Solutions to Lee's *Smooth Manifolds*

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Lee 1-1

Let X be the set of all points $(x, y) \in \mathbb{R}^2$ such that $y = 1$ or $y = -1$. Let M be the quotient of X under the equivalence generated by $(x, 1) \sim (x, -1)$ for all $x \neq 0$. Then M is locally Euclidean and second countable, but not Hausdorff. (This space is called the line with two origins.)

Proof. First we will show that M is locally Euclidean. Let $\pi : X \to M$ be the quotient map. The sets $X^+ = \{(x, 1), x \in \mathbb{R}\}$ and $X^- = \{(x, -1), x \in \mathbb{R}\}$ are open subsets of X that are clearly homemorphic to $\mathbb{R}$. The map $\pi \restriction_{X^+}$ is one-to-one. Also, π is continuous, so the map $\pi \restriction_{X^+}$ is continuous. Let $U = (a, b) \times \{1\} \subset X^+$. The set $\pi \restriction_{X^+} (U)$ is open in M if and only if $\pi^{-1}\pi \restriction_{X^+} (U)$ is open in X. If $0 \in (a, b)$ then $\pi^{-1}\pi \restriction_{X^+} (U) = (a, b) \times \{1\} \cup (a, 0) \times \{-1\} \cup (0, b) \times \{-1\}$, which is open. Otherwise, $\pi^{-1}\pi \restriction_{X^+} (U) = (a, b) \times \{1\} \cup (a, b) \times \{-1\}$, which is open. Thus $\pi \restriction_{X^+} (U)$ is in fact open, and since elements like U form a basis for the topology of X^+, we conclude that $\pi \restriction_{X^+}$ is an open map. Thus $\pi \restriction_{X^+}$ maps U^+ homeomorphically onto its image.

With a few symbols switched, the same argument goes forward to show that $\pi \restriction_{X^-}$ maps X^- homeomorphically onto its image. Since the two spaces X^+ and X^- partition X, the images $\pi \restriction_{X^-} (X^-)$ and $\pi \restriction_{X^+} (X^+)$ cover M. Since X^+ and X^- are both homemorphic to $\mathbb{R}$, the space M is locally homeomorphic to $\mathbb{R}$.

We may construct charts as follows: $\varphi_1 : \pi(X^+) \to \mathbb{R}$ defined by $\varphi_1 = \pi_x \circ (\pi \restriction_{X^+})^{-1}$ where π_x denotes the map $(x, y) \mapsto x$, and $\varphi_2 : \pi(X^-) \to \mathbb{R}$ defined by $\varphi_2 = \pi_x \circ (\pi \restriction_{X^-})^{-1}$.

Observe that $\mathbb{R}$ is second countable—just take the collection $\mathcal{F}$ of open intervals with rational endpoints. We have two charts φ_1 and φ_2 that together cover M. Take $\mathcal{C} = \{U | U = \varphi_1^{-1}(V), V \in \mathcal{F}\} \cup \{U | U = \varphi_2^{-1}(V), V \in \mathcal{F}\}$. Then $\mathcal{C}$ is clearly a countable basis for the topology.

Now we show that M is not Hausdorff. Let $U \subset M$ be any open set containing $\pi(0, 1)$, and let $V \subset M$ be any open set containing $\pi(0, -1)$. Then $\pi^{-1}(U)$ is an open subset of X containing $(0, 1)$. It must therefore contain $(-\epsilon, 0) \times \{1\}$ for ϵ small enough. But if $\pi^{-1}(U)$ contains $(-\epsilon, 0) \times \{1\}$ it must also contain $(-\epsilon, 0) \times \{-1\}$ because $\pi((-\epsilon, 0) \times \{1\}) = \pi((-\epsilon, 0) \times \{-1\})$. Now we note that $\pi^{-1}(V)$ is open in X, and it contains $(0, -1)$, so it must contain a set of the form $(\delta, \delta) \times \{-1\}$ for some δ small enough. This shows that $\pi^{-1}(U)$ intersects $\pi^{-1}(V)$. Then U intersects V, and that shows the space isn't Hausdorff. □

Lee 1-2

The disjoint union of uncountably many copies of $\mathbb{R}$ is locally Euclidean and Hausdorff, but not second countable.

Proof. Let Γ be an uncountable ordinal index, and let $X = \bigsqcup_{\lambda\in\Gamma} \mathbb{R}_\lambda$, where $\mathbb{R}_\lambda = \mathbb{R}$. Then each element $x \in X$ is contained in some $\mathbb{R}_\lambda$, which is homeomorphic to $\mathbb{R}$, and so X is locally Euclidean.

Now let any two points $x, y \in X$ be given, distinct. If $x \in \mathbb{R}_\lambda$ and $y \in \mathbb{R}_\gamma$ where $\lambda \neq \gamma$, then $\mathbb{R}_\lambda$ and $\mathbb{R}_\gamma$ are disjoint open sets about x and y. If both $x, y \in \mathbb{R}_\lambda$ there exist disjoint open sets about x and y because $\mathbb{R}$ is Hausdorff.

Now we show that X is not second countable. Suppose $\mathcal{F}$ is a basis for the topology of X. For each $\lambda \in \Gamma$, let U_λ denote the open unit ball centered at the origin. By definition, for each U_λ there is a basis element $B_\lambda \in \mathcal{F}$ such that $0 \in B_\lambda \subset U_\lambda$. There is at least one such B_λ for each $\lambda \in \Gamma$, so by the axiom of choice, there is a choice function $\Phi : \Gamma \to \mathcal{F}$. Suppose $\Phi(\lambda) = \Phi(\gamma)$. Then $\Phi(\lambda) \subset U_\gamma$ and $\Phi(\lambda) \subset U_\lambda$, but if $\gamma \neq \lambda$ then U_λ is disjoint from U_γ. Thus $\lambda = \gamma$. This shows that the function Φ is injective.

Now, if $\mathcal{F}$ is countable, then there is an injection $\phi : \mathcal{F} \to \mathbb{N}$, and then $\phi \circ \Phi : \Gamma \to \mathbb{N}$ is an injection, proving that Γ is countable, contrary to assumption. Thus $\mathcal{F}$ is not countable.

□

Lee 1-3

Let M be a nonempty topological manifold of dimension $n \geq 1$. If M has a smooth structure, then it has uncountably many distinct smooth structures.

Proof. Fix a chart (φ, U) in a smooth structure $\mathcal{F}$ on M. Assume without loss of generality that φ maps U onto the open unit ball $\mathbb{B}^n$. Let $p = \varphi^{-1}(0)$, and define $f_\alpha : \mathbb{B}^n \to \mathbb{B}^n$ by $u \mapsto \dfrac{|u|^{1/\alpha}u}{|u|}$ for $\alpha \geq 1$. It is easily seen that f_α^{-1} is just $u \mapsto \dfrac{|u|^{\alpha}u}{|u|}$, and both f_α and its inverse are everywhere continuous. Of course, for $\alpha = 1$ this is just the identity.

In spherical coordinates, f_α is just

$$(\theta_1, \theta_2, \ldots, \theta_{n-1}, r) \mapsto (\theta_1, \theta_2, \ldots, \theta_{n-1}, r^{1/\alpha}),$$

and it is readily seen that all partial derivatives of all orders exist except at $r = 0$, where the first derivative does not exist.

We now construct continuum-many smooth structures. Let $\mathcal{F}_\alpha$ contain the chart $f_\alpha \circ \varphi$. Observe that for any chart $(\psi, V) \in \mathcal{F}$, if $p \notin V$ then ψ is smoothly compatible with $f_\alpha \circ \varphi$. With that in mind, we take a closed ball S about 0 in $\varphi(U)$. Then $\varphi^{-1}(S)$ is a closed set in M with p in its interior. For every chart $(\psi, V) \in \mathcal{F}$, let $\mathcal{F}_\alpha$ contain $\psi \restriction (\varphi^{-1}(S))^c$—that is, we restrict all the other charts to the complement of our closed coordinate ball in M. These maps $\psi \restriction (\varphi^{-1}(S))^c$ are then smoothly compatible with $f_\alpha \circ \varphi$ because they do not overlap at p. Thus $\mathcal{F}_\alpha$ is a smooth atlas, and it can be extended to a smooth structure. Let us just call the induced smooth structure $\mathcal{F}_\alpha$.

Now we show that our continuum-many smooth structures are distinct. Let any $\alpha, \beta \geq 1$ be given. Assume $\alpha > \beta$. The chart $f_\alpha \circ \varphi \in \mathcal{F}_\alpha$ and the chart $f_\beta \circ \varphi \in \mathcal{F}_\beta$ have the transition map

$$f_\alpha \circ \varphi \circ \varphi^{-1} \circ f_\beta^{-1} = f_\alpha \circ f_\beta^{-1}$$

which is $u \mapsto \dfrac{|u|^{\beta/\alpha}u}{|u|}$. This map is not differentiable at 0, and both charts send p to 0, so they are not smoothly compatible, and all of the $\mathcal{F}_\alpha$ are distinct smooth structures.

□

Lee 1-4

If k is an integer between 0 and $\min(m, n)$, the set of $m \times n$ matrices whose rank is at least k is an open submanifold of $M(m \times n, \mathbb{R})$. This is not true if "at least k" is replaced by "equal to k".

Proof. Suppose A is a matrix of rank at least k. Then A contains a $k \times k$ submatrix of nonzero determinant. The determinant function is polynomial in the $k \times k$ entries of the polynomial. Thus it is a continuous map from $\mathbb{R}^{k^2}$ into $\mathbb{R}$, and the preimage of the complement of zero is open. Thus there exists an open ball B around this submatrix inside which the determinant is nonzero. The other entries of the matrix A outside this submatrix can be changed aribrarily without reducing the rank of A so long as the submatrix remains in B. This shows that the set of matrices whose rank is at least k is open, so it is an open submanifold.

To show that the statement with "equal to k" fails, we provide an example. Consider

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

in $M(2 \times 2, \mathbb{R})$ with $k = 1$. If any of the entries of this matrix is perturbed by an arbitrarily small ϵ, the resulting matrix will have rank 2, so there is no open set about this matrix contained in the set of rank 1 matrices. □

Remark: My professor did not consider my above example in $M(2 \times 2, \mathbb{R})$ a sufficiently general answer to the exercise. If you agree with that, this reasoning can be generalized to arbitrary dimension. On logical grounds, I maintain that my answer suffices.

Lee 1-5

Let $N = (0, \dots, 0, 1)$ be the north pole in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, and let $S = -N$ be the south pole. Define *stereographic projection* $\sigma : \mathbb{S}^n \backslash N \to \mathbb{R}^n$ by

$$\sigma(x_1, \dots, x_{n+1}) = \frac{(x_1, \dots, x_{n+1})}{1 - x_{n+1}}.$$

Let $\tilde{\sigma}(x) = -\sigma(-x)$ for $x \in \mathbb{S}^n \backslash (S)$.

(a) For any $x \in \mathbb{S}^n \backslash (N)$, the point $\sigma(x)$ is the point where the line through N and x intersects the linear subspace where $x_{n+1} = 0$, identified with $\mathbb{R}^n$ in the obvious way. Similarly, $\tilde{\sigma}(x)$ is the point where the line through S and x intersects the same subspace.

Proof. The line through x and N is given by

$$\begin{aligned} \ell(t) &= [(x_1, \dots, x_{n+1}) - (0, \dots, 0, 1)]t + (0, \dots, 0, 1) \\ &= (x_1, \dots, x_{n+1} - 1)t + (0, \dots, 0, 1) \\ &= (x_1 \cdot t, \dots, (x_{n+1} - 1) \cdot t + 1). \end{aligned}$$

The component $(x_{n+1} - 1) \cdot t + 1$ is 0 when $t = t_0 = \frac{1}{1 - x_{n+1}}$, and $\ell(t_0) = \frac{1}{1 - x_{n+1}}(x_1, \dots, x_n, 0) = \sigma(x)$ under the aforementioned identification.

The statement about $\tilde{\sigma}$ follows in the same way. □

(b) The map σ is bijective.

Proof. Suppose $\frac{(x_1, \dots, x_n)}{1 - x_{n+1}} = (u_1, \dots, u_n)$. Let $\omega = 1 - x_{n+1}$ for convenience of notation. Then $(x_1, \dots, x_n) = (\omega u_1, \dots, \omega u_n)$. Since $x = (x_1, \dots, x_{n+1})$ is on the unit sphere, we have $x_{n+1}^2 = 1 - \omega^2 |u|^2$, or $(1 - \omega)^2 = 1 - \omega^2 |u|^2$. Thus

$$(1 + |u|^2)\omega^2 - 2\omega = 0.$$

Thus $\omega = \frac{2}{1 + |u|^2}$. (The solution $\omega = 0$ means $x_{n+1} = 1$, which means x is N, not in the domain of σ.) Then $x_{n+1} = \frac{|u|^2 - 1}{1 + |u|^2}$, and we have

$$(x_1, \dots, x_n, x_{n+1}) = \left(\frac{2u_1}{1 + |u|^2}, \dots, \frac{2u_n}{1 + |u|^2}, \frac{|u|^2 - 1}{1 + |u|^2}\right).$$

This shows that σ is injective. This function is also a right inverse of σ, so σ is onto as well. □

(c) The atlas consisting of the two charts $(\mathbb{S}^n \backslash \{N\}, \sigma)$ and $(\mathbb{S}^n \backslash \{S\}, \tilde{\sigma})$ defines a smooth structure on $\mathbb{S}^n$. (The coordinates defined by σ and $\tilde{\sigma}$ are called stereographic coordinates.)

Proof. It's a straightforward computation that $\tilde{\sigma} \circ \sigma^{-1} : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$ is the map

$$x \mapsto \frac{x}{|x|^2},$$

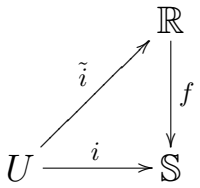
which is a smooth rational function.

Also, $\sigma \circ (\tilde{\sigma})^{-1} = (\tilde{\sigma} \circ \sigma^{-1})^{-1} = \tilde{\sigma} \circ \sigma^{-1}$, which is also a straightforward computation. □

Lee 1-6

Let $\mathbb{S}$ denote the unit circle in $\mathbb{C}$. An *angle function* on a subset $U \subset \mathbb{S}$ is a continuous function $\theta : U \to \mathbb{R}$ such that $e^{i\cdot\theta(p)} = p$ for all $p \in U$. There exists an angle function θ on an open $U \subset \mathbb{S}$ if and only if $U \neq \mathbb{S}$. For any such θ, (U, θ) is a smooth coordinate chart for $\mathbb{S}$ with its standard smooth structure.

Proof. Observe that $f : \mathbb{R} \to \mathbb{S}$ defined by $\theta \mapsto e^{i\theta}$ is a convering map. Let $i : U \to \mathbb{S}$ denote inclusion. The desired angle function $\theta : U \to \mathbb{R}$ is a lift $\tilde{i}$ of the inclusion map i with respect to f.



Let $p_0 \in U$ and let $x \in \mathbb{R}$ be any point such that $e^{ix} = i(p)$. By the lifting criterion, there exists a lift $\tilde{i}$ such that $\tilde{i}(p) = x$ if and only if $i_*\pi_1(U, p) \subset (f)_*\pi_1(\mathbb{R}, x)$.

Now, $\mathbb{R}$ is simply connected, so $\pi_1(\mathbb{R}, x)$ is the trivial group, and so $(f)_*\pi_1(\mathbb{R}, x)$ is the trivial group. However, $i_*\pi_1(U, p) \cong \pi_1(U, p)$, and $\pi_1(U, p)$ has a nontrivial element if an only if $U = \mathbb{S}$. (Else the connected component containing p is simply connected.) Thus the desired lift $\tilde{i}$ exists if and only if $U \neq \mathbb{S}$.

Now we show that (U, θ) is a smooth coordinate chart for $\mathbb{S}$ with its standard smooth structure. We must show that it is smoothly compatible an atlas for the standard smooth structure. These are the maps:

$$\begin{aligned}
&\varphi_x^+ : U_x^+ \to (-1,1) \subset \mathbb{R} \quad \text{where } U_x^+ = \{x + yi \in \mathcal{C} | y > 0\} \quad \text{defined } \varphi_x^+(x+yi) = x \\
&\varphi_x^- : U_x^- \to (-1,1) \subset \mathbb{R} \quad \text{where } U_x^- = \{x + yi \in \mathcal{C} | y < 0\} \quad \text{defined } \varphi_x^-(x+yi) = x \\
&\varphi_y^+ : U_y^+ \to (-1,1) \subset \mathbb{R} \quad \text{where } U_y^+ = \{x + yi \in \mathcal{C} | x > 0\} \quad \text{defined } \varphi_y^+(x+yi) = y \\
&\varphi_y^- : U_y^- \to (-1,1) \subset \mathbb{R} \quad \text{where } U_y^- = \{x + yi \in \mathcal{C} | x < 0\} \quad \text{defined } \varphi_y^-(x+yi) = y.
\end{aligned}$$

Observe that $\varphi_x^\pm \circ \theta^{-1}$ is the map $x \mapsto \cos(x)$, which is smooth, and $\varphi_y^\pm \circ \theta^{-1}$ is the map $x \mapsto \sin(x)$, also smooth. We must also show that $\theta \circ (\varphi_x^\pm)^{-1}$ is smooth. It is a standard result from calculus that the inverse functions $\sin^{-1}(x)$ and $\cos^{-1}(x)$ are differentiable and the derivatives are $\frac{d}{dx}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}$ and $\frac{d}{dx}\cos^{-1}(x) = \frac{-1}{\sqrt{1-x^2}}$, both of which are clearly smooth on $(-1, 1)$. This completes the proof. □

Lee 2-5

Let M be a nonempty smooth manifold of dimension $n \geq 1$. Then $C^\infty(M)$ is infinite-dimensional.

Proof. We will construct an infinite linearly independent set. Let a point $p \in M$ be given, and choose any chart (U, φ) about p such that φ maps U onto $\mathbb{R}^n$ and $\varphi(p) = 0$. Let $B_M = \varphi^{-1}(B_E)$ where B_E is the closed ball of radius 1 centered at the origin in $\mathbb{R}^n$. Define $f_k : B_E \to \mathbb{R}$ for $k = 1 \ldots \infty$ by $f_k(u) = (|u|^2)^k = (u_1^2 + \cdots u_n^2)^k$. Each of these functions f_k is polynomial in the coordinates, so f_k is smooth. Then $\varphi^{-1} \circ f_k : B_M \to \mathbb{R}$ is smooth. By the extension lemma (Lemma 2.27 in Lee), f_k can be extended to $\tilde{f}_k : M \to \mathbb{R}$ such that $\tilde{f}_k \restriction B_M = f_k$.

We have constructed an infinite set of smooth functions on M, $\{\tilde{f}_1, \tilde{f}_2 \ldots, \tilde{f}_k, \ldots\}$. It remains to show that this set is linearly independent. However, it is quite easy to see. Assume only the u_1 coordinate is nonzero and we have a linear combination of $\tilde{f}_k s$ identically equal to zero. A linear combination of $\tilde{f}_k s$ is a polynomial in u_1 having at most a finite number of roots unless all the coefficients are zero.

□

Lee 2-7

Let M be a connected smooth manifold, and let $\pi : \tilde{M} \to M$ be a topological covering map. Then there is only one smooth structure on $\tilde{M}$ such that π is a smooth covering map.

Remark: In the following proof, I have made an attempt to carefully explain how to deal with the domain issues. A more experienced reader might simply say something like, "we can choose a neighborhood small enough so that all of the domain issues work out nicely," and that appears to be true. But making the details hazy would defeat my purpose. I highly recommend reading this proof with a pen and paper and making a drawing to keep all the notation straight. It is currently beyond my ability to produce a book-quality diagram.

Proof. Lee proves the existence part of this, so we will only address the uniqueness. Suppose $\tilde{M}$ has two smooth structures $\mathcal{F}$ and $\mathcal{H}$ such that π is smooth with respect to both $\mathcal{F}$ and $\mathcal{H}$. Let $(U, \tilde{\varphi}) \in \mathcal{F}$ and $(W, \tilde{\psi}) \in \mathcal{H}$ be such that $U \cap W \neq \emptyset$. We wish to show that $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ is smooth on $\tilde{\varphi}(U \cap W)$. Fix any $p \in \tilde{\varphi}(U \cap W)$ and say $q = \tilde{\varphi}(p)$.

Take a chart (V, ϕ) of M at $\pi(q)$. Now take a smooth local section $\sigma_{\mathcal{H}} : \mathcal{O} \to \tilde{M}$ of π with respect to the smooth structure $\mathcal{H}$ and such that q is in the image of $\sigma_{\mathcal{H}}$. Then $\mathcal{O}$ is a neighborhood of $\pi(q)$. Let $\tilde{V} = V \cap \mathcal{O} \subset M$.

Since π is a covering map, there exists a neighborhood $\tilde{V}_0$ of $\pi(q)$ that is evenly covered by π. Then $\tilde{V}_0 \cap \tilde{V} \cap \sigma_{\mathcal{H}}^{-1}(W)$ is evenly covered by π, and exactly one of the connected components of $\pi^{-1}(\tilde{V}_0 \cap \tilde{V} \cap \sigma_{\mathcal{H}}^{-1}(W))$ contains q, and this connected component is open. Call it U_0. Then let $N = \tilde{\varphi}^{-1}(U_0)$. This is the appropriate neighborhood on which to show that $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ is smooth.

We have assumed that π is smooth with respect to both smooth structures on $\tilde{M}$, so the map $\phi \circ \pi \circ \tilde{\varphi}^{-1} : N \to \mathbb{R}^n$ is smooth. The map $\sigma_{\mathcal{H}}$ is smooth with respect to $\mathcal{H}$, and this implies that $\tilde{\psi} \circ \sigma_{\mathcal{H}} \circ \phi^{-1} \restriction \phi(\tilde{V}_0 \cap \tilde{V} \cap \sigma_{\mathcal{H}}^{-1}(W))$ is smooth. Then the composition $\tilde{\psi} \circ \sigma_{\mathcal{H}} \circ \phi^{-1} \circ \phi \circ \pi \circ \tilde{\varphi}^{-1} \restriction N$ is smooth.

Observe that $\sigma_{\mathcal{H}} \circ \pi \neq id_{\tilde{M}}$ in general, but because of our careful choice of domain, we will have $\sigma_{\mathcal{H}} \circ \pi = id$ in the following:

$$\begin{aligned} & \tilde{\psi} \circ \sigma_{\mathcal{H}} \circ \phi^{-1} \circ \phi \circ \pi \circ \tilde{\varphi}^{-1} \restriction N \\ = \; & \tilde{\psi} \circ \sigma_{\mathcal{H}} \circ \pi \circ \tilde{\varphi}^{-1} \restriction N \\ = \; & \tilde{\psi} \circ \tilde{\varphi}^{-1} \restriction N. \end{aligned}$$

Thus $\tilde{\psi} \circ \tilde{\varphi}^{-1} \restriction N$ is smooth, so $\tilde{\psi} \circ \tilde{\varphi}^{-1}$ is smooth at p. □

Lee 2-11

Let G be a connected Lie group, and let $U \subset G$ be any neighborhood of the identity. Then every element of G can be written as a finite product of elements of U; that is, U generates G.

Proof. To begin, observe that U^{-1} is open because the inversion map $g \mapsto g^{-1}$ is continuous and this map is its own continuous inverse. Also, U^{-1} intersects U in at least the identity, e, so let $W = U \cap U^{-1}$, and note that W is open. Also note that $W^{-1} = W$.

We will show that $H = \bigcup_{n=1}^{\infty} W^n$ is an open subgroup of G. Each W^n is open because group multiplication is continuous and continuously invertible. Thus H is open. It is obvious that H is closed under products, since for any $x, y \in H$ we have $x \in W^n$ and $y \in W^m$ for some n and m. Then $xy \in W^{n+m} \subset H$. Also, say $x = w_1 \cdot w_2 \cdots w_n$, where each $w_i \in W$. Then $x^{-1} = w_n^{-1} \cdot \cdots \cdot w_1^{-1} \in W^n \subset H$. Thus H is a subgroup of G.

Since H is an open subgroup, and multiplication by any fixed element is a homemorphism on G, the cosets of H are all open and—as is always the case—disjoint; that is, the cosets of H are an open partition of G, but G is connected. Whence $H = G$. Thus W generates G, and since $W \subset U$, we have shown that U generates G. □

Lee 2-12

Let G be a connected Lie group, and let G_0 denote the connected component of G containing the identity. Then (a) G_0 is the only connected open subgroup of G, and (b) each connected component of G is diffeomorphic to G_0.

Proof. First observe that G_0 is a connected component and G is locally path-connected because it is a topological manifold. Thus G_0 is open and G_0 is path-connected. We must show that G_0 is a group. Let $x, y \in G_0$. Then there exists a path $\gamma : I \to G_0$ from e, the identity, to x. Also, left multiplication by y, denoted ℓ_y is continuous by assumption, and the composition $\ell_y \circ \gamma$ is a path from y to yx. Since $y \in G_0$ and G_0 is connected, this proves that yx is in G_0. Now take $\ell_{x^{-1}} \circ \gamma$, where $\ell_{x^{-1}}$ is left translation by x^{-1}. This is a path from x^{-1} to e, which shows that $x^{-1} \in G_0$.

We have shown that G_0 is an open subgroup of G. We still need to show that it is the only connected open subgroup. Suppose $H \subset G$ is a connected subgroup. Then it contains the identity and so it is entirely contained in G_0. Suppose $H \subset G_0$ is an open subgroup of G_0. Then H contains a neighborhood of the identity. Now we simply observe that G_0 is a connected Lie group and apply Exercise 2-11, which tells us that H generates G_0. Thus $H = G_0$.

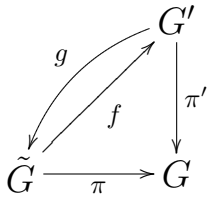
For part (b), observe that the cosets of G_0 are diffeomorphic to G_0 because left translation by any element is smooth and smoothly invertible. It follows they are all open and connected. And it is always true that they are disjoint, so each coset of G_0 is a connected component of G. □

Lee 2-13

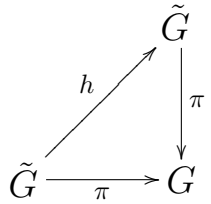
Let G be a connected Lie group. The universal covering group $\tilde{G}$ constructed in Lee Theorem 2.13 is unique in the following sense: if G' is any simply connected Lie group that admits a smooth covering map $\pi' : G' \to G$ that is also a Lie group homomorphism, then there exists a Lie group isomorphism $f : \tilde{G} \to G'$ such that $\pi' \circ f = \pi$.

Proof. Let $\tilde{e}$ denote the identity of $\tilde{G}$ and e the identity of G, and let e' denote the identity of G'. We will use the lifting criterion, which is proved in Lee's *Introduction to Topological Manifolds*. We have that $\tilde{G}$ is simply connected, π is continuous, and π' is a covering map. Since $\pi'(e') = e = \pi(\tilde{e})$, there exists a continuous lift $f : \tilde{G} \to G'$ of π with respect to π' such that $f(\tilde{e}) = e'$ and $\pi = \pi' \circ f$.

Similarly, G' is simply connected, π' is continuous, and π is a covering map, so there exists a continuous lift $g : G' \to \tilde{G}$ of π' with respect to π satisfying $\pi' = \pi \circ g$ and $g(e') = \tilde{e}$.

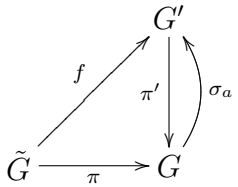


Let $h = g \circ f$. Then observe that $\pi \circ h = \pi \circ g \circ f = \pi' \circ f = \pi$ and $h(\tilde{e}) = \tilde{e}$. Thus h is a lift of π with respect to π. The identity $\mathrm{id}_{\tilde{G}}$ is also such a lift, and $h(\tilde{e}) = \mathrm{id}_{\tilde{G}}(\tilde{e})$. By the unique lifting property (again, see Lee's *Introduction to Topological Manifolds*), $h = \mathrm{id}_{\tilde{G}}$.



By the same reasoning with letters changed, we can see that $f \circ g = \mathrm{id}_{G'}$. That f has a left inverse shows that f is injective. That it has a right inverse shows that it is surjective. Since both are the same continuous map g, we have that f is a homeomorphism taking $\tilde{e}$ to e'.

We now show that f is smooth. Let $a \in \tilde{G}$. Then there exists a smooth local section $\sigma_a : U \subset G \to G'$ of π' such that $f(a) \in \sigma_a(U)$.



Thus $\sigma_a(p) = f(a)$ for some p. Also $(\pi' \circ \sigma_a)(p) = p$ by definition of smooth local section, and $(\pi' \circ \sigma_a)(p) = \pi'(f(a)) = \pi(a)$ by the commutativity of the diagram, so $p = \pi(a) \in U$.

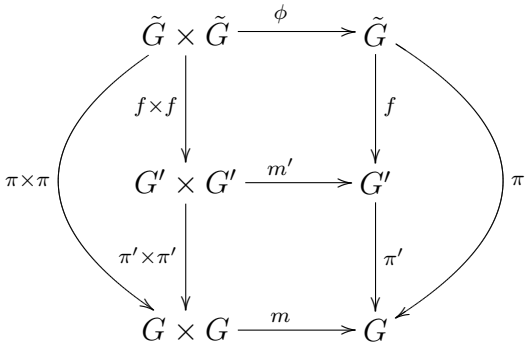
Now, since $\pi = \pi' \circ f$, we restrict to the neighborhood $\pi^{-1}(U)$ of a and we have $\sigma_a \circ \pi \restriction \pi^{-1}(U) = \sigma_a \circ \pi' \circ f \restriction \pi^{-1}(U) = f \restriction \pi^{-1}(U)$. We see that locally f is a composition of smooth maps π and σ_a. This shows that f is smooth.

Observe that f is a homeomorphism, so it is invertible, and all of the preceeding argument goes forward the same if we switch $\tilde{G}$ and G' and write f^{-1} in place of f. Thus f^{-1} is smooth also.

Now we must show that f is a group homomorphism. The group G' is equipped with a product map $m' : G' \times G' \to G'$. The commutativity of the diagram

$$\begin{array}{ccc} G' \times G' & \xrightarrow{m'} & G' \\ \downarrow{\scriptstyle \pi' \times \pi'} & & \downarrow{\scriptstyle \pi'} \\ G \times G & \xrightarrow{m} & G \end{array}$$

is nothing more than our assumption that π' is a Lie group homomorphism. Observe that $m' \circ (f \times f)(\tilde{e} \times \tilde{e}) = m'(e' \times e') = e' = f(\tilde{e})$. Since f is a diffeomorphism, it is a covering map, and by the lifting criterion there exists a lift $\phi : \tilde{G} \times \tilde{G} \to \tilde{G}$ of the map $m' \circ (f \times f)$ with respect to f such that $f \circ \phi = m' \circ (f \times f)$ and $\phi(\tilde{e} \times \tilde{e}) = \tilde{e}$, as in the diagram.



Now we see that

$$\begin{aligned} & \pi \circ \phi \\ = \ & \pi' \circ f \circ \phi \\ = \ & \pi' \circ m' \circ (f \times f) \\ = \ & m \circ (\pi' \times \pi') \circ (f \times f) \\ = \ & m \circ (\pi \times \pi) \end{aligned}$$

so that ϕ is a lift of $m \circ (\pi \times \pi)$ satisfying $\phi(\tilde{e} \times \tilde{e}) = \tilde{e}$. By the uniqueness of this lift, ϕ is the product map $\tilde{m}$ defined on $\tilde{G}$. We now have that the following diagram is commutative.

$$\begin{array}{ccc} \tilde{G} \times \tilde{G} & \xrightarrow{\tilde{m}} & \tilde{G} \\ \downarrow{\scriptstyle f \times f} & & \downarrow{\scriptstyle f} \\ G' \times G' & \xrightarrow{m'} & G' \end{array}$$

That means exactly that f is a group homomorphism. Thus f is a Lie group isomorphism as desired.

□

Lee 3-4

Let $C \subset \mathbb{R}^2$ be the unit circle and let $S \subset \mathbb{R}^2$ be the boundary of the square of side 2 centered at the origen:

$$S = \{(x,y) : \ \max(|x|,|y|) = 1.$$

There is a homeomorphism $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F(C) = S$, but there is no diffeomorphism with the same property.

Proof. We begin by defining a map $f : [0, \pi/2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{\cos(x)} & : x \in [0, \pi/4) \\ \frac{1}{\sin(x)} & : x \in [\pi/4, \pi/2] \end{cases}.$$

It is easily seen that $f(x)$ is continuous on its domain and that $f(0) = f(\pi/2)$. Define $f(\theta) = f(\theta \bmod \pi/2)$ for any $\theta \in [0, 2\pi]$. This extension f is continuous by the pasting lemma.

Now define the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ by $(\theta, r) \mapsto (\theta, r \cdot f(\theta))$. This map is obviously continuous and continuously invertible, since the inverse is just $(\theta, r) \mapsto (\theta, r \cdot \frac{1}{f(\theta)})$. That this map sends the circle to the square is elementary trigonometry.

Suppose now that we have a diffeomorphism F on $\mathbb{R}^2$ sending the circle to the square. Then we may take a smooth parameterization of the circle $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (\cos(t), \sin(t))$, and observe that the composition $F \circ \gamma : \mathbb{R} \to \mathbb{R}^2$ is a smooth parameterization of the square. For convenience of notation, let g denote this composition.

The map g is a diffeomorphism, and so there is an interval (a, b) that g maps diffeomorphically onto a neighborhood about the point $(1, 1)$. Let $c \in (a, b)$ be the point where $g(c) = (1, 1)$. Assume (a, c) gets mapped into the right side of the square and (c, b) gets mapped into the top side.

Let $\{e_1, e_2\}$ be the standard orthogonal basis in $\mathbb{R}^2$. Observe that for any point $t_0 \in (a, c)$ we have

$$\langle \frac{d}{dt}\Big|_{t=t_0} g(t), e_1 \rangle = \langle \lim_{h\to 0} \frac{g(t_0+h) - g(t_0)}{h}, e_1 \rangle = \lim_{h\to 0} \frac{g_1(t_0+h) - g_1(t_0)}{h} = 0.$$

Similarly, for every $t_0 \in (c, b)$ we have

$$\langle \frac{d}{dt}\Big|_{t=t_0} g(t), e_2 \rangle = \langle \lim_{h\to 0} \frac{g(t_0+h) - g(t_0)}{h}, e_2 \rangle = \lim_{h\to 0} \frac{g_2(t_0+h) - g_2(t_0)}{h} = 0.$$

By continuity of the inner product,

$$\langle \frac{d}{dt}\Big|_{t=c} g(t), e_1 \rangle = 0$$

and

$$\langle \frac{d}{dt}\Big|_{t=c} g(t), e_2 \rangle = 0.$$

Thus $\frac{d}{dt}\Big|_{t=c} g(t) = 0$. However, $\frac{d}{dt}\Big|_{t=c} \gamma(t) \neq 0$ and $\frac{d}{dt}\Big|_{t=c} g(t) = F_* \frac{d}{dt}\Big|_{t=c} g(t)$. Since F is a diffeomorphism, its pushforward F_* is an isomorphism. It cannot send a nonzero vector to the zero vector. We have arrived at a contradiction, so no such diffeomorphism F can exist.

□