Solutions to Manifolds (CM433Z/CMMS18)

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Problem: Show that the induced topology indeed satisfies the definition of a topology.

Solution: Let $\mathcal{U} = \{U_i\}$ be the topology of Y and $X \subset Y$. The induced topology is $\mathcal{V} = \{U_i \cap X | U_i \in \mathcal{U}\}$.

i) $\emptyset = \emptyset \cap X$ so since $\emptyset \in \mathcal{U}$ we have $\emptyset \in \mathcal{V}$.

ii) $X = X \cap Y$ so since $Y \in \mathcal{U}$ we have $X \in \mathcal{V}$

iii) Take $V_i = U_i \cap X \in \mathcal{V}$ then

$$\begin{aligned} V_1 \cap V_2 \cap ... \cap V_n &= (U_1 \cap X) \cap (U_2 \cap X) \cap ... \cap (U_n \cap X) \\ &= (U_1 \cap U_2 \cap ... \cap U_n) \cap X \\ &\in \mathcal{V} \end{aligned} \tag{0.1}$$

iv) For an arbtirary number of V_i's:

$$\bigcup_i V_i = \bigcup_i (U_i \cap X) = \left(\bigcup_i U_i\right) \cap X \in \mathcal{V} \tag{0.2}$$

Problem: Why aren't closed subsets of $\mathbb{R}^n$, *e.g.* a disk with boundary or a line in $\mathbb{R}^2$, along with the identity map charts (note that in its own induced topology any subset of $\mathbb{R}^n$ is an open set)?

Solution: If $C \subset \mathbb{R}^n$ is closed we may still view it as an open set in its own induced topology. The identity map $id : C \to \mathbb{R}^n$ is certainly continuous (in inverse image of an

open set is open by definition of the induced topology). It is also clearly 1-1 and hence is a bijection onto its image. However consider $id^{-1} : id(C) \subset \mathbb{R}^n \to C$ this has

$$(id^{-1})^{-1}(C) = C \tag{0.3}$$

but since C is open in its own induced topology and closed in the topology of $\mathbb{R}^n$ we see that id^{-1} is not continuous (as $(id^{-1})^{-1}$ of an open set is closed). Thus id is not a homeomorphism.

Problem: What is $\mathbb{R}P^1$?

Solution: By definition $\mathbb{R}P^1 = \{(x,y) \in \mathbb{R}^2 - (0,0) | (x,y) \sim (\lambda x, \lambda y), \lambda \in \mathbb{R} - 0\}$. A point $(x,y) \in \mathbb{R}^2 - (0,0)$ defines a line through the origin. The point $(\lambda x, \lambda y)$ with $\lambda \neq 0$ will define the same line as (x,y). Thus $\mathbb{R}P^1$ is the space of lines through the origin.

On the other and a line through the origin is specified by the angle (roughly $\theta = \arctan(y/x)$) it makes with the postive x-axis. Since (x,y) and $(-x,-y)$ define the same line this angle is indentified modulo π rather than 2π. Thus $\mathbb{R}P^1$ can be indentified with a circle.

To make this more precises one should construct a diffeomorhpism from $\mathbb{R}P^1$ to S^1. Try this.

Problem: Show that the following:

$$\begin{aligned} U_1 &= \{(x,y) \in S^1 | y > 0\}\ , \qquad \phi_1(x,y) = x \\ U_2 &= \{(x,y) \in S^1 | y < 0\}\ , \qquad \phi_2(x,y) = x \\ U_3 &= \{(x,y) \in S^1 | x > 0\}\ , \qquad \phi_3(x,y) = y \\ U_4 &= \{(x,y) \in S^1 | x < 0\}\ , \qquad \phi_4(x,y) = y \end{aligned} \tag{0.4}$$

are a set of charts which cover S^1.

Solution:

It should be clear that all the U_i are open and cover S^1 and that the ϕ_i continuous with $\phi_i(U_i) = (-1,1)$. Their inverses are

$$\begin{aligned} \phi_1^{-1}(\theta) &= (\theta, \sqrt{1-\theta^2}) \\ \phi_2^{-1}(\theta) &= (\theta, -\sqrt{1-\theta^2}) \\ \phi_3^{-1}(\theta) &= (\sqrt{1-\theta^2}, \theta) \\ \phi_4^{-1}(\theta) &= (-\sqrt{1-\theta^2}, \theta) \end{aligned} \tag{0.5}$$

which are continuous for $\theta \in (-1,1)$ hence they are homeomorphisms (onto their image).

Next we must check that $\phi_i \circ \phi_j^{-1} : (-1,1) \to (-1,1)$ are $\mathcal{C}^\infty$ for all non-intersecting pairs. Thus we must check that

$$\begin{aligned}
\phi_1 \circ \phi_3^{-1}(\theta) &= \sqrt{1-\theta^2}\,, \qquad \phi_3 \circ \phi_1^{-1}(\theta) = \sqrt{1-\theta^2} \\
\phi_1 \circ \phi_4^{-1}(\theta) &= -\sqrt{1-\theta^2}\,, \qquad \phi_4 \circ \phi_1^{-1}(\theta) = \sqrt{1-\theta^2} \\
\phi_2 \circ \phi_3^{-1}(\theta) &= \sqrt{1-\theta^2}\,, \qquad \phi_3 \circ \phi_2^{-1}(\theta) = -\sqrt{1-\theta^2} \\
\phi_2 \circ \phi_4^{-1}(\theta) &= -\sqrt{1-\theta^2}\,, \qquad \phi_4 \circ \phi_2^{-1}(\theta) = -\sqrt{1-\theta^2}
\end{aligned} \tag{0.6}$$

These are all $\mathcal{C}^\infty$ since $\theta \in (-1,1)$.

Problem: Show that the 2-sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2+z^2 = 1\}$ is a 2-dimensional manifold.

Solution: The hint was to consider sterographic projection. This requires using two charts

$$U_S = \{(x,y,z) \in S^2 | z < 1\} \qquad \text{and} \qquad U_N = \{(x,y,z) \in S^2 | z > -1\} \tag{0.7}$$

these are clearly open and cover S^2. In each chart one constructs $\phi_{N/S} : U_{N/S} \to \mathbb{R}^2$ by taking a straight line through either the south pole $(0,0,-1)$ or notth pole $(0,0,1)$ and then through the point $p \in U_{N/S}$. These lines are defined by the equation

$$X(\lambda) = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix} + \lambda \begin{pmatrix} x \\ y \\ z \mp 1 \end{pmatrix} \tag{0.8}$$

so that $X(0)$ is either the north or south pole and $X(1)$ is a point on S^2. We define $\phi_{N/S}(p)$ to be the point in the (x,y)-plane where the line intersects $z=0$. Note that this construction works on all of U_N and on all of U_S respectively but not on all of S^2. Explicitly one has

$$\begin{aligned}
\phi_S(x,y,z) &= \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \\
\phi_N(x,y,z) &= \left(\frac{x}{1+z}, \frac{y}{1+z}\right)
\end{aligned} \tag{0.9}$$

By construction these maps are injective as they define a unique line and this will intersect the $z=0$ plane at a unique point. They are continuous and their inverses (to construct them consider the line through $(u,v,0)$ and $(0,0,\pm 1)$ and see where it intersects S^2):

$$\begin{aligned}
\phi_S^{-1}(u,v) &= \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right) \\
\phi_N^{-1}(u,v) &= \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2}\right)
\end{aligned} \tag{0.10}$$

These are also continuous. Finally we must simply observe that

$$\begin{aligned}\phi_S \circ \phi_N^{-1}(u,v) &= \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right) \\ \phi_N \circ \phi_S^{-1}(u,v) &= \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2}\right)\end{aligned} \tag{0.11}$$

are C^∞ on $\phi_N(U_N \cap U_S) = \phi_S(U_N \cap U_S) = \mathbb{R} - (0,0)$.

Problem: Consider the circle S^1 as above. Show that $f : S^1 \to \mathbb{R}$ with $f(x,y) = x^2 + y$ is $\mathcal{C}^\infty$.

Solution: We can take the coordinates above. We need to consider $f \circ \phi_i^{-1}(\theta) : \phi_i(U_i) \to \mathbb{R}$:

$$\begin{aligned}f \circ \phi_1^{-1}(\theta) &= \theta^2 + \sqrt{1-\theta^2} \\ f \circ \phi_2^{-1}(\theta) &= \theta^2 - \sqrt{1-\theta^2} \\ f \circ \phi_3^{-1}(\theta) &= 1 - \theta^2 + \theta \\ f \circ \phi_4^{-1}(\theta) &= 1 - \theta^2 + \theta\end{aligned} \tag{0.12}$$

clearly all these functions are C^∞ on $\phi_i(U_i) = (-1,1)$.

Problem: Show that $f : S^1 \to S^1$ defined by $f(e^{2\pi i\theta}) = e^{2\pi i n\theta}$ is $\mathcal{C}^\infty$ for any n.

Solution: Again we choose the same charts and note that $\theta \in [0,1]$. First observe that

$$\begin{aligned}f \circ \phi_1^{-1}(\theta) &= e^{in \arctan(\sqrt{1-\theta^2}/\theta)} \\ f \circ \phi_2^{-1}(\theta) &= e^{-in \arctan(\sqrt{1-\theta^2}/\theta)} \\ f \circ \phi_3^{-1}(\theta) &= e^{in \arctan(\theta/\sqrt{1-\theta^2})} \\ f \circ \phi_4^{-1}(\theta) &= e^{-in \arctan(\theta/\sqrt{1-\theta^2})}\end{aligned} \tag{0.13}$$

Note that this is well defined since $\theta \neq 0$ on $\phi_1(U_1), \phi_2(U_2)$ and $\theta \neq \pm 1$ on $\phi_3(U_3), \phi_4(U_4)$. For any function χ we also have that

$$\begin{aligned}\phi_1(e^{in\chi}) &= \cos(n\chi) \\ \phi_2(e^{in\chi}) &= \cos(n\chi) \\ \phi_3(e^{in\chi}) &= \sin(n\chi) \\ \phi_4(e^{in\chi}) &= \sin(n\chi)\end{aligned} \tag{0.14}$$

thus we see that $\phi_i \circ f \circ \phi_j^{-1}(\theta)$ has the form:

$$\begin{array}{l} \cos(n \arctan(\sqrt{1-\theta^2}/\theta) \\ \pm \sin(n \arctan(\sqrt{1-\theta^2}/\theta) \\ \cos(n \arctan(\theta/\sqrt{1-\theta^2}) \\ \pm \sin(n \arctan(\theta/\sqrt{1-\theta^2}) \end{array} \tag{0.15}$$

And these are all $\mathcal{C}^\infty$ on the appropriate range of θ.

Problem: Show that the charts of two diffeomorphic manifolds are in a one to one correspondence.

Solution: Let $\{U_i, \phi_i\}$ and $\{V_a, \psi_a\}$ be differential structures for $\mathcal{M}$ and $\mathcal{N}$ respectively and $f : \mathcal{M} \to \mathcal{N}$ a diffeomorphism.

First we show that $\{f(U_i), \phi_i \circ f^{-1}\}$ is a set of charts that cover $\mathcal{N}$. We note that these cover $\mathcal{N}$:

$$\cup_i f(U_i) = f(\cup_i U_i) = f(\mathcal{M}) = \mathcal{N} \tag{0.16}$$

Furthermore $\phi_i \circ f^{-1}$ are clearly homeomorphisms (bijective, continuous with the inverse continuous).

Similarly $\{f^{-1}(V_a), \psi_a \circ f\}$ is a set of charts that cover $\mathcal{M}$.

Now on $V_a \cap f(U_i)$ we have that

$$\psi_a \circ f^{-1} \circ \phi_i^{-1} : \phi_i(V_a \cap f(U_i)) \to \psi_a(V_a \cap f(U_i)) \tag{0.17}$$

and this is $\mathcal{C}^\infty$ as f is $\mathcal{C}^\infty$ (recall the definition). Thus the charts $\{f(U_i), \phi_i \circ f^{-1}\}$ are compatible with the charts $\{V_a, \psi_a\}$. Since we take the differential structure to be maximal we find that the charts $\{f(U_i), \phi_i \circ f^{-1}\}$ must be included in the differential struture $\{V_a, \psi_a\}$ of $\mathcal{N}$.

Similarly $\{f^{-1}(V_a), \psi_a \circ f\}$ are compatible with the charts $\{U_i, \phi_i\}$ and since we assume the differential structure to be maximal it follows that the $\{f^{-1}(V_a), \psi_a \circ f\}$ are included in $\{U_i, \phi_i\}$.

Thus it follows that $\{V_a, \psi_a\}$ and $\{U_i, \phi_i\}$ are in a one-to-one correspondence with each other.

Problem: Show that the set of diffeomorphisms from a manifold to itself forms a group under composition.

Solution: Suppose $f, g : \mathcal{M} \to \mathcal{M}$ are diffeomorphims. Then

$$\phi_i \circ (f \circ g) \circ \phi_j^{-1} = \phi_i \circ (f \circ \phi_k^{-1} \circ \phi_k \circ g) \circ \phi_j^{-1} = (\phi_i \circ f \circ \phi_k^{-1}) \circ (\phi_k \circ g \circ \phi_j^{-1}) \tag{0.18}$$

is $\mathcal{C}^\infty$ for all i, j, k since f and g are C^∞. Similarly for $g^{-1} \circ f^{-1}$. Furthermore $f \circ g$ is a bijection if both f and g are. Thus $f \circ g$ is a diffeomorphism.

Clearly $id : \mathcal{M} \to \mathcal{M}$ is a diffeomorhism and by definition (the properties are symmetric between f and f^{-1}) if f is a diffeomorphism then so is f^{-1}.

Problem: What goes wrong if try to define $(X \cdot Y)(f) = X(f) \cdot Y(f)$?

Solution: With this defintion we find that

$$\begin{aligned} X \cdot Y(f+g) &= (X(f)+X(g))(Y(f)+Y(g)) \\ &= X(f)Y(f) + X(f)Y(g) + X(g)Y(f) + X(f)Y(g) \\ &= X \cdot Y(f) + X \cdot Y(g) + X(f)Y(g) + X(g)Y(f) \end{aligned} \tag{0.19}$$

and the last two terms are unwanted.

If we try the same trick that we used for the commutator and define $[X,Y](f) = X(f)Y(f) - Y(f)X(f)$ then this clearly vanishes identically.

Problem: Show that, if in a particular coordinate system,

$$X = \sum_\mu X^\mu(x) \frac{\partial}{\partial x^\mu}\Big|_p \,, \qquad Y = \sum_\mu Y^\mu(x) \frac{\partial}{\partial x^\mu}\Big|_p \tag{0.20}$$

then

$$[X,Y] = \sum_\mu \sum_\nu (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) \frac{\partial}{\partial x^\nu}\Big|_p \tag{0.21}$$

Solution: We simply calculate:

$$\begin{aligned} X(Y)(f) &= X\left(\sum_\nu Y^\nu \frac{\partial}{\partial x^\nu}(f \circ \phi_i^{-1}) \circ \phi_i\right) \\ &= \sum_\mu \left(X^\mu \frac{\partial}{\partial x^\mu}\left(\sum_\nu Y^\nu \frac{\partial}{\partial x^\nu}(f \circ \phi_i^{-1}) \circ \phi_i \circ \phi_i^{-1}\right) \circ \phi_i\right) \\ &= \sum_\mu \sum_\nu \left(X^\mu \partial_\mu Y^\nu \frac{\partial}{\partial x^\nu}(f \circ \phi_i^{-1}) + X^\mu Y^\nu \frac{\partial^2}{\partial x^\nu \partial x^\mu}(f \circ \phi_i^{-1})\right) \circ \phi_i \end{aligned} \tag{0.22}$$

Since the second term is symmetric in X^μ and Y^ν we find that

$$X(Y)(f) - Y(X)(f) = \sum_\mu \sum_\nu \left(X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu\right) \frac{\partial}{\partial x^\nu}(f \circ \phi_i^{-1}) \circ \phi_i \tag{0.23}$$

and we prove the theorem.

Problem: Show that for three vector fields X, Y, Z on $\mathcal{M}$ the jacobi identity holds:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \tag{0.24}$$

Solution: Here we simply expand things out

$$\begin{aligned}[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]] &= X([Y,Z])-[Y,Z](X)+Y([Z,X]) \\ &\quad -[Z,X](Y)+Z([X,Y])-[X,Y](Z) \\ &= X(Y(Z))-X(Z(Y))-Y(Z(X))+Z(Y(X)) \\ &\quad +Y(Z(X))-Y(X(Z))-Z(X(Y))+X(Z(Y)) \\ &\quad +Z(X(Y))-Z(Y(X))-X(Y(Z))+Y(X(Z)) \\ &= 0 \end{aligned} \tag{0.25}$$

where we used the linearity properties of vectors, *i.e.* $X(Y+Z)=X(Y)+X(Z)$.

Problem: Consider a manifold with a local coordinate system $\phi_i=(x^1,...,x^n)$.
i) Show that $\left[\frac{\partial}{\partial x^\mu}\Big|_p,\frac{\partial}{\partial x^\nu}\Big|_p\right]=0$
ii) Evaluate $\left[\frac{\partial}{\partial x^1}\Big|_p,\varphi(x^1,x^2)\frac{\partial}{\partial x^2}\Big|_p\right]$ where $\varphi(x^1,x^2)$ is a $\mathcal{C}^\infty$ function of x^1,x^2.

Solution: In the first case we find

$$\begin{aligned}\left[\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial x^\nu}\right](f) &= \frac{\partial}{\partial x^\mu}\Big|_p\left(\frac{\partial}{\partial x^\nu}(f\circ\phi_i^{-1})\circ\phi_i\right)-\frac{\partial}{\partial x^\nu}\Big|_p\left(\frac{\partial}{\partial x^\mu}(f\circ\phi_i^{-1})\circ\phi_i\right) \\ &= \frac{\partial^2}{\partial x^\nu\partial x^\mu}(f\circ\phi_i^{-1})\circ\phi_i-\frac{\partial^2}{\partial x^\mu\partial x^\nu}(f\circ\phi_i^{-1})\circ\phi_i \\ &= 0 \end{aligned} \tag{0.26}$$

And in the second case:

$$\begin{aligned}\left[\frac{\partial}{\partial x^\mu},\varphi\frac{\partial}{\partial x^\nu}\right](f) &= \frac{\partial}{\partial x^\mu}\Big|_p\left(\varphi\frac{\partial}{\partial x^\nu}(f\circ\phi_i^{-1})\circ\phi_i\right)-\varphi\frac{\partial}{\partial x^\nu}\Big|_p\left(\frac{\partial}{\partial x^\mu}(f\circ\phi_i^{-1})\circ\phi_i\right) \\ &= \varphi\frac{\partial^2}{\partial x^\nu\partial x^\mu}(f\circ\phi_i^{-1})\circ\phi_i+\left(\frac{\partial}{\partial x^\mu}\Big|_p\varphi\right)\frac{\partial}{\partial x^\nu}\Big|_p(f) \\ &\quad -\varphi\frac{\partial^2}{\partial x^\mu\partial x^\nu}(f\circ\phi_i^{-1})\circ\phi_i \\ &= \left(\frac{\partial}{\partial x^\mu}\Big|_p\varphi\right)\frac{\partial}{\partial x^\nu}\Big|_p(f) \end{aligned} \tag{0.27}$$

so

$$\left[\frac{\partial}{\partial x^\mu},\varphi\frac{\partial}{\partial x^\nu}\right]=\left(\frac{\partial}{\partial x^\mu}\Big|_p\varphi\right)\frac{\partial}{\partial x^\nu}\Big|_p \tag{0.28}$$

Problem: Show that the dual space is a vector space and has the same dimension as the original vector space (if it is finite dimensional).

Solution: Let V be a vector space with a basis $e^1,...,e^n$. The space of linear maps $L:V\to\mathbb{R}$ is clearly a vector space (you can add them and multiply them by elements of $\mathbb{R}$). A linear map is defined by its action on a basis. Hence if $L(e^i)=c^i$ then

$$L(\sum v_ie^i)=\sum v_iL(e^i)=\sum v_ic^i \tag{0.29}$$

Let L_j be the linear map defined by

$$L_j(e^i) = \delta^i_j \tag{0.30}$$

Clearly there are n such maps, one for each value of j. Furthermore any other L can be written as

$$L = \sum c^i L_i \tag{0.31}$$

so that $L(v) = \sum v_i c^i$. To prove that they are linearly independent we assume that

$$\sum d^i L_i = 0 \tag{0.32}$$

Then we consider this acting on each vector e_j to find $d^j = 0$. Thus they are indeed linearly independent.

Problem: Prove that, if, in a local coordinate system $(x^1, ..., x^n)$,

$$X = \sum_\mu X^\mu(x) \frac{\partial}{\partial x^\mu}\Big|_p \,, \qquad \omega = \sum_\mu \omega_\mu dx^\mu\Big|_p \tag{0.33}$$

then

$$\mathcal{L}_X \omega = \sum_\mu \sum_\nu \left(\partial_\nu \omega_\mu X^\nu + \omega_\nu \partial_\mu X^\nu\right) dx^\mu\Big|_p \tag{0.34}$$

Solution: Here we just do the calculation noting that

$$\sigma^\mu(\epsilon, p) = x^\mu(p) + \epsilon X^\mu + ... \tag{0.35}$$

near $\epsilon = 0$ and hence that

$$\begin{aligned} (\sigma^\star \omega)_\mu &= \sum \omega_\nu(\sigma) \frac{\partial \sigma^\nu}{\partial x^\mu} \\ &= \sum (\omega_\nu + \epsilon \partial_\lambda \omega_\nu X^\lambda)(\delta^\nu_\mu + \epsilon \partial_\mu X^\nu) + ... \\ &= \sum (\omega_\mu + \epsilon \partial_\lambda \omega_\mu X^\lambda + \epsilon \partial_\mu X^\nu \omega_\nu) + ... \end{aligned} \tag{0.36}$$

$$\begin{aligned} \mathcal{L}_X \omega &= \frac{d}{dt}\left(\sigma(t,p)^\star \omega\Big|_p\right) \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon}\left(\sum \epsilon \partial_\lambda \omega_\mu X^\lambda + \epsilon \partial_\mu X^\nu \omega_\nu\right) dx^\mu \\ &= \sum \left(X^\nu \partial_\nu \omega_\mu + \partial_\mu X^\nu \omega_\nu\right) dx^\mu \end{aligned} \tag{0.37}$$

Problem: Show that if, in a local coordinate system

$$T = \sum A^{\mu_1 \mu_2 ... \mu_r}{}_{\nu_1 \nu_2 ... \nu_s} \frac{\partial}{\partial x^{\mu_1}}\Big|_p \otimes ... \otimes \frac{\partial}{\partial x^{\mu_r}}\Big|_p \otimes dx^{\nu_1}\Big|_p \otimes ... \otimes dx^{\nu_s}\Big|_p \tag{0.38}$$

is an (r,s)-tensor,

$$X_i = \sum X_i^\mu \frac{\partial}{\partial x^\mu}\Big|_p \,, \qquad i = 1, ..., s \tag{0.39}$$

are s vectors and

$$\omega^I = \sum \omega^I_\nu dx^\nu\Big|_p \,, \qquad I = 1, ..., r \tag{0.40}$$

are r co-vectors, then

$$T(\omega^1, ..., \omega^r, X_1, ..., X_s) = \sum A^{\mu_1 ... \mu_r}{}_{\nu_1 ... \nu_s} \omega^1_{\mu_1} ... \omega^1_{\mu_r} X_1^{\nu_1} ... X_1^{\nu_s} \tag{0.41}$$

Solution: Here we simply evaluate using linearity:

$$\begin{aligned} T(\omega^1, ..., \omega^r, X_1, ..., X_s) &= T\left(\sum \omega^1_{\mu_1} dx^{\mu_1}\Big|_p, ..., \sum X_1^{\nu_1} \frac{\partial}{\partial x^{\nu_1}}\Big|_p, ...\right) \\ &= \sum \omega^1_{\mu_1} ... \omega^r_{\mu_r} X_1^{\nu_1} ... X_1^{\nu_s} T\left(dx^{\mu_1}\Big|_p, ..., \frac{\partial}{\partial x^{\nu_1}}\Big|_p, ...\right) \\ &= \sum A^{\mu_1 ... \mu_r}{}_{\nu_1 ... \nu_s} \omega^1_{\mu_1} ... \omega^r_{\mu_r} X_1^{\nu_1} ... X_1^{\nu_s} \end{aligned} \tag{0.42}$$

Problem: Show that if T is an (r,s)-tensor, $(U_1, \phi_1 = (x^1, ..., x^n))$ and $(U_2, \phi_2 = (y^1, ..., y^n))$ are two local coordinates charts with $U_1 \cap U_2 \neq \emptyset$ such that

$$T = \sum A^{\mu_1 \mu_2 ... \mu_s}{}_{\nu_1 \nu_2 ... \nu_s} \frac{\partial}{\partial x^{\mu_1}}\Big|_p \otimes ... \otimes \frac{\partial}{\partial x^{\mu_r}}\Big|_p \otimes dx^{\nu_1}\Big|_p \otimes ... \otimes dx^{\nu_s}\Big|_p \tag{0.43}$$

and

$$T = \sum B^{\mu_1 \mu_2 ... \mu_s}{}_{\nu_1 \nu_2 ... \nu_s} \frac{\partial}{\partial y^{\mu_1}}\Big|_p \otimes ... \otimes \frac{\partial}{\partial y^{\mu_r}}\Big|_p \otimes dy^{\nu_1}\Big|_p \otimes ... \otimes dy^{\nu_s}\Big|_p \tag{0.44}$$

then

$$\begin{aligned} B^{\mu_1 \mu_2 ... \mu_r}{}_{\nu_1 \nu_2 ... \nu_s} &= \sum_{\lambda_1=1}^n ... \sum_{\lambda_r=1}^n \sum_{\rho_1=1}^n ... \sum_{\rho_s=1}^n A^{\rho_1 \rho_2 ... \rho_s}{}_{\lambda_1 \lambda_2 ... \lambda_s} \\ &\quad \times \left(\frac{\partial}{\partial x^{\rho_1}}\Big|_p y^{\mu_1}\right) ... \left(\frac{\partial}{\partial x^{\rho_r}}\Big|_p y^{\mu_r}\right) \\ &\quad \times \left(\frac{\partial}{\partial y^{\nu_1}}\Big|_p x^{\lambda_1}\right) ... \left(\frac{\partial}{\partial y^{\nu_s}}\Big|_p x^{\lambda_s}\right) \end{aligned} \tag{0.45}$$

where $y^\mu(x)$ is understood to be the component of the transition function $\phi_2 \circ \phi_1^{-1}$ and $x^\mu(y)$ is understood to be the components of the transition function $\phi_1 \circ \phi_2^{-1}$.

Solution: Here we note that in the y-coordinate chart

$$B^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s} = T\left(dy^{\mu_1}\Big|_p, ..., \frac{\partial}{\partial y^{\nu_1}}\Big|_p, ...\right) \tag{0.46}$$

We know from previous calculations that

$$dy^{\mu_1}\Big|_p = \sum \frac{\partial y^{\mu_1}}{\partial x^{\lambda_1}} dx^{\lambda_1}\Big|_p \ , \qquad \frac{\partial}{\partial y^{\nu_1}}\Big|_p = \sum \frac{\partial x^{\rho_1}}{\partial y^{\nu_1}} \frac{\partial}{\partial x^{\rho_1}}\Big|_p \tag{0.47}$$

Thus, using linearity, we find

$$\begin{aligned} B^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s} &= T\left(dy^{\mu_1}\Big|_p, ..., \frac{\partial}{\partial y^{\nu_1}}\Big|_p, ...\right) \\ &= T\left(\sum \frac{\partial y^{\mu_1}}{\partial x^{\lambda_1}} dx^{\lambda_1}\Big|_p, ..., \sum \frac{\partial x^{\rho_1}}{\partial y^{\nu_1}} \frac{\partial}{\partial x^{\rho_1}}\Big|_p ...\right) \\ &= \sum \frac{\partial y^{\mu_1}}{\partial x^{\lambda_1}} ... \frac{\partial x^{\rho_1}}{\partial y^{\nu_1}} ... T\left(dx^{\lambda_1}\Big|_p, ..., \frac{\partial}{\partial x^{\rho_1}}\Big|_p ...\right) \\ &= \sum \frac{\partial y^{\mu_1}}{\partial x^{\lambda_1}} ... \frac{\partial y^{\mu_r}}{\partial x^{\lambda_r}} \frac{\partial x^{\rho_1}}{\partial y^{\nu_1}} ... \frac{\partial x^{\rho_s}}{\partial y^{\nu_s}} A^{\lambda_1\ldots\lambda_r}{}_{\rho_1\ldots\rho_s} \end{aligned} \tag{0.48}$$

Problem: If

$$\begin{aligned} \omega &= A_{12} dx^1 \wedge dx^2 + A_{34} dx^3 \wedge dx^4 \\ \eta &= B_{123} dx^1 \wedge dx^2 \wedge dx^3 + B_{125} dx^1 \wedge dx^2 \wedge dx^5 \end{aligned} \tag{0.49}$$

then what is $\omega \wedge \eta$?

Solution: We find

$$\begin{aligned} \omega \wedge \eta &= \left(A_{12} dx^1 \wedge dx^2 + A_{34} dx^3 \wedge dx^4\right) \wedge \left(B_{123} dx^1 \wedge dx^2 \wedge dx^3 + B_{125} dx^1 \wedge dx^2 \wedge dx^5\right) \\ &= A_{34} B_{125} dx^3 \wedge dx^4 \wedge dx^1 \wedge dx^2 \wedge dx^5 \\ &= A_{34} B_{125} dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \wedge dx^5 \end{aligned} \tag{0.50}$$

where we used the fact that $dx^\lambda \wedge dx^\lambda = 0$.

Problem: Prove that if $\omega \in \Omega^p(\mathcal{M})$ and $\eta \in \Omega^q(\mathcal{M})$ then $\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega$.

Solution: By linearity it is sufficient to consider

$$\omega = dx^1 \wedge ... \wedge dx^p \ , \qquad \eta = dx^{p+1} \wedge ... \wedge dx^{p+q} \tag{0.51}$$

Then we find

$$\begin{aligned}
\omega \wedge \eta &= dx^1 \wedge ... \wedge dx^p \wedge dx^{p+1} \wedge ... \wedge dx^{p+q} \\
&= (-1)^p dx^{p+1} \wedge dx^1 \wedge ... \wedge dx^p \wedge dx^{p+2} \wedge ... \wedge dx^{p+q} \\
&= (-1)^{p+p} dx^{p+1} \wedge dx^{p+2} \wedge dx^1 \wedge ... \wedge dx^p \wedge dx^{p+3} \wedge ... \wedge dx^{p+q} \\
&= (-1)^{pq} dx^{p+1} \wedge ... \wedge dx^{p+q} \wedge dx^1 \wedge ... \wedge dx^p \\
&= (-1)^{pq} \eta \wedge \omega
\end{aligned} \tag{0.52}$$

Problem: Prove that if $\omega \in \Omega^p(\mathcal{M})$ and $\eta \in \Omega^q(\mathcal{M})$ then $d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^p \omega \wedge d\eta$.

Solution: In components we have

$$\begin{aligned}
d(\omega \wedge \eta) &= \frac{1}{p!q!} \sum \partial_\lambda \left(\omega_{\mu_1 ... \mu_p} \eta_{\nu_1 ... \nu_q} \right) dx^\lambda \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} \wedge dx^{\nu_1} \wedge ... \wedge dx^{\nu_q} \\
&= \frac{1}{p!q!} \sum \left(\partial_\lambda \omega_{\mu_1 ... \mu_p} \eta_{\nu_1 ... \nu_q} + \omega_{\mu_1 ... \mu_p} \partial_\lambda \eta_{\nu_1 ... \nu_q} \right) \\
&\qquad dx^\lambda \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} \wedge dx^{\nu_1} \wedge ... \wedge dx^{\nu_q} \\
&= \frac{1}{p!q!} \sum \partial_\lambda \omega_{\mu_1 ... \mu_p} \eta_{\nu_1 ... \nu_q} dx^\lambda \wedge dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} \wedge dx^{\nu_1} \wedge ... \wedge dx^{\nu_q} \\
&\quad + (-1)^p \frac{1}{p!q!} \sum \omega_{\mu_1 ... \mu_p} \partial_\lambda \eta_{\nu_1 ... \nu_q} dx^{\mu_1} \wedge ... \wedge dx^{\mu_p} \wedge dx^\lambda \wedge dx^{\nu_1} \wedge ... \wedge dx^{\nu_q} \\
&= (d\omega) \wedge \eta + (-1)^p \omega \wedge d\eta
\end{aligned} \tag{0.53}$$

Problem: Prove the following. Let $f : \mathcal{M} \to \mathcal{N}$, $(y^1, ..., y^n)$ be local coordinates on $V \subset \mathcal{N}$ and $(x^1, ..., x^m)$ local coordinates on $U \cap f^{-1}(V) \subset \mathcal{M}$. If

$$\omega = \sum_{\nu=1}^{n} \omega_{\nu_1, ..., \nu_p}(f(p)) dy^{\nu_1} \Big|_{f(p)} \wedge ... \wedge dy^{\nu_p} \Big|_{f(p)} \tag{0.54}$$

then

$$f^\star \omega = \sum_{\mu=1}^{m} \sum_{\nu=1}^{n} \omega_{\nu_1 ... \nu_p}(f(p)) \frac{\partial}{\partial x^{\mu_1}} (y^{\nu_1} \circ f) ... \frac{\partial}{\partial x^{\mu_p}} (y^{\nu_p} \circ f) dx^{\mu_1} \Big|_p \wedge ... \wedge dx^{\mu_p} \Big|_p \tag{0.55}$$

Solution: Recall that we defined

$$f^\star \omega(X_1, ..., X_p) = \omega(f_\star X_1, ..., f_\star X_p) \tag{0.56}$$

Now we saw before that

$$(f_\star X)^\mu = \sum X^\nu \frac{\partial}{\partial x^\nu} (y^\mu \circ f) \tag{0.57}$$

Hence

$$\begin{aligned}(f^\star\omega)_{\nu_1\ldots\nu_p} &= (f^\star\omega)\left(\frac{\partial}{\partial x^{\nu_1}},...,\frac{\partial}{\partial x^{\nu_p}}\right)\\ &= \omega\left(f_\star\frac{\partial}{\partial x^{\nu_1}},...,f_\star\frac{\partial}{\partial x^{\nu_p}}\right)\\ &= \sum\omega_{\mu_1..\mu_p}\left(f_\star\frac{\partial}{\partial x^{\nu_1}}\right)^{\mu_1}...\left(f_\star\frac{\partial}{\partial x^{\nu_p}}\right)^{\nu_p}\\ &= \sum\omega_{\mu_1..\mu_p}\frac{\partial}{\partial x^{\nu_1}}(y^{\mu_1}\circ f)...\frac{\partial}{\partial x^{\nu_p}}(y^{\mu_p}\circ f)\end{aligned} \tag{0.58}$$

Problem: Show that if

$$\omega = pdx + qdy \tag{0.59}$$

then

$$C^\star\omega = \left(p(C(t))\frac{dx\circ C}{dt} + p(C(t))\frac{dy\circ C}{dt}\right)dt \tag{0.60}$$

Solution: We have just seen that

$$(C^\star\omega)_\mu = \sum\frac{\partial}{\partial x^\mu}(y^\nu\circ C)\omega_\nu(C) \tag{0.61}$$

Thus in this case we simply have $C^\star\omega = (C^\star\omega)_t dt$ with

$$\begin{aligned}(C^\star\omega)_t &= \frac{\partial}{\partial t}(x\circ C)\omega_x(C(t)) + \frac{\partial}{\partial t}(y\circ C)\omega_y(C)\\ &= p(C(t))\frac{dx\circ C}{dt} + p(C(t))\frac{dy\circ C}{dt}\end{aligned} \tag{0.62}$$

Problem: Consider the manifold $\mathbb{R}^2 - \{(0,0)\}$ and the 1-form

$$\omega = \frac{ydx}{x^2+y^2} - \frac{xdy}{x^2+y^2} \tag{0.63}$$

What is

$$\int_C\omega \tag{0.64}$$

along the curve $C(t) = (2+\cos(2\pi t), 2+\sin(2\pi t))$. Next consider the 2-form

$$\omega = \frac{dx\wedge dy}{x^2+y^2} \tag{0.65}$$

What is

$$\int_C\omega \tag{0.66}$$

where $C : I_2 \to \mathbb{R}^2 - \{(0,0)\}$ is given by $C(t_1, t_2) = (t_1 + 1)(\cos(2\pi t_2), \sin(2\pi t_2))$.

Solution: Our definition of the integral tells us that

$$\begin{aligned}
\int_C \omega &= \int_0^1 C^\star \omega \\
&= -2\pi \int_0^1 \left(\frac{(2+\sin(2\pi t))\sin(2\pi t)}{(2+\sin(2\pi t))^2 + (2+\cos(2\pi t))^2} + \frac{(2+\cos(2\pi t))\cos(2\pi t)}{(2+\sin(2\pi t))^2 + (2+\cos(2\pi t))^2} \right) dt \\
&= -2\pi \int_0^1 \left(\frac{\sin(2\pi t)}{2+\sin(2\pi t)} + \frac{(2+\cos(2\pi t))\cos(2\pi t)}{(2+\sin(2\pi t))^2} \right) \frac{1}{1+\left(\frac{2+\cos(2\pi t)}{2+\sin(2\pi t)}\right)^2} dt \\
&= \arctan\left(\frac{2+\cos(2\pi t)}{2+\sin(2\pi t)} \right) \Big|_0^1 \\
&= 0
\end{aligned} \tag{0.67}$$

For the second example we find

$$\begin{aligned}
\int_C \omega &= \int_0^1 \int_0^1 C^\star \omega \\
&= \int_0^1 \int_0^1 \frac{1}{(C^x)^2 + (C^y)^2} \left(\frac{\partial C^x}{\partial t_1}\frac{\partial C^y}{\partial t_2} - \frac{\partial C^y}{\partial t_1}\frac{\partial C^x}{\partial t_2} \right) dt_1 dt_2 \\
&= 2\pi \int_0^1 \int_0^1 \frac{1}{(1+t_1)^2} \left(\cos^2(2\pi t_2) + \sin^2(2\pi t_2) \right) dt_1 dt_2 \\
&= 2\pi \int_0^1 \frac{dt}{(1+t)^2} \\
&= -2\pi \frac{1}{1+t} \Big|_0^1 \\
&= \pi
\end{aligned} \tag{0.68}$$

Problem: Consider the manifold $\mathbb{R}^2 - \{(0,0)\}$ and the 1-form

$$\omega = \frac{ydx}{x^2+y^2} - \frac{xdy}{x^2+y^2} \tag{0.69}$$

Show that this is closed. Is it exact?

Solution: To show that it is closed we calculate

$$\begin{aligned}
d\omega &= \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2} \right) dy \wedge dx - \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} \right) dx \wedge dy \\
&= \left(\frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} \right) dy \wedge dx - \left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} \right) dx \wedge dy
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{x^2-y^2}{(x^2+y^2)^2}\right) dy\wedge dx - \left(\frac{y^2-x^2}{(x^2+y^2)^2}\right) dx\wedge dy \\
&= 0
\end{aligned} \tag{0.70}
$$

We saw in the lectures that

$$\int_C \omega = -1 \tag{0.71}$$

where $C(t) = (\cot(2\pi t), \sin(2\pi t))$. Note that $\partial C = C(2\pi) - C(0) = 0$ hence if $\omega = d\eta$ for some η then

$$\int_C \omega = \int_C d\eta = \int_{\partial C} \eta = 0 \tag{0.72}$$

this is a contradiction hence η can't exist, *i.e.* ω is not exact.

Problem: Show that if

$$X = \sum_\mu X^\mu \frac{\partial}{\partial x^\mu}\Big|_p \tag{0.73}$$

is a vector field expanded in some local coordinate chart $\phi_i(p) = (x^1(p), ..., x^n(p))$ and we define

$$dX = \sum_\mu \partial_\nu X^\mu dx^\nu\Big|_p \otimes \frac{\partial}{\partial x^\mu}\Big|_p \tag{0.74}$$

then dX will not be a tensor. However check that if $\omega = \sum \omega_\mu dx^\mu|_p$ is a co-vector (1-form) then

$$d\omega = \sum_\mu (\partial_\nu \omega_\mu - \partial_\mu \omega_\nu) dx^\nu|_p \otimes dx^\mu|_p \tag{0.75}$$

is a tensor. Hint: consider how things look in a different coordinate chart $\psi(p) = (y^1, ..., y^n)$.

Solution: Consider another coordinate system $\psi(p) = (y^1, ..., y^n)$. In this coordinate system we have

$$X = \sum X'^\nu \frac{\partial}{\partial y^\nu}\ , \qquad X'^\nu = \sum \frac{\partial y^\nu}{\partial x^\rho} X^\rho \tag{0.76}$$

so that

$$
\begin{aligned}
\frac{\partial}{\partial y^\nu} X'^\mu &= \sum \frac{\partial x^\lambda}{\partial y^\nu} \frac{\partial}{\partial x^\lambda}\left(\frac{\partial y^\mu}{\partial x^\rho} X^\rho\right) \\
&= \sum \frac{\partial x^\lambda}{\partial y^\nu}\frac{\partial y^\mu}{\partial x^\rho}\partial_\lambda X^\rho + \sum \frac{\partial x^\lambda}{\partial y^\nu}\frac{\partial^2 y^\nu}{\partial x^\rho \partial x^\lambda} X^\rho
\end{aligned} \tag{0.77}
$$

The first term is okay, *i.e.* it obeys the tensorial transformation rule. But the second term is wrong. Thus it is not a tensor.

For a 1-form one we have

$$\omega = \sum \omega'_\mu dy^\mu\ , \qquad \omega'_\mu = \sum \frac{\partial x^\rho}{\partial y^\mu}\omega_\rho \tag{0.78}$$

the same steps give

$$\frac{\partial}{\partial y^\nu}\omega'_\mu = \sum \frac{\partial x^\lambda}{\partial y^\nu}\frac{\partial x^\rho}{\partial y^\mu}\partial_\lambda \omega_\rho + \sum \frac{\partial^2 x^\rho}{\partial y^\mu \partial y^\nu}\omega_\rho \tag{0.79}$$

Thus again this is not a tensor on behalf of the second term. However the second term is manifestly symmetric in the indices μ and ν. Hence if we construct the exterior derivative, whose compondents are,

$$\partial_\mu \omega_\nu - \partial_\nu \omega_\mu \tag{0.80}$$

then this will transform as a tensor.

Problem: Prove that in a local coordinate system the curvature tensor is

$$\begin{aligned} R &= \sum R_{\mu\nu\lambda}{}^\rho dx^\mu\Big|_p \otimes dx^\nu\Big|_p \otimes dx^\lambda\Big|_p \otimes \frac{\partial}{\partial x^\rho}\Big|_p \\ &= \sum \left(-\partial_\mu \Gamma^\rho_{\nu\lambda} + \partial_\nu \Gamma^\rho_{\mu\lambda} + \Gamma^\sigma_{\mu\lambda}\Gamma^\rho_{\nu\sigma} + \Gamma^\sigma_{\nu\lambda}\Gamma^\rho_{\mu\sigma}\right) dx^\mu\Big|_p \otimes dx^\nu\Big|_p \otimes dx^\lambda\Big|_p \otimes \frac{\partial}{\partial x^\rho}\Big|_p \end{aligned} \tag{0.81}$$

Solution: To evaluate the components of R in a coordinate chart we can simply evaluate

$$\begin{aligned} R_{\mu\nu\rho}{}^\lambda &= R\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial x^\rho}, dx^\lambda\right) \\ &= \left(-D_\mu D_\nu\left(\frac{\partial}{\partial x^\rho}\right) + D_\nu D_\mu\left(\frac{\partial}{\partial x^\rho}\right)\right)(dx^\lambda) \\ &= \sum\left(-D_\mu\left(\Gamma^\sigma_{\nu\rho}\left(\frac{\partial}{\partial x^\sigma}\right)\right) + D_\nu\left(\Gamma^\sigma_{\mu\rho}\left(\frac{\partial}{\partial x^\sigma}\right)\right)\right)(dx^\lambda) \\ &= \sum\left(-\partial_\mu\Gamma^\sigma_{\nu\rho}\frac{\partial}{\partial x^\sigma} - \Gamma^\sigma_{\nu\rho}\Gamma^\tau_{\mu\sigma}\frac{\partial}{\partial x^\tau} + \partial_\nu\Gamma^\sigma_{\mu\rho}\frac{\partial}{\partial x^\sigma} + \Gamma^\sigma_{\mu\rho}\Gamma^\tau_{\nu\sigma}\frac{\partial}{\partial x^\tau}\right)(dx^\lambda) \end{aligned} \tag{0.82}$$

from here we can read off the desired formula.

Problem: Convince yourself that in a local coordinate system

$$\begin{aligned} D_\lambda T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} &= \partial_\mu T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} \\ &\quad + \sum \Gamma^{\mu_i}_{\lambda\rho} T^{\mu_1\ldots\rho\ldots\mu_r}{}_{\nu_1\ldots\nu_s} \\ &\quad - \sum \Gamma^\rho_{\lambda\nu_i} T^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\rho\ldots\nu_s} \end{aligned} \tag{0.83}$$

Solution: We start with the definition

$$\begin{aligned} D_X T(\omega^1,...,\omega^r,Y_1,...,Y_s) &= X(T(\omega^1,...,\omega^r,Y_1,...,Y_s)) \\ &\quad -\sum_i T(\omega^1,...,D_X\omega^i,...,\omega^r,Y_1,...,Y_s) \\ &\quad -\sum_i T(\omega^1,...,\omega^r,Y_1,...,D_X Y_i,...,Y_s) \end{aligned} \tag{0.84}$$

Just as with the one-form case in the notes we find that, in a coordinate system, with

$$\begin{aligned} X = \frac{\partial}{\partial x^\lambda}\ , \qquad Y^i &= \frac{\partial}{\partial x^{\nu_i}}\ , \qquad \omega_j = dx^{\mu_j} \\ T^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_r} &= T\left(dx^{\mu_1},...,dx^{\mu_r},\frac{\partial}{\partial x^{\nu_1}},...,\frac{\partial}{\partial x^{\nu_s}}\right) \end{aligned} \tag{0.85}$$

so that

$$D_X Y^i = \sum \Gamma^\rho_{\lambda\nu_i}\frac{\partial}{\partial x^\rho}\ , \qquad D_X\omega_j = -\sum \Gamma^{\mu_j}_{\lambda\rho}dx^\rho \tag{0.86}$$

the second expression follows from the definition of $D_X\omega$ given in the notes. Evaluating the definition using these expressions gives

$$\begin{aligned} D_\lambda T^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s} &= \partial_\mu T^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\nu_s} \\ &\quad +\sum_i \Gamma^{\mu_i}_{\lambda\rho}T^{\mu_1\ldots\rho\ldots\mu_r}{}_{\nu_1\ldots\nu_r} \\ &\quad -\sum_i \Gamma^\rho_{\lambda\nu_i}T^{\mu_1\cdots\mu_r}{}_{\nu_1\ldots\rho\ldots\nu_r} \end{aligned} \tag{0.87}$$

Problem: Consider $\mathbb{R}^2$ and cover it (or more precisely all but the origin) with polar coordinates show that the Euclidean metric is

$$g = dr\otimes dr + r^2 d\theta\otimes d\theta \tag{0.88}$$

Solution: We start with the Cartesian metric in 2-dimensions

$$g = dx\otimes dx + dy\otimes dy = \sum \delta_{\mu\nu}dx^\mu\otimes dx^\nu \tag{0.89}$$

To construct the metric in polar coordinates we note that, on $\mathbb{R}^2 - \{0,0\}$,

$$x = r\cos\theta\ , \qquad y = r\sin\theta \tag{0.90}$$

are the transition functions $y^\mu(x^\nu)$. In the new coordinates

$$g'_{\mu\nu} = \sum \frac{\partial x^\lambda}{\partial y^\mu}\frac{\partial x^\rho}{\partial y^\nu}\delta_{\lambda\rho} \tag{0.91}$$

where the y's are the polar coordinates. Thus we find

$$\begin{aligned} g'_{rr} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial r} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial r} = \cos^2\theta + \sin^2\theta = 1 \\ g'_{r\theta} &= \frac{\partial x}{\partial r}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial r}\frac{\partial y}{\partial \theta} = -r\cos\theta\sin\theta + r\sin\theta\cos\theta = 0 \\ g'_{\theta\theta} &= \frac{\partial x}{\partial \theta}\frac{\partial x}{\partial \theta} + \frac{\partial y}{\partial \theta}\frac{\partial y}{\partial \theta} = r^2\sin^2\theta + r^2\cos^2\theta = r^2 \end{aligned} \tag{0.92}$$

and hence

$$g' = \sum g'_{\mu\nu} dy^\mu \otimes dy^\nu = dr \otimes dr + r^2 d\theta \otimes d\theta \tag{0.93}$$

Problem: Show that in a local coordinate system $(x^1, ..., x^n)$ the coefficients of the Levi-Civita connection are

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{0.94}$$

Solution: From the lectures we saw that

$$\begin{aligned} 2g(D_Z Y, X) &= Z(g(X,Y)) + Y(g(Z,X)) - X(g(Y,Z)) \\ &\quad -g([Y,Z],X) - g([Z,X],Y) + g([X,Y],Z) \end{aligned} \tag{0.95}$$

So let us choose our usual coordinate system and

$$X = \frac{\partial}{\partial x^\mu}\ , \qquad Y = \frac{\partial}{\partial x^\nu}\ , \qquad Z = \frac{\partial}{\partial x^\rho}\ , \qquad \text{and} \qquad g = \sum g_{\mu\nu} dx^\mu \otimes dx^\nu \tag{0.96}$$

The above equation becomes

$$\begin{aligned} 2g\left(\Gamma^\lambda_{\rho\nu}\frac{\partial}{\partial x^\lambda}, \frac{\partial}{\partial x^\mu}\right) = \frac{\partial}{\partial x^\rho}\left(g\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right)\right) &+ \frac{\partial}{\partial x^\nu}\left(g\left(\frac{\partial}{\partial x^\rho}, \frac{\partial}{\partial x^\mu}\right)\right) \\ &- \frac{\partial}{\partial x^\mu}\left(g\left(\frac{\partial}{\partial x^\nu}, \frac{\partial}{\partial x^\rho}\right)\right) \end{aligned} \tag{0.97}$$

so this becomes

$$2g_{\lambda\mu}\Gamma^\lambda_{\rho\nu} = \partial_\rho g_{\mu\nu} + \partial_\nu g_{\rho\mu} - \partial_\mu g_{\nu\rho} \tag{0.98}$$

or equivalently

$$\Gamma^\lambda_{\rho\nu} = \frac{1}{2}g^{\lambda\mu}(\partial_\rho g_{\mu\nu} + \partial_\nu g_{\mu\rho} - \partial_\mu g_{\nu\rho}) \tag{0.99}$$