a winner.

Example VI.7.2. Let R be a commutative ring. Let M and N be R-modules, and let $P^+_\bullet$ be an R-projective resolution of M. Let $r \in R$, and let $\mu^N_r : N \to N$ be given by $n \mapsto rn$. The induced map

$$\mathrm{Tor}^R_i(M, \mu^N_r)\colon \mathrm{Tor}^R_i(M,N) \to \mathrm{Tor}^R_i(M,N)$$

is given by multiplication by r. In particular, the special case $r = 1$ shows that

$$\operatorname{Tor}_i^R(M, \mathbb{1}_N) = \mathbb{1}_{\operatorname{Tor}_i^R(M,N)} \colon \operatorname{Tor}_i^R(M, N) \to \operatorname{Tor}_i^R(M, N)$$

for all $i \in \mathbb{Z}$.

Here is a result that we do not have time to prove.

Theorem VI.7.3. *Let R be a commutative ring. Let M be an R-module, and let $g \colon N \to N'$ be an R-module homomorphism. For each integer i, the map*

$$\operatorname{Tor}_i^R(M, g) \colon \operatorname{Tor}_i^R(M, N) \to \operatorname{Tor}_i^R(M, N')$$

is independent of the choice of projective resolution of M. In other words, if $P_\bullet^+$ and $Q_\bullet^+$ are R-projective resolutions of M then there is a commutative diagram

$$\begin{array}{ccc} \operatorname{H}_i(P_\bullet \otimes_R N) & \xrightarrow{\operatorname{H}_i(P_\bullet \otimes_R g)} & \operatorname{H}_i(P_\bullet \otimes_R N') \\ \cong \downarrow & & \downarrow \cong \\ \operatorname{H}_i(Q_\bullet \otimes_R N) & \xrightarrow{\operatorname{H}_i(Q_\bullet \otimes_R g)} & \operatorname{H}_i(Q_\bullet \otimes_R N') \end{array}$$

where the unspecified vertical isomorphisms are from Theorem IV.4.4.

Here is the functoriality of this version of Tor.

Proposition VI.7.4. *Let R be a commutative ring. Let M be an R-module, and let $g \colon N \to N'$ and $g' \colon N' \to N''$ be R-module homomorphisms. Then the following diagram commutes*

$$\begin{array}{ccc} \operatorname{Tor}_i^R(M, N) & \xrightarrow{\operatorname{Tor}_i^R(M,g)} & \operatorname{Tor}_i^R(M, N') \\ & \searrow^{\operatorname{Tor}_i^R(M,g'g)} & \downarrow \operatorname{Tor}_i^R(M,g') \\ & & \operatorname{Tor}_i^R(M, N'') \end{array}$$

for each integer i, that is, we have $\operatorname{Tor}_i^R(M, g'g) = \operatorname{Tor}_i^R(M, g') \operatorname{Tor}_i^R(M, g)$.

PROOF. Exercise VI.7.17. □

Definition VI.7.5. Let R be a commutative ring. Let $f \colon M \to M'$ be an R-module homomorphism, and let N be an R-module. Let $P_\bullet^+$ be an R-projective resolution of M, and let $Q_\bullet^+$ be an R-projective resolution of M'. Let $F_\bullet \colon P_\bullet \to Q_\bullet$ be a lifting of f, that is, a chain map as in Proposition VI.3.2; see Example VI.3.5. Consider the chain map

$$F_\bullet \otimes_R N \colon P_\bullet \otimes_R N \to Q_\bullet \otimes_R N$$

from Definition VI.4.4(a) and Proposition VI.4.5. Since this is a chain map, it induces R-module homomorphisms on corresponding homology modules

$$\operatorname{H}_i(F_\bullet \otimes_R N) \colon \underbrace{\operatorname{H}_i(P_\bullet \otimes_R N)}_{\operatorname{Tor}_i^R(M,N)} \to \underbrace{\operatorname{H}_i(Q_\bullet \otimes_R N))}_{\operatorname{Tor}_i^R(M',N)}$$

so we set

$$\operatorname{Tor}_i^R(f, N) = \operatorname{H}_i(F_\bullet \otimes_R N) \colon \operatorname{Tor}_i^R(M, N) \to \operatorname{Tor}_i^R(M', N).$$

In general, these maps are a pain to compute. However, the next example is always a winner.

Example VI.7.6. Let R be a commutative ring. Let M and N be R-modules, and let $P_\bullet^+$ be an R-projective resolution of M. Let $r \in R$, and let $\mu_r^M \colon M \to M$ be given by $m \mapsto rm$. The induced map

$$\operatorname{Tor}_i^R(\mu_r^M, N) \colon \operatorname{Tor}_i^R(M,N) \to \operatorname{Tor}_i^R(M,N)$$

is given by multiplication by r. In particular, the special case $r = 1$ shows that

$$\operatorname{Tor}_i^R(\mathbb{1}_M, N) = \mathbb{1}_{\operatorname{Tor}_i^R(M,N)} \colon \operatorname{Tor}_i^R(M,N) \to \operatorname{Tor}_i^R(M,N)$$

for all $i \in \mathbb{Z}$. See Exercise VI.7.19.

Here is a result that we do not have time to prove.

Theorem VI.7.7. *Let R be a commutative ring. Let $f \colon M \to M'$ be an R-module homomorphism, and let N be an R-module. For each integer i, the map*

$$\operatorname{Tor}_i^R(f, N) \colon \operatorname{Tor}_i^R(M,N) \to \operatorname{Tor}_i^R(M,N)$$

is independent of (1) the choice of projective resolutions of M and M' and (2) the choice of lifting of f. In other words, assume that $P_\bullet^+$ and $\widetilde{P}_\bullet^+$ are R-projective resolutions of M, that $Q_\bullet^+$ and $\widetilde{Q}_\bullet^+$ are R-projective resolutions of M', and that $F_\bullet \colon P_\bullet \to Q_\bullet$ and $\widetilde{F}_\bullet \colon \widetilde{P}_\bullet \to \widetilde{Q}_\bullet$ are liftings of f; then there is a commutative diagram

$$\begin{array}{ccc} \operatorname{H}_i(P_\bullet \otimes_R N) & \xrightarrow{\operatorname{H}_i(F_\bullet \otimes_R N)} & \operatorname{H}_i(Q_\bullet \otimes_R N) \\ \cong \downarrow & & \cong \downarrow \\ \operatorname{H}_i(\widetilde{P}_\bullet \otimes_R N) & \xrightarrow{\operatorname{H}_i(\widetilde{F}_\bullet \otimes_R N)} & \operatorname{H}_i(\widetilde{Q}_\bullet \otimes_R N) \end{array}$$

where the unspecified vertical isomorphisms are from Theorem IV.4.4.

Here is the functoriality of this version of Tor.

Proposition VI.7.8. *Let R be a commutative ring, and let N be an R-module. Let $f \colon M \to M'$ and $f' \colon M' \to M''$ be R-module homomorphisms. Then the following diagram commutes*

$$\begin{array}{ccc} \operatorname{Tor}_i^R(M,N) & \xrightarrow{\operatorname{Tor}_i^R(f,N)} & \operatorname{Tor}_i^R(M',N) \\ & \searrow^{\operatorname{Tor}_i^R(f'f,N)} & \downarrow{\scriptstyle \operatorname{Tor}_i^R(f',N)} \\ & & \operatorname{Tor}_i^R(M'',N) \end{array}$$

for each integer i, that is, we have $\operatorname{Tor}_i^R(f'f, N) = \operatorname{Tor}_i^R(f', N) \operatorname{Tor}_i^R(f, N)$.

Proof. Exercise VI.7.20. □

Remark VI.7.9. Let R be a commutative ring. As a consequence of Propositions VI.7.4 and VI.7.8, we have $\operatorname{Tor}_i^R(0, N) = 0$ and $\operatorname{Tor}_i^R(M, 0) = 0$, whenever 0 is a zero-map. See Exercises VI.7.18 and VI.7.21.

Now, we show how the Tor-maps can be computed via projective resolutions on the other side.

Remark VI.7.10. Let R be a commutative ring. Let N be an R-module, and let $Q_\bullet^+$ be an R-projective resolution of N. Let $f\colon M \to M'$ be an R-module homomorphism, and consider the chain map

$$f \otimes_R Q_\bullet \colon M \otimes_R Q_\bullet \to M' \otimes_R Q_\bullet$$

from Definition VI.4.1(b) and Proposition VI.4.2. Since this is a chain map, it induces R-module homomorphisms on corresponding homology modules

$$\mathrm{H}_i(f \otimes_R Q_\bullet) \colon \underbrace{\mathrm{H}_i(M \otimes_R Q_\bullet)}_{\mathrm{Tor}_i^R(M,N)} \to \underbrace{\mathrm{H}_i(M' \otimes_R Q_\bullet)}_{\mathrm{Tor}_i^R(M',N)}.$$

Here is a result that we do not have time to prove.

Theorem VI.7.11. *Let R be a commutative ring. Let N be an R-module, and let $f\colon M \to M'$ be an R-module homomorphism. For each integer i, the map $\mathrm{Tor}_i^R(M,N) \to \mathrm{Tor}_i^R(M',N)$ from Remark VI.7.10 is independent of the choice of projective resolution of N, and it is equivalent to the map $\mathrm{Tor}_i^R(f,N)$ from Definition VI.7.5. That is, if $P_\bullet$ is an R-projective resolution of N then there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{H}_i(M \otimes_R P_\bullet) & \xrightarrow{\mathrm{H}_i(f \otimes_R P_\bullet)} & \mathrm{H}_i(M' \otimes_R P_\bullet) \\ \cong \downarrow & & \cong \downarrow \\ \mathrm{Tor}_i^R(M,N) & \xrightarrow{\mathrm{Tor}_i^R(f,N)} & \mathrm{Tor}_i^R(M',N) \end{array}$$

where the unspecified vertical isomorphisms are from Theorem IV.4.8.

Remark VI.7.12. Let R be a commutative ring. Let $g\colon N \to N'$ be an R-module homomorphism, and let M be an R-module. Let $P_\bullet^+$ be an R-projective resolution of N, and let $Q_\bullet^+$ be an R-projective resolution of N'. Let $G_\bullet\colon P_\bullet \to Q_\bullet$ be a lifting of g, that is, a chain map as in Proposition VI.3.2; see Example VI.3.5. Consider the chain map

$$M \otimes_R G_\bullet \colon M \otimes_R P_\bullet \to M \otimes_R Q_\bullet$$

from Definition VI.4.4(b) and Proposition VI.4.5. Since this is a chain map, it induces R-module homomorphisms on corresponding homology modules

$$\mathrm{H}_i(M \otimes_R G_\bullet) \colon \underbrace{\mathrm{H}_i(M \otimes_R P_\bullet)}_{\mathrm{Tor}_i^R(M,N)} \to \underbrace{\mathrm{H}_i(M \otimes_R Q_\bullet)}_{\mathrm{Tor}_i^R(M,N')}.$$

Here is another result that we do not have time to prove.

Theorem VI.7.13. *Let R be a commutative ring. Let $g\colon N \to N'$ be an R-module homomorphism, and let M be an R-module. For each integer i, the map $\mathrm{Tor}_i^R(M,N) \to \mathrm{Tor}_i^R(M,N')$ from Remark VI.7.12 is independent of the choice of projective resolutions of N and N'; it is independent of the lifting of g; and it is equivalent to the map $\mathrm{Tor}_i^R(M,g)$ from Definition VI.7.1. That is, let $P_\bullet^+$ be an R-projective resolution of N, and let $Q_\bullet^+$ be an R-projective resolution of N'. If*

$G_\bullet \colon P_\bullet \to Q_\bullet$ is a lifting of g, then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{H}_i(M \otimes_R P_\bullet) & \xrightarrow{\mathrm{H}_i(M\otimes_R G_\bullet)} & \mathrm{H}_i(M \otimes_R Q_\bullet) \\ \cong\downarrow & & \cong\downarrow \\ \mathrm{Tor}_i^R(M,N) & \xrightarrow{\mathrm{Tor}_i^R(M,g)} & \mathrm{Tor}_i^R(M,N') \end{array}$$

where the unspecified vertical isomorphisms are from Theorem IV.4.8.

Lemma VI.7.14. *Let R be a commutative ring, and let M and N be R-modules. Then $\mathrm{Ann}_R(M) \cup \mathrm{Ann}_R(N) \subseteq \mathrm{Ann}_R(\mathrm{Tor}_i^R(M,N))$ for all i.*

PROOF. Exercise. □

Remark VI.7.15. Let R be a commutative ring, and let M and N be R-modules. Let $I, J \subseteq R$ be ideals such that $IM = 0$ and $JN = 0$. Lemma V.5.8 implies that $(I+J)\,\mathrm{Tor}_i^R(M,N) = 0$. Because of this, Remark I.5.10 implies that $\mathrm{Tor}_i^R(M,N)$ has the structure of an $R/(I+J)$-module, the structure of an R/I-module, and the structure of an R/J-module via the formula $\overline{r}z = rz$. Furthermore, $\mathrm{Tor}_i^R(M,N)$ is finitely generated over R if and only if it is finitely generated over R/I, and similarly over R/J and $R/(I+J)$.

Exercises.

Exercise VI.7.16. Verify the facts in Example VI.7.2.

Exercise VI.7.17. Prove Proposition VI.7.4.

Exercise VI.7.18. Let R be a commutative ring. Let M be an R-module, and let $g \colon N \to N'$ be an R-module homomorphism.

(a) Without using Proposition VI.7.4, prove that, if $g = 0$, then $\mathrm{Tor}_i^R(M,g) = 0$ for all indices $i \in \mathbb{Z}$.
(b) Now, using Proposition VI.7.4, prove that, if $g = 0$, then $\mathrm{Tor}_i^R(M,g) = 0$ for all indices $i \in \mathbb{Z}$.
(c) Prove that, if g is an isomorphism, then $\mathrm{Tor}_i^R(M,g)$ is an isomorphism for all indices $i \in \mathbb{Z}$.

Exercise VI.7.19. Verify the facts in Example VI.7.6.

Exercise VI.7.20. Prove Proposition VI.7.8.

Exercise VI.7.21. Let R be a commutative ring. Let $f \colon M \to M'$ be an R-module homomorphism, and let N be an R-module.

(a) Without using Proposition VI.7.8, prove that, if $f = 0$, then $\mathrm{Tor}_i^R(f,N) = 0$ for all indices $i \in \mathbb{Z}$.
(b) Now, using Proposition VI.7.8, prove that, if $f = 0$, then $\mathrm{Tor}_i^R(f,N) = 0$ for all indices $i \in \mathbb{Z}$.
(c) Prove that, if f is an isomorphism, then $\mathrm{Tor}_i^R(f,N)$ is an isomorphism for all indices $i \in \mathbb{Z}$.

Exercise VI.7.22. Let R be a commutative ring. Let $f \colon M \to M'$ and $g \colon N \to N'$ be R-module homomorphisms. Prove that the following diagram commutes for each

index $i \in \mathbb{Z}$:

$$\begin{array}{ccc} \mathrm{Tor}_i^R(M', N) & \xrightarrow{\mathrm{Tor}_i^R(M', g)} & \mathrm{Tor}_i^R(M', N') \\ \Big\downarrow{\scriptstyle \mathrm{Tor}_i^R(f, N)} & & \Big\downarrow{\scriptstyle \mathrm{Tor}_i^R(f, N')} \\ \mathrm{Tor}_i^R(M, N) & \xrightarrow{\mathrm{Tor}_i^R(M, g)} & \mathrm{Tor}_i^R(M, N'). \end{array}$$

Exercise VI.7.23. Prove Lemma VI.7.14

CHAPTER VII

Ext, Tor, and Homological Dimensions September 8, 2009

The goal of this chapter is to show how Ext-vanishing and Tor-vanishing are related to projective dimension, injective dimension, and flat dimension.

VII.1. Assumptions

Fact VII.1.1. Let R be a commutative ring. Given an R-module N and an exact sequence of R-modules

$$0 \to M' \xrightarrow{f'} M \xrightarrow{f} M'' \to 0$$

there are three long exact sequences: the first one is for $\operatorname{Ext}^i_R(N,-)$

$$\begin{aligned}
0 \to \operatorname{Hom}_R(N,M') &\xrightarrow{\operatorname{Hom}_R(N,f')} \operatorname{Hom}_R(N,M) \xrightarrow{\operatorname{Hom}_R(N,f)} \operatorname{Hom}_R(N,M'') \\
\longrightarrow \operatorname{Ext}^1_R(N,M') &\xrightarrow{\operatorname{Ext}^1_R(N,f')} \operatorname{Ext}^1_R(N,M) \xrightarrow{\operatorname{Ext}^1_R(N,f)} \operatorname{Ext}^1_R(N,M'') \to \cdots \\
\cdots \longrightarrow \operatorname{Ext}^i_R(N,M') &\xrightarrow{\operatorname{Ext}^i_R(N,f)} \operatorname{Ext}^i_R(N,M) \xrightarrow{\operatorname{Ext}^1_R(N,f)} \operatorname{Ext}^i_R(N,M'') \to \cdots
\end{aligned}$$

the second one is for $\operatorname{Ext}^i_R(-,N)$

$$\begin{aligned}
0 \to \operatorname{Hom}_R(M'',N) &\xrightarrow{\operatorname{Hom}_R(f,N)} \operatorname{Hom}_R(M,N) \xrightarrow{\operatorname{Hom}_R(f',N)} \operatorname{Hom}_R(M',N) \\
\longrightarrow \operatorname{Ext}^1_R(M'',N) &\xrightarrow{\operatorname{Ext}^1_R(f,N)} \operatorname{Ext}^1_R(M,N) \xrightarrow{\operatorname{Ext}^1_R(f',N)} \operatorname{Ext}^1_R(M',N) \to \cdots \\
\cdots \longrightarrow \operatorname{Ext}^i_R(M'',N) &\xrightarrow{\operatorname{Ext}^i_R(f,N)} \operatorname{Ext}^i_R(M,N) \xrightarrow{\operatorname{Ext}^i_R(f',N)} \operatorname{Ext}^i_R(M',N) \to \cdots
\end{aligned}$$

and the third one is for $\operatorname{Tor}^R_i(-,N)$

$$\begin{aligned}
\cdots \to \operatorname{Tor}^R_n(M',N) &\xrightarrow{\operatorname{Tor}^R_n(f',N)} \operatorname{Tor}^R_n(M,N) \xrightarrow{\operatorname{Tor}^R_n(f,N)} \operatorname{Tor}^R_n(M'',N) \to \cdots \\
\cdots \to \operatorname{Tor}^R_1(M',N) &\xrightarrow{\operatorname{Tor}^R_1(f',N)} \operatorname{Tor}^R_1(M,N) \xrightarrow{\operatorname{Tor}^R_1(f,N)} \operatorname{Tor}^R_1(M'',N) \\
\longrightarrow M' \otimes_R N &\xrightarrow{f' \otimes_R N} M \otimes_R N \xrightarrow{f \otimes_R N} M'' \otimes_R N \longrightarrow 0.
\end{aligned}$$

See Theorems VIII.2.1, VIII.2.2, and VIII.2.3.

VII.2. Depth and Dimension

We start by recalling some facts from dimension theory.

Definition VII.2.1. Let R be a commutative ring, and let M be an R-module. The *Krull dimension* of M is

$$\dim_R(M) = \sup\{n \geqslant 0 \mid \text{there is a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \operatorname{Supp}_R(M)\}.$$

The *Krull dimension* of R is the Krull dimension of R, considered as an R-module:

$$\dim(R) = \sup\{n \geqslant 0 \mid \text{there is a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \operatorname{Spec}(R)\}.$$

Fact VII.2.2. Let R be a commutative ring. If M is a non-zero R-module, then

$$0 \leqslant \dim_R(M) \leqslant \dim(R).$$

If $I \subsetneq R$ is an ideal, then

$$\dim(R/I) = \dim_R(R/I) \leqslant \dim(R).$$

If $U \subseteq R$ is a multiplicatively closed subset, then

$$\dim_{U^{-1}R}(U^{-1}M) \leqslant \dim_R(M) \qquad \dim(U^{-1}R) \leqslant \dim(R).$$

Proposition V.3.9 implies that, if R is noetherian and M is finitely generated, then

$$\begin{aligned}
\dim_R(M) &= \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Supp}_R(M)\} \\
&= \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}_R(M)\} \\
&= \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Min}_R(M)\} \\
\dim(R) &= \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\} \\
&= \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}_R(R)\} \\
&= \sup\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Min}_R(R)\}.
\end{aligned}$$

Fact VII.2.3. If k is a field, then the polynomial ring $k[X_1, \dots, X_n]$ has dimension n, as do the localized polynomial ring $k[X_1, \dots, X_n]_{(X_1,\dots,X_n)}$ and the power series ring $k[\![X_1, \dots, X_n]\!]$. Moreover, if R is a commutative noetherian ring, then

$$\dim(A[X_1, \dots, X_n]) = \dim(A) + n = \dim(A[\![X_1, \dots, X_n]\!]).$$

These equalities are non-trivial. See Example X.1.4.

In general, the quantity $\dim(R)$ need not be finite, even if R is noetherian. However, we do have the following.

Fact VII.2.4. If $(R, \mathfrak{m})$ is a commutative noetherian local ring, then

$$\dim(R) \leqslant \nu_R(\mathfrak{m}) < \infty$$

where $\nu_R(\mathfrak{m})$ is the minimal number of generators for $\mathfrak{m}$. See Remark X.1.1 for more information.

The goal of this section is to establish the inequality $\operatorname{depth}_R(M) \leqslant \dim_R(M)$. See Theorem VII.2.7. For this, we need the following two lemmas.

Lemma VII.2.5. *Let R be a commutative ring, let M be an R-module, and let t be an integer.*

(a) *Consider an exact sequence of R-module homomorphisms*

$$0 \to N' \to N \to N'' \to 0.$$

If $\operatorname{Ext}^t_R(N'', M) = 0 = \operatorname{Ext}^t_R(N', M)$, then $\operatorname{Ext}^t_R(N, M) = 0$.

(b) *Consider a chain of submodules $N = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_n = 0$. If $\operatorname{Ext}^t_R(N_j/N_{j+1}, M) = 0$ for $j = 0, \dots, n-1$, then $\operatorname{Ext}^t_R(N, M) = 0$.*

PROOF. (a) Part of the long exact sequence in $\operatorname{Ext}_R(-,M)$ associated to the given exact sequence has the following form

$$\underbrace{\operatorname{Ext}^t_R(N'',M)}_{=0} \to \operatorname{Ext}^t_R(N,M) \to \underbrace{\operatorname{Ext}^t_R(N',M)}_{=0}.$$

It follows that $\operatorname{Ext}^t_R(N,M)=0$.

(b) We proceed by induction on n. In the first base case $n=1$ we have $N=N_0\cong N_0/0=N_0/N_1$, so the conclusion is automatic from the assumptions.

Base case $n=2$. In this case, the given filtration yields an exact sequence

$$0\to \underbrace{N_1}_{\cong N_1/N_2} \to N \to \underbrace{N/N_1}_{=N_0/N_1} \to 0.$$

Using the assumption $\operatorname{Ext}^t_R(N_1/N_2,M)=0=\operatorname{Ext}^t_R(N_0/N_1,M)$, part (a) implies that $\operatorname{Ext}^t_R(N,M)=0$.

Induction step. Assume that $n\geqslant 3$ and that the result holds for modules with filtrations of length $n-1$. The module N_1 has a filtration of length $n-1$

$$N_1\supset N_2\supset\cdots\supset N_n=0.$$

Our assumptions imply that $\operatorname{Ext}^t_R(N_j/N_{j+1},M)=0$ for $j=1,\dots,n-1$, so our induction hypothesis implies that $\operatorname{Ext}^t_R(N_1,M)=0$. Furthermore, we have

$$0=\operatorname{Ext}^t_R(N_0/N_1,M)\cong\operatorname{Ext}^t_R(N/N_1,M)$$

so the base case $n=2$ applied to the filtration $N=N_0\supset N_1\supset 0$ yields the desired conclusion $\operatorname{Ext}^t_R(N,M)=0$. □

The proof of the following shows how to use regular elements in an induction argument. It also shows how to use prime filtrations to give you regular elements.

Lemma VII.2.6. *Let $(R,\mathfrak{m})$ be a commutative noetherian local ring, and let M and N be non-zero finitely generated R-modules. Set $l=\operatorname{depth}_R(M)$ and $r=\dim_R(N)$. Then $\operatorname{Ext}^i_R(N,M)=0$ for all $i<l-r$.*

PROOF. Proceed by induction on r.

Base case: $r=0$. In this case, we have $\dim_R(N)=0$ and so $\operatorname{Supp}_R(N)=\{\mathfrak{m}\}$. Hence, the result follows from Theorem V.5.11, as discussed above.

For the induction step, assume that $r\geqslant 1$ and that, for every finitely generated R-module $N'\neq 0$ with $\dim_R(N')<r$, we have $\operatorname{Ext}^i_R(N',M)=0$ for all indices $i<l-\dim_R(N')$. It follows that $\operatorname{Ext}^i_R(N',M)=0$ for all $i<l-r+1$ since $l-r<l-\dim_R(N')$.

First, consider the special case where $N\cong R/\mathfrak{p}$ for some $\mathfrak{p}\in\operatorname{Spec}(R)$. Since $\dim_R(N)=r\geqslant 1$, we have $\mathfrak{p}\subsetneq\mathfrak{m}$. Let $x\in\mathfrak{m}-\mathfrak{p}$, and consider the exact sequence

$$0\to N\xrightarrow{x} N\to N/xN\to 0.$$

Note that we have $N/xN\cong R/(x,\mathfrak{p})$. Since $\mathfrak{p}$ is prime and $x\in\mathfrak{m}-\mathfrak{p}$, it is straightforward to show that $\dim_R(N/xN)<\dim_R(N)$. (Actually, we have equality here, but that's harder to show and we don't need it here.) Hence, by our induction hypothesis, we know that $\operatorname{Ext}^i_R(N/xN,M)=0$ for all $i<l-r+1$. Hence, for each $i<l-r$, the portion of the long exact sequence in $\operatorname{Ext}^i_R(-,M)$ associated to the displayed sequence has the form

$$0\to\operatorname{Ext}^i_R(N,M)\xrightarrow{x}\operatorname{Ext}^i_R(N,M)\to 0.$$

It follows that $\operatorname{Ext}^i_R(N,M) = x\operatorname{Ext}^i_R(N,M)$. Since $x \in \mathfrak{m}$ and $\operatorname{Ext}^i_R(N,M)$ is finitely generated, Nakayama's Lemma implies $\operatorname{Ext}^i_R(N,M) = 0$.

For the general case, take a prime filtration $N = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_n = 0$ so that $N_j/N_{j+1} \cong R/\mathfrak{p}_j$ for some $\mathfrak{p}_j \in \operatorname{Spec}(R)$ for $j = 0, \dots, n-1$. We have seen previously that

$$\dim(R/\mathfrak{p}_j) = \dim_R(N_j/N_{j+1}) \leqslant \dim_R(N_j) \leqslant \dim_R(N) = r.$$

Hence, our induction hypothesis works with Case 1 to imply $\operatorname{Ext}^i_R(N_j/N_{j+1}, M) = 0$ for all $i \leqslant l - r$ and for $j = 0, \dots, n-1$. Lemma VII.2.5(b) implies that $\operatorname{Ext}^i_R(N,M) = 0$ for all $i < l - r$, as desired. □

Theorem VII.2.7. *Let $(R, \mathfrak{m})$ be a commutative noetherian local ring, and let M be a non-zero finitely generated R-module.*

(a) *For each $\mathfrak{p} \in \operatorname{Ass}_R(M)$ we have* $\operatorname{depth}_R(M) \leqslant \dim(R/\mathfrak{p})$
(b) *We have* $\operatorname{depth}_R(M) \leqslant \dim_R(M)$.

Proof. (a) For each $\mathfrak{p} \in \operatorname{Ass}_R(M)$ we have

$$\operatorname{Ext}^0_R(R/\mathfrak{p}, M) \cong \operatorname{Hom}_R(R/\mathfrak{p}, M) \neq 0$$

see Example V.2.9. Lemma VII.2.6 implies that $\operatorname{Ext}^i_R(R/\mathfrak{p}, M) = 0$ for all indices $i < \operatorname{depth}_R(M) - \dim(R/\mathfrak{p})$. It follows that $\operatorname{depth}_R(M) - \dim(R/\mathfrak{p}) \leqslant 0$ and hence $\operatorname{depth}_R(M) \leqslant \dim_R(M)$.

(b) From Proposition V.3.9 one deduces the equality in the next sequence

$$\dim_R(M) = \max\{\dim(R/\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Ass}_R(M)\} \geqslant \operatorname{depth}_R(M)$$

and the inequality is from part (a). □

Exercises.

Exercise VII.2.8. Verify the properties in Fact VII.2.2.

Exercise VII.2.9. Let R be a commutative ring.

(a) Given an exact sequence $0 \to M' \to M \to M'' \to 0$ of R-module homomorphisms, prove that $\dim_R(M) = \sup\{\dim_R(M'), \dim_R(M'')\}$.
(b) Given an R-module M with a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = 0$, prove that $\dim_R(M) = \sup\{\dim_R(M_0/M_1), \dots, \dim_R(M_{n-2}/M_{n-1}), \dim_R(M_{n-1})\}$.
(c) Show that the versions of parts (a) and (b) for depth fail.

Exercise VII.2.10. Let k be a field. Calculate the dimensions of the following rings:

(a) $k[X]/(X^2)$
(b) $k[X,Y]/(XY)$
(c) $k[X,Y]/(X^2, Y^3)$.

VII.3. Ext and Projective Dimension

The projective dimension of a module M is the length of the shortest projective resolution of M. More technically, we have the following.

Definition VII.3.1. Let R be a commutative ring, and let M be an R-module. The *projective dimension* of M is

$$\operatorname{pd}_R(M) = \inf\left\{ n \geqslant 0 \;\middle|\; \begin{array}{c} M \text{ has a projective resolution } P_\bullet \\ \text{such that } P_i = 0 \text{ for all } i > n. \end{array} \right\}.$$

In other words, we have $\operatorname{pd}_R(M) \leqslant n$ if there is an exact sequence

$$0 \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

and we have $\operatorname{pd}_R(M) = \infty$ if M does not have a bounded projective resolution.

Remark VII.3.2. Let R be a commutative ring, and let M be an R-module. Then $\operatorname{pd}_R(M) \geqslant 0$, and M is projective if and only if $\operatorname{pd}_R(M) = 0$.

Lemma VII.3.3. *Let R be a commutative ring, and let M be an R-module. If $U \subseteq R$ is a multiplicatively closed subset, then $\operatorname{pd}_{U^{-1}R}(U^{-1}M) \leqslant \operatorname{pd}_R(M)$.*

Proof. Assume without loss of generality that $n = \operatorname{pd}_R(M) < \infty$. Then there is an exact sequence

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

such that each P_i is a projective R-module.

Each localization $U^{-1}P_i$ is a projective $U^{-1}R$-module; see Exercise III.1.21(c). Also, the following localized sequence is exact

$$0 \to U^{-1}P_n \to \cdots \to U^{-1}P_0 \to U^{-1}M \to 0$$

and consists of $U^{-1}R$-module homomorphisms. It follows that $\operatorname{pd}_{U^{-1}R}(U^{-1}M) \leqslant n = \operatorname{pd}_R(M)$. □

Here is a useful lemma:

Lemma VII.3.4. *Let R be a commutative ring, and consider an exact sequence of R-module homomorphisms $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$. If $\operatorname{Ext}^1_R(M'', M') = 0$, then the given sequence splits.*

Proof. The long exact sequence in $\operatorname{Ext}^i_R(-, M')$ associated to the given sequence begins as follows because $\operatorname{Ext}^1_R(M'', M') = 0$:

$$0 \to \operatorname{Hom}_R(M'', M') \xrightarrow{\operatorname{Hom}_R(g,M')} \operatorname{Hom}_R(M, M') \xrightarrow{\operatorname{Hom}_R(f,M')} \operatorname{Hom}_R(M', M') \to 0.$$

It follows that $\operatorname{Hom}_R(f, M')$ is surjective, so there is an element $h \in \operatorname{Hom}_R(M, M')$ such that $\mathbb{1}_{M'} = \operatorname{Hom}_R(f, M')(h)$, that is, such that $\mathbb{1}_{M'} = hf$. The map h gives the desired splitting of the sequence. □

The next lemma give a first connection between Ext and projective dimension.

Lemma VII.3.5. *Let R be a commutative ring, and let M be an R-module. Then M is projective if and only if $\operatorname{Ext}^1_R(M, -) = 0$.*

Proof. If M is projective, then $\operatorname{Ext}^i_R(M, N) = 0$ for each R-module N and for all $i \geqslant 1$, by Proposition IV.3.8(a).

Conversely, assume that $\operatorname{Ext}^1_R(M, -) = 0$, and consider an exact sequence

$$0 \to M' \xrightarrow{f} P \xrightarrow{g} M \to 0$$

where P is projective. Since $\operatorname{Ext}^1_R(M, M') = 0$, Lemma VII.3.4 shows that the sequence splits, and hence $M \oplus M' \cong P$. Since P is projective, it follows that M is projective. □

The next two results are called "dimension-shifting" in the literature. It would be more proper to call it "degree-shifting".

Lemma VII.3.6. *Let R be a commutative ring, and let N be an R-module. Consider an exact sequence of R-module homomorphisms*

$$0 \to M' \xrightarrow{f} P \xrightarrow{g} M \to 0. \tag{VII.3.6.1}$$

If $\operatorname{Ext}^i_R(P,N) = 0$ for all $i \geqslant 1$, for instance if P is projective, then $\operatorname{Ext}^i_R(M',N) \cong \operatorname{Ext}^{i+1}(M,N)$ for all $i \geqslant 1$.

Proof. Part of the long exact sequence in $\operatorname{Ext}_R(-,N)$ associated to the sequence (VII.3.6.1) has the following form:

$$\underbrace{\operatorname{Ext}^i_R(P,N)}_{=0} \to \operatorname{Ext}^i_R(M',N) \xrightarrow{\eth^i} \operatorname{Ext}^{i+1}_R(M,N) \to \underbrace{\operatorname{Ext}^{i+1}_R(P,N)}_{=0}.$$

It follows that $\eth^i$ is an isomorphism. □

Lemma VII.3.7. *Let R be a commutative ring, and let M and N be R-modules. Let $n \geqslant 1$, and consider an exact sequence of R-module homomorphisms*

$$0 \to K_n \xrightarrow{g_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_1} P_0 \xrightarrow{h_0} M \to 0.$$

If $\operatorname{Ext}^i_R(P_j,N) = 0$ for all $i \geqslant 1$ and all $j = 1,\dots,n-1$, for instance if each P_j is projective, then $\operatorname{Ext}^i_R(K_n,N) \cong \operatorname{Ext}^{i+n}(M,N)$ for all $i \geqslant 1$.

Proof. By induction on n. The base case $n = 1$ is Lemma VII.3.6.

Induction step: assume that $n \geqslant 2$ and that the result holds for sequences of length $n-1$. The given sequence yields two exact sequences

$$0 \to K_1 \xrightarrow{g_1} P_0 \xrightarrow{h_0} M \to 0 \tag{VII.3.7.1}$$

and

$$0 \to K_n \xrightarrow{g_n} P_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} P_1 \xrightarrow{h_1} K_1 \to 0. \tag{VII.3.7.2}$$

In the following sequence, the first isomorphism follows from the induction hypothesis applied to (VII.3.7.2)

$$\operatorname{Ext}^i_R(K_n,N) \cong \operatorname{Ext}^{i+n-1}(K_1,N) \cong \operatorname{Ext}^{i+n}(M,N)$$

and the second isomorphism follows from the base case applied to (VII.3.7.1). □

The next result is a souped-up version of Lemma VII.3.5. For its proof, recall that the *cokernel* of an R-module homomorphism $f\colon N \to N'$ is the quotient $\operatorname{Coker}(f) = N'/\operatorname{Im}(f)$.

Theorem VII.3.8. *Let R be a commutative ring, and let M be an R-module. For an integer $n \geqslant 0$, the following conditions are equivalent:*

(i) $\operatorname{pd}_R(M) \leqslant n$;
(ii) $\operatorname{Ext}^i_R(M,-) = 0$ *for all* $i > n$;
(iii) $\operatorname{Ext}^{n+1}_R(M,-) = 0$;
(iv) *For each projective resolution $P_\bullet$ of M, the module $K_n = \operatorname{Coker}(\partial^P_{n+1})$ is projective; and*
(v) *For some projective resolution $P_\bullet$ of M, the module $K_n = \operatorname{Coker}(\partial^P_{n+1})$ is projective.*

In particular, we have $\operatorname{pd}_R(M) = \sup\{n \geqslant 0 \mid \operatorname{Ext}^n_R(M,-) \neq 0\}$.

PROOF. (i) $\Longrightarrow$ (ii). Assume that $\operatorname{pd}_R(M) \leqslant n$, so that M has a projective resolution $P_\bullet$ such that $P_i = 0$ for all $i > n$. It follows that, for each R-module N and each $i > n$, we have $\operatorname{Hom}_R(P_\bullet, N)_{-i} = 0$ and hence $\operatorname{Ext}^i_R(M,N) = \operatorname{H}_{-i}(\operatorname{Hom}_R(P_\bullet, N)) = 0$.

(ii) $\Longrightarrow$ (iii). This is logically trivial.

(iii) $\Longrightarrow$ (iv). Let $P_\bullet$ be a projective resolution of M. If $n = 0$, then Lemma VII.3.5 implies that $M \cong \operatorname{Coker}(\partial^P_1)$ is projective.

Now, assume that $n \geqslant 1$, and consider the exact sequence

$$0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

For each R-module N, Lemma VII.3.7 implies

$$\operatorname{Ext}^1_R(K_n, N) \cong \operatorname{Ext}^{1+n}_R(M,N) = 0.$$

Lemma VII.3.5 implies that K_n is projective.

(iv) $\Longrightarrow$ (v). This follows because M has a projective resoltuion.

(v) $\Longrightarrow$ (i). Let $P_\bullet$ be a projective resolution of M such that the module $K_n = \operatorname{Coker}(\partial^P_{n+1})$ is projective. It follows that the next sequence

$$0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

is an augmented projective resolution of length $\leqslant n$ and hence $\operatorname{pd}_R(M) \leqslant n$. □

Corollary VII.3.9. *Let R be a commutative ring, and consider the following exact sequence of R-module homomorphisms*

$$0 \to M_3 \to M_2 \to M_1 \to 0.$$

If two of the M_i have finite projective dimension over R, then so does the third one.

PROOF. We will prove this in the case where $\operatorname{pd}_R(M_3), \operatorname{pd}_R(M_2) < \infty$. The other cases are similar. Assume that $\operatorname{pd}_R(M_3), \operatorname{pd}_R(M_2) < n$. Theorem VII.3.8 implies that for every R-module N and every $i \geqslant n$, we have

$$\operatorname{Ext}^i_R(M_3, N) = 0 = \operatorname{Ext}^i_R(M_2, N).$$

From the long exact sequence in $\operatorname{Ext}^i_R(-,N)$, we conclude that $\operatorname{Ext}^i_R(M_1,N) = 0$ for all $i > n$. Another application of Theorem VII.3.8 shows that $\operatorname{pd}_R(M_1) \leqslant n$. □

Corollary VII.3.10. *Let R be a commutative ring, and let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules. Then $\operatorname{pd}_R(\coprod_\lambda M_\lambda) = \sup\{\operatorname{pd}_R(M_\lambda) \mid \lambda \in \Lambda\}$.*

PROOF. By Exercise VI.2.12(a), we have $\operatorname{Ext}^i_R(\coprod_\lambda M_\lambda, N) \cong \prod_\lambda \operatorname{Ext}^i_R(M_\lambda, N)$ for each R-module N and each index i. In particular, we have $\operatorname{Ext}^i_R(\coprod_\lambda M_\lambda, N) = 0$ for all N and all $i \geqslant n$ if and only if $\operatorname{Ext}^i_R(M_\lambda, N) = 0$ for all N and all $i \geqslant n$. The result now follows from Theorem VII.3.8. □

Compare the next result to Exercise III.1.21(c) and Corollary VII.4.3.

Corollary VII.3.11. *Let R be a commutative noetherian ring, and let M be a finitely generated R-module. Given in integer $n \geqslant 1$, the following conditions are equivalent:*

(i) $\operatorname{pd}_R(M) < n$;
(ii) $\operatorname{pd}_{U^{-1}R}(U^{-1}M) < n$ *for each multiplicatively closed subset* $U \subseteq R$;
(iii) $\operatorname{pd}_{R_\mathfrak{p}}(M_\mathfrak{p}) < n$ *for each prime ideal* $\mathfrak{p} \subsetneq R$; *and*
(iv) $\operatorname{pd}_{R_\mathfrak{m}}(M_\mathfrak{m}) < n$ *for each maximal ideal* $\mathfrak{m} \subsetneq R$.

Hence, there are equalities

$$\begin{aligned}\operatorname{pd}_R(M) &= \sup\{\operatorname{pd}_{U^{-1}R}(U^{-1}M) \mid U \subseteq R \text{ is multiplicatively closed}\}\\ &= \sup\{\operatorname{pd}_{R_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \text{ is a prime ideal of } R\}\\ &= \sup\{\operatorname{pd}_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \text{ is a maximal ideal of } R\}.\end{aligned}$$

PROOF. The implication (i) $\implies$ (ii) follows from Lemma VII.3.3, and the implications (ii) $\implies$ (iii) $\implies$ (iv) are routine.

(iv) $\implies$ (i). Assume that $\operatorname{pd}_{R_\mathfrak{m}}(M_\mathfrak{m}) < n$ for each maximal ideal $\mathfrak{m} \subsetneq R$. To prove that $\operatorname{pd}_R(M) < n$, it suffices to show that $\operatorname{Ext}^n_R(M,N) = 0$ for every R-module N. For each maximal ideal $\mathfrak{m} \subset R$, the isomorphism in the following sequence is from Theorem VI.2.7(b)

$$\operatorname{Ext}^n_R(M,N)_\mathfrak{m} \cong \operatorname{Ext}^n_{R_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) = 0$$

and the vanishing follows from the assumption $\operatorname{pd}_{R_\mathfrak{m}}(M_\mathfrak{m}) < n$. Exercise I.4.25 implies that $\operatorname{Ext}^n_R(M,N) = 0$, as desired.

The final equalities follow from the equivalence of (i)–(iv). □

We next show how Theorem VII.3.8 can be improved for finitely generated modules over noetherian local rings; see Theorem VII.3.14. First we need the following two lemmas.

Lemma VII.3.12. *Let $(R,\mathfrak{m},k)$ be a commutative noetherian local ring. Let M be a finitely generated R-module with minimal generating sequence $x_1,\dots,x_n \in M$. Then $\operatorname{Hom}_R(M,k) \cong k^n \cong M/\mathfrak{m}M \cong M \otimes_R k$.*

PROOF. Let $\overline{x_i}$ denote the residue of x_i in $M/\mathfrak{m}M$. Nakayama's Lemma implies that the sequence $\overline{x_1},\dots,\overline{x_n}$ is a basis for $M/\mathfrak{m}M$ as a k-vector space. This explains the last isomorphism in the next sequence:

$$M \otimes_R k \cong M \otimes_R R/\mathfrak{m} \cong M/\mathfrak{m}M \cong k^n.$$

The first isomorphism is by definition, and the second is from Exercise II.4.14.

Let $\tau\colon M \to M/\mathfrak{m}M$ denote the canonical surjection. The left-exactness of $\operatorname{Hom}_R(-,k)$ implies that the induced map

$$\operatorname{Hom}_R(\tau,k)\colon \operatorname{Hom}_R(M/\mathfrak{m}M,k) \to \operatorname{Hom}_R(M,k)$$

is injective. We claim that it is also surjective. To see this, let $\xi \in \operatorname{Hom}_R(M,k)$. Observe that $\mathfrak{m}M \subseteq \operatorname{Ker}(\xi)$ because

$$\xi(\mathfrak{m}M) = \mathfrak{m}\xi(M) \subseteq \mathfrak{m}k = 0.$$

It follows (using the universal mapping property for quotients) that the function $\overline{\xi}\colon M/\mathfrak{m}M \to k$ given by $\overline{\xi}(\overline{x}) = \xi(x)$ is a well-defined R-module homomorphism such that $\overline{\xi}\circ\tau = \xi$. In other words, we have $\xi = \operatorname{Hom}_R(\tau,k)(\overline{\xi})$, as desired.

The previous paragraph explains the first isomorphism in the next sequence:

$$\operatorname{Hom}_R(M,k) \cong \operatorname{Hom}_R(M/\mathfrak{m}M,k) \cong \operatorname{Hom}_R(k^n,k) \cong \operatorname{Hom}_R(k,k)^n \cong k^n$$

The second isomorphism is from the first paragraph, the third isomorphism is from Exercise I.3.3(c), and the fourth isomorphism follows from the standard fact $\operatorname{Hom}_R(k,k) \cong k$. □

Lemma VII.3.13. *Let $(R,\mathfrak{m},k)$ be a commutative noetherian local ring, and let M be a finitely generated R-module. If $\operatorname{Ext}^1_R(M,k) = 0$, then M is projective.*

PROOF. Let $x_1, \dots, x_n \in M$ be a minimal generating sequence for M. The map $\tau\colon R^n \to M$ given by the formula $\tau(\sum_i r_i e_i) = \sum_i r_i x_i$ is a well-defined R-module epimorphism. Set $K = \operatorname{Ker}(\tau)$ and note that K is finitely generated since R is noetherian. It suffices to show that $K = 0$.

Consider the exact sequence

$$0 \to K \to R^n \xrightarrow{\tau} M \to 0.$$

Part of the associated long exact sequence in $\operatorname{Ext}^i_R(-.k)$ has the following form:

$$0 \to \underbrace{\operatorname{Hom}_R(M,k)}_{\cong k^n} \xrightarrow{\tau^*} \underbrace{\operatorname{Hom}_R(R^n,k)}_{\cong k^n} \to \underbrace{\operatorname{Hom}_R(K,k)}_{\cong k^m} \to \underbrace{\operatorname{Ext}^1_R(M,k)}_{=0} \quad \text{(VII.3.13.1)}$$

where m is the minimal number of generators for K.

Since τ^* is a linear transformation between finite dimensional vector spaces of the same rank, the fact that τ^* is injective implies that it is an isomorphism. From the exact sequence (VII.3.13.1) we conclude that

$$0 = k^m \cong \operatorname{Hom}_R(K,k) \cong K/\mathfrak{m}K.$$

Hence, Nakayama's Lemma implies that $K = 0$, as desired. □

Theorem VII.3.14. *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring, and let M be a finitely generated R-module. For an integer $n \geqslant 0$, the following conditions are equivalent:*

(i) $\operatorname{pd}_R(M) \leqslant n$;
(ii) $\operatorname{Ext}^i_R(M,k) = 0$ *for all* $i > n$;
(iii) $\operatorname{Ext}^{n+1}_R(M,k) = 0$.

It follows that there is an equality $\operatorname{pd}_R(M) = \sup\{i \geqslant 0 \mid \operatorname{Ext}^i_R(M,k) \neq 0\}$.

PROOF. The implication (i) $\implies$ (ii) follows from Theorem VII.3.8, and the implication (ii) $\implies$ (iii) is routine.

(iii) $\implies$ (i) Let $P_\bullet$ be a projective resolution of M such that each P_i is finitely generated. If $n = 0$, then Lemma VII.3.13 implies that M is projective, and hence $\operatorname{pd}_R(M) = 0$ as desired.

Now, assume that $n \geqslant 1$, and consider the exact sequence

$$0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

where $K_n = \operatorname{Im}(\partial^P_n)$. Lemma VII.3.7 implies that

$$\operatorname{Ext}^1_R(K_n,k) \cong \operatorname{Ext}^{1+n}_R(M,k) = 0.$$

From Lemma VII.3.13 we conclude that K_n is projective, so Theorem VII.3.8 implies $\operatorname{pd}_R(M) \leqslant n$. □

Exercises.

Exercise VII.3.15. Prove that the inequality $\operatorname{pd}_{U^{-1}R}(U^{-1}M) \leqslant \operatorname{pd}_R(M)$ from Lemma VII.3.3 can be strict.

Exercise VII.3.16. Finish the proof of Corollary VII.3.9.

Exercise VII.3.17. Let R be a commutative ring, and let M be an R-module. Let $\pi\colon P \to M$ be an R-module epimorphism wherein P is projective. Show that, if $\operatorname{pd}_R(M) < \infty$, then

$$\operatorname{pd}_R(\operatorname{Ker}(\pi)) = \begin{cases} \operatorname{pd}_R(M) - 1 & \text{if } \operatorname{pd}_R(M) \geqslant 1 \\ 0 & \text{if } \operatorname{pd}_R(M) = 0. \end{cases}$$

Exercise VII.3.18. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring, and let M and N be non-zero finitely generated R-modules. Show that, if $r = \operatorname{pd}_R(M) < \infty$, then $\operatorname{Ext}^r_R(M, N) \neq 0$. [Hint: Use Theorem VII.3.14 with the long exact sequence in $\operatorname{Ext}^i_R(M, -)$ associated to the short exact sequence $0 \to \mathfrak{m}N \to N \to N/\mathfrak{m}N \to 0$.

Exercise VII.3.19. Let R be a commutative noetherian ring, and let M be a finitely generated R-module. For an integer $n \geqslant 0$, prove that the following conditions are equivalent:

(i) $\operatorname{pd}_R(M) \leqslant n$;
(ii) $\operatorname{Ext}^i_R(M, R/\mathfrak{m}) = 0$ for all $i > n$ and for every maximal ideal $\mathfrak{m} \subsetneq R$;
(iii) $\operatorname{Ext}^{n+1}_R(M, R/\mathfrak{m}) = 0$ for every maximal ideal $\mathfrak{m} \subsetneq R$.

Exercise VII.3.20. Finish the proof of Corollary VII.3.11.

VII.4. Tor and Projective Dimension

Lemma VII.4.1. *Let $(R, \mathfrak{m}, k)$ be a commutative local noetherian ring, and let M be a finitely generated R-module. The following conditions are equivalent:*

(i) *M is free;*
(ii) *M is projective;*
(iii) *M is flat;*
(iv) *$\operatorname{Tor}^R_i(M, -) = 0$ for all $i \geqslant 1$; and*
(v) *$\operatorname{Tor}^R_1(M, k) = 0$.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) have been covered. The implication (iii) $\Longrightarrow$ (iv) is in Proposition IV.4.7(b), and (iv) $\Longrightarrow$ (v) is trivial. (Note that none of these implications require R to be local or noetherian nor do they require M to be finitely generated.)

(v) $\Longrightarrow$ (i). Since M is finitely generated, there is an exact sequence

$$0 \to K \xrightarrow{f} R^b \xrightarrow{g} M \to 0$$

where $b = \mu_R(M)$. The associated long exact sequence in $\operatorname{Tor}^R_i(-, k)$ starts with

$$\underbrace{\operatorname{Tor}^R_1(M, k)}_{=0} \to K \otimes_R k \xrightarrow{f \otimes_R k} \underbrace{R^b \otimes_R k}_{\cong k^b} \xrightarrow{g \otimes_R k} \underbrace{M \otimes_R k}_{\cong k^b} \to 0.$$

As in the proof of Lemma IX.1.1, Nakayama's Lemma implies that $g \otimes_R k$ is an isomorphism. From the second exact sequence we conclude that $K \otimes_R k = 0$. Since K is finitely generated, Nakayama's Lemma implies that $K = 0$. Thus, the first exact sequence implies $M \cong R^b$. □

The next example shows that the module M needs to be finitely generated in order for the implication (v) $\Longrightarrow$ (i) from Lemma VII.4.1 to hold.

Example VII.4.2. Let p be a prime number, and set $R = \mathbb{Z}_{(p)}$. Then R is a noetherian local domain with maximal ideal $\mathfrak{m} = pR$ and residue field $k = R/pR \cong \mathbb{Z}/p\mathbb{Z}$. The quotient field of R is $\mathbb{Q}$. It is straightforward to show that $\mathbb{Q}$ is not a free R-module. (Check that $\mathbb{Q}$ is not cyclic and that every pair of elements $a, b \in \mathbb{Q}$ is linearly dependent over R.)

We show that $\operatorname{Tor}_i^R(\mathbb{Q}, k) = 0$ for every index i. The Koszul complex

$$K_\bullet = \qquad 0 \to R \xrightarrow{p} R \to 0$$

is a projective resolution of k. The complex $\mathbb{Q} \otimes_R K_\bullet$ has the following form:

$$\mathbb{Q} \otimes_R K_\bullet = \qquad 0 \to \mathbb{Q} \xrightarrow{p} \mathbb{Q} \to 0.$$

This complex is exact, so its homology modules are all 0, that is $\operatorname{Tor}_i^R(\mathbb{Q}, k) \cong \operatorname{H}_i(\mathbb{Q} \otimes_R K_\bullet) = 0$ for all i.

The next result compares to Exercise III.1.21(c) and Corollary VII.3.11.

Corollary VII.4.3. *Let R be a commutative noetherian ring, and let M be a finitely generated R-module. The following conditions are equivalent:*

(i) *M is projective as an R-module;*
(ii) *$M_\mathfrak{p}$ is free as an $R_\mathfrak{p}$-module for each prime ideal $\mathfrak{p} \subsetneq R$; and*
(iii) *$M_\mathfrak{m}$ is free as an $R_\mathfrak{m}$-module for each maximal ideal $\mathfrak{m} \subsetneq R$.*

Proof. (i) $\implies$ (ii). Assume that M is projective as an R-module, and fix a prime ideal $\mathfrak{p} \subsetneq R$. Exercise III.1.21(c) implies that $M_\mathfrak{p}$ is projective as an $R_\mathfrak{p}$-module. Since $R_\mathfrak{p}$ is noetherian and $M_\mathfrak{p}$ is finitely generated over $R_\mathfrak{p}$, Lemma VII.4.1 implies that $M_\mathfrak{p}$ is free as an $R_\mathfrak{p}$-module.

(ii) $\implies$ (iii). This follows from the fact that every maximal ideal is prime.

(iii) $\implies$ (i). This is a consequence of Corollary VII.3.11. □

The next lemma is another "dimension-shifting" result, which is proved like Lemma VII.3.7.

Lemma VII.4.4. *Let R be a commutative ring, and let M and N be R-modules. Let $n \geqslant 1$, and consider the following exact sequence of R-module homomorphisms*

$$0 \to K_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0.$$

If $\operatorname{Tor}_i^R(F_j, N) = 0$ for all $i \geqslant 1$ and all $j = 0, \dots, n-1$, for instance if each F_j is flat, then $\operatorname{Tor}_i^R(K_n, N) \cong \operatorname{Tor}_{i+n}^R(M, N)$ for all $i \geqslant 1$. □

The proof of the following result is almost identical to the proof of the Ext-characterization of projective dimension in Theorem VII.3.14.

Theorem VII.4.5. *Let $(R, \mathfrak{m}, k)$ be a commutative local noetherian ring, and let M be a finitely generated R-module. Let $n \geqslant 0$. The following conditions are equivalent:*

(i) *$\operatorname{pd}_R(M) \leqslant n$;*
(ii) *$\operatorname{Tor}_i^R(M, -) = 0$ for all $i > n$; and*
(iii) *$\operatorname{Tor}_{n+1}^R(M, k) = 0$.*

In particular, we have $\operatorname{pd}_R(M) = \sup\{n \geqslant 0 \mid \operatorname{Tor}_n^R(M, k) \neq 0\}$. □

Exercises.

Exercise VII.4.6. Prove Lemma VII.4.4

Exercise VII.4.7. Prove Theorem VII.4.5

VII.5. Ext and Injective Dimension

Injective dimension behaves slightly differently from projective dimension because, in general, injective modules are not finitely generated. However, there is an Ext-characterization that is similar.

The injective dimension of a module M is the length of the shortest injective resolution of M. More technically, we have the following.

Definition VII.5.1. Let R be a commutative ring, and let M be an R-module. The *injective dimension* of M is

$$\operatorname{id}_R(M) = \inf\left\{n \geqslant 0 \;\middle|\; \begin{array}{l} M \text{ has an injective resolution } I_\bullet \\ \text{such that } I_{-j} = 0 \text{ for all } j > n. \end{array}\right\}.$$

In other words, we have $\operatorname{id}_R(M) \leqslant n$ if there is an exact sequence

$$0 \to M \to I_0 \to I_{-1} \to \cdots \to I_{1-n} \to I_{-n} \to 0$$

and we have $\operatorname{id}_R(M) = \infty$ if M does not have a bounded injective resolution.

Remark VII.5.2. Let R be a commutative ring, and let M be an R-module. Then $\operatorname{id}_R(M) \geqslant 0$, and M is injective if and only if $\operatorname{id}_R(M) = 0$.

The next result is proved like Lemma VII.3.3, using Proposition III.1.19.

Lemma VII.5.3. *Let R be a commutative noetherian ring. If M is an R-module and $U \subseteq R$ is a multiplicatively closed subset, then* $\operatorname{id}_{U^{-1}R}(U^{-1}M) \leqslant \operatorname{id}_R(M)$. □

The next lemma give a first connection between Ext and injective dimension. It is proved like Lemma VII.3.5.

Lemma VII.5.4. *Let R be a commutative ring, and let M be an R-module. Then M is injective if and only if* $\operatorname{Ext}^1_R(-, M) = 0$. □

The following "dimension-shifting" result is proved like Lemma VII.3.7.

Lemma VII.5.5. *Let R be a commutative ring, and let M and N be R-modules. Let $n \geqslant 1$, and consider an exact sequence of R-module homomorphisms*

$$0 \to M \to I_0 \xrightarrow{g_n} I_1 \to \cdots \to I_{n-1} \to C_n \to 0.$$

If $\operatorname{Ext}^i_R(N, I_j) = 0$ *for all $i \geqslant 1$ and all $j = 1, \dots, n-1$, for instance if each I_j is injective, then* $\operatorname{Ext}^i_R(N, C_n) \cong \operatorname{Ext}^{i+n}(N, M)$ *for all $i \geqslant 1$.* □

The next result is a souped-up version of Lemma VII.3.5. It is proved like Theorem VII.3.8.

Theorem VII.5.6. *Let R be a commutative ring, and let M be an R-module. For an integer $n \geqslant 0$, the following conditions are equivalent:*

(i) $\operatorname{id}_R(M) \leqslant n$;
(ii) $\operatorname{Ext}^i_R(-, M) = 0$ *for all $i > n$;*
(iii) $\operatorname{Ext}^{n+1}_R(-, M) = 0$;
(iv) *For each injective resolution $I_\bullet$ of M, the module $C_n = \operatorname{Ker}(\partial^I_{-n})$ is injective;*

(v) *For some injective resolution $I_\bullet$ of M, the module $C_n = \operatorname{Ker}(\partial^I_{-n})$ is injective.*

In particular, we have $\operatorname{id}_R(M) = \sup\{n \geqslant 0 \mid \operatorname{Ext}^n_R(-, M) \neq 0\}$. □

The next two results are proved like Corollaries VII.3.9 and VII.3.10.

Corollary VII.5.7. *Let R be a commutative ring, and consider the following exact sequence of R-module homomorphisms*

$$0 \to M_3 \to M_2 \to M_1 \to 0.$$

If two of the M_i have finite injective dimension over R, then so does the third one.

Corollary VII.5.8. *Let R be a commutative ring, and let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules. Then* $\operatorname{id}_R(\prod_\lambda M_\lambda) = \sup\{\operatorname{id}_R(M_\lambda) \mid \lambda \in \Lambda\}$.

The version of Theorem VII.3.14 for injective dimension is somewhat different. We prove it in Theorem VII.5.11 below. In preparation, we need the following homological version of Baer's criterion.

Lemma VII.5.9. *Let R be a commutative ring, and let M be an R-module. Then M is injective if and only if* $\operatorname{Ext}^1_R(R/\mathfrak{a}, M) = 0$ *for each ideal $\mathfrak{a} \subseteq R$.*

Proof. One implication follows from Theorem VII.5.6. For the converse, assume that $\operatorname{Ext}^1_R(R/\mathfrak{a}, M) = 0$ for each ideal $\mathfrak{a} \subseteq R$. We show that M satisfies Baer's criterion; see Corollary III.1.4. Fix an ideal $\mathfrak{a} \subseteq R$. Let $\iota\colon \mathfrak{a} \to R$ denote the inclusion, and consider the exact sequence

$$0 \to \mathfrak{a} \xrightarrow{\iota} R \to R/\mathfrak{a} \to 0.$$

The long exact sequence in $\operatorname{Ext}^i_R(-, M)$ associated to this sequence begins as follows:

$$0 \to \operatorname{Hom}_R(R/\mathfrak{a}, M) \to \operatorname{Hom}_R(R, M) \xrightarrow{\operatorname{Hom}_R(\iota, M)} \operatorname{Hom}_R(\mathfrak{a}, M) \to \underbrace{\operatorname{Ext}^i_R(R/\mathfrak{a}, M)}_{=0}.$$

It follows that $\operatorname{Hom}_R(\iota, M)$ is surjective, as desired. □

The next result is a souped-up version of Theorem VII.5.6.

Theorem VII.5.10. *Let R be a commtuative ring, and let M be an R-module. For an integer $n \geqslant 0$, the following conditions are equivalent:*

(i) $\operatorname{id}_R(M) \leqslant n$;
(ii) $\operatorname{Ext}^i_R(-, M) = 0$ *for all $i > n$;*
(iii) $\operatorname{Ext}^{n+1}_R(N, M) = 0$ *for each finitely generated R-module N; and*
(iv) $\operatorname{Ext}^{n+1}_R(R/\mathfrak{a}, M) = 0$ *for each ideal $\mathfrak{a} \subseteq R$.*

In particular, we have $\operatorname{id}_R(M) = \sup\{n \geqslant 0 \mid \operatorname{Ext}^n_R(R/\mathfrak{a}, M) \neq 0$ *for some* $\mathfrak{a} \subseteq R\}$.

Proof. The implication (i) $\implies$ (ii) is in Theorem VII.5.6. The implication (ii) $\implies$ (iii) is trivial, and the implication (iii) $\implies$ (iv) follows from the fact that $R/\mathfrak{a}$ is finitely generated.

(iv) $\implies$ (i) Assume that $\operatorname{Ext}^{n+1}_R(R/\mathfrak{a}, M) = 0$ for each ideal $\mathfrak{a} \subseteq R$. Let $I_\bullet$ be an injective resolution of M, and set $C_n = \operatorname{Ker}(\partial^I_n)$. Lemma VII.5.5 implies that

$$\operatorname{Ext}^1_R(R/\mathfrak{a}, C_n) \cong \operatorname{Ext}^{n+1}_R(R/\mathfrak{a}, M) = 0$$

for each ideal $\mathfrak{a} \subseteq R$. Lemma VII.5.9 implies that C_n is injective, so $\operatorname{id}_R(M) \leqslant n$ by Theorem VII.5.6. □

Here is the version of Theorem VII.3.14 for injective dimension. Note that the ring is not assumed to be local, and M is not assumed to be finitely generated.

Theorem VII.5.11. *Let R be a commtuative noetherian ring, and let M be an R-module. For an integer $n \geqslant 0$, the following conditions are equivalent:*

(i) $\mathrm{id}_R(M) \leqslant n$;
(ii) $\mathrm{Ext}^i_R(-, M) = 0$ *for all* $i > n$;
(iii) $\mathrm{Ext}^{n+1}_R(N, M) = 0$ *for each finitely generated R-module N; and*
(iv) $\mathrm{Ext}^{n+1}_R(R/\mathfrak{p}, M) = 0$ *for each prime ideal* $\mathfrak{p} \subsetneq R$.

In particular, $\mathrm{id}_R(M) = \sup\{n \geqslant 0 \mid \mathrm{Ext}^n_R(R/\mathfrak{p}, M) \neq 0 \text{ for some } \mathfrak{p} \in \mathrm{Spec}(R)\}$.

Proof. In light of Theorem VII.5.10, it suffices to prove that (iv) $\Longrightarrow$ (iii). Assume that $\mathrm{Ext}^{n+1}_R(R/\mathfrak{p}, M) = 0$ for each prime ideal $\mathfrak{p} \subsetneq R$, and let N be a finitely generated R-module. Since R is noetherian and N is finitely generated, there is a filtration

$$N = N_0 \supset N_1 \supset N_2 \supset \cdots \supset N_n = 0$$

such that for $i = 0, \dots, n-1$ we have $N_i/N_{i+1} \cong R/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i \subsetneq R$. Our assumption implies that $\mathrm{Ext}^{n+1}_R(R/\mathfrak{p}_i, M) = 0$ for $i = 0, \dots, n-1$, and Lemma VII.2.5(b) says that $\mathrm{Ext}^{n+1}_R(N, M) = 0$, as desired. □

Corollary VII.5.12. *Let R be a commutative noetherian ring, and let $\{M_\lambda\}_{\lambda \in \Lambda}$ be a set of R-modules. Then* $\mathrm{id}_R(\coprod_\lambda M_\lambda) = \sup\{\mathrm{id}_R(M_\lambda) \mid \lambda \in \Lambda\}$. □

The next result compares to Proposition III.1.19.

Corollary VII.5.13. *Let R be a commutative noetherian ring, and let M be an R-module.*

(a) *Given in integer $n \geqslant 1$, the following conditions are equivalent:*
  (i) $\mathrm{id}_R(M) < n$;
  (ii) $\mathrm{id}_{U^{-1}R}(U^{-1}M) < n$ *for each multiplicatively closed subset $U \subseteq R$;*
  (iii) $\mathrm{id}_{R_\mathfrak{p}}(M_\mathfrak{p}) < n$ *for each prime ideal $\mathfrak{p} \subsetneq R$; and*
  (iv) $\mathrm{id}_{R_\mathfrak{m}}(M_\mathfrak{m}) < n$ *for each maximal ideal $\mathfrak{m} \subsetneq R$.*

(b) *There are equalities*

$$\begin{aligned}\mathrm{id}_R(M) &= \sup\{\mathrm{id}_{U^{-1}R}(U^{-1}M) \mid U \text{ is a multiplicatively closed subset of } R\}\\ &= \sup\{\mathrm{id}_{R_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \text{ is a prime ideal of } R\}\\ &= \sup\{\mathrm{id}_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \text{ is a maximal ideal of } R\}.\end{aligned}$$

(c) *The following conditions are equivalent:*
  (i) *M is injective as an R-module;*
  (ii) *the localization $U^{-1}M$ is injective as an $U^{-1}R$-module for each multiplicatively closed subset $U \subseteq R$;*
  (iii) *the localization $M_\mathfrak{p}$ is injective as an $R_\mathfrak{p}$-module for each prime ideal $\mathfrak{p} \subsetneq R$; and*
  (iv) *the localization $M_\mathfrak{m}$ is injective as an $R_\mathfrak{m}$-module for each maximal ideal $\mathfrak{m} \subsetneq R$.*

Proof. (a) The implication (i) $\Longrightarrow$ (ii) follows from Lemma VII.5.3, and the implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are routine.

(iv) $\Longrightarrow$ (i). Assume that $\mathrm{id}_{R_\mathfrak{m}}(M_\mathfrak{m}) < n$ for each maximal ideal $\mathfrak{m} \subsetneq R$. To prove that $\mathrm{id}_R(M) < n$, it suffices to show that $\mathrm{Ext}^n_R(N, M) = 0$ for every finitely

generated R-module N. For each maximal ideal $\mathfrak{m} \subset R$, the isomorphism in the following sequence is from Theorem VI.2.7(b)

$$\operatorname{Ext}^n_R(N,M)_\mathfrak{m} \cong \operatorname{Ext}^n_{R_\mathfrak{m}}(N_\mathfrak{m}, M_\mathfrak{m}) = 0$$

and the vanishing follows from the assumption $\operatorname{id}_{R_\mathfrak{m}}(M_\mathfrak{m}) < n$. Exercise I.4.25 implies that $\operatorname{Ext}^n_R(N,M) = 0$, as desired.

(b) This follows from part (a).

(c) This is the special case $n = 1$ of part (a). □

Exercises.

Exercise VII.5.14. Prove Lemma VII.5.3.

Exercise VII.5.15. Prove Lemma VII.5.4.

Exercise VII.5.16. Prove Lemma VII.5.5.

Exercise VII.5.17. Prove Theorem VII.5.6.

Exercise VII.5.18. Prove Corollary VII.5.7.

Exercise VII.5.19. Prove Corollary VII.5.8.

Exercise VII.5.20. Prove Corollary VII.5.12.

Exercise VII.5.21. Let R be a principal ideal domain, and let M be an R-module. Prove that $\operatorname{id}_R(M) \leqslant 1$ and $\operatorname{pd}_R(M) \leqslant 1$.

Exercise VII.5.22. Complete the proof of Corollary VII.5.13.

VII.6. Tor and Flat Dimension

The flat dimension of a module M is the length of the shortest flat resolution of M. More technically, we have the following.

Definition VII.6.1. Let R be a commutative ring, and let M be an R-module. A *flat resolution* of M over R or an *R-flat resolution* of M is an exact sequence of R-module homomorphisms

$$F^+_\bullet = \cdots \xrightarrow{\partial^F_2} F_1 \xrightarrow{\partial^F_1} F_0 \xrightarrow{\tau} \underbrace{M}_{\text{degree } -1} \to 0$$

such that each F_i is a flat R-module. The *truncated flat resolution of M associated to $F^+_\bullet$* is the R-complex

$$F_\bullet = \cdots \xrightarrow{\partial^F_2} F_1 \xrightarrow{\partial^F_1} F_0 \to 0.$$

The *flat dimension* of M is

$$\operatorname{fd}_R(M) = \inf\left\{ n \geqslant 0 \;\middle|\; \begin{array}{c} M \text{ has a flat resolution } F_\bullet \\ \text{such that } F_j = 0 \text{ for all } j > n. \end{array} \right\}.$$

In other words, we have $\operatorname{fd}_R(M) \leqslant n$ if there is an exact sequence

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0$$

and we have $\operatorname{fd}_R(M) = \infty$ if M does not have a bounded flat resolution.

Remark VII.6.2. Let R be a commutative ring, and let M be an R-module. Then $\operatorname{fd}_R(M) \geqslant 0$, and M is flat if and only if $\operatorname{fd}_R(M) = 0$.

Lemma VII.6.3. *Let R be a commutative ring, and let M be an R-module. Then* $\mathrm{fd}_R(M) \leqslant \mathrm{pd}_R(M)$.

PROOF. Assume without loss of generality that $n = \mathrm{pd}_R(M) < \infty$, and consider an augmented projective resolution

$$0 \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0.$$

Exercise II.3.10(b) implies that each P_i is flat, so this resolution shows that we have $\mathrm{fd}_R(M) \leqslant n = \mathrm{pd}_R(M)$. □

The next result is proved like Lemma VII.3.3, using Exercise II.3.11.

Lemma VII.6.4. *Let R be a commutative ring. If M is an R-module and $U \subseteq R$ is a multiplicatively closed subset, then* $\mathrm{fd}_{U^{-1}R}(U^{-1}M) \leqslant \mathrm{fd}_R(M)$. □

The next lemma give a first connection between Tor and flat dimension. It is proved like Lemma VII.3.5.

Lemma VII.6.5. *Let R be a commutative ring, and let M be an R-module. Then M is flat if and only if* $\operatorname{Tor}_1^R(-, M) = 0$. □

Theorem VII.6.6. *Let R be a commutative ring, and consider an exact sequence*

$$0 \to M' \to M \to M'' \to 0$$

of R-module homomorphisms. Assume that M'' is flat. Then M' is flat if and only if M is flat.

PROOF. Assume that M is flat. To show that M' is flat, it suffices to show that $\operatorname{Tor}_1^R(N, M') = 0$ for every R-module N. Since M and M'' are flat, Proposition IV.4.7(b) implies that $\operatorname{Tor}_1^R(N, M) = 0 = \operatorname{Tor}_2^R(N, M'')$. Part of the long exact sequence in $\operatorname{Tor}^R(N, -)$ associated to the given sequence has the form

$$\underbrace{\operatorname{Tor}_R^2(N, M'')}_{=0} \to \operatorname{Tor}_R^1(N, M') \to \underbrace{\operatorname{Tor}_R^1(N, M)}_{=0}$$

and it follows that $\operatorname{Tor}_1^R(N, M') = 0$. □

The next result is a souped-up version of Lemma VII.6.5. It is proved like Theorem VII.3.8, using Lemma VII.4.4.

Theorem VII.6.7. *Let R be a commutative ring, and let M be an R-module. For an integer $n \geqslant 0$, the following conditions are equivalent:*

(i) $\mathrm{fd}_R(M) \leqslant n$;
(ii) $\operatorname{Tor}_i^R(-, M) = 0$ *for all* $i > n$;
(iii) $\operatorname{Tor}_{n+1}^R(-, M) = 0$;
(iv) *For each flat resolution $F_\bullet$ of M, the module $K_n = \operatorname{Coker}(\partial_{n+1}^F)$ is flat;*
(v) *For some flat resolution $F_\bullet$ of M, the module $K_n = \operatorname{Coker}(\partial_{n+1}^F)$ is flat.*

In particular, we have $\mathrm{fd}_R(M) = \sup\{n \geqslant 0 \mid \operatorname{Tor}_n^R(-, M) \neq 0\}$. □

The next two results are proved like Corollaries VII.3.9 and VII.3.10.

Corollary VII.6.8. *Let R be a commutative ring, and consider the following exact sequence of R-module homomorphisms*

$$0 \to M_3 \to M_2 \to M_1 \to 0.$$

If two of the M_i have finite flat dimension over R, then so does the third one. □

Corollary VII.6.9. *Let R be a commutative ring, and let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules. Then* $\operatorname{fd}_R(\coprod_\lambda M_\lambda) = \sup\{\operatorname{fd}_R(M_\lambda) \mid \lambda \in \Lambda\}$. □

Theorems VII.4.5 and VII.6.7 combine with Corollary V.4.9 to produce the following. It compares with Exercise II.3.9.

Corollary VII.6.10. *Let R be a commutative noetherian local ring, and let M be a non-zero finitely generated R-module. Then* $\operatorname{fd}_R(M) = \operatorname{pd}_R(M)$. *In particular, M is flat if and only if it is projective if and only if it is free.* □

The next result are versions of VII.5.9– VII.5.12 for flat dimension, with similar proofs.

Lemma VII.6.11. *Let R be a commutative ring, and let M be an R-module. Then M is flat if and only if* $\operatorname{Tor}_1^R(R/\mathfrak{a}, M) = 0$ *for each ideal $\mathfrak{a} \subseteq R$.* □

Theorem VII.6.12. *Let R be a commtuative ring, and let M be an R-module. For an integer $n \geqslant 0$, the following conditions are equivalent:*

(i) $\operatorname{fd}_R(M) \leqslant n$;
(ii) $\operatorname{Tor}_i^R(-, M) = 0$ *for all $i > n$;*
(iii) $\operatorname{Tor}_{n+1}^R(N, M) = 0$ *for each finitely generated R-module N; and*
(iv) $\operatorname{Tor}_{n+1}^R(R/\mathfrak{a}, M) = 0$ *for each ideal $\mathfrak{a} \subseteq R$.*

Hence, we have $\operatorname{fd}_R(M) = \sup\{n \geqslant 0 \mid \operatorname{Tor}_n^R(R/\mathfrak{a}, M) \neq 0$ *for some $\mathfrak{a} \subseteq R$}$\}$. □

Theorem VII.6.13. *Let R be a commtuative noetherian ring, and let M be an R-module. For an integer $n \geqslant 0$, the following conditions are equivalent:*

(i) $\operatorname{fd}_R(M) \leqslant n$;
(ii) $\operatorname{Tor}_i^R(-, M) = 0$ *for all $i > n$;*
(iii) $\operatorname{Tor}_{n+1}^R(N, M) = 0$ *for each finitely generated R-module N; and*
(iv) $\operatorname{Tor}_{n+1}^R(R/\mathfrak{p}, M) = 0$ *for each prime ideal $\mathfrak{p} \subsetneq R$.*

Hence, $\operatorname{fd}_R(M) = \sup\{n \geqslant 0 \mid \operatorname{Tor}_n^R(R/\mathfrak{p}, M) \neq 0$ *for some $\mathfrak{p} \in \operatorname{Spec}(R)$*$\}$. □

Corollary VII.6.14. *Let R be a commutative noetherian ring, and let $\{M_\lambda\}_{\lambda\in\Lambda}$ be a set of R-modules. Then* $\operatorname{fd}_R(\prod_\lambda M_\lambda) = \sup\{\operatorname{fd}_R(M_\lambda) \mid \lambda \in \Lambda\}$. □

The next result is proved like Corollary VII.5.13. It compares to III.2.11(b).

Corollary VII.6.15. *Let R be a commutative ring, and let M be an R-module.*

(a) *Given in integer $n \geqslant 1$, the following conditions are equivalent:*
  (i) $\operatorname{fd}_R(M) < n$;
  (ii) $\operatorname{fd}_{U^{-1}RU^{-1}}(U^{-1}M) < n$ *for each multiplicatively closed subset $U \subseteq R$;*
  (iii) $\operatorname{fd}_{R_\mathfrak{p}}(M_\mathfrak{p}) < n$ *for each prime ideal $\mathfrak{p} \subsetneq R$; and*
  (iv) $\operatorname{fd}_{R_\mathfrak{m}}(M_\mathfrak{m}) < n$ *for each maximal ideal $\mathfrak{m} \subsetneq R$.*

(b) *There are equalities*

$$\begin{aligned}\operatorname{fd}_R(M) &= \sup\{\operatorname{fd}_{U^{-1}RU^{-1}}(U^{-1}M) \mid U \text{ is a multiplicatively closed subset of } R\}\\ &= \sup\{\operatorname{fd}_{R_\mathfrak{p}}(M_\mathfrak{p}) \mid \mathfrak{p} \text{ is a prime ideal of } R\}\\ &= \sup\{\operatorname{fd}_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \text{ is a maximal ideal of } R\}.\end{aligned}$$

(c) *The following conditions are equivalent:*
  (i) *M is flat as an R-module;*
  (ii) *$U^{-1}M$ is flat as a $U^{-1}R$-module for each multiplcatively closed subset $U \subseteq R$;*

(iii) *$M_{\mathfrak{p}}$ is flat as an $R_{\mathfrak{p}}$-module for each prime ideal $\mathfrak{p} \subsetneq R$; and*
(iv) *$M_{\mathfrak{m}}$ is flat as an $R_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m} \subsetneq R$.* □

Exercises.

Exercise VII.6.16. Prove Lemma VII.6.4

Exercise VII.6.17. Prove Lemma VII.6.5

Exercise VII.6.18. Prove Theorem VII.6.7

Exercise VII.6.19. Prove Corollary VII.6.8

Exercise VII.6.20. Prove Corollary VII.6.9

Exercise VII.6.21. Prove Corollary VII.6.10

Exercise VII.6.22. Prove Lemma VII.6.11

Exercise VII.6.23. Prove Theorem VII.6.12

Exercise VII.6.24. Prove Theorem VII.6.13

Exercise VII.6.25. Prove Corollary VII.6.9

Exercise VII.6.26. Prove Corollary VII.6.15

CHAPTER VIII

Long Exact Sequences September 8, 2009

In this chapter, we construct the long exact sequences in Ext and Tor. We also construct the mapping cone of a chain map and use it to build the Koszul complex.

VIII.1. General Long Exact Sequences

The long exact sequences in Ext and Tor are special cases of the long exact sequence associated to a short exact sequence of chain maps.

Definition VIII.1.1. Let R be a commutative ring. A diagram of chain maps

$$0 \to M'_\bullet \xrightarrow{F_\bullet} M_\bullet \xrightarrow{G_\bullet} M''_\bullet \to 0$$

is a *short exact sequence* of chain maps if it is exact in each degree, that is, if each sequence

$$0 \to M'_i \xrightarrow{F_i} M_i \xrightarrow{G_i} M''_i \to 0$$

is exact.

Remark VIII.1.2. Let R be a commutative ring. A short exact sequence of chain maps is a diagram of R-module homomorphisms

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \partial^{M'}_{i+1}} & & \downarrow{\scriptstyle \partial^{M}_{i+1}} & & \downarrow{\scriptstyle \partial^{M''}_{i+1}} & & \\
0 & \longrightarrow & M'_i & \xrightarrow{F_i} & M_i & \xrightarrow{G_i} & M''_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial^{M'}_{i}} & & \downarrow{\scriptstyle \partial^{M}_{i}} & & \downarrow{\scriptstyle \partial^{M''}_{i}} & & \\
0 & \longrightarrow & M'_{i-1} & \xrightarrow{F_{i-1}} & M_{i-1} & \xrightarrow{G_{i-1}} & M''_{i-1} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial^{M'}_{i-1}} & & \downarrow{\scriptstyle \partial^{M}_{i-1}} & & \downarrow{\scriptstyle \partial^{M''}_{i-1}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

in which every square commutes, every column is a chain complex, and every row is exact. We will construct examples below.

Remark VIII.1.3. There is a more general notion of an exact sequence of chain maps (not only short exact sequences) but we will not need this notion here.

Here is the mother of all long exact sequences. The proof is quite long. The reader may wish to use the diagram in Remark VIII.1.2 to follow along with the various steps.

Theorem VIII.1.4. *Let R be a commutative ring, and fix a short exact sequence of chain maps*

$$0 \to M'_\bullet \xrightarrow{F_\bullet} M_\bullet \xrightarrow{G_\bullet} M''_\bullet \to 0.$$

There is a sequence of R-module homomorphisms

$$\{\eth_i \colon \mathrm{H}_i(M''_\bullet) \to \mathrm{H}_{i-1}(M'_\bullet) \mid i \in \mathbb{Z}\}$$

making the following sequence exact:

$$\cdots \xrightarrow{\eth_{i+1}} \mathrm{H}_i(M'_\bullet) \xrightarrow{\mathrm{H}_i(F_\bullet)} \mathrm{H}_i(M_\bullet) \xrightarrow{\mathrm{H}_i(G_\bullet)} \mathrm{H}_i(M''_\bullet) \xrightarrow{\eth_i} \mathrm{H}_{i-1}(M'_\bullet) \xrightarrow{\mathrm{H}_{i-1}(F_\bullet)} \cdots .$$

Proof. We complete the proof in nine steps.

Step 1. We construct $\eth_i$. Let $\xi \in \mathrm{H}_i(M''_\bullet)$ be given. The definition

$$\mathrm{H}_i(M''_\bullet) = \mathrm{Ker}(\partial_i^{M''})/\operatorname{Im}(\partial_{i+1}^{M''})$$

implies that there is an element $\alpha \in \mathrm{Ker}(\partial_i^{M''})$ such that $\xi = \overline{\alpha}$. The map G_i is surjective, so there is an element $\beta \in M_i$ such that $G_i(\beta) = \alpha$. Since $G_\bullet$ is a chain map, we have the first equality in the next sequence:

$$G_{i-1}(\partial_i^M(\beta)) = \partial_i^{M''}(G_i(\beta)) = \partial_i^{M''}(\alpha) = 0.$$

The second equality is from the definition of β, and the third equality is from the condition $\alpha \in \mathrm{Ker}(\partial_i^{M''})$. It follows that $\partial_i^M(\beta) \in \mathrm{Ker}(G_{i-1}) = \operatorname{Im}(F_{i-1})$, so there is an element $\gamma \in M'_{i-1}$ such that $F_{i-1}(\gamma) = \partial_i^M(\beta)$. We define

$$\eth_i(\xi) = \overline{\gamma} \in \mathrm{Ker}(\partial_{i-1}^{M'})/\operatorname{Im}(\partial_i^{M'}) = \mathrm{H}_{i-1}(M'_\bullet).$$

Step 2. We show that $\eth_i$ is well-defined. The first thing we need to show is that $\gamma \in \mathrm{Ker}(\partial_{i-1}^{M'})$. For this, we compute as follows:

$$F_{i-2}(\partial_{i-1}^{M'}(\gamma)) = \partial_{i-1}^M(F_{i-1}(\gamma)) = \partial_{i-1}^M(\partial_i^M(\beta))) = 0.$$

The first equality is from the fact that $F_\bullet$ is a chain map. The second equality is from the definition of γ. The third equality is from the fact that $M_\bullet$ is an R-complex. Since the map F_{i-2} is injective, we conclude that $\partial_{i-1}^{M'}(\gamma) = 0$, that is, that $\gamma \in \mathrm{Ker}(\partial_{i-1}^{M'})$, as desired.

The second thing to show is that $\overline{\gamma} \in \mathrm{H}_{i-1}(M'_\bullet)$ is independent of the choices made in Step 1. To this end, let $\alpha, \alpha' \in \mathrm{Ker}(\partial_i^{M''})$ such that $\overline{\alpha} = \xi = \overline{\alpha'}$ in $\mathrm{H}_i(M''_\bullet)$. Let $\beta, \beta' \in M_i$ such that $G_i(\beta) = \alpha$ and $G_i(\beta') = \alpha'$. And let $\gamma, \gamma' \in M'_{i-1}$ such that $F_{i-1}(\gamma) = \partial_i^M(\beta)$ and $F_{i-1}(\gamma') = \partial_i^M(\beta')$. We need to show that $\overline{\gamma} = \overline{\gamma'}$ in $\mathrm{H}_{i-1}(M'_\bullet) = \mathrm{Ker}(\partial_{i-1}^{M'})/\operatorname{Im}(\partial_i^{M'})$. That is, we need to show that $\gamma - \gamma' \in \operatorname{Im}(\partial_i^{M'})$. That is, we need to find an element $\omega \in M'_i$ such that $\partial_i^{M'}(\omega) = \gamma - \gamma'$.

By assumption, we have $\overline{\alpha} = \overline{\alpha'}$ in $\mathrm{H}_i(M'_\bullet) = \mathrm{Ker}(\partial_i^{M''})/\operatorname{Im}(\partial_{i+1}^{M''})$. This implies that $\alpha - \alpha' \in \operatorname{Im}(\partial_{i+1}^{M''})$, so there is an element $\eta \in M''_{i+1}$ such that $\alpha - \alpha' = \partial_{i+1}^{M''}(\eta)$. The map G_{i+1} is surjective, so there is an element $\nu \in M_{i+1}$ such that $G_{i+1}(\nu) = \eta$. To continue, we compute:

$$G_i(\beta - \beta' - \partial_{i+1}^M(\nu)) = G_i(\beta) - G_i(\beta') - G_i(\partial_{i+1}^M(\nu)) = \alpha - \alpha' - (\alpha - \alpha') = 0$$

This computation shows that $\beta - \beta' - \partial_{i+1}^M(\nu) \in \mathrm{Ker}(G_i) = \operatorname{Im}(F_i)$, so there is an element $\omega \in M'_i$ such that $F_i(\omega) = \beta - \beta' - \partial_{i+1}^M(\nu)$. The fact that F_i is an

R-module homomorphism explains the first equality in the next sequence:

$$\begin{aligned} F_{i-1}(\partial_i^{M'}(\omega) - (\gamma - \gamma')) &= F_{i-1}(\partial_i^{M'}(\omega)) - F_{i-1}(\gamma) + F_{i-1}(\gamma') \\ &= \partial_i^M(F_i(\omega)) - \partial_i^M(\beta) + \partial_i^M(\beta') \\ &= \partial_i^M(\beta - \beta' - \partial_{i+1}^M(\nu)) - \partial_i^M(\beta) + \partial_i^M(\beta') \\ &= \partial_i^M(\beta - \beta' - \partial_{i+1}^M(\nu) - \beta + \beta') \\ &= -\partial_i^M(\partial_{i+1}^M(\nu)) \\ &= 0 \end{aligned}$$

The second equality follows from the fact that $F_\bullet$ is a chain map, with the defining properties of γ and γ'. The third equality is from the definition of ω. The fourth equality is from the linearity of ∂_i^M. The fifth equality is routine, and the sixth equality comes from the fact that $M_\bullet$ is an R-complex. Since F_{i-1} is injective, we conclude that $\partial_i^{M'}(\omega) - (\gamma - \gamma') = 0$, that is, that $\partial_i^{M'}(\omega) = \gamma - \gamma'$. This concludes Step 2.

Step 3. We show that $\eth_i$ is an R-module homomorphism. Let $\xi, \xi' \in \mathrm{H}_i(M''_\bullet)$ and $r \in R$. Let $\alpha, \alpha' \in \mathrm{Ker}(\partial_i^{M''})$ such that $\overline{\alpha} = \xi$ and $\overline{\alpha'} = \xi'$ in $\mathrm{H}_i(M''_\bullet)$. Let $\beta, \beta' \in M_i$ such that $G_i(\beta) = \alpha$ and $G_i(\beta') = \alpha'$. And let $\gamma, \gamma' \in M'_{i-1}$ such that $F_{i-1}(\gamma) = \partial_i^M(\beta)$ and $F_{i-1}(\gamma') = \partial_i^M(\beta')$. Notice that $\alpha + \alpha' \in \mathrm{Ker}(\partial_i^{M''})$ and that $\overline{\alpha + \alpha'} = \xi + \xi'$ in $\mathrm{H}_i(M''_\bullet)$. Furthermore, we have $\beta + \beta' \in M_i$ such that $G_i(\beta + \beta') = \alpha + \alpha'$. Also, we have $\gamma + \gamma' \in M'_{i-1}$ and

$$F_{i-1}(\gamma + \gamma') = F_{i-1}(\gamma) + F_{i-1}(\gamma') = \partial_i^M(\beta) + \partial_i^M(\beta') = \partial_i^M(\beta + \beta').$$

This explains the first equality in the next sequence:

$$\eth_i(\xi + \xi') = \overline{\gamma + \gamma'} = \overline{\gamma} + \overline{\gamma'} = \eth_i(\xi) + \eth_i(\xi').$$

Similarly, we have $r\alpha \in \mathrm{Ker}(\partial_i^{M''})$ and $\overline{r\alpha} = r\xi$ in $\mathrm{H}_i(M''_\bullet)$. Furthermore, we have $r\beta \in M_i$ such that $G_i(r\beta) = r\alpha$. Also, we have $r\gamma \in M'_{i-1}$ and

$$F_{i-1}(r\gamma) = rF_{i-1}(\gamma) = r\partial_i^M(\beta) = \partial_i^M(r\beta).$$

This explains the first equality in the next sequence:

$$\eth_i(r\xi) = \overline{r\gamma} = r\overline{\gamma} = r\eth_i(\xi).$$

This concludes Step 3.

Step 4. We show that $\mathrm{Im}(\mathrm{H}_i(F_\bullet)) \subseteq \mathrm{Ker}(\mathrm{H}_i(G_\bullet))$. Let $\delta \in \mathrm{H}_i(M''_\bullet)$, and let $\rho \in \mathrm{Ker}(\partial_i^{M''})$ such that $\delta = \overline{\rho}$. In the next sequence, the first two equalities are by definition:

$$\mathrm{H}_i(G_\bullet)(\mathrm{H}_i(F_\bullet)(\delta)) = \mathrm{H}_i(G_\bullet)(\overline{F_i(\rho)}) = \overline{G_i(F_i(\rho))} = \overline{0} = 0.$$

The third equality comes from the exactness of the original sequence of chain maps. This completes Step 4. (Here is a quicker proof: $\mathrm{H}_i(G_\bullet)\,\mathrm{H}_i(F_\bullet) = \mathrm{H}_i(G_\bullet F_\bullet) = \mathrm{H}_i(0) = 0$.)

Step 5. We show that $\mathrm{Im}(\mathrm{H}_i(F_\bullet)) \supseteq \mathrm{Ker}(\mathrm{H}_i(G_\bullet))$. Let

$$\delta \in \mathrm{Ker}(\mathrm{H}_i(G_\bullet)) \subseteq \mathrm{H}_i(M_\bullet)$$

and let $\rho \in \mathrm{Ker}(\partial_i^M)$ such that $\delta = \overline{\rho}$. The condition $\overline{\rho} \in \mathrm{Ker}(\mathrm{H}_i(G_\bullet))$ implies that $0 = \mathrm{H}_i(G_\bullet)(\overline{\rho}) = \overline{G(\rho)}$ in $\mathrm{H}_i(M''_\bullet) = \mathrm{Ker}(\partial_i^{M''})/\,\mathrm{Im}(\partial_{i+1}^{M''})$. Hence, we have

$G(\rho) \in \mathrm{Im}(\partial^{M''}_{i+1})$, so there is an element $\mu \in M''_{i+1}$ such that $G(\rho) = \partial^{M''}_{i+1}(\mu)$. The map G_{i+1} is surjective, so there is an element $\sigma \in M_{i+1}$ such that $G_{i+1}(\sigma) = \mu$, and this explains the third equality in the next sequence:

$$\begin{aligned} G_i(\rho - \partial^M_{i+1}(\sigma)) &= G_i(\rho) - G_i(\partial^M_{i+1}(\sigma)) \\ &= G_i(\rho) - \partial^{M''}_{i+1}(G_{i+1}(\sigma)) \\ &= G_i(\rho) - \partial^{M''}_{i+1}(\mu) \\ &= G_i(\rho) - G_i(\rho) \\ &= 0. \end{aligned}$$

It follows that $\rho - \partial^M_{i+1}(\sigma) \in \mathrm{Ker}(G_i) = \mathrm{Im}(F_i)$ so there is an element $\tau \in M'_i$ such that $F_i(\tau) = \rho - \partial^M_{i+1}(\sigma)$.

Claim: $\tau \in \mathrm{Ker}(\partial^{M'}_i)$. It suffices to show that $F_{i-1}(\partial^{M'}_i(\tau)) = 0$, since F_{i-1} is injective. So, we compute:

$$\begin{aligned} F_{i-1}(\partial^{M'}_i(\tau)) &= \partial^M_i(F_i(\tau)) = \partial^M_i(\rho - \partial^M_{i+1}(\sigma)) \\ &= \partial^M_i(\rho) - \partial^M_i(\partial^M_{i+1}(\sigma)) = 0. \end{aligned}$$

The first equality comes from the fact that $F_\bullet$ is a chain map. The second equality follows from our choice of τ. The third equality is from the additivity of ∂^M_i. The fourth equality comes from the conditions $\rho \in \mathrm{Ker}(\partial^M_i)$ and $\partial^M_i \partial^M_{i+1} = 0$.

The elements ρ and $\partial^M_{i+1}(\sigma)$ are both in $\mathrm{Ker}(\partial^M_i)$. Hence, they represent elements in $\mathrm{H}_i(M_\bullet)$. Similarly, the element τ represents an element in $\mathrm{H}_i(M'_\bullet)$. Hence, each equality in the next sequence is by definition:

$$\mathrm{H}_i(F_\bullet)(\overline{\tau}) = \overline{F_i(\tau)} = \overline{\rho - \partial^M_{i+1}(\sigma)} = \overline{\rho} - \overline{\partial^M_{i+1}(\sigma)} = \overline{\rho} = \delta.$$

It follows that $\delta \in \mathrm{Im}(\mathrm{H}_i(F_\bullet))$, and Step 5 is complete.

Step 6. We show that $\mathrm{Im}(\mathrm{H}_i(G_\bullet)) \subseteq \mathrm{Ker}(\eth_i)$. Let $\zeta \in \mathrm{H}_i(M_\bullet)$, and fix an element $\beta \in \mathrm{Ker}(\partial^M_i)$ such that $\zeta = \overline{\beta}$ in $\mathrm{H}_i(M_\bullet)$. We show that $\eth_i(\mathrm{H}_i(G_\bullet)(\overline{\beta})) = 0$. With $\alpha = G_i(\beta)$, we have $\mathrm{H}_i(G_\bullet)(\overline{\beta}) = \overline{G_i(\beta)} = \overline{\alpha}$. To compute $\eth_i(\mathrm{H}_i(G_\bullet)(\overline{\beta})) = \eth_i(\overline{\alpha})$, we need to find an element $\gamma \in \mathrm{Ker}(\partial^{M'}_{i-1})$ such that $F_{i-1}(\gamma) = \partial^M_i(\beta)$. However, we have $\beta \in \mathrm{Ker}(\partial^M_i)$ by assumption, so $\partial^M_i(\beta) = 0 = F_{i-1}(0)$. Thus, we may set $\gamma = 0$ to find $\eth_i(\mathrm{H}_i(G_\bullet)(\overline{\beta})) = \eth_i(\overline{\alpha}) = \overline{\gamma} = \overline{0} = 0$ as desired.

Step 7. We show that $\mathrm{Im}(\mathrm{H}_i(G_\bullet)) \supseteq \mathrm{Ker}(\eth_i)$. Let $\xi \in \mathrm{Ker}(\eth_i) \subseteq \mathrm{H}_i(M''_\bullet)$, and choose an element $\alpha \in \mathrm{Ker}(\partial^{M''}_i)$ such that $\xi = \overline{\alpha}$. Fix an element $\beta \in M_i$ such that $G_i(\beta) = \alpha$, and an element $\gamma \in M'_{i-1}$ such that $F_{i-1}(\gamma) = \partial^M_i(\beta)$. We then have $0 = \eth_i(\xi) = \overline{\gamma} \in \mathrm{H}_{i-1}(M'_\bullet) = \mathrm{Ker}(\partial^{M'}_{i-1})/\mathrm{Im}(\partial^{M'}_i)$. It follows that $\gamma \in \mathrm{Im}(\partial^{M'}_i)$, so there is an element $\omega \in M'_i$ such that $\partial^{M'}_i(\omega) = \gamma$.

Observe that $\beta - F_i(\omega) \in \mathrm{Ker}(\partial^M_i)$. Indeed, in the following sequence, each step is by definition:

$$\begin{aligned} \partial^M_i(\beta - F_i(\omega)) &= \partial^M_i(\beta) - \partial^M_i(F_i(\omega)) = \partial^M_i(\beta) - F_{i-1}(\partial^{M'}_i(\omega)) \\ &= \partial^M_i(\beta) - F_{i-1}(\omega) = \partial^M_i(\beta) - \partial^M_i(\beta) = 0. \end{aligned}$$

It follows that $\beta - F_i(\omega)$ represents an element of $\mathrm{H}_i(M_\bullet)$. Furthermore, we have the following sequence of equalities wherein the third equality is from the condition

$G_iF_i = 0$, and the other equalities are by definition:

$$\mathrm{H}_i(G_\bullet)(\overline{\beta - F_i(\omega)}) = \overline{G_i(\beta - F_i(\omega))} = \overline{G_i(\beta) - G_i(F_i(\omega))} = \overline{G_i(\beta)} = \overline{\alpha} = \xi.$$

It follows that $\xi \in \mathrm{Im}(\mathrm{H}_i(G_\bullet))$, and Step 7 is complete.

Step 8. We show that $\mathrm{Im}(\eth_i) \subseteq \mathrm{Ker}(\mathrm{H}_{i-1}(F_\bullet))$. Let $\xi \in \mathrm{H}_i(M''_\bullet)$, and fix an element $\alpha \in \mathrm{Ker}(\partial_i^{M''})$ such that $\xi = \overline{\alpha}$. We need to show that $\mathrm{H}_{i-1}(F_\bullet)(\eth_i(\overline{\alpha})) = 0$. Choose an element $\beta \in M_i$ such that $G_i(\beta) = \alpha$, and an element $\gamma \in M'_{i-1}$ such that $F_{i-1}(\gamma) = \partial_i^M(\beta)$. We then have

$$\mathrm{H}_{i-1}(F_\bullet)(\eth_i(\overline{\alpha})) = \mathrm{H}_{i-1}(F_\bullet)(\overline{\gamma}) = \overline{F_{i-1}(\gamma)} = \overline{\partial_i^M(\beta)} = 0$$

as desired.

Step 9. We show that $\mathrm{Im}(\eth_i) \supseteq \mathrm{Ker}(\mathrm{H}_{i-1}(F_\bullet))$. Let $\lambda \in \mathrm{Ker}(\mathrm{H}_{i-1}(F_\bullet))$, and fix an element $\gamma \in \mathrm{Ker}(\partial_{i-1}^{M'})$ such that $\lambda = \overline{\gamma}$ in $\mathrm{H}_{i-1}(M'_\bullet)$. By assumption, we have

$$0 = \mathrm{H}_{i-1}(F_\bullet)(\lambda) = \mathrm{H}_{i-1}(F_\bullet)(\overline{\gamma}) = \overline{F_{i-1}(\gamma)}$$

in $\mathrm{H}_{i-1}(M_\bullet)$. It follows that $F_{i-1}(\gamma) \in \mathrm{Im}(\partial_i^M)$ so there is an element $\beta \in M_i$ such that $F_{i-1}(\gamma) = \partial_i^M(\beta)$. Set $\alpha = G_i(\beta)$.

Observe that $\alpha \in \mathrm{Ker}(\partial_i^{M''})$. Indeed, by definition we have

$$\partial_i^{M''}(\alpha) = \partial_i^{M''}(G_i(\beta)) = G_{i-1}(\partial_i^M(\beta)) = G_{i-1}(F_{i-1}(\gamma)) = 0.$$

It follows that α represents an element of $\mathrm{H}_i(M''_\bullet)$, and furthermore that

$$\eth_i(\overline{\alpha}) = \overline{\gamma} = \lambda$$

which implies that $\lambda \in \mathrm{Im}(\eth_i)$, as desired.

This completes the proof of the theorem. □

Corollary VIII.1.5 (Snake Lemma). *Let R be a commutative ring, and consider the following commutative diagram of R-module homomorphisms with exact rows:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M'_1 & \xrightarrow{F_1} & M_1 & \xrightarrow{G_1} & M''_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_1^{M'}} & & \downarrow{\scriptstyle \partial_1^{M}} & & \downarrow{\scriptstyle \partial_1^{M''}} & & \\
0 & \longrightarrow & M'_0 & \xrightarrow{F_0} & M_0 & \xrightarrow{G_0} & M''_0 & \longrightarrow & 0.
\end{array}$$

Then there is an exact sequence

$$0 \longrightarrow \mathrm{Ker}(\partial_1^{M'}) \xrightarrow{\overline{F_1}} \mathrm{Ker}(\partial_1^{M}) \xrightarrow{\overline{G_1}} \mathrm{Ker}(\partial_1^{M''}) \xrightarrow{\eth_1} \mathrm{Coker}(\partial_1^{M'}) \xrightarrow{\overline{F_0}} \mathrm{Coker}(\partial_1^{M}) \xrightarrow{\overline{G_0}} \mathrm{Coker}(\partial_1^{M''}) \longrightarrow 0.$$

PROOF. The given commutative diagram extends to the following short exact sequence of chain maps:

$$0 \longrightarrow M'_\bullet \xrightarrow{F_\bullet} M_\bullet \xrightarrow{G_\bullet} M''_\bullet \longrightarrow 0$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M'_1 & \xrightarrow{F_1} & M_1 & \xrightarrow{G_1} & M''_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_1^{M'}} & & \downarrow{\scriptstyle \partial_1^{M}} & & \downarrow{\scriptstyle \partial_1^{M''}} & & \\
0 & \longrightarrow & M'_0 & \xrightarrow{F_0} & M_0 & \xrightarrow{G_0} & M''_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

The desired exact sequence of kernels and cokernels is precisely the long exact sequence guaranteed by Theorem VIII.1.4. For instance, we have $\mathrm{H}_1(M_\bullet) = \mathrm{Ker}(\partial_1^M)$ and $\mathrm{H}_0(M_\bullet) = \mathrm{Coker}(\partial_1^M)$ □

Exercises.

Exercise VIII.1.6. (Functoriality of long exact sequences) Let R be a commutative ring, and consider the following diagram of chain maps:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M'_\bullet & \xrightarrow{F_\bullet} & M_\bullet & \xrightarrow{G_\bullet} & M''_\bullet & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_\bullet} & & \downarrow{\scriptstyle g_\bullet} & & \downarrow{\scriptstyle h_\bullet} & & \\
0 & \longrightarrow & N'_\bullet & \xrightarrow{H_\bullet} & N_\bullet & \xrightarrow{K_\bullet} & N''_\bullet & \longrightarrow & 0.
\end{array}$$

Assume that, for each integer i, the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M'_i & \xrightarrow{F_i} & M_i & \xrightarrow{G_i} & M''_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle h_i} & & \\
0 & \longrightarrow & N'_i & \xrightarrow{H_i} & N_i & \xrightarrow{K_i} & N''_i & \longrightarrow & 0.
\end{array}$$

Show that the following diagram of long exact sequences commutes:

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\eth_{i+1}^M} & \mathrm{H}_i(M'_\bullet) & \xrightarrow{\mathrm{H}_i(F_\bullet)} & \mathrm{H}_i(M_\bullet) & \xrightarrow{\mathrm{H}_i(G_\bullet)} & \mathrm{H}_i(M''_\bullet) & \xrightarrow{\eth_i^M} & \mathrm{H}_{i-1}(M'_\bullet) & \xrightarrow{\mathrm{H}_{i-1}(F_\bullet)} & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{H}_i(f_\bullet)} & & \downarrow{\scriptstyle \mathrm{H}_i(g_\bullet)} & & \downarrow{\scriptstyle \mathrm{H}_i(h_\bullet)} & & \downarrow{\scriptstyle \mathrm{H}_{i-1}(f_\bullet)} & & \\
\cdots & \xrightarrow{\eth_{i+1}^N} & \mathrm{H}_i(N'_\bullet) & \xrightarrow{\mathrm{H}_i(H_\bullet)} & \mathrm{H}_i(N_\bullet) & \xrightarrow{\mathrm{H}_i(K_\bullet)} & \mathrm{H}_i(N''_\bullet) & \xrightarrow{\eth_i^N} & \mathrm{H}_{i-1}(N'_\bullet) & \xrightarrow{\mathrm{H}_{i-1}(H_\bullet)} & \cdots .
\end{array}$$

Exercise VIII.1.7. Let R be a commutative ring, and consider the following commutative diagram of R-module homomorphisms with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_1' & \xrightarrow{F_1} & M_1 & \xrightarrow{G_1} & M_1'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial_1^{M'}} & & \downarrow{\scriptstyle \partial_1^{M}} & & \downarrow{\scriptstyle \partial_1^{M''}} & & \\ 0 & \longrightarrow & M_0' & \xrightarrow{F_0} & M_0 & \xrightarrow{G_0} & M_0'' & \longrightarrow & 0. \end{array}$$

Assume that $\partial_1^{M'}$ is surjective, and show that ∂_1^{M} is surjective if and only if $\partial_1^{M''}$ is surjective.

Exercise VIII.1.8. Prove the following generalized Snake Lemma without using Theorem VIII.1.4 or Corollary VIII.1.5: Let R be a commutative ring, and consider the following commutative diagram of R-module homomorphisms with exact rows:

$$\begin{array}{ccccccc} & & M_1' & \xrightarrow{F_1} & M_1 & \xrightarrow{G_1} & M_1'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial_1^{M'}} & & \downarrow{\scriptstyle \partial_1^{M}} & & \downarrow{\scriptstyle \partial_1^{M''}} & & \\ 0 & \longrightarrow & M_0' & \xrightarrow{F_0} & M_0 & \xrightarrow{G_0} & M_0''. & & \end{array}$$

Then there is an exact sequence

$$\begin{array}{ccccccc} \operatorname{Ker}(\partial_1^{M'}) & \xrightarrow{\overline{F_1}} & \operatorname{Ker}(\partial_1^{M}) & \xrightarrow{\overline{G_1}} & \operatorname{Ker}(\partial_1^{M''}) & & \\ & & & \overset{\eth_1}{\searrow} & & & \\ & & \operatorname{Coker}(\partial_1^{M'}) & \xrightarrow{\overline{F_0}} & \operatorname{Coker}(\partial_1^{M}) & \xrightarrow{\overline{G_0}} & \operatorname{Coker}(\partial_1^{M''}). \end{array}$$

Exercise VIII.1.9. Use Exercise VIII.1.8 to give an alternate proof of Theorem VIII.1.4.

VIII.2. Long Exact Sequences in Ext and Tor

In this section, we derive the long exact sequences in Ext and Tor.

Theorem VIII.2.1. *Let R be a commutative ring, and let N be an R-module. Given an exact sequence of R-module homomorphisms*

$$0 \to K \xrightarrow{g} M \xrightarrow{f} C \to 0$$

there is a long exact sequence (in $\operatorname{Ext}^i_R(N,-)$)

$$\begin{aligned} 0 \to \operatorname{Hom}_R(N,K) &\xrightarrow{\operatorname{Hom}_R(N,g)} \operatorname{Hom}_R(N,M) \xrightarrow{\operatorname{Hom}_R(N,f)} \operatorname{Hom}_R(N,C) \\ \to \operatorname{Ext}^1_R(N,K) &\xrightarrow{\operatorname{Ext}^1_R(N,g)} \operatorname{Ext}^1_R(N,M) \xrightarrow{\operatorname{Ext}^1_R(N,f)} \operatorname{Ext}^1_R(N,C) \to \cdots \\ \cdots \to \operatorname{Ext}^i_R(N,K) &\xrightarrow{\operatorname{Ext}^i_R(N,g)} \operatorname{Ext}^i_R(N,M) \xrightarrow{\operatorname{Ext}^i_R(N,f)} \operatorname{Ext}^i_R(N,C) \to \cdots. \end{aligned}$$

Proof. Let $P_\bullet$ be a projective resolution of N. Proposition VI.4.2 implies that the following sequences are chain maps

$$\begin{aligned} \operatorname{Hom}(P_\bullet, g)\colon \operatorname{Hom}(P_\bullet, K) &\to \operatorname{Hom}(P_\bullet, M) \\ \operatorname{Hom}(P_\bullet, f)\colon \operatorname{Hom}(P_\bullet, M) &\to \operatorname{Hom}(P_\bullet, C). \end{aligned}$$

Together, these chain maps form a short exact sequence

$$0 \longrightarrow \mathrm{Hom}(P_\bullet, K) \xrightarrow{\mathrm{Hom}(P_\bullet, g)} \mathrm{Hom}(P_\bullet, M) \xrightarrow{\mathrm{Hom}(P_\bullet, f)} \mathrm{Hom}(P_\bullet, C) \longrightarrow 0$$

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \mathrm{Hom}(\partial_i^P, K)} & & \downarrow{\scriptstyle \mathrm{Hom}(\partial_i^P, M)} & & \downarrow{\scriptstyle \mathrm{Hom}(\partial_i^P, C)} & & \\
0 & \longrightarrow & \mathrm{Hom}(P_i, K) & \xrightarrow{\mathrm{Hom}(P_i, g)} & \mathrm{Hom}(P_i, M) & \xrightarrow{\mathrm{Hom}(P_i, f)} & \mathrm{Hom}(P_i, C) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Hom}(\partial_{i+1}^P, K)} & & \downarrow{\scriptstyle \mathrm{Hom}(\partial_{i+1}^P, M)} & & \downarrow{\scriptstyle \mathrm{Hom}(\partial_{i+1}^P, C)} & & \\
0 & \longrightarrow & \mathrm{Hom}(P_{i+1}, K) & \xrightarrow{\mathrm{Hom}(P_{i+1}, g)} & \mathrm{Hom}(P_{i+1}, M) & \xrightarrow{\mathrm{Hom}(P_{i+1}, f)} & \mathrm{Hom}(P_{i+1}, C) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Hom}(\partial_{i+2}^P, K)} & & \downarrow{\scriptstyle \mathrm{Hom}(\partial_{i+2}^P, M)} & & \downarrow{\scriptstyle \mathrm{Hom}(\partial_{i+2}^P, C)} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

The exactness of each row follows from the assumption that each P_i is projective. Apply Theorem VIII.1.4 to this short exact sequence to derive the desired long exact sequence. □

Theorem VIII.2.2. *Let R be a commutative ring, and let N be an R-module. Given an exact sequence of R-module homomorphisms*

$$0 \to K \xrightarrow{g} M \xrightarrow{f} C \to 0$$

there is a long exact sequence (in $\mathrm{Ext}^i_R(-, N)$)

$$\begin{aligned}
0 \to \mathrm{Hom}_R(C, N) &\xrightarrow{\mathrm{Hom}_R(f,N)} \mathrm{Hom}_R(M, N) \xrightarrow{\mathrm{Hom}_R(g,N)} \mathrm{Hom}_R(K, N) \\
&\to \mathrm{Ext}^1_R(C, N) \xrightarrow{\mathrm{Ext}^1_R(f,N)} \mathrm{Ext}^1_R(M, N) \xrightarrow{\mathrm{Ext}^1_R(g,N)} \mathrm{Ext}^1_R(K, N) \to \cdots \\
\cdots \to \mathrm{Ext}^i_R(C, N) &\xrightarrow{\mathrm{Ext}^i_R(f,N)} \mathrm{Ext}^i_R(M, N) \xrightarrow{\mathrm{Ext}^i_R(g,N)} \mathrm{Ext}^i_R(K, N) \to \cdots .
\end{aligned}$$

Proof. Let $I_\bullet$ be an injective resolution of N. As in the proof of Theorem VIII.2.1, we use Proposition VI.4.2 to show that the following is a short exact

sequence of chain maps:

$$0 \longrightarrow \operatorname{Hom}(C, I_\bullet) \xrightarrow{\operatorname{Hom}(f, I_\bullet)} \operatorname{Hom}(M, I_\bullet) \xrightarrow{\operatorname{Hom}(g, I_\bullet)} \operatorname{Hom}(K, I_\bullet) \longrightarrow 0$$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow{\scriptstyle \operatorname{Hom}(C,\partial^I_{j+1})} & & \downarrow{\scriptstyle \operatorname{Hom}(M,\partial^I_{j+1})} & & \downarrow{\scriptstyle \operatorname{Hom}(K,\partial^I_{j+1})} \\
0 & \longrightarrow & \operatorname{Hom}(C, I_j) & \xrightarrow{\operatorname{Hom}(f, I_j)} & \operatorname{Hom}(M, I_j) & \xrightarrow{\operatorname{Hom}(g, I_j)} & \operatorname{Hom}(K, I_j) \longrightarrow 0 \\
 & & \downarrow{\scriptstyle \operatorname{Hom}(C,\partial^I_j)} & & \downarrow{\scriptstyle M,\operatorname{Hom}(M,\partial^I_j)} & & \downarrow{\scriptstyle \operatorname{Hom}(K,\partial^I_j)} \\
0 & \longrightarrow & \operatorname{Hom}(C, I_{j-1}) & \xrightarrow{\operatorname{Hom}(f, I_{j-1})} & \operatorname{Hom}(M, I_{j-1}) & \xrightarrow{\operatorname{Hom}(g, I_{j-1})} & \operatorname{Hom}(K, I_{j-1}) \longrightarrow 0 \\
 & & \downarrow{\scriptstyle \operatorname{Hom}(C,\partial^I_{j-1})} & & \downarrow{\scriptstyle M,\operatorname{Hom}(M,\partial^I_{j-1})} & & \downarrow{\scriptstyle \operatorname{Hom}(K,\partial^I_{j-1})} \\
 & & \vdots & & \vdots & & \vdots
\end{array}$$

The desired long exact sequence comes from Theorem VIII.1.4. □

The next two results are proved similarly, using a projective resolution and the tensor product.

Theorem VIII.2.3. *Let R be a commutative ring, and let N be an R-module. Given an exact sequence of R-module homomorphisms*

$$0 \to K \xrightarrow{g} M \xrightarrow{f} C \to 0$$

there is a long exact sequence (in $\operatorname{Tor}_i^R(-, N)$)

$$\cdots \to \operatorname{Tor}_i^R(K, N) \xrightarrow{\operatorname{Tor}_i^R(g,N)} \operatorname{Tor}_i^R(M, N) \xrightarrow{\operatorname{Tor}_i^R(f,N)} \operatorname{Tor}_i^R(C, N) \to \cdots$$

$$\cdots \to \operatorname{Tor}_1^R(K, N) \xrightarrow{\operatorname{Tor}_1^R(g,N)} \operatorname{Tor}_1^R(M, N) \xrightarrow{\operatorname{Tor}_1^R(f,N)} \operatorname{Tor}_1^R(C, N) \to$$

$$\longrightarrow K \otimes_R N \xrightarrow{g \otimes_R N} M \otimes_R N \xrightarrow{f \otimes_R N} C \otimes_R N \longrightarrow 0.$$

Theorem VIII.2.4. *Let R be a commutative ring, and let N be an R-module. Given an exact sequence of R-module homomorphisms*

$$0 \to K \xrightarrow{g} M \xrightarrow{f} C \to 0$$

there is a long exact sequence (in $\operatorname{Tor}_i^R(N, -)$)

$$\cdots \to \operatorname{Tor}_i^R(N, K) \xrightarrow{\operatorname{Tor}_i^R(N,g)} \operatorname{Tor}_i^R(N, M) \xrightarrow{\operatorname{Tor}_i^R(N,f)} \operatorname{Tor}_i^R(N, C) \to \cdots$$

$$\cdots \to \operatorname{Tor}_1^R(N, K) \xrightarrow{\operatorname{Tor}_1^R(N,g)} \operatorname{Tor}_1^R(N, M) \xrightarrow{\operatorname{Tor}_1^R(N,f)} \operatorname{Tor}_1^R(N, C) \to$$

$$\longrightarrow N \otimes_R K \xrightarrow{N \otimes_R g} N \otimes_R M \xrightarrow{N \otimes_R f} N \otimes_R C \longrightarrow 0.$$

Exercises.

Exercise VIII.2.5. Complete the proof of Theorem VIII.2.2.

Exercise VIII.2.6. Prove Theorems VIII.2.3 and VIII.2.4.

Exercise VIII.2.7. (Functoriality of long exact sequences) Let R be a commutative ring, and let $\phi\colon L \to N$ be an R-module homomorphism. Given an exact sequence of R-module homomorphisms

$$0 \to K \xrightarrow{g} M \xrightarrow{f} C \to 0$$

show that there are commutative diagrams of long exact sequences:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \operatorname{Ext}^i_R(N,K) & \longrightarrow & \operatorname{Ext}^i_R(N,M) & \longrightarrow & \operatorname{Ext}^i_R(N,C) & \longrightarrow & \operatorname{Ext}^{i+1}_R(N,K) \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \operatorname{Ext}^i_R(L,K) & \longrightarrow & \operatorname{Ext}^i_R(L,M) & \longrightarrow & \operatorname{Ext}^i_R(L,C) & \longrightarrow & \operatorname{Ext}^{i+1}_R(L,K) \longrightarrow \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \operatorname{Ext}^i_R(C,L) & \longrightarrow & \operatorname{Ext}^i_R(M,L) & \longrightarrow & \operatorname{Ext}^i_R(K,L) & \longrightarrow & \operatorname{Ext}^{i+1}_R(C,L) \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \operatorname{Ext}^i_R(C,N) & \longrightarrow & \operatorname{Ext}^i_R(M,N) & \longrightarrow & \operatorname{Ext}^i_R(K,N) & \longrightarrow & \operatorname{Ext}^{i+1}_R(C,N) \longrightarrow \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \operatorname{Tor}^R_i(L,K) & \longrightarrow & \operatorname{Tor}^R_i(L,M) & \longrightarrow & \operatorname{Tor}^R_i(L,C) & \longrightarrow & \operatorname{Tor}^R_{i-1}(L,K) \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \operatorname{Tor}^R_i(N,K) & \longrightarrow & \operatorname{Tor}^R_i(N,M) & \longrightarrow & \operatorname{Tor}^R_i(N,C) & \longrightarrow & \operatorname{Tor}^R_{i-1}(N,K) \longrightarrow \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \operatorname{Tor}^R_i(K,L) & \longrightarrow & \operatorname{Tor}^R_i(M,L) & \longrightarrow & \operatorname{Tor}^R_i(C,L) & \longrightarrow & \operatorname{Tor}^R_{i-1}(K,L) \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & \operatorname{Tor}^R_i(K,N) & \longrightarrow & \operatorname{Tor}^R_i(M,N) & \longrightarrow & \operatorname{Tor}^R_i(C,N) & \longrightarrow & \operatorname{Tor}^R_{i-1}(K,N) \longrightarrow \cdots .
\end{array}$$

where the vertical maps are induced by ϕ.

VIII.3. Horseshoe Lemmas

Lemma VIII.3.1. *Let R be a commutative ring, and consider a short exact sequence of R-module homomorphisms*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0.$$

Let $\tau'\colon P' \to M'$ and $\tau''\colon P'' \to M''$ be surjections where P' and P'' are projective. There is a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P' & \xrightarrow{\epsilon} & P' \oplus P'' & \xrightarrow{\pi} & P'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau'} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau''} & & \\
0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where ϵ and π are the natural injection and surjection, respectively.

PROOF. Use the fact that P'' is projective to find an R-module homomorphism $h\colon P'' \to M$ making the following diagram commute:

$$\begin{array}{ccccc} & & P'' & & \\ & \swarrow^{h} & \downarrow{\scriptstyle \tau''} & & \\ M & \xrightarrow{g} & M'' & \longrightarrow & 0. \end{array}$$

Define $\tau\colon P' \oplus P'' \to M$ by the formula

$$\tau(p', p'') = (f(\tau'(p')), h(p'')).$$

Check that τ is an R-module homomorphism and that τ makes the desired diagram commute. Since τ' and τ'' are surjective, a diagram chase (or the Snake Lemma or the Short Five Lemma) shows that τ is also surjective. □

Lemma VIII.3.2 (Horseshoe Lemma)**.** *Let R be a commutative ring, and consider a short exact sequence of R-module homomorphisms*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0.$$

Let $P'_\bullet$ be a projective resolution of M', and let $P''_\bullet$ be a projective resolution of M''. There is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \partial_2^{P'}} & & \downarrow{\scriptstyle \partial_2^{P}} & & \downarrow{\scriptstyle \partial_2^{P''}} & & \\
0 & \longrightarrow & P'_1 & \xrightarrow{F_1} & P_1 & \xrightarrow{G_1} & P''_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_1^{P'}} & & \downarrow{\scriptstyle \partial_1^{P}} & & \downarrow{\scriptstyle \partial_1^{P''}} & & \\
0 & \longrightarrow & P'_0 & \xrightarrow{F_0} & P_0 & \xrightarrow{G_0} & P''_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau'} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau''} & & \\
0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

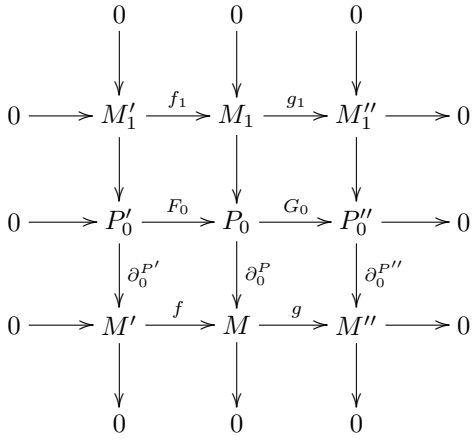
such that the middle column is an augmented projective resolution of M.

Remark VIII.3.3. Note that each row of the diagram (except the bottom row) is split since each P''_i is projective.

PROOF. Use Lemma VIII.3.1 to construct a commutative diagram with exact rows and columns

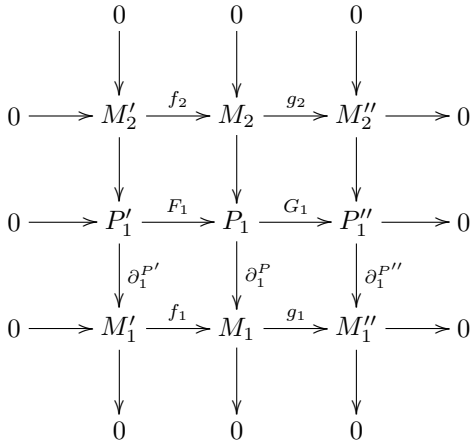
$$\begin{array}{ccccccccc}
0 & \longrightarrow & P'_0 & \xrightarrow{F_0} & P'_0 \oplus P''_0 & \xrightarrow{G_0} & P''_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau'} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau''} & & \\
0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where F_0 and G_0 are the natural injection and surjection, respectively. Set $M_1' = \mathrm{Ker}(\tau')$ and $M_1 = \mathrm{Ker}(\tau)$ and $M_1'' = \mathrm{Ker}(\tau'')$. The Snake Lemma shows that the following commutative diagram has exact rows and exact columns

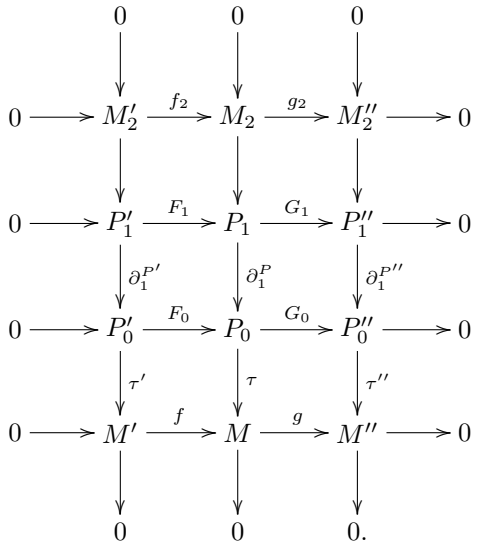


where the unlabeled vertical maps are the inclusions, and the maps f_1 and g_1 are induced by F_0 and G_0, respectively.

Repeat this process using the new sequence $0 \to M_1' \xrightarrow{f_1} M_1 \xrightarrow{g_1} M_1'' \to 0$ to obtain a commutative diagram with exact rows and exact columns



and splice these two diagrams together to obtain the next commutative diagram with exact rows and exact columns



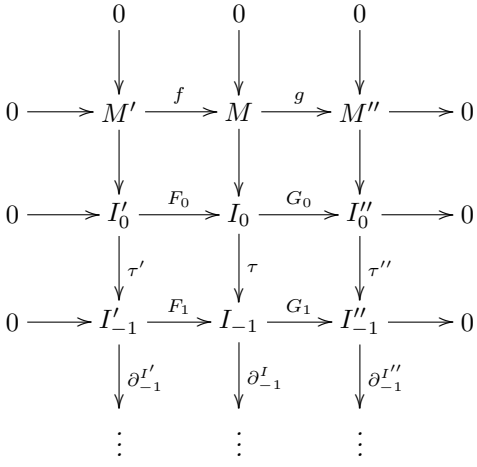
Continue this process inductively to construct the desired diagram. □

The next result is an injective version of Lemma VIII.3.2 with a similar proof.

Lemma VIII.3.4 (Horseshoe Lemma)**.** *Let R be a commutative ring, and consider a short exact sequence of R-module homomorphisms*

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0.$$

Let $I'_\bullet$ be an injective resolution of M', and let $I''_\bullet$ be an injective resolution of M''. There is a commutative diagram with exact rows



such that the middle column is an augmented injective resolution of M. □

Exercises.

Exercise VIII.3.5. Complete the proof of Lemma VIII.3.2.

Exercise VIII.3.6. Prove Lemma VIII.3.4.

Exercise VIII.3.7. Use Lemma VIII.3.4 to reprove Theorem VIII.2.1.

Exercise VIII.3.8. Use Lemma VIII.3.2 to reprove Theorem VIII.2.2.

Exercise VIII.3.9. Use Lemma VIII.3.2 to prove Theorems VIII.2.3 and VIII.2.4.

VIII.4. Mapping Cones

In this section, we discuss the mapping cone of a chain map, which gives another important short exact sequence of chain maps. We begin with a definition.

Definition VIII.4.1. Let R be a commutative ring, and let $X_\bullet$ be an R-complex. The *suspension* or *shift* of $X_\bullet$ is the sequence $\Sigma X_\bullet$ defined as $(\Sigma X)_i = X_{i-1}$ and $\partial_i^{\Sigma X} = -\partial_{i-1}^X$.

Remark VIII.4.2. Let R be a commutative ring, and let $X_\bullet$ be an R-complex. Diagramatically, we see that $\Sigma X_\bullet$ is essentially obtained by shifting $X_\bullet$ one degree to the left:

$$X_\bullet = \cdots \xrightarrow{\partial_{i+1}^X} X_i \xrightarrow{\partial_i^X} X_{i-1} \xrightarrow{\partial_{i-1}^X} \cdots$$
$$\Sigma X_\bullet = \cdots \xrightarrow{-\partial_i^X} X_{i-1} \xrightarrow{-\partial_{i-1}^X} X_{i-2} \xrightarrow{-\partial_{i-2}^X} \cdots$$

It follows readily that $\Sigma X_\bullet$ is an R-complex and that there is an isomorphism $\mathrm{H}_n(\Sigma X_\bullet) = \mathrm{H}_{n-1}(X_\bullet)$ for each n.

Definition VIII.4.3. Let R be a commutative ring, and let $f_\bullet \colon X_\bullet \to Y_\bullet$ be a chain map. The *mapping cone* of $f_\bullet$ is the sequence $\mathrm{Cone}(f_\bullet)$ defined as follows:

$$\mathrm{Cone}(f_\bullet) = \cdots \to \begin{matrix} Y_i \\ \oplus \\ X_{i-1} \end{matrix} \xrightarrow{\begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix}} \begin{matrix} Y_{i-1} \\ \oplus \\ X_{i-2} \end{matrix} \xrightarrow{\begin{pmatrix} \partial_{i-1}^Y & f_{i-2} \\ 0 & -\partial_{i-2}^X \end{pmatrix}} \begin{matrix} Y_{i-2} \\ \oplus \\ X_{i-3} \end{matrix} \to \cdots$$

In other words, we have

$$\begin{aligned} \mathrm{Cone}(f)_i &= Y_i \oplus X_{i-1} \\ \partial_i^{\mathrm{Cone}(f)} &\colon Y_i \oplus X_{i-1} \to Y_{i-1} \oplus X_{i-2} \\ \partial_i^{\mathrm{Cone}(f)} \left(\begin{smallmatrix} y_i \\ x_{i-1} \end{smallmatrix}\right) &= \begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix} \left(\begin{smallmatrix} y_i \\ x_{i-1} \end{smallmatrix}\right) = \begin{pmatrix} \partial_i^Y(y_i) + f_{i-1}(x_{i-1}) \\ -\partial_{i-1}^X(x_{i-1}) \end{pmatrix} \\ &= \begin{pmatrix} \partial_i^Y(y_i) + f_{i-1}(x_{i-1}) \\ -\partial_{i-1}^X(x_{i-1}) \end{pmatrix} \end{aligned}$$

Proposition VIII.4.4. *Let R be a commutative ring, and let $f_\bullet \colon X_\bullet \to Y_\bullet$ be a chain map. The sequence* $\mathrm{Cone}(f_\bullet)$ *is an R-complex.*

Proof. It is straightforward to show that each map $\partial_i^{\mathrm{Cone}(f)}$ is an R-module homomorphism. Since $X_\bullet$ and $Y_\bullet$ are R-complexs, we have $\partial_{i-2}^X \partial_{i-1}^X = 0$ and

$\partial^Y_{i-1}\partial^Y_i = 0$ for each i. Since $f_\bullet$ is a chain map, we have $\partial^Y_{i-1}f_{i-1} = f_{i-2}\partial^X_{i-1}$ for each i. These facts give the last equality in the following computation:

$$\begin{aligned}\partial^{\mathrm{Cone}(f)}_{i-1}\partial^{\mathrm{Cone}(f)}_i &= \begin{pmatrix} \partial^Y_{i-1} & f_{i-2} \\ 0 & -\partial^X_{i-2}\end{pmatrix}\begin{pmatrix} \partial^Y_i & f_{i-1} \\ 0 & -\partial^X_{i-1}\end{pmatrix} \\ &= \begin{pmatrix} \partial^Y_{i-1}\partial^Y_i & \partial^Y_{i-1}f_{i-1}-f_{i-2}\partial^X_{i-1} \\ 0 & \partial^X_{i-2}\partial^X_{i-1}\end{pmatrix} = \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right).\end{aligned}$$

This shows that $\partial^{\mathrm{Cone}(f)}_{i-1}\partial^{\mathrm{Cone}(f)}_i = 0$ and hence the desired result. □

Proposition VIII.4.5. *Let R be a commutative ring, and let $f_\bullet\colon X_\bullet \to Y_\bullet$ be a chain map.*

(a) *For each i, let $\epsilon_i\colon Y_i \to \mathrm{Cone}(f)_i$ be given by*

$$\epsilon_i(y_i) = \left(\begin{smallmatrix}\mathbb{1}_{Y_i}\\0\end{smallmatrix}\right)\left(\begin{smallmatrix}y_i\end{smallmatrix}\right) = \left(\begin{smallmatrix}y_i\\0\end{smallmatrix}\right).$$

Then the sequence $\epsilon_\bullet\colon Y_\bullet \to \mathrm{Cone}(f_\bullet)$ is a chain map.

(b) *For each i, let $\tau_i\colon \mathrm{Cone}(f)_i \to (\Sigma X)_i$ be given by*

$$\tau_i\left(\begin{smallmatrix}y_i\\x_{i-1}\end{smallmatrix}\right) = \left(\begin{smallmatrix}0 & \mathbb{1}_{X_{i-1}}\end{smallmatrix}\right)\left(\begin{smallmatrix}y_i\\x_{i-1}\end{smallmatrix}\right) = x_{i-1}.$$

Then the sequence $\tau_\bullet\colon \mathrm{Cone}(f_\bullet) \to \Sigma X_\bullet$ is a chain map.

(c) *The following sequence is exact:*

$$0 \to Y_\bullet \xrightarrow{\epsilon_\bullet} \mathrm{Cone}(f_\bullet) \xrightarrow{\tau_\bullet} \Sigma X_\bullet \to 0.$$

(d) *In the long exact sequence on homology associated to the exact sequence in part* (c)*, the connecting map $\eth_i\colon \mathrm{H}_i(\Sigma X_\bullet) \to \mathrm{H}_{i-1}(Y_\bullet)$ is the same as the map $\mathrm{H}_{i-1}(f_\bullet)\colon \mathrm{H}_{i-1}(X_\bullet) \to \mathrm{H}_{i-1}(Y_\bullet)$.*

Proof. (b) It is straightforward to show that each map τ_i is an R-module homomorphism. We need to show that $\tau_{i-1}\circ\partial^{\mathrm{Cone}(f)}_i = \partial^{\Sigma X}_i \circ \tau_i$ for each i. We use the matrix notation:

$$\begin{aligned}\tau_{i-1}\circ\partial^{\mathrm{Cone}(f)}_i &= \left(\begin{smallmatrix}0 & \mathbb{1}_{X_{i-1}}\end{smallmatrix}\right)\begin{pmatrix}\partial^Y_i & f_{i-1}\\ 0 & -\partial^X_{i-1}\end{pmatrix} \\ &= \left(\begin{smallmatrix}0\circ\partial^Y_i+\mathbb{1}_{X_{i-1}}\circ 0 & 0\circ f_{i-1}-\mathbb{1}_{X_{i-1}}\partial^X_{i-1}\end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix}0 & -\partial^X_{i-1}\end{smallmatrix}\right) \\ \partial^{\Sigma X}_i\circ\tau_i &= \left(\begin{smallmatrix}-\partial^X_{i-1}\end{smallmatrix}\right)\left(\begin{smallmatrix}0 & \mathbb{1}_{X_{i-1}}\end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix}-\partial^X_{i-1}\circ 0 & -\partial^X_{i-1}\circ\mathbb{1}_{X_{i-1}}\end{smallmatrix}\right) \\ &= \left(\begin{smallmatrix}0 & -\partial^X_{i-1}\end{smallmatrix}\right)\end{aligned}$$

Note that this computation explains the need for the sign in $\partial^{\Sigma X}_i$.

(a) Similar to (and easier than) part (b).

(c) By definition, we need to show that, for each i, the sequence

$$0 \to Y_i \xrightarrow{\epsilon_i} \mathrm{Cone}(f)_i \xrightarrow{\tau_i} (\Sigma X)_i \to 0.$$

From the definitions, this sequence is the same as

$$0 \to Y_i \xrightarrow{\left(\begin{smallmatrix}\mathbb{1}_{Y_i}\\0\end{smallmatrix}\right)} \begin{matrix}Y_i\\ \oplus \\ X_{i-1}\end{matrix} \xrightarrow{\left(\begin{smallmatrix}0 & \mathbb{1}_{X_{i-1}}\end{smallmatrix}\right)} X_{i-1} \to 0$$

and hence is exact.

(d) Recall the steps for evaluating $\eth_i$:

1. Let $x \in \operatorname{Ker}(\partial_i^{\Sigma X})$.
2. Find an element $a \in \operatorname{Cone}(f)_i$ such that $\tau_i(a) = x$.
3. Find an element $b \in Y_{i-1}$ such that $\epsilon_{i-1}(b) = \partial_i^{\operatorname{Cone}(f)}(a)$.
4. Set $\eth_i(\overline{x}) = \overline{b} \in \operatorname{H}_{i-1}(Y_\bullet)$.

We work through these steps to verify the desired equality:

1. Let $x \in \operatorname{Ker}(\partial_i^{\Sigma X}) = \operatorname{Ker}(\partial_{i-1}^{X})$.
2. The element $a = \left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right)$ satisfies

$$a = \left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right) \in \operatorname{Cone}(f)_i = \begin{matrix} Y_i \\ \oplus \\ X_{i-1} \end{matrix} \qquad \text{and} \qquad \tau_i(a) = \left(\begin{smallmatrix} 0 & \mathbb{1}_{X_{i-1}} \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right) = x$$

3. The element $b = f_{i-1}(x)$ satisfies

$$\partial_i^{\operatorname{Cone}(f)}(a) = \left(\begin{smallmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right) = \left(\begin{smallmatrix} f_{i-1}(x) \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} b \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \mathbb{1}_{Y_i} \\ 0 \end{smallmatrix}\right)\left(b\right) = \epsilon_{i-1}(b).$$

4. We have $\eth_i(\overline{x}) = \overline{b} = \overline{f_{i-1}(x)} = \operatorname{H}_{i-1}(f_\bullet)(\overline{x})$, as desired. □

Definition VIII.4.6. Let R be a commutative ring, and let $f_\bullet \colon X_\bullet \to Y_\bullet$ be a chain map. The chain map $f_\bullet$ is a *quasiisomorphism* if, for each index i, the induced map $\operatorname{H}_i(f_\bullet) \colon \operatorname{H}_i(X_\bullet) \to \operatorname{H}_i(Y_\bullet)$ is an isomorphism.

Example VIII.4.7. Let R be a commutative ring, and let $f_\bullet \colon X_\bullet \to Y_\bullet$ be a chain map. If $f_\bullet$ is an isomorphism, then it is a quasiisomorphism; see Exercise VI.1.7(a). The converse fails in general; see, e.g., Exercise VIII.4.12.

Here is one of the useful properties of the mapping cone.

Proposition VIII.4.8. *Let R be a commutative ring. A chain map $f_\bullet \colon X_\bullet \to Y_\bullet$ is a quasiisomorphism if and only if its mapping cone* $\operatorname{Cone}(f_\bullet)$ *is exact.*

Proof. For the first implication, assume that $\operatorname{Cone}(f_\bullet)$ is exact, that is, that $\operatorname{H}_i(\operatorname{Cone}(f_\bullet)) = 0$ for each integer i. Using Proposition VIII.4.5 (d), a piece of the long exact sequence associated to the mapping cone has the following form

$$\underbrace{\operatorname{H}_{i+1}(\operatorname{Cone}(f_\bullet))}_{=0} \to \operatorname{H}_i(X_\bullet) \xrightarrow{\operatorname{H}_i(f_\bullet)} \operatorname{H}_i(X_\bullet) \to \underbrace{\operatorname{H}_i(\operatorname{Cone}(f_\bullet))}_{=0}$$

and it follows readily that the map $\operatorname{H}_i(f_\bullet)$ is an isomorphism for each i, that is, that $f_\bullet$ is a quasiisomorphism.

For the converse, assume that $f_\bullet$ is a quasiisomorphism. Another piece of the long exact sequence associated to the mapping cone has the following form

$$\operatorname{H}_i(X_\bullet) \xrightarrow[\cong]{\operatorname{H}_i(f_\bullet)} \operatorname{H}_i(X_\bullet) \to \operatorname{H}_i(\operatorname{Cone}(f_\bullet)) \to \operatorname{H}_{i-1}(X_\bullet) \xrightarrow[\cong]{\operatorname{H}_{i-1}(f_\bullet)} \operatorname{H}_{i-1}(X_\bullet)$$

and it is straightforward to show that this implies that $\operatorname{H}_i(\operatorname{Cone}(f_\bullet))$ for each i, that is, that $\operatorname{Cone}(f_\bullet)$ is exact. □

We obtain Schanuel's Lemma as a consequence of the mapping cone construction, after the following lemma.

Lemma VIII.4.9. *Let R be a commutative ring, and consider the following exact sequence of R-module homomorphisms:*

$$0 \to K_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} P_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_1} P_0 \to 0.$$

If the modules $P_0, \dots, P_{n-1}$ are projective, then so is K_n.

PROOF. We proceed by induction on n. If $n = 1$, then the given exact sequence has the form

$$0 \to K_1 \xrightarrow{f_1} P_0 \to 0$$

so $K_1 \cong P_0$, which is projective.

If $n = 2$, then this sequence has the form

$$0 \to K_2 \to P_1 \to P_0 \to 0.$$

Since P_0 is projective, this exact sequence splits, so $K_2 \oplus P_0 \cong P_1$. Since P_1 is projective, we conclude that K_2 is projective. This completes the base case.

Inductive step. Assume that $n \geqslant 3$, and that the result holds for sequences of length $n-1$. Since P_0 is projective, the given exact sequence splits into two exact sequences

$$0 \to K_2 \to P_1 \to P_0 \to 0$$

and

$$0 \to K_n \xrightarrow{f_n} P_{n-1} \xrightarrow{f_{n-1}} P_{n-2} \xrightarrow{f_{n-2}} \cdots \xrightarrow{f_3} P_2 \xrightarrow{f_2} K_2 \to 0$$

with $K_2 = \operatorname{Im}(f_2) = \operatorname{Ker}(f_1)$. By the base case, we conclude that K_2 is projective and our inductive hypothesis implies that K_n is projective. □

Lemma VIII.4.10 (Schanuel's Lemma)**.** *Let R be a commutative ring, and let M be an R-module. Consider two exact sequences*

$$0 \to K \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

and

$$0 \to L \to Q_{n-1} \to \cdots \to Q_0 \to M \to 0$$

such that each P_i and Q_i is projective. Then K is projective if and only if L is projective.

PROOF. If $n = 0$, then the result is trivial since $L \cong M \cong K$. So assume that $n \geqslant 1$. By symmetry, it suffices to assume that K is projective and show that L is projective. Proposition VI.3.2 provides a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & K & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_{n-1}} & & & & \downarrow{\scriptstyle F_0} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & L & \longrightarrow & Q_{n-1} & \longrightarrow & \cdots & \longrightarrow & Q_0 & \longrightarrow & M & \longrightarrow & 0.
\end{array}$$

(Note that Proposition VI.3.2 also assumes that L is projective, but this is not needed in the proof.) Truncating these complexes yields a chain map

$$\begin{array}{ccccccccccc}
P'_\bullet & = & 0 & \longrightarrow & K & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & 0 \\
\downarrow{\scriptstyle F_\bullet} & & & & \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_{n-1}} & & & & \downarrow{\scriptstyle F_0} & & \\
Q'_\bullet & = & 0 & \longrightarrow & L & \longrightarrow & Q_{n-1} & \longrightarrow & \cdots & \longrightarrow & Q_0 & \longrightarrow & 0.
\end{array}$$

Since each of the original complexes is exact, the chain map $F_\bullet$ is a quasiisomorphism. Thus, Proposition VIII.4.8 implies that $\operatorname{Cone}(F_\bullet)$ is exact.

$$\operatorname{Cone}(F_\bullet) = \quad 0 \to K \to L \oplus P_{n-1} \to Q_{n-1} \oplus P_{n-2} \to \cdots \to Q_1 \oplus P_0 \to Q_0 \to 0$$

Since the modules $L \oplus P_{n-1}$, $Q_{n-1} \oplus P_{n-2}, \dots$, $Q_1 \oplus P_0$, and Q_0 are projective, Lemma VIII.4.9 implies that L is projective. □

Exercises.

Exercise VIII.4.11. Let R be a commutative ring, and let $x \in R$. Prove that the following chain map

$$\begin{array}{ccccccccccc} M_\bullet & & 0 & \longrightarrow & R^1 & \xrightarrow{\left(\begin{smallmatrix}1\\-1\\x\end{smallmatrix}\right)} & R^3 & \xrightarrow{\left(\begin{smallmatrix}x&0&-1\\0&x&1\end{smallmatrix}\right)} & R^2 & \longrightarrow & 0 \\ {\scriptstyle F_\bullet}\downarrow & & & & \downarrow & & \downarrow{\scriptstyle (1\,1\,0)} & & \downarrow{\scriptstyle (1\,1)} & & \\ N_\bullet & & 0 & \longrightarrow & 0 & \longrightarrow & R^1 & \xrightarrow{(x)} & R^1 & \longrightarrow & 0 \end{array}$$

is a quasiisomorphism. (Be sure to verify that $M_\bullet$ and $N_\bullet$ are R-complexes and that $F_\bullet$ is a chain map.)

Exercise VIII.4.12. Let R be a commutative ring, and let $X_\bullet$ be an R-complex. Show that the following conditions are equivalent:

(i) The natural chain map $0 \to X_\bullet$ is a quasiisomorphism;
(ii) The complex $X_\bullet$ is exact; and
(iii) the natural chain map $X_\bullet \to 0$ is a quasiisomorphism.

Exercise VIII.4.13. Let R be a commutative ring, and let

$$0 \to M'_\bullet \xrightarrow{F_\bullet} M_\bullet \xrightarrow{G_\bullet} M''_\bullet \to 0$$

be a short exact sequence of chain maps.

(i) Prove that $M'_\bullet$ is exact if and only if $G_\bullet$ is a quasiisomorphism.
(ii) Prove that $M''_\bullet$ is exact if and only if $F_\bullet$ is a quasiisomorphism.

Exercise VIII.4.14. Let R be a commutative ring. Given a commutative diagram of chain maps

$$\begin{array}{ccc} X_\bullet & \xrightarrow{f_\bullet} & Y_\bullet \\ {\scriptstyle g_\bullet}\downarrow & & \downarrow{\scriptstyle h_\bullet} \\ X'_\bullet & \xrightarrow{f'_\bullet} & Y'_\bullet \end{array}$$

show that there is an induced chain map $\operatorname{Cone}(f_\bullet) \to \operatorname{Cone}(f'_\bullet)$.

Exercise VIII.4.15. Prove that, in the proof of Lemma VIII.4.10, the chain map $F_\bullet$ is a quasiisomorphism.

Exercise VIII.4.16. Let R be a commutative ring, and consider the following exact sequence of R-module homomorphisms:

$$0 \to I_n \xrightarrow{f_n} I_{n-1} \xrightarrow{f_{n-1}} \cdots \xrightarrow{f_2} I_1 \xrightarrow{f_1} C_0 \to 0.$$

If the modules $I_1, \dots, I_n$ are injective, then so is C_0.

Exercise VIII.4.17. (Schanuel's Lemma) Let R be a commutative ring, and let M be an R-module. Consider two exact sequences of R-module homomorphisms

$$0 \longrightarrow M \xrightarrow{i} I_t \xrightarrow{f_t} I_{t-1} \xrightarrow{f_{t-1}} \cdots \xrightarrow{f_2} I_1 \xrightarrow{f_1} I_0 \xrightarrow{\pi} C \longrightarrow 0$$
$$0 \longrightarrow M \xrightarrow{j} J_t \xrightarrow{g_t} J_{t-1} \xrightarrow{g_{t-1}} \cdots \xrightarrow{g_2} J_1 \xrightarrow{g_1} J_0 \xrightarrow{\tau} D \longrightarrow 0$$

where each I_j and J_i is injective. Prove that C is injective if and only if D is injective.

Exercise VIII.4.18. Let R be a commtuative ring, and let $F_\bullet\colon X_\bullet \to Y_\bullet$ be a chain map. Prove that there are isomorphisms

$$\operatorname{Cone}(F_\bullet \otimes_R M) \cong \operatorname{Cone}(F_\bullet) \otimes_R M$$
$$\operatorname{Cone}(M \otimes_R F_\bullet) \cong M \otimes_R \operatorname{Cone}(F_\bullet)$$
$$\operatorname{Cone}(\operatorname{Hom}_R(M, F_\bullet)) \cong \operatorname{Hom}_R(M, \operatorname{Cone}(F_\bullet))$$
$$\operatorname{Cone}(\operatorname{Hom}_R(F_\bullet, M)) \cong \Sigma \operatorname{Hom}_R(\operatorname{Cone}(F_\bullet), M).$$

VIII.5. Ext, Tor, and Resolutions

In this section, we prove that Ext and Tor are independent of the choice of projective resolutions. We also prove that Tor can be computed using flat resolutions.

Lemma VIII.5.1. *Let R be a commutative ring, and let $P_\bullet$ be an exact sequence of projective R-modules such that $P_i = 0$ for all $i < i_0$. For each R-module N, the sequence $\operatorname{Hom}_R(P_\bullet, N)$ is exact.*

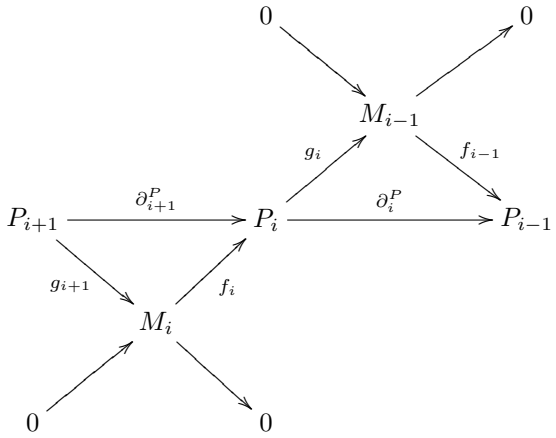
Proof. For each integer i, set $M_i = \operatorname{Ker}(\partial^P_i) = \operatorname{Im}(\partial^P_{i+1})$. This yields an exact sequence

$$0 \to M_i \xrightarrow{f_i} P_i \xrightarrow{g_i} M_{i-1} \to 0 \tag{VIII.5.1.1}$$

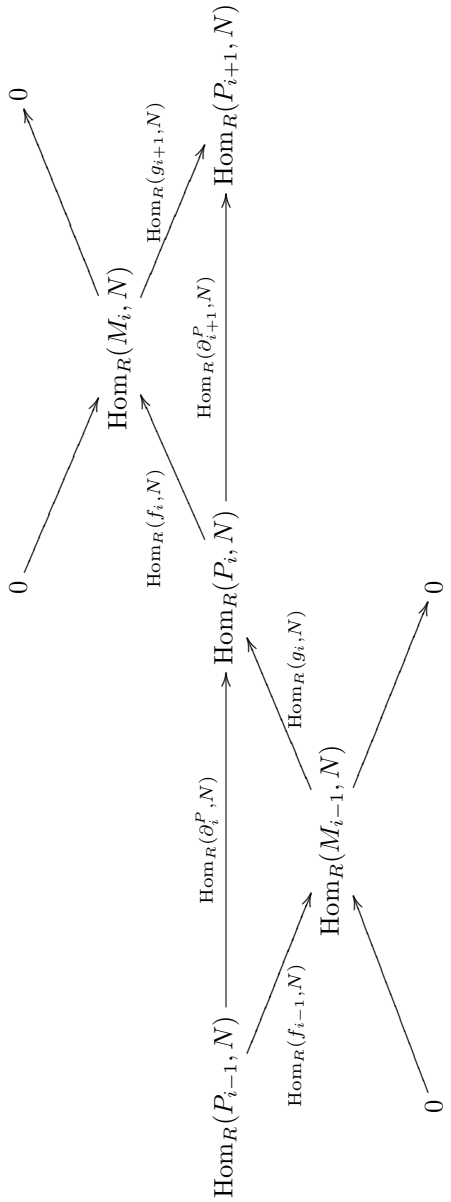
for each i such that $f_{i-1}g_i = \partial^P_i$. Here f_i is the inclusion map, and g_i is induced by ∂^P_i. Since $P_{i_0-1} = 0$, it is straightforward to show that $M_{i_0} = P_{i_0}$. In particular, the module M_{i_0} is projective. Since P_i is also projective, an induction argument using the sequences (VIII.5.1.1) implies that M_i is projective for all $i \geqslant i_0$. It follows that each sequence (VIII.5.1.1) splits. From this, it is straightforward to conclude that the induced sequence

$$0 \to \operatorname{Hom}_R(M_{i-1}, N) \xrightarrow{\operatorname{Hom}_R(g_i,N)} \operatorname{Hom}_R(P_i, N) \xrightarrow{\operatorname{Hom}_R(f_i,N)} \operatorname{Hom}_R(M_i, N) \to 0$$

is exact. Since the following diagram commutes and has exact diagonals



it follows that the induced diagram commutes and has exact diagonals



A diagram chase shows that center row in this diagram is also exact. □

Here is Theorem IV.3.5.

Theorem VIII.5.2. *Let R be a commutative ring, and let M and N be R-modules. The modules $\operatorname{Ext}^i_R(M,N)$ are independent of the choice of projective resolution of M. In other words, if $P^+_\bullet$ and $Q^+_\bullet$ are projective resolutions of M, then there is an R-module isomorphism $\operatorname{H}_{-i}(\operatorname{Hom}_R(P_\bullet, N)) \cong \operatorname{H}_{-i}(\operatorname{Hom}_R(Q_\bullet, N))$ for each index i.*

PROOF. Let $F_\bullet \colon P_\bullet \to Q_\bullet$ be a lifting of the identity map $\mathbb{1}_M \colon M \to M$ as in Proposition VI.3.2. Using Exercise VI.3.10, it is straightforward to show that $F_\bullet$ is a

quasiisomorphism. Proposition VIII.4.8 implies that the mapping cone $\operatorname{Cone}(F_\bullet)$ is exact. Furthermore, the complex $\operatorname{Cone}(F_\bullet)$ consists of projective R-modules (since $P_\bullet$ and $Q_\bullet$ do) and satisfies $\operatorname{Cone}(F_\bullet)_i = 0$ for all $i < -1$. Thus, Lemma VIII.5.1 implies that the complex $\operatorname{Hom}_R(\operatorname{Cone}(F_\bullet), N)$ is exact. It follows readily that the next complex is also exact

$$\Sigma \operatorname{Hom}_R(\operatorname{Cone}(F_\bullet), N) \cong \operatorname{Cone}(\operatorname{Hom}_R(F_\bullet, N)).$$

The isomorphism comes from Exercise VIII.4.18. From this, we conclude that the following chain map is a quasiisomorphism

$$\operatorname{Hom}_R(F_\bullet, N)\colon \operatorname{Hom}_R(Q_\bullet, N) \to \operatorname{Hom}_R(P_\bullet, N).$$

By definition, this says that each induced map

$$\operatorname{H}_{-i}(\operatorname{Hom}_R(F_\bullet, N))\colon \operatorname{H}_{-i}(\operatorname{Hom}_R(Q_\bullet, N)) \xrightarrow{\cong} \operatorname{H}_{-i}(\operatorname{Hom}_R(P_\bullet, N))$$

is an isomorphism, as desired. □

The next five results are proved similarly.

Lemma VIII.5.3. *Let R be a commutative ring, and let $I_\bullet$ be an exact sequence of injective R-modules such that $I_j = 0$ for all $j > j_0$. For each R-module M, the sequence $\operatorname{Hom}_R(M, I_\bullet)$ is exact.* □

Here is part of Theorem IV.3.10.

Theorem VIII.5.4. *Let R be a commutative ring, and let M and N be R-modules. If $I^+_\bullet$ and $J^+_\bullet$ are injective resolutions of N, then there is an R-module isomorphism $\operatorname{H}_{-i}(\operatorname{Hom}_R(M, I_\bullet)) \cong \operatorname{H}_{-i}(\operatorname{Hom}_R(M, J_\bullet))$ for each index i.* □

Lemma VIII.5.5. *Let R be a commutative ring, and let $P_\bullet$ be an exact sequence of projective R-modules such that $P_i = 0$ for all $i < i_0$. For each R-module N, the sequences $P_\bullet \otimes_R N$ and $N \otimes_R P_\bullet$ are exact.* □

Here is Theorem IV.4.4.

Theorem VIII.5.6. *Let R be a commutative ring, and let M and N be R-modules. The modules $\operatorname{Tor}^R_i(M, N)$ are independent of the choice of projective resolution of M. In other words, if $P^+_\bullet$ and $Q^+_\bullet$ are projective resolutions of M, then there is an R-module isomorphism $\operatorname{H}_i(P_\bullet \otimes_R N) \cong \operatorname{H}_i(Q_\bullet \otimes_R N)$ for each index i.* □

Here is part of Theorem IV.4.8.

Theorem VIII.5.7. *Let R be a commutative ring, and let M and N be R-modules. If $P^+_\bullet$ and $Q^+_\bullet$ are projective resolutions of N, then there is an R-module isomorphism $\operatorname{H}_i(M \otimes_R P_\bullet) \cong \operatorname{H}_i(M \otimes_R Q_\bullet)$ for each index i.* □

We end this section by showing that Tor can be computed using flat resolutions. Note that the proof uses Corollary IV.4.9; this uses the fact that Tor is balanced IV.4.8, which we have not proved yet.

Lemma VIII.5.8. *Let R be a commutative ring, and let $F_\bullet$ be an exact sequence of flat R-modules such that $F_i = 0$ for all $i < i_0$. For each R-module N, the sequences $F_\bullet \otimes_R N$ and $N \otimes_R F_\bullet$ are exact.*

PROOF. Since there is an isomorphism $F_\bullet \otimes_R N \cong N \otimes_R F_\bullet$, it suffices to show that $F_\bullet \otimes_R N$ is exact.

For each integer i, set $M_i = \operatorname{Ker}(\partial_i^F) = \operatorname{Im}(\partial_{i+1}^F)$. This yields an exact sequence

$$0 \to M_i \xrightarrow{f_i} F_i \xrightarrow{g_i} M_{i-1} \to 0 \tag{VIII.5.8.1}$$

for each i such that $f_{i-1}g_i = \partial_i^F$. Here f_i is the inclusion map, and g_i is induced by ∂_i^F. Since $F_{i_0-1} = 0$, it is straightforward to show that $M_{i_0} = F_{i_0}$. In particular, the module M_{i_0} is flat. Since F_i is also flat, an induction argument (using Theorem VII.6.6 and the sequences (VIII.5.8.1)) implies that M_i is flat for all $i \geqslant i_0$. It follows that $\operatorname{Tor}_1^R(M_{i-1}, -) = 0$, so the long exact sequence in $\operatorname{Tor}^R(-, N)$ associated to (VIII.5.8.1) starts as follows:

$$0 \to M_i \otimes_R N \xrightarrow{f_i \otimes_R N} F_i \otimes_R N \xrightarrow{g_i \otimes_R N} M_{i-1} \otimes_R N \to 0.$$

The proof concludes like the proof of Lemma VIII.5.1. □

Theorem VIII.5.9. *Let R be a commutative ring, and let M and N be R-modules. The modules $\operatorname{Tor}_i^R(M, N)$ can be computed using a flat resolution of M. In other words, if $F_\bullet^+$ is a flat resolution of M, then there is an R-module isomorphism $\operatorname{H}_i(F_\bullet \otimes_R N) \cong \operatorname{Tor}_i^R(M, N)$ for each index i.*

PROOF. Argue as in the proof of Theorem VIII.5.2, using the fact that Proposition VI.3.2 allows for a lift $G_\bullet \colon P_\bullet \to F_\bullet$. □

The last result of this section is proved like Theorem VIII.5.9.

Theorem VIII.5.10. *Let R be a commutative ring, and let M and N be R-modules. If $P_\bullet^+$ is a projective resolution of N and $F_\bullet^+$ is a flat resolution of N, then there is an R-module isomorphism $\operatorname{H}_i(M \otimes_R P_\bullet) \cong \operatorname{H}_i(M \otimes_R F_\bullet)$ for each index i.* □

Exercises.

Exercise VIII.5.11. Complete the proof of Lemma VIII.5.1.

Exercise VIII.5.12. Complete the proof of Theorem VIII.5.2.

Exercise VIII.5.13. Prove Lemma VIII.5.3.

Exercise VIII.5.14. Prove Theorem VIII.5.4.

Exercise VIII.5.15. Prove Lemma VIII.5.5.

Exercise VIII.5.16. Prove Theorem VIII.5.6.

Exercise VIII.5.17. Prove Theorem VIII.5.7.

Exercise VIII.5.18. Complete the proof of Lemma VIII.5.8.

Exercise VIII.5.19. Prove Theorem VIII.5.9.

Exercise VIII.5.20. Prove Theorem VIII.5.10.

VIII.6. Koszul Complexes

We begin the section with a motivating example.

Example VIII.6.1. Let R be a commutative ring, and let M be an R-module. Consider M as an R-complex concentrated in degree 0:

$$M_\bullet = \quad 0 \to M \to 0.$$

For each $r \in R$, the map $\mu^r_\bullet \colon M_\bullet \to M_\bullet$ given by $\mu^r_i(m) = rm$ is an R-module homomorphism and a chain map. The associated mapping cone is isomorphic to the R-complex

$$\operatorname{Cone}(\mu^r_\bullet) = \qquad 0 \to M \xrightarrow{r} M \to 0.$$

It follows that r is M-regular if and only if $\operatorname{H}_i(\operatorname{Cone}(\mu^r_\bullet)) = 0$ for all $i \neq 0$ and $\operatorname{H}_0(\operatorname{Cone}(\mu^r_\bullet)) \neq 0$.

The complex $\operatorname{Cone}(\mu^r_\bullet)$ is a Koszul complex on one element. We will now construct more general Koszul complexes and show that they have the ability to detect regular sequences of longer length.

Definition VIII.6.2. Let R be a commutative ring, and let M be an R-module. For each $r \in R$, set $(0 :_M r) = \{m \in M \mid rm = 0\}$.

Remark VIII.6.3. Let R be a commutative ring, and let M be an R-module. For each $r \in R$, the set $(0 :_M r)$ is an R-submodule of M. In fact, it is the largest R-submodule $M' \subseteq M$ such that $rM' = 0$.

Proposition VIII.6.4. *Let R be a commutative ring, and $X_\bullet$ an R-complex. Let $r \in R$, and let $\mu^r_\bullet \colon X_\bullet \to X_\bullet$ be the homothety $\mu_i(x_i) = rx_i$. Consider the short exact sequence*

$$0 \to X_\bullet \xrightarrow{\epsilon_\bullet} \operatorname{Cone}(\mu_\bullet) \xrightarrow{\tau_\bullet} \Sigma X_\bullet \to 0 \tag{VIII.6.4.1}$$

from Proposition VIII.4.5.

(a) *For each i, the connecting map $\eth_i \colon \operatorname{H}_i(\Sigma X_\bullet) \to \operatorname{H}_{i-1}(X_\bullet)$ in the long exact sequence associated to* (VIII.6.4.1) *is the homothety $\operatorname{H}_{i-1}(X_\bullet) \xrightarrow{r} \operatorname{H}_{i-1}(X_\bullet)$.*

(b) *For each i, there is an exact sequence*

$$0 \to \operatorname{H}_i(X_\bullet)/r\operatorname{H}_i(X_\bullet) \to \operatorname{H}_i(\operatorname{Cone}(\mu^r_\bullet)) \to (0 :_{\operatorname{H}_{i-1}(X_\bullet)} r) \to 0.$$

Proof. (a) Proposition VIII.4.5(d) implies that $\eth_i(\overline{x}) = \overline{\mu^r_{i-1}(x)} = \overline{rx} = r\overline{x}$.

(b) By part (a) the long exact sequence for (VIII.6.4.1) has the form

$$\cdots \to \operatorname{H}_i(X_\bullet) \xrightarrow{r} \operatorname{H}_i(X_\bullet) \xrightarrow{\operatorname{H}_i(\epsilon_\bullet)} \operatorname{H}_i(\operatorname{Cone}(\mu_\bullet)) \xrightarrow{\operatorname{H}_i(\tau_\bullet)} \operatorname{H}_{i-1}(X_\bullet) \xrightarrow{r} \operatorname{H}_{i-1}(X_\bullet) \cdots .$$

This induces an exact sequence

$$0 \to \operatorname{Im}(\operatorname{H}_i(\epsilon_\bullet)) \to \operatorname{H}_i(\operatorname{Cone}(\mu_\bullet)) \to \operatorname{Im}(\operatorname{H}_i(\tau_\bullet)) \to 0. \tag{VIII.6.4.2}$$

The exactness of the long exact sequence provides isomorphisms

$$\operatorname{Im}(\operatorname{H}_i(\epsilon_\bullet)) \cong \operatorname{H}_i(X_\bullet)/r\operatorname{H}_i(X_\bullet)$$

and

$$\operatorname{Im}(\operatorname{H}_i(\tau_\bullet)) = \operatorname{Ker}(\operatorname{H}_{i-1}(X_\bullet) \xrightarrow{r} \operatorname{H}_{i-1}(X_\bullet)) = (0 :_{\operatorname{H}_{i-1}(X_\bullet)} r).$$

Substituting into the sequence (VIII.6.4.2) yields the desired exact sequence. □

Definition VIII.6.5. Let R be a commutative ring, and let M be an R-module. Let $\mathbf{x} = x_1, \dots, x_n \in R$. We build the *Koszul complex* $K_\bullet(\mathbf{x}; M)$ by induction on n.

Base case: $n = 1$. In this case $K_\bullet(x_1; M)$ is the complex from Example VIII.6.1:

$$K_\bullet(x_1; M) = (0 \to M \xrightarrow{x_1} M \to 0) = \operatorname{Cone}(\mu_{x_1} \colon M \to M).$$

Inductive step: Assume that $n \geqslant 2$ and that $K_\bullet(x_1, \dots, x_{n-1}; M)$ has been constructed. Let $\mu^{x_n}_\bullet \colon K_\bullet(x_1, \dots, x_{n-1}; M) \to K_\bullet(x_1, \dots, x_{n-1}; M)$ be the homothety given by $\mu^{x_n}_i(k_i) = x_n k_i$, and set

$$K_\bullet(\mathbf{x}; M) = K_\bullet(x_1, \dots, x_{n-1}, x_n; M) = \operatorname{Cone}(\mu^{x_n}_\bullet).$$

For each i, we write $\mathrm{H}_i(\mathbf{x}; M) = \mathrm{H}_i(K_\bullet(\mathbf{x}; M))$. When $M = R$, we write $K_\bullet(\mathbf{x}) = K_\bullet(\mathbf{x}; R)$ and $\mathrm{H}_i(\mathbf{x}) = \mathrm{H}_i(K_\bullet(\mathbf{x}))$.

Example VIII.6.6. Let R be a commutative ring, and let M be an R-module. Let $x, y, z \in R$. It is straightforward to show the following

$$K_\bullet(x; M) \cong \qquad 0 \to M \xrightarrow{x} M \to 0$$

and

$$K_\bullet(x, y; M) \cong \qquad 0 \to M \xrightarrow{\left(\begin{smallmatrix} y \\ -x \end{smallmatrix}\right)} M^2 \xrightarrow{(\begin{smallmatrix} x & y \end{smallmatrix})} M \to 0$$

and

$$K_\bullet(x, y, z; M) \cong \qquad 0 \to M \xrightarrow{\left(\begin{smallmatrix} z \\ -y \\ x \end{smallmatrix}\right)} M^3 \xrightarrow{\left(\begin{smallmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{smallmatrix}\right)} M^3 \xrightarrow{(\begin{smallmatrix} x & y & z \end{smallmatrix})} M \to 0$$

using the definition of the Koszul complex.

See Corollary VIII.6.17 for more general versions of the next two examples.

Example VIII.6.7. Let A be a commutative ring, and consider the polynomial ring $R = A[X]$ in one variable. Using the description of $K_\bullet(X)$ from Example VIII.6.6, we conclude that $\mathrm{H}_0(X) \cong R/(X)R \cong A$ and that $\mathrm{H}_i(X) = 0$ when $i \neq 0$.

Example VIII.6.8. Let A be a commutative ring, and consider the polynomial ring $R = A[X, Y]$ in two variables. Example VIII.6.6 shows the following:

$$K_\bullet(X, Y; M) \cong \qquad 0 \to R \xrightarrow{\left(\begin{smallmatrix} Y \\ -X \end{smallmatrix}\right)} R^2 \xrightarrow{(\begin{smallmatrix} X & Y \end{smallmatrix})} R \to 0.$$

We compute the homologies here.

It is straightforward to show that $\mathrm{H}_0(X, Y) \cong R/(X, Y)R \cong A$ and that $\mathrm{H}_i(X, Y) = 0$ when $i \geqslant 3$ or $i \leqslant -1$. It is also straightforward to verify the steps in the next sequence:

$$\mathrm{H}_2(X, Y) \cong \{r \in R \mid Xr = 0 = Yr\} = 0.$$

We claim that $\mathrm{H}_2(X, Y) = 0$. To this end, let $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \operatorname{Ker}(\partial_1^{K(X,Y)})$. We show that $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \operatorname{Im}(\partial_1^{K(X,Y)})$. Our assumption on $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$ says that

$$0 = \begin{pmatrix} X & Y \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = Xf + Yg.$$

It follows that $Xf = -Yg$. Using the fact that the ring R is a free A-module with basis $\{X^i Y^j \mid i, j \geqslant 0\}$, it is straightforward to show that this implies that $f \in YR$

and $g \in XR$. Hence, there are polynomials $f', g' \in R$ such that $f = Yf'$ and $g = Xg'$. Thus, we have

$$XYf' = Xf = -Yg = XY(-g').$$

Since XY is not a zero-divisor on R we have $f' = -g'$, and hence

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} Yf' \\ Xg' \end{pmatrix} = \begin{pmatrix} -Yg' \\ Xg' \end{pmatrix} = \begin{pmatrix} Y \\ -X \end{pmatrix} (-g') \in \operatorname{Im}(\partial_1^{K(X,Y)}).$$

This is the desired conclusion.

Example VIII.6.9. Let k be a field, and set $R = k[x,y]/(xy)$. Recall that every element $r \in R$ has a unique representation of the form

$$r = a + x\textstyle\sum_i b_i x^i + y\sum_j c_j y^j$$

with $a, b_i, c_j \in k$.

We show that

$$\mathrm{H}_i(x,y) \cong \begin{cases} R/(x,y) & \text{if } i = 0,1 \\ 0 & \text{if } i \neq 0,1. \end{cases}$$

From Example VIII.6.6 we have

$$K_\bullet(x,y) \cong \left(0 \to R \xrightarrow{\left(\begin{smallmatrix} y \\ -x \end{smallmatrix}\right)} R^2 \xrightarrow{(x\ y)} R \to 0\right).$$

This implies

$$\mathrm{H}_0(x,y) \cong R/(x,y).$$

It remains to check the cases $i = 1, 2$.

For $\mathrm{H}_1(x,y)$ we compute

$$\operatorname{Ker}\left(R^2 \xrightarrow{(x\ y)} R\right) = \left\{ \left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right) \in R^2 \mid 0 = (x\ y)\left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right) = xr + ys \right\}.$$

We claim that this kernel is generated by the columns $\left(\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right)$. One checks readily that these columns are in the kernel. To see that everything in the kernel can be written in terms of these columns, let $\left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right)$ be in the kernel, and write

$$r = a + x\textstyle\sum_i b_i x^i + y\sum_j c_j y^j$$

and

$$s = d + x\textstyle\sum_i e_i x^i + y\sum_j f_j y^j.$$

The equation $xr + ys = 0$ translates to

$$\begin{aligned} 0 &= x\left(a + x\textstyle\sum_i b_i x^i + y\sum_j c_j y^j\right) + y\left(d + x\sum_i e_i x^i + y\sum_j f_j y^j\right) \\ &= x\left(a + x\textstyle\sum_i b_i x^i\right) + y\left(d + y\sum_j f_j y^j\right) \end{aligned}$$

so $a = 0 = b_i = d = f_j$. This yields

$$\left(\begin{smallmatrix} r \\ s \end{smallmatrix}\right) = \left(\begin{smallmatrix} y\sum_j c_j y^j \\ x\sum_i e_i x^i \end{smallmatrix}\right) = \textstyle\sum_j c_j y^j \left(\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right) + \sum_i e_i x^i \left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right).$$

This establishes the claim.

It follows that

$$\mathrm{H}_1(x,y) = \frac{\operatorname{Ker}\left(R^2 \xrightarrow{\left(\begin{smallmatrix} x & y \end{smallmatrix}\right)} R\right)}{\operatorname{Im}\left(R \xrightarrow{\left(\begin{smallmatrix} y \\ -x \end{smallmatrix}\right)} R^2\right)} = \frac{\left\langle \left(\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right) \right\rangle}{\left\langle \left(\begin{smallmatrix} y \\ -x \end{smallmatrix}\right) \right\rangle} = \frac{\left\langle \left(\begin{smallmatrix} y \\ -x \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right) \right\rangle}{\left\langle \left(\begin{smallmatrix} y \\ -x \end{smallmatrix}\right) \right\rangle}$$

This quotient is cyclic, generated by the coset $\overline{\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right)}$. Hence, there is a surjection $\tau\colon R \to \mathrm{H}_1(x,y)$ given by $r \mapsto r\overline{\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right)}$. To complete the computation of $\mathrm{H}_1(x,y)$ we need only show that $\operatorname{Ker}(\tau) = (x,y)$.

To see that $\operatorname{Ker}(\tau) \supseteq (x,y)$, we check the generators of (x,y):

$$\tau(y) = y\overline{\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right)} = \overline{\left(\begin{smallmatrix} 0 \\ yx \end{smallmatrix}\right)} = \overline{\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)} = 0.$$

$$\tau(x) = x\overline{\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right)} = -x\overline{\left(\begin{smallmatrix} y \\ -x \end{smallmatrix}\right)} = -x0 = 0.$$

It is straightforward to show that $0 \neq \overline{\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right)} = 1\overline{\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right)}$, so $1 \notin \operatorname{Ker}(\tau)$. It follows that $R \supsetneq \operatorname{Ker}(\tau) \supseteq (x,y)$. Since (x,y) is a maximal ideal, we have $\operatorname{Ker}(\tau) = (x,y)$, as desired.

An easier computation shows that

$$\mathrm{H}_2(x,y) \cong \operatorname{Ker}\left(R \xrightarrow{\left(\begin{smallmatrix} y \\ -x \end{smallmatrix}\right)} R^2\right) = \{r \in R \mid xr = 0 = yr\} = 0.$$

Note that similar arguments can be used to compute $\mathrm{H}_i(x,y)$ when $R = k[\![x,y]\!]/(xy)$ or $R = k[x,y]_{(x,y)}/(xy)$.

Here are some basic properties of the Koszul complex.

Proposition VIII.6.10. *Let R be a commutative ring, and let M be an R-module. Let $\mathbf{x} = x_1, \dots, x_n \in R$. Show that, for each integer i, we have*

$$K_i(\mathbf{x}; M) \cong M^{\binom{n}{i}}.$$

Note that this implies that $K_i(\mathbf{x}; M) = 0 = \mathrm{H}_i(\mathbf{x}; M)$ when either $i > n$ or $i < 0$.

Proof. We argue by induction on n. The base case $n = 1$ is contained in Example VIII.6.6.

Assume that $n \geqslant 2$ and that the result holds for sequences of length $n-1$. Set $L_\bullet = K_\bullet(x_1, \dots, x_{n-1}; M)$. The induction hypothesis implies that $L_i \cong M^{\binom{n-1}{i}}$ for every integer i. The definition of $K_\bullet(\mathbf{x}; M)$ as a mapping cone explains the first isomorphism in the next sequence

$$K_i(\mathbf{x}; M) \cong L_i \oplus L_{i-1} \cong M^{\binom{n-1}{i}} \oplus M^{\binom{n-1}{i-1}} \cong M^{\binom{n-1}{i}+\binom{n-1}{i}} \cong M^{\binom{n}{i}}.$$

The second isomorphism is from the induction hypothesis, the third isomorphism is standard, and the fourth isomorphism is from the standard "Pascal's triangle" recurrence relation for binomial coefficients. □

Proposition VIII.6.11. *Let R be a commutative ring, and let M be an R-module. Let $\mathbf{x} = x_1, \dots, x_n \in R$.*

(a) *Under the identifications $K_\bullet(\mathbf{x}; M)_0 \cong M$ and $K_\bullet(\mathbf{x}; M)_1 \cong M^n$, one has $\partial_1^{K(\mathbf{x};M)} = (x_1 \ x_2 \ \cdots \ x_n)$.*

(b) *Under the identifications* $K_\bullet(\mathbf{x};M)_n \cong M$ *and* $K_\bullet(\mathbf{x};M)_{n-1} \cong M^n$*, one has*
$$\partial_n^{K(\mathbf{x};M)} = \begin{pmatrix} x_n \\ -x_{n-1} \\ \vdots \\ (-1)^{n-1}x_1 \end{pmatrix}.$$
(c) *There is an isomorphism* $\mathrm{H}_0(\mathbf{x};M) \cong M/\mathbf{x}M$*.*
(d) *There is an isomorphism* $\mathrm{H}_n(\mathbf{x};M) \cong \cap_{i=1}^n (0 :_M x_i)$*.*

Proof. (a) and (b) The identifications $K_\bullet(\mathbf{x};M)_0 \cong M$ and $K_\bullet(\mathbf{x};M)_1 \cong M^n$ and $K_\bullet(\mathbf{x};M)_n \cong M$ and $K_\bullet(\mathbf{x};M)_{n-1} \cong M^n$ come from Proposition VIII.6.10. We argue by induction on n. The base case $n = 1$ follows from the description of $K_\bullet(x_1;M)$ in Example VIII.6.6. Assume that $n \geqslant 2$ and that the result holds for sequences of length $n-1$. In particular, we have
$$\partial_1^{K(x_1,\dots,x_{n-1};M)} = (x_1 \ x_2 \ \cdots \ x_{n-1})\colon M^{n-1} \to M$$
$$\partial_n^{K(x_1,\dots,x_{n-1};M)} = \begin{pmatrix} x_{n-1} \\ -x_{n-2} \\ \vdots \\ (-1)^{n-2}x_1 \end{pmatrix} \colon M \to M^{n-1}$$

Let $\mu_\bullet^{x_n}\colon K_\bullet(x_1,\dots,x_{n-1};M) \to K_\bullet(x_1,\dots,x_{n-1};M)$ denote the homothety defined by multiplication by x_n. Under the given identifications, this chain map has the following form:
$$\begin{array}{ccccccccccc}
0 & \longrightarrow & M & \xrightarrow{\begin{pmatrix} x_{n-1} \\ -x_{n-2} \\ \vdots \\ (-1)^{n-2}x_1 \end{pmatrix}} & M^{n-1} & \longrightarrow \cdots \longrightarrow & M^{n-1} & \xrightarrow{\begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} \end{pmatrix}} & M & \longrightarrow 0 \\
 & & \downarrow{\scriptstyle x_n} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle x_n} & & \\
0 & \longrightarrow & M & \xrightarrow[\begin{pmatrix} x_{n-1} \\ -x_{n-2} \\ \vdots \\ (-1)^{n-2}x_1 \end{pmatrix}]{} & M^{n-1} & \longrightarrow \cdots \longrightarrow & M^{n-1} & \xrightarrow[\begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} \end{pmatrix}]{} & M & \longrightarrow 0.
\end{array}$$

The definition of the Koszul complex as a mapping cone implies that the terms of $K_\bullet(\mathbf{x};M)_n$ in degrees -1 to 1 are given in the top row of the following diagram:
$$\begin{array}{ccccccc}
\cdots & \longrightarrow & \begin{matrix} M^{n-1} \\ \oplus \\ M \end{matrix} & \xrightarrow[\partial_1^{K(\mathbf{x};M)}]{\begin{pmatrix} (x_1 & x_2 & \cdots & x_{n-1}) & x_n \\ (0 & 0 & \cdots & 0) & 0 \end{pmatrix}} & \begin{matrix} M \\ \oplus \\ 0 \end{matrix} & \longrightarrow & 0 \\
 & & \cong\downarrow & & \downarrow\cong & & \\
\cdots & \longrightarrow & M^n & \xrightarrow{\begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} & x_n \end{pmatrix}} & M & \longrightarrow 0.
\end{array}$$

The vertical isomorphism are the natural ones. It is straightforward to check that this diagram commutes. Hence, the map $\partial_1^{K(\mathbf{x};M)}$ has the desired form, and part (a) is established.

The terms of $K_\bullet(\mathbf{x};M)_n$ in degrees $n-1$ to $n+1$ are given in the top row of the next diagram:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \begin{matrix}0\\ \oplus\\ M\end{matrix} & \xrightarrow[\partial_n^{K(\mathbf{x};M)}]{\begin{pmatrix}\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix} & -\begin{pmatrix}x_n\\x_{n-1}\\-x_{n-2}\\\vdots\\(-1)^{n-2}x_1\end{pmatrix}\end{pmatrix}} & \begin{matrix}M\\ \oplus\\ M^{n-1}\end{matrix} & \longrightarrow & \cdots \\
 & & \cong\downarrow & & \downarrow\cong & & \\
0 & \longrightarrow & M & \xrightarrow[\begin{pmatrix}x_n\\-x_{n-1}\\\vdots\\(-1)^{n-1}x_1\end{pmatrix}]{} & M^n & \longrightarrow & \cdots .
\end{array}$$

The vertical isomorphism are the natural ones. It is straightforward to check that this diagram commutes. Hence, the map $\partial_n^{K(\mathbf{x};M)}$ has the desired form, and part (b) is established.

(c) The first and third isomorphisms in the following sequence are by definition:

$$\mathrm{H}_0(\mathbf{x};M) \cong \mathrm{Coker}(\partial_1^{K(\mathbf{x};M)}) \cong M/\operatorname{Im}\begin{pmatrix}x_1 & x_2 & \cdots & x_n\end{pmatrix} = M/(\mathbf{x})M.$$

Part (a) explains the second isomorphism.

(d) The first and third isomorphisms in the following sequence are by definition:

$$\mathrm{H}_n(\mathbf{x};M) \cong \mathrm{Ker}(\partial_n^{K(\mathbf{x};M)}) \cong \{m \in M \mid x_i m = 0 \text{ for } i = 1,\dots,n\} = \cap_{i=1}^n (0 :_M x_i).$$

Part (b) explains the second isomorphism. □

Proposition VIII.6.12. *Let R be a commutative ring, and let M be an R-module. Let $\mathbf{x} = x_1,\dots,x_n \in R$. For each i, there is an exact sequence*

$$0 \to \frac{\mathrm{H}_i(x_1,\dots,x_{n-1};M)}{x_n \mathrm{H}_i(x_1,\dots,x_{n-1};M)} \to \mathrm{H}_i(\mathbf{x};M) \to (0 :_{\mathrm{H}_{i-1}(x_1,\dots,x_{n-1};M)} x_n) \to 0.$$

Proof. This is Proposition VIII.6.4 with $X_\bullet = K_\bullet(x_1,\dots,x_{n-1};M)$. □

Proposition VIII.6.13. *Let R be a commutative ring, and let M be an R-module. Let $\mathbf{x} = x_1,\dots,x_n \in R$. For each i and j, there exists an integer $m_{i,j}$ such that $x_i^{m_{i,j}} \mathrm{H}_j(\mathbf{x};M) = 0$. (In fact $m_{i,j} = 2^{i+1}$ satisfies this condition.)*

Proof. We proceed by induction on n.

Base case $n = 1$. The Koszul complex $K_\bullet(x_1;M)$ has the following form by definition:

$$K_\bullet(x_1;M) = \qquad 0 \to M \xrightarrow{x_1} M \to 0.$$

It follows that the only non-zero homology modules are the following:

$$\mathrm{H}_0(x_1;M) \cong M/x_1M \qquad\qquad \mathrm{H}_0(x_1;M) \cong (0 :_M x_1)$$

Each of these modules is annihilated by x_1, so $m_{1,j} = 1$ works in this case.

Inductive step. Assume that $n \geqslant 2$ and that the result holds for the homology modules $\mathrm{H}_j(x_1,\dots,x_{n-1};M)$, that is, that there are integers $p_{i,j}$ such that $x_i^{p_{i,j}} \mathrm{H}_j(x_1,\dots,x_{n-1};M) = 0$ for $i = 1,\dots,n-1$ and $j = 0,\dots,n-1$. It follows that $x_i^{p_{i,j}}$ annihilates the submodule $(0 :_{\mathrm{H}_{i-1}(x_1,\dots,x_{n-1};M)} x_n)$ and the quotient module $\mathrm{H}_i(x_1,\dots,x_{n-1};M)/x_n \mathrm{H}_i(x_1,\dots,x_{n-1};M)$. By definition, the element x_n also annihilates these modules. Set $p_{n,j} = 1$ for $j = 0,\dots,n-1$. Set $p_{i,j} = 1$ for $i = 1,\dots,n$ and $j = -1, n$.

For $i = 0,\dots,n$ Proposition VIII.6.12 yields an exact sequence

$$0 \to \frac{\mathrm{H}_j(x_1,\dots,x_{n-1};M)}{x_n \mathrm{H}_j(x_1,\dots,x_{n-1};M)} \to \mathrm{H}_j(\mathbf{x};M) \to (0 :_{\mathrm{H}_{j-1}(x_1,\dots,x_{n-1};M)} x_n) \to 0.$$

For $i = 1,\dots,n$, we have $x_i^{p_{i,j}} \mathrm{H}_j(x_1,\dots,x_{n-1};M)/x_n \mathrm{H}_j(x_1,\dots,x_{n-1};M) = 0$ and $x_i^{p_{i,j-1}}(0 :_{\mathrm{H}_{j-1}(x_1,\dots,x_{n-1};M)} x_n) = 0$. Hence, Exercise VIII.6.21 implies that $x_i^{p_{i,j}+p_{i,j-1}} \mathrm{H}_j(\mathbf{x};M) = 0$, as desired. □

Remark VIII.6.14. Let R be a commutative ring, and let M be an R-module. Let $\mathbf{x} = x_1,\dots,x_n \in R$. It is less straightforward to show that, for each i and j, one has $x_i \mathrm{H}_j(\mathbf{x};M) = 0$. That is, in Proposition VIII.6.13, the integer $m_{i,j} = 1$ has the given properties.

Theorem VIII.6.15. *Let R be a commutative ring, and let M be an R-module. If $\mathbf{x} = x_1,\dots,x_n \in R$ is an M-regular sequence, then $\mathrm{H}_i(\mathbf{x};M) = 0$ for all $i \neq 0$.*

Proof. We proceed by induction on n. The base case $n = 1$ follows from Example VIII.6.1.

For the induction step, assume that $n \geqslant 2$ and that the result holds for regular sequences of length $n-1$. In particular, since the sequence $x_1,\dots,x_{n-1}$ is M-regular, we have $\mathrm{H}_i(x_1,\dots,x_{n-1};M) = 0$ for all $i \neq 0$. Also, from Proposition VIII.6.10 we know that $\mathrm{H}_i(\mathbf{x};M) = 0$ for $i < 0$.

We claim that $(0 :_{\mathrm{H}_{i-1}(x_1,\dots,x_{n-1};M)} x_n) = 0$ for all $i \geqslant 1$. Since $\mathbf{x}$ is M-regular, we know that x_n is $M/(x_1,\dots,x_{n-1})M$-regular. By Proposition VIII.6.11, we have

$$\mathrm{H}_0(x_1,\dots,x_{n-1};M) \cong M/(x_1,\dots,x_{n-1})M$$

so x_n is $\mathrm{H}_0(x_1,\dots,x_{n-1};M)$-regular. Hence, we have $(0 :_{\mathrm{H}_0(x_1,\dots,x_{n-1};M)} x_n) = 0$. (This is the case $i = 1$.) When $i \geqslant 2$, we have $\mathrm{H}_{i-1}(x_1,\dots,x_{n-1};M) = 0$, so $(0 :_{\mathrm{H}_{i-1}(x_1,\dots,x_{n-1};M)} x_n) = 0$. this establishes the claim.

Now we use the exact sequence from Proposition VIII.6.12:

$$0 \to \frac{\mathrm{H}_i(x_1,\dots,x_{n-1};M)}{x_n \mathrm{H}_i(x_1,\dots,x_{n-1};M)} \to \mathrm{H}_i(\mathbf{x};M) \to (0 :_{\mathrm{H}_{i-1}(x_1,\dots,x_{n-1};M)} x_n) \to 0.$$

When $i \geqslant 1$, we have $\mathrm{H}_i(x_1,\dots,x_{n-1};M) = 0$, so

$$\mathrm{H}_i(x_1,\dots,x_{n-1};M)/x_n \mathrm{H}_i(x_1,\dots,x_{n-1};M) = 0.$$

The claim implies that $(0 :_{\mathrm{H}_{i-1}(x_1,\dots,x_{n-1};M)} x_n) = 0$ for each $i \geqslant 1$, and we conclude from the displayed exact sequence that $\mathrm{H}_i(x_1,\dots,x_{n-1},x_n;M) = 0$ for $i \geqslant 1$, as desired. □

Theorem VIII.6.16. *Let R be a commutative ring, and let $\mathbf{x} = x_1, \dots, x_n \in R$ be an R-regular sequence.*

(a) *The Koszul complex $K_\bullet(\mathbf{x})$ is an R-free resolution of $R/(\mathbf{x})$.*
(b) *We have $\operatorname{Ext}^i_R(R/(\mathbf{x}), R/(\mathbf{x})) \cong [R/(\mathbf{x})]^{\binom{n}{i}}$ for each index i.*
(c) *We have $\operatorname{pd}_R(R/(\mathbf{x})) = n$.*

PROOF. (a) By Theorem VIII.6.15 we know that $\operatorname{H}_i(K_\bullet(\mathbf{x})) = 0$ for all $i \neq 0$. Proposition VIII.6.11 shows that $\operatorname{H}_0(K_\bullet(\mathbf{x})) \cong R/(\mathbf{x})$. Proposition VIII.6.10 shows that $K_i(\mathbf{x})$ is a (finitely generated) free R-module for each i, and that $K_i(\mathbf{x}) = 0$ when $i < 0$. It follows that $K_\bullet(\mathbf{x})$ is an R-free resolution of $R/(\mathbf{x})$.

(b) We use the projective resolution $K_\bullet(\mathbf{x})$. Proposition VIII.6.11(b) shows that this resolution has the following shape:

$$\cdots \xrightarrow{\partial^{K(\mathbf{x})}_{i+2}} R^{\binom{n}{i+1}} \xrightarrow{\partial^{K(\mathbf{x})}_{i+1}} R^{\binom{n}{i}} \xrightarrow{\partial^{K(\mathbf{x})}_{i}} R^{\binom{n}{i-1}} \xrightarrow{\partial^{K(\mathbf{x})}_{i-1}} \cdots .$$

Exercise VIII.6.20 shows that the matrices representing the differentials in this complex have only 0 and $\pm x_j$ entries. It follows that the relevant piece of the complex $\operatorname{Hom}_R(K_\bullet(\mathbf{x}), R/(\mathbf{x}))$ has the following form.

$$\operatorname{Hom}_R(R^{\binom{n}{i-1}}, R/(\mathbf{x})) \xrightarrow{(\partial^{K(\mathbf{x})}_{i})^*} \operatorname{Hom}_R(R^{\binom{n}{i}}, R/(\mathbf{x})) \xrightarrow{(\partial^{K(\mathbf{x})}_{i+1})^*} \operatorname{Hom}_R(R^{\binom{n}{i+1}}, R/(\mathbf{x}))$$

Under the isomorphisms $\operatorname{Hom}_R(R^j, R/(\mathbf{x})) \cong (R/(\mathbf{x}))^j$, the relevant piece of the complex $\operatorname{Hom}_R(K_\bullet(\mathbf{x}), R/(\mathbf{x}))$ has the following form:

$$[R/(\mathbf{x})]^{\binom{n}{i-1}} \xrightarrow[=0]{(\partial^{K(\mathbf{x})}_{i})^*} [R/(\mathbf{x})]^{\binom{n}{i}} \xrightarrow[=0]{(\partial^{K(\mathbf{x})}_{i+1})^*} [R/(\mathbf{x})]^{\binom{n}{i-1}}.$$

The displayed differential is 0 because each $x_i \in (\mathbf{x})$. Taking homology, we have

$$\operatorname{Ext}^i_R(R/(\mathbf{x}), R/(\mathbf{x})) \cong [R/(\mathbf{x})]^{\binom{n}{i}} / \operatorname{Im}(0) \cong [R/(\mathbf{x})]^{\binom{n}{i}}$$

as desired.

(c) The fact that that $K_i(\mathbf{x}) = 0$ when $i > n$ implies $\operatorname{pd}_R(R/(\mathbf{x})) \leqslant n$. Part (b) yields the isomorphism in the next display

$$\operatorname{Ext}^n_R(R/(\mathbf{x}), R/(\mathbf{x})) \cong R/(\mathbf{x}) \neq 0$$

and the non-vanishing holds because $\mathbf{x}$ is R-regular. Theorem VII.3.8 implies that $\operatorname{pd}_R(R/(\mathbf{x})) \geqslant n$, and hence the desired equality. □

Corollary VIII.6.17. *Let A be a commutative ring, and let $X_1, \dots, X_n$ be a list of independent variables. Set $R = A[X_1, \dots, X_n]$ or $A[X_1, \dots, X_n]_{(X_1, \dots, X_n)}$ or $A[\![X_1, \dots, X_n]\!]$. Then $R/(X_1, \dots, X_n) \cong A$ and $\operatorname{pd}_R(A) = n$.*

PROOF. In each case, the isomorphism $R/(X_1, \dots, X_n) \cong A$ is standard, and the sequence $X_1, \dots, X_n$ is R-regular. Hence, the computation $\operatorname{pd}_R(A) = n$ follows from Theorem VIII.6.16(c). □

Lemma VIII.6.18. *Let R be a commutative noetherian ring, and let M be a finitely generated non-zero R-module. Let $\mathbf{x} = x_1, \dots, x_n \in R$ be an R-regular sequence.*

(a) *One has $\operatorname{Tor}^R_i(R/(\mathbf{x}), M) \cong \operatorname{H}_i(\mathbf{x}; M)$ for all i.*
(b) *If $\mathbf{x}$ is also M-regular, then $\operatorname{Tor}^R_i(R/(\mathbf{x}), M) = 0$ for all $i \geqslant 1$.*

PROOF. (a) Theoerem VIII.6.16(VIII.6.16) implies that the Koszul complex $K_\bullet(\mathbf{x})$ is a free resolution of $R/(\mathbf{x})$ over R. Hence, we have the first isomorphism in the following sequence

$$\operatorname{Tor}_i^R(R/(\mathbf{x}), M) \cong \operatorname{H}_i(K_\bullet(\mathbf{x}) \otimes_R M) \cong \operatorname{H}_i(\mathbf{x}; M).$$

The second isomorphism is by definition

(b) When $\mathbf{x}$ is M-regular, the vanishing $\operatorname{H}_i(\mathbf{x}; M) = 0$ for $i \geqslant 1$ is from Theorem VIII.6.15. □

Exercises.

Exercise VIII.6.19. Verify the claims of Example VIII.6.6.

Exercise VIII.6.20. Let R be a commutative ring, and let M be an R-module. Let $\mathbf{x} = x_1, \dots, x_n \in R$. Show that, for each i, the differential $\partial_i^{K(\mathbf{x};M)} \colon M^{\binom{n}{i}} \to M^{\binom{n}{i-1}}$ can be represented by a matrix whose entries are $0, \pm x_j$.

Exercise VIII.6.21. Let R be a commutative ring, and let $x, y \in R$. Consider an exact sequence of R-module homomorphisms

$$0 \to A \to B \to C \to 0.$$

Show that, if $xA = 0 = yC$, then $xyB = 0$.

Exercise VIII.6.22. Let R be a commutative ring, and let M be an R-module. Let $\mathbf{x} = x_1, \dots, x_n \in R$. Show that there is an isomorphism of R-complexes

$$K_\bullet(\mathbf{x}) \otimes_R M \cong K_\bullet(\mathbf{x}; M).$$

Exercise VIII.6.23. Let R be a commutative ring, and let M be an R-module. Let $\mathbf{x} \in R$ be an R-regular sequence. Prove that $\operatorname{Tor}_i^R(R/(\mathbf{x}), M) \cong \operatorname{H}_n(\mathbf{x}; M)$ for each index i.

Exercise VIII.6.24. Let R be a commutative ring, and let M be an R-module. Let $\mathbf{x} = x_1, \dots, x_n \in R$.

(a) If F is a flat R-module, show that there is an isomorphism $\operatorname{H}_i(\mathbf{x}; M \otimes_R F) \cong \operatorname{H}_i(\mathbf{x}; M) \otimes_R F$ for each index i.
(b) Let $U \subseteq R$ be a multiplicatively closed set. Show that there is an isomorphism $\operatorname{H}_i(\mathbf{x}; U^{-1}M) \cong U^{-1} \operatorname{H}_i(\mathbf{x}; M)$ for each index i. Conclude that, if $x_j \in U$ for some j, then $\operatorname{H}_i(\mathbf{x}; U^{-1}M) = 0$ for all i.

Exercise VIII.6.25. (Challenge exercise: Koszul complexes are exterior algebras) Let R be a commutative ring, and let $\mathbf{x} = x_1, \dots, x_n \in R$.

Set $L_0 = R$ with basis 1. Set $L_1 = R^n$ with basis $e_1, \dots, e_n$. For $i = 2, \dots, n$ let L_i denote the free R-module whose basis is the following set of formal symbols:

$$\{e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i} \mid 1 \leqslant j_1 < j_2 < \cdots < j_i \leqslant n\}.$$

Let $L_\bullet$ be the sequence

$$L_\bullet = \quad 0 \to L_n \xrightarrow{\partial_n^L} L_{n-1} \xrightarrow{\partial_{n-1}^L} \cdots \xrightarrow{\partial_1^L} L_0 \to 0$$

with maps defined on bases as follows:

$$\begin{aligned} i=1: \qquad & \partial_1^L \colon R^n \to R \\ & e_j \mapsto x_j \\ i>1: \qquad & \partial_i^L \colon L_i \to L_{i-1} \end{aligned}$$

$$e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i} \mapsto \sum_{l=1}^{j} (-1)^{l+1} x_l e_{j_1} \wedge \cdots \wedge e_{j_{l-1}} \wedge e_{j_{l+1}} \wedge \cdots \wedge e_{j_i}$$

(a) Write out the sequence $L_\bullet$ in the cases $n = 1, 2, 3$ writing the maps ∂_i^L as matrices. Compare your answer to the complexes in Example VIII.6.6.
(b) Prove that $L_\bullet$ is an R-complex.
(c) Prove that $L_i \cong R^{\binom{n}{i}}$ for each index i.
(d) Prove that $L_\bullet$ is isomorphic to the Koszul complex $K_\bullet(\mathbf{x})$.
(e) Prove that $L_\bullet$ is independent of the order of the sequence $\mathbf{x}$: if $\mathbf{x}'$ is a rearrangement of the sequence $\mathbf{x}$ and $L'_\bullet$ is constructed using the sequence $\mathbf{x}'$, then $L'_\bullet \cong L_\bullet$.
(f) Prove that $K_\bullet(\mathbf{x})$ is independent of the order of the sequence $\mathbf{x}$: if $\mathbf{x}'$ is a rearrangement of the sequence $\mathbf{x}$ then $K_\bullet(\mathbf{x}') \cong K_\bullet(\mathbf{x})$.

VIII.7. Epilogue: Tor and Torsion

Here we describe the connection between Tor and torsion. This material is not needed for the sequel. See Section IV.5 for definitions. (This material is taken from Rotman [**4**].)

Lemma VIII.7.1. *Let R be an integral domain with field of fractions K, and let M be an R-module.*

(a) *If M is torsion, then $(K/R) \otimes_R M = 0 = K \otimes_R M$ and $\operatorname{Tor}_1^R(K/R, M) \cong M$.*
(b) *We have $\operatorname{Tor}_i^R(K/R, M) = 0$ for all $i \geqslant 2$.*
(c) *If M is torsion-free, then the natural map $M \to K \otimes_R M$ is a monomorphism, and $\operatorname{Tor}_1^R(K/R, M) = 0$.*

Proof. We use the exacts sequence

$$0 \to R \to K \to K/R \to 0 \tag{VIII.7.1.1}$$

throughout the proof.

(a) The vanishing $(K/R) \otimes_R M = 0 = K \otimes_R M$ follows from the fact that K/R and K divisible, because M is torsion.

For the isomorphism, consider the following piece of the long exact sequence in $\operatorname{Tor}_i^R(-, M)$ associated to the short exact sequence (VIII.7.1.1):

$$\underbrace{\operatorname{Tor}_1^R(K, M)}_{=0} \to \operatorname{Tor}_1^R(K/R, M) \to \underbrace{R \otimes_R M}_{\cong M} \to \underbrace{K \otimes_R M}_{=0}.$$

The vanishing $\operatorname{Tor}_1^R(K, M) = 0$ is due to the fact that K is flat; see Lemma VI.2.1 and Propositions II.2.9(d) and IV.4.7(b). The desired isomorphism now follows.

(b) Again, we consult part of the long exact sequence in $\operatorname{Tor}_i^R(-, M)$ associated to the short exact sequence (VIII.7.1.1):

$$\underbrace{\operatorname{Tor}_i^R(K, M)}_{=0} \to \operatorname{Tor}_i^R(K/R, M) \to \underbrace{\operatorname{Tor}_{i-1}^R(R, M)}_{=0}.$$

The vanishing follows from the fact that K and R are flat, since $i \geqslant 2$. The desired vanishing now follows.

(c) Let $m \in M$ be an element of the kernel of the map $M \to K \otimes_R M$. Let $P = (0)R$, which is prime and satisfies $M_P \cong K \otimes_R M$. It follows that m is in the kernel of the natural map $M \to M_P$, that is, that there is an element $s \in R \smallsetminus P = R \smallsetminus \{0\}$ such that $sm = 0$. Since M is torsion-free, we conclude that $m = 0$, so the map $M \to K \otimes_R M$ has trivial kernel.

The module $K \otimes_R M$ is a K-module. Since K is a field, we have $K \otimes_R M \cong K^{(\Lambda)}$ for some set Λ. Since K is a flat R-module, Exercise II.3.9 implies that $K^{(\Lambda)}$ is flat, that is, that $K \otimes_R M$ is flat.

There is an exact sequence

$$0 \to M \to K \otimes_R M \to (K \otimes_R M)/M \to 0$$

and we consider the associated long exact sequence in $\operatorname{Tor}_i^R(K/R, -)$:

$$\underbrace{\operatorname{Tor}_2^R(K/R, (K \otimes_R M)/M)}_{=0} \to \operatorname{Tor}_1^R(K/R, M) \to \underbrace{\operatorname{Tor}_1^R(K/R, K \otimes_R M)}_{=0}.$$

The vanishing $\operatorname{Tor}_2^R(K/R, (K \otimes_R M)/M) = 0$ is from part (b), and the fact that $K \otimes_R M$ is flat implies that $\operatorname{Tor}_1^R(K/R, K \otimes_R M) = 0$. The desired vanishing now follows. □

Theorem VIII.7.2. *Let R be an integral domain with field of fractions K, and let M be an R-module. There is an R-module isomorphism $\psi\colon \operatorname{Tor}_1^R(K/R, M) \xrightarrow{\cong} \operatorname{t}(M)$.*

Proof. Set $M' = M/\operatorname{t}(M)$, and recall that Remark IV.5.6 implies that $\operatorname{t}(M)$ is torsion, and M' is torsion-free. Consider the exact sequence

$$0 \to \operatorname{t}(M) \to M \to M' \to 0$$

and the associated long exact sequence in $\operatorname{Tor}_i^R(K/R, -)$:

$$\underbrace{\operatorname{Tor}_2^R(K/R, M')}_{=0} \to \underbrace{\operatorname{Tor}_1^R(K/R, \operatorname{t}(M))}_{\cong \operatorname{t}(M)} \to \operatorname{Tor}_1^R(K/R, M) \to \underbrace{\operatorname{Tor}_1^R(K/R, M')}_{=0}.$$

The vanishings are from Lemma VIII.7.1, parts (b) and (c). Lemma VIII.7.1(a) yields the isomorphism $\operatorname{Tor}_1^R(K/R, \operatorname{t}(M)) \cong \operatorname{t}(M)$. This exact sequence provides the desired isomorphism. □

Exercises.

Exercise VIII.7.3. Let R be an integral domain with field of fractions K, and let M be an R-module.

(a) Prove that there is an exact sequence

$$0 \to \operatorname{t}(M) \to M \to K \otimes_R M \to (K/R) \otimes_R M \to 0.$$

(b) Prove that M is torsion if and only if $K \otimes_R M = 0$.

Exercise VIII.7.4. Let R be an integral domain with field of fractions K, and let M and N be R-modules.

(a) Prove that, if N is torsion, then $\operatorname{Tor}_i^R(M, N)$ is torsion for all $i \geqslant 0$. (Hint: proceed by induction on i, using dimension-shifting.)

(b) Prove that $\operatorname{Tor}_i^R(M,N)$ is torsion for all $i \geqslant 1$. (Hint: First prove the case where N is torsion-free, using the exact sequence

$$0 \to N \to K \otimes_R N \to (K \otimes_R N)/N \to 0.$$

For the general case, use the exact sequence $0 \to \mathrm{t}(M) \to M \to M/\mathrm{t}(M) \to 0$.)

CHAPTER IX

Depth and Homological Dimensions September 8, 2009

One goal of this chapter is to prove the Auslander-Buchsbaum formula: If R is a local noetherian ring and M is a non-zero finitely generated R-module of finite projective dimension, then $\mathrm{pd}_R(M) = \mathrm{depth}(R) - \mathrm{depth}_R(M)$. See Theorem IX.2.3.

IX.1. Projective Dimension and Regular Sequences

This section contains preparatory lemmas for use in the proof of the Auslander-Buchsbaum formula. We begin with a useful consequence of Nakayama's Lemma.

Lemma IX.1.1. *Let $(R, \mathfrak{m}, k)$ be a commutative local ring, and let M be a non-zero finitely generated R-module. Let $m_1, \dots, m_n \in M$ be a minimal generating sequence for M, and let $f\colon R^n \to M$ be given by $f(\sum_i r_i e_i) = \sum_i r_i m_i$. Then $\mathrm{Ker}(f) \subseteq \mathfrak{m}R^n$.*

Proof. Note that Nakayama's Lemma implies that the residues

$$\overline{m_1}, \dots, \overline{m_n} \in M/\mathfrak{m}M \cong M \otimes_R k$$

form a basis for $M/\mathfrak{m}M$ as a k-vector space. The map f is surjective by definition. Furthermore, tensoring with k yields an isomorphism $\overline{f}\colon k^n \to M \otimes_R k$, because the $\overline{m_1}, \dots, \overline{m_n} \in M/\mathfrak{m}M \cong M \otimes_R k$ form a k-basis for $M/\mathfrak{m}M$. Consider the following commutative diagram:

$$\begin{array}{ccc} R^n & \xrightarrow{f} & M \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\tau} \\ k^n & \xrightarrow[\cong]{\overline{f}} & M \otimes_R k \end{array}$$

where the vertical maps are the natural surjections. Chase the diagram to see that $\pi(\mathrm{Ker}(f)) \subseteq \mathrm{Ker}(\overline{f}) = 0$. It follows that

$$\mathrm{Ker}(f) \subseteq \pi^{-1}(\pi(\mathrm{Ker}(f))) \subseteq \pi^{-1}(0) = \mathfrak{m}R^n$$

as desired. □

Here is a useful application of the long exact sequence from Theorem VIII.1.4.

Lemma IX.1.2. *Let R be a commutative ring, and let M be a non-zero finitely generated R-module. Let $\mathbf{x} = x_1, \dots, x_n \in R$ be a sequence that is R-regular and M-regular, and let $P_\bullet$ be a free resolution of M over R. Then the complex $P_\bullet \otimes_R R/(\mathbf{x})$ is a free resolution of $M/(\mathbf{x})M$ over $R/(\mathbf{x})$.*

PROOF. Note that each module P_i is of the form $R^{(\Lambda_i)}$ for some set Λ_i. It follows that

$$(P_\bullet \otimes_R R/(\mathbf{x}))_i = P_i \otimes_R R/(\mathbf{x}) \cong R^{(\Lambda_i)} \otimes_R R/(\mathbf{x}) \cong (R/(\mathbf{x}))^{(\Lambda_i)}.$$

In particular, each module $(P_\bullet \otimes_R R/(\mathbf{x}))_i$ is a free $R/(\mathbf{x})$-module, and we have $(P_\bullet \otimes_R R/(\mathbf{x}))_i = 0$ when $i < 0$. Furthermore, since each map ∂_i^P is an R-module homomorphism, we conclude that the induced map

$$\partial_i^{P \otimes_R R/(\mathbf{x})} = \partial_i^P \otimes_R R/(\mathbf{x})$$

is an $R/(\mathbf{x})$-module homomorphism.

To complete the proof, it suffices to show that

$$\mathrm{H}_i(P_\bullet \otimes_R R/(\mathbf{x})) \cong \begin{cases} M/(\mathbf{x})M & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

We proceed by induction on n.

Base case: $n = 1$. Consider the sequence

$$0 \to R \xrightarrow{x_1} R \to R/(x_1) \to 0$$

which is exact because x is R-regular. Tensoring with $P_\bullet$ yields the next sequence of chain complexes:

$$0 \to P_\bullet \otimes_R R \xrightarrow{x_1} P_\bullet \otimes_R R \to P_\bullet \otimes_R R/(x_1) \to 0.$$

This is isomorphic to the following sequence

$$0 \to P_\bullet \xrightarrow{x_1} P_\bullet \to P_\bullet \otimes_R R/(x_1) \to 0. \tag{IX.1.2.1}$$

Because x_1 is R-regular, it is P_i-regular for each i. Hence, the sequence (IX.1.2.1) is exact. In small degrees, the associated long exact sequence looks like

$$0 \to \mathrm{H}_1(P_\bullet \otimes_R R/(x_1)) \to \mathrm{H}_0(P_\bullet) \xrightarrow{x_1} \mathrm{H}_0(P_\bullet) \to \mathrm{H}_0(P_\bullet \otimes_R R/(x_1)) \to 0.$$

Because $P_\bullet$ is a resolution of M, this has the form

$$0 \to \mathrm{H}_1(P_\bullet \otimes_R R/(x_1)) \to M \xrightarrow{x_1} M \to \mathrm{H}_0(P_\bullet \otimes_R R/(x_1)) \to 0.$$

Since x_1 is M-regular, it follows that $\mathrm{H}_1(P_\bullet \otimes_R R/(x_1)) = 0$. This sequence also shows that $\mathrm{H}_0(P_\bullet \otimes_R R/(x_1)) \cong M/x_1M$. For $i \geqslant 2$, the long exact sequence associated to (IX.1.2.1) has the form

$$0 \to \mathrm{H}_i(P_\bullet \otimes_R R/(x_1)) \to 0$$

so $\mathrm{H}_i(P_\bullet \otimes_R R/(x_1)) = 0$. This completes the base case.

The induction step is straightforward. □

Lemma IX.1.3. *Let R be a commutative ring, and let M be a non-zero R-module. Let $\mathbf{x} = x_1, \dots, x_n \in R$ be a sequence that is R-regular and M-regular.*

(a) *For each $R/\mathbf{x}R$-module N and each $i \geqslant 0$, there is an isomorphism*

$$\mathrm{Ext}^i_{R/\mathbf{x}R}(M/\mathbf{x}M, N) \cong \mathrm{Ext}^i_R(M, N).$$

(b) *Assume that R is noetherian and local and that M is finitely generated. Then $\mathrm{pd}_{R/\mathbf{x}R}(M/\mathbf{x}M) = \mathrm{pd}_R(M)$. In particular $M/\mathbf{x}M$ is free over $R/\mathbf{x}R$ if and only if M is free over R.*

PROOF. (a) Let $P_\bullet$ be a free resolution of M over R. Lemma IX.1.2 implies that the complex $P_\bullet \otimes_R R/\mathbf{x}R$ is a free resolution of $M/\mathbf{x}M \cong M \otimes_R R/\mathbf{x}R$ over $R/\mathbf{x}R$. In the following sequence, the first isomorphism is Hom-tensor adjointness VI.1.9(a) and the second isomorphism is from Remark I.5.3 and Exercise VI.4.8:

$$\operatorname{Hom}_{R/\mathbf{x}R}(P_\bullet \otimes_R R/\mathbf{x}R, N) \cong \operatorname{Hom}_R(P_\bullet, \operatorname{Hom}_{R/\mathbf{x}R}(R/\mathbf{x}R, N)) \cong \operatorname{Hom}_R(P_\bullet, N).$$

Since $P_\bullet \otimes_R R/\mathbf{x}R$ is a free resolution of $M/\mathbf{x}M$ over $R/\mathbf{x}R$ and $P_\bullet$ a free resolution of M over R, we have

$$\begin{aligned}\operatorname{Ext}^i_{R/\mathbf{x}R}(M/\mathbf{x}M, N) &\cong \operatorname{H}_{-i}(\operatorname{Hom}_{R/\mathbf{x}R}(P_\bullet/\mathbf{x}P_\bullet, N)) \\ &\cong \operatorname{H}_{-i}(\operatorname{Hom}_R(P_\bullet, N)) \cong \operatorname{Ext}^i_R(M, N)\end{aligned}$$

as desired.

(b) Let k denote the residue field of R, which is also the residue field of $R/(\mathbf{x})$. Part (a) explains the second equality in the next sequence

$$\begin{aligned}\operatorname{pd}_{R/\mathbf{x}R}(M/\mathbf{x}M) &= \sup\{i \geqslant 0 \mid \operatorname{Ext}^i_{R/\mathbf{x}R}(M/\mathbf{x}M, k) \neq 0\} \\ &= \sup\{i \geqslant 0 \mid \operatorname{Ext}^i_R(M, k) \neq 0\} \\ &= \operatorname{pd}_R(M)\end{aligned}$$

while the other equalities are from Theorem VII.3.14. Corollary V.4.9 shows that M is free if and only if it has projective dimension 0, so we conclude that $M/\mathbf{x}M$ is free over $R/\mathbf{x}R$ if and only if M is free over R. □

The special case $M_1 = R$ and $M_2 = M$ in the next result shows how you can determine when sequences that are M-regular and R-regular exist.

Lemma IX.1.4. *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring, and let $M_1, \dots, M_n$ be non-zero finitely generated R-modules. If $\operatorname{depth}(M_i) \geqslant d$ for each i, then there is a sequence $\mathbf{x} = x_1, \dots, x_d \in \mathfrak{m}$ that is M_i-regular for each i.*

PROOF. Set $M = M_1 \oplus \cdots \oplus M_n$. Our hypotheses on depth imply that $\operatorname{Ext}^j_R(k, M_i) = 0$ for each i and each $j < d$. Hence, Exercise VI.2.12 implies $\operatorname{Ext}^j_R(k, M) = 0$ for each $j < d$, that is, $\operatorname{depth}_R(M) \geqslant d$. Thus, there is an M-regular sequence $\mathbf{x} = x_1, \dots, x_d \in \mathfrak{m}$, and one checks readily that this sequence is M_i-regular for each i. □

Exercises.

Exercise IX.1.5. Complete the proof of Lemma IX.1.2.

Exercise IX.1.6. Construct examples showing that the sequence $\mathbf{x}$ must be both R-regular and M-regular in Lemmas IX.1.2 and IX.1.3(b).

IX.2. The Auslander-Buchsbaum Formula

We begin with the base case of our proof of the Auslander-Buchsbaum Formula.

Lemma IX.2.1. *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring, and let M be a non-zero finitely generated R-module. If $\operatorname{depth}(R) = 0$ and $\operatorname{pd}_R(M) < \infty$, then M is free.*

PROOF. Let $n \geqslant 0$, and assume that $\operatorname{pd}_R(M) \leqslant n$. We show by induction on n that M is free. The case $n = 0$ is Corollary V.4.9.

Base case: $n = 1$. Let $m_1, \dots, m_n \in M$ be a minimal generating sequence for M. Let $f\colon R^n \to M$ be given by $f(\sum_i r_i e_i) = \sum_i r_i m_i$. Lemma IX.1.1 implies that $K = \operatorname{Ker}(f) \subseteq \mathfrak{m}R^n$. Since $\operatorname{pd}_R(M) < \infty$, the sequence

$$0 \to K \to R^n \xrightarrow{f} M \to 0$$

shows that K is projective by Exercise VII.3.17. Corollary V.4.9 implies that there is an isomorphism $R^m \cong K$ for some $m \geqslant 0$. Combine this with the inclusion $K \subseteq R^n$ to find an R-module monomorphism $g\colon R^m \to R^n$ such that $\operatorname{Im}(g) = K \subseteq \mathfrak{m}R^n$. Let $e_1, \dots, e_m \in R^m$ be a basis.

Suppose that $m \neq 0$. Then g is represented by an $n \times m$ matrix $(a_{i,j})$. The columns of this matrix are elements of $K \subseteq \mathfrak{m}R^n$. Hence, each $a_{i,j} \in \mathfrak{m}$. Since $\operatorname{depth}(R) = 0$, there is an element $0 \neq r \in R$ such that $\mathfrak{m}r = 0$. It follows that $a_{i,j}r = 0$ for all i, j. Hence, we have $0 \neq re_1 \in \operatorname{Ker}(g) = 0$, a contradiction. It follows that $m = 0$, and thus $\operatorname{Ker}(f) \cong R^m = 0$. Hence, f is an isomorphism and M is free. This completes the base case.

Induction step. Assume that $n > 1$ and that, whenever N is a finitely generated R-module such that $\operatorname{pd}_R(N) < n$, we know that N is free. Suppose that M is not free. Then $\operatorname{pd}_R(M) \geqslant 1$. Let $f\colon R^n \to M$ be an R-module epimorphism, and set $K = \operatorname{Ker}(f) \subseteq R^n$. Since $\operatorname{pd}_R(M) < \infty$, the sequence

$$0 \to K \to R^n \xrightarrow{f} M \to 0$$

shows that $\operatorname{pd}_R(K) = \operatorname{pd}_R(M) - 1 < n$ by Exercise VII.3.17. Hence, our induction hypothesis implies that K is free. Thus, the displayed sequence implies that $\operatorname{pd}_R(M) \leqslant 1$. The case $n = 1$ implies that M is free, a contradiction. Thus, M is free, as desired. □

Example IX.2.2. Let k be a field, and let $R = k[\![x, y]\!]/(x^2, xy)$. If M is an R-module of finite projective dimension, then M is free. The same conclusion holds if R is replaced by any local artinian ring, e.g., $k[x_1, \dots, x_n]/I$ where I is a proper ideal containing a power of each of the variables.

Theorem IX.2.3 (Auslander-Buchsbaum Formula)**.** *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring, and let M be a non-zero finitely generated R-module. If $\operatorname{pd}_R(M) < \infty$, then $\operatorname{pd}_R(M) = \operatorname{depth}(R) - \operatorname{depth}_R(M)$.*

PROOF. By induction on $d = \operatorname{depth}(R)$. The base case $\operatorname{depth}(R) = 0$ is contained in Lemma IX.2.1: If $\operatorname{depth}(R) = 0$, then M is free, so $\operatorname{pd}_R(M) = 0$ and $\operatorname{depth}_R(M) = \operatorname{depth}(R) = 0$.

Inductively, assume that $d \geqslant 1$ and that the result holds for all finitely generated modules over all commutative noetherian local rings of depth $< d$.

Case 1: $\operatorname{depth}_R(M) \geqslant 1$. Since $\operatorname{depth}(R) \geqslant 1$, Lemma IX.1.4 implies that there is an element $x \in \mathfrak{m}$ that is R-regular and M-regular. In the following sequence, the first equality is from Lemma IX.1.3(b)

$$\begin{aligned}
\operatorname{pd}_R(M) &= \operatorname{pd}_{R/xR}(M/xM) \\
&= \operatorname{depth}(R/xR) - \operatorname{depth}_{R/xR}(M/xM) \\
&= [\operatorname{depth}(R) - 1] - [\operatorname{depth}_R(M) - 1] \\
&= \operatorname{depth}(R) - \operatorname{depth}_R(M).
\end{aligned}$$

The second equality is by induction, because the fact that x is R-regular implies $\operatorname{depth}(R/xR) = \operatorname{depth}(R) - 1$. The third equality is a basic property of depth, using the fact that x is R-regular and M-regular.

Case 2. $\operatorname{depth}_R(M) = 0$. Since $\operatorname{depth}(R) \geqslant 1$, in this case we know that M is not free. (If it were, it would have $0 = \operatorname{depth}_R(M) = \operatorname{depth}(R) \geqslant 1$.) Let $f\colon R^n \to M$ be a surjection, and set $K = \operatorname{Ker}(f) \subseteq R^n$. Since $\operatorname{pd}_R(M) < \infty$, the exact sequence

$$0 \to K \to R^n \xrightarrow{f} M \to 0$$

shows that $\operatorname{pd}_R(K) = \operatorname{pd}_R(M) - 1 < n$ by Exercise VII.3.17. Since $\operatorname{depth}(R) \geqslant 1$, we have $\operatorname{depth}_R(R^n) \geqslant 1$. In particular, we have $\operatorname{Hom}_R(k, R^n) \cong \operatorname{Ext}^0_R(k, R^n) = 0$. Thus, the long exact sequence in $\operatorname{Ext}^i_R(k, -)$ associated to the displayed sequence starts as

$$0 \to \operatorname{Hom}_R(k, K) \to 0 \to \operatorname{Hom}_R(k, M) \to \operatorname{Ext}^1_R(k, K).$$

By assumption, we have $\operatorname{depth}_R(M) = 0$ and hence $\operatorname{Hom}_R(k, M) \neq 0$. The displayed sequence implies $\operatorname{Ext}^1_R(k, K) \neq 0 = \operatorname{Hom}_R(k, K)$, and it follows that

$$\operatorname{depth}_R(K) = 1 = \operatorname{depth}_R(M) + 1.$$

This explains the third equality in the next sequence:

$$\operatorname{pd}_R(M) = \operatorname{pd}_R(K) + 1 = \operatorname{depth}(R) - \operatorname{depth}_R(K) + 1 = \operatorname{depth}(R) - \operatorname{depth}_R(M).$$

the second equality is from Case 1. □

Corollary IX.2.4. *Let R be a commutative noetherian local ring, and let M be a non-zero finitely generated R-module. If $\operatorname{pd}_R(M) < \infty$, then there is an inequality $\operatorname{depth}_R(M) \leqslant \operatorname{depth}(R)$.*

Proof. The Auslander-Buchsbaum formula implies

$$0 \leqslant \operatorname{pd}_R(M) = \operatorname{depth}(R) - \operatorname{depth}_R(M)$$

and the desired conclusion follows directly. □

Example IX.2.5. Let k be a field, and let $R = k[\![x, y, z]\!]/(xz, yz)$. The element $x - z$ is R-regular and $R/(x - z) \cong k[\![x, y]\!]/(x^2, xy)$. Since $\operatorname{depth}(R/(x - z)) = 0$, we conclude that $x - z$ is a maximal R-sequence in $\mathfrak{m} = (x, y, z)R$. Thus, $\operatorname{depth}(R) = 1$.

The R-module $M = R/z \cong k[\![x, y]\!]$ has depth 2, so Corollary IX.2.4 that $\operatorname{pd}_R(M) = \infty$.

Corollary IX.2.6. *Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring, and let M be a non-zero finitely generated R-module. If $\operatorname{pd}_R(M) < \infty$, then there are inequalities $\operatorname{pd}_R(M) \leqslant \operatorname{depth}(R) \leqslant \dim(R)$.*

Proof. The Auslander-Buchsbaum formula implies

$$\operatorname{pd}_R(M) = \operatorname{depth}(R) - \operatorname{depth}_R(M) \leqslant \operatorname{depth}(R)$$

because $\operatorname{depth}_R(M) \geqslant 0$. The inequality $\operatorname{depth}(R) \leqslant \dim(R)$ is contained in Theorem VII.2.7(b). □

Exercises.

Exercise IX.2.7. Verify the facts from Example IX.2.2.

IX.3. Depth and Flat Ring Homomorphisms

The goal of this section is to prove Theorem IX.3.6, which explains the relation between $\operatorname{depth}(R)$ and $\operatorname{depth}(S)$ when $\varphi\colon R \to S$ is a flat local ring homomorphism between commutative local noetherian rings. (The definition of a local ring homomorphism is in III.5.4.) Much of the material for this section comes from [**1**].

We require the following result, called Krull's intersection theorem, which we do not have time to prove; see [**3**, (8.10)].

Theorem IX.3.1 (Krull)**.** *Let $(S,\mathfrak{n})$ be a commutative noetherian local ring. If N is a finitely generated S-module, then $\cap_{i=0}^{\infty}\mathfrak{a}^i N = 0$ for each ideal $\mathfrak{a} \subseteq \mathfrak{n}$.*

Lemma IX.3.2. *Let $\varphi\colon (R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ be a flat local ring homomorphism between commutative noetherian rings, and let M be a finitely generated non-zero R-module. If $\mathbf{y} = y_1,\dots,y_n \in \mathfrak{n}$ is an $S/\mathfrak{m}S$-regular sequence, then $\mathbf{y}$ is $S\otimes_R M$-regular and S-regular, and the composition $R \to S \to S/\mathbf{y}S$ is flat and local.*

Proof. Claim 1: For each integer $j \geqslant 0$, there is an S-module isomorphism $\alpha_j\colon S\otimes_R(\mathfrak{m}^j M) \xrightarrow{\cong} \mathfrak{m}^j(S\otimes_R M)$ such that $\alpha_j(s\otimes(xm)) = x(s\otimes m)$ for all $s\in S$, all $x\in\mathfrak{m}^j$, and all $m\in M$. Let $\epsilon_j\colon \mathfrak{m}^j M \to M$ be the inclusion. Since S is flat, the induced map

$$S\otimes_R \epsilon_j\colon S\otimes_R(\mathfrak{m}^j M) \to S\otimes_R M$$

is an S-module monomorphism. By definition, we have

$$(S\otimes_R\epsilon_j)(s\otimes(xm)) = s\otimes(xm) = x(s\otimes m)$$

for all $s\in S$, all $x\in\mathfrak{m}^j$, and all $m\in M$. From this, it follows readily that $\operatorname{Im}(S\otimes_R\epsilon_j) = \mathfrak{m}^j(S\otimes_R M)$, so the map $S\otimes_R\epsilon_j$ induces an isomorphism with the desired properties.

Claim 2: For each integer $j\geqslant 0$, there is an R-module isomorphism

$$(\mathfrak{m}^j M)/(\mathfrak{m}^{j+1}M) \cong k^{t_j}$$

for some integer $k_t \geqslant 0$. The R-module $\mathfrak{m}^j M$ is finitely generated, so Exercise V.4.13 implies that the $R/\mathfrak{m}$-module

$$(\mathfrak{m}^j M)/(\mathfrak{m}^{j+1}M) = (\mathfrak{m}^j M)/[\mathfrak{m}(\mathfrak{m}^j M)]$$

is finitely generated. Since $R/\mathfrak{m} = k$ is a field, this module has the form k^{t_j}, as claimed.

Claim 3: For each integer $j\geqslant 0$, there is an S-module isomorphism

$$[\mathfrak{m}^j(S\otimes_R M)]/[\mathfrak{m}^{j+1}(S\otimes_R M)] \cong (S/\mathfrak{m}S)^{t_j}$$

where $t_j = \dim_k((\mathfrak{m}^j M)/(\mathfrak{m}^{j+1}M))$. Consider the exact sequence

$$0 \to \mathfrak{m}^{j+1}M \to \mathfrak{m}^j M \to (\mathfrak{m}^j M)/(\mathfrak{m}^{j+1}M) \to 0.$$

Since S is flat over R, the top row of the following diagram is exact

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S\otimes_R(\mathfrak{m}^{j+1}M) & \longrightarrow & S\otimes_R(\mathfrak{m}^j M) & \longrightarrow & S\otimes_R\left(\dfrac{\mathfrak{m}^j M}{\mathfrak{m}^{j+1}M}\right) & \longrightarrow & 0 \\
 & & \cong\downarrow{\scriptstyle \alpha_{j+1}} & & \cong\downarrow{\scriptstyle \alpha_j} & & \cong\downarrow{\scriptstyle \exists\beta_j} & & \\
0 & \longrightarrow & \mathfrak{m}^{j+1}(S\otimes_R M) & \longrightarrow & \mathfrak{m}^j(S\otimes_R M) & \longrightarrow & \dfrac{\mathfrak{m}^j(S\otimes_R M)}{\mathfrak{m}^{j+1}(S\otimes_R M)} & \longrightarrow & 0
\end{array}$$

is exact. The bottom row is the natural exact sequence induced by the inclusion $\mathfrak{m}^{j+1}(S \otimes_R M) \subseteq \mathfrak{m}^j(S \otimes_R M)$. It is straightforward to show that the left-most square in this diagram commutes, where α_{j+1} and α_j are the isomorphisms from Claim 1. It follows that there is an S-module isomorphism β_j making the right-most square commute. This explains the first isomorphism in the next sequence

$$\frac{\mathfrak{m}^j(S \otimes_R M)}{\mathfrak{m}^{j+1}(S \otimes_R M)} \cong S \otimes_R \left(\frac{\mathfrak{m}^j M}{\mathfrak{m}^{j+1} M}\right) \cong S \otimes_R (k^{t_j}) \cong (S \otimes_R k)^{t_j} \cong (S/\mathfrak{m}S)^{t_j}.$$

The second isomorphism is from Claim 2, and the other isomorphisms are justified as in the proof of Lemma III.5.12.

We now prove the result by induction on n.

Base case: $n = 1$. Since M is finitely generated over R, the base-changed module $S \otimes_R M$ is finitely generated over S. Let $\xi \in S \otimes_R M$ such that $\xi \neq 0$, and suppose that $y_1\xi = 0$. Krull's intersection theorem implies that

$$0 = \cap_{i=0}^{\infty}(\mathfrak{m}S)^i(S \otimes_R M) = \cap_{i=0}^{\infty}\mathfrak{m}^i(S \otimes_R M)$$

so the condition $0 \neq \xi \in S \otimes_R M$ implies that there is an integer $i \geqslant 0$ such that $\xi \in \mathfrak{m}^i(S \otimes_R M) \smallsetminus \mathfrak{m}^{i+1}(S \otimes_R M)$. The element

$$\overline{\xi} \in \mathfrak{m}^i(S \otimes_R M)/\mathfrak{m}^{i+1}(S \otimes_R M) \cong (S/\mathfrak{m}S)^{t_j}$$

is nonzero and is annihilated by y_1. However, since y_1 is $S/\mathfrak{m}S$-regular, it is also $(S/\mathfrak{m}S)^{t_j}$-regular, and this is a contradiction. Thus, y_1 is $S \otimes_R M$-regular.

Note that the special case $M = R$ implies that y_1 is S-regular.

Set $\overline{S} = S/y_1S$. To prove that the composition $R \to S \to \overline{S}$ is flat and local, it suffices to show that $\overline{S}$ is flat as an R-module. Corollary III.2.5 shows that it suffices to consider an arbitrary short exact sequence

$$0 \to M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \to 0$$

of finitely generated R-modules and show that the induced sequence

$$0 \to \overline{S} \otimes_R M_1 \xrightarrow{\overline{S}\otimes_R f} \overline{S} \otimes_R M_2 \xrightarrow{\overline{S}\otimes_R g} \overline{S} \otimes_R M_3 \to 0$$

is exact. Since S is flat over R, the sequence

$$0 \to S \otimes_R M_1 \xrightarrow{S\otimes_R f} S \otimes_R M_2 \xrightarrow{S\otimes_R g} S \otimes_R M_3 \to 0$$

is exact. Since y_1 is S-regular and $S \otimes_R M_3$ regular, Lemma VIII.6.18 implies that $\operatorname{Tor}_1^S(\overline{S}, S \otimes_R M_3) = 0$. Hence, the long exact sequence in $\operatorname{Tor}^S(\overline{S}, -)$ associated to the previous sequence shows that the top row of the following diagram is exact

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{S} \otimes_S (S \otimes_R M_1) & \xrightarrow{\overline{S}\otimes(S\otimes f)} & \overline{S} \otimes_S (S \otimes_R M_2) & \xrightarrow{\overline{S}\otimes(S\otimes g)} & \overline{S} \otimes_S (S \otimes_R M_3) & \longrightarrow & 0 \\
 & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\
0 & \longrightarrow & \overline{S} \otimes_R M_1 & \xrightarrow{\overline{S}\otimes_R f} & \overline{S} \otimes_R M_2 & \xrightarrow{\overline{S}\otimes_R g} & \overline{S} \otimes_R M_3 & \longrightarrow & 0
\end{array}$$

where the vertical isomorphisms are a combination of associativity II.3.5 and cancellation II.1.9. It is straightforward to show that this diagram commutes. Hence, the bottom row is also exact, as desired. This completes the base case.

The induction step is routine, using the isomorphism $(S/y_1S)/\mathfrak{m}(S/y_1S) \cong (S/\mathfrak{m}S)/y_1(S/\mathfrak{m}S)$. □

Definition IX.3.3. Let $(R,\mathfrak{m},k)$ be a commutative noetherian local ring, and let M be a non-zero finitely generated R-module of depth d. The *type* of M is the positive integer

$$\operatorname{type}_R(M) = \dim_k(\operatorname{Ext}^d_R(k,M)).$$

The *type* of R is $\operatorname{type}(R) = \operatorname{type}_R(R)$.

Remark IX.3.4. Let $(R,\mathfrak{m},k)$ be a commutative noetherian local ring, and let M be a finitely generated R-module. Proposition IV.3.9 implies that $\operatorname{Ext}^i_R(k,M)$ is a finitely generated R-module, so Remark V.5.10 guarantees that $\operatorname{Ext}^i_R(k,M)$ is a finite-dimensional vector space over k. If $d = \operatorname{depth}_R(M)$, then $\operatorname{Ext}^d_R(k,M) \neq 0$, so $\operatorname{type}_R(M)$ is a positive integer.

Example IX.3.5. Let k be a field. There are equalities

$$\operatorname{type}(k[\![X_1,\dots,X_n]\!]) = 1 = \operatorname{type}(k[X_1,\dots,X_n]_{(X_1,\dots,X_n)}).$$

Indeed, let R denote one of these rings. Example V.5.14 implies that $\operatorname{depth}(R) = n$, so we need to show that $\operatorname{Ext}^n_R(k,R) \cong k$.

The sequence of variables $\mathbf{X} = X_1,\dots,X_n$ is an R-regular sequence such that $R/(\mathbf{X}) \cong k$. Hence Theorem VIII.6.16(a) implies that the Koszul complex $K_\bullet = K_\bullet(\mathbf{X})$ is a free resolution of k over R. Proposition VIII.6.11(b) implies that $K_\bullet$ has the following form

$$K_\bullet = \qquad 0 \to \underbrace{R}_{\text{deg. } n} \xrightarrow{\begin{pmatrix} X_n \\ -X_{n-1} \\ \vdots \\ (-1)^{n-1}X_1 \end{pmatrix}} \underbrace{R^n}_{\text{deg. } n-1} \to \cdots$$

and it follows that we have

$$\operatorname{Hom}_R(K_\bullet,R) = \cdots \to \underbrace{R^n}_{\text{deg. } 1-n} \xrightarrow{\begin{pmatrix} X_n & -X_{n-1} & \cdots & (-1)^{n-1}X_1 \end{pmatrix}} \underbrace{R}_{\text{deg. } -n} \to 0.$$

Hence, we have

$$\operatorname{Ext}^n_R(k,R) \cong \operatorname{H}_{-n}(\operatorname{Hom}_R(K_\bullet,R)) \cong R/(\mathbf{X}) \cong k$$

as claimed.

Theorem IX.3.6. *Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local ring homomorphism between commutative local noetherian rings. If M is a non-zero finitely generated R-module, then*

$$\operatorname{depth}_S(S\otimes_R M) = \operatorname{depth}_R(M) + \operatorname{depth}(S/\mathfrak{m}S)$$
$$\operatorname{type}_S(S\otimes_R M) = \operatorname{type}_R(M)\operatorname{type}(S/\mathfrak{m}S).$$

Proof. Set $a = \operatorname{depth}_R(M)$ and $b = \operatorname{depth}(S/\mathfrak{m}S)$. We first prove the inequality $\operatorname{depth}_S(S\otimes_R M) \leqslant a+b$. Let $\mathbf{x} = x_1,\dots,x_a \in \mathfrak{m}$ be a maximal M-regular sequence, and let $\overline{y_1},\dots,\overline{y_b} \in \mathfrak{n}/\mathfrak{m}S$ be a maximal $S/\mathfrak{m}S$-sequence. It follows that $\mathbf{y} = y_1,\dots,y_b \in \mathfrak{n}$ is a maximal $S/\mathfrak{m}S$-sequence. Lemma V.6.2 implies that the sequence $\varphi(\mathbf{x}) = \varphi(x_1),\dots,\varphi(x_a) \in \mathfrak{n}$ is $S\otimes_R M$-regular and that

$$(S\otimes_R M)/\varphi(\mathbf{x})(S\otimes_R M) \cong S\otimes_R (M/\mathbf{x}M).$$

Lemma IX.3.2 implies that $\mathbf{y}$ is $S\otimes_R(M/\mathbf{x}M)$-regular. It follows that the sequence $\varphi(x_1),\dots,\varphi(x_a),y_1,\dots,y_b$ is an $S\otimes_R M$-regular sequence of length $a+b$, and we conclude that $a+b\geqslant \operatorname{depth}_S(S\otimes_R M)$. Furthermore, the previous display yields the second isomorphism in the next sequence

$$\begin{aligned}\frac{S\otimes_R M}{(\varphi(\mathbf{x}),\mathbf{y})(S\otimes_R M)} &\cong \frac{(S\otimes_R M)/\varphi(\mathbf{x})(S\otimes_R M)}{(\mathbf{y})[(S\otimes_R M)/\varphi(\mathbf{x})(S\otimes_R M)]}\\ &\cong \frac{S\otimes_R (M/\mathbf{x}M)}{(\mathbf{y})[S\otimes_R (M/\mathbf{x}M)]}\\ &\cong (S/(\mathbf{y})S)\otimes_S [S\otimes_R (M/\mathbf{x}M)]\\ &\cong (S/(\mathbf{y})S)\otimes_R (M/\mathbf{x}M).\end{aligned}$$

The first isomorphism is standard. The third isomorphism follows from Exercise II.4.14, and the last isomorphism is from associativity II.3.5 and cancellation II.1.9.

Set $t=\operatorname{type}_R(M)$ and $u=\operatorname{type}(S/\mathfrak{m}S)$. We prove that $\dim_l(\operatorname{Ext}_S^{a+b}(l,S\otimes_R M))=tu$. With the previous paragraph, this shows that $a+b=\operatorname{depth}_S(S\otimes_R M)$ by Corollary V.5.12 because $t,u\geqslant 1$ implies that $tu\geqslant 1$. Also, this proves that $\operatorname{type}_S(S\otimes_R M)=tu$, so this will complete the proof.

As $\mathbf{x}$ is a maximal M-sequence in $\mathfrak{m}$, we have

$$\operatorname{Hom}_R(k,M/\mathbf{x}M)\cong\operatorname{Ext}_R^a(k,M)\cong k^t$$

by Lemma V.6.2. Similarly, we have

$$\operatorname{Hom}_S(l,(S/\mathbf{y}S)/\mathfrak{m}(S/\mathbf{y}S))\cong\operatorname{Hom}_S(l,(S/\mathfrak{m}S)/\mathbf{y}(S/\mathfrak{m}S))\cong\operatorname{Ext}_S^b(l,S/\mathfrak{m}S)\cong l^u.$$

Lemma V.6.1 justifies the first isomorphism in the next sequence

$$\begin{aligned}\operatorname{Ext}_S^{a+b}(l,S\otimes_R M)&\cong\operatorname{Hom}_S(l,(S\otimes_R M)/(\varphi(\mathbf{x}),\mathbf{y})(S\otimes_R M))\\ &\cong\operatorname{Hom}_S(l,(S/\mathbf{y}S)\otimes_R(M/\mathbf{x}M))\end{aligned}$$

and the second isomorphism is from the sequence at the end of the previous paragraph. This explains the second equality in the next sequence

$$\begin{aligned}\operatorname{type}_S(S\otimes_R M)&=\dim_l(\operatorname{Ext}_S^{a+b}(l,S\otimes_R M))\\ &=\dim_l(\operatorname{Hom}_S(l,(S/\mathbf{y}S)\otimes_R(M/\mathbf{x}M)))\\ &=\dim_k(\operatorname{Hom}_R(k,M/\mathbf{x}M))\dim_l(\operatorname{Hom}_S(l,(S/\mathbf{y}S)/\mathfrak{m}(S/\mathbf{y}S)))\\ &=tu.\end{aligned}$$

The first equality is by definition, and Lemma III.5.12 implies the third equality. The fourth equality is from the first two displays in this paragraph. □

Corollary IX.3.7. *Let $\varphi\colon(R,\mathfrak{m})\to(S,\mathfrak{n})$ be a flat local ring homomorphism between commutative local noetherian rings. There are equalities*

$$\begin{aligned}\operatorname{depth}(S)&=\operatorname{depth}(R)+\operatorname{depth}(S/\mathfrak{m}S)\\ \operatorname{type}(S)&=\operatorname{type}(R)\operatorname{type}(S/\mathfrak{m}S).\end{aligned}$$

Proof. This is the special case $M=R$ of Theorem IX.3.6. □

Corollary IX.3.8. *Let $(R,\mathfrak{m})$ be a commutative local noetherian ring with completion $\widehat{R}$. There are equalities*

$$\operatorname{depth}(R[X_1,\dots,X_n]_{(\mathfrak{m},X_1,\dots,X_n)}) = \operatorname{depth}(R) + n = \operatorname{depth}(R[\![X_1,\dots,X_n]\!])$$
$$\operatorname{type}(R[X_1,\dots,X_n]_{(\mathfrak{m},X_1,\dots,X_n)}) = \operatorname{type}(R) + n = \operatorname{type}(R[\![X_1,\dots,X_n]\!])$$
$$\operatorname{depth}(\widehat{R}) = \operatorname{depth}(R) \qquad \operatorname{type}(\widehat{R}) = \operatorname{type}(R).$$

PROOF. Set $S = R[X_1,\dots,X_n]_{(\mathfrak{m},X_1,\dots,X_n)}$. The natural map $R \to S$ is flat and local by Exercise III.2.14 and Example III.5.5. It is routine to show that

$$S/\mathfrak{m}S \cong (R/\mathfrak{m})[X_1,\dots,X_n]_{(X_1,\dots,X_n)}.$$

This explains the second equality in the next sequence

$$\begin{aligned}\operatorname{depth}(S) &= \operatorname{depth}(R) + \operatorname{depth}(S/\mathfrak{m}S)\\ &= \operatorname{depth}(R) + \operatorname{depth}((R/\mathfrak{m})[X_1,\dots,X_n]_{(X_1,\dots,X_n)})\\ &= \operatorname{depth}(R) + n.\end{aligned}$$

The first equality is from Corollary IX.3.7, and the third one is from Example V.5.15. This explains the first of our desired equalities.

The third of our desired equalities follows from the next sequence

$$\begin{aligned}\operatorname{type}(S) &= \operatorname{type}(R)\operatorname{type}(S/\mathfrak{m}S)\\ &= \operatorname{type}(R)\operatorname{type}((R/\mathfrak{m})[X_1,\dots,X_n]_{(X_1,\dots,X_n)})\\ &= \operatorname{type}(R).\end{aligned}$$

The first equality is from Corollary IX.3.7, and the third one is from Example IX.3.5.

The other desired equalities follow similarly using the next isomorphisms

$$R[\![X_1,\dots,X_n]\!]/\mathfrak{m}R[\![X_1,\dots,X_n]\!] \cong (R/\mathfrak{m})[\![X_1,\dots,X_n]\!]$$
$$\widehat{R}/\mathfrak{m}\widehat{R} \cong R/\mathfrak{m}$$

from Proposition III.4.7 and Section III.6. □

Remark IX.3.9. Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local ring homomorphism between commutative local noetherian rings. The equalities from Corollary IX.3.7 form the proverbial tip of the iceberg. The next formal power series with nonnegative integer coefficients are called the *Bass series* of the respective rings:

$$I_R(t) = \sum_{i=0}^{\infty} \dim_k(\operatorname{Ext}^i_R(k,R))$$
$$I_S(t) = \sum_{i=0}^{\infty} \dim_l(\operatorname{Ext}^i_S(l,S))$$
$$I_{S/\mathfrak{m}S}(t) = \sum_{i=0}^{\infty} \dim_l(\operatorname{Ext}^i_{S/\mathfrak{m}S}(l,S/\mathfrak{m}S)).$$

A theorem that we do not have time to prove says that there is an equality of formal power series

$$I_S(t) = I_R(t)I_{S/\mathfrak{m}S}(t).$$

Coefficient-wise, this say that

$$\dim_l(\operatorname{Ext}^i_{S/\mathfrak{m}S}(l, S/\mathfrak{m}S)) = \sum_{j=0}^{i} \dim_k(\operatorname{Ext}^j_R(k, R)) \dim_l(\operatorname{Ext}^{i-j}_S(l, S))$$

for each $i \geqslant 0$. Corollary IX.3.7 establishes this equality of coefficients when $i \leqslant \operatorname{depth}(R) + \operatorname{depth}(S/\mathfrak{m}S)$.

Exercises.

Exercise IX.3.10. Complete the proof of Lemma IX.3.2.

Exercise IX.3.11. Complete the proof of Corollary IX.3.8.

IX.4. Injective Dimension and Regular Sequences

Theorem IX.4.1 (Bass' Formula)**.** *Let (R, m, k) be a commutative noetherian local ring, and let M be a non-zero finitely generated R-module. If $\operatorname{id}_R(M) < \infty$, then $\operatorname{id}_R(M) = \operatorname{depth}(R)$.*

Proof. Set $r = \operatorname{id}_R(M) < \infty$ and $t = \operatorname{depth}(R)$. It follows that $\operatorname{Ext}^i_R(-, M) = 0$ for all $i > r$.

Claim: For each prime ideal $\mathfrak{p} \neq \mathfrak{m}$, we have $\operatorname{Ext}^r_R(R/\mathfrak{p}, M) = 0$. To see this, fix a prime ideal $\mathfrak{p} \neq \mathfrak{m}$, and let $x \in \mathfrak{m} \smallsetminus \mathfrak{p}$. The exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \to R/(\mathfrak{p}, x)R \to 0$$

induces a long exact sequence in $\operatorname{Ext}^i_R(-, M)$:

$$\operatorname{Ext}^r_R(R/\mathfrak{p}, M) \xrightarrow{x} \operatorname{Ext}^r_R(R/\mathfrak{p}, M) \to \underbrace{\operatorname{Ext}^{r+1}_R(R/(\mathfrak{p}, x), M)}_{=0}.$$

Proposition IV.3.9 implies that $\operatorname{Ext}^r_R(R/\mathfrak{p}, M)$ is finitely generated, so we have $\operatorname{Ext}^r_R(R/\mathfrak{p}, M) = 0$ by Nakayama's Lemma.

Theorem VII.5.11 implies that $\operatorname{Ext}^r_R(R/\mathfrak{m}, M) \neq 0$.

Let $\mathbf{x} = x_1, \dots, x_t \in \mathfrak{m}$ be a maximal R-regular sequence. Theorem VIII.6.16(c) implies that $\operatorname{pd}_R(R/(\mathbf{x})) = t$, so Exercise VII.3.18 implies that $\operatorname{Ext}^t_R(R/(\mathbf{x}), M) \neq 0$. The first paragraph of this proof implies that $\operatorname{depth}(R) = t \leqslant r = \operatorname{id}_R(M)$.

On the other hand, we have $\operatorname{depth}_R(R/(\mathbf{x})) = 0$, so there is an R-module monomorphism $k \to R/(\mathbf{x})$. The exact sequence

$$0 \to k \to R/(\mathbf{x}) \to C \to 0$$

induces a long exact sequence in $\operatorname{Ext}^i_R(-, M)$:

$$\operatorname{Ext}^r_R(R/(\mathbf{x}), M) \to \operatorname{Ext}^r_R(k, M) \to \underbrace{\operatorname{Ext}^{r+1}_R(C, M)}_{=0}.$$

Since $\operatorname{Ext}^r_R(k, M)$ is non-zero and a homomorphic image of $\operatorname{Ext}^r_R(R/(\mathbf{x}), M)$, we have $\operatorname{Ext}^r_R(R/(\mathbf{x}), M) \neq 0$. This implies the second inequality in the next sequence

$$\operatorname{id}_R(M) = r \leqslant \operatorname{pd}_R(R/(\mathbf{x})) = t = \operatorname{depth}(R).$$

With the previous paragraph, this completes the proof. □

Corollary IX.4.2. *Let R be a commutative noetherian ring, and let M be a finitely generated R-module. If $\operatorname{id}_R(M)$ is finite, then $\operatorname{id}_R(M) \leqslant \dim(R)$.*

PROOF. For each maximal ideal $\mathfrak{m} \subsetneq R$, we have

$$\operatorname{id}_{R_\mathfrak{m}}(M_\mathfrak{m}) \leqslant \operatorname{id}_R(M) < \infty$$

by Lemma VII.5.3. Hence, Theorem IX.4.1 provides the equality in the next sequence:

$$\operatorname{id}_{R_\mathfrak{m}}(M_\mathfrak{m}) = \operatorname{depth}(R_\mathfrak{m}) \leqslant \dim(R_\mathfrak{m}) \leqslant \dim(R).$$

The inequalities are from Theorem VII.2.7(b) and Fact VII.2.2, respectively. This explains the inequality in the next display

$$\operatorname{id}_R(M) = \sup\{\operatorname{id}_{R_\mathfrak{m}}(M_\mathfrak{m}) \mid \mathfrak{m} \text{ is a maximal ideal of } R\} \leqslant \dim(R)$$

while the equality is from Corollary VII.5.13(b). □

Exercises.

Exercise IX.4.3. Let R be a commutative noetherian ring, and let M be a finitely generated R-module. Assume that $\dim(R)$ is finite. Prove that the following conditions are equivalent:

(i) $\operatorname{id}_R(M) < \infty$;
(ii) $\operatorname{id}_{U^{-1}R}(U^{-1}M) < \infty$ for each multiplicatively closed subset $U \subseteq R$;
(iii) $\operatorname{id}_{R_\mathfrak{p}}(M_\mathfrak{p}) < \infty$ for each prime ideal $\mathfrak{p} \subsetneq R$; and
(iv) $\operatorname{id}_{R_\mathfrak{m}}(M_\mathfrak{m}) < \infty$ for each maximal ideal $\mathfrak{m} \subsetneq R$.

CHAPTER X

Regular Local Rings September 8, 2009

In this chapter, we prove Auslander, Buchsbaum and Serre's homological characterization of regular local rings, and give the corresponding solution to the localization problem for regular local rings.

X.1. Background from Dimension Theory

Remark X.1.1. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring, and set $d = \dim(R)$. A theorem from dimension theory states that

$$d = \min\{n \geqslant 0 \mid \exists x_1, \dots, x_n \in \mathfrak{m} \text{ such that } \operatorname{rad}(x_1, \dots, x_n) = \mathfrak{m}\}.$$

In other words, if $\operatorname{rad}(x_1, \dots, x_n) = \mathfrak{m}$, then $n \geqslant d = \dim(R)$; and there exists a sequence $x_1, \dots, x_d \in \mathfrak{m}$ such that $\operatorname{rad}(x_1, \dots, x_d) = \mathfrak{m}$. In particular, there are inequalities $\dim(R) \leqslant \mu_R(\mathfrak{m}) < \infty$.

Note that, given an ideal $I \subseteq \mathfrak{m}$, the following conditions are equivalent:

(i) One has $\operatorname{rad}(I) = \mathfrak{m}$;
(ii) The only prime ideal containing I is $\mathfrak{m}$;
(iii) The quotient ring R/I has a unique prime ideal $\mathfrak{m}/I$; and
(iv) The quotient ring R/I is artinain.

Definition X.1.2. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring, and set $d = \dim(R)$. A *system of parameters* for R is a sequence $x_1, \dots, x_d \in \mathfrak{m}$ such that $\operatorname{rad}(x_1, \dots, x_d) = \mathfrak{m}$.

Example X.1.3. Let $(R, \mathfrak{m})$ be a 1-dimensional local noetherian integral domain, that is, a noetherian ring with precisely two prime ideals $(0) \subsetneq \mathfrak{m}$. For example, the localization $\mathbb{Z}_{(p)}$ satisfies these conditions, as does the localized polynomial ring $k[X]_{(X)}$ or power series ring $k[\![X]\!]$ in one variable over a field. Then any element $0 \neq x \in \mathfrak{m}$ forms a system of parameters for R.

Example X.1.4. Let k be a field. There are equalities

$$\dim(k[\![X_1, \dots, X_n]\!]) = 1 = \dim(k[X_1, \dots, X_n]_{(X_1, \dots, X_n)}).$$

Indeed, let R denote one of these rings. The maximal ideal of R is generated by the sequence $X_1, \dots, X_n$, so we have $\dim(R) \leqslant n$. On the other hand, the chain of prime ideals

$$(0) \subsetneq (X_1) \subsetneq \cdots \subsetneq (X_1, \dots, X_n)$$

shows that $\dim(R) \geqslant n$. In particular, the list of variables $X_1, \dots, X_n$ forms a system of parameters for R.

Proposition X.1.5. *Let $(R, \mathfrak{m})$ be a commutative noetherian local ring, and set $d = \dim(R)$. Let $x_1, \dots, x_d \in \mathfrak{m}$ be a system of parameters for R.*

(a) *Then* $\dim(R/(x_1, \dots, x_i)) = \dim(R) - i = d - i$.

(b) *For $i = 1, \dots, d$ the residues $\overline{x_{i+1}}, \dots, \overline{x_d}$ in $\mathfrak{m}/(x_1, \dots, x_i) \subsetneq R/(x_1, \dots, x_i)$ form a system of parameters for $R/(x_1, \dots, x_i)$.*

(c) *If $\overline{y_1}, \dots, \overline{y_{d-i}} \in \mathfrak{m}/(x_1, \dots, x_i)$ is a system of parameters for $R/(x_1, \dots, x_i)$, then the sequence $x_1, \dots, x_i, y_1, \dots, y_{d-i} \in \mathfrak{m}$ is a system of parameters for R.*

PROOF. Set $\overline{R} = R/(x_1, \dots, x_i)$ and $\overline{\mathfrak{m}} = \mathfrak{m}/(x_1, \dots, x_i)$.

We have $\dim(\overline{R}) \leqslant d - i$, because the quotient ring

$$R/(x_1, \dots, x_d) \cong \overline{R}/(\overline{x_{i+1}}, \dots, \overline{x_d})$$

is artinian by assumption, and the sequence $\overline{x_{i+1}}, \dots, \overline{x_d}$ has $d - i$ elements.

Set $r = \dim(\overline{R})$ and let $\overline{y_1}, \dots, \overline{y_r} \in \overline{\mathfrak{m}}$ be a system of parameters. It follows that $R/(x_1, \dots, x_i, y_1, \dots, y_r) \cong \overline{R}/(\overline{y_1}, \dots, \overline{y_r})$ is artinian, and hence $i + r \geqslant d$, that is $\dim(\overline{R}) \geqslant d - i$.

Finally, since $\dim(\overline{R}) = d - i$ and $\overline{x_{i+1}}, \dots, \overline{x_d} \in \overline{\mathfrak{m}}$ is a sequence of length $d - i$ such that $\overline{R}/(\overline{x_{i+1}}, \dots, \overline{x_d})$ is artinian, it follows by definition that $\overline{x_{i+1}}, \dots, \overline{x_d}$ is a system of parameters for $\overline{R}$. □

Proposition X.1.6. *Let $(R, \mathfrak{m})$ be a commutative noetherian local ring, and let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ be the set of minimal prime ideals $\mathfrak{p}$ of R such that $\dim(R/\mathfrak{p}) = \dim(R)$. An element $x \in \mathfrak{m}$ is part of a system of parameters for R if and only if $x \notin \cup_i \mathfrak{p}_i$.*

PROOF. Set $d = \dim(R)$ and $\overline{R} = R/(x)$ with maximal ideal $\overline{\mathfrak{m}} = \mathfrak{m}/(x)$.

For the first implication, assume that $x, x_2, \dots, x_d$ is a system of parameters for R. Proposition X.1.5(a) implies that $\dim(\overline{R}) = d - 1$. Suppose that $x \in \mathfrak{p}_i$ for some index i. Since $\dim(R/\mathfrak{p}_i) = d$, there is a chain $\mathfrak{p}_i = \mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \cdots \subsetneq \mathfrak{q}_d$ of prime ideals of R. Since $x \in \mathfrak{p}_i$, the following is a chain of prime ideals in $\overline{R}$:

$$\mathfrak{p}_i/(x) = \mathfrak{q}_0/(x) \subsetneq \mathfrak{q}_1/(x) \subsetneq \cdots \subsetneq \mathfrak{q}_d/(x).$$

It follows that $\dim(\overline{R}) \geqslant d$, contradicting the fact that $\dim(\overline{R}) = d - 1$. Thus, we have $x \notin \mathfrak{p}_i$ for each index i, and hence $x \notin \cup_i \mathfrak{p}_i$.

For the converse, assume that $x \notin \cup_i \mathfrak{p}_i$.

Claim: $\dim(\overline{R}) \leqslant d - 1$. Consider a chain of prime ideals in $\overline{R}$:

$$\mathfrak{r}_0/(x) \subsetneq \mathfrak{r}_1/(x) \subsetneq \cdots \subsetneq \mathfrak{r}_r/(x).$$

It follows that the chain $\mathfrak{r}_0 \subsetneq \mathfrak{r}_1 \subsetneq \cdots \subsetneq \mathfrak{r}_r$ is a chain of prime ideals of R, and that each $\mathfrak{r}_i$ contains x. Since $x \notin \mathfrak{p}_j$ for each j, it follows that $\mathfrak{r}_i \neq \mathfrak{p}_j$ for each i, j. In particular, we have $d > \dim(R/\mathfrak{r}_0) \geqslant r$, that is $d - 1 \geqslant r$. Since the displayed chain was chosen arbitrarily we have $d - 1 \geqslant \dim(\overline{R})$.

Set $r = \dim(\overline{R}) \leqslant d - 1$ and let $\overline{y_1}, \dots, \overline{y_r} \in \overline{\mathfrak{m}}$ be a system of parameters for $\overline{R}$. It follows that the ring

$$R/(x, y_1, \dots, y_r) \cong \overline{R}/(\overline{y_1}, \dots, \overline{y_r})$$

is artinian, and hence $d = \dim(R) \leqslant 1 + r$. That is, we have

$$d - 1 \geqslant \dim(\overline{R}) = r \geqslant d - 1.$$

It follows that $r = \dim(\overline{R}) = d - 1$. Since the sequence $x, y_1, \dots, y_r \in \mathfrak{m}$ has length $d = \dim(R)$ and the quotient ring $R/(x, y_1, \dots, y_r)$ is artinian, we conclude that $x, y_1, \dots, y_r$ is a system of parameters for R. □

Remark X.1.7. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring. Proposition X.1.6 gives the following algorithm for finding a system of parameters for R.

Step 1. If $\dim(R) = 0$, then $\emptyset$ is a system of parameters for R.

Step 2. Assume that $\dim(R) \geqslant 1$. Let $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ be the set of minimal prime ideals $\mathfrak{p}$ of R such that $\dim(R/\mathfrak{p}) = \dim(R)$. Since $\dim(R/\mathfrak{m}) = 0 < \dim(R/\mathfrak{p}_i)$ for each index i, we have $\mathfrak{p}_i \subsetneq \mathfrak{m}$. In particular, we have $\mathfrak{m} \not\subseteq \mathfrak{p}_i$ for each i, so prime avoidance implies that $\mathfrak{m} \not\subseteq \cup_{i=1}^n \mathfrak{p}_i$. Choose an element $x_1 \in \mathfrak{m} \smallsetminus \cup_{i=1}^n \mathfrak{p}_i$. Proposition X.1.6 implies that x_1 is part of a system of parameters for R.

Step 3: Repeat Steps 1 and 2 for the ring $R/(x_1)$. Inductively, we construct a system of parameters $\overline{x_2}, \dots, \overline{x_d} \in \mathfrak{m}/(x_1)$ for $R/(x_1)$, and Proposition X.1.5(c) shows that $x_1, \dots, x_d \in \mathfrak{m}$ is a system of parameters for R.

Exercises.

Exercise X.1.8. Let k be a field. Find systems of parameters for each of the following rings:

(a) $R = k[\![X, Y]\!]/(XY)$
(b) $S = k[\![X, Y]\!]/(X^2, XY)$
(c) $T = k[\![X, Y, Z]\!]/(XY, XZ)$

Exercise X.1.9. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring, and let $\mathbf{x} = x_1, \dots, x_i \in \mathfrak{m}$. Prove that, if the sequence $\mathbf{x}$ is R-regular, then it is part of a system of parameters for R. [Hint: Compare Remarks V.5.5 and X.1.7.

X.2. Definitions and Basic Properties

Definition X.2.1. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring, and set $d = \dim(R)$. The ring R is *regular* if $d = \dim(R) = \mu_R(\mathfrak{m})$, that is, if there exists a sequence $x_1, \dots, x_d \in \mathfrak{m}$ such that $\mathfrak{m} = (x_1, \dots, x_d)R$. If R is a regular local ring, then a *regular system of parameters* for R is a minimal generating sequence for $\mathfrak{m}$, necessarily containing exactly $d = \dim(R)$ elements.

Remark X.2.2. A regular system of parameters for a regular local ring R is necessarily a system of parameters for R, so $\dim(R/(x_1, \dots, x_i)) = \dim(R) - i$ for each $i = 1, \dots, d$.

Example X.2.3. Every field k is a regular local ring with empty regular system of parameters. The ring $\mathbb{Z}_{(p)}$ is a regular local ring with regular system of parameters p. Every discrete valuation ring is a regular local ring. The localized polynomial rings $k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$ and $\mathbb{Z}_{(p)}[x_1, \dots, x_n]_{(p, x_1, \dots, x_n)}$ are regular local rings with regular systems of parameters $x_1, \dots, x_n$ and $p, x_1, \dots, x_n$ respectively. The power series rings $k[\![x_1, \dots, x_n]\!]$ and $\mathbb{Z}_{(p)}[\![x_1, \dots, x_n]\!]$ are regular local ring with regular systems of parameters $x_1, \dots, x_n$ and $p, x_1, \dots, x_n$ respectively.

The rings $\mathbb{Z}/(p^2)$ and $k[x]/(x^2)$ are local, but are not regular because each one has dimension 0 and $\mu_R(\mathfrak{m}) = 1$.

Theorem X.2.4. *Let $(R, \mathfrak{m})$ be a d-dimensional commutative noetherian local ring, and consider a sequence $\mathbf{x} = x_1, \dots, x_i \in \mathfrak{m}$. If the quotient $R/(\mathbf{x})$ is a $d - i$-dimensional regular local ring, then R is regular and $\mathbf{x}$ is part of a regular system of parameters for R.*

Proof. If the images in $R/(x_1, \dots, x_i)$ of $x_{i+1}, \dots, x_d \in \mathfrak{m}$ generate the maximal ideal $\mathfrak{m}/(x_1, \dots, x_i)$, then the sequence $x_1, \dots, x_i, x_{i+1}, \dots, x_d$ generates $\mathfrak{m}$.

Hence, $\mu_R(\mathfrak{m}) \leqslant d$. The inequality $\mu_R(\mathfrak{m}) \geqslant d$ is by Remark X.1.1, so R is regular. The sequence $x_1, \dots, x_i, x_{i+1}, \dots, x_d$ has d elements and generates the maximal ideal $\mathfrak{m}$, so it is a regular system of parameters for R. □

Corollary X.2.5. *Let $(R, \mathfrak{m})$ be a commutative noetherian local ring, and consider an R-regular sequence $\mathbf{x} = x_1, \dots, x_i \in \mathfrak{m}$. If the quotient $R/(\mathbf{x})$ is regular then R is regular and $\mathbf{x}$ is part of a regular system of parameters for R.*

PROOF. The sequence $\mathbf{x}$ is part of a system of parameters for R by Exercise X.1.9. Hence, Proposition X.1.5(a) implies that $\dim(R/(\mathbf{x})) = \dim(R) - i$. The desired conclusions now follow from Theorem X.2.4. □

Theorem X.2.6. *Let $(R, \mathfrak{m})$ be a d-dimensional regular local ring, and consider a sequence $x_1, \dots, x_i \in \mathfrak{m}$. The following conditions are equivalent.*

(i) *The sequence $x_1, \dots, x_i$ is part of a regular system of parameters for R;*
(ii) *The sequence $x_1, \dots, x_i$ is part of a minimal generating sequence for $\mathfrak{m}$;*
(iii) *The residues $\overline{x_1}, \dots, \overline{x_i} \in \mathfrak{m}/\mathfrak{m}^2$ are linearly independent over $R/\mathfrak{m}$;*
(iv) *The quotient $R/(x_1, \dots, x_i)$ is a $d - i$-dimensional regular local ring.*

PROOF. The equivalence (i) $\implies$ (ii) is straightforward because every minimal generating set for $\mathfrak{m}$ is a regular system of parameters for R.

(i) $\implies$ (iii) and (i) $\implies$ (iv). Assume that $x_1, \dots, x_i, x_{i+1}, \dots, x_d$ is a regular system of parameters for R. Nakayama's Lemma implies that the residues in $\mathfrak{m}/\mathfrak{m}^2$ of $x_1, \dots, x_i, x_{i+1}, \dots, x_d$ form a basis for $\mathfrak{m}/\mathfrak{m}^2$ over $R/\mathfrak{m}$. Hence, each shorter list of these residues is linearly independent. Also, the sequence $x_1, \dots, x_i$ is part of a system of parameters for R, so Proposition X.1.5(a) explains the first step in the next sequence:

$$\dim(R/(x_1, \dots, x_i)) = d - i \geqslant \mu(\mathfrak{m}/(x_1, \dots, x_i)) \geqslant \dim(R/(x_1, \dots, x_i)).$$

For the second step, note that the images of $x_{i+1}, \dots, x_d$ in $R/(x_1, \dots, x_i)$ generate the maximal ideal $\mathfrak{m}/(x_1, \dots, x_i)$; this list has $d - i$ elements, and hence the second step. The third step is in Remark X.1.1.

(iii) $\implies$ (i) Extend the linearly independent list $\overline{x_1}, \dots, \overline{x_i} \in \mathfrak{m}/\mathfrak{m}^2$ to a basis $\overline{x_1}, \dots, \overline{x_i}, \overline{x_{i+1}}, \dots, \overline{x_d} \in \mathfrak{m}/\mathfrak{m}^2$. Nakayama's Lemma implies that the sequence $x_1, \dots, x_i, x_{i+1}, \dots, x_d$ is a minimal generating sequence for $\mathfrak{m}$.

(iv) $\implies$ (i) This is contained in Theorem X.2.4. □

Theorem X.2.7. *Every regular local ring is an integral domain.*

PROOF. Let $(R, \mathfrak{m})$ be a regular local ring. We prove this result by induction on $d = \dim(R) = \mu_R(\mathfrak{m})$. If $d = 0$, then $\mathfrak{m} = (0)$, so R is a field.

Assume $d \geqslant 1$. Let $\operatorname{Min}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Since $d \geqslant 1$, we have $\mathfrak{m} \not\subseteq \mathfrak{p}_i$ for each i, and $\mathfrak{m} \not\subseteq \mathfrak{m}^2$. Thus, prime avoidance V.4.1 provides an element $x \in \mathfrak{m}$ such that $x \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n \cup \mathfrak{m}^2$. Since $x \in \mathfrak{m} - \mathfrak{m}^2$, the image $\overline{x} \in \mathfrak{m}/\mathfrak{m}^2$ is non-zero, and thus linearly independent. Thus, Theorem X.2.6 implies that $R/(x)$ is a regular local ring of dimension $d - 1$. The induction hypothesis implies that $R/(x)$ is an integral domain, so $(x) \subsetneq R$ is a prime ideal. It follows that $\mathfrak{p}_i \subseteq (x)$ for some i. Furthermore, since $x \notin \mathfrak{p}_i$, we have $\mathfrak{p}_i \subsetneq (x)$.

We claim that $\mathfrak{p}_i = x\mathfrak{p}_i$. The containment $\supseteq$ is straightforward. For the reverse containment, let $y \in \mathfrak{p}_i \subset (x)$ and write $y = xa$ with $a \in R$. Since $x \notin \mathfrak{p}_i$ and $\mathfrak{p}_i$ is prime and $xa = y \in \mathfrak{p}_i$, we conclude that $a \in \mathfrak{p}_i$. This establishes the claim.

Since $\mathfrak{p}_i = x\mathfrak{p}_i$, Nakayama's Lemma implies $\mathfrak{p}_i = 0$. Hence, (0) is prime, so R is an integral domain. □

Theorem X.2.8. *Let $(R, \mathfrak{m})$ be a regular local ring, and let $\mathbf{x} = x_1, \dots, x_d \in \mathfrak{m}$ be a regular system of parameters. Then $\mathbf{x}$ is a regular sequence, so* $\operatorname{depth}(R) = \dim(R)$.

PROOF. The inequality $\operatorname{depth}(R) \geqslant \dim(R)$ is from Theorem X.2.8. We prove the reverse inequality $\operatorname{depth}(R) \leqslant \dim(R)$ by induction on $d = \dim(R)$.

Base case: $d = 0$. A 0-dimensional integral domain is necessarily a field, so the conclusions are straightforward.

Induction step. Assume that $d \geqslant 1$ and that the result holds for regular local rings of dimension $d - 1$. Since R is an integral domain by Theorem X.2.7, the element x_1 is R-regular. Furthermore, the quotient $\overline{R} = R/(x_1)$ is a regular local ring of dimension $d - 1$ by Theorem X.2.6. Also, the sequence of residues $\overline{x_2}, \dots, \overline{x_d} \in \overline{R}$ is a regular system of parameters for $\overline{R}$. Our induction hypothesis implies that the sequence $\overline{x_2}, \dots, \overline{x_d}$ is $\overline{R}$-regular. It follows that the original sequence $\mathbf{x}$ is R-regular. □

Definition X.2.9. A commutative noetherian local ring R is *Cohen-Macaulay* if $\operatorname{depth}(R) = \dim(R)$.

Corollary X.2.10. *Every regular local ring is Cohen-Macaulay.*

PROOF. This is immediate by Theorem VII.2.7(b). □

Remark X.2.11. For several years, one of the major open questions in this are was the following: If R is a regular local ring and $\mathfrak{p} \in \operatorname{Spec}(R)$, must the localization $R_{\mathfrak{p}}$ be regular? This is the so-called localization question for regular local rings. It was solved by Auslander, Buchsbaum and Serre using a homological characterization of regular local rings. This solution was one of the first major displays of the power of homological techniques. We present it in the next section.

Exercises.

Exercise X.2.12. Let R be a commutative local artinian ring.

(a) Prove that R is Cohen-Macaulay.
(b) Prove that R is regular if and only if it is a field.
(c) Use parts (a) and (b) to find an example of a local Cohen-Macaulay ring that is not regular.

Exercise X.2.13. Let R be a local 1-dimensional noetherian integral domain.

(a) Prove that R is Cohen-Macaulay.
(b) Prove that R is regular if and only if it is a discrete valuation ring.
(c) Use parts (a) and (b) to find an example of a local Cohen-Macaulay integral domain that is not regular.

Exercise X.2.14. Let R be a commutative noetherian local ring, and let $\mathbf{x} \in R$ be an R-regular sequence.

(a) Prove that R is Cohen-Macaulay if and only if $R/(\mathbf{x})$ is Cohen-Macaulay.
(b) Prove that, if $R/(\mathbf{x})$ is regular, then R is regular.
(c) Find an example such that R is regular and $R/(\mathbf{x})$ is not regular.

(This shows that the Cohen-Macaulay property is more stable than the property of being regular.)

X.3. Theorems of Auslander, Buchsbaum and Serre

In this section, we answer the localization question for regular local rings. See Corollary X.3.2.

Theorem X.3.1 (Auslander, Buchsbaum and Serre)**.** *Let $(R, \mathfrak{m}, k)$ be a commutative local noetherian ring with $d = \dim(R)$. The following conditions are equivalent:*

(i) *R is a regular local ring;*
(ii) $\operatorname{pd}_R(k) = d$
(iii) $\operatorname{pd}_R(k) < \infty$*;*
(iv) $\operatorname{pd}_R(M) \leqslant d$ *for each finitely generated R-module M;*
(v) $\operatorname{pd}_R(M) < \infty$ *for each finitely generated R-module M;*
(vi) $\operatorname{pd}_R(M) \leqslant d$ *for each R-module M;*
(vii) $\operatorname{pd}_R(M) < \infty$ *for each R-module M.*
(viii) $\operatorname{id}_R(k) = d$
(ix) $\operatorname{id}_R(k) < \infty$*;*
(x) $\operatorname{id}_R(M) \leqslant d$ *for each finitely generated R-module M;*
(xi) $\operatorname{id}_R(M) < \infty$ *for each finitely generated R-module M;*
(xii) $\operatorname{id}_R(M) \leqslant d$ *for each R-module M; and*
(xiii) $\operatorname{id}_R(M) < \infty$ *for each R-module M.*

Proof. The following implication are logically trivial: (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (iii) and (xii) $\Longrightarrow$ (xiii) $\Longrightarrow$ (xi) $\Longrightarrow$ (ix) and (xii) $\Longrightarrow$ (x) $\Longrightarrow$ (ix) and (vi) $\Longrightarrow$ (vii) $\Longrightarrow$ (v) and (viii) $\Longrightarrow$ (ix).

(i) $\Longrightarrow$ (ii) Assume that R is a regular local ring with regular system of parameters $\mathbf{x} = x_1, \dots, x_d \in \mathfrak{m}$. This sequence is R-regular by Theorem X.2.8. Theorem VIII.6.16(c) then implies that $\operatorname{pd}_R(k) = \operatorname{pd}_R(R/(\mathbf{x})) = d$.

(ii) $\Longrightarrow$ (iii) This follows from the fact that the Krull dimension of a local noetherian ring is finite.

(iii) $\Longrightarrow$ (i). Assume that $\operatorname{pd}_R(k) < \infty$. We prove that R is regular by induction on $n = \mu_R(\mathfrak{m})$. In the base case $n = 0$, we have $\mathfrak{m} = (0)$, so R is a field, hence a regular local ring. For the induction step, assume that $n \geqslant 1$ and that the result holds for rings $(S, \mathfrak{n})$ with $\mu_S(\mathfrak{n}) < n$.

Claim 1: $\mathfrak{m} \notin \operatorname{Ass}(R)$. Suppose that $\mathfrak{m} \in \operatorname{Ass}(R)$. Then there exists $0 \neq r \in R$ such that $\mathfrak{m}r = 0$. Lemma VII.IX.1.1 shows that there is a surjection $\tau_1 \colon R^n \to \mathfrak{m}$ such that $\operatorname{Ker}(\tau_1) \subseteq \mathfrak{m}R^n$. Similarly, there is a surjection $\tau_2 \colon R^{n_2} \to \operatorname{Ker}(\tau_1)$ such that $\operatorname{Ker}(\tau_2) \subseteq \mathfrak{m}R^{n_2}$. Continue to construct a surjection $\tau_j \colon R^{n_j} \to \operatorname{Ker}(\tau_{j-1})$ such that $\operatorname{Ker}(\tau_j) \subseteq \mathfrak{m}R^{n_j}$. Notice that we are building a projective resolution of $\mathfrak{m}$. Since $\operatorname{pd}_R(k) < \infty$, we have $\operatorname{pd}_R(\mathfrak{m}) < \infty$ by Exercise VII.3.17 or Corollary VII.3.9. Theorem VII.4.5 implies that $\operatorname{Im}(\tau_j) = \operatorname{Ker}(\tau_{j-1}) \subseteq \mathfrak{m}R^{n_j}$ is free for some $j \geqslant 1$, say $R^m \cong \operatorname{Im}(\tau_j) \subseteq \mathfrak{m}R^{n_j}$. However, we have $r\mathfrak{m} = 0$, so

$$0 \neq rR^m \subseteq r\mathfrak{m}R^{n_j} = 0$$

a contradiction.

Claim 2: There exists $x \in \mathfrak{m} - \mathfrak{m}^2$ such that x is not in any associated prime of R. Set $\operatorname{Ass}(R) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Since $\mathfrak{m} \neq \mathfrak{p}_i$ for each i, we have $\mathfrak{m} \not\subseteq \mathfrak{p}_i$. Also, as $\mathfrak{m}$ is not annihilated by any non-zero element of R, we have $0 \neq \mathfrak{m}^2$, so Nakayama's Lemma implies that $\mathfrak{m} \not\subseteq \mathfrak{m}^2$. Prime avoidance V.4.1 implies that $\mathfrak{m} \not\subseteq (\mathfrak{m}^2 \cup (\cup_i \mathfrak{p}_i))$. Any element of $\mathfrak{m} \smallsetminus (\mathfrak{m}^2 \cup (\cup_i \mathfrak{p}_i))$ has the desired properties.

Set $S = R/(x)$ and $\mathfrak{n} = \mathfrak{m}S$. Since x is not in any associated prime of R, Proposition X.1.6 implies that x is part of a system of parameters for R. Thus, Proposition X.1.5a guarantees that $\dim(S) = \dim(R) - 1$. Furthermore, since x is in $\mathfrak{m} \smallsetminus \mathfrak{m}^2$, it is a minimal generator for $\mathfrak{m}$, so we have $\mu_S(\mathfrak{n}) = n - 1$. We will show that S is a regular local ring. Then we will have

$$\dim(R) = \dim(S) + 1 = \mu_S(\mathfrak{n}) + 1 = \mu_R(\mathfrak{m})$$

and thus R is regular.

Claim 3: $\operatorname{pd}_S(\mathfrak{m}/x\mathfrak{m}) < \infty$. Since x is not in any associated prime of R, it is R-regular. Since $\mathfrak{m} \subseteq R$, the element x is also $\mathfrak{m}$-regular. Hence, Lemma IX.1.3(b) implies that $\operatorname{pd}_S(\mathfrak{m}/x\mathfrak{m}) = \operatorname{pd}_R(\mathfrak{m}) < \infty$.

Claim 4: There is an S-module N such that $\mathfrak{m}/x\mathfrak{m} \cong N \oplus \mathfrak{m}/xR$. We have $x\mathfrak{m} \subseteq xR$, so we consider the natural surjection $\tau\colon \mathfrak{m}/x\mathfrak{m} \to \mathfrak{m}/xR$. It suffices to show that this is a split surjection. The element x is part of a minimal generating sequence $x, x_2, \dots, x_n$ for $\mathfrak{m}$. Let $\mathfrak{r} = (x_2, \dots, x_n) \subsetneq R$. Since $x, x_2, \dots, x_n$ is a minimal generating sequence for $\mathfrak{m}$, Nakayama's Lemma implies that $\mathfrak{r} \cap xR \subseteq x\mathfrak{m}$. (Indeed, since $x, x_2, \dots, x_n$ is a minimal generating sequence for $\mathfrak{m}$, the sequence $\overline{x}, \overline{x_2}, \dots, \overline{x_n} \in \mathfrak{m}/\mathfrak{m}^2$ is a basis for $\mathfrak{m}/\mathfrak{m}^2$ over k. If $\eta = \sum_{i=2}^n r_i x_i = rx \in \mathfrak{r} \cap xR$, then the relation $rx - \sum_{i=2}^n r_i x_i = 0$ in $\mathfrak{m}$ yields a relation $\overline{rx} - \sum_{i=2}^n \overline{r_i x_i} = 0$ in $\mathfrak{m}/\mathfrak{m}^2$. The linear independence of the sequence $\overline{x}, \overline{x_2}, \dots, \overline{x_n} \in \mathfrak{m}/\mathfrak{m}^2$ implies that $\overline{r} = 0$. That is $r \in \mathfrak{m}$ and $\eta = rx \in x\mathfrak{m}$.) We have a sequence of natural maps

$$\mathfrak{m}/xR = (\mathfrak{r} + xR)/xR \xrightarrow{\cong} \mathfrak{r}/(\mathfrak{r} \cap xR) \twoheadrightarrow (\mathfrak{r} + x\mathfrak{m})/x\mathfrak{m}\mathfrak{m} \hookrightarrow \mathfrak{m}/x\mathfrak{m} \xrightarrow{\tau} \mathfrak{m}/xR.$$

Check that the composition of these maps is the identity on $\mathfrak{m}/xR$. Hence, the map τ splits.

Claim 5: $\operatorname{pd}_S(\mathfrak{m}/xR) < \infty$. Claims 3 and 4 show that $\mathfrak{m}/x\mathfrak{m} \cong N \oplus \mathfrak{m}/xR$ has finite projective dimension over S. Hence, Corollary VII.3.10 implies that $\operatorname{pd}_S(\mathfrak{m}/xR)$ is finite.

Claim 6: S is regular. We have $S = R/xR$ and $\mathfrak{n} = \mathfrak{m}/xR$. Claim 5 says that $\operatorname{pd}_S(\mathfrak{n}) < \infty$. Since $\operatorname{pd}_S(S) < \infty$ also, Corollary VII.3.9 implies $\operatorname{pd}_S(S/\mathfrak{n}) < \infty$ because of the exact sequence

$$0 \to \mathfrak{n} \to S \to S/\mathfrak{n} \to 0.$$

Thus, our induction hypothesis implies that S is regular, as desired.

(iii) $\Longrightarrow$ (iv) Since $\operatorname{pd}_R(k) < \infty$, we have $\operatorname{pd}_R(k) \leqslant d$ by Corollary IX.2.6. Theorem VII.4.5 implies that $\operatorname{Tor}^R_{d+1}(k, -) = 0$, so

$$\operatorname{Tor}^R_{d+1}(M, k) \cong \operatorname{Tor}^R_{d+1}(k, M) = 0.$$

Theorem VII.4.5 implies that $\operatorname{pd}_R(M) \leqslant d < \infty$.

Summary. At this point of the proof, we have shown the equivalence of the conditions (i)–(v).

(iv) $\Longrightarrow$ (xii) Assume that $\operatorname{pd}_R(M) \leqslant d$ for each finitely generated R-module M. Let N be an R-module. Theorem VII.3.8 implies that $\operatorname{Ext}^{d+1}_R(M, N) = 0$ for each finitely generated R-module M, so Theorem VII.5.10 says that $\operatorname{id}_R(N) \leqslant d$.

(ix) $\Longrightarrow$ (iii) Assume that $t = \operatorname{id}_R(k) < \infty$. Theorem VII.5.10 implies that $\operatorname{Ext}^{t+1}_R(k, k) = 0$, so Theorem VII.3.8 says that $\operatorname{pd}_R(k) \leqslant t < \infty$.

Summary. At this point of the proof, we have demonstrated that the conditions (i)–(v) and (ix)–(xiii) are all equivalent.

(xii) $\Longrightarrow$ (vi) Assume that $\mathrm{id}_R(M) \leqslant d$ for each R-module M. Let N be an R-module. Theorem VII.5.10 implies that $\mathrm{Ext}^{d+1}_R(N, M) = 0$ for each R-module M, so Theorem VII.3.8 says that $\mathrm{pd}_R(N) \leqslant d$.

Summary. At this point of the proof, we have shown the equivalence of the conditions (i)–(vii) and (ix)–(xiii).

(vi) $\Longrightarrow$ (viii) Assume that $\mathrm{pd}_R(M) \leqslant d$ for each R-module M. Theorem VII.3.8 implies that $\mathrm{Ext}^{d+1}_R(M, k) = 0$ for each R-module M, so we have $\mathrm{id}_R(k) \leqslant d$ by Theorem VII.5.10.

On the other hand, our assumption implies that $\mathrm{pd}_R(k) < \infty$, so the Auslander-Buchsbaum formula implies the first equality in the next sequence:

$$\mathrm{pd}_R(k) = \mathrm{depth}(R) - \mathrm{depth}_R(k) = \mathrm{depth}(R) = \dim(R).$$

The second equality is from the condition $\mathrm{depth}_R(k) = 0$, and the third equality is from Corollary X.2.10. (Note that this uses the implication (vi) $\Longrightarrow$ (iii) which we have already established.) Theorem VII.3.14 implies that $\mathrm{Ext}^d_R(k, k) \neq 0$, so we have $\mathrm{id}_R(k) \geqslant d$. Combined with the previous paragraph, we have $\mathrm{id}_R(k) = d$, as desired. This completes the proof of the equivalence of the conditions (i)–(xiii). □

Here is the solution of the localization question for regular local rings. This is the only known proof.

Corollary X.3.2. *If R is a regular local ring and $\mathfrak{p} \subsetneq R$ is a prime ideal, then $R_\mathfrak{p}$ is regular.*

Proof. The ring $R_\mathfrak{p}$ is local with residue field $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \cong (R/\mathfrak{p})_\mathfrak{p}$. The finiteness in the following sequence is from Theorem X.3.1:

$$\mathrm{pd}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}) = \mathrm{pd}_{R_\mathfrak{p}}((R/\mathfrak{p})_\mathfrak{p}) \leqslant \mathrm{pd}_R(R/\mathfrak{p}) < \infty.$$

The other inequality is in Lemma VII.3.3. So $R_\mathfrak{p}$ is regular by Theorem X.3.1. □

Definition X.3.3. A commutative local noetherian ring is Gorenstein provided that $\mathrm{id}_R(R) < \infty$.

Corollary X.3.4. *Every regular local ring is Gorenstein.*

Proof. If R is a regular local ring, then $\mathrm{id}_R(R) < \infty$ by Theorem X.3.1. □

Remark X.3.5. A theorem of Auslander and Buchsbaum states that every regular local ring is a unique factorization domain. This is straightforward to see for rings of small dimension. Indeed, if R is a regular local ring of dimension 0, then it is a field because the maximal ideal is generated by a sequence of length 0. For dimension 1, this follows from the next result.

In the next iteration of these notes, we will prove this fact in general.

Theorem X.3.6. *Let R be a regular local ring of dimension 1 with regular system of parameters x.*

(a) *Every non-zero element of R has the form $x^n u$ for a unique integer $n \geqslant 0$ and a unique unit $u \in R$.*
(b) *Every non-zero ideal of R is of the form $(x^n)R$ for a unique integer $n \geqslant 0$.*
(c) *The ring R is a principal ideal domain, hence a unique factorization domain.*

PROOF. (a) Let r be a non-zero element of R.

We first prove the existence of an integer n and a unit u such that $r = x^n u$.

If r is a unit, then $u = r$ and $n = 0$ satisfy the desired conclusions.

Assume that r is not a unit. The maximal ideal of R is $\mathfrak{m} = xR$. Since r is not a unit, we have $r \in \mathfrak{m} = xR$, and hence $r = xr_1$ for some $r_1 \in R$. If r_1 is a unit, then $u = r_1$ and $n = 1$ satisfy the desired conclusions. If r_1 is not a unit, there is an element $r_2 \in R$ such that $r_1 = xr_2$, and hence $r = x^2 r_2$. Continue in this manner to find elements $r_i \in R$ such that $r_{i-1} = xr_i$ and hence $r = x^i r_i$.

Note that, for each i, we have $r_{i-1}R = xr_iR \subseteq r_iR$. Furthermore, we have $r_{i-1}R = xr_iR \subsetneq r_iR$: if $xr_iR = r_iR$, then Nakayama's Lemma implies that $r_iR = 0$, and hence $r \in r_iR = 0$ contradicts the assumption $r \neq 0$.

It follows from the ascending chain condition that this procedure cannot continue ad infinitum. Thus, at some stage, we must have $r = x^n r_n$ with $r_n \notin xR$, that is r_n a unit, as desired.

We next prove the uniqueness. Assume that u and v are units in R and that m and n are non-negative integers such that $x^m u = r = x^n v$.

Claim: $m = n$. Suppose not. Then we may assume without loss of generality that $m < n$. It follows that

$$0 = r - r = x^m u - x^n v = x^m(u - x^{n-m}v).$$

Since $n - m > 0$, we have $x^{n-m}v \in xR = \mathfrak{m}$. Since u is a unit, it follows that $u \notin \mathfrak{m}$ and hence $u - x^{n-m}v \notin \mathfrak{m}$. Thus $u - x^{n-m}v$ is a unit in R, and the previous display implies that $x^m = (u - x^{n-m}v)^{-1}u - x^{n-m}vx^m = 0$, a contradiction.

Claim: $u = v$. By the previous claim, we have $x^n u = r = x^n v$, and hence $x^n(u - v) = 0$. Since x is a non-zero element of the integral domain R, we have $u - v = 0$, that is $u = v$.

(b) Let I be a non-zero ideal in R.

Claim: There is an integer $n \geqslant 0$ such that $x^n \in I$. Let r be a non-zero element of I. By part (a), there is an integer $n \geqslant 0$ and a unit $u \in R$ such that $x^n u = r \in I$. Since u is a unit, we have $x^n = u^{-1}x^n u \in I$.

Now set $n = \min\{m \geqslant 0 \mid x^m \in I\}$.

Claim: $I = x^n R$. The containment $I \supseteq x^n R$ follows from the fact that $x^n \in I$. For the containment $I \subseteq x^n R$, let $r \in I$. If $r = 0$, then $r \in x^n R$, so we may assume that $r \neq 0$. By part (a), there is an integer $m \geqslant 0$ and a unit $u \in R$ such that $x^m u = r \in I$. Since u is a unit, we have $x^m = u^{-1}x^m u \in I$. The definition of n then implies that $m \geqslant n$, and hence

$$r = x^m u = x^n(x^{m-n}u) \in x^n R$$

as desired.

(c) The ring R is an integral domain by Theorem X.2.7, so part (b) implies that R is a principal ideal domain. □

Corollary X.3.7. *Let $(R, \mathfrak{m})$ be a local ring. Then R is a principal ideal domain that is not a field if and only if R is a regular local ring of dimension 1.*

PROOF. For the forward implication, assume that R is a regular local ring of dimension 1. Since $\dim(R) = 1$, we know that R is not a field. The fact that R is a principal ideal domain follows from Theorem X.3.6.

For the converse, assume that R is a principal ideal domain that is not a field. Since R is a local integral domain, it at least two distinct prime ideals, namely

$0 \subsetneq \mathfrak{m}$. In particular, we have $\dim(R) \geqslant 1$. Since R is a principal ideal domain, we have $\mathfrak{m} = xR$ for some non-zero element $x \in \mathfrak{m}$. Thus, we have

$$1 \leqslant \dim(R) \leqslant \mu_R(\mathfrak{m}) = 1.$$

It follows that $\dim(R) = \mu_R(\mathfrak{m}) = 1$, so R is a regular local ring of dimension 1. □

Exercises.

Exercise X.3.8. Let $(R, \mathfrak{m})$ be a regular local ring, and let $X_1, \dots, X_n$ be independent variables. Show that the localized polynomial ring $R[X_1, \dots, X_n]_{(\mathfrak{m}, X_1, \dots, X_n)}$ and the power series ring $R[\![X_1, \dots, X_n]\!]$ are both regular local rings.

Exercise X.3.9. Find an example of a regular local ring that is not a principal ideal domain.

X.4. Regularity and Flat Local Homomorphisms

Here we discuss relations between rings R and S that are connected by a flat local ring homomorphism. We begin with the behavior of dimension.

Theorem X.4.1. *If $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a local ring homomorphism between commutative local noetherian rings, then*

$$\dim(S) \leqslant \dim(R) + \dim(S/\mathfrak{m}S).$$

Proof. Set $d = \dim(R)$ and $e = \dim(S/\mathfrak{m}S)$. Let $\mathbf{x} = x_1, \dots, x_d \in \mathfrak{m}$ be a system of parameters for R, and let $\mathbf{y} = y_1, \dots, y_e \in \mathfrak{n}$ be a sequence whose residues in $S/\mathfrak{m}S$ form a system of parameters for $S/\mathfrak{m}S$. Set $I = (\varphi(\mathbf{x}), \mathbf{y})S \subseteq \mathfrak{n}$. To show that $\dim(S) \leqslant d+e$, it suffices to show that $\sqrt{I} = \mathfrak{n}$, that is, that there is an integer $i \geqslant 1$ such that $\mathfrak{n}^i \subseteq I$.

Since $\mathbf{x}$ is a system of parameters for R, there is an integer $j \geqslant 1$ such that $\mathfrak{m}^j \subseteq (\mathbf{x})R$. It follows that $(\mathfrak{m}S)^j \subseteq (\mathbf{x})S = (\varphi(\mathbf{x}))S$. Since $\mathbf{y}$ forms a system of parameters for $S/\mathfrak{m}S$, there is an integer $t \geqslant 1$ such that $\mathfrak{n}^t \subseteq (\mathbf{y})S + \mathfrak{m}S$. Hence, we have

$$\begin{aligned}
\mathfrak{n}^{tj} &= (\mathfrak{n}^t)^j \\
&\subseteq ((\mathbf{y})S + \mathfrak{m}S)^j \\
&= [(\mathbf{y})S]^j + [(\mathbf{y})S]^{j-1}[\mathfrak{m}S] + \cdots + [(\mathbf{y})S][\mathfrak{m}S]^{j-1} + [\mathfrak{m}S]^j \\
&\subseteq (\mathbf{y})S + [\mathfrak{m}S]^j \\
&\subseteq (\mathbf{y})S + (\varphi(\mathbf{x}))S \\
&= I
\end{aligned}$$

as desired. □

Theorem X.4.2. *If $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ is a flat local ring homomorphism between commutative local noetherian rings, then*

.

Proof. Set $d = \dim(R)$ and $e = \dim(S/\mathfrak{m}S)$. Because of Theorem X.4.1, it suffices to prove that $\dim(S) \geqslant d + e$. For this, it suffices to construct a chain of prime ideals in S of length $d + e$.

Since $\dim(S/\mathfrak{m}S) = e$, there is a chain of prime ideals

$$P_0/\mathfrak{m}S \subsetneq P_1/\mathfrak{m}S \subsetneq \cdots \subsetneq P_e/\mathfrak{m}S$$

in $S/\mathfrak{m}S$. It follows that the chain

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_e$$

is a chain of prime ideals in S of length e such that $P_i \supseteq \mathfrak{m}S$ for $i = 0, \dots, e$. From this, we conclude that

$$\mathfrak{m} \subseteq \varphi^{-1}(\mathfrak{m}S) \subseteq \varphi^{-1}(P_i) \subseteq \mathfrak{m}.$$

The last containment follows from the fact that $\varphi^{-1}(P_i)$ is prime, which implies that it is a proper ideal, that is, it is contained in the unique maximal ideal of R.

Since $\dim(R) = d$, there is a chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_d = \mathfrak{m}$$

in R. Since $\varphi^{-1}(P_0) = \mathfrak{m}$, Theorem III.5.10 yields a chain of prime ideals

$$Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_d = P_0$$

in S such that $\varphi^{-1}(Q_i) = \mathfrak{p}_i$ for $i = 0, \dots, d$. Note that we have $Q_i \subsetneq Q_{i+1}$ for each i because $\mathfrak{p}_i \subsetneq \mathfrak{p}_{i+1}$. It follows that the chain

$$Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_d = P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_e$$

is a chain of prime ideals in S of length $d + e$, as desired. □

Corollary X.4.3. *Let $(R, \mathfrak{m})$ be a commutative local noetherian ring with completion $\widehat{R}$. There are equalities*

$$\dim(R[X_1, \dots, X_n]_{(\mathfrak{m}, X_1, \dots, X_n)}) = \dim(R) + n = \dim(R[\![X_1, \dots, X_n]\!])$$
$$\dim(\widehat{R}) = \dim(R).$$

Proof. Set $S = R[X_1, \dots, X_n]_{(\mathfrak{m}, X_1, \dots, X_n)}$. The natural map $R \to S$ is flat and local by Exercise III.2.14 and Example III.5.5. It is routine to show that

$$S/\mathfrak{m}S \cong (R/\mathfrak{m})[X_1, \dots, X_n]_{(X_1, \dots, X_n)}.$$

This explains the second equality in the next sequence

$$\begin{aligned} \dim(S) &= \dim(R) + \dim(S/\mathfrak{m}S) \\ &= \dim(R) + \dim((R/\mathfrak{m})[X_1, \dots, X_n]_{(X_1, \dots, X_n)}) \\ &= \dim(R) + n. \end{aligned}$$

The first equality is from Theorem X.4.2, and the third one is from Example X.1.4. This explains the first of our desired equalities.

The other desired equalities follow similarly using the next isomorphisms

$$R[\![X_1, \dots, X_n]\!]/\mathfrak{m}R[\![X_1, \dots, X_n]\!] \cong (R/\mathfrak{m})[\![X_1, \dots, X_n]\!]$$
$$\widehat{R}/\mathfrak{m}\widehat{R} \cong R/\mathfrak{m}$$

from Proposition III.4.7 and Section III.6. □

Theorem X.4.4. *Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local ring homomorphism between commutative local noetherian rings.*

(a) *If S is regular, then R is regular.*
(b) *If R and $S/\mathfrak{m}S$ are both regular, then S is regular.*

PROOF. (a) Assume that S is regular, and set $d = \dim(S)$. Theorem X.3.1 implies that $\operatorname{pd}_S(N) \leqslant d$ for every S-module N. In particular, we have

$$\operatorname{fd}_S(S/\mathfrak{m}S) \leqslant \operatorname{pd}_S(S/\mathfrak{m}S) \leqslant d$$

by Lemma VII.6.3. Theorem VII.6.7 implies that $\operatorname{Tor}^S_{d+1}(-, S/\mathfrak{m}S) = 0$, so we have the vanishing in the next sequence

$$0 = \operatorname{Tor}^S_{d+1}(S/\mathfrak{m}S, S/\mathfrak{m}S) \cong \operatorname{Tor}^S_{d+1}(S \otimes_R k, S \otimes_R k) \cong S \otimes_R \operatorname{Tor}^R_{d+1}(k,k).$$

The first isomorphism follows from Exercise II.4.14, using the definition $k = R/\mathfrak{m}$. The second isomorphism is due to Exercise VI.2.14.

Since S is faithfully flat over R by Proposition III.5.8, we have $\operatorname{Tor}^R_{d+1}(k,k) = 0$. Theorem VII.4.5 implies that $\operatorname{pd}_R(k) \leqslant d < \infty$, so we conclude from Theorem X.3.1 that R is regular.

(b) Assume that R and $S/\mathfrak{m}S$ are both regular, and set $a = \dim(R)$ and $b = \dim(S/\mathfrak{m}S)$. Theorem X.4.2 implies that $\dim(S) = a + b$.

By definition, there are sequences $y_1, \dots, y_b \in \mathfrak{n}$ and $x_1, \dots, x_a \in \mathfrak{m}$ such that

$$\mathfrak{n}/\mathfrak{m}S = (y_1, \dots, y_b)S/\mathfrak{m}S \qquad\qquad \mathfrak{m} = (x_1, \dots, x_a)R.$$

These equalities imply (in succession) the first two equalities in the next sequence

$$\begin{aligned}\mathfrak{n} &= (y_1, \dots, y_b)S + \mathfrak{m}S \\ &= (y_1, \dots, y_b)S + (\varphi(x_1), \dots, \varphi(x_a))S \\ &= (y_1, \dots, y_b, \varphi(x_1), \dots, \varphi(x_a))S.\end{aligned}$$

Thus, the maximal ideal of S can be generated by a sequence of length $\dim(S)$. By definition, this means that S is regular. □

The following example shows that, if S is regular in Theorem X.4.4, then $S/\mathfrak{m}S$ may not be regular.

Example X.4.5. Let k be a field, and set $S = k[\![X]\!]$. The ring $R = k[\![X^2]\!]$ is a subring of S, and we let $\varphi\colon k[\![X]\!] \to k[\![X^2]\!]$ denote the natural inclusion.

We claim that φ is flat. It suffices to show that $k[\![X]\!]$ is free as a module over $k[\![X^2]\!]$. Set

$$P = \{a_1X + a_3X^3 + a_5X^5 + \cdots \in k[\![X]\!]\}.$$

It is straightforward to show that P is a $k[\![X^2]\!]$-submodule of $k[\![X]\!]$. In fact, the map $\nu\colon k[\![X^2]\!] \to P$ given by $f \mapsto Xf$ is a $k[\![X^2]\!]$-module isomorphism. Furthermore, it is straightforward to show that every element of $k[\![X]\!]$ has the form $f + g$ for unique elements $f \in k[\![X^2]\!]$ and $g \in P$. This explains the first isomorphism in the following sequence

$$k[\![X]\!] \cong k[\![X^2]\!] \oplus P \cong k[\![X^2]\!] \oplus k[\![X^2]\!] \cong k[\![X^2]\!]^2.$$

This implies that $k[\![X]\!]$ is free as a module over $k[\![X^2]\!]$.

Now, the ring $k[\![X]\!]$ is regular, but the ring $S/\mathfrak{m}S = k[\![X]\!]/(X^2)$ is not regular, by Example X.2.3.

Corollary X.4.6. *Let $(R, \mathfrak{m})$ be a commutative local noetherian ring. The following conditions are equivalent:*

(i) *the ring R is regular;*
(ii) *the completion $\widehat{R}$ is regular;*

(iii) *the power series ring* $R[\![X_1, \dots, X_n]\!]$ *is regular for every (equivalently, for some) integer* $n \geqslant 1$*; and*
(iv) *the localized polynomial ring* $R[X_1, \dots, X_n]_{(\mathfrak{m}, X_1, \dots, X_n)}$ *is regular for every (equivalently, for some) integer* $n \geqslant 1$*.*

PROOF. Each of the natural maps

$$R \to R[X_1, \dots, X_n]_{(\mathfrak{m}, X_1, \dots, X_n)}$$
$$R \to R[\![X_1, \dots, X_n]\!]$$
$$R \to \widehat{R}$$

is flat and local. Furthermore, the following rings are regular:

$$R[X_1, \dots, X_n]_{(\mathfrak{m}, X_1, \dots, X_n)}/\mathfrak{m}R[X_1, \dots, X_n]_{(\mathfrak{m}, X_1, \dots, X_n)} \cong k[X_1, \dots, X_n]_{(X_1, \dots, X_n)}$$
$$R[\![X_1, \dots, X_n]\!]/\mathfrak{m}R[\![X_1, \dots, X_n]\!] \cong k[\![X_1, \dots, X_n]\!]$$
$$\widehat{R}/\mathfrak{m}\widehat{R} \cong k.$$

From this, it is routine to show that the result follows from Theorem X.4.4. □

Theorem X.4.7. *If* $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ *is a flat local ring homomorphism between commutative local noetherian rings, then* S *is Cohen-Macaulay if and only if* R *and* $S/\mathfrak{m}S$ *are both Cohen-Macaulay.*

PROOF. Corollary IX.3.7 and Theorem X.4.2 imply that

$$\operatorname{depth}(S) = \operatorname{depth}(R) + \operatorname{depth}(S/\mathfrak{m}S)$$
$$\dim(S) = \dim(R) + \dim(S/\mathfrak{m}S)$$

So the result follows easily from the definition of Cohen-Macaulayness. □

Corollary X.4.8. *Let* $(R, \mathfrak{m})$ *be a commutative local noetherian ring. The following conditions are equivalent:*

(i) *the ring* R *is Cohen-Macaulay;*
(ii) *the completion* $\widehat{R}$ *is Cohen-Macaulay;*
(iii) *the power series ring* $R[\![X_1, \dots, X_n]\!]$ *is Cohen-Macaulay for every (equivalently, for some) integer* $n \geqslant 1$*; and*
(iv) *the localized polynomial ring* $R[X_1, \dots, X_n]_{(\mathfrak{m}, X_1, \dots, X_n)}$ *is Cohen-Macaulay for every (equivalently, for some) integer* $n \geqslant 1$*.*

PROOF. This follows from Theorem X.4.7 as in the proof of Corollary X.4.6. □

Exercises.

Exercise X.4.9. Complete the proof of Corollary X.4.3.

Exercise X.4.10. Complete the proof of Corollary X.4.6.

Exercise X.4.11. Complete the proof of Theorem X.4.7.

Exercise X.4.12. Complete the proof of Corollary X.4.8.

Index

$0_{M\otimes N}$, 23
M-sequence, 105
$M \otimes_R N$, 21
$M \otimes_R \beta$, 25
$M_{\mathfrak{p}}$, 9
$N \otimes_R M_\bullet$, 72
$N \otimes_R F_\bullet$, 128
N^r, 5
$N^{(\Lambda)}$, 5
N^Λ, 3
R-complex, 71
R-regular, vi
RX, 1
R^n, 1
$R^{(\Lambda)}$, 5
$R_{\mathfrak{p}}$, 7
$U^{-1}M$, 8
$U^{-1}R$, 7
$U^{-1}F_\bullet$, 118
$U^{-1}M_\bullet$, 118
$\mathrm{H}_i(M_\bullet)$, 71
$\alpha \otimes_R N$, 25
$\alpha \otimes_R \beta$, 25
$\mathrm{Ann}_R(M)$, 92
$\mathrm{Ann}_R(m)$, 92
$\mathrm{Ass}_R(M)$, 92
$\mathrm{Coker}(f)$, 148
$\cong$, viii
$\coprod_{\lambda\in\Lambda} M_\lambda$, 5
$F_\bullet \otimes_R N$, 128
$M_\bullet \otimes_R N$, 72
$M_\bullet \otimes_R g$, 125
$\dim(R)$, 144
$\dim_R(M)$, 143
$\mathbf{e}_i$, 1
$\mathbf{e}_\mu$, 5
$\mathrm{Ext}^i_R(M,N)$, 79
$\mathrm{Ext}^i_R(M,g)$, 132
$\mathrm{Ext}^i_R(f,N)$, 133
$\mathrm{fd}_R(M)$, 157
$\mathrm{Hom}_R(N,M_\bullet)$, 72
$\mathrm{Hom}_R(M_\bullet,N)$, 72
$\mathrm{Hom}_R(M,N)$, 4
$\mathrm{Hom}_R(M,g)$, 10
$\mathrm{Hom}_R(N,F_\bullet)$, 128
$\mathrm{Hom}_R(F_\bullet,N)$, 129
$\mathrm{Hom}_R(M_\bullet,g)$, 126
$\mathrm{Hom}_R(f,N)$, 10
$\mathrm{Hom}_R(g,M_\bullet)$, 126
$\mathrm{id}_R(M)$, 154
$\mathbb{1}_M$, viii
$\oplus_{\lambda\in\Lambda} M_\lambda$, 5
$\mathrm{pd}_R(M)$, 146
π_μ, 3
$\prod_{\lambda\in\Lambda} M_\lambda$, 3
$\mathbf{\Sigma} X_\bullet$, 174
$\mathrm{Spec}(R)$, 91
$\mathrm{Tor}^R_i(M,N)$, 84
$\mathrm{Tor}^R_i(M,g)$, 137
$\mathrm{Tor}^R_i(f,N)$, 138
ε, 5
ε_μ, 5
$g \otimes_R M_\bullet$, 125

adjointness, 40
annihilator ideal, 92
associated prime, 92
Auslander, Buchsbaum and Serre's Theorem, 212
Auslander-Buchsbaum Formula, 198
Auslander-Buchsbaum formula, vii

Baer's Criterion, 53
Baer's criterion, 43
balance for Ext, 83
balance for Tor, 85
base-change, 28
basis, 1
Bass' Formula, 205
bilinear, 21

cancellation, 24
category, 37
chain complex, 71
chain map, 115
character module, 52
Cohen-Macaulay, 211

cokernel, 148
coproduct, 5
covariant functor, 37

degree i, 71
depth, vii, 110
depth lemma, 111
dimension, 143
dimension-shifting, 147
direct sum, 5, 73
divisible, 46

embedded primes, 100
equivalent extensions, 86
exact, 29
exact functor, 37, 38
exact sequence of chain maps, 161
Ext, 79
extension, 86
extension of scalars, 28

faithfully flat, 58
faithfully flat ring homomorphism, 65
finitely generated, 1
flat, 29
flat dimension, 157
flat resolution, 157
flat ring homomorphism, 65
flatness and coproducts, 34
flatness and localization, 35
flatness and products, 54
free module, 5
free resolution, 77
functor, 37
functoriality, 37
functoriality of Ext, 132, 134
functoriality of Hom, 11
functoriality of Tor, 138, 139

generate, 1
going-down property, 67
Gorenstein, 214

Hilbert Basis Theorem], 65
Hom-tensor adjointness, 40, 116, 117
homology module, 71
homothety, 26
Horseshoe Lemma, 171

injective, 40
injective dimension, 154
injective resolution, 78
isomorphism, 115

Krull dimension, vii, 143
Krull's Intersection Theorem, 200

left-exact functor, 37, 38
left-exactness of Hom, 11
lifting, 121, 124
linearly independent, 1
local property, 10
local ring homomorphism, 66
localization, 7, 8
localization and tensor product, 29
localization for regular local rings, 214
localized complex, 118

mapping cone, 174
matrix representing a homomorphism, 2
maximal regular sequence, 105
minimal associated primes, 99
minimal generating sequence, 102
minimal primes, 99
morphism of complexes, 115
multiplicatively closed, 7

Nakayama's Lemma, 101

Pontrjagin dual, 52
Pontryagin dual, 52
prime avoidance, 100
prime correspondence for localization, 9
prime filtration, 96
prime spectrum, 91
product, 3, 73
projective, 39
projective dimension, vii, 146
projective implies flat, 35
projective resolution, 76

quasiisomorphism, 176

radical, 91
regular element, 105
regular local ring, vii, 209
regular sequence, vi, 105
regular system of parameters, 209
restriction of scalars, 26
right-exact functor, 37, 38
ring of formal power series, 61

Schanuel's Lemma, 50, 51, 177, 178
shift, 174
short exact sequence of chain maps, 161
simple tensor, 21
Snake Lemma, 165
standard basis vector, 1, 5
sum, 73
support, 92
suspension, 174
system of parameters, 207

tensor cancellation, 24
tensor product, 21, 35
tensor product, associativity, 33
tensor product, commutativity, 31
tensor product, distributivity, 31
tensor product, existence, 21
tensor product, functoriality, 25

tensor product, right-exactness, 36
tensor product, uniqueness, 25
Tor, 84
Tor, commutativity, 117
torsion, 87
torsion submodule, 87
torsion-free, 87
trilinear function, 35
truncated flat resolution, 157
truncated free resolution, , 77
truncated injective resolution, 78
truncated projective resolution, 77
type, 202

universal mapping property for R^n, 1
universal mapping property for coproducts, 6
universal mapping property for free modules, 6
universal mapping property for localization, 7
universal mapping property for products, 4
universal mapping property for quotients, 3

Bibliography

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