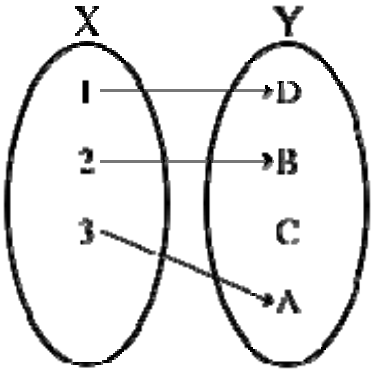
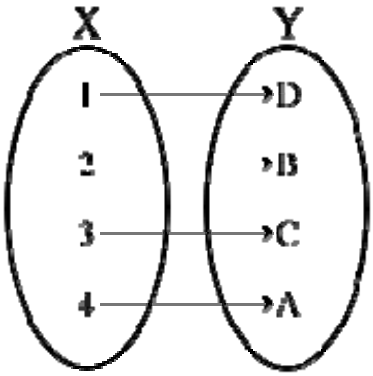
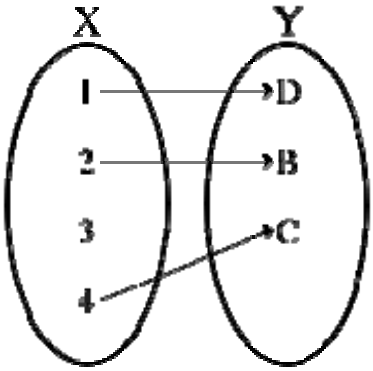
Injective function



An injective function (injection)



Another injective function (this one is a bijection)



A **non**-injective function (this one happens to be a surjection)

In mathematics, an **injective function** is a function which associates distinct arguments with distinct values. More precisely, a function *f* is said to be **injective** if it maps distinct *x* in the domain to distinct *y* in the codomain, such that $f(x) = y$.

Put another way, f is injective if $f(a) = f(b)$ implies $a = b$ (or $a \neq b$ implies $f(a) \neq f(b)$), for any a, b in the domain.

An injective function is called an **injection**, and is also said to be an **information-preserving** or **one-to-one function** (the latter is not to be confused with *one-to-one correspondence*, i.e. a bijective function).

A function f that is *not* injective is sometimes called **many-to-one**. (However, this terminology is also sometimes used to mean "single-valued", i.e. each argument is mapped to at most one value.)

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Examples and counter-examples

- For any set X, the identity function on X is injective.
- The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = 2x + 1$ is injective.
- The function $g : \mathbf{R} \to \mathbf{R}$ defined by $g(x) = x^2$ is *not* injective, because (for example) $g(1) = 1 = g(-1)$. However, if g is redefined so that its domain is the non-negative real numbers $[0,+\infty)$, then g is injective.
- The exponential function is injective (but not surjective as no value maps to a negative number).
- The natural logarithm function is injective.
- The function $g : \mathbf{R} \to \mathbf{R}$ defined by $g(x) = x^n - x$ is not injective, since, for example, $g(0) = g(1)$.

  More generally, when X and Y are both the real line $\mathbf{R}$, then an injective function $f : \mathbf{R} \to \mathbf{R}$ is one whose graph is never intersected by any horizontal line more than once.

  Injections can be undone

  Functions with left inverses (often called sections) are always injections. That is to say, for $f : X \to Y$, if there exists a function $g : Y \to X$ such that, for every

  $x \in X$

  $g(f(x)) = x$

(*f* can be undone by *g*)

then *f* is injective. Conversely, every injection with non-empty domain has a left inverse.

Note that *g* may not be a complete inverse of *f* because the composition in the other order, *f* o *g*, may not be the identity on *Y*. In other words, a function that can be undone or "*reversed*", such as *f*, is not necessarily invertible (bijective). Injections are "*reversible*" but not always invertible.

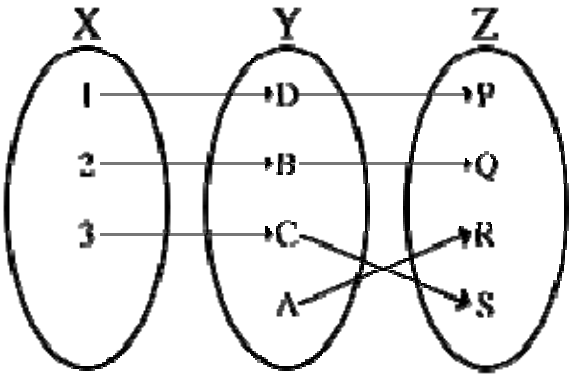
Although it is impossible to reverse a non-injective (and therefore information-losing) function, you can at least obtain a "quasi-inverse" of it, that is a multiple-valued function.

Injections may be made invertible

In fact, to turn an injective function $f : X \to Y$ into a bijective (hence invertible) function, it suffices to replace its codomain *Y* by its actual range $J = f(X)$. That is, let $g : X \to J$ such that $g(x) = f(x)$ for all *x* in *X*; then *g* is bijective. Indeed, *f* can be factored as $\mathrm{incl}_{J,Y} \circ g$, where $\mathrm{incl}_{J,Y}$ is the inclusion function from *J* into *Y*.

Other properties

- If *f* and *g* are both injective, then *f* o *g* is injective.

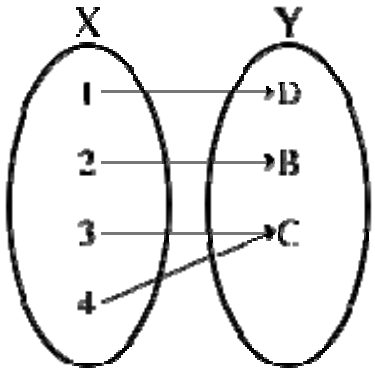


The composition of two injective functions is injective.

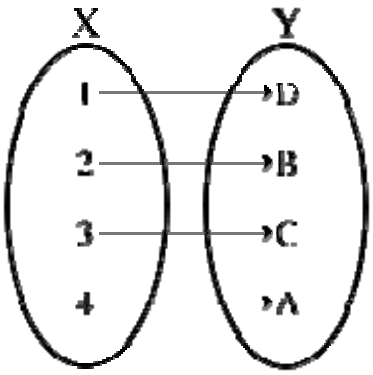
- If *g* o *f* is injective, then *f* is injective (but *g* need not be).

- $f : X \to Y$ is injective if and only if, given any functions $g, h : W \to X$, whenever $f \circ g = f \circ h$, then $g = h$. In other words, injective functions are precisely the monomorphisms in the category **Set** of sets.
- If $f : X \to Y$ is injective and A is a subset of X, then $f^{-1}(f(A)) = A$. Thus, A can be recovered from its image $f(A)$.
- If $f : X \to Y$ is injective and A and B are both subsets of X, then $f(A \cap B) = f(A) \cap f(B)$.
- Every function $h : W \to Y$ can be decomposed as $h = f \circ g$ for a suitable injection f and surjection g. This decomposition is unique up to isomorphism, and f may be thought of as the inclusion function of the range $h(W)$ of h as a subset of the codomain Y of h.
- If $f : X \to Y$ is an injective function, then Y has at least as many elements as X, in the sense of cardinal numbers.
- If both X and Y are finite with the same number of elements, then $f : X \to Y$ is injective if and only if f is surjective.
- Every embedding is injective.

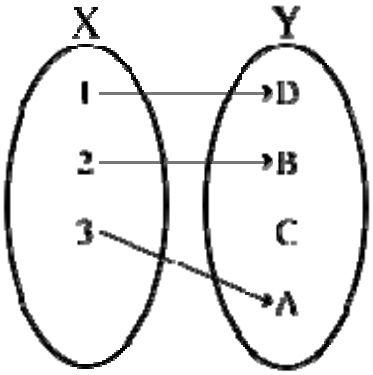
Surjective function



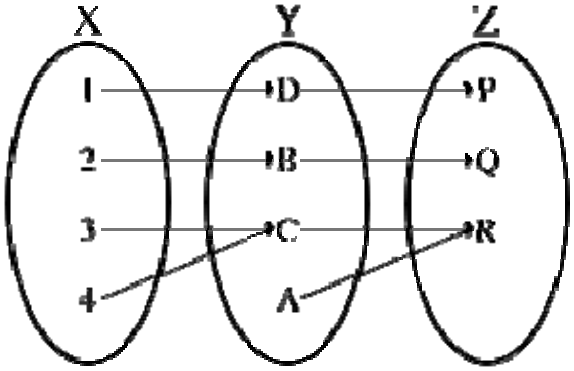
A surjective function.



Another surjective function. (This one happens to be a bijection)



A **non**-surjective function. (This one happens to be an injection)



Surjective composition: the first function need not be surjective.

In mathematics, a function f is said to be **surjective** if its values span its whole codomain; that is, for every y in the codomain, there is at least one x in the domain such that $f(x) = y$.

Said another way, a function $f: X \to Y$ is surjective if and only if its range $f(X)$ is equal to its codomain Y. A surjective function is called a **surjection**, and also said to be **onto**.

- **Examples and a counterexample**

- For any set X, the identity function id_X on X is surjective.
- The function f: **R** → **R** defined by $f(x) = 2x + 1$ is surjective (and even bijective), because for every real number y we have $f(x) = y$ where x is $(y - 1)/2$.
- The natural logarithm function ln: $(0,+\infty)$ → **R** is surjective.
- The function f: **Z** → **{0,1,2,3}** defined by $f(x) = x$ **mod** 4 is surjective.
- The function g: **R** → **R** defined by $g(x) = x^2$ is *not* surjective, because (for example) there is no real number x such that $x^2 = -1$. However, if the codomain is defined as $[0,+\infty)$, then g is surjective.

- **There always exists a function "reversible" by a surjection**

Every function with a right inverse is a surjection. The converse is equivalent to the axiom of choice. That is, assuming the axiom of choice, a function $f: X \to Y$ is surjective if and only if there exists a function $g: Y \to X$ such that, for every

$y \in Y$

$f(g(y)) = y$

(g can be undone by f)

that is a function g such that f o g equals the identity function on Y (cf. with definition of inverse function).

Note that g may not be a complete inverse of f because the composition in the other order, g o f, may not be the identity on X. In

other words, f can undo or "*reverse*" g, but not necessarily can be reversed by it. Surjections are not always invertible (bijective).

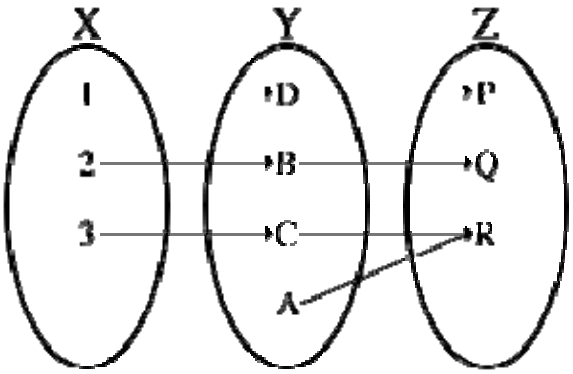
For example, in the first illustration, there is some function g such that $g(C) = 4$. There is also some function f such that $f(4) = C$. It doesn't matter that $g(C)$ can also equal 3; it only matters that f "reverses" g.

Other properties

- If f and g are both surjective, then $f \circ g$ is surjective.
- If $f \circ g$ is surjective, then f is surjective (but g need not be).
- $f: X \to Y$ is surjective if and only if, given any functions $g,h: Y \to Z$, whenever $g \circ f = h \circ f$, then $g = h$. In other words, surjective functions are precisely the epimorphisms in the category **Set** of sets.
- If $f: X \to Y$ is surjective and B is a subset of Y, then $f(f^{-1}(B)) = B$. Thus, B can be recovered from its preimage $f^{-1}(B)$.
- For any function $h: X \to Z$ there exists a surjection $f: X \to Y$ and injection $g: Y \to Z$ such that $h = g \circ f$. To see this, define Y to be the sets $h^{-1}(z)$ where z is in Z. These sets are disjoint and partition X. Then f carries each x to the element of Y which contains it, and g carries each element of Y to the point in Z to which h sends its points. Then f is surjective since it is a projection map, and g is injective by definition.
- By collapsing all arguments mapping to a given fixed image, every surjection induces a bijection defined on a quotient of its domain. More precisely, every surjection $f: A \to B$ can be factored as a projection followed by a bijection as follows. Let $A/\sim$ be the equivalence classes of A under the following equivalence relation: $x \sim y$ if and only if $f(x) = f(y)$. Equivalently, $A/\sim$ is the set of all preimages under f. Let $P(\sim) : A \to A/\sim$ be the projection map which sends each x in A to its equivalence class $[x]_\sim$, and let $f_P : A/\sim \to B$ be the well-defined function given by $f_P([x]_\sim) = f(x)$. Then $f = f_P \circ P(\sim)$.

- If $f: X \to Y$ is a surjective function, then X has at least as many elements as Y, in the sense of cardinal numbers.
- If both X and Y are finite with the same number of elements, then $f : X \to Y$ is surjective if and only if f is injective.
-

Bijection



Bijective composition: the first function need not be surjective and the second function need not be injective.

A function is **bijective** if it is both injective and surjective. A bijective function is a **bijection** (**one-to-one correspondence**). A function is bijective if and only if every possible image is mapped to by exactly one argument. This equivalent condition is formally expressed as follows.

The function

$f : A \to B$

is bijective iff for all

$b \in B$

, there is a unique

$a \in A$

such that $f(a) = b$.

- A function $f : A \to B$ is bijective if and only if it is invertible, that is, there is a function $g: B \to A$ such that g o f = identity function on A

and f o g = identity function on B. This function maps each image to its unique preimage.

- The composition of two bijections is again a bijection, but if g o f is a bijection, then it can only be concluded that f is injective and g is surjective. (See the figure at right and the remarks above regarding injections and surjections.)
- The bijections from a set to itself form a group under composition, called the symmetric group.

Cardinality

Suppose you want to define what it means for two sets to "have the same number of elements". One way to do this is to say that two sets "have the same number of elements" if and only if all the elements of one set can be paired with the elements of the other, in such a way that each element is paired with exactly one element. Accordingly, we can define two sets to "have the same number of elements" if there is a bijection between them. We say that the two sets have the same cardinality.

Examples

It is important to specify the domain and codomain of each function since by changing these, functions which we think of as the same may have different *jectivity*.

Injective and surjective (bijective)

- For every set A the identity function id_A and thus specifically .
- and thus also its inverse .
- The exponential function and thus also its inverse the natural logarithm

Injective and non-surjective

Properties

- For every function f, subset A of the domain and subset B of the codomain we have $A \subset f^{-1}(fA)$ and $f(f^{-1}B) \subset B$. If f is injective we have $A = f^{-1}(fA)$ and if f is surjective we have $f(f^{-1}B) = B$.
- For every function $h : A \to C$ we can define a surjection $H : A \to h(A) : a \to h(a)$ and an injection $I : h(A) \to C : a \to a$. It follows that $h = I \circ H$. This decomposition is unique up to isomorphism.